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## Adventures in (thermal) Wonderland

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INSTITUT  
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NNT : 2024IPPAX065

Thèse de doctorat



ΑΡΙΣΤΟΤΕΛΕΙΟ  
ΠΑΝΕΠΙΣΤΗΜΙΟ  
ΘΕΣΣΑΛΟΝΙΚΗΣ



# Adventures in (thermal) Wonderland

Aspects of Carrollian physics, asymptotically flat spacetimes and thermal field theory

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée au Centre de Physique Théorique de l'École polytechnique et au  
Laboratoire de Physique Théorique de l'Université Aristote de Thessalonique

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Thèse présentée et soutenue à Thessalonique, le 23 septembre 2024, par

**MATTHIEU BERNARD JEAN VILATTE**

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# Adventures in (thermal) Wonderland

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Matthieu Valatte

3rd October 2024



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À mes parents et ma petite sœur,



# Abstract

The work we present in this thesis is structured around the concepts of field theories and geometry, which are applied to gravity and thermalisation.

On the gravity side, our work aims at shedding new light on the asymptotic structure of the gravitational field in the context of asymptotically flat spacetimes, using information encoded on the conformal boundary. The latter is a null hypersurface on which Carrollian physics instead of relativistic physics is at work. A Carroll structure on a manifold is a degenerate metric and a vector field spanning the kernel of the latter. This vector selects a particular direction which can be the starting point for describing Carroll structures in a split frame. We first elaborate on the geometry one can construct on such a manifold in this frame, including a comprehensive study of connections and (conformal isometries). Effective actions can be defined on a Carrollian background. Canonical momenta conjugate to the geometry or the connection are introduced, and the variation of the action shall give their conservation equations, upon which isometric charges can be reached.

Carrollian physics is also known to emerge as the vanishing speed of light of relativistic physics. This limit usually exhibits more Carrollian descendants than what might be expected from a naive intrinsic analysis, as shown in the explicit examples of Carrollian fluids, Carrollian scalar fields (for which two actions, electric and magnetic arise in the limit) and the Carrollian Chern-Simons action. The richness of the limiting procedure is due to this versatility in describing a palette of degrees of freedom. This turns out to be an awesome tool in studying the relationship between asymptotically anti de Sitter (AdS) and flat spacetimes.

Metrics on asymptotically flat spacetimes can be expressed as an infinite expansion in a gauge, covariant with respect to their null boundaries. This slight extension of the Newman-Unti gauge is shown to be valid also in AdS, which allows to take the flat limit in the bulk i.e. the Carrollian limit on the boundary, while preserving this covariance feature. We demonstrate that the infinite solution space of Ricci-flat spacetimes actually arises from the Laurent expansion of the AdS boundary energy-momentum tensor. These replicas obey at each order Carrollian dynamics (flux/balance laws). Focusing our attention to Petrov algebraically special spacetimes (for which the infinite expansion resums), we use the Carrollian flux/balance laws together with the conservation of the energy-momentum and Cotton tensors to build two dual towers of bulk charges from a purely boundary perspective. Among them we recover the mass and angular momentum multipolar moments for the Kerr-Taub-NUT family. The covariant gauge is also the appropriate framework to

unveil the action of hidden symmetries of gravity on the null boundary. In this thesis we study exhaustively the case of Ehlers'  $SL(2, \mathbb{R})$  symmetry.

On the side of thermal field theory we see that while at infinite temperature a CFT is described by its spectrum and the OPE coefficients, additional data is needed in the thermal case. These are the average values of primary operators, completely determined up to a constant coefficient. Numerical simulations, duality with black-hole states in AdS or spectral analyses are the methods usually employed to uncover the latter. Our work features a new breadth. Starting from two coupled harmonic oscillators, we show that they are related to conformal ladder graphs of fishnet theories. This observation is the first step for setting a new correspondence between thermal partition functions and graphs.

# Résumé en Français

Le travail que nous présentons dans cette thèse est structuré autour de la notion de théorie des champs et de géométrie, qui sont appliquées à la gravité et la thermalisation.

En gravité, notre travail donne un éclairage nouveau sur la structure asymptotique du champ gravitationnel dans le contexte des espace-temps asymptotiquement plats, ceci en utilisant l'information codée sur leur bord conforme. Ce dernier est une hypersurface de genre lumière sur laquelle émerge la physique carrollienne au lieu de la physique relativiste. Une structure carrollienne sur une variété est constituée une métrique dégénérée et un champ de vecteurs couvrant le noyau de cette dernière. Ce vecteur sélectionne une direction particulière qui peut être le point de départ de la description des structures carrolliennes dans un cadre séparé. Nous développons d'abord la géométrie carrollienne, y compris une étude complète des connexions et isométries (conformes). Des actions effectives peuvent vivre sur un arrière-plan carrollien. Les moments canoniques conjugués à la géométrie ou à la connexion peuvent être définis, et la variation de l'action donnera leurs équations de conservation, à partir desquelles les charges isométriques peuvent être bâties.

La physique carrollienne émerge également lorsque la vitesse de la lumière tend vers zéro. Cette limite donne généralement plus de descendants carrolliens que ce qui est attendu après une analyse intrinsèque, comme le montrent les exemples explicites des fluides carrolliens, des champs scalaires carrolliens (pour lesquels deux actions, électrique et magnétique, apparaissent dans la limite) et du tenseur de Cotton carrollien. La richesse de la limite est due à sa possibilité de décrire plus de degrés de liberté, ce qui s'avère être un outil fondamental dans l'étude de la relation entre les espace-temps asymptotiquement anti de Sitter et plats.

Les espace-temps asymptotiquement plats peuvent être écrits comme une expansion infinie dans une jauge covariante par rapport à leur bord nul. Cette légère extension de la jauge de Newman-Unti est également valable dans anti-de Sitter, ce qui permet de prendre la limite plate dans le bulk, équivalente à la limite carrollienne sur le bord. Nous démontrons que l'espace des solutions infini des espace-temps Ricci-plats provient en fait du développement en série de Laurent du tenseur énergie-impulsion d'AdS. Ces répliques obéissent à chaque ordre une dynamique carrollienne (lois de flux). Dans le cadre des espaces algébriquement spéciaux de Petrov (pour lesquels le développement infini se resomme), nous utilisons les lois de flux carrolliennes ainsi que la conservation des tenseurs énergie-impulsion et de Cotton pour construire, du point de vue du bord, deux tours duales de charges du bulk. Parmi elles, nous retrouvons l'expansion multipolaire de la masse et du

moment angulaire pour la famille Kerr-Taub-NUT. La jauge covariante est également le cadre approprié pour dévoiler l'action des symétries cachées de la gravité sur le bord nul. Dans ce travail, nous étudions le cas de la symétrie  $SL(2, \mathbb{R})$  d'Ehlers.

Du côté de la théorie thermique des champs, nous travaillons sur l'ensemble minimal de données nécessaires pour les décrire à température finie. Alors qu'à température infinie toutes les valeurs moyennes des opérateurs primaires s'annulent, leurs valeurs non nulle dans le cas thermique constituent les données supplémentaires qu'il faut calculer pour caractériser la théorie. Les simulations numériques, la dualité avec un trou noir dans AdS ou une analyse spectrale sont généralement les méthodes employées pour trouver la valeur de ces coefficients. Notre travail propose une nouvelle approche à ce problème en montrant, à partir de deux oscillateurs harmoniques couplés, que ces coefficients sont en fait liés à des graphes conformes de théories de type «fishnet». A partir de cette observation, nous avons établi une correspondance entre les fonctions de partition thermique et ces graphes.

## Περίληψη στα ελληνικά

Η εργασία που παρουσιάζω σε αυτή τη διατριβή αφορά μελέτες πάνω στην βαρύτητα τύπου *Carroll* και τις σύμμορφες θεωρίες πεδίου σε μη μηδενική θερμοκρασία.

Από την πλευρά της βαρύτητας, η εργασία μου στοχεύει στο να ρίξει νέο φως στην ασυμπτωτική δομή του βαρυτικού πεδίου σε ασυμπτωτικά επίπεδους χωρόχρονους, χρησιμοποιώντας πληροφορίες που κωδικοποιούνται στο σύμμορφο όριο του. Το τελευταίο όριο είναι μια μηδενική υπερεπιφάνεια στην οποία λειτουργεί η φυσική τύπου *Carroll* αντί της σχετικιστικής φυσικής. Η γεωμετρική *Carroll* σε μια πολλαπλότητα βασίζεται σε μια εκφυλισμένη μετρική και ένα διανυσματικό πεδίο που εκτείνεται στον πυρήνα της τελευταίας. Το διανυσματικό πεδίο αυτό επιλέγει μια συγκεκριμένη κατεύθυνση η οποία μπορεί να αποτελέσει το σημείο εκκίνησης για την περιγραφή δομών *Carroll*. Μελετήσαμε αρχικά τη γεωμετρία που μπορεί κανείς να κατασκευάσει σε μια τέτοια πολλαπλότητα συμπεριλαμβανομένης μιας ολοκληρωμένης μελέτης των συνοχών και των σύμμορφων ισομετριών τους. Ενεργές δράσεις τύπου *Carroll* διαφόρων πεδίων μπορούν να οριστούν σε αυτό το υπόβαθρο. Εισάγονται κανονικές ορμές συζυγείς με τη γεωμετρία ή την συνοχή, και η μεταβολή της δράσης θα δώσει τις εξισώσεις διατήρησής τους και έτσι να μελετηθούν διατηρήσιμα φορτία που συνδέονται με ισομετρίες.

Η φυσική τύπου *Carroll* προκύπτει στο όριο όπου η ταχύτητα του φωτός μηδενίζεται. Στο όριο αυτό εμφανίζονται συνήθως περισσότερες φυσικές καταστάσεις από ότι θα ανέμενε κανείς. Αυτό φαίνεται στα παραδείγματα των ρευστών τύπου *Carroll*, των βαθμωτών πεδίων τύπου *Carroll* (για τα οποία προκύπτουν δύο δράσεις, η ηλεκτρική και η μαγνητική), και της δράσης *Chern – Simons* τύπου *Carroll*, το όριο *Carroll* εμπεριέχει μια ενδιαφέρουσα ευελιξία στην περιγραφή διαφορετικών βαθμών ελευθερίας. Αυτό αποδεικνύεται ένα σημαντικό εργαλείο για τη μελέτη της σχέσης μεταξύ ασυμπτωτικά *Anti – De Sitter* και επίπεδων χωροχρόνων.

Οι μετρικές σε ασυμπτωτικά επίπεδους χωροχρόνους μπορούν να εκφραστούν σε μια βαθμίδα που είναι συναλλοίωτη σε σχέση με το φωτοειδές όριο τους. Αυτή η βαθμίδα είναι μια μικρή επέκταση της γνωστής βαθμίδας *Newman – Unti* και αποδεικνύεται ότι ισχύει και στον χώρο *AdS*. Η βαθμίδα αυτή μας επιτρέπει να πάρουμε το επίπεδο όριο του χωρόχρονου - το οποίο αντιστοιχεί στο όριο τύπου *Carroll* στο όριο του - με συναλλοίωτο τρόπο. Δείνουμε ότι ο άπειρος χώρος λύσεων των επίπεδων χωροχρόνων *Ricci* προκύπτει στην πραγματικότητα από το ανάπτυγμα *Laurent* του ταυνοστή ενέργειας-ορμής του ορίου *anti – de Sitter*. Οι λύσεις αυτές, σε κάθε τάξη, υπακούουν σε δυναμική τύπου *Carroll* και συγκεκριμένα σε νόμους ροής/ισσοροπίας. Επικεντρώνοντας την προσοχή μας στους αλγεβρικά ειδικούς χωροχρόνους *Petrou* για τους οποίους το ανάπτυγμα τύπου *Laurent* μπορεί να αθροισθεί, χρησιμοποιούμε τους νόμους τύπου *Carroll*



ροής/ισορροπίας μαζί με τη διατήρηση των τανυστών ενέργειας-ορμής και τον τανυστή *Cotton* για να κατασκευάσουμε από το σύνορο του χωρόχρονου δύο άπειρες σειρές διατηρήσιμων φορτίων της μετρικής. Ανάμεσά τους συναντούμε τις πολυπολικές ροπές μάζας και στροφορμής για την οικογένεια μετρικών *Kerr – Taub – NUT*. Επιπλέον, η παραπάνω συναλλοίωτη βαθμίδα αποδεικνύεται ότι είναι το κατάλληλο πλαίσιο για να αποκαλυφθούν κρυμμένες συμμετρίες της βαρύτητας στο όριο τύπου *Carroll*. Σε αυτή τη διατριβή και σε αυτό το πλαίσιο μελετάμε αναλυτικά την περίπτωση της συμμετρίας  $SL(2, \mathbb{R})$  του *Ehlers* και την εμφάνισή της στο όριο τύπου *Carroll*.

Σε ένα ξεχωριστό κομμάτι της διατριβής μου μελέτησα σύμμορφες θεωρίες πεδίου σε μη μηδενική θερμοκρασία. Ενώ σε μηδενική θερμοκρασία μια σύμμορφη θεωρία πεδίου περιγράφεται από το φάσμα των τελεστών της και από τους συντελεστές του τελεστικού αναπτύγματος *OPE*, στη θερμική περίπτωση απαιτούνται πρόσθετα δεδομένα. Αυτά είναι παράμετροι που αντιστοιχούν στις θερμικές μέσες τιμές των πρωτογενών σύμμορφων τελεστών της θεωρίας. Οι μέθοδοι που μέχρι τώρα χρησιμοποιούνται για την μελέτη των παραμέτρων αυτών είναι αριθμητικές προσομοιώσεις τύπου *bootstrap*, δυαδικότητα τύπου *AdS/CFT* με καταστάσεις φασματικές αναλύσεις. Η εργασία μου δίνει μια εντελώς νέα διάσταση στις παραπάνω μελέτες. Ξεκινώντας από δύο συζευγμένους αρμονικούς ταλαντωτές δείχνω ότι οι παραπάνω παράμετροι σχετίζονται με σύμμορφα διαγράμματα *Feynman* θεωριών τύπου *fishnet*. Πιστεύω ότι αυτή η παρατήρηση είναι το πρώτο βήμα για μια εντελώς απροσδόκητη νέα αντιστοιχία ανάμεσα σε θερμικές σύμμορφες συναρτήσεις και διαγραμμάτων *Feynman*.

# Nomenclature

- $(A, B, C, \dots)$  are  $d+2$ -spacetime indices labeling  $\{r, t, \mathbf{x}\}$  in Chapter 3 and are arbitrary frame indices in Appendix A,
- $(\mu, \nu, \dots)$  are  $d + 1$  spacetime indices labeling  $\{t, \mathbf{x}\}$ ,
- $(a, b, \dots)$  are Cartan's frame spatial indices in Chapter 3 and Appendix ?? and B.2. In these chapters the time index is denoted by 0,
- $(i, j, \dots)$  are spatial indices labeling  $\{\mathbf{x}\}$ ,
- $\mathbf{v}$  is the field of observers,  $\tau$  the clock form which contains the Ehresmann connection  $b_i$ ,
- Weak Carroll structures contains  $\mathbf{v}$  and a metric  $a_{ij}$ . Ruled weak Carroll structures contains an Ehresmann on top of that,
- $\bar{\nabla}$  denote an arbitrary connection,  $\hat{\nabla}$  denotes the Carrollian connection without spatial torsion. It is spatially Levi-Civita,
- Strong Carroll structures (resp. ruled) are weak (resp. ruled) structures equipped with a Carrollian connection,
- $\frac{1}{\Omega} \hat{D}_t$  is the Carroll-covariant temporal derivative. Its Weyl-extension is denoted  $\frac{1}{\Omega} \hat{\mathcal{D}}_t$ ,
- $\hat{\nabla}_i$  is the Carroll-covariant spatial derivative. Its Weyl-extension is denoted  $\hat{\mathcal{D}}_i$ ,
- $c$  is the velocity of light while  $k = \sqrt{-\frac{\Lambda}{3}}$  is the effective velocity of light on the timelike boundary of AdS (with  $\Lambda$  the cosmological constant),
- $\mathbf{u}$  is a timelike congruence normalized at  $-k^2$ ,
- Abbreviations: AdS = anti-de Sitter, FG = Fefferman-Graham, CNU = Covariant Newman-Unti, RP = Randers-Papapetrou, OPE = Operator Product Expansion.



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# Introduction

The most important and useful tool to handle the laws of Nature is symmetry, as it helps characterizing any physical system via the construction of charges (e.g. the electric charge) and constants of motion. Symmetries can be divided into two parts, those which act the same way at every point of spacetime, dubbed *global*, which are true symmetries of the theory; and those which act in a different way from one spacetime point to another, called *local* (or *gauge*) symmetries, which only exist due to the redundancy in the description of the degrees of freedom of the theory. After the seminal works of Lagrange and Hamilton, Noether derived her famous first and second theorems [1] stating the equivalence between symmetries and conserved charges. The first theorem is dealing with global symmetries and associate to them a charge living on a co-dimension one hypersurface while the second one relates gauge symmetries to charges defined on co-dimension two hypersurfaces, as the current one can build from the first theorem vanishes on-shell (see e.g. [2, 3] for reviews). The main example is electromagnetism, gauge theory based on the Abelian group  $U(1)$  whose charge, the electric charge, lives on two-dimensional spheres. Having the set of all symmetries and charges carried by a system is essential for its faithful description and is the first step towards its quantization. Indeed the canonical quantization procedure, due to Dirac, allows to translate the symplectic structure and the Poisson brackets generated by the classical charges into correlators of quantum operators.

A gauge theory which was at the center of the attention during the last few years is gravity. Mediated by a spin-2 gauge boson, the graviton, which propagates two polarizations in four spacetimes dimensions, its gauge group corresponds to the group of diffeomorphisms of coordinates used to label points in space-time. Interest in gravity grew at all physical scales. In the ultra-violet regime, therefore at microscopical scales, semi-classical approaches like the Hawking radiation [4] led to the so-called black hole information paradox [5] which states that after a black hole evaporation, information about the initial state would be permanently lost, which is in conflict with the unitarity of quantum mechanics. Such a statement calls for a quantization of General Relativity (or any theory of gravity). In the infrared regime, consequently at macroscopic scales, the study of gravitational waves dazzled a lot of attention [6–13], as the latter constitutes a way to probe astrophysical phenomena such as the merging of two supermassive black holes.

Investigating the infrared structure of gravity demands to consider the entire spacetime, and not just a subpart centered on a region of interest. A careful treatment of infinities, the boundary of the spacetime, is then necessary. The notion of boundary appears after a conformal compac-



tification [14] which brings infinities to a finite distance, at the price of working in an unphysical manifold, which however has the virtue of preserving the causal structure. Going back to symmetry principles, gravity being a gauge theory, its charges are encoded in co-dimension two hypersurfaces from the point of view of the interior of spacetime (dubbed from now on the bulk) so on co-dimension one hypersurfaces from the point of view of the boundary. Therefore, one is tempted to reinterpret the gauge symmetries of the bulk close to its boundary as global symmetries of the latter. This is, with the black hole entropy analysis [15–17] showing that it scales like the area of the horizon instead of the interior volume, one of the strongest mathematical suggestion that gravity is holographic. An other observation due to Brown and Henneaux [18] revealed that asymptotic symmetries of Anti-de Sitter spacetime in three dimensions, equipped with suitable boundary conditions, are made of two copies of the Virasoro algebra (typical feature of two dimensional Conformal Field Theory) with central charge  $c = 3l/2G$  (with  $l$  the AdS radius and  $G$  Newton’s constant), suggesting that gravity in AdS may be equivalently described by a two-dimensional CFT in which gravity plays no role. This paradigm is generally known under the name of “Holography”.

Holography can be henceforth thought of as a way to describe the gravitational degrees of freedom of the bulk in terms of non-gravitational boundary data. On the one hand, the zeroth step of such a duality corresponds to being able to reconstruct the full bulk gravitational metric from degrees of freedom defined on the boundary. On the other hand, the ultimate achievement would be a complete dictionary between two theories translating into the equivalence between the bulk and boundary quantum partition functions. This happens within the AdS/CFT correspondence of Maldacena [19] and Witten [20] where gravitational degrees of freedom in a given region of AdS can be equivalently described by a conformal quantum field theory without gravity living on its (conformal) timelike boundary, once Dirichlet boundary conditions have been imposed. Explicit realizations of the duality are Type IIB String Theory on  $AdS_5 \times S^5$  and  $\mathcal{N} = 4$  Super Yang-Mills with  $U(N)$  gauge group; or Vasiliev higher spin gravity [21, 22] and the  $O(N)$  vector model at large  $N$  [23, 24]. This AdS/CFT correspondence has been found to be very useful in areas such as thermal CFT (see e.g. [25]), condensed matter physics (see the Lecture Notes [26]) in addition to high energy physics. A spin-off of AdS/CFT is the so-called fluid/gravity correspondence [27–29] where classical AdS spacetimes are described by a boundary theory in the hydrodynamic regime. In this correspondence one misses the microscopic description of the dual theory, as the long wavelengths approximation is assumed, but re-expressing Einstein’s equations in terms of fluid conservation laws make them easier to handle. The bulk metric is then expressed in terms of fluid quantities (see [30] for explicit examples of bulk reconstruction).

The holographic paradigm constitutes a very prolific area of research and this thesis aims at developing some neighboring aspects, arising from natural questions still left opened. Our first angle of studies concerns the extension of the AdS/CFT duality, valid when the cosmological constant is negative, to the case where the latter vanishes. In this case the bulk is now an asymptotically flat spacetime i.e. a spacetime which goes back to Minkowski at infinity. They are of great interest as they constitute the toy model of an isolated gravitational system emitting radiation, *in fine* captured by an observer at astrophysical scales away from the source. Such an observer could be the

LIGO and VIRGO detectors, making this model a fundamental tool to handle gravitational wave physics. The structure of gravity away from its sources will then be of primordial importance. Several roads can be followed to unveil this structure, but the mostly used one is the gauge fixing procedure and asymptotic symmetries (see [31] for a pedagogical review). After conformal compactification a coordinate system adapted to the problem one wants to solve is selected. In the case of gravitational wave physics, one chooses a coordinate system which allows to follow null-rays, which in the conformally compactified manifold ends at a locus dubbed *null infinity*.<sup>1</sup> This is what Bondi, van der Burg and Metzner [32] and at the same time Sachs [33] studied in their seminal works which have led to two major advances. On the one hand, they showed that gravitational radiation is not an artefact of the linearized theory of gravity (which was the framework used by Einstein to demonstrate the existence of gravitational waves). On the other hand they revealed that contrary to the expectations, instead of Poincaré, the asymptotic symmetries<sup>2</sup> of asymptotically flat spacetimes near null infinity form an infinite dimensional enhancement of the latter, dubbed the *BMS group*. If the holographic principle can be extended to flat spacetimes, the symmetry analysis tells us that the dual theory living on the null boundary should be a BMS-invariant field theory (see e.g. [34]). This seminal analysis and the unexpected conclusions it drew signed the renaissance of classical general relativity, mainly in the context of asymptotically flat spacetimes. The geometric structure of the gravitational field close to infinities was unveiled in [35] where the first covariant definition of gravitational radiation was given. In a series of papers [36–39], the path towards what could be a quantized version of gravity was paved, as the radiative degrees of freedom (therefore the graviton itself) were clearly identified at null infinity as a part of the affine connection induced by the bulk on that codimension-one hypersurface. In addition, a unified treatment of null and spatial infinity (necessary as the latter cut the former into two parts, past and future null infinity) was proposed.

Many years later and not so long ago, the study of the infrared structure of gravity has undergone a revival thanks to the observation that asymptotic symmetries constitute one of a tripartite web of dualities, ubiquitous in gauge theories [40]. This “infrared triangle” is composed for the two other edges by memory effects [9, 11, 41–43] on the one hand, which are unalterable modifications of physical quantities (like the spacing between two inertial observers) due to the passage of gravitational radiation, and by soft theorems on the other hand. Soft theorems were first enunciated by Weinberg [44] and state that whenever a soft particle, i.e. a particle whose momenta tends to vanish, is inserted into a scattering, the structure of the S-matrix in inverse powers of the soft momenta when the latter goes to zero, is universal. Works on this infrared triangle include e.g. [45–48]. Eventually, the IR structure of gravity is much richer than expected and, in view of finding a flat analogue to the AdS/CFT duality, has to be exposed. This thesis aims therefore at presenting some aspects of the asymptotic structure of gravity, keeping holography in mind.

The path we follow in this work uses a fundamental result. The BMS algebra is actually isomorphic to the algebra of conformal isometries of a Carrollian manifold [49]. The latter is the conformal extension of the Carroll<sup>3</sup> algebra, which is the İnönü-Wigner contraction of Poincaré

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<sup>1</sup>Massive particles would end at *time-like infinity* while *space-like infinity* is the locus where all Cauchy surfaces ends.

<sup>2</sup>These are the symmetries of the metric once the latter approaches the boundary of the compactified manifold.

<sup>3</sup>Named in honour of Lewis Carroll’s *Through the Looking Glass* in which the Red Queen told Alice that “in her

when the velocity of light  $c$  vanishes [50, 51]. This unusual limit of the Poincaré group reveals new and somehow unexpected physical behaviors which are typical features of Wonderland, the Carrollian world. As the light cone shrinks on the time axis, any kind of motion and interaction seem to be prohibited, even if the particles have momentum. The relativity at work in the Carrollian world is completely different from the one of special relativity and is the exact dual of the one we can experience in our daily life i.e. Newtonian or Galilean relativity. In the latter, boosts do not affect time as time is taken to be absolute. Conversely, Carrollian relativity does not allow for boosts affecting space, giving then an absolute status to space. However amusing this new physics seemed and even though it was long thought of as just a mathematical curiosity, interesting applications to systems undergoing Carroll symmetry were found. Some of them will be presented in the main content of this manuscript but let us mention in particular that Carrollian physics, in the hydrodynamic regime, was shown to be applicable to the description of Ricci-flat spacetimes belonging to the algebraically special class [52]. This flat extension of the fluid/gravity duality demonstrated that for such spacetimes, Einstein dynamics in the bulk can be completely recast into the conservation laws of a dual Carrollian fluid [53]. This was one of the major advances in the realm of flat bulk reconstruction from a null boundary perspective. Such a reconstruction can actually be extended for more general spacetimes, once a suitable gauge is chosen, as we shall see in this thesis. Eventually, it gives hope in the existence of a flat version of AdS holography, perhaps reachable from a limit of the latter.

The equivalence between asymptotic symmetries near null infinity and the conformal Carroll algebra is actually not a surprise. Future null infinity, denoted  $\mathcal{I}^+$ , which is the conformal boundary of asymptotically flat spacetimes, has the topology  $\mathbb{R} \times S^d$  (in  $(d + 2)$ -bulk dimensions).<sup>4</sup> Most importantly it is naturally equipped with a vector and a degenerate metric whose kernel is spanned by the former. The non-degenerate part of the metric is  $d$ -dimensional and describe the  $d$ -spheres. This is a geometrical structure which is known under the name of Carroll structures. First described in [54] and much later in e.g. [55, 56], these structures are ubiquitous in physics as they describe any manifold whose metric is degenerate. Hence, apart from null infinity, any embedded null hypersurface, like the horizon of a black hole [57], can be described by Carrollian geometry and is the setting for the associated physics. It turns out that the conformal isometries of a flat conformal Carroll structure reproduce exactly the BMS group, explaining *a posteriori* the result of Bondi, Sachs and collaborators. Carrollian physics is therefore set to play a prominent role in the quest for “Flat holography”. The main question we would like to answer in this work is: what perspective on the bulk dynamics can the Carrollian physics living on the null boundary give us?

Going back to AdS, the holographic paradigm, beyond its microscopic aspects, provides a way to handle the process of thermalization. More precisely, a black hole state in AdS can be described by a boundary conformal field theory at finite temperature (see e.g. [25]), the latter being the Hawking temperature of the black hole. Both in view of a better understanding of the AdS/CFT correspondence and in the aim of anticipating a possible flat limit at finite temperature, which would lead

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country it take all the running one can do to stay at the same place". In other words, nothing moves in the Carrollian world, as the maximal allowed velocity, namely the one of light, tends to zero.

<sup>4</sup>More precisely any open subset of  $\mathcal{I}^+$  possesses this topology.

us towards the Thermal Wonderland, having a better control on thermal field theory constitute a direction worth unveiling. How to describe precisely a conformal field theory at finite temperature? This is the second topic of this thesis in which we present a work related to the description of thermalization from a pure field theoretical viewpoint. That is to say that we seek the minimal set of data needed to fully describe a thermal theory. Part of these data corresponds to thermal average of primary operators. We observed that the latter are actually related to a certain class of conformal ladder graphs, and established a correspondence between these two *a priori* distinct realms.

The two directions detailed in this thesis emanate from holography. They both belong to a wider and ambitious plan which aims at finding the flat counterpart of the AdS gauge/gravity duality. Recall that this work does not explicitly deals with holography, but more with geometry, bulk reconstruction and field theory. Holography in AdS and its putative flat extension serve as binders between the inspected themes.

## Outline of the manuscript and summary of achievements

This thesis is divided into three parts.

Part I is devoted to the presentation of Carrollian geometry and Carrollian field theories from two distinct albeit complementary approaches. In Chapter 1 we present the notion of Carroll structures, Carroll connections, Carroll isometries and Carrollian conservation laws in a formalism which takes advantage of the null direction generated by the kernel of the metric. This *split formalism* is complementary to the covariant formalism of [54]. This first Chapter is a comprehensive presentation of the subject and it aims at being as concise as possible. As the Carroll group emerges from the  $c \rightarrow 0$  limit of the Poincaré group, Carrollian theories can be obtained from a limit of relativistic theories. In Chapter 2 we present how the  $c \rightarrow 0$  limit is performed and highlight its power with three examples: Carrollian hydrodynamics, the Carrollian scalar field and the Carrollian Cotton tensor, which is a geometric tensor built upon third derivatives of the metric.

Part II is dedicated to applications of Carrollian physics in the context of gravity in asymptotically flat spacetimes. Even though many important results were derived in the Bondi gauge (or related gauges like Newman-Unti [58] or partial Bondi [59]), it does not decompose the bulk metric into boundary tensors; preventing any attempts at giving a boundary perspective on the bulk dynamics. As our objective is to look for what the Carrollian boundary can tell us about gravity in bulk, we first present in Chapter 3 a new gauge, which is a covariant version of Newman-Unti, which expresses the quantities required to construct the bulk metric as tensors living on  $\mathcal{I}^+$ . A method to get a Ricci-flat spacetimes from AdS is also provided; it provides in particular an explanation for the origin of the infinite set of functions composing the flat solution space, rooted in the AdS boundary energy-momentum tensor. Armed with the general expression of asymptotically flat spacetimes in Covariant Newman-Unti gauge we present two situations in which the Carrollian boundary can give us a new perspective on bulk dynamics. In Chapter 4 we study the relationship between bulk and boundary isometries and show how bulk charges can emerge from

a pure boundary analysis. In Chapter 5 we leave asymptotic symmetries aside and consider another type of symmetries gravity can encompass: hidden symmetries. A wide range of such symmetries, mainly appearing through dimensional reduction, were found. In this work, we focus on one of them, namely Ehlers  $SL(2, \mathbb{R})$  symmetry [60], and show how it acts on the Carrollian boundary data. We will show that, even though the symmetry acts non-locally in the bulk, on the boundary everything becomes local, therefore much simpler to handle. The latter action will also reveals an interesting electric-magnetic duality, through an interplay between the Bondi mass and the NUT charge.

Part III presents in details the open question of thermal averages in thermal field theories and exhibits how the correspondence between thermal partition functions and conformal ladder graphs can help. It is a self-contained part which can be read independently from the rest of the manuscript.

In a Conclusion we summarize our findings and discuss several open directions to consider in this quest of finding the putative flat analogue of the gauge/gravity duality, and we discuss steps to pave the way through it.

## List of publications

During the completion of this doctoral thesis, the following articles have been published

- M. Karydas, S. Li, A.C. Petkou and M. Vilatte, *Conformal Graphs as Twisted Partition Functions*, Phys. Rev. Lett. **132** (2024) 23, [arXiv:2312.00135](#).
- A. Campoleoni, A. Delfante, S. Pekar, P.M. Petropoulos, D. Rivera-Betancour and M. Vilatte, *Flat from AdS*, JHEP **12** (2023) 078, [arXiv:2309.15182](#).
- N. Mittal, P.M. Petropoulos, D. Rivera-Betancour and M. Vilatte, *Ehlers, Carroll, charges and dual charges*, JHEP **07** (2023) 065, [arXiv:2212.14062](#).
- D. Rivera-Betancour and M. Vilatte, *Revisiting the Carrollian scalar field*, Phys. Rev. D **106** (2022) 162, [arXiv:2207.01647](#).

The following works were in preparation when the manuscript was being written

- M. Karydas, S. Li, A.C. Petkou and M. Vilatte, *Thermal Field theories and conformal ladder graphs*.
- A. Fiorucci, S. Pekar, P.M. Petropoulos and M. Vilatte, *Galilean versus Carrollian connections*.

This manuscript is self-contained and deals mostly with the content of the published articles, which are appended at the end. Part of the unpublished results are displayed in Chapter 1 and Chapter 6.

# **Part I**

## **Carrollian geometry and Carrollian physics**



# Chapter 1

## Intrinsic Carrollian geometry in the split formalism

In differential geometry, the distance between two infinitesimally close points is measured with a covariant rank-two tensor dubbed the metric. When the latter is non-degenerate, it defines a scalar product between vectors as well as an isomorphism (usually called the *musical isomorphism*) between vectors and covectors. The manifold is said to be Riemannian. The metric, together with its inverse (which always exists due to the non-degeneracy) are used to lower and raise indices of components of tensors, once a basis is chosen. Once a metric is given, it is always possible to equip the tangent space with an affine connection, uniquely built upon it: this is the Levi-Civita connection. When working in a coordinate basis, the geometry of Riemannian manifold can be expressed in a diffeomorphism covariant way. The variation of an action with respect to the latter yields the covariant conservation of the energy-momentum tensor of the considered theory, while the boundary terms disclose the conserved current upon which a conserved charge can be reached. All these considerations are closely related to the underlying assumption that the metric is invertible. What happens if this is no longer the case i.e. if the metric possesses a degenerate direction? The geometry associated to such a metric is called a *Carroll structure*.

Carroll structures were originally introduced in [54] as manifolds endowed with a doublet  $(q_{\mu\nu}, n^\mu)$  made of a metric  $q_{\mu\nu}$  whose kernel is spanned by the nowhere vanishing vector  $n^\mu$  i.e.  $q_{\mu\nu}n^\nu = 0$ , everything being written in a covariant way under general diffeomorphisms. However due to the degeneracy of the metric, one cannot freely raise and lower indices like in a plain relativistic set-up; the index at hand has to be transverse with respect to  $n^\mu$ . This makes the manipulation of quantities and the contractions between tensors harder, in addition to the fact that such a way of writing tensors does not take advantage of the natural splitting between the direction spanned by the kernel (which could reasonably be identified with the *Carrollian time*) and the others (the *spatial directions*).

More recently, in a series of papers [52, 53, 61–63] on Carrollian structures and Carrollian fluids and their use in flat holography, an alternative way of describing the Carrollian geometry has been



developed: the time/space split frame or *split formalism*. One coordinate is aligned with the degenerate direction while the others label points on the base space. In this Chapter we aim at reviewing this formalism and show its numerous advantages.

After describing the notion of Carroll structure in the split formalism (Sec. 1.1), we will deal with Carrollian connections. We shall give a comprehensive description of them (Sec. 1.2), from the most general one to the one that resembles the most the relativistic Levi-Civita connection, with emphasis on the ones widely used in the literature. In Sec. 1.3 we present how Carrollian dynamics emerges from requiring Carrollian invariance of an action for a theory defined on a Carroll structure. The way to encompass Weyl transformations (in view of applications to null infinity) is explained in Sec. 1.4 while Sec. 1.5 deals with the notion of isometries together with the one of currents and charges. For later purposes (see Chapter 3), it will be useful to describe the structure in a Cartan orthonormal frame. To this end Sec. B.1 aims then at reviewing Carroll structures in this framework. The second part of this work will deal with three dimensional Carroll structures, mostly described in holomorphic coordinates. We gather in Sec. 1.6 all the important geometric results. Our approach will be to work in coordinates; a coordinate-free description of Carroll structures can be found in the Appendix of [56] and in [49, 55, 64]. Apart from the study of Carrollian connection in Sec. 1.2 and the associated conservation equations (at the end of Sec. 1.3), the material presented in this Chapter mostly reviews the already cited literature and [65, 66].

This Chapter is complemented by several Appendices. Appendix A gathers in one place all useful formulae, the reader is advised to go back to it as much as needed. As the splitted formalism is not the most common one in the literature, we relate it with the covariant approach of [54] and latter e.g. [67–69] in Appendix C. Finally in Appendix D we say a few words about the structure dual to a Carrollian manifold: Newton-Cartan spacetimes.

## 1.1 Weak Carroll structures

A *weak Carroll structure* [49, 54, 63, 65] on a  $(d + 1)$ -dimensional manifold of the form  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  is a doublet made of a degenerate metric and a nowhere-vanishing vector field which spans the kernel of the latter. Such a spacetime can be seen as a one-dimensional fiber bundle over the Riemannian base space  $\mathcal{S}$ , which will be equipped with a genuine Riemannian, hence non-degenerate, metric. The fibers are identified with the (Carrollian) time direction.

One can always choose a coordinate system  $(t, \mathbf{x})$  where  $t$  is aligned with the fibers and  $\mathbf{x} = (x^i)$  for  $i = 1, \dots, d$  are local coordinates on  $\mathcal{S}$ . This allows to write the Carrollian metric as

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu = 0 \times dt^2 + a_{ij}(t, \mathbf{x}) dx^i dx^j \quad (1.1.1)$$

where  $a_{ij}$  is Riemannian of signature  $(+, \dots, +)$ . The kernel of  $q_{\mu\nu}$  is spanned by the *field of observers*

(also dubbed Carrollian vector field) which reads

$$\frac{1}{\Omega(t, \mathbf{x})} \partial_t := \mathbf{v} \quad (1.1.2)$$

where  $\Omega$  is finite and non-zero for  $\mathbf{v}$  to be nowhere-vanishing.

**Remark** Note that our definition of Carroll structure encompassing a nowhere-vanishing Carroll vector field fits with the historical definitions of [54] and [49], but is not the most general. Extended Carroll structures with vanishing Carroll vector field for a finite number of loci on the fibre are found to be relevant to define a notion of Carroll black holes [70] or to describe null infinity of asymptotically flat spacetime [71, 72].

This structure naturally introduces an *Ehresmann connection*, background gauge field  $\mathbf{b} = b_i(t, \mathbf{x}) dx^i$ , contained in the dual form of the Carrollian vector field. The latter is called the *clock form* and reads<sup>1</sup>

$$\boldsymbol{\tau} = \Omega dt - b_i dx^i, \quad (1.1.3)$$

such that  $\iota_{\mathbf{v}} \boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{v}) = 1$ , the plus sign being conventional.<sup>2</sup> Endowing a weak Carroll structure with an Ehresmann connection makes it a *ruled* Carroll structure, as the clock form defines the notion of ruler.

**Remark** From a geometrical viewpoint, the role of the clock form is to split the tangent space into a direct sum  $T\mathcal{M} = \text{Span}(\mathbf{v}) \oplus H$  where

$$H = \{X \in \Gamma(T\mathcal{M}) / \iota_X \boldsymbol{\tau} = 0\}. \quad (1.1.4)$$

The Ehresmann connection  $\mathbf{b}$  encodes all the obstruction to the unicity of this splitting. Vectors along  $\text{Span}(\mathbf{v})$  are dubbed *longitudinal* while vectors belonging to  $H$  are *transverse*. While this terminology makes a lot of sense in the covariant approach to Carroll structures, the use of the split frame renders it a bit anecdotic.

We can see that the duality relation  $\boldsymbol{\tau}(\mathbf{v}) = 1$  does not constrain the value of  $b_i$ . Hence another type of transformation, acting on the Ehresmann connection, is authorized. Denoting by  $\lambda$  its parameter, it acts like a shift  $\delta_\lambda b_i = \lambda_i$ , therefore  $\lambda$  has to be a transverse covector. This is often called a local Carroll boost in the literature but here we will prefer the term *shift symmetry*.

<sup>1</sup>Most of the literature, especially the one in covariant formalism refers to  $\boldsymbol{\tau}$  instead of  $\mathbf{b}$  when dealing with the Ehresmann connection. Note however that as the time leg is constrained by the duality condition,  $\mathbf{b}$  is the only arbitrary piece, hence the genuine degree of freedom.

<sup>2</sup>Note that this frame is not the most general, as one could have chosen  $\tilde{\mathbf{v}} = \frac{1}{\Omega}(\partial_t + w^i \partial_i)$  dual to  $\tilde{\boldsymbol{\tau}} = \Omega dt - b_i(dx^i - w^i dt)$ . Here  $\iota_X$  is the interior product with a vector  $\mathbf{v}$ . However, what will be said in the following does not depend on the expression of  $\mathbf{v}$  in a coordinate basis.

This formalism, which completely separates time from space, is motivated by the natural desire that objects should transform covariantly under the restricted class of general diffeomorphism that preserves this split. These are the so-called *Carrollian diffeomorphisms* and are of the form

$$t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}), \quad (1.1.5)$$

together with the Jacobians

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \quad (1.1.6)$$

Asking the weak Carroll structure  $(ds^2, \mathbf{v})$  to be invariant under (1.1.5) leads to the following transformation laws

$$\Omega' = \frac{\Omega}{J}, \quad a'_{ij} = (J^{-1})_i^k (J^{-1})_j^l a_{kl}, \quad b'_k = (J^{-1})_k^i \left( b_i + \frac{\Omega}{J} j_i \right), \quad (1.1.7)$$

i.e.  $\Omega$  is a scalar density for time-time diffeomorphisms,  $b_i$  a connection (as it should) and  $a_{ij}$  a rank-2 covariant tensor. The coordinate basis  $\{\partial_\mu\}$  (where  $\mu = t, i$ ) does not have a vector-like transformation law under (1.1.5)

$$\partial'_t = \frac{1}{J} \partial_t \quad \text{and} \quad \partial'_j = (J^{-1})_j^i \left( \partial_i - \frac{j_i}{J} \partial_t \right), \quad (1.1.8)$$

so we are tempted to introduce as basis and dual basis  $(A = 0, a)$

$$\mathcal{B} = \{\hat{e}_A\} = \{\mathbf{v}, \hat{\partial}_i\} \quad \text{and} \quad \mathcal{B}^* = \{\hat{\theta}^A\} = \{\boldsymbol{\tau}, dx^i\} \quad (1.1.9)$$

where  $\hat{\partial}_i := \partial_i + \frac{b_i}{\Omega} \partial_t$ . They transform as

$$\{\mathbf{v}, \hat{\partial}_i\} \rightarrow \left\{ \mathbf{v}, (J^{-1})_i^j \hat{\partial}_j \right\} \quad \text{and} \quad \{\boldsymbol{\tau}, dx^i\} \rightarrow \left\{ \boldsymbol{\tau}, J_j^i dx^j \right\} \quad (1.1.10)$$

under (1.1.5). From now on indices in the basis (1.1.9) will be denoted  $(\hat{t}, \hat{i})$ . Note that the metric  $a_{ij}$  is non-degenerate and thus can be used, along with its inverse  $a^{ij}$ , to raise and lower indices. In the bases at hand the Carrollian metric reads

$$ds^2 = -0 \times \boldsymbol{\tau}^2 + a_{ij} dx^i dx^j. \quad (1.1.11)$$

**Remark** One needs to be careful with the difference between indices  $(t, i)$  and  $(\hat{t}, \hat{i})$ . For example let us consider a vector  $\boldsymbol{\xi} = \xi^t \partial_t + \xi^i \partial_i$ . In the bases (1.1.9) its components read

$$\boldsymbol{\xi} := \xi^{\hat{t}} \mathbf{v} + \xi^{\hat{i}} \hat{\partial}_i = (\Omega \xi^t - \xi^i b_i) \mathbf{v} + \xi^i \hat{\partial}_i. \quad (1.1.12)$$

In this expression  $\xi^{\hat{t}}$  is a Carrollian scalar (contrary to  $\xi^t$  which was a scalar density) and  $\xi^{\hat{i}} = \xi^i$  the components of a vector. As the spatial components coincide in both frames, we

will have the tendency in the following to drop the hat on top of the spatial indices, as an abuse of notations. Let's now consider a form  $\mathbf{B} = B_t dt + B_i dx^i$ . In the basis (1.1.9) we get

$$\mathbf{B} := B_t \boldsymbol{\tau} + B_i dx^i = \frac{B_t}{\Omega} \boldsymbol{\tau} + \left( B_i + \frac{b_i}{\Omega} B_t \right) dx^i \quad (1.1.13)$$

with  $B_i$  and  $B_{\hat{i}}$  respectively Carrollian scalar and the components of a covector.

At the level of the shift symmetry the bases at hand transform as

$$\delta_\lambda \mathbf{v} = 0 \quad , \quad \delta_\lambda \hat{\partial}_i = \lambda_i \mathbf{v} \quad \delta_\lambda \boldsymbol{\tau} = -\lambda_i dx^i \quad \text{and} \quad \delta_\lambda dx^i = 0. \quad (1.1.14)$$

The form basis carries non-holonomy coefficients

$$d\boldsymbol{\tau} = \varphi_i dx^i \wedge \boldsymbol{\tau} - \hat{\omega}_{ij} dx^i \wedge dx^j \quad (1.1.15)$$

i.e. using (A.0.3)

$$\hat{C}_{ii}^{\hat{i}} = \varphi_i \quad \text{and} \quad \hat{C}_{ij}^{\hat{i}} = 2\hat{\omega}_{ij} \quad (1.1.16)$$

where

$$\varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega) \quad (1.1.17)$$

$$\hat{\omega}_{ij} = \partial_{[i} b_{j]} + b_{[i} \varphi_{j]} \quad (1.1.18)$$

are respectively dubbed the *acceleration* and the *vorticity* [53]. Note that (1.1.15) is equivalent to the non commutativity of the basis vectors

$$[\mathbf{v}, \hat{\partial}_i] = \varphi_i \mathbf{v} \quad [\hat{\partial}_i, \hat{\partial}_j] = 2\hat{\omega}_{ij} \mathbf{v}. \quad (1.1.19)$$

$\varphi_i$  and  $\hat{\omega}_{ij}$  are genuine Carrollian tensors (respectively one and two-forms) and transform under shift symmetry as

$$\delta_\lambda \varphi_i = \mathbf{v}(\lambda_i) \quad \text{and} \quad \delta_\lambda \hat{\omega}_{ij} = \hat{\partial}_{[i} \lambda_{j]} + \lambda_{[i} \varphi_{j]}, \quad (1.1.20)$$

i.e. like connections.

Note that the acceleration is also related to the Lie derivative of the clock form along the field of observers i.e.

$$\varphi_i = (\mathcal{L}_{\mathbf{v}} \boldsymbol{\tau})_i. \quad (1.1.21)$$

Let us finish with an important discussion about the vorticity. In this thesis we consider (in view of physical applications) Carroll structures only on manifolds of the type  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ . However this is not the most general situation. Once we are given an Ehresmann connection, the tangent space of  $\mathcal{M}$  gets splitted into  $\text{Span}(\mathbf{v}) \oplus H$  with  $H$  defined in (1.1.4). One may legitimately wonder when  $H$  is itself the tangent space of a codimension one hypersurface of  $\mathcal{M}$ . When this is the case,  $\mathcal{M}$  is actually foliated by a succession of hypersurface modeled on the base space  $\mathcal{S}$ , and one says that  $H$

is an integrable distribution. For this to hold, Frobenius criterion should be satisfied for the clock form i.e.  $\tau \wedge d\tau$  should vanish, which finally implies  $\hat{\omega}_{ij} = 0$ . Carroll structures with vanishing vorticity are then always of the form  $\mathbb{R} \times \hat{S}$  with  $T\hat{S} = H$ .

**Remark** For completeness, note that another type of transformations a Carroll structure can encompass is related to the rescalings of the field of observers  $\mathbf{v} \rightarrow f(t, \mathbf{x})\mathbf{v}$  with  $f$  an arbitrary function, while the degenerate metric and the clock form are left invariant. This shall not be confused with Weyl transformations that affect both the field of observers and the metric (see Sec. 1.4). This is dubbed a *vertical boost* in [73]. We will not consider such transformations in this thesis.

## 1.2 Classifying all possible connections

In this Section we will clarify the statement according which there is no equivalent of a Levi-Civita connection on a weak Carroll structure. In general, after decomposing the most general connection in the bases (1.1.9), one imposes constraints on it, like the compatibility with the degenerate metric, or the absence of some components of the torsion; all choices restricting drastically the shape the connection coefficients can take. In this paragraph we shall consider a completely unconstrained connection  $\bar{\nabla}$  that we will comprehensively study. We will then classify the connections mostly used in the literature depending on the physical manifold at hand.

From now on, let  $(\mathcal{M}, a_{ij}, b_i, \Omega)$  be a ruled Carroll structure and let  $\bar{\nabla}$  be a connection on  $\mathcal{M}$ .

### 1.2.1 Generalities

In general terms, in the bases at hand, one can decompose  $\bar{\nabla}$  in the following way

$$\begin{aligned}
\bar{\nabla}_{\mathbf{v}}\mathbf{v} &= \hat{\gamma}\mathbf{v} + \hat{\rho}^i\hat{\partial}_i & \bar{\nabla}_{\mathbf{v}}\boldsymbol{\tau} &= -\hat{\gamma}\boldsymbol{\tau} - \hat{\delta}_i dx^i \\
\bar{\nabla}_{\mathbf{v}}\hat{\partial}_i &= \hat{\delta}_i\mathbf{v} + \hat{\gamma}^j{}_i\hat{\partial}_j & \bar{\nabla}_{\mathbf{v}}dx^i &= -\hat{\rho}^i\boldsymbol{\tau} - \hat{\gamma}^i{}_j dx^j \\
\bar{\nabla}_{\hat{\partial}_i}\mathbf{v} &= \hat{\alpha}_i\mathbf{v} + \hat{\kappa}_i^j\hat{\partial}_j & \bar{\nabla}_{\hat{\partial}_i}\boldsymbol{\tau} &= -\hat{\alpha}_i\boldsymbol{\tau} - \hat{\beta}_{ij} dx^j \\
\bar{\nabla}_{\hat{\partial}_i}\hat{\partial}_j &= \hat{\beta}_{ij}\mathbf{v} + \hat{\gamma}^k{}_{ij}\hat{\partial}_k & \bar{\nabla}_{\hat{\partial}_i}dx^j &= -\hat{\kappa}_i^j\boldsymbol{\tau} - \hat{\gamma}^j{}_{ik} dx^k.
\end{aligned} \tag{1.2.1}$$

Hence we have that the most general connection on can put on a Carrollian structure is decomposed into of the following  $(d + 1)^3$  degrees of freedom,  $\hat{\gamma}$ ,  $\hat{\rho}^i$ ,  $\hat{\delta}_i$ ,  $\hat{\alpha}_i$ ,  $\hat{\gamma}^j{}_i$ ,  $\hat{\kappa}_i^j$ ,  $\hat{\beta}_{ij}$  and  $\hat{\gamma}^k{}_{ij}$ .

Before going on, it is legitimate to ask about the transformation rules of these new objects under the transformations we exhibited in the last section, Carrollian diffeomorphisms and shift symmetry. Denoting with a prime the quantities obtained after the change of coordinates we have by definition

$$\bar{\nabla}'_{\mathbf{v}'}\hat{\partial}'_i := \hat{\delta}'_i\mathbf{v}' + \hat{\gamma}'^j{}_i\hat{\partial}'_j. \tag{1.2.2}$$

Using (1.1.10) we get<sup>3</sup>

$$\hat{\delta}'_i = (J^{-1})_i^j \hat{\delta}_j \quad \text{and} \quad \hat{\gamma}'^j_i = J^j_k (J^{-1})_i^l \hat{\gamma}^k_l \quad (1.2.3)$$

i.e. the components of a Carrollian one-form and a rank-(1, 1) tensor. Applying the same method to  $\bar{\nabla}'_{\hat{\delta}'_i}$  we find

$$\hat{\beta}'_{ij} = (J^{-1})_i^k (J^{-1})_j^l \hat{\beta}_{kl} \quad \text{and} \quad \hat{\gamma}'^k_{ij} = J^k_n (J^{-1})_i^l (J^{-1})_j^m \hat{\gamma}^n_{lm} + J^k_n (J^{-1})_i^l \hat{\delta}_l \left( (J^{-1})_j^n \right) \quad (1.2.4)$$

i.e.  $\hat{\beta}'_{ij}$  is a Carrollian rank-2 tensor and  $\hat{\gamma}'^k_{ij}$  are Carrollian connection coefficients. With  $\bar{\nabla}_{\mathbf{v}'}$  we get

$$\hat{\gamma}' = \hat{\gamma} \quad \text{and} \quad \hat{\rho}'^i = J^i_j \hat{\rho}^j \quad (1.2.5)$$

i.e.  $\hat{\gamma}$  is a genuine scalar while  $\hat{\rho}^i$  is a Carrollian vector, while  $\bar{\nabla}_{\hat{\delta}'_i}$  yields

$$\hat{\alpha}'_i = (J^{-1})_i^j \hat{\alpha}_j \quad \text{and} \quad \hat{\kappa}'^j_i = J^j_k (J^{-1})_i^l \hat{\kappa}_l^k \quad (1.2.6)$$

so  $\hat{\alpha}_i$  are indeed the components of a co-vector and  $\hat{\kappa}_i^j$  a rank-(1, 1) tensor. From now on we will adopt a new notation for the time-covariant derivative

$$\bar{\nabla}_{\mathbf{v}} := \frac{1}{\Omega} \bar{D}_t. \quad (1.2.7)$$

**Remark** For a general connection  $\Gamma_{\mu\nu}^\lambda$  in  $(d+1)$ -dimensions, given the Jacobian  $J^\mu_\nu$ , the finite transformation law under diffeomorphism is

$$\Gamma'_{\mu\nu}{}^\rho = J^\rho_\gamma (J^{-1})_\mu{}^\alpha (J^{-1})_\nu{}^\beta \Gamma_{\alpha\beta}^\gamma + (J^{-1})_\mu{}^\alpha J^\rho_\gamma \mathbf{e}_\alpha \left[ (J^{-1})^\gamma_\nu \right], \quad (1.2.8)$$

with  $\mathbf{e}_\alpha$  a basis vector. In our bases (1.1.9) the Jacobian takes the matrix form  $J^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & J^i_j(\mathbf{x}) \end{pmatrix}$  so we recover (1.2.3), (1.2.4), (1.2.5) and (1.2.6) from (1.2.8), as it should.

**Remark** As genuine Carrollian tensors,  $\gamma$ ,  $\hat{\delta}_i$ ,  $\hat{\alpha}_i$ ,  $\hat{\rho}^i$ ,  $\hat{\gamma}_{ij}$ ,  $\hat{\kappa}_{ij}$  and  $\hat{\beta}_{ij}$  could be consistently set equal to Carrollian tensors of the same rank without breaking Carroll diffeomorphism covariance. It is also consistent to set them to zero, and we shall pick later a particular connection for which this happens. However such choices may not be shift invariant.

Applying the same method on (1.2.1), using (1.1.14) we get the transformations of the coefficients under shift symmetry

$$\delta_\lambda \hat{\gamma} = -\hat{\rho}^i \lambda_i \quad (1.2.9a)$$

---

<sup>3</sup>The tensorial character of  $\hat{\gamma}'^j_i$  comes from the use of Carrollian diffeomorphisms, so is rooted in the time-independence of the Jacobian  $J^i_j$ . Otherwise a term like  $\mathbf{v} \left( (J^{-1})_i^k \right) J^j_k$  would have appear. This has to be linked with the further use of this tensor as Carrollian *temporal* connection, while  $\hat{\gamma}^k_{ij}$  will play the role of Carrollian *spatial* connection, see (1.3.2).

$$\delta_\lambda \hat{\rho}^i = 0 \quad (1.2.9b)$$

$$\delta_\lambda \hat{\delta}_i = \lambda_i \hat{\gamma} - \hat{\gamma}_i^j \lambda_j + \mathbf{v}(\lambda_i) \quad (1.2.9c)$$

$$\delta_\lambda \hat{\alpha}_i = \lambda_i \hat{\gamma} - \hat{\kappa}_i^j \lambda_j \quad (1.2.9d)$$

$$\delta_\lambda \hat{\gamma}_i^j = \lambda_i \hat{\rho}^j \quad (1.2.9e)$$

$$\delta_\lambda \hat{\kappa}_i^j = \lambda_i \hat{\rho}^j \quad (1.2.9f)$$

$$\delta_\lambda \hat{\beta}_{ij} = -\hat{\gamma}_{ij}^k \lambda_k + \lambda_j \hat{\alpha}_i + \lambda_i \hat{\delta}_j + \hat{\delta}_i(\lambda_j) \quad (1.2.9g)$$

$$\delta_\lambda \hat{\gamma}_{ij}^k = \lambda_j \hat{\kappa}_i^k + \lambda_i \hat{\gamma}_j^k \quad (1.2.9h)$$

showing that  $\hat{\delta}_i$  and  $\hat{\beta}_{ij}$  are respectively the Carrollian temporal and spatial shift connections.

**Remark** Putting together (1.2.5) and (1.2.9b) we conclude that  $\hat{\rho}^i$  is both a Carrollian tensor and a shift-invariant quantity. Hence it can be consistently set to zero. Once set to zero,  $\hat{\kappa}_j^i$  is also a shift-invariant Carrollian tensor, which we can also set to zero. Therefore we have that neither  $\hat{\rho}^i$  nor  $\hat{\kappa}_j^i$  play any role in building a Carrollian connection. The study in Cartan's orthonormal frame that we present in Sec. B.1 will show that these degrees of freedom are actually connections for Galilean boosts, transformations that do not play any role on a Carroll structure.

The torsion two-form of such a connection reads

$$\hat{\mathcal{T}}^i = \frac{1}{2}(\hat{\delta}_i - \hat{\alpha}_i - \varphi_i)\boldsymbol{\tau} \wedge dx^i + (\hat{\beta}_{[ij]} - \hat{\omega}_{ij})dx^i \wedge dx^j \quad (1.2.10a)$$

$$\hat{\mathcal{T}}^i = \frac{1}{2}(\hat{\gamma}_j^i - \hat{\kappa}_j^i)\boldsymbol{\tau} \wedge dx^j + \hat{\gamma}_{[kj]}^i dx^k \wedge dx^j. \quad (1.2.10b)$$

This implies the following commutation rules on a scalar

$$\left[ \frac{1}{\Omega} \bar{D}_i, \bar{\nabla}_i \right] f = (\hat{\alpha}_i + \varphi_i - \hat{\delta}_i)\mathbf{v}(f) + (\hat{\kappa}_i^j - \hat{\gamma}_i^j)\hat{\delta}_j(f) \quad (1.2.11a)$$

$$[\bar{\nabla}_i, \bar{\nabla}_j] f = 2(\hat{\omega}_{ij} - \hat{\beta}_{[ij]})\mathbf{v}(f) - 2\hat{\gamma}_{[ij]}^k \hat{\delta}_k(f). \quad (1.2.11b)$$

We see that it is possible to completely cancel the torsion with the following requirements (which are legitimate as these are tensorial equalities)<sup>4</sup>

$$\hat{\delta}_i = \hat{\alpha}_i + \varphi_i \quad , \quad \hat{\beta}_{[ij]} = \hat{\omega}_{ij} \quad , \quad \hat{\gamma}_j^i = \hat{\kappa}_j^i \quad , \quad \hat{\gamma}_{[ij]}^k = 0. \quad (1.2.12)$$

Note that the symmetric part of  $\hat{\beta}_{(ij)}$  is not constrained by these requirements, hence it is a genuine ambiguity of the connection. This extends to general connections the well known result that any two torsion-free Carrollian connections are related by a symmetric, transverse (w.r.t. the field of observers) rank-2 tensor, see Appendix C for the proof in covariant formalism. Under a shift

<sup>4</sup>Due to the first of these constraint we see that there exists an infinite number of such connections.

transformation the torsion two-form behaves like

$$\delta_\lambda \mathcal{T}^i = -\lambda_j \mathcal{T}^j \quad (1.2.13a)$$

$$\delta_\lambda \mathcal{T}^i = 0. \quad (1.2.13b)$$

Hence, asking for no torsion at all, imposing (1.2.12), is a shift-invariant statement.

**Proposition 1.0.1.** Endowing a Carrollian structure with a completely torsion-free connection is a Carroll diffeomorphism invariant as well as a Carroll-shift invariant statement.

Actually, to preserve Carroll diffeomorphisms and shifts it is sufficient to cancel only the spatial part of the torsion. This statement will no longer hold when dealing with Carrollian strong connections, see Sec. 1.2.2. The Riemann curvature matrix-valued two-form reads

$$\begin{aligned} \mathcal{R}^i_{\hat{i}} &= \frac{1}{2} \left( \mathbf{u}(\hat{\alpha}_i) + \hat{\kappa}_i^j \hat{\delta}_j - (\hat{\partial}_i + \varphi_i) \hat{\gamma} - \hat{\rho}^j \hat{\beta}_{ij} - \varphi_i \hat{\gamma} \right) \boldsymbol{\tau} \wedge dx^i \\ &\quad + \left( \hat{\partial}_{[i} \hat{\alpha}_{j]} + \hat{\kappa}_{[j}^k \hat{\beta}_{i]k} - \hat{\omega}_{ij} \hat{\gamma} \right) dx^i \wedge dx^j, \end{aligned} \quad (1.2.14a)$$

$$\begin{aligned} \mathcal{R}^i_{\hat{i}} &= \frac{1}{2} \left( \mathbf{u}(\hat{\beta}_{ji}) + \hat{\beta}_{ji} \hat{\gamma} + \hat{\gamma}_{ji}^k \hat{\delta}_k - \hat{\partial}_j (\hat{\delta}_i) - \hat{\delta}_i \hat{\alpha}_j - \hat{\gamma}_i^k \hat{\beta}_{jk} - \varphi_j \hat{\delta}_i \right) \boldsymbol{\tau} \wedge dx^j \\ &\quad + \left( \hat{\partial}_{[j} (\hat{\beta}_{k]i}) + \hat{\beta}_{[k|i|} \hat{\alpha}_{j]} + \hat{\gamma}_{[k|i|}^l \hat{\beta}_{j]l} - \hat{\omega}_{jk} \hat{\delta}_i \right) dx^j \wedge dx^k \end{aligned} \quad (1.2.14b)$$

$$\begin{aligned} \mathcal{R}^i_{\hat{i}} &= \frac{1}{2} \left( \mathbf{u}(\hat{\kappa}_j^i) + \hat{\alpha}_j \hat{\rho}^i + \hat{\kappa}_j^k \hat{\gamma}_k^i - \hat{\partial}_j (\hat{\rho}^i) - \hat{\gamma} \hat{\kappa}_j^i - \hat{\rho}^k \hat{\gamma}_{kj}^i - \varphi_j \hat{\rho}^i \right) \boldsymbol{\tau} \wedge dx^j \\ &\quad + \left( \hat{\partial}_{[j} (\hat{\kappa}_{k]}^i) + \hat{\alpha}_{[k} \hat{\kappa}_{j]}^i + \hat{\kappa}_{[k}^l \hat{\gamma}_{j]l}^i - \hat{\omega}_{jk} \hat{\rho}^i \right) dx^j \wedge dx^k \end{aligned} \quad (1.2.14c)$$

$$\begin{aligned} \mathcal{R}^i_{\hat{i}} &= \frac{1}{2} \left( \mathbf{u}(\hat{\gamma}_{kj}^i) + \hat{\beta}_{kj} \hat{\rho}^i + \hat{\gamma}_{kj}^l \hat{\gamma}_l^i - \hat{\partial}_k (\hat{\gamma}_j^i) - \hat{\delta}_j \hat{\kappa}_k^i - \hat{\gamma}_j^l \hat{\gamma}_{kl}^i - \varphi_k \hat{\gamma}_j^i \right) \boldsymbol{\tau} \wedge dx^k \\ &\quad + \left( \hat{\partial}_{[k} (\hat{\gamma}_{l]j}^i) + \hat{\kappa}_{[k}^l \hat{\beta}_{l]j} + \hat{\gamma}_{[k|m|}^l \hat{\gamma}_{l]j}^m - \hat{\omega}_{kl} \hat{\gamma}_j^i \right) dx^k \wedge dx^l \end{aligned} \quad (1.2.14d)$$

and gives the commutator of two covariant derivatives on a vector, see (A.o.10b). One could study the tensorial and shift properties of the latter but as such general connections are marginal in the literature this would bring us far from our goal.

For the moment we haven't imposed any constraints on the connection, we have just seen that there exists a set of hypotheses allowing to completely cancel the torsion. Our analysis hence restricts to the decomposition of a general connection into longitudinal parts (with only  $\hat{t}$  indices), purely transverse parts (with only Latin indices) and mixed parts.

**Remark** As we shall see in the next section, Carrollian connections were originally demanded to satisfy some constraints (e.g. the preservation of the weak structure). However as shown in e.g. [73, 74] the connection induced on a black-hole horizon by the ambient Levi-Civita one does not fulfill the latter requirement, rendering our previous analysis worth being done. The same remark applies to the case of stretched horizons [75].



## 1.2.2 Strong Carroll connections

Let us start with the connection  $\bar{\nabla}$ . In [49] a *Carrollian connection* is defined to be a connection that preserves the weak Carroll structure. In this paragraph we shall impose the latter constraint and study its consequences.

**Hypothesis 1 :** Compatibility with the field of observers i.e.  $\bar{\nabla}\mathbf{v} = 0$

This implies that  $\hat{\gamma} = 0$ ,  $\hat{\alpha}_i = 0$ ,  $\hat{\rho}^i = 0$  and  $\hat{\kappa}_i^j = 0$ , so we are left with

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}}\mathbf{v} &= 0 & \bar{\nabla}_{\mathbf{v}}\boldsymbol{\tau} &= -\hat{\delta}_i dx^i \\ \bar{\nabla}_{\mathbf{v}}\hat{\delta}_i &= \hat{\delta}_i \mathbf{v} + \hat{\gamma}_i^j \hat{\delta}_j & \bar{\nabla}_{\mathbf{v}}dx^i &= -\hat{\gamma}_j^i dx^j \\ \bar{\nabla}_{\hat{\delta}_i}\mathbf{v} &= 0 & \bar{\nabla}_{\hat{\delta}_i}\boldsymbol{\tau} &= -\hat{\beta}_{ij} dx^j \\ \bar{\nabla}_{\hat{\delta}_i}\hat{\delta}_j &= \hat{\beta}_{ij} \mathbf{v} + \hat{\gamma}_{ij}^k \hat{\delta}_k & \bar{\nabla}_{\hat{\delta}_i}dx^j &= -\hat{\gamma}_{ik}^j dx^k. \end{aligned} \tag{1.2.15}$$

Everything is well-behaved under Carrollian diffeomorphisms and the transformations under shift (1.2.9c), (1.2.9e), (1.2.9h) and (1.2.9g) reduces to

$$\delta_\lambda \hat{\delta}_i = \mathbf{v}(\lambda_i) - \hat{\gamma}_i^j \lambda_j \tag{1.2.16a}$$

$$\delta_\lambda \hat{\gamma}_i^j = 0 \tag{1.2.16b}$$

$$\delta_\lambda \hat{\gamma}_{ij}^k = \lambda_i \hat{\gamma}_j^k \tag{1.2.16c}$$

$$\delta_\lambda \hat{\beta}_{ij} = \hat{\delta}_i(\lambda_j) + \lambda_i \delta_j - \lambda_k \hat{\gamma}_{ij}^k. \tag{1.2.16d}$$

Hence, given the last equations and the Remark below Eq. (1.2.9h), imposing Hypothesis 1 is a Carroll diffeomorphism and shift invariant requirement.

**Hypothesis 2 :** Compatibility with the metric,  $\bar{\nabla}a_{ij} = 0$

Using App. (A.0.8) we get only two non-identically zero conditions<sup>5</sup>

$$\frac{1}{\Omega} \bar{D}_i a_{ij} = \mathbf{v}(a_{ij}) - 2\hat{\gamma}_{(ij)} = 0 \quad \text{and} \quad \bar{\nabla}_k a_{ij} = \hat{\delta}_k a_{ij} - \hat{\gamma}_{ki}^l a_{lj} - \hat{\gamma}_{kj}^l a_{il} = 0 \tag{1.2.17}$$

which are solved imposing first

$$\hat{\gamma}_{(ij)} = \frac{1}{2} \mathbf{v}(a_{ij}) := \xi_{ij} + \frac{\theta}{d} a_{ij} \tag{1.2.18}$$

where  $\xi_{ij}$  is a Carrollian tensor dubbed the *geometric shear* and

$$\theta = \mathbf{v}(\ln \sqrt{a}) \tag{1.2.19}$$

<sup>5</sup>Our conventions for (anti)-symmetrisation are as follows (for a rank-2 tensor)  $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$  and  $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ .

is a Carrollian scalar called the *expansion*. This combination is often called in the literature the *extrinsic curvature* as it measures how the metric of the base space  $\mathcal{S}$  changes along the geodesic tubes generated by  $\mathbf{v}$ . The second condition in (1.2.17) is solved using the standard technique ( $\bar{\nabla}_k a_{ij} - \bar{\nabla}_i a_{jk} - \bar{\nabla}_j a_{ki} = 0$ ) and gives

$$\hat{\gamma}^k_{(ij)} = \frac{1}{2} a^{kl} (\hat{\partial}_i a_{jl} + \hat{\partial}_j a_{li} - \hat{\partial}_l a_{ij}) \quad (1.2.20)$$

i.e. genuine Christoffels symbols for the purely spatial part of the connection. A connection that satisfied Hypotheses 1 and 2 is dubbed a *strong Carroll connection* and the structure  $(a_{ij}, \mathbf{v}, \bar{\nabla})$  is then called a *strong Carroll structure* (ruled if we add a  $b_i$ ). We see that not all the residual degrees of freedom we got after requiring the first hypothesis are constrained for a strong connection. Actually we have the following

**Proposition 1.0.2.** A strong Carrollian connection leaves free  $\hat{\delta}_i$ ,  $\hat{\beta}_{ij}$ ,  $\hat{\gamma}_{[ij]}$  and  $\hat{\gamma}^k_{[ij]}$  i.e.  $\frac{d(d+1)^2}{2}$  degrees of freedom.

For completeness, let us see how our strong connection acts on Carrollian tensors. Let  $\mathbf{V} = v^i \mathbf{v} + v^i \hat{\delta}_i$  and  $\omega = \omega_i \boldsymbol{\tau} + \omega_i dx^i$  be respectively a vector and a one-form. Given (1.2.15) together with (1.2.18), (1.2.20) and the Leibniz rule we get

$$\begin{aligned} \frac{1}{\Omega} \bar{D}_i v^{\hat{i}} &= \mathbf{v}(v^{\hat{i}}) + v^{\hat{i}} \hat{\delta}_i & \frac{1}{\Omega} \bar{D}_i \omega_{\hat{i}} &= \mathbf{v}(\omega_{\hat{i}}) \\ \bar{\nabla}_i v^{\hat{i}} &= \hat{\partial}_i v^{\hat{i}} + \hat{\beta}_{ij} v^j & \bar{\nabla}_i \omega_{\hat{i}} &= \hat{\partial}_i \omega_{\hat{i}} \\ \frac{1}{\Omega} \bar{D}_i v^i &= \mathbf{v}(v^i) + \hat{\gamma}^{ij} v_j & \frac{1}{\Omega} \bar{D}_i \omega_i &= \mathbf{v}(\omega_i) - \omega^j \hat{\gamma}_{ij} - \omega_i \hat{\delta}_i \\ \bar{\nabla}_j v^i &= \hat{\partial}_j v^i + \hat{\gamma}^i_{jk} v^k & \bar{\nabla}_j \omega_i &= \hat{\partial}_j \omega_i - \hat{\gamma}^k_{ij} \omega_k - \hat{\beta}_{ij} \omega_i. \end{aligned} \quad (1.2.21)$$

Let's end this paragraph describing some examples of Carroll strong connections. Indeed given the remaining pieces, depending on the degrees of freedom one chooses to discard, the resulting connection will get drastically different characteristics.

### Carroll special connection

Using the general formulae (1.2.10a) and (1.2.10b) we get for the torsion one-form in the case at hand

$$\hat{\mathcal{T}}^i = (\hat{\beta}_{[ij]} - \hat{\omega}_{ij}) dx^i \wedge dx^j + (\hat{\delta}_i - \varphi_i) \boldsymbol{\tau} \wedge dx^i \quad (1.2.22a)$$

$$\hat{\mathcal{T}}^i = \hat{\gamma}^i_j \boldsymbol{\tau} \wedge dx^j + \hat{\gamma}^i_{[kj]} dx^k \wedge dx^j = a^{ik} (\hat{\gamma}_{(kj)} + \hat{\gamma}_{[kj]}) \boldsymbol{\tau} \wedge dx^j + \hat{\gamma}^i_{[kj]} dx^k \wedge dx^j. \quad (1.2.22b)$$

From this formula we can make several remarks. First, because  $\hat{\gamma}_{(kj)}$  is related to the geometry in virtue of (1.2.18) we have the following result

**Proposition 1.0.3.** It is not possible to endow a general weak Carrollian structure with a completely torsionfree strong connection. The unavoidable piece of torsion is related to the extrinsic

curvature via the geometric shear  $\xi_{ij}$  and the expansion  $\theta$  and reads

$$\hat{\mathcal{T}}^{\hat{t}} = 0 \quad \text{and} \quad \hat{\mathcal{T}}^i = a^{ik} \left( \xi_{ij} + \frac{\theta}{d} a_{ij} \right) \boldsymbol{\tau} \wedge dx^j. \quad (1.2.23)$$

This statement is not shift invariant due to (1.2.13a), (1.2.13b) and  $\mathcal{T}^{\hat{t}}$  being zero.

**Remark** This is in perfect agreement with what was found in the covariant formalism, see Remark (C.o.11) and [56], where it was shown in a coordinate independent way that only invariant Carroll structures (the ones for which  $\mathcal{L}_{\mathbf{v}}q = 0$ ) can admit torsion-free connections.

**Remark** In the case of  $\mathcal{J}^+$  is it always possible to go to a frame in which the extrinsic curvature vanishes, by making use of a Weyl transformation. Thus one can always endow  $\mathcal{J}^+$  with a completely torsion-free strong connection.

The maximum we can do is to ask all the pieces in (1.2.22a) and (1.2.22b) but  $\hat{\gamma}_{(ij)}$  to vanish i.e.

$$\hat{\delta}_i = \varphi_i \quad , \quad \hat{\beta}_{[ij]} = \hat{\omega}_{ij} \quad , \quad \hat{\gamma}_{[ij]} = 0 \quad , \quad \text{and} \quad \hat{\gamma}^i_{[kj]} = 0 \quad (1.2.24)$$

which is perfectly fine given the transformations (1.2.3) and (1.2.4).<sup>6</sup> We call such a connection a *special Carrollian connection*. We have the result

**Proposition 1.0.4.** A special Carrollian connection leaves only free a rank-2 transverse and symmetric tensor,  $\hat{\beta}_{(ij)}$  i.e.  $\frac{d(d+1)}{2}$  degrees of freedom.

**Remark** This is precisely in this tensor that Ashtekar identified the two polarization of the graviton (see [76] for a review in modern notations), relating then  $\hat{\beta}_{(ij)}$  (more precisely its trace-free part) with the Bondi shear. Indeed in four-dimensional bulk spacetimes i.e. three-dimensional null infinity, the trace-free part of  $\hat{\beta}_{ij}$  is both unconstrained and can hold two degrees of freedom. These are exactly the two main features of the shear tensor.

### Split-frame compatible connection

As we saw in (1.2.15), strong connections do not leave the clock form invariant *a priori*. This will ultimately blur the time/space splitting. This is neither a caveat nor an issue, however one may want to consider a connection that completely preserves the splitting. In that case it is asked that  $\bar{\nabla} \boldsymbol{\tau} \propto \boldsymbol{\tau}$  and thus  $\hat{\delta}_i = 0$  and  $\hat{\beta}_{ij} = 0$ . Hence, because of (1.2.22a) and (1.2.22b), the torsion is supported by geometric quantities ( $\varphi_i$  and  $\hat{\omega}_{ij}$ ) and then cannot be generally set to zero. This is an important result that for general Carrollian structures, there is no special strong connection that fully preserves the time/space splitting. Moreover such connections cannot be Carroll shift invariant as  $\hat{\delta}_i$  and  $\hat{\beta}_{ij}$  gauge the latter transformation, see (1.2.16a) and (1.2.16d).

<sup>6</sup>Note however that this is not a shift invariant statement as  $\mathcal{T}^{\hat{t}}$  is non vanishing.

## 1.3 Strong Carroll structures and conservation equations

In this Section we introduce the connection used in [52, 53, 61, 65] which is a connection without ambiguities and which is the closest analogue to a Levi-Civita connection. Equipped with this connection we shall consider a generic effective action whose variation under a Carrollian diffeomorphism will give rise to the notion of conjugate momenta and the emergence of their associated conservation equations.

### 1.3.1 The hat connection $\hat{\nabla}$

Due to their tensorial properties, one can also choose to set

$$\hat{\delta}_i = 0, \quad \hat{\beta}_{ij} = 0, \quad \hat{\gamma}_{[ij]} = 0 \quad \text{and} \quad \hat{\gamma}^i_{[kj]} = 0 \quad (1.3.1)$$

leading to a torsionfull connection but with no spatial torsion.<sup>7</sup> This makes it a Levi-Civita connection on the spatial sections  $\mathcal{S}$ . We shall denote it  $\hat{\nabla}$  and use it from now on.<sup>8</sup> We give the action on a vector field, as then one can deduce the rest from the Leibniz rule

$$\frac{1}{\Omega} \hat{D}_i V^i = \frac{1}{\Omega} \partial_i V^i + \hat{\gamma}^i_j V^j \quad \text{and} \quad \hat{\nabla}_j V^i = \hat{\delta}_j V^i + \hat{\gamma}^i_{(jk)} V^k. \quad (1.3.2)$$

The Carrollian metric is also preserved by the temporal covariant derivative,<sup>9</sup> that is

$$\frac{1}{\Omega} \hat{D}_i a_{ij} = 0. \quad (1.3.3)$$

Acting on scalars the covariant derivatives do not commute

$$\left[ \frac{1}{\Omega} \hat{D}_i, \hat{\nabla}_i \right] f = \varphi_i \frac{1}{\Omega} \partial_i f - \hat{\gamma}_i^j \hat{\delta}_j f \quad (1.3.4a)$$

$$[\hat{\nabla}_i, \hat{\nabla}_j] f = 2\omega_{ij} \frac{1}{\Omega} \partial_i f. \quad (1.3.4b)$$

We see appearing the intrinsic torsion ( $\hat{\gamma}_{(ij)}$ ,  $\varphi_i$  and  $\omega_{ij}$ ) that is left non zero by our choices (1.3.1).

The commutators of Carrollian covariant derivatives on a vector define Carrollian curvature tensors<sup>10</sup>

$$\begin{aligned} [\hat{\nabla}_k, \hat{\nabla}_l] V^i &= \left( \hat{\delta}_k \hat{\gamma}^i_{lj} - \hat{\delta}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj} \right) V^j + [\hat{\delta}_k, \hat{\delta}_l] V^i \\ &= \hat{r}^i_{jkl} V^j + \hat{\omega}_{kl} \frac{2}{\Omega} \partial_i V^i, \end{aligned} \quad (1.3.5)$$

where  $\hat{r}^i_{jkl}$  is called the *Riemann–Carroll* tensor. The *Ricci–Carroll* tensor and the *Ricci–Carroll*

<sup>7</sup>Here  $\mathcal{T}^i = -\omega_{ij} dx^i \wedge dx^j - \varphi_i \tau \wedge dx^i$  and  $\mathcal{T}^i = \hat{\gamma}^i_j \tau \wedge dx^j$  so  $T^k_{ij} = 0$ .

<sup>8</sup>In Section 2.1.1 we shall see that  $\hat{\nabla}$  is the connection inherited from the relativistic ascendant.

<sup>9</sup>This can be easily shown by direct computation.

<sup>10</sup>Note that the second term of the second line is due to torsion.

scalar curvature are defined as

$$\hat{r}_{ij} = \hat{r}_{ikj}^k \neq \hat{r}_{ji}, \quad \hat{r} = a^{ij} \hat{r}_{ij}. \quad (1.3.6)$$

Similarly, space and time derivatives do not commute

$$\left[ \frac{1}{\Omega} \hat{D}_t, \hat{\nabla}_i \right] V^j = \varphi_i \left( \left( \frac{1}{\Omega} \hat{D}_t + \theta \right) V^j - \hat{\gamma}_k^j V^k \right) - \hat{\gamma}_i^k \hat{\nabla}_k V^j - d\hat{r}_{ik}^j V^k \quad (1.3.7)$$

with

$$\hat{r}_{ik}^j = \frac{1}{d} \left( \theta \varphi_i \delta_k^j + \hat{\nabla}_i \hat{\gamma}_k^j - \frac{1}{\Omega} \partial_t \hat{\gamma}_{ik}^j \right), \quad \hat{r}_{jk}^j = \hat{r}_k = \frac{1}{d} \left( \hat{\nabla}_j \hat{\gamma}_k^j - \hat{\partial}_k \theta \right), \quad (1.3.8)$$

further Carrollian curvature tensors mixing space and time.

### 1.3.2 Action, momenta, and conservation equations

In this section we consider the connection  $\hat{\nabla}$  and an effective Carrollian action, which is a functional of the geometric pieces and of matter fields (collectively denoted by  $\Phi$ ),

$$S = S[a_{ij}, b_i, \Omega; \Phi] = \int dt d^d x \sqrt{a} \Omega \mathcal{L} \quad (1.3.9)$$

where  $\mathcal{L}$  is the Lagrangian density. From the relativistic intuition we know that varying the action under a general diffeomorphism and asking it to be invariant implies the on-shell conservation of the energy-momentum tensor, seen as the conjugate momenta of the metric. We shall define here the notion of Carrollian momenta and derive from Carrollian diffeomorphism invariance of  $S$  their conservation equations. Throughout the Section  $\xi = \xi^i(t, \mathbf{x}) \mathbf{v} + \xi^i(\mathbf{x}) \hat{\partial}_i$  parameterizes an infinitesimal Carrollian diffeomorphism. Note already that  $\xi^i$  is time-independent.

#### Lie derivatives and divergences

As  $a_{ij}$ ,  $\mathbf{v}$ ,  $\boldsymbol{\tau}$  are tensors under Carrollian diffeomorphisms the latter will act with the Lie derivative

$$\delta_\xi a_{ij} = -\mathcal{L}_\xi a_{ij} \quad , \quad \delta_\xi \mathbf{v} = -\mathcal{L}_\xi \mathbf{v} \quad , \quad \delta_\xi \boldsymbol{\tau} = -\mathcal{L}_\xi \boldsymbol{\tau}, \quad (1.3.10)$$

where the minus sign is conventional. Using (A.o.16a) and (A.o.24a) we get

$$\mathcal{L}_\xi a_{ij} = 2\xi^i \hat{\gamma}_{(ij)} + 2\hat{\nabla}_{(i} \xi^k a_{j)k} \quad (1.3.11a)$$

$$\mathcal{L}_\xi \mathbf{v} = - \left( \frac{1}{\Omega} \partial_t \xi^i + \varphi_i \xi^i \right) \mathbf{v} \quad (1.3.11b)$$

$$\mathcal{L}_\xi \boldsymbol{\tau} = \left( \frac{1}{\Omega} \partial_t \xi^i + \varphi_i \xi^i \right) \boldsymbol{\tau} + \left( (\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \omega_{ji} \right) dx^i. \quad (1.3.11c)$$

Given that

$$\mathcal{L}_\xi \mathbf{v} = \mathcal{L}_\xi(\Omega^{-1} \partial_t) = -\mathcal{L}_\xi(\ln \Omega) \mathbf{v} \quad \text{and} \quad \mathcal{L}_\xi \boldsymbol{\tau} = \mathcal{L}_\xi(\ln \Omega) \boldsymbol{\tau} + (\mathcal{L}_\xi b_i - b_i \mathcal{L}_\xi(\ln \Omega)) dx^i \quad (1.3.12)$$

we infer<sup>11</sup>

$$\mathcal{L}_\xi(\ln \Omega) = \left( \frac{1}{\Omega} \partial_t \xi^i + \varphi_i \xi^i \right) \quad \text{and} \quad \mathcal{L}_\xi b_i = b_i \left( \frac{1}{\Omega} \partial_t \xi^i + \varphi_j \xi^j \right) + \left( (\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \hat{\omega}_{ji} \right). \quad (1.3.13)$$

**Remark** In the basis  $\mathcal{B}$  the total metric (1.1.11) takes the matrix form  $q_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & a_{ij} \end{pmatrix}$  and we have

$\mathcal{L}_\xi q_{ii} = 0$ ,  $\mathcal{L}_\xi q_{ii} = \mathcal{L}_\xi q_{ii} = 0$  together with  $\mathcal{L}_\xi q_{ij} = \mathcal{L}_\xi a_{ij}$ . Hence Carrollian diffeomorphisms preserve the form (1.1.11), i.e. the time/space splitting, as advertised.

In the process of varying an action one has to deal with integrations by parts. Here we shall write the Carrollian equivalent of the well-known relativistic formula (valid for a Levi-Civita connection)  $\sqrt{-g} \nabla_\mu V^\mu = \partial_\mu(\sqrt{-g} V^\mu)$  with  $g = \det(g_{\mu\nu})$  the determinant of the Riemannian metric and  $V^\mu$  a vector. Note that because we are integrating over  $t$  and  $x^i$  with a volume element  $\sqrt{a} \Omega$  the good temporal and spatial boundary terms should be built upon the ordinary spatial derivative  $\partial_i$  and not  $\hat{\partial}_i$ . Manipulating the temporal and spatial divergences, i.e. respectively  $\mathbf{v}(f)$  and  $\hat{\nabla}_i V^i$ , with  $f$  a Carrollian scalar and  $V^i$  the components of a Carrollian vector, we get

$$\sqrt{a} \Omega \mathbf{v}(f) = -\sqrt{a} \Omega \partial_t f + \partial_t(\sqrt{a} f) \quad (1.3.14a)$$

$$\sqrt{a} \Omega \hat{\nabla}_i V^i = -\sqrt{a} \Omega \varphi_i V^i + \partial_t(\sqrt{a} b_i V^i) + \partial_i(\sqrt{a} \Omega V^i). \quad (1.3.14b)$$

**Remark** In deriving the formula for the spatial divergence we made use of the equality

$$\hat{\nabla}_i V^i = \frac{1}{\sqrt{a}} \hat{\partial}_i(\sqrt{a} V^i) \quad (1.3.15)$$

which is valid because the connection  $\hat{\nabla}$  is spatially Levi-Civita. One then has to be careful when using another connection. In particular this is the reason why we choose to re-express the Lie derivatives (originally written with ordinary derivatives) in terms of the connection  $\hat{\nabla}$ .

### Varying the effective action

*Carrollian momenta* are conjugate variables to the geometrical data. They are the Carrollian equivalents of the relativistic energy-momentum tensor and are defined as

$$\Pi^{ij} = \frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a_{ij}}, \quad \Pi^i = \frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta b_i}, \quad \Pi = -\frac{1}{\sqrt{a}} \left( \frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S}{\delta b_i} \right). \quad (1.3.16)$$

<sup>11</sup>Note that because  $\Omega$  and  $b_i$  are not tensorial under Carrollian diffeomorphisms it is an abuse of notation to use the symbol  $\mathcal{L}_\xi$  on them. One should then see (1.3.13) as definitions.

They<sup>12</sup> are dubbed respectively the *energy-stress tensor*, the *energy current* and the *energy density* and their definition guarantees their tensorial properties under Carrollian diffeomorphisms.<sup>13</sup>

**Remark** Asking for a Carrollian theory to be shift invariant will impose that no physical quantity should depend on  $b_i$ , hence  $\Pi^i = 0$ . However not all theories behave like that (as we will see in the case of the magnetic scalar field in Sec. 2.3.1). This is in particular the case for radiating systems for which the energy-flux  $\Pi^i$  will be non zero as it will encode part of the radiation (like in Robinson-Trautman spacetimes, Sec. 2.3.3).

Taking the variation of the action gives

$$\delta_\xi S = \int dt d^d x \sqrt{a} \Omega \left( \frac{1}{2} \Pi^{ij} \delta_\xi a_{ij} + \Pi^i \delta_\xi b_i - \frac{1}{\Omega} (\Pi + b_i \Pi^i) \delta_\xi \Omega \right). \quad (1.3.17)$$

Using (1.3.11a), (1.3.13) together with (1.3.14a) and (1.3.14b) and  $\xi^i = \xi^i(\mathbf{x})$  we get<sup>14</sup>

$$\begin{aligned} \delta_\xi S = \int dt d^d x \sqrt{a} \Omega & \left\{ -\xi^i \left[ \left( \frac{1}{\Omega} \partial_t + \theta \right) \Pi + (\hat{\nabla}_i + 2\varphi_i) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} \right] \right. \\ & + \xi^i \left[ (\hat{\nabla}_j + \varphi_j) \Pi^j_i + 2\Pi^j \hat{\omega}_{ji} + \Pi \varphi_i \right] \left. \right\} \\ & + \int dt d^d x \left\{ \partial_t \left[ \sqrt{a} (\xi^i (\Pi + b_i \Pi^i) - \xi^j b_i \Pi^i_j) \right] \right. \\ & \left. + \partial_i \left[ \sqrt{a} \Omega (\xi^i \Pi^i - \xi^j \Pi^i_j) \right] \right\} \end{aligned} \quad (1.3.19)$$

which leads to the conservation equations<sup>15</sup>

$$\boxed{\left( \frac{1}{\Omega} \partial_t + \theta \right) \Pi + (\hat{\nabla}_i + 2\varphi_i) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} \hat{=} 0}, \quad (1.3.20)$$

and

$$\boxed{(\hat{\nabla}_j + \varphi_j) \Pi^j_i + 2\Pi^j \hat{\omega}_{ji} + \Pi \varphi_i \hat{=} - \left( \frac{1}{\Omega} \partial_t + \theta \right) P_i} \quad (1.3.21)$$

where an extra piece, the *momentum*  $P_i$ , appears because of the time independence of  $\xi^i$ . Indeed  $\sqrt{a} \Omega \xi^i (\frac{1}{\Omega} \partial_t + \theta) P_i = \partial_t (\sqrt{a} \Omega \xi^i P_i)$  and this is a genuine boundary term.<sup>16</sup> For an explicit theory,

<sup>12</sup>We thus have  $\frac{1}{\sqrt{a}\Omega} \frac{\delta S}{\delta \Omega} = -\frac{1}{\Omega} (\Pi + b_i \Pi^i)$ .

<sup>13</sup>See Section 2.2 for more details about the relationship between the Carrollian momenta and the relativistic energy-momentum tensor.

<sup>14</sup>To help the reader we give one of the intergrations by part to perform

$$-\Pi^{ij} \hat{\nabla}_i \xi_j = -\hat{\nabla}_i (\Pi^i_j \xi^j) + \xi^j \hat{\nabla}_i \Pi^i_j = \xi^j (\hat{\nabla}_i + \varphi_i) \Pi^i_j - \frac{1}{\sqrt{a}\Omega} \partial_i (\sqrt{a} \Omega \Pi^i_j \xi^j) - \frac{1}{\sqrt{a}\Omega} \hat{\partial}_t (\sqrt{a} b_i \Pi^i_j \xi^j) \quad (1.3.18)$$

where we have used (1.3.14b).

<sup>15</sup>On-shellness will be denoted with a hat on top of the equal sign.

<sup>16</sup>In the most general frame where  $\mathbf{v} = \frac{1}{\Omega} (\partial_t + \omega^i \partial_i)$  the momentum is obtained through variation via  $P_i = \frac{1}{\sqrt{a}\Omega} \frac{\delta S}{\delta \omega^i}$  and appears naturally in the equation when requiring general diffeomorphism invariance.

these two equations must be valid on-shell and this determines  $P_i$  (see Section 2.3.2 for the example of the scalar field).

### Equation for a $U(1)$ current

Let's assume that our action is further invariant under a local  $U(1)$  symmetry associated with a gauge field  $\mathbf{B} = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ . Under a gauge transformation of parameter  $\Lambda$  the gauge field changes as  $\delta_\Lambda B = \partial_t \Lambda$  and  $\delta_\Lambda B_i = \partial_i \Lambda$ . The conjugate momenta are dubbed the *charge density* and the *charge current*

$$\varrho = \frac{1}{\sqrt{a}} \left( \frac{\delta S}{\delta B} - \frac{b_i}{\Omega} \frac{\delta S}{\delta B_i} \right), \quad (1.3.22a)$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \frac{\delta S}{\delta B_i} \quad (1.3.22b)$$

so that  $\frac{\delta S}{\delta B} = \sqrt{a} (\varrho + b_i N^i)$ . The gauge variation of the action is here

$$\delta_\Lambda S = \int dt d^d x \sqrt{a} \left( (\varrho + b_i N^i) \delta_\Lambda B + \Omega N^i \delta_\Lambda B_i \right) \quad (1.3.23)$$

$$\begin{aligned} &= - \int dt d^d x \sqrt{a} \left( (\varrho + b_i N^i) \partial_t \Lambda + \Omega N^i \partial_i \Lambda \right) \\ &= \int dt d^d x \sqrt{a} \Omega \Lambda \left( \frac{1}{\Omega} \partial_t \varrho + \theta \varrho + (\hat{\nabla}_i + \varphi_i) N^i \right) \\ &\quad - \int dt d^d x \left\{ \partial_t \left( \sqrt{a} \Lambda (\varrho + b_i N^i) \right) + \partial_i \left( \sqrt{a} \Lambda \Omega N^i \right) \right\}. \end{aligned} \quad (1.3.24)$$

Invariance of  $S$  leads to a Carrollian continuity equation<sup>17</sup>

$$\boxed{\left( \frac{1}{\Omega} \partial_t + \theta \right) \varrho + (\hat{\nabla}_i + \varphi_i) N^i \doteq 0.} \quad (1.3.25)$$

These equations (1.3.20), (1.3.21) together with (1.3.25) were obtained in [53, 65].

### Turning on the ambiguities

One may finally wonder what are the conservation equations if we turn on all the possible ambiguities of a strong Carrollian connection, that is  $\hat{\delta}_i, \hat{\beta}_{ij}, \hat{\gamma}_{[ij]} := \hat{\lambda}_{ij}$  and  $\hat{\gamma}^k_{[ij]} := \hat{\rho}^k_{ij}$  to<sup>18</sup> which we

<sup>17</sup>This is the Carrollian equivalent of the relativistic divergence of a vector  $\nabla_\mu V^\mu = 0$  thanks to (1.3.14a) and (1.3.14b).

<sup>18</sup>Recall that our convention for antisymmetrization is  $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$ .



associate the conjugate momenta<sup>19</sup>

$$\begin{aligned} \zeta^i &= \frac{1}{\sqrt{a}\Omega} \frac{\delta S}{\delta \hat{\delta}_i} & \Lambda^{ij} &= \frac{2}{\sqrt{a}\Omega} \frac{\delta S}{\delta \hat{\lambda}_{ij}} \\ B^{ij} &= \frac{2}{\sqrt{a}\Omega} \frac{\delta S}{\delta \hat{\beta}_{ij}} & Z_k^{ij} &= \frac{2}{\sqrt{a}\Omega} \frac{\delta S}{\delta \hat{\rho}_{ij}^k} \end{aligned} \quad (1.3.26)$$

with  $\lambda^{ij} = \Lambda^{[ij]}$  and  $Z_k^{ij} = Z_k^{[ij]}$ . All the object being genuine Carrollian tensors, Carrollian diffeomorphisms acts only with the Lie derivative, giving

$$-\delta_\xi \hat{\delta}_i = \mathcal{L}_\xi \hat{\delta}_i = \xi^i \frac{1}{\Omega} \hat{D}_i \hat{\delta}_i + \xi^j \hat{\nabla}_j \hat{\delta}_i + \hat{\delta}_j \hat{\nabla}_i \xi^j + \xi^i \hat{\gamma}_{(ij)} \hat{\delta}^j + \xi^i \hat{\lambda}_{ij} \hat{\delta}^j + 2\xi^j \hat{\rho}_{ji}^k \hat{\delta}_k \quad (1.3.27a)$$

$$\begin{aligned} -\delta_\xi \hat{\beta}_{ij} &= \mathcal{L}_\xi \hat{\beta}_{ij} = \xi^i \frac{1}{\Omega} \hat{D}_i \hat{\beta}_{ij} + \xi^k \hat{\nabla}_k \hat{\beta}_{ij} + \hat{\beta}_{kj} \hat{\nabla}_i \xi^k + \hat{\beta}_{ik} \hat{\nabla}_j \xi^k + \xi^i \left( \hat{\gamma}_{(ik)} \hat{\beta}_{ij}^k + \hat{\gamma}_{(jk)} \hat{\beta}_{ij}^k \right) \\ &+ \xi^i \left( \hat{\lambda}_{ik} \hat{\beta}_{ij}^k + \hat{\lambda}_{jk} \hat{\beta}_{ij}^k \right) + 2\xi^k \hat{\rho}_{kj}^l \hat{\beta}_{il} + 2\xi^k \hat{\rho}_{ki}^l \hat{\beta}_{lj} \end{aligned} \quad (1.3.27b)$$

$$\begin{aligned} -\delta_\xi \hat{\lambda}_{ij} &= \mathcal{L}_\xi \hat{\lambda}_{ij} = \xi^i \frac{1}{\Omega} \hat{D}_i \hat{\lambda}_{ij} + \xi^k \hat{\nabla}_k \hat{\lambda}_{ij} + \hat{\lambda}_{kj} \hat{\nabla}_i \xi^k + \hat{\lambda}_{ik} \hat{\nabla}_j \xi^k + \xi^i \left( \hat{\gamma}_{(ik)} \hat{\lambda}_{ij}^k + \hat{\gamma}_{(jk)} \hat{\lambda}_{ij}^k \right) \\ &+ \xi^i \left( \hat{\lambda}_{ik} \hat{\lambda}_{ij}^k + \hat{\lambda}_{jk} \hat{\lambda}_{ij}^k \right) + 2\xi^k \hat{\rho}_{kj}^l \hat{\lambda}_{il} + 2\xi^k \hat{\rho}_{ki}^l \hat{\lambda}_{lj} \end{aligned} \quad (1.3.27c)$$

$$\begin{aligned} -\delta_\xi \hat{\rho}_{ij}^k &= \mathcal{L}_\xi \hat{\rho}_{ij}^k = \xi^i \frac{1}{\Omega} \hat{D}_i \hat{\rho}_{ij}^k + \xi^l \hat{\nabla}_l \hat{\rho}_{ij}^k + \hat{\rho}_{lj}^k \hat{\nabla}_i \xi^l + \hat{\rho}_{il}^k \hat{\nabla}_j \xi^l - \hat{\rho}_{ij}^l \hat{\nabla}_l \xi^k \\ &+ \xi^i \left( \hat{\gamma}_{(il)} \hat{\rho}_{ij}^{kl} + \hat{\gamma}_{(jl)} \hat{\rho}_{ij}^{kl} - \hat{\gamma}^{(kl)} \hat{\rho}_{lij} \right) + \xi^i \left( \hat{\lambda}_{il} \hat{\rho}_{ij}^{kl} + \hat{\lambda}_{jl} \hat{\rho}_{ij}^{kl} - \hat{\lambda}^{kl} \hat{\rho}_{lij} \right) \\ &+ 2\xi^l \left( \hat{\rho}_{li}^m \hat{\rho}_{mj}^k + \hat{\rho}_{lj}^m \hat{\rho}_{im}^k - \hat{\rho}_{lm}^k \hat{\rho}_{ij}^l \right) \end{aligned} \quad (1.3.27d)$$

where we have used (A.o.18a), (A.o.20a) and (A.o.22). We can now take the variation of the action  $S[a_{ij}, b_i, \Omega, \hat{\delta}_i, \hat{\beta}_{ij}, \hat{\lambda}_{ij}, \hat{\rho}^k ; \Phi]$  evaluated on a Carrollian diffeomorphism

$$\begin{aligned} \delta_\xi S &= \int dt d^d x \sqrt{a} \Omega \left( \frac{1}{2} \Pi^{ij} \delta_\xi a_{ij} + \Pi^i \delta_\xi b_i - \frac{1}{\Omega} (\Pi + \Pi^i b_i) \delta_\xi \Omega + \zeta^i \delta_\xi \hat{\delta}^i \right. \\ &\left. + \frac{1}{2} B^{ij} \delta_\xi \hat{\beta}_{ij} + \frac{1}{2} \Lambda^{ij} \delta_\xi \hat{\lambda}_{ij} + \frac{1}{2} Z_k^{ij} \delta_\xi \hat{\rho}_{ij}^k \right). \end{aligned} \quad (1.3.28)$$

After a tedious but straightforward computation we get

$$\begin{aligned} \left( \frac{1}{\Omega} \partial_t + \theta \right) \Pi &+ \left( \hat{\nabla}_i + 2\varphi_i \right) \Pi^i + \Pi^{ij} \hat{\gamma}_{(ij)} + \zeta^i \frac{1}{\Omega} \hat{D}_i \hat{\delta}_i + \frac{1}{2} B^{ij} \frac{1}{\Omega} \hat{D}_i \hat{\beta}_{ij} + \frac{1}{2} \Lambda^{ij} \frac{1}{\Omega} \hat{D}_i \hat{\lambda}_{ij} \\ &+ \frac{1}{2} Z_k^{ij} \frac{1}{\Omega} \hat{D}_i \hat{\rho}_{ij}^k + \zeta^i \hat{\delta}^j \left( \hat{\gamma}_{(ij)} + \hat{\lambda}_{ij} \right) + \frac{1}{2} B^{ij} \left( \hat{\gamma}_{(ik)} \hat{\beta}_{ij}^k + \hat{\gamma}_{(jk)} \hat{\beta}_{ij}^k + \hat{\lambda}_{ik} \hat{\beta}_{ij}^k + \hat{\lambda}_{jk} \hat{\beta}_{ij}^k \right) \\ &+ \frac{1}{2} \Lambda^{ij} \left( \hat{\gamma}_{(ik)} \hat{\lambda}_{ij}^k + \hat{\gamma}_{(jk)} \hat{\lambda}_{ij}^k + \hat{\lambda}_{ik} \hat{\lambda}_{ij}^k + \hat{\lambda}_{jk} \hat{\lambda}_{ij}^k \right) \\ &+ \frac{1}{2} Z_k^{ij} \left( \hat{\gamma}_{(il)} \hat{\rho}_{ij}^{kl} + \hat{\gamma}_{(jl)} \hat{\rho}_{ij}^{kl} - \hat{\gamma}^{(kl)} \hat{\rho}_{lij} + \hat{\lambda}_{il} \hat{\rho}_{ij}^{kl} + \hat{\lambda}_{jl} \hat{\rho}_{ij}^{kl} - \hat{\lambda}^{kl} \hat{\rho}_{lij} \right) = 0 \end{aligned} \quad (1.3.29)$$

<sup>19</sup>Actually, all these pieces enter the relativistic hypermomentum (see App. A), usually denoted  $\Delta_\rho^{\mu\nu}$  with here  $\zeta^i = \Delta_i^i$ ,  $\hat{B}^{ij} = \Delta_i^{ij}$ ,  $\hat{\lambda}_i^j = \Delta_i^{ij}$  and  $\hat{Z}_k^{ij} = \Delta_k^{ij}$ .

and

$$\begin{aligned}
& \left( \hat{\nabla}_i + \varphi_i \right) \left[ \Pi^i_k + \zeta^i \hat{\delta}_k + \frac{1}{2} \hat{\beta}_{kj} B^{ij} + \frac{1}{2} \hat{\beta}_{jk} B^{ji} + \hat{\lambda}_{kj} \Lambda^{ij} + \hat{\rho}^l_{kj} Z_l^{ij} - \frac{1}{2} \hat{\rho}^i_{lj} Z_k^{lj} \right] \\
& + 2\Pi^i \hat{\omega}_{ik} + \Pi \varphi_k - \zeta^i \hat{\nabla}_k \hat{\delta}_i - \frac{1}{2} B^{ij} \hat{\nabla}_k \hat{\beta}_{ij} - \frac{1}{2} \Lambda^{ij} \hat{\nabla}_k \hat{\lambda}_{ij} - \frac{1}{2} Z_l^{ij} \hat{\nabla}_k \hat{\rho}^l_{ij} \\
& - 2\zeta^i \hat{\delta}_j \hat{\rho}^j_{ki} - B^{ij} \left( \hat{\beta}_{lj} \hat{\rho}^l_{ki} + \hat{\beta}_{il} \hat{\rho}^l_{kj} \right) - \Lambda^{ij} \left( \hat{\lambda}_{lj} \hat{\rho}^l_{ki} + \hat{\lambda}_{il} \hat{\rho}^l_{kj} \right) \\
& - Z_l^{ij} \left( \hat{\rho}^m_{ki} \hat{\rho}^l_{mj} + \hat{\rho}^m_{kj} \hat{\rho}^l_{im} - \hat{\rho}^l_{km} \hat{\rho}^m_{ij} \right) = - \left( \frac{1}{\Omega} \partial_t + \theta \right) P_k.
\end{aligned} \tag{1.3.30}$$

The two previous equations represent the implication of Carroll diffeomorphism invariance imposed on an action whose associated strong Carroll structure possesses the most general strong Carroll connection, without any other requirements but the preservation of the weak structure. As we have already discussed the irrelevance of the other degrees of freedom (which anyway would spoil the strong character of the connection), we will not display the conservation equations associated to the connection  $\bar{\nabla}$  of Sec. 1.2.<sup>20</sup>

## 1.4 Weyl covariance and its implications

When a Carroll structure lies on the null boundary of an asymptotically flat spacetime, it naturally inherits Weyl covariance from the bulk conformal compactification procedure, see e.g. [52]. In this Chapter where our study of Carrollian structure is intrinsic and irrespective of any bulk spacetime, we shall impose by hand covariance under Weyl transformations and adapt the previously developed framework to encompass the latter. This is the aim of the present Section.

### 1.4.1 Generalities

#### Conformal compactification and Weyl covariance

When dealing with null infinity in the context of asymptotically flat spacetimes, it turns out that the conformal compactification calls not for Carroll structures but for *conformal Carroll structures* i.e. structures which can encompass rescalings of the metric and the field of observers. From an intrinsic viewpoint we just add, *ex nihilo*, to the transformations a weak Carroll structure can support, a new one, dubbed *Weyl transformations*, whose action yields

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij} \quad , \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i \quad , \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega \tag{1.4.1}$$

<sup>20</sup>A motivated reader can however try to compute them as an exercise.

with  $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$  an arbitrary function. We read that these quantities possess a Weyl-weight  $-2$ ,  $-1$  and  $-1$  respectively. This implies at the level of the bases (1.1.9)

$$\{\mathbf{v}, \hat{\partial}_i\} \rightarrow \{\mathcal{B}\mathbf{v}, \hat{\partial}_i\} \quad \text{and} \quad \{\boldsymbol{\tau}, dx^i\} \rightarrow \{\mathcal{B}^{-1}\boldsymbol{\tau}, dx^i\}. \quad (1.4.2)$$

Hence the whole previous framework needs to be enhanced to include Weyl covariance. This is what we will do in the next paragraph.

**Remark** The metric  $a_{ij}$  being weighted, in the Weyl–Carroll framework the position of spatial indices is crucial. Raising an index increases the weight by 2 when lowering decreases it by 2.

### Weyl Carroll covariant derivative

Weyl-covariance under Weyl transformations requires the introduction of a *Weyl–Carroll* connection built on  $\varphi_i$  and  $\theta$  which transform as

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B} \quad \text{and} \quad \theta \rightarrow \mathcal{B}\theta - d\mathbf{v}(\mathcal{B}). \quad (1.4.3)$$

The Weyl–Carroll connection 1-form then takes the form<sup>21</sup> [53]

$$\mathbf{A} = \frac{1}{d}\theta\boldsymbol{\tau} + \varphi_i dx^i. \quad (1.4.4)$$

The Carrollian vorticity  $\hat{\omega}_{ij}$  and the Carrollian shear  $\xi_{ij}$  are Weyl-covariant of weight  $-1$ . See [62] for a discussion on Weyl-connections in the context of AdS/CFT holography.

The Weyl–Carroll space and time covariant derivatives are torsionless and metric-compatible. They are built upon our strong covariant derivatives  $\frac{1}{Q}\hat{D}_i$  and  $\hat{\nabla}$ . For a weight- $w$  scalar function  $\Phi$ , a vector with weight- $w$  components  $V^l$  and a form of weight- $w$  components  $W_l$ , the action is [53]

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w\varphi_j \Phi, \quad (1.4.5a)$$

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1)\varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i, \quad (1.4.5b)$$

$$\hat{\mathcal{D}}_j W_l = \hat{\nabla}_j W_l + (w+1)\varphi_j W_l + \varphi_l W_j - a_{jl} W_i \varphi^i. \quad (1.4.5c)$$

The Weyl–Carroll spatial derivative does not alter the weight, and one checks that  $\hat{\mathcal{D}}_j a_{kl} = 0$ . Especially we get for a rank-2 and weight- $w$  tensor

$$\hat{\mathcal{D}}_j t_{kl} = \hat{\nabla}_j t_{kl} + (w+2)\varphi_j t_{kl} + \varphi_k t_{jl} + \varphi_l t_{kj} - a_{jl} t_{ki} \varphi^i - a_{jk} t_{il} \varphi^i \quad (1.4.6a)$$

$$\hat{\mathcal{D}}_j t^{kl} = \hat{\nabla}_j t^{kl} + (w-2)\varphi_j t^{kl} + \varphi^l t^k_j + \varphi^k t_j^l - \delta_j^l t^{ik} \varphi_i - \delta_j^k t^{il} \varphi_i. \quad (1.4.6b)$$

<sup>21</sup>Note that this is the form one gets when taking the flat limit from the AdS Weyl connection appearing in the covariant Newmann–Unti gauge, see Sec. 3.3.1.

**Remark** Note that there is a pattern in the second factor of the right-hand-side with the weight. It is always of the form  $(w + p - q)$  for a  $p$ -covariant and  $q$ -contravariant tensor of weight- $w$ .

Regarding time, one defines [53]

$$\frac{1}{\Omega} \hat{\mathcal{D}}_i \Phi = \frac{1}{\Omega} \hat{D}_i \Phi + \frac{w}{d} \theta \Phi = \mathbf{v}(\Phi) + \frac{w}{d} \theta \Phi, \quad (1.4.7a)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_i V^l = \frac{1}{\Omega} \hat{D}_i V^l + \frac{w-1}{d} \theta V^l = \mathbf{v}(V^l) + \frac{w}{d} \theta V^l + \xi^l_i V^i, \quad (1.4.7b)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_i W_l = \frac{1}{\Omega} \hat{D}_i W_l + \frac{w+1}{d} \theta W_l = \mathbf{v}(W_l) + \frac{w}{d} \theta W_l - \xi_{lj} W^j. \quad (1.4.7c)$$

and all are of weight  $w + 1$ . Similarly for any tensor by Leibniz rule e.g. for a rank-2 tensor<sup>22</sup> of weight- $w$ ,

$$\frac{1}{\Omega} \hat{\mathcal{D}}_i t_{kl} = \frac{1}{\Omega} \hat{D}_i t_{kl} + \frac{w+2}{d} \theta t_{kl} = \mathbf{v}(t_{kl}) + \frac{w}{d} \theta t_{kl} - \xi_{kj} t^j_l - \xi_{lj} t^j_k. \quad (1.4.8)$$

In particular we find  $\hat{\mathcal{D}}_i a_{kl} = 0$ .

We now close this paragraph with the Weyl–Carroll curvature tensors, appearing in the commutation of Weyl–Carroll covariant derivatives. We find

$$[\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j] \Phi = \frac{2}{\Omega} \hat{\omega}_{ij} \hat{\mathcal{D}}_i \Phi + w \Omega_{ij} \Phi, \quad (1.4.9a)$$

$$[\hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l] V^i = \left( \hat{\mathcal{R}}^i_{jkl} - 2\xi^i_j \hat{\omega}_{kl} \right) V^j + \frac{2}{\Omega} \hat{\omega}_{kl} \hat{\mathcal{D}}_i V^i + w \Omega_{kl} V^i, \quad (1.4.9b)$$

where we have introduced the following Carrollian, weight-0 Weyl-covariant tensors

$$\begin{aligned} \hat{\mathcal{R}}^i_{jkl} &= \hat{r}^i_{jkl} - \delta^i_j \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i + \delta^i_k \hat{\nabla}_l \varphi_j - \delta^i_l \hat{\nabla}_k \varphi_j \\ &\quad + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) - \left( \delta^i_k a_{jl} - \delta^i_l a_{jk} \right) \varphi_m \varphi^m + \left( \delta^i_k \varphi_l - \delta^i_l \varphi_k \right) \varphi_j, \end{aligned} \quad (1.4.10a)$$

$$\Omega_{ij} = \hat{\delta}_i \varphi_j - \hat{\delta}_j \varphi_i - \frac{2}{d} \hat{\omega}_{ij} \theta. \quad (1.4.10b)$$

Additionally, we define traces as:

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k_{ikj}, \quad \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (1.4.11)$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1) \left( 2\hat{\nabla}_i \varphi^i - (d-2) \varphi_i \varphi^i \right). \quad (1.4.12)$$

Observe that the Weyl-covariant Carroll–Ricci tensor is not symmetric:  $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2} \Omega_{ij}$ . Finally, we recall that

$$\left[ \frac{1}{\Omega} \hat{\mathcal{D}}_i, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j_i \hat{\mathcal{D}}_j \Phi, \quad (1.4.13)$$

<sup>22</sup>Note the same pattern for the second term than for the spatial derivatives, of the form  $(w + p - q)$ .

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_i \varphi_i - \frac{1}{d} (\hat{\partial}_i + \varphi_i) \theta \quad (1.4.14)$$

are the components of a Weyl-covariant weight-1 Carrollian curvature one-form.

### 1.4.2 Weyl covariant conservation equations

Let's first recap the Weyl weight of the relevant quantities.<sup>23</sup> In the Table, an (a) means an inhomogeneous transformation like in (1.4.3).

Quantity	$\Pi^{ij}$	$\Pi^i$	$P^i$	$\Pi$	$a_{ij}$	$b_i$	$\Omega$	$\omega_{ij}$	$\xi_{ij}$	$\theta$	$\varphi_i$
Weight $w$	$d + 3$	$d + 2$	$d + 2$	$d + 1$	$-2$	$-1$	$-1$	$-1$	$-1$	$1 (a)$	$0 (a)$

(1.4.15)

The procedure is then to start from (1.3.20) and (1.3.21), go from usual covariant derivatives to Weyl–Carroll ones and ask for all terms proportionnal either to  $\theta$  (for the time equation) or  $\varphi_i$  (for the spatial one) to vanish, as they are inhomogeneous under Weyl transformations (see (1.4.3)). This will give the Carrollian equivalent to the usual tracefree condition implied by Weyl invariance in the relativistic framework. We find

$$\boxed{\frac{1}{\Omega} \hat{\mathcal{D}}_i \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} \hat{=} 0} \quad (1.4.16)$$

and

$$\boxed{\hat{\mathcal{D}}_i \left( \Pi^i_k \right) + 2 \Pi^j \omega_{jk} + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_i \delta_{ki} + \xi_{ki} \right) P^i \hat{=} 0,} \quad (1.4.17)$$

together with the condition

$$\boxed{\Pi^i_i \hat{=} \Pi.} \quad (1.4.18)$$

Starting from (1.3.25) we get

$$\boxed{\frac{1}{\Omega} \hat{\mathcal{D}}_i \rho + \hat{\mathcal{D}}_i N^i \hat{=} 0} \quad (1.4.19)$$

for a  $U(1)$  current.

### Completely free strong connection

We start by completing the table of weights (1.4.15) with the new ambiguities ( $\hat{\delta}_i, \hat{\beta}_{ij}, \hat{\gamma}_{[ij]} := \hat{\lambda}_{ij}$  and  $\hat{\gamma}^k_{[ij]} := \hat{\rho}^k_{ij}$ ) and their momenta (1.3.26). The method for finding the weights under Carrollian diffeomorphisms is the following. The defining properties of the strong Carrollian connection, namely the preservation of the field of observers and the metric, should be valid irrespective of the

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<sup>23</sup>The weight of the canonical momenta are found by demanding that the total action in  $(d + 1)$ -dimensions is of weight 0.

Weyl frame at hand. This gives

$$\hat{\gamma} \rightarrow \mathcal{B}\hat{\gamma} + \mathbf{v}(\mathcal{B}) \quad (1.4.20a)$$

$$\hat{\rho}^i \rightarrow \mathcal{B}^2 \hat{\rho}^i \quad (1.4.20b)$$

$$\hat{\alpha}_i \rightarrow \hat{\alpha}_i + \hat{\partial}_i(\ln \mathcal{B}) \quad (1.4.20c)$$

$$\hat{\kappa}_i^j \rightarrow \mathcal{B} \hat{\kappa}_i^j \quad (1.4.20d)$$

for the preservation of the vector, and

$$\hat{\gamma}_j^i \rightarrow \mathcal{B}\hat{\gamma}_j^i + \mathbf{v}(\mathcal{B})a_{ij} \quad (1.4.21a)$$

$$\hat{\gamma}_{(ij)}^k \rightarrow \hat{\gamma}_{(ij)}^k - \hat{\partial}_i(\ln \mathcal{B})\delta_j^k - \hat{\partial}_j(\ln \mathcal{B})\delta_i^k - \hat{\partial}^k(\ln \mathcal{B})a_{ij}. \quad (1.4.21b)$$

It therefore remains to fix the transformation rules for  $\hat{\delta}_i$  and  $\hat{\beta}_{ij}$ . We use for that the expression of the torsion (B.1.27) and (B.1.28) where we see that the non-torsion conditions (1.2.12) implies that

$$\hat{\delta}_i \rightarrow \hat{\delta}_i \quad (1.4.22a)$$

$$\hat{\beta}_{ij} \rightarrow \mathcal{B}^{-1} \hat{\beta}_{ij}. \quad (1.4.22b)$$

The complete table of weights then reads

Quantity	$\hat{\delta}_i$	$\hat{\zeta}^i$	$\hat{\beta}_{ij}$	$B^{ij}$	$\hat{\lambda}_{ij}$	$\hat{\Lambda}^{ij}$	$\hat{\rho}_{ij}^k$	$\hat{Z}_k^{ij}$
Weight $w$	0	$d+1$	-1	$d+2$	-1	$d+2$	0	$d+1$

(1.4.23)

After a very long and tedious but straightforward computation we get for the conservation equations

$$\begin{aligned} & \frac{1}{\Omega} \hat{\mathcal{D}}_i \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} + \frac{1}{2} B^{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_i \hat{\beta}_{ij} + \frac{1}{2} \Lambda^{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_i \hat{\lambda}_{ij} + \frac{1}{2} Z_k^{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_i \hat{\rho}_{ij}^k \\ & + \frac{1}{2} B^{ij} \left( \xi_{ik} \hat{\beta}_j^k + \xi_{jk} \hat{\beta}_i^k + \hat{\lambda}_{ik} \hat{\beta}_j^k + \hat{\lambda}_{jk} \hat{\beta}_i^k \right) + \frac{1}{2} \Lambda^{ij} \left( \xi_{ik} \hat{\lambda}_j^k + \xi_{jk} \hat{\lambda}_i^k + \hat{\lambda}_{ik} \hat{\lambda}_j^k + \hat{\lambda}_{jk} \hat{\lambda}_i^k \right) \\ & + \frac{1}{2} Z_k^{ij} \left( \xi_{il} \hat{\rho}_{ij}^{kl} + \xi_{jl} \hat{\rho}_{ij}^{kl} - \xi^{kl} \hat{\rho}_{lij} + \hat{\lambda}_{il} \hat{\rho}_{ij}^{kl} + \hat{\lambda}_{jl} \hat{\rho}_{ij}^{kl} - \hat{\lambda}^{kl} \hat{\rho}_{lij} \right) = 0, \end{aligned} \quad (1.4.24)$$

and

$$\begin{aligned} & \hat{\mathcal{D}}_i \left[ \Pi_k^i + \frac{1}{2} \hat{\beta}_{kj} B^{ij} + \frac{1}{2} \hat{\beta}_{jk} B^{ji} + \hat{\lambda}_{kj} \Lambda^{ij} + \hat{\rho}_{kj}^l Z_l^{ij} - \frac{1}{2} \hat{\rho}_{lj}^i Z_k^{lj} \right] \\ & + 2 \Pi^i \omega_{ik} - \frac{1}{2} B^{ij} \hat{\mathcal{D}}_k \hat{\beta}_{ij} - \frac{1}{2} \Lambda^{ij} \hat{\mathcal{D}}_k \hat{\lambda}_{ij} - \frac{1}{2} Z_l^{ij} \hat{\mathcal{D}}_k \hat{\rho}_{ij}^l \\ & - B^{ij} \left( \hat{\beta}_{lj} \hat{\rho}_{ki}^l + \hat{\beta}_{il} \hat{\rho}_{kj}^l \right) - \Lambda^{ij} \left( \hat{\lambda}_{lj} \hat{\rho}_{ki}^l + \hat{\lambda}_{il} \hat{\rho}_{kj}^l \right) \\ & - Z_l^{ij} \left( \hat{\rho}_{ki}^m \hat{\rho}_{mj}^l + \hat{\rho}_{kj}^m \hat{\rho}_{im}^l - \hat{\rho}_{km}^l \hat{\rho}_{ij}^m \right) \hat{=} - \left( \frac{1}{\Omega} \hat{\mathcal{D}}_i \delta_{ki} + \xi_{ki} \right) P^i. \end{aligned} \quad (1.4.25)$$

together with the Weyl condition

$$\boxed{\Pi^i_{;i} + \frac{1}{2}\hat{\beta}_{ij}B^{ij} + \frac{1}{2}\hat{\lambda}_{ij}\Lambda^{ij} + \frac{1}{2}Z_k^{ij}\hat{\rho}^k_{;ij} = \Pi.} \quad (1.4.26)$$

Recall that (1.4.24) and (1.4.25) result from the Carroll diffeomorphism invariance plus Weyl covariance of the effective action, nothing more.

**Remark** One might also want to Weyl covariantise the conservation equations obtained with the completely free connection  $\hat{\nabla}$ . However, as Weyl covariance appears in a physical situation (on  $\mathcal{I}^+$ ) for which the Carrollian structure is endowed with a strong Carrollian connection, there is no point, or at least no relevant applications known to us to these Weyl covariant equations.

## 1.5 Killing vectors, charges and conservation

This section deals with the notion of isometries and charges on a Carroll structure. This will also be the occasion to encounter for the first time the Carroll algebra and its conformal extension. It is highly inspired by [65, 66].

### Carroll-Killing equations and the Carroll algebra

Isometries are by definition [64, 65] generated by vector fields  $\xi$  satisfying

$$\mathcal{L}_\xi a_{ij} = 0 \quad \text{and} \quad \mathcal{L}_\xi \mathbf{v} = 0. \quad (1.5.1)$$

This gives in the frame at hand the Carroll-Killing equations

$$\boxed{\hat{\nabla}_{(i}\xi_{j)} + \xi^{\hat{t}}\hat{\gamma}_{(ij)} = 0 \quad , \quad \mathbf{v}\left(\xi^{\hat{t}}\right) + \varphi_i\xi^i = 0.} \quad (1.5.2)$$

These equations reflect the invariance of the weak Carroll structure and usually possess an infinite number of solutions, dubbed *Carrollian Killing vectors*. One may also ask the connection to be left invariant by  $\xi$ ,<sup>24</sup> then the solution space of (1.5.2) may be drastically reduced. However, we do not ask the clock form to be left invariant and for  $\xi$  a Carrollian Killing (1.3.13) becomes

$$\mathcal{L}_\xi \Omega = 0 \quad \text{and} \quad \mathcal{L}_\xi b_i = (\hat{\partial}_i - \varphi_i)\xi^{\hat{t}} - 2\xi^j\hat{\omega}_{ji}. \quad (1.5.3)$$

For example let us go in the case  $a_{ij} = \delta_{ij}$ , constant  $b_i$  and  $\Omega = 1$ , which is the *flat ruled Carrollian*

<sup>24</sup>Using in particular (1.3.27a), (1.3.27b), (1.3.27c) and (1.3.27d).

structure. There, equations (1.5.2) have an infinite number of solutions

$$\xi = \left( \Omega_i^j x^i + X^j \right) \partial_j + f(\mathbf{x}) \partial_t \quad (1.5.4)$$

where  $\Omega_{ij} = \Omega_i^k \delta_{kj}$  are constants and antisymmetric (generating the  $\mathfrak{so}(d)$  Lie algebra of rotations),  $X^j$  are constants (generating the spatial translations) and  $f(\mathbf{x})$  is an arbitrary function of space. If the flat Carroll connection is also invariant<sup>25</sup> under  $\xi$  then one has  $f = T - B_i x^i$  where  $T$  generates times translations and  $B_i$  Carroll boosts. One then recovers<sup>26</sup>

$$\xi_{\text{carr}} = \left( \Omega_i^j x^i + X^j \right) \partial_j + \left( T - B_i x^i \right) \partial_t \quad (1.5.7)$$

i.e. the  $(d+1)(d+2)/2$  solutions generating the Carroll algebra  $\text{carr}(d+1)$ .<sup>27</sup> The vector  $\xi_{\text{carr}}$  forms a differential representation of the latter with generators  $(J_{ij}, P_i, B_j, H)$  ( $i, j = 1, \dots, d$ ) associated with  $(\Omega_{ij}, X_i, B_i, T)$  respectively. The non-vanishing commutators are

$$[J_{ij}, P_k] = \delta_{kj} P_i - \delta_{ki} P_j \quad (1.5.8a)$$

$$[J_{ij}, B_k] = \delta_{kj} B_i - \delta_{ki} B_j \quad (1.5.8b)$$

$$[J_{ij}, J_{kl}] = \delta_{ik} J_{jl} + \delta_{jl} J_{ki} - \delta_{il} J_{jk} - \delta_{jk} J_{li} \quad (1.5.8c)$$

$$[B_i, P_j] = \delta_{ij} H. \quad (1.5.8d)$$

The Hamiltonian is then a central element and the last commutator on the one hand betrays the presence of an Heisenberg subalgebra  $(H, P_i, B_i)$  while on the other hand it prevents the Carroll algebra from being centrally extended, contrary to its dual ‘‘Galilean’’ algebra whose central extension yields the Bargmann algebra and the notion of mass.<sup>28</sup> Carroll boosts commute hence there is no Thomas precession in a Carroll structure. Note that at the level of the Ehresmann connection we get when  $\xi$  belongs to the Carroll algebra

$$\mathcal{L}_{\xi_{\text{carr}}} b_i = - \left( B_i + \Omega_i^j b_j \right) \neq 0 \quad (1.5.9)$$

exhibiting a constant shift related to the boost parameter  $B_i$  as well as a  $\mathfrak{so}(d)$  rotation. This justifying *a posteriori* the mostly used terminology employed to refer to this transformation. Also note

<sup>25</sup>The connection  $\hat{\nabla}$  being Levi-Civita on the spatial sections we can use the identity

$$\delta_{\xi} \hat{\gamma}_{(ij)}^k = \frac{1}{2} a^{kl} \left( 2 \hat{\nabla}_{(i} (\delta_{\xi} a_{j)l}) - \hat{\nabla}_l (\delta_{\xi} a_{ij}) \right) \quad (1.5.5)$$

which implies that  $f(\mathbf{x})$  has to be at most linear in  $\mathbf{x}$ .

<sup>26</sup>The Carroll group acts on coordinates as

$$\begin{cases} t' = t + B_i x^i + t_0 \\ x'^k = R_i^k x^i + x_0^k \end{cases} \quad (1.5.6)$$

with  $R \in SO(d)$  and  $B_i$  a constant covector.

<sup>27</sup>This algebra can also be obtained as the Inönü-Wigner contraction of the Poincaré algebra when  $c \rightarrow 0$ , see e.g. [50].

<sup>28</sup>In  $(1+1)$ -dimensions the Carroll and Galilean algebras are isomorphic to the Heisenberg algebra, upon exchanging the role of time and space, hence the two-dimensional Carroll algebra can be centrally extended.



that Carroll boosts are broken if one asks the Ehreman connection to be preserved by  $\xi$  in addition to the weak structure.

**Remark** Actually, to go from (1.5.4) to (1.5.7) instead of asking the flat connection to be preserved one can ask the ruled flat Carroll structure to be preserved. This implies that  $b_i$  (which is constant) has to be shifted only by a constant, implying that  $\xi^t$  is at most linear in  $\mathbf{x}$ . Hence the full flat Carroll structure including a constant Ehresmann is preserved only by the Carroll algebra.

### Carrollian Killing charges

In a relativistic framework, if  $\xi$  is a Killing field of the spacetime  $\mathcal{M}$  and  $T_{\mu\nu}$  the energy-momentum tensor of the theory, the current defined as

$$I_\mu = \xi^\nu T_{\mu\nu} \quad (1.5.10)$$

has zero divergence and (recall that  $\mathcal{S}$  is the  $d$ -dimensional spatial section of  $\mathcal{M}$  with induced metric of determinant  $\sigma$  and outwards pointing normal  $n^\mu$ )

$$Q_I = \int_{\mathcal{S}} d^d x \sqrt{\sigma} n_\mu I^\mu \quad (1.5.11)$$

is conserved. On a Carrollian structure a current has a scalar component  $\kappa$  as well as a Carrollian-vector set of components  $K^i$  which are build upon the Carrollian momenta, and the Carrollian Killing field

$$\kappa = \xi^i P_i - \xi^t \Pi \quad K^i = \xi^j \Pi_j^i - \xi^t \Pi^i. \quad (1.5.12)$$

**Remark** In the relativistic framework, varying an action  $S = S[g_{\mu\nu}; \Phi]$  functional of the metric  $g_{\mu\nu}$  and of a collection of matter fields  $\Phi$ , one gets after variation under a general diffeomorphism and integration by parts

$$\delta_\xi S \hat{=} - \int_{\mathcal{M}} d^{d+1} x \sqrt{-g} \xi^\mu \nabla_\nu T_\mu^\nu + \int_{\mathcal{M}} d^{d+1} x \sqrt{-g} \nabla_\nu \left( \xi^\mu T_\mu^\nu \right) \quad (1.5.13)$$

where the first term gives the conservation equation while the conserved current is displayed in the second term (i.e. the boundary term). Applying the same method in the Carrollian setup we read the expressions (1.5.12) from the boundary terms of (1.3.20) and (1.3.21).

The divergence of the current  $(\kappa, K^i)$  takes the form

$$\mathcal{H} = \left( \frac{1}{\Omega} \partial_t + \theta \right) \kappa + \left( \hat{\nabla}_i + \varphi_i \right) K^i = -\Pi^i (\mathcal{L}_\xi \tau)_i \quad (1.5.14)$$

implying that, contrary to the relativistic case,

**Proposition 1.0.5.** A plain Carrollian Killing field does not generically provide a conservation law in Weyl-invariant Carrollian dynamics.

Conservation is subject either to the cancellation of the energy flux, which translates local Carroll-shift invariance (requirement not even satisfied in the flat Carroll structure) or to the Carrollian Killing field satisfying

$$\mathcal{L}_\xi \tau = 0. \quad (1.5.15)$$

We dubbed the latter *strong Carrollian Killings* [66]. Note that nothing prevents *a priori*  $\mathcal{K}$  to be a boundary term, but this is to be appreciated case-by-case, see in particular the next remark on potential flow below. Strong Killings form a subalgebra of the algebra of Carroll–Killing field.<sup>29</sup> Preserving the clock form is actually a boost dependent statement, by virtue of (1.1.14). Hence the associated charges would be conserved only in a particular boost frame.

**Remark** It is too strong to ask that  $\mathcal{K}$  identically vanishes for conservation to occur, as  $\mathcal{K}$  can actually just be a divergence. In the flat instance we get  $\mathcal{K} = \Pi^i (B_i + \Omega_i^j b_j)$  using (1.5.7) and (1.5.9).  $U_i = B_i + \Omega_i^j b_j$  is then a constant covector. If there exist two functions  $\phi(t, \mathbf{x})$  and  $\tilde{\phi}_i(t, \mathbf{x})$  such that  $\mathcal{K} = \partial_t(\phi) U^i + \partial_i(\tilde{\phi}) U^i$  then it is easy to remark that, using (1.3.14a),(1.3.14b) and knowing that  $\sqrt{a} = 1$  and  $\Omega = 1$ ,

$$\left(\frac{1}{\Omega} \partial_t + \theta\right) (\kappa - \phi_i U^i) + (\hat{\nabla}_i + \varphi_i) (K^i - \phi U^i) = 0. \quad (1.5.16)$$

This defines a *potential flow conserved current* for boosts and rotations, built upon  $\tilde{\kappa} = \kappa - \phi_i U^i$  and  $\tilde{K}^i = (K^i - \phi U^i)$ .

The charge associated with the current  $(\kappa, \mathbf{K})$  is an integral at fixed  $t$  over the base manifold  $\mathcal{S}$  of the Carrollian structure<sup>30</sup>

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa + b_i K^i), \quad (1.5.17)$$

whose time evolution reads

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial \mathcal{S}} \star \mathbf{K} \Omega, \quad (1.5.18)$$

where  $\star \mathbf{K}$  is the  $\mathcal{S}$ -Hodge dual of  $K_i dx^i$ . This shows that for vanishing divergence  $\mathcal{K}$ , this is conserved if one can ignore the boundary term owing to adequate fall-off or boundary conditions on the fields, as expected.

**Remark** One may say that the presence of  $b_i$  in (1.5.17) breaks Carroll diffeomorphism covariance down to a covariance under the diffeomorphisms that preserve the form of the integrand.

<sup>29</sup>When the Carrollian structure is the null boundary of an asymptotically flat spacetime, plain Killings of the bulk become strong Carrollian Killings on the boundary, see [66] and Sec. 4.2.

<sup>30</sup>This formula is not a guess but comes naturally when seeing the Carrollian structure as coming from a relativistic ascendant, see [61] and Sec. 2.1.1.

These are *Aristotelian diffeomorphisms*  $t' = t'(t)$  and  $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ . This is actually not a caveat as it only translates a feature of the hypersurface we have chosen to compute the charge, that is  $\mathcal{S}$ . For example for a Carroll structure whose clock form is closed  $\tau = d\psi$  one may choose as space-like hypersurface  $\Sigma_\psi$  such that  $\psi(t, \mathbf{x}) = \text{cst}$  and obtain  $Q_K = \int_{\Sigma_\psi} d^d x \sqrt{a} \kappa$ . Nevertheless all choices of spacelike hypersurfaces lead to the same charge.

**Remark** If we are given a  $U(1)$  current satisfying (1.3.25) then one can build the conserved charge

$$Q_N = \int_S d^d x \sqrt{a} \left( \rho + b_i N^i \right). \quad (1.5.19)$$

Taking  $\mathcal{V} \subset \mathcal{S}$  such that  $\partial\mathcal{V}$  is time-independent we get for the time evolution of the  $U(1)$  charge

$$\frac{dQ_N}{dt} = - \int_{\mathcal{V}} d^d x \partial_i \left( \sqrt{a} \Omega N^i \right) = - \int_{\partial\mathcal{V}} \Omega \star \mathbf{N}. \quad (1.5.20)$$

### Conformal Carrollian isometries and the BMS algebra

Conformal isometries are generated by a vector field  $\xi$  satisfying

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij} \quad \text{and} \quad \mathcal{L}_\xi \mathbf{v} = \mu \mathbf{v} \quad (1.5.21)$$

where

$$\lambda(t, \mathbf{x}) = \frac{2}{d} \left( \hat{\nabla}_i \xi^i + \theta \xi^i \right) \quad \text{and} \quad \mu(t, \mathbf{x}) = - \left( \frac{1}{\Omega} \partial_i \xi^i + \varphi_i \xi^i \right). \quad (1.5.22)$$

An extra condition is usually imposed for reaching an operational definition of conformal Killing vectors. The guideline for this is Weyl covariance, imposed by hand on a Carrollian structure but coming from the bulk in a holographic perspective (See Sec. 1.4). A desirable feature of conformal Killings is to be insensitive to Weyl rescalings of the metric. Under (1.4.1) we find

$$\lambda \mapsto \lambda - 2\xi^i \hat{\partial}_i (\ln \mathcal{B}) \quad \text{and} \quad \mu \mapsto \mu + \xi^i \hat{\partial}_i (\ln \mathcal{B}). \quad (1.5.23)$$

The Weyl-invariant combination then reads

$$2\mu + \lambda \quad (1.5.24)$$

which can be set to zero. It physically means that conformal isometry should treat time and space the same way, via this combination adapted to the Weyl weights of the metric and the field of observers. To solve (1.5.21) for  $\xi$  we first plug the value of  $\lambda(t, \mathbf{x})$  back into (1.5.21). This yields

$$\hat{\nabla}_{(i} \xi_{j)} - \frac{2}{d} \hat{\nabla}_k \xi^k a_{ij} = -2\xi^i \xi_{ij} \quad (1.5.25)$$

while (1.5.24) gives a second equation

$$\tilde{\nabla}_i \xi^i - d\varphi_i \xi^i = d \frac{1}{\Omega} \partial_t \xi^i - \theta \xi^i. \quad (1.5.26)$$

A conformal Carrollian Killing field is then a solution of (1.5.25) and (1.5.26). This is a very hard system to solve without any further assumptions. However given that most of the applications (especially in holography) concern shearless Carroll structures (i.e.  $\xi_{ij} = 0$ ) this will be our framework from now on. We first note the important result<sup>31</sup>

**Proposition 1.0.6.** In a weak Carrollian structure  $(a_{ij}, \mathbf{v})$ , the vanishing of the geometric shear,  $\xi_{ij} = 0$  is equivalent to the factorization of time-dependence of the metric

$$a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \tilde{a}_{ij}(\mathbf{x}). \quad (1.5.27)$$

Under this hypothesis one can perform a Weyl rescaling by a function  $\mathcal{B}(t, \mathbf{x}) = e^{\sigma(t, \mathbf{x})}$  that remove the time dependence of the metric, leading to simpler equations whose general solution is

$$\xi_{T,Y} = \left( T(\mathbf{x}) - Y^i(\mathbf{x}) \hat{\partial}_i C(t, \mathbf{x}) + \frac{1}{d} C(t, \mathbf{x}) \tilde{\nabla}_i Y^i(\mathbf{x}) \right) e^{-\sigma(t, \mathbf{x})} \mathbf{v} + Y^i(t, \mathbf{x}) \hat{\partial}_i \quad (1.5.28)$$

with  $\tilde{\nabla}_i$  the Levi-Civita connection for  $\tilde{a}_{ij}$ ,  $T(\mathbf{x})$  an arbitrary function and  $Y^i(\mathbf{x})$  satisfying

$$\tilde{\nabla}_{(i} Y_{j)} = \frac{1}{d} \tilde{\nabla}_k Y^k \tilde{a}_{ij}. \quad (1.5.29)$$

Also we have

$$C(t, \mathbf{x}) := \int^t d\tau e^{-\sigma(\tau, \mathbf{x})} \Omega(t, \mathbf{x}). \quad (1.5.30)$$

Hence in the shearless case the algebra of conformal Carrollian Killing vectors<sup>32</sup> is just

$$\mathfrak{ccarr}_{d+1}^* = \mathfrak{conf}(\tilde{a}_{ij}) \ltimes \mathfrak{s} \quad (1.5.31)$$

where  $\mathfrak{s}$  stands for the supertranslations algebra (generated by  $T(\mathbf{x})$ ) and the other term is the algebra of conformal isometries of  $\tilde{a}_{ij}$ . The commutator of two such vector fields yields

$$[\xi_{T,Y}, \xi_{\hat{T}, \hat{Y}}] = \xi_{M_Y(\hat{T}) - M_Y(T), [Y, \hat{Y}]} \quad (1.5.32)$$

where the operator  $M_Y$  acts on any function  $\Phi(t, \mathbf{x})$  as

$$M_Y(\Phi) := Y^i \hat{\partial}_i(\Phi) - \frac{1}{d} \Phi \tilde{\nabla}_i Y^i. \quad (1.5.33)$$

<sup>31</sup>The proof is simple. If  $\xi_{ij} = 0$  then from (1.2.18) one gets an PDE on  $a_{ij}(t, \mathbf{x})$  whose most general solution is (1.5.27).

<sup>32</sup>We denote it with a star to make the difference with the flat structure case which is *in fine* related to the BMS algebra.

For example, let us consider again the standard flat Carroll spacetime ( $\tilde{a}_{ij} = \delta^{ij}$ ,  $\Omega = 1$  and constant  $b_i$ ). Equations (1.5.21) and (1.5.24) possess an infinite number of solutions, which for a strong Carroll structure with flat connection read

$$\xi_{\text{ccarr}} = Y^j(\mathbf{x})\hat{\partial}_j + \left( T(\mathbf{x}) + \frac{t}{d}\hat{\partial}_i Y^i \right) \partial_t \quad (1.5.34)$$

with  $T(\mathbf{x})$  an arbitrary function generating the so-called *supertranslations* and  $Y^i(\mathbf{x})\partial_j$  being conformal Killing fields of Euclidean  $d$ -dimensional space, generating  $\mathfrak{so}(d+1, 1)$ . This is the conformal Carroll algebra  $\text{ccarr}(d+1) \equiv \mathfrak{so}(d+1, 1) \ltimes \mathfrak{s}$ , also known as the  $\text{BMS}_{d+2}$  for Bondi–van der Burg–Metzner–Sachs. Note that this is the usual BMS algebra which could be extended to the version of [77] at the expense of giving up integrability at 0 and  $\infty$ .<sup>33</sup> However as  $Y(\mathbf{x})$  has to be a conformal Killing of the sphere we cannot recover the generalized version of the BMS algebra proposed in [78].

Assuming the existence of a conformal isometry, the conservation equations (1.3.20) and (1.3.21) can be used for computing the Carrollian scalar  $\mathcal{K}$ , corresponding to the divergence of the current built upon  $\xi$  (as in (1.5.34)) and Carrollian momenta,

$$\mathcal{K} = \Pi \left( \frac{\lambda}{2} + \mu \right) - \Pi^i \left( (\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \hat{\omega}_{ji} \right). \quad (1.5.35)$$

The defining equation (1.5.24) for conformal Killing vectors on Carrollian spacetimes expectantly arises in (1.5.35), but is insufficient to ensure  $\mathcal{K} = 0$ . As anticipated,

**Proposition 1.0.7.** A plain conformal Killing field does not generically provide a conservation law in Weyl-invariant Carrollian dynamics.

The conditions to satisfy for conservation to occur are the same than for a plain Killing. Hence there also exist a notion of *strong conformal Carrollian Killing fields*, for which the associated charge is always conserved.

**Remark** Actually the most general extra condition is not (1.5.24) but rather  $2\mu + z\lambda = 0$  with  $z$  a constant dubbed the dynamical exponent [55, 64] such that  $\mathbf{v} \rightarrow \mathcal{B}^z \mathbf{v}$ . The associated algebra exhibits a level  $N = 2/z$ :  $\text{ccarr}_N(d+1)$ . Strictly speaking  $\text{ccarr}(d+1) \equiv \text{ccarr}_2(d+1)$  is  $\text{BMS}_{d+2}$  for  $d = 1, 2$  only.

Under a conformal isometry the clock form behaves as

$$\mathcal{L}_\xi \tau = \left( \partial_i \left( T - Y^j b_j \right) + \frac{b_i}{d} \partial_j Y^j + \frac{t}{d} \partial_i \partial_j Y^j \right) dx^i. \quad (1.5.36)$$

The associated current is not conserved since  $\mathcal{K}$  in (1.5.35) does not generically vanish, unless  $\partial_j Y^j = C_0$  and  $T = T_0 + Y^j b_j - \frac{C_0}{d} b_i x^i$  with  $C_0$  and  $T_0$  constants. This excludes the  $d$  special con-

<sup>33</sup>We are here moving from  $S^2$  to  $\mathbb{C}$  via the stereographic projection.

formal transformations of  $\mathfrak{so}(d+1, 1)$  and leaves the supertranslations with the time translation as unique freedom, leading to a symmetry subgroup of finite dimension  $\frac{d^2}{2} + \frac{d}{2} + 2$ .

**Remark** Note that some authors, like the ones of [79] call conformal Carroll algebra the  $c \rightarrow 0$  limit of the usual conformal algebra  $\mathfrak{so}(2, d-1)$  which would hence be finite-dimensional and isomorphic to the Poincaré algebra, contrary to (1.5.31).

**Remark** It is also possible that  $\Pi^i \left( (\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \hat{\omega}_{ji} \right)$  with  $\xi$  a conformal Killing field of a Carrollian manifold  $\mathcal{M}$  might be a boundary term, possibly leading to a conserved Carrollian current in the same way than for plain Killings and potential flows, see for example the magnetic scalar field on a Robinson-Trautman background in Sec. 2.3.3.

## 1.6 Three dimensional Carroll structures in holomorphic coordinates

When the dimension of the base space  $\mathcal{S}$  is two i.e. when  $d = 2$ , it is convenient to use complex spatial coordinates  $\zeta$  and  $\bar{\zeta}$ . With the permission of the authors of [52] and [66], we paraphrase here the appendices of these references, summarizing the useful formulas in this coordinate system as we will use many of them in Chapters 3 and 5. Using Carrollian diffeomorphisms (1.1.7), the metric of the Carrollian geometry on the two-dimensional surface  $\mathcal{S}$  can be recast in conformally flat form,

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \quad (1.6.1)$$

with  $P = P(t, \zeta, \bar{\zeta})$  a real function, under the necessary and sufficient condition that the Carrollian shear  $\xi_{ij}$  displayed in (1.2.18) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data:  $\Omega(t, \zeta, \bar{\zeta})$  and

$$\mathbf{b} = b_\zeta(t, \zeta, \bar{\zeta}) d\zeta + b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) d\bar{\zeta} \quad (1.6.2)$$

with  $b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) = \bar{b}_\zeta(t, \zeta, \bar{\zeta})$ .

### Hodge duality in two-dimensions

For  $d = 2$ , the  $\mathcal{S}$ -Hodge duality is induced by  $\eta_{ij} = \sqrt{a}\epsilon_{ij}$ . Our conventions are the ones of Ref. [52], namely  $\epsilon_{12} = -1$ ,  $\eta_{\zeta\bar{\zeta}} = \frac{-i}{P^2}$ ,  $\sqrt{a} = \frac{i}{P^2}$  and therefore the spatial volume form reads  $\frac{1}{2}\eta_{ij}dx^i \wedge dx^j = d^2x \sqrt{a} = \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$ . Note that  $\eta^{il}\eta_{jl} = \delta_j^i$  and  $\eta^{ij}\eta_{ij} = 2$  so that

$$\boldsymbol{\tau} \wedge dx^i \wedge dx^j = \eta^{ij} dt d^2x \sqrt{a} \Omega = -i \eta^{ij} dt \frac{d\zeta \wedge d\bar{\zeta}}{P^2} \Omega. \quad (1.6.3)$$

This duality is involutive on Carrollian vectors as well as on two-index symmetric and traceless Carrollian tensors:

$$*V_i = \eta^l{}_i V_l, \quad *W_{ij} = \eta^l{}_i W_{lj}, \quad (1.6.4)$$

in particular

$$*\omega = \frac{1}{2} \eta^{ij} \omega_{ij} \iff \omega_{ij} = *\omega \eta_{ij}. \quad (1.6.5)$$

This fully antisymmetric form can be used to recast some of the expressions introduced in Sec 1.3.1.

The first-derivative Carrollian tensors are the acceleration (1.1.17), the expansion (1.2.19) and the scalar vorticity (1.6.5):

$$\varphi_\zeta = \partial_t \frac{b_\zeta}{\Omega} + \hat{\partial}_\zeta \ln \Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln \Omega, \quad (1.6.6a)$$

$$\theta = -\frac{2}{\Omega} \partial_t \ln P, \quad *\omega = \frac{i\Omega P^2}{2} \left( \hat{\partial}_\zeta \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_\zeta}{\Omega} \right) \quad (1.6.6b)$$

with

$$\hat{\partial}_\zeta = \partial_\zeta + \frac{b_\zeta}{\Omega} \partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega} \partial_t. \quad (1.6.7)$$

Curvature scalars and vector are second-derivative (see (1.3.8))<sup>34</sup>

$$\hat{K} = P^2 (\hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta + \hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}}) \ln P, \quad \hat{A} = iP^2 (\hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta - \hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}}) \ln P, \quad (1.6.8a)$$

$$\hat{r}_\zeta = \frac{1}{2} \hat{\partial}_\zeta \left( \frac{1}{\Omega} \partial_t \ln P \right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2} \hat{\partial}_{\bar{\zeta}} \left( \frac{1}{\Omega} \partial_t \ln P \right), \quad (1.6.8b)$$

and we also quote

$$*\varphi = iP^2 (\hat{\partial}_\zeta \varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}} \varphi_\zeta), \quad (1.6.9a)$$

$$\hat{\nabla}_k \varphi^k = P^2 \left[ \hat{\partial}_\zeta \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \partial_t \frac{b_\zeta}{\Omega} + (\hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta) \ln \Omega \right]. \quad (1.6.9b)$$

## Weyl-Carroll derivatives

We also remind for convenience some expressions for the determination of Weyl–Carroll covariant derivatives. If  $\Phi$  is a weight- $w$  scalar function

$$\hat{\mathcal{D}}_\zeta \Phi = \hat{\partial}_\zeta \Phi + w \varphi_\zeta \Phi, \quad \hat{\mathcal{D}}_{\bar{\zeta}} \Phi = \hat{\partial}_{\bar{\zeta}} \Phi + w \varphi_{\bar{\zeta}} \Phi. \quad (1.6.10)$$

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<sup>34</sup>We also quote for completeness

$$\hat{K} = K + P^2 \left[ \partial_\zeta \frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}} \frac{b_\zeta}{\Omega} + \partial_t \frac{b_\zeta b_{\bar{\zeta}}}{\Omega^2} + 2 \frac{b_{\bar{\zeta}}}{\Omega} \partial_\zeta + 2 \frac{b_\zeta}{\Omega} \partial_{\bar{\zeta}} + 2 \frac{b_\zeta b_{\bar{\zeta}}}{\Omega^2} \partial_t \right] \partial_t \ln P$$

with  $K = 2P^2 \partial_\zeta \partial_{\bar{\zeta}} \ln P$  the ordinary Gaussian curvature of the two-dimensional metric (1.6.1).

For weight- $w$  form components  $V_\zeta$  and  $V_{\bar{\zeta}}$  the Weyl–Carroll derivatives read

$$\hat{\mathcal{D}}_\zeta V_\zeta = \hat{\nabla}_\zeta V_\zeta + (w + 2)\varphi_\zeta V_\zeta, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} + (w + 2)\varphi_{\bar{\zeta}} V_{\bar{\zeta}}, \quad (1.6.11a)$$

$$\hat{\mathcal{D}}_\zeta V_{\bar{\zeta}} = \hat{\nabla}_\zeta V_{\bar{\zeta}} + w\varphi_\zeta V_{\bar{\zeta}}, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_\zeta = \hat{\nabla}_{\bar{\zeta}} V_\zeta + w\varphi_{\bar{\zeta}} V_\zeta, \quad (1.6.11b)$$

while the Carrollian covariant derivatives are simply

$$\hat{\nabla}_\zeta V_\zeta = \frac{1}{P^2} \hat{\partial}_\zeta (P^2 V_\zeta), \quad \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} = \frac{1}{P^2} \hat{\partial}_{\bar{\zeta}} (P^2 V_{\bar{\zeta}}), \quad \hat{\nabla}_\zeta V_{\bar{\zeta}} = \hat{\partial}_\zeta V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}} V_\zeta = \hat{\partial}_{\bar{\zeta}} V_\zeta. \quad (1.6.12)$$

Finally,

$$\hat{\mathcal{D}}_k \hat{\mathcal{D}}^k \Phi = P^2 (\hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w\Phi (\hat{\partial}_\zeta \varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \varphi_\zeta) + 2w (\varphi_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \varphi_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w\varphi_\zeta \varphi_{\bar{\zeta}} \Phi)). \quad (1.6.13)$$

The weight-1 curvature one-form (1.4.14) is

$$\hat{\mathcal{R}}_\zeta = \frac{1}{\Omega} \partial_t \varphi_\zeta - \frac{1}{2} (\hat{\partial}_\zeta + \varphi_\zeta) \theta, \quad \hat{\mathcal{R}}_{\bar{\zeta}} = \frac{1}{\Omega} \partial_t \varphi_{\bar{\zeta}} - \frac{1}{2} (\hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}}) \theta. \quad (1.6.14)$$

and the Carroll-Ricci

$$\hat{\mathcal{R}}_{ij} = \hat{s}_{ij} + \hat{\mathcal{H}} a_{ij} + \hat{\mathcal{A}} \eta_{ij}, \quad (1.6.15)$$

where we have introduced two weight-2 Weyl-covariant scalar Gauss–Carroll curvatures:

$$\hat{\mathcal{H}} = \frac{1}{2} a^{ij} \hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2} \eta^{ij} \hat{\mathcal{R}}_{ij} = \hat{A} - * \varphi. \quad (1.6.16)$$

## 1.7 Discussion

This concludes our study of Carrollian structures in split formalism. Weak Carroll structures contains a degenerate metric and a field of observers. If one considers an Ehresmann connection the structure is said to be ruled. Adding on top of that a compatible affine connection makes it a strong Carroll structure. The split formalism provides a complementary approach to the covariant approach as it completely makes use of the natural splitting between the Carrollian time direction and the spatial ones. We have been as comprehensive as possible while fitting with the historical definitions of [54] and much later [49] which considers only nowhere vanishing field of observers. As explained in a remark such a definition is unable to encompass Carroll black hole horizons [70], it should then be extended and consequences at the level of the connection and the conservation equations should be studied. This is to be done in future endeavor. Weyl covariance can be added to the geometry at the price of modifying slightly the formalism. Finally we ended up presenting the notion of isometries and charges of Carroll structures.

The two main physical applications of Carrollian geometry are null infinity  $\mathcal{I}^+$  and black hole horizons. Null infinity is a manifold that makes use of all transformations displayed in this Chapter: Carrollian diffeomorphisms, shift symmetry and also Weyl transformations, as the structure ap-



pearing on it is conformal. In that instance, as  $\mathcal{S}^+$  is related to a bulk, the dynamics of the latter will impose constraints on its geometry. In particular the geometric shear identically vanishes ( $\xi_{ij} = 0$ ) and it is always possible to perform a Weyl rescaling that cancels the expansion (1.2.19) (at the expense of reducing the residual Weyl symmetry to time-independent rescalings though). The connection on  $\mathcal{S}^+$  is induced by the bulk Levi-Civita and can always be chosen to be torsionless [38] as one can cancel the extrinsic curvature. The conformal isometries of null infinity are (1.5.31) which as expected coincide with the BMS algebra in one dimension higher. Any open subset of  $\mathcal{S}^+$  has the topology  $\mathbb{R} \times S^d$  in  $(d + 2)$ -bulk dimensions. As we will be mostly dealing with the case  $d = 2$  in the second part of this thesis our discussion about holomorphic coordinates was worth doing.

The case of black hole horizons is also of great interest for physical applications.<sup>35</sup> It was shown in [57] that Einstein's equations once projected on the horizon (which is a null hypersurface) can be mapped to plain Carrollian dynamics of the form (1.3.20) and (1.3.21). What makes black hole horizon peculiar is the vanishing of the energy flux  $\Pi^i$  hence the invariance under shift symmetry. The case of the connection is ambivalent as on the one hand it is also induced by the ambient spacetime Levi-Civita while on the other hand it usually possesses torsion as the expansion scalar is generically non zero. While null infinity is a null hypersurface located at conformal infinity, black hole horizons are at finite distance and one can always build a coordinate system with a time-like coordinate  $\rho$  such that the horizon is located at  $\rho = 0$  while every hypersurfaces  $\rho = \text{cst} > 0$  are timelike. These are the *Gaussian null coordinates* [80] which were used first in [57] and latter in [75] to approach physics at the horizon via a succession of Lorentzian spacetimes on which our intuition is more efficient. This procedure actually reduces to finding Carrollian physics at the horizon via a limit (here  $\rho \rightarrow 0$ ). This limiting procedure is a powerful tool to handle Carrollian theories taking advantage of what is already known in relativistic physics. Delving deeper in such considerations is our objective for the next Chapter.

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<sup>35</sup>Emphasis should be made once again on the fact that we consider only black holes in the bulk which are relativistic and not Carroll black holes like in [70].

## Chapter 2

# From relativistic systems to Carrollian physics: the $c \rightarrow 0$ limit

Carrollian structures were studied from an intrinsic geometric point of view in the last Chapter. Nothing has been said about real “physical” aspects neither about explicit field theories on a Carroll structure. As explained in the Introduction, the shrinking of the light cone that occurs in the Carrollian world forbids any motion, rendering *a priori* particularly hard the deep understanding of basic relativistic or Newtonian concepts such as particles and interactions. Carroll symmetries have nonetheless a wide range of applications: description of null hypersurfaces [57, 63, 75, 81–83], the fluid/gravity correspondence [53, 84–86] (spin-off of the AdS/CFT correspondence where the dual CFT is treated within the hydrodynamic approximation), condensed matter systems such as fractons [87–90], Hall effects [91] or fermions [92]. The main question is thus, how can we get insights about Carrollian physics? A practical method to get a Carrollian system from a well-known relativistic one: the  $c \rightarrow 0$  limit (a discussion about this procedure is proposed in [93]). As we will see in this Chapter it has the advantage of delving directly into physical considerations, which complement the intrinsic analysis. However, the limit has to be supplemented with group-theoretical considerations like representations of the Carroll algebra [94] to classify all possible particles or the co-adjoint orbit method to build geometric actions (see e.g. [34]).

As the Carroll algebra is the  $c \rightarrow 0$  contraction of the Poincaré algebra, it is expected to get Carrollian structures and Carrollian theories from the limit of vanishing speed of light of relativistic metrics and systems. We first present how to parametrize a relativistic metric in such a way as to recover in the limit a Carrollian structure of the type of (1.1.1) and (1.1.2). This is the Randers–Papapetrou frame, presented in Sec. 2.1. Developing any relativistic quantity in powers of  $c^2$  in such a frame coincides with a decomposition of tensors under the group of Carrollian diffeomorphisms (1.1.5), ensuring that the  $c \rightarrow 0$  limit will give rise to Carrollian quantities. The power of this tool is demonstrated first by considering the Carrollian counterpart of hydrodynamics in Sec. 2.2. We shall define what is a Carrollian fluid, what kind of equations it obeys and conclude with some examples. As we will explain, the limiting procedure is actually the only known way till now to

deal with such fluids. In Sec. 2.3 we study an explicit field theory: the Carrollian scalar field. In Sec. 2.4 we present the relativistic and Carrollian Cotton tensors, together with the procedure to find the latter as the  $c \rightarrow 0$  limit of the former. The decomposition under Carrollian diffeomorphisms will show that starting from a unique relativistic action or tensor, one usually gets two or more Carrollian descendent which do not always have an intrinsic origin. This is in agreement with the observation of [49] in the case of electromagnetism, latter generalised for scalars and  $p$ -form gauge theories in [95]. The limiting procedure hence gives us access to more information. Works on Carrollian field theories include [69, 79, 96] for scalar fields or [97–102] in the case of gravity and higher-spin theories.

This Chapter, mostly inspired from [53, 66, 103, 104], is completed by Appendix B.2 where the  $c \rightarrow 0$  limit of relativistic structures in Cartan’s frame yields the Carroll structure displayed in Appendix B.1.

## 2.1 Relativistic ascendant of a Carroll structure

In this Section we explain how one can get weak and strong Carroll structures from relativistic spacetimes<sup>1</sup> written in an adapted parametrisation. We then show how the definition of the Carrollian charge (1.5.17) arises from the relativistic one. We also elaborate on the link between relativistic and Carrollian isometries. Hence, here we are no longer living intrinsically on a Carrollian manifold.

### 2.1.1 The Randers-Papapetrou frame

Let  $\mathcal{M}$  be a  $(d + 1)$ -dimensional Riemannian spacetime. Its relativistic metric can always be put in the Randers-Papapetrou (RP for short) form

$$ds^2 := g_{\mu\nu} dx^\mu dx^\nu = -c^2 \left( \Omega dt - b_i dx^i \right)^2 + a_{ij} dx^i dx^j = -c^2 \tau^2 + a_{ij} dx^i dx^j \quad (2.1.1)$$

where all parameters are functions of all coordinates  $(t, \mathbf{x})$ . Assuming the  $c$ -dependence to be explicit (i.e. neither  $\Omega$  nor  $b_i$  nor  $a_{ij}$  depend on  $c$ ) one see that the  $c \rightarrow 0$  limit of this metric yields a weak Carroll structure as defined in (1.1.1) and (1.1.2).<sup>2</sup> This frame is invariant under Carrollian diffeomorphism (1.1.5) given the transformation rules (1.1.7). Here again one can have  $(\hat{t}, \hat{i})$  coordinates

<sup>1</sup>The adjective *relativistic* refers to pseudo-Riemannian spacetimes. Following Jean-Marc Lévy-Leblond the latter is actually a misnomer because Carrollian dynamics hence Carrollian manifolds are also relativistic, albeit with a different relativity group (which discards boosts affecting space).

<sup>2</sup>In [105] the authors considered the case when  $a_{ij}$  explicitly depends on  $c$ . We shall latter explain what this hypothesis changes in the framework we will develop.

when using  $\tau$  as temporal form. The covariant basis is then

$$\mathcal{B}_{\text{RP}} = \left\{ \frac{c}{\Omega} \partial_0, \hat{\partial}_i := \partial_i + \frac{c b_i}{\Omega} \partial_0 \right\}. \quad (2.1.2)$$

In this frame the components of a vector  $\xi = \xi^\mu \partial_\mu$  and its form  $\xi_\mu$  (related via the musical isomorphism  $\xi_\mu = g_{\mu\nu} \xi^\nu$ ) read

$$\xi^{\hat{t}} = \Omega \xi^t - b_i \xi^i, \quad \xi^{\hat{i}} = \xi^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i + b_i \xi_{\hat{t}}. \quad (2.1.3)$$

**Remark** One may wonder how such a parametrisation (2.3.2) came out. Paraphrasing [53] we note on the one hand that, while Galilean fundamental objects are particles moving along lines, their Carrollian counterpart (or dual) could be thought as extended objects like space-filling branes (i.e. a  $d$ -dimensional hypersurface  $\mathcal{V} \subset \mathcal{M}$ ) whose action is the one of Dirac, Born and Infeld

$$\mathcal{S}_{\text{DBI}} = \int_{\mathcal{V}} d^d x \sqrt{h} \quad (2.1.4)$$

with  $h_{ij}$  the induced metric on  $\mathcal{V}$

$$h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}. \quad (2.1.5)$$

Here  $g_{\mu\nu}$  and  $x^\mu$  ( $\mu = 0, 1, \dots, d$ ) are the metric and local coordinates on  $\mathcal{M}$  while  $y^i$  ( $i = 1, \dots, d$ ) are local coordinates on  $\mathcal{V}$ . On the other hand, seeking an action intrinsic to  $\mathcal{V}$  which is invariant under the Carroll group (1.5.6) we find

$$\mathcal{S}_t = \int_{\mathcal{V}} d^d x \frac{1}{2} h^{ij} (\partial_i \varphi - b_i) (\partial_j \varphi - b_j) \quad (2.1.6)$$

where  $b_i$  plays the role of an inverse-velocity. This means that the brane should be described by a scalar field  $\varphi(\mathbf{x})$  with action (2.1.6). Expanding  $\mathcal{S}_{\text{DBI}}$  in powers of  $c^2$  one sees that its strict Carrollian limit yields  $\mathcal{S}_t$  if and only if  $g_{\mu\nu}$  is in the Randers-Papapetrou parametrisation, justifying the use of the latter.

Within the RP frame, relativistic tensors can be directly reduced under Carrollian diffeomorphisms. Any tensor with a **lower** time index component transforms as a Carrollian density hence gives a scalar upon division by  $\Omega$  while its components with **upper** spatial indices transforms as Carrollian tensors. In the case of a vector  $\mathbf{u} = u^\mu \partial_\mu$  its  $d + 1$  components transforms as

$$u'_0 = \frac{u_0}{J}, \quad u'^i = J^i_k u^k. \quad (2.1.7)$$

Therefore the position of the indices before taking the  $c \rightarrow 0$  limit is very important. We are now ready to study the Levi-Civita connection in the frame (2.3.2) to see what it gives after the  $c \rightarrow 0$  limit.

As a genuine relativistic spacetime,  $\mathcal{M}$  can be endowed with the Levi-Civita connection, the unique torsionfree and metric-compatible connection. Using (A.0.5) we get

$$\begin{aligned}\Gamma_{\hat{t}\hat{t}}^{\hat{t}} &= 0 & \Gamma_{\hat{t}\hat{t}}^{\hat{t}} &= \varphi_i & \Gamma_{\hat{t}\hat{t}}^{\hat{t}} &= 0 & \Gamma_{ij}^{\hat{t}} &= \frac{1}{2c^2}\mathbf{v}(a_{ij}) + 2\hat{\omega}_{ij} & \Gamma_{\hat{t}\hat{t}}^i &= c^2\varphi^i \\ \Gamma_{\hat{t}\hat{t}}^i &= \Gamma_{ij}^i &= \frac{1}{2}a^{ik}\mathbf{v}(a_{jk}) - c^2\hat{\omega}^i_j & & \Gamma_{(jk)}^i &= \frac{1}{2}a^{il}(\hat{\partial}_j a_{jl} + \hat{\partial}_j a_{il} - \hat{\partial}_l a_{ij}) & \Gamma_{[jk]}^i &= 0.\end{aligned}\quad (2.1.8)$$

We thus first see that the limit is smooth only for invariant Carroll structures i.e. those for which the extrinsic curvature vanishes completely

$$\frac{1}{2}\mathbf{v}(a_{ij}) = 0. \quad (2.1.9)$$

Then we can take the Carrollian limit and we reach a connection which in terms of the variables (1.2.1) reads

$$\begin{aligned}\hat{\gamma} &= 0, & \hat{\delta}_i &= \varphi_i, & \hat{\alpha}_i &= 0, & \hat{\beta}_{ij} &= \hat{\omega}_{ij}, \\ \hat{\rho}^i &= 0, & \hat{\gamma}^i_j &= \kappa_j^i = 0, & \hat{\gamma}^k_{(ij)} &= \Gamma^k_{(ij)}, & \hat{\gamma}^k_{[ij]} &= 0.\end{aligned}\quad (2.1.10)$$

Hence, we see that, spatially, we get the hat connection  $\hat{\nabla}$  we introduced in Sec. 1.3. However we notice two restrictions: i) such a connection is valid only for invariant Carrollian geometries and ii) the temporal connection  $\hat{\gamma}^i_j$  vanishes whereas the ambiguity  $\hat{\delta}_i$  is set equal to the acceleration  $\varphi_i$ . Therefore the hat connection is a slight extension of the one we get in the limit, the extension being such that the intrinsic connection can also be considered on a structure with extrinsic curvature. In the following we will then work with the hat connection.

### 2.1.2 Isometries, charges and their Carrollian limits

We consider here a vector  $\xi = \xi^{\hat{t}}\mathbf{v} + \xi^i\hat{\delta}_i$  and a relativistic energy-momentum tensor  $T^{\mu\nu}$  which is then by definition symmetric  $T^{\mu\nu} = T^{\nu\mu}$  and divergence-free  $\nabla_\mu T^\mu_\nu = 0$ . The relativistic conserved current is then  $I^\mu = \xi^\nu T^\mu_\nu$  and its divergence reads

$$\nabla_\mu I^\mu = -T_{00} \left( \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) + 2T_0^i \left( (\hat{\delta}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \hat{\omega}_{ji} - \frac{1}{c^2 \Omega} a_{ij} \partial_t \xi^j \right) + T^{ij} \left( \hat{\nabla}_{(i} \xi_{j)} + \xi^{\hat{t}} \hat{\gamma}_{(ij)} \right). \quad (2.1.11)$$

This is a relativistic expression although written in terms of Carrollian variables. It vanishes if and only if

$$\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0 \quad (2.1.12a)$$

$$(\hat{\delta}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \hat{\omega}_{ji} - \frac{1}{c^2 \Omega} a_{ij} \partial_t \xi^j = 0 \quad (2.1.12b)$$

$$\hat{\nabla}_{(i} \xi_{j)} + \xi^{\hat{t}} \hat{\gamma}_{(ij)} = 0. \quad (2.1.12c)$$

Taking the  $c \rightarrow 0$  limit of these equations leads to (1.5.2) together with the strong Killing condition  $\mathcal{L}_\xi \tau = 0$  (this is the second equation in (2.1.12a)) while finiteness at zero  $c$  asks for  $\partial_t \xi^j = 0$  i.e.  $\xi$  generates a Carrollian diffeomorphism. We thus see why only strong Carrollian Killings can give rise to a conserve charge. This computation is the application in RP frame of a more general one than can be performed in any arbitrary frame (see (B.1.68) for the case of Cartan's frame).

Given a conserved current  $I^\mu$  one can always build a conserved charge via the formula

$$Q = \int_{\Sigma_t} d^d x \sqrt{\sigma} n_\mu I^\mu \quad (2.1.13)$$

where the integration is performed on a time-like hypersurface  $\Sigma_t$  (identified with a surface  $t = cst$ ) whose induced metric is  $\sigma_{\mu\nu}$  and whose outwards pointing normal vector is  $n_\mu$ . As shown in [61] in the RP frame one gets

$$\sqrt{\sigma} = \sqrt{a} + \mathcal{O}(c^2) \quad , \quad n_0 = c\Omega + \mathcal{O}(c^3) \quad , \quad I^0 = \frac{c}{\Omega} \left( \mathcal{I} + b_i \mathcal{I}^i \right) \quad (2.1.14)$$

with  $\mathcal{I} = I^\mu \partial_\mu = \mathcal{I} \frac{c}{\Omega} \partial_t + \mathcal{I}^i \hat{\partial}_i$  and  $n_i = 0$ . Hence the  $c \rightarrow 0$  limit of (2.1.13) gives  $Q \rightarrow c^2 Q_{\text{Carr}}$  with

$$Q_{\text{Carr}} = \int_{\Sigma_t} d^d x \sqrt{a} \left( \mathcal{I} + b_i \mathcal{I}^i \right) \quad (2.1.15)$$

which is exactly (1.5.17). This is a first example of the power of using the RP frame in order to find Carrollian formulas for quantities: go to the RP frame and take carefully the  $c \rightarrow 0$  limit of the well-known relativistic definition. The next example will be the Cotton tensor in Sec. 2.4.

## 2.2 Carrollian fluids

In this Section we aim at explaining how the conservation equations (1.3.20) and (1.3.21) arise from the expansion of the relativistic energy-momentum tensor in RP frame. This will be the occasion to present the notion of Carrollian fluids (whenever the energy-momentum tensor can be interpreted as the one of a fluid). After briefly reviewing relativistic hydrodynamics, we carefully expand the relativistic fluid equations in order to derive their Carrollian counterpart. We end up with two examples of Carrollian fluids. This section is inspired by [53, 65, 103].

### 2.2.1 Reviewing relativistic hydrodynamics

Relativistic fluids are used to described out-of-equilibrium phenomena when the typical length scale of the perturbations are large compared to the typical kinetic scales (like the mean free path for example). When living on a relativistic spacetime  $(\mathcal{M}, g_{\mu\nu}, \nabla)$  with  $\nabla$  the Levi-Civita connection of  $g_{\mu\nu}$ , these fluids are usually described by means of a four-velocity congruence  $\mathbf{u} = u^\mu \partial_\mu$  used to decompose their energy-momentum tensor into thermodynamic quantities, as local thermal equi-

librium is assumed. While perfect fluids possess only an energy density  $\varepsilon$  and a pressure  $p$ , it is customary to add terms resulting from friction (the symmetric stress-tensor  $\tau_{\mu\nu}$ ) and thermal conduction (the heat current  $q^\mu$ ) when dealing with viscous fluids. The energy-momentum tensor for such viscous fluids then reads

$$T^{\mu\nu} = (\varepsilon + p) \frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2} \quad (2.2.1)$$

and its covariant conservation

$$\nabla_\mu T^{\mu\nu} = 0, \quad (2.2.2)$$

yields a set of  $d + 1$  equations, the temporal one being the conservation of energy while the  $d$  spatial ones are called *Euler* equations. The dynamical variables are  $\varepsilon$ ,  $p$  and the spatial components of the velocity  $u^i$ ,<sup>3</sup> as hydrodynamics assumes constitutive relations expressing the stress and the heat current as derivative expansions on the velocity, the temperature (and the chemical potential if we have a further conserved current). In its current form the system (2.2.2) contains  $d + 1$  equations for  $d + 2$  variables, it can be closed by setting a relationship between the energy density and the pressure

$$\varepsilon = \varepsilon(p). \quad (2.2.3)$$

This is called an *equation of state*. It is customary to assume a conformal equation of state, i.e. a relation of the form

$$\varepsilon = dp. \quad (2.2.4)$$

The tensors  $q^\mu$  and  $\tau^{\mu\nu}$  are transverse with respect to the velocity congruence  $\mathbf{u}$  i.e.

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad (2.2.5)$$

which means that only  $q^i$  and  $\tau^{ij}$  are relevant. We finally find that

$$u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu. \quad (2.2.6)$$

Summarizing, we have

- that  $\varepsilon$  is the energy density per unit of proper volume as measured by an observer at velocity  $\mathbf{u}$ ,
- that  $p$  is the local-equilibrium thermodynamic pressure which obeys (2.2.3),
- that  $q^i$  and  $\tau^{ij}$  capture the physical properties of the out of equilibrium state. They are often expressed as expansions in temperatures, chemical potential and derivatives of the velocity. This constitute the constitutive relations. At first order in derivatives one can write

$$q_\mu^{(1)} = -\kappa h_\mu{}^\nu \left( \partial_\nu T + \frac{T}{c^2} a_\nu \right) \quad (2.2.7)$$

---

<sup>3</sup>This is due to the normalisation  $\|\mathbf{u}\|^2 = -c^2$  which fixes the temporal component in terms of the spatial ones.

$$\tau_{\mu\nu}^{(1)} = -2\eta\sigma_{\mu\nu} - \zeta h_{\mu\nu}\Theta \quad (2.2.8)$$

where we see appearing, on top of the temperature  $T$  and the transport coefficients  $\kappa, \eta, \zeta$ , kinematical quantities related to the velocity  $\mathbf{u}$

$$\left\{ \begin{array}{ll} a_\mu = u^\nu \nabla_\nu u_\mu & \text{acceleration} \\ \Theta = \nabla_\mu u^\mu & \text{relativistic expansion} \\ \sigma_{\mu\nu} = \sigma_{\nu\mu} = \nabla_{[\mu} u_{\nu]} + \frac{1}{c^2} u_{[\mu} a_{\nu]} - \frac{1}{d} \Theta h_{\mu\nu} & \text{shear} \\ \omega_{\mu\nu} = -\omega_{\nu\mu} = \nabla_{[\mu} u_{\nu]} + \frac{1}{c^2} u_{[\mu} a_{\nu]} & \text{vorticity} \end{array} \right. \quad (2.2.9)$$

with  $h_{\mu\nu}$  the projector onto the space transverse to the velocity field

$$h_{\mu\nu} = \frac{u^\mu u^\nu}{c^2} + g_{\mu\nu}. \quad (2.2.10)$$

The unknown coefficients in (2.2.7) and (2.2.8) characterize the transport phenomena occurring in the fluid.  $\kappa$  is the thermal conductivity while  $\eta$  and  $\zeta$  are respectively the shear and bulk viscosity. The latter is related to the change of volume of the fluid while the former describes the resistance applied by a layer of fluid onto the one which is above it. These coefficients are determined either if one has access to a microscopic theory either phenomenologically.

## 2.2.2 The emergence of Carrollian fluids

What is a Carrollian fluid? Following the relativistic or Galilean paradigm, this should be a many-body state close to thermodynamic equilibrium. None of these concepts has been defined in the Carrollian regime, and the primitive absence of motion for point-like objects makes it even harder to state. Leaving these deep issues aside, we will assume that the Carrollian equivalent of a fluid exists, and focus mostly on their dynamics in terms of conservation laws, similar to the existing ones in the previously quoted instances. As a preamble to their exhibition, it may be useful to recall the various paths one can follow to derive the Galilean fluid equations a.k.a. the Navier-Stokes equations.

- The first way is to consider a mesoscopic element of fluid on which we apply Newton's second law i.e. the fundamental principle of Galilean dynamics (see [106]). Following this path for Carroll would require first to agree on what would the fundamental law of Carrollian dynamics be and this is still under scrutiny.
- An other path is via a kinetic theory *à la* Boltzmann, where the small fluid element made of  $N$  particles is described by a function of time, positions and momenta (in the sense of momentum  $\mathbf{p} = m\mathbf{v}$  with  $m$  the mass) of all particles which satisfies the Vlasov equation in virtue of Liouville theorem. Taking momenta with respect to the average velocity of the latter equation give rise to the continuity and Euler equations, to which one has just to add viscous terms to get Navier-Stokes. However this approach gives a special role to the time direction



which in the Carrollian case is degenerate, rendering it more complex than it seems.

- Apart from these “physical” ways of deriving the Navier-Stokes equation, one can also think of simply taking the  $c \rightarrow \infty$  limit of the relativistic equations, as Galilean fluids are just the non-relativistic approximation of the latter. It has been done in particular in [53] following [106], where the authors followed the dual path to find the Carrollian fluid equations i.e. by taking the  $c \rightarrow 0$  limit of  $\nabla_\mu T^{\mu\nu} = 0$ . Presenting this method is our objective in the following paragraph.

In order to proceed with the fluid equations we need to assume a behavior at small  $c$  for the relativistic energy-momentum tensor

$$\begin{cases} T^{ij} = \frac{1}{c^2} \tilde{\Pi}^{ij} + \Pi^{ij} + \mathcal{O}(c^2) \\ -\frac{\varepsilon}{\Omega} T^i_0 = \Pi^i + c^2 P^i + \mathcal{O}(c^4) \\ \frac{1}{\Omega^2} T_{00} = \Pi + \mathcal{O}(c^2). \end{cases} \quad (2.2.11)$$

Contrary to the non-relativistic (i.e. Galilean) case, there are no physical principles to guide us towards the best scaling in powers of  $c^2$ . The idea of the authors of [53] was to choose the scaling dictated in [52] when using Carrollian fluids to describe holographically solutions of asymptotically flat Einstein gravity. The scalings (2.2.11) are the minimal scalings needed to describe at least Petrov-type D solutions of Einstein’s equations and also encompass Robinson-Trautman spacetimes. More on Petrov classification will be said in Chapter 4.

**Remark** The expansion (2.2.11) translates into an expansion at the level of the thermodynamic quantities introduced in the last section

$$\varepsilon = \eta + \mathcal{O}(c^2) \quad (2.2.12a)$$

$$p = \bar{\omega} + \mathcal{O}(c^2) \quad (2.2.12b)$$

$$q^i = Q^i + c^2 \pi^i + \mathcal{O}(c^4) \quad (2.2.12c)$$

$$\tau^{ij} = -\frac{\Sigma^{ij}}{c^2} - \Xi^{ij} + \mathcal{O}(c^2), \quad (2.2.12d)$$

The Carrollian momenta are then equal to

$$\Pi^i = Q^i, \quad P^i = \pi^i, \quad \Pi = \eta, \quad \tilde{\Pi}^{ij} = -\Sigma^{ij}, \quad \Pi^{ij} = \bar{\omega} a^{ij} - \Xi^{ij}. \quad (2.2.13)$$

The relativistic equations (conservation of the energy-momentum tensor) should be presented as

$$\nabla_\mu T^\mu_0 = 0, \quad \nabla_\mu T^{\mu i} = 0. \quad (2.2.14)$$

Under Carrollian diffeomorphisms (1.1.5), (1.1.6), (2.2.14) transform as

$$\nabla'_\mu T'^\mu_0 = \frac{1}{J} \nabla_\mu T^\mu_0, \quad \nabla'_\mu T'^{\mu i} = J^i_l \nabla_\mu T^{\mu l}, \quad (2.2.15)$$

which means that the temporal and spatial equations do not mix under these transformations. This was to be expected as the Randers-Papapetrou frame allows for this splitting. Plugging the behavior (2.2.12) into (2.2.14) we can expand the relativistic equations in powers of  $c^2$

$$\frac{c}{\Omega} \nabla_\mu T^\mu_0 = \frac{\mathcal{F}}{c^2} + \mathcal{E} + \mathcal{O}(c^2), \quad (2.2.16a)$$

$$\nabla_\mu T^{\mu j} = \frac{\mathcal{H}^j}{c^2} + \mathcal{G}^j + \mathcal{O}(c^2). \quad (2.2.16b)$$

where each coefficient of the expansion provides a Carrollian tensor. What we have done with this expansion is then a reduction of the covariant relativistic equation under Carrollian diffeomorphisms. In these expressions (2.2.16a) and (2.2.16b) we read

$$\mathcal{E} = - \left( \frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi - \left( \hat{\nabla}_i + 2\varphi_i \right) \Pi^i - \Pi^{ij} \hat{\gamma}_{ij}, \quad (2.2.17a)$$

$$\mathcal{F} = -\tilde{\Pi}^{ij} \hat{\gamma}_{ij}, \quad (2.2.17b)$$

$$\mathcal{G}^j = \left( \hat{\nabla}_i + \varphi_i \right) \Pi^{ij} + 2\Pi_i \hat{\omega}^{ij} + \Pi \varphi^j + \left( \frac{1}{\Omega} \hat{D}_t + \theta \right) P^j + P_i \hat{\gamma}^{ij}. \quad (2.2.17c)$$

$$\mathcal{H}^j = \left( \hat{\nabla}_i + \varphi_i \right) \tilde{\Pi}^{ij} + \left( \frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi^j + \Pi_i \hat{\gamma}^{ij}, \quad (2.2.17d)$$

At zero  $c$  the Carrollian fluid equations are then  $\mathcal{E} = \mathcal{F} = 0$  and  $\mathcal{G}^i = \mathcal{H}^i = 0$ .

**Remark** Asking all coefficients associated with non positive powers of  $c^2$  in the expansions (2.2.16a) and (2.2.16b) to vanish is legitimate as we work in the limit  $c \rightarrow 0$ . If we were working at a fixed and finite value of  $c^2$  the whole infinite expansion would have been set to zero, and not the coefficients separately.

**Remark** There are instances for which expansions of the type of (2.2.16a) and (2.2.16b) are finite and exact. One then talks about a decomposition in powers of  $c^2$ . In that case the development is valid irrespective of the finite value of  $c^2$  and all coefficients, even those with positive powers of the parameter, are required to vanish. An explicit example will be given in Sec. 2.4 with the Cotton tensor.

It is now time to pause and think a bit more about the equations (2.2.17a), (2.2.17b), (2.2.17c) and (2.2.17d).

- Equation  $\mathcal{E} = 0$  is scalar and mimic eq. (1.3.20) found through variation. This is the energy conservation.
- Equation  $\mathcal{G}^i = 0$  is the vectorial counterpart of  $\mathcal{E} = 0$ . It is a momentum equation which is the parallel of (1.3.21). It can therefore be obtained through variation. This is a general fact that when comparing what one can find by the limiting procedure and by an intrinsic analysis, the intrinsic case sits at order 1 in the expansion prior to the limit.

- Equation  $\mathcal{F} = 0$  is non-dynamical. It is valid either if the momenta  $\tilde{\Pi}^{ij}$  is absent, which is a “physical” requirement, or if the extrinsic curvature of the Carroll structure vanishes, which is a geometric constraint. The latter is fulfilled when the metric  $a_{ij}$  does not depend on time.
- The equation  $\mathcal{H}^i = 0$  is a new dynamical equation on the heat current. It involves the time derivative of the vector  $\Pi^i$  together with the gradient of the rank-two tensor  $\Pi^{ij}$ . We are therefore tempted to interpret it as a kind of “Carrollian continuity equation”. Note that according to the current knowledge, this equation is not associated to any symmetry and thus cannot be obtained through variation.

It is also important to notice that, contrary to the intrinsic case, the momentum  $P_i$  is no longer undetermined as it can be expressed explicitly in terms of the thermodynamic variables. In conclusion, the expansion of the energy-momentum tensor and its conservation equations in powers of  $c^2$  leads to a set of couples of equations ( $\mathcal{E} = 0$  with  $\mathcal{G}^i = 0$  and  $\mathcal{F} = 0$  with  $\mathcal{H}^i = 0$ ). Through variation, one obtain the couple sitting at order 1 in (2.2.16a) and (2.2.16b). The other couples are unreachable by this means excepts if one assumes a similar Laurent expansion of the Carroll structure, field of observers and metric, see e.g. [105].

**Remark** One could have assume an expansion more general than (2.2.11), with more degrees of freedom at negative powers of  $c^2$ . This would have lead to a similar multiplication of equations, see [65] for a discussion on that topic.

**Remark** Equations (2.2.17a), (2.2.17b), (2.2.17c) and (2.2.17d) can be re-expressed in a Weyl-covariant way using tools developed in Sec. 1.4. We find

$$\mathcal{E} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi - \hat{\mathcal{D}}_i \Pi^i - \Pi^{ij} \xi_{ij}, \quad (2.2.18a)$$

$$\mathcal{F} = -\tilde{\Pi}^{ij} \xi_{ij}, \quad (2.2.18b)$$

$$\mathcal{G}_j = \hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_i \delta_j^i + \xi^i_j \right) P_i, \quad (2.2.18c)$$

$$\mathcal{H}_j = \hat{\mathcal{D}}_i \tilde{\Pi}^i_j + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_i \delta_j^i + \xi^i_j \right) \Pi_i. \quad (2.2.18d)$$

The trace condition would be in that case

$$\tilde{\Pi}^i_i = 0 \quad \text{and} \quad \Pi^i_i = \Pi. \quad (2.2.19)$$

We conclude this paragraph with a little discussion about the fate of Carrollian charges as described in Sec. 1.5 when additional degrees of freedom i.e. momenta are present. Suppose that  $\xi$  is the generator of a Carrollian diffeomorphism (1.1.5). It can be used to create two currents out<sup>4</sup> of  $\Pi^{ij}$ ,

<sup>4</sup>Of course if more momenta were present, more currents would be available.

$\tilde{\Pi}^{ij}$ ,  $\Pi^i$ ,  $P^i$  and  $\Pi$

$$\begin{cases} \kappa = \xi^i P_i - \xi^i \Pi \\ \tilde{\kappa} = \xi^i \Pi_i \\ K^i = \xi^j \Pi_j^i - \xi^i \Pi^i \\ \tilde{K}^i = \xi^j \tilde{\Pi}_j^i, \end{cases} \quad (2.2.20)$$

If  $\xi$  is a (conformal) Carrollian Killing field, and assuming all momenta on-shell i.e. Eqs. (2.2.18a), (2.2.18b), (2.2.18c) and (2.2.18d) (with (2.2.19) satisfied in the conformal instance), one finds the following Carrollian divergences (the conformal weights of  $\kappa$  and  $\tilde{\kappa}$  are  $d$ , those of  $K^i$  and  $\tilde{K}^i$ ,  $d + 1$ , and  $-1$  for  $\nu_i$ )<sup>5</sup>

$$\begin{cases} \mathcal{K} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \tilde{\kappa} + \hat{\mathcal{D}}_i \tilde{K}^i = 0 \\ \mathcal{H} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \kappa + \hat{\mathcal{D}}_i K^i = \Pi^i \nu_i, \end{cases} \quad (2.2.21)$$

with  $\nu_i = (\hat{\partial}_i - \varphi_i) \xi^i + 2\xi^j \hat{\omega}_{ij}$ . Two charges can be defined following (1.5.17):  $Q_{\tilde{K}}$  and  $Q_K$ . The former is always conserved, whereas the latter isn't for generic isometries unless the field configuration has vanishing energy flux  $\Pi^i$ , i.e. if local Carroll-boost invariance is unbroken (which will not be the case for radiating spacetimes, see Chapter 5).

## Comparison with Galilean fluids: the case of 1 + 1-dimensions

To get more insights on the physical content of the Carrollian fluid equations, namely Eqs. (2.2.17a), (2.2.17b), (2.2.17c) and (2.2.17d), one can compare with the Galilean analysis. The latter serves as a guide in absence of a microscopic description. To make the relation between Galilean and Carrollian fluid, we restrict the following comments to the (1 + 1)-dimensional case because in this dimension the Carrollian and Galilean algebras are isomorphic. This translates into interesting duality relations between the fluid equations. We will mostly follow [107] (see also [108, 109] for considerations on the stability of the equations). The coordinates on the (1 + 1)-dimensional spacetime are  $(t, x)$  and will be denoted by an index 0 (for time) and  $x$  (for space).

Galilean structures (dubbed Newton-Cartan spacetimes) are described in any dimension in Appendix D. We recall here the principal notions which will be useful when writing the fluid equations. Many notations are common with the Carrollian case so an index ‘‘G’’ will be associated to the Galilean quantities, whenever an ambiguity exists. The Galilean clock form is  $\tau_G = \Omega_G dt$ , dual to the field of observers  $\mathbf{v}_G = \frac{1}{\Omega_G} (\partial_t + w^x \partial_x) + \alpha^x \partial_x$ . The degenerate cometric reads  $\partial_s^2 = a^{-1} \partial_x^2$ .

Galilean fluid equations are obtained either from first principles (as recalled in the introduction of this section) either from the  $c \rightarrow \infty$  limit of the relativistic equations. Following this latter path,

<sup>5</sup>Notice that these are the specific conformal weights ensuring the Carroll divergence in (1.5.14) be identical to the Weyl–Carroll divergence. For  $\{\kappa, K^i\}$  of general weights  $(w, w + 1)$  we find instead  $(\frac{1}{\Omega} \partial_t + \theta) \kappa + (\hat{\nabla}_i + \varphi_i) K^i = \frac{1}{\Omega} \hat{\mathcal{D}}_i \kappa + \hat{\mathcal{D}}_i K^i + (d - w) \left( \frac{\theta}{d} \kappa + \varphi_i K^i \right)$ .

we need to assume a scaling in powers of  $c^2$  for the energy-momentum tensor

$$\begin{cases} T_{xx} = \Pi_{xx} + \mathcal{O}(c^2) \\ c\Omega T_x^0 = c^2 P_x + \Pi_x + \mathcal{O}(c^4) \\ \Omega^2 T^{00} = c^2 \rho + \Pi + \mathcal{O}(c^2). \end{cases} \quad (2.2.22)$$

In (2.2.22)  $\rho$  is the mass density and  $P_x$  is the momentum (it contains in particular the combination  $\rho\alpha_x$  i.e. the mass density times the velocity). Similarly  $\Pi$  is the total energy-density and  $\Pi_{xx}$  the total energy-stress tensor. Note that these scalings are suggested by physical consideration, contrary to the Carrollian ones (2.2.11) which were dictated only by the fluid/gravity correspondence.

The large- $c$  expansion of the relativistic fluid equations leads to

$$c\nabla_\mu T^{\mu 0} = c^2 C + \mathcal{E}_G + \mathcal{O}\left(\frac{1}{c^2}\right) \quad (2.2.23a)$$

$$\nabla_\mu T^\mu{}_1 = c^2 \mathcal{N} + \mathcal{M} + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (2.2.23b)$$

The Galilean fluid equations are then  $C = \mathcal{E}_G = 0$  and  $\mathcal{N} = \mathcal{M} = 0$  with

$$\mathcal{E}_G = \left(\frac{1}{\Omega}(\partial_t + w^x \partial_x) + \theta^w\right) \Pi + \left(\hat{\nabla}_x + 2\varphi_x\right) \Pi^x + \theta^w \Pi^x{}_x, \quad (2.2.24a)$$

$$\mathcal{M} = \left(\frac{1}{\Omega}\hat{D}_t + 2\theta^w\right) P_x + \left(\hat{\nabla}_x + \varphi_x\right) \Pi^x{}_x + \varphi_x \Pi, \quad (2.2.24b)$$

$$C = \left(\frac{1}{\Omega}(\partial_t + w^x \partial_x) + \theta^w\right) \rho + \left(\hat{\nabla}_x + 2\varphi_x\right) P^x, \quad (2.2.24c)$$

$$\mathcal{N} = \varphi_x \rho. \quad (2.2.24d)$$

In these expressions  $\varphi_x = \partial_x \ln \Omega_G$ ,  $\theta^w$  is given in (D.2.20), the Galilean covariant temporal derivative  $\frac{1}{\Omega}\hat{D}_t$  is displayed in (D.2.21) and the spatial covariant derivative is built in (D.2.11).

Delving deeper into the structure of these equations we find that

- $C = 0$  is the continuity equation. It is associated to the conservation of the mass.
- $\mathcal{E}_G = 0$  is the energy equation.
- $\mathcal{M} = 0$  is the momentum (Euler) equation.
- $\mathcal{N}$  is more exotic and translates into either a physical requirement ( $\rho = 0$ ) either a geometric identity namely the closure of the Galilean clock form  $\tau_G$ . The first option of the alternative is reached for a fluid with massless carriers like photons. The second option implies the existence of an absolute time  $t_N$  such that  $\tau = dt_N$ . It corresponds to the instance of ordinary fluids.

We therefore notice the following duality relations between Galilean fluid equations and their Carrollian counterparts. In the first instance the constraint equation  $\mathcal{N}_i = 0$  calls for the absolute character of time, while for Carroll structures with absolute space,  $\mathcal{F} = 0$  is automatically satisfied. This is reminiscent of the dual roles played by space and time in Galilean and Carrollian geometries. The duality can be thrust further. Following [107] and given the dual roles of time and space we can associate the Galilean energy equation  $\mathcal{E}_G$  to the momentum Carrollian equation  $\mathcal{G}$ . Similarly the Galilean momenta equation  $\mathcal{M}$  is associated to the Carrollian energy equation  $\mathcal{E}$ . Finally the Carrollian equation  $\mathcal{H}$  which was lacking interpretation is dual to the Galilean continuity equation  $\mathcal{C}$ . However in the Carrollian case the continuity is associated with the directions transverse to the fluid velocity and not the longitudinal ones like for ordinary fluids. This dual role between time and space gives the impression that concepts such that the entropy would be more related to dissipative terms, leading to a kind of “magnetic entropy”, yet to define and understand.

To conclude we observed that giving a physical sense to the Carrollian fluid equations is not an easy task, as long as a microscopic description is not available. Comparing with the Galilean case we were able to exhibit a complete duality between time and space, which translates into a duality between the fluid equations. Let us now give an explicit example of Carrollian fluid. As the third part of this thesis deals with thermalization, we present the notion of “thermal” Carrollian fluids.

### 2.2.3 Example: a “thermal” Carrollian fluid

We consider a three-dimensional manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  where the two-dimensional surface  $\mathcal{S}$  is described by means of complex holomorphic coordinates  $(\zeta, \bar{\zeta})$

$$d\ell^2 = \frac{2}{P^2(t, \zeta, \bar{\zeta})} d\zeta d\bar{\zeta} \quad (2.2.25)$$

which is a case in which the geometric shear  $\xi_{ij}$  vanishes. We choose the frame such that  $b_i = 0$  and  $\Omega = 1$  i.e.  $\varphi_i = 0$  and  $\hat{\omega}_{ij} = 0$ . On this surface we put a Carrollian fluid at rest ( $\beta^i = 0$ ) whose non vanishing momenta are  $\Pi = \varepsilon$ ,  $\Pi^i = Q^i$ ,  $\Pi^{ij} = p a^{ij}$  and  $\tilde{\Pi}^{ij} = -\Sigma^{ij}$  where  $\varepsilon$  and  $p$  are related by a conformal equation of state  $\varepsilon(t, \zeta, \bar{\zeta}) = 2p(t, \zeta, \bar{\zeta})$ . The conformal behavior of the fluid imposes that  $\Sigma^{ij}$  appearing in (2.2.12) satisfies  $\Sigma^{\zeta\bar{\zeta}} = 0$ .

The Carrollian fluid equations (2.2.18a), (2.2.18b), (2.2.18c) and (2.2.18d) reduce, for the fluid at hand, to

$$\mathcal{E} = 3\varepsilon\partial_t \ln P - \partial_t \varepsilon - \partial_i Q^i = 0 \quad (2.2.26a)$$

$$\mathcal{G}_i = \partial_i p \quad (2.2.26b)$$

$$\mathcal{H}^i = \partial_t Q^i - 2Q^i \partial_t \ln P - \partial_j \Sigma^{ij} = 0, \quad (2.2.26c)$$

the others being identically zero due to the absence of absence of certain momenta and the vanishing of the geometric shear. The momentum equation  $\mathcal{G}_i = 0$  imposes that  $p$  (and the energy density

because of the equation of state (2.2.3)) only depends on time. To give an interpretation to the energy equation  $\mathcal{E} = 0$  we should first pause and think.

In first order relativistic hydrodynamics one relates the heat current of the fluid to the gradient of the temperature via Fourier's law. We may think of doing something similar for a Carrollian fluid i.e. we seek a quantity  $T^C$  which could be considered as temperature and such that

$$\mathbf{Q} = -\mathbf{grad} \left( \kappa^C T^C \right), \quad (2.2.27)$$

with  $\kappa^C$  the Carrollian "thermal" conductivity. However, as there is no kinetic theory in this set-up, we should not think about temperature as coming from the motion of particles but rather as the motion (better, the change) of the background geometry itself. The latter is encoded into the Gauss curvature which for the two-dimensional surface (2.2.25) reads

$$K = \Delta \ln P \quad (2.2.28)$$

with  $\Delta = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta}$  the Laplacian operator in  $(\zeta, \bar{\zeta})$  coordinates. We may then define the *Carrollian temperature* as

$$\kappa^C T(t, \zeta, \bar{\zeta}) = \kappa' K(t, \zeta, \bar{\zeta}). \quad (2.2.29)$$

Here,  $\kappa'$  is a constant introduced for dimensional reasons. Indeed, the temperature has Weyl weight 1 so, because of (2.2.29), the product  $\kappa^C T^C$  is weight 2. Given (2.2.29) the equation  $\mathcal{E} = 0$  in (2.2.26) reads

$$3\varepsilon \partial_t \ln P - \partial_t \varepsilon + \kappa' \Delta K = 0, \quad (2.2.30)$$

where we explicitly see that the fluid is purely geometrical. If one defines a mass such that  $M(t) = \frac{\varepsilon(t)}{4\kappa'}$  and set  $\kappa' = \frac{1}{16\pi G}$  we get

$$\Delta \Delta \ln P + 12M \partial_t \ln P - 4\partial_t M = 0, \quad (2.2.31)$$

which is the Robinson-Trautman equation [110]. The Robinson–Trautman equations appears then both as a heat equation in conformal Carrollian fluids and in four-dimensional Einstein gravity. This is an example of the fluid/gravity duality [27], spin-off of the AdS/CFT in which an asymptotically AdS spacetime is associated to a conformal fluid on the boundary. This duality was extended to the flat case in [52]. In the case treated here, the two-dimensional conformal Carrollian fluid studied here originates from flat Robinson–Trautman spacetime holography.

This concludes our analysis of Carroll geometry in Randers–Papapetrou frame and its applications to Carrollian fluids as the  $c \rightarrow 0$  limit of relativistic fluids. Actually hydrodynamics is not the only instance in which such a limit allows to build Carrollian theories from relativistic ones. Some of them were studied e.g. in [49, 95] and also in [103] where we elaborated on the case of the scalar field.

## 2.3 The Carrollian scalar field

In this Section we illustrate part of the previously developed formalism in the simple case of the scalar field, as developed in [103]. We will follow the limiting procedure approach even though one could also define intrinsically on a Carroll structure Carroll diffeomorphisms invariant actions. Other references on the Carrollian scalar field, though in different formalisms, include [69, 95, 96].

### 2.3.1 Electric and magnetic actions

#### The set-up

Let  $\mathcal{M}$  a  $(d + 1)$ -dimensional Riemannian manifold. We start with the action for a relativistic scalar field  $\Phi$

$$S = - \int_{\mathcal{M}} dt d^d x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right), \quad (2.3.1)$$

on a Papapetrou-Randers background

$$ds^2 = -c^2 \tau^2 + a_{ij} dx^i dx^j. \quad (2.3.2)$$

Assuming the following expansion in powers of  $c^2$  for the potential  $V$

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + \mathcal{O}(c^2), \quad (2.3.3)$$

we can decompose the relativistic action in powers of  $c^2$

$$S = \frac{1}{c^2} S_e + S_m + \mathcal{O}(c^2) \quad (2.3.4)$$

with  $S_e$  and  $S_m$  the Carrollian actions with Lagrangian densities

$$\mathcal{L}_e = \frac{1}{2} \left( \frac{1}{\Omega} \partial_t \Phi \right)^2 - V_e(\Phi), \quad (2.3.5a)$$

$$\mathcal{L}_m = -\frac{1}{2} a^{ij} \hat{\partial}_i \Phi \hat{\partial}_j \Phi - V_m(\Phi). \quad (2.3.5b)$$

The indices “e” and “m” stand for *electric* and *magnetic*. The terminology comes from [49] where the Carrollian limit of electromagnetism was studied. Latter it was used in other theories like Yang-Mills and gravity in [95, 101]. Note that both actions are invariant under Carrollian diffeomorphisms. Hence they are, according to our terminology, genuine Carrollian actions.<sup>6</sup> Indeed the Randers-Papapetrou form of the metric behaves covariantly under Carrollian diffeomorphisms, hence the decomposition of any relativistic tensor as a (usually truncated) Laurent expansion in  $c^2$ , provides

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<sup>6</sup>The actions associated with the  $\mathcal{O}(c^2)$  terms are non-dynamical as no kinetic term appears at this order, see the explicit example of the conformally coupled scalar in the following.



a Carrollian tensor at each order. Phrased in more mathematical terms, the expansion in powers of  $c^2$ , amounts to reducing the representations of the full diffeomorphism group, with respect to the subgroup of Carrollian diffeomorphism.

**Remark** The decomposition (2.3.4) crucially depends on the assumption (2.3.3). One can show that most of the possible potential terms, and among them the conformal coupling that we shall shortly see, fit in this hypothesis.

**Remark** With a strict  $c \rightarrow 0$  limit we can only reach one of the two actions,  $S_m$ . A rescaling of the field  $\Phi$  is necessary to get the other one, see e.g. [95]. In this work and also e.g. in [79] the terminology "magnetic" was reserved to a two-field formulation of the theory whose Lagrangian density reads

$$\mathcal{L} = \pi \dot{\phi} - \frac{1}{2} \partial_i \phi \partial^i \phi \quad (2.3.6)$$

with  $\pi$  the Hamiltonian conjugate variable to the field  $\phi$ . Hence calling "magnetic" the Lagrangian (2.3.5b) refers more to its subleading position in the expansion (2.3.4).

**Remark** In [68] the authors recovered the electric and magnetic (two-fields) actions also with an expansion in  $c^2$ . However, contrary to (2.3.1) and (2.3.3) where the geometry and the potential were expanded without touching the dynamical field, in [68] the field  $\phi$  itself is expanded around  $c = 0$

$$\phi = c^\Delta (\phi_0 + c^2 \phi_1 + c^4 \phi_2 + \dots) , \quad (2.3.7)$$

for some  $\Delta$ . The field  $\phi_0$  and  $\phi_1$  plays respectively the role of  $\phi$  and  $\pi$  in (2.3.6).

## Conformal coupling

For a scalar field  $\Phi$  of weight  $w = \frac{d-1}{2}$ ,

$$V(\Phi) = \frac{d-1}{8d} R \Phi^2. \quad (2.3.8)$$

is a conformal coupling. Conformality imposes to translate covariant derivatives into Weyl-covariant derivatives. In our setup this can be done using a congruence  $\mathbf{u}$  dual to  $\boldsymbol{\tau}$ , from which we define the Weyl-connection

$$A = \frac{1}{c^2} \left( \mathbf{a} - \frac{\Theta}{d} \mathbf{u} \right) \quad (2.3.9)$$

with  $\mathbf{a}$  and  $\Theta$  the acceleration and expansion of  $\mathbf{u}$  as defined in (2.2.9). The Weyl-covariant derivative acts then on scalars  $f$  of weight  $w$  as

$$\mathcal{D}_\lambda f = \nabla_\lambda f + w A_\lambda f \quad (2.3.10)$$

and on one-forms  $v_\mu$  as

$$\mathcal{D}_\lambda v_\mu = \nabla_\lambda v_\mu + (w+1) A_\lambda v_\mu + A_\mu v_\lambda - g_{\mu\lambda} A^\rho v_\rho. \quad (2.3.11)$$

The energy-momentum tensor of the conformally coupled theory<sup>7</sup>

$$\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi + \frac{d-1}{4d} (G_{\mu\nu} \Phi^2 + g_{\mu\nu} \square \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2) \\
&= \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi - \frac{1}{2} g_{\mu\nu} \mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi \\
&\quad + \frac{d-1}{4d} \left( \left( \mathcal{R}_{(\mu\nu)} - \frac{\mathcal{R}}{2} g_{\mu\nu} \right) \Phi^2 + g_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}^\alpha \Phi^2 - \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi^2 \right)
\end{aligned} \tag{2.3.13}$$

is traceless when  $\Phi$  is on-shell (i.e.  $\square \Phi = 0$ ) and with a Weyl-weight of  $d-1$ . Conformal couplings are very useful in cosmology as well as when dealing with null infinity (see below where our theories will be put on the null boundary of Robinson-Trautman spacetimes).

As a consequence of diffeomorphism invariance, the energy-momentum tensor obeys a Weyl-covariant conservation equation, when the field  $\Phi$  is on-shell

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu} \doteq 0. \tag{2.3.14}$$

Note that the usual covariant divergence matches the Weyl-Carroll one thanks to the weight of  $T^{\mu\nu}$  (use (1.4.5a) and Leibniz rule). Finally the equations of motion reads

$$-\mathcal{D}_\mu \mathcal{D}^\mu \Phi + \frac{d-1}{4d} \mathcal{R} \Phi = 0. \tag{2.3.15}$$

We now need to recast (2.3.8) in the form (2.3.3), decomposing first the Ricci scalar

$$R = \frac{1}{c^2} \left( \frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) + \hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i + c^2 \hat{\omega}_{ij} \hat{\omega}^{ij}. \tag{2.3.16}$$

This leads to

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + c^2 V_{\text{nd}}(\Phi) \tag{2.3.17}$$

with

$$V_e(\Phi) = \frac{d-1}{8d} \left( \frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) \Phi^2, \tag{2.3.18a}$$

$$V_m(\Phi) = \frac{d-1}{8d} \left( \hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i \right) \Phi^2, \tag{2.3.18b}$$

$$V_{\text{nd}}(\Phi) = \frac{d-1}{8d} \hat{\omega}_{ij} \hat{\omega}^{ij} \Phi^2. \tag{2.3.18c}$$

In the last expression the index “nd” stands for “non-dynamical.” The reason is that when the ex-

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<sup>7</sup>Here  $G_{\mu\nu}$  is the Einstein tensor,  $\mathcal{R}_{\mu\nu}$  and  $\mathcal{R}$  are the Weyl-Ricci and scalars defined by

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-1) \left( \nabla_\nu A_\mu + A_\mu A_\nu - g_{\mu\nu} A_\lambda A^\lambda \right) + g_{\mu\nu} \nabla_\lambda A^\lambda - F_{\mu\nu}, \tag{2.3.12a}$$

$$\mathcal{R} = R + 2d \nabla_\lambda A^\lambda - d(d-1) A_\lambda A^\lambda. \tag{2.3.12b}$$

pression (2.3.17) of the potential is used in the relativistic action (2.3.1), it produces the Carrollian electric and magnetic actions – with some boundary terms dropped here<sup>8</sup>

$$S_e = \int dt d^d x \sqrt{a} \Omega \left( \frac{1}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \right), \quad (2.3.19a)$$

$$S_m = \int dt d^d x \sqrt{a} \Omega \left( -\frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi - \frac{d-1}{8d} \hat{\mathcal{R}} \Phi^2 \right), \quad (2.3.19b)$$

as well as a third one  $S_{\text{nd}} = -\int dt d^d x \sqrt{a} \Omega \frac{d-1}{8d} \hat{\omega}_{ij} \hat{\omega}^{ij} \Phi^2$ , which has no kinetic term for  $\Phi$ . The Carrollian equations of motion in the two non trivial cases read

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi + \frac{d-1}{4d} \xi_{ij} \xi^{ij} \Phi = 0 \quad \text{electric}, \quad (2.3.20a)$$

$$-\hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \hat{\mathcal{R}} \Phi = 0 \quad \text{magnetic}, \quad (2.3.20b)$$

and are Weyl-covariant of weight  $\frac{d+1}{2}$ . This discussion highlights how we get much more information with a Carrollian reduction with respect to a purely intrinsic study.

### 2.3.2 Momenta

In (1.3.16) we showed how to define conjugate momenta on a weak Carroll structure, once we are given an action. For the electric and magnetic actions at hand we find

$$\begin{cases} \Pi_e^{ij} = \frac{a^{ij}}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 + \frac{d-1}{4d} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t (\xi^{ij} \Phi^2) - a^{ij} \left( \frac{1}{2} \xi_{lk} \xi^{lk} \Phi^2 + \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 \right) \right) \\ \Pi_e^i = 0 \\ \Pi_e = \frac{1}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2, \end{cases} \quad (2.3.21)$$

and

$$\begin{cases} \Pi_m^{ij} = \hat{\mathcal{D}}^i \Phi \hat{\mathcal{D}}^j \Phi - \frac{a^{ij}}{2} \hat{\mathcal{D}}_l \Phi \hat{\mathcal{D}}^l \Phi + \frac{d-1}{4d} \left( \left( \hat{\mathcal{R}}^{(ij)} - \frac{\hat{\mathcal{R}}}{2} a^{ij} \right) \Phi^2 + a^{ij} \hat{\mathcal{D}}_l \hat{\mathcal{D}}^l \Phi^2 - \hat{\mathcal{D}}^{(i} \hat{\mathcal{D}}^{j)} \Phi^2 \right) \\ \Pi_m^i = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left( \hat{\mathcal{D}}^i \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 - \hat{\mathcal{D}}_j (\xi^{ij} \Phi^2) \right) \\ \Pi_m = \frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left( \frac{\hat{\mathcal{R}}}{2} \Phi^2 - \hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi^2 \right). \end{cases} \quad (2.3.22)$$

For the non-dynamical action we get

$$\begin{cases} \Pi_{\text{nd}}^{ij} = \frac{d-1}{4d} \left( 2 \hat{\omega}^{li} \hat{\omega}_l^j - \frac{a^{ij}}{2} \hat{\omega}_{lk} \hat{\omega}^{lk} \right) \Phi^2 \\ \Pi_{\text{nd}}^i = \frac{d-1}{4d} \hat{\mathcal{D}}_j (\hat{\omega}^{ji} \Phi^2) \\ \Pi_{\text{nd}} = \frac{3(d-1)}{8d} \hat{\omega}_{ij} \hat{\omega}^{ij} \Phi^2. \end{cases} \quad (2.3.23)$$

<sup>8</sup>On the relativistic side we find:  $\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{d-1}{8d} R \Phi^2 = \frac{1}{2} \mathcal{D}^\mu \Phi \mathcal{D}_\mu \Phi + \frac{d-1}{8d} \mathcal{R} \Phi^2 - \frac{d-1}{4\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu \Phi^2)$ .

They all obey the Weyl trace condition  $\Pi_i^i = \Pi$ . Both electric and magnetic set satisfy (1.3.14a) and (1.3.14b) and assuming the fields to be on-shell we get the expression for the momenta

$$P_e^i = \Pi_m^i \quad (2.3.24a)$$

$$P_m^i = \Pi_{\text{nd}}^i. \quad (2.3.24b)$$

**Remark** Eqs. (2.3.24a) and (2.3.24b) set a relationship between the electric and magnetic dynamics and this comes from the limiting procedure. Actually both theories (2.3.5a) and (2.3.5b) are genuine Carrollian theories that could have been studied from an intrinsic point of view. Getting them from the limit means that they come from the same relativistic parent action (2.3.1). The relativistic energy-momentum tensor of this theory reads

$$\begin{cases} T^{ij} = \frac{1}{c^2} \Pi_e^{ij} + \Pi_m^{ij} + c^2 \Pi_{\text{nd}}^{ij} \\ -\frac{c}{\Omega} T_0^i = \Pi_m^i + c^2 \Pi_{\text{nd}}^i \\ \frac{1}{\Omega^2} T_{00} = \frac{1}{c^2} \Pi_e + \Pi_m + c^2 \Pi_{\text{nd}}, \end{cases} \quad (2.3.25)$$

so comparing with the general result (2.2.11) we understand Eqs. (2.3.24a) and (2.3.24b). We could have also expanded the relativistic conservation of energy–momentum  $\nabla_\mu T^\mu_\nu = 0$  and recollect the Carrollian conservation equations for the electric, the magnetic and the non-dynamical cases. In this process, like in the case of fluids, the role of momenta  $P^i$  of the leading quadruplet of momenta (the electric one) is played by the energy-flux of the subleading quadruplet (the magnetic one), and so on and so forth.

### 2.3.3 Charges on a Robinson-Trautman background

Robinson–Trautman (RT for short) spacetimes are four-dimensional, time-dependent Ricci-flat solutions of algebraically special Petrov type. They describe radiative configurations which settle down in the far future into a Schwarzschild black hole. The original solution was first described in [110, 111]. Latter they have been discussed in the framework of AdS/CFT in [30, 112–114], and further in flat holography in [52, 59]. Their null boundary is a Carrollian manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ , where  $\mathcal{S}$  is equipped with a conformally flat  $d = 2$  metric

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}, \quad (2.3.26)$$

where  $P = P(t, \zeta, \bar{\zeta})$  obeys a fourth-order partial-differential equation known as Robinson–Trautman’s equation

$$\Delta \Delta \ln P + 12M \partial_t \ln P - 4\partial_t M = 0, \quad (2.3.27)$$

with  $M(t)$  the mass aspect.

The field of observers and the clock form are ( $\Omega = 1, b_i = 0$ )

$$\mathbf{v} = \partial_t, \quad \boldsymbol{\tau} = dt. \quad (2.3.28)$$

Hence, one can compute the basic geometric data<sup>9</sup>

$$\theta = -2\partial_t \ln P, \quad \varphi_i = 0, \quad \hat{\omega}_{ij} = 0, \quad \xi_{ij} = 0, \quad \hat{\mathcal{R}} = 4P^2 \partial_{\bar{z}} \partial_z \ln P. \quad (2.3.29)$$

From now on our Carrollian scalar fields, electric and magnetic, will live on this RT background. As this Section aims at illustrating the theoretical framework previously developed in Chapter 1, will shall now compute some Carrollian charges, built upon the symmetries of RT spacetimes.

### Asymptotic symmetries

Robinson–Trautman solutions have no isometries but they have asymptotic symmetries, and these are actually reflected in the conformal isometries of the Carrollian boundary. Following Sec (1.5), we recall that the conformal Killing fields of  $\mathcal{M}$  are expressed in terms of an arbitrary real function  $T(\zeta, \bar{\zeta})$ , which encodes the *supertranslations* and the conformal Killing vectors  $\mathbf{Y} = Y^\zeta \partial_\zeta + Y^{\bar{\zeta}} \partial_{\bar{\zeta}}$  of  $d\bar{\ell}^2 = 2d\zeta d\bar{\zeta}$ , which is flat space. When  $\mathbf{Y}$  are *superrotations*, we find that they can be written as any combination of  $\ell_m + \bar{\ell}_m$  or  $i(\ell_m - \bar{\ell}_m)$  with

$$\ell_m = -\zeta^{m+1} \partial_\zeta, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1} \partial_{\bar{\zeta}}, \quad (2.3.30)$$

obeying Witt  $\oplus$  Witt:

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m - n)\bar{\ell}_{m+n}. \quad (2.3.31)$$

In this representation,  $\mathfrak{so}(3, 1)$  is generated by  $n = 0, \pm 1$ . The conformal Killing fields of  $\mathcal{M}$  are

$$\xi_{T,Y} = (T - M_Y(C)) \frac{1}{P} \partial_t + Y^i \partial_i, \quad (2.3.32)$$

where

$$C(t, \zeta, \bar{\zeta}) = \int^t d\tau P(\tau, \zeta, \bar{\zeta}), \quad (2.3.33)$$

and  $M_Y$  is an operator acting on scalar functions  $f(t, \zeta, \bar{\zeta})$  as:

$$M_Y(f) = Y^k \partial_k f - \frac{f}{2} \partial_k Y^k. \quad (2.3.34)$$

The structure  $\mathfrak{so}(3, 1) \oplus$  supertranslations – or (Witt  $\oplus$  Witt)  $\oplus$  supertranslations – is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']} \quad (2.3.35)$$

---

<sup>9</sup>Recal that our conventions are  $\sqrt{a} = \frac{i}{p^2}$  and  $\varepsilon_{\zeta\bar{\zeta}} = 1$ , see Sec. 1.6.

as seen in Sec. 1.5.

**Remark** The existence of conformal Killing fields for the Carrollian structure at hand is remarkable. Actually, the relativistic ascendant of this structure  $ds^2 = -c^2 dt^2 + \frac{2}{p^2} d\zeta d\bar{\zeta}$ , appearing as the conformal time-like boundary of AdS-Robinson–Trautman spacetimes, has generically no conformal Killings. In particular, it is not conformally flat because it has a non-zero Cotton tensor – see [30, 112–114].

## Electric theory

The electric equation of motion (2.3.20a) reads as follows in the three-dimensional Carrollian spacetime (2.3.26)

$$\partial_t \frac{1}{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0. \quad (2.3.36)$$

Its general solution is given in terms of two arbitrary functions  $f(\zeta, \bar{\zeta})$  and  $g(\zeta, \bar{\zeta})$

$$\Phi = \sqrt{P} (Cf + g). \quad (2.3.37)$$

In terms of these arbitrary functions, the conformal Killing fields (2.3.32) reads

$$\kappa_{eT,Y} = P^2 \left[ Y^i \left( \frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} - \frac{1}{4} \partial_i \left( Y^i C f^2 \right) \right], \quad (2.3.38)$$

which leads to the charges

$$Q_{eT,Y} = -i \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} \left( Y^i \left( \frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} \right) - \frac{1}{4} \int_{\partial\mathcal{S}} \star \mathbf{Y} C f^2 P^2. \quad (2.3.39)$$

On-shell, the time dependence is exclusively encoded in the last term through  $P$  (and  $C$ ). This is a flux at infinity, and thus it vanishes upon appropriate fall-off behaviour of the field  $f$ . Therefore, the charges are indeed conserved, as expected for a theory with vanishing energy flux  $\Pi_e^i = 0$ .

The infinite number of conserved charges is a consequence of the separation of time and space imposed by Carrollian symmetry. The field equation (2.3.36) contains no spatial derivative, hence every locus  $(\zeta, \bar{\zeta})$  provides a decoupled degree of freedom. This can even happen when the equations of motion contain both temporal and spatial derivatives as e.g. in the magnetic conformally stationary scalar field – see below. This infinity is rooted in the fact that the conformal extension of the Carroll algebra is infinite dimensional.

## "Magnetic" theory

The magnetic equation of motion (2.3.20b) is

$$4\partial_\zeta\partial_{\bar{\zeta}}\Phi = \Phi\partial_\zeta\partial_{\bar{\zeta}}\ln P. \quad (2.3.40)$$

We are here in the canonical situation where the magnetic energy flux  $\Pi_m^i$  is non-vanishing. Conservation will therefore occur under the following alternative: either  $\xi$  is a strong Killing field, either we take a field configuration (i.e. a solution of (2.3.40)) such that  $\Pi_m^i = 0$ . Let's study separately the two cases.

**Strong Killing.** This requirement yields

$$T = SP + M_Y(C), \quad (2.3.41)$$

where  $S$  is a function of time only. Since  $P$  and  $C$  are time-dependent while  $T$  isn't, Eq. (2.3.41) restricts severely the allowed subset of  $\mathcal{S}$ -conformal Killings  $\mathbf{Y}$ , which may even be empty. Assuming this set is not empty, due to the vanishing of the magnetic momentum  $P_m^i$  (as a consequence of  $\varpi_{ij} = 0$ ), Eq. (1.5.12) leads to a single conserved charge built upon  $\kappa_{mS} = -S\Pi_m$  with  $\Pi_m$  given in (2.3.22)

$$Q_{mS} = -S \int_{\mathcal{S}} \frac{d\zeta d\bar{\zeta}}{P^2} \Pi_m. \quad (2.3.42)$$

This charge is the total energy, but it turns out to vanish here as, on-shell,  $\Pi_m$  reads (Eqs. (2.3.20b) and (2.3.22)),

$$\Pi_m = \frac{1}{2d} \hat{\mathcal{D}}_i \left( \Phi \hat{\mathcal{D}}^i \Phi \right), \quad (2.3.43)$$

irrespective of the dimension and of the geometric background. In the case under consideration where  $b_i = 0$  and  $\varphi_i = 0$ ,  $\Pi_m = \frac{P^2}{4} \left[ \partial_\zeta (\Phi \partial_{\bar{\zeta}} \Phi) + \partial_{\bar{\zeta}} (\Phi \partial_\zeta \Phi) \right]$ , which is a divergence. Hence  $Q_{mS}$  receives only an  $\mathcal{S}$ -boundary contribution, vanishing under appropriate fall-off or boundary conditions.

**Remark** Vanishing magnetic conserved charges actually appear in any dimensions  $d$  as long as the Ehresmann connection is set to zero. Indeed, when  $b_i = 0$ , the vorticity vanishes, leading therefore to  $Q_m = - \int_{\mathcal{S}} d^d x \sqrt{a} \xi^i \Pi_m$ . For Killing fields obeying the extra condition  $\mathcal{L}_\xi \tau = 0$ , using (2.3.43) we find that the on-shell integral is again a boundary term.

**Vanishing energy-flux.** When  $\Pi_m^i$  given in (2.3.22) vanishes, the magnetic charges are all conserved, as inferred by Eq. (1.5.14). This occurs in particular (the Carrollian geometric shear vanishes here, see (2.3.29)) for conformally stationary scalars obeying  $\frac{1}{Q} \hat{\mathcal{D}}_t \Phi \equiv \sqrt{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0$ , thus of the form  $\Phi = \sqrt{P} g(\zeta, \bar{\zeta})$ , where  $g(\zeta, \bar{\zeta})$  is further determined by solving the magnetic equation of motion (2.3.40). The latter<sup>10</sup> may not be solvable in a general Robinson–Trautman background  $P(t, \zeta, \bar{\zeta})$

<sup>10</sup>With  $\Phi = \sqrt{P} g(\zeta, \bar{\zeta})$ , Eq. (2.3.40) reads:  $4P\partial_\zeta\partial_{\bar{\zeta}}g + 2(\partial_\zeta P\partial_{\bar{\zeta}}g + \partial_{\bar{\zeta}} P\partial_\zeta g) + g\partial_\zeta\partial_{\bar{\zeta}}P = 0$  (also valid if  $P$  is traded for  $C$ ).

under the present ansatz. If it is, the conserved magnetic charges are found using Eqs. (1.5.12) and (2.3.32). On-shell, these lead to

$$\kappa_{mT,Y} = -\xi^t \Pi_m = \frac{P^2}{2} (M_Y(C) - T) (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g), \quad (2.3.44)$$

which are integrated as

$$Q_{mT} = \frac{i}{2} \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} T (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) - \frac{1}{4} \int_{\partial \mathcal{S}} \star \mathbf{X} P^2 \quad (2.3.45)$$

with

$$\begin{cases} X^\zeta = C \left( Y^\zeta (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^{\bar{\zeta}} \left( 3 (\partial_{\bar{\zeta}} g)^2 - g \partial_{\bar{\zeta}}^2 g \right) \right) - \frac{1}{2} Y^{\bar{\zeta}} g^2 \partial_{\bar{\zeta}}^2 C \\ X^{\bar{\zeta}} = C \left( Y^{\bar{\zeta}} (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^\zeta \left( 3 (\partial_\zeta g)^2 - g \partial_\zeta^2 g \right) \right) - \frac{1}{2} Y^\zeta g^2 \partial_\zeta^2 C. \end{cases} \quad (2.3.46)$$

As in the electric case (see Eq. (2.3.39)), the time dependence is confined into a boundary term, which ultimately drops, taking with it all the dependence on the  $\mathfrak{so}(3, 1)$  vectors  $\mathbf{Y}$ . For a conformally stationary scalar field in Robinson–Trautman background, the magnetic charges are non-zero and conserved on-shell without restriction on the Carrollian conformal Killing vector  $\xi_{T,Y}$  (the energy flux vanishes), but they only depend on its supertranslation component  $T(\zeta, \bar{\zeta})$ .

This ends our study of the Carrollian scalar field, which had the double virtue of showing us first, how useful the Carrollian reduction method can be and second, how to treat the conservation of charges in an explicit Carrollian theory.

## 2.4 The Carrollian Cotton tensor in three dimensions

In this section we introduce an important character when dealing with the null infinity of asymptotically flat spacetimes, the Cotton tensor. Presenting its main features and especially how to get from its relativistic definition the Carrollian equivalent would be the occasion to show again the power of the Randers-Papapetrou frame in taking the  $c \rightarrow 0$  limit. Inspired by [52, 66] and especially [104] we make a comprehensive review of the state of the art regarding the Carrollian counterpart of the Cotton tensor.

### 2.4.1 Reviewing the relativistic Cotton tensor

The Cotton tensor can be defined on any  $(d + 1)$ -dimensional Riemannian manifold  $(\mathcal{M}, g_{\mu\nu}, \nabla)$  endowed with a metric  $g_{\mu\nu}$  and a Levi-Civita connection  $\nabla$ . It carries three indices and is partially antisymmetric. In the special case of three dimensions (i.e.  $d = 2$ ) it can be Hodge dualized into a



two-index symmetric tensor

$$C_{\mu\nu} = \eta_{\mu}^{\rho\sigma} \nabla_{\rho} \left( R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right) \quad (2.4.1)$$

where  $S_{\mu\nu} = R_{\mu\nu} - R/4 g_{\mu\nu}$  is the usual Schouten tensor,  $g_{\mu\nu}$  is of signature  $(- + +)$ , determinant  $g$  and  $\eta_{\mu\nu\rho} = \sqrt{-g} \epsilon_{\mu\nu\rho}$  ( $\epsilon_{012} = 1$ ). The vanishing of the Cotton tensor is equivalent to conformal flatness of the associated manifold.

This tensor is Weyl-covariant with weight  $d+1$ . In three dimensions, due to the absence of Weyl tensor (the Cotton itself is playing its role) and the first Bianchi identity, it is covariantly conserved and Weyl-conserved<sup>11</sup>

$$\nabla_{\rho} C^{\rho}_{\nu} = \mathcal{D}_{\rho} C^{\rho}_{\nu} = 0. \quad (2.4.2)$$

From its symmetry and (2.4.2) we observe a lot of similarities between this tensor and a usual energy-momentum tensor. The main difference lies in that the Cotton tensor is conserved off-shell while the energy-momentum conservation requires the equations of motion to be fulfilled.

**Remark** The Cotton is actually the energy-momentum tensor obtained through the variation of the (gravitational) Chern-Simons action with respect to the metric i.e.

$$C_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{CS}}{\delta g^{\mu\nu}}, \quad (2.4.3)$$

with

$$S_{CS} = \frac{1}{2c} \int_{\mathcal{M}} \text{Tr} \left( \boldsymbol{\omega} \wedge d\boldsymbol{\omega} + \frac{2}{3} \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \boldsymbol{\omega} \right) \quad (2.4.4)$$

where  $\boldsymbol{\omega}$  is the Levi-Civita spin connection. The trace is defined as  $\text{Tr}(\boldsymbol{\omega} \wedge d\boldsymbol{\omega}) = \omega^{\mu}_{\nu} \wedge d\omega^{\nu}_{\mu}$  and similarly for the second term. This action is built upon general diffeomorphisms or local Lorentz transformations.

## 2.4.2 Carrollian descendants from the limiting procedure

In all this section we are on a  $(2 + 1)$ -Riemannian manifold or a  $(2 + 1)$ -Carrollian structure. Recall that the two-dimensional Hodge duality (denoted by an asterisk  $*$ ) on such a structure was introduced in Sec. 1.6.

### Decomposing the relativistic Cotton

Finding the Carrollian descendants of the relativistic Cotton tensor constitute the *paragon* of how to use the Randers-Papapetrou frame to get Carrollian descendants of a relativistic quantity. The procedure is canonical and recalls what we saw in Sec 2.2 when we dealt with the energy-momentum

<sup>11</sup>The Weyl-covariant conservation is due to the weight,  $d + 1$ , which is the only one for which covariant and Weyl-covariant derivatives agree.

tensor and its conservation equations. We start by decomposing in powers of  $c$  the relativistic Cotton tensor on a Randers-Papapetrou background with the Levi-Civita connection  $\nabla$ . It is straightforward: Carrollian scalars and vectors emerge from  $C^{\hat{t}\hat{t}}$  and  $C^{\hat{t}i}$ , while  $C^{ij} - \frac{C^{\hat{t}\hat{t}}}{2}a^{ij}$  leads to symmetric and traceless Carrollian tensors. They are readily decomposed in powers of  $c$  as follows

$$\frac{1}{c}C^{\hat{t}\hat{t}} = c^2\gamma + c_{\text{Cot}} + \frac{\zeta}{c^2} + \frac{\tau}{c^4}, \quad (2.4.5a)$$

$$C^{\hat{t}i} = c^2\psi^i + \chi^i + \frac{z^i}{c^2}, \quad (2.4.5b)$$

$$\frac{C^{\hat{t}\hat{t}}a^{ij}}{2c} - \frac{C^{ij}}{c} = \Psi^{ij} + \frac{X^{ij}}{c^2} + \frac{Z^{ij}}{c^4}. \quad (2.4.5c)$$

With this, any Carrollian structure supplied with the connection at hand, is naturally endowed with ten Weyl-covariant Carrollian Cotton descendants. The former method, used to reach Carrollian counterparts of relativistic, is dubbed a *Carrollian reduction*.

**Remark** Note that this decomposition in powers of  $c^2$  is *exact*, contrary to what we saw in Sec. 2.2 when we assumed an expansion of the energy-momentum tensor. This is logical as the Cotton tensor is coming only from the geometry.

These descendants are

- four weight-3 scalars<sup>12</sup>

$$\gamma = 8 * \hat{\omega}^3, \quad c_{\text{Cot}} = \left( \hat{\mathcal{D}}_l \hat{\mathcal{D}}^l + 2\hat{\mathcal{H}} \right) * \hat{\omega}, \quad \zeta = \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \xi^{ij}, \quad \tau = * \xi_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \xi^{ij}; \quad (2.4.6)$$

- three weight-2 forms

$$\psi_i = 3\eta_{ji} \hat{\mathcal{D}}^j * \hat{\omega}^2, \quad (2.4.7a)$$

$$\chi_i = \frac{1}{2}\eta_{ji} \hat{\mathcal{D}}^j \hat{\mathcal{H}} + \frac{1}{2}\hat{\mathcal{D}}_i \hat{\mathcal{A}} - 2 * \hat{\omega} \left( \hat{\mathcal{R}}_i + 2\hat{\mathcal{D}}^j \xi_{ij} \right) + 3\hat{\mathcal{D}}^j (* \hat{\omega} \xi_{ij}), \quad (2.4.7b)$$

$$z_i = \frac{1}{2}\eta_{ij} \hat{\mathcal{D}}^j \xi^2 - \hat{\mathcal{D}}^j \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi_{ij} - * \xi_{ij} \hat{\mathcal{D}}_k \xi^{jk}; \quad (2.4.7c)$$

- three weight-1 traceless and symmetric two-index covariant tensors

$$\Psi_{ij} = -2 * \hat{\omega}^2 * \xi_{ij} + \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \hat{\omega} - \frac{1}{2}a_{ij} \hat{\mathcal{D}}^k \hat{\mathcal{D}}_k * \hat{\omega} - \eta_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t * \hat{\omega}^2, \quad (2.4.8a)$$

$$X_{ij} = \frac{1}{2}\eta_{ki} \hat{\mathcal{D}}^k \left( \hat{\mathcal{R}}_j + \hat{\mathcal{D}}^l \xi_{jl} \right) + \frac{1}{2}\eta_{kj} \hat{\mathcal{D}}_i \left( \hat{\mathcal{R}}^k + \hat{\mathcal{D}}^l \xi^{kl} \right) - \frac{3}{2}\hat{\mathcal{A}} \xi_{ij} - \hat{\mathcal{H}} * \xi_{ij} + 3 \frac{* \hat{\omega}}{\Omega} \hat{\mathcal{D}}_t \xi_{ij}, \quad (2.4.8b)$$

$$Z_{ij} = 2 * \xi_{ij} \xi^2 - \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi_{ij}. \quad (2.4.8c)$$

<sup>12</sup>The subscript 'Cot' in  $c_{\text{Cot}}$  is here to differentiate between this Cotton density and the velocity of light  $c$ .

**Remark** The limiting procedure is, at the time this section was written, the only way to define a Carrollian equivalent to the Cotton tensor, even though the work started in [104] about Chern-Simons theory paved the way towards an intrinsic definition. However, as the limit always gives more information than intrinsic definitions, it is likely that the later would only capture  $c_{\text{Cot}}$ ,  $\chi^i$  and  $\Psi^{ij}$ .

**Remark** The Carrollian limit of relativistic Chern-Simons theories has been studied in [104] where the authors showed that the Carrollian reduction method gives rise to four different Carrollian Chern-Simons dynamics. This has to be compared with other methods to get Carrollian actions, like algebra gauging [101] or strict  $c \rightarrow 0$  limit [93].

## Conservation equations

As for the conservation equation (2.4.2), the covariant conservation of the Cotton tensor supplies the following Carrollian decompositions

$$\nabla_\rho C^\rho_{\hat{0}} = \mathcal{D}_\rho C^\rho_{\hat{0}} = c^2 \mathcal{D}_{\text{Cot}} + \mathcal{E}_{\text{Cot}} + \frac{\mathcal{F}_{\text{Cot}}}{c^2} + \frac{\mathcal{W}_{\text{Cot}}}{c^4} = 0, \quad (2.4.9)$$

and

$$\frac{1}{c} \nabla_\rho C^{\rho i} = \frac{1}{c} \mathcal{D}_\rho C^{\rho i} = c^2 I_{\text{Cot}}^i + \mathcal{G}_{\text{Cot}}^i + \frac{\mathcal{H}_{\text{Cot}}^i}{c^2} + \frac{\mathcal{X}_{\text{Cot}}^i}{c^4} = 0. \quad (2.4.10)$$

All identities are Weyl-covariant with

$$\mathcal{D}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \gamma - \hat{\mathcal{D}}_i \psi^i, \quad (2.4.11a)$$

$$\mathcal{E}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t c_{\text{Cot}} - \hat{\mathcal{D}}_i \chi^i + \Psi_{ij} \xi^{ij}, \quad (2.4.11b)$$

$$\mathcal{F}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \zeta - \hat{\mathcal{D}}_i z^i + X_{ij} \xi^{ij}, \quad (2.4.11c)$$

$$\mathcal{W}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \tau + Z_{ij} \xi^{ij}, \quad (2.4.11d)$$

and

$$I_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \gamma + 2 * \hat{\omega} * \psi^i, \quad (2.4.12a)$$

$$\mathcal{G}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i c_{\text{Cot}} - \hat{\mathcal{D}}_j \Psi^{ij} + 2 * \hat{\omega} * \chi^i + \frac{1}{\Omega} \hat{\mathcal{D}}_i \psi^i + \psi_j \xi^{ij}, \quad (2.4.12b)$$

$$\mathcal{H}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \zeta - \hat{\mathcal{D}}_j X^{ij} + 2 * \hat{\omega} * z^i + \frac{1}{\Omega} \hat{\mathcal{D}}_i \chi^i + \chi_j \xi^{ij}, \quad (2.4.12c)$$

$$\mathcal{X}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \tau - \hat{\mathcal{D}}_j Z^{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_i z^i + z_j \xi^{ij}. \quad (2.4.12d)$$

The four couples of equations  $\{\mathcal{D}_{\text{Cot}} = 0, I_{\text{Cot}}^i = 0\}$ ,  $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^i = 0\}$ ,  $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^i = 0\}$  and  $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^i = 0\}$  originate from the different orders in  $c^2$  in which the conservation of the Cotton tensor (2.4.2) decomposes. These are *purely geometrical identities fulfilled on any three-*

*dimensional Carroll structure*  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ . Moreover, they are typical Carrollian conservation equations obtained as a consequence of general covariance applied to a Weyl-invariant action  $S = \int_{\mathcal{M}} dt d^d x \mathcal{L}$  defined on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ , see (1.4.16) and (1.4.17). Hence one can associate Carrollian momenta  $\Pi_{\text{Cot}}, \Pi_{\text{Cot}}^i, P_{\text{Cot}}^j, \Pi_{\text{Cot}}^{ij}$  and even  $\tilde{\Pi}_{\text{Cot}}, \tilde{\Pi}_{\text{Cot}}^i$  and  $\tilde{\Pi}_{\text{Cot}}^{ij}$  to the Cotton tensor

$$\Pi_{\text{Cot}} = c_{\text{Cot}}, \quad \Pi_{\text{Cot}}^i = \chi^i, \quad P_{\text{Cot}}^j = \psi^j, \quad \Pi_{\text{Cot}}^{ij} = \frac{c_{\text{Cot}}}{2} a^{ij} - \Psi^{ij}, \quad (2.4.13a)$$

$$\tilde{\Pi}_{\text{Cot}} = \zeta, \quad \tilde{\Pi}_{\text{Cot}}^i = z^i, \quad \tilde{P}_{\text{Cot}}^i = \chi^i, \quad \tilde{\Pi}^{ij} = -X^{ij}, \quad (2.4.13b)$$

comparing Eqs. (2.4.12a), (2.4.12b), (2.4.12c) and (2.4.12d) from which current and charges could be defined (see Sec. 4.1 for an application in Ricci-flat gravity). From (2.4.11d) we could complete the list (2.4.13) with more subleading momenta (denoted with a bar)

$$\bar{\Pi}_{\text{Cot}} = \tau \quad \text{and} \quad \bar{\Pi}_{\text{Cot}}^{ij} = Z^{ij}, \quad (2.4.14)$$

and so on and so forth.

**Remark** To be more precise, the variation of the action defined directly on the Carrollian structure will give access to  $\mathcal{E}_{\text{Cot}}$  and  $\mathcal{G}_{\text{Cot}}^i$  as they are at  $\mathcal{O}(1)$  in the decomposition in powers of  $c^2$  of (2.4.2). The other equations, even though they possess the same shape, are coming from the limiting procedure and give rise to subleading momenta. This is the multiplication of degrees of freedom described in Sec. 2.2.

### Conformally flat Carroll structures

As the Cotton tensor indicates in three dimensions the lack of conformal flatness one may wonder what would happen in the Carrollian case. Recall that a Carrollian structure  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ , having first a fiber bundle structure and second a much wider freedom in the choice of an affine connection, this notion might be severely different. Actually, we should make a clear distinction between conformal flatness of the Riemannian base space  $\mathcal{S}$  and what would be the equivalent for the fibers. Several cases could be separately treated.

- **Vanishing geometric shear.** In virtue of 1.0.6, when  $\xi_{ij} = 0$  we can write  $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \tilde{a}_{ij}(\mathbf{x})$  and as  $\tilde{a}_{ij}(\mathbf{x})$  is a two dimensional metric, it is necessarily proportional to  $\delta_{ij}$ , hence  $\mathcal{S}$  is conformally flat. As a consequence, we infer from (2.4.6), (2.4.7c) and (2.4.8c) that the Carrollian Cotton pieces  $\zeta, \tau, z_i$  and  $Z_{ij}$  vanishes.
- **Vanishing vorticity.** If  $\omega_{ij} = 0$  then, at the level of the clock form we get from (1.1.15)

$$d\tau = \varphi_i dx^i \wedge \tau \quad (2.4.15)$$

which implies, in virtue of Fröbenius criterion, that  $\tau$  is an exact form. Hence, choosing appropriately the time coordinate, one can always cancel the Ehresmann connection  $b_i$ , yielding

$$\tau = \Omega(t, \mathbf{x})dt. \quad (2.4.16)$$

This is the equivalent of what could be called conformal flatness along the fibers. As a consequence, we infer from (2.4.6), (2.4.7a) and (2.4.8a) that the Carrollian Cotton pieces  $\gamma$ ,  $c_{\text{Cot}}$ ,  $\psi_i$  and  $\Upsilon_{ij}$  vanishes.

Merging these two cases one is tempted to define as total Carrollian conformal flatness a Carrollian structure with vanishing vorticity and geometric shear. Note that even in that case two Carrollian Cotton descendent, namely  $\chi^i$  and  $X^{ij}$  still do not vanish, blurring the exact equivalence between Carrollian conformal flatness and vanishing of all Carrollian Cotton pieces.

## 2.5 Some last remarks

As we saw the limiting procedure is among the most powerful tools at our disposal to understand better the structure if not the meaning of Carrollian systems and theories. However a lot of questions are still unanswered and we would like to close this Chapter with some of them.

Getting the Randers–Papapetrou frame (2.3.2) from the limit of a brane-like action (2.1.4) suggests that Carroll fundamental excitations should not be thought like point-particles but rather instantonic space-filling objects whose dynamics is encoded directly in the geometry they live on. From such a conception one could try to build a fundamental principle of Carrollian dynamics (dual to Newton’s second law) and use it to build a “kinetic-like” theory of Carroll particles, leading ultimately to a physical notion of Carrollian fluids and conservation equations (not mentioning also an interpretation for the thermodynamic variables appearing in the fluid energy-momentum tensor). If Galilean/Newtonian particles are points, then Carrollian ones should be waves. In [115] the authors nicely discussed the interplay between a Galilean position space and a Carrollian momentum space and vice-versa. In fact the whole Galilean/Carrollian duality should be thrust further.

Another remark deals with the continuity equation. In Galilean systems the continuity is an extra equation, arising on top of the energy balance and momentum Euler equations. It betrays an extra symmetry, which is associated the mass conservation. In the parent Carrollian system, the extra equation is  $\mathcal{H}_t = 0$ . No symmetry emerges in the Carrollian algebra such as the Galilean central extension (Bargmann algebra). What would be the relevant underlying Carrollian structure, if any, that would yield this equation, without referring to the zero- $c$  limit?

This concludes our part about Carroll geometry and Carrollian physics either from an intrinsic point of view or through the  $c \rightarrow 0$  limit. Our attention will now be turned to applications of Carrollian physics in the context of asymptotically flat spacetimes, for which Carrollian physics appears on their null boundary.

## **Part II**

# **Carrollian perspective on asymptotically flat gravity**



## Chapter 3

# The Covariant Newman-Unti gauge

Studying the asymptotic structure of the gravitational field but also more generally of any gauge theory is an old subject which in the context of gravity saw his torch rekindle with the advent of the AdS/CFT correspondence in conjunction with the increasing interest in gravitational radiation. Indeed, any attempt of extending the framework of validity of AdS/CFT must resonate with a clear control of asymptotic symmetries underlying that framework, be it flat or de Sitter space, with gravity or gauge theories such as higher spins. When dealing with the asymptotic structure of a gauge theory one means firstly to impose “suitable” boundary conditions, strong enough to discard unphysical situations but weak enough to allow for simple solutions (like Schwarzschild in gravity), secondly to compute the asymptotic symmetries and their algebra to finally get their associated charges. Charged gauge transformations will ultimately map two inequivalent solutions of the theory at hand.

Several paths towards asymptotics of gravity exist. The first one is due to Penrose and is purely geometric [14, 116], with conformal compactifications in a coordinate independent formalism as a basis. While its obvious main advantage is the fact of being gauge independent, this approach has the drawback of rendering very hard the choice and definition of boundary conditions, to such an extent that various authors first work in coordinates and then try to find the Penrose equivalent to their findings. The second path is to work in Hamiltonian formalism (like e.g. in [18, 117, 118]) where a coordinate system splitting time (more generally an evolution parameter) and space is chosen without any gauge fixing. Gauge theories are usually better described by means of the Lagrangian formalism, where gauge invariance can be used to set the system in a convenient form. In gravity this amounts to a choice of coordinate system, a way to write the metric. Depending on the problem we want to address, adapted coordinate systems can be defined. This is the third path, the gauge fixing approach to asymptotics of gravity. In this case one fixes the form of the metric in a way that is always reachable by a diffeomorphism. This form should allow for residual gauge transformations, namely such that the chosen form is preserved. The metric is then written in terms of arbitrary functions on which one imposes fall-offs at infinity: these are the boundary conditions. The asymptotic symmetries are the residual diffeomorphisms that also preserve the boundary conditions. If their



associated charges are non-vanishing, then their action alters the physical configuration. They are dubbed strong asymptotic symmetries. The set of functions necessary to describe the metric, given the gauge fixing and the fall-off conditions, and after Einstein's equations have been imposed, is the solution space. These functions obey a set of evolution equations (some of those, if not all, are the celebrated flux-balance equations).

The structure of gravity of asymptotically AdS and flat spacetimes away from sources is often probed in different gauges. In AdS the analysis is usually performed in the Fefferman-Graham gauge, as it was shown in [119, 120] that any conformally compact manifold (like the Wick-rotated version of AdS) can be charted with such coordinates. The analysis reveals a finite solution space expressed in terms of tensorial quantities with respect to the boundary together with a set of flux/balance equations. Despite being very powerful in AdS, the Fefferman-Graham gauge cannot be extended to asymptotically flat spacetimes. The latter are usually described in a gauge that supports a lightlike coordinate, a radial coordinate and angles. Paragons are the Bondi [32, 33] and Newman-Unti [58, 121] gauges identified by the affine character of the radial coordinate in Newman-Unti. The analysis reveals for asymptotic symmetries the BMS group. The solution space turns out to be infinite, with an infinite tower of consistency relations involving the deep degrees of freedom.<sup>1</sup> This is the first difference with the AdS case while the second one is that the solution space is not expressed in terms of boundary-covariant quantities, which makes harder any attempt to use the latter to give a new perspective on the bulk dynamics.

Can we explain how the infinite flat solution space arises from the finite AdS one? As explained in the Introduction, Carrollian physics naturally emerges on the null boundary of asymptotically flat spacetimes. This null boundary is coming from the timelike boundary of AdS when the cosmological constant is taken to zero. Hence it is legitimate to think that a  $\lambda \rightarrow 0$  limit of the AdS analysis can give us an answer to this question. However, for the limit to have a chance to work, two things are necessary. The first one is a gauge in which both asymptotically AdS and flat spacetimes are expressed in terms of boundary (relativistic or Carrollian) tensors. This will be a covariantized version of the Newman-Unti gauge, hence the name of the present Chapter. The second one is a way to take the limit at the level of the AdS solution space, flux/balance laws and line element. This amounts to taking once again the Carrollian limit of a relativistic system, and as we saw in the previous Chapter, it is required to expand quantities in a Laurent series in powers of the velocity of light (here the effective velocity on the boundary, related to the bulk cosmological constant). This will ultimately give rise to more Carrollian descendants than relativistic quantities and this is in these replicas that the infinite flat solution space finds its origin. Finally, we should be also able to recover the flux-balance laws and consistency relations, though with slightly different definitions for the quantities, requiring the limit to be smooth.

Our plan for this Chapter is the following. In Section 3.1 we briefly recap the analysis of asymptotically AdS and flat spacetimes respectively in Fefferman-Graham and Bondi/Newman-Unti gauge and we show how they fit in the derivative expansion framework. In Section 3.2 we build the Cov-

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<sup>1</sup>The adjective deep refers to the power in radial coordinate of the degree of freedom in the fall-offs. In the following they shall be called *Chthonians* in reference of the Greek God of depths.

ariant Newman-Unti gauge for AdS spacetimes upon the usual one plus a velocity congruence like in the derivative expansion. As infinity is not a plain but rather a conformal manifold, Weyl transformations should be supported in our analysis. The finite AdS solution space and the flux/balance laws are also derived within this new framework. In Section 3.3 we explain the flat limit: how to expand the quantities, how to find the subleading degrees of freedom from replicas of the AdS energy-momentum tensor and how to derive the flat flux/balance equations either from the AdS ones, either from requiring finiteness of the line element. We conclude in Section 3.4 with a discussion about the deep structure of gravity in this new gauge. Unless specified we work in four bulk dimensions i.e. three-dimensional boundary and two-dimensional celestial sphere.

### 3.1 A web of gauges

In this Section we review the main gauges used to probe the asymptotic structure of AdS and flat spacetimes, highlighting their advantages and drawbacks. From this analysis, the way to build a conformal boundary covariant gauge will become clear. The following discussion is adapted to the four-dimensional case from [85, 86] where the authors investigated this web of gauges and their relations in three bulk dimensions.

#### Fefferman-Graham gauge

The description of asymptotically (locally) AdS spacetimes is well-known in the literature [119, 120, 122] (see [3, 31] for reviews). We recap here for completeness its basic aspects. The analysis is based on a radial coordinate  $\rho$  (dubbed the *holographic coordinate*) parameterizing a family of timelike hypersurfaces ( $\rho = \text{cst}$ ) and for which the conformal boundary is located at  $\rho = 0$ . The metric takes the form

$$ds_{\text{FG}}^2 = \frac{d\rho^2}{(k\rho)^2} + g_{ab}(\rho, x^c) dx^a dx^b, \quad (3.1.1)$$

where the  $x^a$  for  $a = 1, 2, 3$  are local coordinates on the hypersurfaces orthogonal to  $\partial_\rho$  and where the constant  $k$  is related to the cosmological constant  $\Lambda$  (negative for AdS spacetimes) through

$$\Lambda = -3k^2. \quad (3.1.2)$$

The only dynamical field is thus  $g_{ab}(\rho, x^c)$ . To unveil its equation of motion it is customary to assume an expansion in powers of the radial coordinate

$$g_{ab}(\rho, x^c) = \frac{1}{\rho^2} g_{ab}^{(-2)}(x^c) + \frac{1}{\rho} g_{ab}^{(-1)}(x^c) + g_{ab}^{(0)}(x^c) + \rho g_{ab}^{(1)}(x^c) + \mathcal{O}(\rho^2), \quad (3.1.3)$$

where the logarithmic terms are discarded as they only appear for odd bulk dimensions [123, 124]. Einstein's equations for the metric (3.1.1) with ansatz (3.1.3) leads to the following results

- $g_{ab}^{(-2)}$  and  $g_{ab}^{(1)}$  are the only elements of the solution space, as the other coefficient of the expansion (3.1.3) can be expressed in terms of the latter,
- $g_{ab}^{(-2)}$  is a symmetric rank-2 tensor corresponding to the Lorentzian *boundary metric*,
- $g_{ab}^{(1)}$  is a symmetric rank-2 tensor constrained by Einstein's equations to be traceless and covariantly conserved. If one introduces  $T_{ab} = \frac{3k}{16\pi G} g_{ab}^{(1)}$  then

$$D_a T^{ab} = 0 \quad \text{and} \quad T_a^a = 0, \quad (3.1.4)$$

with  $D_a$  the Levi-Civita connection associated to  $g_{ab}^{(-2)}$ .

The solution space is then made of 11 arbitrary functions, 6 in  $g_{ab}^{(-2)}$ , 5 in  $g_{ab}^{(1)}$  and the flux/balances laws are encoded into the conservation of the *boundary energy-momentum tensor*  $T_{ab}$ .

The main advantage of the Fefferman-Graham gauge is its covariance with respect to the conformal boundary. It can also be modified to make it Weyl-covariant [62,125–127]. However as the cosmological constant appear in the denominator in (3.1.1), one cannot extend it to encompass asymptotically flat spacetimes. This is possible with another gauge dubbed the Bondi gauge.

### Bondi gauge for asymptotically AdS spacetimes

This gauge, introduced in [32,33,128] for asymptotically flat spacetimes, and later studied in [129,130],<sup>2</sup> is valid irrespective of the value of the cosmological constant. In the case of asymptotically globally AdS spacetimes it was studied e.g. in [133,134] while the introduction of leaky boundary conditions in [122,135] allowed for encompassing also asymptotically locally AdS spacetimes. The gauge is built in the following way. Take  $u$  a coordinate parametrizing a set of null hypersurfaces ( $u = \text{cst}$ ) i.e.  $g^{uu} = 0$ . On each hypersurfaces we take local coordinates  $x^A$  ( $A = 1, 2$ ) which parameterize two-spheres and are such that their directional derivative along the normal vanishes i.e.  $g^{uA} = 0$ . To follow a light ray from its emission point to the boundary we take a radial coordinate  $r$  such that  $r \rightarrow \infty$  is the boundary and such that  $r$  is the luminosity distance i.e.

$$\partial_r \left( \frac{\det g_{AB}}{r^4} \right) = 0. \quad (3.1.5)$$

In the coordinates  $(u, r, x^A)$  the metric reads<sup>3</sup>

$$ds_{\text{Bondi}}^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + g_{AB} \left( dx^A - U^A du \right) \left( dx^B - U^B du \right) \quad (3.1.6)$$

where  $V, \beta, U^B$  and  $g_{AB}$  are arbitrary functions of all coordinates. We impose the following boundary conditions

<sup>2</sup>See also [59,131,132] for a partial enhancement and its consequences on asymptotic symmetries and charges

<sup>3</sup>Lowering the conditions  $g^{uu} = 0$  and  $g^{uA} = 0$  gives  $g_{rr} = 0$  and  $g_{rA} = 0$ .

- $g_{AB} = r^2 q_{AB} + r C_{AB} + D_{AB} + r^{-1} E_{AB} + r^{-2} F_{AB} + \mathcal{O}(r^{-3})$ ,
- $\frac{V}{r} = \frac{\Lambda}{3} e^{2\beta_0} r^2 + \dots + \frac{2M}{r} + \mathcal{O}(r^{-2})$ ,
- $\beta = \beta_0 + \mathcal{O}(r^{-1})$ ,
- $U^A = U_0^A + \dots + \frac{1}{r^3} (N^A + \dots) + \mathcal{O}(r^{-4})$ .

Solving Einstein's equations with negative cosmological constant for the ansatz (3.1.6) with the previous fall-offs leads to a solution space made of  $\beta_0, U_0^A, q_{AB}, M, N^A, E_{AB}$  i.e. 11 arbitrary functions of all boundary coordinates  $(u, x^A)$ . Among them  $M$  and  $N^A$  satisfy dynamical equations where  $u$  plays the role of the evolutionary parameter. These are the flux/balance laws. The function  $M$  is called the *mass aspect* and  $N^A$  is the *angular momentum aspect*. The tensor  $q_{AB}$  is the "boundary metric" which is here the metric on the two-dimensional slices of the boundary. In [122] it was left free, allowing radiation to pass through the boundary even in AdS (contrary to previous works where Dirichlet conditions were assumed).

The Bondi solution space is then equivalent to the one of Fefferman-Graham albeit expressed in a different form. The diffeomorphism between the latter and Bondi was performed in three dimensions in [85, 86] and in four dimensions in [122].

### Bondi gauge for asymptotically flat spacetimes

In the case of asymptotically flat spacetimes the ansatz for the metric remains the same (3.1.6) but Einstein's equations without cosmological constant give rise to an infinite solution space made of the 11 functions of AdS plus an infinite tower of degrees of freedom which corresponds to the symmetric and traceless projection of the tensors  $E_{AB}, F_{AB}$  etc... appearing in the expansion of  $g_{AB}$ . In either cases, AdS and flat, the Bondi gauge works. However it is not covariant with respect to the boundary geometry, contrary to Fefferman-Graham. For example performing a Carrollian diffeomorphism  $u' = u'(u, x^A)$  will make terms proportional to  $du dx^A$  appear while they are forbidden by definition of the gauge. In addition it cannot encompass Weyl transformations which is a pillar in holography as what appears on the null boundary in a conformal Carroll structure. Our objective now is to design a gauge which borrow the good features of both Bondi and Fefferman-Graham i.e. be Weyl-covariant, boundary-covariant and valid both for AdS and flat spacetimes. To reach that goal we can find inspiration in the fluid/gravity *derivative expansion* (DE for short) [27, 52]

$$ds_{\text{DE}}^2 = \frac{2}{k^2} \mathbf{u} (dr + rA) + r^2 ds_{\text{bry}}^2 + \dots \quad (3.1.7)$$

where the dots denote an infinite expansion in powers of  $r$ . This is an incomplete gauge fixing where the additional degrees of freedom are packaged in a boundary timelike congruence  $\mathbf{u}$  normalized to<sup>4</sup>  $\|\mathbf{u}\|^2 = -k^2$  and which corresponds on to the velocity of the dual fluid describing the asymptotically

<sup>4</sup>This justifies that on the boundary  $k$  plays the role of the effective velocity of light.

AdS bulk. One then replace  $du$  by  $\mathbf{u}$ . Fixing this congruence amounts to fixing a gauge (e.g. Bondi where  $\mathbf{u} = \exp(2\beta)du$ ) hence the derivative expansion is a two-parameter family of true-gauges among which sit the usual ones. This new degree of freedom will make all quantities boundary-covariant, as we shall see in the next section. Eq. (3.1.7) also naturally contains a Weyl connection  $A$ , which is another of its advantages. The main tool of this parametrization is also a null affine coordinate  $r$ , which is not the case in Bondi gauge but rather in Newman-Unti gauge. The good gauge to covariantize should then be the latter.

We thus finally recall for completeness that the *Newman-Unti* line element reads

$$ds_{\text{bulk}}^2 = \frac{V}{r} du^2 - 2du dr + g_{AB} \left( dx^A - U^A du \right) \left( dx^B - U^B du \right), \quad (3.1.8)$$

The main results previously described in Bondi stays valid and the comprehensive review of that gauge is available in [121].

## 3.2 Building a covariant gauge in AdS

In this Section we explain how to make the Newman-Unti gauge covariant under boundary Weyl transformations and boundary diffeomorphisms in AdS. With the new ansatz for the metric, we solve Einstein's equations in AdS and write the explicit solution up to  $\frac{1}{r^2}$ -order. This resolution will show that like in Fefferman-Graham, the solution space is made of the energy-momentum tensor, though in a split form with respect to the congruence  $\mathbf{u}$ , and by the boundary geometry through the Cotton tensor (instead of the metric).

### 3.2.1 Covariantising the Newman-Unti gauge

The Newman-Unti gauge (3.1.8) is stable neither under boundary diffeomorphisms  $x \rightarrow x'$ , nor under Weyl rescalings  $r \rightarrow r\mathcal{B}(x)$ . To reach a covariantized version we trade  $-k^2 du$  for a boundary one-form  $\mathbf{u} = u_\mu dx^\mu$ , which is an invariant object dual to a time-like vector field normalized at  $-k^2$ . As it will become manifest in Sec. 3.3.1, where the AdS time-like conformal boundary will become a null manifold equipped with a Carrollian structure in the limit  $k \rightarrow 0$ , our parameterization is such that  $k$  plays the role of the effective boundary velocity of light.

**Remark** Introducing this normalized but otherwise arbitrary boundary congruence turns on two degrees of freedom. This a relaxation of the original Newman-Unti gauge fixing. Incomplete gauge fixings might produce enhancements of asymptotic symmetries and materialize in extra charges — not always integrable or conserved. They have been investigated mostly in three bulk dimensions in [84–86, 136–141]. It was shown there that adding an arbitrary congruence combined with the freedom of choosing the boundary metric restores the boundary local Lorentz symmetry (related to the hydrodynamic frame invariance) and its realization

as bulk diffeomorphisms, augmenting the asymptotic symmetry group [84–86, 140, 141].

As local Lorentz invariance is manifest in Cartan’s orthonormal frame, we choose express the boundary metric as

$$ds^2 = \eta_{AB}\theta^A\theta^B = -\left(\theta^{\hat{0}}\right)^2 + \delta_{ab}\theta^a\theta^b, \quad (3.2.1)$$

and set

$$\mathbf{u} = -k\theta^{\hat{0}}. \quad (3.2.2)$$

From now on we use  $A, B, \dots \in \{\hat{0}, \hat{1}, \hat{2}\}$  as boundary “flat” indices with  $a, b, \dots \in \{\hat{1}, \hat{2}\}$ . The dual frame vectors are  $\{e_A\} = \{e_{\hat{0}}, e_a\}$  with  $\theta^B(e_A) = \delta_A^B$ . A possible parameterization of the frame, which we will not use explicitly though, is displayed in Eqs. (B.2.2), (B.2.3) and (B.2.4).

The boundary congruence  $\mathbf{u}$  provides also the appropriate tool for addressing Weyl invariance, as from any congruence one can build a Weyl connection as in (2.3.9). The bulk geometry should be insensitive to a Weyl rescaling of the boundary metric (weight  $-2$ ) and of the boundary velocity form (weight  $-1$ )

$$ds^2 \rightarrow \frac{ds^2}{\mathcal{B}^2}, \quad \mathbf{u} \rightarrow \frac{\mathbf{u}}{\mathcal{B}}, \quad (3.2.3)$$

which should be *in fine* reabsorbed into a redefinition of the radial coordinate:  $r \rightarrow r\mathcal{B}$ . As we already saw many times we just need to introduce a Weyl connection one-form  $A = A_\mu dx^\mu$  transforming as

$$A \rightarrow A - d \ln \mathcal{B}. \quad (3.2.4)$$

This suggests the following amendment to the Newman–Unti gauge

$$-dudr \rightarrow \frac{\mathbf{u}}{k^2}(dr + rA), \quad (3.2.5)$$

which is indeed Weyl-invariant, as well as being boundary-general-covariant. Modifying the rest of the framework and in particular covariant derivatives to encompass Weyl transformations will be addressed in Sec. 3.2.2.

We can recast (3.1.8), ignoring the logarithms, in terms of boundary tensors according to their transversality with respect to the congruence  $\mathbf{u}$  as well as their conformal weights. The bulk line element reads

$$\begin{aligned} ds_{\text{bulk}}^2 &= 2\frac{\mathbf{u}}{k^2}(dr + rA) + r^2 ds^2 + r\mathcal{C}_{AB}\theta^A\theta^B + \frac{1}{k^4}\mathcal{F}_{AB}\theta^A\theta^B \\ &+ \sum_{s=1}^{\infty} \frac{1}{r^s} \left( f_{(s)} \frac{\mathbf{u}^2}{k^4} + 2\frac{\mathbf{u}}{k^2} f_{(s)A}\theta^A + f_{(s)AB}\theta^A\theta^B \right), \end{aligned} \quad (3.2.6)$$

where all boundary tensors are now defined in the orthonormal frame at hand and depend only on boundary coordinates  $(u, x^a)$ . This is the *covariant Newman–Unti gauge* (CNU for short) which we recall is an incomplete gauge fixing due to the arbitrariness in  $\mathbf{u}$ .

In the expression (3.2.6),  $f_{(s)}$  are *boundary scalars*,  $f_{(s)A}$  *boundary transverse vectors*,  $f_{(s)A}u^A = 0$ ,

and  $f_{(s)AB}$  boundary symmetric and transverse tensors,  $f_{(s)AB}u^A = 0$ . Their Weyl weights are  $s + 2$ .<sup>5</sup> At order  $r^2$  we recognize the boundary metric  $ds^2$ , which is a free piece of data in the spirit of [84–86, 122, 141]. As long as the bulk metric (3.2.6) is off-shell, the boundary symmetric tensors  $\mathcal{F}_{AB}$  (weight 2) and  $\mathcal{C}_{AB}$  (weight 1) have no reason to be transverse with respect to  $\mathbf{u}$ . The latter is the shear of the affine null geodesic congruence tangent to  $\partial_r$ . It is also known as the “Bondi” shear even though, strictly speaking, the latter is defined in the BMS gauge (in the expansion  $G_{ij} = r^2 q_{ij} + \mathcal{O}(r)$  the two-dimensional metric  $q_{ij}$  is fixed to be the round sphere). This usual Bondi shear is related to the one introduced here by an inhomogeneous transformation.

Imposing Einstein’s equations will determine all the boundary tensors introduced so far in terms of basic independent functions that define the solution space. As we will see, this set of functions includes  $\mathbf{u}$  as well as the boundary metric  $ds^2$  and a rank-two symmetric, traceless and conserved tensor coinciding with the energy–momentum tensor of the Fefferman–Graham gauge.

**Remark** Logarithms of the radial coordinate are not always required when probing the space of solutions of Einstein’s equations. In some cases, like, e.g., when choosing the Fefferman–Graham gauge in odd spacetime dimensions, they are necessary. In other cases, like, e.g., in asymptotically flat spacetimes in the Bondi gauge, they describe an independent sector of the solution space that might be added or not to the polynomial expansion. A thorough investigation of the larger space of solutions including logarithms and analyzing its interplay with residual symmetries was done e.g. in [3, 122, 135, 142, 143]). Note also the recent work [132] in which the authors work in the partial Bondi gauge [59, 131], and define the notion of “logarithmically-asymptotically-flat” spacetimes, as logarithmic terms are implied by tails in the displacement memory effect.

**Remark** In [122] the authors have shown that logarithms are generated in AdS by the traceless projection of the order 1 tensor in the expansion of the sphere metric,  $D_{AB}$  (see just after (3.1.6)) which can either be taken into account or not, as two different subclass of solutions are described. In this thesis we treat the latter case.

We now drive our attention towards one of the two main advantages of this new gauge, the consideration of Weyl covariance.

### 3.2.2 Encompassing Weyl covariance

The covariant Newman–Unti gauge (3.2.6) naturally encompasses Weyl covariance. As any boundary Weyl transformation can be reabsorbed by a redefinition of the radial coordinate  $r$  it is desirable to list the available tensors with the correct conformal weights at each order  $s$  of the radial expansion, as they are the only ones that can appear at the selected order. One also has to translate

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<sup>5</sup>This shows the power of the orthonormal frame, all unknown functions possess the same Weyl weight.

covariant derivatives into Weyl-covariant derivatives. This is what we shall do now. The reader will recognize a formalism already discussed in Sec. 1.4 but now from a relativistic point of view.

The Weyl connection one-form  $\mathbf{A} = A_A \theta^A$  is built as usual on the congruence  $\mathbf{u} = u_A \theta^A$  via

$$\mathbf{A} = \frac{1}{k^2} \left( \mathbf{a} - \frac{\Theta}{2} \mathbf{u} \right), \quad (3.2.7)$$

which transforms as anticipated in (3.2.4). Kinematical variables of the congruence  $\mathbf{u}$  were already displayed in (2.2.9) and (2.2.10). Note however that we are here in a Cartan orthonormal frame (3.2.1).

Using the Weyl connection  $\mathbf{A}$  we get Weyl covariant derivatives  $\mathcal{D}_A$  acting on a weight- $w$  tensor as e.g. a scalar  $\Phi$

$$\mathcal{D}_A \Phi = e_A(\Phi) + w A_A \Phi, \quad (3.2.8)$$

or a form  $v_A$

$$\mathcal{D}_B v_A = \nabla_B v_A + w A_B v_A + A_A v_B - \eta_{AB} A^C v_C. \quad (3.2.9)$$

The resulting tensors have weight  $w+1$ . When dealing with Weyl covariance in orthonormal frame, the metric components have weight zero. Hence for any tensor, covariant and contravariant components have the same weights. The coframe form elements, however, have weight  $-1$ , whereas the frame vectors have weight  $+1$ . If a weight- $w$  tensor has  $p$  contravariant and  $q$  covariant indices, its Weyl-covariant derivative reads

$$\begin{aligned} \mathcal{D}_C K_{B\dots}^{A\dots} &= \nabla_C K_{B\dots}^{A\dots} + (w+p-q) A_C K_{B\dots}^{A\dots} \\ &+ \left( \eta_{CD} A^A - \delta_C^A A_D - \delta_D^A A_C \right) K_{B\dots}^{D\dots} + \dots \\ &- \left( \eta_{CB} A^D - \delta_C^D A_B - \delta_B^D A_C \right) K_{D\dots}^{A\dots} - \dots \end{aligned}$$

and this has now weight  $w+1$ . See App. B.1 for the Carrollian analogue.

The form field  $\mathbf{u}$  has weight  $-1$  i.e.  $u_A$  are weight-zero, whereas  $\omega_{AB}$  and  $\sigma_{AB}$  have all weight 1. The explicit form of  $\mathbf{A}$  (3.2.7) is obtained by demanding

$$\mathcal{D}_A u^A = 0 \quad \text{and} \quad u^C \mathcal{D}_C u_A = 0. \quad (3.2.10)$$

The Weyl covariant derivative is metric-compatible with effective torsion

$$\mathcal{D}_C \eta_{AB} = 0, \quad (3.2.11a)$$

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) \Phi = w \Phi F_{AB}, \quad (3.2.11b)$$

where

$$F = \frac{1}{2} F_{AB} \theta^A \wedge \theta^B = d\mathbf{A} \quad (3.2.12)$$



is Weyl-invariant ( $F_{AB}$  are weight-2). Metric compatibility and (3.2.10) imply

$$u^C \mathcal{D}_C h_{AB} = 0, \quad (3.2.13)$$

inferring that the operator  $u^C \mathcal{D}_C$  respects transversality.

Commuting the Weyl-covariant derivatives acting on vectors, one defines the Weyl covariant Riemann tensor

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) V^C = \mathcal{R}^C{}_{DAB} V^D + (w+1) V^C F_{AB} \quad (3.2.14)$$

( $V^C$  are weight- $w$  whereas  $V = V^C e_C$  has weight  $w+1$ ) and the usual subsequent quantities. In three (boundary) spacetime dimensions, the covariant Ricci and the scalar (both weight-2) curvatures read

$$\mathcal{R}_{AB} = R_{AB} + \nabla_B A_A + A_A A_B + \eta_{AB} \left( \nabla_C A^C - A_C A^C \right) - F_{AB}, \quad (3.2.15a)$$

$$\mathcal{R} = R + 4 \nabla_A A^A - 2 A_A A^A, \quad (3.2.15b)$$

where  $R_{AB}$  is the Ricci tensor of the boundary Levi-Civita connection and  $R$  the corresponding scalar curvature. The Weyl-invariant Schouten tensor is

$$\mathcal{S}_{AB} = \mathcal{R}_{AB} - \frac{1}{4} \mathcal{R} \eta_{AB} = R_{AB} - \frac{1}{4} R \eta_{AB} + \nabla_B A_A + A_A A_B - \frac{1}{2} A_C A^C \eta_{AB} - F_{AB}. \quad (3.2.16)$$

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2} \omega_{AB} dx^A \wedge dx^B = \frac{1}{2} \left( du + \frac{1}{k^2} u \wedge a \right), \quad (3.2.17)$$

as well as the Hodge dual of this form, which is proportional to  $u$ :

$$k \gamma u = \star \omega \quad \Leftrightarrow \quad k \gamma u_A = \frac{1}{2} \epsilon_{ABC} \omega^{BC}. \quad (3.2.18)$$

In this expression  $\gamma$  is a scalar of weight 1.

In three spacetime dimensions and in the presence of a vector field  $u$ , one naturally defines a fully antisymmetric two-index tensor<sup>6</sup>

$$\hat{\eta}_{AB} = -\frac{u^C}{k} \epsilon_{CAB}, \quad (3.2.19)$$

obeying

$$\hat{\eta}_{AC} \hat{\eta}_B{}^C = h_{AB}, \quad \hat{\eta}^{AB} \hat{\eta}_{AB} = 2. \quad (3.2.20)$$

This is the relativistic equivalent of the Carrollian tensor  $\eta_{\zeta\bar{\zeta}}$  introduced in Sec. 1.6. With this tensor

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<sup>6</sup>This hatted two-index tensor should not be confused with Minkowski metric.

the vorticity reads

$$\omega_{AB} = k^2 \gamma \hat{\eta}_{AB}. \quad (3.2.21)$$

The two-index tensor  $\hat{\eta}_{AB}$  defines a duality map within the space of symmetric, transverse (with respect to  $\mathbf{u}$ ) and traceless tensors. If  $V^A$  is transverse, so is

$$*V^A = \hat{\eta}^B{}_A V_B. \quad (3.2.22)$$

Similarly with a symmetric, transverse and traceless tensor  $W_{AB}$

$$*W_{AB} = \hat{\eta}^C{}_A W_{CB} \quad (3.2.23)$$

is symmetric, transverse and traceless. We can also define a scalar quantity from a rank-2 antisymmetric tensor, using (3.2.19) e.g.

$$*\omega = \frac{1}{2} \eta^{AB} \omega_{AB}, \quad (3.2.24)$$

which yields the *vorticity scalar*.

### 3.2.3 Einstein's equations for asymptotically AdS spacetimes

#### AdS energy-momentum and Cotton tensor

As we said, metrics of AlAdS spacetimes are completely described by means of the boundary metric and the boundary energy-momentum tensor. What is new in our CNU gauge is that the former will appear also through its third order derivatives via another tensor presented in Sec. 2.4, the Cotton tensor.

Having at hand the time-like congruence  $\mathbf{u}$  such that  $||\mathbf{u}||^2 = -k^2$  we use it to decompose the energy-momentum tensor in a hydrodynamic way (see Sec. 2.2)

$$T_{AB} = (\varepsilon + p) \frac{u_A u_B}{k^2} + p \eta_{AB} + \tau_{AB} + \frac{u_A q_B}{k^2} + \frac{u_B q_A}{k^2}. \quad (3.2.25)$$

Such a decomposition allows us to re-use all the formalism developed in Sec. 2.2 and we refer the reader to it e.g. for the transversality properties of  $q_A$  and  $\tau_{AB}$ .

**Remark** Even though the energy-momentum tensor is written in a fluid form, we are not doing hydrodynamics here. What we have chosen is a timelike congruence under which we decompose  $T_{AB}$  like in (3.2.25), this is just a way to package differently the degrees of freedom. Hydrodynamics (we recall) is more, as all thermodynamic quantities are to be expanded in powers of the velocity and the temperature with transport coefficients. These are the constitutive relations. The latter are not obeyed everywhere in the Einstein solution space. The subspace where this happens is the realm of fluid/gravity correspondence [27, 144]. For convenience, we will nonetheless refer to  $\mathbf{u}$  as the “velocity field” and decompose the energy-momentum

tensor accordingly.

**Remark** Whenever the energy–momentum tensor gets a fluid interpretation, the congruence at hand is identified with the fluid lines and its arbitrariness corresponds to the relativistic hydrodynamic-frame invariance [65, 106, 145]. This feature is however strictly *local* because the bulk diffeomorphisms associated with the boundary hydrodynamic-frame transformations are possibly charged (see [84–86] for a discussion in three bulk dimensions).

In CNU gauge the other important character is the Cotton tensor (see later (3.2.38)). Together with the energy-momentum tensor they both enter the bulk metric, playing dual, electric versus magnetic, roles in various instances, as e.g. in the bulk Weyl tensor. Indeed it has been shown in [146, 147] that in the  $1/r$ -expansion of the latter appears at leading order ( $1/r^3$ ) a combination of the form  $T_{AB} + \frac{w}{k}C_{AB}$  with a constant coefficient  $w$  depending on Newton’s constant  $G$ . The Cotton tensor possessing the same symmetries than the energy-momentum tensor (it is also traceless) and satisfying the same equation ( $\nabla_A C^{AB} = \mathcal{D}_A C^{AB} = 0$ ) it is reasonable to decompose it along the congruence  $\mathbf{u}$  like in (3.2.25)

$$\frac{1}{k}C_{AB} = \frac{3c}{2} \frac{u_A u_B}{k^2} + \frac{c}{2} \eta_{AB} - \frac{c_{AB}}{k^2} + \frac{u_A c_B}{k^2} + \frac{u_B c_A}{k^2}. \quad (3.2.26)$$

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$c = \frac{1}{k^3} C_{AB} u^A u^B, \quad (3.2.27)$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor*  $c_{AB}$  and the *Cotton current*  $c_A$  (also weight-3) are purely transverse

$$c_A{}^A = 0, \quad u^A c_{AB} = 0, \quad u^A c_A = 0, \quad (3.2.28)$$

and obey

$$c_{AB} = -k h^C{}_A h^D{}_B C_{CD} + \frac{ck^2}{2} h_{AB}, \quad c_B = -c u_B - \frac{u^A C_{AB}}{k}. \quad (3.2.29)$$

The Cotton density, current and stress tensor can also be expressed as ordinary or Weyl derivatives of the curvature using its definition as derivative of the Schouten, see (2.4.1)

$$c = \frac{1}{k^2} u^B \hat{\eta}^{DC} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}), \quad (3.2.30a)$$

$$c_B = \hat{\eta}^{CD} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}) - c u_B, \quad (3.2.30b)$$

$$c_{AB} = -h^E{}_A \left( k \epsilon_B{}^{CD} - u_B \hat{\eta}^{CD} \right) \mathcal{D}_C (\mathcal{S}_{ED} + F_{ED}) + \frac{ck^2}{2} h_{AB}. \quad (3.2.30c)$$

We conclude this paragraph displaying the Weyl weights of all relevant quantities

Quantity	$T_{AB}$	$\varepsilon$	$p$	$q_A$	$\tau_{AB}$	$c$	$c_A$	$c_{AB}$
Weight $w$	3	3	3	3	3	3	3	3

(3.2.31)

therefore the table of weight is very easy due to the choice of orthonormal frame.

### Solving the equations order by order

We shall now plug this ansatz (3.2.6) into the bulk Einstein equations

$$\mathcal{E}_{MN} \equiv R_{MN}^{\text{bulk}} - \frac{1}{2}R^{\text{bulk}}g_{MN}^{\text{bulk}} - 3k^2g_{MN}^{\text{bulk}} = 0, \quad (3.2.32)$$

where we use  $M, N, \dots \in \{r, \text{boundary}\}$  as bulk indices, and solve them order by order.

- **Order  $r$ .** This equation will relate the would-be Bondi shear  $\mathcal{C}_{AB}$  to the shear of the congruence  $\mathbf{u}$  via

$$k^2\mathcal{C}_{AB} = -2\sigma_{AB}, \quad (3.2.33)$$

which set a relation between a gravitational quantity and a purely geometric quantity. On-shell, the Bondi shear becomes transverse and traceless (as  $u^A\sigma_{AB} = 0$  and  $\sigma_A^A = 0$ ). One can define in this framework the equivalent to the Bondi news tensor, taking the appropriate derivative of the shear

$$\mathcal{N}_{AB} = u^C\mathcal{D}_C\mathcal{C}_{AB}. \quad (3.2.34)$$

This tensor is manifestly boundary-covariant (by contrast with all other gauges and in particular the Bondi gauge), Weyl-invariant, symmetric, traceless and transverse.

- **Order 1.** The equations (3.2.32) fix the form of the tensor  $\mathcal{F}_{AB}$  which ends up being related to the boundary Weyl-invariant Schouten tensor

$$\begin{aligned} \mathcal{F}_{AB} &= 2u^C(\mathcal{S}_{C(A} + F_{C(A)}u_{B)}) + \mathcal{D}_A u^C\mathcal{D}_B u^C \\ &= 2u_{(A}\mathcal{D}_C(\sigma_{B)}^C + \omega_{B)}^C) - \frac{\mathcal{R}}{2}u_A u_B + (\sigma^2 + k^4\gamma^2)h_{AB} + 2\omega_{(A}^C\sigma_{B)C}, \end{aligned} \quad (3.2.35)$$

where

$$\gamma^2 = \frac{1}{2k^4}\omega_{AB}\omega^{AB}, \quad \sigma^2 = \frac{1}{2}\sigma_{AB}\sigma^{AB}. \quad (3.2.36)$$

Note that at this stage, the only independent and free data are those defining the boundary geometry.

- **Orders  $1/r$  and  $1/r^2$ .** At order  $\frac{1}{r}$  new information is expected to come up in the form of a boundary conformal energy–momentum tensor. In the Fefferman–Graham gauge it appears in one block  $T_{AB}$  whereas in the CNU gauge the energy–momentum enters through its decomposition with respect to the congruence  $\mathbf{u}$ , i.e.  $\varepsilon$ ,  $q_A$  and  $\tau_{AB}$ . Furthermore, it comes accompanied with the transverse-dual of the Cotton current and stress,  $*c_A$  and  $*c_{AB}$  which is yet another motivation to split the energy–momentum tensor as discussed earlier. This trait is new, both compared to the Fefferman–Graham gauge, where the Cotton tensor does not appear explicitly at any order,<sup>7</sup> and with respect to standard Bondi or Newman–Unti gauges.

<sup>7</sup>This is the Schouten tensor that appears in [148, 149].

The functions to be determined are  $f_{(1)}$ ,  $f_{(1)A}$  and  $f_{(1)AB}$ , which must have conformal weight 3. This leaves little freedom, given the available tensors. We find that

$$f_{(1)} \frac{\mathbf{u}^2}{k^4} + 2 \frac{\mathbf{u}}{k^2} f_{(1)A} \theta^A + f_{(1)AB} \theta^A \theta^B = \frac{8\pi G}{k^4} \left( \varepsilon \mathbf{u}^2 + \frac{4}{3} \mathbf{u} \Delta q + \frac{2k^2}{3} \Delta \tau \right) \quad (3.2.37)$$

with  $\Delta q = \Delta q_A \theta^A$  and  $\Delta \tau = \Delta \tau_{AB} \theta^A \theta^B$  defined as

$$\Delta q_A = q_A - \frac{1}{8\pi G} *c_A, \quad \Delta \tau_{AB} = \tau_{AB} + \frac{1}{8\pi G k^2} *c_{AB}. \quad (3.2.38)$$

The functions  $\varepsilon$ ,  $q_A$  and  $\tau_{AB}$  are the unknown functions that which merely parameterize the line element. Retrospectively we see that they can be packaged in a symmetric and traceless tensor  $T_{AB}$  as in (3.2.25). Moreover (3.2.42) and, as we shall see shortly, Einstein's equations demand the conservation of this  $T_{AB}$  built upon these unknowns. Thus we are tempted to identify this tensor with the boundary energy-momentum tensor as in the Fefferman-Graham gauge.

We now focus on the  $\frac{1}{r^2}$  contribution to the line element (3.2.6), i.e., on

$$f_{(2)} \frac{\mathbf{u}^2}{k^4} + 2 \frac{\mathbf{u}}{k^2} f_{(2)A} \theta^A + f_{(2)AB} \theta^A \theta^B, \quad (3.2.39)$$

where  $f_{(2)}$ ,  $f_{(2)A}$  and  $f_{(2)AB}$  must have conformal weight 4. The analogy with the Fefferman-Graham expansion suggests that no new free boundary functions should appear without spoiling Einstein's equations, they can just be derivatives or contraction with a weight-1 tensor or the energy-momentum components. Indeed, upon imposing (3.2.33) and (3.2.35), one finds

$$\begin{cases} \mathcal{E}_{rr} = -\frac{3}{r^5} \eta_{AB} f_{(1)}^{AB} - 6 \left( \eta_{AB} f_{(2)}^{AB} + \frac{3}{2k^2} \sigma_{AB} f_{(1)}^{AB} \right) \frac{1}{r^6} + \mathcal{O} \left( \frac{1}{r^7} \right) \\ k \mathcal{E}_{r\hat{0}} = \left( -f_{(2)} - 2k^2 \eta_{AB} f_{(2)}^{AB} + \frac{1}{2} h_{AB} \mathcal{D}^A f_{(1)}^B - \frac{5}{2} \sigma_{AB} f_{(1)}^{AB} + c\gamma \right) \frac{1}{r^4} + \mathcal{O} \left( \frac{1}{r^5} \right) \\ \mathcal{E}_{ra} = \left( 2f_{(2)a} - \frac{3}{2} h_{aB} \mathcal{D}_C f_{(1)}^{BC} + \frac{1}{k^2} (\sigma_{aB} + 4\omega_{aB}) f_{(1)}^B \right) \frac{1}{r^4} + \mathcal{O} \left( \frac{1}{r^5} \right) \\ \mathcal{E}^{ab} = \left( -f_{(2)} h^{ab} + c\gamma h^{ab} + 4\omega_C^{(a} f_{(1)}^{b)C} + 2k^2 \hat{\eta}_C^a \hat{\eta}_D^b f_{(2)}^{CD} - 2u^C \mathcal{D}_C f_{(1)}^{ab} \right. \\ \left. + \hat{\eta}_C^a \hat{\eta}_D^b \mathcal{D}^{(C} f_{(1)}^{D)} + \frac{1}{k^2} (c \hat{\eta}_C^a \sigma^{Cb} - f_{(1)} \sigma^{ab}) + 4\sigma_C^{(a} f_{(1)}^{b)C} \right) \frac{1}{r^2} + \mathcal{O} \left( \frac{1}{r^3} \right) \end{cases} \quad (3.2.40)$$

for the constraint Einstein's equations.<sup>8</sup> These equations confirm the absence of any new free function. When rewritten in terms of the basic quantities parameterizing the space of solutions, the three coefficients in (3.2.39) read

$$f_{(2)} = \frac{8\pi G}{3k^2} \left( \sigma_{CD} \Delta \tau^{CD} + \mathcal{D}_C \Delta q^C \right) + c\gamma, \quad (3.2.41a)$$

$$f_{(2)A} = -\frac{8\pi G}{3k^4} \sigma_{AC} \Delta q^C + \frac{4\pi G}{k^2} \left( h_{AC} \mathcal{D}_D \Delta \tau^{CD} + \frac{8}{3} \gamma * \Delta q_A \right), \quad (3.2.41b)$$

<sup>8</sup>This terminology is borrowed from the Hamiltonian analysis of Einstein's gravity.

$$\begin{aligned}
f_{(2)AB} = & -\frac{4\pi G}{k^4} \left( \frac{4}{3} u^C \mathcal{D}_C \Delta \tau_{AB} + \frac{2}{3} h_{AC} h_{BD} \mathcal{D}^{(C} \Delta q^{D)} - \frac{1}{3} h_{AB} h^{CD} \mathcal{D}_C \Delta q_D + 2\sigma_{(A}^C \Delta \tau_{B)C} \right) \\
& - \frac{1}{2k^4} (8\pi G \varepsilon \sigma_{AB} - c * \sigma_{AB}) + \frac{32\pi G}{3k^2} \gamma * \Delta \tau_{AB}.
\end{aligned} \tag{3.2.41c}$$

These expressions contain all possible combinations of the shear and of the vorticity together with adequately projected Weyl covariant derivatives of the energy–momentum and Cotton tensors,<sup>9</sup> carrying the right tensorial structure and conformal weight. Substituting Eqs. (3.2.41a), (3.2.41b), (3.2.41c) into the remaining Einstein’s equations (3.2.32) one obtains

$$\frac{k}{8\pi G} \mathcal{E}_{\hat{0}\hat{0}} = \frac{1}{r^2} \mathcal{D}_B T^B_{\hat{0}} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \frac{k}{8\pi G} \mathcal{E}_{\hat{0}a} = \frac{1}{r^2} \mathcal{D}_B T^B_a + \mathcal{O}\left(\frac{1}{r^3}\right) \tag{3.2.42}$$

(since  $T_{AB}$  is traceless,  $\mathcal{D}_A \equiv \nabla_A$ , the Levi–Civita boundary connection for the frame metric  $\eta_{AB}$ ). The omitted terms contain the tensors  $f_{(3)}$ ,  $f_{(3)A}$  and  $f_{(3)AB}$ . This confirms that no additional constraints are imposed on the quantities parameterizing the solution space identified at the previous orders, i.e., the velocity field, the boundary metric (frame in the present formalism) and the boundary energy–momentum tensor. We therefore conclude that

$$\boxed{\text{AdS solution space} = \{u^A, ds_{\text{bry}}^2, \varepsilon, q_A, \tau_{AB}\}}. \tag{3.2.43}$$

- **Subleading orders** The above pattern can be repeated again and again at the cost of a substantial growth in admissible terms. The third order would be interesting as it is expected to host the Newman–Penrose charges in the flat limit, according to the analysis of [150–154]. However as showed for the order  $1/r^2$  no additional degrees of freedom will appear in the solution space.

**Remark** There is a set of hypotheses under which the line element (3.2.6) can be resummed in a finite and closed expression. This is the case when the bulk Weyl tensor is tuned to select the subclass of solution of Einstein’s equations which are Petrov-algebraically special. Imposing

$$\sigma_{AB} = 0, \quad \Delta q_A = 0, \quad \Delta \tau_{AB} = 0, \tag{3.2.44}$$

which implies that

$$f_{(s)A} = 0, \quad f_{(s)AB} = 0 \tag{3.2.45}$$

and

$$f_{(2s+1)} = (-)^s 8\pi G \varepsilon \gamma^{2s}, \quad f_{(2s+2)} = (-)^s c \gamma^{2s+1}. \tag{3.2.46}$$

The boundary metric is still a free variable, but only the energy density  $\varepsilon(\mathbf{x})$  remains from the energy–momentum tensor, whose heat current and stress are fixed by those of the Cotton

$$q_A = \frac{1}{8\pi G} *c_A, \quad \tau_{AB} = -\frac{1}{8\pi G k^2} *c_{AB}. \tag{3.2.47}$$

<sup>9</sup>This is in line with the identification of the CNU gauge with a derivative expansion.

As a consequence, assuming that  $\nabla_A T^{AB} = 0$  is satisfied, one finds for the line element

$$ds_{\text{res. Einstein}}^2 = 2 \frac{\mathbf{u}}{k^2} (dr + rA) + r^2 ds^2 + \frac{\mathcal{F}}{k^4} + \frac{\mathbf{u}^2}{k^4 \rho^2} (8\pi G \varepsilon r + c\gamma) \quad (3.2.48)$$

with

$$\rho^2 = r^2 + \gamma^2 \quad (3.2.49)$$

and  $\mathcal{F} = \mathcal{F}_{AB} \theta^A \theta^B$  given in (3.2.35) imposing zero geometric shear  $\sigma_{AB}$ . The Petrov analysis of (3.2.48) has been discussed in Refs. [52, 114]. and the flat counterpart will be discussed in Sec. 4.1.

## 3.3 Asymptotically flat spacetimes as a limit of Anti de Sitter

### 3.3.1 Laurent expansion and the flat limit

In this subsection we take the  $k \rightarrow 0$  limit of the AdS case. After recalling how a Carroll structure emerges from the timelike boundary of AdS we expand the unknowns of (3.2.6) i.e. the energy-momentum tensor, in a Laurent series about  $k = 0$ , allowing us to take the limit in the conservation equation. From this we get the first two flux/balance laws while the others are found requiring finiteness at zero  $k$ . Remember that we are in four-dimensional bulk spacetime with indices  $(\hat{u}, a)$  where  $a$  ranges from 1 to 3.

#### From Riemann to Carroll

The Carrollian limit of the boundary geometry is reached as follows

$$\mathbf{v} = \lim_{k \rightarrow 0} k \mathbf{e}_0 \quad , \quad \boldsymbol{\tau} = - \lim_{k \rightarrow 0} \frac{\mathbf{u}}{k^2} = \lim_{k \rightarrow 0} \frac{\boldsymbol{\theta}^0}{k} \quad , \quad \hat{\mathbf{e}}_a = \lim_{k \rightarrow 0} \mathbf{e}_a \quad , \quad \hat{\boldsymbol{\theta}}^a = \lim_{k \rightarrow 0} \boldsymbol{\theta}^a \quad (3.3.1)$$

so that the Carrollian degenerate metric spells

$$d\ell^2 = \lim_{k \rightarrow 0} ds^2 = \delta_{ab} \hat{\boldsymbol{\theta}}^a \hat{\boldsymbol{\theta}}^b. \quad (3.3.2)$$

The kernel of the degenerate metric (3.3.2) is the *field of observers*  $\mathbf{v}$ , and  $\boldsymbol{\tau}$  is its dual *clock form* embracing also the Ehresmann connection, as explained in Sec. B.1. The frame and coframe,  $\{\mathbf{v}, \hat{\mathbf{e}}_a\}$  and  $\{\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}^a\}$  obey

$$\boldsymbol{\tau}(\mathbf{v}) = 1, \quad \hat{\boldsymbol{\theta}}^a(\hat{\mathbf{e}}_b) = \delta_b^a, \quad \hat{\boldsymbol{\theta}}^a(\mathbf{v}) = 0, \quad \boldsymbol{\tau}(\hat{\mathbf{e}}_a) = 0. \quad (3.3.3)$$

The Carrollian geometric data are part of the solution space of Ricci-flat spacetimes in the flat covariant Newman–Unti gauge. Compared to the standard flat Newman–Unti gauge, the extra

piece of data is the clock form  $\boldsymbol{\tau}$ , which echoes the velocity congruence of the AdS relative.<sup>10</sup> More accurately, the additional piece of information carried by the covariant Newman–Unti gauge is the boundary vorticity  $*\hat{\omega}$ , see (3.2.24).

The vanishing- $k$  limit of the AdS-boundary Weyl connection  $A$  is straightforward due to its  $k$ -independence. We actually recover the connection given in (1.4.4)

$$A = \varphi_a \hat{\boldsymbol{\theta}}^a + \frac{\theta}{2} \boldsymbol{\tau} \quad (3.3.4)$$

with  $\varphi_a$  and  $\theta$  defined in (B.1.11) and (B.1.31). Therefore, the first two terms in (3.2.6) have a well-defined limit without the need of imposing Einstein’s equations.

We proceed now with the next two terms namely the order  $r$  and 1 which we recall are transverse. The order  $r$  plays an essential role in gravitational physics. Indeed, Einstein’s equation (3.2.33), reproduced here for the spatial components — the only non-zero due transversality combined with our choice of congruence  $\boldsymbol{u}$ ,

$$k^2 \mathcal{C}_{ab} = -2\sigma_{ab}, \quad (3.3.5)$$

implies that  $\sigma_{ab} = 0$  at vanishing  $k$ . As explained in Eq. (B.2.12), the latter translates in Carrollian terms into

$$k^2 \mathcal{C}_{ab} = -2\xi_{ab}, \quad (3.3.6)$$

where  $\xi_{ab}$  is defined in (B.1.31) as the traceless component of the extrinsic curvature. In the flat instance we find that the geometrical shear  $\xi_{ab}$  of the boundary Carrollian geometry must vanish. On the other hand, the dynamical shear  $\mathcal{C}_{ab}$  is completely unconstrained and carries two degrees of freedom which are identified with the two polarizations of the graviton. No equation makes  $\mathcal{C}_{ab}$  evolve, but it will source the evolution of other degrees of freedom. From the Bondi shear  $\mathcal{C}_{ab}$  one can introduce the Bondi news which is another traceless Carrollian tensor obtained as the Carrollian limit of Eq. (3.2.34)

$$\hat{\mathcal{N}}_{ab} = \hat{\mathcal{D}}_v \mathcal{C}_{ab}. \quad (3.3.7)$$

The relationship between (3.3.7) and the news defined in other classical gauges will be unveiled in Sec. 3.3.2.

In summary, till the order  $r$ , the Ricci-flat bulk metric reads

$$ds_{\text{Ricci-flat}}^2|_r = -\boldsymbol{\tau} \left[ 2dr + r \left( 2\varphi_a \hat{\boldsymbol{\theta}}^a + \theta \boldsymbol{\tau} \right) \right] + r^2 dt^2 + r \mathcal{C}_{ab} \hat{\boldsymbol{\theta}}^a \hat{\boldsymbol{\theta}}^b, \quad (3.3.8)$$

where  $\mathcal{C}_{ab}(u, \mathbf{x})$  is an arbitrary traceless Carrollian tensor.

To handle the limit of the order 1 term we will design a method which constitutes the main achievement of the work [155]. The trick is to make a *partially on-shell* flat limit, using the equality between the shears (3.3.6). Trading now the geometric shear  $\xi_{ab}$  for  $-\frac{k^2}{2} \mathcal{C}_{ab}$  may most of the time

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<sup>10</sup>In the usual Newman–Unti gauge the Ehresmann connection is zero, which amounts in the split notations to set  $b_i = 0$ .



not have any effect on the term under consideration, which will still drop out; however it may sometimes cancel a divergence, making the term appear in the Carrollian case. When dealing with the flux/balance equations we will prove that the radiation, source of the latter, appears in this way. For the moment we illustrate this method with the tensor  $\mathcal{F} = \mathcal{F}_{AB}\theta^A\theta^B$ , which can be expanded in Carrollian terms as

$$\begin{aligned}\frac{\mathcal{F}}{k^4} &= \frac{\xi^2}{k^4}d\ell^2 + \frac{1}{k^2}\left(3\xi^2\tau^2 - 2\hat{\mathcal{D}}_b\xi^b{}_a\tau\hat{\theta}^a - 2*\hat{\omega}*\xi_{ab}\hat{\theta}^a\hat{\theta}^b\right) \\ &\quad + *\hat{\omega}^2d\ell^2 + 2*\hat{\mathcal{D}}_a*\hat{\omega}\tau\hat{\theta}^a - \hat{\mathcal{K}}\tau^2 - 5k^2*\hat{\omega}^2\tau^2 \\ &= \left(\frac{\mathcal{C}^2}{4} + *\hat{\omega}^2\right)d\ell^2 - \hat{\mathcal{K}}\tau^2 + \hat{\mathcal{D}}_b\mathcal{C}^b{}_a\tau\hat{\theta}^a + 2*\hat{\mathcal{D}}_a*\hat{\omega}\tau\hat{\theta}^a \\ &\quad + *\hat{\omega}*\mathcal{C}_{ab}\hat{\theta}^a\hat{\theta}^b + k^2\left(\frac{3}{4}\mathcal{C}^2 - 5*\hat{\omega}^2\right)\tau^2\end{aligned}\tag{3.3.9}$$

with  $\mathcal{C}^2 = \frac{1}{2}\mathcal{C}^{ab}\mathcal{C}_{ab}$  and quantities like  $\xi^2$ ,  $*\hat{\omega}$ ,  $\hat{\mathcal{K}}$  defined in Sec. B.1.

**Remark** Recall that in (3.3.9) we just decomposed the relativistic quantities in a Carrollian fashion, no limit has been taken yet. So the result, even though written in terms of Carrollian quantities, is still genuinely relativistic.

Some terms drop in the zero- $k$  limit but no divergence occurs and we are left with a piece in the line element, which now contains explicitly the Bondi shear

$$\lim_{k\rightarrow 0}\frac{\mathcal{F}}{k^4} = \left(\frac{\mathcal{C}^2}{4} + *\hat{\omega}^2\right)d\ell^2 - \hat{\mathcal{K}}\tau^2 + \hat{\mathcal{D}}_b\mathcal{C}^b{}_a\tau\hat{\theta}^a + 2*\hat{\mathcal{D}}_a*\hat{\omega}\tau\hat{\theta}^a + *\hat{\omega}*\mathcal{C}_{ab}\hat{\theta}^a\hat{\theta}^b.\tag{3.3.10}$$

**Remark** The authors of Ref. [122] were the first to propose the idea of substituting the Bondi for the geometric shear with the accompanying power of the cosmological constant. In their work it turned out to be necessary to reach integrable charges.

## Expanding the energy momentum tensor

Apart from the replacement between the two shears our second hypothesis is that the energy-momentum tensor, which we recall constitutes the other part of the AdS solution space, is analytic in powers of  $k^2$ . It can thus be written as a Laurent series about  $k = 0$

$$\varepsilon = \sum_{n\in\mathbb{Z}}k^{2n}\varepsilon_{(n)},\tag{3.3.11a}$$

$$q^a = \sum_{n\geq 2}\frac{\zeta_{(n)}^a}{k^{2n}} + \frac{\zeta^a}{k^2} + Q^a + k^2\pi^a + \sum_{n\geq 2}k^{2n}\pi_{(n)}^a,\tag{3.3.11b}$$

$$\tau^{ab} = -\sum_{n\geq 3}\frac{\zeta_{(n)}^{ab}}{k^{2n}} - \frac{\zeta^{ab}}{k^4} - \frac{\Sigma^{ab}}{k^2} - \Xi^{ab} - k^2E^{ab} - \sum_{n\geq 2}k^{2n}E_{(n)}^{ab}.\tag{3.3.11c}$$

Each function in these series is a Carrollian tensor (scalar, vector, or symmetric and traceless two-tensor) that is possibly one of the boundary degrees of freedom, which we call *Chthonian* to recall they encode the asymptotically flat Einstein dynamics probing the bulk metric *in depth* from the boundary. These tensors are expected to obey flux-balance equations, which are Carrollian avatars of vacuum Einstein's equations, and that we will attain using the anti-de Sitter dynamics and imposing a regular behaviour at zero  $k$ .

**Remark** We have no proof for this analyticity. The latter is a working framework which leads to a consistent description of asymptotically flat spacetimes. Retrospectively we thus consider this assumption to be physically relevant. The rules are simple: given the limit of the geometry (3.3.1) and the expansion (3.3.11a), (3.3.11b) and (3.3.11c), impose regularity at  $k = 0$  after trading  $\xi_{ab}$  for  $-\frac{k^2}{2}\mathcal{C}_{ab}$ . This process starts with  $\frac{1}{r}$ , since this is the first term sensitive to the energy-momentum tensor, but the substitution of  $\mathcal{C}_{ab}$  will be performed systematically, everywhere  $\xi_{ab}$  appears.

At order  $\frac{1}{r}$  we should probe (3.2.37), which spells

$$f_{(1)}\frac{u^2}{k^4} + 2\frac{u}{k^2}f_{(1)a}\theta^a + f_{(1)ab}\theta^a\theta^b = 8\pi G \left( \varepsilon\tau^2 - \frac{4}{3}\tau\frac{\Delta q_a}{k^2}\hat{\theta}^a + \frac{2}{3}\frac{\Delta\tau_{ab}}{k^2}\hat{\theta}^a\hat{\theta}^b \right) \quad (3.3.12)$$

with  $\varepsilon$  given in (3.3.11a) and

$$\begin{aligned} \frac{\Delta q^a}{k^2} &= \sum_{n \geq 2} \frac{\zeta_{(n)}^a}{k^{2n+2}} + \frac{1}{k^4} \left( \zeta^a - \frac{*z^a}{8\pi G} \right) + \frac{1}{k^2} \left( Q^a - \frac{* \chi^a}{8\pi G} \right) + \left( \pi^a - \frac{* \psi^a}{8\pi G} \right) \\ &\quad + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a, \end{aligned} \quad (3.3.13)$$

$$\begin{aligned} \frac{\Delta\tau^{ab}}{k^2} &= -\sum_{n \geq 3} \frac{\zeta_{(n)}^{ab}}{k^{2n+2}} - \frac{1}{k^6} \left( \zeta^{ab} - \frac{*Z^{ab}}{8\pi G} \right) - \frac{1}{k^4} \left( \Sigma^{ab} - \frac{*X^{ab}}{8\pi G} \right) - \frac{1}{k^2} \left( \Xi^{ab} - \frac{* \Psi^{ab}}{8\pi G} \right) \\ &\quad - E^{ab} - \sum_{n \geq 2} k^{2n-2} E_{(n)}^{ab}, \end{aligned} \quad (3.3.14)$$

where we have used (3.3.11b), (3.3.11c), the definitions (3.2.38) of  $\Delta q^a$  and  $\Delta\tau^{ab}$ , as well as the Carrollian Cotton tensors  $z^a$ ,  $\chi^a$ ,  $\psi^a$ ,  $Z^{ab}$ ,  $X^{ab}$ ,  $\Psi^{ab}$  displayed in (2.4.7c), (2.4.7b), (2.4.7a), (2.4.8c), (2.4.8b) and (2.4.8a). Finiteness in the flat limit sets up two sorts of requirements on the Carrollian descendants of the energy-momentum tensor.

- Infinite subsets of Laurent coefficients are required to vanish:

$$\begin{cases} \varepsilon_{(n)} = 0 & \forall n < 0 \\ \zeta_{(n)}^a = 0 & \forall n \geq 2 \\ \zeta_{(n)}^{ab} = 0 & \forall n \geq 3; \end{cases} \quad (3.3.15)$$

- Five Laurent coefficients are locked in terms of the dual of Carroll Cotton tensors

$$\zeta^a = \frac{*z^a}{8\pi G}, \quad Q^a = \frac{* \chi^a}{8\pi G}, \quad \zeta^{ab} = \frac{*Z^{ab}}{8\pi G}, \quad \Sigma^{ab} = \frac{*X^{ab}}{8\pi G}, \quad \Xi^{ab} = \frac{* \Psi^{ab}}{8\pi G}. \quad (3.3.16)$$

Hence a finite subset of energy–momentum Carrollian descendants *are not independent but are instead of geometric nature, determined by the boundary Carroll structure via its Cotton tensor.*

These were the only constraints at  $\frac{1}{r}$  order. Defining

$$N^a = *\psi^a - 8\pi G\pi^a, \quad (3.3.17)$$

we recast the order- $\frac{1}{r}$  term (3.3.12) in the flat limit as

$$\begin{aligned} \lim_{k \rightarrow 0} \left( f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)a} \theta^a + f_{(1)ab} \theta^a \theta^b \right) &= 8\pi G \varepsilon_{(0)} \boldsymbol{\tau}^2 + \frac{4}{3} \boldsymbol{\tau} N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \\ &\equiv \hat{f}_{(1)} \boldsymbol{\tau}^2 - 2\boldsymbol{\tau} \hat{f}_{(1)a} \hat{\theta}^a + \hat{f}_{(1)ab} \hat{\theta}^a \hat{\theta}^b. \end{aligned} \quad (3.3.18)$$

Note that the Chthonian Carrollian tensors  $\varepsilon_{(n \geq 1)}$ ,  $\pi_{(n \geq 2)}^a$  and  $E_{(n \geq 2)}^{ab}$  are absent. We should not interpret this as a sign that those aren't genuine degrees of freedom. Some of them may appear in the line element in the next orders and therefore participate in the dynamics. Only when one is guaranteed that a Laurent coefficient is absent from the line element at any order, can we declare it is irrelevant and set it consistently to zero. We should thus wait for the analysis at order- $\frac{1}{r^2}$ .

### 3.3.2 The flat flux/balance equations and the infinite solution space

#### Order $1/r$ and flux-balance equations

Given our method, this dynamics is encoded in the zero- $k$  limit of anti-de Sitter Einstein's equations and in the finiteness requirement of the line element. As the flat limit of the line element up to  $\frac{1}{r}$ -order is finite we shall consider now the energy–momentum conservation  $\nabla_A T^{AB} = \mathcal{D}_A T^{AB} = 0$ . Our treatment consists in the four steps summarized below.

1. In the frame (3.3.1), we consider  $\mathcal{D}_A T^{AB} = 0$  recast in Carrollian terms, see Eqs. (B.2.18a) and (B.2.18b). We re-display them here

$$\mathcal{L} = \hat{\mathcal{D}}_v \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab} = 0, \quad (3.3.19a)$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \hat{\omega}^{ba} + \frac{1}{k^2} \left( \hat{\mathcal{D}}_v q^a + \xi^{ab} q_b \right) = 0. \quad (3.3.19b)$$

2. We insert in these equations the variables  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  in their expanded forms (3.3.11a), (3.3.11b) and (3.3.11c), taking into account the finiteness requirements (3.3.15) and (3.3.16).

3. The requirements (3.3.15) and (3.3.16) bring the Cotton tensor inside the boundary energy-momentum conservation equations  $\mathcal{L} = 0$  and  $\mathcal{T}^a = 0$ . Therefore we must also exploit the Cotton identities  $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$  and  $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$  set in Eqs. (2.4.11a), (2.4.11b), (2.4.11c), (2.4.11d), (2.4.12a), (2.4.12b), (2.4.12c), (2.4.12d) and recast for our needs as

$$\hat{\mathcal{D}}_b * \Psi^{ab} + 2 * \hat{\omega} \chi^a = \hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab}, \quad (3.3.20a)$$

$$\hat{\mathcal{D}}_b * X^{ab} + 2 * \hat{\omega} z^a - \hat{\mathcal{D}}_v * \chi^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab}, \quad (3.3.20b)$$

$$\hat{\mathcal{D}}_b * Z^{ab} - \hat{\mathcal{D}}_v * z^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab}. \quad (3.3.20c)$$

With this we reach the following longitudinal equation

$$\begin{aligned} \mathcal{L} &= k^2 \hat{\mathcal{D}}_a \pi^a + \sum_{n \geq 2} k^{2n} \hat{\mathcal{D}}_a \pi_{(n)}^a - \xi_{ab} \left( k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) \\ &+ \hat{\mathcal{D}}_v \varepsilon_{(0)} + \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}_v \varepsilon_{(n)} - \frac{1}{8\pi G} \left( * \Psi^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * \chi^a \right) \\ &- \frac{1}{8\pi G k^2} \left( * X^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * z^a \right) - \frac{1}{8\pi G k^4} * Z^{ab} \xi_{ab}, \end{aligned} \quad (3.3.21)$$

and

$$\begin{aligned} \mathcal{T}^a &= -\hat{\mathcal{D}}_b \left( k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) + 2 * \hat{\omega} \left( k^2 * \pi^a + \sum_{n \geq 2} k^{2n} * \pi_{(n)}^a \right) + \frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} \\ &+ \frac{1}{2} \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}^a \varepsilon_{(n)} + \hat{\mathcal{D}}_v \left( \pi^a + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a \right) + \xi^a_b \left( \pi^b + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^b \right) \\ &- \frac{1}{8\pi G} \left( \hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab} \right) \\ &- \frac{1}{8\pi G k^2} \left( \frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab} \right) - \frac{1}{8\pi G k^4} \left( \frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab} \right), \end{aligned} \quad (3.3.22)$$

for the transverse equation.

4. Lastly we substitute the geometric Carrollian shear for the Bondi shear with a power of the cosmological constant:  $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$  inside Eqs. (3.3.21) and (3.3.22). Some singular terms in the last lines of (3.3.21) and (3.3.22) get regularized at vanishing  $k$ . Instead of a divergence we obtain a bunch of finite terms, all coming from the Carrollian Cotton tensor.

The flat limit of the boundary energy-momentum conservation can now be safely taken and yields

$$\lim_{k \rightarrow 0} \mathcal{L} = \hat{\mathcal{D}}_v \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a - \frac{1}{16\pi G} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \mathcal{N}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b + \frac{1}{2} \mathcal{C}_{ab} \hat{\mathcal{D}}_v \mathcal{N}^{ab} \right), \quad (3.3.23)$$

and

$$\begin{aligned}
\lim_{k \rightarrow 0} \mathcal{T}^a &= \frac{1}{2} \hat{\mathcal{D}}_b \left( \delta^{ab} \varepsilon_{(0)} + \frac{1}{8\pi G} \eta^{ab} c_{(0)} \right) + \hat{\mathcal{D}}_b \left( \pi^a - \frac{1}{8\pi G} * \psi^a \right) \\
&+ \frac{1}{16\pi G} \left[ \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \hat{\omega} * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) \\
&\left. + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left( \mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.3.24)
\end{aligned}$$

**Remark** Observe that all terms accounting for gravitational radiation are coming from the Cotton tensor, through the use of (3.2.33). This method therefore shows the central role of the latter.

We can finish our computation, inserting (3.3.16) which relates the Carrollian Cotton descendants to replicas of the energy-momentum tensor, into (3.3.23) and (3.3.24). This yields an alternative form of the flux/balance equations

$$\hat{\mathcal{D}}_v \varepsilon_{(0)} + \hat{\mathcal{D}}_a Q^a = \frac{1}{16\pi G} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{E}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{R}}_b + \frac{1}{2} \mathcal{E}_{ab} \hat{\mathcal{D}}_v \hat{\mathcal{N}}^{ab} \right) \quad (3.3.25)$$

and

$$\begin{aligned}
\frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} - \hat{\mathcal{D}}_b \Xi^{ab} + 2 * \hat{\omega} * Q^a + \hat{\mathcal{D}}_v \pi^a &= -\frac{1}{16\pi G} \left[ \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \hat{\omega} * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} \\
&\left. + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left( \mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.3.26)
\end{aligned}$$

These two equations are particular examples of the Carrollian conservation equations (with locked connection i.e. no ambiguities) that we saw in (1.3.20) and (1.3.21) but *with a right-hand side though*. This is thus a *flux-balance equation*, where the source is captured by the bulk gravitational radiation encoded in the shear and the news.<sup>11</sup> Notice that the above momentum  $\pi^a$  coincides with  $P^a$  in (1.3.21) and is dynamical, whereas the traceless Carrollian stress  $\Xi^{ab}$  is  $-\Upsilon^{ab}$  in (B.1.49), (B.1.50), and is dictated by the Cotton due to (3.3.16); similarly  $Q^a$  here is the energy flux  $\Pi^a$  of (B.1.49), (B.1.50), also locked by the Cotton in (3.3.16).

**Remark** Note that even if the Bondi shear  $\mathcal{E}_{ab}$  or news  $\hat{\mathcal{N}}_{ab}$ , vanish local Carroll boost (aka shift symmetry) is broken due to the presence of a non-zero energy flux  $\Pi^a$ . This breaking originates from the bulk gravitational radiation, which in the covariant Newman–Unti gauge does not originate solely in the news (3.2.34) but is also carried by the Carrollian energy flux

<sup>11</sup>It was argued by Ashtekar in [38] (then reviewed in [76]) that the radiative degrees of freedom should be contained in the boundary connection, more precisely in  $\hat{\beta}_{ab}$  defined in (1.2.1).

$\Pi^a = Q^a = \frac{1}{8\pi G} * \chi^a$  therefore by the Cotton tensor. The paragon are Robinson–Trautman spacetimes for which in the present gauge, the gravitational radiation is exclusively rooted in this Carrollian Cotton descendant, see [52].

**Remark** The Cotton tensor also satisfies  $\nabla_A C^{AB} = 0$  off-shell, after the  $k \rightarrow 0$  limit we get two equations

$$\hat{\mathcal{D}}_\nu c_{(0)} + \hat{\mathcal{D}}_a \chi^a = 0, \quad (3.3.27a)$$

$$\frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \hat{\omega} * \chi^a + \hat{\mathcal{D}}_\nu \psi^a = 0. \quad (3.3.27b)$$

which play dual roles with respect to Eqs. (3.3.25) and (3.3.26), because the energy density  $\varepsilon_{(0)} = c_{\text{Cot}}$  of Sec. 2.4 carries information on the mass of the source, while  $c_{(0)}$  captures its NUT charge (monopole-like magnetic mass). The two sets of equations are not symmetric though: Eq. (3.3.27a) for instance is driven exclusively by the Cotton vector  $\chi^a$  — as opposed to its Carroll-dual  $*\chi^a$  entering the electric-mass equation (3.3.25) through  $Q^a = \frac{1}{8\pi G} * \chi^a$ .

### Setting a dictionary between the various gauges

The flux/balance equations displayed in (3.3.23) and (3.3.24) corresponds to the boundary and Weyl covariant enhancement of the ones already derived in the seminal BMS papers [32, 33] and reviewed in [130] by brute force computation for the Bondi mass aspect mass  $M_{\text{Bondi}}$  and the angular momentum aspect  $N_{\text{Bondi}}^A$ . In [156] the authors found them using symmetry arguments based on the BMS-Weyl asymptotic symmetry group of asymptotically flat spacetimes. They defined the notion of tensors and pseudo-tensors under the latter group (tensors are homogeneous while pseudo-tensors can admit linear anomalies), sorted the pseudo-tensors with respect to their spin and found the right combination of the latter and their derivatives such that the whole thing is tensorial (i.e. no longer just pseudo-tensorial). These combinations are the flux/balance equations for the covariant mass  $\mathcal{M}$ , the angular momentum  $\mathcal{P}^A$  and the covariant stress-tensor  $\mathcal{E}_{AB}$ , where we have used the notations of [156].

What should be then worth doing now it to set up the dictionary between our work [155] and the latter two [130, 156]. Our first exercise is to introduce the concept of covariant Bondi mass and angular momentum. For the mass we can define

$$M = 4\pi G \varepsilon_{(0)} - \frac{1}{8} \mathcal{E}^{ab} \hat{\mathcal{N}}_{ab}. \quad (3.3.28)$$

This definition is reached from Eq. (2.39) of [122] valid in anti-de Sitter, at  $k = 0$ . It coincides with (42) of [156] upon identifying  $\mathcal{M}$  of this reference with our  $4\pi G \varepsilon_{(0)}$ . What distinguishes the energy density  $4\pi G \varepsilon_{(0)}$  and the mass  $M$  is a radiative contribution which makes  $M$  transforms unhomogeneously under BMS-Weyl (while  $4\pi G \varepsilon_{(0)}$  transforms homogeneously). As studying the action of asymptotic symmetries on the solution space is not part of the agenda of this thesis the reader is asked to look at the aforementioned references for more details on that point. For the

angular momentum aspect the definition (3.3.17) is enough.

We can also try to define a magnetic-mass aspect starting from anti-de Sitter, where the behaviour of the bulk Weyl tensor in the gauge used here exhibits the complex-mass combination  $\tau = -c + 8\pi i G \varepsilon$  (see [147]) with  $\varepsilon$  the AdS-boundary energy density and  $c$  the Cotton scalar. We thus define the *complex mass aspect* of Ricci-flat spacetimes in CNU gauge as

$$\hat{\tau} = \lim_{k \rightarrow 0} \tau = -2\nu + 8\pi i G \varepsilon_{(0)}, \quad (3.3.29)$$

where

$$\nu = \frac{1}{2} \lim_{k \rightarrow 0} c = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab} - \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} \quad (3.3.30)$$

is the *magnetic-mass aspect* reached using (2.4.5a) and (2.4.6) upon substitution of  $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$ . Inspired by (3.3.28) we subtract the radiative contribution. This defines the *NUT aspect*

$$N = \nu + \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab}, \quad (3.3.31)$$

where  $c_{(0)} = \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}^a + 2\hat{\mathcal{K}} \right) * \hat{\omega}$  is one of the four Carroll Cotton scalars displayed in (2.4.6). Our definitions for  $\nu$  and  $N$  match with  $-\tilde{\mathcal{M}}$  and  $-\tilde{M}$  of [156], Eqs. (53) and (55), for  $c_{(0)} = 0$  (no magnetic monopole mass).

With the above definitions, Eqs. (3.3.27a), (3.3.25) and (3.3.26) become<sup>12</sup>

$$\hat{\mathcal{D}}_\nu N = -\frac{1}{2} \hat{\mathcal{D}}_a \chi^a - \frac{1}{4} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \hat{\mathcal{N}}^{ab} - * \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \right), \quad (3.3.32a)$$

$$\hat{\mathcal{D}}_\nu M = -\frac{1}{2} \hat{\mathcal{D}}_a * \chi^a + \frac{1}{4} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b - \frac{1}{2} \hat{\mathcal{N}}_{ab} \hat{\mathcal{N}}^{ab} \right), \quad (3.3.32b)$$

$$\begin{aligned} \hat{\mathcal{D}}_\nu N^a - \hat{\mathcal{D}}^a M + * \hat{\mathcal{D}}^a N &= \frac{1}{2} \left[ \mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \hat{\omega} * \mathcal{C}^{ab} \hat{\mathcal{R}}_b - \frac{1}{2} * \hat{\mathcal{D}}^a \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c * \mathcal{C}^{bc} \right. \\ &\quad \left. - \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{C}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{C}_{bc} \right) + \mathcal{C}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{C}^{ac} \hat{\mathcal{N}}_{bc} \right) \right]. \end{aligned} \quad (3.3.32c)$$

The first equation phrases the loss process of the NUT aspect sustained by the Carroll-dual news  $* \hat{\mathcal{N}}_{ab}$  and the Carroll Cotton current  $\chi^a$ . It is actually a *geometric identity* associated with the Carroll structure — as is (3.3.27b), which could have been reexpressed as well in terms of the NUT aspect. The last two flux-balance equations (3.3.32b) and (3.3.32c) for the electric-mass and angular-momentum aspects are *genuinely dynamical* and coincide with Eqs. (2.53) and (2.50) of Ref. [122], where the approach to asymptotic flatness via a limit of vanishing cosmological constant was proposed, or else with (4.50) and (4.49) of [130], obtained in a plain Ricci-flat context.

**Remark** We observe that even-though the loss phenomena affects both the electric “Bondi” mass

<sup>12</sup>All these computations call for abundant use of the Weyl-covariant-derivative commutators presented in the appendix, Eqs. (B.1.61a), (B.1.61b), (B.1.61c), (B.1.64a), (B.1.64b) and (B.1.64c).

and the magnetic NUT charge, the latter is not affected neither by  $\hat{\mathcal{C}}_{ab}$  nor  $\hat{\mathcal{N}}_{ab}$  i.e. by gravitational radiation. This is in line with the important distinction raised between these two quantities e.g. in [157].

Let us recap in a Table the links between the CNU gauge [155], the Bondi gauge in Bondi aspects [130] and the Bondi gauge in BMS-Weyl covariant aspects [156]. The main message is the equality between the dynamical shears.

CNU gauge [155]	Bondi [130]	Bondi BMS-Weyl [156]
$\tau$	$\exp(2\beta_0)du$	$du$
$*\hat{\omega}, \varphi_a \neq 0$	0	0
$\mathcal{C}_{ab}$	$C_{ab}$	$\mathcal{C}_{ab}$
$\hat{\mathcal{N}}^{ab}$	$N_{\text{trace-free}}^{ab} - \frac{1}{2}C^{ab}$	$\mathcal{N}^{ab}$
$4\pi G\varepsilon_{(0)}$	$M_{\text{Bondi}} - \frac{1}{8}\mathcal{C}_{ab}\hat{\mathcal{N}}^{ab}$	$\mathcal{M}$
$N^a$	$N_{\text{Bondi}}^a + \frac{1}{4}\left(\mathcal{C}^{ab}\hat{\nabla}^c\mathcal{C}_{bc} + \frac{3}{8}\hat{\nabla}^a(\mathcal{C}^{bc}\mathcal{C}_{bc})\right)$	$\mathcal{P}^a$
$E^{ab}$	$-\frac{3}{16\pi G}\left(E_{\text{Bondi}}^{ab} - \frac{1}{16}\mathcal{C}^{ab}\mathcal{C}^{cd}\mathcal{C}_{cd}\right)$	$-\frac{1}{16\pi G}\mathcal{T}^{ab}$

(3.3.33)

Note that we have already introduced in this table the stress  $E^{ab}$  which appears at order- $\frac{1}{r}$  and whose evolution equation will now be derived by taking the  $k \rightarrow 0$  limit of the subleading order  $\frac{1}{r^2}$ , requiring regularity. Comparing our momenta with the BMS-Weyl covariant one of [158] we therefore conclude that our CNU provides an appropriate framework to handle asymptotically flat spacetimes in a boundary covariant fashion, at least in the regime in which logarithmic terms of the radial coordinate are neglected.

### Order $1/r^2$ and the flat line element

The Carrollian symmetric and traceless two-tensor  $E_{ab}$ , descendant of the AdS-boundary stress, enters the line element at order  $\frac{1}{r}$ . However, the fundamental Carrollian energy-momentum conservation equations (3.3.23) and (3.3.24) fail to capture its dynamics. Hence we have to go to the next subleading order. The bulk metric, including the term (3.2.39) with the  $f_{(2)}$ s as in (3.2.41a) (3.2.41b) and (3.2.41c), is on-shell, as we have assumed the energy-momentum tensor to be conserved. However, this term is due to exhibit divergences at vanishing  $k$ . Removing them will impose conditions involving the Chthonian degrees of freedom as well as their longitudinal derivatives appearing explicitly in (3.2.41c). This is the way that flat flux-balance equations are recovered in the transition from anti-de Sitter to asymptotically flat spacetimes.

We just need to repeat the same method again, take the  $k \rightarrow 0$  limit trading  $\xi_{ab}$  for  $-\frac{k^2}{2}\mathcal{C}_{ab}$ . Let us open the study with the scalar contribution  $f_{(2)}$ , Eq. (3.2.41a) whose limit yields

$$\lim_{k \rightarrow 0} f_{(2)} = 2 * \hat{\omega} \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \equiv \hat{f}_{(2)}. \quad (3.3.34)$$



Next we consider the transverse vector  $f_{(2)a}\theta^a$  in (3.2.41b):

$$\lim_{k \rightarrow 0} f_{(2)a} = -\frac{1}{6} N^b \mathcal{C}_{ba} - \frac{4}{3} * \hat{\omega} * N_a - 4\pi G \hat{\mathcal{D}}_b E^b{}_a \equiv \hat{f}_{(2)a}. \quad (3.3.35)$$

Neither the limit (3.3.34) nor (3.3.35) introduce any new Chthonian degree of freedom or impose any further condition on their evolution. As we will now see, the situation is different for the transverse tensor (3.2.41c)  $f_{(2)ab}\theta^a\theta^b$ . Using the numerous tools developed in this Chapter, we find<sup>13</sup>

$$\begin{aligned} f_{(2)ab} &= \frac{1}{k^2} \left( \frac{16\pi G}{3} \hat{\mathcal{D}}_v E_{ab} + \frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b \rangle} + 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} - \frac{\nu}{2} * \mathcal{C}_{ab} \right) \\ &+ 2\pi G \left( \frac{8}{3} \hat{\mathcal{D}}_v E_{(2)ab} - \frac{4}{3} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \varepsilon_{(1)} \mathcal{C}_{ab} - 2 \mathcal{C}_{(a}{}^c E_{b)c} \right) - 2 * \hat{\omega}^3 * \mathcal{C}_{ab} \\ &+ \mathcal{O}(k^2). \end{aligned} \quad (3.3.36)$$

This expansion in powers of  $k^2$  possesses interesting features, characteristic of the Chthonian orders

- The flat limit is singular unless the order- $\frac{1}{k^2}$  contribution to  $f_{(2)ab}$  is absent i.e.

$$\boxed{\hat{\mathcal{D}}_v E_{ab} = \frac{3}{16\pi G} \left( -\frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b \rangle} - 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} + \frac{\nu}{2} * \mathcal{C}_{ab} \right)}, \quad (3.3.37)$$

which is the new Carrollian *flux-balance equation* for  $E_{ab}$ . This equation matches with Eq. (4e) of [156] given the dictionary established in (C.o.18).

- Assuming Eq. (3.3.37) is fulfilled, the limit can be taken

$$\begin{aligned} \lim_{k \rightarrow 0} f_{(2)ab} &= \frac{16\pi G}{3} \left( \hat{\mathcal{D}}_v E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{4} \mathcal{C}_{(a}{}^c E_{b)c} \right) - 2 * \hat{\omega}^3 * \mathcal{C}_{ab} \\ &\equiv \hat{f}_{(2)ab}, \end{aligned} \quad (3.3.38)$$

and provides the last piece of the order- $\frac{1}{r^2}$  term in the Ricci-flat line element.

- New Chthonian degrees of freedom enter the bulk metric at this order:  $E_{(2)ab}$ ,  $\pi_{(2)a}$  and  $\varepsilon_{(1)}$  in the form of a symmetric and traceless Carrollian tensor

$$F_{ab} = \hat{\mathcal{D}}_v E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{8\pi G} * \hat{\omega}^3 * \mathcal{C}_{ab}. \quad (3.3.39)$$

Their dynamics is unknown at this stage but will be unravelled in the course of the analysis at order  $\frac{1}{r^3}$ .

We will close this paragraph exhibiting the explicit Ricci flat metric at the considered order. To

<sup>13</sup>We define the symmetric and traceless part of a Carrollian two-tensor  $s_{ab}$  as  $s_{\langle ab \rangle} = s_{(ab)} - \frac{1}{d} s_c{}^c \delta_{ab}$  (here  $d = 2$ ).

this end we use the results (3.2.6), (3.3.8), (3.3.10), (3.3.18), (3.3.34), (3.3.35), (3.3.38) and (3.3.39)

$$\begin{aligned}
ds_{\text{Ricci-flat}}^2 = & -\tau \left[ 2dr + \left( 2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \hat{\omega} - \hat{\mathcal{D}}_b \mathcal{C}_a^b \right) \hat{\theta}^a + \left( r\theta + \hat{\mathcal{H}} \right) \tau \right] \\
& + \left( r^2 + * \hat{\omega}^2 + \frac{\mathcal{C}^2}{4} \right) d\ell^2 + (r\mathcal{C}_{ab} + * \hat{\omega} * \mathcal{C}_{ab}) \hat{\theta}^a \hat{\theta}^b \\
& + \frac{1}{r} \left( 8\pi G \varepsilon_{(0)} \tau^2 + \frac{4}{3} \tau N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \right) \\
& + \frac{1}{r^2} \left( 2 * \hat{\omega} \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \right) \tau^2 + \frac{1}{r^2} \tau \left( \frac{1}{3} N^b \mathcal{C}_{ba} + \frac{8}{3} * \hat{\omega} * N_a + 8\pi G \hat{\mathcal{D}}_b E_a^b \right) \hat{\theta}^a \\
& + \frac{1}{r^2} \left( \frac{16\pi G}{3} F_{ab} - 4\pi G \mathcal{C}_{(a}{}^c E_{b)c} \right) \hat{\theta}^a \hat{\theta}^b + \mathcal{O} \left( \frac{1}{r^3} \right). \tag{3.3.40}
\end{aligned}$$

This solution to vacuum Einstein's equations is built upon the following boundary Carrollian data

- a generic Carrollian structure without geometric shear  $\xi_{ab} = 0$  (but arbitrary Ehresmann connection providing  $\varphi_a$  and  $*\hat{\omega}$ )
- a dynamical shear  $\mathcal{C}_{ab}$  left free via the Carrollian limit of (3.2.33), which sources the flux-balance laws for the mass and the angular momentum,
- an energy density  $\varepsilon_{(0)}$  i.e. a Bondi mass  $M$ , a heat current  $N_a$  aka the Bondi angular momentum aspect and a stress  $E_{ab}$ , all satisfying the flux-balance equations (3.3.32b), (3.3.32c) and (3.3.37) (recall that the magnetic equivalent for the NUT charge  $N$  is a geometric identity and not a flux-balance law),
- three more degrees of freedom  $E_{(2)ab}$ ,  $\pi_{(2)a}$  and  $\varepsilon_{(1)}$  encoded in  $F_{ab}$  (3.3.39) with evolution equations yet to be uncovered but expected to arise at order- $\frac{1}{r^3}$ .

This concludes the presentation of our method to find the structure of asymptotically flat spacetimes from the one of AdS.

## 3.4 Outlook and discussion

In this Chapter we described a method to reach the infinite solution space of asymptotically flat spacetimes from a smooth  $k \rightarrow 0$  limit of the finite AdS one. This was not expected as these two types of solutions of Einstein gravity support completely distinct behavior regarding gravitational radiation, which passes through the null boundary in the former, while being reflected in the latter (under usual boundary conditions though). However we were able to design an incomplete gauge fixing, adapted from Newman-Unti, built upon a timelike congruence  $\mathbf{u}$ , which allows to treat both cases in a boundary covariant way, the covariance surviving the flat limit. This is our first achievement.

The second one is the flat limit itself. Sending the cosmological constant to zero in the bulk translates into a Carrollian limit on the pseudo-Riemannian boundary of AdS. The limit is carefully taken following three steps. The Bondi shear  $\mathcal{C}_{ab}$  is traded on-shell for the geometric shear  $\sigma_{ab}$ . Then the AdS energy-momentum tensor  $T^{AB}$  decomposed with respect to  $\mathbf{u}$  is Laurent-expanded in powers of  $k^2 = -\frac{\Lambda}{3}$  about  $k^2 = 0$ , supplying an infinite set of replicas which account for the infinite Chthonian flat degrees of freedom. Finally the evolution equations (flux/balance laws) are obtained through the limit of the AdS conservation of energy and momentum, or by requiring finiteness of the line element. On top of that, some replicas of the energy-momentum tensor are locked in terms of the Cotton tensor, hence the geometry, reducing the total number of arbitrary variables.

Our analysis showed that even in flat space the boundary energy-momentum tensor plays a central role in reconstructing the bulk. Besides the Bondi shear, the solution space is made of the Carrollian energy density, angular momentum, stress together with an infinite tower of Chthonian tensors. The status of the latter is still to be uncovered. Are they relevant and if yes to what extend? What are their dynamics? What do they encode? These are old interrogations that our covariant gauge may help to answer.<sup>14</sup> One case of special interest is the order- $\frac{1}{r^3}$  where we expect to find, as already said, the ten Newman-Penrose charges [150–154]. The method presented here should then be push further by first finding the relevant tensors in AdS and then taking the flat limit which should give, in addition to the flat line element, the flux/balance law for the first Chthonian tensor  $F_{ab}$  in (3.3.39).

Another achievement of the present analysis concerns the AdS origin of the gravitational radiation of the flat instance. Indeed we show that all source terms in the flux/balance laws (3.3.23) and (3.3.24) are rooted in the decomposition of the boundary AdS Cotton tensor. Invisible in usual gauges, the CNU gives shows the central role it plays in the description of asymptotically flat spacetimes.

The CNU gauge has been introduced here with the flat limit in mind. The analysis usually continues with the computation of the asymptotic symmetry algebra and group in view of finding the action of the latter on the solution space. Such a study is of great help, for example, in defining a physical notion of gravitational radiation as it is expected that the tensor supporting it transforms homogeneously under the asymptotic symmetry algebra (see e.g. [161] for the definition of the physical news tensor in Bondi gauge). Other works use the transformations laws of the solution space to define a physical notion of angular momentum [162–165]. One should also address the question of logarithmic terms. How do they fit in the analysis presented in this Chapter and how this is related to previous works like [59, 132]? This has to be unveiled in the near future.

The boundary covariance of the CNU gauge allows to study asymptotically flat spacetimes from a completely new perspective, namely the one of the Carrollian boundary. The next Chapter aims at illustrating this statement in the restricted context of algebraically special spacetimes.

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<sup>14</sup>Note also that an analysis in double-light cone coordinate may also prove useful, see e.g. [159, 160].

# Chapter 4

## Algebraically special spacetimes and a Carrollian perspective on charges

The covariant Newman-Unti gauge provides an adequate framework to investigate asymptotically flat spacetimes, rooted in their null boundary and associated with a free congruence. This was the conclusion of the previous chapter. The aim of the present is to illustrate this statement by showing how to recover e.g. gravitational multipoles like in [166] but from a Carrollian perspective. As our purpose is only to give explicit examples, we will not deal with the complete metric in CNU gauge (3.3.40) but rather with a simpler, resummed version, obtained once suitable conditions are assumed on the boundary data. This will be the flat analogue of (3.2.48) obtained in AdS. Like in this instance, it captures Petrov-algebraically special spacetimes. This subclass of solutions of Einstein gravity should be thought of as a playground to better understand the interplay between bulk and boundary dynamics. Extending the analysis to the general case should be rather direct and is part of our future agenda.

The quantities we want to study from a Carrollian point of view are charges built upon isometries. One should then relate bulk isometries to isometries of the Carrollian boundary. If the latter are strong (i.e. they preserve the clock form) they will be associated with conserved charges. We would like to clarify a point at this stage. In gravity like in any gauge theory one can associate to a gauge symmetry a charge defined on a codimension-two hypersurface using Noether theorem. These charges are computed using usual methods like the covariant phase space formalism (see [167] or [3] for reviews). The latter are called surface charges, even though they are purely related to the bulk gauge symmetries. The question we want to address here deals with bulk isometries i.e. rigid symmetries and their associated charges. We want to compute the latter focusing exclusively on the boundary Carroll structure, and compare the result with the one procured by traditional bulk methods.

This Chapter is organised as follows. Sec. 4.1 is aimed at presenting the set of algebraically special solutions in CNU gauge, together with a reminder on the Petrov classification. Then, as the charge analysis and Chapter 5 will extensively use the subclass of stationary solutions, we shall

study them in details. In Section 4.2 we exhibit a way to relate bulk isometries and their associated charges to the ones we learnt to construct in a Carroll structure in Sec. 1.5. Thanks to the resummation of the line element, two infinite towers of charges can be constructed either from the energy-momentum tensor or from the Cotton tensor. We compute them explicitly in the case of Kerr-Taub-NUT spacetimes and observe that they correspond to gravitational multipoles. Recall that we are still in four dimensions and that we shall describe the two-dimensional base space of the Carrollian boundary in holomorphic coordinates, for which Sec. 1.6 contained all useful information.

## 4.1 Algebraically special solutions

In this Section we focus on a sub-class of solutions of asymptotically flat gravity for which the infinite expansion (3.3.40) is resummed into a finite form. After presenting the hypotheses under which this occurs, we will comprehensively study the sub-sector of time-independent solutions, as they will constitute our framework in the next Chapter.

### 4.1.1 Generalities

#### Resumming the radial expansion

Resummation of (3.3.40) occurs when suitable conditions are imposed on the boundary data. These conditions are the following

- the vanishing of the dynamical shear  $\mathcal{C}_{ab}(t, \mathbf{x})$  which implies via (3.3.28) that the Bondi mass aspect reads  $M = 4\pi G \varepsilon_{(0)}$ ,
- all Chthonian functions should vanish, like e.g.  $E_{ab}$ ,  $E_{(2)ab}$  or  $\pi_{(2)a}$ ,
- the angular momentum aspect  $N^a$  is set to zero. This fixes the fluid heat current  $\pi^a$  as a component of the dual Cotton tensor

$$\pi^a = \frac{1}{8\pi G} * \psi^a, \quad (4.1.1)$$

leaving  $\varepsilon_{(0)}$  as only arbitrary component of the energy-momentum tensor.

The bulk spacetime is then entirely described by means of the Carrollian boundary geometry (metric, field of observers and Ehresmann connection) plus the the energy density of the associated Carrollian fluid and its line element reads

$$ds_{\text{res. Ricci-flat}}^2 = -\boldsymbol{\tau} \left[ 2dr + \left( 2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \hat{\omega} \right) \hat{\theta}^a + \left( r\theta + \hat{\mathcal{K}} \right) \boldsymbol{\tau} \right]$$

$$+ (r^2 + *\omega^2) d\ell^2 + \frac{1}{r^2 + *\omega^2} (8\pi G \varepsilon_{(0)} r + *\omega c_{(0)}) \boldsymbol{\tau}^2. \quad (4.1.2)$$

This captures all algebraically special Ricci-flat spacetimes (see below for the proof) provided  $\varepsilon_{(0)}$  obeys (3.3.25) and (3.3.26) which now read

$$\hat{\mathcal{D}}_{\mathbf{v}} \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a = 0, \quad (4.1.3a)$$

$$\hat{\mathcal{D}}_a \varepsilon_{(0)} - \frac{1}{8\pi G} * \hat{\mathcal{D}}_a c_{(0)} = 0, \quad (4.1.3b)$$

where we recall that  $\varepsilon_{(0)}$  is the fluid energy density while  $c_{(0)}$  and  $\chi^i$  are pieces of the Carrollian Cotton tensor displayed in (2.4.6) and in (2.4.7b). These Carrollian fluid equations, which guarantee Ricci-flatness of the line element (4.1.2), are now genuine conservation equations without source terms (2.2.18a), (2.2.18b), (2.2.18c) and (2.2.18d), where the momenta are

$$\Pi = \varepsilon_{(0)}, \quad \Pi^a = \frac{1}{8\pi G} * \chi^a, \quad P^a = \frac{1}{8\pi G} * \psi^a, \quad \tilde{\Pi}^{ab} = -\frac{1}{8\pi G} * X^{ab}, \quad \Pi^{ab} = \frac{\varepsilon_{(0)}}{2} a^{ab} - \frac{1}{8\pi G} * \Psi^{ab} \quad (4.1.4)$$

together with  $\tilde{\Pi} = 0$  and  $\tilde{\Pi}^a = 0$ . Recall that similar equations holds for the Cotton tensor, with momenta displayed in Eq. (2.4.13).

**Remark** Equations (4.1.3a) and (4.1.3b) coincide with Eqs. (29.16) and (29.15) of [110].<sup>1</sup> The latter are rather complicated and it is remarkable they are translated into simple conservation equations such as (4.1.3a) and (4.1.3b). Reaching this conclusion would have been inconceivable without the null boundary perspective and the Carrollian tools, which are the appropriate language for asymptotically flat spacetimes.

From the above Eqs. (4.1.3a) and (4.1.3b) as well as Eq. (2.4.11b) one can observe that the energy density  $\varepsilon_{(0)}$  and the Carrollian Cotton scalar  $c_{(0)}$  play dual roles. This will be formulated concretely in Sec. 5.2 in the context of the boundary action of the hidden Ehlers symmetry. Anticipating what will be done in this Chapter, we introduce the following Carrollian complex scalar  $\hat{\tau}(t, \mathbf{x})$  and vector  $\hat{\chi}^a(t, \mathbf{x})$

$$\hat{\tau} = -c_{(0)} + 8\pi i G \varepsilon_{(0)}, \quad (4.1.5a)$$

$$\hat{\chi}^a = \chi^a - i * \chi^a. \quad (4.1.5b)$$

The aforementioned equations are thus recast as

$$\hat{\mathcal{D}}_{\mathbf{v}} \hat{\tau} = \hat{\mathcal{D}}_a \hat{\chi}^a, \quad \hat{\mathcal{D}}_a \hat{\tau} \hat{\mathcal{D}}^a \hat{\tau} = 0, \quad (4.1.6)$$

$$\hat{\mathcal{D}}_a \hat{\tau} \hat{\mathcal{D}}^a \hat{\tau} = 8 \left( 2 * \omega * \chi_b + \hat{\mathcal{D}}_{\mathbf{v}} \psi_b - \hat{\mathcal{D}}^a \Psi_{ab} \right) \left( 2 * \omega * \chi^b + \hat{\mathcal{D}}_{\mathbf{v}} \psi^b - \hat{\mathcal{D}}_k \Psi^{cb} \right).$$

<sup>1</sup>For that purpose, the following identifications are necessary, in Papapetrou–Randers frame and complex coordinates  $\mathbf{x} = \{\zeta, \bar{\zeta}\}$  with  $d\ell^2 = \frac{2}{p^2(u, \zeta, \bar{\zeta})} d\zeta d\bar{\zeta}$ ,  $\mathbf{v} = \frac{1}{\Omega} \partial_u$ ,  $\boldsymbol{\tau} = \Omega du - b_a dx^a$  and  $\hat{\mathbf{e}}_a = \hat{\partial}_a = \partial_a + \frac{b_a}{\Omega} \partial_u$ ,

$\theta^a = dx^a$ :  $\Omega = 1$ ,  $b_\zeta = -L$ ,  $*\omega = -\Sigma$ ,  $\hat{\tau} = 2(M + im)$ , whereas their radial coordinate is  $\hat{r} = r - r_0$  with  $r_0(u, \zeta, \bar{\zeta})$  the origin in the affine parameter of the geodesic congruence tangent to  $\partial_r$ .

Acting with a second spatial derivative on (4.1.3b) and using (1.4.9a), we finally obtain

$$\hat{\mathcal{D}}^a \hat{\mathcal{D}}_a \hat{\tau} = 2i \left( \hat{\mathcal{D}}_\nu * \hat{\omega} \hat{\tau} - \hat{\mathcal{A}} \hat{\tau} \right). \quad (4.1.7)$$

This equation will be of major importance when studying stationary spacetimes in Sec. 4.1.2.

**Remark** The algebraically special character of a spacetime comes from the Petrov classification of the Weyl tensor  $C_{\mu\nu\rho\sigma}$ , which reads in  $(3+1)$ -dimensions

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2} (R_{\mu\sigma} g_{\nu\rho} - R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma} - R_{\nu\sigma} g_{\mu\rho}) + \frac{1}{6} R (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}). \quad (4.1.8)$$

Spacetimes are differentiated regarding the *principal null directions* their Weyl tensor possesses. A principal null direction is a null vector field  $\mathbf{k}$  which satisfies

$$k_{[\gamma} C_{\mu]\nu\rho[\sigma} k_{\kappa]} k^\nu k^\rho = 0. \quad (4.1.9)$$

There exist at most four such vectors. The Petrov class the spacetime at hand belongs to depends on the number and multiplicity of its principal null direction (PND). While counting them should not be an issue, computing their multiplicity requires a null tetrad  $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$  in which the metric reads  $ds^2 = -2\mathbf{k} \cdot \mathbf{l} + 2\mathbf{m} \cdot \bar{\mathbf{m}}$  and upon which the *Newman-Penrose Weyl scalar* are computed. The latter read

$$\Psi_0 = C_{\mu\nu\rho\sigma} k^\mu m^\nu k^\rho m^\sigma \quad (4.1.10a)$$

$$\Psi_1 = C_{\mu\nu\rho\sigma} k^\mu l^\nu k^\rho m^\sigma \quad (4.1.10b)$$

$$\Psi_2 = C_{\mu\nu\rho\sigma} k^\mu m^\nu \bar{m}^\rho l^\sigma \quad (4.1.10c)$$

$$\Psi_3 = C_{\mu\nu\rho\sigma} k^\mu l^\nu \bar{m}^\rho l^\sigma \quad (4.1.10d)$$

$$\Psi_4 = C_{\mu\nu\rho\sigma} \bar{m}^\mu l^\nu \bar{m}^\rho l^\sigma \quad (4.1.10e)$$

and, when  $\mathbf{k}$  is a PND, the five of them are related by a fourth-order algebraic equation

$$\Psi_0 - 4E\Psi_1 + 6E^2\Psi_2 - 4E^3\Psi_3 + E^4\Psi_4 = 0, \quad (4.1.11)$$

to solve for  $E$ . The number of roots of (4.1.11) and their multiplicities is exactly what we were seeking. The complete Petrov classification is then displayed in the next Table.

Petrov type	Number of roots	Multiplicities	Newman-Penrose coefficients
I	4	(1, 1, 1, 1)	$\Psi_1 = \Psi_3 = 0$
II	3	(2, 1, 1)	$\Psi_0 = \Psi_1 = \Psi_3 = 0$
III	2	(3, 1)	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0$
D	2	(2, 2)	$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$
N	1	(4)	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$
O	0	None	$\Psi_i = 0$ for $i = 0, \dots, 5$

(4.1.12)

An algebraically special spacetime is then a spacetimes which is not type I which means that at least  $\Psi_0$  and  $\Psi_1$  both have to vanish. Well-known solutions of Type D includes the Plebanski-Demianski [168] family among which sits the Schwarzschild and Taub-NUT solutions. Type O only include the vacuum.

**Remark** The Newman-Penrose Weyl scalars [169] contain useful information to characterise the spacetime at hand. If one remembers that the Weyl tensor contains the propagating degrees of freedom, i.e. the gravitational radiation, in a null tetrad  $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$  where  $\mathbf{k}$  labels incoming null rays and  $\mathbf{l}$  outgoing null rays, the  $\Psi$ s get the following interpretation

- $\Psi_0$  and  $\Psi_1$  encode the incoming gravitational radiation coming from  $\mathcal{I}^-$ ,
- $\Psi_3$  and  $\Psi_4$  encode the outgoing gravitational radiation passing through  $\mathcal{I}^+$ ,
- $\Psi_2$  is a complex number whose real part is related to the mass of the source of gravitation while its imaginary part corresponds to the NUT charge or magnetic mass (the latter is the  $n$  parameter of the Taub-NUT solution [170, 171]).

In terms of the solution space in CNU gauge, given its relation to the Bondi gauge and using [131] we get that  $E_{ab} \propto \Psi_0^{(0)}$ ,  $N_a \propto \Psi_1^{(0)}$ ,  $\varepsilon_{(0),\nu} \propto \Psi_2^{(0)}$ ,  $\mathcal{D}_a \hat{\mathcal{N}}^{ab} \propto \Psi_3^{(0)}$  and  $\mathcal{D}_\nu \hat{\mathcal{N}}^{ab} \propto \Psi_4^{(0)}$ . The upper 0 index refers to the leading coefficient in the  $\frac{1}{r}$ -expansion of the Newman-Penrose scalars.

The Goldberg-Sachs theorem (see e.g. Theorem 7.1 of [110]) states that *if a gravitational field possesses a shearfree geodesic null congruence  $\mathbf{k}$  and if  $R_{ab}k^a k^b = 0$  then this field is algebraically special and  $\mathbf{k}$  is a degenerate eigendirection i.e.  $\Psi_0 = \Psi_1 = 0$* . The cancellation of these two Newmann-Penrose coefficients shows that such spacetimes cannot encompass incoming radiation (in the fluid language this corresponds to  $E_{ab} = 0$  and  $N_a = 0$  which fits precisely in our hypotheses). In our case we use the null, geodesic, and shear-free bulk congruence tangent to  $\partial_r$ . The latter is part of the canonical null tetrad parallelly transported along  $\partial_r$  (thanks to the affine nature of  $r$ ) introduced in [52], which coincides with that of [110], Eq. (29.13a), as well as with the original Ref. [169]. In complex coordinates  $\zeta$  and  $\bar{\zeta}$  the null tetrad reads

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left( \frac{8\pi G \varepsilon_{(0)} r + * \hat{\omega} c_{(0)}}{r^2 + * \hat{\omega}^2} - r\theta - \hat{\mathcal{K}} \right) \partial_r + \nu \\ \mathbf{m} = \frac{P}{r - i * \hat{\omega}} \left( \hat{\partial}_{\bar{\zeta}} + (* \hat{\mathcal{D}}_{\bar{\zeta}} * \hat{\omega} - r \varphi_{\bar{\zeta}}) \partial_r \right) \end{cases} \quad (4.1.13)$$

with  $\mathbf{k} \cdot \mathbf{l} = -1$ ,  $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$  and  $\text{ds}_{\text{res. Ricci-flat}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$ . Generically,  $\mathbf{k}$  is a multiplicity-two principal null direction of the Weyl tensor, and using the tetrad at hand we find the following Weyl complex scalars:  $\Psi_0 = \Psi_1 = 0$  (as expected) and

$$\Psi_2 = \frac{i\hat{r}}{2(r - i * \hat{\omega})^3}, \quad \Psi_3 = \frac{iP\chi_{\zeta}}{(r - i * \hat{\omega})^2} + \text{O}\left(\frac{1}{(r - i * \hat{\omega})^3}\right), \quad \Psi_4 = \frac{iX_{\zeta}^{\bar{\zeta}}}{r - i * \hat{\omega}} + \text{O}\left(\frac{1}{(r - i * \hat{\omega})^2}\right). \quad (4.1.14)$$



Note that all  $\Psi$ s are expressed using the Carrollian descendants of the boundary Cotton tensor — as well as their derivatives in the higher-order terms. This is not a surprise as the Cotton tensor is nothing but the three-dimensional analogue of the Weyl tensor. Hence the bulk information about radiation is directly translated on the Carrollian boundary via the Cotton tensor, as already observed in the last Chapter.

The precise Petrov type of the solution at hand can be determined using the following Carrollian tensors [114] (with the notations of Sec. 2.2)

$$\varepsilon^\pm = \varepsilon_{(0)} \pm \frac{i}{8\pi G} c_{(0)}, \quad (4.1.15a)$$

$$Q_i^\pm = Q_i \pm \frac{i}{8\pi G} \chi_i, \quad (4.1.15b)$$

$$\Sigma_{ij}^\pm = \Sigma_{ij} \pm \frac{i}{8\pi G} X_{ij}. \quad (4.1.15c)$$

These Carrollian geometric tensors encompass information also stored in the Newman-Penrose scalars (4.1.10). The way they determine the Petrov type will be presented now with an explicit example.

### Example of resumable spacetime

We have already encountered this example when dealing with the charges of the Carrollian scalar field as well as when presenting a simple Carrollian fluid. This is the four-dimensional Robinson-Trautman solution described in coordinates  $(r, t, \zeta, \bar{\zeta})$ . As we saw in Sec. 2.2 these solutions are described by a vorticity-free Carrollian fluid as  $b_i = 0$  and  $\Omega = 1$ . The resumable metric (4.1.2) can then be written as

$$ds_{\text{RT}}^2 = -2dt(dr + Hdt) + 2\frac{r^2}{P^2}d\zeta d\bar{\zeta}, \quad (4.1.16)$$

with  $P(t, \zeta, \bar{\zeta})$  satisfying the Robinson-Trautman equation (2.3.27) and where

$$2H = -2r\partial_t \ln P + K - \frac{2M(t)}{r}, \quad (4.1.17)$$

with again  $K$  the Gaussian curvature (see (2.2.28)). In addition two descendants of the Cotton tensor are non vanishing,  $\chi$  and  $\mathbf{X}$ , and read

$$\chi = \frac{i}{2} (\partial_\zeta K d\zeta - \partial_{\bar{\zeta}} K d\bar{\zeta}), \quad \mathbf{X} = \frac{i}{2} (\partial_\zeta (P^2 \partial_t \partial_\zeta \ln P) d\zeta^2 - \partial_{\bar{\zeta}} (P^2 \partial_t \partial_{\bar{\zeta}} \ln P) d\bar{\zeta}^2). \quad (4.1.18)$$

In the coordinates at hand the tensors (4.1.15) read

$$\varepsilon^+ = \frac{M(t)}{4\pi G}, \quad \mathbf{Q}^+ = -\frac{1}{8\pi G} \partial_\zeta K d\zeta, \quad \Sigma^+ = -\frac{1}{4\pi G P^2} \partial_\zeta (P^2 \partial_t \partial_\zeta \ln P) d\zeta^2, \quad (4.1.19)$$

and the following classification was derived in [114]

- **Type II.** generic tensors,
- **Type III.**  $\varepsilon^+ = 0$  and  $\mathbf{div} Q^+ = 0$ ,
- **Type N.**  $\varepsilon^+ = 0$  and  $Q_\zeta^+ = 0$ ,
- **Type D.**  $2Q_\zeta^+ Q_{\bar{\zeta}}^+ = 3\varepsilon^+ \Sigma_{\zeta\bar{\zeta}}^+$  and vanishing traceless part of the first derivative of  $Q$ .

### 4.1.2 Stationary solutions

To illustrate the power of the boundary perspective on asymptotically flat spacetimes, we will study in a more systematic and comprehensive way an important subclass of spacetimes, the stationary ones. A nice feature of the latter is that the timelike character of the Killing field is maintained even in the asymptotic region when  $r \rightarrow \infty$ . This allows us to pick as Killing a vector that will ultimately coincide with the field of observers  $\mathbf{v}$  of the Carrollian boundary (see (1.1.2)) and hence with the fibers of the Carroll structure. Using the freedom given by Weyl transformations (1.4.1) we will eventually choose this Killing to be aligned with  $\partial_t$  by setting  $\Omega = 1$ . This set-up is of course far from being general as it for example forbids any solutions in which the timelike Killing field becomes spacelike in the asymptotic region, like the  $C$ -metric (see [110, 168, 172]). Anyway it will be enough for our purposes and extending the forthcoming analysis to more general spacetimes is part of our future agenda.

When  $\partial_t$  is a Killing field and when  $\Omega$  is taken equal to one, all the Carrollian data present in (4.1.2) is time-independent. Moreover, at the level of the geometry  $\theta = 0$  and  $\varphi_i = 0$  which have for main consequence to cancel the differences between Carroll and Weyl-Carroll covariant derivatives (as the Weyl connection (1.4.4) identically vanishes). The only non vanishing geometrical data are the vorticity  $*\hat{\omega}$ , the Gauss curvature  $\hat{\mathcal{K}}$  and pieces of the Carrollian Cotton tensor. They read

$$*\hat{\omega} = \frac{iP^2}{2} (\partial_\zeta b_{\bar{\zeta}} - \partial_{\bar{\zeta}} b_\zeta), \quad (4.1.20a)$$

$$\hat{\mathcal{K}} = \hat{K} = K = \Delta \ln P, \quad (4.1.20b)$$

$$c_{(0)} = (\Delta + 2K) *\hat{\omega}, \quad (4.1.20c)$$

$$\chi_\zeta = \frac{i}{2} \partial_\zeta K, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \partial_{\bar{\zeta}} K, \quad (4.1.20d)$$

$$\psi_\zeta = 3i\partial_\zeta *\hat{\omega}^2, \quad \psi_{\bar{\zeta}} = -3i\partial_{\bar{\zeta}} *\hat{\omega}^2, \quad (4.1.20e)$$

$$\Psi_{\zeta\zeta} = \frac{1}{P^2} \partial_\zeta (P^2 \partial_\zeta *\hat{\omega}), \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \frac{1}{P^2} \partial_{\bar{\zeta}} (P^2 \partial_{\bar{\zeta}} *\hat{\omega}), \quad (4.1.20f)$$

where we recall that  $\Delta f = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta f$ . To these one should add the energy density  $\varepsilon_{(0)}$ , as well as another scalar

$$\hat{\omega} = \frac{P^2}{2} (\partial_\zeta b_{\bar{\zeta}} + \partial_{\bar{\zeta}} b_\zeta), \quad (4.1.21)$$

which corresponds to  $\frac{1}{2} \nabla_i b^i$  and should not be confused with  $*\hat{\omega} = -\frac{1}{2} \nabla_i *b^i$  displayed explicitly in (4.1.20a) (its definition as the two-dimensional Hodge dual of the vorticity was displayed in (3.2.24)).

These two real twist scalars are adroitly combined into the *complex Carrollian twist*

$$\hat{\omega} = *\omega + i\bar{\omega}, \quad (4.1.22)$$

in a combination that recall the definition of  $\hat{\tau}$  in (4.1.5a).

Together with the geometry are the equations of motion (4.1.3a) and (4.1.3b) that we prefer to treat in the form (4.1.6) with  $\hat{\tau}$  in view of the subsequent Ehlers analysis to be done in Chapter 5. They are recast as

$$\Delta K = 0, \quad (4.1.23a)$$

$$\partial_{\zeta} \hat{\tau} = 0, \quad (4.1.23b)$$

which first imply that the curvature is a harmonic function i.e.

$$K(\zeta, \bar{\zeta}) = \frac{1}{2} \left( \hat{k}(\zeta) + \hat{k}(\bar{\zeta}) \right), \quad (4.1.24)$$

with  $\hat{k}(\zeta)$  a function and  $\hat{k}$  is the function whose values are complex-conjugate to  $\hat{k}$ . For further use we also introduce another harmonic function

$$K^*(\zeta, \bar{\zeta}) = \frac{1}{2i} \left( \hat{k}(\zeta) - \hat{k}(\bar{\zeta}) \right) \quad (4.1.25)$$

which is the imaginary part of  $\hat{k}(\zeta)$ .

**Remark** Although  $\hat{k}(\zeta)$  may seem arbitrary, remember that  $K = \Delta \ln P$  (see (4.1.20b)) so *in fine* the freedom is rather limited. Actually only one non constant solution has been exhibited so far [110]:  $K = -3(\zeta + \bar{\zeta})$  associated with  $P = (\zeta + \bar{\zeta})^{3/2}$ .

Using (4.1.23b) and the definition (4.1.5a) we infer that  $-c_{(0)}$  is the real part of an arbitrary holomorphic function  $\hat{\tau}(\zeta)$  while  $8\pi G\varepsilon_{(0)}$  is its imaginary part. As (4.1.7) reduces in the situation at hand to  $\Delta \hat{\tau} = 0$ , both  $c(\zeta, \bar{\zeta})$  and  $\varepsilon(\zeta, \bar{\zeta})$  are holomorphic functions. Finally, given the Cotton density  $c_{(0)}$  and the curvature  $K$  one can solve (4.1.20c) for  $*\omega$  and eventually  $b_{\zeta}, b_{\bar{\zeta}}$  (which is the complex conjugate of the former). This is what means solving Einstein's equations in the present case. Two main situations can be distinguished, regarding the curvature  $K$  which can be either constant or not.

**Non-constant  $K$ .** Apart from the explicit solution exhibited above, even though this is the generic situation, that case is in practice very obscure regarding the interpretation of the associated bulk geometry. Anyway if one has a  $P$  associated with a non-constant curvature one gets for Ehresmann connection

$$b_{\zeta}(\zeta, \bar{\zeta}) = \frac{i\hat{\tau}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})\partial_{\bar{\zeta}}\hat{k}(\bar{\zeta})}. \quad (4.1.26)$$

**Constant  $K$ .** In this case  $\hat{k}(\zeta)$  is also constant, hence due to the derivative in the denominator

of (4.1.26), that solution is no longer valid. This case is the most common situation as it captures three standard instances: spherical, flat or hyperbolic foliations. For example the Kerr-Taub-NUT family that we shall study more in details in the following belongs to that class. One should now distinguish between zero and non-zero curvature. Before that, note that one can always parametrize the function  $P$  (which is harmonic i.e.  $\Delta P = 0$ ) in term of two real constants  $A$  and  $D$  plus one complex constant  $B$  as

$$P(\zeta, \bar{\zeta}) = A\zeta\bar{\zeta} + B\zeta + \bar{B}\bar{\zeta} + D, \quad (4.1.27)$$

which leads to a curvature of the form

$$K = 2(AD - B\bar{B}). \quad (4.1.28)$$

- **If  $K \neq 0$ .** As  $-c_{(0)}$  is the real part of  $\hat{\tau}$ , it can be expressed as  $c_{(0)}(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$ . In this case (4.1.20c) can be solved as

$$*\omega(\zeta, \bar{\zeta}) = \frac{c_{(0)}(\zeta, \bar{\zeta})}{2K} + i \left( \bar{f}(\bar{\zeta}) \partial_{\bar{\zeta}} \ln P(\zeta, \bar{\zeta}) - f(\zeta) \partial_{\zeta} \ln P(\zeta, \bar{\zeta}) + \frac{1}{2} (\partial_{\zeta} f(\zeta) - \partial_{\bar{\zeta}} \bar{f}(\bar{\zeta})) \right) \quad (4.1.29)$$

with  $f(\zeta)$  an arbitrary holomorphic function. Such a solution is reached with the following Ehresmann connection

$$b_{\zeta}(\zeta, \bar{\zeta}) = -\frac{\bar{\zeta}(\hat{\tau}_0 + i\hat{\tau}(\zeta))}{2K(B\zeta + D)P(\zeta, \bar{\zeta})} + \frac{\bar{f}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})} \quad (4.1.30a)$$

$$b_{\bar{\zeta}}(\zeta, \bar{\zeta}) = \frac{i\zeta\hat{\tau}(\zeta)}{2K(\bar{B}\bar{\zeta} + D)P(\zeta, \bar{\zeta})} + \frac{f(\zeta)}{P^2(\zeta, \bar{\zeta})}, \quad (4.1.30b)$$

with  $\hat{\tau}_0$  a real constant.

- **If  $K = 0$ .** This case can be attained with  $A = B = 0$  so that  $P = D$  i.e. a constant. Given an arbitrary holomorphic function  $c_{(0)}(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$  and another arbitrary holomorphic function  $Z(\zeta)$  we find for the vorticity

$$*\omega(\zeta, \bar{\zeta}) = \frac{i}{2} (Z(\zeta) - \bar{Z}(\bar{\zeta})) - \frac{1}{4P^2} \left( \bar{\zeta} \int^{\zeta} dz \hat{\tau}(z) + \zeta \int^{\bar{\zeta}} d\bar{z} \hat{\tau}(\bar{z}) \right), \quad (4.1.31)$$

and for the Ehresmann

$$b_{\zeta}(\zeta, \bar{\zeta}) = \frac{1}{P^2} \int^{\bar{\zeta}} d\bar{z} \bar{Z}(\bar{z}) - \frac{\bar{\zeta}^2}{4P^4} \int^{\zeta} dz (\hat{\tau}_0 + i\hat{\tau}(z)) \quad (4.1.32a)$$

$$b_{\bar{\zeta}}(\zeta, \bar{\zeta}) = \frac{1}{P^2} \int^{\zeta} dz Z(z) + \frac{i\zeta^2}{4P^4} \int^{\bar{\zeta}} d\bar{z} \hat{\tau}(\bar{z}), \quad (4.1.32b)$$

with  $\hat{\tau}_0$  a real integration constant.

Note that the last two cases have in common the instance where  $c_{(0)} = K = 0$ , realized with vanishing  $\hat{\tau}$  and constant  $P$ .

**Remark** As  $c_{(0)}$  is a purely geometrical quantity on the conformal Carrollian boundary, which is also Weyl covariant (i.e. it transforms without inhomogeneous terms), one may argue that we could have used a Weyl rescaling to set it to a constant. However such a transformation would have set  $\Omega$  back to a non constant value, hence  $\varphi_i$  would no longer have been zero. As we have chosen to get rid of the latter it is no longer possible to use Weyl rescalings to simplify the value of  $c_{(0)}$ .

**Kerr-Taub-NUT family.** There is a useful subclass of constant curvature stationary solutions. It is described by means of a mass parameter  $M$  and a NUT charge (also dubbed “magnetic mass”)  $n$ . Again two cases emerge, vanishing or non-vanishing  $K$ , realised with vanishing or non-vanishing  $A$ .

- For non-vanishing  $K$ , the holomorphic function  $\hat{\tau}$  is

$$\hat{\tau} = 2i(M + iKn), \quad (4.1.33)$$

where  $M$  is the mass and  $n$  the nut charge, both constants. The holomorphic function  $f(\zeta)$  reads

$$f(\zeta) = ia\zeta \quad (4.1.34)$$

with  $a$  the Kerr angular velocity. Using Eqs. (4.1.29), (4.1.30a) and (4.1.30b) with  $\hat{\tau}_0 = 2M$  we find:

$$b_\zeta(\zeta, \bar{\zeta}) = -i\bar{\zeta} \left( \frac{a}{P^2} - \frac{n}{DP} \right) \quad (4.1.35)$$

and

$$*\hat{\omega}(\zeta, \bar{\zeta}) = n + a - \frac{2Da}{P}, \quad (4.1.36)$$

where  $P = A\zeta\bar{\zeta} + D$  and  $K = 2AD$ .

- For  $K = 0$  (i.e.  $P = D$  constant), we use Eqs. (4.1.31), (4.1.32a) and (4.1.32b) with  $\hat{\tau}_0 = 2M$ ,<sup>2</sup>

$$\hat{\tau} = 2iM \quad (4.1.37)$$

and

$$Z = ia. \quad (4.1.38)$$

This leads to

$$b_\zeta(\zeta, \bar{\zeta}) = -i\frac{\bar{\zeta}a}{P^2} \quad (4.1.39)$$

and

$$*\hat{\omega} = -a. \quad (4.1.40)$$

---

<sup>2</sup>Both for vanishing and non-vanishing  $K$ ,  $\hat{\tau}_0$  has been tuned so as to ensure that  $M$  does not appear in  $b_\zeta$ , displayed in (4.1.35) and (4.1.39). There is no principle behind this choice, it is simply in line with standard conventions for the Kerr-Taub-NUT family. As a consequence,  $\hat{\omega}$  defined in (4.1.21) vanishes.

Despite this absence of magnetic charges, the solution at hand belongs formally to the Taub–NUT family (see Ref. [172], §12.3.2).

This concludes our study of stationary solutions that can be written as a resumable metric expressed in terms of Carrollian data (4.1.2). We now turn to the analysis of the isometrics of such spacetimes but from a boundary rather than from a bulk perspective.

## 4.2 Carrollian perspective on bulk isometric charges

In this Section, we discuss to what extent the Carrollian boundary can give us a new perspective on the determination and the computation of bulk gravitational charges. The latter contain fundamental information about the content of the spacetime at hand, and they allow to distinguish between two diffeomorphic solutions. After showing how to relate bulk isometries to boundary isometries, we discuss how expanding back the resummed metric (4.1.2) leads to two towers of charges, electric and magnetic, coming from the covariant conservation of the energy-momentum and Cotton tensors respectively.

### Translating bulk isometries into (strong) boundary isometries

Recall that the  $(d+2)$ -dimensional bulk spacetime (4.1.2) is described in orthonormal frame by means of

$$\begin{aligned} e_i &\equiv \mathbf{v}, & e_a &\equiv \hat{\partial}_a = \hat{e}_a, & e_r &\equiv \partial_r, \\ \theta^{\hat{u}} &\equiv \tau, & \theta^a &\equiv \hat{\theta}^a, & \theta^{\hat{r}} &\equiv dr. \end{aligned} \quad (4.2.1)$$

The components for the bulk metric (4.1.2) read (in order to avoid cluttering, we keep the “hat” on the time indices only, where potential ambiguity exists)

$$\begin{aligned} g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G \varepsilon r + *\hat{\omega}c) - r\theta - \hat{\mathcal{K}}, & g_{\hat{t}a} &= *\hat{\mathcal{D}}_a *\hat{\omega} - r\varphi_a, & g_{\hat{t}r} &= -1, \\ g_{ra} &= 0, & g_{rr} &= 0, & g_{ab} &= \rho^2 \delta_{ab}, \end{aligned} \quad (4.2.2)$$

with

$$\rho^2 = r^2 + (*\hat{\omega})^2. \quad (4.2.3)$$

We give ourselves a Killing field of (4.1.2) which is assumed to have no legs along the radial direction  $e_r$

$$\xi = \xi^{\hat{t}}(t, x^a) \mathbf{v} + \xi^b(t, x^a) e_b \quad (4.2.4)$$

which of course does not constitute the more general case but is enough for our purpose. The Lie derivative of the bulk metric then reads

$$\begin{aligned}
\mathcal{L}_\xi g_{rr} &= 0, & \mathcal{L}_\xi g_{ri} &= \mu, & \mathcal{L}_\xi g_{ra} &= \nu_a, \\
\mathcal{L}_\xi g_{ab} &= 2\rho^2 \left( \hat{\nabla}_{(a} \xi^c \delta_{b)c} + \xi^i \hat{\gamma}_{ab} \right) - 2g_{i(a} \nu_{b)} + \delta_{ab} \xi (*\hat{\omega}^2), \\
\mathcal{L}_\xi g_{ia} &= -g_{ia} \mu - g_{ii} \nu_a - r \left( \xi(\varphi_a) + \varphi_b \hat{\partial}_a \xi^b \right) + \xi \left( *\hat{\mathcal{D}}_a *\hat{\omega} \right) + \left( *\hat{\mathcal{D}}_b *\hat{\omega} \right) \hat{\partial}_a \xi^b + \rho^2 \delta_{ab} \mathbf{v}(\xi^b), \\
\mathcal{L}_\xi g_{ii} &= -2g_{ii} \mu + 2g_{ia} \mathbf{v}(\xi^a) - \xi \left( \frac{1}{\rho^2} (8\pi G \epsilon r + *\hat{\omega} c) - r\theta - \hat{\mathcal{K}} \right)
\end{aligned} \tag{4.2.5}$$

with  $\mu(t, \mathbf{x})$  and  $\nu_a(t, \mathbf{x})$  given by

$$\mu(t, \mathbf{x}) = -\mathbf{v}(\xi^i) - \xi^a \varphi_a \tag{4.2.6}$$

$$\nu_a(t, \mathbf{x}) = -\hat{e}_a(\xi^i) + \varphi_a \xi^i + 2\xi^b \hat{\omega}_{ba}. \tag{4.2.7}$$

Observe that everything is expressed in terms of boundary geometric objects.

Since the Killing components are  $r$ -independent, the above Lie derivative vanishes if and only if the coefficients of every power of  $r$  do. The independent conditions we reach for this to occur are

$$\mathbf{v}(\xi^a) = 0 \tag{4.2.8}$$

and (1.5.2) together with (1.5.15), which therefore map the bulk Killing field (4.2.4) onto a boundary Carrollian strong Killing vector (see Sec. 1.5). Some apparent extra conditions such as  $\xi(*\hat{\omega}^2) = 0$  or  $\xi(\varphi_a) + \varphi_b e_a \xi^b = 0$  are the vanishing of  $\xi$ -Lie derivatives of some Carrollian tensors, which is guaranteed by the strong Killing requirement on  $\xi$ . Hence the important result to remember is that *A bulk Killing field of the form (4.2.4) translates into a strong Carrollian Killing on the null boundary.*

## Towers of electric and magnetic charges

Charges are often described using the asymptotic symmetry group of the bulk spacetime,  $\text{BMS}_4$  in the case of asymptotically flat gravity in four dimensions. Due to gravitational radiation they are not conserved in general, the main example being the Bondi mass aspect as seen in (3.3.28). From a boundary perspective this group translates into the infinite set of conformal Carrollian isometries (1.5.31), which in principle allows for an alternative construction of the charges, from a purely boundary perspective, whose (non) conservation is encoded into the Carrollian Cotton descendants (see Sec. 2.4) and the shear  $\hat{\mathcal{C}}_{ab}$ .

The Carrollian way of constructing charges has been described in Sec. 1.5. It requires, on top of the conformal Carrollian vector  $\xi$ , a set of momenta  $\Pi, \Pi^a, \Pi^{ab}, P^a$  satisfying the Carrollian dynamics encoded in (1.3.20) and (1.3.21). With this at hand, (1.5.12) gives the current and (1.5.17) the charge. These are the features a Ricci-flat metric described in terms of Carrollian variables (3.3.40) (or (4.1.2) for the resummed version) should possess for the bulk charges to be described with a boundary approach. Focusing here on the resumable instance (4.1.2) we see that this is indeed the

case. Expanding the  $\frac{1}{\rho^2}$  in powers of  $r$  we get for the first few orders

$$\begin{aligned} ds_{\text{res. Ricci-flat}}^2 &= \cdots + \frac{1}{r^2 + *\hat{\omega}^2} (8\pi G \varepsilon_{(0)} r + *\hat{\omega} c_{(0)}) \boldsymbol{\tau}^2 \\ &= \cdots + \frac{1}{r^2} \left[ 8\pi G \varepsilon r + *\hat{\omega} c - 8\pi G \varepsilon \frac{(*\hat{\omega})^2}{r^2} - \frac{(*\hat{\omega})^3 c}{r^3} + \cdots \right] \boldsymbol{\tau}^2. \end{aligned} \quad (4.2.9)$$

We see two patterns emerging (here  $s \in \{1, 2, \dots\}$ )

- **at every order**  $\mathcal{O}(\frac{1}{r^{2s+1}})$  we get Carrollian dynamics with “fluid-like” momenta  $\Pi_{(s)}$ ,  $\Pi_{(s)}^i$ ,  $P_{(s)}^i$ ,  $\tilde{\Pi}_{(s)}^{ij}$  and  $\Pi_{(s)}^{ij}$ . The  $s = 0$  terms are displayed in (4.1.4) while going to deeper order is made through the multiplication of the latter by  $(*\hat{\omega})^{2s}$ . Every such set of momenta together with the Carrollian conformal Killings (1.5.29), which depends on a supertranslation  $T$  and a superrotation  $Y$ , lead to currents  $\kappa_{(s)}$ ,  $K_{(s)}^i$ ,  $\tilde{\kappa}_{(s)}$ ,  $\tilde{K}_{(s)}^i$  and charges  $Q_{(s)T,Y}$  and  $\tilde{Q}_{(s)T,Y}$ , following (1.5.17) and (2.2.20). Their conservation or evolution encoded in (1.5.18) depends on  $\mathcal{H}_{(s)}$ ,  $\tilde{\mathcal{H}}_{(s)}$  in (2.2.21). The set associated with  $s = 0$  corresponds to the fluid momenta (4.1.4) and its charges are *leading*; the sets with  $s \geq 1$  reveal the *subleading charges*. Moreover, all these charges should be referred to as **electric** because their conservation, if valid, occurs on-shell.<sup>3</sup> Using the Carroll–Bianchi identities (4.2.28a), (4.2.28b) and (4.2.28c), we find the divergences (2.2.21), which contribute to the time evolution of the charges computed as in (1.5.17), using (1.5.18):

$$\tilde{\mathcal{H}}_{(s)} = -s *\hat{\omega}^{2s-2} \left( *\hat{\omega} \hat{\mathcal{A}} \tilde{\kappa} + \frac{1}{3} \tilde{K}^i *\psi_i \right), \quad (4.2.10a)$$

$$\mathcal{H}_{(s)} = -\frac{*\hat{\omega}^{2s}}{8\pi G} *\chi^i \left( \hat{\mathcal{D}}_i \xi^i - 2\xi^j \hat{\omega}_{ji} \right) - s *\hat{\omega}^{2s-2} \left( *\hat{\omega} \hat{\mathcal{A}} \kappa + \frac{1}{3} K^i *\psi_i \right) \quad (4.2.10b)$$

with currents

$$\begin{cases} \kappa = \frac{1}{8\pi G} \xi^i *\psi_i - \xi^i \varepsilon_{(0)} \\ \tilde{\kappa} = \frac{1}{8\pi G} \xi^i *\chi_i \\ K^i = \frac{\varepsilon_{(0)}}{2} \xi^i - \frac{1}{8\pi G} \left( \xi^j *\Psi_j^i + \xi^i *\chi^i \right) \\ \tilde{K}^i = -\frac{1}{8\pi G} \xi^j *X_j^i, \end{cases} \quad (4.2.11)$$

and the Killing components  $\xi^i$  and  $\tilde{\xi}^i$  read off in (1.5.28) and displayed again here for facilitate the reading

$$\xi_{T,Y} = \left( T(\mathbf{x}) - Y^i(\mathbf{x}) \hat{\partial}_i C(t, \mathbf{x}) + \frac{1}{d} C(t, \mathbf{x}) \nabla_i Y^i(\mathbf{x}) \right) e^{-\sigma(t, \mathbf{x})} \mathbf{v} + Y^i(t, \mathbf{x}) \hat{\partial}_i. \quad (4.2.12)$$

Regarding the charges and their evolution, only  $\tilde{Q}_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} (\tilde{\kappa} + b_j \tilde{K}^j) \equiv \tilde{Q}_{T,Y}$  are *always* conserved (i.e. for all  $T$  and  $Y$ ). These charges are purely geometric because they are

<sup>3</sup>Remember that they come from the Carrollian descendant of the relativistic energy-momentum tensor, which is covariantly conserved only on-shell.



integrals over<sup>4</sup>  $\mathcal{S}$ , the base space of the Carrollian boundary,

$$\tilde{Q}_{T,Y} = -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} * \xi^i \left( \chi_i - b_j X_i^j \right), \quad (4.2.13)$$

which do not involve the energy density  $\varepsilon_{(0)}$ , as opposed to  $Q_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} (\kappa + b_j K^j) \equiv Q_{T,Y}$  spelled as

$$\begin{aligned} Q_{T,Y} &= -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( 8\pi G \varepsilon_{(0)} + b_i * \chi^i \right) \\ &+ \frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( * \psi_i + 4\pi G \varepsilon_{(0)} b_i - b_j * \Psi_i^j \right). \end{aligned} \quad (4.2.14)$$

The latter are conserved for strong Carrollian Killings. Other charges might also be conserved for specific Carrollian conformal Killings, or depending on the configuration. The full tower of leading and subleading electric charges  $\{Q_{(s)T,Y}\}$  and  $\{\tilde{Q}_{(s)T,Y}\}$  are obtained by multiplying the integrand of (4.2.13) and the one of (4.2.14) by  $(*\hat{\omega})^{2s}$ .

- **at every order**  $\mathcal{O}\left(\frac{1}{r^{2s+2}}\right)$  the expansion of (4.1.2) reveals *off-shell* Carrollian dynamics for the Carrollian Cotton tensor with momenta  $\Pi_{\text{Cot}(s)}$ ,  $\Pi_{\text{Cot}(s)}^i$ ,  $P_{\text{Cot}(s)}^i$ ,  $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$  and  $\Pi_{\text{Cot}(s)}^{ij}$ . The  $s = 0$  terms are displayed in (2.4.13) while going to deeper order is made through the multiplication of the latter by  $(*\hat{\omega})^{2s}$ . Here also currents  $\kappa_{\text{Cot}(s)}$ ,  $K_{\text{Cot}(s)}^i$ ,  $\tilde{\kappa}_{\text{Cot}(s)}$ ,  $\tilde{K}_{\text{Cot}(s)}^i$ , and finally magnetic charges  $Q_{\text{Cot}(s)T,Y}$  and  $\tilde{Q}_{\text{Cot}(s)T,Y}$  can be defined. They are dubbed **magnetic** as their conservation arise off-shell. In the situation at hand the leading magnetic currents read

$$\begin{cases} \kappa_{\text{Cot}} = \xi^i \psi_i - \xi^i c \\ \tilde{\kappa}_{\text{Cot}} = \xi^i \chi_i \\ K_{\text{Cot}}^i = \frac{c}{2} \xi^i - \xi^j \Psi_j^i - \xi^i \chi^i \\ \tilde{K}_{\text{Cot}}^i = -\xi^j X_j^i, \end{cases} \quad (4.2.15)$$

Their divergences (2.2.21) take the form

$$\tilde{\mathcal{H}}_{\text{Cot}(s)} = -s * \hat{\omega}^{2s-2} \left( * \hat{\omega} \hat{\mathcal{L}} \tilde{\kappa}_{\text{Cot}} + \frac{1}{3} \tilde{K}_{\text{Cot}}^i * \psi_i \right), \quad (4.2.16a)$$

$$\mathcal{H}_{\text{Cot}(s)} = - * \hat{\omega}^{2s} \chi^i \left( \hat{\mathcal{D}}_i \xi^i - 2\xi^j \hat{\omega}_{ji} \right) - s * \hat{\omega}^{2s-2} \left( * \hat{\omega} \hat{\mathcal{L}} \kappa_{\text{Cot}} + \frac{1}{3} K_{\text{Cot}}^i * \psi_i \right). \quad (4.2.16b)$$

These determine the evolution (1.5.18) of the charges (1.5.17), from which we learn that  $\tilde{Q}_{\text{Cot}(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \tilde{\kappa}_{\text{Cot}} + b_j \tilde{K}_{\text{Cot}}^j \right) \equiv \tilde{Q}_{\text{Cot}T,Y}$  are always conserved

$$\tilde{Q}_{\text{Cot}T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( \chi_i - b_j X_i^j \right). \quad (4.2.17)$$

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<sup>4</sup>We use the property  $V^i * W_i = - * V^i W_i$  – see (1.6.4).

For strong Carrollian Killing fields,  $Q_{\text{Cot}(0)T,Y} = \int_S d^2x \sqrt{a} \left( \kappa_{\text{Cot}} + b_j K_{\text{Cot}}^j \right) \equiv Q_{\text{Cot}T,Y}$  given by

$$Q_{\text{Cot}T,Y} = - \int_S d^2x \sqrt{a} \xi^i \left( c + b_i \chi^i \right) + \int_S d^2x \sqrt{a} \xi^i \left( \psi_i + \frac{c}{2} b_i - b_j \Psi_i^j \right) \quad (4.2.18)$$

are also conserved off-shell, as other magnetic charges are in specific situations. The full tower of leading and subleading electric charges  $\{Q_{\text{Cot}(s)T,Y}\}$  and  $\{\tilde{Q}_{\text{Cot}(s)T,Y}\}$  are obtained by multiplying the integrand of (4.2.17) and the one of (4.2.18) by  $(*\hat{\omega})^{2s}$ .

- **electric and magnetic towers are related.** As the limit from AdS spacetimes sets a relationship between the energy-momentum tensor and the Cotton tensor (see (3.3.16) and (4.1.1)) it should not be a surprise that the electric and magnetic towers of charge described above possess a non-empty intersection:  $\tilde{Q}_{(s)T,Y}$  and  $\tilde{Q}_{\text{Cot}(s)T,Y}$  generally coincide. This is because in  $d = 2$ , if  $\xi^i$  are the spatial components of a conformal Killing field, so are  $*\xi^i$ . Hence the set of all  $\xi^i$ s is identical to that of  $*\xi^i$ s. The associated charges could be called “self-dual,” and in total three distinct towers emerge: the self-dual  $\{\tilde{Q}_{(s)T,Y}\} \equiv \{\tilde{Q}_{\text{Cot}(s)T,Y}\}$ , the electric  $\{Q_{(s)T,Y}\}$  and the magnetic  $\{Q_{\text{Cot}(s)T,Y}\}$ . The  $*\hat{\omega}^{2s}$  insertion pattern grants the subleading towers with the status of *multipolar moments* (see the original works [166, 173–175] as well as [176] for a modern perspective).

Note finally that among the above charges associated to Carrollian Conformal Killings, which are not always conserved, one can find the ones that are associated to bulk isometries, if present. As discussed in the last subsection, these isometries are mapped to the boundary as strong Carrollian isometries. We shall now illustrate the previous discussion with an explicit example.

### Charge analysis of the Kerr-Taub-NUT family

We consider again the Kerr-Taub-NUT family described in the previous subsection from Eq. (4.1.33). The Carrollian approach to the computation of bulk charges enables us to find the gravitational mass and angular momentum charges and their multipolar expansion. As we consider stationary spacetimes,  $\xi = \partial_t$  is a Killing field. The non-tilde electric and magnetic currents then read

$$\kappa = -\varepsilon_{(0)}, \quad K^i = -\frac{1}{8\pi G} * \chi^i, \quad \kappa_{\text{Cot}} = -c_{(0)}, \quad K_{\text{Cot}}^i = -\chi^i, \quad (4.2.19)$$

from which we find the leading ( $s = 0$ ) charges

$$Q_{\text{em}} = \int_S \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (8\pi G \varepsilon_{(0)} + \hat{\omega}K), \quad Q_{\text{mm}} = \int_S \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (-c_{(0)} + *\hat{\omega}K), \quad (4.2.20)$$

where the indices refers to electric mass (em) and magnetic mass (mm) because these charges are associated to  $\partial_t$ . They are nicely combined into a complex mass charge

$$Q_{\text{m}} = Q_{\text{mm}} + iQ_{\text{em}} = \int_S \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\omega}K). \quad (4.2.21)$$

**Remark** The integrals can be performed by setting  $\zeta = Ze^{i\Phi}$ , where  $0 \leq \Phi < 2\pi$  and  $Z = \sqrt{2} \tan \frac{\Theta}{2}$ ,  $0 < \Theta < \pi$  for  $\mathbb{S}^2$ ;  $Z = \frac{R}{\sqrt{2}}$ ,  $0 < R < +\infty$  for  $\mathbb{E}_2$ ;  $Z = \sqrt{2} \tanh \frac{\Psi}{2}$ ,  $0 < \Psi < +\infty$  for  $\mathbb{H}_2$ .

To get the subleading ( $s \geq 1$ ) we insert the appropriate power of  $(*\hat{\omega})^{2s}$ . this leads for the higher- $s$  mass multipole moment

$$Q_{m(s)} = \int_S \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\omega}K) (*\hat{\omega})^{2s}. \quad (4.2.22)$$

In the instance of the  $K = 1$  Kerr–Taub–NUT family displayed in Eqs. (4.1.33), (4.1.34), (4.1.35), (4.1.36) with  $A = \frac{1}{2}$  and  $D = 1$ , we find

$$Q_{m(s)} = 4\pi i (M + in) \left( \frac{(n+a)^{2s+1} - (n-a)^{2s+1}}{a(2s+1)} \right), \quad (4.2.23)$$

where we recall that  $M$  is the Bondi mass (constant here),  $n$  the NUT charge and  $a$  the angular velocity.

For this set of solutions  $\eta = i(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}})$  is also a Killing field (spacelike) which has the virtue of being pushed forward into a strong Carrollian isometry on the null boundary. Therefore the above discussion with  $\partial_t$  can be repeated with  $\eta$ . Again the “tilde” (Eqs. (4.2.13) and (4.2.17)) are zero whereas the “non-tilde” (see. (4.2.11) and (4.2.15)) are combined in the complex higher- $s$  *angular-momentum multipole moments*

$$Q_{r(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} 6\zeta\bar{\zeta} \left( \frac{n+iM}{P^2} (a-nP) \left( n+a - \frac{2a}{P} \right)^{2s} - \frac{2a}{P^2} \left( n+a - \frac{2a}{P} \right)^{2s+1} \right) \quad (4.2.24)$$

with  $P = 1 + \frac{1}{2}\zeta\bar{\zeta}$ , which are non-zero if one of the parameters  $a$  or  $n$  is present. We find for example

$$Q_{r(0)} = -8\pi [a(n+iM) + 3n(n-iM)]. \quad (4.2.25)$$

Expressions (4.2.23) and (4.2.25), found from a boundary analysis, are in line with the results obtained in Refs. [166,173–175] (see also [177], where the electric part of  $Q_{r(0)}$  is given) using standard methods restricted to bulk dynamics. They provide conserved moments since the divergences (4.2.10b) and (4.2.16b) vanish.

**Remark** There is another simple charge that we could have built for Kerr–Taub–NUT spacetimes.

However before presenting this construction we need to complete the three-dimensional Carrollian analysis in holomorphic coordinates presented in Sec. 1.6. The Weyl–Ricci tensor can be decomposed as

$$\hat{\mathcal{R}}_{ij} = \hat{s}_{ij} + \hat{\mathcal{K}} a_{ij} + \hat{\mathcal{A}} \eta_{ij}, \quad (4.2.26)$$

where we have introduced two weight-2 Weyl-covariant scalar Gauss–Carroll curvatures

$$\hat{\mathcal{K}} = \frac{1}{2} a^{ij} \hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2} \eta^{ij} \hat{\mathcal{R}}_{ij} = \hat{A} - *\varphi, \quad (4.2.27)$$

and  $*\varphi = \frac{1}{2}\eta^{ij}\varphi_{ij}$ . Recall that  $\hat{\eta}_{ij}$  is defined above (1.6.3). These obey Carroll-Bianchi identities

$$\frac{2}{\Omega}\hat{\mathcal{D}}_t*\hat{\omega} + \hat{\mathcal{A}} = 0, \quad (4.2.28a)$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{K}} - a^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j - \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j\xi^{ij} = 0, \quad (4.2.28b)$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{A}} + \eta^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j = 0. \quad (4.2.28c)$$

The identities (4.2.28b) and (4.2.28c) resemble Carrollian divergence of a current  $(\kappa, K^i)$  with couples  $\{\hat{\mathcal{K}}, -\hat{\mathcal{R}}^i - \hat{\mathcal{D}}_j\xi^{ij}\}$  and  $\{\hat{\mathcal{A}}, -*\hat{\mathcal{R}}^i\}$  of weights (2, 3). This allows to define *electric and magnetic curvature charges* as in Eqs. (1.5.17)

$$Q_{\text{ec}} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \hat{\mathcal{K}} - b_i \left( \hat{\mathcal{R}}^i + \hat{\mathcal{D}}_j \xi^{ij} \right) \right), \quad Q_{\text{mc}} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \hat{\mathcal{A}} - b_i * \hat{\mathcal{R}}^i \right). \quad (4.2.29)$$

Following (1.5.18), we find

$$\frac{dQ_{\text{ec}}}{dt} = \int_{\partial\mathcal{S}} * \left( \hat{\mathcal{R}} + \hat{\mathcal{D}} \cdot \boldsymbol{\xi} \right) \Omega, \quad \frac{dQ_{\text{mc}}}{dt} = - \int_{\partial\mathcal{S}} \hat{\mathcal{R}} \Omega. \quad (4.2.30)$$

Upon regular behaviour, the boundary terms vanish and the curvature charges are both conserved. For a Kerr-Taub-NUT spacetime the electric curvature charge reads<sup>5</sup>

$$Q_{\text{ec}} = \int_S \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} K. \quad (4.2.31)$$

When divided by the volume of  $\mathcal{S}$ , this is simply the average Gauss curvature. Note in passing that the charges defined here are extensive, hence the integrals may reveal convergence issues, in particular when  $\mathcal{S}$  is non-compact. Normalizing by  $\text{Vol} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$  is the simplest way to fix this divergence. Alternatively,  $\mathcal{S}$  could be compactified – quotiented by a discrete isometry group.

This concludes our study of algebraically special spacetimes in CNU gauge. It should be emphasised that the boundary analysis of bulk gravitational charges is also in its early stages. What we found is that bulk isometries are supported on the boundary by strong Carrollian Killing fields. Given such a boundary Killing vector, one can built two infinite towers of charges contracting the latter with pieces of the bulk energy-momentum and Cotton tensor and their conservation properties originate from those of the bulk tensor. A precise contact with charges discovered and discussed from a purely bulk perspective in [150–154, 163, 178] should be made. For that a precise translation of our findings in Newman-Penrose formalism has to be established.

It is now time to leave asymptotic symmetries by considering the case of hidden symmetries of

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<sup>5</sup>Remember that here  $\xi_{ij} = 0$ , and the geometry is  $t$ -independent with vanishing  $\theta, \varphi_i, \hat{\mathcal{A}}, \hat{\mathcal{R}}_i$  as well as  $X_{ij}$ . Also  $\hat{\mathcal{K}} = K$ .

gravity and their action on the Carrollian boundary.

## Chapter 5

# Hidden symmetries of gravity on the Carrollian boundary

Hidden symmetries of gravity usually arise after a dimensional reduction of the theory. The story started in the late fifties with the seminal work by Ehlers [60] in the context of four-dimensional Ricci-flat Einstein gravity. Reducing such a spacetime to three dimensions, along the orbits of a Killing field, which should be either timelike or spacelike, reveals a hidden  $SL(2, \mathbb{R})$  symmetry acting on the reduced Einstein's equations. With the advent of supergravity, a deeper and wider understanding of dimensional reduction was reached, which disclosed a broader class of hidden-symmetry groups, among them the exceptional ones (see e.g. [179–182] and more recently [183]). Hidden symmetries can be used to design solution-generating techniques. In particular, following Geroch [184], Ehlers symmetry can be used to generate a new Ricci-flat spacetime from a given one, the new one possessing the same Killing than the one used for the reduction. In general this involves a non-local transformation in the bulk (in the sense that the metric of the two spacetimes are not related by an algebraic transformation). This observation was soon generalized to more complicated situations like e.g. with two commuting Killings allowing for a two-dimensional reduction [185–187], each time with new and bigger hidden symmetry groups. This paved the way towards the analysis of noticeable and somehow unexpected integrability properties of subsets of solutions of Einstein gravity [188–192] (see in particular [193–195] for a comprehensive discussion of the stationary, axially symmetric case).

A legitimate set of questions one may ask when casting a solution to vacuum Einstein's equations from a Carrollian boundary perspective is: what is the action of hidden symmetries on the Carrollian boundary data and how do the corresponding charges behave under this action? This is the central question we treated in [66] and which we want to present in this Chapter. In order to acquire a clear view of the physical phenomenon while keeping the technical level reasonable, we have restricted our analysis to the case of Ricci-flat metrics (4.1.2), hence algebraically special spacetimes, studied in Chapter 4, for which the computations are tractable.

Sec. 5.1 is first devoted to the presentation of Ehlers symmetry from the bulk perspective, with

an explicit example, before showing next, in Sec. 5.2 what happens to the Carrollian boundary data. In Sec. 5.3 we make the link with the charge analysis of Chapter 4 as we present the action of Ehlers' symmetry on the latter. Our analysis is centered on three-dimensional Carrollian structures which we shall describe using holomorphic coordinates. We therefore remind the reader that Sec. 1.6 contains all useful results.

## 5.1 Ehlers' hidden symmetry and Geroch's method

In this Section we review the hidden symmetry exhibited by Ehlers in [60] and further studied by Geroch in [184, 185], where he designed a new method to generate solutions of Ricci-flat gravity. We also give an explicit example to highlight its power.

### 5.1.1 The solution-generating technique

Let  $(\mathcal{M}, g, \xi)$  be a 4-dimensional manifold endowed with a Lorentzian metric  $g$  having a Killing field  $\xi$  everywhere either spacelike or timelike. Along the Section we denote by capital Latin letters  $M, N, \dots \in 0, 1, 2, 3$  the bulk indices. As we will go in the timelike case in Sec. 5.2 this will be our framework from now on. We define the norm and the twist of  $\xi$  as

$$\lambda = \xi^M \xi_M \tag{5.1.1a}$$

$$\omega_M = \sqrt{-g} \epsilon_{MNPQ} \xi^N \nabla^P \xi^Q. \tag{5.1.1b}$$

We further assume this spacetime to be Ricci flat i.e.  $R_{MN} = 0$ . Thus one sees that the twist is closed thus locally exact i.e. there exist a scalar  $\omega$  such that

$$\omega = d\omega. \tag{5.1.2}$$

One can construct a 3-dimensional space by quotienting  $\mathcal{M}$  by  $\text{Orb}(\xi)$  the orbits of the Killing field

$$\mathcal{S} = \mathcal{M} / \text{Orb}(\xi). \tag{5.1.3}$$

**Remark** In the case where  $\xi$  is orthogonal to an hypersurface  $\Sigma$ ,  $\mathcal{S} = \Sigma$  i.e. a 3-dimensional manifold where by each point, one and only one orbit of  $\xi$  passes through. In general,  $\xi$  is not hypersurface orthogonal, thus endowing  $\mathcal{S}$  with a metric and a covariant derivative requires more work.

Following Geroch, we define  $h$  the metric on  $\mathcal{S}$  by

$$h_{MN} = g_{MN} - \frac{\xi_M \xi_N}{\lambda}. \tag{5.1.4}$$

Then one can state a one to one correspondence between tensors in  $\mathcal{S}$  and  $\xi$ -invariant and transverse tensors in  $\mathcal{M}$  namely tensors satisfying  $\iota_\xi T = 0$  and  $\mathcal{L}_\xi T = 0$  (see the appendix of [184] for more details), from which one defines the  $\mathcal{S}$ -covariant derivative

$$\mathcal{D}_Q T_{M_1 \dots M_p}^{N_1 \dots N_q} = h_Q^L h_{M_1}^{Q_1} \dots h_{M_p}^{Q_p} h_{R_1}^{N_1} \dots h_{R_q}^{N_q} \nabla_L T_{Q_1 \dots Q_p}^{R_1 \dots R_q} \quad (5.1.5)$$

with  $\nabla_M$  the  $\mathcal{M}$ -Levi-Civita connection. It is a fact that  $\mathcal{D}_M$  coincides with the unique Levi-Civita covariant derivative one can construct on  $\mathcal{S}$ . This sets a relationship<sup>1</sup> between the Riemann tensor on  $\mathcal{M}$  and the one on  $\mathcal{S}$

$$\mathcal{R}_{MNPQ} = h_{[M}^S h_{N]}^T h_{[P}^U h_{Q]}^V \left( R_{STUV} + \frac{2}{\lambda} (\nabla_S \xi_T \nabla_U \xi_V + \nabla_S \xi_U \nabla_T \xi_V) \right). \quad (5.1.6)$$

Using (5.1.5) and (5.1.6), Einstein's equations on  $\mathcal{M}$  can be recast in terms of  $(h, \omega, \lambda)$  seen as fields on  $\mathcal{S}$

$$\tilde{\mathcal{R}}_{MN} = -2(\tau - \bar{\tau})^{-1} \tilde{\mathcal{D}}_{(M} \tau \tilde{\mathcal{D}}_{N)} \bar{\tau} \quad (5.1.7a)$$

$$\tilde{\mathcal{D}}^2 \tau = 2(\tau - \bar{\tau})^{-1} \tilde{\mathcal{D}}_M \tau \tilde{\mathcal{D}}_N \bar{\tau} \tilde{h}^{MN} \quad (5.1.7b)$$

where

$$\tau = \omega + i\lambda. \quad (5.1.8)$$

$\tilde{\mathcal{D}}_M$  and  $\tilde{\mathcal{R}}_{MN}$  are the Levi-Civita covariant derivative and Ricci tensor with respect to  $\tilde{h}_{AB} = \lambda h_{AB}$ . Eqs. (5.1.7) are the fundamental tools to build new solutions from a given one. They are telling us when a new dataset  $(h'_{AB}(h_{AB}, \lambda, \omega), \lambda'(\lambda, \omega), \omega'(\lambda, \omega))$  can be lifted to a new four-dimensional Ricci-flat manifold  $\mathcal{M}'$ .

The last step is to go back to four dimensions and build the new spacetime. We can define a new four dimensional metric  $g'$  on  $\mathcal{M}$  with Killing vector  $\xi' = \eta \lambda'$  (normalized such that  $\iota_\xi \eta = 1$ ) as

$$g'_{MN} = h'_{MN} + \frac{\xi'_M \xi'_N}{\lambda'} \quad (5.1.9)$$

Here we have defined  $\eta$  using the fact that for a curl-free skew-symmetric field  $F'$  on  $S$ , the pull back  $F_{MN}$  on  $\mathcal{M}$  is closed, i.e.,

$$F'_{MN} := \frac{1}{(-\lambda')^{3/2}} \sqrt{-g} \varepsilon'_{MNP} \mathcal{D}^P \omega' = d\eta', \quad (5.1.10)$$

with  $\varepsilon'_{MNP}$  the Levi-Civita tensor on  $\mathcal{S}$ . On top of this method to generate solutions of Ricci-flat gravity, Ehlers noticed an additional symmetry which forms its homonymous group. The process of going back to four dimensions given a three dimensional metric  $h_{MN}$  on  $\mathcal{S}$  is dubbed an *oxidation*.

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<sup>1</sup>This is a generalisation of the Gauss-Codazzi equation.



## The Ehlers group

This group is nothing else than the group of transformations preserving the fundamental set of equations in  $\mathcal{S}$  (5.1.7). The form of the latter suggest to consider

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \quad (5.1.11)$$

where the real numbers  $\alpha, \beta, \gamma$  and  $\delta$  satisfy  $\alpha\delta - \beta\gamma = 1$ . One gets a transformed triplet in terms of the old one which is still solution of these equations. This new triplet reads

$$h'_{MN} = [(\omega\gamma + \delta)^2 + \gamma^2\lambda^2]h_{MN} \quad (5.1.12a)$$

$$\lambda' = \frac{\lambda}{(\omega\gamma + \delta)^2 + \gamma^2} \quad (5.1.12b)$$

$$\omega' = \frac{(\omega\alpha + \beta)(\omega\gamma + \delta) + \alpha\gamma\lambda^2}{(\omega\gamma + \delta)^2 + \gamma^2}. \quad (5.1.12c)$$

Hence, the procedure described above exhibits a hidden  $SL(2, \mathbb{R})$  symmetry in four-dimensional Ricci-flat spacetimes. Given a Ricci-flat solution of vacuum Einstein's equations, the other ones are obtained via an  $SL(2, \mathbb{R})$  transformation of  $(\lambda, \omega)$ . The  $SL(2, \mathbb{R})$  is hidden in the four-dimensional perspective, but explicit in the three-dimensional model, materialized here in Eqs. (5.1.7).

**Remark** The Ehlers transformations can actually be divided into two class. Part of the group is in fact visible in four dimensions because it acts as four-dimensional diffeomorphisms; part is creating genuinely different Ricci-flat solutions. This can be illustrated in the concrete example of Schwarzschild–Taub–NUT solutions with mass  $M$  and nut charge  $n$  that we will show below. The compact subgroup of rotations  $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SO(2) \subset SL(2, \mathbb{R})$  induces rotations of angle  $2\chi$  in the parameter space  $(M, n)$ , while non-compact transformations  $\begin{pmatrix} \alpha & \beta \\ 0 & \frac{1}{\alpha} \end{pmatrix} \in N \subset SL(2, \mathbb{R})$  act homothetically,  $(M, n) \rightarrow \left(\frac{M}{\alpha}, \frac{n}{\alpha}\right)$ . As a conclusion, among all the allowed transformations in the space  $\mathcal{S}$  only a sub part of it are relevant. Giving rise to a pure gauge or an actual transformation depends on the Killing vector chosen to quotient the manifold  $\mathcal{M}$  (when several choices are possible).

### 5.1.2 From Schwarzschild to Taub-NUT

To illustrate the power of this method, we propose to work out the calculations showing how the Taub-NUT solution is obtained from a Schwarzschild spacetime. We recall that in coordinates  $(t, r, \theta, \phi)$  the Taub-NUT metric reads

$$ds^2 = -f(r)(dt^2 + 2n(1 - \cos\theta)d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2) \quad (5.1.13)$$

with

$$f(r) = \frac{1}{r^2 + n^2} [r^2 - n^2 - 2mr + k^2(r^4 + 6n^2r^2 - 3n^4)] \quad (5.1.14)$$

where  $k = 0, +1, -1$  corresponds to flat, dS and AdS cases and  $m$  and  $n$  are respectively the mass and NUT parameter.

We start our analysis with the usual form of Schwarzschild's metric in  $(t, r, \theta, \phi)$  coordinates (with  $r_s = 2m$ )

$$ds_{\text{Schwarzschild}}^2 = - \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (5.1.15)$$

which possesses four Killing fields:  $\partial_t$  and an  $\mathfrak{so}(3)$  algebra for the spherical symmetry. Here we choose  $\xi$  to be the timelike Killing vector  $\partial_t$ . We use equation (5.1.1a) to find the norm and scalar twist of this Killing field as

$$\lambda = g_{tt} = - \left(1 - \frac{r_s}{r}\right), \quad \omega = 0. \quad (5.1.16)$$

Using (5.1.4), the metric on  $\mathcal{S}$  is given by

$$ds_{\mathcal{S}}^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (5.1.17)$$

Now we have the triplet  $(h, \lambda, \omega)$  at hand. The form of (5.1.7) suggests that we can take  $\tilde{h}' (= \lambda' h')$  to be equal to  $\tilde{h} (= \lambda h)$ . Then, we perform an  $SO(2) \subset SL(2, \mathbb{R})$  transformation on  $\tau = \omega + i\lambda$  parametrized by an angle  $\chi^2$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix}. \quad (5.1.18)$$

We get

$$\begin{aligned} \lambda' &= \frac{\lambda}{\cos^2 \chi + \lambda^2 \sin^2 \chi} \\ \omega' &= \frac{\sin \chi \cos \chi (1 - \lambda^2)}{\cos^2 \chi + \lambda^2 \sin^2 \chi} \\ h'_{MN} &= (\cos^2 \chi + \lambda^2 \sin^2 \chi) h_{MN}. \end{aligned} \quad (5.1.19)$$

Finally using equation (5.1.10), we get that the field  $\eta$ , necessary to reconstruct the transformed four-dimensional metric, is such that

$$\nabla_{[M} \eta_{N]} = \frac{1}{(-\lambda)^{3/2}} \tilde{h}^{PQ} \epsilon_{MNP} \partial_Q \omega', \quad (5.1.20)$$

where  $\epsilon_{abc} = \frac{1}{\sqrt{-\lambda}} \epsilon_{abcd} \xi^d$ . This equation can be solved component-wise to give,

$$\nabla_{[r} \eta_{\theta]} = 0, \quad \nabla_{[r} \eta_{\phi]} = - \frac{1}{(-\lambda)^{3/2}} h'^{\theta\theta} \partial_{\theta} \omega', \quad \nabla_{[\theta} \eta_{\phi]} = \frac{1}{(-\lambda)^{3/2}} h'^{rr} \partial_r \omega' \quad (5.1.21)$$

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<sup>2</sup>We are then making what was called an ‘‘actual’’ transformation.

The latter  $\eta$  is in  $\Gamma(T\mathcal{S})$  so now we promote it to a vector field on  $\mathcal{M}'$  the new four dimensional manifold using the the normalization condition  $\iota_\xi \eta = 1$ . An ansatz for  $\eta$  is then,

$$\eta_M = (1, 0, 0, a(\theta)) \quad (5.1.22)$$

and  $F$  is skew symmetric and substituting it in (5.1.21), we get

$$a(\theta) = -2r_s \cos \theta \sin 2\chi. \quad (5.1.23)$$

Hence the new metric  $g'$  of equation (5.1.9) from Schwarzschild geometry is

$$\begin{aligned} ds'^2 &= \lambda' dt^2 - \frac{1}{\lambda'} dr^2 + \frac{\lambda}{\lambda'} r^2 d\theta^2 + \lambda' \left( 2adt + \left( \frac{\lambda}{\lambda'^2} r^2 \sin^2 \theta + a^2 \right) d\phi \right) d\phi \\ &= \lambda' (dt + ad\phi)^2 - \frac{1}{\lambda'} dr^2 + \frac{\lambda}{\lambda'} r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \quad (5.1.24)$$

Clearly if we choose

$$\lambda' = -f(r), \quad \sin 2\chi = \frac{n}{r_s} \quad (5.1.25)$$

then (5.1.24) is a Taub-NUT metric (5.1.13). A crucial point to note here is that not all Killing vectors lead to a new geometry. Indeed, had we chosen the spacelike Killing vector like  $\partial_\phi$  instead, we would have obtained a metric that is diffeomorphic to the Schwarzschild metric. This would have been a “pure gauge” transformation. Note also that all Killings of the spherical  $\mathfrak{so}(3)$  algebra would have led to pure gauge transformations.

**Remark** Parametrizing the  $SO(2)$  transformation with an angle  $\chi$ ,  $g'_{\mu\nu}(\chi)$  obtained after transformation is a solution of vacuum Einstein’s equations. All these metrics define a one-parameter family of solutions.

**Remark** It was shown in [185] that when  $g_{\mu\nu}$  has two commuting Killings  $\xi^\mu$  and  $\psi^\mu$ , after applying this algorithm with  $\xi^\mu$ ,  $\psi^\mu$  remains a Killing of the new metric. Thus, one can reapply the algorithm with  $\psi^\mu$  to generate a two-parameter family of solutions  $g'_{\mu\nu}(\chi, \chi')$  from the original one. Given the property of conversing the original Killings, one may wonder what happens if one applies once again the algorithm to  $g'_{\mu\nu}$ . In general, according to Geroch, new solutions are reached, thus we create a three-parameter family of solution, and so on and so forth such that all asymptotically flat, stationary, axisymmetric vacuum solutions of Einstein’s equations can be generated from Minkowski spacetime via Geroch’s algorithm (see [185] for more details).

## 5.2 Action of the Ehlers transformation on the Carrollian boundary

In this Section we unravel the action of the Ehlers group (5.1.11) on the boundary Carrollian observables, using the expression of the resummed bulk Ricci-flat metric (4.1.2) assumed to possess a time-like Killing vector field,  $\xi = \partial_t$  and  $\Omega = 1$ .

In order to proceed, we follow the steps for the Geroch reduction described in Sec. 5.1, i.e. determine  $\tau$  as defined in (5.1.8) for the metric (4.1.2) with  $\lambda$  and  $\omega$  given in (5.1.1a) and (5.1.2). These should be expanded in inverse powers of  $r$  and thus deliver the boundary ingredients together with their transformations following (5.1.11).

**Remark** One has to be careful when performing the Geroch reduction. Indeed as we want to study its action on the Carrollian boundary data, we should identify quantities before and after the transformation which defines the novel Ricci-flat solution. Nothing guarantees that the new metric will assume again the resummed form (4.1.2) in CNU gauge. Actually we will see that it doesn't and a redefinition of the radial coordinate is necessary to bring it back into the original gauge, before the identification can be safely made.

It is convenient for the present task to adopt the Cartan frame defined in (4.2.1), leading to the bulk metric components

$$\begin{aligned} g_{it} &= \frac{1}{\rho^2} (8\pi G\epsilon r + *\omega c) - K, & g_{ti} &= *\partial_i *\omega, & g_{tr} &= -1, \\ g_{ri} &= 0, & g_{rr} &= 0, & g_{ij} &= \rho^2 a_{ij} \end{aligned} \quad (5.2.1)$$

obtained using (4.2.2), assuming  $t$ -independence and  $\Omega = 1$ . In this expression  $*\omega$ ,  $K$  and  $c$  are given in Eqs. (4.1.20a), (4.1.20b) and (4.1.20c). The Killing form reads:

$$\xi = - \left( K - \frac{1}{\rho^2} (8\pi G\epsilon r + *\omega c) \right) \tau + *\partial_i *\omega dx^i - dr, \quad (5.2.2)$$

with norm

$$\lambda = \frac{8\pi G\epsilon r + *\omega c}{\rho^2} - K. \quad (5.2.3)$$

For the twist potential we get

$$\omega = \frac{8\pi G\epsilon *\omega - cr}{\rho^2} + K^*, \quad (5.2.4)$$

with  $K^*$  introduced in (4.1.25). On-shellness is implemented here through boundary dynamics as summarized in Sec. 4.1.2, i.e. in Eqs. (4.1.23b), (4.1.24) and (4.1.25). Inserting the above results into Eqs. (5.1.8) and using (4.1.5a), we find

$$\tau = \frac{\hat{\tau}}{r + i *\omega} - i \hat{k}. \quad (5.2.5)$$

The Geroch reduced and rescaled metric (2.1.5) is

$$\tilde{h}_{AB}dx^A dx^B = - \left( dr - * \partial_k * \bar{\omega} dx^k \right)^2 + \lambda \rho^2 a_{ij} dx^i dx^j. \quad (5.2.6)$$

With this,  $\tau$  given in (5.2.5) unsurprisingly solves the reduced Einstein's equations (5.1.7).

The Ehlers transformation rules are (5.1.11) and the invariance of  $\tilde{h}_{AB}$ . From these follows the rest of the construction, i.e. the transformation of  $h_{AB}$  and the oxidation toward  $g'_{AB}$ . Using for convenience holomorphic and antiholomorphic coordinates as introduced in Sec. 1.6, expression (5.2.6) is recast as follows

$$\tilde{h}_{AB}dx^A dx^B = - \left( dr - i \partial_\zeta * \bar{\omega} d\zeta + i \partial_{\bar{\zeta}} * \bar{\omega} d\bar{\zeta} \right)^2 + \frac{(\tau - \bar{\tau})(r + i * \bar{\omega})(r - i * \bar{\omega})}{i P^2} d\zeta d\bar{\zeta}. \quad (5.2.7)$$

Combining (5.1.11) with (5.2.5), we obtain the following boundary transformations

$$\hat{\tau}' = - \frac{\hat{\tau}}{(\gamma \hat{k} + i \delta)^2}, \quad (5.2.8a)$$

$$\hat{k}' = i \frac{\alpha \hat{k} + i \beta}{\gamma \hat{k} + i \delta}, \quad (5.2.8b)$$

$$\hat{\omega}' = \hat{\omega} + \frac{\gamma \hat{\tau}}{\gamma \hat{k} + i \delta} \quad (5.2.8c)$$

and

$$P' = \frac{P}{|\gamma \hat{k} + i \delta|}, \quad (5.2.9)$$

as well as the radial shift<sup>3</sup>

$$r' = r + \frac{i}{2} \left( \frac{\gamma \hat{\tau}}{\gamma \hat{k} + i \delta} - \frac{\gamma \hat{\tau}}{\gamma \hat{k} - i \delta} \right). \quad (5.2.10)$$

These transformation rules leave indeed (5.2.7) invariant. As advertised earlier, they are *local*, providing a direct transformation (5.2.9) of the boundary metric. The transformation of the energy density  $\varepsilon_{(0)}$  is obtained from (5.2.8a) using (4.1.5a)

$$8\pi G \varepsilon'_{(0)} = \frac{8\pi G \varepsilon_{(0)} \left( (\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) - 2c_{(0)} \gamma K (\gamma K^* + \delta)}{\left( \gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (5.2.11)$$

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<sup>3</sup>One could alternatively adopt a new radial coordinate defined as  $\tilde{r} = r + \bar{\omega}$  that is invariant under Möbius transformations. This is actually mandatory in order to reach boundary  $SL(2, \mathbb{R})$ -covariant tensors from the bulk, as we will discuss in Sec. 5.3. It furthermore coincides with the radial coordinate of Ref. [110] §29 provided  $r_0 = -\bar{\omega}$ .

The transformation of  $c_{(0)}$  is inferred similarly

$$c'_{(0)} = \frac{c_{(0)} \left( (\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) + 16\pi G \varepsilon_{(0)} \gamma K (\gamma K^* + \delta)}{\left( \gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (5.2.12)$$

All these rules are compatible with the relations Eqs. (4.1.2ob) and (4.1.2oc). Finally the transformations of the Carrollian Cotton descendents  $\chi_\zeta$  and  $\Psi_{\zeta\bar{\zeta}}$  are reached using the above results combined with Eqs. (4.1.2od), (4.1.2oe) and (4.1.2of). The transformation of the Ehresmann connection is obtained directly from the expressions reached for the latter in (4.1.3oa), (4.1.3ob) (4.1.32a) and (4.1.3ob). To this end, observe that in the constant- $\hat{k}$  instance,  $A$ ,  $B$ ,  $\bar{B}$  and  $D$  transform with a factor  $\frac{1}{|\gamma\hat{k}+i\delta|}$  in order to comply with (5.2.9). Similarly,  $f(\zeta)$  and  $Z(\zeta)$ , introduced in (4.1.29) and (4.1.31), must be respectively invariant and transforming as

$$Z'(\zeta) = Z(\zeta) + i \frac{\gamma \hat{\tau}(\zeta)}{\gamma \hat{k} + i\delta}, \quad (5.2.13)$$

so that (5.2.8c) be fulfilled.

Going back to Geroch's solution-generating method, once the  $SL(2, \mathbb{R})$  transformation is performed on the boundary, the reconstruction of the new Ricci-flat solution is straightforward using the boundary-to-bulk formula (4.1.2), expressed with primed data – except for the unaltered boundary coordinates  $\{t, \zeta, \bar{\zeta}\}$ . This is equivalent to the oxidation procedure operated from three to four dimensions along the lines of Eqs. (5.1.10), (5.1.9) with

$$\eta' = \tau' - \frac{1}{\lambda'} (dr' - i\partial_\zeta * \bar{\omega}' d\zeta + i\partial_{\bar{\zeta}} * \bar{\omega}' d\bar{\zeta}), \quad \tau' = dt - b'_\zeta d\zeta - b'_{\bar{\zeta}} d\bar{\zeta}, \quad (5.2.14)$$

finally leading to (5.1.9), which assumes the form (4.1.2) primed. The new bulk Killing vector  $\xi' = \lambda' \eta'$  is again  $\partial_t$ .

**Example.** Consider again the Kerr–Taub–NUT family treated at the end of Sec. 4.1.2 with  $P = \frac{1}{2}\zeta\bar{\zeta} + 1$ ,  $K = 1$  and  $K^* = 0$  (this was not explicitly required). These choices are stable only under  $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SL(2, \mathbb{R})$ . For this transformation, using (5.2.8a) we find  $M' + in' = (M + in)e^{-2i\chi}$  hence a rotation in the plane generated by the mass and magnetic-mass aspects. From a solution containing just a mass, Ehlers' transformations generate a NUT aspect. This is the boundary explanation of how Taub-NUT emerges from Schwarzschild. Note that computations are much easier on the boundary contrary to what they were in the bulk.

We conclude repeating that the main message of our study in the simple set-up of stationary spacetimes is thus that the non-local action of the Ehlers transformation in the bulk translates into a local action on the Carrollian boundary.

### 5.3 Action on boundary charges

Carrollian charges have been introduced in Sec. 4.2 and further discussed for stationary and algebraic spacetimes. Two generic charges were found and computed:  $Q_{\text{ec}}$  and  $Q_{\text{m}}$ . The former is purely geometric and stands for the integrated curvature  $K$  of the celestial sphere, see (4.2.31); the latter carries genuine dynamical information captured in the electric and magnetic masses (recall that  $\varepsilon_{(0)}$  and  $c_{(0)}$  satisfies flux/balance laws (4.1.3a) and (4.1.3b)). Actually  $Q_{\text{ec}}$  is invariant under Ehlers'  $SL(2, \mathbb{R})$  as it can be inferred from (5.2.8b) and (5.2.8c). The mass charge  $Q_{\text{m}}$  is not but its transformation (see (5.2.8a), (5.2.8b) and (5.2.8c)) suggests that it might belong to some  $SL(2, \mathbb{R})$  multiplet or, more accurately, that it may be modified to this end. Indeed, a slight amendment to the charge  $Q_{\text{m}}$ , namely

$$Q'_{\text{m}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + 2\hat{\omega}K), \quad (5.3.1)$$

is  $SL(2, \mathbb{R})$ -invariant.<sup>4</sup>

It is then legitimate to wonder whether one can make these charges fit into  $SL(2, \mathbb{R})$  multiplets. Actually in [184], Geroch obtained an  $SL(2, \mathbb{R})$  triplet of  $\mathcal{M}$  two-forms by oxidizing the following two-form triplet on  $\mathcal{S}$

$$V_1 = \frac{1}{(\tau - \bar{\tau})^2} \star_{\tilde{h}}^3 (d\tau + d\bar{\tau}), \quad V_2 = \frac{1}{(\tau - \bar{\tau})^2} \star_{\tilde{h}}^3 (\bar{\tau}d\tau + \tau d\bar{\tau}), \quad V_3 = \frac{1}{(\tau - \bar{\tau})^2} \star_{\tilde{h}}^3 (\bar{\tau}^2 d\tau + \tau^2 d\bar{\tau}), \quad (5.3.2)$$

where “ $\star_{\tilde{h}}^3$ ” stands for the three-dimensional Hodge-dual on  $\mathcal{S}$  equipped with  $\tilde{h}_{AB}$  displayed in (5.2.7). Upon integration over the celestial sphere of  $\mathcal{M}$ , (5.3.2) leads to surface charges. One may wonder whether one can find this triplet from a Carrollian boundary perspective.

In general terms, in order to build a  $SL(2, \mathbb{R})$  two-form triplet from Carrollian considerations, one needs a Carrollian two-form  $\mathbf{v}$  transforming under  $SL(2, \mathbb{R})$  as

$$\mathbf{v} \rightarrow \mathbf{v}' = -\mathbf{v} \left( \gamma \hat{k} - i\delta \right)^2, \quad (5.3.3)$$

from which we construct the triplet, a symmetric rank-two tensor, transforming as

$$\begin{pmatrix} \mathbf{v}'_3 & \mathbf{v}'_2 \\ \mathbf{v}'_2 & \mathbf{v}'_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad (5.3.4)$$

where

$$\mathbf{v}_1 = \mathbf{v}, \quad \mathbf{v}_2 = i\hat{k}\mathbf{v}, \quad \mathbf{v}_3 = -\hat{k}^2\mathbf{v}. \quad (5.3.5)$$

The same holds for the complex-conjugate triplet  $\bar{\mathbf{v}}_1 = \bar{\mathbf{v}}$ ,  $\bar{\mathbf{v}}_2 = -i\hat{k}\bar{\mathbf{v}}$  and  $\bar{\mathbf{v}}_3 = -\hat{k}^2\bar{\mathbf{v}}$ . An  $SL(2, \mathbb{R})$  triplet of charges is thus reached as

$$Q_I = \int_S \mathbf{v}_I, \quad I = 1, 2, 3, \quad (5.3.6)$$

<sup>4</sup>There is a factor of 2 of difference with respect to (4.2.21) (in front of  $\hat{\omega}$ ).

and  $Q \equiv Q_1 Q_3 - Q_2^2$  is invariant under Ehlers' transformations, from properties of the determinant.

The above general procedure can be applied in the explicit case of stationary resumable spacetimes. We can exhibit two two-forms satisfying the properties displayed in Eq. (5.3.3), namely

$$\mathbf{x} = -\frac{\hat{\tau}}{2(\hat{k} + \hat{k})} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}, \quad \mathbf{y} = -\left(\frac{P}{\hat{k} + \hat{k}}\right)^2 \partial_\zeta \hat{k} \partial_{\bar{\zeta}} \hat{\omega} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}. \quad (5.3.7)$$

These lead along (5.3.6) to two triplets of charges.

What is remarkable is that the asymptotic limit of Geroch's two-form triplet (5.3.2) coincides with that designed earlier from Carrollian boundary considerations. This statement is captured in the following result

$$\lim_{\tilde{r} \rightarrow \infty} \begin{pmatrix} V_3 & V_2 \\ V_2 & V_1 \end{pmatrix} = \begin{pmatrix} -\hat{k}^2(\mathbf{x} + \mathbf{y}) - \hat{k}^2(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) \\ i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & \mathbf{x} + \mathbf{y} + \bar{\mathbf{x}} + \bar{\mathbf{y}} \end{pmatrix}, \quad (5.3.8)$$

where  $\tilde{r} = r + \hat{\omega}$  was introduced in footnote 3 as an  $SL(2, \mathbb{R})$ -invariant radial coordinate, which must be used here in order to guarantee that the limit preserves the  $SL(2, \mathbb{R})$  behaviour. This results sets again a relationship amongst the charges introduced in Sec. 4.2 using purely boundary methods and those computed directly by standard bulk techniques.

## 5.4 Outlook and discussion

Our main motivation in this Chapter was to develop the necessary tools for exhibiting the action of the Ehlers' group on the three-dimensional Carrollian boundary. We have restricted our analysis to stationary, algebraically special spacetimes, whose timelike isometry is aligned with the fiber of the Carrollian boundary. As a consequence the bulk reduction translates on the boundary into a reduction along the fibers which bring us to the base space of the Carroll structure, therefore on a genuine Riemannian manifold. The Carrollian metric reduces to the one of the base space and the field of observers becomes an irrelevant direction. The last important point finally is that after performing an Ehlers transformation, the bulk line element stays in CNU gauge, up to a radial shift.

Extending the previous analysis to more general situations requires an utmost care. Other choices of Killing fields for the reduction make the Ehlers transformation blur the algebraically special character of the spacetime at hand, forbidding the identification of quantities before and after its action. Moreover, performing the bulk reduction along an isometry which is not  $\partial_t$  requires to understand Carrollian reductions along a direction distinct from the fiber. This is an uncharted territory of mathematics, which is relevant to study in its own right, before delving into its application to the topic at hand.

Finally, a legitimate question would be to repeat the analysis of hidden symmetries but in AdS spacetimes. It should therefore be emphasized that hidden symmetries *à la* Ehlers are generally



absent (or restricted) in AdS [146,147] and the issue of their boundary realization could only be raised in the framework we have elaborated in this thesis (with the CNU gauge), allowing to reconstruct Ricci-flat spacetimes from null boundary dynamics. Furthermore, as opposed to its action on the bulk metric, the boundary action of the hidden symmetry is local – the bulk non-locality is rooted to the radial coordinate.

## **Part III**

### **Some aspects of thermal field theory**



## Chapter 6

# Thermal correlators, twisted partition functions and fishnet graphs

Symmetries are, like in asymptotically flat gravity, one of the most powerful tools to handle and get a better control on field theories, as they impose a set of consistency relations operators and correlation functions have to satisfy. The paragon is conformal invariance i.e. invariance under transformations that preserves the angles. It uniquely fixes (up to an overall multiplicative constant) the form of the two and three-points functions, which in virtue of Wick's theorem applied to Gaussian theories, completely determine the shape of all possible correlators. One then talks about a Conformal Field Theory (CFT) [196]. CFTs are ubiquitous in Nature, as they are used to describe second order phase transitions and corresponds to IR and UV fixed points of Quantum Field Theories (QFT) thanks to scale invariance. In a sense there exist only CFTs, as any QFT can just be seen as a deformation of a CFT by a set of relevant operators. First studied in two-dimensions where the conformal group is infinite dimensional, the effect of conformal invariance on field theories in  $d > 2$  was comprehensively analysed in [197]. What matters when defining a CFT is no longer the correlation functions, which are already fixed, but however the spectrum of operators together with their scaling dimension (their eigenvalue under the action of the dilation operator  $D$ ) and the multiplicative factor of their three-point functions. Hence these multiplicative factors are of primary importance in describing the CFT at hand and one should have the easiest possible way to compute them.

Euclidean CFTs are basically defined on flat space with the Euclidean metric  $\delta_{\mu\nu}$ , a setup in which there is no notion of temperature. However, as they are relevant for the study of system with matter, extending the framework to encompass temperature  $T$  is crucial. Given an Euclidean field theory living on a flat background, say  $\mathbb{R}^d$ , the procedure to obtain a thermal CFT from a one at infinite temperature is to compactify on a circle of diameter  $\beta = \frac{1}{T}$  one of the dimensions, the theory living then on  $S^1_\beta \times \mathbb{R}^{d-1}$ . These two backgrounds are however not related by conformal transformations, unless  $d = 2$ . Therefore, one expects to deduce all the features of the thermal CFT from the original one in two-dimensions, whereas in  $d > 2$  some additional data will be needed. The latter are

encoded into correlation functions at finite  $T$ , dubbed now thermal correlators. The general result for a massless scalar theory yields for the two-point function a formula of the type [198]

$$\langle \varphi(r, \cos \theta) \varphi(0, 0) \rangle = \frac{1}{r^{2\Delta}} \sum_{\{Q_s\}} a_{Q_s} \left( \frac{r}{\beta} \right)^{\Delta_{Q_s}} C_s^\nu(\cos \theta) \quad (6.0.1)$$

where the sum runs over all spin- $s$  operators  $Q_s$  appearing in the conformal operator product expansion (OPE)  $\varphi \times \varphi$ . Also  $\Delta = \frac{1}{2}(d - 2)$ . We work in Euclidean coordinates  $(\tau, \mathbf{x})$  with  $\tau \sim \tau + \beta$ ,  $r^2 = \tau^2 + |\mathbf{x}|^2$ ,  $\cos \theta = \frac{\tau}{r}$  and denote by  $C_s^\nu$  the Gegenbauer polynomial depending on  $\nu = \frac{d}{2} - 1$ . Here, the only unknowns are the  $a_{Q_s}$  which, as we shall see in the main content, are related to the expectation values of the  $Q_s$ , higher spin trace-free operators. These are the data one needs to describe the thermal CFT. The massless scalar theory being still a plain CFT, one may wonder what happens if the CFT is deformed by a set of relevant operators like the mass or the  $U(1)$  charge. It turns out that the resulting theory is generically not a CFT, unless some conditions known under the name of “gap equations” fix the deformation parameters to a precise value that restore conformal invariance [199]. Anyway one can still expand the thermal part of the 2-point function in Gegenbauers (6.0.1) and look for the  $a_{Q_s}$ . Several options are at our disposal to compute them: the AdS/CFT correspondence (see e.g. [25]) is of primary use: one recast the thermal correlator calculation into a gravity computation on a background endowed with an AdS black hole state of Hawking temperature  $T$ . Spectral analysis and inversion-like formulas (see [198]) are also useful and standard ways to compute these 1-point functions. However there seems to be no guiding principle to understand where do these numbers come from and how one could follow their evolution along the Renormalisation Group flow generated by the deformations.

Diving deeper into the structure of the mathematical expression for the lower spin  $a_{Q_s}$ , the ubiquity of a certain class of single-valued polylogarithms was described in [200] after having been first noticed in the case of the  $O(N)$  vector model at large  $N$  in [201]. On the other hand, these functions often appears in the context of ladder graphs [202–204] and fishnet theories [205, 206]. One may then legitimately wonder if there exists a relationship between thermal expectation values and conformal ladder graphs. This is precisely what we shall study in this part of the thesis. In addition to the establishment of a correspondence between these two *a priori* distinct realm of field theory, it turns out that, starting from the very elementary model of a couple of harmonic oscillators, one can deduce from its partition function a set of differential operators which allows one to construct the full tower of  $L$ -loops conformal ladder graphs from the tree level one. One may also eventually resum the full expansion in loops and get the exact correlator of the fishnet theory, using our correspondence in the other way. This paves the way to a yet completely unexplored web of relations between graphs and thermal averages.

The plan of this section is as follows. In Section 6.1 we recall some basics notions on CFT. In Sec. 6.2 we present thermal field theories, general and then conformal, with some remarks on how to recover a CFT after a deformation. All these preliminary notions will be used in Section 6.3 where the precise correspondence between thermal partition functions and conformal ladder graphs is stated. We end up in Section 6.4 with some applications of the latter to the construction of higher

loops ladder graphs and their resummation. We finally conclude with some extensions of our work and perspectives for the (near) future. The new results of this Chapter are mostly based on [200] and [207] by the author and collaborators.

## 6.1 Crash course on Conformal Field Theories

This Section aims at reviewing the basic ingredients we may use in the future Sections of this Chapter. We will mostly base ourselves on [208].

Let us start we study of conformal invariance in dimensions strictly greater than 2 i.e. we take  $d \geq 3$  and our background shall be  $\mathbb{R}^{p,q}$  (with  $d = p + q$ ) with coordinates  $x = x^\mu$  and Lorentzian metric  $g_{\mu\nu}$ ,  $\mu, \nu = 0, 1, \dots, d$  of signature  $(p, q)$ .

### Conformal transformations

Let  $\phi$  be a differentiable map between two metrics  $\phi : g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x')$ . Note that both the point at which the metric is calculated and the metric itself are assumed to change under such a transformation. The special case for which  $g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x)$  with  $\Omega(x)$  an arbitrary function of the coordinates is called a *Weyl transformation* and  $\Omega(x)$  is the *Weyl factor*. Conformal transformations are more. These are by definition **changes of coordinates**  $x \rightarrow x'$  such that  $g'_{\mu\nu}(x') = \Omega^2(x)g_{\mu\nu}(x)$  holds.<sup>1</sup> Given the tensorial character of the metric under diffeomorphisms we get that a conformal transformation should satisfy

$$g_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Omega^2(x)g_{\mu\nu} , \quad (6.1.1)$$

from which we clearly see that angles are preserved by such transformations.

We seek now the infinitesimal form of conformal transformation in the simplest case of flat space i.e.  $g_{\mu\nu} = \eta_{\mu\nu}$ . Assuming

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu(x) + \mathcal{O}(\varepsilon) \quad (6.1.2)$$

we get the conformal Killing equation at linear order

$$\partial_{(\mu} \varepsilon_{\nu)} = \frac{1}{d} (\partial_\rho \varepsilon^\rho) \eta_{\mu\nu} \quad (6.1.3)$$

whose most general solution reads

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \quad (6.1.4)$$

with  $a_\mu$ ,  $b_{\mu\nu}$  and  $c_{\mu\nu\rho} = c_{\mu\rho\nu}$  constant coefficients. Note that one can show that  $b_{(\mu\nu)} \propto \eta_{\mu\nu}$ . The

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<sup>1</sup>Indeed Weyl transformations have nothing to do with changes of coordinates, they just rescale the metric by an overall multiplicative factor.

various terms in (6.1.4) can be interpreted as follows.<sup>2</sup>

- $a_\mu$  corresponds to a usual translations generated by the operator  $P_\mu = -i\partial_\mu$ ,
- as  $b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$  with  $m_{\mu\nu} = m_{[\mu\nu]}$  one identifies the latter with usual Lorentz transformations generated by  $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$  while the former corresponds to a scale transformation (i.e.  $x'^\mu = (1 + \alpha + \mathcal{O}(\alpha^2))x^\mu$ ) generated by  $D = -ix^\mu\partial_\mu$ ,
- $c_{\mu\nu\rho}$  corresponds to a new type of transformations, dubbed *special conformal transformations* acting as (with  $b_\nu = \frac{1}{d}c_{\mu\nu}^\mu$ )

$$x'^\nu = x^\mu + 2x^\nu b_\nu x^\mu - x^\nu x_\nu b^\mu + \mathcal{O}(b^2) \quad (6.1.8)$$

generated by the operator  $K_\mu = -i(2x_\mu x^\nu\partial_\nu - x_\nu x^\nu\partial_\mu)$ . We display for completeness the finite version of a special conformal transformation

$$x'^\mu = \frac{x^\mu - x_\nu x^\nu b^\mu}{1 - 2b_\nu x^\nu + b_\nu b^\nu x_\rho x^\rho}. \quad (6.1.9)$$

The set of operators  $\{P_\mu, L_{\mu\nu}, D, K_\mu\}$  forms the *conformal algebra*  $\mathfrak{so}(p+1, q+1)$ . Its (non-zero) commutators read

$$[D, P_\mu] = iP_\mu \quad (6.1.10a)$$

$$[D, K_\mu] = -iK_\mu \quad (6.1.10b)$$

$$[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \quad (6.1.10c)$$

$$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \quad (6.1.10d)$$

$$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \quad (6.1.10e)$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \quad (6.1.10f)$$

meaning that as expected the  $L_{\mu\nu}$  form a Lorentz algebra with  $P_\mu$  and  $K_\mu$  transforming under the vectorial representation of the latter. This algebra is  $\frac{(d+2)(d+1)}{2}$  dimensional.

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<sup>2</sup>To find the expression of the generator, given a transformation

$$x'^\mu = x^\mu + \varepsilon_a \frac{\delta x^\mu}{\delta \varepsilon_a} \quad (6.1.5a)$$

$$\phi'(x') = F(\phi(x)) \quad (6.1.5b)$$

with  $F$  a function, we define the generator of the transformation as

$$\phi'(x) - \phi(x) := -i\varepsilon_a G_a \phi(x) \quad (6.1.6)$$

so that

$$iG_a \phi = \frac{\delta x^\mu}{\delta \varepsilon_a} \partial_\mu \phi - \frac{\delta F}{\delta \varepsilon_a}. \quad (6.1.7)$$

**Remark** Given that  $p + q = d$  one can redefine the generator as

$$J_{\mu\nu} := L_{\mu\nu} \quad , \quad J_{-1\mu} := \frac{1}{2}(P_\mu - K_\mu) \quad , \quad J_{0\mu} := \frac{1}{2}(P_\mu + K_\mu) \quad , \quad J_{-10} := D \quad (6.1.11)$$

so that they satisfy the algebra

$$[J_{AB}, J_{CD}] = i(\eta_{BC}L_{AD} + \eta_{AD}L_{BC} - \eta_{AC}L_{BD} - \eta_{BD}L_{AC}) \quad , \quad (6.1.12)$$

when  $\eta_{AB}$  is taken to be  $\text{diag}(-1, -1, 1, \dots, 1)$ .

One could then pursue by studying the representations of the conformal algebra, but this very canonical notion is not part of our agenda. Instead let's focus on the transformations of fields under a scale transformation.

### Scaling dimensions, (quasi)-primary fields, descendants

Let  $\Phi$  be a field, under a finite scale transformation  $x' = \lambda x$  it is assumed to behave like

$$\Phi(\lambda x) = \lambda^{-\Delta_\Phi} \Phi(x) \quad . \quad (6.1.13)$$

The real number  $\Delta_\Phi \in \mathbb{R}$  is called the *scaling dimension* of  $\Phi$ . For example if  $\Phi = \phi$  an ordinary massless scalar field, given that the action  $S = \int d^d x \partial_\mu \phi \partial^\mu \phi$ , should be invariant one finds that

$$\Delta_\phi = \frac{d}{2} - 1 \quad . \quad (6.1.14)$$

Under a general conformal transformation  $x \rightarrow x'$  of Jacobian  $|\frac{\partial x'}{\partial x}|$  our scalar field  $\phi$  changes like

$$\phi(x) \rightarrow \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_\phi/d} \phi(x) \quad , \quad (6.1.15)$$

and such a field is then called a *quasi-primary* field.

**Remark** The notion of primary field comes from the representation theory of the conformal algebra. Without delving too much on that, one defines a *primary field* as a representation  $\Phi$  of the conformal algebra that is annihilated by  $K_\mu$  once inserted at  $x = 0$  that is  $K_\mu \Phi(0) = 0$ . The latter operator is then seen as a lowering operator whereas the  $P_\mu$  is the raising operator which allows to build *descendants fields* from a conformal primary. As  $P_\mu$  is realized as  $-i\partial_\mu$ , descendants fields are just derivatives of the primary field.

**Remark** In general this Jacobian can be expressed like a scale factor  $\Omega(x)$  that multiplies a rotation matrix  $M^\mu_\nu$ , i.e.

$$|\partial x'^\mu / \partial x^\nu| = \Omega(x) M^\mu_\nu(x) \quad . \quad (6.1.16)$$



Hence we can deduce that the equivalent of (6.1.15) for a spin  $s$ -field  $\psi_s$  of scaling dimension  $\Delta_\psi$  is

$$\psi_s(x) \rightarrow \psi'_s(x') = \Omega^{-\Delta_\psi}(x) R[M^\mu{}_\nu(x)] \psi_s(x) \quad (6.1.17)$$

with  $R$  the spin- $s$  irreducible representation of  $SO(d)$ .

**Remark** One has to be careful with scale invariance and conformal invariance. A theory may be scale invariant without being conformal invariant. Actually a CFT possesses a traceless energy-momentum tensor while in a scale invariant theory this trace is equal to the divergence of a local current  $T^\mu{}_\mu = \partial_\mu K^\mu$  but with some additional requirements on  $K^\mu$  like the good scaling dimension (plus e.g. in  $d = 2$  it should not be itself the derivative of a scalar). If there are no suitable candidates for  $K^\mu$  then scale invariance is equivalent to conformal invariance. Note that the converse is not true as there exist theories (like a bi-scalar quadratically coupled in  $d = 4 - \epsilon$ ) which possesses such a current while being conformally invariant. There is a set of requirements which allows to directly go from scale to conformal invariance. In  $d = 2$  and  $d = 4$ , Lorentz invariance and unitarity suffice [209, 210] while for  $d = 3$  and  $d \geq 5$  this set is yet unknown, whereas no counter examples have been found [211].

Once we are given a primary field, one can compute its correlators or  $n$ -point functions. Let  $S[\Phi_i]$  be the action of a general Lorentzian CFT whose matter content is denoted by a set of fields  $\Phi_i$ ,  $i \in I$ . Introducing the path-integral partition function  $Z$

$$Z = \int \mathcal{D}[\Phi_i] e^{iS[\Phi_i]} \quad (6.1.18)$$

one computes the  $n$ -point function with the formula

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}[\Phi_i] \phi(x_1) \dots \phi(x_n) e^{iS[\Phi_i]} . \quad (6.1.19)$$

Assuming conformal invariance of the action and given that all  $\phi_j$ ,  $j = 1, \dots, n$  are quasi-primaries of dimension  $\Delta_i$  we find that the  $n$ -point function should behave like

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \left| \frac{\partial x}{\partial x'} \right|_{x=x_1}^{\Delta_1/d} \dots \left| \frac{\partial x}{\partial x'} \right|_{x=x_n}^{\Delta_n/d} \langle \phi_1(x'_1) \dots \phi'_n(x'_n) \rangle \quad (6.1.20)$$

under a conformal transformation. This equation (6.1.20) drastically constrain the shape of low  $n$  correlators. Actually, thanks to translation invariance, all 1-point functions should be zero except for the unity operator

$$\langle \phi(x) \rangle = \begin{cases} 1 & \text{if } \phi = Id \\ 0 & \text{otherwise .} \end{cases} \quad (6.1.21)$$

Eq. (6.1.20) also completely fixes the shape of the 2 and 3-point correlators which respectively are

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \begin{cases} \frac{d_{12}}{|x_1-x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{otherwise} \end{cases} \quad (6.1.22)$$

with  $d_{12}$  a constant, and

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{\lambda_{123}}{x_{12}^{\Delta-2\Delta_3} x_{23}^{\Delta-2\Delta_1} x_{13}^{\Delta-2\Delta_2}} \quad (6.1.23)$$

with  $x_{ij} = |x_i - x_j|$ ,  $\Delta = \Delta_1 + \Delta_2 + \Delta_3$  and  $\lambda_{123}$  a constant. Hence up to a numerical factor, the shape of these correlators is completely fixed by conformal invariance. For the 4-point function we can only write the correlators up to an arbitrary function  $f$  that depends on conformal invariant quantities called *conformal ratios*

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} := u \quad \text{and} \quad \frac{x_{12}^2 x_{34}^2}{x_{23}^2 x_{14}^2} := v \quad (6.1.24)$$

to<sup>3</sup> give

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = f(u, v) \prod_{i<j}^4 x_{ij}^{\Delta/3-\Delta_i-\Delta_j} \quad (6.1.26)$$

with<sup>4</sup> again  $\Delta = \sum_{i=1}^4 \Delta_i$ . Recall that this is what we can achieve using only what conformal invariance tells us. As we shall see, the notion of operator product expansion will help to compute explicitly higher point correlators.

**Remark** In computing a 4-point function as in (6.1.26) it is always possible to use conformal invariance to fix some of the points to a chosen value. With a translation we can set  $x_1 = 0$ , then with the combination of a rotation and a scale transformation we set  $x_2 = 1$  (the rotation is around the origin) and finally with a special conformal transformation  $x_3 = +\infty$ . As dilata-tion would not bring anything else we see that one can always fix three of the coordinates, which would considerably simplify the computation of the correlator.

## Operator-state correspondence

We recall first in this paragraph the process of *radial quantisation*. Quantisation requires a notion of evolution along a parameter that is often took to be the time, as one needs to compute equal-time commutation relations. Geometrically speaking it means that our  $d$ -dimensional background is cut

<sup>3</sup>One can express the conformal ratios in terms of complex variables  $z, \bar{z}$  related to  $u$  and  $v$  as

$$v = z\bar{z} \quad \text{and} \quad \frac{u}{v} = (1-z)(1-\bar{z}). \quad (6.1.25)$$

<sup>4</sup>An additional symmetry dubbed *crossing symmetry* and related to the OPE imposes that  $f(u, v) = \left(\frac{u}{v}\right)^\Delta f(v, u)$ .

(one says foliated) by  $(d - 1)$ -dimensional sub-manifolds corresponding to  $t = cst$ . In (Euclidean) CFT, given the dilation operator ( $D = r\partial_r$ ) and the invariance under scale transformations, instead of foliating with respect to time, we choose  $S^{d-1}$  spheres centered at the origin.<sup>5</sup> Moving from one sphere to another is done with the  $D$  operator and we classify states by their scaling dimension and their spin

$$D|\Delta, s\rangle = i\Delta|\Delta, s\rangle \quad (6.1.27a)$$

$$L_{\mu\nu}|\Delta, s\rangle = R[(\Sigma_{\mu\nu})]|\Delta, s\rangle \quad (6.1.27b)$$

with  $R[\Sigma_{\mu\nu}]$  the spin- $s$  representation of the Lorentz algebra. This whole process amounts to write the metric of  $\mathbb{R}^d$  as

$$ds^2 = dr^2 + r^2 d\Omega_{d-2}^2 \quad (6.1.28)$$

with  $d\Omega_{d-1}^2$  the metric of the unit round  $(d - 1)$ -sphere. Introducing  $\tau := \log r$  we get

$$ds^2 = e^{2\tau} (d\tau^2 + d\Omega_{d-1}^2) \quad (6.1.29)$$

which allows to write the evolution operator between the spheres as

$$U = e^{iD\tau} . \quad (6.1.30)$$

Note in passing that this change of coordinate leading to the overall scale factor  $e^{2\tau}$  shows that  $\mathbb{R}^d$  and  $\mathbb{R} \times S^{d-1}$  are conformally equivalent. This is a fact that we will use in the next Section. Note finally that radial quantization is strictly equivalent to ordinary quantization with respect to time but on a cylinder.

Within this framework, the *operator-state correspondence* states that, given the vacuum state  $|0\rangle$  of dimension and spin 0, a state  $|\Delta\rangle$  of scaling dimension  $\Delta$  is equivalent to the insertion at the origin of a local quasi-primary operator of that dimension. In other words

$$\langle \phi(x_1)\phi(x_2)\dots O_\Delta(0) \rangle = \langle 0|\phi(x_1)\phi(x_2)\dots |\Delta\rangle , \quad (6.1.31)$$

which will make us from now on identify states with local operators.

**Remark** It is possible to give constraints on the scaling dimension  $\Delta$ . Requiring unitarity<sup>6</sup> (i.e. positive norm states) we get the well-known *unitarity bounds*

$$\Delta_{s=0} \geq \frac{d}{2} - 1 \quad (6.1.32a)$$

$$\Delta_{s=1/2} \geq \frac{d-1}{2} \quad (6.1.32b)$$

$$\Delta_{s \geq 1} \geq d - 2 + s , \quad (6.1.32c)$$

<sup>5</sup>Translation-invariance ensures that quantising from any other point would give the same correlators.

<sup>6</sup>In Euclidean signature one should instead talk about reflexion positivity.

with  $s$  the spin of the field.

### Wick's theorem, operator product expansion

Two additional notions are very useful when dealing with CFT. When the theory is Gaussian one can always reduce the computation of any  $n$ -points functions to lower point functions using Wick's theorem which states that<sup>7</sup>

$$\langle \phi_1(x_1)\phi_2(x_2) \dots \phi_{2n-1}(x_{2n-1})\phi_{2n}(x_{2n}) \rangle = \sum_{\sigma} \langle \phi_{\sigma(1)}(x_{\sigma(1)})\phi_{\sigma(2)}(x_{\sigma(2)}) \rangle \dots \langle \phi_{\sigma(n-1)}(x_{\sigma(n-1)})\phi_{\sigma(n)}(x_{\sigma(n)}) \rangle \quad (6.1.33)$$

where the sum runs over all permutations  $\sigma \in \mathfrak{S}_{2n}$  of  $1, \dots, 2n$ .

The other notion is the *operator product expansion* which states that the product of two operators  $O_i$  and  $O_j$  evaluated at neighboring points  $x$  and  $y$  can always be re-expressed as a sum of operators i.e.

$$O_i(x)O_j(y) = \sum_k \lambda_{ijk} C_O(y, \partial_y) O_k(y) \quad (6.1.34)$$

where  $C_O(y, \partial_y)$  can be thought as an expansion in powers of  $\partial_y$  and where  $\lambda_{ijk}$  are the three-point function coefficients (6.1.23). Hence we see that in such an OPE appears the primary states and their descendent, which are just derivatives of the latter. It can be proven that in the OPE between a quasi-primary  $O$  and itself only appears itself and its descendants.

**Remark** Recall that applying the OPE inside a correlation function is not innocuous at all. OPEs only work in a certain regime (refers as *neighboring points* before), i.e. in the context of radial quantization one should select two operators  $O_i$  and  $O_j$  (where the hat means a fixed value of the index) which can be surrounded by a sphere whose interior is flat that excludes all other operators. This is trivial on  $\mathbb{R}^d$  but not for more complicated manifolds. See [198] for discussions on that issue.

### What characterises a CFT?

So far one could see that the fundamental quantities that describe a CFT are its spectrum of primary operators of dimension  $\Delta$  (from which one can deduce its descendent) and the normalisation constants of the two and three-point functions  $d_{ij}$  and  $\lambda_{ijk}$  (the latter appearing also in the OPE of the corresponding fields). However it is always possible to renormalise the two-point function coefficient to 1 when the fields are the same i.e.  $d_{ii} = 1$  but once this is fixed it is not possible to fix its three-point analogue. Therefore the fundamental data describing the CFT at hand is the spectrum of operators and its OPE coefficients

$$\{O_{\Delta}, \lambda_{ijk}\}. \quad (6.1.35)$$

<sup>7</sup>We write it for an even number of fields for simplicity, in the odd case a three point function would appear.

From this set of data, using (6.1.34) one can deduce the shape of all  $n$ -point functions and hence solve the theory. This was for CFTs without temperature. It is actually possible to extend the framework so that it encompasses this notion. This is the purpose of the next paragraph.

**Remark** This whole paragraph was about CFT in  $d \geq 3$  where the conformal algebra and groups are finite dimensional. In two dimensions the conformal algebra is infinite dimensional which allows for a much richer analysis exhibiting nice algebraic structures like the Witt and Virasoro algebras. Everything we described in higher dimensions can be redone in  $d = 2$  but as we will mostly sit in higher odd dimensions in the rest of this Chapter it is not worth spending too much time on that topic. In addition, some words about two dimension will be given in the thermal case.

## 6.2 Thermal Field Theories

We focus here on how to add temperature in field theory, focusing first on the case of one-dimensional field theory a.k.a. quantum mechanics. Using conformal invariance we then show that in two-dimensions, CFTs and thermal CFT are in one-to-one correspondence, meaning that no additional data is required to describe the  $2d$  thermal CFT. In higher dimensions problems arise but we left that issue for the next Section as our advertised correspondence with conformal graphs will help solving it.

### 6.2.1 Generalities and useful techniques

#### Thermal quantum mechanics

When dealing with thermal systems in statistical physics and then in quantum mechanics, the notion of ensemble is fundamental. We recall that the two main ensembles are the *canonical* and the *grand canonical*. In the former the system is assumed to be at equilibrium with a reservoir at temperature  $T := \frac{1}{\beta}$  and with a fixed number of particles, hence it exchanges energy with the reservoir while its volume  $V$  and its number of particles  $N$  remains constant. In the latter the system exchanges both energy and particles hence both the volume and the chemical potential  $\mu$  of the particles are fixed. In both ensembles the fundamental quantity is the *partition function*  $Z$ , quantity from which one can deduce all the thermodynamics of the system. In the canonical ensemble it reads<sup>8</sup>

$$Z = \text{Tr}(e^{-\beta\hat{H}}) \tag{6.2.1}$$

with  $\hat{H}$  the Hamiltonian operator and the trace is taken over the whole Hilbert space.<sup>9</sup> Taking derivatives of (6.2.1) (more precisely  $\ln Z$ , this will have a big impact in the following) we respectively

<sup>8</sup>The quantity  $e^{-\beta\hat{H}}$  is often called the *density operator* and denoted by  $\hat{\rho}$ .

<sup>9</sup>If  $\mathcal{E}_H = \{|\psi_i\rangle\}$  is the Hilbert space and  $\hat{A}$  and operator acting on it we have  $\text{Tr}(\hat{A}) = \sum_i \langle \psi_i | \hat{A} | \psi_i \rangle$ .

get the free energy  $F$ , the pressure  $P$ , the entropy  $S$  or the number of particles  $N$

$$F = -T \ln Z, \quad P = T \frac{\partial \ln Z}{\partial V}, \quad S = -\frac{\partial F}{\partial T}, \quad N = T \frac{\partial \ln Z}{\partial \mu}, \quad (6.2.2)$$

while the energy reads  $E$

$$E = \frac{1}{Z} \text{Tr}(\hat{H} e^{-\beta \hat{H}}). \quad (6.2.3)$$

This was for the canonical ensemble. One gets the grand-canonical replacing  $\hat{H}$  by  $\hat{H} - \mu \hat{N}$  with  $\hat{N}$  the operator that counts the number of particles.

Given an operator  $\hat{A}$  we define its *thermal expectation value* at temperature  $\beta$  as

$$\langle \hat{A} \rangle_\beta = \frac{1}{Z} \text{Tr}(\hat{A} e^{-\beta \hat{H}}) \quad (6.2.4)$$

This is of course, like in the rest of this paragraph, a one-dimensional definition. The easiest system one can study in that dimension is the thermal harmonic oscillator. Given the importance it will get in our correspondence Section 6.3 we shall spend some time detailing it. For a massive harmonic oscillator of frequency  $\omega$  and mass also  $m$  the Hamiltonian read (with  $\hbar = 1$ )

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2 := \omega \left( \hat{N} + \frac{1}{2} \right) \quad (6.2.5)$$

where  $\hat{p}$  and  $\hat{x}$  are respectively the momentum and position operators while  $\hat{N}$  is the number operator.<sup>10</sup> The partition function is then

$$Z_{\text{H.O.}} = \frac{1}{2 \sinh \frac{\beta \omega}{2}} \quad (6.2.6)$$

from which we get

$$F = \frac{\omega}{2} + \frac{1}{\beta} \ln(1 - e^{-\beta \omega}) \quad (6.2.7a)$$

$$S = -\ln(1 - e^{-\beta \omega}) + \omega \beta \frac{1}{e^{\beta \omega} - 1} \quad (6.2.7b)$$

$$E = \omega \left( \frac{1}{2} + \frac{1}{1 - e^{-\beta \omega}} \right). \quad (6.2.7c)$$

Note the contribution of the zero-temperature case in the energy. The Hamiltonian (6.2.5) can be enhanced to encompass a chemical potential. This will be our fundamental model in Section 6.3. This model can also be studied by path-integral quantisation [212] but we won't give details on that.

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<sup>10</sup>Recall that  $\hat{N} = \hat{a}^\dagger \hat{a}$  in terms of ladder operators.

## Imaginary time formalism and Matsubara frequencies

In this section we go to field theory and we shall show some of the usual techniques when dealing with temperature. The first one comes under the name of *imaginary time formalism*. Let  $S = i \int dt d^{d-1}x \mathcal{L}$  be an action for a  $d$ -dimensional theory<sup>11</sup> in Lorentzian signature at infinite temperature with coordinates  $(t, \mathbf{x})$ . One can add temperature performing the following steps

- Wick rotate the time, defining  $\tau := it$ , hence the name of imaginary time formalism. This brings the theory to Euclidean signature,
- replace the Lorentzian action by the Euclidean one  $S \rightarrow S_E = - \int dt dx L_E$  with  $L_E = -\mathcal{L}(\tau = it)$ ,
- require periodicity on  $\tau$  i.e.  $\tau \sim \tau + \beta$  with  $\beta := \frac{1}{T}$  where  $T$  is the temperature. This compactifies the time direction on a circle of radius  $\beta$  a.k.a.  $S^1_\beta$
- compute the partition function using the Euclidean path integral,  $Z[\Phi] = \int \mathcal{D}[\Phi] e^{-S_E[\Phi]}$ .

Carrying out the compactification on a circle will have tremendous consequences when the theory lives in  $d > 2$ . We will come back to that issue in the next Section.

**Remark** Note that compactifying a dimension implies an explicit breaking of translation invariance, hence one expects the thermal expectation values of operators  $\langle O \rangle_\beta$  to be in general non-zero.

The coordinates are now  $(\tau, \mathbf{x})$ . When it comes to compute thermal correlators of a field  $\phi(\tau, \mathbf{x})$  (let's take it scalar for simplicity) a useful trick is to Fourier transform. As the time direction is now compact the Fourier transform is no longer an integral but rather a discrete (albeit infinite) sum

$$\phi(\tau, \mathbf{x}) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \tilde{\phi}(\omega_n, \mathbf{x}) e^{-i\omega_n \tau} \quad (6.2.8)$$

where the  $\omega_n$  are called *Matsubara frequencies*. What are the values these frequencies can take? It turns out that the imaginary time formalism comes with its own set of conditions. One of them is that all fields should be either periodic or anti-periodic (depending on their commutation relations) in imaginary time i.e.

$$\Phi(0, \mathbf{x}) = \pm \Phi(\beta, \mathbf{x}) . \quad (6.2.9)$$

This is the *Kubo-Martin-Schwinger relation* (KMS for short) and it implies that the Matsubara frequencies should be

$$\omega_n = \frac{2\pi n}{\beta} \quad \text{for commuting fields} \quad (6.2.10)$$

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<sup>11</sup> $(d-1+1)$ ...

$$\omega_n = \frac{2\pi(n + 1/2)}{\beta} \quad \text{for anti-commuting fields.} \quad (6.2.11)$$

From (6.2.8) one can also Fourier transform along the spatial dimensions to get

$$\phi(\tau, \mathbf{x}) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}}{(2\pi)^{d-1}} \tilde{\phi}(\omega_n, \mathbf{p}) e^{-i\omega_n \tau - i\mathbf{p} \cdot \mathbf{x}}. \quad (6.2.12)$$

We shall now illustrate these concepts with an explicit example.

### Massless free scalar field in $d$ -dimensions

In this paragraph we denote by  $(x) = (x^\mu)$ ,  $\mu = 0, 1, \dots, d-1$  the coordinates  $(\tau, \mathbf{x})$  and we set  $\beta = 1$ . The Lagrangian of this simple theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi. \quad (6.2.13)$$

Let's compute the thermal propagator i.e. the thermal 2-points function, by Fourier transform and using polar coordinates

$$\begin{aligned} \langle \phi(x) \phi(0) \rangle_\beta &:= g_d(r, \cos \theta) & (6.2.14) \\ &= \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}}{(2\pi)^{d-1}} \frac{e^{-i\omega_n \tau - i\mathbf{p} \cdot \mathbf{x}}}{\omega_n^2 + \mathbf{p}^2} \\ &= \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \sum_{n=-\infty}^{+\infty} \frac{1}{[(\tau + n)^2 + \mathbf{x}^2]^{d/2-1}} \\ &= \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \left[ \frac{1}{r^{d-2}} + \sum_{n=1}^{+\infty} \left( \frac{1}{(r^2 + n^2 + 2rn \cos \theta)^{d/2-1}} + \frac{1}{(r^2 + n^2 - 2rn \cos \theta)^{d/2-1}} \right) \right] \\ &= \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \frac{1}{r^{d-2}} \left[ 1 + \sum_{n=1}^{+\infty} 2\zeta(d-2+2n) r^{d-2+2n} C_{2n}^{d/2-1}(\cos \theta) \right], \end{aligned}$$

where<sup>12</sup> we have introduced the widely used notations  $r^2 = \tau^2 + \mathbf{x}^2$  and  $\cos \theta = \frac{\tau}{r}$  (while  $\sin \theta = \frac{|\mathbf{x}|}{r}$ ). In this expression also appear the well-known Gegenbauer polynomials  $C_n^\nu(\cos \theta)$ . Note also that this expression reassembles the one of (6.0.1) which means that we have in that case

$$a_{\mathcal{Q}_s} = 2\zeta(d-2+2n) \propto \langle \mathcal{Q}_s \rangle_\beta \quad (6.2.15)$$

where the  $\mathcal{Q}_s$  form an infinite set of higher-spin operators with dimensions  $\Delta_{\mathcal{Q}_s} = d-2+s$  and even spin  $s = 2p$ . Hence the massless scalar field, which is always a CFT, is a theory for which the

<sup>12</sup>In (6.2.14)  $\Gamma(x)$  stands for the Euler Gamma function.



thermal expectation values are explicitly computable. We also find that

$$\square_d g_d(r, \cos \theta) = 0, \quad (6.2.16)$$

with  $\square_d$  the Laplacian in  $d$ -dimensions. This is a typical feature of a theory whose spectrum is uniquely composed by higher-spin operators.

**Remark** The Gegenbauer polynomials form a set of orthogonal functions defined on  $[-1, 1]$  and generated by

$$\frac{1}{(1 - 2xt + t^2)^\nu} = \sum_{n=0}^{+\infty} C_n^\nu(x) t^n \quad (6.2.17)$$

with  $0 \leq |x| < 1$ ,  $|t| < 1$  and  $\nu > 0$ . They are particular solutions of the differential equation on  $y(x)$

$$(1 - x^2)y'' - (2\nu + 1)xy' + n(n + 2\nu)y = 0 \quad (6.2.18)$$

and their orthogonality relation reads

$$\int_{-1}^1 C_n^\nu(x) C_m^\nu(x) (1 - x^2)^{\nu - \frac{1}{2}} dx = 0. \quad (6.2.19)$$

**Remark** Actually the  $Q_s$  in (6.2.15) are nothing but the infinite tower of higher spin currents one can always construct from a scalar field (see e.g. the Section 5 of [213]). As the latter was real in our case, only even spins appears.

**Remark** In a more general theory, on top of the higher-spin operators, it appears in the OPE  $\phi \times \phi$  *higher-twist* operator of the form

$$\mathcal{O}_{n,s} = \phi \partial^{\mu_1} \dots \partial^{\mu_s} \square^{2n} \phi \quad (6.2.20)$$

with  $\mu_1, \dots, \mu_s \in \{0, 1, \dots, d - 1\}$  and  $n$  the twist number. When dealing with the two-point function written as (6.2.25), in virtue of (6.2.16), the part annihilated by the Laplacian in  $d$ -dimension is the one containing only the zero-twist higher-spin operators.

This ends our brief review of the main techniques used in thermal field theory, not necessarily conformal. We shall now delve into the case of CFT, study how conformal invariance can help us describing the thermal theory, and point out the main questions to be solved, among them the one our correspondence illuminates.

## 6.2.2 From CFT to thermal CFT

Thanks to conformal invariance, when two backgrounds  $\mathcal{M}$  and  $\mathcal{M}'$  are related by a conformal transformation, one can get any information about the CFT on the latter given what is known on the former. This is of precious help when thermalisation arises.

## The two-dimensional case

We first consider a theory living on  $\mathbb{R}^2$  whose metric in polar coordinates reads

$$ds^2 = dr^2 + r^2 d\theta^2, \quad (6.2.21)$$

with  $r \geq 0$  and  $\theta \in [0, 2\pi)$ . Introducing a new coordinate  $\rho$  such that  $r = Le^{\rho/L}$  one re-expresses the metric as

$$ds^2 = e^{2\rho/L} (d\rho^2 + L^2 d\theta^2) \quad (6.2.22)$$

with  $\rho \in [0, L]$ . (6.2.22) is conformally equivalent to a metric on  $S_\beta^1 \times \mathbb{R}$  with  $\beta = L$  and  $e^{2\rho/L}$  as conformal factor. (6.2.21) and (6.2.22) being conformally related, one can deduce from the correlators on  $\mathbb{R}^2$  namely<sup>13</sup>

$$\langle \mathcal{O}(r_1, \theta) \mathcal{O}(r_2, \theta) \rangle = \frac{1}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta)^{\Delta_{\mathcal{O}}}} \quad (6.2.23)$$

the form of the correlators on  $S_\beta^1 \times \mathbb{R}$

$$\langle \mathcal{O}(\rho_1, \theta) \mathcal{O}(\rho_2, 0) \rangle = \frac{1}{L^{2\Delta_{\mathcal{O}}}} \frac{1}{(2 \cosh \frac{\rho_1 - \rho_2}{L} - 2 \cos \theta)^{\Delta_{\mathcal{O}}}} \quad (6.2.24)$$

i.e. from the theory on the flat background we determine completely the theory on the thermal geometry (6.2.22). This is an important result that in  $d = 2$ , after thermalisation there are no additional data required to describe the CFT, the spectrum and the OPE coefficients are sufficient.

**Remark** One may ask about the fate of one-point functions which are all vanishing on  $\mathbb{R}^2$  but may non vanish on  $S_\beta^1 \times \mathbb{R}$  due to the breaking of translation invariance. This seems in contradiction with the previous statement that no additional data is needed in the thermal case. Actually non vanishing thermal one-point functions are determined by the anomalous behavior under conformal transformation of the operators of the non-thermal CFT. This is in particular the case of the energy-momentum tensor. See [214, 215] and latter [199] for comments on that.

## What about higher dimensions?

If we start now from a CFT living on  $\mathbb{R}^d$  with  $d > 2$  we see that the same change of variable  $r = Le^{\rho/L}$  allows to conformally relate this background to  $\mathbb{R} \times S^{d-1}$ . However this is not a thermal geometry and actually it can be shown that  $\mathbb{R}^d$  cannot be related to  $S_\beta^1 \times \mathbb{R}^{d-1}$  by a conformal transformation. Therefore thermal correlators in dimensions higher than two should depend on extra parameters with respect to the infinite temperature CFT. In this thesis we will focus mostly on what can be said about thermal 2-points functions.

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<sup>13</sup>This is just (6.1.22) but in  $(r, \theta)$  coordinates.

It as been shown in [198] that in the case of scalar fields one can always expand the thermal CFT 2-point function like in (6.2.14) i.e. in the basis of Gegenbauer polynomials

$$\langle \varphi(r, \cos \theta) \varphi(0, 0) \rangle = \frac{1}{r^{2\Delta}} \sum_{\{O_s\}} a_{O_s} \left( \frac{r}{\beta} \right)^{\Delta_{O_s}} C_s^\nu(\cos \theta) \quad (6.2.25)$$

with  $\nu = d/2 - 1$ . In (6.2.25) the sum runs over all operators  $O_s$  appearing in the OPE  $\phi \times \phi$ .<sup>14</sup> The latter are spin- $s$  irreducible representation of  $SO(d)$ . It can also be shown that the coefficient  $a_{O_s}$  is related to the thermal expectation value of  $O_s$  via<sup>15</sup>

$$a_{O_s} = \frac{s!}{2^s (\nu)_s} \frac{\lambda_{\phi\phi O}}{d_{O_s}} b_{O_s} \quad (6.2.27)$$

and<sup>16</sup>

$$\langle O_s \rangle_\beta = b_{O_s} (\hat{e}_{\mu_1} \dots \hat{e}_{\mu_s} - \text{traces}) \quad (6.2.28)$$

where  $d_{O_s}$  is the normalisation factor of  $\langle O_s O_s \rangle$ ,  $\lambda_{\phi\phi O}$  is the one of  $\langle \phi\phi O_s \rangle$  and with  $\hat{e}$  a unit vector. The  $b_{O_s}$  are the new parameters we need to determine in order to describe our thermal CFT (we already found them in the case of the free massless scalar field in (6.2.15)). Recall that as translation-invariance is broken there are to be non-zero.<sup>17</sup> Note that the expression

$$\left( \frac{r}{\beta} \right)^{\Delta_{O_s}} C_s^\nu(\cos \theta) \quad (6.2.29)$$

is universal and can then receive the sweet name of “*thermal (conformal) blocks*” by analogy with the conformal blocks in higher point functions in infinite temperature CFT.

**Remark** In [198] the authors pointed out a possible spectral analysis in seeking an expression for the  $a_{O_s}$ . Introducing a new variable  $\Delta$  one can rewrite (6.2.25) as

$$g(r, \cos \theta) = \sum_s \oint_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} \frac{d\Delta}{2\pi i} a(\Delta, s) \frac{C_s^\nu(\cos \theta)}{r^{2\Delta_\phi - \Delta}} \quad (6.2.30)$$

with  $\varepsilon > 0$  small and  $a(\Delta, s) \sim -\frac{a_{O_s}}{\Delta - \Delta_{O_s}}$ . Using (6.2.19) one finally gets, setting  $x = \cos \theta$

$$a(\Delta, s) = \frac{1}{N_{\nu, s}} \int_0^1 \frac{dr}{r^{\Delta - 2\Delta_\phi + 1}} \int_{-1}^1 dx (1 - x^2)^{\nu - 1/2} C_s^\nu(x) g(r, x), \quad (6.2.31)$$

<sup>14</sup>Such an expansion is only valid in the OPE regime which implies  $|x| = \sqrt{\tau^2 + \mathbf{x}^2} < \beta$ . Otherwise there are no sphere with flat interior to apply the OPE.

<sup>15</sup>Recall the expression of the Pochhammer symbol

$$(x)_n = x(x+1) \dots (x+n-1). \quad (6.2.26)$$

<sup>16</sup>To be rigorous one should read  $O_{\mu(s)}$  instead of  $O_s$  with  $\mu(s)$  a multi-index completely symmetrised.

<sup>17</sup>Only for quasi-primary operators, descendent still have a vanishing expectation value [198].

with  $N_s$  an overall normalisation factor [199]

$$N_{\nu,s} = \frac{2^{1-2\nu} \pi \Gamma(s+2\nu)}{(s+\nu)\Gamma(s+1)\Gamma^2(\nu)}. \quad (6.2.32)$$

The latter equation (6.2.31) is also called an (*Euclidean inversion formula*). It can be extended to Lorentzian signature, see [198].

### What characterises a thermal CFT?

Adding temperature requires to know the spectrum of operators of the theory with their scaling dimensions and their OPE coefficients, but the main difference lies in the thermal one point functions of quasi-primary operators, which contain an arbitrary coefficient  $b_O$  (see (6.2.27)). Hence the set of data that fully describes a thermal CFT is

$$\{\Delta_O, \lambda_{OOO}, b_O\}. \quad (6.2.33)$$

### 6.2.3 Deforming the CFT and evolution along the RG flow

So far we were dealing with theories that remains conformally-invariant in the thermal instance. However, as we recall in the introduction of this Chapter, CFTs are always RG fixed points of QFTs. In other words, QFTs are deformations of CFT by relevant operators such as the mass with parameter  $m$  or an  $U(1)$  charge with parameter  $\mu$ , the chemical potential. One may then legitimately wonder what is the fate of the new data  $a_{O_s}$  once a free CFT gets deformed.

Let's go back to the case of the scalar field which we now consider to be massive, the theory is hence deformed by the operator  $\phi^2$ . (6.2.14) gets modified into

$$\begin{aligned} g_d(r, \cos \theta; m) &= \sum_{n=-\infty}^{+\infty} \int \frac{d^{d-1} \mathbf{p}}{(2\pi)^{d-1}} \frac{e^{-i\omega_n \tau - i\mathbf{p} \cdot \mathbf{x}}}{\omega_n^2 + \mathbf{p}^2 + m^2} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{m}{r}\right)^{d/2-1} K_\nu[mr] + \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{+\infty} \left[ \left( \frac{m}{[(k-\zeta)(k-\bar{\zeta}]^{d/2}} \right)^\nu K_\nu[m((k-\zeta)(k-\bar{\zeta})] \right. \\ &\quad \left. + \left( \frac{m}{[(k+\zeta)(k+\bar{\zeta}]^{d/2}} \right)^\nu K_\nu[m((k+\zeta)(k+\bar{\zeta})] \right] \end{aligned} \quad (6.2.34)$$

with  $\nu = d/2 - 1$  again,  $\zeta = \tau + i|\mathbf{x}|$  so  $\bar{\zeta} = \tau - i|\mathbf{x}|$  and  $K_\nu$  the modified Bessel function of the second kind. The first term in the second line corresponds to the usual zero temperature result. Note that one can also expand  $g_d(r, \cos \theta; m)$  in Gegenbauer polynomials and get a formula similar to (6.2.25).

In that case one gets for higher even spins ( $s > 0$ ) with a spectral analysis [199]

$$a(\Delta_s) = \frac{1}{2^{2s+\frac{d-3}{2}}} \frac{1}{s!} \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(d-2+s)} \sum_{n=0}^{\frac{d-3}{2}+s} \frac{2^{n+1} (d-3+2s-n)!}{n! (\frac{d-3}{2}+s-n)!} m^n \text{Li}_{d-2+s-n}(e^{-m}) \quad (6.2.35)$$

where we see appearing for the first time *polylogarithms*  $\text{Li}_n(z)$ , ubiquitous functions in graph (e.g. [216]) and number theory (e.g. [217]). They are defined by

$$\text{Li}_n(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^n}, \quad (6.2.36)$$

with  $n \in \mathbb{C}$  the order and  $z$ , complex number of modulus 1, the argument. Note that in particular that  $\text{Li}_1(z) = -\ln(1-z)$  and

$$\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} dt. \quad (6.2.37)$$

The linear combination appearing in (6.2.35) has the special property of being *single-valued*.

**Remark** The massive scalar field is generically not a CFT so we should not expect any consequence of conformal invariance to hold. However, in odd dimensions  $d = 2L + 1$  ( $L = 0, 1, 2, 3, \dots$ ), the Bessel functions become polynomials and it is always possible to tune the mass parameter  $m$  to get a CFT [199]. Actually it is expected that, in the deformed theory, the operator  $\phi^2$  do not appear in the spectrum, leaving its place to its so-called *shadow operator*. Such an operator is often denoted by  $\sigma$  and has a scaling dimension  $\Delta_\sigma$  such that  $\Delta_\sigma + \Delta_{\phi^2} = d$  i.e.  $\Delta_\sigma = 2$ . When the thermal theory is a CFT the shadows do not belong to the spectrum contrary to  $\phi^2$ . Expanding on the one hand the zero temperature result and on the other hand the thermal part, both in power of  $(mr)$  we can find the coefficient that multiplies the power of  $\frac{1}{r}$  corresponding to  $\langle \phi^2 \rangle$  and ask for the coefficient in front to vanish. This leads to the *gap equation* of [199] to solve of  $m$

$$\sum_{n=0}^{L-1} \frac{2^{n+1} (2(k-1)-n)!}{n! (k-1-n)!} m_{\text{th}}^n \text{Li}_{2k-1-n}(e^{-m_{\text{th}}}) = -\frac{1}{2\sqrt{\pi}} m_{\text{th}}^{2k-1} \Gamma(-k + \frac{1}{2}), \quad (6.2.38)$$

with  $m_{\text{th}}$  the special value of the parameter that leads to conformal invariance of the deformed theory. For example in  $d = 3$  one finds that the mass should be adjusted to

$$m_{\text{th}}^{d=3} = 2 \ln \left( \frac{1 + \sqrt{5}}{2} \right). \quad (6.2.39)$$

The general picture is now clear. In a zero temperature CFT we know the shape of the correlators (6.1.22), (6.1.23) and (6.1.26). One can deform this CFT and study its evolution along the RG flow but still be able to recover a CFT upon fine-tuning of the parameters. When thermalisation arises, a new set of data, contained in the  $a_{\mathcal{O}}$ , is needed to describe the CFT, which again can also be deformed

as in (6.2.34). Several questions then arise

- Where do these  $a_O$  come from? Is there an underlying principle explaining their values?
- Is it possible to write the thermal 2-point function in a CFT form?
- How does  $a_O$  evolve along the RG flow i.e. what happen when  $a_O(0) \rightarrow a_O(m)$ ?

This are our objectives for the remaining of the Chapter and an innocent but crucial observation will help us in that quest.

### Towards conformal graphs

As we pointed out before (6.2.36), the single-valued polylogarithms are graphical functions (in the sense of [216]) which were found to appear in [202] when computing conformal ladder graph for 3 and 4-point functions of a massless  $\varphi^3$  theory at  $L$ -loops and in odd dimensions. It seems then that those two *a priori* distinct area of field theory, on the one hand thermal correlators and on the other hand conformal ladder graphs, are related through these mathematical functions. This is the observation made in [200] and pursued in our work [207] where we went further, showing that conformal graphs are actually related to thermal partition functions.

## 6.3 Conformal graphs as twisted partition functions

In this Section we establish our correspondence [207] between partition functions, thermal correlators in odd dimensions  $d = 2L + 1$  and  $L$ -loops conformal ladder graphs. After reviewing our fundamental model (from which everything will be deduced) we shall give some details about the structure of conformal fishnet theories, mainly the model of [206] which is at the heart of our work. After precisely stating the correspondence we finish with some of its present and future applications.

### 6.3.1 Twisted scalar fields in $d = 2L + 1$ dimensions

#### The model: relativistic Bose gas

In this paragraph our background is  $S^1_\beta \times \mathbb{R}^{d-1}$  described by means of the (Euclidean) coordinates  $(\tau, \mathbf{x})$  where  $\tau$  is the imaginary time. The temperature is still denoted by  $T = \frac{1}{\beta}$ . We mostly follow [200] and our work [207].

We consider the thermal partition function of a massive complex scalar field  $\phi$  coupled to a  $U(1)$  gauge field  $A_\mu = (\mu, \vec{0})$  with  $\mu$  a complex imaginary chemical potential<sup>18</sup>

$$Z_d(\beta; m, \mu) = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} e^{-S_E[\phi, A_\mu]} \quad (6.3.1)$$

with Euclidean action<sup>19</sup>

$$S_E = \int_0^\infty d\tau \int d^{d-1}x \left[ |(\partial_\tau - i\mu)\phi|^2 + |\vec{\partial}\phi|^2 + m^2|\phi|^2 \right], \quad (6.3.2)$$

and periodic boundary conditions

$$\phi(\tau + \beta, x) = \phi(\tau, x). \quad (6.3.3)$$

**Remark** Note that the parameter  $\mu$  can equivalently be identified to a twist i.e. if one introduces the field  $\tilde{\phi}$  such that

$$\tilde{\phi}(\tau, x) = e^{-i\mu\tau} \phi(\tau, x), \quad (6.3.4)$$

then it satisfies *twisted boundary conditions*

$$\phi(\tau + \beta, \mathbf{x}) = e^{i\beta\nu} \phi(\tau, \mathbf{x}). \quad (6.3.5)$$

Such a theory (6.3.2) is Gaussian, (6.3.1) can hence be computed exactly with usual QFT techniques [218, 219]. However for latter purposes we shall derive it from a different perspective. We start by the easiest case of  $d = 1$  i.e. we have a thermal quantum mechanical system. Using the techniques developed in Sec. (6.2.1) we should be able to write  $Z_{d=1}$  like in (6.2.1) for a yet to determine Hamiltonian  $\hat{H}$ . An easy way to proceed is to start from the **real time** action (before eventually performing a ‘‘Wick rotation’’)

$$S := \int dt \mathcal{L} = \int dt \left( |\dot{\phi}|^2 + i\mu \left( (\partial_t \phi) \bar{\phi} - \phi (\partial_t \bar{\phi}) \right) + (\mu^2 - m^2) |\phi|^2 \right), \quad (6.3.6)$$

compute its canonical momenta

$$\pi_\phi := \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \bar{\phi} + i\mu \bar{\phi} \quad , \quad \pi_{\bar{\phi}} := \frac{\partial \mathcal{L}}{\partial(\partial_t \bar{\phi})} = \partial_t \phi - i\mu \phi, \quad (6.3.7)$$

to deduce the Hamiltonian

$$\hat{H} = \hat{\pi}^\dagger \hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} + i\mu \left( \hat{\pi} \hat{\phi} - \hat{\phi}^\dagger \hat{\pi}^\dagger \right), \quad (6.3.8)$$

with  $\hat{\pi} = \hat{\pi}_\phi$ ,  $\hat{\pi}^\dagger = \hat{\pi}_{\bar{\phi}}$  and canonical commutation relations  $[\hat{\phi}, \hat{\pi}] = i$  and  $[\hat{\phi}^\dagger, \hat{\pi}^\dagger] = i$ . Introducing

<sup>18</sup>Note that in the context of the AdS/CFT correspondence, the charge of the black hole state translates into the chemical potential of the dual CFT.

<sup>19</sup>This is the usual action with gauge derivative  $D_\mu = \partial_\mu - igA_\mu$  of coupling  $g$  set to one in our model.

ladder operators  $\hat{a}$ ,  $\hat{b}$  (and h.c.) via

$$\hat{\phi} = \frac{1}{\sqrt{2m}}(\hat{a}^\dagger + \hat{b}) \quad , \quad \hat{\pi} = i\sqrt{\frac{m}{2}}(\hat{b}^\dagger - \hat{a}) \quad , \quad (6.3.9)$$

with commutation relations  $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$  one can re-write (6.3.8) as

$$\hat{H} = m(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1) + \mu(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \quad . \quad (6.3.10)$$

We recognise the Hamiltonian of two-coupled harmonic oscillators twisted by a chemical potential  $\mu$ . Recalling the number operators  $\hat{N}_1 = \hat{a}^\dagger \hat{a}$  and  $\hat{N}_2 = \hat{b}^\dagger \hat{b}$  we get the total Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \{|n_1\rangle \otimes |n_2\rangle / n_1, n_2 \in \mathbb{N}\} \quad . \quad (6.3.11)$$

In terms of the usual position  $\hat{x}_1, \hat{x}_2$  and momentum operators  $\hat{p}_1, \hat{p}_2$  the ladder operators read

$$\hat{a} = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 - i\hat{x}_2) + (\hat{p}_2 + i\hat{p}_1)) \quad , \quad \hat{b} = \frac{1}{2\sqrt{m}}(m(\hat{x}_1 + i\hat{x}_2) - (\hat{p}_2 - i\hat{p}_1)) \quad , \quad (6.3.12)$$

which yields for the real Hamiltonian

$$\hat{H} = \underbrace{\frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2)}_{\hat{H}_0} + m^2 \underbrace{\frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2)}_{\mathcal{O}} + \mu \underbrace{(\hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2)}_{\mathcal{Q}} \quad . \quad (6.3.13)$$

This is the Hamiltonian of a free theory  $\hat{H}_0$  deformed by a mass operator  $\mathcal{O}$  and a charge operator  $\mathcal{Q}$ . All this was in real time formalism, to go back to the thermal theory we perform the ‘‘Wick rotation’’  $\mu \rightarrow -i\mu$  and express  $Z_{d=1}$  like in (6.2.1)

$$Z_{d=1} = \text{Tr}_{\mathcal{H}} \left[ e^{-\beta(\hat{H}_0 + m^2 \mathcal{O})} e^{-i\beta\mu \mathcal{Q}} \right] \quad , \quad (6.3.14)$$

which can be explicitly computed

$$Z_{d=1} = \frac{e^{-\beta m}}{(1 - e^{-\beta m - i\beta\mu})(1 - e^{-\beta m + i\beta\mu})} \quad . \quad (6.3.15)$$

It is actually more convenient to unpack  $m$  and  $\mu$  in a unique complex variable  $z$

$$z = e^{-\beta m - i\beta\mu} \iff \bar{z} = e^{-\beta m + i\beta\mu} \quad , \quad (6.3.16)$$

and to compute the logarithm of the partition function

$$\ln Z_{d=1} = \ln |z| - \ln |1 - z| - \ln |1 - \bar{z}| \quad , \quad (6.3.17)$$



which can be put into an integral form

$$\ln Z_{d=1} = \int_0^z \frac{dz'}{1-z'} + \int_0^{\bar{z}} \frac{dz'}{1-z'} - \int_{|z|}^1 \frac{dz'}{z'}. \quad (6.3.18)$$

We can now compute the  $d$ -dimensional partition function, however restricting ourselves to the odd-dimensional case i.e. we set  $d = 2L + 1$  with  $L = 0, 1, 2, \dots$  (so  $\ln Z_{d=1} := \ln Z_0$ ). This is actually a standard textbook computation. We assume that our system lives in a  $(d - 1)$ -dimensional cubic box of volume  $V_{d-1} = \ell^{d-1}$  with quantized momentum  $\vec{p} = \left(\frac{2\pi}{\ell}n_1, \dots, \frac{2\pi}{\ell}n_{d-1}\right) = \frac{2\pi}{\ell}\vec{n}$ . The number of modes having momenta inside the spherical shell bounded by  $|\vec{p}|$  and  $|\vec{p}| + d|\vec{p}|$  in  $d = 2L + 1$  dimensions is

$$dn = \left(\frac{\ell^2}{4\pi^2}\right)^L |\vec{p}|^{2L-1} d|\vec{p}| \int d\Omega_{2L}, \quad (6.3.19)$$

with  $\int d\Omega_{2L} = 2\pi^L/\Gamma(L)$ . Using then the dispersion relation  $\omega^2 = \vec{p}^2 + m^2$ , for  $\rho_L(\omega; m) \equiv dn/d\omega$  we obtain

$$\ln Z_L = \frac{2\alpha^2\beta^2}{(L-1)!} \int_m^\infty \omega d\omega (\omega^2 - m^2)^{L-1} \ln Z_0; \quad (6.3.20)$$

with  $\alpha^2 = \ell^2/4\pi\beta^2$ . Hence we have written  $\ln Z_L$  as that of a  $d = 2L + 1$ -dimensional relativistic thermal gas of bosons

$$\ln Z_L = \int d\omega \rho_L(\omega; m) \ln Z_0, \quad (6.3.21)$$

with density of state

$$\rho_L(\omega; m) = \frac{2\alpha^2\beta^2}{(L-1)!} \omega (\omega^2 - m^2)^{L-1}. \quad (6.3.22)$$

(6.3.20) will actually allow us to write  $\ln Z_L$  in a particularly convenient form. Observing that

$$\frac{(\omega^2 - m^2)^{L-1}}{(L-1)!} = \int_m^\infty dm' 2m' \frac{(\omega^2 - m'^2)^{L-2}}{(L-2)!} \quad (6.3.23)$$

we see that (6.3.21) can be written, for  $L > 1$ , as iterated integrals

$$\ln Z_L = \frac{\ell^{2L}}{(2\pi)^L} \int_m^\infty dm_1 m_1 \int_{m_1}^\infty dm_2 m_2 \cdots \int_{m_{L-1}}^\infty dm_L m_L \ln Z_0(\beta; m_L, \mu), \quad (6.3.24)$$

i.e.

$$\boxed{\ln Z_L = (-2\alpha^2)^L \prod_{i=0}^{L-1} \left[ \int_0^{w_{i+1}} \frac{dw_i}{w_i} \ln w_i \right] \ln Z_0}, \quad (6.3.25)$$

where  $\ln Z_0$  is taken to be a function of  $z_0, \bar{z}_0$  with  $z_0 = w_0 e^{-i\beta\mu}$ ,  $w_L = |z|$  and the integrals are performed in the order  $w_0 \mapsto w_1 \dots \mapsto w_L$ . The conclusion is that integrating the  $d = 1$  harmonic oscillator partition function actually generates the higher dimensional ones.

**Remark** In all these derivations we have used an imaginary chemical potential. This may seem odd but actually, the imaginary character of this deformation parameter allows for numerical

studies like Monte-Carlo to converge. A complex fermionic determinant arises in the case of a real chemical potential occurring a sign problem in the algorithm which breaks down. It has been shown however that working with an imaginary chemical potential and then analytically continue it to real values does not blur any information (see e.g. [220]). For more works on imaginary chemical potential see e.g. [221–224].

**Remark** Computations of relativistic Bose-gas integral and their expansion at high and low temperature carried out in full generality in [225].

### The emergence of single-valued polylogarithms

Using (6.3.25) and knowing (6.3.18) we see that  $\ln Z_L$  coincides with the class of iterated integrals giving rise to single-valued polylogarithms

$$\ln Z_L = \alpha^{2L} \frac{(-1)^L L!}{2(2L+1)!} (2 \log |z|)^{2L+1} + \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)! n!} 2\Re[\text{Li}_{2L+1-n}(z)], \quad (6.3.26)$$

where we have carefully perform the regularization of the zero point energy, subtracting the zero temperature partition function at  $m = \mu = 0$ . The thermal expectation value of the  $U(1)$ -charge operator associated to the chemical potential  $\mu$ , which reads  $Q = \phi^\dagger \overleftrightarrow{D}_\tau \phi$ , can also be expressed in terms of polylogarithms, however not with a sum but rather a difference

$$\langle Q \rangle_L = \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)! n!} 2i\Im[\text{Li}_{2L-n}(z)]. \quad (6.3.27)$$

This difference of polylogarithms has been related to graphical functions in [216].

Given the iterated integral structure of (6.3.25) one expects differential relations between the partition functions in different odd dimensions. We introduce the differential operators

$$\hat{\mathbf{D}} = \frac{1}{\beta^2} \frac{\partial}{\partial m^2} = \frac{1}{2 \ln |z|} (z \partial_z + \bar{z} \partial_{\bar{z}}), \quad (6.3.28a)$$

$$\hat{\mathbf{L}} = \frac{i}{\beta} \frac{\partial}{\partial \mu} = (z \partial_z - \bar{z} \partial_{\bar{z}}). \quad (6.3.28b)$$

Explicit calculations yield the following set of first order differential equations [200]

$$\langle O \rangle_L = -\beta \hat{\mathbf{D}} \ln Z_L = \beta \alpha^2 \ln Z_{L-1} \quad (6.3.29a)$$

$$\langle Q \rangle_L = \hat{\mathbf{L}} \ln Z_L = -\hat{\mathbf{D}} \cdot \langle Q \rangle_{L+1} / \alpha^2. \quad (6.3.29b)$$

Notice that  $\hat{\mathbf{D}}$  acts on  $\ln Z_L$  and  $\langle Q \rangle_L$  as a dimension lowering operator. Introducing the Laplacian in the variables  $m$  and  $\mu$  as

$$\hat{\Delta} = 4\beta^2 z \bar{z} \partial_z \partial_{\bar{z}} = \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial \mu^2}, \quad (6.3.30)$$

we further find

$$\hat{\Delta} f_L(z, \bar{z}) = -4\beta^2 L \alpha^2 f_{L-1}(z, \bar{z}), \quad (6.3.31)$$

for  $f_L(z, \bar{z}) = \{\ln Z_L, \langle \mathcal{Q} \rangle_L\}$ . We can combine (6.3.31) with (6.3.29a), (6.3.29b) to obtain the second order equation

$$[m^2 \hat{\Delta} - 4L\beta^2 m^2 \hat{\mathbf{D}}] f_L(z, \bar{z}) = 0. \quad (6.3.32)$$

Notice that  $m^2 \hat{\Delta}$  is the Laplacian on the upper half plane  $\mathbb{H}_2$  with coordinates  $m, \mu$  and  $2\beta^2 m^2 \hat{\mathbf{D}} = m(\partial/\partial m)$  is the radial derivative. Equation (6.3.32) is reminiscent of similar results for partition functions in [226] where the a connection to the huge literature of string scattering amplitudes [227–229] was noted. Another interpretation of (6.3.32) is as the Laplace-Beltrami operator of  $AdS_{2L+2}$  with metric

$$ds^2 = \frac{1}{m^2} \left( dm^2 + d\mu^2 + \sum_{i=1}^{2L} dx^i dx^i \right), \quad (6.3.33)$$

acting on functions of just  $m$  and  $\mu$ . Since  $m$  and  $\mu$  parametrize relevant deformations of a free CFT, such an interpretation may be related to RG flow.

As we already said, single-valued polylogarithms also appear in the realm of conformal ladder graphs and fishnet theories that we shall now discuss.

### 6.3.2 Twisted partition functions as conformal ladder graphs

Recall that  $d = 2L + 1$  is the dimension of the thermal correlator, while the coordinates are  $(\tau, \mathbf{x})$ . When dealing with conformal graph, we will denote the dimension of the latter  $D$  and the coordinates  $(x)$  to make a clear distinction between the two.

#### An observation

We consider the conformal integral in  $D = 4$

$$I(x_1, x_2, x_3, x_4) = \frac{1}{\pi^2} \int d^4x \frac{1}{(x-x_1)^2(x-x_2)^2(x-x_3)^2(x-x_4)^2} \quad (6.3.34)$$

which reads, in terms of the variables (6.1.24)

$$I(x_1, x_2, x_3, x_4) = \frac{1}{x_{13}^2 x_{24}^2} \Phi(v, u), \quad (6.3.35)$$

and corresponds to a graph of the form of Fig. (6.1). Using conformal invariance we can take the limit of that graph fixing the values of three points, namely  $x_1 \rightarrow 0$ ,  $x_3 \rightarrow \infty$  and  $x_4 \rightarrow 1$ , and leaving  $x_2 \rightarrow z$  with  $z$  defined in (6.1.25). It yields, upon rescaling

$$\lim_{(x_1, x_2, x_3, x_4) \rightarrow (0, z, \infty, 1)} x_3^2 I(x_1, x_2, x_3, x_4) = \frac{1}{\pi^2} \int d^4x \frac{1}{x^2 (x-z)^2 (x-1)^2} := \Phi_4^{(1)}(z, \bar{z}) \quad (6.3.36)$$

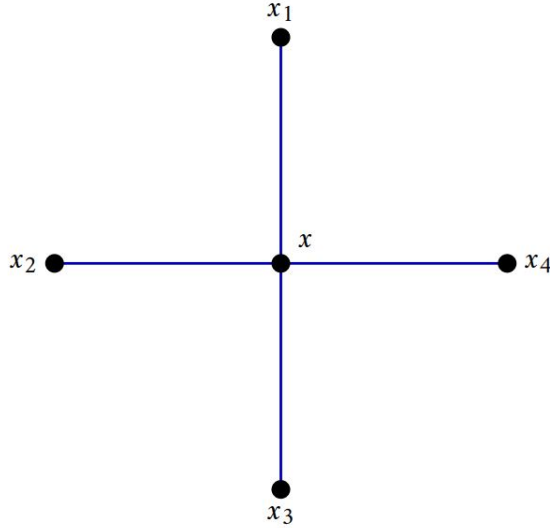


Figure 6.1: *The 4-point function (6.3.35).*

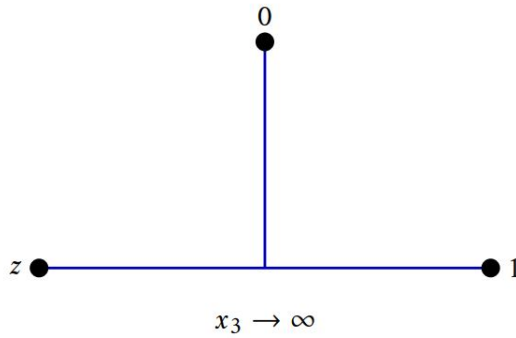


Figure 6.2: *Conformal limit of the 4-points function (6.3.36).*

where in  $\Phi_4^{(1)}$  the upper index refers to the number of vertical lines in the graph 6.1 (i.e. here we work at tree level) while the lower index refers to the dimension of the underlying theory. The associated graph is then Fig. (6.2).

It turns out that (6.3.36) can be written in terms of our beloved polylogarithms

$$\Phi_4^{(1)}(z, \bar{z}) = \frac{1}{z - \bar{z}} 4i [\text{Li}_2(z) - \text{Li}_2(\bar{z}) + \ln |z| (\ln |1 - z| - \ln |1 - \bar{z}|)] . \quad (6.3.37)$$

One may then see a relation between our thermal expectation values  $\langle O \rangle$  and  $\langle Q \rangle$  and the graph Fig. 6.1. And this is actually true; comparing (6.3.27) with (6.3.37) we find

$$\Phi_4^{(1)}(z, \bar{z}) = \frac{1}{a^2} \frac{1}{z - \bar{z}} \langle Q \rangle_1 , \quad (6.3.38)$$

setting the stage to a correspondence glimpsed in [200].

**Remark** Equation (6.3.37) can actually be expressed in terms of the Bloch-Wigner function

$$D(z) = \Im[\text{Li}_2(z) + \ln|z| \ln|1-z|] \quad (6.3.39)$$

that gives the volume of an ideal tetrahedron in three-dimensional hyperbolic space  $\mathbb{H}^3$  with vertices attached to the boundary  $\partial\mathbb{H}^3$  [230]. Note also that  $\langle Q \rangle_0$  itself has a geometric interpretation. Indeed,

$$\langle Q \rangle_0 = \frac{z - \bar{z}}{(1-z)(1-\bar{z})}. \quad (6.3.40)$$

and setting  $z = e^{i\phi}(b/a)$  with  $\cos\phi = (a^2 + b^2 - 1)/2ab$  we find that  $\langle Q \rangle_0/4i = \frac{1}{2}ab \sin\phi$  gives the area of a triangle whose side lengths are  $a, b$  and  $1$ , and  $\phi$  the angle between  $a$  and  $b$ . Then (6.3.38) gives the volume of an ideal hyperbolic tetrahedron as an integral of the area of a triangle. One then wonders if there is a geometric interpretation for the higher order iterated integrals in (6.3.24).

Going to loop level and computing loop-diagrams coming from (6.1) we showed in [207] that

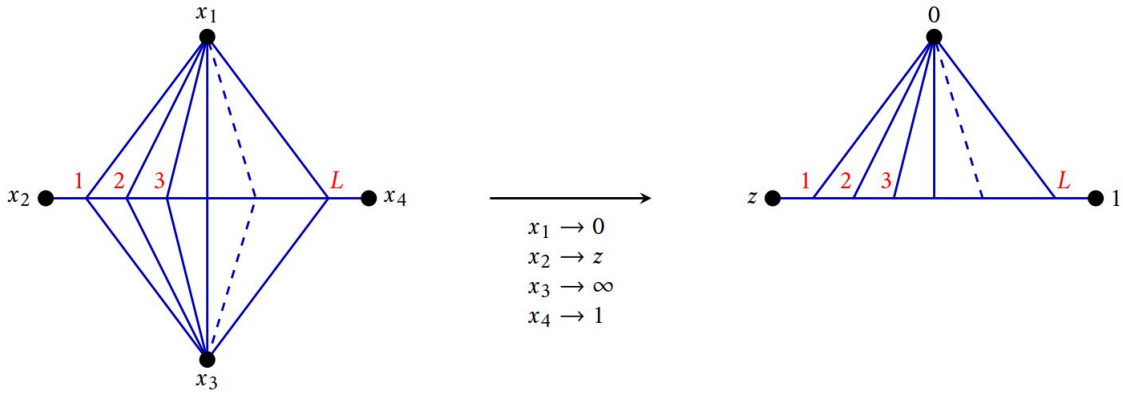


Figure 6.3:  $L$ -loops expansion of the graph  $I$ .

$$\Phi_4^{(L)}(z, \bar{z}) = \frac{1}{L!} \frac{1}{z - \bar{z}} \langle Q \rangle_L, \quad (6.3.41)$$

showing indeed that this particular thermal expectation value in various odd dimensions  $d = 2L + 1$  corresponds to a  $L$ -loop conformal ladder graph in  $D = 4$ . But what is the theory leading to the graphs 6.3? It actually comes from a special type of bi-scalars theories dubbed *fishnet theories*, on which we say a few words now.

## Fishnet theories

These are particular limits of the generalised bi-scalar theory in  $D$ -dimensions first formulated in [205] (see also [206]) with Lagrangian

$$\mathcal{L} = N_c \text{Tr} \left[ \phi_1^\dagger (-\partial^2)^\omega \phi_1 + \phi_2^\dagger (-\partial^2)^{\frac{D-2\omega}{2}} \phi_2 + a_{D,\omega}^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right]. \quad (6.3.42)$$

$\phi_{1,2}$  belong to the adjoint of  $SU(N_c)$ ,  $\omega \in \left(0, \frac{D}{2}\right)$  and coupling  $a_{D,\omega}^2$  is classically dimensionless. The fractional box operator ( $\square = \partial_\mu \partial^\mu$ ) is defined as

$$(\partial_\mu \partial^\mu)^\gamma := \frac{(-4)^\gamma \Gamma\left(\frac{D}{2} + \gamma\right)}{\pi^{D/2} \Gamma(-\gamma)} \int d^D x \frac{f(y)}{|x-y|^{D+2\gamma}}, \quad (6.3.43)$$

when acting of a scalar field  $f(x)$ , leading a (position-space) propagator of the form (by definition solution of  $(\partial_\mu \partial^\mu)^\gamma G(x) = \delta^{(D)}(x)$ )

$$G(x-y) = \frac{\Gamma\left(\frac{D}{2} + \gamma\right)}{4^\gamma \pi^{D/2} \Gamma(\gamma) |x-y|^{D-2\gamma}}. \quad (6.3.44)$$

In (6.3.42) the scaling dimensions of the operators  $\phi_1$  and  $\phi_2$  are

$$\Delta_{\phi_1} = \frac{D-2\omega}{2} \quad \text{and} \quad \Delta_{\phi_2} = \omega, \quad (6.3.45)$$

where we see the two relevant label of the fishnet theory, its dimension  $D$  and its order  $\omega$ . We now consider the four-point function function

$$G_{D,\omega}^{(L)}(\{x_i\}) = \langle \text{Tr} \left[ \phi_2^L(x_1) \phi_1(x_3) \phi_2^{\dagger L}(x_2) \phi_1^\dagger(x_4) \right] \rangle, \quad (6.3.46)$$

whose leading  $N_c$  contribution comes from a unique  $L$ -loop conformal ladder graph.<sup>20</sup> Using again

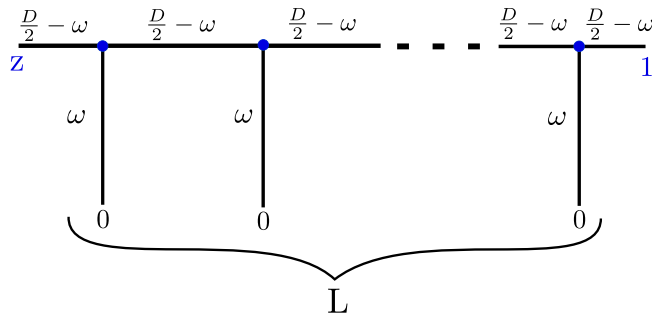


Figure 6.4: The graph contributing to  $G_{D,\omega}^{(L)}$ .

conformal invariance we can reduce the computation of (6.3.46) to the one of the graph 6.4 which

<sup>20</sup>The trace acts on internal  $SU(N_c)$  indices.

is exactly (6.3).

Hence for  $D = 4, \omega = 1$  the model coincides with the original four-dimensional conformal fishnet CFT introduced in [205], and then  $G_{4,1}^{(L)}$  is proportional to the Davydychev-Usuykina  $L$ -loop conformal ladder graphs [202, 203]. Up to overall normalizations and using (6.3.27) we verify that

$$\boxed{\tilde{G}_{4,1}^{(L)}(z, \bar{z}) = \frac{1}{L!} \frac{1}{z - \bar{z}} \langle \mathcal{Q} \rangle_L(z, \bar{z})}, \quad (6.3.47)$$

when we set  $a_{4,1}^2 = \alpha^2$ . In writing (6.3.47) we have identified: i) the variable  $z$  representing conformal ratios on the l.h.s. with the modular-like parameter  $z$  of the thermal QFT on the r.h.s. and ii) the number of loops  $L$  on the l.h.s. with  $L = (d - 1)/2$  on the r.h.s. . We thus have found in fishnet graphs the underlying theory for  $\Phi_4^{(L)}(z, \bar{z})$  showed that the latter is hence a spin-1 operator.

Let's study now the case of the mass operator, namely  $\langle \mathcal{O} \rangle_L$ .

### What about $\langle \mathcal{O} \rangle_L$ ?

Giving more computational details we start with (6.3.46) at  $L$ -loops i.e.

$$G_{D,\omega}^{(L)}(\{x_i\}) = \int \prod_{i=1}^L d^D u_i \prod_{n=1}^{L+1} \frac{1}{|u_{n-1} - u_n|^{D-2\omega}} \prod_{n=1}^L \left( \frac{1}{|x_1 - u_n|^{2\omega}} \frac{1}{|u_n - x_2|^{2\omega}} \right), \quad (6.3.48)$$

with  $u_0 := x_3$  and  $u_L := x_4$ . Due to conformal invariance, in terms of  $z, \bar{z}$  variables we get

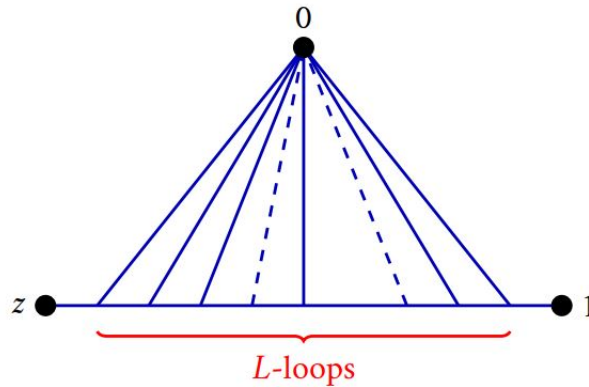


Figure 6.5: Conformal simplification of the integral (6.3.46).

$$G_{D,\omega}^{(L)}(\{x_i\}) = \frac{1}{(x_{12}^2)^{L\omega} (x_{34}^2)^{\frac{D-2\omega}{2}}} \frac{(1-z)(1-\bar{z})}{z-\bar{z}} I_L(z, \bar{z}), \quad (6.3.49)$$

where  $I_L(z, \bar{z})$  can be computed in the conformal limit like in (6.3.36) (see the graph 6.5). For  $D = 2, \omega = 1$  the model (6.3.42) is singular as  $G_{2,1}^{(L)}$  would seem to vanish.<sup>21</sup> Nevertheless, a nonzero result can be obtained if we define the effective coupling

$$\tilde{a}_{D,\omega} = a_{D,\omega} \frac{1}{\Gamma(D/2 - \omega)}, \quad (6.3.50)$$

that remains finite as  $D \mapsto 2, \omega \mapsto 1$ . Then, following the graph-building techniques introduced in [231–234] we can show that the appropriately normalised four-point function of Fig. 6.5 is given by

$$\tilde{G}_{2,1}^{(L)}(z, \bar{z}) = \tilde{a}_{2,1}^{2L} \sum_{m \in \mathbb{Z}} \int d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}}. \quad (6.3.51)$$

Since  $|z| < 1$  we compute the integrals above using contour intergration. When  $m \neq 0$  we can close the contour from below and pick up the residues in the lower half complex plane. We obtain

$$\begin{aligned} \sum_{m \neq 0} \int d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}} &= -\frac{2\pi i}{L!} \sum_{m \neq 0} \left(\frac{z}{\bar{z}}\right)^{\frac{m}{2}} \frac{d^L}{d\nu^L} \frac{(z\bar{z})^{i\nu}}{(\nu - \frac{i|m|}{2})^{L+1}} \Big|_{\nu = -\frac{i|m|}{2}} \\ &= \frac{2\pi}{L!} \sum_{m \neq 0} \left[ (z/\bar{z})^{m/2} \frac{d^L}{d\nu^L} \frac{e^{2i \log |z| \nu}}{i(\nu - \frac{i|m|}{2})^{L+1}} \Big|_{\nu = -\frac{i|m|}{2}} \right] \\ &= \frac{2\pi}{L!} \sum_{m \neq 0} \left[ (z/\bar{z})^{m/2} \sum_{n=0}^L \binom{L}{n} \left( \frac{d^n}{d\nu^n} e^{2i \log |z| \nu} \frac{d^{L-n}}{d\nu^{L-n}} \frac{1}{i(\nu - \frac{i|m|}{2})^{L+1}} \right) \Big|_{\nu = -\frac{i|m|}{2}} \right] \\ &= \frac{2\pi}{L!} \sum_{m \neq 0} \left[ (z/\bar{z})^{m/2} \sum_{n=0}^L \frac{L!}{(L-n)!n!} (2i \log |z|)^n (z\bar{z})^{|m|/2} (-1)^{L-n} \times \right. \\ &\quad \left. \times \frac{1}{i(-i|m|)^{2L+1-n}} \frac{(2L-n)!}{L!} \right] \\ &= \frac{2\pi}{L!} \sum_{m \neq 0} \left[ (z/\bar{z})^{m/2} \sum_{n=0}^L \frac{(2L-n)!}{(L-n)!n!} (-1)^n (2 \log |z|)^n (z\bar{z})^{|m|/2} \frac{1}{|m|^{2L+1-n}} \right] \\ &= \frac{2\pi}{L!} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)!n!} 2\Re [Li_{2L+1-n}(z)] \quad (6.3.52) \\ &= \frac{2\pi}{\alpha^{2L} L!} \ln Z_L^{\beta \neq 0}. \end{aligned}$$

For  $m = 0$  the contour integral appears to be zero, but there is a pole on the real axis which we need to circumvent by a small semicircle  $C_\epsilon$  between  $-\epsilon$  and  $\epsilon$  run counterclockwise. The contour integral is now zero since it does not include any poles, Taking the Cauchy principal value we obtain

$$- \int_{C_\epsilon} d\nu \frac{|z|^{2i\nu}}{\nu^{2L+2}} = -i \int_\pi^{2\pi} d\theta \frac{\exp(2i\epsilon \log |z| e^{i\theta})}{\epsilon^{2L+1} e^{i(2L+1)\theta}}. \quad (6.3.53)$$

<sup>21</sup>The  $x$ -space two-point function with Lagrangian  $\mathcal{L} = \phi(-\partial^2)^a \phi$  in  $d = 2L + 1$ -dimensions is  $C_\phi^L(a)/x^{2L+1-2a}$ , with  $C_\phi^L(a) = \Gamma(L + 1/2 - a)/\Gamma(a) 4^a \pi^{L+1/2}$ .



For  $\epsilon \mapsto 0$  we encounter  $2L + 1$  divergent terms, which we discard, and a finite contribution which reads

$$-i \int_{\pi}^{2\pi} d\theta \frac{(2i \log |z|)^{2L+1}}{(2L+1)!} = (-)^L \pi \frac{(2 \log |z|)^{2L+1}}{(2L+1)!}. \quad (6.3.54)$$

Putting together (6.3.52) and (6.3.54) we finally obtain

$$\boxed{G_{2,1}^{(L)}(z, \bar{z}) = \frac{2\pi}{L!} \ln Z_L(z, \bar{z})}, \quad (6.3.55)$$

when we set  $\tilde{a}_{2,1}^2 = \alpha^2$ . This is one of the main results of [207].<sup>22</sup> This then relates the  $D = 2, \omega = 1$  fishnet to a spin-0 operator.

**Remark** The differential operators  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{D}}$  together with the relations (6.3.29a) and (6.3.29b) give relations among the graphs. Actually, acting with  $\hat{\mathbf{L}}$  on both sides of (6.3.55) and using (6.3.29b) we see that the ladder graphs of the four-dimensional CFT are derivatives of the corresponding ladder graphs of the two-dimensional CFT. This dimension-shift property between conformal ladder graphs generalises to all even dimensions.

## The correspondence

We have then set up a relationship between two *a priori* distinct areas of field theory. On the one hand we have thermal expectation values of deformation operator (mass  $m$  and chemical potential  $\mu$  packaged in  $z, \bar{z}$ ) of a  $d = 2L + 1$  dimensional theory, which can all be obtained from the partition function of a twisted harmonic oscillator in one-dimension. On the other hand we have fishnet conformal graphs in ( $D = 2, \omega = 1$ ) and ( $D = 4, \omega = 1$ ) computed at  $L$ -loops and at coordinates points  $z, \bar{z}$ . Everything is summarised in the next Table.

Graphs	Thermal 1-point functions
$D$	?
$L$ -loops	$d = 2L + 1$
$x_i = (0, 1, z, \infty)$	$z = e^{-\beta m - i\beta\mu}$
coupling $a_{4,1}^2$	$\alpha^2 \frac{l^2}{4\pi\beta^2}$
$I_L(z, \bar{z})_{D=2, \omega=1}$	$\ln Z_L(z, \bar{z})$
$I_L(z, \bar{z})_{D=4, \omega=1}$	$\langle Q \rangle_L$

(6.3.56)

The question mark in the first row correspond to a quantity not defined yet, which we will introduce in Section 6.4.1.

<sup>22</sup>Notice that the leading “zero temperature” contributions in (6.3.55) and (6.3.14) arise after the subtraction of a finite number of divergent terms.

## Back to the thermal scalar two-point function

The thermal two-point function of a massive and twisted scalar field, like any other, can be written in the form (6.0.1) with  $a_{\mathcal{O}_s}$  thermal expectation values of spin- $s$  quasi-primary operators appearing in the OPE  $\phi \times \phi$ , which is, when keeping things as general as possible, made of higher-spin, higher-twist operators. The sector annihilated by the Laplacian  $\square_d$ , as we saw in (6.2.16), is the one where zero-twist higher-spins operators lie. Focusing on that part and expanding (6.2.34) in Gegenbauers we obtain

$$a_{\mathcal{O}_s}^L = \frac{\Gamma(L - \frac{1}{2})}{\Gamma(L + s - \frac{1}{2})} \frac{1}{(4\pi)^L 2^{2s}} \sum_{n=0}^{L-1+s} \frac{2^n (\beta m)^n (2L - 2 + s - n)!}{n! (L - 1 + s - n)!} \times [\text{Li}_{2L-1+s-n}(z) + (-1)^s \text{Li}_{2L-1+s-n}(\bar{z})] . \quad (6.3.57)$$

If the theory were a CFT we would associate the coefficients  $a_{\mathcal{O}_s}^L$  with thermal one-point functions of conformal quasi-primary operators. For generic values of  $m$  and  $\mu$  this is more complicated. For example,  $a_{\mathcal{O}_2}^L$  represents the contribution of a rank-2 symmetric traceless tensor which is *not* the the energy momentum tensor of the massive theory since the latter has nonzero trace. Nevertheless, the coefficients  $a_{\mathcal{O}_0}^L$  and  $a_{\mathcal{O}_1}^L$  do represent the thermal one-point functions of the operators  $\mathcal{O}$  and  $\mathcal{Q}$  as they have been independently calculated in (6.3.29a), (6.3.27). Also these are the only necessary building blocks for computing the one-point functions of all higher spin fields. Explicitly we have

$$a_{\mathcal{O}_0}^L = \frac{1}{(4\pi)^L \beta \alpha^{2L}} \langle \mathcal{O} \rangle_L, \quad a_{\mathcal{O}_1}^L = \frac{1}{(4\pi)^L \alpha^{2L}} \frac{1}{2} \langle \mathcal{Q} \rangle_L . \quad (6.3.58)$$

Using (6.3.29a), (6.3.29b) we see that for  $z = \bar{z} = 1$  the above reduce to (6.2.15) as they should. The novel result is that *all* coefficients  $a_{\mathcal{O}_s}^L$  with  $s \geq 2$  are related to  $L$ -loop conformal graphs by virtue of the following recursion relations shown by brute force calculations

$$a_{\mathcal{O}_{s+2}}^L = \frac{2\pi}{2L-1} a_{\mathcal{O}_s}^{L+1} + \frac{(m\beta)^2}{(2L-1+2s)(2L+1+2s)} a_{\mathcal{O}_s}^L \quad (6.3.59)$$

Consequently, the part of the twisted thermal two-point function (6.2.34) that is annihilated by the  $d$ -dimensional Laplacian<sup>23</sup> is a generating function for (linear combinations) of  $L$ -loop conformal ladder graphs.

As we have related  $a_{\mathcal{O}_0}^L$  and  $a_{\mathcal{O}_1}^L$  to spin-0 and spin-1 operators one may legitimately wonder if this is also the case for higher spins. Let's consider the spin-2 case i.e. the energy-momentum tensor  $t_{\mu\nu}$ . Eq. (6.3.59) actually corresponds to a standard hydrodynamic result. From the twisted partition function (6.3.21) with Hamiltonian  $H = H_0 + m^2 \mathcal{O} + i\mu \mathcal{Q}$  we can derive

$$\langle H \rangle_L = \frac{d-1}{\beta} \ln Z_L + 2m^2 \langle \mathcal{O} \rangle_L + i\mu \langle \mathcal{Q} \rangle_L, \quad (6.3.60)$$

where  $\langle H \rangle_L = -\langle t_{\tau\tau} \rangle_L$ . When the CFT gets deformed by  $m$  and  $\mu$  the energy-momentum tensor

<sup>23</sup>Which we recall is the one corresponding to zero-twist higher-spin operators.

is no longer traceless, however it is possible to build from it a traceless spin-2 operator  $\mathcal{T}_{\mu\nu}$  with  $\mathcal{T}_{\tau\tau} = t_{\tau\tau} + 2m^2\mathcal{O}/d + i\mu\mathcal{Q}$ . Then (6.3.60) becomes

$$-\langle\mathcal{T}_{\tau\tau}\rangle_L = \frac{d-1}{\beta} \ln Z_L + 2m^2 \frac{d-1}{d} \langle\mathcal{O}\rangle_L. \quad (6.3.61)$$

The general relation connecting  $a_{\mathcal{O}_2}^L$  with the  $\mathcal{T}_{\tau\tau}$  is <sup>24</sup>

$$\frac{(4\pi\alpha^2)^L}{\beta} a_{\mathcal{O}_2}^L = \frac{2g_{\phi^\dagger\phi\mathcal{T}}}{(d-1)(d-2)C_{\mathcal{T}}} \langle T_{00}\rangle_L = -\frac{C_{\phi}^L(1)S_L}{2(d-1)} \langle\mathcal{T}_{\tau\tau}\rangle. \quad (6.3.62)$$

Using then (6.3.58), (6.3.29a) and (6.3.29b) we can verify that (6.3.61) coincides with (6.3.59). We believe that similar arguments relating tracefull and traceless higher spin operators of the massive free scalar theory can provide a physical understanding for (6.3.59) for general  $s$ , but this is yet to be understood.

## Summarising

We have connected two seemingly unrelated quantities: twisted partition functions of a massive free complex scalar field in  $d = 2L + 1$  dimensions, and four-point conformal  $L$ -loop ladder graphs. The reason for such a relationship is that they both satisfy the same sets of differential equations. For the partition functions these are given by (6.3.32). For the conformal ladder graphs they are the differential equations discussed in number of earlier works on conformal integrals (i.e. Eq. 2.15 in [235]). We do not have however an explanation for this common property.

Our results draw a unifying picture for the thermal one-point functions  $a_{\mathcal{O}_s}^L$  in massive free complex scalar theories. This is depicted in Fig. (6.6). By the algebraic relations (6.3.59) they are all ultimately given by  $a_{\mathcal{O}_0}^L$  or  $a_{\mathcal{Q}_1}^L$ , and then by the action of the differential operators  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{L}}$  to the  $\langle\mathcal{O}\rangle_0$  and  $\langle\mathcal{Q}\rangle_0$  of the harmonic oscillator model (6.3.14).

## 6.4 Some applications

### 6.4.1 Constructing fishnet graphs from partition functions in $D > 4$

Starting from  $\ln Z_0$  (see (6.3.17)) we constructed  $\ln Z_L$  for  $L = 0, 1, \dots$  and from it  $\langle\mathcal{O}\rangle_L$  and  $\langle\mathcal{Q}\rangle_L$  which are the conformal ladder graphs in  $D = 2$  and  $D = 4$  respectively. The differential relations we exhibited in (6.3.29a) and (6.3.29b) corresponds then to relations between graphs at different loop level. The second order equation (6.3.32) is brand new and should call for further investigations

<sup>24</sup>We use the standard free CFT results for the three-point function coupling  $g_{\phi^\dagger\phi\mathcal{T}}$ , the normalization of the two-point function of  $\mathcal{T}_{\mu\nu}$ ,  $C_{\mathcal{T}}$  and we take into account that a complex scalar corresponds to two real scalars.  $S_L = 2\pi^{L+1/2}/\Gamma(L+1/2)$  is the surface of the  $2L+1$ -dimensional unit sphere.

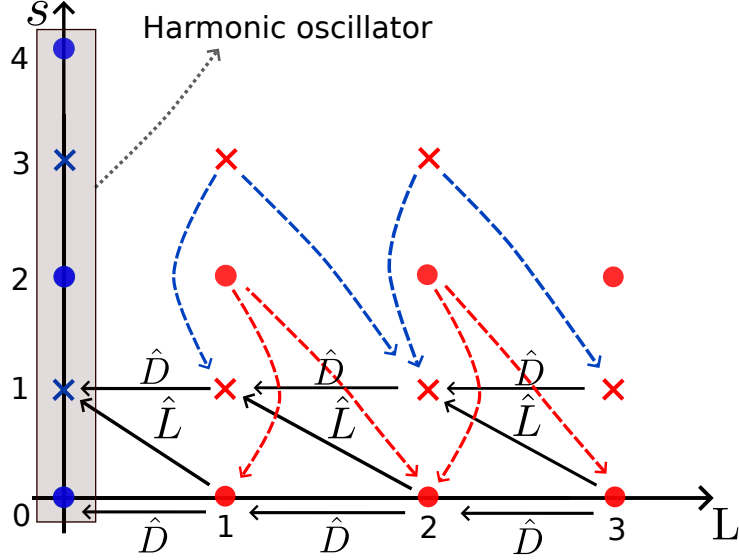


Figure 6.6: Differential (solid lines) and algebraic (dashed lines) relationships among the  $a_{O_s}^L$ .

which are not part of our present agenda though. However we want now to show how one can construct the graphs in  $D > 4$  using the  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{L}}$  operators. For that purpose we need an additional operator.

We see with (6.3.29a) that  $\hat{\mathbf{D}}$  lowers the number of loops  $L$ , hence its inverse  $\hat{\mathbf{D}}^{-1}$  that we will denote  $\hat{\mathbf{d}}$  raises  $L$ . Its expression is

$$\hat{\mathbf{d}} = 2\beta^2 \int_m^\infty \omega d\omega, \quad (6.4.1)$$

and it satisfies

$$\hat{\mathbf{d}}\langle \hat{\mathcal{O}} \rangle_L = -\frac{1}{\alpha^2} \langle \mathcal{O} \rangle_{L+1}. \quad (6.4.2)$$

Hence from  $\ln Z_0$  we construct  $\ln Z_L$  by a repeated action of the  $\hat{\mathbf{d}}$  operator. Now recall that we constructed the  $L$ -loops,  $D = 2$  conformal ladder graph from  $\ln Z_0$  in such a way

$$\hat{\mathbf{L}} \ln Z_0 = \langle \mathcal{Q} \rangle = \langle \hat{N}_1 - \hat{N}_2 \rangle = \frac{z - \bar{z}}{(1-z)(1-\bar{z})}, \quad (6.4.3)$$

with  $\hat{N}_i$  the number operators introduced after (6.3.10). Their difference is related to spin and will be denoted by

$$\hat{N}_1 - \hat{N}_2 := \hat{S}. \quad (6.4.4)$$

Introducing a new notation we write

$$\hat{\mathbf{L}} \ln Z_0 = \langle \mathcal{Q} \rangle = (z - \bar{z}) q_0^{(1)} \implies q_0^{(1)} = \frac{1}{|1-z|^2} \quad (6.4.5)$$

where we can think of  $q_0^{(1)}$  as the massless free two-point function for a scalar field in  $D = 4$  (which would then have scaling dimension  $\Delta_\phi = 2$ ). It is actually better to think of it as the “singular” part

of a bi-scalar theory of the type (6.3.42)

$$\langle \phi_1^{\dagger L}(x_1) \phi_2^{\dagger}(x_2) \phi_1^L(x_3) \phi_2(x_4) \rangle_{L=0} \xrightarrow{(x_1, x_2, x_3, x_4) \rightarrow (0, z, \infty, 1)} \frac{1}{|1-z|^{D-2}} \quad (6.4.6)$$

leading to the last of (6.4.5) when  $D = 4$ . This was for tree level, but as we just noticed acting with  $\hat{\mathbf{d}}$  we get the higher loops

$$\langle \mathcal{Q} \rangle_L = (-\alpha^2)^L \hat{\mathbf{d}}^L [(z - \bar{z}) q_0^{(1)}] := (z - \bar{z}) q_L^{(1)}, \quad (6.4.7)$$

so that the low index of  $q$  represents the number of loops. To understand the upper index let's generalise this construction

$$\begin{aligned} \left( \frac{1}{z - \bar{z}} \hat{\mathbf{L}} \right)^k \ln Z_0 &= \frac{1}{(z - \bar{z})^k} (\hat{\mathbf{L}} - k + 1)(\hat{\mathbf{L}} - k + 2) \dots \hat{\mathbf{L}} \ln Z_0 \\ &= \frac{1}{z - \bar{z}} \langle \hat{\mathbf{S}}(\hat{\mathbf{S}} - 1) \dots (\hat{\mathbf{S}} - k + 1) \rangle_{L=0} \\ &= \frac{1}{|1-z|^{2k}} := q_0^{(k)}, \end{aligned} \quad (6.4.8)$$

ans setting  $k = \frac{D}{2} - 1$  we can identify  $q_0^{(k)}$  with the singular part of the four-point function in  $D$ -dimensions with  $D > 4$ . The number  $k$  should then be identify to a kind of “spin”, the would-be spin of the operator whose thermal expectation value is related to these new graphs. Acting with a slightly modified version of  $\hat{\mathbf{d}}$  we obtain the  $L$ -loop graphs

$$q_{L+1}^{(k)} = \mathcal{D}^{(k)} q_L^{(k)} \quad (6.4.9)$$

with

$$\mathcal{D}^{(k)} = \frac{1}{(z - \bar{z})} [\hat{\mathbf{d}}(z - \bar{z})^k]. \quad (6.4.10)$$

Therefore we can actually construct conformal ladder graphs from the partition function of the twisted harmonic oscillator (6.3.14), in all even dimension  $D = 2k + 2$  with  $k \in \mathbb{N}$ . This general construction is summarised in Fig. 6.7.

## 6.4.2 Resummation of ladder graphs

Equations such as (6.3.29a) and (6.3.29b) lead naturally to the resummation of infinite series. For example, by virtue of (6.3.29a) the infinite product  $Z = \prod_{n=0}^{\infty} Z_n$  satisfies the inhomogenous first order equation

$$(\hat{\mathbf{D}} + \alpha^2) \ln Z = -\frac{1}{\beta} \langle \mathcal{O} \rangle_0. \quad (6.4.11)$$

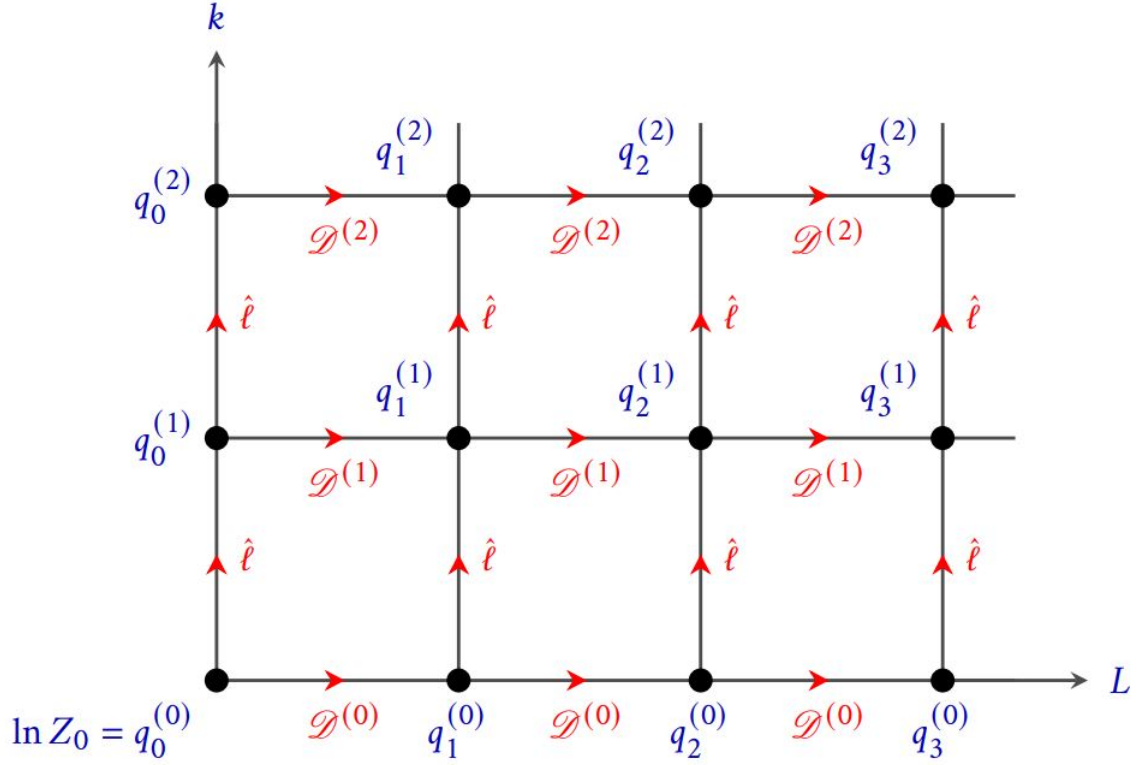


Figure 6.7: Generation of ladder graphs from  $\ln Z_0$  with repeated action of  $\hat{\ell} = \frac{\hat{\mathbf{L}}}{z-\bar{z}}$  and  $\mathcal{D}^{(k)}$ .

This can be integrated to

$$\ln Z = -\beta e^{-\beta^2 \alpha^2 m^2} \int^{m^2} e^{\beta^2 \alpha^2 \tilde{m}^2} \langle O \rangle_0 d\tilde{m}^2. \quad (6.4.12)$$

An analogous result can be derived for the  $\langle Q \rangle = \sum_{n=0}^{\infty} \langle Q \rangle_n$ . Given (6.3.47) and (6.3.55) these are all-loop Borel summations of conformal ladder graphs. Indeed applying  $\hat{\mathbf{L}}$  to (6.4.12) gives

$$(\hat{\mathbf{D}} + \alpha^2) \langle Q \rangle = \hat{\mathbf{D}} \langle Q \rangle_0 \quad (6.4.13)$$

where  $\langle Q \rangle = \sum_{L=0}^{\infty} \langle Q \rangle_L$ . By virtue of (6.3.47) this sum can be Borel transformed into the Broadhurst-Davydychev infinite sum of the  $L$ -loop conformal ladder graphs in four-dimensions [204], see also [236, 237]. Indeed the solution of the first order equation (6.4.13) is

$$\langle Q \rangle = \beta^2 e^{-\beta^2 \alpha^2 m^2} \int^{m^2} e^{\beta^2 \alpha^2 \tilde{m}^2} \hat{\mathbf{D}} \langle Q \rangle_0 d\tilde{m}^2, \quad (6.4.14)$$

and can be thought of as a series of the form  $\langle Q \rangle \equiv A(z) = \sum_{n=0}^{\infty} a_n z^n$  with  $z = \alpha^2$ . Its Borel transform series  $\mathcal{B}[A](t) := \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$  is given by the contour integral

$$\mathcal{B}[A](t) = \frac{1}{2\pi i} \int_C \frac{dz}{z} e^z A(t/z) \quad (6.4.15)$$

where  $C$  is the Hankel contour.<sup>25</sup> Using the following integral representation of the Bessel function

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{2\pi i} \int_C dt \frac{1}{t^{\nu+1}} e^{t - \frac{z^2}{4t}}, \quad (6.4.16)$$

we obtain

$$\begin{aligned} \mathcal{B}[Q](t) &= \beta^2 \frac{1}{2\pi i} \int^{m^2} d\tilde{m}^2 (\hat{\mathbf{D}}\langle Q \rangle_0) \int_C \frac{du}{u} e^{u - \frac{t\beta^2}{u}(m^2 - \tilde{m}^2)} \\ &= \beta^2 \int^{m^2} d\tilde{m}^2 J_0(2\beta\sqrt{t(m^2 - \tilde{m}^2)}) (\hat{\mathbf{D}}\langle Q \rangle_0) \end{aligned} \quad (6.4.17)$$

Using then

$$\hat{\mathbf{D}}\langle Q \rangle_0 = \frac{i}{2\beta m} \frac{\sinh(\beta m) \sin(\beta\mu)}{(\cosh(\beta m) - \cos(\beta\mu))^2}, \quad (6.4.18)$$

and setting  $t = -\frac{\kappa^2}{4}$ ,  $\beta\tilde{m} = \eta$ ,  $\ell = 2\beta m$  and putting the lower bound of the integral to be  $+\infty$ , (6.4.17) coincides with Eq. 15 of [204]. This is one of the main proof/application in favor of our new correspondence.

## 6.5 Outlook

In this chapter we exhibited a new correspondence between twisted partition functions in odd dimensions, thermal expectation values and conformal ladder graphs. All relevant quantities on one side (dimension, deformation parameters, twist) have been found to get a partner on the other side (number of loops, coordinate, dimension of the fishnet). Differential identities between the expectation values translates into unexpected relations between graphs at different loop level. Our correspondence proved useful in generating all bi-scalar fishnet conformal ladder graphs from the partition function of the harmonic oscillator, and also allows to resum them in a close expression. In  $D = 2$  and  $D = 4$  we showed that all loop graphs can be related to a spin 0 and a spin 1 operator constructed from the harmonic oscillator. It is then natural to generalize and seek the higher-spin operators whose thermal average give the conformal ladder graphs in  $D > 4$ .

Our results focused on bosonic fields, therefore a similar analysis starting from the fermionic harmonic oscillator should be performed to see what kind of graphs can be obtained. It is worth mentioning that this work opens several news paths and relations between areas of theoretical physics. In particular it offers a complementary approach to the AdS/CFT correspondence for the computation of thermal correlators. Also the relations between thermal one point function and geometry through the relationship with hypergeometry should be thrust further. Hence we believe to have opened a path towards a very exciting direction of research with this correspondence.

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<sup>25</sup>We use the convention that  $C$  starts at  $\infty - i\epsilon$  with  $\epsilon > 0$ , then encircles  $(0,0)$  counterclockwise and ends up to  $-\infty + i\epsilon$ .

# Conclusion

This thesis was an opportunity to explore two directions of research, aimed at extending and exploring aspects of the AdS/CFT gauge/gravity duality. The first direction points towards flat-space holography and Carrollian physics, whereas the second is related to thermal field theory.

In the first two parts of this manuscript we have presented some elements of Carrollian geometry and Carrollian physics, together with applications to the study of gravity in asymptotically flat spacetimes. The discussion proposed at the end of each chapter was already taking stock of the performed analysis. Here we would like to summarize broadly the results and present the open questions, from the simplest to the more conceptual, and the future prospects, from the short-term projects to the long-term foreseen analyses.

We have started with Carroll structures and spent some time detailing the construction of affine connections, isometries and charges. This framework was also extended to include Weyl covariance. After this intrinsic analysis, we presented how Carrollian theories emerge from the vanishing speed of light limit of relativistic theories. With explicit examples such as Carrollian hydrodynamics, Carrollian scalar field and Carrollian Cotton tensor, we ascertain that expanding in powers of  $c^2$  prior to the limit itself can possibly replicate the Carrollian theories that the intrinsic approach yields.

We then turned our attention to applications of Carrollian physics at null infinity in asymptotically flat spacetimes. As a prerequisite to this, a gauge analysis was necessary. A desirable feature in view of future holographic applications is that the gauge should be covariant with respect to the Carrollian boundary. Chapter 3 was devoted to the construction of such a gauge. Valid both in the AdS and flat instances, we were able to take the flat limit in the bulk i.e. the Carrollian limit on the boundary. This revealed that the infinite solution space of the flat case can actually be generated by the expansion of the AdS boundary energy-momentum tensor in powers of the bulk cosmological constant. Demanding the limit be smooth for the AdS flux-balance laws and for the line element gave access to their flat counterparts. The last two chapters were devoted to show explicit situations in which the Carrollian boundary can give a new perspective on Einstein dynamics. Restricting our analysis to the subclass of Petrov-algebraically special spacetimes, we presented firstly how bulk charges and gravitational multipoles can be constructed from purely boundary considerations. Secondly we explored the action of hidden symmetries of gravity on the Carrollian boundary data. We exhibited a local action of the symmetry on the boundary while in the bulk the latter is



highly non-local.

In the third part of this thesis we drove our attention towards thermal field theories. We showed that thermalization implies non-vanishing averages of primary operators. These values are new data which, together with the scaling dimensions and the OPE coefficients, constitute the complete description of the field theory. A way to compute these averages was presented, based on a new correspondence with Feynman graphs of fishnet theories.

The results of this work call for natural extensions. The geometry of Carroll structures is by now well-established, in two complementary formalisms, covariant and split. Dimensional reduction is a major concept that remains to be defined in general terms, especially along directions which do not coincide with the field of observers. As already advocated at the end of the fifth chapter, such a framework is needed for generalising our preliminary study of hidden symmetries to bigger symmetry groups. Among many interesting questions, the validity of the boundary-action locality and its potential dependence on the symmetry group, or the organization of the towers of charges (multipoles) are of outmost importance.

Carrollian field theories can be obtained starting from relativistic theories via a limiting procedure that is well understood and complements the intrinsic analysis, handling possibly more degrees of freedom. The expansion method has been successfully applied to the Chern-Simons action [104] with the consequences on the Cotton tensor we have analyzed and exploited in [155]. Building Carrollian Chern-Simons actions intrinsically from a gauge symmetry principle is an open question with diverse applications. The expansion method could also be explored to probe the behavior of a relativistic theory (as e.g. relativistic hydrodynamics) in the neighborhood of an embedded null hypersurface, such as a black-hole horizon or an ultra-relativistic domain.

Leaving aside the study of Carrollian geometry and its direct physical applications, and going back to gravity, the main extensions of the work we presented in Chapter 3 are threefold. First, the covariant Newman-Unti gauge in four dimensions with flat or anti de Sitter asymptotics should be studied in detail: residual diffeomorphisms, asymptotic symmetries, charges etc. with emphasis on the status of hydrodynamic frame transformations accessible thanks to the congruence  $\mathbf{u}$ . The latter undergoes local Lorentz (or Carroll) transformations on the boundary, which, according to the three-dimensional analysis of [85, 86], translates into bulk diffeomorphisms. The latter being possibly charged, a change of fluid velocity may relate two nonequivalent physical situations. The second extension is related to the Chthonian orders in the radial expansion (the ones after the order  $\frac{1}{r^2}$ ). As already mentioned earlier, the Newman-Penrose charges arise at the next order, and our goal would be to reach them in a boundary covariant fashion, translating thereby the analysis developed in [150–152]. Regarding the Chthonian degrees of freedom an important question resurface. Which of them are really relevant (in the sense that they explicitly appear in the line element)? Understanding better the new gauge at hand and comparing our analysis with previous works in other gauges, especially the recent ones in partial Bondi gauge [59, 131] or in double-null coordinates [159, 160] should help us addressing this issue. The third extension is connected to higher dimensions. When the boundary is three-dimensional, we showed the prominent role played by the Cotton tensor, es-

pecially regarding gravitational radiation. In higher dimensions, both the boundary Weyl-covariant Cotton and the Weyl tensor enter the bulk line element, carrying distinct information. Furthermore, gravitational radiation is not captured by the same order in the radial expansion. Hence, part of our results is clearly specific to the four-dimensional bulk case, whereas others, such as the validity of the covariant Newman-Unti gauge, are generic. A higher-dimensional analysis is worth undertaking.

The results gathered in this manuscript demonstrate that a wealth of information on asymptotically flat spacetimes is drawn from AdS asymptotics. However one should refrain in speculating on a flat analogue of the AdS gauge/gravity duality, as deep questions are still to be answered, and many of them may require a long time to be settled. What is a (conformal) Carrollian field theory? How to quantize a (conformal) theory on a space with a degenerate direction? Is there an analogue to the state-operator correspondence or of operator product expansion? How to define primary operators, descendants? Is there a radial quantization? More related to flat holography one may wonder what would be the fundamental observables of the dual Carrollian field theory. What would be the role of the replicas of the energy-momentum tensor? And most importantly, can such a duality between Ricci-flat bulks and Carrollian boundaries qualify as holography, as an infinite set of data is required to reconstruct the bulk? The limiting procedure developed in Chapter 3 sets the hope that the flat limit of the AdS/CFT is on a firmer ground. Working out explicit situations may be the good path to handle these questions.

Coming back to thermal field theories, in addition to a better understanding of the correspondence we have presented, some results turn useful in the realm of standard holography. Indeed, the holographic dual of a black-hole state in AdS is a thermal CFT. This motivates our work for its potential AdS/CFT applications. At the same time, it opens Pandora's box for the would be flat/Carroll CFT correspondence. This asserts even more basic questions, the first being the definition of the temperature in Carrollian systems, long before being ready to address questions on thermal correlators in Carrollian CFTs. The geometric definition of temperature given in the second chapter may serve as a guide. However, it is legitimate to wonder if we can reach a physical i.e. microscopic definition for a Carrollian temperature. Thermodynamics needs to be rethought from start and an explicit example of Carrollian systems has to be worked out. Finally, in line with the correspondence discovered in [207], one may wonder if thermal Carrollian partition functions are related to Feynman graphs – assuming that all these words are given a precise meaning. All these questions are foreseeable long-term explorations that demand for the Carrollian translation of many basic concepts of relativistic and Galilean physics.

In conclusion this thesis is an invitation to explore the realm of Carrollian physics and asymptotically flat gravity with a small but noticeable excursion in thermal field theories. Having set Carrollian geometry under control and equipped the bulk with a boundary-covariant gauge, the appropriate tools are now at our disposal to start addressing the hard core of holography in flat spacetimes, hoping in the longer term to add temperature and ultimately investigate the Thermal Wonderland.



# Appendices



# Appendix A

## Useful formulae

### Frames, connection and related objects

Let  $\mathcal{M}$  a  $(d + 1)$ -dimensional manifold whose tangent and cotangent spaces are describe by means of the bases

$$\mathcal{B} = \{\mathbf{e}_A\} \quad \text{and} \quad \mathcal{B}^* = \{\boldsymbol{\theta}^A\}, \quad (\text{A.o.1})$$

which are such that  $\boldsymbol{\theta}^A(\mathbf{e}_B) = \delta_B^A$ . At the level of the Lorentzian metric<sup>1</sup> we get

$$ds^2 = g_{AB}\boldsymbol{\theta}^A\boldsymbol{\theta}^B. \quad (\text{A.o.2})$$

One usually has

$$[\mathbf{e}_A, \mathbf{e}_B] = C_{AB}^C \mathbf{e}_C \quad (\text{A.o.3})$$

where  $C_{AB}^C$  are the non-holonomy coefficients. One can easily show that equation (A.o.3) is equivalent to

$$d\boldsymbol{\theta}^C + \frac{1}{2}C_{AB}^C \boldsymbol{\theta}^A \wedge \boldsymbol{\theta}^B = 0. \quad (\text{A.o.4})$$

$\mathcal{M}$  can also be endowed with a general connection  $\nabla$  that does not need to be related to the metric. Its coefficients are defined by

$$\nabla_{\mathbf{e}_A} \mathbf{e}_B := \nabla_A \mathbf{e}_B = \Gamma_{AB}^C \mathbf{e}_C. \quad (\text{A.o.5})$$

Note that then Cartan's spin connection is defined by

$$\boldsymbol{\omega}^A{}_B = \Gamma_{CB}^A \boldsymbol{\theta}^C. \quad (\text{A.o.6})$$

Given a connection  $\Gamma_{BC}^A$  one can built three tensors<sup>2</sup>

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<sup>1</sup>Our signature convention is the mostly plus one.

<sup>2</sup>Here we give only the components in the frame at hand, this is derived from the very general frame independant definitions of those tensors.

- the torsion tensor

$$S_{BC}^A = 2\Gamma_{[BC]}^A - C_{BC}^A, \quad (\text{A.o.7})$$

- the non-metricity tensor

$$Q_{CAB} := \nabla_C g_{AB} = \mathbf{e}_C(g_{AB}) - \Gamma_{CA}^D g_{DB} - \Gamma_{CB}^D g_{AD}, \quad (\text{A.o.8})$$

- the Riemann curvature tensor

$$R^E_{CAB} = \mathbf{e}_A(\Gamma_{BC}^E) + \Gamma_{BC}^D \Gamma_{AD}^E - \mathbf{e}_B(\Gamma_{AC}^E) - \Gamma_{AC}^D \Gamma_{BD}^E - C_{AB}^D \Gamma_{DC}^E. \quad (\text{A.o.9})$$

These tensors can of course be read from the commutator of two covariant derivatives built upon  $\Gamma_{BC}^A$  acting on a scalar and on a vector

$$[\nabla_A, \nabla_B] f = -S_{AB}^C \nabla_C f \quad (\text{A.o.10a})$$

$$[\nabla_A, \nabla_B] W^C = R^C_{DAB} W^D - S_{AB}^D \nabla_D W^C. \quad (\text{A.o.10b})$$

One can also repackage the information stored in the torsion and curvature tensors into a vector-valued and a matrix-valued two-form respectively

$$\hat{\mathcal{T}}^A = \hat{S}_{BC}^A \boldsymbol{\theta}^B \wedge \boldsymbol{\theta}^C \quad (\text{A.o.11a})$$

$$\mathcal{R}^A_B = \frac{1}{2} R^A_{BCD} \boldsymbol{\theta}^C \wedge \boldsymbol{\theta}^D. \quad (\text{A.o.11b})$$

Note finally that when dealing with a Levi-Civita connection that is a torsion-free metric-compatible connection, the symmetric part of the coefficients reads

$$\widehat{\Gamma}_{(AB)}^C = \frac{1}{2} g^{CD} \left( \mathbf{e}_A(g_{BD}) + \mathbf{e}_B(g_{DA}) - \mathbf{e}_D(g_{AB}) + g_{AE} C_{DB}^E + g_{BE} C_{DA}^E \right), \quad (\text{A.o.12})$$

while the antisymmetric part is just the non-holonomy i.e.  $\widehat{\Gamma}_{[AB]}^C = \frac{1}{2} C_{AB}^C$ .

It is also frequent to build vectors upon (A.o.7) and (A.o.8). These are the torsion pseudo-vector

$$\hat{T}_A = \epsilon_{ABCD} T^{BCD}, \quad (\text{A.o.13})$$

and the non metricity vector and dual vector

$$Q_A = Q_{ABC} g^{BC} \quad \text{and} \quad \hat{Q}_C = Q_{ABC} g^{AB}. \quad (\text{A.o.14})$$

The difference between a general connection  $\Gamma_{AB}^C$  and the Levi-Civita connection is encoded into the *distorsion tensor*  $N_{ABC}$  which reads

$$N_{ABC} = \frac{1}{2} (Q_{BCA} + Q_{CAB} - Q_{ABC}) - (S_{ABC} + S_{ACB} - S_{BCA}). \quad (\text{A.o.15})$$

The fundamental tool that we will need in the main text and especially when varying the Carrollian action, is the Lie derivative.

## Lie derivatives

By virtue of the Leibniz rule and the possibility to decompose any tensor fields in the bases  $\mathcal{B}$  and  $\mathcal{B}^*$  we just need the action of the Lie derivative on the latter and on a scalar field. Let  $\xi$  a vector field<sup>3</sup>  $\xi = \xi^A \mathbf{e}_A$ , and let  $f$  a scalar field. We have

$$\mathcal{L}_\xi f = \xi^A \mathbf{e}_A(f) \quad (\text{A.o.16a})$$

$$\mathcal{L}_\xi \mathbf{e}_A = (\xi^D C^B_{DA} - \mathbf{e}_A(\xi^B)) \mathbf{e}_B \quad (\text{A.o.16b})$$

$$\mathcal{L}_\xi \theta^A = (\xi^C C^A_{BC} + \mathbf{e}_B(\xi^A)) \theta^B. \quad (\text{A.o.16c})$$

Now for a vector  $\mathbf{B} = B^A \mathbf{e}_A$  and a one form  $\omega = \omega_A \theta^A$  we have

$$\mathcal{L}_\xi \mathbf{B} = \mathcal{L}_\xi B^A \mathbf{e}_A + B^A \mathcal{L}_\xi \mathbf{e}_A \quad (\text{A.o.17a})$$

$$\mathcal{L}_\xi \omega = \mathcal{L}_\xi \omega_A \theta^A + \omega_A \mathcal{L}_\xi \theta^A \quad (\text{A.o.17b})$$

which leads to

$$\mathcal{L}_\xi B^A = \xi^C \mathbf{e}_C(B^A) - B^C \mathbf{e}_C(\xi^A) + \xi^B B^C C^A_{BC} \quad (\text{A.o.18a})$$

$$\mathcal{L}_\xi \omega_B = \xi^A \mathbf{e}_A(\omega_B) + \omega_A \mathbf{e}_B(\xi^A) + \omega_A \xi^C C^A_{BC}. \quad (\text{A.o.18b})$$

The same procedure gives for a rank-two covariant tensor  $\mathbf{K} = K_{AB} \theta^A \otimes \theta^B$  and for a rank-two contravariant tensor  $\mathbf{K} = K^{AB} \mathbf{e}_A \otimes \mathbf{e}_B$

$$\mathcal{L}_\xi \mathbf{K} = \mathcal{L}_\xi K_{AB} \theta^A \otimes \theta^B + K_{AB} \mathcal{L}_\xi \theta^A \otimes \theta^B + K_{AB} \theta^A \otimes \mathcal{L}_\xi \theta^B, \quad (\text{A.o.19a})$$

$$\mathcal{L}_\xi \mathbf{K} = \mathcal{L}_\xi K^{AB} \mathbf{e}_A \otimes \mathbf{e}_B + K^{AB} \mathcal{L}_\xi \mathbf{e}_A \otimes \mathbf{e}_B + K^{AB} \mathbf{e}_A \otimes \mathcal{L}_\xi \mathbf{e}_B, \quad (\text{A.o.19b})$$

and thus

$$\mathcal{L}_\xi K_{AB} = \xi^C \mathbf{e}_C(K_{AB}) + K_{CB} \mathbf{e}_A(\xi^C) + K_{AC} \mathbf{e}_B(\xi^C) + (K_{CB} C^C_{AD} + K_{AC} C^C_{BD}) \xi^D \quad (\text{A.o.20a})$$

$$\mathcal{L}_\xi K^{AB} = \xi^C \mathbf{e}_C(K^{AB}) - K^{CB} \mathbf{e}_C(\xi^A) - K^{AC} \mathbf{e}_C(\xi^B) + (K^{AC} C^B_{DC} + K^{CB} C^A_{CD}) \xi^D. \quad (\text{A.o.20b})$$

Finally for a mixed tensor  $\mathbf{N} = N^A{}_B \mathbf{e}_A \otimes \theta^B$  we get

$$\mathcal{L}_\xi N^A{}_B = \xi^C \mathbf{e}_C(N^A{}_B) + N^A{}_C \mathbf{e}_B(\xi^C) - N^C{}_B \mathbf{e}_C(\xi^A) + N^C{}_B \xi^D C^A_{DC} + N^A{}_D \xi^C C^D_{BC}. \quad (\text{A.o.21})$$

<sup>3</sup>In the following we shall often identify vectors and tensors with their components in a given basis.



The last Lie derivative that we used in the main content was the one for a mix type tensor of the form  $\mathbf{M} = M^A_{BC} \mathbf{e}_A \otimes \theta^B \otimes \theta^C$  for which we have

$$\mathcal{L}_\xi M^A_{BC} = \xi^D \mathbf{e}_D(M^A_{BC}) + M^A_{EC} \mathbf{e}_B(\xi^E) + M^A_{BE} \mathbf{e}_C(\xi^E) - M^E_{AB} \mathbf{e}_E(\xi^A) \quad (\text{A.o.22})$$

$$+ (M^E_{BC} C^A_{DE} + M^A_{EC} C^E_{BD} + M^A_{BE} C^E_{CD}) \xi^D. \quad (\text{A.o.23})$$

**Remark** Once expressed using ordinary derivatives, one may need to translate the Lie derivative in covariant derivatives. In the simple case where the connection  $\nabla$  is metric compatible but with torsion we have the following identities

$$\mathcal{L}_\xi g_{AB} = 2\nabla_{(A} \xi_{B)} - 2g_{(A|C} S^C_{B)D} \xi^D \quad (\text{A.o.24a})$$

$$\mathcal{L}_\xi B^A = \xi^C \nabla_C B^A - B^C \nabla_C \xi^A + S^A_{BC} B^B \xi^C. \quad (\text{A.o.24b})$$

These formulae (A.o.24a) and (A.o.24b) are independent from the exact form of the connection and in particular of its ambiguities.

Equipped with the Lie derivative we can take variations of an effective action under diffeomorphisms.

**Remark** At the level of the connection coefficients the action of a diffeomorphisms yields

$$\begin{aligned} \delta_\xi \Gamma^A_{BC} &= \xi^D \mathbf{e}_D(\Gamma^A_{BC}) - \Gamma^A_{DC} \mathbf{e}_B(\xi^D) - \Gamma^A_{BD} \mathbf{e}_C(\xi^D) + \Gamma^D_{BC} \mathbf{e}_D(\xi^A) \quad (\text{A.o.25}) \\ &+ \left( \Gamma^E_{BC} C^A_{DE} + \Gamma^A_{EC} C^E_{BD} + \Gamma^E_{CD} \right) \xi^D - \mathbf{e}_A \left( \mathbf{e}_B(\xi^A) \right) \end{aligned}$$

where we notice the inhomogeneous last term characteristic for a connection.

## General variations and "conservation" equations

Let  $S$  an action functional and  $\xi$  a general diffeomorphism. Such a transformation acts on tensorial quantities with the Lie derivative and the assumption that the action should be invariant will lead to an equation valid on-shell.

In the following the action will depend first on the background metric  $g_{AB}$ , on a gauge field  $B^A$  and on a set of matter fields collectively denoted by  $\Phi$  so  $S = S[g_{AB}, B^A; \Phi]$ . We first assume the connection  $\nabla$  to be just metric compatible but torsionfull. The conjugate momenta are

$$T^{AB} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{AB}} \quad \text{and} \quad J_A = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B^A} \quad (\text{A.o.26})$$

and are dubbed the *energy-momentum tensor* and the *gauge current*. Note that  $g = \det g_{AB}$ . Varying

$S$  with respect to  $\xi$  we get

$$\delta_\xi S = \int d^{d+1}x \sqrt{-g} \left( \frac{1}{2} T_{AB} \mathcal{L}_\xi g^{AB} + J_A \mathcal{L}_\xi B^A + \text{E.O.M.} \frac{\delta S}{\delta \Phi} \right), \quad (\text{A.o.27})$$

where appears the equations of motion (E.O.M). Using (A.o.24a), (A.o.24b) and performing some integration by parts<sup>4</sup> we are left with, on shell<sup>5</sup>

$$\begin{aligned} \delta_\xi S \doteq & \int d^{d+1}x \sqrt{-g} \left( \nabla_A (T^A_C + B^A J_C + J_A \nabla_C B^A + (T^A_B + B^A J_B) S^B_{AC}) \right) \xi^C \\ & - \int d^{d+1}x e_A [\sqrt{-g} (T^A_B \xi^B + J_B \xi^B B^A)]. \end{aligned} \quad (\text{A.o.28})$$

Upon vanishing of the boundary term we get the on-shell conservation equation

$$\boxed{\hat{\nabla}_A (T^A_C + B^A J_C) + J_A \hat{\nabla}_C B^A + (T^A_B + B^A J_B) S^B_{AC} \doteq 0}. \quad (\text{A.o.29})$$

Now let's take a completely general connection, such that we can associate to it a conjugate momenta dubbed *the hypermomentum*

$$\Delta_C^{AB} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta \Gamma^C_{AB}}. \quad (\text{A.o.30})$$

We consider an action functional  $S = S[g_{AB}, \Gamma^C_{AB}; \Phi]$  and  $\xi$  a general diffeomorphism. In the following the connection is denoted  $\nabla$  and its Riemann tensor is  $R^A_{BCD}$ .

**Remark** By virtue of the definitions of the hypermomentum and of the energy-momentum tensor, if the connection is assumed to be completely independent from the metric then the two variations should commute, leading to

$$\frac{\partial}{\partial g_{AB}} \left( \sqrt{-g} \Delta_C^{DE} \right) = \sqrt{-g} \frac{\partial T_{AB}}{\partial \Gamma^C_{DE}}. \quad (\text{A.o.31})$$

Hence it is expected that for an explicit action whose matter content does not couple to the connection (e.g. scalar fields) the associated hypermomentum is metric independent.

**Remark** Note that in general the connection is not completely independent from the metric, imposing some constraints to be satisfied. Some Lagrange multipliers may then be needed in the process of varying the action.

Varying  $S$  with respect to  $\xi$  leads to the following equation

$$\sqrt{-g} \left( 2 \hat{\nabla}_A T^A_B - \Delta^{CAD} R_{CADB} \right) + \hat{\nabla}_A \hat{\nabla}_B (\sqrt{-g} \Delta_B^{AD}) + 2 S_{AB}{}^C \hat{\nabla}_D (\sqrt{-g} \Delta_C^{AD}) = 0 \quad (\text{A.o.32})$$

<sup>4</sup>We use the well known formula valid only for a Levi-Civita connection  $\sqrt{-g} \nabla_A V^A = e_A (\sqrt{-g} V^A)$  for any vector field  $V^A$ .

<sup>5</sup>On-shellness is denoted by a  $\doteq$  symbol.

with  $\hat{\nabla}$  the Levi-Civita derivative (A.o.12) and  $\tilde{\nabla}_A = \nabla_A - 2T_A$  is the *modified covariant derivative* (where  $T_A$  is introduced in (A.o.13)). Introducing the *modified energy-momentum tensor*

$$t^A_B = T^A_B - \frac{1}{2\sqrt{-g}} \hat{\nabla}_C(\sqrt{-g}\Delta_B^{AC}) \quad (\text{A.o.33})$$

(A.o.32) can be written as

$$\boxed{\frac{1}{\sqrt{-g}} \hat{\nabla}_A(\sqrt{-g}t^A_B) = \frac{1}{2}\Delta^{CAD}R_{CADB} + \frac{1}{2}Q_{BAD}T^{AD} + 2S_{BAD}t^{AD}}. \quad (\text{A.o.34})$$

Works on hypermomentum include [238, 239] and reference therein. We conclude with this expression our compilation of useful formulae.

# Appendix B

## Carroll structures in Cartan's frame

### B.1 Intrinsic geometry

In this Appendix we give some details on Carrollian geometry in a Cartan orthonormal frame. We denote with Latin letters from the beginning of the alphabet frame indices which range from 1 to  $d$ . The Carroll structure is still  $(d + 1)$ -dimensional and defined on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ .

#### B.1.1 Frame, coframe, non holonomy

The aim of a Cartan orthonormal frame is to decompose Carrollian tensors under local  $\mathfrak{so}(d)$  rotations instead of Carrollian diffeomorphisms (1.1.5). The weak Carroll structure is then given by

$$ds^2 = 0 \times \boldsymbol{\tau} + \delta_{ab} \hat{\boldsymbol{\theta}}^a \hat{\boldsymbol{\theta}}^b \quad (\text{B.1.1})$$

whose one-dimensional kernel is spanned by  $\mathbf{v}$ . The frame and coframe,  $\{\mathbf{v}, \hat{\boldsymbol{e}}_a\}$  and  $\{\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}^a\}$  obey

$$\boldsymbol{\tau}(\mathbf{v}) = 1, \quad \hat{\boldsymbol{\theta}}^a(\hat{\boldsymbol{e}}_b) = \delta_b^a, \quad \hat{\boldsymbol{\theta}}^a(\mathbf{v}) = 0, \quad \boldsymbol{\tau}(\hat{\boldsymbol{e}}_a) = 0, \quad (\text{B.1.2})$$

where  $\boldsymbol{\tau}$  is still the clock form. When needed, the time index will be denoted  $\hat{u}$  to make a clear and complete distinction w.r.t. the split notations.<sup>1</sup> Latin indices, lowered and raised using  $\delta_{ab}$  and its inverse  $\delta^{ab}$ , represent tensorial indices with respect to the subgroup of the Carroll group generating spatial rotations, while quantities without indices are scalars.

Our frames at hand undergo two types of transformations, the (infinitesimal) *local rotations* parameterized by  $w_a{}^b$  (with  $w_{ab} = -w_{ba}$ )<sup>2</sup> and the *local boosts* parameterized by  $\lambda_a$ ; they act as

$$\delta_w \mathbf{v} = 0, \quad \delta_w \boldsymbol{\tau} = 0, \quad \delta_w \hat{\boldsymbol{e}}_a = w_a{}^b \hat{\boldsymbol{e}}_b \quad \delta_w \hat{\boldsymbol{\theta}}^a = -w_b{}^a \hat{\boldsymbol{\theta}}^b \quad (\text{B.1.3})$$

<sup>1</sup>Note then that  $\boldsymbol{\theta}^{\hat{u}} = \boldsymbol{\tau}$ .

<sup>2</sup>The linearized form of the rotation would then be  $B_a{}^b = \delta_a^b + w_a{}^b$ .

$$\delta_\lambda \mathbf{v} = 0, \quad \delta_\lambda \boldsymbol{\tau} = -\lambda_a \hat{\boldsymbol{\theta}}^a, \quad \delta_\lambda \hat{\mathbf{e}}_a = \lambda_a \mathbf{v}, \quad \delta_\lambda \hat{\boldsymbol{\theta}}^a = 0. \quad (\text{B.1.4})$$

The latter are coming from the  $c \rightarrow 0$  limit of the relativistic Lorentz transformations applied to an orthonormal frame.

**Remark** Let's give more details on this last point. Let  $\{\boldsymbol{\theta}^A\}$  with  $A = 0, a$  and  $a = 1, \dots, d$  an orthonormal frame i.e.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{AB} \boldsymbol{\theta}^A \boldsymbol{\theta}^B, \quad (\text{B.1.5})$$

the dual basis being denoted by  $\{\mathbf{e}_A\}$ . While general diffeomorphisms acts on the coordinate bases, Lorentz transformations  $\Lambda^A_B$  are at work on the frame

$$\boldsymbol{\theta}'^A = \Lambda^A_B \boldsymbol{\theta}^B \quad \text{and} \quad \mathbf{e}'_A = (\Lambda^{-1})^B_A \mathbf{e}_B. \quad (\text{B.1.6})$$

Any Lorentz transformation can be written as a combination of a boost parameterized by a vector  $\lambda_a$  (whose norm is  $\lambda^2 = \delta_{ab} \lambda^a \lambda^b$ ) and a rotation parameterized by a matrix  $B^a_b$

$$\Lambda^A_B = \begin{pmatrix} \Gamma & -c\Gamma \lambda_c B^c_b \\ -c\Gamma \lambda^a & B^a_b + (\Gamma - 1) \frac{\lambda_c \lambda^a B^c_b}{\lambda^2} \end{pmatrix} \quad (\text{B.1.7})$$

whose inverse reads

$$(\Lambda^{-1})^A_B = \begin{pmatrix} \Gamma & c\Gamma \lambda_b \\ c\Gamma \lambda^c B^a_c & -B^a_b + (\Gamma - 1) \frac{\lambda_c \lambda^a w^c_b}{\lambda^2} \end{pmatrix}. \quad (\text{B.1.8})$$

Here the Lorentz factor  $\Gamma$  reads  $\Gamma = \left(\sqrt{1 - c^2 \lambda^2}\right)^{-1} = 1 + \mathcal{O}(c^2)$ . In order to get the Carrollian equivalent one has to take the  $c \rightarrow 0$  limit of (B.1.6), informed with

$$\mathbf{v} = \lim_{c \rightarrow 0} c \mathbf{e}_0, \quad \boldsymbol{\tau} = \lim_{c \rightarrow 0} \frac{\boldsymbol{\theta}^0}{c}, \quad \hat{\mathbf{e}}_a = \lim_{c \rightarrow 0} \mathbf{e}_a, \quad \hat{\boldsymbol{\theta}}^a = \lim_{c \rightarrow 0} \boldsymbol{\theta}^a \quad (\text{B.1.9})$$

together with the  $c$ -independence of the boost parameter and knowing that  $B^a_b = \delta^a_b - w^a_b$  at linear level. For example it gives for the Carrollian clock form

$$\boldsymbol{\theta}'^0 = \Lambda^0_0 \boldsymbol{\theta}^0 + \Lambda^0_a \boldsymbol{\theta}^a \implies \boldsymbol{\tau}' = \boldsymbol{\tau} - \lambda_a \boldsymbol{\theta}^a \quad (\text{B.1.10})$$

that is  $\boldsymbol{\tau}$  is insensitive to local rotations but undergoes Carrollian boosts. Performing the same analysis for the rest of the frame and coframe we get (B.1.3) and (B.1.4).

These frames (B.1.2) are non holonomous by virtue of

$$d\boldsymbol{\tau} - \varphi_a \hat{\boldsymbol{\theta}}^a \wedge \boldsymbol{\tau} + \hat{\omega}_{ab} \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b = 0 \quad \text{and} \quad d\hat{\boldsymbol{\theta}}^c + \hat{c}^c_a \hat{\boldsymbol{\theta}}^a \wedge \boldsymbol{\tau} + \frac{1}{2} \hat{c}^c_{ab} \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b = 0 \quad (\text{B.1.11})$$

or equivalently

$$[\mathbf{v}, \hat{\mathbf{e}}_a] = \varphi_a \mathbf{v} - \hat{c}^c_a \hat{\mathbf{e}}_c, \quad \text{and} \quad [\hat{\mathbf{e}}_a, \hat{\mathbf{e}}_b] = 2\hat{\omega}_{ab} \mathbf{v} + \hat{c}^c_{ab} \hat{\mathbf{e}}_c \quad (\text{B.1.12})$$

with<sup>3</sup>  $\hat{\omega}_{(ab)} = 0$  and  $\hat{c}^c_{(ab)} = 0$ . We thus read

$$C^{\hat{u}}_{\hat{u}a} = \varphi_a \quad , \quad C^{\hat{u}}_{ab} = 2\hat{\omega}_{ab} \quad , \quad C^c_{\hat{u}a} = -\hat{c}^c_a \quad , \quad C^c_{ab} = \hat{c}^c_{ab}. \quad (\text{B.1.14})$$

as non-holonomy coefficients. Under linearized rotations of the form  $B^a_b = \delta^a_b - \omega_b^a$  (i.e.  $B_a^b = \delta_a^b + \omega_a^b$ ) these geometric quantities transform as

$$\delta_w \varphi_a = \omega_a^b \varphi_b \quad (\text{B.1.15a})$$

$$\delta_w \hat{\omega}_{ab} = \omega_a^c \hat{\omega}_{cb} + \omega_b^c \hat{\omega}_{ac} \quad (\text{B.1.15b})$$

$$\delta_w \hat{c}^c_d = \omega_d^b \hat{c}^c_b - \omega_a^c \hat{c}^a_d - \mathbf{v}(\omega_d^c) \quad (\text{B.1.15c})$$

$$\delta_w \hat{c}^c_{ab} = \left( \omega_a^d \hat{c}^c_{db} + \omega_b^d \hat{c}^c_{ad} - \omega_d^c \hat{c}^d_{ab} \right) + 2 \hat{\mathbf{e}}_{[a}(\omega_{b]}^c). \quad (\text{B.1.15d})$$

which, from a group-theoretical viewpoint, shows that the latter two quantities are the gauge connections associated with the local rotations while from a geometrical viewpoint they represent the non-closure of the coframe  $\hat{\Theta}^a$ . They do not appear in [75, 105] where the authors assume  $d\hat{\Theta}^a = 0$ . At the level of the local boosts the geometry transforms as on the frame while they give

$$\delta_\lambda \varphi_a = \mathbf{v}(\lambda_a) + \lambda_c \hat{c}^c_a \quad , \quad (\text{B.1.16a})$$

$$\delta_\lambda \hat{\omega}_{ab} = \hat{\mathbf{e}}_{[a}(\lambda_{b]}) + \lambda_{[a} \varphi_{b]} - \frac{1}{2} \lambda_c \hat{c}^c_{ab} \quad , \quad (\text{B.1.16b})$$

$$\delta_\lambda \hat{c}^a_b = 0 \quad , \quad (\text{B.1.16c})$$

$$\delta_\lambda \hat{c}^c_{ab} = 2 \hat{c}^c_{[a} \lambda_{b]} \quad . \quad (\text{B.1.16d})$$

This time,  $\varphi_a$  and  $\hat{\omega}_{ab}$  play the role of temporal and spatial connections under local Carroll boosts.

## B.1.2 Adding a connection

We now equip the structure again with a connection that we first decompose into

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}} \mathbf{v} &= \hat{\gamma} \mathbf{v} + \hat{\rho}^a \hat{\mathbf{e}}_a & \bar{\nabla}_a \mathbf{v} &= \hat{\alpha}_a \mathbf{v} + \hat{\kappa}_a^b \hat{\mathbf{e}}_b \\ \bar{\nabla}_{\mathbf{v}} \hat{\mathbf{e}}_a &= \hat{\delta}_a \mathbf{v} + \hat{\gamma}^b_a \hat{\mathbf{e}}_b & \bar{\nabla}_a \hat{\mathbf{e}}_b &= \hat{\beta}_{ab} \mathbf{v} + \hat{\gamma}^c_{ab} \hat{\mathbf{e}}_c \\ \bar{\nabla}_{\mathbf{v}} \boldsymbol{\tau} &= -\hat{\gamma} \boldsymbol{\tau} - \hat{\delta}_a \hat{\Theta}^a & \bar{\nabla}_a \boldsymbol{\tau} &= -\hat{\alpha}_a \boldsymbol{\tau} - \hat{\beta}_{ab} \hat{\Theta}^b \\ \bar{\nabla}_{\mathbf{v}} \hat{\Theta}^a &= -\hat{\rho}^a \boldsymbol{\tau} - \hat{\gamma}^a_b \hat{\Theta}^b & \hat{\nabla}_a \hat{\Theta}^b &= -\hat{\kappa}_a^b \boldsymbol{\tau} - \hat{\gamma}^b_{ac} \hat{\Theta}^c. \end{aligned} \quad (\text{B.1.17})$$

<sup>3</sup>There is a geometric formula for  $\varphi_a$  and  $\hat{c}^a_b$ . Using Cartan's magic formula we get

$$\mathcal{L}_{\mathbf{v}} \boldsymbol{\tau} = -\varphi_a \hat{\Theta}^a \quad \text{and} \quad \mathcal{L}_{\mathbf{v}} \hat{\Theta}^a = \hat{c}^a_b \hat{\Theta}^b. \quad (\text{B.1.13})$$

Note that this coincides with (C.0.4) of the covariant formalism.

Under (infinitesimal) local rotations the various components behave as<sup>4</sup>

$$\delta_w \hat{\gamma} = 0, \quad (\text{B.1.18a})$$

$$\delta_w \hat{\rho}^a = -w_b^a \hat{\rho}^b, \quad (\text{B.1.18b})$$

$$\delta_w \hat{\alpha}_a = w_a^b \hat{\alpha}_b, \quad (\text{B.1.18c})$$

$$\delta_w \hat{\kappa}_a^b = w_a^c \hat{\kappa}_c^b - w_c^b \hat{\kappa}_a^c, \quad (\text{B.1.18d})$$

$$\delta_w \hat{\delta}_a = w_a^b \hat{\delta}_b, \quad (\text{B.1.18e})$$

$$\delta_w \hat{\beta}_{ab} = w_a^c \hat{\beta}_{cb} + w_b^c \hat{\beta}_{ac}, \quad (\text{B.1.18f})$$

$$\delta_w \hat{\gamma}^a_b = -w_c^a \hat{\gamma}^c_b + w_b^c \hat{\gamma}^a_c + \mathbf{v}(w_b^a) = \hat{\nabla}_{\mathbf{v}} w_b^a, \quad (\text{B.1.18g})$$

$$\delta_w \hat{\gamma}^c_{ab} = w_a^d \hat{\gamma}^c_{db} + w_b^d \hat{\gamma}^c_{ad} - w_d^c \hat{\gamma}^d_{ab} + \hat{e}_a(w_b^c) = w_a^d \hat{\gamma}^c_{db} + \hat{\nabla}_a w_b^c, \quad (\text{B.1.18h})$$

where (B.1.18g) and (B.1.18h) shows that  $\hat{\gamma}^a_b$  and  $\hat{\gamma}^c_{ab}$  are respectively a temporal and spatial rotation connection. Under local boost we get

$$\delta_\lambda \hat{\gamma} = -\hat{\rho}^a \lambda_a \quad (\text{B.1.19a})$$

$$\delta_\lambda \hat{\rho}^a = 0 \quad (\text{B.1.19b})$$

$$\delta_\lambda \hat{\alpha}_a = \lambda_a \hat{\gamma} - \hat{\kappa}_a^b \lambda_b \quad (\text{B.1.19c})$$

$$\delta_\lambda \hat{\delta}_a = \lambda_a \hat{\gamma} - \hat{\gamma}^b_a \lambda_b + \mathbf{v}(\lambda_a) = \lambda_a \hat{\gamma} + \hat{\nabla}_{\mathbf{v}} \lambda_a \quad (\text{B.1.19d})$$

$$\delta_\lambda \hat{\beta}_{ab} = -\hat{\gamma}^c_{ab} \lambda_c + \lambda_b \hat{\alpha}_a + \lambda_a \hat{\delta}_b + \hat{e}_a(\lambda_b) = \lambda_b \hat{\alpha}_a + \lambda_a \hat{\delta}_b + \hat{\nabla}_a \lambda_b \quad (\text{B.1.19e})$$

$$\delta_\lambda \hat{\kappa}_a^b = \lambda_a \hat{\rho}^b \quad (\text{B.1.19f})$$

$$\delta_\lambda \hat{\gamma}^b_a = \lambda_a \hat{\rho}^b \quad (\text{B.1.19g})$$

$$\delta_\lambda \hat{\gamma}^c_{ab} = \lambda_b \hat{\kappa}_a^c + \lambda_a \hat{\gamma}^c_b \quad (\text{B.1.19h})$$

where (B.1.19d) and (B.1.19e) shows that  $\hat{\delta}_a$  and  $\hat{\beta}_{ab}$  are respectively a temporal and spatial Carrollian boosts connection.

The Carrollian affine-connection one-form reads (see (A.o.6))

$$\begin{aligned} \bar{\omega}^{\hat{u}}_{\hat{u}} &= \hat{\gamma} \boldsymbol{\tau} + \hat{\alpha}_a \hat{\boldsymbol{\theta}}^a & \bar{\omega}^a_{\hat{u}} &= \hat{\rho}^a \boldsymbol{\tau} + \hat{\kappa}_b^a \hat{\boldsymbol{\theta}}^b \\ \bar{\omega}^{\hat{u}}_a &= \hat{\delta}_a \boldsymbol{\tau} + \hat{\beta}_{ba} \hat{\boldsymbol{\theta}}^b & \bar{\omega}^a_b &= \hat{\gamma}^a_b \boldsymbol{\tau} + \hat{\gamma}^a_{cb} \hat{\boldsymbol{\theta}}^c \end{aligned} \quad (\text{B.1.20})$$

which means that the spin connection varies under local rotations as

$$\delta_w \bar{\omega}^0_0 = 0, \quad (\text{B.1.21a})$$

$$\delta_w \bar{\omega}^0_a = w_a^b \bar{\omega}^0_b, \quad (\text{B.1.21b})$$

$$\delta_w \bar{\omega}^a_0 = -w_b^a \bar{\omega}^b_0, \quad (\text{B.1.21c})$$

$$\delta_w \bar{\omega}^a_b = -w_c^a \bar{\omega}^c_b + w_b^c \bar{\omega}^a_c + \left( \mathbf{v}(w_b^a) \boldsymbol{\tau} + \hat{e}_c(w_b^a) \hat{\boldsymbol{\theta}}^c \right). \quad (\text{B.1.21d})$$

<sup>4</sup>As it will drastically simplify the variation of the action under rotation later, we already re-express the variations in terms of the hat derivative defined in (B.1.34).

while under boosts it changes like<sup>5</sup>

$$\delta_\lambda \bar{\omega}^0_0 = -\lambda_a \bar{\omega}^a_0, \quad (\text{B.1.22a})$$

$$\delta_\lambda \bar{\omega}^0_a = \lambda_a \bar{\omega}^0_0 + \left( \mathbf{v}(\lambda_a) - \hat{\gamma}^b_a \lambda_b \right) \boldsymbol{\tau} + \left( \hat{\mathbf{e}}_b(\lambda_a) - \hat{\gamma}^c_{ba} \lambda_c \right) \hat{\boldsymbol{\theta}}^b, \quad (\text{B.1.22b})$$

$$\delta_\lambda \bar{\omega}^a_0 = 0, \quad (\text{B.1.22c})$$

$$\delta_\lambda \bar{\omega}^a_b = \lambda_b \bar{\omega}^a_0. \quad (\text{B.1.22d})$$

Hence we see that  $\omega^{\hat{u}}_a$  encodes the Carrollian boosts while  $\omega^a_b$  encodes the local rotations.

**Remark** If we were to study a Galilean structure in Cartan's frame instead of a Carrollian one (see App. D), the Galilean equivalent of the boosts would act on the bases  $\{\mathbf{v}, \hat{\mathbf{e}}_a\}$  and  $\{\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}^a\}$  like

$$\delta_\lambda \mathbf{v} = \lambda^a \hat{\mathbf{e}}_a, \quad \delta_\lambda \boldsymbol{\tau} = 0, \quad \delta_\lambda \hat{\mathbf{e}}_a = 0, \quad \delta_\lambda \hat{\boldsymbol{\theta}}^a = -\lambda^a \boldsymbol{\tau}. \quad (\text{B.1.23})$$

Studying the transformation properties of the connection (B.1.17) we would have found that  $\hat{\rho}^a$  and  $\hat{\kappa}_b^a$  and hence  $\omega^a_{\hat{u}}$  are the components encoding the Galilean boosts. For the interpretation of the remaining two pieces  $\hat{\gamma}$  and  $\hat{\alpha}_a$  see later.

We can now specify further our connection first by asking the preservation of the field of observers  $\bar{\nabla}_{\mathbf{v}} \mathbf{v} = 0$  and  $\bar{\nabla}_a \mathbf{v} = 0$  we are left with

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}} \hat{\mathbf{e}}_a &= \hat{\delta}_a \mathbf{v} + \hat{\gamma}^b_a \hat{\mathbf{e}}_b & \bar{\nabla}_{\mathbf{v}} \boldsymbol{\tau} &= -\hat{\delta}_a \hat{\boldsymbol{\theta}}^a & \bar{\nabla}_{\mathbf{v}} \hat{\boldsymbol{\theta}}^a &= -\hat{\gamma}^a_b \hat{\boldsymbol{\theta}}^b \\ \bar{\nabla}_a \hat{\mathbf{e}}_b &= \hat{\beta}_{ab} \mathbf{v} + \hat{\gamma}^c_{ab} \hat{\mathbf{e}}_c & \bar{\nabla}_a \boldsymbol{\tau} &= -\hat{\beta}_{ab} \hat{\boldsymbol{\theta}}^b & \bar{\nabla}_a \hat{\boldsymbol{\theta}}^b &= -\hat{\gamma}^b_{ac} \hat{\boldsymbol{\theta}}^c, \end{aligned} \quad (\text{B.1.24})$$

from which we can infer the vector compatible Carrollian affine-connection one-form

$$\bar{\omega}^{\hat{u}}_{\hat{u}} = 0, \quad \bar{\omega}^{\hat{u}}_a = \hat{\delta}_a \boldsymbol{\tau} + \hat{\beta}_{ba} \hat{\boldsymbol{\theta}}^b, \quad \bar{\omega}^a_{\hat{u}} = 0, \quad \bar{\omega}^a_b = \hat{\gamma}^a_b \boldsymbol{\tau} + \hat{\gamma}^a_{cb} \hat{\boldsymbol{\theta}}^c. \quad (\text{B.1.25})$$

Metric-compatibility being equivalent to  $\bar{\omega}_{(ab)} = 0$  we get two conditions

$$\hat{\gamma}_{(ab)} = 0 \quad \text{and} \quad \hat{\gamma}_{[a|c|b]} = 0 \quad (\text{B.1.26})$$

where the symmetrisation acts on the two extreme indices. Hence we get the following result

**Proposition B.o.1.** In Cartan's frame, a strong Carrollian connection leaves free  $\hat{\delta}_a, \hat{\beta}_{ab}, \hat{\gamma}_{[ab]}$  and  $\hat{\gamma}_{[a|c|b]}$ .

The torsion two-form then reads

$$\bar{\mathcal{T}}^{\hat{u}} = \left( \hat{\delta}_a - \varphi_a \right) \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^a + \left( \hat{\beta}_{[ab]} - \bar{\omega}_{ab} \right) \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b \quad (\text{B.1.27})$$

<sup>5</sup>Here we already used the hat connection  $\bar{\nabla}$  defined in (B.1.34).



$$\bar{\mathcal{T}}^a = \delta^{bc} (\hat{\gamma}_{[ba]} + \hat{c}_{[ba]} + \hat{c}_{(ba)}) \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^a + \left( \hat{\gamma}_{ab}^c - \frac{1}{2} \hat{c}_{ab}^c \right) \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b \quad (\text{B.1.28})$$

Now it is time to make choices

- we could ask for the *minimal possible torsion* which imposes

$$\hat{\delta}_a = \varphi_a, \quad \hat{\beta}_{[ab]} = \hat{\omega}_{ab}, \quad \hat{\gamma}_{[ab]} = -\hat{c}_{ab}, \quad \hat{\gamma}_{bc}^a = \frac{1}{2} (\hat{c}_{bc}^a + \hat{c}_{b\ c}^a + \hat{c}_{c\ b}^a), \quad (\text{B.1.29})$$

which is perfectly fine as we are equating quantities having, in virtue of (B.1.15), the same tensorial properties. The remaining torsion is then

$$\hat{\mathcal{T}}^{\hat{u}} = 0 \quad \text{and} \quad \hat{\mathcal{T}}^c = +\delta^{ca} \hat{c}_{(ab)} \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^b \quad (\text{B.1.30})$$

so the torsion is held by a symmetric rank-two tensor  $\hat{c}_{(ab)}$  which can further be decomposed into a symmetric and traceless part  $\xi_{ab}$  and a pure trace part  $\theta$  via

$$\hat{c}_{(ab)} = \xi_{ab} + \frac{\theta}{d} \hat{\delta}_{ab}. \quad (\text{B.1.31})$$

Hence in the frame formulation this tensor  $\hat{c}_{(ab)}$  plays the role of the extrinsic curvature. If and only if it vanishes one can build a completely torsion free Carrollian connection. Note that everything is coherent because in the frame the metric is chosen to be constant so it has no *usual* extrinsic curvature, the latter is then supported by the lack of closeness of the orthonormal forms  $\hat{\boldsymbol{\theta}}^a$  in the temporal direction  $\boldsymbol{\tau}$ . Once (B.1.32) imposed, the remaining degrees of freedom are encoded into  $\hat{\beta}_{(ab)}$ .

- alternatively one could ask for a preservation of the time and space splitting i.e.  $\hat{\delta}_a = 0$  and  $\hat{\beta}_{ab} = 0$  while minimizing the torsion demanding  $\hat{\gamma}_{[ab]} = -\hat{c}_{[ab]}$ . The remaining torsion is then

$$\hat{\mathcal{T}}^{\hat{u}} = \varphi_a \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^a - \hat{\omega}_{ab} \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b \quad \text{and} \quad \hat{\mathcal{T}}^c = \delta^{ca} \hat{c}_{(ab)} \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^b. \quad (\text{B.1.32})$$

This connection, denoted  $\hat{\nabla}_a$  from now on is the equivalent of  $\hat{\nabla}_i$  in coordinate frame. This is the one we will use from now on.

## The connection $\hat{\nabla}$

The covariant time and space derivatives with  $\hat{\nabla}$  act on Carrollian scalars  $f$  as time and space directional derivatives i.e.

$$\hat{\nabla}_{\mathbf{v}} f = \mathbf{v}(f) \quad \text{and} \quad \hat{\nabla}_a f = \hat{e}_a(f). \quad (\text{B.1.33})$$

For Carrollian vectors  $\zeta = \zeta^a \hat{e}_a$  and forms  $\zeta = \zeta_a \hat{\boldsymbol{\theta}}^a$  we obtain:

$$\hat{\nabla}_a \zeta^b = \hat{e}_a(\zeta^b) + \hat{\gamma}_{ac}^b \zeta^c \Leftrightarrow \hat{\nabla}_a \zeta_b = \hat{e}_a(\zeta_b) - \hat{\gamma}_{ab}^c \zeta_c, \quad (\text{B.1.34a})$$

$$\hat{\nabla}_{\mathbf{v}} \zeta^a = \mathbf{v}(\zeta^a) - \hat{c}^{[ab]} \zeta_b \Leftrightarrow \hat{\nabla}_{\mathbf{v}} \zeta_a = \mathbf{v}(\zeta_a) - \hat{c}_{[ab]} \zeta^b, \quad (\text{B.1.34b})$$

and we extend these actions to higher order tensors using the Leibniz rule.

We can finally determine the curvature of the Carrollian connection under consideration using Cartan's formula in Appendix A

$$\hat{\mathcal{R}}^{\hat{u}}{}_b = 0 \quad \text{and} \quad \hat{\mathcal{R}}^a{}_b = \hat{R}^a{}_{cb} \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^c + \frac{1}{2} \hat{R}^a{}_{bcd} \hat{\boldsymbol{\theta}}^c \wedge \hat{\boldsymbol{\theta}}^d \quad (\text{B.1.35})$$

with

$$\hat{R}^a{}_{bcd} = \hat{e}_c(\hat{\gamma}^a{}_{db}) - \hat{e}_d(\hat{\gamma}^a{}_{cb}) + \hat{\gamma}^e{}_{db} \hat{\gamma}^a{}_{ce} - \hat{\gamma}^e{}_{cb} \hat{\gamma}^a{}_{de} - \hat{c}^e{}_{cd} \hat{\gamma}^a{}_{eb} + 2\hat{\omega}_{cd} \hat{\gamma}_{[eb]} \delta^{ae}, \quad (\text{B.1.36a})$$

$$\hat{R}^a{}_{cb} = -\left(\hat{\nabla}^a + \varphi^a\right) \hat{\gamma}_{(bc)} + \left(\hat{\nabla}_b + \varphi_b\right) \hat{\gamma}_{(cd)} \delta^{ad}. \quad (\text{B.1.36b})$$

One can trace the above and yield the Carroll-Ricci tensor and the Carroll scalar curvature

$$\hat{R}_{cd} = \hat{R}^a{}_{cad} \quad \text{and} \quad \hat{R} = \delta^{cd} \hat{R}_{cd}. \quad (\text{B.1.37})$$

As usual these components of the Riemann tensor allows one to derive the commutation rules of our connection  $\hat{\nabla}$ .

### B.1.3 Momenta and conservation equations

Let  $S$  be an (effective) action functional. It depends on the geometry encoded into  $\{\mathbf{v}, \hat{e}_a\}$  and on a set of matter fields collectively denoted by  $\Phi$

$$S = S[\mathbf{v}, \hat{e}_a; \Phi] = \int \boldsymbol{\mu} \mathcal{L} \quad (\text{B.1.38})$$

with  $\boldsymbol{\mu}$  the volume form  $\boldsymbol{\mu} = \frac{1}{2} \varepsilon_{ab} \hat{\boldsymbol{\theta}}^a \wedge \hat{\boldsymbol{\theta}}^b \wedge \boldsymbol{\tau}$  and  $\mathcal{L}$  the (effective) Lagrangian density. Note then that we consider the connection as being completely fixed with no residual degrees of freedom i.e.  $\hat{\delta}_a = 0, \hat{\beta}_{ab} = 0, \hat{\gamma}_{[ab]} = 0, \hat{\gamma}^c{}_{[ab]} = 0$ . The (non-evaluated) variation of the action yields

$$\delta S = \int \boldsymbol{\mu} (U \delta \mathbf{v} + U^a \delta \hat{e}_a) \quad (\text{B.1.39})$$

where one should see the conjugate momenta  $U$  and  $U^a$ . They are respectively a scalar-valued and a vector-valued one-form which can thus be decomposed as

$$U = U_{\hat{u}} \boldsymbol{\tau} + U_a \hat{\boldsymbol{\theta}}^a \quad \text{and} \quad U^a = U^a_{\hat{u}} \boldsymbol{\tau} + U^a_b \hat{\boldsymbol{\theta}}^b. \quad (\text{B.1.40})$$

Given our three transformations at hand, local rotations, local boosts and variation along a vector, requiring invariance of the action under each of them will yield constraints and equations on the momenta.

- **Boosts.** Let  $\lambda_a$  be the boost parameter.

$$\delta_\lambda S = 0 \iff \int \mu \lambda_a U^a(\mathbf{v}) = 0 \quad (\text{B.1.41})$$

which implies

$$\boxed{U^a_{\hat{u}} = 0} \quad (\text{B.1.42})$$

- **Rotations.** Let  $w_{ab} = -w_{ba}$  be the rotation parameter.

$$\delta_w S = 0 \iff \int \mu w_a{}^b U^a_b = 0 \quad (\text{B.1.43})$$

which implies

$$\boxed{U^{ab} = U^{ba}} \quad (\text{B.1.44})$$

by anti-symmetry of  $w_{ab}$ .

- **Diffeomorphisms.** Let  $\xi = \xi^a \hat{e}_a$  be a vector. Under its action the frame vectors change with the Lie derivative

$$\mathcal{L}_\xi \mathbf{v} = -(\mathbf{v}(\xi^0) + \xi^a \varphi_a) \mathbf{v} - (\mathbf{v}(\xi^a) - \xi^a \hat{c}^b{}_a) \hat{e}_b, \quad (\text{B.1.45a})$$

$$\mathcal{L}_\xi \boldsymbol{\tau} = (\mathbf{v}(\xi^0) + \xi^a \varphi_a) \boldsymbol{\tau} + (\hat{e}_b(\xi^0) + 2\xi^c \hat{\omega}_{bc} - \xi^0 \varphi_b) \hat{\boldsymbol{\theta}}^b, \quad (\text{B.1.45b})$$

$$\mathcal{L}_\xi \hat{e}_a = -(\hat{e}_a(\xi^0) + 2\xi^b \hat{\omega}_{ab} - \xi^0 \varphi_a) \mathbf{v} - (\hat{e}_a(\xi^c) + \xi^0 \hat{c}^c{}_a + \xi^b \hat{c}^c{}_{ab}) \hat{e}_c, \quad (\text{B.1.45c})$$

$$\mathcal{L}_\xi \hat{\boldsymbol{\theta}}^a = (\mathbf{v}(\xi^a) - \hat{c}^a{}_b \xi^b) \boldsymbol{\tau} + (\hat{e}_b(\xi^a) + \xi^c \hat{c}^a{}_{bc} + \xi^0 \hat{c}^a{}_b) \hat{\boldsymbol{\theta}}^b. \quad (\text{B.1.45d})$$

where we have related the formulae to our connection  $\hat{\nabla}$ . The last step before varying the action is to display the divergence equations (1.3.14a) and (1.3.14b) in Cartan's frame. In the following,  $f$ ,  $V^a$  and  $V^a_b$  are respectively a scalar, a vector and a rank-(1, 1) tensor under rotations

$$\mu f \hat{\nabla}_\mathbf{v} \xi^{\hat{u}} = -\mu \xi^{\hat{u}} (\hat{\nabla}_\mathbf{v} + \theta) f + \text{b.t.} \quad (\text{B.1.46})$$

$$\mu V_a \hat{\nabla}_\mathbf{v} \xi^a = -\mu \xi^a (\hat{\nabla}_\mathbf{v} + \theta) V_a + \text{b.t.}$$

and

$$\mu V^a \hat{\nabla}_a \xi^{\hat{u}} = -\mu \xi^{\hat{u}} (\hat{\nabla}_a + \varphi_a) V^a + \text{b.t.} \quad (\text{B.1.47})$$

$$\mu V^a_b \hat{\nabla}_a \xi^b = -\mu \xi^b (\hat{\nabla}_a + \varphi_a) V^a_b + \text{b.t.}$$

where b.t. means “up to boundary terms” that we do not display here (there are available in the hybrid frame, see (1.3.14a) and (1.3.14b)). For completeness we display the explicit computation

for the first equation (recall that as  $\xi^{\hat{u}}$  is a scalar  $\hat{\nabla}_{\mathbf{v}}(\xi^{\hat{u}}) \equiv \mathbf{v}(\xi^{\hat{u}})$ )

$$\begin{aligned}
\mu f \mathbf{v}(\xi^0) &= \mathcal{L}_{\mathbf{v}}(\mu f \xi^0) - \xi^0 \mathcal{L}_{\mathbf{v}}(\mu f) \\
&= \mathbf{d} \left( \frac{1}{2} \varepsilon_{ab} \hat{\theta}^a \wedge \hat{\theta}^b f \xi^0 \right) - \xi^0 \frac{1}{2} \varepsilon_{ab} \mathbf{d}(\hat{\theta}^a \wedge \hat{\theta}^b f) \\
&= \text{b.t.} - \mu \xi^0 \mathbf{v}(f) - \xi^0 f \varepsilon_{ab} \mathbf{d}\hat{\theta}^a \wedge \hat{\theta}^b \\
&= \text{b.t.} - \mu \xi^0 \mathbf{v}(f) - \mu \xi^0 f \hat{c}^a_a \\
&= \text{b.t.} - \mu \xi^0 [\mathbf{v}(f) + \theta f] .
\end{aligned} \tag{B.1.48}$$

Asking the effective action to be invariant under such a transformation yields the conservation equations in Cartan's frame

$$\boxed{\left( \hat{\nabla}_{\mathbf{v}} + \theta \right) U_{\hat{u}} + \left( \hat{\nabla}_a + 2\varphi_a \right) U^a_{\hat{u}} - \hat{c}_{(ab)} U^{ab} = 0} \tag{B.1.49}$$

and

$$\boxed{\left( \hat{\nabla}_a + \varphi_a \right) U^a_b - U_{\hat{u}} \varphi_b - 2U^a_{\hat{u}} \hat{\omega}_{ab} = - \left( \hat{\nabla}_{\mathbf{v}} + \theta \right) U_b} \tag{B.1.50}$$

which, comparing with (1.3.20) and (1.3.21), leads to the following identifications

$$U_{\hat{u}} = \Pi \quad , \quad U_a = -P_a \quad , \quad U^a_{\hat{u}} = \Pi^a \quad , \quad U^{ab} = -\Pi^{ab}. \tag{B.1.51}$$

Note that the latter are consistent with the properties (B.1.42) and (B.1.44).

One could then turn on the ambiguities of the connection and perform a similar analysis than the one at the end of Sec. (1.3.2).

## B.1.4 Weyl covariance

Weyl transformations parametrised by an arbitrary function  $\mathcal{B}(t, \mathbf{x})$  are added by hand into the Carroll structure. They act on the frame and coframe as

$$\mathbf{v} \rightarrow \mathcal{B} \mathbf{v} \quad , \quad \hat{\mathbf{e}}_a \rightarrow \mathcal{B} \hat{\mathbf{e}}_a \quad , \quad \boldsymbol{\tau} \rightarrow \mathcal{B}^{-1} \boldsymbol{\tau} \quad , \quad \hat{\theta}^a \rightarrow \mathcal{B}^{-1} \hat{\theta}^a \tag{B.1.52}$$

meaning that the frame vectors are weight 1 while the coframe forms are weight  $-1$ . The other geometric quantities behave like

$$\hat{c}^a_b \rightarrow \mathcal{B} \hat{c}^a_b - \mathbf{v}(\mathcal{B}) \delta_a^b \quad , \quad \hat{c}^c_{ab} \rightarrow \mathcal{B} \hat{c}^c_{ab} - \hat{\mathbf{e}}_{[a}(\mathcal{B}) \delta_{b]}^c \tag{B.1.53}$$

$$\varphi_a \rightarrow \mathcal{B} \varphi_a + \hat{\mathbf{e}}_a(\mathcal{B}) \quad , \quad \hat{\omega}_{ab} \rightarrow \mathcal{B} \hat{\omega}_{ab}. \tag{B.1.54}$$

The various components of the connection are splitted in two categories regarding Weyl trans-

formations. Part of them are covariant and their weights are displayed in the next Table.

Quantity	$\hat{\delta}_a$	$\hat{\rho}^a$	$\hat{\beta}_{ab}$	$\hat{\kappa}_b^a$	(B.1.55)
Weight $w$	1	1	1	1	

The remaining ones are still weight 1 but possess an anomalous part.

$$\hat{\gamma} \rightarrow \mathcal{B}\hat{\gamma} + \mathbf{v}(\mathcal{B}) \quad , \quad \hat{\alpha}_a \rightarrow \mathcal{B}\hat{\alpha}_a + \hat{e}_a(\mathcal{B}) \quad (\text{B.1.56})$$

$$\hat{\gamma}_b^a \rightarrow \mathcal{B}\hat{\gamma}_b^a + \mathbf{v}(\mathcal{B})\delta_b^a \quad , \quad \hat{\gamma}_{ab}^c \rightarrow \mathcal{B}\hat{\gamma}_{ab}^c + \hat{e}_a(\mathcal{B})\delta_b^c. \quad (\text{B.1.57})$$

**Remark** It is now time to recap which part of the spin-connection is a connection under which transformations. Using (B.1.18g), (B.1.18h), (B.1.19d), (B.1.19e), (B.1.56) and (B.1.57) we have that

- $\omega^{\hat{u}}_{\hat{u}}$  gauges tracefree Weyl transformations seen as the generator  $u\partial_u \in GL(d+1)$ ,
- $\omega^a_{\hat{u}}$  gauges Galilean boosts seen as the generator  $u\partial_a \in GL(d+1)$ ,
- $\omega^{\hat{u}}_a$  gauges Carrollian boosts seen as the generator  $x^a\partial_u \in GL(d+1)$ ,
- $\omega^a_b$  gauges both local rotations seen as the generator  $x^a\partial_b - x^b\partial_a \in GL(d+1)$ .

**Remark** As one may have already noticed, when working in Cartan's orthonormal frame, the Weyl weights are slightly modified with respect to the ones in hybrid frame  $(\hat{t}, i)$ . The main virtue of the former frame is that all geometric quantities, whether belonging to the metric or the connection, have the same weight (up to anomalous parts).

Following the pattern adopted for the affine connection  $\hat{\nabla}$ , we introduce here a Weyl connection that respects the time and space splitting, associated with two Weyl-covariant derivatives. These act on weight- $w$  Carrollian tensors and produce Carrollian tensors of weight  $w+1$ . The Weyl connection is encoded in  $\theta$  and  $\varphi_a$ , see (B.1.11), and the Weyl-covariant derivatives are defined as follows:

- on scalars

$$\hat{\mathcal{D}}_{\mathbf{v}}\Phi = \mathbf{v}(\Phi) + \frac{w}{d}\theta\Phi, \quad \hat{\mathcal{D}}_a\Phi = \hat{e}_a(\Phi) + w\varphi_a\Phi; \quad (\text{B.1.58})$$

- on vectors  $\mathbf{v} = v^a\hat{e}_a$

$$\hat{\mathcal{D}}_{\mathbf{v}}v^a = \hat{\nabla}_{\mathbf{v}}v^a + \frac{w}{d}\theta v^a, \quad \hat{\mathcal{D}}_av^b = \hat{\nabla}_av^b + w\varphi_av^b + \varphi^bv_a - \delta_a^bv^c\varphi_c; \quad (\text{B.1.59})$$

- on rank-2 tensors  $\mathbf{t} = t_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$

$$\hat{\mathcal{D}}_{\mathbf{v}}t_{ab} = \hat{\nabla}_{\mathbf{v}}t_{ab} + \frac{w}{d}\theta t_{ab}, \quad (\text{B.1.60a})$$

$$\hat{\mathcal{D}}_ct_{ab} = \hat{\nabla}_ct_{ab} + w\varphi_ct_{ab} + \varphi_at_{cb} + \varphi_bt_{ac} - \delta_{ac}t_{db}\varphi^d - \delta_{cb}t_{ad}\varphi^d. \quad (\text{B.1.60b})$$

Using Leibniz' rule one obtains the generalization for any conformal tensor.

The Riemann–Carroll–Weyl curvature is a weight-2 tensor defined through the commutator of the Carrollian Weyl derivatives acting on Carrollian scalars  $\Phi$ , vectors  $v^c$  or 2-tensors  $t^{cd}$  of weight  $w$

$$[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] \Phi = 2\hat{\omega}_{ab} \hat{\mathcal{D}}_v \Phi + w\Omega_{ab} \Phi, \quad (\text{B.1.61a})$$

$$[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] v^c = \hat{\mathcal{S}}^c_{dab} v^d + 2\hat{\omega}_{ab} \hat{\mathcal{D}}_v v^c + w\Omega_{ab} v^c, \quad (\text{B.1.61b})$$

$$[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] t^{cd} = \hat{\mathcal{S}}^c_{eab} t^{ed} + \hat{\mathcal{S}}^d_{eab} t^{ce} + 2\hat{\omega}_{ab} \hat{\mathcal{D}}_v t^{cd} + w\Omega_{ab} t^{cd}, \quad (\text{B.1.61c})$$

where

$$\Omega_{ab} = \hat{e}_a(\varphi_b) - \hat{e}_b(\varphi_a) - \hat{c}^c_{ab} \varphi_c - \frac{2}{d} \hat{\omega}_{ab} \theta \quad (\text{B.1.62})$$

is yet another weight-2 Carrollian tensor. From the Riemann–Weyl–Carroll tensor, we define

$$\hat{\mathcal{S}}_{cd} = \hat{\mathcal{S}}^a_{cad}, \quad \hat{\mathcal{R}} = \delta^{cd} \hat{\mathcal{S}}_{cd}, \quad (\text{B.1.63})$$

all weight-2.

We can further consider time and space derivatives:

$$[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a] \Phi = -\xi^b_a \hat{\mathcal{D}}_b \Phi + w\hat{\mathcal{R}}_a \Phi, \quad (\text{B.1.64a})$$

$$[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a] v^b = -\hat{\mathcal{S}}^b_{ac} v^c - \xi^c_a \hat{\mathcal{D}}_c v^b + w\hat{\mathcal{R}}_a v^b, \quad (\text{B.1.64b})$$

$$[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a] t^{bc} = -\hat{\mathcal{S}}^b_{ad} t^{dc} - \hat{\mathcal{S}}^c_{ad} t^{bd} - \xi^d_a \hat{\mathcal{D}}_d t^{bc} + w\hat{\mathcal{R}}_a t^{bc}, \quad (\text{B.1.64c})$$

revealing a clear pattern for any Carrollian conformal tensor. In these expressions

$$\hat{\mathcal{S}}^c_{ab} = -\hat{\mathcal{S}}^c_{ba} = \hat{\mathcal{D}}^c \xi_{ab} - \hat{\mathcal{D}}_b \xi^c_a + \delta^c_a \hat{\mathcal{R}}_b - \delta_{ab} \hat{\mathcal{R}}^c \quad (\text{B.1.65})$$

and  $\hat{\mathcal{R}}_a$  are weight-two tensors. Note that in Cartan frame, both the shear  $\xi_{ab}$  and the vorticity  $\hat{\omega}_{ab}$  have weight one, regardless of the position of the indices. In natural frame  $\xi_{ij}$  and  $\hat{\omega}_{ij}$  have weight  $-1$ , but raising an index augments the weight by two units.

Within the Weyl-covariant framework in Cartan's frame, the conservation equations with the connection  $\hat{\mathcal{D}}$  (i.e. the equivalent of (1.4.16) and (1.4.17)) takes the form

$$\hat{\mathcal{D}}_v \Pi + \hat{\mathcal{D}}_a \Pi^a + \Upsilon^a_b \xi^b_a = 0, \quad (\text{B.1.66})$$

$$\frac{1}{2} \hat{\mathcal{D}}_a \Pi + \hat{\mathcal{D}}_b \Upsilon^b_a + 2 * \hat{\omega} * \Pi^a + \hat{\mathcal{D}}_v P_a + \xi^b_a P_b = 0. \quad (\text{B.1.67})$$

with the momenta  $\Pi^a_b = \Upsilon^a_b + \frac{1}{2} \Pi \delta^a_b$ .

## B.1.5 Isometries and charges

A plain Killing field is require to preserve the weak Carroll structure (i.e.  $\mathcal{L}_\xi \mathbf{v} = 0$  and  $\mathcal{L}_\xi \delta_{ab} = 0$ ) which leads to the Carroll-Killing frame equations

$$\begin{cases} \hat{\nabla}_{(a} \xi_{b)} + 2\xi^{\hat{u}} \hat{c}_{(ab)} = 0 \\ \mathbf{v}(\xi^{\hat{u}}) + \xi^a \varphi_a = 0 \\ \mathbf{v}(\xi^a) - \hat{c}^a_b \xi^b = 0. \end{cases} \quad (\text{B.1.68})$$

**Remark** The last condition selects a peculiar type of diffeomorphisms which actually are the Carrollian diffeomorphisms. Hence  $\mathbf{v}(\xi^a) - \hat{c}^a_b \xi^b = 0$  can be taken as their defining property.

**Remark** In Cartan's frame strong Killings are asked to preserve the clock form  $\tau$  yielding

$$\hat{e}_a(\xi^{\hat{u}}) - \varphi_a \xi^{\hat{u}} + 2\omega_{ab} \xi^b = 0. \quad (\text{B.1.69})$$

One then defines a charge current by a couple  $(\kappa, K^a)$  contracting the momenta with the components of a Carroll-Killing field

$$\kappa = -\xi^{\hat{u}} \Pi + \xi^a P_a \quad \text{and} \quad K^a = -\xi^{\hat{u}} \Pi^a + \xi^b \Pi^a_b \quad (\text{B.1.70})$$

Its divergence reads

$$\left( \hat{\nabla}_{\mathbf{v}} + \theta \right) \kappa + \left( \hat{\nabla}_a + \varphi_a \right) K^a = (\mathcal{L}_\xi \tau)_a \Pi^a \quad (\text{B.1.71})$$

leading to the same alternative that in the hybrid frame. Finally the charge takes the following form

$$Q_{(\kappa, K)} = \int_S \mu \kappa. \quad (\text{B.1.72})$$

## B.2 The $c \rightarrow 0$ limit

We turn now to Carroll structures describes in a Cartan orthonormal frame and obtained from the limit. As this Appendix is just displayed for completeness, with the permission of the authors of [155] we reproduce here the Appendix A of that reference.

Here capital Latin indices  $A, B, \dots$  range over  $0, 1, \dots, d$  while small Latin indices  $a, b, \dots$  over  $1, 2, \dots$ . We can reach a Carroll structure like (A.0.2) starting from a pseudo-Riemannian spacetime in the following parametrization

$$ds^2 = \eta_{AB} \theta^A \theta^B = -\left(\theta^{\hat{0}}\right)^2 + \delta_{ab} \theta^a \theta^b = -c^2 \left(\hat{\theta}^{\hat{u}}\right)^2 + \delta_{ab} \hat{\theta}^a \hat{\theta}^b, \quad (\text{B.2.1})$$

where we have assumed that all  $c$ -dependence is explicit i.e.  $\theta^a = \hat{\theta}^a$  while  $\theta^{\hat{0}} = c \hat{\theta}^{\hat{u}}$ . The relationship between the relativistic congruence  $\mathbf{u}$  and the Carrollian fibre attributes, field of observers and

clock form, is  $\mathbf{v} = \mathbf{u} = \hat{e}_u$  for the former and  $\boldsymbol{\tau} = -\frac{\mathbf{u}}{c^2} = \hat{\boldsymbol{\theta}}^u$  for the latter.

When the Carrollian frame, coframe and degenerate metric are parameterized as

$$e_{\hat{0}} = \frac{\gamma}{k} (\partial_u + v^i \partial_i) \quad \Leftrightarrow \quad \theta^{\hat{0}} = k \left( \frac{du}{\gamma} - \Delta_i (dx^i - v^i du) \right), \quad (\text{B.2.2})$$

$$e_a = e_a^i (\partial_i + \gamma \Delta_i (\partial_u + v^j \partial_j)) \quad \Leftrightarrow \quad \theta^a = e^a_i (dx^i - v^i du) \quad (\text{B.2.3})$$

then relativistic metric reads

$$\begin{aligned} ds^2 &= -c^2 \left( \frac{du}{\gamma} - \Delta_i (dx^i - v^i du) \right)^2 + \Gamma_{ij}^2 (dx^i - v^i du) (dx^j - v^j du), \\ &= -\frac{c^2}{\gamma^2} \left( du^2 - 2\gamma \Delta_i du (dx^i - v^i du) \right) + \left( \Gamma_{ij}^2 - c^2 \Delta_i \Delta_j \right) (dx^i - v^i du) (dx^j - v^j du), \end{aligned} \quad (\text{B.2.4})$$

where the normalized vector congruence is

$$\mathbf{u} = \gamma (\partial_u + v^i \partial_i). \quad (\text{B.2.5})$$

We will not explicitly operate with this frame, which coincides at  $v^i = 0$  with the Papapetrou–Randers form employed in Refs. [52, 53, 65, 66, 84, 104], where  $\Omega = \frac{1}{\gamma}$ ,  $b_i = \Delta_i$  and  $a_{ij} = \Gamma_{ij}^2$ .

The pseudo-Riemannian manifold is naturally equipped with a Levi–Civita connection. We would like to express the latter in terms of the Carrollian tensors appearing in Eqs. (A.0.6) and (B.1.11). This procedure is to provide the suitable tools for reaching the  $c \rightarrow 0$  limit in relativistic dynamical equations such as  $\nabla_a T^{ab} = 0$ . We reckon that in the parameterization of  $\{d\theta^A\} = \{d\theta^{\hat{0}}, d\theta^a\}$ , Eq. (B.1.11) holds

$$d\theta^{\hat{0}} - \varphi_a \theta^a \wedge \theta^{\hat{0}} + c \hat{\omega}_{ab} \theta^a \wedge \theta^b = 0, \quad d\theta^c + \frac{1}{c} \hat{\gamma}^c_a \theta^a \wedge \theta^{\hat{0}} + \frac{1}{2} \hat{c}^c_{ab} \theta^a \wedge \theta^b = 0. \quad (\text{B.2.6})$$

Thus the Levi–Civita affine connection one-form reads:

$$\begin{aligned} \omega_{ab} &= - \left( c \hat{\omega}_{ab} + \frac{1}{c} \hat{\gamma}_{[ab]} \right) \theta^{\hat{0}} + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\theta}^c \\ &= - (c^2 \hat{\omega}_{ab} + \hat{\gamma}_{[ab]}) \boldsymbol{\tau} + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\boldsymbol{\theta}}^c = -c^2 \hat{\omega}_{ab} \boldsymbol{\tau} + \hat{\omega}_{ab}, \end{aligned} \quad (\text{B.2.7})$$

and

$$\omega^{\hat{0}}_a = \varphi_a \theta^{\hat{0}} - c \hat{\omega}_{ab} \theta^b + \frac{1}{c} \hat{\gamma}_{(ab)} \theta^b = -c \left( -\varphi_a \boldsymbol{\tau} + \hat{\omega}_{ab} \hat{\boldsymbol{\theta}}^b \right) + \frac{1}{c} \hat{\gamma}_{(ab)} \hat{\boldsymbol{\theta}}^b. \quad (\text{B.2.8})$$

It has zero torsion and the curvature reads

$$\begin{aligned} \mathcal{R}^{\hat{0}}_a &= \left[ \frac{1}{c} \left( \hat{\nabla}_v \hat{\gamma}_{(ab)} + \hat{\gamma}_{(ac)} \hat{\gamma}_{(bd)} \delta^{cd} \right) - c \left( \hat{\omega}_a^c \hat{\gamma}_{(cb)} + \hat{\omega}_b^c \hat{\gamma}_{(ca)} + \hat{\nabla}_{(a} \varphi_{b)} + \varphi_a \varphi_b \right) \right. \\ &\quad \left. + c^3 \hat{\omega}_a^c \hat{\omega}_{bc} \right] \boldsymbol{\tau} \wedge \hat{\boldsymbol{\theta}}^b + \frac{1}{2} \left[ \frac{1}{c} \left( \hat{R}_{bac} - \varphi_b \hat{\gamma}_{(ac)} + \varphi_c \hat{\gamma}_{(ab)} \right) \right. \end{aligned}$$



$$-c \left( \hat{\nabla}_a \hat{\omega}_{bc} + \varphi_a \hat{\omega}_{bc} + \varphi_b \hat{\omega}_{ac} - \varphi_c \hat{\omega}_{ab} \right) \Big] \hat{\theta}^b \wedge \hat{\theta}^c, \quad (\text{B.2.9a})$$

$$\begin{aligned} \mathcal{R}^a_b &= \hat{\mathcal{R}}^a_b + \delta^{ad} \left[ \varphi_d \hat{\gamma}_{(cb)} - \varphi_b \hat{\gamma}_{(cd)} + c^2 \left( \hat{\nabla}_c \hat{\omega}_{db} + \varphi_c \hat{\omega}_{db} + \varphi_d \hat{\omega}_{cb} - \varphi_b \hat{\omega}_{cd} \right) \right] \boldsymbol{\tau} \wedge \hat{\theta}^c \\ &+ \frac{1}{2} \delta^{ae} \left[ \frac{1}{c^2} \left( \gamma_{(ec)} \gamma_{(bd)} - \gamma_{(ed)} \gamma_{(bc)} \right) - \gamma_{(ec)} \hat{\omega}_{bd} + \gamma_{(ed)} \hat{\omega}_{bc} \right. \\ &\left. - \gamma_{(bd)} \hat{\omega}_{ec} + \gamma_{(bc)} \hat{\omega}_{ed} + k^2 \left( 2 \hat{\omega}_{eb} \hat{\omega}_{cd} - \hat{\omega}_{ed} \hat{\omega}_{bc} + \hat{\omega}_{ec} \hat{\omega}_{bd} \right) \right] \hat{\theta}^c \wedge \hat{\theta}^d, \end{aligned} \quad (\text{B.2.9b})$$

where we have used the Carrollian expressions available in (B.1.35), (B.1.36a) and (B.1.36b).

**Remark** Note that at  $\Delta_i = 0$  in (B.2.4), one recovers the boundary frame of bulk Newman–Unti anti-de Sitter gauge, and

$$d\theta^{\hat{0}} = \varphi_a \theta^a \wedge \theta^{\hat{0}}, \quad (\text{B.2.10})$$

which resonates with the Carrollian relative (B.1.11). Hence the boundary vorticity vanishes following Eq. (B.2.6).

We would like now to make the contact with the Carrollian descendants. The relativistic congruence is  $\mathbf{u} = -c\theta^{\hat{0}}$ . Given the connection, we can determine its kinematical properties: the expansion  $\Theta$ , the acceleration  $a_A$ , the shear  $\sigma_{AB}$  and the vorticity  $\omega_{AB}$  as defined in Eqs. (2.2.9). The latter tensors are all transverse (and traceless for the shear) and have thus non-vanishing components in spatial directions only (indices  $a, b, \dots$ ). We find

$$\Theta = \theta, \quad a_a = c^2 \varphi_a, \quad (\text{B.2.11})$$

and

$$\sigma_{ab} = \xi_{ab} = \hat{\gamma}_{(ab)} - \frac{\theta}{d} \delta_{ab}, \quad \omega_{ab} = c^2 \hat{\omega}_{ab}. \quad (\text{B.2.12})$$

Anticipating the next Chapter we can furthermore determine the Weyl connection we will encounter (3.2.7) (where we must trade the 2 for  $d$ )

$$\mathbf{A} = \varphi_a \hat{\theta}^a + \frac{\theta}{d} \boldsymbol{\tau}, \quad (\text{B.2.13})$$

and its curvature (3.2.12):

$$\mathbf{F} = d\mathbf{A} = \frac{1}{2} \Omega_{ab} \hat{\theta}^a \wedge \hat{\theta}^b + \hat{\mathcal{R}}_a \boldsymbol{\tau} \wedge \hat{\theta}^a, \quad (\text{B.2.14})$$

where  $\Omega_{ab}$  and  $\hat{\mathcal{R}}_a$  are defined in Eqs. (B.1.62) and (B.1.64a) — explicitly

$$\hat{\mathcal{R}}_a = \hat{\nabla}_v \varphi_a + \xi_{ab} \varphi^b - \frac{1}{d} \hat{e}_a(\theta). \quad (\text{B.2.15})$$

All the above quantities are relativistic, but expressed in terms of the Carrollian descendants describing the properties of the manifold reached at vanishing- $c$ .

We can finally convey the relativistic conservation equations  $\nabla_A T^{AB} = 0$  for an arbitrary

energy–momentum tensor  $T^{AB}$ , stated in Carrollian language. Given the choice of congruence, the transverse heat current and stress tensor have only spatial components:  $q^a$  and  $\tau^{ab}$ . We then define as usual the longitudinal and transverse components of the conservation equations,

$$\mathcal{L} = -u^B \nabla_C T^C_B = -c \nabla_C T^C_{\hat{0}} = -\nabla_C T^C_{\hat{u}}, \quad \mathcal{T}^a = e^a_B \nabla_C T^{CB} = \nabla_C T^{Ca}, \quad (\text{B.2.16})$$

and explicitly find

$$\mathcal{L} = \mathbf{v}(\varepsilon) + \theta\varepsilon + \left( \hat{\nabla}_a + 2\varphi_a \right) q^a + \left( \xi_{ab} + \frac{\theta}{d} \delta_{ab} \right) \left( \tau^{ab} + p\delta^{ab} \right), \quad (\text{B.2.17a})$$

$$\mathcal{T}^a = \left( \hat{\nabla}_b + \varphi_b \right) \left( \tau^{ab} + p\delta^{ab} \right) + \varphi^a \varepsilon + 2q_b \hat{\omega}^{ba} + \frac{1}{c^2} \left( \hat{\nabla}_v q^a + \frac{d+1}{d} \theta q^a + \xi^{ab} q_b \right). \quad (\text{B.2.17b})$$

In the conformal case, assuming thus  $\varepsilon = dp$  and  $\tau_a^a = 0$  and canonical conformal weights  $d+1$  for  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  (we are in Cartan' frame and the weights do not depend on the position of the indices), these equations are recast as:

$$\mathcal{L} = \hat{\mathcal{D}}_v \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab}, \quad (\text{B.2.18a})$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \hat{\omega}^{ba} + \frac{1}{c^2} \left( \hat{\mathcal{D}}_v q^a + \xi^{ab} q_b \right). \quad (\text{B.2.18b})$$

As discussed extensively in Refs. [53, 65], the outcome of the Carrollian limit depends on the behaviour of  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  with respect to  $c$ . The equations at hand will be conceivably multiplied, leading to replicas. The same phenomenon occurs in the Galilean limit with the emergence of the continuity equation out of the relativistic longitudinal equation, besides the energy equation.

## Useful formulas

We close this appendix with some formulas that are useful when considering the zero- $c$  limit. In the following, we reduce the Riemannian Levi–Civita and Weyl covariant derivatives in terms of the Carrollian connections introduced earlier.

**Levi–Civita** We will present the vector and the rank-two tensor

$$\mathbf{V} = V^A \mathbf{e}_A \quad \text{— } V^a \text{ provide the components of a Carrollian vector and } V_{\hat{u}} = cV_{\hat{0}} = -cV^{\hat{0}} \text{ a Carrollian scalar}$$

$$\begin{cases} c^2 \nabla_{\hat{0}} V^{\hat{0}} = c\mathbf{v} \left( V^{\hat{0}} \right) + c^2 \varphi_a V^a \\ c \nabla_{\hat{0}} V^b = \hat{\nabla}_v V^b + cV^{\hat{0}} \varphi^b + c^2 V^a \hat{\omega}_a^b \\ c \nabla_a V^{\hat{0}} = c\hat{e}_a \left( V^{\hat{0}} \right) + \left( \xi_{ab} + \frac{\theta}{d} \delta_{ab} + c^2 \hat{\omega}_{ab} \right) V^b \\ \nabla_a V^b = \hat{\nabla}_a V^b + \frac{1}{c} \left( \xi_a^b + \frac{\theta}{d} \delta_a^b + c^2 \hat{\omega}_a^b \right) V^{\hat{0}}; \end{cases} \quad (\text{B.2.19})$$

$$\mathbf{T} = T^{AB} \mathbf{e}_A \otimes \mathbf{e}_B \quad \text{— } T^{ab} \text{ are farther interpreted as components of a Carrollian rank-two}$$

tensor,  $T_{\hat{u}}{}^a = cT_{\hat{0}}{}^a = -cT^{\hat{0}a}$  and  $T^a_{\hat{u}} = cT^a_{\hat{0}} = -cT^{a\hat{0}}$  those of Carrollian vectors, while  $T_{\hat{u}\hat{u}} = c^2T^{\hat{0}\hat{0}} = c^2T^{\hat{0}\hat{0}}$  gives a Carrollian scalar

$$\begin{cases} c^3\nabla_{\hat{0}}T^{\hat{0}\hat{0}} = c^2\nabla_{\hat{\nu}}(T^{\hat{0}\hat{0}}) + c^3\varphi_a(T^{a\hat{0}} + T^{\hat{0}a}) \\ c^2\nabla_{\hat{0}}T^{b\hat{0}} = c\hat{\nabla}_{\hat{\nu}}T^{b\hat{0}} + c^2\varphi^bT^{\hat{0}\hat{0}} + c^2\varphi_aT^{ba} + c^3\hat{\omega}_a{}^bT^{a\hat{0}} \\ c\nabla_{\hat{0}}T^{ab} = \hat{\nabla}_{\hat{\nu}}T^{ab} + c(\varphi^aT^{\hat{0}b} + \varphi^bT^{a\hat{0}}) + c^2(T^{ac}\hat{\omega}_c{}^b + T^{cb}\hat{\omega}_c{}^a) \\ c\nabla_aT^{b\hat{0}} = c\hat{\nabla}_aT^{b\hat{0}} + \left(\xi_{ac} + \frac{\theta}{d}\delta_{ac} + c^2\hat{\omega}_{ac}\right)T^{bc} + \left(\xi_a{}^b + \frac{\theta}{d}\delta_a{}^b + c^2\hat{\omega}_a{}^b\right)T^{\hat{0}\hat{0}} \\ c^2\nabla_aT^{\hat{0}\hat{0}} = c^2\hat{\nabla}_a(T^{\hat{0}\hat{0}}) + c\left(\xi_{ac} + \frac{\theta}{d}\delta_{ac} + c^2\hat{\omega}_{ac}\right)T^{c\hat{0}} + c\left(\xi_a{}^c + \frac{\theta}{d}\delta_a{}^c + c^2\hat{\omega}_a{}^c\right)T^{\hat{0}c} \\ \nabla_aT^{bc} = \hat{\nabla}_aT^{bc} + \frac{1}{c}\left(\xi_a{}^b + \frac{\theta}{d}\delta_a{}^b + c^2\hat{\omega}_a{}^b\right)T^{\hat{0}c} + \frac{1}{c}\left(\xi_a{}^c + \frac{\theta}{d}\delta_a{}^c + c^2\hat{\omega}_a{}^c\right)T^{b\hat{0}}; \end{cases} \quad (\text{B.2.20})$$

**Weyl** similarly

$$\mathbf{V} = V^A\mathbf{e}_A$$

$$\begin{cases} c^2\mathcal{D}_{\hat{0}}V^{\hat{0}} = c\hat{\mathcal{D}}_{\hat{\nu}}V^{\hat{0}} \\ c\mathcal{D}_{\hat{0}}V^b = \hat{\mathcal{D}}_{\hat{\nu}}V^b + c^2V^a\hat{\omega}_a{}^b \\ c\mathcal{D}_aV^{\hat{0}} = c\hat{\mathcal{D}}_aV^{\hat{0}} + (\xi_{ab} + c^2\hat{\omega}_{ab})V^b \\ \mathcal{D}_aV^b = \hat{\mathcal{D}}_aV^b + \frac{1}{c}(\xi_a{}^b + c^2\hat{\omega}_a{}^b)V^{\hat{0}}; \end{cases} \quad (\text{B.2.21})$$

$$\mathbf{T} = T^{AB}\mathbf{e}_A \otimes \mathbf{e}_B$$

$$\begin{cases} c^3\mathcal{D}_{\hat{0}}T^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_{\hat{\nu}}(c^2T^{\hat{0}\hat{0}}) \\ c^2\mathcal{D}_{\hat{0}}T^{\hat{0}b} = \hat{\mathcal{D}}_{\hat{\nu}}(cT^{\hat{0}b}) + c^3T^{\hat{0}a}\hat{\omega}_a{}^b \\ c\mathcal{D}_{\hat{0}}T^{ab} = \hat{\mathcal{D}}_{\hat{\nu}}T^{ab} + c^2(T^{cb}\hat{\omega}_c{}^a + T^{ac}\hat{\omega}_c{}^b) \\ c^2\mathcal{D}_aT^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_a(c^2T^{\hat{0}\hat{0}}) + (\xi_{ab} + c^2\hat{\omega}_{ab})cT^{b\hat{0}} + (\xi_{ab} + c^2\hat{\omega}_{ab})cT^{\hat{0}b} \\ c\mathcal{D}_aT^{\hat{0}b} = \hat{\mathcal{D}}_a(cT^{\hat{0}b}) + (\xi_{ac} + c^2\hat{\omega}_{ac})T^{cb} + (\xi_a{}^b + c^2\hat{\omega}_a{}^b)T^{\hat{0}\hat{0}} \\ \mathcal{D}_aT^{bc} = \hat{\mathcal{D}}_aT^{bc} + \frac{1}{c}(\xi_a{}^b + c^2\hat{\omega}_a{}^b)T^{\hat{0}c} + \frac{1}{c}(\xi_a{}^c + c^2\hat{\omega}_a{}^c)T^{b\hat{0}}. \end{cases} \quad (\text{B.2.22})$$

# Appendix C

## Bridging the split and covariant formalisms

In this Appendix we give some details about the covariant approach to Carrollian geometry and we set up a dictionary with the splitted formalism.<sup>1</sup> When dealing with the covariant formalism, spatial indices are denoted by a Latin index  $i, j, \dots$  and the temporal one by 0.

### Weak Carroll structures

Let  $\mathcal{C}$  be a smooth manifold. The Carrollian structure is denoted by  $(g_{\mu\nu}, n^\mu)$  with  $g_{\mu\nu}n^\nu = 0$  and  $n^\mu$  nowhere vanishing. Associated clock forms are one-forms  $\tau_\mu$  obeying

$$\tau_\mu n^\mu = -1. \quad (\text{C.o.1})$$

from which one can define the transverse cometric  $G^{\mu\nu}$  as the unique contravariant symmetric tensor solving

$$G^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho + n^\mu \tau_\rho, \quad (\text{C.o.2a})$$

$$G^{\mu\nu} \tau_\nu = 0. \quad (\text{C.o.2b})$$

**Remark** Note that there exist infinitely many cometrics to  $g_{\mu\nu}$ , precisely because of its degeneracy. However, once a clock form (hence an Ehresmann connection) is fixed, there is a unique solution to (C.o.2a) and (C.o.2b).

These objects are not uniquely defined as the clock form is subjected to the following gauge invariance

$$\delta_\lambda \theta_\mu = \lambda_\mu, \quad \delta_\lambda G^{\mu\nu} = n^\mu \lambda^\nu + n^\nu \lambda^\mu, \quad (\text{C.o.3})$$

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<sup>1</sup>It is a pleasure to thank Adrien Fiorucci for numerous discussions about the covariant formalism and for authorising me to display for completeness some of the notions he will detail in his Lectures Notes [240].

with covector parameter  $\lambda_\mu$  such that  $n^\mu \lambda_\mu = 0$  i.e. the shift acts only on the purely transverse part of the clock form (i.e. the Ehresmann connection), like in the split formalism with  $\lambda_i$ . Once a clock form is given (the Carrollian structure is said *ruled*), horizontal spaces are spanned by vectors  $V^\mu$  obeying  $\theta_\mu V^\mu = 0$ . The components of the related Ehresmann curvature  $d\theta$  are the Carrollian acceleration  $\varphi_\mu$  and vorticity  $\hat{\omega}_{\mu\nu}$  given by

$$\varphi_\mu = \mathcal{L}_n \theta_\mu, \quad \hat{\omega}_{\mu\nu} = \partial_{[\mu} \theta_{\nu]} + \theta_{[\mu} \varphi_{\nu]}, \quad (\text{C.o.4})$$

with  $\varphi_\mu n^\mu = 0$  and  $\hat{\omega}_{\mu\nu} n^\mu = \hat{\omega}_{\mu\nu} n^\nu = 0$ .<sup>2</sup>

The horizontal projector is defined by

$$\Pi^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu \theta_\nu = G^{\mu\rho} g_{\rho\nu}. \quad (\text{C.o.5})$$

Finally the extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n g_{\mu\nu}. \quad (\text{C.o.6})$$

It is a transverse symmetric tensor and is shift-invariant. In the following Table we sketch the dictionary between this covariant formalism and the time/space split one.

Covariant	$g_{\mu\nu}$	$n^\mu$	$\theta_\mu$	$G^{\mu\nu}$	$\varphi_\mu$	$\hat{\omega}_{\mu\nu}$	$\Pi^\mu{}_\nu$	$K_{\mu\nu}$
Splitting	$a_{ij}$	$\mathbf{v}/\Omega$	$\boldsymbol{\tau}$	$a^{ij}$	$\varphi_i$	$\hat{\omega}_{ij}$	$\delta_j^i$	$\hat{\gamma}_{(ij)}$

(C.o.7)

### Strong Carroll structures

Connections are required to preserve the weak Carrollian structure  $(g_{\mu\nu}, n^\mu)$  i.e. we ask

$$\nabla_\rho g_{\mu\nu} = 0 \quad \text{and} \quad \nabla_\mu n^\nu = 0. \quad (\text{C.o.8})$$

It can be shown that the coefficients of the most general strong Carrollian connection are given by the coefficients

$$\Gamma^\rho{}_{\mu\nu} = \gamma^\rho{}_{\mu\nu} - n^\rho \partial_\mu \theta_\nu - G^{\rho\alpha} K_{\mu\alpha} \theta_\nu + n^\rho a_{\mu\nu}^{(n)} + G^{\rho\alpha} A_{\alpha\mu\nu}^{(G)} \quad (\text{C.o.9})$$

where  $a_{\mu\nu}^{(n)} := \nabla_\mu \theta_\nu$  is a tensor (non-necessarily symmetric) satisfying  $a_{\mu\nu}^{(n)} n^\nu = 0$  and  $A_{\alpha\mu\nu}^{(G)}$  satisfies

$$n^\alpha A_{\alpha\mu\nu}^{(G)} = 0, \quad A_{\alpha\mu\nu}^{(G)} n^\nu = 0, \quad A_{\alpha(\mu\nu)}^{(G)} = A_{\mu[\nu\alpha]}^{(G)} + A_{\nu[\mu\alpha]}^{(G)}. \quad (\text{C.o.10})$$

**Remark** If  $\nabla$  is a solution of (C.o.8), one can derive from (C.o.6) the following identity:

$$K_{\mu\nu} = B_{(\mu\nu)\alpha} n^\alpha, \quad B_{\mu\nu\alpha} \equiv g_{\mu\beta} T^\beta{}_{\nu\alpha}, \quad (\text{C.o.11})$$

where  $T^\rho{}_{\mu\nu} \equiv 2\Gamma^\rho{}_{[\mu\nu]}$  is the torsion of the connection  $\nabla$  (since the indices  $\mu, \nu, \rho \dots$  refer to a

---

<sup>2</sup>Coherent with the splitted formulae having only spatial legs  $\varphi_i$  and  $\hat{\omega}_i$ .

coordinate basis). Hence there is always a unavoidable part of torsion related to the extrinsic curvature  $K_{\mu\nu}$ . We recover in covariant formalism the result of Theorem (1.0.3) given that  $\hat{\gamma}_{(ij)} \Leftrightarrow K_{\mu\nu}$ .

On the other hand the most general torsionfree connection reads, when  $K_{\mu\nu} = 0$ ,

$$\Gamma^\rho_{\mu\nu} = \gamma^\rho_{\mu\nu} - n^\rho \partial_{(\mu} \theta_{\nu)} - n^\rho \theta_{(\mu} \varphi_{\nu)} \quad (\text{C.o.12})$$

where  $\gamma^\rho_{\mu\nu} = \frac{1}{2} G^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$  are the Carroll Christoffel symbols.

**Remark** It turns out that the requirements (C.o.8) do not fix uniquely the Carrollian connection, even with stronger (and simplifying) hypotheses such as the absence of torsion. Indeed, consider  $\nabla$  and  $\nabla'$ , two symmetric (*i.e.* torsion-free) connections solving (C.o.8). For any one-form  $\alpha$ , one has

$$(\nabla'_\mu - \nabla_\mu)\alpha_\nu = \Delta\Gamma^\rho_{\mu\nu}\alpha_\rho, \quad \Delta\Gamma^\rho_{\mu\nu} = \Delta\Gamma^\rho_{(\mu\nu)}. \quad (\text{C.o.13})$$

where  $\Delta\Gamma^\rho_{\mu\nu}$ , being the difference of two connections, is a tensor. By virtue of the (C.o.8), any contraction like  $\Delta\Gamma^\rho_{\mu\nu} g_{\rho\sigma}$  must vanish:<sup>3</sup> therefore, one can write  $\Delta\Gamma^\rho_{\mu\nu} = n^\rho \Sigma_{\mu\nu}$  for some symmetric tensor  $\Sigma_{\mu\nu}$ . Given a clock form  $\theta_\mu$ , because of (C.o.1), one has  $\nabla_\mu(\theta_\nu n^\nu) = 0 = \nabla'_\mu(\theta_\nu n^\nu)$  which translates into

$$(\nabla'_\mu - \nabla_\mu)(\theta_\nu n^\nu) = n^\nu \Delta\Gamma^\rho_{\mu\nu} \theta_\rho = n^\mu n^\rho \Sigma_{\mu\nu} \theta_\rho = -n^\mu \Sigma_{\mu\nu} = 0 \quad (\text{C.o.14})$$

using (C.o.8) and (C.o.13). As a result, two torsion-free metric-compatible Carrollian connections  $\nabla$  and  $\nabla'$  may differ by some ambiguous terms aligned with the field of observers and involving a traverse symmetric tensor, *i.e.*

$$\Gamma'^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + n^\mu \Sigma_{\nu\rho}, \quad \Sigma_{\mu\nu} n^\nu = 0, \quad (\text{C.o.15})$$

which is the covariant version of Proposition 1.0.4. Adding the latter ambiguity in (C.o.12) we get

$$\Gamma^\rho_{\mu\nu} = \gamma^\rho_{\mu\nu} - n^\rho \partial_{(\mu} \theta_{\nu)} - n^\rho \theta_{(\mu} \varphi_{\nu)} + n^\rho \Sigma_{\mu\nu}. \quad (\text{C.o.16})$$

Using (1.0.4) it is then easy to relate  $\Sigma_{\mu\nu}$  with  $\hat{\beta}_{(ij)}$  of the splitting formalism.

## Mapping the ambiguities

We go back to (C.o.9) where we see that  $\Sigma_{\mu\nu}$  is contained into  $a_{\mu\nu}^{(n)}$ . Let's count the degrees of freedom. In  $a_{\mu\nu}^{(n)} = \nabla_\mu \theta_\nu$  there are<sup>4</sup>  $d^2 + d$  and in  $A_{\alpha\mu\nu}^{(G)}$  there are, given (C.o.10),  $\frac{d(d^2-1)}{2}$ . Furthermore the total

<sup>3</sup>Use Leibniz rule on  $g_{\mu\nu}$  given (C.o.13).

<sup>4</sup>Being a rank-2 tensor transverse on one index.

torsion of the most general Carrollian connection (C.o.9) reads

$$T^\rho{}_{\mu\nu} = 2\Gamma^\rho{}_{[\mu\nu]} = \widetilde{T}^\rho{}_{\mu\nu} + 2n^\rho a_{[\mu\nu]}^{(n)} - 2G^{\rho\alpha} A_{\alpha[\mu\nu]}^{(G)}, \quad (\text{C.o.17})$$

so we are tempted to make the following identifications (up to multiplicative factors)

Covariant	$a_{ij}^{(n)}$	$a_{0i}^{(n)}$	$A_{i0j}^{(G)}$	$A_{ikj}^{(G)}$
Splitting	$\hat{\beta}_{ij}$	$\hat{\delta}_i$	$\hat{\gamma}_{[ij]}$	$\hat{\gamma}^k{}_{[ij]}$

(C.o.18)

which show that the time/space split formalism also allows to recover all the well-known ambiguities first discovered in the covariant one. Note that  $\nabla$  acts non trivially on the cometric

$$\nabla_\mu G^{\nu\rho} = 2n^{(\mu} G^{\rho)\alpha} \nabla_\mu \theta_\alpha \quad (\text{C.o.19})$$

hence its value crucially depends on the ambiguity  $a_{\mu\nu}^{(n)}$ .

## Projected connections and splitted connection $\hat{\nabla}$

Even though the use of covariant notations blurs the separation between time and space, the Carrollian manifold can still be seen as a 1-dimensional fiber bundle over a base space. One may then wonder what happens to the connection when projected either along the fibers or onto the base space.

Using the projector (C.o.5) one defines the *spatially projected connection* (aka horizontal connection) as

$$\hat{\nabla}_\mu T^\nu{}_\rho \equiv \Pi^\alpha{}_\mu \Pi^\nu{}_\beta \Pi^\gamma{}_\rho \nabla_\alpha T^\beta{}_\gamma \quad (\text{C.o.20})$$

and the *vertical connection* as

$$\nabla_\perp T^\nu{}_\rho = n^\alpha \Pi^\nu{}_\beta \Pi^\gamma{}_\rho \nabla_\alpha T^\beta{}_\gamma. \quad (\text{C.o.21})$$

The commutators on scalars are given by

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]f = 2\hat{\omega}_{\mu\nu} n^\alpha \partial_\alpha f \quad (\text{C.o.22a})$$

$$[\nabla_\perp, \hat{\nabla}_\mu]f = K_\mu{}^\nu \hat{\nabla}_\nu f + \varphi_\mu \nabla_\perp f. \quad (\text{C.o.22b})$$

Hence using the dictionary (C.o.18) it can be easily shown (see [240]) that (C.o.20) reproduces our splitted  $\hat{\nabla}_i$  while (C.o.21) is exactly  $\frac{1}{\Omega} \hat{D}_i$ .

## Energy momentum tensor

In the covariant formalism the effective action on the Carrollian structure is a functional of the degenerate metric and the Carrollian vector  $S = S[g_{\mu\nu}, n^\mu]$ . The conjugate momenta are defined as

$$T_\mu^{(n)} = -\frac{1}{\Omega_V} \frac{\delta S}{\delta n^\mu} \quad \text{and} \quad T_{\mu\nu}^{(G)} = -\frac{2}{\Omega_V} \frac{\delta S}{\delta G^{\mu\nu}}, \quad (\text{C.o.23})$$

and the *covariant Carrollian energy-momentum tensor* is defined as a combination of the two [69]

$$T^\mu{}_\nu \equiv -n^\mu T_\nu^{(n)} - G^{\mu\rho} T_{\nu\rho}^{(G)}. \quad (\text{C.o.24})$$

**Remark** The Carrollian volume element is defined as the strictly positive scalar density of weight one  $\Omega_V$  solution of [54]

$$\mathcal{G}^{\mu\nu} = \Omega_V^2 n^\mu n^\nu, \quad (\text{C.o.25})$$

where

$$\mathcal{G}^{\mu\nu} = \frac{1}{(d-1)!} \epsilon^{\mu\alpha_1 \dots \alpha_{d-1}} \epsilon^{\nu\beta_1 \dots \beta_{d-1}} g_{\alpha_1\beta_1} \dots g_{\alpha_{d-1}\beta_{d-1}} \quad (\text{C.o.26})$$

represents the matrix of minors of  $g_{\mu\nu}$  ( $\epsilon_{\mu_1 \dots \mu_d}$  is the numerically invariant Levi-Civita symbol in  $d$  dimensions), which is correctly a rank-one symmetric matrix obeying  $\mathcal{G}^{\mu\nu} g_{\nu\rho} = 0$  because of the degeneracy of the metric. Under the gauge transformation (C.o.3), one has  $\delta_\lambda \Omega_V = 0$ . One can show that

$$\Omega_V^2 = \theta_\mu \theta_\nu \mathcal{G}^{\mu\nu} = \det(M_{\mu\nu}), \quad M_{\mu\nu} \equiv \theta_\mu \theta_\nu + g_{\mu\nu}. \quad (\text{C.o.27})$$

General diffeomorphism invariance of the action gives on-shell

$$\delta_\xi S = - \int_{\mathcal{M}} d^d x \Omega_V \left( T_\mu^{(n)} \delta_\xi n^\mu + \frac{1}{2} T_{\mu\nu}^{(G)} \delta_\xi G^{\mu\nu} \right). \quad (\text{C.o.28})$$

i.e. after having discarded the boundary terms [69]

$$\partial_\mu (\Omega_V T^\mu{}_\nu) = \Omega_V \left( T_\mu^{(n)} \partial_\nu n^\mu + \frac{1}{2} T_{\mu\rho}^{(G)} \partial_\nu G^{\mu\rho} \right), \quad (\text{C.o.29})$$

which is the Carrollian avatar of the formula  $\nabla_\mu T^\mu{}_\nu = 0$  in Lorentzian theories. Components of the covariant energy-momentum tensor can be identified with the splitted momenta (1.3.16), as displayed in the next Table.

Splitted	$\Pi_j^i$	$\Pi^i$	$P_i$	$\Pi$	(C.o.30)
Covariant	$T_j^i$	$T_0^i$	$T_i^0$	$T_0^0$	

This concludes our dictionary between the covariant and the splitted formalisms. The interested reader is advised to have a look at the upcoming Lectures Notes [240] for more details.





# Appendix D

## A few words on Newton-Cartan geometries

This appendix is aimed at being the Galilean counterpart of Sec. 1.1, 1.2 and 1.3.2. We shall give the very basics definitions of a weak and strong Galilean (or Newton-Cartan) structures, highlighting the major differences with respect to the Carrollian ones, and we end up showing the conservation equations arising from invariance of the action. We restrict ourselves here to study the split frame whereas everything can also be performed in Cartan's orthonormal frame. This Appendix is inspired by [53] and [65].

### D.1 Weak Newton-Cartan structures

A *weak Newton-Cartan structure* is a  $(d + 1)$ -dimensional manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  endowed with a degenerate **cometric** whose kernel is generated by a nowhere-vanishing **one-form**  $\tau$  dubbed the *clock form*. It is always convenient to take a coordinate system  $(t, \mathbf{x})$  such that the clock-form is proportional to the coordinate differential  $dt$  i.e. such that it exists a function  $\Omega = \Omega(t, \mathbf{x})$  such that

$$\tau = \Omega dt, \tag{D.1.1}$$

and such that  $\mathbf{x}$  are local coordinates on  $\mathcal{S}$ . Actually, Newton-Cartan structures imposes that a notion of absolute time can be define at every points of the manifold  $\mathcal{M}$ . This is possible if and only if  $\Omega$  is space-independent because then

$$\theta = \int \tau = \int \Omega(t) dt \tag{D.1.2}$$

defines and *absolute time*.<sup>1</sup> In the  $(t, \mathbf{x})$  coordinates the degenerate cometric takes the following form

$$\partial_s^2 = a^{ij}(t, \mathbf{x})\partial_i\partial_j. \quad (\text{D.1.3})$$

Like in the Carrollian case there exist a dual of the clock-form, the *field of observers* whose form is

$$\mathbf{v} = \frac{1}{\Omega} \left( \partial_t + w^i \partial_i \right) \quad (\text{D.1.4})$$

where  $w^i$  is an additional degree of freedom with which the triple  $(a^{ij}, \Omega, w^i)$  forms a *ruled* Newton-Cartan structure.

**Remark** In the ruled case the subspace  $\mathcal{S}$  plays the role of a  $d$ -dimensional Newtonian spacetime endowed with a Riemannian metric  $a_{ij}(t, \mathbf{x})$  and observed from a frame with respect to which the locally inertial frame as velocity  $\mathbf{w} = w^i \partial_i$ .

One has the normalisation condition

$$\boldsymbol{\tau}(\mathbf{v}) = 1. \quad (\text{D.1.5})$$

Like in the Carrollian instance (D.1.5) does not uniquely fix the form of the dual variable, here the field of observers. Hence there also exist a *Galilean shift symmetry*, also dubbed Galilean boosts in the literature, that acts on  $w^i$  like

$$\delta_\lambda w^i = \Omega \lambda^i \quad (\text{D.1.6})$$

while preserving (D.1.5). The presence of  $\Omega$  is conventionnal and added for the Galilean covariant bases to have a tractable transformation law.

The frame at hand is adapted for the reduction of tensors under *Galilean diffeomorphisms*, the ones leaving time absolute

$$t' = t'(t) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}). \quad (\text{D.1.7})$$

The associated Jacobians are

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial t}, \quad J_j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \quad (\text{D.1.8})$$

together with the transformation laws of the ruled structure, obtained by asking the co-line element and the clock form to be invariant under (D.1.7)

$$\Omega' = \frac{\Omega}{J}, \quad a'_{ij} = (J^{-1})_i^k (J^{-1})_j^l a_{kl}, \quad w'^k = \frac{1}{J} \left( J^k_i w^i + j^k \right). \quad (\text{D.1.9})$$

The covariant vector and form basis are in this case

$$\mathcal{B} = \{\hat{\mathbf{e}}_A\} = \{\mathbf{v}, \partial_i\} \quad \text{and} \quad \mathcal{B}^* = \{\hat{\boldsymbol{\theta}}^A\} = \{\boldsymbol{\tau}, \widehat{\mathbf{d}x}^i\} \quad (\text{D.1.10})$$

---

<sup>1</sup>It is then called a *torsionless* Newton-Cartan spacetime.

with  $\widehat{dx}^i = dx^i - w^i dt$ . In these bases the indices shall be denoted by  $(\hat{t}, \hat{i})$ .

**Remark** Given a vector  $\xi = \xi^t \partial_t + \xi^i \partial_i$ . In the bases (D.1.10) its components reads

$$\xi := \xi^{\hat{t}} \mathbf{v} + \xi^{\hat{i}} \partial_i = \Omega \xi^t \mathbf{v} + (\xi^i - \xi^t w^i) \partial_i. \quad (\text{D.1.11})$$

Let's now consider a form  $\mathbf{B} = B_t dt + B_i dx^i$ . In the basis (D.1.10) we get

$$\mathbf{B} := B_{\hat{t}} \boldsymbol{\tau} + B_{\hat{i}} \widehat{dx}^i = \frac{1}{\Omega} (B_t + w^i B_i) \boldsymbol{\tau} + B_i \widehat{dx}^i \quad (\text{D.1.12})$$

with  $B_{\hat{t}}$  and  $B_{\hat{i}}$  respectively Galilean scalar and covector. In the following, we will have the tendency to drop the hat on top of the spatial indices as the components are the same for one-forms, so one should be careful.

At the level of the shift symmetry the bases at hand transform as

$$\delta_\lambda \mathbf{v} = \lambda^i \partial_i, \quad \delta_\lambda \partial_i = 0, \quad \delta_\lambda \boldsymbol{\tau} = 0 \quad \text{and} \quad \delta_\lambda \widehat{dx}^i = -\lambda^i \boldsymbol{\tau}. \quad (\text{D.1.13})$$

The form basis carries non-holonomy coefficients

$$d\boldsymbol{\tau} = 0 \quad \text{and} \quad \widehat{dx}^i = \frac{1}{\Omega} \partial_j w^i \boldsymbol{\tau} \wedge \widehat{dx}^j \quad (\text{D.1.14})$$

i.e. using (A.0.3)

$$\hat{C}_{ij}^i = -\frac{1}{\Omega} \partial_j w^i := -A_j^i \quad (\text{D.1.15})$$

with  $A_j^i$  the *Galilean frame acceleration*. Note that (D.1.14) is equivalent to the non commutativity of the basis vectors

$$[\mathbf{v}, \partial_i] = -A_j^i \partial_j, \quad [\hat{\partial}_i, \hat{\partial}_j] = 0. \quad (\text{D.1.16})$$

$A_j^i$  is a genuine Galilean tensor whose transformation under Galilean shift reads

$$\delta_\lambda A_j^i = \partial_j \lambda^i. \quad (\text{D.1.17})$$

**Remark** Such intrinsic Newton-Cartan spacetimes can be obtained via the  $c \rightarrow +\infty$  limit of a relativistic metric written in the Zermelo frame (dual to the Randers-Papapetrou frame used in the Carrollian instance, see Sec. 2.1.1)

$$ds^2 = -c^2 \Omega^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt) \quad (\text{D.1.18})$$

together with the inverse metric

$$\partial_s^2 = -\frac{1}{c^2 \Omega^2} (\partial_t + w^i \partial_i) + a^{ij} \partial_i \partial_j. \quad (\text{D.1.19})$$

Remember that first the  $c$ -dependence is explicit and second that here  $\Omega = \Omega(t)$  only, which makes (D.1.18) slightly different from the well-known ADM parametrisation.

**Remark** The frame at hand is not the most general one. Asking the clock form to be aligned only along the temporal direction is a restriction that allows to break general covariance down to Galilean covariance, which is the one we get starting from a Zermelo background. In the most general case the clock form reads

$$\tau = \Omega(t)dt - b_i(t, \mathbf{x})dx^i \quad (\text{D.1.20})$$

with a new degree of freedom  $b_i$ , Galilean covector.

## D.2 Newton-Cartan's connections

In this Section we construct the most general Galilean strong connection, that is a metric and clock form-compatible connection.

### Generalities

Very generally, in the bases at hand, one can decompose  $\bar{\nabla}$  in such a way

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}}\mathbf{v} &= \hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\mathbf{v} + \hat{\Gamma}_{\hat{t}\hat{t}}^i\partial_i & \bar{\nabla}_{\mathbf{v}}\tau &= -\hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\tau - \hat{\Gamma}_{\hat{t}\hat{t}}^i\widehat{dx}^i \\ \bar{\nabla}_{\mathbf{v}}\partial_i &= \hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\mathbf{v} + \hat{\Gamma}_{\hat{t}\hat{t}}^j\partial_j & \bar{\nabla}_{\mathbf{v}}\widehat{dx}^i &= -\hat{\Gamma}_{\hat{t}\hat{t}}^i\tau - \hat{\Gamma}_{\hat{t}\hat{t}}^i\widehat{dx}^j \\ \bar{\nabla}_{\partial_i}\mathbf{v} &= \hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\mathbf{v} + \hat{\Gamma}_{\hat{t}\hat{t}}^j\partial_j & \bar{\nabla}_{\partial_i}\tau &= -\hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\tau - \hat{\Gamma}_{\hat{t}\hat{t}}^j\widehat{dx}^j \\ \bar{\nabla}_{\partial_i}\partial_j &= \hat{\Gamma}_{\hat{t}\hat{t}}^{\hat{t}}\mathbf{v} + \hat{\Gamma}_{\hat{t}\hat{t}}^k\partial_k & \bar{\nabla}_{\partial_i}\widehat{dx}^j &= -\hat{\Gamma}_{\hat{t}\hat{t}}^j\tau - \hat{\Gamma}_{\hat{t}\hat{t}}^j\widehat{dx}^k. \end{aligned} \quad (\text{D.2.1})$$

We shall from now on give a different name to all connection coefficients i.e.

$$\begin{aligned} \bar{\nabla}_{\mathbf{v}}\mathbf{v} &= \hat{\gamma}\mathbf{v} + \hat{\rho}^i\partial_i & \bar{\nabla}_{\mathbf{v}}\tau &= -\hat{\gamma}\tau - \hat{\delta}_i\widehat{dx}^i \\ \bar{\nabla}_{\mathbf{v}}\partial_i &= \hat{\delta}_i\mathbf{v} + \hat{\gamma}_i^j\partial_j & \bar{\nabla}_{\mathbf{v}}\widehat{dx}^i &= -\hat{\rho}^i\tau - \hat{\gamma}_i^j\widehat{dx}^j \\ \bar{\nabla}_{\partial_i}\mathbf{v} &= \hat{\alpha}_i\mathbf{v} + \hat{\kappa}_i^j\partial_j & \bar{\nabla}_{\partial_i}\tau &= -\hat{\alpha}_i\tau - \hat{\beta}_{ij}\widehat{dx}^j \\ \bar{\nabla}_{\partial_i}\partial_j &= \hat{\beta}_{ij}\mathbf{v} + \hat{\gamma}_{ij}^k\partial_k & \bar{\nabla}_{\partial_i}\widehat{dx}^j &= -\hat{\kappa}_i^j\tau - \hat{\gamma}_{ik}^j\widehat{dx}^k. \end{aligned} \quad (\text{D.2.2})$$

Hence we have the following result<sup>2</sup>

**Proposition D.o.1.** The most general connection one can put on a Carrollian structure is made of the following  $(d+1)^3$  degrees of freedom,  $\hat{\gamma}$ ,  $\hat{\rho}^i$ ,  $\hat{\delta}_i$ ,  $\hat{\alpha}_i$ ,  $\hat{\gamma}_i^j$ ,  $\hat{\kappa}_i^j$ ,  $\hat{\beta}_{ij}$  and  $\hat{\gamma}_{ij}^k$ .

<sup>2</sup>This is just a time/space splitting of the usual  $(d+1)^3$  degrees of freedom of a unconstrained connection.

All these degrees of freedom are genuine Galilean tensors, as their index structure suggests it, except for  $\hat{\gamma}^i_j$  and  $\hat{\gamma}^k_{ij}$  which are temporal and spatial Galelian connections transforming as

$$\hat{\gamma}'^i_j = J^i_k (J^{-1})^l_j \hat{\gamma}^k_l - \underbrace{(J^{-1})^k_j \mathbf{v} \left( J^i_k \right)} \quad (\text{D.2.3})$$

$$\hat{\gamma}'^k_{ij} = J^k_n (J^{-1})^l_i (J^{-1})^m_j \hat{\gamma}^n_{lm} + J^k_n (J^{-1})^l_i \partial_l \left( (J^{-1})^n_j \right), \quad (\text{D.2.4})$$

where the second term in the transformation of  $\hat{\gamma}^i_j$  vanishes for Galilean diffeomorphism, rendering the latter tensorial.

**Remark** As genuine Galilean tensors,  $\gamma$ ,  $\hat{\delta}_i$ ,  $\hat{\alpha}_i$ ,  $\hat{\rho}^i$ ,  $\hat{\kappa}_{ij}$  and  $\hat{\beta}_{ij}$  can be consistently set equal to Galilean tensors of the same rank. Otherwise one would break Galilean diffeomorphism covariance. It is also consistent to set them to zero, and we shall pick later a particular connection for which it happens.

Applying the same method on (D.2.2), using (D.1.13) we get the transformations of the coefficients under shift symmetry

$$\delta_\lambda \hat{\gamma} = \lambda^i \hat{\alpha}_i + \hat{\delta}_i \lambda^i \quad (\text{D.2.5a})$$

$$\delta_\lambda \hat{\rho}^i = -\lambda^i \hat{\gamma} - \lambda^j \hat{\kappa}_j^i + \hat{\gamma}^i_j \lambda^j + \mathbf{v}(\lambda^i) \quad (\text{D.2.5b})$$

$$\delta_\lambda \hat{\delta}_i = \lambda^i \hat{\beta}_{ij} \quad (\text{D.2.5c})$$

$$\delta_\lambda \hat{\alpha}_i = \lambda^j \hat{\beta}_{ji} \quad (\text{D.2.5d})$$

$$\delta_\lambda \hat{\gamma}^j_i = -\lambda^i \hat{\delta}_j - \lambda^k \hat{\gamma}^i_{kj} \quad (\text{D.2.5e})$$

$$\delta_\lambda \hat{\kappa}_i^j = -\lambda^j \hat{\alpha}_i + \hat{\gamma}^j_{ik} \lambda^k + \partial_i \lambda^j \quad (\text{D.2.5f})$$

$$\delta_\lambda \hat{\beta}_{ij} = 0 \quad (\text{D.2.5g})$$

$$\delta_\lambda \hat{\gamma}^k_{ij} = -\lambda^j \hat{\beta}_{ik} \quad (\text{D.2.5h})$$

showing that, as found in the last Chapter,  $\hat{\rho}^i$  and  $\hat{\kappa}_j^i$  are the Galilean boosts temporal and spatial connections. On the other hand, as  $\hat{\delta}_i$  and  $\hat{\beta}_{ij}$  encodes Carrollian boosts, they can be completely discarded and this will be consistent with the transformations laws (D.2.5c) and (D.2.5g).<sup>3</sup> One could then conclude that they do not play any role in Galilean connections.

The torsion one-form of such a connection reads

$$\hat{\mathcal{T}}^i = \frac{1}{2} (\hat{\delta}_i - \hat{\alpha}_i) \boldsymbol{\tau} \wedge \widehat{\text{d}x}^i + \hat{\beta}_{[ij]} \widehat{\text{d}x}^i \wedge \widehat{\text{d}x}^j \quad (\text{D.2.6a})$$

$$\hat{\mathcal{T}}^i = \frac{1}{2} \left( \hat{\gamma}^i_j - \hat{\kappa}_j^i + A_j^i \right) \boldsymbol{\tau} \wedge \widehat{\text{d}x}^j + \hat{\gamma}^i_{[kj]} \widehat{\text{d}x}^k \wedge \widehat{\text{d}x}^j. \quad (\text{D.2.6b})$$

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<sup>3</sup>Indeed  $\hat{\beta}_{ij}$  being both a Galilean tensor and a boost invariant quantity, it can be consistently set to zero, which makes  $\hat{\delta}_i$  also both a tensor and a boost invariant quantity, hence set to zero also.

This implies the following commutation rules on a scalar

$$\left[ \frac{1}{\Omega} \bar{D}_i, \bar{\nabla}_i \right] f = (\hat{\alpha}_i - \hat{\delta}_i) \mathbf{v}(f) + (\hat{\chi}_i^j - \hat{\gamma}_i^j) \partial_j (f) \quad (\text{D.2.7a})$$

$$[\bar{\nabla}_i, \bar{\nabla}_j] f = -2\hat{\beta}_{[ij]} \mathbf{v}(f) - 2\hat{\gamma}_{[ij]}^k \partial_k (f) . \quad (\text{D.2.7b})$$

We see that it is possible to completely cancel the torsion with the following requirements (which are legitimate as these are tensorial equalities)

$$\hat{\delta}_i = \hat{\alpha}_i \quad , \quad \hat{\beta}_{[ij]} = 0 \quad , \quad \hat{\gamma}_j^i + A_j^i = \hat{\chi}_j^i \quad , \quad \hat{\gamma}_{[ij]}^k = 0. \quad (\text{D.2.8})$$

Note that the symmetric part of  $\hat{\beta}_{(ij)}$  is not constrained by these requirements, hence it is a genuine ambiguity of the connection. One could then compute the Riemann tensor and study further the properties of the completely unconstrained connection, but to keep things short we shall now construct from (D.2.2) a strong Galilean connection.

### Strong connections

Like in the Carrollian case, we ask the fundamental doublet  $(\boldsymbol{\tau}, a^{ij})$  to be left invariant by a strong connection. Asking the clock form to be preserved yields

$$\hat{\gamma} = 0 \quad , \quad \hat{\alpha}_i = 0 \quad , \quad \hat{\delta}_i = 0 \quad \text{and} \quad \hat{\beta}_{ij} = 0. \quad (\text{D.2.9})$$

Hence we have that a clock-form compatible Galilean connection is described in terms of  $\hat{\rho}^i, \hat{\chi}_j^i, \hat{\gamma}_j^i$  and  $\hat{\gamma}_{ij}^k$ .

Imposing the cometric compatibility leads to

$$\hat{\gamma}^{(ij)} = -\frac{1}{2\Omega} \partial_l a^{ij} \quad \text{and} \quad \hat{\gamma}_{jk}^i = \frac{1}{2} a^{il} (\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}) \quad (\text{D.2.10})$$

while  $\hat{\gamma}^{[ij]}$  and  $\hat{\gamma}_{jk}^i$  are not fixed. Hence the result is

**Proposition D.o.2.** The most general strong Galilean connection leaves free  $\hat{\rho}^i, \hat{\chi}_j^i, \hat{\gamma}^{[ij]}$  and  $\hat{\gamma}_{jk}^i$  i.e.  $\frac{d(d+1)^2}{2}$  degrees of freedom.

To further constrain this connection one can cancel the purely spatial part of the torsion that is set  $\hat{\gamma}_{[jk]}^i = 0$ , giving rise to a Levi-Civita  $\hat{\gamma}$  connection on the Riemannian subspace  $\mathcal{S}$ . However, because of (D.2.10) which relates  $\hat{\gamma}^{(ij)}$  to the geometry and due to (D.2.6b), one sees that it is also impossible in the Galilean framework to build a torsion-free strong connection.

## Our Galilean connection

The connection we are using is the one that arises from the  $c \rightarrow +\infty$  limit of the Levi-Civita connection in Zermelo frame. As we are still separating time from space, let's start with the spatial part of the connection

$$\gamma_{jk}^i = \frac{a^{il}}{2} (\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}). \quad (\text{D.2.11})$$

The associated covariant derivative is spelled  $\hat{\nabla}_i$ , as opposed to  $\nabla_i$ , the spatial component of the Levi-Civita covariant derivative  $\nabla_\mu$  defined on the ascendent pseudo-Riemannian spacetime. This connection is torsionless

$$\hat{t}_{ij}^k = 2\gamma_{[ij]}^k = 0, \quad (\text{D.2.12})$$

and metric-compatible

$$\hat{\nabla}_i a_{jk} = 0 \quad (\text{D.2.13})$$

as being just a sub-connection of  $\bar{\nabla}$  defined in the last paragraph. Its Riemann, Ricci and scalar curvature tensors are defined as usual  $d$ -dimensional Levi-Civita curvature tensors would be on  $\mathcal{S}$ . Note however their independence with respect to time

$$[\hat{\nabla}_k, \hat{\nabla}_l] V^i = \left( \partial_k \gamma_{lj}^i - \partial_l \gamma_{kj}^i + \gamma_{km}^i \gamma_{lj}^m - \gamma_{lm}^i \gamma_{kj}^m \right) V^j = \hat{r}_{jkl}^i V^j. \quad (\text{D.2.14})$$

For the temporal part of the connection we need an observation. Galilean tensors can be constructed from an object which is not a vector but rather transforming like a connection,

$$A'^k = \frac{1}{J} \left( J_i^k A^i + j^k \right). \quad (\text{D.2.15})$$

Indeed

$$\frac{1}{\Omega} \hat{\nabla}^{(k} A^{l)} - \frac{1}{2\Omega} \partial_t a^{kl} = -\frac{1}{2\Omega} \left( \mathcal{L}_{\mathbf{A}} a^{kl} + \partial_t a^{kl} \right) \quad (\text{D.2.16})$$

( $\mathcal{L}_{\mathbf{A}}$  is the Lie derivative along  $\mathbf{A} = A^i \partial_i$ ) and

$$\frac{1}{\Omega} \hat{\nabla}_{(k} A_{l)} + \frac{1}{2\Omega} \partial_t a_{kl} = \frac{1}{2\Omega} \left( \mathcal{L}_{\mathbf{A}} a_{kl} + \partial_t a_{kl} \right) \quad (\text{D.2.17})$$

have tensorial transformation rules, and their trace is a scalar.<sup>4</sup> We can apply this to  $\mathbf{w}$  and define

$$\hat{\gamma}^w_{ij} = \frac{1}{\Omega} \left( \hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right). \quad (\text{D.2.18})$$

This quantity is purely geometrical (and emerges in the large- $c$  expansion of the relativistic-spacetime Levi-Civita connection in Zermelo frame). From this tensor, one defines their traceless relatives

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<sup>4</sup>Observe that neither  $\frac{1}{\Omega} \partial_t$  nor  $\frac{1}{\Omega} \mathcal{L}_{\mathbf{A}}$  acting on Galilean tensors give separately tensors.



and the traces: the *geometric Galilean shear*

$$\xi^w_{ij} = \frac{1}{\Omega} \left( \hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right) - \frac{1}{d} a_{ij} \theta^w, \quad (\text{D.2.19})$$

and the *geometric Galilean expansion*

$$\theta^w = \frac{1}{\Omega} \left( \partial_t \ln \sqrt{a} + \hat{\nabla}_i w^i \right). \quad (\text{D.2.20})$$

The temporal, metric-compatible covariant derivative  $\hat{D}$  (again emerging in the Galilean expansion of the spacetime Levi–Civita covariant derivative in the time direction of a Zermelo frame) is then built upon  $\gamma^w_{ij}$ . For a scalar function  $\Phi$  it is simply

$$\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt} = e_i(\Phi) = \frac{1}{\Omega} \partial_t \Phi + \frac{w^j}{\Omega} \partial_j \Phi, \quad (\text{D.2.21})$$

whereas for vectors one finds<sup>5</sup>

$$\begin{aligned} \frac{1}{\Omega} \frac{\hat{D}V^i}{dt} &= \frac{1}{\Omega} \partial_t V^i + \frac{w^j}{\Omega} \partial_j V^i - V^j \partial_j \frac{w^i}{\Omega} + \hat{\gamma}^{wi}_j V^j \\ &= \frac{1}{\Omega} \left( \partial_t V^i + \mathcal{L}_{\mathbf{w}} V^i \right) + \hat{\gamma}^{wi}_j V^j. \end{aligned} \quad (\text{D.2.22})$$

More generally, the Leibniz rule leads to

$$\frac{1}{\Omega} \frac{\hat{D}K^{i\dots}_{j\dots}}{dt} = \frac{1}{\Omega} \left( \partial_t K^{i\dots}_{j\dots} + \mathcal{L}_{\mathbf{w}} K^{i\dots}_{j\dots} \right) + \hat{\gamma}^{wi}_k K^{k\dots}_{j\dots} + \dots - \hat{\gamma}^{wk}_j K^{i\dots}_{k\dots} - \dots, \quad (\text{D.2.23})$$

and as a consequence

$$\frac{1}{\Omega} \frac{\hat{D}a^{ij}}{dt} = \frac{1}{\Omega} \frac{\hat{D}a_{ij}}{dt} = 0. \quad (\text{D.2.24})$$

Space and time Galilean covariant derivatives do not commute. They define a Galilean tensor, rooted in the Riemann tensor of the ascendent relativistic spacetime at finite velocity of light. We find

$$\left[ \frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] \Phi = -\hat{\gamma}^{wk}_i \partial_k \Phi, \quad (\text{D.2.25a})$$

$$\left[ \frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] V^j = -\hat{\gamma}^{wk}_i \hat{\nabla}_k V^j + \hat{r}^j_{ik} V^k, \quad (\text{D.2.25b})$$

and similarly for higher-rank Galilean tensors, where

$$\hat{r}^j_{ik} = \frac{1}{\Omega} \left( \partial_t \gamma^j_{ik} + \hat{\nabla}_i \hat{\nabla}_k w^j - \hat{\nabla}_i \hat{\gamma}^{wj}_k + w^l \hat{r}^j_{kli} \right). \quad (\text{D.2.26})$$

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<sup>5</sup>When writing  $\mathcal{L}_{\mathbf{w}}$  one sees  $\mathbf{w}$  as a vector and apply the usual formula e.g.  $\mathcal{L}_{\mathbf{w}} \mathbf{A} = [\mathbf{w}, \mathbf{A}]$  for a vector  $\mathbf{A} = A^i \partial_i$ .

### D.3 Galilean conservation equations

In this section we are equipped with the connection  $\hat{\nabla}$  and an effective Galilean action,  $S[a^{ij}, w^i, \Omega; \Phi] = \int dt d^d x \sqrt{a} \Omega \mathcal{L}$  where  $\mathcal{L}$  is the Lagrangian density. We shall define here the notion of Galilean momenta and derive from Galilean diffeomorphism invariance of  $S$  their conservation equations. Throughout the Section  $\xi = \xi^i(t) \mathbf{v} + \xi^i(t, \mathbf{x}) \partial_i$  parameterizes a linear Galilean diffeomorphism. Note already the independence of  $\xi^i$  with respect to space.

#### Lie derivatives and divergences

The variation under diffeomorphisms is implemented through the Lie derivative (the minus sign is conventional)

$$-\delta_\xi a^{ij} = \mathcal{L}_\xi a^{ij} = -2 \left( \hat{\nabla}^{(i} \xi^{j)} + \hat{\gamma}^{wij} \xi^i + \frac{1}{\Omega} w^{(i} a^{j)k} \partial_k \xi^i \right), \quad (\text{D.3.1})$$

where the last term drops for Galilean diffeomorphisms. Furthermore

$$\mathcal{L}_\xi \mathbf{v} = -\frac{1}{\Omega} \left( \partial_t \xi^i + \mathcal{L}_w \xi^i \right) \mathbf{v} - \frac{1}{\Omega} \left( \partial_t \xi^i + \mathcal{L}_w \xi^i \right) \partial_i, \quad (\text{D.3.2})$$

from which, we infer

$$-\delta_\xi \Omega = \mathcal{L}_\xi \Omega = \partial_t \xi^i + \mathcal{L}_w \xi^i, \quad \delta_\xi w^i = -\mathcal{L}_\xi w^i = \partial_t \xi^i + \mathcal{L}_w \xi^i. \quad (\text{D.3.3})$$

Notice also the action on the clock form

$$\mathcal{L}_\xi \boldsymbol{\tau} = \frac{1}{\Omega} \left( \partial_t \xi^i + \mathcal{L}_w \xi^i \right) \boldsymbol{\tau} = \frac{1}{\Omega} \frac{\hat{D} \xi^i}{dt} \boldsymbol{\tau} = \mu \boldsymbol{\tau}, \quad (\text{D.3.4})$$

where we introduced

$$\mu(t, \mathbf{x}) = \frac{1}{\Omega} \frac{\hat{D} \xi^i}{dt}. \quad (\text{D.3.5})$$

#### Varying the effective action

*Galilean momenta* are conjugate variables to the geometry. They are the Galilean equivalent of the relativistic energy-momentum tensor and are defined as the *energy-stress tensor*, the *momentum* and the *energy density*

$$\Pi_{ij} = -\frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a^{ij}}, \quad (\text{D.3.6a})$$

$$P_i = -\frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta w^i}, \quad (\text{D.3.6b})$$

$$\Pi = -\frac{1}{\sqrt{a} \Omega} \left( \Omega \frac{\delta S}{\delta \Omega} - \frac{w^i}{\Omega} \frac{\delta S}{\delta w^i} \right), \quad (\text{D.3.6c})$$

which can likewise be combined as  $\frac{\delta S}{\delta \Omega} = -\sqrt{a} \left( \Pi + \frac{w^i}{\Omega} P_i \right)$ .

**Remark** Asking for a Galilean theory to be shift invariant will impose that no physical results should depend on  $w^i$ , hence  $P_i = 0$ . However not all theories behave like that so one has to be careful.

Varying the effective action then yields

$$\delta S = - \int dt \Omega \int d^d x \sqrt{a} \left( \frac{1}{2} \Pi_{ij} \delta a^{ij} + P_i \delta \frac{w^i}{\Omega} + \left( \Pi + \frac{w^i}{\Omega} P_i \right) \delta \ln \Omega \right). \quad (\text{D.3.7})$$

so under a Galilean diffeomorphism we get using (D.3.1), (D.3.4) and (D.3.2)

$$\begin{aligned} \delta_\xi S &= \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^i \left[ \frac{1}{\Omega} \frac{\hat{D}\Pi}{dt} + \theta^w \Pi + \Pi_{ij} \hat{\gamma}^{wij} \right] \right. \\ &\quad \left. + \xi^i \left[ \frac{1}{\Omega} \frac{\hat{D}P_i}{dt} + \theta^w P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} \right] \right\} \\ &\quad + \int dt d^d x \left\{ \partial_t \left( \sqrt{a} \left( \Pi \xi^i - P_j \xi^j \right) \right) \right. \\ &\quad \left. + \partial_i \left( \sqrt{a} w^i \left( \Pi \xi^i - P_j \xi^j \right) - \sqrt{a} \Omega \Pi^i_j \xi^j \right) \right\}. \end{aligned} \quad (\text{D.3.8})$$

Requiring that  $\delta_\xi S$  vanishes and ignoring the boundary terms (last two lines in Eq. (D.3.8)), we reach two equations. The momentum equation is the simplest because  $\xi^j$  being functions of both  $t$  and  $\mathbf{x}$ , their factor must vanish

$$\boxed{\left( \frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} = 0.} \quad (\text{D.3.9})$$

The energy equation is more subtle because  $\xi^i$  depends on  $t$  only. As a consequence it is enough to require that its factor be the Galilean divergence of a vector

$$\boxed{\left( \frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) \Pi + \Pi_{ij} \hat{\gamma}^{wij} = -\hat{\nabla}_i \Pi^i,} \quad (\text{D.3.10})$$

where  $\Pi^i$  is undetermined a priori. Indeed,  $\sqrt{a} \Omega \xi^i \hat{\nabla}_i \Pi^i = \partial_i \left( \sqrt{a} \Omega \xi^i \Pi^i \right)$ , which leads to a boundary term and vanishes inside the integral. One can interpret  $\Pi^i$  as the *energy current* (also *energy flux*).<sup>6</sup>

<sup>6</sup>This new degree of freedom can arise as a variation when going to the most general frame (D.1.20)

$$\Pi^i = -\frac{1}{\Omega \sqrt{a}} \frac{\delta S}{\delta b_i}. \quad (\text{D.3.11})$$

## Equation for a $U(1)$ current

Let's assume that our action is invariant under a local  $U(1)$  associated with a gauge field  $B = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ . Under a gauge transformation of parameter  $\lambda$  the gauge field changes as  $\delta_\lambda B = \partial_t \lambda$  and  $\delta_\lambda B_i = \partial_i \lambda$ . The conjugate momenta are now the *matter density* and the *matter current*

$$\rho = -\frac{1}{\sqrt{a}} \frac{\delta S}{\delta B}, \quad (\text{D.3.12a})$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \left( w^i \frac{\delta S}{\delta B} - \frac{\delta S}{\delta B_i} \right) \quad (\text{D.3.12b})$$

with  $\frac{\delta S}{\delta B_i} = -\sqrt{a} (\Omega N^i + \rho w^i)$ , and

$$\delta S = - \int dt d^d x \sqrt{a} \left( \rho \delta B + (\Omega N^i + \rho w^i) \delta B_i \right) \quad (\text{D.3.13})$$

for the matter sector. The gauge variation of the action reads

$$\begin{aligned} \delta_\lambda S &= \int dt d^d x \sqrt{a} \left( \rho \partial_t \lambda + (\Omega N^i + \rho w^i) \partial_i \lambda \right) \\ &= - \int dt d^d x \sqrt{a} \Omega \lambda \left( \frac{1}{\Omega} \hat{D} \rho + \theta^w \rho + \hat{\nabla}_i N^i \right) \\ &\quad + \int dt d^d x \left\{ \partial_t \left( \sqrt{a} \lambda \rho \right) + \partial_i \left( \sqrt{a} \lambda (\Omega N^i + \rho w^i) \right) \right\}. \end{aligned} \quad (\text{D.3.14})$$

Invariance of  $S$  leads to the Galilean continuity equation

$$\boxed{\left( \frac{1}{\Omega} \hat{D} + \theta^w \right) \rho + \hat{\nabla}_i N^i = 0.} \quad (\text{D.3.15})$$

These equations (D.3.10), (D.3.9) together with (D.3.15) were obtained in [53, 65]. The reader is advised to look at these references for more details on isometries and charges.

**Remark** Weyl-covariance can also be accommodated in Newton-Cartan structures. Like in intrinsic Carrollian structure the latter is imposed by hand. We assume that under the ruled Newton-Cartan structure undergo an additional type of transformations, dubbed *Weyl transformations*, parametrised by a function  $\mathcal{B}$  and acting as

$$a^{ij} \rightarrow \mathcal{B}^2 a^{ij}, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad w^i \rightarrow w^i, \quad w_i \rightarrow \frac{1}{\mathcal{B}^2} w_i. \quad (\text{D.3.16})$$

Since  $\Omega$  is a function of  $t$  only, the second of (D.3.16) imposes  $\mathcal{B} = \mathcal{B}(t)$ . Weyl-invariance requirement of an effective action  $S$  leads to following weights for the Galilean momenta, the energy-stress tensor  $\Pi_{ij}$  has weight  $d - 1$ , the momentum  $P_i$ ,  $d$ , and the energy density  $\Pi$ ,  $d + 1$ . The energy flux  $\Pi_i$  introduced in (D.3.10) has also weight  $d$ . The Weyl trace condition

reads

$$\Pi_i^i = \Pi. \tag{D.3.17}$$

On the matter sector, the gauge fields  $B$  and  $B_i$  are weight-zero, whereas  $\varrho$  is weight- $d$  and  $N_i$ ,  $d - 1$ . It remains to build a Weyl-covariant derivative (for which no spatial Weyl connection exist, as a major difference with respect to the Carrollian instance) and rewrite the conservation equations (D.3.10), (D.3.9) and (D.3.15) with them. This would bring us far from our goal in writing this Appendix, the interested reader should see [65].

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# **Publications**



## Conformal Graphs as Twisted Partition Functions

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We show that a class of  $L$ -loop conformal ladder graphs are intimately related to twisted partition functions of free massive complex scalars in  $d = 2L + 1$  dimensions. The graphs arise as four-point functions in certain two- and four-dimensional conformal fishnet models. The twisted thermal two-point function of the scalars becomes a generator of conformal ladder graphs for all loops. We argue that this correspondence is seeded by a system of two decoupled harmonic oscillators twisted by an imaginary chemical potential. We find a number of algebraic and differential relations among the conformal graphs that mirror the underlying free dynamics.

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*Introduction and summary.*—In [1], one of the authors observed that the logarithm of the partition function  $Z_L$  of a free massive complex scalar  $\phi(x)$ , twisted by the global  $U(1)$  charge along the thermal circle in  $d = 2L + 1$  dimensions, is given in terms of a class of single-valued polylogarithms. The latter functions are ubiquitous in multiloop quantum field theory (QFT) calculations (see, e.g., [2] for a recent review), and their intriguing mathematical properties have been discussed in a number of works [3,4]. The twisting parameter  $\mu$  corresponds to an imaginary chemical potential for the Abelian “charge” operator  $\mathcal{Q} = \phi^\dagger \overleftrightarrow{\mathcal{D}}_\tau \phi$ , with  $\mathcal{D}_\tau = \partial_\tau - i\mu$ , which together with  $\mathcal{O} = |\phi|^2$  can be viewed as integrable relevant deformations of the massless free theory. From  $\ln Z_L$  we can calculate the thermal one-point functions  $\langle \mathcal{O} \rangle_L$  and  $\langle \mathcal{Q} \rangle_L$ , respectively, and it was shown in [1] that  $\langle \mathcal{Q} \rangle_L$  is essentially given by the  $L$ -loop Davydychev-Ustyukina conformal ladder graph [5,6].

We show here that  $\ln Z_L$  itself is also given by an  $L$ -loop conformal ladder graph that evaluates a certain four-point function of the singular two-dimensional conformal fishnet model of Kazakov and Olivucci [7]. Consequently, the differential equations satisfied by  $\langle \mathcal{O} \rangle_L$  and  $\langle \mathcal{Q} \rangle_L$  presented in [1] become differential relations among four-point ladder graphs of conformal fishnet models in two and four dimensions. The observations above prompt us to consider

the twisted thermal two-point function  $\langle \phi^\dagger(x)\phi(0) \rangle_L$ . When  $m = \mu = 0$  this is expanded in thermal conformal blocks with constant coefficients corresponding to the thermal one-point functions of conformal quasiprimary operators with definite dimension and spin [see (24) later on]. We show that for nonzero values of  $m$  and  $\mu$  the above two-point function can also be expanded in terms of thermal conformal blocks, but with coefficients now given by single-valued polylogarithms. The latter are recursively related to linear combinations of  $\langle \mathcal{O} \rangle_L$  and  $\langle \mathcal{Q} \rangle_L$ , and hence of conformal ladder graphs. In other words  $\langle \phi^\dagger(x)\phi(0) \rangle_L$  is a generating function of all-loop conformal ladder graphs. Some implications of our results and a number of future directions are discussed.

*From relativistic Bose gases to single-valued polylogarithms.*—We firstly rederive the results in [1] from a new perspective. Consider the following twisted partition function of two decoupled harmonic oscillators with unit mass and common frequency  $m$ :

$$Z_0 = \text{Tr}_{\mathcal{H}_{1,2}} [e^{-\beta(H_0 + m^2 \mathcal{O})} e^{-i\beta\mu \mathcal{Q}}]. \quad (1)$$

This can be viewed as a deformation of the free Hamiltonian  $H_0 = (\hat{p}_1^2 + \hat{p}_2^2)/2$  by the operators  $\mathcal{O} = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2)$  and  $\mathcal{Q} = \hat{p}_2 \hat{x}_1 - \hat{p}_1 \hat{x}_2$  [8]. The twisting parameter  $\mu$  acts effectively as an imaginary chemical potential for  $\mathcal{Q}$ .  $Z_0$  is the grand canonical partition function. Using the complex variable  $z = e^{-\beta m - i\beta\mu}$  one finds

$$\ln Z_0 = \int_0^z \frac{dz'}{1-z'} + \int_0^{\bar{z}} \frac{dz'}{1-z'} - \int_{|z|}^1 \frac{dz'}{z'}. \quad (2)$$

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From (2) we can construct the logarithm of the partition function of a free charged scalar field in  $d$  dimensions with mass  $m$  and twisting parameter  $\mu$  [9] as that of a  $d = 2L + 1$ -dimensional relativistic thermal gas (see Appendix B for details),

$$\ln Z_L = \int d\omega \rho_L(\omega; m) \ln Z_0. \quad (3)$$

The calculations are considerably simpler for integer  $L$  ( $d$  odd) to which we restrict from now on.

After some straightforward manipulations (3) can be brought into the form of an iterated integral for  $L > 1$  as

$$\ln Z_L = (-2\alpha^2)^L \prod_{i=0}^{L-1} \left[ \int_0^{w_{i+1}} \frac{dw_i}{w_i} \ln w_i \right] \ln Z_0, \quad (4)$$

where  $\ln Z_0$  is taken to be a function of  $z_0, \bar{z}_0$  with  $z_0 = w_0 e^{-i\beta\mu}$ ,  $w_L = |z|$ , and the integrals are performed in the order  $w_0 \mapsto w_1 \dots \mapsto w_L$ . Here,  $\alpha^2 = \ell^2/4\pi\beta^2$  is a dimensionless parameter. By virtue of (2) we see that (4) coincides with the class of iterated integrals that give rise to single-valued polylogarithms [4]. We obtain [10]

$$\begin{aligned} \ln Z_L = & \alpha^{2L} \frac{(-1)^L L!}{2(2L+1)!} (2 \log |z|)^{2L+1} \\ & + \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)! n!} 2\Re[Li_{2L+1-n}(z)], \end{aligned} \quad (5)$$

$$\langle \mathcal{Q} \rangle_L = \alpha^{2L} \sum_{n=0}^L \frac{(2L-n)! (-2 \log |z|)^n}{(L-n)! n!} 2i\Im[Li_{2L-n}(z)]. \quad (6)$$

The formulas above correspond to the class of single-valued polylogarithms discussed in many places in the literature. The functions (6) correspond to the graphical functions nicely discussed in [4,11]. However, to our knowledge the functions (5) have not been discussed in terms of graphical functions until now. Below we show that they correspond to conformal ladder graphs of a two-dimensional conformal field theory (CFT).

It is useful to introduce the differential operators

$$\hat{\mathbf{D}} = \frac{1}{\beta^2} \frac{\partial}{\partial m^2} = \frac{1}{2 \ln |z|} (z\partial_z + \bar{z}\partial_{\bar{z}}), \quad (7)$$

$$\hat{\mathbf{L}} = \frac{i}{\beta} \frac{\partial}{\partial \mu} = (z\partial_z - \bar{z}\partial_{\bar{z}}). \quad (8)$$

Explicit calculations yield the following set of first order differential equations [1]:

$$\langle \mathcal{O} \rangle_L = -\beta \hat{\mathbf{D}} \ln Z_L = \beta \alpha^2 \ln Z_{L-1}, \quad (9)$$

$$\langle \mathcal{Q} \rangle_L = \hat{\mathbf{L}} \ln Z_L = -\hat{\mathbf{D}} \cdot \langle \mathcal{Q} \rangle_{L+1} / \alpha^2. \quad (10)$$

Notice that  $\hat{\mathbf{D}}$  acts on  $\ln Z_L$  and  $\langle \mathcal{Q} \rangle_L$  as a dimension lowering operator. Introducing the Laplacian in the variables  $m$  and  $\mu$  as

$$\hat{\Delta} = 4\beta^2 z\bar{z}\partial_z\partial_{\bar{z}} = \frac{\partial^2}{\partial m^2} + \frac{\partial^2}{\partial \mu^2}, \quad (11)$$

we further find

$$\hat{\Delta} f_L(z, \bar{z}) = -4\beta^2 L \alpha^2 f_{L-1}(z, \bar{z}) \quad (12)$$

for  $f_L(z, \bar{z}) = \{\ln Z_L, \langle \mathcal{Q} \rangle_L\}$ . We can combine (12) with (9), (10) to obtain the second order equation

$$[m^2 \hat{\Delta} - 4L\beta^2 m^2 \hat{\mathbf{D}}] f_L(z, \bar{z}) = 0. \quad (13)$$

Notice that  $m^2 \hat{\Delta}$  is the Laplacian on the upper half plane  $\mathbb{H}_2$  with coordinates  $m, \mu$ , and  $2\beta^2 m^2 \hat{\mathbf{D}} = m(\partial/\partial m)$  is the radial derivative. Equation (13) is reminiscent of similar results for partition functions in [12] where the connection to the huge literature of string scattering amplitudes [13–15] was noted. Another interpretation of (13) is as the Laplace-Beltrami operator of  $AdS_{2L+2}$  with metric

$$ds^2 = \frac{1}{m^2} \left( dm^2 + d\mu^2 + \sum_{i=1}^{2L} dx^i dx^i \right) \quad (14)$$

acting on functions of just  $m$  and  $\mu$ . Since  $m$  and  $\mu$  parametrize relevant deformations of a free CFT, such an interpretation may be related to RG flow.

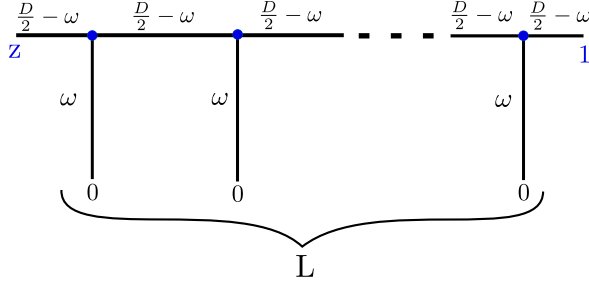
*Conformal graphs as thermal partition functions.*—We will now show that formulas (5) and (6) arise in an apparently unrelated context: as four-point correlators in conformal fishnet models. The latter are particular limits of the generalized biscalar theory in  $D$  dimensions introduced in [7] with Lagrangian

$$\mathcal{L} = N_c \text{Tr} [\phi_1^\dagger (-\partial^2)^\omega \phi_1 + \phi_2^\dagger (-\partial^2)^{\frac{D-2\omega}{2}} \phi_2 + a_{D,\omega}^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2]. \quad (15)$$

$\phi_{1,2}$  belong to the adjoint of  $SU(N_c)$ ,  $\omega \in [0, (D/2)]$  and coupling  $a_{D,\omega}^2$  is classically dimensionless. We consider the four-point function

$$G_{D,\omega}^{(L)}(\{x_i\}) = \langle \text{Tr} [\phi_2^L(x_1) \phi_1(x_3) \phi_2^{\dagger L}(x_2) \phi_1^\dagger(x_4)] \rangle, \quad (16)$$

whose leading  $N_c$  contribution comes from a unique  $L$ -loop conformal ladder graph. It is well known that due to


 FIG. 1. The graph contributing to  $G_{D,\omega}^{(L)}$ .

conformal invariance  $G_{D,\omega}^{(L)}$  depends on two conformal ratios, or equivalently a complex variable  $z$ , and can be represented by an integral of the form depicted in Fig. 1. For  $D = 4$ ,  $\omega = 1$  the model coincides with the original four-dimensional conformal fishnet CFT introduced in [16], and then  $G_{4,1}^{(L)}$  is proportional to the Davydychev-Usuykina  $L$ -loop conformal ladder graphs [5,6]. Up to overall normalizations and using (9) we verify that

$$\tilde{G}_{4,1}^{(L)}(z, \bar{z}) = \frac{1}{L!} \frac{1}{z - \bar{z}} \langle \mathcal{Q} \rangle_L(z, \bar{z}) \quad (17)$$

when we set  $a_{4,1}^2 = \alpha^2$ . In writing (17) we have identified (i) the variable  $z$  representing conformal ratios on the lhs with the modularlike parameter  $z$  of the thermal QFT on the rhs and (ii) the number of loops  $L$  on the lhs with  $L = (d - 1)/2$  on the rhs.

For  $D = 2$ ,  $\omega = 1$  the model (15) is singular as  $G_{2,1}^{(L)}$  would seem to vanish [17]. Nevertheless, a nonzero result can be obtained if we define the effective coupling

$$\tilde{a}_{D,\omega} = a_{D,\omega} \frac{1}{\Gamma(D/2 - \omega)}, \quad (18)$$

which remains finite as  $D \mapsto 2$ ,  $\omega \mapsto 1$ . Then, following the graph-building techniques introduced in [18–21] we can show that the appropriately normalized four-point function of Fig. 1 is given by

$$\tilde{G}_{2,1}^{(L)}(z, \bar{z}) = \tilde{a}_{2,1}^{2L} \sum_{m \in \mathbb{Z}} \int d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}}. \quad (19)$$

Since  $|z| < 1$  we compute the integrals above using contour integration. When  $m \neq 0$  we can close the contour from below and pick up the residues in the lower half complex plane. We obtain

$$\begin{aligned} & \sum_{m \neq 0} \int d\nu \frac{(z\bar{z})^{i\nu} (z/\bar{z})^{m/2}}{(\frac{m^2}{4} + \nu^2)^{L+1}} \\ &= \frac{2\pi}{L!} \sum_{n=0}^L \frac{(2L-n)! (-2\log|z|)^n}{(L-n)! n!} 2\Re[Li_{2L+1-n}(z)]. \end{aligned} \quad (20)$$

For  $m = 0$  the contour integral appears to be zero, but there is a pole on the real axis. Taking the Cauchy principal value we obtain

$$- \int_{C_\epsilon} d\nu \frac{|z|^{2i\nu}}{\nu^{2L+2}} = -i \int_{-\pi}^{2\pi} d\theta \frac{\exp(2i\epsilon \log|z| e^{i\theta})}{e^{2L+1} e^{i(2L+1)\theta}}. \quad (21)$$

For  $\epsilon \mapsto 0$  we encounter  $2L + 1$  divergent terms, which we discard, and a finite contribution that reads

$$-i \int_{-\pi}^{2\pi} d\theta \frac{(2i \log|z|)^{2L+1}}{(2L+1)!} = (-)^L \pi \frac{(2 \log|z|)^{2L+1}}{(2L+1)!}. \quad (22)$$

Putting together (20) and (22) we finally obtain

$$G_{2,1}^{(L)}(z, \bar{z}) = \frac{2\pi}{L!} \ln Z_L(z, \bar{z}) \quad (23)$$

when we set  $\tilde{a}_{2,1}^2 = \alpha^2$ . This is one of the main results of the present work. Notice that the leading “zero temperature” contributions in (19) and (4) arise after the subtraction of a finite number of divergent terms. Acting with  $\hat{\mathbf{L}}$  on both sides of (23) and using (10) we see that the ladder graphs of the four-dimensional CFT are derivatives of the corresponding ladder graphs of the two-dimensional CFT. This dimension-shift property between conformal ladder graphs generalizes to all even dimensions.

*Twisted thermal one-point functions and multiloop conformal graphs.*—The thermal one-point functions  $\langle \mathcal{O} \rangle_L$  and  $\langle \mathcal{Q} \rangle_L$  appear in the expansion of the thermal two-point function  $\langle \phi^\dagger(x) \phi(0) \rangle = g^{(L)}(\tau, \mathbf{x})$ . This motivates us to ask whether thermal one-point functions of higher spin operators are also related to conformal ladder graphs. It is usually highly nontrivial to calculate thermal one-point functions in a generic QFT. However, for a CFT with a complex scalar  $\phi(x)$  having dimension  $\Delta_\phi$  in  $d = 2L + 1$  we have [22]

$$g^{(L)}(\tau, \mathbf{x}) = \sum_{\mathcal{O}_s} a_{\mathcal{O}_s}^L \left( \frac{r}{\beta} \right)^{\Delta_{\mathcal{O}_s}} \frac{C_s^\nu(\cos \theta)}{r^{2\Delta_\phi}}, \quad (24)$$

where  $\nu = d/2 - 1$ . The main assumption behind (24) is the existence of a conformal operator product expansion at zero temperature such that  $\phi^\dagger \times \phi$  can be expanded in a sum of quasiprimary operators  $\mathcal{O}_s$  with definite spins  $s$  and scaling dimensions  $\Delta_{\mathcal{O}_s}$ . The latter are represented by symmetric, traceless rank- $s$  tensors, and their one-point functions depend on a single parameter that is proportional to the coefficient  $a_{\mathcal{O}_s}^L$ . For example, for free massless complex scalars when  $\Delta_\phi = L - 1/2$  one obtains [23–25]

$$a_{\mathcal{O}_s}^L = 2C_\phi^L(1) \zeta(2L - 1 + s), \quad s = 0, 2, 4, \dots \quad (25)$$

In that case, only symmetric and conserved higher-spin operators with dimensions  $\Delta_{\mathcal{O}_s} = d - 2 + s$  and even spin

$s$  appear in (24). Each term in the sum (24) is of the form  $r^s C_s^{L-1/2}(\cos\theta)$  and we find  $\square_d G^{(L)}(x) = 0$  with  $\square_d$  the  $d$ -dimensional Laplacian. This is the usual free field theory result away from the origin.

In nontrivial CFTs the operator spectrum, their scaling dimensions, and  $a_{\mathcal{O}_s}^L$  change in a way determined by the dynamics; hence, the thermal two-point function does not satisfy a simple equation in general, although the form of the expansion (24) remains the same. The latter property is not expected to be true in a generic QFT. Nevertheless, remarkably, the thermal two-point functions of the complex scalars  $\phi(x)$  in the massive free theory with partition function (4) do admit an expansion of the form (24) and contain a part that is annihilated by the  $d$ -dimensional Laplacian, albeit with different coefficients  $a_{\mathcal{O}_s}^L$  from (25). This might not be surprising as the theory is Gaussian; nevertheless, the theory is not generically a CFT.  $g^{(L)}(\tau, \mathbf{x})$  is obtained as the Fourier transform of the unit normalized momentum space two-point function with twisted boundary conditions [9] on  $S_\beta^1$ . We obtain (setting  $\beta = 1$  for simplicity)

$$g^{(L)}(\tau, \mathbf{x}) = \frac{1}{(2\pi)^\nu} \sum_{n=-\infty}^{\infty} e^{i\mu n} \left[ \frac{m}{|X_n|} \right]^\nu K_\nu(m|X_n|), \quad (26)$$

with  $X_n = (\tau - n, \mathbf{x})$  and  $K_\nu$  the modified Bessel functions. The coefficients  $a_{\mathcal{O}_s}^L$  can be calculated from (26) using the inversion method of [24], as it was done in [25], but taking now care that the two-point function is complex so that the discontinuities along the cuts in the positive and negative  $r$  axis are complex conjugates [26]. We focus on the part of  $G^{(L)}(\tau, \bar{x})$  that is annihilated by the  $d$ -dimensional Laplacian, namely to the contribution of the would-be higher-spin currents with dimensions  $\Delta_{\mathcal{O}_s} = 2L - 1 + s$ . We obtain

$$\begin{aligned} a_{\mathcal{O}_s}^L &= \frac{\Gamma(L - \frac{1}{2})}{\Gamma(L + s - \frac{1}{2})(4\pi)^L 2^{2s}} \\ &\times \sum_{n=0}^{L-1+s} \frac{2^n (\beta m)^n (2L - 2 + s - n)!}{n! (L - 1 + s - n)!} \\ &\times [Li_{2L-1+s-n}(z) + (-1)^s Li_{2L-1+s-n}(\bar{z})]. \end{aligned} \quad (27)$$

If the theory were a CFT we would associate the coefficients  $a_{\mathcal{O}_s}^L$  with thermal one-point functions of conformal quasiprimary operators. For generic values of  $m$  and  $\mu$  this is more complicated. For example,  $a_{\mathcal{O}_2}^L$  represents the contribution of a rank-2 symmetric traceless tensor that is *not* the energy momentum tensor of the massive theory since the latter has nonzero trace. Nevertheless, the coefficients  $a_{\mathcal{O}_0}^L$  and  $a_{\mathcal{O}_1}^L$  do represent the thermal one-point functions of the operators  $\mathcal{O}$  and  $\mathcal{Q}$  as they have been independently calculated in (9), (6). Explicitly we have

$$a_{\mathcal{O}_0}^L = \frac{1}{(4\pi)^L \beta \alpha^{2L}} \langle \mathcal{O} \rangle_L, \quad a_{\mathcal{O}_1}^L = \frac{1}{(4\pi)^L \alpha^{2L} 2} \langle \mathcal{Q} \rangle_L. \quad (28)$$

Using (9), (10) we see that for  $z = \bar{z} = 1$  the above reduce to (25) as they should. The novel result is that *all* coefficients  $a_{\mathcal{O}_s}^L$  with  $s \geq 2$  are related to  $L$ -loop conformal graphs by virtue of the following recursion relations shown by brute force calculations:

$$a_{\mathcal{O}_{s+2}}^L = \frac{2\pi}{2L-1} a_{\mathcal{O}_s}^{L+1} + \frac{(m\beta)^2}{(2L-1+2s)(2L+1+2s)} a_{\mathcal{O}_s}^L. \quad (29)$$

Consequently, the part of the twisted thermal two-point function (26) that is annihilated by the  $d$ -dimensional Laplacian is a generating function for (linear combinations) of  $L$ -loop conformal ladder graphs.

Our (29) implies that we can associate a ‘‘spin’’ to a certain combination of  $L$ - and  $L-1$ -loop conformal ladder graphs. This is evident for  $s=0$ , and it can be generalized for all  $s$ . We do not yet have an understanding of this ‘‘spin’’ from the point of view of the conformal graphs, but from the thermal field theory point of view it can be given a physical interpretation in terms of the underlying free field theory dynamics. However, we believe that they have a simpler underlying physical interpretation. For example, (29) corresponds to a standard thermodynamics relationship for  $s=2$ . To see that, note that from the twisted partition function  $Z_L$  with Hamiltonian of the form  $H = H_0 + m^2 \mathcal{O} + i\mu \mathcal{Q}$  one can derive the following general result:

$$\langle H \rangle_L = \frac{d-1}{\beta} \ln Z_L + 2m^2 \langle \mathcal{O} \rangle_L + i\mu \langle \mathcal{Q} \rangle_L, \quad (30)$$

where  $\langle H \rangle_L = -\langle t_{\tau\tau} \rangle_L$  with  $t_{\mu\nu}$  the energy momentum tensor of theory. For nonzero  $m$  and  $\mu$  this is not traceless, but for the massless free complex scalar with imaginary chemical potential we can construct a traceless spin-2 operator  $\mathcal{T}_{\mu\nu}$  with  $\mathcal{T}_{\tau\tau} = t_{\tau\tau} + 2m^2 \mathcal{O}/d + i\mu \mathcal{Q}$ . Then (30) becomes

$$-\langle \mathcal{T}_{\tau\tau} \rangle_L = \frac{d-1}{\beta} \ln Z_L + 2m^2 \frac{d-1}{d} \langle \mathcal{O} \rangle_L. \quad (31)$$

The general relation connecting  $a_{\mathcal{O}_2}^L$  with the  $\mathcal{T}_{\tau\tau}$  is [27]

$$\begin{aligned} \frac{(4\pi\alpha^2)^L}{\beta} a_{\mathcal{O}_2}^L &= \frac{2g_{\phi^\dagger\phi T}}{(d-1)(d-2)C_T} \langle T_{00} \rangle_L \\ &= -\frac{C_\phi^L(1)S_L}{2(d-1)} \langle \mathcal{T}_{\tau\tau} \rangle. \end{aligned} \quad (32)$$

Using then (28), (9), and (10) we can verify that (31) coincides with (29). We believe that similar arguments

relating trace-full and traceless higher-spin operators of the massive free scalar theory can provide a physical understanding for (29) for general  $s$ .

*Discussion and outlook.*—In this Letter, we have connected two seemingly unrelated quantities: twisted partition functions of a massive free complex scalar field in  $d = 2L + 1$  dimensions, and four-point conformal  $L$ -loop ladder graphs. The reason for such a relationship is that they both satisfy the same sets of differential equations. For the partition functions these are given by (13). For the conformal ladder graphs they are the differential equations discussed in number of earlier works on conformal integrals [i.e., Eq. (2.15) in [28]]. This common property begs for a deeper explanation.

Our results draw a unifying picture for the thermal one-point functions  $a_{\mathcal{O}_s}^L$  in massive free complex scalar theories. This is depicted in Fig. 2. By the algebraic relations (29) they are all ultimately given by  $a_{\mathcal{O}_0}^L$  or  $a_{\mathcal{O}_1}^L$ , and then by the action of the differential operators  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{L}}$  to the  $\langle \mathcal{O} \rangle_0$  and  $\langle \mathcal{Q} \rangle_0$  of the harmonic oscillator model (1).

There are many questions that arise from our observations. It would be interesting to understand the possible relationship of our results to the integrability of fishnet models. It would also be interesting to connect our results to works that relate partition functions and string amplitudes. Another question would be to connect our approach to studies of nonintegrable deformations of thermal CFTs (i.e., see Refs. [29–31] for interesting recent works).

We close with some remarks. Our iterated integral formula (4) when applied to  $\langle \mathcal{Q} \rangle_L$  gives for  $L = 1$

$$\langle \mathcal{Q} \rangle_1 = (-2\alpha^2) \int_0^{|z|} \frac{d|z'|}{|z'|} \ln |z'| \langle \mathcal{Q} \rangle_0 = 4i\alpha^2 D(z), \quad (33)$$

where  $D(z) = \Im[Li_2(z) + \ln |z| \ln(1 - z)]$  is the celebrated Bloch-Wigner function that gives the volume of an ideal tetrahedron in three-dimensional hyperbolic space  $\mathbb{H}^3$  with

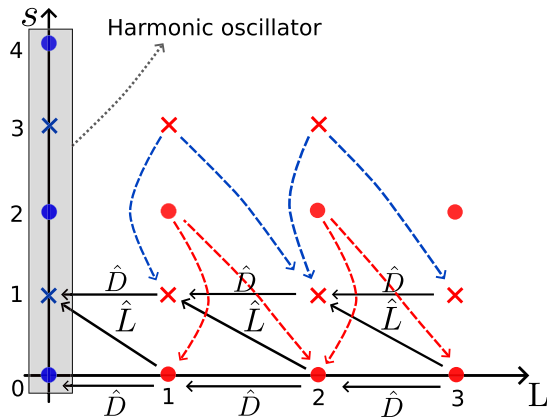


FIG. 2. Differential (solid lines) and algebraic (dashed lines) relationships among the  $a_{\mathcal{O}_s}^L$ .

vertices in  $\partial\mathbb{H}^3$  [32]. It is then amusing to note that  $\langle \mathcal{Q} \rangle_0$  itself has a geometric interpretation. Indeed,

$$\langle \mathcal{Q} \rangle_0 = \frac{z - \bar{z}}{(1 - z)(1 - \bar{z})}, \quad (34)$$

and setting  $z = e^{i\phi}(b/a)$  with  $\cos \phi = (a^2 + b^2 - 1)/2ab$  we find that  $\langle \mathcal{Q} \rangle_0/4i = \frac{1}{2}ab \sin \phi$  gives the area of a triangle whose side lengths are  $a$ ,  $b$  and  $1$ , and  $\phi$  the angle between  $a$  and  $b$ . Then (33) gives the volume of an ideal hyperbolic tetrahedron as an integral of the area of a triangle. One then wonders if there is a geometric interpretation for the higher order iterated integrals in (4). We should further note that  $\langle \mathcal{O} \rangle_0$  also has an interpretation as an area, but we are not aware of a nice geometric interpretation of  $\langle \mathcal{O} \rangle_1$ .

Another observation is that  $\langle \mathcal{Q} \rangle_0 = -8\pi \rho_{m_0 \mapsto m_1 + m_2}^{D=4}$  where  $\rho_{m_0 \mapsto m_1 + m_2}^D$  is the  $D$ -dimensional  $1 \mapsto 2$  decay phase space of relativistic massive particles. Since

$$\rho_{m_0 \mapsto m_1 + m_2}^{D=4} = \frac{1}{8\pi} \sqrt{\lambda(1, m_1^2/m_0^2, m_2^2/m_0^2)} \quad (35)$$

with  $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$  the Källén triangle function, we see that if we set  $a = m_1/m_0$  and  $b = m_2/m_0$ , then  $\langle \mathcal{Q} \rangle_0$  represents the phase space for a virtual process with  $\lambda < 0$  [33]. Then, our (4) is reminiscent of Eq. (7) of [34], which gives a recurrent relationship for higher dimensional  $1 \mapsto 2$  relativistic phase spaces.

We further note that equations such as (9) and (10) lead naturally to the resummation of infinite series. For example, by virtue of (9) the infinite product  $Z = \prod_{n=0}^{\infty} Z_n$  satisfies the inhomogeneous first order equation

$$(\hat{\mathbf{D}} + \alpha^2) \ln Z = -\frac{1}{\beta} \langle \mathcal{O} \rangle_0. \quad (36)$$

This can be integrated to

$$\ln Z = -\beta e^{-\beta^2 \alpha^2 m^2} \int^{m^2} e^{\beta^2 \alpha^2 \tilde{m}^2} \langle \mathcal{O} \rangle_0 d\tilde{m}^2. \quad (37)$$

An analogous result can be derived for the  $\langle \mathcal{Q} \rangle = \sum_{n=0}^{\infty} \langle \mathcal{Q} \rangle_n$ . Given (17) and (23) these are all-loop Borel summations of conformal ladder graphs [35]. See Appendix A for some additional observations.

Finally, we point out the work [36] where 2-2 scattering amplitudes are given in terms of a dispersive integral over generating functions of knot polynomials [see, e.g., (12) and (23) of that reference] [37]. The latter generating functions written in terms of the variables  $z, \bar{z}$  correspond to thermal averages of certain bilinear operators in a  $q$ -deformed harmonic oscillator, much like our  $\langle \mathcal{O} \rangle_L$  and  $\langle \mathcal{Q} \rangle_L$ . We find the connection of the approach in [36] and our results quite intriguing and we believe that it deserves further study.

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*Appendix A: Further observations.*—Applying  $\hat{\mathbf{L}}$  to (37) gives

$$(\hat{\mathbf{D}} + \alpha^2)\langle Q \rangle = \hat{\mathbf{D}}\langle Q \rangle_0, \quad (\text{A1})$$

where  $\langle Q \rangle = \sum_{L=0}^{\infty} \langle Q \rangle_L$ . By virtue of (17) this sum can be Borel transformed into the Broadhurst-Davydychev infinite sum of the  $L$ -loop conformal ladder graphs in four dimensions [38]; see also [39,40]. Indeed the solution of the first order equation (A1) is

$$\langle Q \rangle = \beta^2 e^{-\beta^2 \alpha^2 m^2} \int^m e^{\beta^2 \alpha^2 \tilde{m}^2} \hat{\mathbf{D}}\langle Q \rangle_0 d\tilde{m}^2 \quad (\text{A2})$$

and can be thought of as a series of the form  $\langle Q \rangle \equiv A(z) = \sum_{n=0}^{\infty} a_n z^n$  with  $z = \alpha^2$ . Its Borel transform series  $\mathcal{B}[A](t) := \sum_{k=0}^{\infty} (a_k/k!) t^k$  is given by the contour integral

$$\mathcal{B}[A](t) = \frac{1}{2\pi i} \int_C \frac{dz}{z} e^z A(t/z), \quad (\text{A3})$$

where  $C$  is the Hankel contour [41]. Using the following integral representation of the Bessel function

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{2\pi i} \int_C dt \frac{1}{t^{\nu+1}} e^{t - \frac{z^2}{4t}}, \quad (\text{A4})$$

we obtain

$$\begin{aligned} \mathcal{B}[Q](t) &= \beta^2 \frac{1}{2\pi i} \int^m d\tilde{m}^2 (\hat{\mathbf{D}}\langle Q \rangle_0) \int_C \frac{du}{u} e^{u - \frac{t^2}{u}(m^2 - \tilde{m}^2)} \\ &= \beta^2 \int^m d\tilde{m}^2 J_0 \left[ 2\beta \sqrt{t(m^2 - \tilde{m}^2)} \right] (\hat{\mathbf{D}}\langle Q \rangle_0). \end{aligned} \quad (\text{A5})$$

Using then

$$\hat{\mathbf{D}}\langle Q \rangle_0 = \frac{i}{2\beta m} \frac{\sinh(\beta m) \sin(\beta \mu)}{[\cosh(\beta m) - \cos(\beta \mu)]^2}, \quad (\text{A6})$$

and setting  $t = -(\kappa^2/4)$ ,  $\beta \tilde{m} = \eta$ ,  $\ell = 2\beta m$  and putting the lower bound of the integral to be  $+\infty$ , (A5) coincides with Eq. 15 of [38].

*Appendix B: The relativistic thermal gas.*—The one-particle density of states  $\rho_L(\omega; m)$  for the relativistic thermal gas in  $d = 2L + 1$  dimensions is found as usual by considering the system in a  $(d-1)$ -dimensional spatial cubic box of volume  $V_{d-1} = \ell^{d-1}$  with quantized momentum  $\vec{p} = [(2\pi/\ell)n_1, \dots, (2\pi/\ell)n_{d-1}] = (2\pi/\ell)\vec{n}$ . The number of modes having momenta inside the spherical shell bounded by  $|\vec{p}|$  and  $|\vec{p}| + d|\vec{p}|$  in  $d = 2L + 1$  dimensions is

$$dn = \left( \frac{\ell^2}{4\pi^2} \right)^L |\vec{p}|^{2L-1} d|\vec{p}| \int d\Omega_{2L}, \quad (\text{B1})$$

with  $\int d\Omega_{2L} = 2\pi^L/\Gamma(L)$ . Using then the dispersion relation  $\omega^2 = \vec{p}^2 + m^2$ , for  $\rho_L(\omega; m) \equiv dn/d\omega$  we obtain

$$\rho_L(\omega; m) = \frac{2\alpha^2 \beta^2}{(L-1)!} \omega(\omega^2 - m^2)^{L-1}, \quad (\text{B2})$$

which when substituted in (3) gives

$$\ln Z_L = \frac{2\alpha^2 \beta^2}{(L-1)!} \int_m^\infty \omega d\omega (\omega^2 - m^2)^{L-1} \ln Z_0, \quad (\text{B3})$$

or alternatively (4) in terms of the real variable  $\omega|z| = e^{-\beta m}$ . We can now apply our differential operators  $\hat{\mathbf{D}}$  and  $\hat{\mathbf{L}}$  to this and obtain the integral representations of all our thermal one-point functions. In particular, applying  $\hat{\mathbf{L}}$  to (B3) we will get the integral representation of the  $L$ -loop conformal ladder graphs given in Eq. (2.20) of [42] for purely imaginary  $\phi$  and up to and overall  $1/L!$ .

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- tensor product Hilbert space  $\mathcal{H}_{1,2} \approx \{|n_1\rangle \otimes |n_2\rangle\}$ ,  $n_1, n_2 = 0, 1, 2, \dots$
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# Flat from anti de Sitter

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**ABSTRACT:** Ricci-flat solutions to Einstein's equations in four dimensions are obtained as the flat limit of Einstein spacetimes with negative cosmological constant. In the limiting process, the anti-de Sitter energy-momentum tensor is expanded in Laurent series in powers of the cosmological constant, endowing the system with the infinite number of boundary data, characteristic of an asymptotically flat solution space. The governing flat Einstein dynamics is recovered as the limit of the original energy-momentum conservation law and from the additional requirement of the line-element finiteness, providing at each order the necessary set of flux-balance equations for the boundary data. This analysis is conducted using a covariant version of the Newman-Unti gauge designed for taking advantage of the boundary Carrollian structure emerging at vanishing cosmological constant and its Carrollian attributes such as the Cotton tensor.

**KEYWORDS:** AdS-CFT Correspondence, Classical Theories of Gravity

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**1 Introduction**

The solution space of Einstein’s equations and the corresponding asymptotic symmetries are severely altered by the presence of a cosmological constant  $\Lambda$ .<sup>1</sup> Firstly, asymptotically flat spacetimes support incoming and outgoing gravitational radiation, which are harder to accommodate in asymptotically anti-de Sitter — unless leaky boundary conditions are assumed [2]. Secondly, the number of free functions on the boundary characterizing the solution space is finite for non-zero  $\Lambda$  and infinite for  $\Lambda = 0$ . Hence, investigating the holographic description of Ricci-flat spacetimes from the limit of Einstein spacetimes with non-vanishing cosmological constant seems at best a futile task, limited to special cases like three spacetime dimensions.

The purpose of the present work is to reconsider this statement in four dimensions and show that expanding the anti-de Sitter energy-momentum tensor in Laurent series in  $k^2 = -\Lambda/3$ , one recovers the full Ricci-flat solution space in a  $1/r$ -expansion together with its evolution dynamics captured in flux-balance equations.

In order to perform the above analysis, a choice of gauge is required, as usual. Baring in mind potential further developments towards flat holography, it is desirable to privilege null infinity in the asymptotically flat instance, which plays the role of a conformal boundary hosting all independent functions of the solution space, often referred to as degrees of freedom in the following. The null boundary is a three-dimensional Carrollian manifold and it is therefore convenient to select a bulk gauge making the corresponding boundary general and Weyl covariance manifest. This has prompted to choose a modified version of the Newman-Unti gauge [3]. This gauge is built upon an incomplete gauge fixing that is expected to lead to an enhancement of the asymptotic symmetries with respect to more customary gauges like the Bondi one, in analogy with what has been observed in three bulk

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<sup>1</sup>See [1] for a recent review and further reading suggestions.



dimensions in several examples of incomplete gauge fixings [4–13]. We will not pursue this direction here, although it has attracted recent interest — see, e.g., [14, 15] where in the latter reference the gauge under consideration was dubbed “differential Newman-Unti” — and we will focus on the comparison between the space of solutions of Einstein’s equations in the asymptotically anti-de Sitter and flat cases in the chosen gauge. Reference [3] has set the stage for the gauge we will describe here, although it was originally circumscribed to the restricted class of algebraically special solutions (for AdS, see [16–22]).

In a nutshell, the starting point is the anti-de Sitter case, where the solution space admittedly consists of the boundary metric and the boundary energy-momentum tensor, which is covariantly conserved as a consequence of bulk Einstein’s equations. The analogue of the “Bondi shear” (sometimes referred to as “dynamical shear” later) is not an independent piece of data because Einstein’s equations require it be proportional with a  $k$ -dependent factor to the geometric shear of the gauge congruence — already part of the solution space. We move to the asymptotically flat instance by sending  $k$  to zero, expanding the energy-momentum tensor in powers of  $k^2$ , trading the geometric shear for the dynamical shear along the lines of [23], and requiring the bulk line element to remain finite. This imposes further evolution equations for the new degrees of freedom at every order in the radial  $1/r$  expansion, which supplement the energy-momentum conservation in the zero- $k$  limit. The resulting infinite set defines the flux-balance equations, which can otherwise be obtained directly by solving Einstein’s equations without cosmological constant.

Besides reaching the correct boundary Ricci-flat dynamics and tracing the AdS origin of the asymptotically flat solution space, the method presented here delivers Carroll-covariant flux-balance equations revealing the entire freedom for the choice of the boundary Carrollian geometry. The pattern involves the general and Weyl-covariant gauge mentioned earlier, which naturally incorporates the Cotton tensor of the anti-de Sitter boundary, or its Carrollian emanations in the asymptotically flat situation (see [24]). This tensor carries information on the gravitational radiation and plays a pivotal role for the description of magnetic charges [25].

We begin our presentation by defining the covariant Newman-Unti gauge for asymptotically anti-de Sitter spaces. Along the way, we review its boundary Weyl covariance, as well as a useful decomposition of the boundary energy-momentum and Cotton tensors. We then provide an on-shell expression of the line element up to order  $1/r^2$ , where  $r$  is a null radial coordinate, infinite at the conformal boundary. The flat limit, following the prescription summarized above, is carried out in the upcoming section, after a precise setting of the boundary Carrollian structure consecutive to the zero- $\Lambda$  limit. We show how the new degrees of freedom resulting from the expansion of the anti-de Sitter energy-momentum tensor are sorted out, how they enter the metric and how flux-balance equations are reached. Two appendices complement the main exposition, bringing about the necessary tools of Carrollian geometry in arbitrary frames (strong Carroll structures, connections, curvatures) as well as showing how such structures can be attained from pseudo-Riemannian geometries at vanishing speed of light. A presentation of the Carrollian descendants of the Cotton tensor closes this article.

## 2 Einstein spacetimes in covariant Newman-Unti gauge

**Choosing a boundary-covariant gauge.** Due to the Fefferman-Graham ambient metric construction [26–28], asymptotically locally anti-de Sitter four-dimensional spacetimes are determined by a set of independent boundary data, namely a three-dimensional metric  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$  and a rank-two tensor  $T = T_{\mu\nu}dx^\mu dx^\nu$ , symmetric ( $T_{\mu\nu} = T_{\nu\mu}$ ), traceless ( $T^\mu{}_\mu = 0$ ) and conserved:<sup>2</sup>

$$\nabla^\mu T_{\mu\nu} = 0. \tag{2.1}$$

This construction is reached by setting a homonymous gauge, imposing fall-offs and solving Einstein’s equations order by order in powers of the radial space-like coordinate.<sup>3</sup> At every order in this expansion, the line element is determined by a tensor  $G_{\mu\nu}^{(s)}$ , fixed by Einstein’s equations in terms of  $g_{\mu\nu} = G_{\mu\nu}^{(-2)}$ ,  $T_{\mu\nu} = \frac{3k}{16\pi G} G_{\mu\nu}^{(1)}$  and their derivatives (here the conformal boundary is located at  $\rho \rightarrow +\infty$ ):

$$ds_{\text{Einstein}}^2 = \frac{d\rho^2}{(k\rho)^2} + \sum_{s \geq -2} \frac{1}{(k\rho)^s} G_{\mu\nu}^{(s)}(x) dx^\mu dx^\nu. \tag{2.2}$$

The conservation (2.1) is itself a consequence of Einstein’s dynamics.

Fefferman-Graham’s gauge is covariant with respect to the three-dimensional pseudo-Riemannian boundary  $\mathcal{M}$ . It can also be modified so as to make it Weyl-covariant [30–33]. However, it does not admit a smooth vanishing- $k$  limit. Alternative gauges are Bondi or Newman-Unti [34–36], valid regardless of the cosmological constant, but not covariant with respect to the boundary. Let us consider for concreteness the Newman-Unti gauge with radial coordinate  $r$ .<sup>4</sup> The line element reads:

$$ds_{\text{bulk}}^2 = \frac{V}{r} du^2 - 2dudr + G_{ij} \left( dx^i - U^i du \right) \left( dx^j - U^j du \right), \tag{2.3}$$

where  $V$ ,  $U_i$  and  $G_{ij}$  are functions of *all* coordinates. They are treated as power series of  $r$ , possibly including logarithms,<sup>5</sup> with coefficients depending on boundary coordinates  $x$ : the retarded time  $u$  and the angles  $\mathbf{x}$ .

The bulk metric (2.3) is stable neither under boundary diffeomorphisms  $x \rightarrow x'$ , nor under Weyl rescalings  $r \rightarrow r\mathcal{B}(x)$ , and these are the features we would like to implement in a “covariantized” version of the gauge at hand. To this end, we trade  $-k^2 du$  for a boundary one-form  $u = u_\mu dx^\mu$ , which is an invariant object dual to a time-like vector field normalized at  $-k^2$ . As it will become manifest in section 3, where the timelike conformal boundary

<sup>2</sup>The covariant derivative  $\nabla$  stands for the boundary Levi-Civita connection. Indices  $\mu, \nu, \dots \in \{0, 1, 2\}$  fill in the boundary natural frame, whereas  $i, j, \dots \in \{1, 2\}$  are associated with spatial sections.

<sup>3</sup>Residual symmetries constrain the possible terms entering each order in the radial expansion, thus simplifying the process of solving Einstein’s equations. The constraints imposed by the boundary Weyl symmetry were studied in the Fefferman-Graham gauge in [29] and will play an important role too in the covariant Newman-Unti gauge discussed in the following.

<sup>4</sup>Both in Bondi and Newman-Unti,  $\partial_r$  is tangent to a null geodesic congruence. In Newman-Unti gauge this congruence is affinely parameterized, in contrast to Bondi. This enables to parallelly transport a canonical null tetrad and make contact with Newman-Penrose formalism [37].

<sup>5</sup>Logarithms also appear in the Fefferman-Graham gauge when the boundary dimension is even [38, 39].

will become a null manifold equipped with a Carrollian structure in the limit  $k \rightarrow 0$ , our parameterization has been chosen such that  $k = \sqrt{-\Lambda/3}$  plays the role of effective boundary velocity of light. Therefore, the previous substitution amounts to choosing a time-like boundary congruence that could be interpreted as a hydrodynamic velocity field, if the boundary energy-momentum tensor were associated with a fluid. This is not necessarily so because the hydrodynamic regime requires constitutive relations, which are not obeyed everywhere in the Einstein solution space. The subspace where this happens is the realm of fluid/gravity correspondence [16, 17]. For convenience, we will nonetheless refer to  $u$  as the “velocity field” and decompose the energy-momentum tensor accordingly.

Introducing a boundary congruence provides also the appropriate tool for addressing Weyl invariance. In the spirit of the Fefferman-Graham ambient construction, the bulk geometry should be insensitive to a Weyl rescaling of the boundary metric (weight  $-2$ ) and of the boundary velocity form (weight  $-1$ )

$$ds^2 \rightarrow \frac{ds^2}{\mathcal{B}^2}, \quad u \rightarrow \frac{u}{\mathcal{B}}, \tag{2.4}$$

which should be reabsorbed into a redefinition of the radial coordinate:  $r \rightarrow r\mathcal{B}$ . This requires to introduce a Weyl connection one-form  $A = A_\mu dx^\mu$  transforming as

$$A \rightarrow A - d \ln \mathcal{B}, \tag{2.5}$$

and suggests the following amendment to the Newman-Unti gauge

$$-dudr \rightarrow \frac{u}{k^2}(dr + rA), \tag{2.6}$$

which is indeed Weyl-invariant, as well as being boundary-general-covariant.

We can follow the suggested pattern and recast (2.3), ignoring the logarithms,<sup>6</sup> avoiding the demarcation of angular and time directions, and reorganizing the expansion in terms of boundary tensors according to their transversality with respect to the congruence  $u$  as well as their conformal weights:

$$ds_{\text{bulk}}^2 = 2\frac{u}{k^2}(dr + rA) + r^2 ds^2 + r\mathcal{C}_{\mu\nu} dx^\mu dx^\nu + \frac{1}{k^4} \mathcal{F}_{\mu\nu} dx^\mu dx^\nu + \sum_{s=1}^{\infty} \frac{1}{r^s} \left( f_{(s)} \frac{u^2}{k^4} + 2\frac{u}{k^2} f_{(s)\mu} dx^\mu + f_{(s)\mu\nu} dx^\mu dx^\nu \right). \tag{2.7}$$

In this expression  $f_{(s)}$  are *boundary scalars*,  $f_{(s)\mu}$  *boundary transverse vectors*,  $f_{(s)\mu} u^\mu = 0$ , and  $f_{(s)\mu\nu}$  *boundary symmetric and transverse tensors*,  $f_{(s)\mu\nu} u^\mu = 0$ . Their conformal weights are  $s + 2$ ,  $s + 1$  and  $s$ . The  $r^2$  term defines the boundary metric  $ds^2$ , which is a

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<sup>6</sup>Logarithms of the radial coordinate might or might not be required depending on the gauge chosen to investigate the space of solutions. In some cases, like, e.g., when choosing the Fefferman-Graham gauge in odd spacetime dimensions, they are necessary to reconstruct the full solution space. In other cases, like, e.g., in asymptotically flat spacetimes in the Bondi gauge, they describe an independent sector of the solution space that might be added or not to the polynomial expansion. Our case fits in the latter class and performing a thorough investigation of the larger space of solutions including logarithms and analyzing its interplay with residual symmetries (see, e.g., [2, 23, 40–42]) is not part of our present agenda.

free piece of data in the spirit of [9–11, 13, 23]. As long as the bulk metric (2.7) is off-shell, the boundary symmetric tensors  $\mathcal{F}_{\mu\nu}$  (weight 0) and  $\mathcal{C}_{\mu\nu}$  (weight  $-1$ ) have no reason to be transverse with respect to  $u$ . The latter is the shear of the affine null geodesic congruence tangent to  $\partial_r$ , aka “Bondi” shear.<sup>7</sup> Imposing Einstein’s equations will determine all the boundary tensors introduced so far in terms of basic independent functions that define the solution space. As we will see, this set of functions includes  $u$  as well as the boundary metric  $ds^2$  and a rank-two symmetric, traceless and conserved tensor coinciding with the energy-momentum tensor of the Fefferman-Graham gauge.

Before moving on to Einstein’s equations, a few comments are worth making to appreciate the covariant Newman-Unti gauge (2.7). Introducing a normalized but otherwise arbitrary boundary congruence amounts to the on-set of two degrees of freedom, i.e. to a relaxation of the original Newman-Unti gauge fixing. Incomplete gauge fixings might produce enhancements of asymptotic symmetries and materialize in extra charges — not always integrable or conserved. They have been investigated mostly in three bulk dimensions [4–13], where the introduction of an arbitrary congruence<sup>8</sup> combined with the freedom of choosing the boundary metric restores the boundary local Lorentz symmetry and its realization as bulk diffeomorphisms, augmenting the asymptotic symmetry group [9–13]. Following [10], an elegant way of taming this information without redundancy is to express the boundary metric in an arbitrary orthonormal Cartan coframe,<sup>9</sup>

$$ds^2 = \eta_{AB}\theta^A\theta^B = -(\theta^{\hat{0}})^2 + \delta_{ab}\theta^a\theta^b, \tag{2.8}$$

and set

$$u = -k\theta^{\hat{0}}. \tag{2.9}$$

The dual frame vectors are  $\{e_A\} = \{e_{\hat{0}}, e_a\}$  with  $\theta^B(e_A) = \delta_A^B$ . A possible parameterization of the frame, which we will not use explicitly though, is displayed in eqs. (A.41), (A.42) and (A.43).

We will not delve into the analysis of asymptotic symmetries in the present note. Due to the partial relaxation of the gauge this complementary task is more intricate and deserves a separate and thorough treatment [9–15].

In order to proceed with the covariant Newman-Unti gauge (2.7) and impose Einstein’s equations, it is desirable to list the available tensors with the correct conformal weights at

<sup>7</sup>Strictly speaking, the Bondi shear is defined in the BMS gauge (in the expansion  $G_{ij} = r^2q_{ij} + \mathcal{O}(r)$  the two-dimensional metric  $q_{ij}$  is fixed to be the round sphere) with a prominent role in the asymptotically flat instance. Normally it is related to the one introduced here by an inhomogeneous transformation.

<sup>8</sup>Whenever the energy-momentum tensor empowers a fluid interpretation, the congruence at hand is interpreted as the fluid lines and its arbitrariness portrays the relativistic hydrodynamic-frame invariance [43–45]. This feature is however strictly *local* because the bulk diffeomorphisms associated with the boundary hydrodynamic-frame transformations are possibly charged. These properties have been thoroughly investigated in two boundary dimensions [9–11, 13] and would undoubtedly deserve a generalization in higher dimensions, which is outside our scope here. It would better fit a broader study where frame orthonormality would be downsized, probing general boundary linear transformations.

<sup>9</sup>We use  $A, B, \dots \in \{\hat{0}, \hat{1}, \hat{2}\}$  as boundary “flat” indices with  $a, b, \dots \in \{\hat{1}, \hat{2}\}$ . The parameterization of the coframe in terms of 8 arbitrary functions, suitable for the Carrollian limit, is provided in appendix A, eq. (A.43).

each order  $s$  of the radial expansion. To achieve this, we need to cope with Weyl covariance and decompose the energy-momentum tensor with respect to the chosen congruence.

**Kinematics, Weyl covariance and transverse duality.** Covariantization with respect to Weyl transformations requires to introduce a connection one-form  $A = A_A \theta^A$ , built on the congruence  $u = u_A \theta^A$ :

$$A = \frac{1}{k^2} \left( a - \frac{\Theta}{2} u \right), \quad (2.10)$$

which transforms as anticipated in (2.5). In this expression  $a = a_A \theta^A$  and  $\Theta$  are the acceleration and expansion of the congruence  $u$ , defined together with the shear and the vorticity as<sup>10</sup>

$$a_A = u^B \nabla_B u_A, \quad \Theta = \nabla_A u^A, \quad (2.11)$$

$$\sigma_{AB} = \nabla_{(A} u_{B)} + \frac{1}{k^2} u_{(A} a_{B)} - \frac{1}{2} \Theta h_{AB}, \quad (2.12)$$

$$\omega_{AB} = \nabla_{[A} u_{B]} + \frac{1}{k^2} u_{[A} a_{B]}, \quad (2.13)$$

where  $h_{AB}$  is the projector onto the space transverse to the velocity field:

$$h_{AB} = \frac{u_A u_B}{k^2} + \eta_{AB} \quad (2.14)$$

(remember we work in an orthonormal Cartan mobile frame — metric displayed in (2.8)). The above vectors are transverse, whereas the tensors are transverse and traceless.

The Weyl connection  $A$  enters the Weyl covariant derivative  $\mathcal{D}_A$  acting on a weight- $w$  tensor as e.g. a scalar  $\Phi$ :

$$\mathcal{D}_A \Phi = e_A(\Phi) + w A_A \Phi, \quad (2.15)$$

or a form  $v_A$ :

$$\mathcal{D}_B v_A = \nabla_B v_A + w A_B v_A + A_A v_B - \eta_{AB} A^C v_C. \quad (2.16)$$

The resulting tensors have weight  $w + 1$ .<sup>11</sup> The form field  $u$  has weight  $-1$  i.e.  $u_A$  are weight-zero, whereas  $\omega_{AB}$  and  $\sigma_{AB}$  have all weight 1. The explicit form of  $A$  (2.10) is obtained by demanding

$$\mathcal{D}_A u^A = 0 \quad \text{and} \quad u^C \mathcal{D}_C u_A = 0. \quad (2.17)$$

<sup>10</sup>Our conventions for (anti-) symmetrization are:  $A_{(AB)} = \frac{1}{2} (A_{AB} + A_{BA})$  and  $A_{[AB]} = \frac{1}{2} (A_{AB} - A_{BA})$ .

<sup>11</sup>Special caution is advised in comparing the present expressions with those appearing e.g. in refs. [3, 45, 46], where a natural frame was used. When dealing with Weyl covariance in orthonormal frame, the metric components have weight zero. Hence for any tensor, covariant and contravariant components have the same weights. The coframe form elements, however, have weight  $-1$ , whereas the frame vectors have weight  $+1$ . If a weight- $w$  tensor has  $p$  contravariant and  $q$  covariant indices, its Weyl-covariant derivative reads:

$$\begin{aligned} \mathcal{D}_C K_{B\dots}^{A\dots} &= \nabla_C K_{B\dots}^{A\dots} + (w + p - q) A_C K_{B\dots}^{A\dots} \\ &+ (\eta_{CD} A^A - \delta_C^A A_D - \delta_D^A A_C) K_{B\dots}^{D\dots} + \dots \\ &- (\eta_{CB} A^D - \delta_C^D A_B - \delta_B^D A_C) K_{D\dots}^{A\dots} - \dots \end{aligned}$$

and this has now weight  $w + 1$ .

The Weyl covariant derivative is metric-compatible with effective torsion:

$$\mathcal{D}_C \eta_{AB} = 0, \tag{2.18}$$

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) \Phi = w \Phi F_{AB}, \tag{2.19}$$

where

$$F = \frac{1}{2} F_{AB} \theta^A \wedge \theta^B = dA \tag{2.20}$$

is Weyl-invariant ( $F_{AB}$  are weight-2). Metric compatibility and (2.17) imply

$$u^C \mathcal{D}_C h_{AB} = 0, \tag{2.21}$$

inferring that the operator  $u^C \mathcal{D}_C$  respects transversality.

Commuting the Weyl-covariant derivatives acting on vectors, one defines the Weyl covariant Riemann tensor

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) V^C = \mathcal{R}^C{}_{DAB} V^D + (w + 1) V^C F_{AB} \tag{2.22}$$

( $V^C$  are weight- $w$  whereas  $V = V^C e_C$  has weight  $w + 1$ ) and the usual subsequent quantities. In three (boundary) spacetime dimensions, the covariant Ricci and the scalar (both weight-2) curvatures read:

$$\mathcal{R}_{AB} = R_{AB} + \nabla_B A_A + A_A A_B + \eta_{AB} (\nabla_C A^C - A_C A^C) - F_{AB}, \tag{2.23}$$

$$\mathcal{R} = R + 4 \nabla_A A^A - 2 A_A A^A, \tag{2.24}$$

where  $R_{AB}$  is the Ricci tensor of the boundary Levi-Civita connection and  $R$  the corresponding scalar curvature. The Weyl-invariant Schouten tensor is

$$\mathcal{S}_{AB} = \mathcal{R}_{AB} - \frac{1}{4} \mathcal{R} \eta_{AB} = R_{AB} - \frac{1}{4} R \eta_{AB} + \nabla_B A_A + A_A A_B - \frac{1}{2} A_C A^C \eta_{AB} - F_{AB}. \tag{2.25}$$

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2} \omega_{AB} dx^A \wedge dx^B = \frac{1}{2} \left( du + \frac{1}{k^2} u \wedge a \right), \tag{2.26}$$

as well as the Hodge dual of this form, which is proportional to  $u$ :

$$k \gamma u = \star \omega \quad \Leftrightarrow \quad k \gamma u_A = \frac{1}{2} \epsilon_{ABC} \omega^{BC}. \tag{2.27}$$

In this expression  $\gamma$  is a scalar of weight 1.

In three spacetime dimensions and in the presence of a vector field  $u$ , one naturally defines a fully antisymmetric two-index tensor:<sup>12</sup>

$$\hat{\eta}_{AB} = -\frac{u^C}{k} \epsilon_{CAB}, \tag{2.28}$$

obeying

$$\hat{\eta}_{AC} \hat{\eta}_B{}^C = h_{AB}, \quad \hat{\eta}^{AB} \hat{\eta}_{AB} = 2. \tag{2.29}$$

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<sup>12</sup>This hatted two-index tensor should not be confused with Minkowski metric.

With this tensor the vorticity reads:

$$\omega_{AB} = k^2 \gamma \hat{\eta}_{AB}. \quad (2.30)$$

The two-index tensor  $\hat{\eta}_{AB}$  defines a duality map within the space of symmetric, transverse (with respect to  $u$ ) and traceless tensors. If  $V^A$  is transverse, so is

$$*V^A = \hat{\eta}^B{}_A V_B. \quad (2.31)$$

Similarly with a symmetric, transverse and traceless tensor  $W_{AB}$ :

$$*W_{AB} = \hat{\eta}^C{}_A W_{CB} \quad (2.32)$$

is symmetric, transverse and traceless.

**The energy-momentum tensor and the Cotton tensor.** Given a normalized congruence  $\|u\|^2 = -k^2$  we can decompose the energy-momentum tensor as in hydrodynamics:

$$T_{AB} = (\varepsilon + p) \frac{u^A u_B}{k^2} + p \eta_{AB} + \tau_{AB} + \frac{u^A q_B}{k^2} + \frac{u^B q_A}{k^2}, \quad (2.33)$$

where

$$\varepsilon = \frac{1}{k^2} T_{AB} u^A u^B \quad (2.34)$$

is the energy density and  $p$  the analogue of a perfect stress. The symmetric viscous stress tensor  $\tau_{AB}$  and the heat current  $q_A$  are purely transverse:

$$u^A \tau_{AB} = 0, \quad u^A q_A = 0, \quad q_B = -\varepsilon u_B - u^A T_{AB}. \quad (2.35)$$

In three dimensions, a conformal energy-momentum tensor has weight-1 covariant components in the coordinate basis, and weight-3 components in the orthonormal frame. Consequently, the pressure and energy density, the heat-current  $q_A$  and the viscous stress tensor  $\tau_{AB}$  have all weight 3. Furthermore, since the splitting of the stress tensor into  $p$  and  $\tau_{AB}$  is arbitrary, we choose to implement the absence of trace as

$$\varepsilon = 2p, \quad \tau_A{}^A = 0. \quad (2.36)$$

Due to the absence of trace, the conservation equation (2.1) can be traded for

$$\mathcal{D}_C T^C{}_B = 0. \quad (2.37)$$

In the gauge under consideration, the energy-momentum tensor comes along with the boundary Cotton tensor. They both enter the bulk metric, playing dual, electric versus magnetic, roles in various instances, as e.g. in the bulk Weyl tensor. The Cotton tensor is generically a three-index tensor with mixed symmetries.<sup>13</sup> In three dimensions, which is

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<sup>13</sup>From the bulk viewpoint, the boundary energy-momentum and Cotton tensors play dual roles. Notice that the energy-momentum tensor in (2.33) has an extra factor of  $k$  with respect to the Cotton tensor in (2.40), due to their different dimensions.

the case for our boundary geometry, the Cotton tensor can be dualized into a two-index, symmetric and traceless tensor:<sup>14</sup>

$$C_{AB} = \epsilon_A^{CD} \mathcal{D}_C (F_{BD} + \mathcal{S}_{BD}) = \epsilon_A^{CD} \nabla_C \left( R_{BD} - \frac{R}{4} \eta_{BD} \right), \quad (2.38)$$

where we recall that  $F_{BD}$  and  $\mathcal{S}_{BD}$  are respectively the Weyl curvature and the Weyl-covariant Schouten tensor defined in (2.20) and (2.25). The Cotton tensor  $C_{AB} \theta^A \theta^B$  is Weyl-covariant of weight 1, and is *identically* conserved:

$$\mathcal{D}_C C^C{}_B = \nabla_C C^C{}_B \equiv 0, \quad (2.39)$$

sharing thereby all properties of the energy-momentum tensor.

Following (2.33) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the congruence  $u$ :

$$\frac{1}{k} C_{AB} = \frac{3c}{2} \frac{u_A u_B}{k^2} + \frac{c}{2} \eta_{AB} - \frac{c_{AB}}{k^2} + \frac{u_{ACB}}{k^2} + \frac{u_{BCA}}{k^2}. \quad (2.40)$$

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$c = \frac{1}{k^3} C_{AB} u^A u^B, \quad (2.41)$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor*  $c_{AB}$  and the *Cotton current*  $c_A$  (also weight-3) are purely transverse:

$$c_A{}^A = 0, \quad u^A c_{AB} = 0, \quad u^A c_A = 0, \quad (2.42)$$

and obey

$$c_{AB} = -k h^C{}_A h^D{}_B C_{CD} + \frac{ck^2}{2} h_{AB}, \quad c_B = -c u_B - \frac{u^A c_{AB}}{k}. \quad (2.43)$$

One can use the definition (2.38) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$c = \frac{1}{k^2} u^B \hat{\eta}^{DC} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}), \quad (2.44)$$

$$c_B = \hat{\eta}^{CD} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}) - c u_B, \quad (2.45)$$

$$c_{AB} = -h^E{}_A \left( k \epsilon_B{}^{CD} - u_B \hat{\eta}^{CD} \right) \mathcal{D}_C (\mathcal{S}_{ED} + F_{ED}) + \frac{ck^2}{2} h_{AB}. \quad (2.46)$$

**Solving Einstein's equations.** Einstein's equations are<sup>15</sup>

$$\mathcal{E}_{MN} \equiv R_{MN}^{\text{bulk}} - \frac{1}{2} R^{\text{bulk}} g_{MN}^{\text{bulk}} - 3k^2 g_{MN}^{\text{bulk}} = 0, \quad (2.47)$$

and we must probe them in the covariant Newman-Unti gauge. Assuming a boundary metric given in (2.8), the bulk line element (2.7) reads:

$$\begin{aligned} ds_{\text{bulk}}^2 &= 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + r \mathcal{C}_{AB} \theta^A \theta^B + \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B \\ &+ \sum_{s=1}^{\infty} \frac{1}{r^s} \left( f_{(s)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(s)A} \theta^A + f_{(s)AB} \theta^A \theta^B \right), \end{aligned} \quad (2.48)$$

<sup>14</sup>We use a plain font for the Cotton  $C_{AB}$  versus a curly font for the shear  $\mathcal{C}_{AB}$ .

<sup>15</sup>We use  $M, N, \dots \in \{r, \text{boundary}\}$  as bulk indices.



where all boundary tensors are now defined in the orthonormal frame at hand.<sup>16</sup> The summation over  $A$  and  $B$  in the last terms of (2.48) is actually reduced to a summation over the transverse components  $a$  and  $b$  thanks to the transversality of  $f_{(s)A}$  and  $f_{(s)AB}$  with respect to the velocity field (2.9).

We will limit here the analysis to the order  $1/r^2$ , which is sufficient for illustrating accurately later the asymptotically flat pattern.

**Order  $r$**  The important output here is that the Bondi shear  $\mathcal{C}_{AB}$  is not free, but settled by the shear of the congruence  $u$ , which is of geometric nature:

$$k^2 \mathcal{C}_{AB} = -2\sigma_{AB}. \quad (2.49)$$

On shell, the Bondi shear is thus manifestly *traceless and transverse* with respect to  $u$ . Anticipating the usage of the present formalism in describing general solutions of vacuum Einstein's equations, we also introduce a *news tensor* (similarly defined in arbitrary dimension). As opposed to the usual definitions, the present tensor is *boundary-covariant, Weyl-invariant, symmetric, traceless and transverse*:

$$\mathcal{N}_{AB} = u^C \mathcal{D}_C \mathcal{C}_{AB}. \quad (2.50)$$

Equation (2.49) will be assumed when moving to the next orders.

**Order 1** Unsurprisingly from the Fefferman-Graham experience, we learn that  $\mathcal{F}_{AB}$  is related to the boundary Weyl-invariant Schouten tensor displayed in eq. (2.25):<sup>17</sup>

$$\begin{aligned} \mathcal{F}_{AB} &= 2u^C \left( \mathcal{L}_{C(A} + F_{C(A)} \right) u_{B)} + \mathcal{D}_A u^C \mathcal{D}_B u^C, \\ &= 2u_{(A} \mathcal{D}_C \left( \sigma_{B)}^C + \omega_{B)}^C \right) - \frac{\mathcal{R}}{2} u_A u_B + \left( \sigma^2 + k^4 \gamma^2 \right) h_{AB} + 2\omega_{(A}^C \sigma_{B)C}, \end{aligned} \quad (2.51)$$

where

$$\gamma^2 = \frac{1}{2k^4} \omega_{AB} \omega^{AB}, \quad \sigma^2 = \frac{1}{2} \sigma_{AB} \sigma^{AB} \quad (2.52)$$

( $\gamma$  was defined alternatively in eq. (2.27)). At this stage, the only independent and free data are those defining the boundary geometry (as stressed in (2.9), the congruence  $u$  is aligned with the observers at rest with respect to (2.8)).

**Orders  $1/r$  and  $1/r^2$**  At order  $1/r$  new information is expected to come up in the form of a boundary conformal energy-momentum tensor. In contrast with the Fefferman-Graham gauge, the energy-momentum enters through its decomposition with respect to the congruence  $u$ , i.e.  $\varepsilon$ ,  $q_A$  and  $\tau_{AB}$ , rather than  $T_{AB}$ . Furthermore, it comes accompanied with the transverse-dual of the Cotton current and stress,  $*c_A$  and  $*c_{AB}$ , see eqs. (2.31), (2.32) and (2.45), (2.46) — yet another motivation to split the

<sup>16</sup>In this frame  $\mathcal{C}_{AB}$  has weight one,  $f_{(s)}$ ,  $f_{(s)A}$  and  $f_{(s)AB}$  have all weight  $s+2$  whereas  $\mathcal{F}_{AB}$  is weight-2.

<sup>17</sup>The tensor defined in (2.51) is slightly different from the analogous tensor  $S_{AB}$  introduced in [3], eq. (2.42). It contains extra shear terms. The reason is that in ref. [3], when writing (2.41), the authors wanted to stress that shear terms were present, but ultimately the shear was vanishing. The present definition accounts for all shear terms.

energy-momentum tensor as discussed earlier. This trait is new, both compared to the Fefferman-Graham gauge, where the Cotton tensor does not appear explicitly at any order, and with respect to standard Bondi or Newman-Unti gauges, where it is present in disguise.<sup>18</sup>

The functions to be determined are  $f_{(1)}$ ,  $f_{(1)A}$  and  $f_{(1)AB}$ , which must have conformal weight 3. This leaves little freedom, given the available tensors. We find:

$$f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)A} \theta^A + f_{(1)AB} \theta^A \theta^B = \frac{8\pi G}{k^4} \left( \varepsilon u^2 + \frac{4}{3} u \Delta q + \frac{2k^2}{3} \Delta \tau \right) \quad (2.53)$$

with  $\Delta q = \Delta q_A \theta^A$  and  $\Delta \tau = \Delta \tau_{AB} \theta^A \theta^B$  defined as

$$\Delta q_A = q_A - \frac{1}{8\pi G} *c_A, \quad \Delta \tau_{AB} = \tau_{AB} + \frac{1}{8\pi G k^2} *c_{AB}. \quad (2.54)$$

The functions  $\varepsilon$ ,  $q_A$  and  $\tau_{AB}$ , which merely parameterize the line element at this stage, can be packaged in a symmetric and traceless tensor  $T_{AB}$  as in (2.33), (2.36) and, as we shall see shortly in (2.60), Einstein's equations demand the conservation of  $T_{AB}$ , thus completing its identification as the boundary energy-momentum tensor as in the Fefferman-Graham gauge.

We now ought to focus on the  $1/r^2$  contribution to the line element (2.48), i.e., on

$$f_{(2)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(2)A} \theta^A + f_{(2)AB} \theta^A \theta^B, \quad (2.55)$$

where  $f_{(2)}$ ,  $f_{(2)A}$  and  $f_{(2)AB}$  must have conformal weight 4. The analogy with the Fefferman-Graham expansion suggests that no new free boundary functions should appear without spoiling Einstein's equations. Indeed, upon imposing (2.49) and (2.51), one finds

$$\begin{cases} \mathcal{E}_{rr} &= -\frac{3}{r^5} \eta_{AB} f_{(1)}^{AB} - 6 \left( \eta_{AB} f_{(2)}^{AB} + \frac{3}{2k^2} \sigma_{AB} f_{(1)}^{AB} \right) \frac{1}{r^6} + \mathcal{O} \left( \frac{1}{r^7} \right) \\ k \mathcal{E}_{r\hat{0}} &= \left( -f_{(2)} - 2k^2 \eta_{AB} f_{(2)}^{AB} + \frac{1}{2} h_{AB} \mathcal{D}^A f_{(1)}^B - \frac{5}{2} \sigma_{AB} f_{(1)}^{AB} + c\gamma \right) \frac{1}{r^4} + \mathcal{O} \left( \frac{1}{r^5} \right) \\ \mathcal{E}_{ra} &= \left( 2f_{(2)a} - \frac{3}{2} h_{aB} \mathcal{D}_C f_{(1)}^{BC} + \frac{1}{k^2} (\sigma_{aB} + 4\omega_{aB}) f_{(1)}^B \right) \frac{1}{r^4} + \mathcal{O} \left( \frac{1}{r^5} \right) \\ \mathcal{E}^{ab} &= \left( -f_{(2)} h^{ab} + c\gamma h^{ab} + 4\omega_C^{(a} f_{(1)}^{b)C} + 2k^2 \hat{\eta}_C^a \hat{\eta}_D^b f_{(2)}^{CD} - 2u^C \mathcal{D}_C f_{(1)}^{ab} \right. \\ &\quad \left. + \hat{\eta}_C^a \hat{\eta}_D^b \mathcal{D}^{(C} f_{(1)}^{D)} + \frac{1}{k^2} (c \hat{\eta}_C^a \sigma^{Cb} - f_{(1)} \sigma^{ab}) + 4\sigma_C^{(a} f_{(1)}^{b)C} \right) \frac{1}{r^2} + \mathcal{O} \left( \frac{1}{r^3} \right) \end{cases} \quad (2.56)$$

for the often referred to as constraint Einstein's equations. These equations fix algebraically all terms at the  $1/r^2$  order in the expansion of the bulk metric, thus confirming the absence of any new free function. When rewritten in terms of the basic

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<sup>18</sup>One could not stress enough the profound versatility of the boundary Cotton tensor. Together with the boundary energy-momentum tensor, they control the asymptotic behaviour of the bulk Weyl tensor, the electric versus magnetic gravitational characteristics, the duality issues, and are natural ingredients in Newman-Penrose formalism. In the flat instance and in the current gauge, the Cotton tensor contributes to the gravitational radiation along with the Bondi shear. A recent presentation of some of these features is available in ref. [25].

quantities parameterizing the space of solutions, the three coefficients in (2.55) read:

$$f_{(2)} = \frac{8\pi G}{3k^2} \left( \sigma_{CD} \Delta \tau^{CD} + \mathcal{D}_C \Delta q^C \right) + c\gamma, \quad (2.57)$$

$$f_{(2)A} = -\frac{8\pi G}{3k^4} \sigma_{AC} \Delta q^C + \frac{4\pi G}{k^2} \left( h_{AC} \mathcal{D}_D \Delta \tau^{CD} + \frac{8}{3} \gamma * \Delta q_A \right), \quad (2.58)$$

$$f_{(2)AB} = -\frac{4\pi G}{k^4} \left( \frac{4}{3} u^C \mathcal{D}_C \Delta \tau_{AB} + \frac{2}{3} h_{AC} h_{BD} \mathcal{D}^{(C} \Delta q^{D)} - \frac{1}{3} h_{AB} h^{CD} \mathcal{D}_C \Delta q_D \right. \\ \left. + 2\sigma_{(A}{}^C \Delta \tau_{B)C} \right) - \frac{1}{2k^4} (8\pi G \varepsilon \sigma_{AB} - c * \sigma_{AB}) + \frac{32\pi G}{3k^2} \gamma * \Delta \tau_{AB}. \quad (2.59)$$

These expressions contain all possible combinations of the shear and of the vorticity together with adequately projected Weyl covariant derivatives of the energy-momentum and Cotton tensors,<sup>19</sup> carrying the right tensorial structure and conformal weight. Substituting eqs. (2.57), (2.58), (2.59) into the remaining Einstein's equations (2.47) one obtains:

$$\frac{k}{8\pi G} \mathcal{E}_{\hat{0}\hat{0}} = \frac{1}{r^2} \mathcal{D}_B T^B{}_{\hat{0}} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \frac{k}{8\pi G} \mathcal{E}_{\hat{0}a} = \frac{1}{r^2} \mathcal{D}_B T^B{}_a + \mathcal{O}\left(\frac{1}{r^3}\right) \quad (2.60)$$

(since  $T_{AB}$  is traceless,  $\mathcal{D}_A \equiv \nabla_A$ , the Levi-Civita boundary connection for the frame metric  $\eta_{AB}$ ). The omitted terms contain the tensors  $f_{(3)}$ ,  $f_{(3)A}$  and  $f_{(3)AB}$ . This confirms that no additional constraints are imposed on the quantities parameterizing the solution space identified at the previous orders, i.e., the velocity field, the boundary metric (frame in the present formalism) and the boundary energy-momentum tensor.

**Higher orders and possible resummation** The above pattern can be repeated ad nauseam at the cost of a substantial growth in admissible terms. The third order would be interesting as it is expected to host the Newman-Penrose charges in the flat limit. This is beyond our motivations, but raises the issue of resummability under conditions of the series (2.7). This question is usually immaterial in Bondi or Newman-Unti gauges, where due to the absence of boundary vorticity<sup>20</sup> simple solutions such as Kerr's are embodied in the form of infinite series. In the covariant Newman-Unti gauge, the explicit appearance of the boundary Cotton tensor allows to tune the bulk Weyl tensor and select algebraically special Einstein spacetimes, for which the series is resumable. This is achieved by imposing

$$\sigma_{AB} = 0, \quad \Delta q_A = 0, \quad \Delta \tau_{AB} = 0, \quad (2.61)$$

which implies that

$$f_{(s)A} = 0, \quad f_{(s)AB} = 0 \quad (2.62)$$

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<sup>19</sup>The covariant Newman-Unti gauge has often been referred to as the *derivative-expansion gauge* for this reason. This was borrowed from the original fluid/gravity literature, where the derivative expansion was inspired by the fluid constitutive relations.

<sup>20</sup>An explicit realization of the boundary frame of Newman-Unti gauge is displayed in (A.41), (A.42) with  $\Delta_i = 0$ , as mentioned earlier in this section. The boundary vorticity always vanishes then as it is proven by comparing (A.45) with (A.46).

and

$$f_{(2s+1)} = (-)^s 8\pi G \varepsilon \gamma^{2s}, \quad f_{(2s+2)} = (-)^s c \gamma^{2s+1}. \quad (2.63)$$

The boundary metric is still a free variable, but only the energy density  $\varepsilon(x)$  remains from the energy-momentum tensor, whose heat current and stress are fixed by those of the Cotton:

$$q_A = \frac{1}{8\pi G} *c_A, \quad \tau_{AB} = -\frac{1}{8\pi G k^2} *c_{AB}. \quad (2.64)$$

As a consequence, assuming that (2.1) is satisfied, one finds

$$ds_{\text{res. Einstein}}^2 = 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + \frac{\mathcal{F}}{k^4} + \frac{u^2}{k^4 \rho^2} (8\pi G \varepsilon r + c\gamma) \quad (2.65)$$

with

$$\rho^2 = r^2 + \gamma^2 \quad (2.66)$$

and  $\mathcal{F} = \mathcal{F}_{AB} \theta^A \theta^B$  given in (2.51) imposing zero shear. The Petrov analysis of (2.65) has been discussed in refs. [3, 20].

### 3 The flat avatars

**First things first.** Handling the flat limit is a triptych. At the first place stands the boundary geometry, which becomes Carrollian as the time-like conformal boundary of asymptotically anti-de Sitter spacetimes is traded for the null infinity of their asymptotically flat relatives. Secondly, the energy-momentum tensor should be expanded in Laurent series with respect to  $k^2$  and embrace all extra degrees of freedom of the flat solution space. Finally comes the bulk line element that should remain finite in the zero- $k$  limit, imposing to this end constraints and evolution equations on the functions defining the solution space, besides the Carrollian limit of the already available eqs. (2.1).

Given the relativistic boundary metric and the velocity field, (2.8) and (2.9), the starting point of our analysis is the bulk line element (2.48), which we reproduce here bearing in mind the transversality properties:

$$ds_{\text{bulk}}^2 = 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + r \mathcal{C}_{ab} \theta^a \theta^b + \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B + \sum_{s=1}^{\infty} \frac{1}{r^s} \left( f_{(s)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(s)a} \theta^a + f_{(s)ab} \theta^a \theta^b \right). \quad (3.1)$$

The Carrollian limit of the boundary geometry is reached as follows:<sup>21</sup>

$$\mu = \lim_{k \rightarrow 0} \frac{u}{k^2} = -\lim_{k \rightarrow 0} \frac{\theta^{\hat{0}}}{k}, \quad \hat{\theta}^a = \lim_{k \rightarrow 0} \theta^a, \quad (3.2)$$

so that the Carrollian degenerate metric spells

$$d\ell^2 = \lim_{k \rightarrow 0} ds^2 = \delta_{ab} \hat{\theta}^a \hat{\theta}^b. \quad (3.3)$$

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<sup>21</sup>Carrollian quantities will often be distinguished with hats. However, in order to avoid cluttering of indices and symbols, we do make the distinction amongst relativistic and Carrollian attributes, only when it is necessary. This will not be the case e.g. for the Bondi shear and news.

For the frame vectors, the prescription is

$$\mathbf{v} = \lim_{k \rightarrow 0} \mathbf{u} = \lim_{k \rightarrow 0} k e_{\hat{0}}, \quad \hat{e}_a = \lim_{k \rightarrow 0} e_a. \quad (3.4)$$

It should be stressed that the limit may not be necessary, because the parameterization of the diads  $\theta^a$  in terms of the natural-coframe components  $dx^\mu$  can be chosen so as not to depend on  $k$ , and that  $\hat{\theta}^0$  could simply be  $dx^0 = k du$  in which case  $\mu = -du$  (some further examples are displayed in appendix A, eqs. (A.3), (A.4), (A.7) and eqs. (A.41), (A.42), (A.43)). This will be definitely our viewpoint here.

The kernel of the degenerate metric (3.3) is the *field of observers*  $\mathbf{v}$ , and  $\mu$  is its dual *clock form* embracing also the Ehresmann connection, as explained in appendix A. All these obey

$$\mu(\mathbf{v}) = -1, \quad \hat{\theta}^a(\hat{e}_b) = \delta_b^a, \quad \hat{\theta}^a(\mathbf{v}) = 0, \quad \mu(\hat{e}_a) = 0. \quad (3.5)$$

The Carrollian geometric data are part of the solution space of Ricci-flat spacetimes in the flat covariant Newman-Unti gauge. Compared to the standard flat Newman-Unti gauge, the extra piece of data is the clock form  $\mu$ , which echoes the velocity congruence of the AdS relative. More accurately, the additional piece of information carried by the covariant Newman-Unti gauge is the boundary vorticity  $*\varpi$ , as discussed in appendix A, footnote 36.

The vanishing- $k$  limit the AdS-boundary Weyl connection  $A$  is readily reached due to its  $k$ -independence. As described explicitly in appendix A, one effortlessly expresses  $A$  in Carrollian terms, eq. (A.53):

$$A = \varphi_a \hat{\theta}^a - \frac{\theta}{2} \mu \quad (3.6)$$

with  $\varphi_a$  and  $\theta$  given in (A.13) or (A.46) and (A.19). Therefore, the first two terms in (3.1) have a well-defined limit without the need of imposing Einstein's equations.

The next term in (3.1) plays an essential role in gravitational physics. Indeed, Einstein's equation (2.49), reproduced here for the spatial components — the only non-zero due transversality combined with our choice of congruence  $\mathbf{u}$ ,

$$k^2 \mathcal{C}_{ab} = -2\sigma_{ab}, \quad (3.7)$$

implies that  $\sigma_{ab} = 0$  at vanishing  $k$ . As explained in eq. (A.52), the latter translates in Carrollian terms into

$$\xi_{ab} = 0, \quad (3.8)$$

where  $\xi_{ab}$  is defined in (A.19) as the traceless component of the extrinsic curvature. On the one hand, the geometrical shear  $\xi_{ab}$  of the boundary Carrollian geometry *must vanish* — an extrinsic-curvature condition for the conformal null boundary. On the other hand, the dynamical shear  $\mathcal{C}_{ab}$  *is free* and carries information on the gravitational radiation. No equation will constrain it or make it evolve, but it will source the evolution of other degrees of freedom.

In summary, till the order  $r$ , the Ricci-flat bulk metric reads:

$$ds_{\text{Ricci-flat}}^2 \Big|_r = \mu \left[ 2dr + r \left( 2\varphi_a \hat{\theta}^a - \theta \mu \right) \right] + r^2 d\ell^2 + r \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b, \quad (3.9)$$

where  $\mathcal{C}_{ab}(u, \mathbf{x})$  is an arbitrary traceless Carrollian tensor, referred to as the Bondi shear (cf. the footnote 7). The Bondi news is another traceless Carrollian tensor obtained as the Carrollian limit of eq. (2.50):

$$\hat{\mathcal{N}}_{ab} = \hat{\mathcal{D}}_{\nu} \mathcal{C}_{ab}. \quad (3.10)$$

In order to pursue the study of the next orders, we must be careful with the zero- $k$  limit. Both in the line element and in the conservation equations (2.1), the geometric shear  $\sigma_{ab} = \xi_{ab}$  must be substituted by  $-\frac{k^2}{2} \mathcal{C}_{ab}$  on account of eq. (3.7) before the limit is taken. Often this won't have any effect and the term in consideration will drop. Sometimes, however, due to the presence of negative powers of  $k$ , finite terms will survive or divergences will impose further requirements.

The first and simplest application of the rule just stated concerns the order-1 term  $\frac{\mathcal{F}}{k^4} = \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B$ . Expressing (2.51) in Carrollian terms we find:

$$\begin{aligned} \frac{\mathcal{F}}{k^4} &= \frac{\xi^2}{k^4} d\ell^2 + \frac{1}{k^2} \left( 3\xi^2 \mu^2 + 2\hat{\mathcal{D}}_b \xi^b{}_a \mu \hat{\theta}^a - 2 * \varpi * \xi_{ab} \hat{\theta}^a \hat{\theta}^b \right) \\ &\quad + * \varpi^2 d\ell^2 - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a - \hat{\mathcal{K}} \mu^2 - 5k^2 * \varpi^2 \mu^2 \\ &= \left( \frac{\mathcal{C}^2}{4} + * \varpi^2 \right) d\ell^2 - \hat{\mathcal{K}} \mu^2 - \hat{\mathcal{D}}_b \mathcal{C}^b{}_a \mu \hat{\theta}^a - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a \\ &\quad + * \varpi * \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b + k^2 \left( \frac{3}{4} \mathcal{C}^2 - 5 * \varpi^2 \right) \mu^2 \end{aligned} \quad (3.11)$$

with  $\mathcal{C}^2 = \frac{1}{2} \mathcal{C}^{ab} \mathcal{C}_{ab}$  and quantities like  $\xi^2$ ,  $*\varpi$ ,  $\hat{\mathcal{K}}$  defined in appendix B. The asterisk stands for the relativistic congruence-transverse or Carrollian-basis duality introduced in eqs. (2.28), (2.29), (2.31), (2.32) or (B.1), (B.2). Some terms drop in the zero- $k$  limit but no divergence occurs and *we are left with a piece in the line element, which now contains explicitly the Bondi shear*:

$$\lim_{k \rightarrow 0} \frac{\mathcal{F}}{k^4} = \left( \frac{\mathcal{C}^2}{4} + * \varpi^2 \right) d\ell^2 - \hat{\mathcal{K}} \mu^2 - \hat{\mathcal{D}}_b \mathcal{C}^b{}_a \mu \hat{\theta}^a - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a + * \varpi * \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b. \quad (3.12)$$

Before moving on to the next order, which uncovers the method of expanding the anti-de Sitter energy-momentum tensor as a mean of reconstructing Ricci-flat spacetimes, it is fair to give credit to the authors of ref. [23], where the pioneering idea of substituting the Bondi for the geometric shear with the accompanying power of the cosmological constant was initiated.

**Order  $1/r$  and the advent of the energy-momentum tensor.** Let us assume that in the course of the bulk flat limit, the boundary energy-momentum tensor is analytic in  $k^2$ . It can thus be represented as a Laurent series about  $k = 0$ :

$$\varepsilon = \sum_{n \in \mathbb{Z}} k^{2n} \varepsilon_{(n)}, \quad (3.13)$$

$$q^a = \sum_{n \geq 2} \frac{\zeta_{(n)}^a}{k^{2n}} + \frac{\zeta^a}{k^2} + Q^a + k^2 \pi^a + \sum_{n \geq 2} k^{2n} \pi_{(n)}^a, \quad (3.14)$$

$$\tau^{ab} = - \sum_{n \geq 3} \frac{\zeta_{(n)}^{ab}}{k^{2n}} - \frac{\zeta^{ab}}{k^4} - \frac{\Sigma^{ab}}{k^2} - \Xi^{ab} - k^2 E^{ab} - \sum_{n \geq 2} k^{2n} E_{(n)}^{ab}. \quad (3.15)$$

Each function in these series (some have been singled out for reasons that will be clarified later) is a Carrollian tensor (scalar, vector, or symmetric and traceless two-tensor) that is possibly one of the boundary degrees of freedom, which we call *Chthonian* to recall they encode the asymptotically flat Einstein dynamics probing the bulk metric *in depth* from the boundary. These tensors are expected to obey flux-balance equations, which are Carrollian avatars of vacuum Einstein's equations, and that we will attain using anti-de Sitter dynamics and imposing a regular behaviour at zero  $k$ .

As an introductory statement, it is important to stress that we have no proof for the proclaimed analyticity. The latter is a working framework, resulting in a consistent description of asymptotically flat spacetimes, and this end justifies the means. The rules are simple: insert (3.13), (3.14), (3.15) in the line element (3.1) at each order, and impose regularity at  $k = 0$  after trading  $\xi_{ab}$  for  $-\frac{k^2}{2}\mathcal{C}_{ab}$ . This process starts with  $1/r$ , since this is the first term sensitive to the energy-momentum tensor, but the substitution of  $\mathcal{C}_{ab}$  will be performed systematically, in the line element, in Einstein's equations, or in the further definition of the complex mass aspect, without raising any order ambiguity.

At order  $1/r$  we should probe (2.53), which spells

$$f_{(1)}\frac{u^2}{k^4} + 2\frac{u}{k^2}f_{(1)a}\theta^a + f_{(1)ab}\theta^a\theta^b = 8\pi G \left( \varepsilon\mu^2 + \frac{4}{3}\mu\frac{\Delta q_a}{k^2}\hat{\theta}^a + \frac{2}{3}\frac{\Delta\tau_{ab}}{k^2}\hat{\theta}^a\hat{\theta}^b \right) \quad (3.16)$$

with  $\varepsilon$  given in (3.13) and

$$\begin{aligned} \frac{\Delta q^a}{k^2} &= \sum_{n \geq 2} \frac{\zeta_{(n)}^a}{k^{2n+2}} + \frac{1}{k^4} \left( \zeta^a - \frac{*z^a}{8\pi G} \right) + \frac{1}{k^2} \left( Q^a - \frac{*X^a}{8\pi G} \right) + \left( \pi^a - \frac{*v^a}{8\pi G} \right) \\ &\quad + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{\Delta\tau^{ab}}{k^2} &= - \sum_{n \geq 3} \frac{\zeta_{(n)}^{ab}}{k^{2n+2}} - \frac{1}{k^6} \left( \zeta^{ab} - \frac{*Z^{ab}}{8\pi G} \right) - \frac{1}{k^4} \left( \Sigma^{ab} - \frac{*X^{ab}}{8\pi G} \right) - \frac{1}{k^2} \left( \Xi^{ab} - \frac{*Psi^{ab}}{8\pi G} \right) \\ &\quad - E^{ab} - \sum_{n \geq 2} k^{2n-2} E_{(n)}^{ab}, \end{aligned} \quad (3.18)$$

where we have used (3.14), (3.15), the definitions (2.54) of  $\Delta q^a$  and  $\Delta\tau^{ab}$ , as well as the Carrollian Cotton tensors  $z^a$ ,  $\chi^a$ ,  $\psi^a$ ,  $Z^{ab}$ ,  $X^{ab}$ ,  $\Psi^{ab}$  displayed in (B.13), (B.14). Finiteness in the flat limit sets up two sorts of requirements on the Carrollian descendants of the energy-momentum tensor.

- Infinite subsets of Laurent coefficients are required to vanish:

$$\begin{cases} \varepsilon_{(n)} = 0 & \forall n < 0 \\ \zeta_{(n)}^a = 0 & \forall n \geq 2 \\ \zeta_{(n)}^{ab} = 0 & \forall n \geq 3; \end{cases} \quad (3.19)$$

- Five Laurent coefficients are locked in terms of the Carroll Cotton tensors defined in (B.17), (B.18), (B.20), (B.21), (B.22):

$$\zeta^a = \frac{*z^a}{8\pi G}, \quad Q^a = \frac{*X^a}{8\pi G}, \quad \zeta^{ab} = \frac{*Z^{ab}}{8\pi G}, \quad \Sigma^{ab} = \frac{*X^{ab}}{8\pi G}, \quad \Xi^{ab} = \frac{*Psi^{ab}}{8\pi G}. \quad (3.20)$$

Hence a finite subset of energy-momentum Carrollian descendants *are not independent but are instead of geometric nature, determined by the boundary Carroll structure via its Cotton tensor.*

No more constraints show on the Chthonian degrees of freedom at  $1/r$  order.

Defining

$$N^a = *\psi^a - 8\pi G\pi^a, \tag{3.21}$$

we recast the order- $1/r$  term (3.16) in the flat limit as:

$$\begin{aligned} \lim_{k \rightarrow 0} \left( f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)a} \theta^a + f_{(1)ab} \theta^a \theta^b \right) &= 8\pi G \varepsilon_{(0)} \mu^2 - \frac{4}{3} \mu N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \\ &\equiv \hat{f}_{(1)} \mu^2 + 2\mu \hat{f}_{(1)a} \hat{\theta}^a + \hat{f}_{(1)ab} \hat{\theta}^a \hat{\theta}^b. \end{aligned} \tag{3.22}$$

The latter expression calls for two remarks. Firstly, the Carrollian tensors  $\varepsilon_{(n \geq 1)}$ ,  $\pi_{(n \geq 2)}^a$  and  $E_{(n \geq 2)}^{ab}$  are absent. We should refrain from interpreting this as a sign that those aren't genuine degrees of freedom. Some of them ought to appear in the line element in the next orders and therefore participate in the dynamics. Only when one is guaranteed that a Laurent coefficient is absent from the line element at any order, can we declare it is irrelevant and set it consistently to zero. The order- $1/r^2$  analysis will significantly underpin this statement.

Secondly comes an important question: *what is the dynamics of the boundary degrees of freedom  $\varepsilon_{(0)}$ ,  $N_a$  and  $E_{ab}$  that remain in the  $1/r$  term of the bulk line element?* Ensuing our philosophy, this dynamics is encoded (i) in the zero- $k$  limit of anti-de Sitter Einstein's equations and (ii) in the finiteness requirement of the line element. The latter has already been exploited at the order under consideration, while the former is the energy-momentum conservation (2.1) on which we will elaborate now. Our treatment consists in the four steps summarized below.

1. In the frame at use, we consider the relativistic energy-momentum tensor conservation equations (2.1) recast in Carrollian terms as in appendix A, eqs. (A.59) and (A.60), which we redisplay here for convenience:

$$\mathcal{L} = \hat{\mathcal{D}}_v \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab} = 0, \tag{3.23}$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \varpi^{ba} + \frac{1}{k^2} \left( \hat{\mathcal{D}}_v q^a + \xi^{ab} q_b \right) = 0. \tag{3.24}$$

2. We insert in these equations the variables  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  in their expanded forms (3.13), (3.14) and (3.15), taking into account the finiteness requirements (3.19) and (3.20).
3. The requirements (3.19) and (3.20) bring the Cotton tensor inside the boundary energy-momentum conservation equations  $\mathcal{L} = 0$  and  $\mathcal{T}^a = 0$ . At this stage we must exploit the Cotton identities  $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$  and  $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$  set in appendix B — see



eqs. (B.25), (B.26), (B.27), (B.28), (B.29), (B.30), (B.31), (B.32) — and recast for our needs as:

$$\hat{\mathcal{D}}_b * \Psi^{ab} + 2 * \varpi \chi^a = \hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab}, \quad (3.25)$$

$$\hat{\mathcal{D}}_b * X^{ab} + 2 * \varpi z^a - \hat{\mathcal{D}}_v * \chi^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab}, \quad (3.26)$$

$$\hat{\mathcal{D}}_b * Z^{ab} - \hat{\mathcal{D}}_v * z^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab}. \quad (3.27)$$

With this we reach the following:

$$\begin{aligned} \mathcal{L} &= k^2 \hat{\mathcal{D}}_a \pi^a + \sum_{n \geq 2} k^{2n} \hat{\mathcal{D}}_a \pi_{(n)}^a - \xi_{ab} \left( k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) \\ &\quad + \hat{\mathcal{D}}_v \varepsilon_{(0)} + \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}_v \varepsilon_{(n)} - \frac{1}{8\pi G} \left( * \Psi^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * \chi^a \right) \\ &\quad - \frac{1}{8\pi G k^2} \left( * X^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * z^a \right) - \frac{1}{8\pi G k^4} * Z^{ab} \xi_{ab}, \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \mathcal{T}^a &= -\hat{\mathcal{D}}_b \left( k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) + 2 * \varpi \left( k^2 * \pi^a + \sum_{n \geq 2} k^{2n} * \pi_{(n)}^a \right) + \frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} \\ &\quad + \frac{1}{2} \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}^a \varepsilon_{(n)} + \hat{\mathcal{D}}_v \left( \pi^a + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a \right) + \xi^a_b \left( \pi^b + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^b \right) \\ &\quad - \frac{1}{8\pi G} \left( \hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab} \right) \\ &\quad - \frac{1}{8\pi G k^2} \left( \frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab} \right) - \frac{1}{8\pi G k^4} \left( \frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab} \right). \end{aligned} \quad (3.29)$$

4. Lastly we express the geometric Carrollian shear as  $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$  inside eqs. (3.28) and (3.29). This is a juggernaut due to the heavy presence of  $\xi_{ab}$  in the Carrollian Cotton tensors  $c_{(1)}$ ,  $c_{(2)}$ ,  $\chi^a$ ,  $z^a$ ,  $\Psi^{ab}$ ,  $X^{ab}$  and  $Z^{ab}$  — see their definitions (B.15), (B.17), (B.18), (B.20), (B.21) and (B.22). This operation regularizes the otherwise singular behaviour of the last lines in (3.28) and (3.29) at vanishing  $k$ , which instead produce a wealth of finite terms, *all rooted in the Carrollian Cotton tensor*.

The flat limit of the boundary energy-momentum conservation can now be safely taken and yields:

$$\lim_{k \rightarrow 0} \mathcal{L} = \hat{\mathcal{D}}_v \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a - \frac{1}{16\pi G} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b + \frac{1}{2} \mathcal{C}_{ab} \hat{\mathcal{D}}_v \hat{\mathcal{N}}^{ab} \right), \quad (3.30)$$

and

$$\begin{aligned}
\lim_{k \rightarrow 0} \mathcal{T}^a &= \frac{1}{2} \hat{\mathcal{D}}_b \left( \delta^{ab} \varepsilon_{(0)} + \frac{1}{8\pi G} \eta^{ab} c_{(0)} \right) + \hat{\mathcal{D}}_v \left( \pi^a - \frac{1}{8\pi G} * \psi^a \right) \\
&+ \frac{1}{16\pi G} \left[ \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) \\
&\left. + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left( \mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.31)
\end{aligned}$$

Equations (3.30) and (3.31) are one of our main achievements and deserve further discussion. We would like to insist that there is neither magic nor ambiguity in reaching them. We have followed a plain zero- $k$  limit informed about the regularity conditions (3.20), which involve the Carrollian Cotton tensor and its identities, and instructed with Einstein's equation  $\xi_{ab} = -\frac{k^2}{2} \mathcal{E}_{ab}$ . Although long and technical, the method reveals the central role of the Cotton tensor: *all terms responsible for the gravitational radiation, involving among others the shear and the news tensors, originate from the Carrollian Cotton tensors.* Because of the vanishing  $\xi_{ab}$ , only six of those remain — see appendix B:  $c_{(-1)}$ ,  $c_{(0)}$ ,  $\psi^a$ , eqs. (B.15), (B.16), and  $\chi^a$ ,  $\Psi^{ab}$ ,  $X^{ab}$  given in (B.35), (B.36), (B.37) for vanishing Carrollian shear. They obey eqs. (B.38), (B.39), (B.40), (B.41) and (B.42).<sup>22</sup> This means in particular that once the bulk flat limit is reached i.e. the boundary Carroll structure has no geometric shear,  $z^a$  and  $Z^{ab}$  vanish. The Carrollian energy-momentum tensors  $\zeta^a$  and  $\zeta^{ab}$  do also vanish by virtue of (3.20). Only  $Q^a$ ,  $\Sigma^{ab}$  and  $\Xi^{ab}$  survive, and (3.30), (3.31) lead to an alternative writing of the conservation equations:

$$\hat{\mathcal{D}}_v \varepsilon_{(0)} + \hat{\mathcal{D}}_a Q^a = \frac{1}{16\pi G} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{E}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{R}}_b + \frac{1}{2} \mathcal{E}_{ab} \hat{\mathcal{D}}_v \hat{\mathcal{N}}^{ab} \right) \quad (3.32)$$

and

$$\begin{aligned}
\frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} - \hat{\mathcal{D}}_b \Xi^{ab} + 2 * \varpi * Q^a + \hat{\mathcal{D}}_v \pi^a &= -\frac{1}{16\pi G} \left[ \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} \\
&\left. + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left( \mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.33)
\end{aligned}$$

This latter form discloses a Carrollian conservation of the type (B.33), (B.34) *with a right-hand side though*. This is thus a *flux-balance equation*, where the source is maintained by the bulk gravitational radiation encoded in the shear and the news. Notice that the above momentum  $\pi^a$  coincides with  $P^a$  in (B.34) and is dynamical, whereas the traceless

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<sup>22</sup>The Carrollian Cotton identities described in appendix B as  $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$  and  $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$  are obtained from the relativistic conservation (2.39) expressed as (B.23) and (B.24). We *must not* insert in these equations  $\xi_{ab} = -\frac{k^2}{2} \mathcal{E}_{ab}$  before taking the flat limit as they are agnostic about bulk Einstein's equations. The Cotton identities at hand are of boundary-geometric nature, and when the Carrollian shear vanishes, they just become simpler by setting  $\xi_{ab} = 0$ .

Carrollian stress  $\Xi^{ab}$  is  $-\Upsilon^{ab}$  in (B.33), (B.34), and is dictated by the Cotton due to (3.20); similarly  $Q^a$  here is the energy flux  $\Pi^a$  of (B.33), (B.34), also locked by the Cotton in (3.20).

Even in absence of Bondi shear  $\mathcal{C}_{ab}$  and news  $\hat{\mathcal{N}}_{ab}$ , the presence of a non-vanishing energy flux betrays the breaking of local Carroll boost invariance (see the end of appendix B) in the boundary Carrollian dynamics associated with Ricci-flat spacetimes. This breaking accounts for bulk gravitational radiation, which in the covariant Newman-Unti gauge does not originate solely in the news (3.10) but is also carried by the Carrollian energy flux  $\Pi^a = Q^a = \frac{1}{8\pi G} * \chi^a$ . In Robinson-Trautman spacetimes and in the present gauge, the gravitational radiation is exclusively rooted in this Carrollian Cotton descendant — see ref. [3].

Observe in passing the Carrollian Cotton identities (B.39) and (B.41), which we replicate here for convenience:

$$\hat{\mathcal{D}}_b c_{(0)} + \hat{\mathcal{D}}_a \chi^a = 0, \tag{3.34}$$

$$\frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_b \psi^a = 0. \tag{3.35}$$

They play dual roles with respect to eqs. (3.32) and (3.33), because the energy density  $\varepsilon_{(0)}$  carries information on the mass of the source, while  $c_{(0)}$  endorses its nut charge (monopole-like magnetic mass) (see e.g. [25] for a recent discussion on these electric-magnetic dual observables). The two sets of equations are dissymmetric though: eq. (3.34) for instance is driven exclusively by the Cotton vector  $\chi^a$  — as opposed to its Carroll-dual  $*\chi^a$  entering the electric-mass equation (3.32) through  $Q^a = \frac{1}{8\pi G} * \chi^a$ . Even though loss phenomena concern both the electric and the magnetic masses, as captured e.g. in eqs. (76) and (80) of [47] — see also appendix D of [48], the time evolution of the nut is not affected by  $\mathcal{C}_{ab}$  and  $\hat{\mathcal{N}}_{ab}$ , whereas that of the mass is, in line with an important distinction between these aspects raised in [49].

A useful exercise, which we will not undertake here, would be to set up a precise dictionary between the gauge at hand and the more conventional Newman-Unti or Bondi gauges, regarding the radiation observables. We can nonetheless take a few steps towards this end using the Carrollian tensor  $N^a$  introduced in (3.21), reminiscent of the Bondi *angular-momentum aspect*,<sup>23</sup> and a Bondi *mass aspect*

$$M = 4\pi G \varepsilon_{(0)} - \frac{1}{8} \mathcal{C}^{ab} \hat{\mathcal{N}}_{ab}. \tag{3.36}$$

This definition is reached from eq. (2.39) of [23] valid in anti-de Sitter, at  $k = 0$ .<sup>24</sup> What distinguishes the energy density  $4\pi G \varepsilon_{(0)}$  and the mass  $M$  is a radiative contribution.

We can attempt to define a magnetic-mass aspect starting from anti-de Sitter, where the behaviour of the bulk Weyl tensor in the gauge used here exhibits the complex-mass

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<sup>23</sup>As for the shear and the news, the physics conveyed by  $N^a$  in the covariant Newman-Unti gauge, is slightly different compared to the standard angular-momentum aspect. For the Kerr geometry, as an example, in the gauge at hand  $N^a = 0$  and the angular momentum is carried by the Carrollian vorticity, as opposed to plain Newman-Unti gauge, where the boundary vorticity is absent (see eq. (A.20)). This hints towards the recent progress in defining a supertranslation-invariant angular momentum and comparing the multiple routes to it (see e.g. [42, 50–53]).

<sup>24</sup>It coincides with (42) of [47] upon identifying  $\mathcal{M}$  of this reference with our  $4\pi G \varepsilon_{(0)}$ .

combination  $\tau = -c + 8\pi i G \varepsilon$  (see [22]) with  $\varepsilon$  the AdS-boundary energy density and  $c$  the Cotton scalar (longitudinal component with respect to the congruence  $u$ ). We thus define the *complex mass aspect* of Ricci-flat spacetimes in covariant Newman-Unti gauge as

$$\hat{\tau} = \lim_{k \rightarrow 0} \tau = -2\nu + 8\pi i G \varepsilon_{(0)}, \quad (3.37)$$

where

$$\nu = \frac{1}{2} \lim_{k \rightarrow 0} c = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab} - \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} \quad (3.38)$$

is the *magnetic-mass aspect* reached using (B.12) and (B.15) upon substitution of  $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$ . Subtracting the radiative contribution as in (3.36), we define the *nut aspect*

$$N = \nu + \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab}, \quad (3.39)$$

where  $c_{(0)} = (\hat{\mathcal{D}}_a \hat{\mathcal{D}}^a + 2\hat{\mathcal{K}}) * \varpi$  is one of the four Carroll Cotton scalars displayed in (B.15).<sup>25</sup> Following the case of asymptotically AdS spacetimes quoted earlier, the behaviour of the bulk Weyl tensor in the Ricci-flat instance does also depend on the complex mass aspect  $\hat{\tau}$ , and we find indeed

$$\Psi_2 = \frac{i\hat{\tau}}{2r^3} + \mathcal{O}(1/r^4). \quad (3.40)$$

The higher-order missing terms in (3.40) are absent in the resumable, algebraically special solutions discussed in refs. [20, 22, 25]. Unsurprisingly, this expression coincides with eq. (68c) of [47].

With the above definitions, eqs. (3.34), (3.32) and (3.33) become:<sup>26</sup>

$$\hat{\mathcal{D}}_\nu N = -\frac{1}{2} \hat{\mathcal{D}}_a \chi^a - \frac{1}{4} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \hat{\mathcal{N}}^{ab} - * \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \right), \quad (3.41)$$

$$\hat{\mathcal{D}}_\nu M = -\frac{1}{2} \hat{\mathcal{D}}_a * \chi^a + \frac{1}{4} \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b - \frac{1}{2} \hat{\mathcal{N}}_{ab} \hat{\mathcal{N}}^{ab} \right), \quad (3.42)$$

$$\begin{aligned} \hat{\mathcal{D}}_\nu N^a - \hat{\mathcal{D}}^a M + * \hat{\mathcal{D}}^a N &= \frac{1}{2} \left[ \mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{C}^{ab} \hat{\mathcal{D}}_b - \frac{1}{2} * \hat{\mathcal{D}}^a \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c * \mathcal{C}^{bc} \right. \\ &\quad \left. - \frac{1}{2} \hat{\mathcal{D}}^b \left( \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{C}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{C}_{bc} \right) + \mathcal{C}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left( \mathcal{C}^{ac} \hat{\mathcal{N}}_{bc} \right) \right]. \end{aligned} \quad (3.43)$$

The first equation phrases the loss process of the nut aspect sustained by the Carroll-dual news  $*\hat{\mathcal{N}}_{ab}$  and the Carroll Cotton current  $\chi^a$ . It is actually a *geometric identity* associated with the Carroll structure — as is (3.35), which could have been reexpressed as well in terms of the nut aspect. The last two flux-balance equations (3.42) and (3.43) for the electric-mass and angular-momentum aspects are *genuinely dynamical* and coincide with eqs. (2.53) and (2.50) of ref. [23], where the approach to asymptotic flatness via a limit

<sup>25</sup>Our definitions for  $\nu$  and  $N$  match with  $-\tilde{\mathcal{M}}$  and  $-\tilde{M}$  of [47], eqs. (53) and (55), for  $c_{(0)} = 0$  (no magnetic monopole mass). This condition pertains to the use of the Bondi gauge in the quoted reference, where no Ehresmann connection exists and thus  $*\varpi$  vanishes (as  $\varphi_a$ ).

<sup>26</sup>All these computations call for abundant use of the Weyl-covariant-derivative commutators presented in the appendix, eqs. (A.31), (A.32), (A.33), (A.36), (A.37) and (A.38).

of vanishing cosmological constant was proposed, or else with (4.50) and (4.49) of [54], obtained in a plain Ricci-flat context.<sup>27</sup>

**Order  $1/r^2$  and the next flux-balance equation.** The Carrollian symmetric and traceless two-tensor  $E_{ab}$ , descendant of the AdS-boundary stress, enters the line element at order  $1/r$ . However, the fundamental Carrollian energy-momentum conservation equations (3.30) and (3.31) fail to capture its dynamics. In a direct search of Ricci-flat spacetimes, Einstein's equations bring their share at each order and this is how the flux-balance equations emerge for the Chthonian degrees of freedom as  $E_{ab}$ . In the present method, Einstein's equations have already been imposed at the considered order. The bulk metric including the term (2.55) with the  $f_{(2)}$ s as in (2.57) (2.58) and (2.59) is thus on-shell — assuming (2.1) is satisfied. However, this term is due to exhibit divergences at vanishing  $k$ . Removing them will impose conditions involving the Chthonian degrees of freedom as well as their longitudinal derivatives appearing explicitly in (2.59). This is how flat *flux-balance equations are recovered in the transition from anti-de Sitter to asymptotically flat spacetimes*, and this is another laudable achievement of this note.

The protocol is by now well established: we ought to follow the four steps enumerated earlier, starting with any tensor  $f_{(2)}$  — and later on with other  $f_{(s)}$ . Let us open the study with the scalar contribution  $f_{(2)}$ , eq. (2.57). With little effort we find:

$$\lim_{k \rightarrow 0} f_{(2)} = 2 * \varpi \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \equiv \hat{f}_{(2)}. \quad (3.44)$$

Next we consider the transverse vector  $f_{(2)a} \theta^a$  in (2.58):

$$\lim_{k \rightarrow 0} f_{(2)a} = -\frac{1}{6} N^b \mathcal{C}_{ba} - \frac{4}{3} * \varpi * N_a - 4\pi G \hat{\mathcal{D}}_b E^b_a \equiv \hat{f}_{(2)a}. \quad (3.45)$$

Neither the limit (3.44) nor (3.45) introduce any new Chthonian degree of freedom or impose any further condition on their evolution. As we will now see, the situation is different for the transverse tensor (2.59)  $f_{(2)ab} \theta^a \theta^b$ . Using the numerous tools developed in this work, we find:<sup>28</sup>

$$\begin{aligned} f_{(2)ab} = & \frac{1}{k^2} \left( \frac{16\pi G}{3} \hat{\mathcal{D}}_\nu E_{ab} + \frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b \rangle} + 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} - \frac{\nu}{2} * \mathcal{C}_{ab} \right) \\ & + 2\pi G \left( \frac{8}{3} \hat{\mathcal{D}}_\nu E_{(2)ab} - \frac{4}{3} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \varepsilon_{(1)} \mathcal{C}_{ab} - 2\mathcal{C}_{(a}{}^c E_{b)c} \right) - 2 * \varpi^3 * \mathcal{C}_{ab} \\ & + \mathcal{O}(k^2). \end{aligned} \quad (3.46)$$

This result meets our expectations and allows us to draw significant conclusions.

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<sup>27</sup>In the quoted section 2.5 of [23]  $\mu = -du$  so that  $\varphi_a = \varpi_{ab} = 0$  (Bondi gauge with  $\exp 2\beta_0 = 1$ ). Furthermore our definition of  $N^a$  is slightly different:  $N^a_{\text{here}} = N^a_{\text{there}} + \frac{1}{4} (\mathcal{C}^{ab} \hat{\nabla}^c \mathcal{C}_{bc} + \frac{3}{8} \hat{\nabla}^a (\mathcal{C}^{bc} \mathcal{C}_{bc}))$  with  $\hat{\nabla}^c$  being actually the ordinary two-dimensional Levi-Civita connection due to the absence of Ehresmann connection in [23] (see eq. (A.29) where the Carroll-Weyl covariant derivative reduces to the ordinary one when  $\varphi_a$  vanishes). This definition is in line with that of [55]. Likewise  $\mathcal{N}^{ab}_{\text{here}} = N^{ab}_{\text{TF there}} - \frac{l_{\text{there}}}{2} \mathcal{C}^{ab}$  with  $l_{\text{there}} = \theta_{\text{here}}$  and for further use we also quote that  $E^{ab}_{\text{here}} = -\frac{3}{16\pi G} (\mathcal{E}^{ab}_{\text{there}} - \frac{1}{16} \mathcal{C}^{ab} \mathcal{C}^{cd} \mathcal{C}_{cd})$ . The comparison with ref. [54] is reviewed in [23].

<sup>28</sup>We define the symmetric and traceless part of a Carrollian two-tensor  $s_{ab}$  as  $s_{(ab)} = s_{(ab)} - \frac{1}{d} s_c{}^c \delta_{ab}$  (here  $d = 2$ ).

- The flat limit is singular unless the order- $1/k^2$  contribution to  $f_{(2)ab}$  is absent i.e.

$$\hat{\mathcal{D}}_\nu E_{ab} = \frac{3}{16\pi G} \left( -\frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b\rangle} - 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} + \frac{\nu}{2} * \mathcal{C}_{ab} \right), \quad (3.47)$$

which is the sought-after Carrollian *flux-balance equation for  $E_{ab}$* , later referred to as  $\text{FBE}_{(1)} = 0$ . This equation matches with eq. (4e) of [47].<sup>29</sup>

- Assuming eq. (3.47) is fulfilled, the limit can be taken

$$\begin{aligned} \lim_{k \rightarrow 0} f_{(2)ab} &= \frac{16\pi G}{3} \left( \hat{\mathcal{D}}_\nu E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b\rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{4} \mathcal{C}_{(a}{}^c E_{b)c} \right) - 2 * \varpi^3 * \mathcal{C}_{ab} \\ &\equiv \hat{f}_{(2)ab}, \end{aligned} \quad (3.48)$$

and provides the last piece of the order- $1/r^2$  term in the Ricci-flat line element.

- New Chthonian degrees of freedom enter the bulk metric at this order:  $E_{(2)ab}$ ,  $\pi_{(2)a}$  and  $\varepsilon_{(1)}$  in the form of a symmetric and traceless Carrollian tensor

$$F_{ab} = \hat{\mathcal{D}}_\nu E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b\rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{8\pi G} * \varpi^3 * \mathcal{C}_{ab}. \quad (3.49)$$

Their dynamics is unknown at this stage but will be unravelled in the course of the analysis at order  $1/r^3$ .

We will close this paragraph exhibiting the explicit Ricci flat metric at the considered order. To this end we use the results (3.1), (3.9), (3.12), (3.22), (3.44), (3.45), (3.48) and (3.49):

$$\begin{aligned} ds_{\text{Ricci-flat}}^2 &= \mu \left[ 2dr + \left( 2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \varpi - \hat{\mathcal{D}}_b \mathcal{C}^b{}_a \right) \hat{\theta}^a - \left( r\theta + \hat{\mathcal{K}} \right) \mu \right] \\ &+ \left( r^2 + * \varpi^2 + \frac{\mathcal{C}^2}{4} \right) d\ell^2 + \left( r \mathcal{C}_{ab} + * \varpi * \mathcal{C}_{ab} \right) \hat{\theta}^a \hat{\theta}^b \\ &+ \frac{1}{r} \left( 8\pi G \varepsilon_{(0)} \mu^2 - \frac{4}{3} \mu N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \right) \\ &+ \frac{1}{r^2} \left( 2 * \varpi \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \right) \mu^2 - \frac{1}{r^2} \mu \left( \frac{1}{3} N^b \mathcal{C}_{ba} + \frac{8}{3} * \varpi * N_a + 8\pi G \hat{\mathcal{D}}_b E^b{}_a \right) \hat{\theta}^a \\ &+ \frac{1}{r^2} \left( \frac{16\pi G}{3} F_{ab} - 4\pi G \mathcal{C}_{(a}{}^c E_{b)c} \right) \hat{\theta}^a \hat{\theta}^b + \mathcal{O}(1/r^3). \end{aligned} \quad (3.50)$$

This solution to vacuum Einstein's equations is built upon the following boundary Carrollian data: (i) a generic Carrollian structure with geometric shear  $\xi_{ab} = 0$  (but arbitrary Ehresmann connection providing  $\varphi_a$  and  $*\varpi$ ); (ii) a dynamical shear  $\mathcal{C}_{ab}$ , utterly free; (iii) an energy density  $\varepsilon_{(0)}$  i.e. a Bondi mass  $M$ , a heat current  $N_a$  aka the Bondi angular momentum aspect and a stress  $E_{ab}$ , all satisfying the flux-balance equations (3.42), (3.43) and (3.47);<sup>30</sup> (iv) three more degrees of freedom  $E_{(2)ab}$ ,  $\pi_{(2)a}$  and  $\varepsilon_{(1)}$  encoded in  $F_{ab}$  (3.49) with evolution equations yet to be uncovered.

<sup>29</sup>For this we use the dictionary for ref. [47] set up in footnotes 24 and 25, together with the relations  $-16\pi G E_{\text{here}}^{ab} = \mathcal{T}_{\text{there}}^{ab}$  and  $N_{\text{here}}^a = \mathcal{P}_{\text{there}}^a$ . Observe that eqs. (58) and (63) of [47] are also compatible with further quantities introduced in [23] and mentioned in footnote 27.

<sup>30</sup>As pointed out earlier the nut aspect  $N$  — equivalently the magnetic mass  $\nu$  — is in essence part of the Carrollian structure and its evolution equation (3.41) is a geometric identity in disguise.

**Recursion and the fate of Chthonian degrees of freedom.** All this has been achieved as the limit of vanishing cosmological constant within general asymptotically anti-de Sitter Einstein spacetimes, where infinitely many flat degrees of freedom originate from the Laurent expansion of the anti-de Sitter boundary energy-momentum tensor about  $k^2 = -\Lambda/3 = 0$  and constrained through evolution equations. Compared to the anti-de Sitter solution space, the extra — Chthonian — functions are  $\{\chi_{(n \geq 2)}\} \equiv \{\varepsilon_{(n-1 \geq 1)}, \pi_{(n \geq 2)}^a, E_{(n \geq 2)}^{ab}\}$ . It is natural to wonder whether these are truly independent functions. Answering this question demands a higher-order analysis but some simple considerations allow to infer that  $\varepsilon_{(n-1)}, \pi_{(n)}^a$  and  $E_{(n)}^{ab}$  could be repackaged in a single symmetric traceless tensor  $F_{(n)}^{ab}$ , having the expected conformal weight.

Indeed, one should recall that the  $\chi_{(n)}$ s are all weight-3 and contribute to the  $f_{(s)}$ s (of weight  $s + 2$ ) through an appropriate number of longitudinal or transverse Weyl-covariant derivatives  $u^C \mathcal{D}_C$  or  $\mathcal{D}_a$ , powers of vorticity  $\omega_{ab}$  or shear  $\sigma_{ab}$ , all raising the weight by one unit (in the Carrollian limit, the latter two bring a factor  $k^2$  with  $*\varpi \hat{\eta}_{ab}$  or  $\mathcal{C}_{ab}$ ). The analysis of Einstein's equations  $\mathcal{E}_{rr}, \mathcal{E}_{r\hat{0}}, \mathcal{E}_{ra}$  and  $\mathcal{E}_{ab}$  in the radial expansion exhibits a remarkable recursion structure for the  $f_{(s)}, f_{(s)}^a$  and  $f_{(s)}^{ab}$  — for  $s = 2$  these equations are sorted in (2.56). The latter are given in terms of quantities of order  $s - 1$  along with one transverse Weyl derivative, one power of vorticity, or one power of shear. Furthermore, the scalar and the vector do not involve any net power of  $k^2$ , whereas the tensor does:  $f_{(s)}^{ab} = \frac{1}{k^2} [u^C \mathcal{D}_C f_{(s-1)}^{ab} + \dots]$ . This shows, on the one hand, that the scalar and vector contributions to the line element remain finite in the Carrollian limit, and do not impose any supplementary constraint. On the other hand, flux-balance equations originate exclusively from the two-index term.

Owing to the fact that  $\chi = \sum_{m \geq 2} k^{2m} \chi_{(m)}$ , the Chthonian degrees of freedom  $\chi_{(m)}$  persist in the Carrollian limit of the  $f$ s if a power of  $k^2$  equal to or more negative than  $-m$  is inherited from the AdS solution. Combined with the above recursive pattern, where in particular negative powers appear solely in the tensor  $f_{(s)}^{ab}$ , this suggests that once a combination of  $\chi_{(m)}$  has emerged, such as  $F^{ab} \equiv F_{(2)}^{ab}$  in eq. (3.49) for  $\chi_{(2)}$  inside (3.46), only this precise expression will appear in the subsequent orders, along with more derivatives, powers of shear and vorticity, and increasing negative powers of  $k^2$ . For instance, this occurs for  $F_{(1)}^{ab} \equiv E_{(1)}^{ab} \equiv E^{ab}$  in  $f_{(1)}^{ab}$  as in eq. (3.22),  $f_{(2)}^{ab}$  as in eq. (3.46), and likewise in higher orders.

This scheme has two consequences. The first is that at order  $(s + 1)$ , one new flux-balance equation  $\text{FBE}_{(s)} = 0$  emerges, for the previously determined combination  $F_{(s)}^{ab}$  of the Chthonian functions  $\chi_{(s)}$ , as it should for global evolution consistency. Schematically this property is captured in the following:

$$f_{(s+1)}^{ab} = \sum_{n=1}^{s-1} \frac{c_{(s,n)}}{k^{2(s-n+1)}} \hat{\mathcal{D}}_v^{s-n} \text{FBE}_{(n)} + \frac{c_{(s,s)}}{k^2} \text{FBE}_{(s)} + \hat{f}_{(s+1)}^{ab} + \mathcal{O}(k^2), \quad (3.51)$$

with  $c_{(s,n)}$  some immaterial coefficients, the new equation being  $\text{FBE}_{(s)} = 0$ . The second consequence is that the triplet  $\chi_{(s)}$  counts as a single Chthonian degree of freedom materialized in  $F_{(s)}^{ab}$ , the one appearing in the line element and obeying a flux-balance equation

revealed at the next order in  $1/r$ . The following generic structure of the solutions underpins the above reasoning.<sup>31</sup>

- $\hat{f}_{(2s+1)}$  contains  $(-)^s 8\pi G \varepsilon_{(0)} * \varpi^{2s}$  and  $\hat{\mathcal{D}}_a \hat{f}_{(2s)}^a$ ;
- $\hat{f}_{(2s+2)}$  contains  $(-)^s 2\nu * \varpi^{2s+1}$  and  $\hat{\mathcal{D}}_a \hat{f}_{(2s+1)}^a$ ;
- $\hat{f}_{(s+1)}^a$  contains  $\hat{\mathcal{D}}_b \hat{f}_{(s)}^{ab}$ ,  $*\varpi * \hat{f}_{(s)}^a$ ,  $\mathcal{C}^a_b \hat{f}_{(s)}^b$ ;
- $\hat{f}_{(s+1)}^{ab} = c_{(s+1,s+1)} F_{(s+1)}^{ab} +$  tensors based on objects of order  $s$ ;
- $F_{(s+1)}^{ab}$  contains  $\hat{\mathcal{D}}_\nu^s E_{(s+1)}^{ab}$ ,  $\hat{\mathcal{D}}_\nu^{s-1} \hat{\mathcal{D}}^{(a} \pi_{(s+1)}^{b)}$ ,  $\varepsilon_{(s)} * \varpi^{s-1} \mathcal{C}^{ab}$ ,  $\dots$ ;
- $\text{FBE}_{(s)} = 0$  is of the form  $\hat{\mathcal{D}}_\nu F_{(s)}^{ab} = \{ \hat{\mathcal{D}}^{(a} \hat{f}_{(s)}^{b)}, \mathcal{C}^{ab} \hat{f}_{(s)}, \dots \}$ ,

where the dots stand for other possible admissible terms. As anticipated, the actual Chthonian degrees of freedom capturing the flat dynamics are the emerging  $F_{(s)}^{ab}$ , which should be substituted for the  $(s-1)$ th derivatives of  $E_{(s)}^{ab}$ ,  $\pi_{(s)}^a$  and  $\varepsilon_{(s-1)}$  delivered by the anti-de Sitter energy-momentum tensor.

A legitimate question one may finally ask in view of our analysis pertains to the existence of other, possibly infinite, sets of Chthonian data originating from a Laurent expansion of the AdS boundary metric (see e.g. [45, 56]). Direct exploration of Ricci-flat solution spaces does not seem to support such an expectation, but a definite answer requires a thorough investigation, which would bring us far from our main goal.

**The flat resummation.** The anti-de Sitter resumable instance presented in eq. (2.65) can be realized in the flat limit, as it was shown in [3]. In this case all Chthonian functions should vanish, together with  $N^a$ ,  $E^{ab}$  and the shear  $\mathcal{C}^{ab}$ , leading ultimately to

$$\begin{aligned} ds_{\text{res. Ricci-flat}}^2 = & \mu \left[ 2dr + \left( 2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \varpi \right) \hat{\theta}^a - \left( r\theta + \hat{\mathcal{X}} \right) \mu \right] \\ & + \left( r^2 + * \varpi^2 \right) d\ell^2 + \frac{1}{r^2 + * \varpi^2} \left( 8\pi G \varepsilon_{(0)} r + * \varpi c_{(0)} \right) \mu^2. \end{aligned} \quad (3.52)$$

This captures all algebraically special Ricci-flat spacetimes provided  $\varepsilon_{(0)}$  obeys (3.32) and (3.33) which now read:

$$\hat{\mathcal{D}}_\nu \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a = 0, \quad (3.53)$$

$$\hat{\mathcal{D}}_a \varepsilon_{(0)} - \frac{1}{8\pi G} * \hat{\mathcal{D}}_a c_{(0)} = 0. \quad (3.54)$$

Equations (3.53) and (3.54) coincide with eqs. (29.16) and (29.15) of [57].<sup>32</sup> The latter are rather complicated and it is remarkable they are tamed into simple conservation equations

<sup>31</sup>We remind that  $F_{(1)}^{ab} \equiv E_{(1)}^{ab} \equiv E^{ab}$ ,  $\pi_{(1)}^a \equiv \pi^a$ , and  $F_{(2)}^{ab} \equiv F^{ab}$ .

<sup>32</sup>For that purpose, the following identifications are necessary, in Papapetrou-Randers frame and complex coordinates  $\mathbf{x} = \{ \zeta, \bar{\zeta} \}$  with  $d\ell^2 = \frac{2}{P^2(u, \zeta, \bar{\zeta})} d\zeta d\bar{\zeta}$ ,  $\nu = \frac{1}{\Omega} \partial_u$ ,  $\mu = -\Omega du + b_a dx^a$  and  $\hat{e}_a = \hat{\partial}_a = \partial_a + \frac{b_a}{\Omega} \partial_u$ ,  $\hat{\theta}^a = dx^a$ :  $\Omega = 1$ ,  $b_\zeta = -L$ ,  $*\varpi = -\Sigma$ ,  $\hat{\tau} = 2(M + im)$ , whereas their radial coordinate is  $\tilde{r} = r - r_0$  with  $r_0(u, \zeta, \bar{\zeta})$  the origin in the affine parameter of the geodesic congruence tangent to  $\partial_r$ .



such as (3.53) and (3.54). Reaching this conclusion would have been inconceivable without the null boundary perspective and the Carrollian tools, which are the appropriate language for asymptotically flat spacetimes.

The algebraically-special nature of the metric (3.52) is proven using the Goldberg-Sachs theorem with the null, geodesic, and shear-free in the resummed instance, bulk congruence tangent to  $\partial_r$ . The latter is part of the canonical null tetrad parallelly transported along  $\partial_r$  (thanks to the affine nature of  $r$ ) introduced in [3], which coincides with that of [57], eq. (29.13a), as well as with the original ref. [37]. In complex celestial-sphere coordinates  $\zeta$  and  $\bar{\zeta}$  (see footnote 32) the null tetrad reads:

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left( \frac{8\pi G \varepsilon_{(0)} r + * \varpi c_{(0)}}{r^2 + * \varpi^2} - r\theta - \hat{\mathcal{H}} \right) \partial_r + \mathbf{v} \\ \mathbf{m} = \frac{P}{r - i * \varpi} \left( \hat{\partial}_{\bar{\zeta}} + (* \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi - r \varphi_{\bar{\zeta}}) \partial_r \right) \end{cases} \quad (3.55)$$

with  $\mathbf{k} \cdot \mathbf{l} = -1$ ,  $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$  and  $ds_{\text{res. Ricci-flat}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$ . Generically,  $\mathbf{k}$  is a multiplicity-two principal null direction of the Weyl tensor, and using the tetrad at hand we find the following Weyl complex scalars:  $\Psi_0 = \Psi_1 = 0$  and

$$\Psi_2 = \frac{i\hat{r}}{2(r - i * \varpi)^3}, \quad \Psi_3 = \frac{iP\chi\zeta}{(r - i * \varpi)^2} + \mathcal{O}(1/(r - i * \varpi)^3), \quad \Psi_4 = \frac{iX_{\zeta}^{\bar{\zeta}}}{r - i * \varpi} + \mathcal{O}(1/(r - i * \varpi)^2). \quad (3.56)$$

Observe that neither  $\Psi_3$  nor  $\Psi_4$  vanish in the instance of Petrov type D solutions, because  $\mathbf{l}$  is *not* a principal null direction. Another tetrad is reached with a Lorentz transformation suitably adjusted for  $\mathbf{l}'$  be a principal direction of multiplicity two whereas  $\mathbf{k}' \propto \mathbf{k}$ , and  $\Psi'_3 = \Psi'_4 = 0$ . Unsurprisingly, all  $\Psi$ s are spelled using the Carrollian descendants of the boundary Cotton tensor — as well as their derivatives in the higher-order terms.

## 4 Outlook

Asymptotically anti-de Sitter and flat spacetimes subject to Einstein equations are distinguished mainly by two features. The first is gravitational radiation escaping at or arriving from null infinity in the flat instance, which is absent under the usual boundary conditions for anti-de Sitter. The second concerns the data required for a faithful depiction of these geometries and of their dynamics imposed by Einstein's equations: a finite versus an infinite number for asymptotically AdS or flat.

In spite of the sharp distinctness of the solution spaces with non-vanishing and zero cosmological constant, the latter can be smoothly reached from the former in a procedure that is the core of this work. It can be outlined in three steps, performed along with the process of sending  $\Lambda$  to zero, which simultaneously transmutes the pseudo-Riemannian conformal boundary of anti-de Sitter into a Carrollian descendant, carrying akin information.

- Bondi's shear  $\mathcal{C}_{AB}$  is substituted on-shell for the geometric shear  $\sigma_{AB}$ .
- The anti-de Sitter boundary energy-momentum tensor  $T_{AB}$  is Laurent-expanded in powers of  $k^2 = -\Lambda/3$  about  $k^2 = 0$ . This supplies an infinite number of replicas, which account for the awaited flat, Chthonian, degrees of freedom.

- The evolution (flux-balance) equations of the — now Carrollian — degrees of freedom are reached using both the limit of the conservation of the energy and momentum, as well as the requirement of finiteness for the line element in the flat limit. The latter (i) selects the Chthonian variables  $\varepsilon_{(n\geq 1)}$ ,  $\pi_{(n\geq 2)}^a$  and  $E_{(n\geq 2)}^{ab}$  besides the standard energy density  $\varepsilon_{(0)}$ , momentum  $\pi^a$  and stress  $E^{ab}$ , and we have argued that genuine degrees of freedom are only the  $F_{(n)}^{ab}$ s; (ii) freezes a few other components of the expanded energy-momentum tensor in terms of the boundary Carrollian Cotton tensors  $(\zeta^a, Q^a, \zeta^{ab}, \Sigma^{ab}, \Xi^{ab})$ ; (iii) delivers the Chthonian dynamics, which is not captured by the energy-momentum conservation but echoes flat Einstein equations.

The technical *tour de force* of our exploration shouldn't shadow the conceptual aftermath of our findings. These bring back the boundary energy and momentum at the center of the asymptotically flat bulk reconstruction, besides the Bondi shear, under the form of a Carrollian energy density, momentum and stress, together with an infinite tower of replicas of the latter. Speculating over a flat extension of AdS gauge/gravity duality, and owing to the key role played by the energy-momentum tensor in the latter, one is led to several unescapable questions. What would the fundamental observables be in the dual Carrollian field theory? What role would the replicas of the energy-momentum sector play? What is the interplay between the Chthonian and the shear/news sector, which has been investigated in celestial holography? Could this correspondence still be qualified as holographic — given the seemingly infinite number of necessary data? Our approach does not yet provide any cue for answering these questions, though it hands some confidence in the zero- $\Lambda$  limit, that could be inquired within the AdS/CFT correspondence. This last point is probably the deepest our analysis conveys.

This is the big picture. Other questions merit equal attention, starting with the ones related to symmetries and charges. What are the asymptotic symmetries in a partially unfixed gauge like the covariant Newman-Unti introduced here? What sort of charges does this extension carry? What is the precise combination of vorticity and angular-momentum aspect that would define the physical angular momentum? How would logarithmic terms in the radial expansion alter the analysis? In answering these questions, one could follow recent works such as, e.g., [10, 13, 15] as well as [42, 50–52, 58]. In particular, one should address the Weyl invariance in conjunction with the boundary local Lorentz (or Carroll) gauge invariance inherited from the onset of a velocity congruence or a clock form in the boundary pseudo-Riemannian (or Carrollian) structure. In a similar fashion as the one presented in this work, a careful analysis would allow to embrace both the anti-de Sitter and flat cases.

Regarding the charges, a thorough comparison of our method with Newman-Penrose's would be a valuable practice, reasonably accessible thanks to the affinely parameterised radial congruence  $\partial_r$  present in the (covariant) Newman-Unti gauge. In the first place, this would allow to extract the famous ten non-vanishing Newman-Penrose conserved charges — we know that these are carried by the Chthonian stress tensor descendant  $F_{ab}$ .<sup>33</sup> Secondly,

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<sup>33</sup>Contact with the Newman-Penrose formalism beyond the algebraically special resumable metrics mentioned at the end of section 3 starts with  $\Psi_0^0 \propto iE_\zeta^{\bar{\zeta}}$ ,  $\Psi_0^1 \propto iF_\zeta^{\bar{\zeta}}$ ,  $\Psi_1^0 \propto iN_\zeta$ ,  $\Psi_2^0 \propto i\hat{r}$  (see (3.40)),

one could recast these charges following their general Carrollian definition, as described in refs. [25, 59], giving credit to this full-fledged boundary method for the charge computation. Lastly, one may deepen concepts such as subleading charges or electric versus magnetic charges and possible dualities involving the Carrollian Cotton tensors, as recently undertaken in [25] from the Carrollian standpoint in the limited framework of resumable, algebraically special Ricci-flat spacetimes, and more generally discussed in refs. [60–75].

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## A Carrollian geometry in Cartan frame and arbitrary dimension

**Frame and covariance.** Carroll structures on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  with a  $d$ -dimensional base  $\mathcal{S}$  were alluded to in section 3. They are equipped with a degenerate metric,

$$d\ell^2 = \delta_{ab} \hat{\theta}^a \hat{\theta}^b, \tag{A.1}$$

as well as a frame and a coframe,  $\{\hat{e}_{\hat{u}} = \mathbf{v}, \hat{e}_a\}$  and  $\{\hat{\theta}^{\hat{u}} = -\mu, \hat{\theta}^a\}$  obeying

$$\mu(\mathbf{v}) = -1, \quad \hat{\theta}^a(\hat{e}_b) = \delta_b^a, \quad \hat{\theta}^a(\mathbf{v}) = 0, \quad \mu(\hat{e}_a) = 0. \tag{A.2}$$

Here  $\mathbf{v}$  is the field of observers, kernel of the degenerate metric, and  $\mu$  the clock form (see e.g. [76]).

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$\Psi_3^0 \propto iP\chi_\zeta$  and  $\Psi_4^0 \propto iX_\zeta^{\bar{\zeta}}$ , where the adopted Carrollian frame is that of footnote 32. The higher-order terms will involve derivatives of the Cotton tensors, of the energy density, the momentum and the stress, as well as the infinite tower of Chthonian replicas  $F_{(s)}^{ab}$ .

A convenient parameterization in terms of  $d + \frac{(d+1)(d+2)}{2}$  functions (i.e. 8 for  $d=2$ ) is<sup>34</sup>

$$\mathbf{v} = \gamma \left( \partial_u + v^i \partial_i \right) \quad \Leftrightarrow \quad \boldsymbol{\mu} = -\frac{du}{\gamma} + \Delta_i \left( dx^i - v^i du \right), \quad (\text{A.3})$$

$$\hat{e}_a = e_a^i \left( \partial_i + \gamma \Delta_i \left( \partial_u + v^j \partial_j \right) \right) \quad \Leftrightarrow \quad \hat{\theta}^a = e_a^i \left( dx^i - v^i du \right) \quad (\text{A.4})$$

with

$$\Gamma_{ij}^2 = \delta_{ab} e_a^i e_b^j \quad \Leftrightarrow \quad \delta_{ab} = e_a^i e_b^j \Gamma_{ij}^2 \quad (\text{A.5})$$

and

$$e_a^i e_a^j = \delta_i^j, \quad e_b^j e_a^j = \delta_a^b, \quad \delta^{ab} e_a^i \Gamma_{ij}^2 = e_b^j, \quad \delta_{ab} e_a^i \Gamma^{2ij} = e_b^j, \quad (\text{A.6})$$

where  $(\Gamma^2)^{ik} \Gamma_{kj}^2 = \delta_j^i$ . Consequently, the degenerate metric assumes the form<sup>35</sup>

$$d\ell^2 = \Gamma_{ij}^2 \left( dx^i - v^i du \right) \left( dx^j - v^j du \right). \quad (\text{A.7})$$

In this specific parameterization, which generalizes that of [10] in arbitrary dimension, the bulk Newman-Unti gauge is recovered by setting  $\Delta_i = 0$  in the boundary frame.<sup>36</sup>

Carrollian tensors have commonly spacetime indices. In the Cartan frame (A.1), (A.2), their tensorial behaviour refers to the local Carroll group, as much as relativistic tensors in an orthonormal Cartan frame are tamed according to the local Lorentz group. Here, the metric being degenerate the spacetime indices cannot be lowered or raised. One way to manage this inconvenience is by introducing a pseudo-inverse [83]. Our strategy has been slightly different, and is hinged on *separating time and space*, since this is natural in Carrollian manifolds due to the fibre structure. In the frame at hand, the method boils down to considering tensors with solely spatial indices, organized in representations of the  $d$ -dimensional orthogonal local group, subgroup of the local Carroll group, and raised or lowered with  $\delta^{ab}$  or  $\delta_{ab}$ . The fibre null-time direction supports scalars without indices.<sup>37</sup> This approach is in line with the boundary reconstruction of Ricci-flat spacetimes, where the longitudinal/transverse decomposition of the fundamental tensors coincides with the time/space reduction of the Carrollian tensors.

A strong Carroll structure comes with a metric-compatible and field-of-observers-compatible connection, which is not unique due to the metric degeneracy. The connection we use defines a parallel transport that respects the time/space splitting mentioned above,

<sup>34</sup>Here  $\gamma$  is an arbitrary function and must not to be confused with (2.27) which is related to the vorticity of the timelike congruence  $u$ .

<sup>35</sup>The degenerate metric is often spelled  $d\ell^2 = q_{\mu\nu} dx^\mu dx^\nu$  in the Carrollian literature, and  $n = n^\mu \partial_\mu$  stands for the field of observers.

<sup>36</sup>The presence of  $\gamma \equiv \exp(-2\beta_0)$ , which persists in the bulk line element as  $-2 \exp(2\beta_0) du dr$ , assesses a slight redefinition of the radial coordinate before reaching stricto sensu Newman-Unti gauge. We are cavalier with this detail because the counting from the point of view of the solution space matches: the contribution of the boundary geometry is  $\frac{(d+1)(d+2)}{2}$ . The same holds for the anti-de Sitter ascendant.

<sup>37</sup>When working in natural frames, as in refs. [3, 9, 24, 25, 45, 46, 85], the tensor structure is based instead on diffeomorphisms. The time/space splitting sought for is realized in Papapetrou-Randers frame, i.e. setting  $v^i = 0$  in the formulas (A.3), (A.4), (A.7), because this frame is stable under the Carrollian subset of diffeomorphisms, consisting of transformations  $u \rightarrow u'(u, \mathbf{x})$  and  $\mathbf{x} \rightarrow \mathbf{x}'(\mathbf{x})$ . Carrollian tensors have again spatial indices and transform with the Jacobian matrices of Carrollian diffeomorphisms.

embracing distinct time and space Carrollian covariant derivatives  $\hat{\nabla}_{\mathbf{v}}$  acting as a scalar and  $\hat{\nabla}_a$  acting as a form. We set for this purpose

$$\hat{\nabla}_{\mathbf{v}}\mathbf{v} = 0, \quad \hat{\nabla}_{\mathbf{v}}\hat{e}_a = \hat{\gamma}_{[ab]}\delta^{bc}\hat{e}_c, \quad \hat{\nabla}_{\hat{e}_a}\mathbf{v} = 0, \quad \hat{\nabla}_{\hat{e}_a}\hat{e}_b = \hat{\gamma}_{ab}^c\hat{e}_c, \quad (\text{A.8})$$

from which we infer the resulting Carrollian affine connection one-form:<sup>38</sup>

$$\hat{\omega}^{\hat{u}}_{\hat{u}} = \hat{\omega}^{\hat{u}}_{\hat{b}} = \hat{\omega}^a_{\hat{u}} = 0, \quad \hat{\omega}^a_{\hat{b}} = \delta^{ac}\hat{\gamma}_{[cb]}\mu + \hat{\gamma}_{cb}^a\hat{\theta}^c \quad (\text{A.9})$$

At this stage  $\hat{\gamma}_{[ab]}$  and  $\hat{\gamma}_{cb}^a$  are arbitrary, although anticipating the next step (metric compatibility), we have imposed antisymmetry for the former.

The covariant time and space derivatives act on Carrollian scalars as time and space directional derivatives. For Carrollian vectors  $\zeta = \zeta^a\hat{e}_a$  and forms  $\zeta = \zeta_a\hat{\theta}^a$  we obtain:

$$\hat{\nabla}_a\zeta^b = \hat{e}_a(\zeta^b) + \hat{\gamma}_{ac}^b\zeta^c \Leftrightarrow \hat{\nabla}_a\zeta_b = \hat{e}_a(\zeta_b) - \hat{\gamma}_{ab}^c\zeta_c, \quad (\text{A.10})$$

$$\hat{\nabla}_{\mathbf{v}}\zeta^a = \mathbf{v}(\zeta^a) - \hat{\gamma}^{[ab]}\zeta_b \Leftrightarrow \hat{\nabla}_{\mathbf{v}}\zeta_a = \mathbf{v}(\zeta_a) - \hat{\gamma}_{[ab]}\zeta^b. \quad (\text{A.11})$$

Under a frame transformation,  $\hat{\gamma}_{[ab]}$  and  $\hat{\gamma}_{cb}^a$  transform as connection coefficients, i.e. with inhomogeneous terms.

Field-of-observers-compatibility is built in (A.8). Metric-compatibility translates in  $\hat{\omega}_{(ab)} = 0$ . This imposes

$$\hat{\gamma}_{(a|c|b)} = 0, \quad (\text{A.12})$$

where the symmetrization acts on the two extreme indices. The latter can be utterly determined by further imposing the absence of torsion in the spatial section,  $T_{ab}^c = 0$ . In order to implement this we can use the following parameterization of the  $d\hat{\theta}^A$ s:

$$d\mu - \varphi_a\hat{\theta}^a \wedge \mu - \varpi_{ab}\hat{\theta}^a \wedge \hat{\theta}^b = 0, \quad d\hat{\theta}^c + \hat{\gamma}^c_a\mu \wedge \hat{\theta}^a + \frac{1}{2}\hat{c}^c_{ab}\hat{\theta}^a \wedge \hat{\theta}^b = 0, \quad (\text{A.13})$$

or equivalently

$$[\mathbf{v}, \hat{e}_a] = \varphi_a\mathbf{v} - \hat{\gamma}^c_a\hat{e}_c, \quad [\hat{e}_a, \hat{e}_b] = 2\varpi_{ab}\mathbf{v} + \hat{c}^c_{ab}\hat{e}_c. \quad (\text{A.14})$$

We have again foreseen the following action by introducing  $\hat{\gamma}_{ab}$  whose antisymmetric part already appears in the affine connection one-form. Hence, the extra condition of the absence of torsion in the spatial section combined with (A.12) delivers

$$\hat{\gamma}_{bc}^a = \frac{1}{2}(\hat{c}^a_{bc} + \hat{c}^a_{cb} + \hat{c}^a_{ca}). \quad (\text{A.15})$$

Let us also point out the useful integrability conditions  $d^2\mu = d^2\hat{\theta}^a = 0$  associated with (A.13):

$$\begin{cases} \hat{\nabla}_{[c}\varpi_{ab]} = \varphi_{[c}\varpi_{ab]} \\ \hat{\nabla}_{\mathbf{v}}\varpi_{ab} + \varpi_a{}^c\hat{\gamma}_{(cb)} - \varpi_b{}^c\hat{\gamma}_{(ca)} = \hat{\nabla}_{[a}\varphi_{b]} \end{cases} \quad (\text{A.16})$$

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<sup>38</sup>Remember that  $\omega^A_B = \Gamma^A_{CB}\theta^C$  with  $\nabla_{e_A}e_B = \Gamma^C_{AB}e_C$ . The torsion and curvature two-forms are  $\mathcal{T}^C = d\theta^C + \omega^C_A \wedge \theta^A = \frac{1}{2}T^C_{AB}\theta^A \wedge \theta^B$  and  $\mathcal{R}^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B = \frac{1}{2}R^A_{BCD}\theta^C \wedge \theta^D$ . Torsion and curvature tensors can alternatively be determined using the commutator of covariant derivatives:  $[\nabla_A, \nabla_B]W^C = R^C_{DAB}W^D - T^D_{AB}\nabla_D W^C$ .

and

$$\begin{cases} \mathbf{v}(\hat{c}^a{}_{bc}) - \hat{\gamma}^a{}_d \hat{c}^d{}_{bc} - 2\hat{c}^a{}_{d[b} \hat{\gamma}^d{}_{c]} + 2\hat{e}_{[b}(\hat{\gamma}^a{}_{c]}) - 2\hat{\gamma}^a{}_{[b} \varphi_{c]} = 0 \\ \hat{e}_{[d}(\hat{c}^a{}_{bc])} - \hat{c}^a{}_{e[b} \hat{c}^e{}_{cd]} + 2\gamma^a{}_{[b} \varpi_{cd]} = 0. \end{cases} \quad (\text{A.17})$$

In summary, our strong Carroll connection is totally determined thanks to the information stored inside the second of eqs. (A.13), by requiring the time-and-space splitting and the absence of spatial torsion. The total torsion is non-zero though and we find:

$$\hat{\mathcal{T}}^{\hat{u}} = \varphi_a \mu \wedge \hat{\theta}^a - \varpi_{ab} \hat{\theta}^a \wedge \hat{\theta}^b, \quad \hat{\mathcal{T}}^a = \delta^{ab} \hat{\gamma}_{(bc)} \hat{\theta}^c \wedge \mu. \quad (\text{A.18})$$

The torsion is thus encoded in three Carrollian tensors (i.e. transforming homogeneously), featuring three properties of the null-time fibre materialized in  $\mathbf{v}$ : the *acceleration*  $\varphi_a$ , the *vorticity*  $\varpi_{ab}$  and the *extrinsic curvature*  $\hat{\gamma}_{(ab)}$ , which can be further decomposed into the *geometric shear*  $\xi_{ab}$  (traceless) and the *expansion*  $\theta$ :

$$\hat{\gamma}_{(ab)} = \xi_{ab} + \frac{\theta}{d} \delta_{ab}. \quad (\text{A.19})$$

We could consistently set the Carrollian torsion to zero. From the bulk perspective, this would significantly impoverish the range of options the covariant Newman-Unti gauge offers for Ricci-flat spacetimes, as discussed in section 3. It is opportune to recall that in the frame-parameterization (A.3), (A.4), ordinary Newman-Unti gauge corresponds to  $\Delta_i = 0$ . In more intrinsic terms, this amounts to setting

$$d\mu = \varphi_a \hat{\theta}^a \wedge \mu \Leftrightarrow [\hat{e}_a, \hat{e}_b] = \hat{c}^c{}_{ab} \hat{e}_c \quad (\text{A.20})$$

i.e. to discarding the vorticity.

We can finally determine the curvature of the Carrollian connection under consideration using Cartan's formula, cf. footnote 38:

$$\hat{\mathcal{R}}^{\hat{u}}{}_{\hat{b}} = 0, \quad \hat{\mathcal{R}}^a{}_{\hat{b}} = \hat{R}^a{}_{cb} \mu \wedge \hat{\theta}^c + \frac{1}{2} \hat{R}^a{}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d \quad (\text{A.21})$$

with

$$\hat{R}^a{}_{bcd} = \hat{e}_c(\hat{\gamma}^a{}_{db}) - \hat{e}_d(\hat{\gamma}^a{}_{cb}) + \hat{\gamma}^e{}_{db} \hat{\gamma}^a{}_{ce} - \hat{\gamma}^e{}_{cb} \hat{\gamma}^a{}_{de} - \hat{c}^e{}_{cd} \hat{\gamma}^a{}_{eb} + 2\varpi_{cd} \hat{\gamma}_{[eb]} \delta^{ae}, \quad (\text{A.22})$$

$$\hat{R}^a{}_{cb} = (\hat{\nabla}^a + \varphi^a) \hat{\gamma}_{(bc)} - (\hat{\nabla}_b + \varphi_b) \hat{\gamma}_{(cd)} \delta^{ad}. \quad (\text{A.23})$$

One can trace the above and yield the Carroll-Ricci tensor and the Carroll scalar curvature:

$$\hat{R}_{cd} = \hat{R}^a{}_{cad}, \quad \hat{R} = \delta^{cd} \hat{R}_{cd}. \quad (\text{A.24})$$

Let us stress anew that the freedom in designing a Carrollian connection is rather wide — see [76–79] or [80–82] for a review — even when conditions like Levi-Civita are imposed, which we haven't. Our guideline has been to ensure that all information ultimately stored in the Carrollian frame, connection, torsion and curvature coincides with that of the relativistic, pseudo-Riemannian ascendant, as we will shortly see:  $\varphi_a$ ,  $\varpi_{ab}$ ,  $\hat{\gamma}_{ab}$  and  $\hat{c}^c{}_{ab}$ .

As a final comment, we would like to mention that Carrollian geometries may have isometries and in particular conformal isometries. The latter play a central role when considering the null conformal boundary, as they mirror bulk asymptotic symmetries. A vector field  $\xi = \xi^{\hat{u}}\mathbf{v} + \xi^a\hat{e}_a$  is a Carrollian Killing if the Lie derivative of the degenerate metric and of the field of observers vanishes. This requirement generates three conditions:

$$\begin{cases} \hat{\nabla}_{(a}\xi_{b)} + \xi^{\hat{u}}\hat{\gamma}_{(ab)} = 0 \\ \mathbf{v}(\xi^{\hat{u}}) + \xi^a\varphi_a = 0 \\ \hat{\nabla}_{\mathbf{v}}\xi_a - \hat{\gamma}_{(ab)}\xi^b = 0. \end{cases} \quad (\text{A.25})$$

In Papapetrou-Randers frame where  $\mathbf{v} = \frac{1}{\Omega}\partial_u$  and the degenerate metric has no time legs, the last condition selects the Carrollian diffeomorphisms,  $\partial_u\xi^i = 0$ . In the Cartan frame at hand all diffeomorphisms are permitted; the Killing fields are nonetheless further constrained. As usual, strong Killing fields must also leave the clock form invariant, which implies

$$\hat{e}_a(\xi^{\hat{u}}) - \varphi_a\xi^{\hat{u}} + 2\varpi_{ab}\xi^b = 0. \quad (\text{A.26})$$

Bulk Killing fields of Ricci-flat spacetimes are mapped onto strong Killings of their null boundary [25].

**Weyl covariance.** Following the pattern adopted for the affine connection, we introduce here a Weyl connection that respects the time and space splitting, associated with two Weyl-covariant derivatives. These act on weight- $w$  Carrollian tensors and deliver Carrollian tensors of weight  $w + 1$ .<sup>39</sup> The Weyl connection is encoded in  $\theta$  and  $\varphi_a$ , see (A.13) and (A.19), and the Weyl-covariant derivatives are defined as follows:

- on scalars

$$\hat{\mathcal{D}}_{\mathbf{v}}\Phi = \mathbf{v}(\Phi) + \frac{w}{d}\theta\Phi, \quad \hat{\mathcal{D}}_a\Phi = \hat{e}_a(\Phi) + w\varphi_a\Phi; \quad (\text{A.27})$$

- on vectors  $\mathbf{v} = v^a\hat{e}_a$

$$\hat{\mathcal{D}}_{\mathbf{v}}v^a = \hat{\nabla}_{\mathbf{v}}v^a + \frac{w}{d}\theta v^a, \quad \hat{\mathcal{D}}_av^b = \hat{\nabla}_av^b + w\varphi_av^b + \varphi^bv_a - \delta_a^bv^c\varphi_c; \quad (\text{A.28})$$

- on rank-2 tensors  $\mathbf{t} = t_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$ :

$$\hat{\mathcal{D}}_ct_{ab} = \hat{\nabla}_ct_{ab} + w\varphi_ct_{ab} + \varphi_at_{cb} + \varphi_bt_{ac} - \delta_{ac}t_{db}\varphi^d - \delta_{cb}t_{ad}\varphi^d, \quad (\text{A.29})$$

$$\hat{\mathcal{D}}_{\mathbf{v}}t_{ab} = \hat{\nabla}_{\mathbf{v}}t_{ab} + \frac{w}{d}\theta t_{ab}. \quad (\text{A.30})$$

Using Leibniz' rule one obtains the generalization for any conformal tensor.

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<sup>39</sup>As already mentioned in footnote 11, when working in a Cartan frame the Weyl properties are slightly modified and there is no contradiction with the results displayed in refs. [3, 9, 24, 25, 45, 46, 85], where a Papapetrou-Randers frame was in use.

The Riemann-Carroll-Weyl curvature is a weight-2 tensor defined through the commutator of the Carrollian Weyl derivatives acting on Carrollian scalars  $\Phi$ , vectors  $v^c$  or 2-tensors  $t^{cd}$  of weight  $w$ :<sup>40</sup>

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] \Phi = 2\varpi_{ab} \hat{\mathcal{D}}_v \Phi + w\Omega_{ab} \Phi, \quad (\text{A.31})$$

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] v^c = \hat{\mathcal{S}}^c_{dab} v^d + 2\varpi_{ab} \hat{\mathcal{D}}_v v^c + w\Omega_{ab} v^c, \quad (\text{A.32})$$

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] t^{cd} = \hat{\mathcal{S}}^c_{eab} t^{ed} + \hat{\mathcal{S}}^d_{cab} t^{ce} + 2\varpi_{ab} \hat{\mathcal{D}}_v t^{cd} + w\Omega_{ab} t^{cd}, \quad (\text{A.33})$$

where

$$\Omega_{ab} = \hat{e}_a(\varphi_b) - \hat{e}_b(\varphi_a) - \hat{c}^c_{ab} \varphi_c - \frac{2}{d} \varpi_{ab} \theta \quad (\text{A.34})$$

is yet another weight-2 Carrollian tensor. From the Riemann-Weyl-Carroll tensor, we define

$$\hat{\mathcal{S}}_{cd} = \hat{\mathcal{S}}^a_{cad}, \quad \hat{\mathcal{R}} = \delta^{cd} \hat{\mathcal{S}}_{cd}, \quad (\text{A.35})$$

all weight-2.

We can further consider time and space derivatives:

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] \Phi = -\xi^b_a \hat{\mathcal{D}}_b \Phi + w\hat{\mathcal{R}}_a \Phi, \quad (\text{A.36})$$

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] v^b = -\hat{\mathcal{S}}^b_{ac} v^c - \xi^c_a \hat{\mathcal{D}}_c v^b + w\hat{\mathcal{R}}_a v^b, \quad (\text{A.37})$$

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] t^{bc} = -\hat{\mathcal{S}}^b_{ad} t^{dc} - \hat{\mathcal{S}}^c_{ad} t^{bd} - \xi^d_a \hat{\mathcal{D}}_d t^{bc} + w\hat{\mathcal{R}}_a t^{bc}, \quad (\text{A.38})$$

revealing a clear pattern for any Carrollian conformal tensor. In these expressions

$$\hat{\mathcal{S}}^c_{ab} = -\hat{\mathcal{S}}^c_{ba} = \hat{\mathcal{D}}^c \xi_{ab} - \hat{\mathcal{D}}_b \xi^c_a + \delta^c_a \hat{\mathcal{R}}_b - \delta_{ab} \hat{\mathcal{R}}^c \quad (\text{A.39})$$

and  $\hat{\mathcal{R}}_a$  are weight-two tensors. Note that in Cartan frame, both the shear  $\xi_{ab}$  and the vorticity  $\varpi_{ab}$  have weight one, regardless of the position of the indices. In natural frame  $\xi_{ij}$  and  $\varpi_{ij}$  have weight  $-1$ , but raising an index augments the weight by two units.

**Relation with a relativistic ascendant.** A Carrollian manifold as described earlier can be reached from a pseudo-Riemannian geometry at zero velocity of light  $k$ . Following the pattern proposed in eqs. (3.2), (3.3) and (3.4), we can express the metric (2.8) of the pseudo-Riemannian ascendant as

$$ds^2 = \eta_{AB} \theta^A \theta^B = -(\theta^{\hat{0}})^2 + \delta_{ab} \theta^a \theta^b = -k^2 (\hat{\theta}^{\hat{u}})^2 + \delta_{ab} \hat{\theta}^a \hat{\theta}^b, \quad (\text{A.40})$$

where we have assumed that all  $k$ -dependence is explicit i.e.  $\theta^a = \hat{\theta}^a$  while  $\theta^{\hat{0}} = k\hat{\theta}^{\hat{u}}$ . The relationship among the relativistic congruence (2.9) and the Carrollian fibre attributes, field of observers and clock form, is  $\mathbf{v} = \mathbf{u} = \hat{e}_{\hat{u}}$  for the former and  $\boldsymbol{\mu} = \frac{\mathbf{u}}{k^2} = -\hat{\theta}^{\hat{u}}$  for the latter.

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<sup>40</sup>The use of  $\hat{\mathcal{S}}$  is unconventional for a curvature, but is intended to avoid confusion with a slightly different definition given as  $\hat{\mathcal{R}}$  in [3, 25, 45, 46].



If the Carrollian frame, coframe and degenerate metric are parameterized as in eqs. (A.3), (A.4) and (A.7), then

$$e_{\hat{0}} = \frac{\gamma}{k} \left( \partial_u + v^i \partial_i \right) \quad \Leftrightarrow \quad \theta^{\hat{0}} = k \left( \frac{du}{\gamma} - \Delta_i \left( dx^i - v^i du \right) \right), \quad (\text{A.41})$$

$$e_a = e_a^i \left( \partial_i + \gamma \Delta_i \left( \partial_u + v^j \partial_j \right) \right) \quad \Leftrightarrow \quad \theta^a = e^a_i \left( dx^i - v^i du \right) \quad (\text{A.42})$$

and the relativistic metric reads:

$$\begin{aligned} ds^2 &= -k^2 \left( \frac{du}{\gamma} - \Delta_i \left( dx^i - v^i du \right) \right)^2 + \Gamma_{ij}^2 \left( dx^i - v^i du \right) \left( dx^j - v^j du \right), \\ &= -\frac{k^2}{\gamma^2} \left( du^2 - 2\gamma \Delta_i du \left( dx^i - v^i du \right) \right) + \left( \Gamma_{ij}^2 - k^2 \Delta_i \Delta_j \right) \left( dx^i - v^i du \right) \left( dx^j - v^j du \right), \end{aligned} \quad (\text{A.43})$$

where the normalized vector congruence is

$$\mathbf{u} = \gamma \left( \partial_u + v^i \partial_i \right). \quad (\text{A.44})$$

We will not explicitly operate with this frame, which coincides at  $v^i = 0$  with the Papapetrou-Randers form employed in refs. [3, 9, 24, 25, 45, 46], where  $\Omega = 1/\gamma$ ,  $b_i = \Delta_i$  and  $a_{ij} = \Gamma_{ij}^2$ .

At  $\Delta_i = 0$ , one recovers the boundary frame of bulk Newman-Unti anti-de Sitter gauge (modulo a remark stated in footnote 36 and valid here), and

$$d\theta^{\hat{0}} = \varphi_a \theta^a \wedge \theta^{\hat{0}}, \quad (\text{A.45})$$

which resonates with the Carrollian relative (A.20). Hence the boundary vorticity vanishes following eq. (A.46) below.

The pseudo-Riemannian manifold is equipped with a Levi-Civita connection. We would like to express the latter in terms of the Carrollian tensors appearing in eqs. (A.9) and (A.13) or (A.14). The purpose of this exercise is to provide the suitable tools for reaching the  $k \rightarrow 0$  limit in relativistic dynamical equations such as (2.1). We reckon that in the parameterization of  $\{d\theta^A\} = \{d\theta^{\hat{0}}, d\theta^a\}$ , eqs. (A.13) and (A.14), hold:

$$d\theta^{\hat{0}} - \varphi_a \theta^a \wedge \theta^{\hat{0}} + k \varpi_{ab} \theta^a \wedge \theta^b = 0, \quad d\theta^c + \frac{1}{k} \hat{\gamma}^c_a \theta^a \wedge \theta^{\hat{0}} + \frac{1}{2} \hat{c}^c_{ab} \theta^a \wedge \theta^b = 0. \quad (\text{A.46})$$

Thus the Levi-Civita affine connection one-form reads:

$$\begin{aligned} \omega_{ab} &= - \left( k \varpi_{ab} + \frac{1}{k} \hat{\gamma}_{[ab]} \right) \theta^{\hat{0}} + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\theta}^c \\ &= \left( k^2 \varpi_{ab} + \hat{\gamma}_{[ab]} \right) \mu + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\theta}^c \\ &= k^2 \varpi_{ab} \mu + \hat{\omega}_{ab}, \end{aligned} \quad (\text{A.47})$$

and

$$\omega^{\hat{0}}_a = \varphi_a \theta^{\hat{0}} - k \varpi_{ab} \theta^b + \frac{1}{k} \hat{\gamma}_{(ab)} \theta^b = -k \left( \varphi_a \mu + \varpi_{ab} \hat{\theta}^b \right) + \frac{1}{k} \hat{\gamma}_{(ab)} \hat{\theta}^b \quad (\text{A.48})$$

with  $\hat{\gamma}_{ab}^c$  as in (A.15). It has zero torsion and the curvature reads:

$$\begin{aligned} \mathcal{R}^{\hat{0}}_a &= \left[ \frac{1}{k} \left( \hat{\nabla}_v \hat{\gamma}_{(ab)} + \hat{\gamma}_{(ac)} \hat{\gamma}_{(bd)} \delta^{cd} \right) - k \left( \varpi_a^c \hat{\gamma}_{(cb)} + \varpi_b^c \hat{\gamma}_{(ca)} + \hat{\nabla}_{(a} \varphi_{b)} + \varphi_a \varphi_b \right) \right. \\ &\quad \left. + k^3 \varpi_a^c \varpi_{bc} \right] \hat{\theta}^b \wedge \mu + \frac{1}{2} \left[ \frac{1}{k} \left( \hat{R}_{bac} - \varphi_b \hat{\gamma}_{(ac)} + \varphi_c \hat{\gamma}_{(ab)} \right) \right. \\ &\quad \left. - k \left( \hat{\nabla}_a \varpi_{bc} + \varphi_a \varpi_{bc} + \varphi_b \varpi_{ac} - \varphi_c \varpi_{ab} \right) \right] \hat{\theta}^b \wedge \hat{\theta}^c, \end{aligned} \quad (\text{A.49})$$

$$\begin{aligned} \mathcal{R}^a_b &= \hat{\mathcal{R}}^a_b + \delta^{ad} \left[ \varphi_d \hat{\gamma}_{(cb)} - \varphi_b \hat{\gamma}_{(cd)} + k^2 \left( \hat{\nabla}_c \varpi_{db} + \varphi_c \varpi_{db} + \varphi_d \varpi_{cb} - \varphi_b \varpi_{cd} \right) \right] \hat{\theta}^c \wedge \mu \\ &\quad + \frac{1}{2} \delta^{ae} \left[ \frac{1}{k^2} \left( \gamma_{(ec)} \gamma_{(bd)} - \gamma_{(ed)} \gamma_{(bc)} \right) - \gamma_{(ec)} \varpi_{bd} + \gamma_{(ed)} \varpi_{bc} \right. \\ &\quad \left. - \gamma_{(bd)} \varpi_{ec} + \gamma_{(bc)} \varpi_{ed} + k^2 (2\varpi_{eb} \varpi_{cd} - \varpi_{ed} \varpi_{bc} + \varpi_{ec} \varpi_{bd}) \right] \hat{\theta}^c \wedge \hat{\theta}^d, \end{aligned} \quad (\text{A.50})$$

where we have used the Carrollian expressions available in (A.21), (A.22) and (A.23).

We would like now to make the contact with the Carrollian descendants. The relativistic congruence is  $u = -k\theta^{\hat{0}}$  see (2.9). Given the connection, we can determine its kinematical properties: the expansion  $\Theta$ , the acceleration  $a_A$ , the shear  $\sigma_{AB}$  and the vorticity  $\omega_{AB}$  as defined in eqs. (2.11), (2.12), (2.13). The latter tensors are all transverse (and traceless for the last two) and have thus non-vanishing components in spatial directions only (indices  $a, b, \dots$ ). We find

$$\Theta = \theta = \hat{\gamma}^c_c, \quad a_a = k^2 \varphi_a, \quad (\text{A.51})$$

and

$$\sigma_{ab} = \xi_{ab} = \hat{\gamma}_{(ab)} - \frac{\theta}{d} \delta_{ab}, \quad \omega_{ab} = k^2 \varpi_{ab}. \quad (\text{A.52})$$

We can furthermore determine the Weyl connection (2.10) (where we must trade the 2 for  $d$ )

$$A = \varphi_a \hat{\theta}^a - \frac{\theta}{d} \mu, \quad (\text{A.53})$$

and its curvature (2.20):

$$F = dA = \frac{1}{2} \Omega_{ab} \hat{\theta}^a \wedge \hat{\theta}^b + \hat{\mathcal{R}}_a \hat{\theta}^a \wedge \mu, \quad (\text{A.54})$$

where  $\Omega_{ab}$  and  $\hat{\mathcal{R}}_a$  are defined in eqs. (A.34) and (A.36) — explicitly

$$\hat{\mathcal{R}}_a = \hat{\nabla}_v \varphi_a + \xi_{ab} \varphi^b - \frac{1}{d} \hat{e}_a(\theta). \quad (\text{A.55})$$

All the above quantities are relativistic, but expressed in terms of the Carrollian descendants describing the properties of the manifold reached at vanishing- $k$ .

We can finally convey the relativistic conservation equations (2.1) for an arbitrary energy-momentum tensor  $T^{AB}$  as in (2.33), stated in Carrollian language. Given the choice of congruence, the transverse heat current and stress tensor have only spatial components:

$q^a$  and  $\tau^{ab}$ . We then define as usual the longitudinal and transverse components of the conservation equations,

$$\mathcal{L} = -u^B \nabla_C T^C_B = -k \nabla_C T^C_{\hat{0}} = -\nabla_C T^C_{\hat{u}}, \quad \mathcal{T}^a = e^a_B \nabla_C T^{CB} = \nabla_C T^{Ca}, \quad (\text{A.56})$$

and explicitly find:

$$\mathcal{L} = \mathbf{v}(\varepsilon) + \theta \varepsilon + \left( \hat{\nabla}_a + 2\varphi_a \right) q^a + \left( \xi_{ab} + \frac{\theta}{d} \delta_{ab} \right) \left( \tau^{ab} + p \delta^{ab} \right), \quad (\text{A.57})$$

$$\mathcal{T}^a = \left( \hat{\nabla}_b + \varphi_b \right) \left( \tau^{ab} + p \delta^{ab} \right) + \varphi^a \varepsilon + 2q_b \varpi^{ba} + \frac{1}{k^2} \left( \hat{\nabla}_b q^a + \frac{d+1}{d} \theta q^a + \xi^{ab} q_b \right). \quad (\text{A.58})$$

In the conformal case, assuming thus  $\varepsilon = dp$  and  $\tau_a^a = 0$  and canonical conformal weights  $d+1$  for  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  (we are in Cartan' frame and the weights do not depend on the position of the indices), these equations are recast as:

$$\mathcal{L} = \hat{\mathcal{D}}_v \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab}, \quad (\text{A.59})$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \varpi^{ba} + \frac{1}{k^2} \left( \hat{\mathcal{D}}_b q^a + \xi^{ab} q_b \right). \quad (\text{A.60})$$

As discussed extensively in refs. [45, 46], the outcome of the Carrollian limit depends on the behaviour of  $\varepsilon$ ,  $q^a$  and  $\tau^{ab}$  with respect to  $k$ . The equations at hand will be conceivably multiplied, leading to replicas. The same phenomenon occurs in the Galilean limit with the emergence of the continuity equation out of the relativistic longitudinal equation, besides the energy equation.

We would like to close this section with some formulas that are useful when considering the zero- $k$  limit, leading in particular to the flux-balance equation (3.47). In the following, we reduce the Riemannian Levi-Civita and Weyl covariant derivatives in terms of the Carrollian connections introduced earlier.

**Levi-Civita** We will present the vector and the rank-two tensor:

$$\mathbf{V} = V^A \mathbf{e}_A \text{ — } V^a \text{ provide the components of a Carrollian vector and } V_{\hat{u}} = kV_{\hat{0}} = -kV^{\hat{0}} \text{ a Carrollian scalar}$$

$$\begin{cases} k^2 \nabla_{\hat{0}} V^{\hat{0}} = k\mathbf{v}(V^{\hat{0}}) + k^2 \varphi_a V^a \\ k \nabla_{\hat{0}} V^b = \hat{\nabla}_v V^b + kV^{\hat{0}} \varphi^b + k^2 V^a \varpi_a^b \\ k \nabla_a V^{\hat{0}} = k \hat{e}_a(V^{\hat{0}}) + \left( \xi_{ab} + \frac{\theta}{d} \delta_{ab} + k^2 \varpi_{ab} \right) V^b \\ \nabla_a V^b = \hat{\nabla}_a V^b + \frac{1}{k} \left( \xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b \right) V^{\hat{0}}; \end{cases} \quad (\text{A.61})$$

$$\mathbf{T} = T^{AB} \mathbf{e}_A \otimes \mathbf{e}_B \text{ — } T^{ab} \text{ are farther interpreted as components of a Carrollian rank-two tensor, } T_{\hat{u}}^a = kT_{\hat{0}}^a = -kT^{\hat{0}a} \text{ and } T^a_{\hat{u}} = kT^a_{\hat{0}} = -kT^{a\hat{0}} \text{ those of Carrollian}$$

vectors, while  $T_{\hat{u}\hat{u}} = k^2 T_{\hat{0}\hat{0}} = k^2 T^{\hat{0}\hat{0}}$  gives a Carrollian scalar

$$\begin{cases} k^3 \nabla_{\hat{0}} T^{\hat{0}\hat{0}} = k^2 \mathbf{v} (T^{\hat{0}\hat{0}}) + k^3 \varphi_a (T^{a\hat{0}} + T^{\hat{0}a}) \\ k^2 \nabla_{\hat{0}} T^{b\hat{0}} = k \hat{\nabla}_{\mathbf{v}} T^{b\hat{0}} + k^2 \varphi^b T^{\hat{0}\hat{0}} + k^2 \varphi_a T^{ba} + k^3 \varpi_a^b T^{a\hat{0}} \\ k \nabla_{\hat{0}} T^{ab} = \hat{\nabla}_{\mathbf{v}} T^{ab} + k (\varphi^a T^{\hat{0}b} + \varphi^b T^{a\hat{0}}) + k^2 (T^{ac} \varpi_c^b + T^{cb} \varpi_c^a) \\ k \nabla_a T^{b\hat{0}} = k \hat{\nabla}_a T^{b\hat{0}} + (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{bc} + (\xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b) T^{\hat{0}\hat{0}} \\ k^2 \nabla_a T^{\hat{0}\hat{0}} = k^2 \hat{e}_a (T^{\hat{0}\hat{0}}) + k (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{c\hat{0}} + k (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{\hat{0}c} \\ \nabla_a T^{bc} = \hat{\nabla}_a T^{bc} + \frac{1}{k} (\xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b) T^{\hat{0}c} + \frac{1}{k} (\xi_a^c + \frac{\theta}{d} \delta_a^c + k^2 \varpi_a^c) T^{b\hat{0}}; \end{cases} \quad (\text{A.62})$$

Weyl similarly:

$$\begin{aligned} \mathbf{V} &= V^A \mathbf{e}_A \\ \begin{cases} k^2 \mathcal{D}_{\hat{0}} V^{\hat{0}} = k \hat{\mathcal{D}}_{\mathbf{v}} V^{\hat{0}} \\ k \mathcal{D}_{\hat{0}} V^b = \hat{\mathcal{D}}_{\mathbf{v}} V^b + k^2 V^a \varpi_a^b \\ k \mathcal{D}_a V^{\hat{0}} = k \hat{\mathcal{D}}_a V^{\hat{0}} + (\xi_{ab} + k^2 \varpi_{ab}) V^b \\ \mathcal{D}_a V^b = \hat{\mathcal{D}}_a V^b + \frac{1}{k} (\xi_a^b + k^2 \varpi_a^b) V^{\hat{0}}; \end{cases} \end{aligned} \quad (\text{A.63})$$

$$\mathbf{T} = T^{AB} \mathbf{e}_A \otimes \mathbf{e}_B$$

$$\begin{cases} k^3 \mathcal{D}_{\hat{0}} T^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_{\mathbf{v}} (k^2 T^{\hat{0}\hat{0}}) \\ k^2 \mathcal{D}_{\hat{0}} T^{\hat{0}b} = \hat{\mathcal{D}}_{\mathbf{v}} (k T^{\hat{0}b}) + k^3 T^{\hat{0}a} \varpi_a^b \\ k \mathcal{D}_{\hat{0}} T^{ab} = \hat{\mathcal{D}}_{\mathbf{v}} T^{ab} + k^2 (T^{cb} \varpi_c^a + T^{ac} \varpi_c^b) \\ k^2 \mathcal{D}_a T^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_a (k^2 T^{\hat{0}\hat{0}}) + (\xi_{ab} + k^2 \varpi_{ab}) k T^{b\hat{0}} + (\xi_{ab} + k^2 \varpi_{ab}) k T^{\hat{0}b} \\ k \mathcal{D}_a T^{\hat{0}b} = \hat{\mathcal{D}}_a (k T^{\hat{0}b}) + (\xi_{ac} + k^2 \varpi_{ac}) T^{cb} + (\xi_a^b + k^2 \varpi_a^b) T^{\hat{0}\hat{0}} \\ \mathcal{D}_a T^{bc} = \hat{\mathcal{D}}_a T^{bc} + \frac{1}{k} (\xi_a^b + k^2 \varpi_a^b) T^{\hat{0}c} + \frac{1}{k} (\xi_a^c + k^2 \varpi_a^c) T^{b\hat{0}}. \end{cases} \quad (\text{A.64})$$

## B The Carrollian Cotton tensors in three dimensions

The Cotton tensor introduced in section 2 can be decomposed in terms of Carrollian descendants, which obey Carrollian identities resulting from (2.39). In Papapetrou-Randers' frame and for vanishing  $\xi_{ab}$  a thorough exhibition is available in appendix C of [25]. The Carrollian Cotton tensor will be investigated from a more general viewpoint in [24]. Here we will summarize its properties in Cartan' frame with  $\xi_{ab} \neq 0$ . Prior to this presentation we need to spend some time on  $d = 2$ .

In three boundary spacetime dimensions, we pointed out that given a congruence  $\mathbf{u}$ , a transverse Hodge duality can be designed mapping transverse vectors to transverse vectors and symmetric, traceless and transverse two-tensors onto similar objects, eqs. (2.28), (2.29), (2.31), (2.32). This procedure is readily extended to a Carroll structure  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  and the duality coincides with the Hodge duality in the 2-dimensional basis  $\mathcal{S}$ : Carrollian vectors are mapped onto Carrollian vectors and Carrollian symmetric and

traceless two-tensors onto the same class. In the relativistic Cartan frame we use here the antisymmetric pseudo-tensor is  $\epsilon_{ABC}$  or  $\epsilon^{ABC}$  with  $\epsilon_{\hat{0}\hat{1}\hat{2}} = -\epsilon^{\hat{0}\hat{1}\hat{2}} = 1$ , whereas using (2.9) in (2.28) we find  $\hat{\eta}_{AB} = -\epsilon_{\hat{0}AB}$  so that only  $\hat{\eta}_{ab}$  is non-zero with  $\hat{\eta}_{\hat{1}\hat{2}} = -1$ . We adopt this convention for the Carrollian object, without introducing any further symbol. Now (2.29) translates into

$$\hat{\eta}_{ac}\hat{\eta}_b^c = \delta_{ab}, \quad \hat{\eta}^{ab}\hat{\eta}_{ab} = 2, \quad (\text{B.1})$$

and (2.31), (2.32) give

$$*v^a = \hat{\eta}^b{}_a v_b, \quad *w_{ab} = \hat{\eta}^c{}_a w_{cb}, \quad (\text{B.2})$$

for Carrollian vectors  $v^a$  and Carrollian symmetric, traceless tensors  $w_{ab}$ . We will often use the following identities, generalizable to any tensor:

$$**v^a = -v^a, \quad *v^a w_a = -v^a *w_a. \quad (\text{B.3})$$

The Carroll-Riemann, Carroll-Ricci and scalar (A.24) curvatures read:

$$\hat{R}_{abcd} = \hat{K}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \hat{R}_{ab} = \hat{K}\delta_{ab}, \quad \hat{R} = 2\hat{K}. \quad (\text{B.4})$$

The Carroll-Weyl-Riemann and Ricci tensors, the Carroll-Weyl-Ricci scalar (see (A.35)) as well as the Carroll-Weyl tensor curvature (A.34) are

$$\hat{\mathcal{S}}_{abcd} = \hat{\mathcal{K}}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \hat{\mathcal{S}}_{ab} = \hat{\mathcal{K}}\delta_{ab}, \quad \Omega_{ab} = -\hat{\mathcal{A}}\hat{\eta}_{ab}, \quad (\text{B.5})$$

expressed in terms of two weight-2 Weyl-covariant scalars:

$$\hat{\mathcal{K}} = \hat{K} + \hat{\nabla}_a \varphi^a, \quad \hat{\mathcal{A}} = *\varpi\theta - *\varphi \quad (\text{B.6})$$

with

$$*\varpi = \frac{1}{2}\hat{\eta}^{ab}\varpi_{ab} \Leftrightarrow \varpi_{ab} = \hat{\eta}_{ab}*\varpi, \quad (\text{B.7})$$

and

$$*\varphi = \frac{1}{2}\hat{\eta}^{ab}\varphi_{ab} \quad \text{where} \quad \varphi_{ab} = \hat{e}_a(\varphi_b) - \hat{e}_b(\varphi_a). \quad (\text{B.8})$$

These obey Carroll-Bianchi identities:

$$2\hat{\mathcal{D}}_\nu *\varpi + \hat{\mathcal{A}} = 0, \quad (\text{B.9})$$

$$\hat{\mathcal{D}}_\nu \hat{\mathcal{K}} - \hat{\mathcal{D}}_a \hat{\mathcal{K}}^a - \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \xi^{ab} = 0, \quad (\text{B.10})$$

$$\hat{\mathcal{D}}_\nu \hat{\mathcal{A}} + \hat{\eta}^{ab}\hat{\mathcal{D}}_a \hat{\mathcal{K}}_b = 0. \quad (\text{B.11})$$

The Carroll reduction of the Cotton tensor is encrypted in the longitudinal, mixed and transverse components (2.41) and (2.43), which encompass several weight-3 Carrollian scalars, vectors and symmetric, traceless two-tensors, dubbed ‘‘Carrollian Cotton tensors.’’ In Cartan’ frame we obtain

$$c = c_{(-1)}k^2 + c_{(0)} + \frac{c_{(1)}}{k^2} + \frac{c_{(2)}}{k^4}, \quad (\text{B.12})$$

$$c^a = k^2\psi^a + \chi^a + \frac{z^a}{k^2}, \quad (\text{B.13})$$

$$c^{ab} = k^2\Psi^{ab} + X^{ab} + \frac{Z^{ab}}{k^2} \quad (\text{B.14})$$

with

- four Carroll scalars:

$$c_{(-1)} = 8 * \varpi^3, \quad c_{(0)} = \left( \hat{\mathcal{D}}_a \hat{\mathcal{D}}^a + 2 \hat{\mathcal{K}} \right) * \varpi, \quad c_{(1)} = \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \xi^{ab}, \quad c_{(2)} = * \xi_{ab} \hat{\mathcal{D}}_v \xi^{ab}; \quad (\text{B.15})$$

- three Carroll vectors:

$$\psi^a = 3 \hat{\eta}^{ba} \hat{\mathcal{D}}_b * \varpi^2, \quad (\text{B.16})$$

$$\chi^a = \frac{1}{2} \hat{\eta}^{ba} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}^a \hat{\mathcal{A}} - 2 * \varpi \left( \hat{\mathcal{R}}^a + 2 \hat{\mathcal{D}}_b \xi^{ab} \right) + 3 \hat{\mathcal{D}}_b \left( * \varpi \xi^{ab} \right), \quad (\text{B.17})$$

$$z^a = \frac{1}{2} \hat{\eta}^{ab} \hat{\mathcal{D}}_b \xi^2 - \hat{\mathcal{D}}_b \hat{\mathcal{D}}_v * \xi^{ab} - * \xi^a_b \hat{\mathcal{D}}_c \xi^{bc}, \quad (\text{B.18})$$

where we defined<sup>41</sup>

$$\xi^2 = \frac{1}{2} \xi^{ab} \xi_{ab} \Leftrightarrow \xi^{ac} \xi_c^b = \xi^2 \delta^{ab}; \quad (\text{B.19})$$

- three Carroll traceless and symmetric two-index tensors:

$$\Psi^{ab} = -2 * \varpi^2 * \xi^{ab} + \hat{\mathcal{D}}^a \hat{\mathcal{D}}^b * \varpi - \frac{1}{2} \delta^{ab} \hat{\mathcal{D}}_c \hat{\mathcal{D}}^c * \varpi - \hat{\eta}^{ab} \hat{\mathcal{D}}_v * \varpi^2, \quad (\text{B.20})$$

$$\begin{aligned} X^{ab} &= \frac{1}{2} \hat{\eta}^{ca} \hat{\mathcal{D}}_c \left( \hat{\mathcal{R}}^b + \hat{\mathcal{D}}_d \xi^{bd} \right) + \frac{1}{2} \hat{\eta}^{cb} \hat{\mathcal{D}}^a \left( \hat{\mathcal{R}}_c + \hat{\mathcal{D}}^d \xi_{cd} \right) \\ &\quad - \frac{3}{2} \hat{\mathcal{A}} \xi^{ab} - \hat{\mathcal{K}} * \xi^{ab} + 3 * \varpi \hat{\mathcal{D}}_v \xi^{ab}, \end{aligned} \quad (\text{B.21})$$

$$Z^{ab} = 2 * \xi^{ab} \xi^2 - \hat{\mathcal{D}}_v \hat{\mathcal{D}}_v * \xi^{ab}. \quad (\text{B.22})$$

As for the conservation equations (2.39), expressing them as in (A.59), (A.60), they yield

$$\mathcal{L}_{\text{Cot}} = -k^3 \mathcal{D}_{\text{Cot}} - k \mathcal{E}_{\text{Cot}} - \frac{\mathcal{F}_{\text{Cot}}}{k} - \frac{\mathcal{W}_{\text{Cot}}}{k^3} = 0, \quad (\text{B.23})$$

$$\mathcal{T}_{\text{Cot}}^a = k^3 \mathcal{I}_{\text{Cot}}^a + k \mathcal{G}_{\text{Cot}}^a + \frac{\mathcal{H}_{\text{Cot}}^a}{k} + \frac{\mathcal{X}_{\text{Cot}}^a}{k^3} = 0 \quad (\text{B.24})$$

with

$$\mathcal{D}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(-1)} - \hat{\mathcal{D}}_a \psi^a, \quad (\text{B.25})$$

$$\mathcal{E}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(0)} - \hat{\mathcal{D}}_a \chi^a + \Psi_{ab} \xi^{ab}, \quad (\text{B.26})$$

$$\mathcal{F}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(1)} - \hat{\mathcal{D}}_a z^a + X_{ab} \xi^{ab}, \quad (\text{B.27})$$

$$\mathcal{W}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(2)} + Z_{ab} \xi^{ab} \quad (\text{B.28})$$

and

$$\mathcal{I}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(-1)} + 2 * \varpi * \psi^a, \quad (\text{B.29})$$

$$\mathcal{G}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_v \psi^a + \xi^{ab} \psi_b, \quad (\text{B.30})$$

$$\mathcal{H}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(1)} - \hat{\mathcal{D}}_b X^{ab} + 2 * \varpi * z^a + \hat{\mathcal{D}}_v \chi^a + \xi^{ab} \chi_b, \quad (\text{B.31})$$

$$\mathcal{X}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(2)} - \hat{\mathcal{D}}_b Z^{ab} + \hat{\mathcal{D}}_v z^a + \xi^{ab} z_b. \quad (\text{B.32})$$

<sup>41</sup>Many identities of this sort are useful:  $\xi^{ac} * \xi_c^b = \xi^2 \hat{\eta}^{ab}$ ,  $* \xi^{ac} * \xi_c^b = \xi^2 \delta^{ab}$ ,  $\varpi^{ac} \varpi_c^b = * \varpi^2 \delta^{ab}$ .

The four couples of equations  $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$ ,  $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$  and  $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$  originate from the different orders in  $k$  in which the conservation of the Cotton tensor (2.39) decomposes. These are *purely geometrical identities fulfilled on any three-dimensional Carroll structure*  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ . Moreover, they are typical Carrollian conservation equations obtained as a consequence of general covariance applied to a Weyl-invariant action  $S = -\frac{1}{2} \int_{\mathcal{M}} \hat{\eta}_{ab} \hat{\theta}^a \wedge \hat{\theta}^b \wedge \mu \mathcal{L}$  defined on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ :

$$\hat{\mathcal{D}}_{\nu} \Pi + \hat{\mathcal{D}}_a \Pi^a + \Upsilon^a_b \xi_a^b = 0, \tag{B.33}$$

$$\frac{1}{2} \hat{\mathcal{D}}_a \Pi + \hat{\mathcal{D}}_b \Upsilon^b_a + 2 * \varpi * \Pi^a + \hat{\mathcal{D}}_{\nu} P_a + \xi_a^b P_b = 0. \tag{B.34}$$

The momenta  $\Pi$ ,  $\Pi^a$ ,  $P_b$  and  $\Pi^a_b = \Upsilon^a_b + \frac{1}{2} \Pi \delta^a_b$  are defined as variations of the action with respect to the triad  $\{\mu, \hat{\theta}^a\}$  (the explicit computation is accessible in ref. [45] for the Papapetrou-Randers frame,<sup>42</sup> where the organizing pattern is the subgroup of Carrollian diffeomorphisms instead of the subgroup of local orthogonal transformations in the tangent space). These are the energy density, the energy flux, the momentum and the stress.

For Carroll structures with vanishing Carrollian shear,  $\xi_{ab} = 0$ , met e.g. at null infinity of asymptotically flat spacetimes, six out of the ten Carroll Cotton tensors survive:  $c_{(-1)}$ ,  $c_{(0)}$ ,  $\psi^a$  as in eqs. (B.15), (B.16) and  $\chi^a$ ,  $\Psi^{ab}$ ,  $X^{ab}$ . Using eqs. (B.17), (B.20), (B.21) we find the simplified expressions of the latter:

$$\chi^a = \frac{1}{2} * \hat{\mathcal{D}}^a \mathcal{K} + \frac{1}{2} \hat{\mathcal{D}}^a \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{K}}^a, \tag{B.35}$$

$$\Psi^{ab} = \hat{\mathcal{D}}^a \hat{\mathcal{D}}^b * \varpi - \frac{1}{2} \delta^{ab} \hat{\mathcal{D}}_c \hat{\mathcal{D}}^c * \varpi - \hat{\eta}^{ab} \hat{\mathcal{D}}_{\nu} * \varpi^2, \tag{B.36}$$

$$X^{ab} = \frac{1}{2} * \hat{\mathcal{D}}^a \hat{\mathcal{K}}^b + \frac{1}{2} \hat{\mathcal{D}}^a * \hat{\mathcal{K}}^b. \tag{B.37}$$

These tensors now obey

$$\mathcal{D}_{\text{Cot}} = -\hat{\mathcal{D}}_{\nu} c_{(-1)} - \hat{\mathcal{D}}_a \psi^a = 0, \tag{B.38}$$

$$\mathcal{E}_{\text{Cot}} = -\hat{\mathcal{D}}_{\nu} c_{(0)} - \hat{\mathcal{D}}_a \chi^a = 0, \tag{B.39}$$

and

$$\mathcal{I}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(-1)} + 2 * \varpi * \psi^a = 0, \tag{B.40}$$

$$\mathcal{G}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_{\nu} \psi^a = 0, \tag{B.41}$$

$$\mathcal{H}_{\text{Cot}}^a = -\hat{\mathcal{D}}_b X^{ab} + \hat{\mathcal{D}}_{\nu} \chi^a = 0. \tag{B.42}$$

On a Carroll manifold in Cartan frame, the degenerate metric is invariant under local Carroll-group transformations. Invariance of the action under its local orthogonal subgroup is in line with a symmetric  $\Pi_{ab}$ ; invariance under local Carroll boosts demands  $\Pi^a = 0$ . This

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<sup>42</sup>Equations (B.33) and (B.34) were obtained for the first time in ref. [46]. They have been recently rediscussed in [84].

is not always met in Carrollian theories approached from relativistic theories at vanishing speed of light (see e.g. [85]) — alternatively it can be imposed by hand as in [86]. The Cotton tensor and the corresponding Chern-Simons dynamics [24] admirably illustrate this feature, which persists in the flux-balance equations of Ricci flat spacetimes, powered by gravitational radiation.

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# Ehlers, Carroll, charges and dual charges

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**ABSTRACT:** We unravel the boundary manifestation of Ehlers' hidden Möbius symmetry present in four-dimensional Ricci-flat spacetimes that enjoy a time-like isometry and are Petrov-algebraic. This is achieved in a designated gauge, shaped in the spirit of flat holography, where the Carrollian three-dimensional nature of the null conformal boundary is manifest and covariantly implemented. The action of the Möbius group is local on the space of Carrollian boundary data, among which the Carrollian Cotton tensor plays a predominant role. The Carrollian and Weyl geometric tools introduced for shaping an appropriate gauge, as well as the boundary conformal group, which is  $BMS_4$ , allow to define electric/magnetic, leading/subleading towers of charges directly from the boundary Carrollian dynamics and explore their behaviour under the action of the Möbius duality group.

**KEYWORDS:** Classical Theories of Gravity, Gauge-Gravity Correspondence, Global Symmetries, Space-Time Symmetries

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**1 Introduction**

Hidden symmetries have a long history in relativistic theories of gravity, which started with the seminal work of Ehlers in the late fifties [1]. It was shown in this article that in the presence of an isometry, vacuum Einstein’s equations were invariant under Möbius transformations. This observation triggered an important activity in several directions. In line with the sixties’ renaissance of general relativity, it opened the way for solution-generating techniques applicable to vacuum Einstein’s equations [2, 3]. This was soon generalized to situations with more commuting Killing fields [4, 5] — and bigger hidden symmetry group, providing the system with remarkable and unexpected integrability properties [6–14]. The underlying deep origin for the above pattern was unravelled with the advent of higher-dimensional supergravity theories, and is rooted in the reduction mechanism. This has revealed a wide class of hidden groups, among which the exceptional play a prominent role (see e.g. [15–17], or [18] for a more recent presentation and further references).

The integrable sector of Einstein’s equations is only a tiny fraction of their solution space. Unveiling the latter, in conjunction with its asymptotic symmetries and conserved charges, has been in the very early agenda of general relativity. It shares features with

gauge theories because of general covariance, and has led Bondi to come out with his homonymous gauge, where a systematic resolution of Einstein’s equations is possible as an expansion in powers of a radial coordinate.<sup>1</sup> This delivers a set of functions of time and angular coordinates, obeying first-order time-evolution equations. For Ricci-flat spacetimes (asymptotically locally flat) this set is infinite, but it is finite for Einstein spacetimes with negative cosmological constant (asymptotically locally anti-de Sitter).

In modern language, the set of functions necessary for reconstructing the solution are said to be defined on the conformal boundary of the spacetime. In the asymptotically flat instance the conformal boundary is null infinity and features a Carrollian three-dimensional hypersurface.<sup>2</sup> Bulk Einstein dynamics is therefore traded for boundary effective Carrollian conformal field dynamics. This statement is accurate when discussing Einstein’s equations. Whether it could be promoted to a holographic principle akin to the better known AdS/CFT involving asymptotically anti-de Sitter spacetimes and conformal field theories defined on their time-like conformal boundary is a timely subject, currently under scrutiny.

How do hidden symmetries such as Ehlers’ act on the Carrollian boundary data? This is the central question we would like to address in the present work. The conformal symmetries of the boundary reflect the asymptotic symmetries of the bulk. These define for instance the  $BMS_4$  algebra (Bondi-van der Burg-Metzner-Sachs [27–29]), which is isomorphic to the conformal Carroll algebra in three dimensions  $\mathfrak{ccarr}(3)$  (see [30, 31]), and emerges upon appropriate fall-off conditions. From this perspective, wondering how the bulk hidden symmetries are embraced by the Carrollian boundary and what their interplay is with  $BMS_4 \equiv \mathfrak{ccarr}(3)$ , is both natural and relevant.

There is yet another motivation for pursuing this analysis. Following Geroch [2, 3], the action of some Ehlers subgroup is a duality rotation in the plane of gravitational electric and gravitational magnetic charges, as are e.g. the mass and the nut charge. Ricci-flat spacetimes possess in fact multiple infinities of charges (not necessarily conserved), incarnated in pairs of electric and magnetic representatives, and originating from the infinitely many independent “subleading” degrees of freedom necessary for reconstructing the bulk solution, as well as the infinitely many generators of the asymptotic symmetry group. This picture has been widely conveyed through the work of Godazgar-Godazgar-Pope [32–34] (see also [35–39]) and amply deserves to be reconsidered in the light of hidden symmetries. The remarkable fact is here that such an analysis can be conducted exclusively on the boundary, where the charges are constructed (see e.g. [40]) using the boundary dynamics combined with the three-dimensional-boundary Carrollian conformal isometries, the latter being always generated by the infinite-dimensional algebra  $BMS_4 \equiv \mathfrak{so}(3, 1) \ltimes$  supertranslations [41]. Translating the Ehlers group on the null boundary forcedly exhibits a mapping among the infinite towers of charges, which is obscured in a bulk approach. The boundary Carrollian geometry provides the most suitable language for clarifying these properties.

<sup>1</sup>Other canonical gauges are Fefferman-Graham or Newman-Unti — see e.g. refs. [19, 20] for a review and more complete reading suggestions on this subject.

<sup>2</sup>The original observation that triggered this “flat-holography” activity is described in refs. [21, 22]. A more systematic analysis in four dimensions was presented in [23], which set the foundations for a Carrollian description of the dual theory, and provides a more complete reference list. Up-to-date developments in this vein are refs. [24–26].

In the present work, we will analyse along the above lines the integrable sector of Ricci-flat spacetimes possessing a time-like Killing field, whose congruence coincides with the boundary Carrollian fiber. This sector is obtained by setting conditions on the boundary data, which ultimately guarantee that the infinite series in powers of the radial coordinates is resummed. The boundary conditions involve the Carrollian boundary Cotton tensor and the Carrollian boundary momenta (see [23]), which both enter the boundary computation of the charges associated with the solution at hand. They rephrase the special structure of bulk Weyl tensor<sup>3</sup> and unsurprisingly lead to algebraic Ricci-flat spacetimes. Although this class leaves interesting cases aside, it captures the main feature of the Ehlers-group boundary manifestation. The latter turns out to be an *algebraic* transformation mixing e.g. the Carrollian Cotton scalar and the Bondi mass aspect. This mixing is transmitted to other boundary observables, including the charges through their boundary expression, and completes the picture of the bulk-and-boundary action of the hidden group. The action of the Möbius group, generically non-local on the four-dimensional Ricci-flat metric, is therefore *local* on the boundary — as it is on the three-dimensional sigma-model of the reduction along the bulk Killing congruence.

The starting point of our study is a reminder on the Ehlers group and the Geroch method, as they emerge in the reduction of Ricci-flat spacetimes along orbits of one-dimensional groups of motions. We next move and describe the bulk-to-boundary relationship for four-dimensional Ricci-flat spacetimes. This requires the use of a gauge (we call it “modified Newman-Unti” or “covariantized”) in which the three-dimensional Carrollian boundary geometry is ostensible. The bulk metric in this gauge is manifestly covariant with respect to the boundary Carrollian diffeomorphisms and to the boundary Weyl transformations. The Carrollian boundary dynamics induced by the bulk Einstein’s equations is the following item in our agenda, which further enables us to define sets of charges and dual charges — electric and magnetic. Finally, using the available tools for a Ricci-flat spacetime enjoying a time-like isometry, we translate the action of the bulk Möbius transformations onto the boundary observables. This analysis is performed for the integrable sector (resummable metrics) and is based on a specific class of time-like Killing fields.

Most of our investigation relies on rather unusual geometric tools, which have been developed recently in the framework of Carroll structures. We have sorted them out in a first appendix, valid for any dimension  $d + 1$ . Carrollian dynamics and conservation properties, necessary for describing the boundary perspective as inherited from bulk Einstein’s equations, is summarized in the second appendix. The third appendix is specific to three dimensions with emphasis on the Carrollian Cotton tensors.

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<sup>3</sup>The interplay between the bulk Weyl tensor, expanded in powers of the radial coordinate, and the boundary Cotton plus energy-momentum tensors for Einstein spacetimes was disclosed in [42–44] — see also [45]. There is no rigorous similar statement for the Ricci-flat instance since the Carrollian relatives of the Cotton tensor have not yet been thoroughly investigated.



## 2 Ehlers and Geroch

We remind here Geroch' generalization of Ehlers' work following [2]. We consider a four-dimensional pseudo-Riemannian manifold  $(\mathcal{M}, g)$  possessing an isometry generated by a time-like<sup>4</sup> Killing vector field  $\xi$ . The latter has norm and twist — here  $A, B, \dots \in \{0, \dots, 3\}$ :

$$\lambda = \xi^A \xi_A, \tag{2.1}$$

$$w_A = \eta_{ABCD} \xi^B \nabla^C \xi^D, \tag{2.2}$$

where  $\eta_{ABCD} = \sqrt{-g} \epsilon_{ABCD}$  ( $\epsilon_{0123} = 1$ ). Assuming the spacetime be Ricci-flat,<sup>5</sup> one shows that the one-form  $w = w_A dx^A$  is closed so locally exact, hence

$$w = d\omega \tag{2.3}$$

with  $\omega$  a scalar function.

We define the three-dimensional space  $\mathcal{S}$  as the quotient  $\mathcal{M}/\text{orb}(\xi)$ . This coset space is not a subspace of  $\mathcal{M}$  unless  $\xi$  is hypersurface-orthogonal, which would imply zero twist with  $\mathcal{S}$  the orthogonal hypersurface. A natural metric on  $\mathcal{S}$  is induced by  $g$  of  $\mathcal{M}$ :

$$h_{AB} = g_{AB} - \frac{\xi_A \xi_B}{\lambda}, \tag{2.4}$$

which defines the projector onto  $\mathcal{S}$  as

$$h^B_A = \delta^B_A - \frac{\xi^B \xi_A}{\lambda}. \tag{2.5}$$

The fully antisymmetric tensor for (2.4) is  $\eta_{ABC} = \frac{-1}{\sqrt{-\lambda}} \eta_{ABCD} \xi^D$ .

Tensors of  $\mathcal{M}$ , transverse and invariant with respect to  $\xi$ , are in one-to-one correspondence with tensors on  $\mathcal{S}$ . If  $T$  is a tensor of  $\mathcal{S}$ , the covariant derivative  $\mathcal{D}$  defined following this correspondence,

$$\mathcal{D}_C T_{A_1 \dots A_p}{}^{B_1 \dots B_q} = h^L_C h^{M_1}_{A_1} \dots h^{M_p}_{A_p} h^{B_1}_{N_1} \dots h^{B_q}_{N_q} \nabla_L T_{M_1 \dots M_p}{}^{N_1 \dots N_q} \tag{2.6}$$

with  $\nabla$  the Levi-Civita connection on  $(\mathcal{M}, g)$ , coincides with the Levi-Civita connection on  $(\mathcal{S}, h)$ . This sets a relationship between the Riemann tensor on  $\mathcal{S}$  and the Riemann tensor on  $\mathcal{M}$ , generalizing thereby the Gauss-Codazzi equations to the instance where  $\xi$  is not-hypersurface orthogonal:

$$\mathcal{R}_{ABCD} = h^P_{[A} h^Q_{B]} h^R_{[C} h^S_{D]} \left( R_{PQRS} + \frac{2}{\lambda} (\nabla_P \xi_Q \nabla_R \xi_S + \nabla_P \xi_R \nabla_Q \xi_S) \right) \tag{2.7}$$

(the calligraphic letters refer to curvature tensors of  $\mathcal{S}$ ).

The Ricci-flat dynamics for  $g_{AB}$  is recast in the present framework in terms of<sup>6</sup>

$$\tilde{h}_{AB} = \lambda h_{AB}, \tag{2.8}$$

---

<sup>4</sup>The described procedure goes through in the same fashion with space-like isometries, but keeping the two options would bring unnecessary multiplication of indices without shedding more light on our purpose.

<sup>5</sup>This property actually holds more generally for Einstein spacetimes [46].

<sup>6</sup>With our conventions, this metric is definite-negative.

as well as  $\omega$  and  $\lambda$  viewed as fields on  $\mathcal{S}$ , packaged in

$$\tau = \omega + i\lambda, \tag{2.9}$$

and obeying the following equations:<sup>7</sup>

$$\begin{aligned} \tilde{\mathcal{R}}_{AB} &= -\frac{2}{(\tau - \bar{\tau})^2} \tilde{\mathcal{D}}_{(A} \tau \tilde{\mathcal{D}}_{B)} \bar{\tau}, \\ \tilde{\mathcal{D}}^2 \tau &= \frac{2}{\tau - \bar{\tau}} \tilde{\mathcal{D}}_M \tau \tilde{\mathcal{D}}_N \tau \tilde{h}^{MN}. \end{aligned} \tag{2.10}$$

The first results from (2.7), while the second is obtained by a direct computation of the  $\mathcal{S}$ -Laplacian acting on  $\tau$ . Here  $\tilde{\mathcal{D}}_A$  and  $\tilde{\mathcal{R}}_{AB}$  are the Levi-Civita covariant derivative and the Ricci tensor associated with the metric  $\tilde{h}_{AB}$  displayed in (2.8).

Equations (2.10) feature two important properties. The first, due to Ehlers [1], is the invariance under transformations maintaining  $\tilde{h}_{AB}$  *unaltered* and mapping  $\tau$  into

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{R}). \tag{2.11}$$

This is the original instance where a *hidden* group,  $\text{SL}(2, \mathbb{R})$ , reveals upon reduction with respect to an isometry. The second, described by Geroch in [2, 3], is the method for reversing the reduction process, and finding a Ricci-flat four-dimensional spacetime with an isometry, starting from any solution of eqs. (2.10) encoded in  $\omega' + i\lambda' = \tau'$  and  $h'_{AB} = \frac{1}{\lambda'} \tilde{h}_{AB}$ . To this end, one shows that the  $\mathcal{S}$ -two-form defined as

$$F'_{AB} = \frac{1}{(-\lambda')^{3/2}} \eta'_{ABC} \mathcal{D}^C \omega' \tag{2.12}$$

is *closed*. Thus, locally

$$F' = d\eta'. \tag{2.13}$$

The one-form field  $\eta'$ , defined on  $\mathcal{S}$ , can be promoted to a field on  $\mathcal{M}$  by adding the necessary exact piece such that its normalization be

$$\xi^A \eta'_A = 1. \tag{2.14}$$

This defines a new Killing field on  $\mathcal{M}$

$$\xi' = \lambda' \eta' \tag{2.15}$$

and the new four-dimensional metric reads:<sup>8</sup>

$$g'_{AB} = h'_{AB} + \frac{\xi'_A \xi'_B}{\lambda'}. \tag{2.16}$$

---

<sup>7</sup>Equations (2.10) can be reached by varying a three-dimensional sigma-model action defined on  $\mathcal{S}$ . This is at the heart of many developments about integrability and hidden symmetries — see the already quoted literature for more information.

<sup>8</sup>The consequence of Möbius transformations on the Weyl tensor has been investigated in ref. [47].

Closing this executive reminder, we would like to add a remark. The  $SL(2, \mathbb{R})$  is hidden from the four-dimensional perspective, but explicit in the three-dimensional sigma-model, materialized here in eqs. (2.10). Nevertheless, part of this group is in fact visible in four dimensions because it acts as four-dimensional diffeomorphisms; part is creating genuinely different Ricci-flat solutions. This can be illustrated in the concrete example of Schwarzschild-Taub-NUT solutions with mass  $M$  and nut charge  $n$ . The compact subgroup of rotations  $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SO(2) \subset SL(2, \mathbb{R})$  induces rotations of angle  $2\chi$  in the parameter space  $(M, n)$ , while non-compact transformations  $\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \in N \subset SL(2, \mathbb{R})$  act homothetically,  $(M, n) \rightarrow (M/\alpha, n/\alpha)$ .

### 3 Ricci-flat spacetimes and Carrollian dynamics

#### 3.1 Bulk reconstruction and resumable Ricci-flat metrics

**Choosing a covariant gauge.** Four-dimensional Ricci-flat metrics are generally obtained as expansions in powers of a radial coordinate, in a designated gauge, usually Bondi or Newman-Unti. Appropriate fall-offs are assumed, and the solution is expressed in terms of an infinite set of functions of time and angles, obeying some evolution equations, mirroring Einstein’s equations (see [19] for details and further references). Can one define a three-dimensional boundary, and describe covariantly this set of functions and their dynamics?

The answer to this question has been known to be positive for a long time in the case of Einstein spacetimes. It is best formulated in the Fefferman-Graham gauge [48, 49] — see also [50] for a Weyl-covariant extension of this gauge. The (conformal) boundary is a three-dimensional pseudo-Riemannian spacetime, and every order in the expansion brings a tensorial object with respect to the boundary geometry. All these are expressed in terms of two independent tensors: the first and second fundamental forms of the boundary, namely the boundary metric and the boundary energy-momentum tensor, which is covariantly conserved with respect to the associated Levi-Civita connection. This conservation translates those of Einstein’s equations that have not been used in the process of taming the expansion.

The boundary covariance of the Fefferman-Graham gauge makes it elegant and suitable for holographic applications in the framework of anti-de Sitter/conformal-field-theory correspondence. Setting up a gauge that is covariant with respect to the boundary is therefore desirable as part of the effort to unravel a similar duality for asymptotically flat spacetimes. In this case, the conformal boundary is at null infinity and is endowed with a Carrollian geometry [21, 22].

Carroll structures [30, 31, 41, 51–56] consist of a  $d+1$ -dimensional manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  equipped with a degenerate metric. The kernel of the metric is a vector field called *field of observers*. We will adopt coordinates  $(t, \mathbf{x})$  and a metric of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \tag{3.1}$$

with kernel

$$\mathbf{v} = \frac{1}{\Omega} \partial_t. \tag{3.2}$$

The coordinate system at hand is adapted to the space/time splitting. It is thus respected by Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{3.3}$$

with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \tag{3.4}$$

The *clock form* is dual to the field of observers with  $\mu(\mathbf{v}) = -1$ :

$$\mu = -\Omega dt + b_i dx^i \tag{3.5}$$

( $\Omega$  and  $b_i$  depend on  $t$  and  $\mathbf{x}$ ) and incorporates an *Ehresmann connection*, which is the background gauge field  $\mathbf{b} = b_i dx^i$ .<sup>9</sup> Carrollian tensors depend on time  $t$  and space  $\mathbf{x}$ . They carry indices  $i, j, \dots$  lowered and raised with  $a_{ij}$  and its inverse  $a^{ij}$ , and transform covariantly under (3.3) with  $J_i^j$  and  $J^{-1j}_i$  defined in (3.4). The basics on Carrollian tensors and Carrollian covariant derivatives are summarized in appendix A. In the following we will focus on  $d = 2$ , corresponding to the three-dimensional conformal null boundary of a four-dimensional asymptotically flat spacetime, and further information on this instance is available in appendix C. The boundary Carrollian covariance is part of the bulk general covariance, as inherited in the boundary geometry.

Fefferman-Graham gauge is only valid for Einstein spacetimes, on the one hand. On the other hand, Bondi and Newman-Unti gauges, applicable to Ricci-flat spacetimes, are not covariant with respect to the boundary, because the spatial section of the three-dimensional null boundary is locked. An alternative, still of the Eddington-Finkelstein type i.e. with a light-like radial direction, was introduced in the framework of fluid/gravity correspondence [60, 61], and made more systematic in the subsequent works both in AdS [62–65] and for Ricci-flat spacetimes [23]. It is a sort of modified and slightly incomplete Newman-Unti gauge [57, 58, 66] (see also [67, 68] for other extensions of the Bondi or Newman-Unti gauges). The time coordinate  $t$  is actually a retarded time (usually spelled  $u$ ) and coincides at the boundary with the Carrollian time used in (3.1), (3.2) and (3.5).

---

<sup>9</sup>A Carroll structure endowed with metric (3.1) and clock form (3.5) is naturally reached in the Carrollian limit ( $c \rightarrow 0$ ) of a pseudo-Riemannian spacetime  $\mathcal{M}$  in Papapetrou-Randers gauge  $ds^2 = -c^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j$ , where all functions are  $x$ -dependent with  $x \equiv (x^0 = ct, \mathbf{x})$ . It should be noticed here that the degenerate metric could generally have components along  $dt$ , which would in turn give  $\partial_i$  components to the field of observers. In this instance, the above Carrollian diffeomorphisms (3.3) play no privileged role, and plain general covariance is at work — without affecting the dynamics presented in appendix B. This option is sometimes chosen (see e.g. [54] for a general approach, or [57–59] for an application to three-dimensional Minkowski spacetime), but it is always possible to single out the time direction supported by the fiber of the Carrollian structure, i.e. distinguish time and spatial sections with no conflict with general covariance.

We can summarize as follows the structure of the four-dimensional Ricci-flat solutions in the advertised gauge, up to order  $1/r^2$  ( $G$  is four-dimensional Newton's constant):

$$\begin{aligned}
 ds_{\text{Ricci-flat}}^2 = & \mu \left[ 2dr - \left( r\theta + \hat{\mathcal{K}} \right) \mu + \left( 2r\varphi_i - 2 * \hat{\mathcal{D}}_i * \varpi - \hat{\mathcal{D}}_j \mathcal{C}_i^j \right) dx^i \right] \\
 & + \mathcal{C}_{ij} \left( r dx^i dx^j - * \varpi * dx^i dx^j \right) + \left( r^2 + * \varpi^2 + \frac{\mathcal{C}_{kl} \mathcal{C}^{kl}}{8} \right) dl^2 \\
 & + \frac{1}{r} \left[ 8\pi G \varepsilon \mu^2 + \frac{32\pi G}{3} \left( \pi_i - \frac{1}{8\pi G} * \psi_i \right) dx^i \mu - \frac{16\pi G}{3} E_{ij} dx^i dx^j \right] \\
 & + \frac{1}{r^2} \left( * \varpi c \mu^2 + \dots \right) + \mathcal{O} \left( \frac{1}{r^3} \right), \tag{3.6}
 \end{aligned}$$

where the star designates a  $d = 2$  Carrollian Hodge duality as defined in eq. (C.1).<sup>10</sup> As anticipated, this expression is neither in Bondi gauge (no determinant condition — see [27, 28]), nor in Newman-Unti ( $g_{rt} = -\Omega \neq -1$  and  $g_{ri} = b_i \neq 0$ , obtained using (3.5) — see [69]).<sup>11</sup> Delving into the details of this gauge would bring us outside the main purpose of the present work. We will rather explain the various ingredients appearing in the above expression and insist on their Carrollian-covariant nature. This includes the account of the required boundary data and the description of the evolution equations they obey so that the bulk metric be Ricci-flat.

All quantities entering expression (3.6) are defined on the conformal boundary and can be sorted as follows (see also the appendices for further information).

**Carrollian geometry.** The conformal boundary itself is part of the solution space. It is materialized in  $a_{ij}$ ,  $b_i$  and  $\Omega$ , accompanied with all attributes such as Carrollian connections and curvature tensors, Carrollian Cotton descendants etc. — see appendices A and C. These are free data, without evolution equations, except for the restriction of *vanishing Carrollian geometric shear as a consequence of Einstein's equations*:  $\xi_{ij} = 0$ .<sup>12</sup>

**Shear.** The *dynamic shear* is a symmetric and traceless Carrollian boundary tensor  $\mathcal{C}_{ij}(t, \mathbf{x})$  not to be confused with the geometric shear  $\xi_{ij}(t, \mathbf{x})$ .<sup>13</sup> It is a boundary emanation of the bulk  $\partial_r$ -congruence shear, and is completely free, although it sources the evolution equations of other tensorial data. The dynamic shear carries information on the bulk gravitational radiation through the symmetric and traceless *Bondi-like news*:

$$\hat{\mathcal{N}}_{ij} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \mathcal{C}_{ij}. \tag{3.7}$$

<sup>10</sup>Referring to the complex coordinates introduced in appendix C, we chose the orientation as inherited from the parent Riemannian spacetime:  $\eta_{0\zeta\bar{\zeta}} = \Omega \sqrt{a} \epsilon_{0\zeta\bar{\zeta}} = \frac{i\Omega}{P^2}$ , where  $x^0 = kt$ .

<sup>11</sup>In all quoted Eddington-Finkelstein type of gauges,  $\partial_r$  is tangent to a null geodesic congruence. In Newman-Unti and in modified Newman-Unti this congruence is affinely parameterized, in contrast to Bondi. In modified Newman-Unti gauge, as opposed to the others,  $\partial_r$  is *not* hypersurface-orthogonal. Indeed, the metric-dual form to  $\partial_r$  is  $\mu$ , which has a twist because of  $\Omega$  and  $b_i$ , the defining features of the gauge at hand:  $\mu \wedge d\mu = * \varpi \eta_{ij} dx^i \wedge dx^j \wedge \mu$  (we have used eqs. (A.5) and (C.3)).

<sup>12</sup>In BMS gauge, one would set  $b_i = 0$ ,  $\Omega = 1$ , and  $a_{ij}$  the round sphere.

<sup>13</sup>In Einstein spacetimes these two shears are proportional with the cosmological constant as a factor. In the asymptotically flat limit, the geometric shear is required to vanish, while the dynamic shear decouples.

With these definitions, the shear and the news are supported by genuine boundary conformal Carrollian-covariant tensors (weight  $-1$  and  $0$ ), hence meeting the advertised expectations.<sup>14</sup>

**Carrollian fluid.** The boundary Carrollian fluid of Ricci-flat spacetimes is the descendant of the relativistic boundary fluid in Einstein spacetimes in the vanishing speed of light limit, supported by the conserved energy-momentum tensor  $T_{\mu\nu}$ . It is described in terms of the energy density  $\varepsilon$ , the heat currents  $Q^i$  and  $\pi^i$ , and the symmetric and traceless stress tensors  $\Sigma^{ij}$  and  $\Xi^{ij}$  [70, 71]. The associated momenta of the fluid dynamics in the sense of appendix B are as follows:

$$\Pi = \varepsilon, \quad \Pi^i = Q^i, \quad P^i = \pi^i, \quad \tilde{\Pi}^{ij} = -\Sigma^{ij}, \quad \Pi^{ij} = \frac{\varepsilon}{2}a^{ij} - \Xi^{ij}. \quad (3.8)$$

As opposed to the relativistic boundary fluid, however, the Carrollian fluid is not free, but sourced by the shear, the news and the Carrollian Cotton descendants. Put differently, its dynamical equations are (B.1), (B.2), (B.3) and (B.4) (at zero  $\xi^{ij}$ ) with a non-vanishing right-hand side. These equations translate part of Einstein's, which furthermore impose<sup>15</sup>

$$Q^i = \frac{1}{8\pi G} * \chi^i, \quad \Sigma^{ij} = \frac{1}{8\pi G} * X^{ij}, \quad \Xi^{ij} = \frac{1}{8\pi G} * \Psi^{ij}. \quad (3.9)$$

Three of the Carrollian fluid data are thus tuned in terms of the boundary geometry through the Carrollian Cotton descendants displayed in eqs. (C.12), (C.14) and (C.15). Only two momenta remain independent ( $\Pi = \varepsilon$  and  $P^i = \pi^i$ ) and subject to two Carrollian-fluid evolution equations ((B.1) and (B.3) with zero  $\xi^{ij}$  and external force,<sup>16</sup> often referred to as *flux-balance equations*) out of the four — the other two are automatically satisfied owing to the Cotton equations (B.1), (B.2), (B.3) and (B.4)

<sup>14</sup>Notice that they do not exactly coincide with the original shear and news defined in BMS gauge. They vanish in Robinson-Trautmann spacetimes expressed in the gauge at hand, which is their defining gauge, although these solutions are radiating.

<sup>15</sup>The presence of a non-vanishing energy flux  $\Pi^i = Q^i$  betrays the breaking of local Carroll boost invariance (see appendix B, footnote 40) in the boundary Carrollian dynamics associated with Ricci-flat spacetimes. This breaking accounts for bulk gravitational radiation, which in the boundary-covariant gauge designed here does not originate solely in the news (3.7) but is also encoded in the Carrollian energy flux  $\Pi^i = Q^i = \frac{1}{8\pi G} * \chi^i$  and the Carrollian stress  $\tilde{\Pi}^{ij} = -\Sigma^{ij} = -\frac{1}{8\pi G} * X^{ij}$  obeying eq. (B.4) or equivalently (C.19). In Robinson-Trautman spacetimes e.g., the gravitational radiation is exclusively rooted in the latter Cotton descendants — see footnote 14 and ref. [23].

<sup>16</sup>We display for completeness these Carrollian equations, which coincide with eqs. (2.53) and (2.50) of ref. [72], once translated from our gauge into the BMS gauge:

$$\begin{aligned} \frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i &= \frac{1}{16\pi G} \left( \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j \hat{\mathcal{N}}^{ij} + \mathcal{C}^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{R}}_j + \frac{1}{2} \mathcal{C}_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\mathcal{N}}^{ij} \right), \\ \hat{\mathcal{D}}_j \Pi^{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t P^i + 2 * \varpi * \Pi^i &= \frac{1}{16\pi G} \left[ \mathcal{C}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{K}} + * \mathcal{C}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{L}} - 4 * \varpi * \mathcal{C}^{ij} \hat{\mathcal{R}}_j - \frac{1}{2} \hat{\mathcal{D}}^j \left( \hat{\mathcal{D}}_j \hat{\mathcal{D}}_k \mathcal{C}^{ik} - \hat{\mathcal{D}}^i \hat{\mathcal{D}}^k \mathcal{C}_{jk} \right) \right. \\ &\quad \left. + \mathcal{C}^{ij} \hat{\mathcal{D}}^k \hat{\mathcal{N}}_{jk} + \frac{1}{2} \hat{\mathcal{D}}^j \left( \mathcal{C}^{ik} \hat{\mathcal{N}}_{jk} \right) - \frac{1}{4} \hat{\mathcal{D}}^i \left( \mathcal{C}^{jk} \hat{\mathcal{N}}_{jk} \right) \right] \end{aligned}$$

with  $\Pi$ ,  $\Pi^i$ ,  $\Pi^{ij}$ ,  $P^i$  as in eqs. (3.8) and (3.9).

with (C.16). These data are related to the Bondi mass and angular momentum aspects,  $M(t, \mathbf{x})$  and  $N^i(t, \mathbf{x})$ :

$$8\pi G\varepsilon = 2M + \frac{1}{4}\mathcal{E}^{jk}\hat{\mathcal{N}}_{jk}, \tag{3.10}$$

$$8\pi G\pi^i = *\psi^i - N^i \tag{3.11}$$

with  $\psi^i$  given in (C.13).

Similarly to the expansion of Einstein spacetimes (in Fefferman-Graham or in the present gauge), fluid-related tensors appear *at every order* and not exclusively for  $1/r$ , as expression (3.6) might suggest.

**Further degrees of freedom.** Contrary to the asymptotically anti-de Sitter case, the above fluid data *are not* the only degrees of freedom besides the boundary geometry. An infinite number of Carrollian tensors are necessary to all orders in the radial expansion, as  $E_{ij}(t, \mathbf{x})$  in (3.6) at order  $1/r$ , which obey Carrollian evolution — flux-balance — equations similar to those already displayed in footnote 16. These are dubbed “Chthonian” degrees of freedom.

We will not elaborate any further on the features of the expansion and the structure of the various evolution equations. The covariantization with respect to boundary Carroll diffeomorphisms and Weyl covariance is a powerful tool,<sup>17</sup> rooted in the bulk general covariance. It can be supplemented with the boundary-fluid hydrodynamic-frame invariance at the expense of giving up radically the complete bulk gauge fixing. This requires a modified and *incomplete* Newman-Unti gauge, and has been performed for three bulk dimensions in refs. [57–59, 66].

**Resumming the series expansion.** In certain circumstances the series (3.6) can be resummed. As advertised in the introduction, this occurs when conditions are imposed on the boundary data, which enforce specific features for the bulk Weyl tensor:

1. the dynamic shear  $\mathcal{E}_{ij}(t, \mathbf{x})$  should vanish, implying in particular the relation  $M = 4\pi G\varepsilon$ ;
2. all non-Carrollian-fluid related degrees of freedom should be discarded, as e.g.  $E_{ij}(t, \mathbf{x})$ ;
3.  $N^i$  in (3.11) should be set to zero, which amounts to demanding the Carrollian momentum  $P^i = \pi^i$  be tuned with respect to a Carrollian Cotton descendant:<sup>18</sup>

$$\pi^i = \frac{1}{8\pi G} *\psi^i. \tag{3.12}$$

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<sup>17</sup>The expression (3.10) matches with eq. (42) of ref. [73], reached through a completely different logical path. Similarly the Carroll Cotton scalar  $c$  given in (C.11) plays here the role of the dual mass aspect, captured in (53) of the quoted reference.

<sup>18</sup>Although eq. (3.12), which secretly tunes the bulk Weyl tensor, bares some resemblance with a self-duality condition, it isn't as the Ricci-flat spacetimes at hand are Lorentzian rather than Euclidean and this option is not available.

In the configuration reached with the above conditions, the remaining degrees of freedom are those describing the boundary Carrollian geometry (metric, field of observers and Ehresmann connection), and the Carrollian-fluid energy density i.e. the Bondi mass aspect. Expression (3.6) is now resummed into an exact Ricci-flat spacetime of algebraically special type:<sup>19</sup>

$$ds_{\text{res. Ricci-flat}}^2 = \mu \left[ 2dr + 2 \left( r\varphi_j - *\hat{\mathcal{D}}_j *\varpi \right) dx^j - \left( r\theta + \hat{\mathcal{K}} \right) \mu \right] + \rho^2 d\ell^2 + \frac{\mu^2}{\rho^2} [8\pi G \varepsilon r + *\varpi c] \quad (3.13)$$

with

$$\rho^2 = r^2 + *\varpi^2. \quad (3.14)$$

Ricci flatness is guaranteed by the Carrollian fluid equations, which are now genuine conservation equations without forcing term (B.1), (B.2), (B.3) and (B.4), where the momenta are (using (3.9) and (3.12))

$$\Pi = \varepsilon, \quad \Pi^i = \frac{1}{8\pi G} *\chi^i, \quad P^i = \frac{1}{8\pi G} *\psi^i, \quad \tilde{\Pi}^{ij} = -\frac{1}{8\pi G} *X^{ij}, \quad \Pi^{ij} = \frac{\varepsilon}{2} a^{ij} - \frac{1}{8\pi G} *\Psi^{ij}. \quad (3.15)$$

The same equations are identically obeyed by the Carrollian Cotton tensors (C.16) and the geometric shear is vanishing. We are therefore left with two independent equations, which are (B.1) and (B.3):

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \varepsilon + \frac{1}{8\pi G} \hat{\mathcal{D}}_i *\chi^i = 0, \quad (3.16)$$

$$\hat{\mathcal{D}}_j \varepsilon - \frac{1}{8\pi G} *\hat{\mathcal{D}}_j c = 0, \quad (3.17)$$

where  $c$  and  $\chi^i$  are given in geometric terms in (C.11) and (C.12), and  $\varepsilon$  is proportional to the Bondi-mass aspect, as stressed in item 1 above. Equations (3.16) and (3.17) are those displayed in footnote 16 with vanishing right-hand side.

From the above eqs. (3.16) and (3.17) as well as eq. (C.17) one can foresee that the energy density  $\varepsilon$  and the Carrollian Cotton scalar  $c$  play dual roles. This will be formulated concretely in section 4 with reference to the boundary action of the Möbius group. Anticipating this argument, we introduce the following Carrollian complex scalar  $\hat{\tau}(t, \mathbf{x})$  and vector  $\hat{\chi}^j(t, \mathbf{x})$ :

$$\hat{\tau} = -c + 8\pi i G \varepsilon, \quad (3.18)$$

$$\hat{\chi}^j = \chi^j - i *\chi^j. \quad (3.19)$$

The aforementioned equations are thus recast as<sup>20</sup>

$$\begin{aligned} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\tau} &= \hat{\mathcal{D}}_j \hat{\chi}^j, & \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 0, \\ \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 8 \left( 2 *\varpi *\chi_i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_i - \hat{\mathcal{D}}^j \Psi_{ji} \right) \left( 2 *\varpi *\chi^i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi^i - \hat{\mathcal{D}}_k \Psi^{ki} \right). \end{aligned} \quad (3.20)$$

<sup>19</sup>Details and examples are available in [23].

<sup>20</sup>The first of eqs. (3.20) is flux-balance, driven exclusively by the Cotton vector  $\hat{\chi}^j$  displayed in (3.19). The loss phenomenon concerns both the mass aspect  $\varepsilon$  and the “magnetic-mass aspect”  $c$ , as captured in eqs. (76) and (80) of [73] — see also appendix D of [75]. As opposed to  $\varepsilon$ , the time evolution (C.17) of the magnetic-mass aspect is not altered by  $\mathcal{C}_{ij}$  and  $\mathcal{N}_{ij}$ , in line with [76].



Acting with a second spatial derivative on (3.17) and using (A.19), we finally obtain

$$\hat{\mathcal{D}}^j \hat{\mathcal{D}}_j \hat{\tau} = 2i \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t * \varpi \hat{\tau} - \hat{\mathcal{A}} \hat{\tau} \right). \quad (3.21)$$

Let us mention for completeness that eqs. (3.16) and (3.17) coincide with eqs. (29.16) and (29.15) of [74].<sup>21</sup> It is remarkable that complicated equations as the latter can actually be tamed into a simple fluid conservation supplemented with a kind of self-duality requirement. It would have been unthinkable to reach such a conclusion without the null boundary analysis performed here and the corresponding Carrollian geometric tools. The latter provide definitely the natural language for unravelling asymptotically flat spacetimes.

A last comment before closing this section concerns the algebraic-special nature of the metric (3.13). This is proven thanks to the Goldberg-Sachs theorem using the null, geodesic and, in the resummed instance, shear-free bulk congruence tangent to  $\partial_r$ . The latter is part of the canonical null tetrad parallelly transported along  $\partial_r$  (thanks to the affine nature of  $r$ ) introduced in [23], which coincides with that of [74], eq. (29.13a), as well as with the original ref. [77]. In complex celestial-sphere coordinates  $\zeta$  and  $\bar{\zeta}$ , see appendix C, the null tetrad reads:

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left( \frac{8\pi G \varepsilon r + * \varpi c}{\rho^2} - r\theta - \hat{\mathcal{K}} \right) \partial_r + \mathbf{v} \\ \mathbf{m} = \frac{P}{r - i * \varpi} \left( \hat{\partial}_{\bar{\zeta}} + (* \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi - r \varphi_{\bar{\zeta}}) \partial_r \right) \end{cases} \quad (3.22)$$

with the usual relations  $\mathbf{k} \cdot \mathbf{l} = -1$ ,  $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$  and  $ds_{\text{res. Ricci-flat}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$ . Generically,  $\mathbf{k}$  is a multiplicity-two principal null direction of the Weyl tensor, and using the tetrad at hand we find the following Weyl complex scalars:<sup>22</sup>

$$\begin{cases} \Psi_0 = \Psi_1 = 0 \\ \Psi_2 = \frac{i\hat{\tau}}{2(r - i * \varpi)^3} \\ \Psi_3 = \frac{iP\chi_{\zeta}}{(r - i * \varpi)^2} + \mathcal{O}(1/(r - i * \varpi)^3) \\ \Psi_4 = \frac{iX_{\zeta}}{r - i * \varpi} + \mathcal{O}(1/(r - i * \varpi)^2). \end{cases} \quad (3.23)$$

Unsurprisingly, all  $\Psi$ s are spelled using the Carrollian descendants of the boundary Cotton tensor — as well as their derivatives in the higher-order terms.

### 3.2 Bulk versus boundary isometries

The geometries under consideration possess at least one Killing vector field. A natural question to address concerns the boundary manifestation of a bulk isometry. At the same time such an analysis provides the recipe for designing bulk isometries from a purely boundary perspective.

<sup>21</sup>For that purpose, the following identifications are necessary (in complex coordinates, as in appendix C):  $b_{\zeta} = -L$ ,  $*\varpi = -\Sigma$ ,  $\hat{\tau} = 2(M + im)$ ,  $\Omega = 1$ ,  $t = u$ , whereas their radial coordinate is  $\tilde{r} = r - r_0$  with  $r_0(t, \zeta, \bar{\zeta})$  the origin in the affine parameter of the geodesic congruence tangent to  $\partial_r$ .

<sup>22</sup>Neither  $\Psi_3$  nor  $\Psi_4$  vanish in the instance of Petrov type D solutions, because  $\mathbf{l}$  is not a principal null direction. Another tetrad is reached with a Lorentz transformation suitably adjusted for  $\mathbf{l}'$  be a principal direction of multiplicity two whereas  $\mathbf{k}' \propto \mathbf{k}$ , and  $\Psi'_3 = \Psi'_4 = 0$ .

We will circumscribe our investigation to vector fields, which have no component along  $\partial_r$ , and whose other components depend only on  $t$  and  $\mathbf{x}$ . We could be more general without much effort assuming e.g. an expansion in inverse powers of  $r$  for the missing component and for the radial dependence of the others. However, this would unnecessarily sophisticate our presentation without shedding more light on our simple and robust conclusion: *the bulk isometries at hand are mapped onto boundary Carrollian diffeomorphisms generated by strong Killing vectors* (a summary on Carrollian isometries is available in appendix B).

It is convenient for the subsequent developments to adopt bulk Cartan frame and coframe aligned with the boundary (3.2), (A.2) and (3.5):

$$\begin{aligned} e_{\hat{t}} \equiv \mathbf{v} &= \frac{1}{\Omega} \partial_t, & e_i \equiv \hat{\partial}_i &= \partial_i + \frac{b_i}{\Omega} \partial_t, & e_{\hat{r}} &\equiv \partial_r, \\ \theta^{\hat{t}} &\equiv -\mu = \Omega dt - b_i dx^i, & \theta^i &\equiv dx^i, & \theta^{\hat{r}} &\equiv dr. \end{aligned} \quad (3.24)$$

The components for the bulk metric (3.13) read (in order to avoid cluttering, we keep the “hat” on the time indices only, where potential ambiguity exists):

$$\begin{aligned} g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - r\theta - \hat{\mathcal{K}}, & g_{\hat{t}i} &= *\hat{\mathcal{D}}_i *\varpi - r\varphi_i, & g_{\hat{t}r} &= -1, \\ g_{ri} &= 0, & g_{rr} &= 0, & g_{ij} &= \rho^2 a_{ij}. \end{aligned} \quad (3.25)$$

Assuming a bulk vector of the form

$$\xi = \xi^{\hat{t}}(t, \mathbf{x}) \partial_{\hat{t}} + \xi^k(t, \mathbf{x}) \partial_k = \xi^{\hat{t}}(t, \mathbf{x}) \mathbf{v} + \xi^k(t, \mathbf{x}) \hat{\partial}_k, \quad (3.26)$$

where  $\xi^{\hat{t}} = \Omega \xi^t - \xi^k b_k$ , we can determine the Lie derivative of the metric:

$$\begin{aligned} \mathcal{L}_\xi g_{rr} &= 0, & \mathcal{L}_\xi g_{r\hat{t}} &= \mu, & \mathcal{L}_\xi g_{ri} &= \nu_i, \\ \mathcal{L}_\xi g_{ij} &= 2\rho^2 \left( \hat{\nabla}_{(i} \xi^k a_{j)k} + \xi^{\hat{t}} \hat{\gamma}_{ij} \right) - 2g_{\hat{t}(i} \nu_{j)} + a_{ij} \xi \left( *\varpi^2 \right), \\ \mathcal{L}_\xi g_{\hat{t}i} &= -g_{\hat{t}i} \mu - g_{\hat{t}\hat{t}} \nu_i - r \left( \xi(\varphi_i) + \varphi_j \hat{\partial}_i \xi^j \right) + \xi \left( *\hat{\mathcal{D}}_i *\varpi \right) + \left( *\hat{\mathcal{D}}_j *\varpi \right) \hat{\partial}_i \xi^j + \rho^2 a_{ij} \frac{1}{\Omega} \partial_t \xi^j, \\ \mathcal{L}_\xi g_{\hat{t}\hat{t}} &= -2g_{\hat{t}\hat{t}} \mu + 2g_{\hat{t}i} \frac{1}{\Omega} \partial_t \xi^i - \xi \left( \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - r\theta - \hat{\mathcal{K}} \right) \end{aligned} \quad (3.27)$$

with  $\mu(t, \mathbf{x})$  and  $\nu_i(t, \mathbf{x})$  given in (B.10). Observe that everything is expressed in terms of boundary Carrollian geometric objects (see appendix A).

Since the Killing components are  $r$ -independent, the above Lie derivative vanishes if and only if the coefficients of every power of  $r$  do. The independent conditions we reach for this to occur are

$$\partial_t \xi^i = 0 \quad (3.28)$$

and (B.12), (B.13) and (B.17), which therefore map the bulk Killing field (3.26) onto a boundary Carrollian strong Killing vector (see appendix B). Some apparent extra conditions such as  $\xi(*\varpi^2) = 0$  or  $\xi(\varphi_i) + \varphi_j \hat{\partial}_i \xi^j = 0$  are the vanishing of  $\xi$ -Lie derivatives of some Carrollian tensors, which is guaranteed by the strong Killing requirement on  $\xi$ .

### 3.3 Towers of charges and dual charges

The Carrollian dynamics emerging on the boundary as a consequence of bulk Einstein’s equations, combined with the always available Carrollian conformal isometry group  $BMS_4$ , enables us to define a variety of charges. These are not necessarily conserved, but even in that instance, their evolution properties are canonical and provide an alternative, tamed picture of the dynamics. Furthermore, they should ultimately pertain to those charges recently discovered and discussed from a bulk perspective [32–39], based as usual on the asymptotic symmetries — also and unsurprisingly  $BMS_4$ , under appropriate fall-off conditions. Making the precise contact with those works would require a translation of our findings into the Newman-Penrose formalism [77] beyond what we have already observed in eqs. (3.23), namely  $\Psi_2^0 = \frac{i}{2}\hat{\tau}$ ,  $\Psi_3^0 = iP\chi_\zeta$  and  $\Psi_4^0 = iX_\zeta\bar{\zeta}$ . This would bring us far from our goal, and we will limit ourselves to pointing out that the ten Newman-Penrose conserved charges vanish here because the spacetimes are algebraically special. These charges would have been otherwise associated with the  $s = 1$  “non-tilde” class introduced below, involving non-zero  $E_{ij}$  and  $N_i$  in the non-algebraic instance.

Ricci-flat metrics, either in the general form (3.6) or in its resummed version (3.13), exhibit two important features for the description of charges. Firstly, every order  $1/r^{2s+1}$  reveals Carrollian dynamics of the type (B.1), (B.2), (B.3) and (B.4) with momenta  $\Pi_{(s)}$ ,  $\Pi_{(s)}^i$ ,  $P_{(s)}^i$ ,  $\tilde{\Pi}_{(s)}^{ij}$  and  $\Pi_{(s)}^{ij}$ , and possibly with right-hand sides — non-conservation. Every such set of momenta together with the Carrollian conformal Killings (C.20) lead to currents  $\kappa_{(s)}$ ,  $K_{(s)}^i$ ,  $\tilde{\kappa}_{(s)}$ ,  $\tilde{K}_{(s)}^i$  and charges  $Q_{(s)T,Y}$  and  $\tilde{Q}_{(s)T,Y}$ , following (B.20) and (B.22). Their conservation or evolution encoded in (B.21) depends on  $\mathcal{K}_{(s)}$ ,  $\tilde{\mathcal{K}}_{(s)}$  in (B.23). The set associated with  $s = 0$  corresponds to the fluid momenta (3.8) and its charges are *leading*; the sets with  $s \geq 1$  reveal the *subleading charges*. Moreover, all these charges should be referred to as *electric* because their conservation, if valid, occurs on-shell.

Secondly, the Carrollian Cotton tensors obey conservation equations (B.1), (B.2), (B.3) and (B.4) with momenta (C.16), leading to two towers of Cotton charges  $Q_{\text{Cot}T,Y}$  and  $\tilde{Q}_{\text{Cot}T,Y}$ , as discussed in appendix C. These charges are *magnetic* as the conservation of the Cotton is an identity valid off-shell.<sup>23</sup> Furthermore, the Carrollian Cotton tensors are not exclusive to  $1/r^2$ : each order  $1/r^{2s+2}$  brings its share of off-shell Carrollian dynamics with momenta  $\Pi_{\text{Cot}(s)}$ ,  $\Pi_{\text{Cot}(s)}^i$ ,  $P_{\text{Cot}(s)}^i$ ,  $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$ ,  $\Pi_{\text{Cot}(s)}^{ij}$ , currents  $\kappa_{\text{Cot}(s)}$ ,  $K_{\text{Cot}(s)}^i$ ,  $\tilde{\kappa}_{\text{Cot}(s)}$ ,  $\tilde{K}_{\text{Cot}(s)}^i$ , and finally magnetic charges  $Q_{\text{Cot}(s)T,Y}$  and  $\tilde{Q}_{\text{Cot}(s)T,Y}$ .

Incidentally, it should be noticed that due to the relationships amongst the fluid and the Cotton (eq. (3.9) in general plus eq. (3.12) in the resumable family), the electric and the magnetic towers have a non-empty intersection:  $\tilde{Q}_{(s)T,Y}$  and  $\tilde{Q}_{\text{Cot}(s)T,Y}$  generally coincide.

Let us for concreteness overview the situation in the resumable instance, eq. (3.13). Expanding the resummed factor  $1/\rho^2$ , we find the following results.

**Electric towers.** These have *sth* momenta  $\Pi_{(s)}$ ,  $\Pi_{(s)}^i$ ,  $P_{(s)}^i$ ,  $\tilde{\Pi}_{(s)}^{ij}$  and  $\Pi_{(s)}^{ij}$  equal to (3.15) multiplied by  $*\varpi^{2s}$ . The same factor will multiply the leading Carrollian current ( $s =$

<sup>23</sup>We borrow here the phrasing *electric* and *magnetic* from refs. [76, 78].

0 i.e.  $\kappa$ ,  $K^i$ ,  $\tilde{\kappa}$ ,  $\tilde{K}^i$ ) and give the sth,  $\kappa_{(s)}$ ,  $K_{(s)}^i$ ,  $\tilde{\kappa}_{(s)}$  and  $\tilde{K}_{(s)}^i$ , following (B.22). Using the Carroll-Bianchi identities (C.6), (C.7) and (C.8), we find the divergences (B.23), which contribute the time evolution of the charges computed as in (B.20), using (B.21):

$$\tilde{\mathcal{K}}_{(s)} = -s * \varpi^{2s-2} \left( * \varpi \hat{\mathcal{A}} \tilde{\kappa} + \frac{1}{3} \tilde{K}^i * \psi_i \right), \quad (3.29)$$

$$\mathcal{K}_{(s)} = -\frac{* \varpi^{2s}}{8\pi G} * \chi^i \left( \hat{\mathcal{D}}_i \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) - s * \varpi^{2s-2} \left( * \varpi \hat{\mathcal{A}} \kappa + \frac{1}{3} K^i * \psi_i \right) \quad (3.30)$$

with

$$\begin{cases} \kappa = \frac{1}{8\pi G} \xi^i * \psi_i - \xi^{\hat{t}} \varepsilon \\ \tilde{\kappa} = \frac{1}{8\pi G} \xi^i * \chi_i \\ K^i = \frac{\varepsilon}{2} \xi^i - \frac{1}{8\pi G} \left( \xi^j * \Psi^i_j + \xi^{\hat{t}} * \chi^i \right) \\ \tilde{K}^i = -\frac{1}{8\pi G} \xi^j * X^i_j, \end{cases} \quad (3.31)$$

and the Killing components  $\xi^{\hat{t}}$  and  $\xi^i$  read off in (C.20) following (B.6).

Regarding the charges and their evolution, only  $\tilde{Q}_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \tilde{\kappa} + b_j \tilde{K}^j \right) \equiv \tilde{Q}_{T,Y}$  are *always* conserved. These charges are purely geometric because they are integrals over  $\mathcal{S}$ <sup>24</sup>

$$\tilde{Q}_{T,Y} = -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} * \xi^i \left( \chi_i - b_j X^j_i \right), \quad (3.32)$$

which do not involve the energy density  $\varepsilon$ , as opposed to  $Q_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \kappa + b_j K^j \right) \equiv Q_{T,Y}$  spelled as

$$Q_{T,Y} = -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{t}} \left( 8\pi G \varepsilon + b_i * \chi^i \right) + \frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( * \psi_i + 4\pi G \varepsilon b_i - b_j * \Psi^j_i \right). \quad (3.33)$$

The latter are conserved for strong Carrollian Killings. Other charges might also be conserved for specific Carrollian conformal Killings, or depending on the configuration.

**Magnetic towers.** The sth magnetic momenta  $\Pi_{\text{Cot}(s)}$ ,  $\Pi_{\text{Cot}(s)}^i$ ,  $P_{\text{Cot}(s)}^i$ ,  $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$  and  $\Pi_{\text{Cot}(s)}^{ij}$  are (C.16) multiplied by  $* \varpi^{2s}$ . As for the electric case, this latter factor will appear in all magnetic currents  $\kappa_{\text{Cot}(s)}$ ,  $K_{\text{Cot}(s)}^i$ ,  $\tilde{\kappa}_{\text{Cot}(s)}$  and  $\tilde{K}_{\text{Cot}(s)}^i$  built out of the leading  $s = 0$ :

$$\begin{cases} \kappa_{\text{Cot}} = \xi^i \psi_i - \xi^{\hat{t}} c \\ \tilde{\kappa}_{\text{Cot}} = \xi^i \chi_i \\ K_{\text{Cot}}^i = \frac{c}{2} \xi^i - \xi^j \Psi^i_j - \xi^{\hat{t}} \chi^i \\ \tilde{K}_{\text{Cot}}^i = -\xi^j X^i_j, \end{cases} \quad (3.34)$$

<sup>24</sup>We use the property  $V^i * W_i = - * V^i W_i$  — see (C.1).

Their divergences (B.23) read:

$$\tilde{\mathcal{K}}_{\text{Cot}(s)} = -s * \varpi^{2s-2} \left( * \varpi \hat{\mathcal{A}} \tilde{\kappa}_{\text{Cot}} + \frac{1}{3} \tilde{K}_{\text{Cot}}^i * \psi_i \right), \quad (3.35)$$

$$\mathcal{K}_{\text{Cot}(s)} = - * \varpi^{2s} \chi^i \left( \hat{\mathcal{D}}_i \xi^{\hat{i}} - 2 \xi^j \varpi_{ji} \right) - s * \varpi^{2s-2} \left( * \varpi \hat{\mathcal{A}} \kappa_{\text{Cot}} + \frac{1}{3} K_{\text{Cot}}^i * \psi_i \right). \quad (3.36)$$

These determine the evolution (B.21) of the charges (B.20), from which we learn that  $\tilde{Q}_{\text{Cot}(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \tilde{\kappa}_{\text{Cot}} + b_j \tilde{K}_{\text{Cot}}^j \right) \equiv \tilde{Q}_{\text{Cot}T,Y}$  are always conserved:

$$\tilde{Q}_{\text{Cot}T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( \chi_i - b_j X^j_i \right). \quad (3.37)$$

For strong Carrollian Killing fields,  $Q_{\text{Cot}(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left( \kappa_{\text{Cot}} + b_j K_{\text{Cot}}^j \right) \equiv Q_{\text{Cot}T,Y}$  given by

$$Q_{\text{Cot}T,Y} = - \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{i}} \left( c + b_i \chi^i \right) + \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left( \psi_i + \frac{c}{2} b_i - b_j \Psi^j_i \right) \quad (3.38)$$

are also conserved off-shell, as other magnetic charges are in specific situations.

Several comments are in order here concerning the above sets of charges obtained for the resumable metrics (3.13). The tower of electric geometric charges  $\tilde{Q}_{(s)T,Y}$ , constructed upon multiplying the integrand of (3.32) by  $*\varpi^{2s}$ , coincides with its magnetic counterpart  $\tilde{Q}_{\text{Cot}(s)T,Y}$  obtained likewise using (3.37). In  $d = 2$ , if  $\xi^i$  are the spatial components of a conformal Killing field, so are  $*\xi^i$ .<sup>25</sup> Hence the set of all  $\xi^i$ 's is identical to that of  $*\xi^i$ 's. The associated charges could be called “self-dual,” and in total three distinct towers emerge: the self-dual  $\left\{ \tilde{Q}_{(s)T,Y} \right\} \equiv \left\{ \tilde{Q}_{\text{Cot}(s)T,Y} \right\}$ , the electric  $\left\{ Q_{(s)T,Y} \right\}$  and the magnetic  $\left\{ Q_{\text{Cot}(s)T,Y} \right\}$  — the last two are reached by inserting  $*\varpi^{2s}$  into the integrals (3.33) and (3.38). The  $*\varpi^{2s}$  insertion pattern grants the subleading towers with the status of *multipolar moments* (see the original works [79–82] as well as [83] for a modern perspective). Making this statement precise would force us to deviate substantially from the analysis of the hidden Möbius group. This could fit more naturally in a comprehensive comparison of the present approach to subleading charges with the rich existing literature quoted earlier. Nonetheless, the pertinence of the proposition will be illustrated in the example of Kerr solution, at the very end of the forthcoming section 3.4.

Among the above towers of BMS<sub>4</sub> charges, always present but not always conserved, one finds those corresponding to the bulk isometries, whenever present. Indeed, as discussed in section 3.2, bulk Killings of the form (3.26) are associated with boundary strong Carrollian Killing vector fields. Combined as previously with the leading and subleading, electric and magnetic momenta, they generate two electric and two magnetic towers of charges:  $\left\{ Q_{(s)}, \tilde{Q}_{(s)}, Q_{\text{Cot}(s)}, \tilde{Q}_{\text{Cot}(s)} \right\}$ . The four leading charges  $\left\{ Q_{(0)}, \tilde{Q}_{(0)}, Q_{\text{Cot}(0)}, \tilde{Q}_{\text{Cot}(0)} \right\}$  are always conserved, but part of them may be trivial or not independent. The subleading are neither necessarily conserved, nor always independent, and have the status of electric and magnetic multipole moments.

<sup>25</sup>The proof of this statement is straightforward in complex coordinates, see footnote 50.

### 3.4 Time-independent solutions

**Reconstruction from the boundary.** In view of the forthcoming Ehlers-Geroch reduction, we will now assume the existence of a time-like Killing vector field  $\xi$  in Ricci-flat solutions of the resumable type (3.13). Such a vector could be generally of the form (3.26). In stationary spacetimes, the field  $\xi$  remains time-like in the asymptotic region. Then, it is possible to choose the field  $\mathbf{v}$  (3.2) of the modified Newman-Unti gauge such that  $\mathbf{v} \equiv \xi$ . Setting further  $\Omega = 1$  brings the Killing to the simple form  $\partial_t$  (see e.g. [77] for a detailed description of the procedure). On the conformal boundary, the time-like Killing congruence thus coincides with the fibre of the Carrollian bundle. This feature is absent for spacetimes where a time-like Killing field exists but becomes space-like in the asymptotic region. Examples of this sort are captured by the Plebański-Demiański family (like the C-metric) [84] (see also [74, 85]), which is algebraically special of Petrov type D.<sup>26</sup> These include the black-hole acceleration parameter, which is responsible for the appearance of another Killing horizon, creating a new asymptotic region where the Killing vector fails to be time-like. Although interesting on its own right — of limited physical use, however — the inclusion of this parameter would render the presentation too convoluted, in particular because the action of the Ehlers group in this instance does not respect the algebraic feature of the spacetime. For the sake of clarity we will restrict our investigation to Killings of the form  $\partial_t$ , aligned with the fiber, i.e. to truly stationary spacetimes, which remain algebraically special under Ehlers transformations.

With the present choice, none of the Carrollian building blocks of  $ds_{\text{res. Ricci-flat}}^2$  depends on  $t$ . As a consequence (see appendices A and C)  $\theta = 0$  and  $\varphi_i = \partial_i \ln \Omega$ . The latter can be set to zero with a time-independent Weyl rescaling, which therefore amounts to setting  $\Omega = 1$ . This is an innocuous gauge fixing that will be assumed here because it allows to severely simplify the dynamics. Backed with time independence, Carrollian Weyl-covariant derivatives become ordinary Levi-Civita derivatives, and the only non-vanishing tensors are the following, in complex coordinates with  $P = P(\zeta, \bar{\zeta})$  — see appendix C:

$$*\varpi = \frac{iP^2}{2} (\partial_\zeta b_{\bar{\zeta}} - \partial_{\bar{\zeta}} b_\zeta), \tag{3.39}$$

$$\hat{\mathcal{K}} = \hat{K} = K = \Delta \ln P, \tag{3.40}$$

$$c = (\Delta + 2K) *\varpi, \tag{3.41}$$

$$\chi_\zeta = \frac{i}{2} \partial_\zeta K, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \partial_{\bar{\zeta}} K, \tag{3.42}$$

$$\psi_\zeta = 3i \partial_\zeta *\varpi^2, \quad \psi_{\bar{\zeta}} = -3i \partial_{\bar{\zeta}} *\varpi^2, \tag{3.43}$$

$$\Psi_{\zeta\zeta} = \frac{1}{P^2} \partial_\zeta (P^2 \partial_\zeta *\varpi), \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \frac{1}{P^2} \partial_{\bar{\zeta}} (P^2 \partial_{\bar{\zeta}} *\varpi), \tag{3.44}$$

where  $\Delta f = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta f$ . To these one should add the energy density (i.e. the Bondi mass aspect)  $\varepsilon$ , as well as another scalar

$$\varpi = \frac{P^2}{2} (\partial_\zeta b_{\bar{\zeta}} + \partial_{\bar{\zeta}} b_\zeta), \tag{3.45}$$

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<sup>26</sup>Their Weyl components are given in eq. (3.23) — see also footnote 22.

which is  $\frac{1}{2}\nabla_i b^i$  and should not be confused with the two-form  $\varpi = \frac{1}{2}\varpi_{ij}dx^i \wedge dx^j$ , i.e. the Hodge-dual of the scalar  $*\varpi = -\frac{1}{2}\nabla_i *b^i$  displayed explicitly in (3.39). These two real twist scalars are adroitly combined into the *complex Carrollian twist*

$$\hat{\varpi} = *\varpi + i\varpi. \tag{3.46}$$

The equations of motion (3.16), (3.17) (or in the form (3.20), (3.21) with  $\hat{\tau}$  defined in (3.18)) are recast as

$$\Delta K = 0, \tag{3.47}$$

$$\partial_\zeta \hat{\tau} = 0. \tag{3.48}$$

The first shows that the curvature is required to be a harmonic function i.e.

$$K(\zeta, \bar{\zeta}) = \frac{1}{2} \left( \hat{k}(\zeta) + \hat{k}(\bar{\zeta}) \right), \tag{3.49}$$

and although  $\hat{k}(\zeta)$  is an arbitrary holomorphic function, the freedom is rather limited as  $K$  must also be the Laplacian of  $\ln P$ . Besides the constant-curvature cases, one solution has been exhibited thus far [74] (up to holomorphic coordinate transformations):  $K = -3(\zeta + \bar{\zeta})$  realized with  $P = (\zeta + \bar{\zeta})^{3/2}$ . We will not specify any particular choice for the moment. For future use, we define the imaginary part of  $\hat{k}(\zeta)$  as another harmonic function

$$K^*(\zeta, \bar{\zeta}) = \frac{1}{2i} \left( \hat{k}(\zeta) - \hat{k}(\bar{\zeta}) \right). \tag{3.50}$$

From eqs. (3.48) and (3.18), we infer that  $-c$  is the real part of an arbitrary holomorphic function  $\hat{\tau}(\zeta)$ , whereas the imaginary part of the latter is  $8\pi G\varepsilon$ ; both are harmonic functions. Given  $c$  and  $K$ , we can proceed with eq. (3.41) and find  $*\varpi$ , from which it is always possible to determine  $b_\zeta$  and  $b_{\bar{\zeta}}$ .

Although the focus of the present work is not to solve Einstein’s equations, we will elaborate for illustrative purposes on the steps we’ve just described, without delving into fine questions like completeness or gauge redundancy of the solutions. Note in passing how remarkably the Carrollian boundary formalism is adapted to the framework of Ricci-flat spacetimes, allowing to convey often complicated expressions in a very elegant manner, and sorting naturally otherwise scattered classes of solutions (the ones we present can be found in various chapters of refs. [74, 85]). Several distinct instances appear, which require a separate treatment.

**Non-constant  $K$ .** This is the generic situation, although in practice the most obscure regarding the interpretation of the bulk geometries. As already mentioned, very few  $P$ s are expected to possess a non-constant harmonic curvature  $K$ , but assuming one has one, accompanied by its holomorphic function  $\hat{k}(\zeta)$ , and making a choice for the arbitrary holomorphic function  $\hat{\tau}(\zeta)$ , eq. (3.41) can be solved for  $*\varpi$ , which is expressed using (3.39) with Ehresmann connection

$$b_\zeta(\zeta, \bar{\zeta}) = \frac{i\hat{\tau}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})\partial_{\bar{\zeta}}\hat{k}(\bar{\zeta})}. \tag{3.51}$$

**Constant  $K$ .** This implies that  $\hat{k}(\zeta)$  is also constant and the above solution is invalid.

The situation at hand is the most common, however, as it captures three standard instances: spherical, flat or hyperbolic foliations. We can parameterize the function  $P$  as follows:

$$P(\zeta, \bar{\zeta}) = A\zeta\bar{\zeta} + B\zeta + \bar{B}\bar{\zeta} + D \quad (3.52)$$

with  $A, D$  arbitrary real constants and  $B$  an arbitrary complex constant, leading to

$$K = 2(AD - B\bar{B}). \quad (3.53)$$

Several cases emerge, which must be treated separately.

**$K \neq 0$ .** Here  $c(\zeta, \bar{\zeta}) = -\frac{\hat{r}(\zeta) + \hat{r}(\bar{\zeta})}{2}$  is an arbitrary (possibly constant) harmonic function, and eq. (3.41) is solved with

$$*\varpi(\zeta, \bar{\zeta}) = \frac{c(\zeta, \bar{\zeta})}{2K} + i \left( \bar{f}(\bar{\zeta}) \partial_{\bar{\zeta}} \ln P(\zeta, \bar{\zeta}) - f(\zeta) \partial_{\zeta} \ln P(\zeta, \bar{\zeta}) + \frac{1}{2} \left( \partial_{\zeta} f(\zeta) - \partial_{\bar{\zeta}} \bar{f}(\bar{\zeta}) \right) \right) \quad (3.54)$$

with  $f(\zeta)$  an arbitrary holomorphic function. It is reached with the following Ehresmann connection ( $\hat{\tau}_0$  is a real constant):

$$b_{\zeta}(\zeta, \bar{\zeta}) = -\frac{\bar{\zeta}(\hat{\tau}_0 + i\hat{r}(\zeta))}{2K(B\zeta + D)P(\zeta, \bar{\zeta})} + \frac{\bar{f}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})}. \quad (3.55)$$

**$K = 0$ .** This instance is obtained with  $A = B = 0$  so that  $P = D$ . Now, given an arbitrary harmonic function  $c(\zeta, \bar{\zeta}) = -\frac{\hat{r}(\zeta) + \hat{r}(\bar{\zeta})}{2}$  and an arbitrary holomorphic function  $Z(\zeta)$ , we find

$$*\varpi(\zeta, \bar{\zeta}) = \frac{i}{2} \left( Z(\zeta) - \bar{Z}(\bar{\zeta}) \right) - \frac{1}{4P^2} \left( \bar{\zeta} \int^{\zeta} dz \hat{r}(z) + \zeta \int^{\bar{\zeta}} d\bar{z} \hat{r}(\bar{z}) \right), \quad (3.56)$$

and ( $\hat{\tau}_0$  is a real integration constant)

$$b_{\zeta}(\zeta, \bar{\zeta}) = \frac{1}{P^2} \int^{\bar{\zeta}} d\bar{z} \bar{Z}(\bar{z}) - \frac{\bar{\zeta}^2}{4P^4} \int^{\zeta} dz (\hat{\tau}_0 + i\hat{r}(z)). \quad (3.57)$$

The last two cases have in common the instance where  $c = K = 0$ , realized with vanishing  $\hat{\tau}$  and constant  $P$ .

As already noticed, all solutions described in a unified fashion here can be found in the earlier quoted literature under distinct labels.<sup>27</sup> Discussing them would take us outside of our objectives. We will only emphasize a notorious subclass, which is the Kerr-Taub-NUT family. For the latter, the curvature  $K$  is constant (3.53) and realized e.g. with  $B = 0$ . Two distinct instances emerge: vanishing and non-vanishing  $K$ , respectively obtained with vanishing and non-vanishing  $A$ .

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<sup>27</sup>It should be stressed that part of the present solution space originates in gauge freedom. In particular,  $c(\zeta, \bar{\zeta})$  being Weyl-covariant of weight 3 (see appendix C), it can always be reabsorbed by a boundary Weyl transformation, which is in turn neutralized by a bulk  $r$ -rescaling. Such a boundary transformation will bring  $\Omega$  back with non-vanishing  $\varphi_i$ , which we have set to zero, and this is the reason we cannot here restrict to constant  $c$  and  $\hat{\tau}$ .



- For non-vanishing  $K$ , the holomorphic function  $\hat{\tau}$  is

$$\hat{\tau} = 2i(M + iKn), \tag{3.58}$$

where  $M$  is the mass and  $n$  the nut charge, both constants. The holomorphic function  $f(\zeta)$  reads

$$f(\zeta) = ia\zeta \tag{3.59}$$

with  $a$  the Kerr angular velocity. Using eqs. (3.54) and (3.55) with  $\hat{\tau}_0 = 2M$  we find:

$$b_\zeta(\zeta, \bar{\zeta}) = -i\bar{\zeta} \left( \frac{a}{P^2} - \frac{n}{DP} \right) \tag{3.60}$$

and

$$* \varpi(\zeta, \bar{\zeta}) = n + a - \frac{2Da}{P}, \tag{3.61}$$

where  $P = A\zeta\bar{\zeta} + D$  and  $K = 2AD$ .

- For  $K = 0$  (i.e.  $P = D$  constant), we use eqs. (3.56) and (3.57) with  $\hat{\tau}_0 = 2M$ ,<sup>28</sup>

$$\hat{\tau} = 2iM \tag{3.62}$$

and

$$Z = ia. \tag{3.63}$$

This leads to

$$b_\zeta(\zeta, \bar{\zeta}) = -i\frac{\bar{\zeta}a}{P^2} \tag{3.64}$$

and

$$* \varpi = -a. \tag{3.65}$$

Observe the absence of nut charge in the present case.<sup>29</sup>

**A remark on the rigidity theorem.** The *rigidity theorem* asserts that under appropriate hypotheses, the isometry group of stationary asymptotically flat spacetimes contains  $\mathbb{R} \times \text{U}(1)$ . This theorem is best presented in refs. [86, 87], where the necessary assumptions are stated more accurately than in the original discussions (see e.g. [88]). Our framework does embrace stationary spacetimes. However, we have been agnostic regarding analyticity or regularity properties, which turn out to be fundamental for the applicability of the theorem at hand. Hence, we have no reason to foresee any additional  $\text{U}(1)$  symmetry in all reconstructed solutions of the present section.

Aside from mathematical rigor, we can recast the conceivable disruption of the rigidity theorem from the boundary perspective, which has been our viewpoint. We have shown in section 3.2 that a bulk Killing field is mapped onto a Carrollian strong Killing on the

<sup>28</sup>Both for vanishing and non-vanishing  $K$ ,  $\hat{\tau}_0$  has been tuned to ensure that  $M$  does not appear in  $b_\zeta$ , displayed in (3.60) and (3.64). There is no principle behind this choice, it is simply in line with standard conventions for the Kerr-Taub-NUT family. As a consequence,  $\varpi$  defined in (3.45) vanishes.

<sup>29</sup>Despite the absence of magnetic charges, the solution at hand belongs formally to the Taub-NUT family (see ref. [85], section 12.3.2).

boundary. The generator of the desired U(1) is of the form (3.26) with no time leg<sup>30</sup> i.e.  $\xi^t = 0$ , and no time dependence in  $\xi^i$  as imposed by (3.28):

$$\xi = - \left( \xi^\zeta b_\zeta + \xi^{\bar{\zeta}} b_{\bar{\zeta}} \right) \partial_t + \xi^\zeta \hat{\partial}_\zeta + \xi^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}}. \quad (3.66)$$

A strong Carrollian Killing field must obey eqs. (B.12), (B.13) and (B.17). Here (B.13) is identically satisfied, whereas (B.12) leads to

$$\partial_\zeta \xi^{\bar{\zeta}} = \partial_{\bar{\zeta}} \xi^\zeta = 0, \quad \partial_\zeta \frac{\xi^\zeta}{P^2} + \partial_{\bar{\zeta}} \frac{\xi^{\bar{\zeta}}}{P^2} = 0. \quad (3.67)$$

Finally (B.17) reads:

$$P^2 \partial_\zeta \left( \xi^\zeta b_\zeta + \xi^{\bar{\zeta}} b_{\bar{\zeta}} \right) + 2i \xi^{\bar{\zeta}} * \varpi = 0 \quad (3.68)$$

plus its complex conjugate.

For arbitrary  $P(\zeta, \bar{\zeta})$ , eqs. (3.67) have no solution, hence no extra Killing field is available. As mentioned earlier in the present section, the  $P$ s with harmonic curvature (required in (3.47)) are very restricted and probably lack the necessary analyticity properties, explaining why the rigidity theorem is not applicable. This indeed happens in the quoted example with  $P = (\zeta + \bar{\zeta})^{3/2}$ .

Alternatively, considering  $P = A\zeta\bar{\zeta} + D$  with constant curvature  $K = 2AD$ , we find three more solutions to the equations (3.67):

$$\xi_1 = i \left( \zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}} \right), \quad (3.69)$$

$$\xi_2 = \frac{i}{2\sqrt{|AD|}} \left( (D - A\zeta^2) \partial_\zeta - (D - A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (3.70)$$

$$\xi_3 = \frac{1}{2\sqrt{|AD|}} \left( (D + A\zeta^2) \partial_\zeta + (D + A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (3.71)$$

closing in  $\mathfrak{so}(3)$ ,  $\mathfrak{e}_2$  and  $\mathfrak{so}(2, 1)$  algebras<sup>31</sup> for positive, zero or negative  $K$ . Using (3.60) and (3.61) one shows that for generic angular velocity  $a$  and nut charge  $n$ , only  $\xi_1$  obeys the strong condition (3.68). This is then promoted to a bulk field generating the rotational U(1) isometry of the Kerr-Taub-NUT family. For vanishing  $a$  and  $n$ , all three Carrollian Killing fields are strong and the bulk Ricci flat solution is fully isotropic — Schwarzschild or A-class metric, see [74, 85].

**Charge analysis.** We would like to close the present section with a brief account on the charges of the Ricci-flat solutions under investigation. Gravitational charges disclose the identity of a background and, as we have proposed in section 3.3, boundary Carrollian geometry supplies alternative techniques for their determination and the study of their conservation. These techniques are still in an incipient stage though, because the contact with the standard methods still needs to be elaborated. Furthermore, non-radiating configurations, in particular stationary and algebraically special, offer a limited playground

<sup>30</sup>We could keep non-vanishing  $\xi^t$  and perform a more thorough analysis. This would not alter the conclusions, which are meant here to illustrate possible boundary faults in the rigidity theorem.

<sup>31</sup>The Lie brackets of the  $\xi_s$  are  $[\xi_1, \xi_2] = \xi_3$ ,  $[\xi_3, \xi_1] = \xi_2$  and  $[\xi_2, \xi_3] = \frac{K}{|K|} \xi_1$ . For vanishing  $K$ ,  $\xi_2 = \frac{i}{2} (\partial_\zeta - \partial_{\bar{\zeta}})$  and  $\xi_3 = \frac{1}{2} (\partial_\zeta + \partial_{\bar{\zeta}})$  are the translation commuting generators of  $\mathfrak{e}_2$ .

in this programme. We would like nevertheless to summarize the situation, in view of the follow-up discussion on Möbius hidden-group action, section 4.2.

The simplest non-vanishing charge is the electric curvature defined in (C.9):<sup>32</sup>

$$Q_{\text{ec}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} K. \tag{3.72}$$

Divided by the volume of  $\mathcal{S}$ , this is simply the average Gauss curvature. Note in passing that the charges defined here are extensive, hence the integrals may reveal convergence issues, in particular when  $\mathcal{S}$  is non-compact. Normalizing with  $\text{Vol} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$  is the simplest way to fix this divergence.<sup>33</sup> Alternatively,  $\mathcal{S}$  could be compactified — quotiented by a discrete isometry group. We will leave this discussion aside, as it would be better addressed within attempts to make sense of Ricci-flat black holes with non-compact horizons (see e.g. Ch. 9 of [85]).

The towers of charges introduced in section 3.3 are slightly simpler in the instance under consideration. Indeed, the Carrollian conformal Killings used in expressions (3.31) and (3.34) are (C.47) with

$$C(t, \zeta, \bar{\zeta}) = tP(\zeta, \bar{\zeta}) \tag{3.73}$$

(see (C.48)). Observe also that  $\tilde{K}^i = \tilde{K}_{\text{Cot}}^i = 0$  so that  $\tilde{\mathcal{K}}_{(s)} = \tilde{\mathcal{K}}_{\text{Cot}(s)} = 0$ . Generically,  $\mathcal{K}_{(s)}$  and  $\mathcal{K}_{\text{Cot}(s)}$  are non-zero though, because the conformal Killing vectors are not necessarily strong and due to the time dependence, here encoded exclusively in their component  $\xi^t$ . The corresponding charges are ultimately expressed as integrals of combinations of  $\hat{k}, \hat{\bar{k}}, \hat{\tau}, \hat{\bar{\tau}}, *\varpi, \varpi$ , and of their derivatives.<sup>34</sup>

For concreteness, we will illustrate the above with the distinctive strong Carrollian conformal Killing field  $\partial_t$ , i.e. the generator of the Ehlers-Geroch bulk three-dimensional reduction. For this Killing field, the “tilde” Carrollian charges vanish. In example, for the leading charges ( $s = 0$  in the coding of section 3.3), we find<sup>35</sup>

$$Q_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (8\pi G\varepsilon + \varpi K), \quad Q_{\text{mm}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (-c + *\varpi K), \tag{3.74}$$

up to boundary terms with respect to (3.33) and (3.38) (and a factor  $-8\pi G$  for the former), handily combined into

$$Q_{\text{m}} = Q_{\text{mm}} + iQ_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\varpi} K). \tag{3.75}$$

The indices stand for magnetic and electric masses. These mass definitions carry some arbitrariness since, as a consequence of time independence, each of the terms in the integrals

<sup>32</sup>Remember that here  $\xi_{ij} = 0$ , and the geometry is  $t$ -independent with vanishing  $\theta, \varphi_i, \mathcal{A}, \hat{\mathcal{R}}_i$  as well as  $X_{ij}$ .

<sup>33</sup>The integrals can be performed by setting  $\zeta = Ze^{i\Phi}$ , where  $0 \leq \Phi < 2\pi$  and  $Z = \sqrt{2} \tan \frac{\Theta}{2}$ ,  $0 < \Theta < \pi$  for  $\mathbb{S}^2$ ;  $Z = \frac{R}{\sqrt{2}}$ ,  $0 < R < +\infty$  for  $\mathbb{E}_2$ ;  $Z = \sqrt{2} \tanh \frac{\Psi}{2}$ ,  $0 < \Psi < +\infty$  for  $\mathbb{H}_2$ .

<sup>34</sup>Although the components  $b_i$  of the Ehresmann connection enter the expression of the Carrollian charges (B.20), upon integration by parts, they are traded for  $*\varpi$  or  $\varpi$ .

<sup>35</sup>Using (3.31) and (3.34) with  $\xi^t = 1$  and  $\xi^i = 0$ , we find  $\kappa = -\varepsilon$ ,  $K^i = -\frac{1}{8\pi G} *\chi^i$ ,  $\kappa_{\text{Cot}} = -c$  and  $K_{\text{Cot}}^i = -\chi^i$ .

provide a separate well-defined charge. We will turn back to this when discussing the action of the Möbius group, in section 4.2.

Following section 3.3, the subleading mass charges associated with the strong Carrollian conformal Killing field  $\partial_t$  are captured in

$$Q_{\text{m}(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\omega}K) * \varpi^{2s} \tag{3.76}$$

and define the higher- $s$  *mass multipole moments*. In the instance of the  $K = 1$  Kerr-Taub-NUT family displayed in eqs. (3.58), (3.59), (3.60), (3.61) with  $A = 1/2$  and  $D = 1$ , we find:

$$Q_{\text{m}(s)} = 4\pi i (M + in) \left( \frac{(n+a)^{2s+1} - (n-a)^{2s+1}}{a(2s+1)} \right). \tag{3.77}$$

For this set of solutions,  $\xi_1$  in (3.69) is a strong Carrollian Killing vector, which brings its own Carrollian rotational charges. Again the “tilde” (eqs. (3.32) and (3.37)) vanish whereas the “non-tilde” (see. (3.31) and (3.34)) are combined in the complex higher- $s$  *angular-momentum multipole moments*

$$Q_{\text{r}(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} 6\zeta\bar{\zeta} \left( \frac{n+iM}{P^2} (a-nP) \left( n+a - \frac{2a}{P} \right)^{2s} - \frac{2a}{P^2} \left( n+a - \frac{2a}{P} \right)^{2s+1} \right) \tag{3.78}$$

with  $P = 1 + \frac{1}{2}\zeta\bar{\zeta}$ , which are non-zero if one rotation parameter  $a$  or  $n$  is present. We find for example:

$$Q_{\text{r}(0)} = -8\pi [a(n+iM) + 3n(n-iM)]. \tag{3.79}$$

Expressions (3.77) and (3.79) are in line with the results obtained in refs. [79–82] (see also [89], where the electric part of  $Q_{\text{r}(0)}$  is given) using standard methods circumscribed to bulk dynamics. They provide conserved moments since the divergences (3.30) and (3.36) vanish.

## 4 Ehlers transformations

### 4.1 Bulk reduction and Möbius action on the boundary

Our next and pivotal task is to unravel the action of the Ehlers group (2.11) on the boundary Carrollian observables, using the expression of the bulk Ricci-flat metric (3.6) assumed to possess a time-like Killing vector field. We will focus in the present work on the restricted class of resumable metrics (3.13), as exploited in section 3.4, i.e. equipped with a time-like Killing field  $\xi = \partial_t$  and  $\Omega = 1$ .

In order to proceed, we are called to follow the steps for the Geroch reduction described in section 2, i.e. determine  $\tau$  as defined in (2.9) for the metric (3.13) with  $\lambda$  and  $\omega$  given in (2.1), (2.2) and (2.3). These should be expanded in inverse powers of  $r$  and thus deliver the boundary ingredients together with their transformations following (2.11). A remark should be made before hand. The Geroch reduction is followed by an oxidation, which defines the novel Ricci-flat solution. Nothing guarantees in this course that the oxidized

metric will assume again the form (3.13). Actually it doesn't and a redefinition of the radial coordinate is necessary to bring it back into the expected original gauge.

It is convenient for the present mission to adopt the Cartan frame defined in (3.24), leading to the bulk metric

$$\begin{aligned}
 g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - K, & g_{\hat{t}i} &= *\partial_i *\varpi, & g_{\hat{t}r} &= -1, \\
 g_{ri} &= 0, & g_{rr} &= 0, & g_{ij} &= \rho^2 a_{ij}
 \end{aligned}
 \tag{4.1}$$

obtained using (3.25), assuming  $t$ -independence and  $\Omega = 1$ . In this expression  $*\varpi$ ,  $K$  and  $c$  are given in eqs. (3.39), (3.40) and (3.41). The Killing form reads:

$$\xi = \left( K - \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) \right) \mu + *\partial_i *\varpi dx^i - dr,
 \tag{4.2}$$

with norm

$$\lambda = \frac{8\pi G\epsilon r + *\varpi c}{\rho^2} - K.
 \tag{4.3}$$

For the twist we use eq. (2.2), expressed as

$$w = -\star(\xi \wedge d\xi),
 \tag{4.4}$$

where “ $\star$ ” stands for the four-dimensional Hodge duality. The latter one-form is exact on-shell and we find the following potential (eq. (2.3)):

$$\omega = \frac{8\pi G\epsilon *\varpi - cr}{\rho^2} + K^*.
 \tag{4.5}$$

On-shellness is implemented here through boundary dynamics as summarized in section 3.4, i.e. in eqs. (3.48), (3.49) and (3.50).

Inserting the above results into eqs. (2.9) and using (3.18), we find

$$\tau = \frac{\hat{\tau}}{r + i*\varpi} - i\hat{k}.
 \tag{4.6}$$

Likewise, we obtain the Geroch reduced and rescaled metric (2.8):

$$\tilde{h}_{AB} dx^A dx^B = - \left( dr - *\partial_k *\varpi dx^k \right)^2 + \lambda \rho^2 a_{ij} dx^i dx^j.
 \tag{4.7}$$

With this,  $\tau$  given in (4.6) unsurprisingly solves the reduced Einstein's equations (2.10).

The premier Ehlers transformation rules are (2.11) and the invariance of  $\tilde{h}_{AB}$ . From these follows the rest of the construction, i.e. the transformation of  $h_{AB}$  and the oxidation toward  $g'_{AB}$ . In the present framework, we have to some extent locked the gauge, via the resummed bulk expression (3.13). Ehlers transformations are not designed a priori to maintain this form, and they are generally expected to require further coordinate transformations. It is rather remarkable that, to this end, a local (i.e. celestial-sphere dependent) shift in the radial coordinate suffices.

Using for convenience holomorphic and antiholomorphic coordinates as introduced in appendix C, expression (4.7) is recast as follows:

$$\tilde{h}_{AB}dx^A dx^B = - \left( dr - i\partial_\zeta * \varpi d\zeta + i\partial_{\bar{\zeta}} * \varpi d\bar{\zeta} \right)^2 + \frac{(\tau - \bar{\tau})(r + i * \varpi)(r - i * \varpi)}{iP^2} d\zeta d\bar{\zeta}. \quad (4.8)$$

Combining (2.11) with (4.6), we obtain the following *boundary* transformations:

$$\hat{\tau}' = - \frac{\hat{\tau}}{(\gamma\hat{k} + i\delta)^2}, \quad (4.9)$$

$$\hat{k}' = i \frac{\alpha\hat{k} + i\beta}{\gamma\hat{k} + i\delta}, \quad (4.10)$$

$$\hat{\omega}' = \hat{\omega} + \frac{\gamma\hat{\tau}}{\gamma\hat{k} + i\delta} \quad (4.11)$$

and

$$P' = \frac{P}{|\gamma\hat{k} + i\delta|}, \quad (4.12)$$

plus the radial shift<sup>36</sup>

$$r' = r + \frac{i}{2} \left( \frac{\gamma\hat{\tau}}{\gamma\hat{k} + i\delta} - \frac{\gamma\hat{\tau}}{\gamma\hat{k} - i\delta} \right). \quad (4.13)$$

These transformation rules leave indeed (4.8) invariant. As advertised earlier, they are *local*, providing a direct transformation (4.12) of the boundary metric. The transformation of the energy density  $\varepsilon$  is obtained from (4.9) using (3.18):

$$8\pi G\varepsilon' = \frac{8\pi G\varepsilon \left( (\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) - 2c\gamma K (\gamma K^* + \delta)}{\left( \gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (4.14)$$

The transformation of  $c$  is inferred similarly:

$$c' = \frac{c \left( (\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) + 16\pi G\varepsilon\gamma K (\gamma K^* + \delta)}{\left( \gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (4.15)$$

All these rules are compatible with eqs. (3.40) and (3.41). Finally the transformations of the Carrollian Cotton tensors are reached using the above results combined with eqs. (3.42), (3.43) and (3.44).

The transformation of the Ehresmann connection is obtained directly from the expressions reached for the latter in (3.51), (3.55) and (3.57). To this end, observe that in the constant- $\hat{k}$  instance,  $A$ ,  $B$ ,  $\bar{B}$  and  $D$  transform with a factor  $1/|\gamma\hat{k} + i\delta|$  in order to comply

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<sup>36</sup>One could alternatively adopt a new radial coordinate defined as  $\tilde{r} = r + \varpi$  that is invariant under Möbius transformations. This is actually mandatory in order to reach boundary  $SL(2, \mathbb{R})$ -covariant tensors from the bulk, as we will discuss in section 4.2. It furthermore coincides with the radial coordinate of ref. [74] section 29 provided  $r_0 = -\varpi$  (origin of the affine parameter along the geodesic congruence tangent to  $\partial_r$  — see footnote 21).

with (4.12). Similarly,  $f(\zeta)$  and  $Z(\zeta)$ , introduced in (3.54) and (3.56), must be respectively invariant and transforming as

$$Z'(\zeta) = Z(\zeta) + i \frac{\gamma \hat{\tau}(\zeta)}{\gamma \hat{k} + i\delta}, \tag{4.16}$$

so that (4.11) be fulfilled.

Let us mention for completeness that once the Möbius transformation is performed on the boundary, the reconstruction of the new Ricci-flat solution is straightforward using the boundary-to-bulk formula (3.13), expressed with primed data — except for the unaltered boundary coordinates  $\{t, \zeta, \bar{\zeta}\}$ . This is equivalent to the oxidation procedure operated from three to four dimensions along the lines of eqs. (2.12), (2.13), (2.14) and (2.15) with

$$\eta' = -\mu' - \frac{1}{\lambda'} \left( dr' - i\partial_\zeta * \varpi' d\zeta + i\partial_{\bar{\zeta}} * \varpi' d\bar{\zeta} \right), \quad \mu' = -dt + b'_\zeta d\zeta + b'_{\bar{\zeta}} d\bar{\zeta}, \tag{4.17}$$

finally leading to (2.16), which assumes the form (3.13) primed. The new bulk Killing vector  $\xi' = \lambda' \eta'$  is again  $\partial_t$ .

In the example of the Kerr-Taub-NUT family treated at the end of section 3.4, the specific choices of  $P = \frac{1}{2}\zeta\bar{\zeta} + 1$ ,  $K = 1$  and  $K^* = 0$  (this was not explicitly demanded) are stable only under  $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ . For this transformation, using (4.9) we find  $M' + in' = (M + in)e^{-2i\chi}$ . Observe that (4.11) will switch on a non-zero  $\varpi$  though, as opposed to its original value in the family at hand (see footnote 28).

## 4.2 Charges and $\text{SL}(2, \mathbb{R})$ multiplets

Carrollian charges have been introduced in section 3.3 and further discussed for stationary and algebraic spacetimes in section 3.4. Two generic charges were found and displayed in (3.72) and (3.75). The former is purely geometric and stands for the integrated curvature of the celestial sphere; the latter carries genuine dynamic information captured in the electric and magnetic masses. It is legitimate to wonder how these quantities behave under Möbius transformations, and possibly tame them in  $\text{SL}(2, \mathbb{R})$  multiplets. Although ideally this programme should be conducted for reductions along generic bulk Killing fields and no special algebraic structure — these would be non-resummable, i.e. of the form (3.6), and labelled by a possibly plethoric set of independent charges — we will pursue it here for illustrative purposes in the restricted framework at hand.

The curvature charge  $Q_{\text{ec}}$  in (3.72) is invariant under Ehlers'  $\text{SL}(2, \mathbb{R})$ , and this is inferred using the transformation laws (4.10) and (4.12). The mass charge  $Q_{\text{m}}$ , eq. (3.75), is not, but its transformation (see (4.9), (4.10) and (4.11)) suggests that it might belong to some  $\text{SL}(2, \mathbb{R})$  multiplet or, more accurately, that it may be modified to this end — we have this freedom owing to time independence. Actually, a slight amendment to the charge  $Q_{\text{m}}$ , namely

$$Q'_{\text{m}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + 2\hat{\omega}K), \tag{4.18}$$

is  $\text{SL}(2, \mathbb{R})$ -invariant. We can even go further and apply the following pattern to generate  $\text{SL}(2, \mathbb{R})$  triplets. Suppose we identify a Carrollian two-form  $\mathbf{v}$  transforming under  $\text{SL}(2, \mathbb{R})$

as

$$\mathbf{v} \rightarrow \mathbf{v}' = -\mathbf{v} \left( \gamma \hat{k} - i\delta \right)^2. \tag{4.19}$$

This allows to design an  $SL(2, \mathbb{R})$  two-form triplet, i.e. a symmetric rank-two tensor, transforming as

$$\begin{pmatrix} \mathbf{v}'_3 & \mathbf{v}'_2 \\ \mathbf{v}'_2 & \mathbf{v}'_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \tag{4.20}$$

where

$$\mathbf{v}_1 = \mathbf{v}, \quad \mathbf{v}_2 = i\hat{k}\mathbf{v}, \quad \mathbf{v}_3 = -\hat{k}^2\mathbf{v}. \tag{4.21}$$

The same holds for the complex-conjugate triplet:  $\bar{\mathbf{v}}_1 = \bar{\mathbf{v}}$ ,  $\bar{\mathbf{v}}_2 = -i\hat{k}\bar{\mathbf{v}}$  and  $\bar{\mathbf{v}}_3 = -\hat{k}^2\bar{\mathbf{v}}$ . An  $SL(2, \mathbb{R})$  triplet of charges is thus reached as

$$Q_I = \int_{\mathcal{S}} \mathbf{v}_I, \quad I = 1, 2, 3, \tag{4.22}$$

and  $Q \equiv Q_1 Q_3 - Q_2^2$  is invariant under Möbius transformations.

The above strategy can be readily applied. Two-forms transforming as in (4.19) can be found, inspired by the structures of the charge (3.75) and of the Carrollian currents (3.31) and (3.34), given the expressions of the Carrollian twist (3.46), the Carrollian curvature (3.40), and the Carrollian Cotton tensors (3.41), (3.42), (3.43) and (3.44). We here exhibit two such Carrollian forms:

$$\mathbf{x} = -\frac{\hat{\tau}}{2(\hat{k} + \hat{k})} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}, \tag{4.23}$$

$$\mathbf{y} = -\left( \frac{P}{\hat{k} + \hat{k}} \right)^2 \partial_\zeta \hat{k} \partial_{\bar{\zeta}} \hat{\omega} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}. \tag{4.24}$$

These lead along (4.22) to two triplets of charges, which do not carry more information than the original (3.72) and (3.75) though — in the constant- $\hat{k}$  paradigm, which is in fact the most generic, these are  $K$ ,  $M$ ,  $n$  and possibly  $a$ , and the second triplet vanishes.

The last item in our Carrollian agenda is to setting the relationship amongst the charges introduced here using purely boundary methods and those computed directly by standard bulk techniques. This sort of question definitely deserves to be addressed in more general situations than ours, i.e. in the presence of a large set of non-trivial surface charges computed e.g. within covariant phase-space formalism [90]. Nonetheless some relevant observations can be made here, in relation with the original discussion on charges of ref. [2], in which the above two-forms (4.23) and (4.24) turn out to play a prominent role.

In ref. [2], an  $SL(2, \mathbb{R})$  triplet of bulk two-forms, leading to surface charges upon integration on the celestial sphere of  $\mathcal{M}$ , is obtained by oxidizing the following two-form triplet of  $\mathcal{S} \equiv \mathcal{M}/\text{orb}(\hat{\xi})$  (eqs. (18) and (16) of the quoted reference):

$$\begin{aligned} V_1 &= \frac{1}{(\tau - \bar{\tau})^2} \star_h^3 (d\tau + d\bar{\tau}), \\ V_2 &= \frac{1}{(\tau - \bar{\tau})^2} \star_h^3 (\bar{\tau}d\tau + \tau d\bar{\tau}), \\ V_3 &= \frac{1}{(\tau - \bar{\tau})^2} \star_h^3 (\bar{\tau}^2 d\tau + \tau^2 d\bar{\tau}), \end{aligned} \tag{4.25}$$



where “ $\star_h^3$ ” stands for the three-dimensional Hodge-dual on  $\mathcal{S}$  equipped with  $\tilde{h}_{AB}$  displayed in (4.8). It is remarkable that the asymptotic limit of this two-form triplet coincides with those designed earlier from Carrollian boundary considerations. This statement is captured in the following result:

$$\lim_{\tilde{r} \rightarrow \infty} \begin{pmatrix} V_3 & V_2 \\ V_2 & V_1 \end{pmatrix} = \begin{pmatrix} -\hat{k}^2(\mathbf{x} + \mathbf{y}) - \hat{k}^2(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) \\ i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & \mathbf{x} + \mathbf{y} + \bar{\mathbf{x}} + \bar{\mathbf{y}} \end{pmatrix}, \quad (4.26)$$

where  $\tilde{r} = r + \varpi$  was introduced in footnote 36 as an  $SL(2, \mathbb{R})$ -invariant radial coordinate, which must be used here in order to guarantee that the limit preserves the  $SL(2, \mathbb{R})$  behaviour.

## 5 Conclusions

When a four-dimensional spacetime geometry is invariant under the action of a one-dimensional group of motions, a reduction can be performed and vacuum Einstein dynamics reveals a symmetry under Möbius transformations. Our main motivation was to exhibit this action from a holographic perspective, namely on the three-dimensional boundary of the Ricci-flat configuration at hand. We have successfully reached this goal for a class of *resummable* or *integrable* metrics, which are algebraic in Petrov’s classification and possess a time-like isometry. All of our findings can be extended to embody any Ricci-flat spacetime possessing an isometry at the expense of an augmented technical difficulty due to (i) the use of generic Killing vectors with Ehlers action ending outside the class of algebraically special, resummable metrics (3.13),<sup>37</sup> and (ii) the presence of an indefinitely increasing number of independent boundary observables transforming under  $SL(2, \mathbb{R})$ . The main features of the boundary  $SL(2, \mathbb{R})$  action are however clearly captured by the simplest case treated here and we will now summarize them.

At the heart of the boundary Möbius transformations one finds the Carrollian Cotton tensors. The latter are a set of descendants of the original boundary pseudo-Riemannian Cotton, reached in the zero-speed-of-light limit. One finds in particular a scalar  $c$ , which is a dual-mass aspect, naturally combined with the Bondi mass aspect, another Carrollian scalar identified with the boundary Carrollian fluid energy density  $\varepsilon$ . The Möbius transformation hence mixes the geometric boundary variables i.e. those which determine the boundary itself with the dynamical variables like the boundary fluid (this is one of the infinite data, made redundant in the resummable situation studied here). Our analysis reveals that this duality transformation on the boundary is *algebraic* i.e. *local* for the metric, Ehresmann connection, field of observers, and for every other Carrollian boundary data. This is an important achievement summarized in eqs. (4.9), (4.10), (4.11) and (4.12), rooted in the decoupling of  $r$  close to the boundary.

An aside message this analysis conveys is the role of the Cotton tensor, which is manifestly dual to the energy/momentum. Before the advent of flat holography and Carrollian

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<sup>37</sup>See e.g. [91], where examples of space-like Killings are displayed with Ehlers groups connecting Petrov special to Petrov general Ricci-flat spacetimes (more recent works in a similar spirit are refs. [92, 93]), and [47] for a mathematical essay on the behaviour of the Weyl tensor under Ehlers’ Möbius group.

physics, the boundary Cotton tensor had been recognized in AdS/CFT as an unavoidable boundary trait carrying information on the bulk magnetic charges such as the nut [63, 64] (see also [94]). The Möbius transformation (4.9) for the  $SO(2) \subset SL(2, \mathbb{R})$  subgroup was actually anticipated as a relationship on the conformal pseudo-Riemannian boundary of four-dimensional Einstein spacetimes [65, 95], in an attempt to relate electric and magnetic solutions to Einstein’s equations. Although such dual solutions exist irrespective of the cosmological constant, the relevant subgroup of Ehlers’ breaks down for  $\Lambda \neq 0$  [46]. The bulk duality relationship fades in this case, but persists asymptotically and reveals on the conformal boundary. What we find here is a  $\Lambda$ -to-zero limit of this relationship.

Notwithstanding their role in boundary Ehlers duality manifestation, the Carrollian Cotton tensors obey off-shell conservation properties and generate towers of magnetic charges, some of them being conserved. This property is not exclusive to Ricci-flat spacetimes and Carrollian boundaries. Einstein bulk spacetimes and pseudo-Riemannian boundaries do provide a conserved Cotton tensor, which contracted with any boundary conformal Killing vector leads to a conserved current, hence a conserved charge. This powerful tool is undermined by the limited — if any — number of conformal isometries on arbitrary three-dimensional Riemannian spacetimes. The remarkable spin-off about Carrollian boundaries is the existence of an infinite-dimensional conformal group, which makes this method of charge determination a serious alternative to the more standard bulk asymptotic techniques. Following the Cotton pattern, electric towers of charges are constructed with the fluid dynamical data, which can only enjoy on-shell conservation — the same would hold in AdS boundaries with the aforementioned limitation. On both electric and magnetic sides, the towers of charges are multiplied ad nauseam, beyond their leading components.

Our present investigation on towers of charges designed from a boundary standpoint is radically novel and deserves a systematic extension. It has been here confined in the integrable case, where the infinite set of observables is redundant and shrinks to the elementary “leading” data — our tentative definition of subleading currents might have turned too naive, hadn’t it reproduced successfully the multipole moments. Moreover, our main goal being primarily on boundary Ehlers action, we have assumed a time-like bulk isometry, which further reduces this set. Besides, the chosen time-like Killing field was aligned with the fibre of the boundary Carrollian structure, which screens the black-hole acceleration parameter and avoids exploring head-on the uncharted subject of Carrollian reductions. The latter is the mathematical tool to be developed for unravelling the bulk-to-boundary relationship of hidden symmetries in Ricci-flat spacetimes. It could encompass bulk reductions along space-like isometries, which are interesting because they leave room for gravitational radiation,<sup>38</sup> probing the interplay between Ehlers Möbius group, time evolution and charge non-conservation. Last, we did not address the question of the charge algebra and its potential central extensions, or discussed other more general related physical aspects. All this calls for a thorough comparison to alternative approaches such as those of refs. [32–39] based on Newman-Penrose formalism — or to applications [83, 96–101].

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<sup>38</sup>The Petrov-algebraic spacetimes (3.13) accommodate axisymmetric time-dependent solutions of the Robinson-Trautman type, whose final state is the C-metric — see. [74] section 28.1.

In the chapter of charges, in spite of the various limitations just stated, we have successfully described the Ehlers Möbius action, and discussed the organisation of available charges in  $SL(2, \mathbb{R})$  multiplets. This enabled us to recover from a Carrollian viewpoint the triplet of Komar charges inferred by Geroch in its original publication [2]. Again, this result should be considered as a first step toward a methodical  $SL(2, \mathbb{R})$  taming of the above towers of electric/magnetic currents and charges in more general situations. These objects should include the boundary attributes of the bulk Weyl tensor, whose behaviour under Möbius transformations has been addressed in [47].

The importance of the boundary covariantization — Carroll and Weyl — is yet another feature we would like to stress, as it hasn't been sufficiently appreciated in the literature. This characteristic is absent from Bondi or Newman-Unti gauges, where the formalism might suggest that the relevant part of the conformal boundary is its two-dimensional spatial section — the celestial sphere. We heavily insist on the three-dimensional and Carrollian nature of the boundary, which is made manifest in the gauge we have been using. In ordinary AdS/CFT holography the Fefferman-Graham gauge is superior for this reason. One should likewise use a truly boundary-covariant gauge in flat holography and take advantage of it, as we modestly did for exhibiting the action of Ehlers' group, or for discussing the charges and their conservation. No boundary approach of this sort would have been possible in the more conventional gauges. Correspondingly, flat holography based on a purely celestial gauge is bound to be incomplete.

It is worth mentioning that Ehlers'  $SL(2, \mathbb{R})$  group is the first and simplest example of a hidden symmetry. As pointed out in the introduction (see the references proposed there), more involved reductions reveal richer symmetries and the underlying dynamics is captured by elegant sigma models in various dimensions. Recasting this knowledge in a holographic fashion, we could possibly learn more, or at least differently, not only about hidden symmetries but also on flat holography. Carrollian reductions might again be the appropriate tool.

On a more speculative tone, our results suggest that a boundary analysis might reveal more general or unexpected duality properties. The paradigm of anti-de Sitter spacetimes, where the  $SL(2, \mathbb{R})$  is broken in the bulk but restored on the boundary, calls for a systematic investigation that would complement the heuristic discussion of ref. [65], and possibly uncover novel instances of boundary duality symmetries, associated e.g. with an asymptotic Killing field rather than a plain reduction along Killing orbits. One could even be more audacious and entertain the idea of a “boundary” analysis for half-flat spaces (this is vaguely motivated by footnote 18), which have attracted some attention in relation with  $w_{1+\infty}$  symmetry (see the original works [104–106] and [107–111] for a recent emanation). The main caveat foreseen here is the absence of Carrollian boundaries in Euclidean gravity, but this could be evaded in the ultra-hyperbolic instance ( $2 + 2$  signature).

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## A Carrollian covariance in arbitrary dimension

Carroll structures on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  were introduced in section 3.1 with emphasis on the covariance properties they enjoy when the time coordinate is aligned with the fiber of the structure. In the present appendix we will elaborate on this subject, treating in particular Carrollian covariant and Weyl-covariant derivatives.

The Carrollian transformations (eqs. (3.3) and (3.4)) are connection-like (non-covariant) for  $\partial_i$  and  $b_i$ , and density-like for  $\partial_t$  and  $\Omega$ :

$$\partial'_j = J^{-1i}{}_j \left( \partial_i - \frac{j_i}{J} \partial_t \right), \quad b'_k = \left( b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k, \quad \partial'_t = \frac{1}{J} \partial_t, \quad \Omega' = \frac{\Omega}{J}. \quad (\text{A.1})$$

The vector fields dual to the forms  $dx^i$  are

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t \quad (\text{A.2})$$

and transform covariantly under (3.3) together with the metric (3.1), and the fields (3.2) and (3.5):

$$\mathbf{v}' = \mathbf{v}, \quad \boldsymbol{\mu}' = \boldsymbol{\mu}, \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j, \quad a^{ij'} = J^i{}_k J^j{}_l a^{kl}. \quad (\text{A.3})$$

The vectors  $\hat{\partial}_i$  and  $\mathbf{v}$  do not commute. They define the *Carrollian vorticity and acceleration*:

$$\left[ \hat{\partial}_i, \hat{\partial}_j \right] = \frac{2}{\Omega} \varpi_{ij} \partial_t, \quad \left[ \mathbf{v}, \hat{\partial}_i \right] = \frac{1}{\Omega} \varphi_i \partial_t, \quad \varpi_{ij} = \partial_{[i} b_{j]} + b_{[i} \varphi_{j]}, \quad \varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega), \quad (\text{A.4})$$

similarly appearing in

$$d\boldsymbol{\mu} = \varpi_{ij} dx^i \wedge dx^j + \varphi_i dx^i \wedge \boldsymbol{\mu}. \quad (\text{A.5})$$

A Carroll structure (strong definition) is also equipped with a metric-compatible and torsionless connection. Due to the degeneration of the metric, such a connection is not unique, but it can be chosen as the connection inherited from the parent relativistic space-time (see footnote 9),

$$\hat{\gamma}^i{}_{jk} = \frac{a^{il}}{2} \left( \hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk} \right), \quad (\text{A.6})$$

obeying  $\hat{\gamma}^k{}_{[ij]} = 0$ ,  $\hat{\nabla}_i a_{jk} = 0$  and leading to the Levi-Civita-Carroll spatial covariant derivative  $\hat{\nabla}_i$ .<sup>39</sup>

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<sup>39</sup>Details on the transformation rules can be found in the appendix A.2 of ref. [70].

The ordinary time-derivative operator  $\frac{1}{\Omega}\partial_t$  acts covariantly on Carrollian tensors. However, it is not metric-compatible because  $a_{ij}$  depend on time and a temporal covariant derivative is defined requiring  $\frac{1}{\Omega}\hat{D}'_t = \frac{1}{\Omega}\hat{D}_t$  and  $\hat{D}_t a_{jk} = 0$ . To this end, we introduce a temporal connection (a sort of extrinsic curvature of the spatial section  $\mathcal{S}$ )

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega}\partial_t a_{ij} = \xi_{ij} + \frac{1}{d}a_{ij}\theta, \quad \theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \quad (\text{A.7})$$

which is a symmetric Carrollian tensor splitting into the *geometric Carrollian shear (traceless)* and the *Carrollian expansion (trace)*. The action of  $\hat{D}_t$  on any tensor is obtained using Leibniz rule plus the action on scalars and vectors:

$$\frac{1}{\Omega}\hat{D}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i_j V^j, \quad \frac{1}{\Omega}\hat{D}_t V_i = \frac{1}{\Omega}\partial_t V_i - \hat{\gamma}_i^j V_j. \quad (\text{A.8})$$

The commutators of Carrollian covariant derivatives define Carrollian curvature tensors:

$$\begin{aligned} [\hat{\nabla}_k, \hat{\nabla}_l] V^i &= \left( \hat{\partial}_k \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj} \right) V^j + [\hat{\partial}_k, \hat{\partial}_l] V^i \\ &= \hat{r}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i, \end{aligned} \quad (\text{A.9})$$

where  $\hat{r}^i_{jkl}$  is the *Riemann-Carroll* tensor. The Ricci-Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k_{ikj} \neq \hat{r}_{ji}, \quad \hat{r} = a^{ij} \hat{r}_{ij}. \quad (\text{A.10})$$

Similarly, space and time derivatives do not commute:

$$\left[ \frac{1}{\Omega} \hat{D}_t, \hat{\nabla}_i \right] V^j = \varphi_i \left( \left( \frac{1}{\Omega} \hat{D}_t + \theta \right) V^j - \hat{\gamma}^j_k V^k \right) - \hat{\gamma}_i^k \hat{\nabla}_k V^j - d \hat{r}^j_{ik} V^k \quad (\text{A.11})$$

with

$$\hat{r}^j_{ik} = \frac{1}{d} \left( \theta \varphi_i \delta^j_k + \hat{\nabla}_i \hat{\gamma}^j_k - \frac{1}{\Omega} \partial_t \hat{\gamma}^j_{ik} \right), \quad \hat{r}^j_{jk} = \hat{r}_k = \frac{1}{d} \left( \hat{\nabla}_j \hat{\gamma}^j_k - \hat{\partial}_k \theta \right), \quad (\text{A.12})$$

further Carrollian curvature tensors.

The boundary geometry — be it pseudo-Riemannian or Carrollian — enjoy conformal properties. Weyl transformations are defined through their action on elementary geometric data

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega \quad (\text{A.13})$$

with  $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$  an arbitrary function. A Weyl-covariant derivative requires an appropriate connection built on  $\varphi_i$  and  $\theta$  defined in (A.4) and (A.7), which transform as

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \rightarrow \mathcal{B} \theta - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (\text{A.14})$$

The Carrollian vorticity  $\varpi_{ij}$  (A.4) and the Carrollian shear  $\xi_{ij}$  (A.7) are Weyl-covariant of weight  $-1$ .

The Weyl-Carroll space and time covariant derivatives are metric-compatible and torsionless. For a scalar function  $\Phi$  and a vector  $V^l$  of weight  $w$ , we find:

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad (\text{A.15})$$

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (\text{A.16})$$

The weights are not altered by the spatial derivative and  $\hat{\mathcal{D}}_j a_{kl} = 0$ . One also defines

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega} \hat{D}_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (\text{A.17})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^l = \frac{1}{\Omega} \hat{D}_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi^l_i V^i, \quad (\text{A.18})$$

both are of weight  $w + 1$ . Furthermore  $\hat{\mathcal{D}}_t a_{kl} = 0$ , using Leibniz rule.

We finally obtain

$$\left[ \hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j \right] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathcal{D}}_t \Phi + w \Omega_{ij} \Phi, \quad (\text{A.19})$$

$$\left[ \hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l \right] V^i = \left( \hat{\mathcal{R}}^i_{jkl} - 2\xi^i_j \varpi_{kl} \right) V^j + \varpi_{kl} \frac{2}{\Omega} \hat{\mathcal{D}}_t V^i + w \Omega_{kl} V^i, \quad (\text{A.20})$$

where

$$\begin{aligned} \hat{\mathcal{R}}^i_{jkl} &= \hat{r}^i_{jkl} - \delta^i_j \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i + \delta^i_k \hat{\nabla}_l \varphi_j - \delta^i_l \hat{\nabla}_k \varphi_j \\ &\quad + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) - \left( \delta^i_k a_{jl} - \delta^i_l a_{jk} \right) \varphi_m \varphi^m + \left( \delta^i_k \varphi_l - \delta^i_l \varphi_k \right) \varphi_j, \end{aligned} \quad (\text{A.21})$$

$$\Omega_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i - \frac{2}{d} \varpi_{ij} \theta \quad (\text{A.22})$$

are weight-0 Weyl-covariant tensors. Tracing them we obtain:

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k_{ikj}, \quad \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (\text{A.23})$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1) \left( 2\hat{\nabla}_i \varphi^i - (d-2) \varphi_i \varphi^i \right), \quad (\text{A.24})$$

of weights zero and 2. The Weyl-covariant Carroll-Ricci tensor is not symmetric,  $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2} \Omega_{ij}$ , and a weight-1 curvature form also appears with

$$\left[ \frac{1}{\Omega} \hat{\mathcal{D}}_t, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j_i \hat{\mathcal{D}}_j \Phi, \quad (\text{A.25})$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{d} \left( \hat{\partial}_i + \varphi_i \right) \theta. \quad (\text{A.26})$$

## B Conformal Carrollian dynamics and charges

A complete account on the subject of dynamics and charges with the present conventions is available in refs. [70, 71]. We summarize here the necessary items, in particular regarding the Weyl-covariant side, which is relevant on the holographic boundaries.

The basics are encoded into four Carrollian momenta, replacing the relativistic energy-momentum tensor, which are obtained by varying some (effective) action with respect to  $a_{ij}$ ,  $b_i$  and  $\Omega$  (the fourth momentum is not necessarily obtained in this way — for details see [71]). These are the energy-stress tensor  $\Pi^{ij}$ , the energy flux  $\Pi^i$ , the energy density  $\Pi$  as well as the momentum  $P^i$ , of conformal weights  $d + 3$ ,  $d + 2$ ,  $d + 1$  and  $d + 2$ . Extra momenta can also emerge as more degrees of freedom may be present. This phenomenon

occurs when studying the small- $c$  limit of a relativistic energy-momentum tensor and the corresponding conservation equations. Keeping things rather minimal with only  $\tilde{\Pi}^{ij}$  the equations read:<sup>40</sup>

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \tag{B.1}$$

$$\tilde{\Pi}^{ij} \xi_{ij} = 0, \tag{B.2}$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) P_i = 0, \tag{B.3}$$

$$\hat{\mathcal{D}}_i \tilde{\Pi}^i_j + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) \Pi_i = 0 \tag{B.4}$$

with

$$\tilde{\Pi}_i^i = 0, \quad \Pi_i^i = \Pi, \tag{B.5}$$

as a consequence of the assumed Weyl invariance.

Equations (B.1), (B.2), (B.3) and (B.4) are the Carrollian emanation of the relativistic conservation equation  $\nabla_\mu T^{\mu\nu} = 0$ . As for the relativistic instance, conformal isometries lead to conserved currents and conserved charges. Let  $\xi$  be a  $d + 1$ -dimensional vector

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \left( \xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left( \partial_i + \frac{b_i}{\Omega} \partial_t \right) = \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^{\hat{i}} \hat{\partial}_i \tag{B.6}$$

restricted to  $\xi^i = \xi^i(\mathbf{x})$ , generator of a one-dimensional group of Carrollian diffeomorphisms on  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ . Its action on the elementary geometric data (3.1), (3.2) and (3.5) is as

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<sup>40</sup>Using the language of fluids,  $\Pi$  appears as the zero- $c$  limit of the relativistic energy density,  $\Pi^i$  and  $P^i$  are the orders one and  $c^2$  of the relativistic heat current, whereas  $\tilde{\Pi}^{ij}$  and  $\Pi^{ij}$  are the orders  $1/c^2$  and one of the relativistic stress. A non-vanishing Carrollian energy flux  $\Pi^i$  breaks local Carroll-boost invariance (see e.g. [102]) and makes its dual variable i.e. the Ehresmann connection  $\mathbf{b} = b_i dx^i$  dynamical. This is neither a surprise nor a caveat. On the one hand, Carrollian dynamics, i.e. dynamics on geometries equipped with a degenerate metric, is often reached as a vanishing- $c$  limit of relativistic dynamics and naturally breaks local Carroll boosts, even when the original relativistic theory is Lorentz-boost invariant. Indeed, invariance under local Lorentz boosts sets symmetry constraints on the components of the relativistic energy-momentum tensor, but not on their behaviour with respect to  $c^2$ , leaving the possibility of persisting energy flux  $\Pi^i$  and “over-stress”  $\tilde{\Pi}^{ij}$  related through eq. (B.4). A similar phenomenon occurs in Galilean theories, defined on spacetimes with a degenerate cometric, where the Galilean momentum is possibly responsible for the breaking of local Galilean-boost invariance. On the other hand, it is fortunate that this happens in the present instance (one of the very few known applications of Carrollian dynamics), when passing from the relativistic boundary of asymptotically anti-de Sitter spacetimes to the Carrollian boundary of their asymptotically flat relatives, as the Carrollian energy flux accounts for non-conservation properties resulting from bulk gravitational radiation, whereas the Ehresmann connection is part of the Ricci-flat solution space.

follows:<sup>41</sup>

$$\mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i}\xi^k a_{j)k} + 2\xi^{\hat{t}}\hat{\gamma}_{ij}, \quad (\text{B.7})$$

$$\mathcal{L}_\xi \mathbf{v} = \mu \mathbf{v}, \quad (\text{B.8})$$

$$\mathcal{L}_\xi \mu = -\mu \mu + \nu \quad (\text{B.9})$$

with  $\mathbf{v} = \nu_i dx^i$  and

$$\begin{aligned} \mu(t, \mathbf{x}) &= -\left(\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i\right), \\ \nu_i(t, \mathbf{x}) &= -\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} + 2\xi^j \varpi_{ji}. \end{aligned} \quad (\text{B.10})$$

Due to the degeneration of the metric on  $\mathcal{M}$ , the variation of the field of observers  $\mathbf{v}$  is not identical to that of the clock form  $\mu$ .

Isometries are generated by Killing fields of the Carrollian type (B.6), required to obey [30, 31, 41, 51]:

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi \mathbf{v} = 0. \quad (\text{B.11})$$

i.e.

$$\hat{\nabla}_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\hat{\gamma}_{ij} = 0, \quad (\text{B.12})$$

$$\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0. \quad (\text{B.13})$$

The clock form is not required to be invariant. Carrollian conformal Killing fields must satisfy

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij} \quad (\text{B.14})$$

with

$$\lambda(t, \mathbf{x}) = \frac{2}{d}\left(\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}}\right). \quad (\text{B.15})$$

This set of partial differential equations is insufficient for defining conformal Killing fields. One usually imposes to tune  $\mu$  versus  $\lambda$  (see [30, 31, 51] for a detailed presentation) so that the scaling of the metric be twice that of the field of observers:

$$2\mu + \lambda = 2\left(\frac{1}{d}\hat{\mathcal{D}}_i \xi^i - \frac{1}{\Omega}\hat{\mathcal{D}}_t \xi^{\hat{t}}\right) = 0 \quad (\text{B.16})$$

(the conformal weight of  $\xi^{\hat{t}}$  is  $-1$ , that of  $\xi^i$  is zero). Again, the clock form is not involved. If one demands the latter be invariant under the action of a Killing field, or aligned with itself under the action of a conformal Killing, which in both cases amounts to setting

$$\nu_i \equiv -\hat{\mathcal{D}}_i \xi^{\hat{t}} + 2\xi^j \varpi_{ji} = 0 \quad (\text{B.17})$$

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<sup>41</sup>The Lie derivative along  $\xi = \xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^i\hat{\partial}_i$  of a general Carrollian tensor reads:

$$\begin{aligned} \mathcal{L}_\xi S_{i\dots}{}^{j\dots} &= \left(\xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^k\hat{\partial}_k\right)S_{i\dots}{}^{j\dots} + S_{k\dots}{}^{j\dots}\hat{\partial}_i \xi^k + \dots - S_{i\dots}{}^{k\dots}\hat{\partial}_k \xi^j - \dots \\ &= \left(\xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^k\hat{\nabla}_k\right)S_{i\dots}{}^{j\dots} + S_{k\dots}{}^{j\dots}\hat{\nabla}_i \xi^k + \dots - S_{i\dots}{}^{k\dots}\hat{\nabla}_k \xi^j - \dots \end{aligned}$$



(this is a conformal rewriting of  $\nu_i$  given in (B.10)), then the corresponding (conformal) isometry generator will be referred to as *(conformal) strong Killing vector field*.<sup>42</sup>

Transformations generated by ordinary Carrollian Killing fields leave invariant geometric markers that are built over the metric  $a_{ij}$  and  $\Omega$ , such as the Carrollian expansion  $\theta$  or the shear  $\xi_{ij}$ , encoded in  $\hat{\gamma}_{ij}$ , see (A.7). The Carrollian vorticity and acceleration given in (A.5) are not left invariant, however, since

$$\mathcal{L}_\xi d\mu = d\nu, \tag{B.18}$$

unless the Carrollian Killing field is strong (vanishing  $\nu$ ). Likewise, curvature invariance does also require the strong condition. This applies in particular to the Carrollian Cotton tensor discussed in  $d = 2$  (see appendix C).

On a Carroll manifold a current has a scalar component  $\kappa$  as well as a Carrollian-vector set of components  $K^i$ . The divergence takes the form (see [40, 71, 103])

$$\mathcal{K} = \left(\frac{1}{\Omega}\partial_t + \theta\right)\kappa + \left(\hat{\nabla}_i + \varphi_i\right)K^i. \tag{B.19}$$

The charge associated with the current  $(\kappa, \mathbf{K})$  is an integral at fixed  $t$  over the basis  $\mathcal{S}$

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} \left(\kappa + b_i K^i\right), \tag{B.20}$$

and obeys the following time evolution:

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial\mathcal{S}} *\mathbf{K} \Omega. \tag{B.21}$$

The last term is of boundary type with  $*\mathbf{K}$  the  $\mathcal{S}$ -Hodge dual of  $K_i dx^i$ . Generally, one can ignore it owing to adequate fall-off or boundary conditions on the fields.

Suppose that  $\xi$  is the generator (B.6) of a Carrollian diffeomorphism. It can be used to create two currents out<sup>43</sup> of  $\Pi^{ij}$ ,  $\tilde{\Pi}^{ij}$ ,  $\Pi^i$ ,  $P^i$  and  $\Pi$  [40, 71]:

$$\begin{cases} \kappa = \xi^i P_i - \xi^t \Pi \\ \tilde{\kappa} = \xi^i \Pi_i \\ K^i = \xi^j \Pi_j^i - \xi^t \Pi^i \\ \tilde{K}^i = \xi^j \tilde{\Pi}_j^i, \end{cases} \tag{B.22}$$

If  $\xi$  is a (conformal) Carrollian Killing field, and assuming all momenta on-shell i.e. eqs. (B.1), (B.2), (B.3) and (B.4) (with (B.5) satisfied in the conformal instance), one finds the following Carrollian divergences (the conformal weights of  $\kappa$  and  $\tilde{\kappa}$  are  $d$ , those of  $K^i$  and  $\tilde{K}^i$ ,  $d + 1$ , and  $-1$  for  $\nu_i$ ):

$$\begin{cases} \tilde{\mathcal{K}} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \tilde{\kappa} + \hat{\mathcal{D}}_i \tilde{K}^i = 0 \\ \mathcal{K} = \frac{1}{\Omega} \hat{\mathcal{D}}_t \kappa + \hat{\mathcal{D}}_i K^i = \Pi^i \nu_i. \end{cases} \tag{B.23}$$

<sup>42</sup>Carroll boosts, which are the archetype isometries of flat Carrollian spacetimes, *are not* generated by strong Killings [71].

<sup>43</sup>We stress here that if more momenta were present, more currents would be available.

Two charges can be defined following (B.20):  $Q_{\hat{K}}$  and  $Q_K$ . The former is conserved, whereas the latter isn't for generic isometries unless the field configuration has vanishing energy flux  $\Pi^i$ , i.e. if local Carroll-boost invariance is unbroken. The breaking of local Carroll-boost invariance hence appears as the trigger of non-conservation laws. This peculiarity was risen in [40, 71] and further illustrated with concrete field realizations in [103]. In four-dimensional Ricci-flat spacetimes, this boundary non-conservation is the consequence of bulk gravitational radiation, as mentioned previously in footnote 40. Observe nevertheless that irrespective of the energy flux  $\Pi^i$ , the (conformal) strong Killings introduced earlier do lead to full conservation properties as a consequence of (B.17).

### C Three dimensions and the Carrollian Cotton tensor

Three-dimensional boundaries ( $d = 2$ ) outline the framework of the Ehlers and Geroch investigation pursued in the main part of this article. Three dimensions have two remarkable properties. At the first place, if the geometric Carrollian shear  $\xi^{ij}$  defined in (A.7) vanishes, which occurs for the Carrollian boundaries of Ricci-flat spacetimes as a consequence of Einstein's equations (see section 3.1), the Carrollian conformal isometry group is infinite-dimensional:  $\text{BMS}_4 \equiv \text{ccarr}(3) \equiv \mathfrak{so}(3,1) \times \text{supertranslations}$  [23, 41]. This potentially generates *infinite towers of charges*, possibly conserved.

Secondly, three-dimensional Carrollian spacetimes possess a *Carrollian Cotton tensor obeying conservation dynamics*. It appears as a set of Carrollian scalars, vectors and tensors emerging in the small- $c$  expansion of the relativistic Cotton  $C_{\mu\nu}$ , which is symmetric, traceless, divergence-free and Weyl-covariant with weight 1. Reference [23] provides a complete account of the Carrollian descendants as they emerge from the pseudo-Riemannian Cotton tensor, in the absence of geometric Carrollian shear. Here we will circumscribe our exhibition to the basic output.

For  $d = 2$ , the  $\mathcal{S}$ -Hodge duality is induced by<sup>44</sup>  $\eta_{ij} = \sqrt{a}\epsilon_{ij}$ . This duality is involutive on Carrollian vectors as well as on two-index symmetric and traceless Carrollian tensors:

$$*V_i = \eta^l{}_i V_l, \quad *W_{ij} = \eta^l{}_i W_{lj}. \tag{C.1}$$

This fully antisymmetric form can be used to recast some of the expressions introduced in appendix A. The Carroll-Ricci tensor (A.10) is decomposed as

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K}a_{ij} + \hat{A}\eta_{ij} \tag{C.2}$$

with

$$\hat{s}_{ij} = 2 * \varpi * \xi_{ij}, \quad \hat{K} = \frac{1}{2} a^{ij} \hat{r}_{ij} = \frac{1}{2} \hat{r}, \quad \hat{A} = \frac{1}{2} \eta^{ij} \hat{r}_{ij} = * \varpi \theta, \quad * \varpi = \frac{1}{2} \eta^{ij} \varpi_{ij}. \tag{C.3}$$

Similarly

$$\hat{\mathcal{R}}_{ij} = \hat{s}_{ij} + \hat{\mathcal{K}}a_{ij} + \hat{\mathcal{A}}\eta_{ij}, \tag{C.4}$$

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<sup>44</sup>We use here the conventions of ref. [23], namely  $\epsilon_{12} = -1$ , convenient when using complex coordinates  $\{\zeta, \bar{\zeta}\}$ . Notice that  $\eta^{il}\eta_{jl} = \delta_j^i$  and  $\eta^{ij}\eta_{ij} = 2$ .

where we have introduced two weight-2 Weyl-covariant scalar Gauss-Carroll curvatures:

$$\hat{\mathcal{K}} = \frac{1}{2}a^{ij}\hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2}\eta^{ij}\hat{\mathcal{R}}_{ij} = \hat{A} - *\varphi. \quad (\text{C.5})$$

These obey Carroll-Bianchi identities:

$$\frac{2}{\Omega}\hat{\mathcal{D}}_t*\varpi + \hat{\mathcal{A}} = 0, \quad (\text{C.6})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{K}} - a^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j - \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j\xi^{ij} = 0, \quad (\text{C.7})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{A}} + \eta^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j = 0. \quad (\text{C.8})$$

Thanks to the identities (C.7) and (C.8), the couples  $\{\hat{\mathcal{K}}, -\hat{\mathcal{R}}^i - \hat{\mathcal{D}}_j\xi^{ij}\}$  and  $\{\hat{\mathcal{A}}, -*\hat{\mathcal{R}}^i\}$  allow to define *electric and magnetic curvature charges* as in eqs. (B.19) and (B.20):

$$Q_{\text{ec}} = \int_{\mathcal{S}} d^2x\sqrt{a} \left( \hat{\mathcal{K}} - b_i \left( \hat{\mathcal{R}}^i + \hat{\mathcal{D}}_j\xi^{ij} \right) \right), \quad Q_{\text{mc}} = \int_{\mathcal{S}} d^2x\sqrt{a} \left( \hat{\mathcal{A}} - b_i *\hat{\mathcal{R}}^i \right). \quad (\text{C.9})$$

Following (B.21), we find

$$\frac{dQ_{\text{ec}}}{dt} = \int_{\partial\mathcal{S}} * \left( \hat{\mathcal{R}} + \hat{\mathcal{D}} \cdot \xi \right) \Omega, \quad \frac{dQ_{\text{mc}}}{dt} = - \int_{\partial\mathcal{S}} \hat{\mathcal{R}} \Omega. \quad (\text{C.10})$$

Upon regular behaviour, the boundary terms vanish and the curvature charges are both conserved.

Besides the various curvature tensors, which are second derivatives of the metric and the Ehresmann connection, one defines third-derivative tensors, the descendants of the relativistic Cotton tensor. We will here limit our presentation to the instance  $\xi_{ij} = 0$ , which is the appropriate framework when solving Einstein's equations in the bulk. This reduces the number of tensors to five, a weight-3 scalar, two weight-2 forms and two weight-1 two-index symmetric and traceless tensors:

$$c = \left( \hat{\mathcal{D}}_l\hat{\mathcal{D}}^l + 2\hat{\mathcal{K}} \right) *\varpi, \quad (\text{C.11})$$

$$\chi_j = \frac{1}{2}\eta^l_j\hat{\mathcal{D}}_l\hat{\mathcal{K}} + \frac{1}{2}\hat{\mathcal{D}}_j\hat{\mathcal{A}} - 2*\varpi\hat{\mathcal{R}}_j, \quad (\text{C.12})$$

$$\psi_j = 3\eta^l_j\hat{\mathcal{D}}_l*\varpi^2, \quad (\text{C.13})$$

$$X_{ij} = \frac{1}{2}\eta^l_j\hat{\mathcal{D}}_l\hat{\mathcal{R}}_i + \frac{1}{2}\eta^l_i\hat{\mathcal{D}}_j\hat{\mathcal{R}}_l, \quad (\text{C.14})$$

$$\Psi_{ij} = \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j*\varpi - \frac{1}{2}a_{ij}\hat{\mathcal{D}}_l\hat{\mathcal{D}}^l*\varpi - \eta_{ij}\frac{1}{\Omega}\hat{\mathcal{D}}_t*\varpi^2. \quad (\text{C.15})$$

As a consequence of the relativistic conservation of the Cotton tensor, its Carrollian descendants obey eqs. (B.1), (B.2),<sup>45</sup> (B.3) and (B.4) with

$$\Pi_{\text{Cot}} = c, \quad \Pi_{\text{Cot}}^i = \chi^i, \quad P_{\text{Cot}}^i = \psi^i, \quad \tilde{\Pi}_{\text{Cot}}^{ij} = -X^{ij}, \quad \Pi_{\text{Cot}}^{ij} = \frac{c}{2}a^{ij} - \Psi^{ij}, \quad (\text{C.16})$$

<sup>45</sup>Equation (B.2), is trivially satisfied due to the vanishing of  $\xi_{ij}$ . If  $\xi_{ij} \neq 0$ , extra Cotton Carrollian descendants are available, and the conservation dynamics is encoded in more momenta and equations — in particular (B.2) is modified.

which read

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c + \hat{\mathcal{D}}_i \chi^i = 0, \tag{C.17}$$

$$\frac{1}{2} \hat{\mathcal{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_j - \hat{\mathcal{D}}_i \Psi^i_j = 0, \tag{C.18}$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \chi_j - \hat{\mathcal{D}}_i X^i_j = 0. \tag{C.19}$$

When the geometric Carrollian shear vanishes, the time dependence in the metric is factorized as  $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \bar{a}_{ij}(\mathbf{x})$ . One then shows [23, 41] that the Carrollian conformal isometry group is the semi-direct product of the conformal group of  $\bar{a}_{ij}(\mathbf{x})$  with the infinite-dimensional supertranslation group. The former is generated by  $Y^i(\mathbf{x})$ , the latter by  $T(\mathbf{x})$ , and the Carrollian conformal Killing fields read:

$$\xi_{T,Y} = \left( T(\mathbf{x}) - Y^i(\mathbf{x}) \hat{\partial}_i C(t, \mathbf{x}) + \frac{1}{2} C(t, \mathbf{x}) \bar{\nabla}_i Y^i(\mathbf{x}) \right) \frac{e^{\sigma(t, \mathbf{x})}}{\Omega} \partial_t + Y^i(\mathbf{x}) \hat{\partial}_i \tag{C.20}$$

with

$$C(t, \mathbf{x}) \equiv \int^t d\tau e^{-\sigma(\tau, \mathbf{x})} \Omega(\tau, \mathbf{x}). \tag{C.21}$$

This result is valid in any dimension. At  $d = 2$ ,  $\bar{a}_{ij}(\mathbf{x})$  is conformally flat and  $Y^i(\mathbf{x})$  generate  $\mathfrak{so}(3, 1)$ .<sup>46</sup>

The conservation of the Carrollian Cotton momenta (C.16) makes it possible to define two infinite towers of Carrollian Cotton charges  $Q_{\text{Cot}T,Y}$  and  $\tilde{Q}_{\text{Cot}T,Y}$  following (B.20), based on the Carrollian Cotton currents  $\kappa_{\text{Cot}}$ ,  $K_{\text{Cot}}^i$ ,  $\tilde{\kappa}_{\text{Cot}}$  and  $\tilde{K}_{\text{Cot}}^i$  (see (3.34)). According to (B.23), the latter are always conserved,<sup>47</sup> whereas the former are only if  $\chi^i \nu_i = -\chi^i (\hat{\mathcal{D}}_i \xi^i - 2\xi^j \varpi_{ji}) = 0$ . This occurs for special geometries ( $\chi^i = 0$ ) or for the subset of strong Carrollian conformal Killing fields ( $\nu_i = 0$ ).

In  $d = 2$ , it is convenient to use complex spatial coordinates  $\zeta$  and  $\bar{\zeta}$ . With the permission of the authors of [23], we reproduce here the appendix of that reference, summarizing the useful formulas in this coordinate system. Using Carrollian diffeomorphisms (3.3), the metric (3.1) of the Carrollian geometry on the two-dimensional surface  $\mathcal{S}$  can be recast in conformally flat form,

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \tag{C.22}$$

with  $P = P(t, \zeta, \bar{\zeta})$  a real function, under the necessary and sufficient condition that the Carrollian shear  $\xi_{ij}$  displayed in (A.7) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data:  $\Omega(t, \zeta, \bar{\zeta})$  and

$$\mathbf{b} = b_\zeta(t, \zeta, \bar{\zeta}) d\zeta + b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) d\bar{\zeta} \tag{C.23}$$

<sup>46</sup>The  $\mathfrak{so}(3, 1)$  factor can also be promoted to superrotations (double Virasoro) if we give up the absolute regularity requirement.

<sup>47</sup>The conformal Killing fields (C.20), (C.21) depend *explicitly* on time. Inside the charges they define, when conserved, this time dependence is confined, on-shell, in a boundary term, and hence drops — see concrete examples in [103].

with  $b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) = \bar{b}_{\zeta}(t, \zeta, \bar{\zeta})$ . Our choice of orientation is inherited from the one adopted for the relativistic boundary (see footnote 10) with  $a_{\zeta\bar{\zeta}} = 1/P^2$  is<sup>48</sup>

$$\eta_{\zeta\bar{\zeta}} = -\frac{i}{P^2}. \quad (\text{C.24})$$

The first-derivative Carrollian tensors are the acceleration (A.4), the expansion (A.7) and the scalar vorticity (A.4), (C.3):

$$\varphi_{\zeta} = \partial_t \frac{b_{\zeta}}{\Omega} + \hat{\partial}_{\zeta} \ln \Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln \Omega, \quad (\text{C.25})$$

$$\theta = -\frac{2}{\Omega} \partial_t \ln P, \quad * \varpi = \frac{i\Omega P^2}{2} \left( \hat{\partial}_{\zeta} \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_{\zeta}}{\Omega} \right) \quad (\text{C.26})$$

with

$$\hat{\partial}_{\zeta} = \partial_{\zeta} + \frac{b_{\zeta}}{\Omega} \partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega} \partial_t. \quad (\text{C.27})$$

Curvature scalars and vector are second-derivative (see (C.3), (A.12)):<sup>49</sup>

$$\hat{K} = P^2 \left( \hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} + \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad \hat{A} = iP^2 \left( \hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} - \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad (\text{C.28})$$

$$\hat{r}_{\zeta} = \frac{1}{2} \hat{\partial}_{\zeta} \left( \frac{1}{\Omega} \partial_t \ln P \right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2} \hat{\partial}_{\bar{\zeta}} \left( \frac{1}{\Omega} \partial_t \ln P \right), \quad (\text{C.29})$$

and we also quote:

$$* \varphi = iP^2 \left( \hat{\partial}_{\zeta} \varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}} \varphi_{\zeta} \right), \quad (\text{C.30})$$

$$\hat{\nabla}_k \varphi^k = P^2 \left[ \hat{\partial}_{\zeta} \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \partial_t \frac{b_{\zeta}}{\Omega} + \left( \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} \right) \ln \Omega \right]. \quad (\text{C.31})$$

Regarding conformal Carrollian tensors we remind the weight-2 curvature scalars (C.5):

$$\hat{\mathcal{H}} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \hat{A} - * \varphi, \quad (\text{C.32})$$

and the weight-1 curvature one-form (A.26):

$$\hat{\mathcal{R}}_{\zeta} = \frac{1}{\Omega} \partial_t \varphi_{\zeta} - \frac{1}{2} \left( \hat{\partial}_{\zeta} + \varphi_{\zeta} \right) \theta, \quad \hat{\mathcal{R}}_{\bar{\zeta}} = \frac{1}{\Omega} \partial_t \varphi_{\bar{\zeta}} - \frac{1}{2} \left( \hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}} \right) \theta. \quad (\text{C.33})$$

The three-derivative Cotton descendants displayed in (C.11)–(C.15) are a scalar

$$c = \left( \hat{\mathcal{D}}_l \hat{\mathcal{D}}^l + 2\hat{\mathcal{H}} \right) * \varpi \quad (\text{C.34})$$

<sup>48</sup>This amounts to setting  $\sqrt{a} = i/P^2$  in coordinate frame and  $\epsilon_{\zeta\bar{\zeta}} = -1$ . The volume form reads  $d^2x \sqrt{a} = \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$ .

<sup>49</sup>We also quote for completeness (useful e.g. in eq. (C.32)):

$$\hat{K} = K + P^2 \left[ \partial_{\zeta} \frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}} \frac{b_{\zeta}}{\Omega} + \partial_t \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} + 2 \frac{b_{\bar{\zeta}}}{\Omega} \partial_{\zeta} + 2 \frac{b_{\zeta}}{\Omega} \partial_{\bar{\zeta}} + 2 \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} \partial_t \right] \partial_t \ln P$$

with  $K = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta} \ln P$  the ordinary Gaussian curvature of the two-dimensional metric (C.22).

of weight 3 ( $*\varpi$  is of weight 1), two vectors

$$\chi_\zeta = \frac{i}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{H}}_\zeta, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{H}}_{\bar{\zeta}}, \quad (\text{C.35})$$

$$\psi_\zeta = 3i \hat{\mathcal{D}}_\zeta * \varpi^2, \quad \psi_{\bar{\zeta}} = -3i \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi^2, \quad (\text{C.36})$$

of weight 2, and two symmetric and traceless tensors

$$X_{\zeta\zeta} = i \hat{\mathcal{D}}_\zeta \hat{\mathcal{H}}_\zeta, \quad X_{\bar{\zeta}\bar{\zeta}} = -i \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{H}}_{\bar{\zeta}}, \quad (\text{C.37})$$

$$\Psi_{\zeta\zeta} = \hat{\mathcal{D}}_\zeta \hat{\mathcal{D}}_\zeta * \varpi, \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi, \quad (\text{C.38})$$

of weight 1. Notice that in holomorphic coordinates a symmetric and traceless tensor  $S_{ij}$  has only diagonal entries:  $S_{\zeta\bar{\zeta}} = 0 = S_{\bar{\zeta}\zeta}$ .

We also remind for convenience some expressions for the determination of Weyl-Carroll covariant derivatives. If  $\Phi$  is a weight- $w$  scalar function

$$\hat{\mathcal{D}}_\zeta \Phi = \hat{\partial}_\zeta \Phi + w \varphi_\zeta \Phi, \quad \hat{\mathcal{D}}_{\bar{\zeta}} \Phi = \hat{\partial}_{\bar{\zeta}} \Phi + w \varphi_{\bar{\zeta}} \Phi. \quad (\text{C.39})$$

For weight- $w$  form components  $V_\zeta$  and  $V_{\bar{\zeta}}$  the Weyl-Carroll derivatives read:

$$\hat{\mathcal{D}}_\zeta V_\zeta = \hat{\nabla}_\zeta V_\zeta + (w+2) \varphi_\zeta V_\zeta, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} + (w+2) \varphi_{\bar{\zeta}} V_{\bar{\zeta}}, \quad (\text{C.40})$$

$$\hat{\mathcal{D}}_\zeta V_{\bar{\zeta}} = \hat{\nabla}_\zeta V_{\bar{\zeta}} + w \varphi_\zeta V_{\bar{\zeta}}, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_\zeta = \hat{\nabla}_{\bar{\zeta}} V_\zeta + w \varphi_{\bar{\zeta}} V_\zeta, \quad (\text{C.41})$$

while the Carrollian covariant derivatives are simply:

$$\hat{\nabla}_\zeta V_\zeta = \frac{1}{P^2} \hat{\partial}_\zeta (P^2 V_\zeta), \quad \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} = \frac{1}{P^2} \hat{\partial}_{\bar{\zeta}} (P^2 V_{\bar{\zeta}}), \quad (\text{C.42})$$

$$\hat{\nabla}_\zeta V_{\bar{\zeta}} = \hat{\partial}_\zeta V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}} V_\zeta = \hat{\partial}_{\bar{\zeta}} V_\zeta. \quad (\text{C.43})$$

Finally,

$$\hat{\mathcal{D}}_k \hat{\mathcal{D}}^k \Phi = P^2 \left( \hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w \Phi \left( \hat{\partial}_\zeta \varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \varphi_\zeta \right) + 2w \left( \varphi_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \varphi_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w \varphi_\zeta \varphi_{\bar{\zeta}} \Phi \right) \right). \quad (\text{C.44})$$

Using complex coordinates, we can recast the conformal Killing vectors of a shear-free Carrollian spacetime  $\mathcal{M}$  in three dimensions, given in eqs. (C.20) and (C.21). These are expressed in terms of an arbitrary real function  $T(\zeta, \bar{\zeta})$ , which encodes the *supertranslations*, and the conformal Killing vectors of flat space  $d\ell^2 = 2d\zeta d\bar{\zeta}$ . The latter are of the form  $Y^\zeta(\zeta) \partial_\zeta + Y^{\bar{\zeta}}(\bar{\zeta}) \partial_{\bar{\zeta}}$ , reached with any combination of  $\ell_m + \bar{\ell}_m$  or  $i(\ell_m - \bar{\ell}_m)$ , where<sup>50</sup>

$$\ell_m = -\zeta^{m+1} \partial_\zeta, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1} \partial_{\bar{\zeta}}, \quad (\text{C.45})$$

obeying the Witt  $\oplus$  Witt algebra:

$$[\ell_m, \ell_n] = (m-n) \ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n) \bar{\ell}_{m+n}, \quad (\text{C.46})$$

---

<sup>50</sup>Notice that combining (C.1) and (C.24), we find  $*(\ell_m + \bar{\ell}_m) = -i(\ell_m - \bar{\ell}_m)$ .

and referred to as *superrotations*. Usually one restricts to  $\mathfrak{so}(3, 1)$ , generated by  $n = 0, \pm 1$ . The conformal Killing fields of  $\mathcal{M}$  are thus

$$\xi_{T,Y} = \left( T - \left( Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) C + \frac{C}{2} \left( \partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right) \right) \frac{1}{P} \mathbf{v} + Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \quad (\text{C.47})$$

with

$$C(t, \zeta, \bar{\zeta}) \equiv \int^t d\tau P(\tau, \zeta, \bar{\zeta}) \Omega(\tau, \zeta, \bar{\zeta}). \quad (\text{C.48})$$

The structure  $\mathfrak{so}(3, 1) \oplus$  supertranslations — or  $(\text{Witt} \oplus \text{Witt}) \oplus$  supertranslations — is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']} \quad (\text{C.49})$$

with

$$M_Y(f) = \left( Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) f - \frac{f}{2} \left( \partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right). \quad (\text{C.50})$$

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## Revisiting the Carrollian scalar field

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We investigate the (conformally coupled) scalar field on a general Carrollian spacetime in arbitrary dimension. The analysis discloses electric and magnetic dynamics. For both, we provide the energy and the momenta of the field, accompanied by their conservation equations. We discuss the conservation and nonconservation properties resulting from the existence of conformal isometries and the associated charges. We illustrate those results for a scalar field propagating on the null boundary of four-dimensional Ricci-flat Robinson-Trautman spacetimes.

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The Carroll group was discovered in the seminal works of Lévy-Leblond [1] and Sen Gupta [2]. Its first application, due to Henneaux, appeared 14 years later [3] and it took almost 50 years for Carrollian physics to emerge as a full-blown research area, ranging from differential geometry to holographic duality. Carrollian physics is meant to embrace phenomena occurring on a Carrollian spacetime, such as hydrodynamics or, at a more fundamental level, field dynamics. The simplest field is a scalar and it has received some attention [4–11].

The aim of the present note is to present the dynamics of a (conformally coupled) scalar on a general Carrollian manifold, tame and illustrate scattered results, and unify two distinct and complementary approaches. The first relies on Carrollian structures and diffeomorphism invariance. The second consists in reaching Carrollian geometry and dynamics from a pseudo-Riemannian relative at vanishing speed of light. The set of features we address includes: (i) electric vs magnetic dynamics; (ii) action and equations of motion; (iii) energy, momentum, and their conservation; (iv) isometries and Noether's theorem. The basic technical tools are listed in the Appendix.

Carroll structures were introduced in [12–14] (see also [15–20]). They consist of a  $(d + 1)$ -dimensional manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$  equipped with a degenerate metric and a vector field, the kernel of the metric. For concreteness, we

will adopt coordinates  $(t, \mathbf{x})$  and degenerate metrics of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \quad (1)$$

with a kernel generated by

$$\mathbf{v} = \frac{1}{\Omega} \partial_t, \quad (2)$$

which defines a field of observers. This coordinate system is adapted to the space/time splitting, which is in turn respected by Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \quad (3)$$

The Carrollian manifold incorporates an Ehresmann connection, which is the background gauge field  $\mathbf{b} = b_i dx^i$  appearing in the dual form of the field of observers (2), defined such as  $\boldsymbol{\mu}(\mathbf{v}) = -1$ :

$$\boldsymbol{\mu} = -\Omega dt + b_i dx^i, \quad (4)$$

the clock form ( $\Omega$  and  $b_i$  depend on  $t$  and  $\mathbf{x}$ ). The vector fields dual to the forms  $dx^i$  are

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t. \quad (5)$$

As shown in the Appendix, they transform covariantly under Carrollian diffeomorphisms (3). A Carroll structure (strong definition) is also equipped with a torsionless and metric-compatible connection. This is not unique, due to the degeneracy of the metric. We use here the connection inherited from the parent relativistic spacetime.

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A Carroll structure endowed with metric (1) and clock form (4) is naturally reached in the Carrollian limit ( $c \rightarrow 0$ ) of a pseudo-Riemannian spacetime  $\mathcal{M}$  in Papapetrou-Randers gauge

$$ds^2 = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j, \quad (6)$$

where all functions are  $x$  dependent with  $x \equiv (x^0 = ct, \mathbf{x})$ . The connection we use on the Carrollian side is given in the Appendix, Eqs. (A4) and (A6). These are parts of the Levi-Civita connection attached to (6), and decomposed in powers of  $c$ .

The dynamics of scalar fields on an arbitrary Carrollian spacetime, limited to two-derivative kinetic terms encompasses two distinct situations dictated by Carrollian covariance. Their Lagrangian densities read

$$\mathcal{L}_e = \frac{1}{2} \left( \frac{1}{\Omega} \partial_t \Phi \right)^2 - V_e(\Phi), \quad (7)$$

$$\mathcal{L}_m = -\frac{1}{2} a^{ij} \hat{\partial}_i \Phi \hat{\partial}_j \Phi - V_m(\Phi), \quad (8)$$

and enter the Carrollian action  $S_C = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega \mathcal{L}$ . The indices “e” and “m” stand for “electric” and “magnetic.” They refer to the origin of these actions in the parent relativistic theory [8,9]. Indeed starting from a relativistic scalar field on a Papapetrou-Randers background (6)

$$S = - \int_{\mathcal{M}} dt d^d x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right), \quad (9)$$

and assuming

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + \mathcal{O}(c^2), \quad (10)$$

we find

$$S = \frac{1}{c^2} S_e + S_m + \mathcal{O}(c^2), \quad (11)$$

with  $S_e$  and  $S_m$  the Carrollian actions with Lagrangian densities (7) and (8). The existence of an expansion (10) for the original relativistic potential in powers of  $c^2$  is a bona fide assumption, necessary to reach two actions invariant under Carrollian diffeomorphisms (3).<sup>1</sup>

Due to the form of the metric (6), and to its subsequent behavior under Carrollian diffeomorphisms, the decomposition of any relativistic tensor as a (usually truncated) Laurent expansion, provides a Carrollian tensor

<sup>1</sup>The actions associated with the  $\mathcal{O}(c^2)$  terms are nondynamical as no kinetic term appears at this order. This will be illustrated in the subsequent analysis of a conformally coupled scalar, see Eq. (16).

for each term.<sup>2</sup> If we insist in reaching a single Carrollian tensor at vanishing  $c$ , then an appropriate rescaling by some power of  $c^2$  is necessary—in order, e.g., to select one out of two options, if only two options are available as in the above scalar-field action (see [8], where this procedure is illustrated in Hamiltonian formalism and for flat spacetime).

An insightful scalar potential for a relativistic curved spacetime in  $d + 1$  dimensions is the following:

$$V(\Phi) = \frac{d-1}{8d} R \Phi^2. \quad (12)$$

For a scalar field  $\Phi$  of weight  $w = \frac{d-1}{2}$ , this is a conformal coupling. Indeed, the relativistic energy-momentum tensor for (9) with (12) has the form  $(\nabla_\mu \Phi = \partial_\mu \Phi)$

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \\ &\quad + \frac{d-1}{4d} (G_{\mu\nu} \Phi^2 + g_{\mu\nu} \square \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2), \\ &= \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi - \frac{1}{2} g_{\mu\nu} \mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi \\ &\quad + \frac{d-1}{4d} \left( \left( \mathcal{R}_{(\mu\nu)} - \frac{\mathcal{R}}{2} g_{\mu\nu} \right) \Phi^2 \right. \\ &\quad \left. + g_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}^\alpha \Phi^2 - \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi^2 \right), \end{aligned} \quad (13)$$

where<sup>3</sup>  $G_{\mu\nu}$  is the Einstein tensor,  $\mathcal{R}_{\mu\nu}$  and  $\mathcal{R}$  the Weyl-covariant Ricci and scalar defined in the Appendix [Eqs. (A37) and (A38)], together with the Weyl-covariant derivative  $\mathcal{D}_\mu$ . This energy-momentum tensor is traceless when  $\Phi$  is on shell, and with a Weyl-covariant of weight  $d-1$ . The action is Weyl invariant (up to boundary terms<sup>4</sup>), whereas the equations of motion can be recast readily with Weyl-covariant attributes:

$$-\mathcal{D}_\mu \mathcal{D}^\mu \Phi + \frac{d-1}{4d} \mathcal{R} \Phi = 0. \quad (14)$$

As a consequence of diffeomorphism invariance, the energy-momentum tensor obeys a Weyl-covariant conservation equation, when the field  $\Phi$  is on-shell:

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu} = 0. \quad (15)$$

<sup>2</sup>Phrased in more mathematical terms, the expansion in powers of  $c^2$ , amounts to reducing the representations of the full diffeomorphism group, with respect to the Carrollian diffeomorphism subgroup.

<sup>3</sup>We thank Konstantinos Siampos for a useful discussion on this topic.

<sup>4</sup>Equation (14) is also simply  $-\square \Phi + \frac{d-1}{4d} R \Phi = 0$ .

The interest for studying relativistic conformally coupled scalar fields is originally found in inflationary models of cosmology.<sup>5</sup> On the Carrollian side the motivation is entrenched in the attempts to generalize the gauge/gravity holographic correspondence for asymptotically flat spacetimes, where the boundary is null infinity, i.e., a Carrollian manifold par excellence.

Inserting inside (12) the Carrollian decomposition of  $R$  as displayed in the Appendix Eq. (A39), leads to

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + c^2 V_{\text{nd}}(\Phi), \quad (16)$$

with

$$V_e(\Phi) = \frac{d-1}{8d} \left( \frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) \Phi^2, \quad (17)$$

$$V_m(\Phi) = \frac{d-1}{8d} (\hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i) \Phi^2, \quad (18)$$

$$V_{\text{nd}}(\Phi) = \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2. \quad (19)$$

In the last expression the index “nd” stands for “non-dynamical.” The reason is that when the expression (16) of the potential is used in the relativistic action (9), it produces the Carrollian electric and magnetic actions—with some boundary terms dropped here<sup>6</sup>

$$S_e = \int dt d^d x \sqrt{a\Omega} \left( \frac{1}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \right), \quad (20)$$

$$S_m = \int dt d^d x \sqrt{a\Omega} \left( -\frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi - \frac{d-1}{8d} \hat{\mathcal{R}} \Phi^2 \right), \quad (21)$$

as well as a third one  $S_{\text{nd}} = -\int dt d^d x \sqrt{a\Omega} \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2$ , which has no kinetic term for  $\Phi$ .<sup>7</sup> The Carrollian equations of motion for the two nontrivial cases are as follows:

<sup>5</sup>See, e.g., [21] where more references are displayed.  
<sup>6</sup>On the relativistic side we find:  $\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{d-1}{8d} R \Phi^2 = \frac{1}{2} \mathcal{D}^\mu \Phi \mathcal{D}_\mu \Phi + \frac{d-1}{8d} \mathcal{R} \Phi^2 - \frac{d-1}{4\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu \Phi^2)$ .

<sup>7</sup>These results coincide with those obtained for  $d=2$  in Ref. [7], where the authors proceed with a thorough investigation of the possible Weyl-compatible terms. The kinetic terms of the electric and magnetic actions, (20) and (21), can also be compared to the corresponding results of [8]. They also agree up to the magnetic constraint introduced in Ref. [8], which would read here  $\Pi_m^i = 0$  [see (31)]. The latter guarantees the invariance of the action under local Carrollian boosts, which we have not required *a priori*—Carrollian invariance features here the covariance under Carrollian diffeomorphisms (3) of a theory defined on a Carrollian spacetime (1) and (2).

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi + \frac{d-1}{4d} \xi_{ij} \xi^{ij} \Phi = 0 \quad \text{electric}, \quad (22)$$

$$-\hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \hat{\mathcal{R}} \Phi = 0 \quad \text{magnetic}, \quad (23)$$

where the detailed expressions for the derivatives and Carrollian tensors are available in the Appendix. These equations are Weyl-covariant of weight  $w = \frac{d+1}{2}$ .

Energy and momenta are part of the agenda when discussing field dynamics. These are conjugate variables to the geometric data, as is  $T_{\mu\nu}$  in (13) for a relativistic theory, and inherit their conservation from the Carrollian diffeomorphism invariance. In Carrollian geometries there is no energy-momentum tensor, but instead an energy-stress tensor  $\Pi^{ij}$ , an energy flux  $\Pi^i$  and an energy density  $\Pi$ , defined as [4,22,23]

$$\Pi^{ij} = \frac{2}{\sqrt{a\Omega}} \frac{\delta S_C}{\delta a_{ij}}, \quad (24)$$

$$\Pi^i = \frac{1}{\sqrt{a\Omega}} \frac{\delta S_C}{\delta b_i}, \quad (25)$$

$$\Pi = -\frac{1}{\sqrt{a}} \left( \frac{\delta S_C}{\delta \Omega} + \frac{b_i \delta S_C}{\Omega \delta b_i} \right), \quad (26)$$

with conformal weights  $d+3$ ,  $d+2$  and  $d+1$ . Requiring Weyl invariance for the action translates into

$$\Pi_i^i = \Pi, \quad (27)$$

which is valid on shell (as the tracelessness of the relativistic energy-momentum tensor).

A momentum  $P_i$  (weight  $d$ ) is also defined but is not conjugate to a geometric variable. It enters the conservation equations that mirror the Carrollian diffeomorphism invariance. For Weyl-invariant dynamics these are [23]

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \quad (28)$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \delta^i_j + \xi^i_j \right) P_i = 0. \quad (29)$$

Conservation equations are satisfied when the field  $\Phi$  is on shell, and this allows us to determine the momentum.

Using Eqs. (24)–(26), we obtain the following energy and momenta for the Carrollian electric and magnetic actions:



$$\begin{cases} \Pi_e^{ij} = \frac{a^{ij}}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 + \frac{d-1}{4d} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t (\xi^{ij} \Phi^2) \right) - a^{ij} \left( \frac{1}{2} \xi_{lk} \xi^{lk} \Phi^2 + \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 \right) \\ \Pi_e^i = 0 \\ \Pi_e = \frac{1}{2} \left( \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \end{cases}, \quad (30)$$

$$\begin{cases} \Pi_{nd}^{ij} = \frac{d-1}{4d} (2\varpi^{li} \varpi_l^j - \frac{a^{ij}}{2} \varpi_{lk} \varpi^{lk}) \Phi^2 \\ \Pi_m^i = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} (\hat{\mathcal{D}}^i \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 - \hat{\mathcal{D}}_j (\xi^{ij} \Phi^2)) \\ \Pi_m = \frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left( \frac{\hat{\mathcal{D}}_t \Phi^2}{2} - \hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi^2 \right) \end{cases} \quad (31)$$

For the nondynamical action, which will turn useful in a short while, we find

$$\begin{cases} \Pi_{nd}^{ij} = \frac{d-1}{4d} (2\varpi^{li} \varpi_l^j - \frac{a^{ij}}{2} \varpi_{lk} \varpi^{lk}) \Phi^2 \\ \Pi_{nd}^i = \frac{d-1}{4d} \hat{\mathcal{D}}_j (\varpi^{ji} \Phi^2) \\ \Pi_{nd} = \frac{3(d-1)}{8d} \varpi_{ij} \varpi^{ij} \Phi^2 \end{cases}. \quad (32)$$

They all obey (27), and conservation equations (28) and (29) are satisfied with the electric momenta, assuming the field be on shell, i.e., obeying (22), and deliver the electric momentum:

$$P_e^i = \Pi_m^i. \quad (33)$$

In a similar fashion for the magnetic dynamics, and using the equation of motion (23), we obtain

$$P_m^i = \Pi_{nd}^i. \quad (34)$$

One might be puzzled at this stage by the interplay Eqs. (33) and (34) seem to entail amongst electric, magnetic, and nondynamics. There is no doubt that electric and magnetic Carrollian scalar dynamics resulting from  $\mathcal{L}_e$  and  $\mathcal{L}_m$  are distinct, and can be studied separately, on any Carrollian background. Likewise, the action  $\mathcal{L}_{nd} = -V_{nd}$  is also Carrollian invariant with bona fide Carrollian momenta, but is nondynamical. What sets a deeper link between these dynamics, which is not visible when treating them directly in the Carrollian framework, is that they all emerge in the “small- $c$  expansion” of a unique relativistic theory for the scalar field. This was one possible guideline for obtaining the Carrollian scalar theories. It can also be applied to the relativistic energy-momentum tensor, and will deliver in a similar expansion<sup>8</sup> the Carrollian momenta:

<sup>8</sup>The wording “expansion” is an abuse because the result is exact here.

$$\begin{cases} T^{ij} = \frac{1}{c^2} \Pi_e^{ij} + \Pi_m^{ij} + c^2 \Pi_{nd}^{ij} \\ -\frac{c}{\Omega} T_0^i = \Pi_m^i + c^2 \Pi_{nd}^i \\ \frac{1}{\Omega^2} T_{00} = \frac{1}{c^2} \Pi_e + \Pi_m + c^2 \Pi_{nd} \end{cases}. \quad (35)$$

The relationship between relativistic and Carrollian dynamics can be thrust further. Following [23,24] we can expand the relativistic conservation of energy momentum (15) and recollect the Carrollian conservation equations for the electric, the magnetic and the nondynamical cases. In this process Eqs. (28) and (29) arise for each case at a different  $c$  order, and their momenta  $P_e^i$  and  $P_m^i$  are naturally determined in terms of the next-order energy fluxes. This explains the above results (33) and (34).

Conserved charges are fundamental ingredients for handling a dynamical system. They often appear as the consequence of symmetries. In a relativistic framework, if  $\xi$  is a Killing field of the spacetime  $\mathcal{M}$ , then the current defined as

$$I_\mu = \xi^\nu T_{\mu\nu} \quad (36)$$

has zero divergence and ( $\mathcal{S}$  is a  $d$ -dimensional spatial section of  $\mathcal{M}$  and  $*I$  the  $\mathcal{M}$ -Hodge dual of  $I = I_\mu dx^\mu$ )

$$Q_I = \int_{\mathcal{S}} *I \quad (37)$$

is conserved. For Weyl-covariant dynamics this applies with conformal Killing fields.

In a Carrollian spacetime a current has a scalar component  $\kappa$  as well as a Carrollian-vector set of components  $K^i$ , and the divergence takes the form

$$\mathcal{K} = \left( \frac{1}{\Omega} \partial_t + \theta \right) \kappa + (\hat{\nabla}_i + \varphi_i) K^i. \quad (38)$$

This result can be inferred<sup>9</sup> as from a relativistic computation, with a current  $I^\mu$  such that

<sup>9</sup>See Ref. [23], where it is also shown how the current components are retrieved without reference to a relativistic ascendent.

$$-\frac{1}{c\Omega}I_0 = \kappa + \mathcal{O}(c^2), \quad I^k = K^k + \mathcal{O}(c^2), \quad (39)$$

leading in a Papapetrou-Randers background (6) to  $\nabla_\mu I^\mu = \mathcal{K} + \mathcal{O}(c^2)$ . Defining a charge associated with the current  $(\kappa, \mathbf{K})$  as an integral at fixed  $t$  over the basis  $\mathcal{S}$  of the Carrollian structure

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa + b_i K^i), \quad (40)$$

we obtain the following time evolution:

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial\mathcal{S}} \star \mathbf{K} \Omega, \quad (41)$$

where  $\star \mathbf{K}$  is the  $\mathcal{S}$ -Hodge dual of  $K_i dx^i$ . For vanishing divergence  $\mathcal{K}$ , this is conserved if one can ignore the boundary term owing to adequate falloff or boundary conditions on the fields. Notice that if  $\mathcal{K}$  happens to be identical to the Carrollian divergence of some potential  $(\phi, \boldsymbol{\phi})$ , then a conserved charge is obtained with  $\kappa - \phi$ ,  $K^i - \phi^i$ .

Suppose that  $\xi$  is the generator of a Carrollian diffeomorphism [see (A24) in the Appendix]. It can be used to create a current out of  $\Pi^{ij}$ ,  $\Pi^i$ ,  $\Pi$ , and  $P^i$  [4,23]:

$$\kappa = \xi^i P_i - \xi^i \Pi, \quad K^i = \xi^j \Pi_j^i - \xi^i \Pi^i. \quad (42)$$

For a Weyl-covariant system [Eq. (27)] with a conformal Killing vector [see the defining conditions in the Appendix, (A30) and (A31)], one obtains this:

$$\mathcal{K} = -\Pi^i ((\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji}). \quad (43)$$

As opposed to the relativistic situation, a conformal Killing field does not provide a conservation law in Weyl-invariant Carrollian dynamics, unless it satisfies (the conformal weight of  $\xi^i$  is  $-1$ , that of  $\xi^i$  zero)

$$(\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji} \equiv \hat{\mathcal{D}}_i \xi^i - 2\xi^j \varpi_{ji} = 0. \quad (44)$$

This last condition amounts to further demanding the clock form (4) be invariant under the action of the conformal Killing [see Eq. (A27) in the Appendix]. In Carrollian dynamics, symmetry is generated by a subalgebra of the conformal isometry algebra.

Electric and magnetic Carrollian scalar fields with conformal coupling have different behavior regarding conservation. The former have vanishing energy flux [see (30)] and lead thus to conserved charges  $Q_e = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa_e + b_i K_e^i)$  with

$$\kappa_e = \xi^i \Pi_{mi} - \xi^i \Pi_e, \quad K_e^i = \xi^j \Pi_{ej}^i, \quad (45)$$

where we have used (33). For the latter,  $Q_m = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa_m + b_i K_m^i)$  with [see (31) and (34)],

$$\kappa_m = \xi^i \Pi_{ndi} - \xi^i \Pi_m, \quad K_m^i = \xi^j \Pi_{mj}^i - \xi^i \Pi_m^i \quad (46)$$

is not conserved since, according to (43),

$$\mathcal{K}_m = -\Pi_m^i ((\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji}). \quad (47)$$

Conservation is attainable for field configurations such that  $\Pi_m^i = 0$ , which translates local Carroll-boost invariance [17]. Following (31), this happens, e.g., when  $\hat{\mathcal{D}}_i \Phi = 0$  in backgrounds with vanishing geometric Carrollian shear [ $\xi_{ij} = 0$ , defined in the Appendix, Eq. (A6)], which is possibly compatible with the magnetic dynamics (23).

Conformal Killing fields on general Carrollian spacetimes are obtained upon solving a set of complicated partial differential equations and this is not an easy task. It is remarkable that when the Carrollian shear [see (A6) in the Appendix]  $\xi^{ij}$  vanishes, the conformal Killing fields are known [19]. Zero shear implies that the time dependence in the metric is factorized:  $a_{ij}(t, \mathbf{x}) = \mathcal{B}^{-2}(t, \mathbf{x}) \tilde{a}_{ij}(\mathbf{x})$ . This drives the conformal algebra of the Carrollian structure to the standard infinite-dimensional semidirect sum of the conformal algebra of  $\tilde{a}_{ij}(\mathbf{x})$  with supertranslations. For conformally flat  $\tilde{a}_{ij}(\mathbf{x})$ , the latter coincides with  $\text{ccarr}(d+1)$ . One recovers in particular  $\text{BMS}_{d+2}$  in  $d = 1$  and  $2$ —possibly in higher dimensions.<sup>10</sup>

The Carrollian spacetimes emerging as null boundaries of asymptotically locally flat solutions to Einstein equations turn out to satisfy the vanishing-shear condition.<sup>11</sup> That makes this class of Carrollian structures particularly appealing and the forthcoming example will illustrate their properties regarding the propagation of a conformally coupled scalar field.

Robinson-Trautman spacetimes are four-dimensional, time-dependent Ricci-flat solutions of algebraically special Petrov type. They describe configurations emitting gravitational radiation and settling down in the far future into a Schwarzschild black hole.<sup>12</sup> Their null boundary is a Carrollian manifold  $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ , where  $\mathcal{S}$  is equipped with a conformally flat  $d = 2$  metric:

<sup>10</sup>The standard conformal Carrollian algebra  $\text{ccarr}(d+1)$  is also referred to as “level-2”  $\text{ccarr}_2(d+1)$ . More general level- $N$  algebras  $\text{ccarr}_N(d+1)$  emerge in the presence of a dynamical exponent  $z = 2/N$ —see footnote 21. For  $d > 2$  the BMS algebra is finite-dimensional, whereas  $\text{ccarr}_N(d+1)$  is not. Infinite-dimensional extensions of the  $\text{BMS}_{d+2}$  require adjustments in the fall-off behaviors and have been considered in the literature (see, e.g., [25] for a recent account and further reading suggestions).

<sup>11</sup>See, e.g., [26], Eq. (3.40) at vanishing  $\Lambda$  (see also [27]). One should not confuse the shear of the boundary Carrollian manifold, with the Bondi shear which is another boundary Carrollian tensor, nonvanishing in general and carrying information about the bulk gravitational radiation.

<sup>12</sup>The original solution is available in [28,29]. Robinson-Trautman spacetimes have been discussed in the framework of AdS/CFT in Refs. [30–33], and further in flat holography in Refs. [27,34].

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}. \quad (48)$$

Here  $P = P(t, \zeta, \bar{\zeta})$  obeys a fourth-order partial-differential equation known as Robinson-Trautman's equation, which also involves the Bondi mass aspect  $M(t)$ .<sup>13</sup> The field of observers and the clock form are ( $\Omega = 1$ ,  $b_i = 0$ )

$$\mathbf{v} = \partial_t, \quad \boldsymbol{\mu} = -dt. \quad (49)$$

Hence, one can compute the basic geometric data<sup>14</sup>:

$$\begin{aligned} \theta &= -2\partial_t \ln P, & \varphi_i &= 0, & \varpi_{ij} &= 0, \\ \xi_{ij} &= 0, & \hat{\mathcal{R}} &= 4P^2 \partial_{\bar{\zeta}} \partial_{\zeta} \ln P. \end{aligned} \quad (50)$$

Although the Robinson-Trautman solutions have no isometries, they have asymptotic symmetries, and these are actually reflected in the conformal isometries of the Carrollian boundary. Following [19], we find that the conformal Killing fields of  $\mathcal{M}$  are expressed in terms of an arbitrary real function  $T(\zeta, \bar{\zeta})$ , which encodes the supertranslations and the conformal Killing vectors  $\mathbf{Y} = Y^\zeta \partial_\zeta + Y^{\bar{\zeta}} \partial_{\bar{\zeta}}$  of  $d\tilde{\ell}^2 = 2d\zeta d\bar{\zeta}$ , which is flat space. The latter generate  $\mathfrak{so}(3, 1)$ —or even a double copy of Witt algebras referred to as “superrotations,” if we are ready to give up invertibility. We find that  $\mathbf{Y}$  is any combination of  $\ell_m + \bar{\ell}_m$  or  $i(\ell_m - \bar{\ell}_m)$  with

$$\ell_m = -\zeta^{m+1} \partial_\zeta, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1} \partial_{\bar{\zeta}}, \quad (51)$$

obeying Witt  $\oplus$  Witt:

$$[\ell_m, \ell_n] = (m - n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m - n)\bar{\ell}_{m+n}. \quad (52)$$

In this representation,  $\mathfrak{so}(3, 1)$  is generated by  $n = 0, \pm 1$ . The conformal Killing fields of  $\mathcal{M}$  are [see (A24) in the Appendix]<sup>15</sup>

$$\xi_{T,Y} = (T - M_Y(C)) \frac{1}{P} \partial_t + Y^i \partial_i, \quad (53)$$

where

<sup>13</sup>We will not specifically use the Robinson-Trautman equation [displayed in the aforementioned literature—footnote 12—e.g., Ref. [33], Eq. (2.35)] in our subsequent analysis, which is thus valid for arbitrary  $P(t, \zeta, \bar{\zeta})$ .

<sup>14</sup>Conventions:  $\sqrt{a} = i/P^2$  and  $\epsilon_{\zeta\bar{\zeta}} = 1$ .

<sup>15</sup>The existence of conformal Killing fields for the Carrollian structure at hand is remarkable. Actually, the relativistic ascendent of this structure  $ds^2 = -c^2 dt^2 + \frac{2}{P^2} d\zeta d\bar{\zeta}$ , appearing as the conformal timelike boundary of AdS (anti-de Sitter)-Robinson-Trautman spacetimes, has generically no conformal Killings. In particular, it is not conformally flat because it has a nonzero Cotton tensor, see [30–33].

$$C(t, \zeta, \bar{\zeta}) = \int^t d\tau P(\tau, \zeta, \bar{\zeta}), \quad (54)$$

and  $M_Y$  is an operator acting on scalar functions  $f(t, \zeta, \bar{\zeta})$  as

$$M_Y(f) = Y^k \partial_k f - \frac{f}{2} \partial_k Y^k. \quad (55)$$

The structure  $\mathfrak{so}(3, 1) \ltimes$  supertranslations—or (Witt  $\oplus$  Witt)  $\ltimes$  supertranslations—is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']}. \quad (56)$$

We are now ready to discuss the dynamics of a conformally coupled scalar field and its conserved charges. The ultimate motivation for this study is flat holography and the possible usefulness of the Carrollian dynamics for describing modes that propagate all the way inside the bulk towards the null boundary of asymptotically flat spacetimes. The electric equation of motion (22) reads as follows in the three-dimensional Carrollian spacetime under consideration:

$$\partial_t \frac{1}{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0. \quad (57)$$

Its general solution is given in terms of two arbitrary functions  $f(\zeta, \bar{\zeta})$  and  $g(\zeta, \bar{\zeta})$ :

$$\Phi = \sqrt{P}(Cf + g). \quad (58)$$

With this, we can compute the energy density  $\Pi_e$  and the electric momentum  $P_e^i$  as in (33), using (30) and (31), and combine them into the scalar component of the current (45) associated with the conformal Killing fields (53):

$$\kappa_{eT,Y} = P^2 \left[ Y^i \left( \frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} - \frac{1}{4} \partial_i (Y^i C f^2) \right]. \quad (59)$$

This leads to the charges

$$\begin{aligned} Q_{eT,Y} &= -i \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} \left( Y^i \left( \frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} \right) \\ &\quad - \frac{1}{4} \int_{\partial\mathcal{S}} \star Y C f^2 P^2. \end{aligned} \quad (60)$$

On shell, the time dependence is exclusively encoded in the last term through  $P$  (and  $C$ ). This is a flux at infinity, and thus vanishes upon appropriate falloff behavior of the field  $f$ . Hence, the charges are indeed conserved.

The infinite number of conserved charges, awkward at first glance, translates the separation of time and space imposed by Carrollian symmetry. The field equation (57) contains no spatial derivative, hence every locus  $(\zeta, \bar{\zeta})$

provides a decoupled degree of freedom. This often happens in Carrollian field theory (as, e.g., in the magnetic conformally stationary scalar field—see below), although the general equations at hand (22) and (23) contain actually both time and space derivatives— $\hat{\mathcal{D}}_t$  and  $\hat{\mathcal{D}}_i$  contain both—making the advertised decoupling less transparent.

The magnetic equation (23) is

$$4\partial_\zeta\partial_{\bar{\zeta}}\Phi = \Phi\partial_\zeta\partial_{\bar{\zeta}}\ln P. \quad (61)$$

According to (47), magnetic charges are conserved with those conformal Killing fields obeying the extra condition (44), which leads to

$$T = SP + M_Y(C), \quad (62)$$

where  $S$  is a function of time only. Since  $P$  and  $C$  are time dependent while  $T$  is not, Eq. (62) restricts severely the allowed subset of  $\mathcal{S}$ -conformal Killings  $Y$ , which may even turn empty. Assuming this set is not empty, due to the vanishing of the magnetic momentum  $P_m^i$  [see (34) with (32)—here  $\varpi_{ij} = 0$ ], Eq. (46) leads to a single conserved charge based on  $\kappa_{mS} = -S\Pi_m$  with  $\Pi_m$  given in (31):  $Q_{mS} = -S \int_{\mathcal{S}} \frac{d\zeta d\bar{\zeta}}{P^2} \Pi_m$ . This charge is nothing but the total energy, but it turns out to vanish here. Indeed, on shell,  $\Pi_m$  reads [Eqs. (23) and (31)] irrespective of the dimension and of the geometric background:

$$\Pi_m = \frac{1}{2d} \hat{\mathcal{D}}_i(\Phi \hat{\mathcal{D}}^i \Phi). \quad (63)$$

In the case under consideration ( $b_i = 0$  and  $\varphi_i = 0$ ),  $\Pi_m = \frac{P^2}{4} [\partial_\zeta(\Phi \partial_{\bar{\zeta}} \Phi) + \partial_{\bar{\zeta}}(\Phi \partial_\zeta \Phi)]$ , which is a divergence. Hence  $Q_{mS}$  receives only an  $\mathcal{S}$ -boundary contribution, vanishing under appropriate falloff or boundary conditions.<sup>16</sup>

It is worth stressing that Eq. (62) is extremely constraining. For instance, if the function  $P(t, \zeta, \bar{\zeta})$  obeys the Robinson-Trautman equation, it can awkwardly entangle time and space dependence (see, e.g., [29]), leaving little room for finding  $T(\zeta, \bar{\zeta})$  and  $Y^i(\zeta, \bar{\zeta})$  that satisfy (62). In the simplest possible instance, which is flat space ( $P = 1$  and  $C = t$ ),<sup>17</sup> the two special conformal transformations of the  $\mathfrak{so}(3, 1)$  are excluded ( $\partial_i Y^i = C_0$  constant), and only constant time translations are allowed ( $T = T_0$  constant, and  $S(t) = T_0 + C_0 t/2$ ); this is a five-dimensional subgroup of the infinite-dimensional BMS<sub>4</sub>.

When  $\Pi_m^i$  given in (31) vanishes, the magnetic charges are all conserved, as inferred by Eq. (47). This occurs in particular [the Carrollian geometric shear vanishes here, see (50)] for conformally stationary scalars obeying  $\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \equiv \sqrt{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0$ , thus of the form  $\Phi = \sqrt{P} g(\zeta, \bar{\zeta})$ , where  $g(\zeta, \bar{\zeta})$  is further determined by solving the magnetic equation of motion (61). The latter<sup>18</sup> may not be solvable in a general Robinson-Trautman background  $P(t, \zeta, \bar{\zeta})$  under the present ansatz. If it is, the conserved magnetic charges are found using Eqs. (46) and (53). On shell, these lead to

$$\kappa_{mTY} = -\xi^t \Pi_m = \frac{P^2}{2} (M_Y(C) - T) (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g), \quad (64)$$

which are integrated as in (40):

$$Q_{mT} = \frac{i}{2} \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} T (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) - \frac{1}{4} \int_{\partial\mathcal{S}} \star X P^2 \quad (65)$$

with

$$\begin{cases} X^\zeta = C(Y^\zeta (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^{\bar{\zeta}} (3(\partial_{\bar{\zeta}} g)^2 - g \partial_{\bar{\zeta}}^2 g)) - \frac{1}{2} Y^{\bar{\zeta}} g^2 \partial_{\bar{\zeta}}^2 C \\ X^{\bar{\zeta}} = C(Y^{\bar{\zeta}} (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^\zeta (3(\partial_\zeta g)^2 - g \partial_\zeta^2 g)) - \frac{1}{2} Y^\zeta g^2 \partial_\zeta^2 C \end{cases}. \quad (66)$$

<sup>16</sup>This property of vanishing scalar-field conserved magnetic charges is actually valid more generally, in any dimension  $d$ , and for a Carrollian background structure with  $b_i = 0$ . Indeed, this implies  $\varpi_{ij} = 0$ , leading therefore to  $Q_m = - \int_{\mathcal{S}} d^d x \sqrt{a} \xi^t \Pi_m$ . For Killing fields obeying the extra condition (44), using (63) we find that the on-shell integral is again a boundary term.

<sup>17</sup>Notice in passing that the general solution of (61) is in this case  $\Phi(t, \zeta, \bar{\zeta}) = f(t, \zeta) + \bar{f}(t, \bar{\zeta})$ , where  $f(t, \zeta)$  is arbitrary.

<sup>18</sup>With  $\Phi = \sqrt{P} g(\zeta, \bar{\zeta})$ , Eq. (61) reads  $4P \partial_\zeta \partial_{\bar{\zeta}} g + 2(\partial_\zeta P \partial_{\bar{\zeta}} g + \partial_{\bar{\zeta}} P \partial_\zeta g) + g \partial_\zeta \partial_{\bar{\zeta}} P = 0$  (also valid if  $P$  is traded for  $C$ ).

As in the electric case [see Eq. (60)], the time dependence is confined into a boundary term, which ultimately drops, taking with it all the dependence on the  $\mathfrak{so}(3, 1)$  vectors  $Y$ . For a conformally stationary scalar field in Robinson-Trautman background, the magnetic charges are nonzero and conserved on shell without restriction on the Carrollian conformal Killing vector  $\xi$  (the energy flux vanishes), but they only depend on its supertranslation component  $T(\zeta, \bar{\zeta})$ .

Concluding, we would like to summarize our results. The present framework is set by a general Carrollian spacetime and the systems under investigation are general covariant with respect to Carrollian diffeomorphisms.

The Carrollian scalar field dynamics is either electric or magnetic. The same holds for a conformally coupled scalar, and the two options are rather different. The electric is “timelike,” whereas the magnetic looks “spacelike,” and they couple to distinct pieces of the Carrollian curvature. We have determined the energy-stress tensor, the energy flux, the energy density, and the momentum in both situations, and shown that Carrollian conformal isometries imply conservation laws in the electric instance but not in the magnetic. The physical reason behind this cleavage is rather easy to understand. A Carrollian (conformal) isometry translates the invariance of the metric and the field of observers, but not that of its dual clock form. Time (supported by the field of observers) and space (associated with the clock form) directions behave differently and this ultimately reveals in the conservation properties of electric versus magnetic dynamics. A similar phenomenon is expected to occur in Newton-Cartan manifolds, where a scalar field will also have electric and magnetic dynamics.<sup>19</sup> Isometries will guarantee conservation laws for the latter, as opposed to the former, because the clock form is invariant under the action of a Killing vector, while the field of observers is not.

The above findings have been illustrated in the case of the null boundary of Robinson-Trautman asymptotically locally flat spacetimes, which are Carrollian with vanishing geometric shear and vorticity. The electric conformally coupled scalar field has been worked out thoroughly, accompanied with its infinite tower of conserved charges. For the magnetic dynamics, we have found that all charges associated with the subalgebra of the conformal Carrollian algebra satisfying the extra conservation condition [Eq. (44)] vanish—i.e., amount to purely boundary terms. Nonvanishing conserved magnetic charges appear for field configurations with  $\Pi_m^i = 0$ , and this happens, e.g., for conformally stationary fields.

From our general discussion one should probably retain the contrast between the infinite tower of conformal Killing fields available in most Carrollian structures and the often lesser conserved Carrollian charges. In this picture one should not underestimate the role of the nonconserved ones, usually infinite in number. When the Carrollian structures are null boundaries of asymptotically flat spacetimes, the presence of nonconserved charges betrays, among others, gravitational radiation.

Even though we have focused our analysis on conformally coupled scalar fields, ordinary scalars share these properties—with Killings instead of conformal Killings.

<sup>19</sup>The magnetic and electric scalar-field actions are, respectively,  $S_m = - \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega (\frac{1}{2} a^{ij} \partial_i \Phi \partial_j \Phi + V_m(\Phi))$  and  $S_e = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega ((\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt})^2 - V_e(\Phi))$  in torsionless Newton-Cartan geometries with degenerate cometric  $a^{ij}$ , clock form  $\Omega dt$ , field of observers  $\frac{1}{\Omega}(\partial_t + w^j \partial_j)$ , and metric-compatible time derivative  $\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt} = \frac{1}{\Omega} \partial_t \Phi + \frac{w^j}{\Omega} \partial_j \Phi$ .

The motivation behind conformal couplings lies in the role these may play in flat holography—for scalar or more general fields. This calls for a better understanding of the classical dynamics, and above all of the quantum properties. The conservation of charges, the associated algebras and the distinction of electric versus magnetic representatives, should ultimately be translated into bulk language. Our example of the null three-dimensional boundary of Robinson-Trautman Ricci-flat spacetimes is meant to illustrate this bridge, although discussed here in a primitive fashion, revealing a generically trivial magnetic conservation as opposed to an infinite set of electric conserved charges. How this reflects flat-holographic properties remains in limbo.

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## APPENDIX: CARROLLIAN MANIFOLDS

Under Carrollian diffeomorphisms (3) with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}, \quad (\text{A1})$$

the transformations are noncovariant (connectionlike) for  $\partial_i$  and  $b_i$ , and densitylike for  $\partial_t$  and  $\Omega$ :

$$\begin{aligned} \partial'_j &= J^{-1i}{}_j \left( \partial_i - \frac{j_i}{J} \partial_t \right), & b'_k &= \left( b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k, \\ \partial'_t &= \frac{1}{J} \partial_t, & \Omega' &= \frac{\Omega}{J}. \end{aligned} \quad (\text{A2})$$

They are covariant for the other objects:

$$\mathbf{v}' = \mathbf{v}, \quad \boldsymbol{\mu}' = \boldsymbol{\mu}, \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j, \quad a^{ij'} = J^i{}_k J^j{}_l a^{kl}. \quad (\text{A3})$$

Carrollian tensors depend on time  $t$  and space  $\mathbf{x}$ , carry indices  $i, j, \dots$  lowered and raised with  $a_{ij}$  and its inverse  $a^{ij}$ , and transform covariantly under Carrollian diffeomorphisms (3) with  $J_i^j$  and  $J^{-1j}{}_i$  defined in (A1). A Levi-Civita-Carroll spatial covariant derivative  $\hat{\nabla}'_i$  is defined with connection coefficients

$$\hat{\gamma}_{jk}^i = \frac{a^{il}}{2} (\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}), \quad (\text{A4})$$

which emerge naturally in the vanishing- $c$  limit of a Levi-Civita connection in the Papapetrou-Randers coordinates (6). This connection is torsionless and metric compatible<sup>20</sup>:  $\hat{\gamma}_{[ij]}^k = 0$ ,  $\hat{\nabla}_i a_{jk} = 0$ . The vectors  $\hat{\partial}_i$  do not commute and define the Carrollian vorticity and acceleration:

$$\begin{aligned} [\hat{\partial}_i, \hat{\partial}_j] &= \frac{2}{\Omega} \varpi_{ij} \partial_t, & \varpi_{ij} &= \partial_{[i} b_{j]} + b_{[i} \varphi_{j]}, \\ \varphi_i &= \frac{1}{\Omega} (\partial_i b_i + \partial_i \Omega). \end{aligned} \quad (\text{A5})$$

The usual time-derivative operator  $\frac{1}{\Omega} \partial_t$  acts covariantly on Carrollian tensors, but it is not metric compatible because  $a_{ij}$  depend on time. A temporal covariant derivative is defined by requiring  $\frac{1}{\Omega} \hat{D}'_t = \frac{1}{\Omega} \hat{D}_t$  and  $\hat{D}_t a_{jk} = 0$ , and is also inherited from the Papapetrou-Randers Levi-Civita connection. To this end, we introduce a temporal connection

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega} \partial_t a_{ij} = \xi_{ij} + \frac{1}{d} a_{ij} \theta, \quad \theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a}, \quad (\text{A6})$$

which is a symmetric Carrollian tensor split into the Carrollian shear (traceless) and the Carrollian expansion (trace). The action of  $\hat{D}_t$  on scalars is  $\partial_t$  whereas on vectors or forms it is defined as

$$\frac{1}{\Omega} \hat{D}_t V^i = \frac{1}{\Omega} \partial_t V^i + \hat{\gamma}^i_j V^j, \quad \frac{1}{\Omega} \hat{D}_t V_i = \frac{1}{\Omega} \partial_t V_i - \hat{\gamma}^j_i V_j. \quad (\text{A7})$$

Generalization to any tensor uses the Leibniz rule.

The commutators of Carrollian covariant derivatives define Carrollian curvature tensors. We keep it minimal here with

$$\begin{aligned} [\hat{\nabla}_k, \hat{\nabla}_l] V^i &= (\hat{\partial}_k \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj}) V^j + [\hat{\partial}_k, \hat{\partial}_l] V^i \\ &= \hat{r}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i. \end{aligned} \quad (\text{A8})$$

In this expression  $\hat{r}^i_{jkl}$  should be called the ‘‘Riemann-Carroll’’ tensor. The Ricci-Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k_{ikj} \neq \hat{r}_{ji}, \quad \hat{r} = a^{ij} \hat{r}_{ij}. \quad (\text{A9})$$

<sup>20</sup>Details on the transformation rules can be found in Appendix A.2 of Ref. [24], together with further useful properties.

Weyl covariance under Weyl transformations

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad (\text{A10})$$

with  $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$  an arbitrary function, requires a Weyl-Carroll connection built on  $\varphi_i$  and  $\theta$  defined in (A5) and (A6), which transform as

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \rightarrow \mathcal{B} \theta - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (\text{A11})$$

The Carrollian vorticity  $\varpi_{ij}$  (A5) and the Carrollian shear  $\xi_{ij}$  (A6) are Weyl covariant of weight  $-1$ .

The Weyl-Carroll space and time covariant derivatives are torsionless and metric compatible. For a weight- $w$  scalar function  $\Phi$  and a vector with weight- $w$  components  $V^l$ , the action is

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad (\text{A12})$$

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (\text{A13})$$

The Weyl-Carroll spatial derivative does not alter the weight, and one checks that  $\hat{\mathcal{D}}_j a_{kl} = 0$ . Regarding time, one defines

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega} \hat{D}_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (\text{A14})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^l = \frac{1}{\Omega} \hat{D}_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi^l_i V^i, \quad (\text{A15})$$

and both are of weight  $w+1$ . Similarly for any tensor by Leibniz rule and in particular we find  $\hat{\mathcal{D}}_t a_{kl} = 0$ .

We now close this paragraph with the Weyl-Carroll curvature tensors, appearing in the commutation of Weyl-Carroll covariant derivatives. We find

$$[\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathcal{D}}_t \Phi + w \Omega_{ij} \Phi, \quad (\text{A16})$$

$$[\hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l] V^i = (\hat{\mathcal{R}}^i_{jkl} - 2\xi_j^i \varpi_{kl}) V^j + \varpi_{kl} \frac{2}{\Omega} \hat{\mathcal{D}}_t V^i + w \Omega_{kl} V^i, \quad (\text{A17})$$

where we have introduced the following Carrollian, weight-0 Weyl-covariant tensors:

$$\begin{aligned} \hat{\mathcal{R}}^i_{jkl} &= \hat{r}^i_{jkl} - \delta^i_j \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i \\ &\quad + \delta^i_k \hat{\nabla}_l \varphi_j - \delta^i_l \hat{\nabla}_k \varphi_j + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) \\ &\quad - (\delta^i_k a_{jl} - \delta^i_l a_{jk}) \varphi_m \varphi^m + (\delta^i_l \varphi_k - \delta^i_k \varphi_l) \varphi_j, \end{aligned} \quad (\text{A18})$$

$$\Omega_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i - \frac{2}{d} \varpi_{ij} \theta. \quad (\text{A19})$$

Additionally, we define traces as

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k{}_{ikj}, \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (\text{A20})$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1)(2\hat{\nabla}_i \varphi^i - (d-2)\varphi_i \varphi^i). \quad (\text{A21})$$

Observe that the Weyl-covariant Carroll-Ricci tensor is not symmetric:  $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2}\Omega_{ij}$ . Finally, we recall that

$$\left[ \frac{1}{\Omega} \hat{\mathcal{D}}_t, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j \hat{\mathcal{D}}_j \Phi, \quad (\text{A22})$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{d} (\hat{\partial}_i + \varphi_i) \theta \quad (\text{A23})$$

are the components of a Weyl-covariant weight-1 Carrollian curvature one-form.

Isometries and conformal isometries are associated with Killing and conformal Killing fields. Carrollian diffeomorphisms (3) are generated by vector fields

$$\begin{aligned} \xi &= \xi^t \partial_t + \xi^i \partial_i = \left( \xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left( \partial_i + \frac{b_i}{\Omega} \partial_t \right) \\ &= \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^{\hat{i}} \hat{\partial}_i \end{aligned} \quad (\text{A24})$$

restricted to  $\xi^i = \xi^i(\mathbf{x})$ . Their action operates with the Lie derivative, and for the geometric data one finds

$$\mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i} \xi^k a_{j)k} + 2\xi^{\hat{t}} \hat{\gamma}_{ij}, \quad (\text{A25})$$

$$\mathcal{L}_\xi \mathbf{v} = \mu \mathbf{v}, \quad (\text{A26})$$

$$\mathcal{L}_\xi \boldsymbol{\mu} = -\mu \boldsymbol{\mu} - ((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^{\hat{j}} \varpi_{ji}) dx^i, \quad (\text{A27})$$

with

$$\mu(t, \mathbf{x}) = -\left( \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^{\hat{i}} \right). \quad (\text{A28})$$

The significant observation is here that due to the degeneration of the metric on  $\mathcal{M}$ , the variation of the clock form  $\mu$  is not identical to that of the field of observers  $v$ . For further use, we also introduce the trace of (A25) divided by  $d$ :

$$\lambda(t, \mathbf{x}) = \frac{2}{d} (\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}}). \quad (\text{A29})$$

Carrollian isometries are Carrollian diffeomorphisms generated by Killing fields, obeying  $\mathcal{L}_\xi a_{ij} = 0$  and  $\mathcal{L}_\xi v = 0$ . In the strong Carroll structure, this requirement is completed with the invariance of the connection. For conformal Carrollian isometries one demands

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij}. \quad (\text{A30})$$

This set of partial differential equations is insufficient for defining conformal Killing vectors and one usually imposes to tune  $\mu$  versus  $\lambda$  so that the scaling of the metric be twice that of the field of observers<sup>21</sup>:

$$2\mu + \lambda = 0. \quad (\text{A31})$$

The projective structure associated with some Carroll connection should also be preserved.

For a pseudo-Riemannian manifold  $\mathcal{M}$  in  $d+1$  dimensions with metric  $g_{\mu\nu}$  (weight-2), a Weyl-covariant derivative  $\mathcal{D}_\mu$  maintains the weight  $w$  of a Weyl-covariant tensor. The corresponding connection uses a (weight-1) vector  $u^\mu$  of norm  $-c^2$ , as well as its expansion  $\Theta$  and acceleration  $a^\mu$ :

$$A = \frac{1}{c^2} \left( a - \frac{\Theta}{d} u \right). \quad (\text{A32})$$

The Weyl covariant derivative is metric compatible with

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) f = w f F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (\text{A33})$$

where the action on a weight- $w$  scalar  $f$  is

$$\mathcal{D}_\lambda f = \nabla_\lambda f + w A_\lambda f. \quad (\text{A34})$$

The action of  $\mathcal{D}_\lambda$  on a weight- $w$  form  $v_\mu$  is

$$\mathcal{D}_\lambda v_\mu = \nabla_\lambda v_\mu + (w+1) A_\lambda v_\mu + A_\mu v_\lambda - g_{\mu\lambda} A^\rho v_\rho, \quad (\text{A35})$$

and we obtain

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) v^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu} v^\sigma + w v^\rho F_{\mu\nu}. \quad (\text{A36})$$

The Weyl-covariant Ricci (weight 0) and scalar (weight 2) curvatures read

<sup>21</sup>One usually considers in the literature  $2\mu + z\lambda = 0$ , where  $z$  is minus the conformal weight of  $\Omega$ , referred to as the dynamical exponent. Here, due to the relationship of the considered Carrollian spacetimes with relativistic parents, the weight of  $\Omega$  is  $-1$ .

$$\begin{aligned} \mathcal{R}_{\mu\nu} = & R_{\mu\nu} + (d-1)(\nabla_\nu A_\mu + A_\mu A_\nu - g_{\mu\nu} A_\lambda A^\lambda) \\ & + g_{\mu\nu} \nabla_\lambda A^\lambda - F_{\mu\nu}, \end{aligned} \quad (\text{A37})$$

$$\mathcal{R} = R + 2d\nabla_\lambda A^\lambda - d(d-1)A_\lambda A^\lambda. \quad (\text{A38})$$

Observe that  $\mathcal{R}_{\mu\nu}$  is not symmetric.

If the metric is of the Papapetrou-Randers form (6), the dependence with respect to the velocity of light  $c$  is explicit. Thus every relativistic tensor, i.e., a tensor with respect to the full diffeomorphism group, can be reduced with respect to the Carrollian subgroup (3), and exhibits a finite number of Carrollian tensors. We find

$$\begin{aligned} R = & \frac{1}{c^2} \left( \frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) + \hat{r} - 2\hat{\nabla}_i \varphi^i \\ & - 2\varphi^i \varphi_i + c^2 \varpi_{ij} \varpi^{ij}, \end{aligned} \quad (\text{A39})$$

$$\mathcal{R} = \frac{1}{c^2} \xi_{ij} \xi^{ij} + \hat{\mathcal{R}} + c^2 \varpi_{ij} \varpi^{ij}. \quad (\text{A40})$$

Actually, the Carroll and Weyl-Carroll connections introduced earlier are also obtained from the ordinary Levy-Civita and the Weyl connections of the pseudo-Riemannian spacetime at hand, in the form of an exact, truncated Laurent expansion.

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**Titre :** Aventures au Pays des Merveilles

**Mots clés :** Symétries asymptotiques, Holographie Carrollienne, Géométrie des espaces de genre lumière, Théories des champs thermiques

**Résumé :** Le travail que nous présentons dans cette thèse est structuré autour de la notion de théorie des champs et de géométrie, qui sont appliquées à la gravité et la thermalisation.

En gravité, notre travail donne un éclairage nouveau sur la structure asymptotique du champ gravitationnel dans le contexte des espace-temps asymptotiquement plats, ceci en utilisant l'information codée sur leur bord conforme. Ce dernier est une hypersurface de genre lumière sur laquelle émerge la physique carrollienne au lieu de la physique relativiste. Une structure carrollienne sur une variété est constituée une métrique dégénérée et un champ de vecteurs couvrant le noyau de cette dernière. Ce vecteur sélectionne une direction particulière qui peut être le point de départ de la description des structures carrolliennes dans un cadre séparé. Nous développons d'abord la géométrie carrollienne, y compris une étude complète des connexions et isométries (conformes). Des actions effectives peuvent vivre sur un arrière-plan carrollien. Les moments canoniques conjugués à la géométrie ou à la connexion peuvent être définis, et la variation de l'action donnera leurs équations de conservation, à partir desquelles les charges isométriques peuvent être bâties.

La physique carrollienne émerge également lorsque la vitesse de la lumière tend vers zéro. Cette limite donne généralement plus de descendants carrolliens que ce qui est attendu après une analyse intrinsèque, comme le montrent les exemples explicites des fluides carrolliens, des champs scalaires carrolliens (pour lesquels deux actions, électrique et magnétique, apparaissent dans la limite) et du tenseur de Cotton carrollien. La richesse de la limite est due à sa possibilité de décrire plus de degrés de liberté, ce qui s'avère être un outil fondamental dans l'étude de la relation entre les espace-temps asymptotiquement anti de Sitter et plats.

Les espace-temps asymptotiquement plats peuvent être écrits comme une expansion infinie dans une jauge covariante par rap-

port à leur bord nul. Cette légère extension de la jauge de Newman-Unti est également valable dans AdS, ce qui permet de prendre la limite plate dans le bulk, équivalente à la limite carrollienne sur le bord. Nous démontrons que l'espace des solutions infini des espace-temps Ricci-plat provient en fait du développement en série de Laurent du tenseur énergie-impulsion d'AdS. Ces répliques obéissent à chaque ordre une dynamique carrollienne (lois de flux). Dans le cadre des espaces algébriquement spéciaux de Petrov (pour lesquels le développement infinie se resomme), nous utilisons les lois de flux carrolliennes ainsi que la conservation des tenseurs énergie-impulsion et de Cotton pour construire, du point de vue du bord, deux tours duales de charges du bulk. Parmi elles, nous retrouvons l'expansion multipolaire de la masse et du moment angulaire pour la famille Kerr-Taub-NUT. La jauge covariante est également le cadre approprié pour dévoiler l'action des symétries cachées de la gravité sur le bord nul. Dans ce travail, nous étudions le cas de la symétrie  $SL(2, R)$  d'Ehlers.

Du côté de la théorie thermique des champs, nous travaillons sur l'ensemble minimal de données nécessaires pour les décrire à température finie. Alors qu'à température infinie toutes les valeurs moyennes des opérateurs primaires s'annulent, leurs valeurs non nulle dans le cas thermique constituent les données supplémentaires qu'il faut calculer pour caractériser la théorie. Les simulations numériques, la dualité avec un trou noir dans AdS ou une analyse spectrale sont généralement les méthodes employées pour trouver la valeur de ces coefficients. Notre travail propose une nouvelle approche à ce problème en montrant, à partir de deux oscillateurs harmoniques couplés, que ces coefficients sont en fait liés à des graphes conformes de théories de type fishnet. A partir de cette observation, nous avons établi une correspondance entre les fonctions de partition thermique et ces graphes.

**Title :** Adventures in (thermal) Wonderland

**Keywords :** Asymptotic symmetries, Carrollian holography, Geometry of null manifolds, Thermal CFT

**Abstract :** The work we present in this thesis is structured around the concepts of field theories and geometry, which are applied to gravity and thermalisation.

On the gravity side, our work aims at shedding new light on the asymptotic structure of the gravitational field in the context of asymptotically flat spacetimes, using information encoded on the conformal boundary. The latter is a null hypersurface on which Carrollian physics instead of relativistic physics is at work. A Carroll structure on a manifold is a degenerate metric and a vector field spanning the kernel of the latter. This vector selects a particular direction which can be the starting point for describing Carroll structures in a split frame. We first elaborate on the geometry one can construct on such a manifold in this frame, including a comprehensive study of connections and (conformal) isometries. Effective actions can be defined on a Carrollian background. Canonical momenta conjugate to the geometry or the connection are introduced, and the variation of the action shall give their conservation equations, upon which isometric charges can be reached.

Carrollian physics is also known to emerge as the vanishing speed of light of relativistic physics. This limit usually exhibits more Carrollian descendants than what might be expected from a naive intrinsic analysis, as shown in the explicit examples of Carrollian fluids, Carrollian scalar fields (for which two actions, electric and magnetic arise in the limit) and the Carrollian Chern-Simons action. The richness of the limiting procedure is due to this versatility in describing a palette of degrees of freedom. This turns out to be an awesome tool in studying the relationship between asymptotically anti de Sitter (AdS) and flat spacetimes.

Metrics on asymptotically flat spacetimes can be expressed as an

infinite expansion in a gauge, covariant with respect to their null boundaries. This slight extension of the Newman-Unti gauge is shown to be valid also in AdS, which allows to take the flat limit in the bulk i.e. the Carrollian limit on the boundary, while preserving this covariance feature. We demonstrate that the infinite solution space of Ricci-flat spacetimes actually arises from the Laurent expansion of the AdS boundary energy-momentum tensor. These replicas obey at each order Carrollian dynamics (flux/balance laws). Focusing our attention to Petrov algebraically special spacetimes (for which the infinite expansion resums), we use the Carrollian flux/balance laws together with the conservation of the energy-momentum and Cotton tensors to build two dual towers of bulk charges from a purely boundary perspective. Among them we recover the mass and angular momentum multipolar moments for the Kerr-Taub-NUT family. The covariant gauge is also the appropriate framework to unveil the action of hidden symmetries of gravity on the null boundary. In this thesis we study exhaustively the case of Ehlers'  $SL(2, R)$  symmetry.

On the side of thermal field theory we see that while at infinite temperature a CFT is described by its spectrum and the OPE coefficients, additional data is needed in the thermal case. These are the average values of primary operators, completely determined up to a constant coefficient. Numerical simulations, duality with black-hole states in AdS or spectral analyses are the methods usually employed to uncover the latter. Our work features a new breadth. Starting from two coupled harmonic oscillators, we show that they are related to conformal ladder graphs of fishnet theories. This observation is the first step for setting a new correspondence between thermal partition functions and graphs.