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Rodrigo Maulen Soto

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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité **MATHEMATIQUES**

Préparée au sein de l'**Université de Caen Normandie**

A Dynamical System Perspective on Stochastic and Inertial Methods for Optimization

Présentée et soutenue par
RODRIGO MAULEN SOTO

Thèse soutenue le 08/11/2024

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A mis padres Rodrigo y Maribel; y a mi hermano Vicente

A mi esposa Franny

A mi familia y amigos

Al Chavo

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Abstract

Motivated by the ubiquity of optimization in many areas of science and engineering, particularly in data science, this thesis exploits the close link between continuous-time dissipative dynamical systems and optimization algorithms to provide a systematic analysis of the global and local behavior of several first- and second-order systems, focusing on convex, stochastic, and infinite-dimensional settings on the one hand, and non-convex, deterministic, and finite-dimensional settings on the other hand. For stochastic convex minimization problems in infinite-dimensional separable real Hilbert spaces, our key proposal is to analyze them through the lens of stochastic differential equations (SDEs) and inclusions (SDIs), as well as their inertial variants. We first consider smooth differentiable convex problems and first-order SDEs, demonstrating almost sure weak convergence towards minimizers under integrability of the noise and providing a comprehensive global and local complexity analysis. We also study composite non-smooth convex problems using first-order SDIs, and show under integrability conditions on the noise, almost sure weak convergence of the trajectory towards a minimizer, with Tikhonov regularization almost sure strong convergence of trajectory to the minimal norm solution. We then turn to developing a unified mathematical framework for analyzing second-order stochastic inertial dynamics via time scaling and averaging of stochastic first-order dynamics, achieving almost sure weak convergence of trajectories towards minimizers and fast convergence of values and gradients. These results are extended to more general second-order SDEs with viscous and Hessian-driven damping, utilizing a dedicated Lyapunov analysis to prove convergence and establish new convergence rates. Finally, we study deterministic non-convex optimization problems and propose several inertial algorithms to solve them derived from second-order ordinary differential equations (ODEs) combining both non-vanishing viscous damping and geometric Hessian-driven damping in explicit and implicit forms. We first prove convergence of the continuous-time trajectories of the ODEs to a critical point under the Kurdyka-Łojasiewicz (KL) property with explicit rates, and generically to a local minimum under a Morse condition. Moreover, we propose algorithmic schemes by appropriate discretization of these ODEs and show that all previous properties of the continuous-time trajectories still hold in the discrete setting under a proper choice of the stepsize.

Keywords: Stochastic Optimization, SGD, Stochastic Differential Equations, Stochastic Differential Inclusions, Itô calculus, Convex Optimization, Inertial Gradient Methods, Viscous Damping, Hessian damping, Łojasiewicz Inequality, KL-inequality, Non-convex optimization, Trajectory convergence, Convergence rates.

Résumé

Motivé par l'omniprésence de l'optimisation dans de nombreux domaines de la science et de l'ingénierie, en particulier dans la science des données, ce manuscrit de thèse exploite le lien étroit entre les systèmes dynamiques dissipatifs à temps continu et les algorithmes d'optimisation pour fournir une analyse systématique du comportement global et local de plusieurs systèmes du premier et du second ordre, en se concentrant sur le cadre convexe, stochastique et en dimension infinie d'une part, et le cadre non convexe, déterministe et en dimension finie d'autre part. Pour les problèmes de minimisation convexe stochastique dans des espaces de Hilbert réels séparables de dimension infinie, notre proposition clé est de les analyser à travers le prisme des équations différentielles stochastiques (EDS) et des inclusions différentielles stochastiques (IDS), ainsi que de leurs variantes inertielles. Nous considérons d'abord les problèmes convexes différentiables lisses et les EDS du premier ordre, en démontrant une convergence faible presque sûre vers les minimiseurs sous hypothèse d'intégrabilité du bruit et en fournissant

une analyse globale et locale complète de la complexité. Nous étudions également des problèmes convexes non lisses composites utilisant des IDS du premier ordre et montrons que, sous des conditions d'intégrabilité du bruit, la convergence faible presque sûre des trajectoires vers les minimiseurs, et avec la régularisation de Tikhonov la convergence forte presque sûre des trajectoires vers la solution de norme minimale. Nous développons ensuite un cadre mathématique unifié pour analyser la dynamique inertielle stochastique du second ordre via la reparamétrisation temporelle et le moyennage de la dynamique stochastique du premier ordre, ce qui permet d'obtenir une convergence faible presque sûre des trajectoires vers les minimiseurs et une convergence rapide des valeurs et des gradients. Ces résultats sont étendus à des EDS plus générales du second ordre avec un amortissement visqueux et Hessien, en utilisant une analyse de Lyapunov spécifique pour prouver la convergence et établir de nouveaux taux de convergence. Enfin, nous étudions des problèmes d'optimisation déterministes non convexes et proposons plusieurs algorithmes inertiels pour les résoudre, dérivés d'équations différentielles ordinaires (EDO) du second ordre combinant à la fois un amortissement visqueux sans vanité et un amortissement géométrique piloté par le Hessien, sous des formes explicites et implicites. Nous prouvons d'abord la convergence des trajectoires en temps continu des EDO vers un point critique pour des objectives vérifiant la propriété de Kurdyka-Łojasiewicz (KL) avec des taux explicites, et génériquement vers un minimum local si l'objective est Morse. De plus, nous proposons des schémas algorithmiques par une discrétisation appropriée de ces EDO et montrons que toutes les propriétés précédentes des trajectoires en temps continu sont toujours valables dans le cadre discret sous réserve d'un choix approprié de la taille du pas.

Mots-clés: Optimisation Stochastique, Équations Différentielles Stochastiques, Inclusions Différentielles Stochastiques, Calcul d'Itô, Optimisation Convexe, Méthodes de Gradient Inertiel, Amortissement Visqueux, Amortissement Hessien, Inégalité de Łojasiewicz, Inégalité KL, Optimisation Non-convexe, Convergence de la trajectoire, Taux de convergence.

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Chapter 1

Introduction

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1.1 Context

The field of optimization is ubiquitous in a wide spectrum of science and applied mathematics. From data processing to machine learning to operations research, the flexibility offered through modeling problems as optimization problems is well established. While many methods exist to solve optimization problems, first-order methods stand out when the problems at hand are extremely large and require only moderately precise solutions; often the case in imaging sciences like computer vision or machine learning where data is collected, and expected to be processed, at a huge scale. First-order methods generally scale well with the problem dimension, in contrast to second (or higher) order methods in which not even a single iteration can be performed because of storage constraints.

In the following sections, we will describe some prototypical dynamical systems and corresponding first-order algorithms for solving different optimization problems. We will provide a historical perspective as well. The purpose of this exposition is to set the stage for describing the contributions of this work. Throughout the rest of the section, we let \mathbb{H} be a real Hilbert space (sometimes taking $\mathbb{H} = \mathbb{R}^d$).

1.1.1 First-order systems

Smooth problems. Consider the minimization problem

$$\min_{x \in \mathbb{H}} f(x), \quad (\mathcal{P}_1)$$

where the objective f satisfies the following standing assumptions:

$$\begin{cases} f \text{ is convex and continuously differentiable with } L\text{-Lipschitz continuous gradient;} \\ \mathcal{S} \stackrel{\text{def}}{=} \operatorname{argmin}(f) \neq \emptyset. \end{cases} \quad (\mathbf{H}_f)$$

Let us recall some classical facts. To solve (\mathcal{P}_1) , a fundamental dynamic to consider is the gradient flow of f , *i.e.* the gradient descent dynamic with initial condition $x_0 \in \mathbb{H}$ given by

$$\begin{cases} \dot{x}(t) = -\nabla f(x(t)), & t > 0; \\ x(0) = x_0. \end{cases} \quad (\mathbf{GF})$$

The gradient system (\mathbf{GF}) is a dissipative dynamical system, whose study dates back to Cauchy [78] in finite dimension. It plays a fundamental role in optimization: it transforms the problem of minimizing f into the study of the asymptotic behavior of the trajectories of (\mathbf{GF}) . This example was the precursor to the rich connection between continuous dissipative dynamical systems and optimization. It is well known since the founding papers of Brezis, Baillon, Bruck in the 1970s that, if the solution set $\operatorname{argmin}(f)$ of (\mathcal{P}_1) is non-empty, then each solution trajectory of (\mathbf{GF}) converges weakly, and its (weak) limit belongs to $\operatorname{argmin}(f)$.

The Euler forward discretization (\mathbf{GF}) , with stepsize sequence $h_k > 0$, is the celebrated gradient descent scheme

$$x_{k+1} = x_k - h_k \nabla f(x_k). \quad (\mathbf{GD})$$

Under (\mathbf{H}_f) , and for $(h_k)_{k \in \mathbb{N}} \subset]0, 2/L[$, then we have both the convergence of the values $f(x_k) - \min f = \mathcal{O}(1/k)$ (in fact even $o(1/k)$), and the weak convergence of iterates $(x_k)_{k \in \mathbb{N}}$ to a point in $\operatorname{argmin}(f)$. This convergence rate can be refined under various additional geometrical properties on the objective f such as error bounds (and the closely related Kurdyka-Łojasiewicz property in the convex case; see [59]).

Non-smooth problems. Weak convergence of trajectory of (\mathbf{GF}) to a point in $\operatorname{argmin}(f)$ is true in a more general setting, simply assuming that the objective function f is proper, lower semicontinuous (lsc) and convex, in which case one has to consider a differential inclusion obtained by replacing the gradient of f in (\mathbf{GF}) by its (convex Fenchel) subdifferential ∂f .

In fact, let us consider the additively structured minimization problem

$$\min_{x \in \mathbb{H}} F(x) \stackrel{\text{def}}{=} f(x) + g(x), \quad (\mathcal{P}_{\text{comp}})$$

where \mathbb{H} is a real Hilbert space, and the objective F satisfies the following standing assumptions:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ g : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \text{ is proper, lower semicontinuous and convex;} \\ \mathcal{S}_F \stackrel{\text{def}}{=} \operatorname{argmin}(F) \neq \emptyset. \end{cases} \quad (\mathbf{H}_F)$$

To solve $(\mathcal{P}_{\text{comp}})$, we consider the subgradient flow, which is the following differential inclusion (DI) starting at $t_0 \geq 0$ with initial condition $x_0 \in \mathbb{H}$:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)), & t > t_0; \\ x(t_0) = x_0. \end{cases} \quad (\mathbf{DI})$$

It is well known that, when the initial data x_0 is in the domain of F , (more generally when it is in its closure), there exists a unique strong global solution of (DI); see [66]. Moreover, if the solution set $\operatorname{argmin}(F)$ of $(\mathcal{P}_{\text{comp}})$ is nonempty then each solution trajectory of (DI) converges weakly, and its weak limit belongs to $\operatorname{argmin}(F)$ [69].

From continuous dynamics to algorithms. The Euler forward (or explicit) discretization of (DI) with stepsize $h_k > 0$ is the subgradient method

$$x_{k+1} \in x_k - h_k \partial F(x_k). \quad (\text{Sub-G})$$

Or equivalently,

$$x_{k+1} = x_k - h_k u_k, \quad \text{where } u_k \in \partial F(x_k) \quad \forall k \in \mathbb{N}. \quad (1.1)$$

The sequence of iterates of (Sub-G) converges if h_k is vanishing but not too fast (typically non-summable but square-summable); see [2]. The pointwise convergence rate on f is however at best $\mathcal{O}(1/\sqrt{k})$, which is very slow [154].

If the Euler backward (or implicit) discretization of (DI) is used, one gets the popular Proximal Point Algorithm (PPA) [142, 180], which takes the form

$$x_{k+1} = \operatorname{prox}_{h_k F}(x_k), \quad (\text{PPA})$$

where, for any $h > 0$, $\operatorname{prox}_{hF} \stackrel{\text{def}}{=} (I + h\partial F)^{-1}$ is called the proximal mapping of F (or the resolvent of ∂F). PPA was shown to converge weakly for any stepsize sequence such that $\inf_k h_k > 0$, and hence is unconditionally stable [180, 46]. Moreover, as it can also be interpreted as a gradient descent on the Moreau envelope of F , its pointwise convergence rate on the objective is $o(1/k)$; see [174].

The PPA algorithm supposes that $\operatorname{prox}_{\gamma F}$ can be computed in closed form, or at least up to good precision. However, even if the proximal mappings of f and g in $(\mathcal{P}_{\text{comp}})$ can be computed in closed form separately, that F in general can be difficult to compute. In addition, the smoothness of f is not exploited. Therefore, a proper numerical scheme should take into account the structure of the problem and the properties of the functions. This suggests that an explicit-implicit (or forward-backward) Euler discretization of (DI) is the right strategy, hence yielding the well-known Forward-Backward splitting (FBS) method [169, 135], whose non-relaxed iteration takes the form

$$x_{k+1} = \operatorname{prox}_{h_k g}(x_k - h_k \nabla f(x_k)) \quad (\text{FBS})$$

where $h_k \in]0, 2/L[$ is the stepsize sequence. The scheme (FBS) achieves full operator splitting: a forward explicit step on f (gradient descent), followed by a backward implicit step on g (proximal point). The iterates of (FBS) converge weakly to a minimizer [46], at the rate $o(1/k)$ on the objective algorithm.

Tikhonov regularization. As discussed above, the trajectories of (DI) converge weakly to an arbitrary minimizer of F . However, strong convergence does not hold in general as the counterexample in [43] shows. One way to go from weak to strong convergence is to use Tikhonov regularization of (DI). In fact, Tikhonov regularization even makes it possible to strongly converge to a particular minimizer: the one of minimal norm.

Given $t_0 > 0$, and a regularization parameter $\varepsilon : [t_0, +\infty[\rightarrow \mathbb{R}_+$, which is a measurable function that vanishes asymptotically in a controlled way, the Tikhonov regularization of (DI) is written:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)) - \varepsilon(t)x(t), & t > t_0; \\ x(t_0) = x_0. \end{cases} \quad (\text{DI-TA})$$

The behavior of (DI-TA) has been studied in depth in the literature; see *e.g.* [84]. Clearly, the idea is to perturb the convex function F with a quadratic function, *i.e.*, to consider the strongly convex function $F_\varepsilon(x) = F(x) + \varepsilon \frac{\|x\|^2}{2}$ with has a unique minimizer x_ε^* , and then to make ε depend

on time and tend to zero sufficiently but not too fast as $t \rightarrow +\infty$. This implies that the Tikhonov regularization parameter $\varepsilon(t)$ induces a hierarchical minimization property: the strong limit of any trajectory exists and it no longer depends on the initial data, it is precisely the minimum norm solution. An abundant literature has been devoted to the asymptotic hierarchical minimization property which results from the introduction of a vanishing viscosity term (in our context the Tikhonov approximation) in (sub)gradient-like dynamics; see *e.g.* [5, 10, 18, 28, 29, 84, 54, 126, 11, 25]. On the algorithmic side, there is also a vast literature on convex minimization algorithms that combine different descent methods (GD, PPA, FBS) with Tikhonov and more general penalty and regularization schemes. The historical evolution can be traced back to [96] where interior point methods were interpreted via a vanishing logarithmic barrier. Some references for the coupling of descent methods and Tikhonov can be found in *e.g.* [72, 31, 30, 20, 13, 27, 35].

1.1.2 Second-order systems

Key role of inertia. As discussed above, the dynamic (GF) is known to yield a convergence rate of $\mathcal{O}(t^{-1})$ (in fact even $o(t^{-1})$) of the values in the convex setting. Second-order inertial dynamical systems have been introduced to provably accelerate the convergence behavior in the convex case. They typically take the form

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t)) = 0, & t > t_0; \\ x(t_0) = x_0; \quad \dot{x}(t_0) = v_0 \end{cases} \quad (\text{IGS}_\gamma)$$

where $t_0 > 0$, $x_0, v_0 \in \mathbb{H}$ and $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ is a time-dependent viscosity coefficient. An abundant literature has been devoted to the study of the inertial dynamics (IGS $_\gamma$). The importance of working with a time-dependent viscosity coefficient to obtain acceleration was stressed by several authors; see *e.g.* [18]. In particular, the case $\gamma(t) = \frac{\alpha}{t}$ was considered by Su, Boyd, and Candès [191], who were the first to show the rate of convergence $\mathcal{O}(t^{-2})$ of the values in the convex setting for $\alpha \geq 3$, thus making the link with the Nesterov accelerated gradient (NAG) method [153] with stepsize $h > 0$,

$$\begin{cases} y_k = x_k + \left(1 - \frac{\alpha}{k}\right)(x_k - x_{k-1}) \\ x_{k+1} = y_k - h\nabla f(y_k). \end{cases} \quad (\text{NAG}_\alpha)$$

Note that a proximal version of (NAG $_\alpha$) was initiated and studied by Güler in his seminal work [108, 107].

The case $\alpha < 3$ leads to the slower rate $\mathcal{O}(t^{-2\alpha/3})$ which cannot be improved in general [7, 26]. For $\alpha > 3$, an even better rate of convergence with little- o instead of big- \mathcal{O} can be obtained together with global weak convergence of the trajectory; see [18, 37] and [79] for algorithm (NAG $_\alpha$).

Another remarkable instance of (IGS $_\gamma$) corresponds to the well-known Heavy Ball with Friction (HBF) method, where $\gamma(t)$ is a constant (say $\gamma_0 > 0$), first introduced (in its discrete and continuous form) by Polyak in [175]. Its discrete form corresponds to the following update rule with stepsize $h > 0$

$$\begin{cases} y_k = x_k + (1 - \gamma_0)(x_k - x_{k-1}) \\ x_{k+1} = y_k - h\nabla f(x_k). \end{cases} \quad (\text{HBF} - \text{Disc})$$

When f is strongly convex, it was shown that the trajectory (and iterations) converges exponentially with an optimal convergence rate if the constant γ_0 is properly chosen as a function of the strong convexity modulus. The convex case was later studied in [4] with a convergence rate on the values of only $\mathcal{O}(t^{-1})$. When f is non-convex, HBF was investigated both for the continuous dynamics in [111, 34, 105, 8] and discrete algorithms in [197, 158, 156].

When the order of the extrapolation and gradient descent operations in (HBF – Disc) are reverted, one gets the so-called Ravine method introduced in 1961 by Gelfand and Tsetlin [102]. The two schemes are largely confused in the literature, but turn out to be different as recently clarified in [32].

Viscous and geometric Hessian driven damping. Because of the inertial aspects and the asymptotic vanishing viscous damping coefficient, (IGS_γ) may exhibit many small oscillations which are not desirable from an optimization point of view. To remedy this, a powerful tool consists in introducing into the dynamic a geometric damping driven by the Hessian of f . This gives the Inertial System with Explicit Hessian Damping which reads

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0, & t > t_0; \\ x(t_0) = x_0; \quad \dot{x}(t_0) = v_0. \end{cases} \quad (\text{ISEHD})$$

Where $\gamma, \beta : [t_0, +\infty[\rightarrow \mathbb{R}_+$. (ISEHD) was proposed in [38] (see also [23]). The second system we consider, inspired by [3] (see also [151] for a related autonomous system) is

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t)) + \beta(t)\dot{x}(t) = 0, & t > t_0; \\ x(t_0) = x_0; \quad \dot{x}(t_0) = v_0. \end{cases} \quad (\text{ISIHD})$$

(ISIHD) stands for Inertial System with Implicit Hessian Damping. The rationale behind the use of the term “implicit” comes from a Taylor expansion of the gradient term (as $t \rightarrow +\infty$ we expect $\dot{x}(t) \rightarrow 0$) around $x(t)$, which makes the Hessian damping appear indirectly in (ISIHD) . Following the physical interpretation of these two ODEs, we call the non-negative parameters γ and β the viscous and geometric damping coefficients, respectively. The two ODEs (ISEHD) and (ISIHD) were found to have a smoothing effect on the energy error and oscillations [3, 152, 23]. Moreover, in [33] they obtain fast convergence rates for the values for the two ODEs when $\gamma(t) = \frac{\alpha}{t}$ ($\alpha > 3$) and $\beta(t) = \beta_0 > 0$.

The time discretization of (ISEHD) with general coefficients has been studied by Attouch, Chbani, Fadili, and Riahi [23]. It provides a rich family of first-order methods for minimizing f . At first glance, the presence of the Hessian may seem to entail numerical difficulties. However, this is not the case as the Hessian intervenes in the above ODE in the form $\nabla^2 f(x(t))\dot{x}(t)$, which is nothing but the time derivative of $t \mapsto \nabla f(x(t))$. This explains why the time discretization of this dynamic provides first-order algorithms. On the contrary, the time-continuous dynamics can be argued to be truly of second-order nature, *i.e.*, close to Newton’s and Levenberg-Marquardt’s dynamics [75]. This understanding suggests that (ISIHD) may represent the nature of first-order algorithms better than (ISEHD) .

From first to second-order systems via time scaling and averaging. The authors in [17] proposed time scaling and averaging to link (GF) and (ISIHD) with a general viscous damping function γ and a properly adjusted geometric damping function β (related to γ). This avoids in particular to go through an intricate and a dedicated Lyapunov analysis for the second-order system. As we will show later (see for instance Chapter 3), a local convergence analysis becomes also easily accessible through this lens while it is barely possible otherwise. The authors specialize their results to the standard case where $\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \frac{t}{\alpha-1}$.

The idea of passing from a first-order system to a second-order one via time scaling is not new. The author of [73] propose time scaling and a tricky change of variables to show that (IGS_γ) is equivalent to an averaged gradient system, *i.e.* the steepest gradient system (GF) where the instantaneous value of $\nabla f(x(t))$ is replaced by some weighted average of the gradients $\nabla f(x(s))$ over all past positions $s \leq t$. See also [104] for more general gradient systems with memory terms involving kernels. This gives rise to an integro-differential equation. The asymptotic behavior of the dynamic associated to this equation and the equivalent second-order dynamic have been investigated in [73].

1.1.3 Non-convex problems

So far, we have focused our discussion on the convex setting. As one may expect, things are far more complicated, however, in the non-convex setting. We will assume here that $\mathbb{H} = \mathbb{R}^d$. We consider

the minimization problem (\mathcal{P}_1) where the objective function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the following assumptions:

$$\begin{cases} f \in C^2(\mathbb{R}^d); \\ \inf f > -\infty. \end{cases} \quad (\text{H}_0)$$

Global convergence. Since the objective function is potentially non-convex, the problem (\mathcal{P}_1) is NP-Hard in general. However, there are tractable methods to ensure theoretical convergence to a critical point, or even to a local minimizer. In this regard, a fundamental dynamic to consider is **(GF)**, for any bounded solution of **(GF)**, using LaSalle’s invariance principle, one can check that $\lim_{t \rightarrow +\infty} \nabla f(x(t)) = 0$. If $d = 1$, any bounded solution of **(GF)** tends to a critical point. For $d \geq 2$ this becomes false in general, as shown in the counterexample by [164]. In order to avoid such behavior, it is necessary to work with functions f that enjoy additional structure which in turn would ensure global convergence of the trajectory to a critical point. Such a prominent property is the Kurdyka-Łojasiewicz (KL) inequality [136, 137, 125]. Roughly speaking, this means that f is sharp up to a reparametrization by a function known as the desingularizing function of f . The KL inequality, including in its non-smooth version, has been successfully used to analyze the asymptotic behavior of various types of dynamical systems [81, 56, 57, 49] and algorithms [59, 97, 155, 42, 119, 1, 56, 16, 158]¹. The importance of the KL inequality comes from the fact that many problems encountered in finite-dimensional optimization involve functions satisfying such an inequality, and it is often elementary to check that the latter is satisfied; *e.g.* real semialgebraic/analytic functions [136, 137], functions definable in an o-minimal structure and more generally tame functions [125, 58]. When this inequality holds, one can ensure global convergence of the trajectory of **(GF)** (and the iterates of **(GD)**) to a critical point with an explicit convergence rate that depends on the KL desingularizing function of f [81, 100, 59, 83], and this is then sharper than standard worst-case rates.

Inertial algorithms. Second-order systems and subsequent inertial algorithms have been studied in the literature for non-convex smooth problems. For instance, [134] proposed a multi-step inertial Forward-Backward algorithm for non-convex optimization and showed its convergence properties under the KL inequality. In [105], the authors consider the case of quasiconvex functions and use an implicit discretization of HBF to derive a proximal point-type algorithm. In [158], an inertial Forward-Backward splitting scheme, coined iPiano—a generalization of HBF, is introduced. In [156], the author presents local convergence results for iPiano. In [157, 61] several abstract convergence theorems are presented and then applied to different versions of iPiano. In [62, 63], inertial algorithms of different nature (Tseng’s type and Forward-Backward, respectively) are studied for non-convex and composite smooth+non-smooth optimization and showed their convergence. To the best of our knowledge, [76] is the only work where the authors study the ODE **(ISEHD)** where the viscous and geometric dampings are positive constants, together with a discrete algorithm.

Trap avoidance. In all works reviewed above, the KL inequality is instrumental to obtain global convergence towards a critical point. However, convergence to critical points is not enough from a minimization perspective. Generic strict saddle point avoidance of descent-like algorithms has been studied by several authors building on the (center) stable manifold theorem [187, Theorem III.7] which finds its roots in the work of Poincaré. Genericity is in general either with respect to initialization or with respect to random perturbation. Note that genericity results for quite general systems, even in infinite dimensional spaces, is an important topic in the dynamical system theory; see *e.g.* [70] and references therein.

First-order descent methods can circumvent strict saddle points provided that they are augmented with unbiased noise whose variance is sufficiently large in each direction. Here, the seminal works of

¹This list is by no means exhaustive.

[65] and [171] allow to establish that the stochastic gradient descent (and more generally the Robbins-Monro stochastic approximation algorithm) avoids strict saddle points almost surely. Those results were extended to the discrete version of HBF by [117] who showed that perturbation allows to escape strict saddle points, and it does so faster than gradient descent. In [98] and [115], the authors analyze a stochastic version of HBF (in continuous-time), showing convergence towards a local minimum under different conditions on the noise.

Recently, there has been active research on how gradient-type descent algorithms escape strict saddle points generically on initialization; see *e.g.* [129, 128, 165, 118] and references therein. In [105], the authors were concerned with HBF and showed that if the objective function is C^2 , coercive and Morse, then generically on initialization, the solution trajectory converges to a local minimum of f . A similar result is also stated in [34]. The algorithmic counterpart of this result was established in [161] who proved that the discrete version of HBF escapes strict saddles points for almost all initializations. Strict saddle avoidance for the second-order ODE (ISEHD) and its discretization was studied in [74, 76], where the author considers the case when the viscous and geometric dampings are positive constants.

1.2 Motivation and objectives

1.2.1 Stochastic convex optimization

Let us consider \mathbb{H}, \mathbb{K} real and separable Hilbert spaces. In many cases, the gradient input is subject to noise, for example, if the gradient cannot be evaluated directly, or due to some other exogenous factor. In such scenario, one can model the associated errors using a stochastic integral with respect to the measure defined by a continuous Itô martingale. This entails the following stochastic differential equation as a stochastic counterpart of (GF):

$$\begin{cases} dX(t) = -\nabla f(X(t))dt + \sigma(t, X(t))dW(t), & t > 0; \\ X(0) = X_0, \end{cases} \quad (\text{SDE})$$

defined over a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where the diffusion (volatility) term $\sigma : \mathbb{R}_+ \times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ (see Section 2.1 for notation) is a measurable function, W is a \mathcal{F}_t -adapted \mathbb{K} -valued Brownian motion (see Section 2.7 for a precise definition), and the initial data X_0 is an \mathcal{F}_0 -measurable \mathbb{H} -valued random variable.

Our first goal is to study the dynamic of (SDE) and its long-time behavior in order to solve (\mathcal{P}_1). To identify the assumptions necessary to hope for such a behavior to occur, remember that when $\mathbb{H} \equiv \mathbb{K}$ and the diffusion term $\sigma = \tilde{\sigma}I_{\mathbb{H}}$, where $\tilde{\sigma}$ is a positive real constant, it is well-known that $X(t)$, in this case, is a continuous-time diffusion process known as Langevin diffusion, and has a unique invariant probability measure $\pi_{\tilde{\sigma}}$ with density $\propto e^{-2f(x)/\tilde{\sigma}^2}$ [52]. In fact, (SDE) can be interpreted as the path-wise solution to the Fokker-Planck equation (see [99]). It is also very well known that the measure $\pi_{\tilde{\sigma}}$ gets concentrated around $\operatorname{argmin}(f)$ as $\tilde{\sigma}$ tends to 0^+ with $\lim_{\tilde{\sigma} \rightarrow 0^+} \pi_{\tilde{\sigma}}(\operatorname{argmin}(f)) = 1$; see *e.g.* [77].

Motivated by this last observation, our work will then mostly focus on the case where $\sigma(\cdot, x)$ vanishes sufficiently fast as $t \rightarrow +\infty$ uniformly in x , and some guarantees will also be provided for uniformly bounded σ . Therefore, throughout the thesis, we assume that σ satisfies:

$$\begin{cases} \sup_{t \geq 0, x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} < +\infty, \\ \|\sigma(t, x') - \sigma(t, x)\|_{\text{HS}} \leq l_0 \|x' - x\|, \end{cases} \quad (\text{H}_\sigma)$$

for some $l_0 > 0$ and for all $t \geq 0, x, x' \in \mathbb{H}$ (where HS is Hilbert-Schmidt norm; see Section 2.1). The Lipschitz continuity assumption is mild and required to ensure the well-posedness of (SDE).

1.2.2 First-order SDE modeling of SGD

To simplify the discussion, let us focus in this section on the finite-dimensional case ($\mathbb{H} = \mathbb{R}^d$, $\mathbb{K} = \mathbb{R}^m$). In various areas of science and engineering, in particular in machine learning, the objective takes the form $f(x) = \mathbb{E}[F(x, \xi)]$, where the expectation is w.r.t. to the random variable ξ ; *e.g.*, denoting a random batch of training data. Although gradient descent algorithm (**GD**) is a classical and simple algorithm to minimize f , the computation of the full gradient is prohibitively expensive if not impossible for large-scale problems. An alternative, the Stochastic Gradient Descent (**SGD**) consists in replacing the full gradient computation by a cheap randomly sampled version, serving as an unbiased estimator. The **SGD** updates the iterates according to

$$x_{k+1} = x_k - h(\nabla f(x_k) + e_k) \quad (\text{SGD})$$

where $h \in \mathbb{R}_+$ is the stepsize and e_k is the random noise term on the gradient at the k -th iteration. As such, (**SGD**) can be viewed as instance of the Robbins-Monro stochastic approximation algorithm [179].

The most popular and simplest form of **SGD** is the single-batch version:

$$x_{k+1} = x_k - h\nabla F(x_k, \xi_k), \quad (\text{SGD}_{\text{SB}})$$

where $(\xi_k)_{k \in \mathbb{N}}$ are i.i.d. random variables with the same distribution as ξ . Of course, (**SGD**_{SB}) is an instance of (**SGD**) with $e_k = \nabla F(x_k, \xi_k) - \nabla f(x_k)$.

SDE as a model of SGD. The SDE continuous-time approach is motivated by its close relation to (**SGD**) or (**SGD**_{SB}). We first note that when the noise e_k in (**SGD**) is $\mathcal{N}(0, \sigma_k I_d)$, (**SDE**) is a better continuous-time model for (**SGD**) than (**GF**), as has been shown recently in [90, Proposition 2.1]. There, the sequence $(x_k)_{k \in \mathbb{N}}$ provided by (**SGD**), with $e_k \sim \mathcal{N}(0, \sigma_k I_d)$, was proved to be accurately approximated by (**SDE**) with $\sigma(t, X(t)) = \sqrt{h}\sigma(t)$ and $\sigma(kh) = \sigma_k I_d$. Using classical numerical analysis tools of SDEs, the approximation error is of order $\mathcal{O}(h)$ which is much better than that of (**GF**) which is only $\mathcal{O}(\sqrt{h})$.

For the standard single-batch **SGD** (**SGD**_{SB}), the argument is more involved. Actually, many recent works (see *e.g.* [140, 131, 163, 113, 186, 133, 190, 196, 127, 132]) have linked algorithm (**SGD**_{SB}) with continuous-time first-order stochastic diffusion dynamics such as (**SDE**). These works show either empirically or theoretically under which conditions (appropriate drift and diffusion terms, regularity of f , etc.) (**SDE**) can be seen as a good approximation model of (**SGD**_{SB}) for fixed stepsize. By a good model we mean that (**SDE**) is a continuum limit of (**SGD**_{SB}) as the stepsize goes to zero, or equivalently that the approximation of (**SGD**_{SB}) via the diffusion process (**SDE**) is precise in some weak sense; see [131, 113, 133].

As a consequence, using (**SDE**) as a proxy of (**SGD**) or (**SGD**_{SB}) allows to capitalize on the wealth of results in the field of SDE's, Itô calculus and measure theory, and this in turn opens the door to new insights in the behavior of (**SGD**) or (**SGD**_{SB}) and to transfer all the convergence results that one can prove for (**SDE**) to (**SGD**). Actually, this is one of the main messages we want to convey in this thesis. Our motivation and results in this thesis are also complementary to those in the literature. Indeed, most, if not all, of works cited above are primarily motivated by the fact that continuous-time SDE approximation of (**SGD**) is a crucial tool to study its trap escaping behavior. Our standpoint is complementary and we argue that the continuous-time perspective offers a deep insight and unveils the key properties of the dynamic of (**SGD**), without being tied to a specific discretization. This, in turn, enlightens the behavior of the sequence generated by some specific algorithm such as (**SGD**). In turn, studying the continuous-time SDE will allow to predict the convergence behavior of **SGD** and other stochastic algorithms.

For the reader convenience, we now give an intuitive explanation of how (**SDE**) could be derived

from **(SGD_{SB})**. Observe from **(SGD_{SB})** that conditioned on x_k ,

$$\mathbb{E}[e_k|x_k] = 0 \text{ and } \Sigma_{\text{SB}}(x_k) \stackrel{\text{def}}{=} \text{Cov}[e_k|x_k] = \mathbb{E}[(\nabla F(x_k, \xi) - \nabla f(x_k))(\nabla F(x_k, \xi) - \nabla f(x_k))^\top | x_k].$$

Let $\Sigma_{\text{SB}}(x_k)^{1/2}$ be the square root of the matrix $\Sigma_{\text{SB}}(x_k)$ and set $(V_k)_{k \in \mathbb{N}}$ a sequence of random variables with zero mean and identity covariance matrix such that $\Sigma_{\text{SB}}(x_k)^{1/2}V_k$ has the same distribution as e_k (conditioned on x_k). Thus, **(SGD_{SB})** can also be written as

$$x_{k+1} = x_k - h\nabla f(x_k) - \sqrt{h}\Sigma_{\text{SB}}(x_k)^{1/2}\sqrt{h}V_k, \quad k \in \mathbb{N}. \quad (1.2)$$

Moreover, the Euler forward time discretization of **(SDE)** with stepsize h is

$$X((k+1)h) = X(kh) - h\nabla f(X(kh)) - \sigma(kh, X(kh))(W((k+1)h) - W(kh)),$$

and since $W((k+1)h) - W(kh)$ has the same distribution as $\sqrt{h}Z_k$, $Z_k \sim \mathcal{N}(0_d, I_d)$, we have

$$X((k+1)h) = X(kh) - h\nabla f(X(kh)) - \sigma(kh, X(kh))\sqrt{h}Z_k. \quad (1.3)$$

Clearly, this discretization resembles **(1.2)** by setting $\sigma(kh, X(kh)) = \sqrt{h}\Sigma_{\text{SB}}(x_k)^{1/2}$ and assuming V_k to be $\mathcal{N}(0_d, I_d)$. This intuitive argument has been made rigorous in [131, 113, 133], where it was proved that

$$|\mathbb{E}[G(X(kh))] - \mathbb{E}[G(x_k)]| = \mathcal{O}(h),$$

for any test function G that is sufficiently smooth and have at most linear asymptotic growth. As a consequence, for an objective f sufficiently smooth whose gradient is Lipschitz-continuous, one has

$$\mathbb{E}[f(x_k) - \min f] = \mathbb{E}[f(X(kh)) - \min f] + \mathcal{O}(h),$$

meaning that any rate proved on $\mathbb{E}[f(X(t))]$ can be directly transferred to $\mathbb{E}[f(x_k)]$. It is worth mentioning here that the work of [131, 133], who also proposed first-order SDEs involving the Hessian. They proved that these first-order SDEs approximate SGD with an error $\mathcal{O}(h^2)$.

The above discussion supposes that the SGD noise e_k is (or close to) Gaussian. Some works have challenged this assumption; see [189] who suggested that SGD noise is heavy-tailed. However, [196] argued that the experimental evidence in [189] made strong assumptions on the nature of the gradient noise which could flag Gaussian distributions as non-Gaussian. [133] showed that non-Gaussian noise is not essential to SGD performance and provide experimental evidence that modeling the SGD noise by a Gaussian one is sufficient to understand its generalization properties.

Concerning **(SDE)**, one can easily infer from [50, Proposition 7.4] that assuming $\sup_{x \in \mathbb{R}^d} \|\sigma(t, x)\|_F = o(1/\sqrt{\log(t)})$, and conditioning on the event that $X(t)$ is bounded, we have almost surely that the set of limits of convergent sequences $X(t_k)$, $t_k \rightarrow +\infty$ is contained in $\text{argmin}(f)$. Using results on asymptotic pseudo-trajectories from [50], the work of [149, 176, 45] analyzed the behavior of the Stochastic Mirror Descent dynamics:

$$\begin{aligned} dY(t) &= -\nabla f(X(t))dt + \sigma(t, X(t))dW(t), \\ X(t) &= Q(\eta Y(t)), \end{aligned} \quad (\text{SMD})$$

where $\mathcal{X} \subset \mathbb{R}^d$ is a closed convex feasible region, f is convex with Lipschitz continuous gradient on \mathcal{X} , $Q : \mathbb{R}^d \rightarrow \mathcal{X}$ is the mirror map induced by some strongly convex entropy, and $\eta > 0$ is a sensitivity parameter. In [149, Theorem 4.1], it is shown that if \mathcal{X} is also assumed bounded, that $\sup_{x \in \mathbb{R}^d} \|\sigma(t, x)\|_F = o(1/\sqrt{\log(t)})$, and Q satisfies some continuity assumptions², then the process $X(t)$ **(SMD)** converges to a point in $\text{argmin}(f)$ almost surely. Similar assumptions can be found in [45] to obtain almost sure convergence on the objective. Let us observe that all these results do not apply to our setting. Indeed, if $\mathcal{X} = \mathbb{R}^d$ (unconstrained problem), $Q(x) = x$ and $\eta = 1$, we recover **(SDE)**.

²Compactness of \mathcal{X} and the condition on $\sigma(\cdot, \cdot)$ are clearly reminiscent of [50, Proposition 7.4], though the latter is not discussed in [149].

Our work does not assume any boundedness whatsoever to establish our results. This comes however at somewhat stronger assumptions on $\sigma(\cdot, \cdot)$.

Also the work of [91], analyzes the behavior of (SDE) for $f \in C^2(\mathbb{R}^d)$ not necessarily convex and which satisfies $\sup_{x \in \mathbb{R}^d} \|\sigma(\cdot, x)\|_F \in L^2(\mathbb{R}_+)$. Conditioning on the event that

$$\limsup_{t \rightarrow +\infty} \|X(t)\| < +\infty,$$

they showed that $\nabla f(X(t)) \rightarrow 0$ almost surely, almost sure convergence of $f(X(t))$, and if the objective f is semialgebraic (and more generally tame), they also showed almost sure convergence of $X(t)$ towards a critical point of f . They also made attempt to get local convergence rates under the Łojasiewicz inequality that are less transparent than ours. Our analysis on the other hand leverages convexity of f to establish stronger results and is made on an infinite dimensional space.

Langevin Monte Carlo. Specializing (1.3) to the case where $d = m$ and $\sigma(\cdot, \cdot) \equiv \sqrt{2}I_d$ gives the following algorithm

$$X((k+1)h) = X(kh) - h\nabla f(X(kh)) + \sqrt{2h}Z_k. \quad (\text{LMC})$$

This algorithm, which is known as Langevin Monte Carlo (see [168]), is a standard sampling scheme, whose purpose is to generate samples from an approximation of a target distribution, in our case, proportional to $e^{-f(x)}$. Under appropriate assumptions on f , when h is small and k is large such that kh is large, the distribution of $X_k \stackrel{\text{def}}{=} X(kh)$ converges in different topologies or is close in various metrics to the target distribution with density $\propto e^{-f(x)}$. Asymptotic and non-asymptotic (with convergence rates) results of this kind have been studied in a number of papers under various conditions; see [87, 88, 93, 94, 80, 116] and references therein. By rescaling the problem, relation between sampling (*i.e.* (LMC)) and optimization (*i.e.* (SGD)) has been also investigated for the strongly convex case in *e.g.* [87].

1.2.3 Stochastic differential inclusions for the non-smooth case

Regarding the non-smooth setting we present the following stochastic differential inclusion (SDI) as a stochastic counterpart of (DI):

$$\begin{cases} dX(t) \in -\partial F(X(t)) + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0, \end{cases} \quad (\text{SDI})$$

where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function that satisfies (H $_{\sigma}$), and W is a \mathbb{K} -valued Brownian motion, and the initial data X_0 is an \mathcal{F}_{t_0} -measurable \mathbb{H} -valued random variable. This dynamic is a stochastic system that aims to minimize F if the diffusion term vanishes sufficiently fast. Also, it is the natural extension to the non-smooth setting of (SDE).

The stochastic counterpart of (DI-TA) (which is the Tikhonov regularization of (SDI)), is the following stochastic differential inclusion with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$):

$$\begin{cases} dX(t) \in -\partial F(X(t)) - \varepsilon(t)X(t) + \sigma(t, X(t))dW(t), & t > t_0; \\ X(t_0) = X_0. \end{cases} \quad (\text{SDI-TA})$$

Our objective in this part is to study the dynamics (SDI) and (SDI-TA) and their long-time behavior in order to solve ($\mathcal{P}_{\text{comp}}$). If the diffusion term vanishes with time, as discussed before, one would expect to solve ($\mathcal{P}_{\text{comp}}$) with our dynamics and obtain for (SDI-TA) the hierarchical minimization property described above. Besides, based on the work of [27], we expect that adding a Tikhonov term as in (SDI-TA) would let us obtain almost sure strong convergence of the trajectory to the minimal norm solution. Moreover, to extend the convergence rates shown in [27, Theorem 5] to the stochastic case. We are not aware if any work studying (SDI) as a proxy of SGD in the non-convex. Indeed, the smoothness assumption is crucial in the existing works.

1.2.4 Second-order SDE modeling of SGD

Let us now consider ($\mathcal{P}_{\text{comp}}$) and second-order methods, we consider the generalization of (ISIHD) to the non-smooth case, which yields the differential inclusion:

$$\begin{cases} \dot{x}(t) = v(t), & t > t_0; \\ \dot{v}(t) \in -[\gamma(t)v(t) + \partial F(x(t) + \beta(t)v(t))], & t > t_0; \\ x(t_0) = x_0, \quad \dot{x}(t_0) = v_0, \end{cases} \quad (\text{ISIHD}_{\text{NS}})$$

where ∂F is the convex subdifferential of F . In this setting, keeping in mind that we want to give a rigorous meaning to (ISIHD_{NS}), we can model the associated errors using a stochastic integral with respect to the measure defined by a continuous Itô martingale. This entails the following stochastic differential inclusion (SDI for short), which is the stochastic counterpart of (ISIHD_{NS}):

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) \in -\gamma(t)V(t)dt - \partial F(X(t) + \beta(t)V(t))dt + \sigma(t, X(t) + \beta(t)V(t))dW(t), & t > t_0; \\ X(t_0) = X_0, \quad V(t_0) = V_0. \end{cases} \quad (\text{S- ISIHD}_{\text{NS}})$$

This SDI is defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$) are the initial data; the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function; and W is a \mathbb{K} -valued Brownian motion (see definition in Section 2.7). When $g \equiv 0$, we recover the stochastic counterpart of (ISIHD) as the following SDE:

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) = -\gamma(t)V(t)dt - \nabla f(X(t) + \beta(t)V(t))dt + \sigma(t, X(t) + \beta(t)V(t))dW(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{S- ISIHD})$$

And when $\beta \equiv 0$, this dynamic will be a stochastic version of the Hamiltonian formulation of (IGS_γ) and it will be described by:

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) = -\gamma(t)V(t)dt - \nabla f(X(t))dt + \sigma(t, X(t))dW(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{IGS}_\gamma - \text{S})$$

To get some intuition, keeping the discussion informal at this stage, let us first identify the assumptions needed to expect that the position state of (IGS_γ - S) “approaches” $\text{argmin}(f)$ in the long run. In the case where $\mathbb{H} = \mathbb{K}$, $\gamma(\cdot) \equiv \gamma > 0$ and $\sigma = \tilde{\sigma}I_{\mathbb{H}}$, where $\tilde{\sigma}$ is a positive real constant. Under mild assumptions one can show that (IGS_γ - S) has a unique invariant distribution $\pi_{\tilde{\sigma}}$ in (x, v) with density proportional to $\exp\left(-\frac{2\gamma}{\tilde{\sigma}^2}\left(f(x) + \frac{\|v\|^2}{2}\right)\right)$, see *e.g.*, [170, Proposition 6.1]. Clearly, as $\tilde{\sigma} \rightarrow 0^+$, $\pi_{\tilde{\sigma}}$ gets concentrated around $\text{argmin}(f) \times \{0_{\mathbb{H}}\}$, with $\lim_{\tilde{\sigma} \rightarrow 0^+} \pi_{\tilde{\sigma}}(\text{argmin}(f) \times \{0_{\mathbb{H}}\}) = 1$. Time-discretized versions of this Langevin diffusion process have been studied in the literature to (approximately) sample from $\propto \exp(-f(x))$ with asymptotic and non-asymptotic convergence guarantees in various topologies and under various conditions; see [80, 139, 89] and references therein. Motivated by these observations and the fact that we aim to exactly solve (\mathcal{P}_1) (resp. ($\mathcal{P}_{\text{comp}}$)), our work will then mainly focus on consider (S - ISIHD) (resp. (S - ISIHD_{NS})) in the case where σ satisfies (\mathbf{H}_σ) (adjusted for a time bigger than t_0).

In the smooth but non-convex case, and motivated again by trap avoidance, the authors in [114, 115] study the convergence behavior of a stochastic discrete heavy ball method from its approximating SDE.

The latter is a randomly perturbed nonlinear oscillator in [114] and a coupled system of nonlinear oscillators in [115]. These SDEs are very different from the ones we study here.

In this part of the work, our goal will be to provide a general mathematical framework for analyzing the convergence properties of $(\mathbf{S} - \text{ISIHD}_{\text{NS}})$ to solve $(\mathcal{P}_{\text{comp}})$. In this context, considering inertial dynamics with a time-dependent vanishing viscosity coefficient γ is a key ingredient to obtain fast convergent methods. More precisely, we study the stochastic dynamics $(\mathbf{S} - \text{ISIHD}_{\text{NS}})$ and its long-time behavior in order to solve $(\mathcal{P}_{\text{comp}})$. We conduct a new analysis using specific and careful arguments that are much more involved than in the deterministic case.

The authors in [17] proposed time scaling and averaging to link (GF) and (ISIHD) with a general viscous damping function γ and a properly adjusted geometric damping function β (related to γ). Our goal is to extend the results of [17] to the stochastic case. Leveraging these techniques with a general function γ and an appropriate β , we will be able to transfer all the results we obtained in Chapter 3 (resp. 4) for (SDE) (resp. (SDI)) to the dynamic $(\mathbf{S} - \text{ISIHD})$ (resp. $(\mathbf{S} - \text{ISIHD}_{\text{NS}})$). This avoids in particular to go through an intricate and a dedicated Lyapunov analysis for $(\mathbf{S} - \text{ISIHD}_{\text{NS}})$. A local convergence analysis becomes also easily accessible through this lens while it is barely possible otherwise. We also specialize our results to a standard case where $\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \frac{t}{\alpha-1}$, being able to show fast convergence results. To handle a more general choice of γ and β , we will have to go through a dedicated Lyapunov analysis for $(\mathbf{S} - \text{ISIHD})$ inspired by the work in [33].

1.2.5 Non-convex deterministic setting for second-order systems

There is an abundant literature regarding the dynamics (ISEHD) and (ISIHD) , either in the exact case or with deterministic errors, but overwhelmingly in the convex case; see [3, 33, 111, 185, 20, 24, 92, 184, 194, 12, 23, 21]). Nevertheless, to the best of our knowledge, except for [76, 74], the non-convex setting is still largely open. Our goal is to fill this gap motivated by examples in machine learning (*e.g.* neural network training with smooth activation functions).

As previously noted, the HBF method, was first introduced by Polyak in [175] where linear convergence was shown for f strongly convex. Convergence rates of HBF taking into account the geometry of the objective can be found in [42] for the convex case and [156] for the non-convex case.

In [110], the authors study the system

$$\ddot{x}(t) + G(\dot{x}(t)) + \nabla f(x(t)) = 0,$$

where $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is such that $\langle G(v), v \rangle \geq c\|v\|^2$ and $\|G(v)\| \leq C\|v\|$, $\forall v \in \mathbb{R}^d$, for some $0 < c \leq C$. They show that when f is real analytic (hence verifies the Łojasiewicz inequality), the trajectory converges to a critical point. To put it in our terms, the analysis is equivalent to study (IGS_γ) in the case where there exists $c, C > 0$ such that $0 < c \leq \gamma(t) \leq C$ for every $t \geq 0$ (see assumption (\mathbf{H}_γ)), letting us to conclude as in [110]. We will extend this result to the systems (ISEHD) and (ISIHD) which will necessitate new arguments.

HBF was also studied in the non-convex case by [49] where it was shown that it can be viewed as a quasi-gradient system. They also proved that a desingularizing function of the objective desingularizes the total energy and its deformed versions. They used this to establish the convergence of the trajectory for instance in the definable case. Global convergence of the HBF trajectory was also shown in [34] when f is a Morse function³.

Our goal is to go far beyond the results presented in [76, 74], considering not only the explicit version (ISEHD) but also the implicit one (ISIHD) , and also allowing a variable (non-vanishing) viscous damping. Even though the choice of the value of the geometric damping is less general than the one

³It turns out that a Morse function satisfies the Łojasiewicz inequality with exponent 1/2; see [15].

presented in [76], we propose a less stringent discretization regarding the stepsize. We also demonstrate the convergence of the dynamic to a critical point under general KL property on the objective. This includes semi-algebraic functions as a special case, which was the only case covered in [76]. Additionally, we show convergence rates under KL inequality for the continuous setting. As mentioned earlier, we propose new algorithms and an independent analysis for the discrete setting, showing convergence of the proposed algorithms to critical points under KL-inequality, and we also exhibit convergence rates of the proposed algorithms under Łojasiewicz inequality.

1.3 Contributions

1.3.1 Chapter 3

We study the convergence properties of the process $X(t)$ and $f(X(t))$ for the stochastic differential equation (SDE) on real separable Hilbert spaces from an optimization perspective, under the assumptions (H_f) and (H_σ) . When the diffusion term is uniformly bounded, we show convergence of $\mathbb{E}[f(X(t)) - \min f]$ to a noise-dominated region both for the convex and strongly convex case. When the diffusion term is square-integrable, we show in Theorem 3.2.3 that $X(t)$ converges almost surely to a solution of (\mathcal{P}_1) , which is a new result to the best of our knowledge. Moreover, in Theorem 3.2.5 and Proposition 3.2.6, we provide new ergodic and pointwise convergence rates of the objective in expectation, again for both the convex and strongly convex case.

Then we turn to a local analysis relying on the Łojasiewicz inequality and its strong ties with error bounds. Since this property is most often satisfied only locally, we deepen the discussion on the long time localization of the process. This is fundamental, because in the recent literature on local convergence properties of stochastic gradient descent, strong assumptions are imposed, such as $X(t)$ or $f(X(t))$ is locally bounded almost surely. Such assumptions are unfortunately unrealistic due to the presence of the Brownian Motion. We manage to circumvent this problem by using arguments from measure theory, in particular Egorov's theorem. In turn, under the Łojasiewicz inequality assumption with exponent $q \geq 1/2$, this allows us to show local convergence rates of the objective and the trajectory itself in expectation over a set of events whose probability is arbitrarily close to 1 (see Theorem 3.3.3).

Table 1.1 summarizes the local and global convergence rates obtained for $\mathbb{E}[f(X(t)) - \min f]$ in Chapter 3. In this table, $\delta > 0$ is a parameter which is intended to be taken arbitrarily close to 0 but bounded away from it, $\sigma_* > 0$ and $\sigma_\infty(\cdot)$ are defined as

$$\|\sigma(t, x)\|_{\text{HS}}^2 \leq \sigma_*^2, \quad \forall t \geq 0, \forall x \in \mathbb{H}, \quad \text{and} \quad \sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}}, \quad (1.4)$$

and $\sigma_\infty(\cdot)$ is a non-increasing function. $\mathbb{L}^q(\mathcal{S})$ is the class of functions satisfying the Łojasiewicz inequality with exponent $q \in [0, 1]$ at each point of \mathcal{S} (see Definition 2.4.7)⁴.

Property of f	Gradient Flow	SDE ($\sigma_\infty(t) \leq \sigma_*$)	SDE ($\sigma_\infty \in L^2(\mathbb{R}_+)$)
Convex	t^{-1}	$t^{-1} + \sigma_*^2$	t^{-1}
μ -Strongly Convex	$e^{-2\mu t}$	$e^{-2\mu t} + \sigma_*^2$	$\max\{e^{-2\mu t}, \sigma_\infty^2(t)\}$
Convex $\cap \mathbb{L}^{1/2}(\mathcal{S})$ (coef. μ)	$e^{-\mu^2 t}$	✗	$\max\{e^{-\mu^2 t}, \sigma_\infty^2(t)\} + \sqrt{\delta}$
Convex $\cap \mathbb{L}^q(\mathcal{S})$, $q \in (\frac{1}{2}, 1)$	$t^{-\frac{1}{2q-1}}$	✗	$t^{-\frac{1}{2q-1}} + \sqrt{\delta}$ ⁵

Table 1.1: Summary of local and global convergence rates obtained for $\mathbb{E}[f(X(t)) - \min f]$.

⁴In finite dimension, semialgebraic functions, and more generally, functions based on the class of analytic functions is a typical family of functions that verify the Łojasiewicz inequality at each point [136, 137].

Although it is natural to think that we can take the limit when δ goes to 0^+ , the time from which these convergence rates are valid depends on δ and increases (potentially to $+\infty$) as δ approaches 0^+ . Assuming only the boundedness of the diffusion and the Łojasiewicz inequality, we could not find better results (cells marked with **✖**) than those presented in the convex case. Since the Łojasiewicz inequality is local, a natural approach would be to localize the process in the long term with high probability. However, it is not clear how to achieve this.

We finally turn to extending some of the preceding results to the structured convex minimization problem ($\mathcal{P}_{\text{comp}}$). This obviously covers the case of constrained minimization of f over a non-empty closed convex set. We take two different routes leading to different SDEs.

The first approach consists in reformulating ($\mathcal{P}_{\text{comp}}$) as finding for zeros of the operator $M_\mu : \mathbb{H} \rightarrow \mathbb{H}$

$$M_\mu(x) = \frac{1}{\mu} (x - \text{prox}_{\mu g}(x - \mu \nabla f(x))),$$

where $\mu > 0$ and $\text{prox}_{\mu g}$ is the proximal mapping of μg . It is well-known that the operator M_μ is cocoercive [19], hence monotone and Lipschitz continuous, and $M_\mu = \nabla f$ when g vanishes. The idea is then to replace the operator ∇f in (SDE) by M_μ leading to an SDE which will have many of the convergence properties obtained in the smooth convex case. This approach is in accordance with the deterministic theory for monotone cocoercive operators (see [55, 6, 19]).

The second approach regularizes the non-smooth component g of the objective function using its Moreau envelope

$$g_\theta(x) = \min_{z \in \mathbb{H}} g(z) + \frac{1}{2\theta} \|x - z\|^2.$$

This leads to studying the dynamic (SDE) with the function $f + g_\theta$, which has a continuous Lipschitz gradient. This approximation method leads to an SDE with non-autonomous drift term. Note, however, that the noise in this case can be considered on the evaluation of $\nabla f(x)$, while it is on $M_\mu(x)$ in the first approach.

1.3.2 Chapter 4

This work significantly extends beyond what is presented in Chapter 3 in two distinct directions: we consider the non-smooth case in a direct way, and we use Tikhonov regularization. The latter makes it possible to pass from weak convergence to strong convergence, and to a particular solution, that of minimal norm.

We first study the properties of the process $X(t)$ and $F(X(t))$ for the stochastic differential inclusion (SDI) on separable real Hilbert spaces from an optimization perspective, under the assumptions (\mathbf{H}_F), (\mathbf{H}_σ) and (\mathbf{H}_λ) (see Section 4.2). When the diffusion term is uniformly bounded, we show convergence of $\mathbb{E}[F(X(t)) - \min F]$ to a noise-dominated region both for the convex and strongly convex case. When the diffusion term is square-integrable, we show in Theorem 4.3.2 that $X(t)$ weakly converges almost surely to a solution of ($\mathcal{P}_{\text{comp}}$), which is a new result to the best of our knowledge. Moreover, in Theorem 4.3.4, we provide new ergodic and pointwise convergence rates of the objective in expectation, again, for both the convex and strongly convex case.

Next, we consider (SDE), obtained by adding a Tikhonov regularization term to (SDI). We show in Theorem 4.4.1 that under certain conditions on the regularization term, $X(t)$ strongly converges almost surely to the minimum norm solution. Then, we show in Theorem 4.4.5 some practical situations where one can obtain an explicit form of the Tikhonov regularizer. Moreover, in Theorem 4.4.8, we show new convergence rates of the objective and the trajectory in expectation for the smooth case.

⁵This is not yet proven, our conjecture is that it is true when $\sigma_\infty = \mathcal{O}((t+1)^{-\frac{q}{2q-1}})$ (see the detailed discussion in Conjecture 3.3.9).

Tables 1.2 summarizes the convergence rates obtained for $\mathbb{E}[F(X(t)) - \min F]$ in Chapter 4. We recall that $\sigma_* > 0$ and $\sigma_\infty(\cdot)$ are defined as

$$\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} \quad \text{where} \quad \|\sigma(t, x)\|_{\text{HS}}^2 \leq \sigma_*^2, \quad \forall t \geq t_0, \forall x \in \mathbb{H}. \quad (1.5)$$

Property of F	(DI)	(SDI ₀) ($\sup_{t \geq t_0} \sigma_\infty(t) \leq \sigma_*$)	(SDI ₀) ($\sigma_\infty \in L^2([t_0, +\infty[))$)
Convex	t^{-1}	$t^{-1} + \sigma_*^2$	t^{-1}
μ -Strongly Convex	$e^{-2\mu t}$	$e^{-\mu t} + \sigma_*^2$	$\max\{e^{-\mu t}, \sigma_\infty^2(t)\}$

Table 1.2: Summary of convergence rates obtained for $\mathbb{E}[F(X(t)) - \min F]$.

In Table 1.3, we summarize the results obtained in the smooth case for the dynamics with Tikhonov regularization, *i.e.*, when $g \equiv 0$. We recall that $\text{EB}^p(\mathcal{S})$ is the local Error Bound inequality defined in (2.11).

Property of f	(DI-TA) ($\varepsilon(t) = t^{-r}, r \in]0, 1[$)	(SDI-TA) ($\varepsilon(t) = t^{-r}, r \in]\frac{2p}{2p+1}, 1[$)
$\text{Convex} \cap \text{EB}^p(\mathcal{S})$	t^{-r}	t^{-r} (whenever $\sigma_\infty^2(t) = \mathcal{O}(t^{-2r})$).

Table 1.3: Summary of convergence rates obtained for $\mathbb{E}[f(X(t)) - \min f]$ for the dynamics with Tikhonov regularization when $\varepsilon(t) = t^{-r}$.

1.3.3 Chapter 5

Our main contributions pertain to the solution trajectories of the dynamics (**S – ISIHD**) and (**S – ISIHD_{NS}**) under integrability conditions on the noise for a general $\gamma(t)$ and a particular choice of $\beta(t)$ ($\beta \equiv \Gamma$, where Γ is defined in (5.1)). They are summarized as follows:

- We show almost sure weak convergence of the trajectory (see Theorem 5.2.2) and convergence rates (see Theorem 5.2.3) in expectation in the case of time-dependent coefficients $\gamma(t)$ and a proper choice of $\beta(t)$. To do so, we transfer, through time scaling and averaging, the results of the Lyapunov analysis of the first-order in-time stochastic (sub-)gradient system studied in Chapters 3-4 to our inertial system (**S – ISIHD**).
- We obtain almost sure and ergodic convergence results which correspond precisely to the best-known results in the deterministic case. In particular, if we let $\alpha > 3, \gamma(t) = \frac{\alpha}{t}, \beta(t) = \frac{t}{\alpha-1}$, then under appropriate assumptions on the diffusion (volatility) term σ , we obtain the rate of convergence $o(1/t^2)$ of the values in almost sure sense (see Corollary 5.2.5), which corresponds to the known result for the accelerated gradient method of Nesterov in the deterministic case.
- We then turn to providing a local analysis with a local linear convergence rate under the Polyak-Łojasiewicz inequality (See Theorem 5.2.6). This is much more challenging in the stochastic case, and even more for second-order systems, as localizing the process in this case is very delicate.
- We also show almost sure strong convergence of the trajectory to the minimal norm solution when adding a Tikhonov regularization to our systems (see Theorem 5.3.1). Moreover, we show convergence rates in expectation for the objective and the trajectory for a particular Tikhonov regularizer (see Theorem 5.3.3).

It is worth observing that since our approach is based on an averaging technique, it will involve Jensen's inequality at some point to get fast convergence rates. In this respect, the convexity condition on the objective function appears unavoidable, at least in this method of proof. It is also worth

mentioning that the approach only makes sense for the implicit form of the Hessian-driven damping. Indeed, as explained above, the explicit form of the Hessian-driven damping has a term involving the time derivative of the (sub)gradient at the trajectory. As the noise, modeled here as an Itô martingale, in practice stems from the (sub)gradient evaluation, this time derivative is meaningless with explicit Hessian-driven damping, as (non-constant) martingales are a.s. not differentiable.

1.3.4 Chapter 6

Our main contributions pertain to the solution trajectories of the dynamic **(S – ISIHD)** for more general parameters than those considered in the previous chapter. They are summarized as follows:

- We will develop a Lyapunov analysis to obtain convergence rates, and integral estimates, in almost sure sense (see Theorem 6.2.7) and in expectation (see Theorem 6.2.8), in the general case of coefficients $\gamma(t)$ and $\beta(t)$.
- We will give two instances where we can find suitable functions for the presented Lyapunov function: when γ is decreasing and vanishing with vanishing derivative, and $\beta(t)$ is constant (see Corollary 6.2.9), and when $\gamma(t) = \frac{\alpha}{t}$, $\beta(t) = \beta_0 + \frac{\gamma_0}{t}$ (see Corollary 6.2.11).
- In the case where the coefficient $\beta(t)$ is zero, we show that we have almost sure weak convergence of the trajectory (see Theorem 6.3.7), convergence rates, and integral estimates (see Theorem 6.3.5). As a special case, we focus on the viscous damping coefficient $\gamma(t) = \frac{\alpha}{t^r}$, $r \in [0, 1]$, $\alpha \geq 1 - r$.

1.3.5 Chapter 7

This chapter analyzes the convergence properties of **(ISEHD)** and **(ISIHD)** when f is smooth and possibly non-convex and with no stochasticity. We consider the case where the time-dependent viscosity coefficient γ is non-vanishing and the geometric damping is constant, *i.e.* $\beta(t) \equiv \beta \geq 0$. We will also propose appropriate discretizations of these dynamics and establish the corresponding convergence guarantees for the resulting discrete algorithms. More precisely, our main contributions can be summarized as follows:

- We provide a Lyapunov analysis of **(ISEHD)** and **(ISIHD)** and show convergence of the gradient to zero and convergence of the values. Moreover, assuming that the objective function is definable, we prove the convergence of the trajectory to a critical point (see Theorem 7.2.1 and 7.3.1). Furthermore, when f is also Morse, using the center stable manifold theorem, we establish a generic convergence of the trajectory to a local minimum of f (see Theorem 7.2.3 and 7.3.2).
- We provide convergence rates of **(ISEHD)** and **(ISIHD)** and show that they depend on the desingularizing function of the Lyapunov energy (see Theorems 7.2.4 and 7.3.3).
- By appropriately discretizing **(ISEHD)** and **(ISIHD)**, we propose algorithmic schemes and prove the corresponding convergence properties, which are the counterparts of the ones shown for the continuous-time dynamics. In particular, assuming that the objective function is definable, we show global convergence of the iterates of both schemes to a critical point of f . Furthermore, a generic strict saddle points (see Definition 7.1.3) avoidance result will also be proved (see Theorem 7.2.7 and 7.3.5).
- Convergence rates for the discrete schemes will also be established under the Łojasiewicz property where the convergence rate will be shown to depend on the exponent of the desingularizing function (see Theorem 7.2.9 and 7.3.6).

We will report a few numerical experiments to support the above findings.

1.4 Outline

This manuscript consists of two parts and eight chapters. Chapter 2 collects the necessary notation and mathematical material used throughout the manuscript. Chapters 3-6 focus on convex optimization problems through first and second-order stochastic differential equations defined over a general real separable Hilbert space, where Chapters 3-4 consider first-order SDEs and Chapters 5-6 second-order ones. Chapter 7 studies second-order Hessian-aware dynamics defined on an Euclidian space in the non-convex and deterministic setting. The final chapter gathers a final discussion, conclusions and draw some directions of future work.

Chapter 2

Mathematical Background

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In this chapter, we collect the necessary notation and mathematical material used in the manuscript.

2.1 Notations and Preliminaries

Vectors and matrices. Throughout, we will use the following shorthand notations: Given $n \in \mathbb{N}$, $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$. We denote by \mathbb{R}_+ the set $[0, +\infty[$. Moreover, we denote \mathbb{R}_+^* and \mathbb{N}^* to refer to $\mathbb{R}_+ \setminus \{0\}$ and $\mathbb{N} \setminus \{0\}$, respectively. The finite-dimensional space \mathbb{R}^d ($d \in \mathbb{N}^*$) is endowed with the canonical scalar product $\langle \cdot, \cdot \rangle$ whose norm is denoted by $\|\cdot\|$. We denote $\mathbb{R}^{n \times d}$ ($n, d \in \mathbb{N}^*$) the space of real matrices of dimension $n \times d$. For $A \in \mathbb{R}^{d \times d}$, A^\top is its transpose, $\lambda_i(A) \in \mathbb{C}$ is its i -th eigenvalue. When A is symmetric then the eigenvalues are real and we denote $\lambda_{\min}(A), \lambda_{\max}(A)$ to be the minimum and maximum eigenvalue of A , respectively. If every eigenvalue of A is positive (resp. negative), we will say that A is positive (resp. negative) definite. We denote by I_d the identity matrix of dimensions $d \times d$ and $0_{n \times d}$ the null matrix of dimensions $n \times d$, respectively.

Hilbert spaces. For any real Hilbert space \mathbb{H} , it is endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and the corresponding norm $\|\cdot\|_{\mathbb{H}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{H}}}$, and we will omit the subscript \mathbb{H} it is clear from the context. \mathbb{B} denotes the closed unitary ball of \mathbb{H} centered at the origin. $I_{\mathbb{H}}$ is the identity operator from \mathbb{H} to \mathbb{H} and $0_{\mathbb{K};\mathbb{H}}$ is the operator that maps every element of \mathbb{K} to $0_{\mathbb{H}}$, where \mathbb{K} is another real Hilbert space. We denote by w-lim (resp. s-lim) the limit for the weak (resp. strong) topology of \mathbb{H} .

For any subset C of \mathbb{H}

$$\text{dist}(x, C) = \inf_{z \in C} \|z - x\|_{\mathbb{H}}$$

is the distance from $x \in \mathbb{H}$ to C . If $C = \emptyset$ we set $\text{dist}(x, C) = +\infty$. For a non-empty closed convex set $C \subseteq \mathbb{H}$, the minimizer in $\text{dist}(x, C)$ exists and is unique for any $x \in \mathbb{H}$, and will denote it $P_C(x)$, the projection of x on C .

Operators. $\mathcal{L}(\mathbb{K}; \mathbb{H})$ will denote the space of bounded linear operators from \mathbb{K} to \mathbb{H} , $\mathcal{L}_1(\mathbb{K})$ is the space of trace-class operators, and $\mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is the space of bounded linear Hilbert-Schmidt operators from \mathbb{K} to \mathbb{H} . For $M \in \mathcal{L}_1(\mathbb{K})$, the trace is defined by

$$\mathrm{tr}(M) \stackrel{\mathrm{def}}{=} \sum_{i \in I} \langle M e_i, e_i \rangle < +\infty,$$

where $I \subseteq \mathbb{N}$ and $(e_i)_{i \in I}$ is an orthonormal basis of \mathbb{K} . Besides, for $M \in \mathcal{L}(\mathbb{K}; \mathbb{H})$, $M^* \in \mathcal{L}(\mathbb{H}; \mathbb{K})$ is the adjoint operator of M , and for $M \in \mathcal{L}_2(\mathbb{K}; \mathbb{H})$,

$$\|M\|_{\mathrm{HS}} \stackrel{\mathrm{def}}{=} \sqrt{\mathrm{tr}(MM^*)} < +\infty$$

is its Hilbert-Schmidt norm (in the finite-dimensional case is equivalent to the Frobenius norm, and is denoted as $\|\cdot\|_F$).

The notation $A : \mathbb{H} \rightrightarrows \mathbb{H}$ means that A is a set-valued operator from \mathbb{H} to \mathbb{H} . A is identified with its graph of denoted by

$$\mathrm{gph} A \stackrel{\mathrm{def}}{=} \{(x, u) \in \mathbb{H} \times \mathbb{H} : u \in Ax\}.$$

Functions. For a function $f : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$, we define $\mathrm{argmin}(f) \stackrel{\mathrm{def}}{=} \{x \in \mathbb{H} : f(x) \leq f(y), \forall y \in \mathbb{H}\}$, when this set is non-empty we define $\min f \stackrel{\mathrm{def}}{=} f(\mathrm{argmin}(f))$. For a (Fréchet) differentiable function $g : \mathbb{H} \rightarrow \mathbb{R}$, we will denote its gradient as ∇g , the set of its critical points as:

$$\mathrm{crit}(g) \stackrel{\mathrm{def}}{=} \{u \in \mathbb{H} : \nabla g(u) = 0\},$$

and when g is twice differentiable, we will denote its Hessian as $\nabla^2 g$. Let $\mathbb{H}_1, \mathbb{H}_2$ be two real Hilbert spaces, for a differentiable function $G : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ we will denote its Jacobian matrix at $x \in \mathbb{H}_1$ as $J_G(x) \in \mathcal{L}(\mathbb{H}_2; \mathbb{H}_1)$.

We denote by $C^s(\mathbb{H})$ the class of s -times continuously differentiable functions on \mathbb{H} . For $L \geq 0$, $C_L^{1,1}(\mathbb{H}) \subset C^1(\mathbb{H})$ is the set of functions on \mathbb{H} whose gradient is L -Lipschitz continuous (a.k.a. Lipschitz-smooth functions). $C_L^2(\mathbb{H})$ is the subset of $C_L^{1,1}(\mathbb{H})$ whose functions are twice differentiable.

For two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ we will denote $f \sim g$ as $t \rightarrow +\infty$, if $\lim_{t \rightarrow +\infty} \frac{f(t)}{g(t)} = 1$. For $a, b \in \mathbb{R}$, such that $a < b$, and $f : \mathbb{H} \rightarrow \mathbb{R}$, the sublevel of f at a height between a and b is denoted $[a < f < b] \stackrel{\mathrm{def}}{=} \{x \in \mathbb{H} : a < f(x) < b\}$. For $1 \leq p \leq +\infty$, $L^p([a, b])$ is the space of measurable functions $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\int_a^b |g(t)|^p dt < +\infty$, with the usual adaptation when $p = +\infty$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $L^p(\Omega; \mathbb{H})$ denotes the (Bochner) space of \mathbb{H} -valued random variables whose p -th moment (with respect to the measure \mathbb{P}) is finite.

Other notations will be explained when they first appear.

2.2 Some convexity analysis

Let us recall some basic results from convex analysis; for a more complete and comprehensive coverage, we refer the reader to [46] and [181] in the finite dimensional case.

A function $f : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous (lsc) if its epigraph is closed. It is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in [0, 1], \forall x, y \in \mathbb{H} \quad (2.1)$$

We denote by $\Gamma_0(\mathbb{H})$ the class of proper lsc and convex functions on \mathbb{H} taking values in the real extended line $\mathbb{R} \cup \{+\infty\}$. For $\mu > 0$, f is μ -strongly convex if the above inequality is strengthened to

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{\mu}{2} \lambda(1 - \lambda) \|x - y\|^2, \quad \forall \lambda \in [0, 1], \forall x, y \in \mathbb{H} \quad (2.2)$$

We denote $\Gamma_\mu(\mathbb{H}) \subset \Gamma_0(\mathbb{H})$ the class of μ -strongly convex functions.

The *subdifferential* of a function $f \in \Gamma_0(\mathbb{H})$ is the set-valued operator $\partial f : \mathbb{H} \rightrightarrows \mathbb{H}$ such that, for every x in \mathbb{H} ,

$$\partial f(x) = \{u \in \mathbb{H} : f(y) \geq f(x) + \langle u, y - x \rangle \quad \forall y \in \mathbb{H}\},$$

which is non-empty for every point in the relative interior of the domain of f . When f is finite-valued, then f is continuous, and $\partial f(x)$ is a non-empty convex and compact set for every $x \in \mathbb{H}$. If f is differentiable, then $\partial f(x) = \{\nabla f(x)\}$. For every $x \in \mathbb{H}$ such that $\partial f(x) \neq \emptyset$, the minimum norm selection of $\partial f(x)$ is the unique element $\{\partial^0 f(x)\} \stackrel{\text{def}}{=} \operatorname{argmin}_{u \in \partial f(x)} \|u\|$, and for a convex set $A \subseteq \mathbb{H}$, we define its remoteness as $\|A\|_- \stackrel{\text{def}}{=} \operatorname{dist}(0, A)$, by definition we have $\|\partial f(x)\|_- = \|\partial^0 f(x)\|$. For any set $C \subset \mathbb{H}$, ι_C is the indicator function of C , which takes the value 0 on C and $+\infty$ otherwise. Additionally, N_C is the normal cone of C .

Proposition 2.2.1. *Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be differentiable. The following are equivalent:*

- (i) f is convex.
- (ii) $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \quad \forall x, y \in \mathbb{H}$.
- (iii) $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathbb{H}$.

If f is twice differentiable, then the previous conditions are equivalent to

- (iv) $\langle \nabla^2 f(x)d, d \rangle \geq 0, \quad \forall x, d \in \mathbb{H}$.

Proposition 2.2.2. *Let $f : \mathbb{H} \rightarrow \mathbb{R}$ be differentiable. The following are equivalent:*

- (i) f is μ -strongly convex.
- (ii) $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^2, \quad \forall x, y \in \mathbb{H}$.
- (iii) $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu\|x - y\|^2, \quad \forall x, y \in \mathbb{H}$.

If f is twice differentiable, then the previous conditions are equivalent to

- (iv) $\langle \nabla^2 f(x)d, d \rangle \geq \mu\|d\|^2, \quad \forall x, d \in \mathbb{H}$.

It is well-known that when $f \in \Gamma_\mu(\mathbb{H})$, then f has a unique global minimizer.

Applying Young's inequality in Proposition 2.2.1(ii), we get the following result.

Proposition 2.2.3. *Assume that $f \in \Gamma_\mu(\mathbb{H})$ is differentiable, then f satisfies the (global) Polyak-Lojasiewicz (see Definition 2.4.7) inequality, i.e.*

$$2\mu(f(x) - \min f) \leq \|\nabla f(x)\|^2, \quad \forall x \in \mathbb{H}.$$

2.3 Lipschitz-smooth functions

A function $F : \mathbb{H} \rightarrow \mathbb{H}$ is Lipschitz with constant L or L -Lipschitz if:

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{H}. \quad (2.3)$$

A function $F : \mathbb{H} \rightarrow \mathbb{H}$ is cocoercive with constant ρ or ρ -cocoercive if:

$$\langle F(x) - F(y), x - y \rangle \geq \rho\|F(x) - F(y)\|^2, \quad \forall x, y \in \mathbb{H}. \quad (2.4)$$

Proposition 2.3.1. *If $F : \mathbb{H} \rightarrow \mathbb{H}$ is ρ -cocoercive, then F is $\frac{1}{\rho}$ -Lipschitz.*

Proof. Direct from the Cauchy-Schwarz inequality. □

The following inequality is at the heart of what is known as the *descent lemma*. It is satisfied by any function in $C_L^{1,1}(\mathbb{H})$, and plays a central role in optimization.

Lemma 2.3.2. *Assume that $f \in C_L^{1,1}(\mathbb{H})$, then*

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2} \|x - y\|^2, \quad \forall x, y \in \mathbb{H}. \quad (2.5)$$

Corollary 2.3.3. *Assume that $f \in C_L^{1,1}(\mathbb{H})$, then*

$$f\left(x - \frac{1}{L} \nabla f(x)\right) - f(x) \leq -\frac{1}{2L} \|\nabla f(x)\|^2, \quad \forall x \in \mathbb{H}, \quad (2.6)$$

and

$$\|\nabla f(x)\|^2 \leq 2L(f(x) - \min f), \quad \forall x \in \mathbb{H}. \quad (2.7)$$

We have seen in Proposition 2.3.1 that cocoercivity implies Lipschitz continuity. The converse is not true unless the operator is the gradient of a convex function. This is known as Baillon-Haddad theorem.

Theorem 2.3.4 ((Baillon-Haddad Theorem) [44, Corollary 10]). *Assume that $f \in \Gamma_0(\mathbb{H})$ and differentiable. Then $f \in C_L^{1,1}(\mathbb{H})$ if, and only if, ∇f is $\frac{1}{L}$ -cocoercive.*

As a consequence, the descent lemma can be refined with a lower bound¹.

Proposition 2.3.5. *Assume that $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$, then*

$$\frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2 \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2} \|x - y\|^2, \quad \forall x, y \in \mathbb{H}.$$

2.4 Kurdyka-Łojasiewicz inequality

An assumption that will be central in our work for the study of our dynamics and algorithms is that the objective function f satisfies the Kurdyka-Łojasiewicz inequality, which roughly speaking means that f is sharp up to a reparametrization. The KL inequality as pioneered in [136, 137, 125] in the smooth case, and extended to the non-smooth case in [57, 58, 56], has been successfully used to analyze the asymptotic behavior of various types of dynamical systems [81, 56, 57, 49] and algorithms, see *e.g.* [59, 97, 155, 42, 119, 1, 16, 158]. The importance of the KL inequality comes from the fact that many problems encountered in optimization involve functions satisfying such an inequality, and it is often elementary to check that the latter is satisfied. In finite dimension, a rich family is provided by semi-algebraic functions, *i.e.*, functions whose graph is defined by some Boolean combination of real polynomial equations and inequalities [86]. KL functions satisfy the Łojasiewicz property with $q \in [0, 1[\cap \mathbb{Q}$; see [136, 137] for the smooth case and [57, 56] for the non-smooth extension. An even more general family is that of definable functions on an o-minimal structure over \mathbb{R} (see [85]) or even tame functions, which corresponds in some sense to an axiomatization of some of the prominent geometrical and stability properties of semi-algebraic geometry [193, 85]. An important result showed that smooth definable functions satisfy the KL inequality at every $\bar{x} \in \mathbb{R}^d$; see [125] for the smooth case and [58] the non-smooth one.

We present now proceed to a precise definition of this notion.

Definition 2.4.1 (Desingularizing function). For $\eta > 0$, we consider

$$\kappa(0, \eta) \stackrel{\text{def}}{=} \{\psi : C^0([0, \eta[) \cap C^1(]0, \eta]) \rightarrow \mathbb{R}_+, \psi' > 0 \text{ on }]0, \eta[, \psi(0) = 0, \text{ and } \psi \text{ concave}\}.$$

Remark 2.4.2. The concavity property of the functions in $\kappa(0, \eta)$ does not appear in the original definition and is added here as it is required for the analysis of global convergence of discrete algorithms; see *e.g.* Subsection 7.2.2-7.3.2.

¹This again highlights the well-known duality between strong convexity and Lipschitz smoothness.

Since we will work either in the convex non-smooth setting or in the non-convex smooth settings, we will present the KL theory adjusted to these contexts rather than the most general version. This approach is taken to avoid introducing more general subdifferentials (Fréchet and limiting ones), which will not be needed in this thesis.

Convex non-smooth case.

Definition 2.4.3 (Kurdyka-Łojasiewicz inequality). Let $f \in \Gamma_0(\mathbb{H})$. f is said to satisfy the KL inequality at $\bar{x} \in \mathbb{H}$ if there exists $r, \eta > 0$ and $\psi \in \kappa(0, \eta)$, such that

$$\psi'(f(x) - f(\bar{x}))\|\partial f(x)\|_- \geq 1, \quad \forall x \in [f(\bar{x}) < f < f(\bar{x}) + \eta]. \quad (2.8)$$

When $\mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f)$ is non-empty, we will say that f satisfies the KL inequality on \mathcal{S} if there exists $r, \eta > 0$ and $\psi \in \kappa(0, \eta)$, such that

$$\psi'(f(x) - \min f)\|\partial f(x)\|_- \geq 1, \quad \forall x \in [\min f < f < \min f + \eta]. \quad (2.9)$$

A notable particular case of this inequality is the known Łojasiewicz inequality, introduced in [136, 137, 138].

Definition 2.4.4 (Łojasiewicz inequality). A function $f \in \Gamma_0(\mathbb{H})$ will satisfy the Łojasiewicz inequality on \mathcal{S} with exponent $q \in]0, 1]$ if f satisfies the KL inequality on \mathcal{S} with $\psi(s) = \frac{1}{\mu(1-q)}s^{1-q}$ for some $\mu > 0$, *i.e.*, if there exists $r > \min f$ such that:

$$\mu(f(x) - \min f)^q \leq \|\partial f(x)\|_-, \quad \forall x \in [\min f < f < r], \quad (2.10)$$

and we will write $f \in \mathbb{L}^q(\mathcal{S})$.

Error bounds have also been successfully applied to various branches of optimization, and in particular to complexity analysis, see [166]. Of particular interest in our setting is the Hölderian error bound.

Definition 2.4.5 (Hölderian error bound). Let $f \in \Gamma_0(\mathbb{H})$. f satisfies a Hölderian (or power-type) error bound inequality on \mathcal{S} with exponent $p \geq 1$, if there exists $\varrho > 0$ and $r > \min f$ such that:

$$f(x) - \min f \geq \varrho \text{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f < f < r], \quad (2.11)$$

and we will write $f \in \text{EB}^p(\mathcal{S})$,

For convex coercive functions, the Hölderian error bound can be globalized, though with a different growth function; see [59, Theorem 3]. Moreover, a deep result due to Łojasiewicz states that for arbitrary continuous semi-algebraic functions, the Hölderian error bound inequality holds on any compact set, and the Łojasiewicz inequality holds at each point; see [136, 137]. In fact, for convex (possibly non-smooth) functions, the Łojasiewicz property and Hölderian error bound are actually equivalent.

Proposition 2.4.6. *Assume that $f \in \Gamma_0(\mathbb{H})$, let $q \in [0, 1[$, $p \stackrel{\text{def}}{=} \frac{1}{1-q} \geq 1$ and $r > \min f$. Then f verifies the Łojasiewicz inequality on \mathcal{S} (2.10) with exponent q at $[\min f < f < r]$ if and only if the Hölderian error bound on \mathcal{S} (2.11) with exponent p holds on $[\min f < f < r]$.*

Proof. Combine [59, Lemma 4 and Theorem 5]. □

The "Polyak-Łojasiewicz"² inequality, widely used in machine learning, is actually a special case of the Łojasiewicz property with exponent $q = \frac{1}{2}$ and is commonly used to prove linear convergence of gradient descent algorithms.

²Though we are not sure that Polyak should be acknowledged for this inequality, and there is a debate exists in the community, there is a consensus in the optimization community that Łojasiewicz is clearly and provably the first to formalize it. Nevertheless, we will still use the acronym PL.

Definition 2.4.7. Let $f \in \Gamma_0(\mathbb{H})$, then f satisfies the PL inequality on \mathcal{S} , if there exists $r > \min f$ and $\mu > 0$ such that

$$2\mu(f(x) - \min f) \leq \|\partial f(x)\|_-^2, \quad \forall x \in [\min f < f < r], \quad (2.12)$$

and we will write $f \in \text{PL}_\mu(\mathcal{S})$.

Remark 2.4.8. If f is μ -strongly convex, then $f \in \text{PL}_\mu(\mathcal{S})$ holds on the entire space (*i.e.* $r = +\infty$); see Proposition 2.2.3.

Remark 2.4.9. According to Proposition 2.4.6, if $f \in \Gamma_0(\mathbb{H}) \cap \text{PL}_\mu(\mathcal{S})$, then f satisfies a Hölderian error bound inequality on \mathcal{S} with exponent $p = 2$.

Non-convex smooth case. Here we stick with the finite-dimensional vase as it is the one that will be used in Chapter 7.

Definition 2.4.10. If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is differentiable and satisfies the KL inequality at $\bar{x} \in \mathbb{R}^d$, then there exists $r, \eta > 0$ and $\psi \in \kappa(0, \eta)$, such that

$$\psi'(f(x) - f(\bar{x}))\|\nabla f(x)\| \geq 1, \quad \forall x \in B(\bar{x}, r) \cap [f(\bar{x}) < f < f(\bar{x}) + \eta]. \quad (2.13)$$

The particular cases of Łojasiewicz inequality is analogous to before.

2.5 The bounded Hausdorff distance

For any $\rho \geq 0$ and any set $C \subset \mathbb{H}$

$$C_\rho \stackrel{\text{def}}{=} C \cap \rho\mathbb{B}.$$

For two sets $C, D \subset \mathbb{H}$, the excess function of C on D is defined as

$$e(C, D) \stackrel{\text{def}}{=} \sup_{x \in C} \text{dist}(x, D),$$

with the natural convention that $e(\emptyset, D) = 0$.

Following [39], for any $\rho \geq 0$, the ρ -Hausdorff distance between C and D is defined as

$$\text{haus}_\rho(C, D) \stackrel{\text{def}}{=} \max(e(C_\rho, D), e(D_\rho, C)).$$

Clearly, for $\rho = +\infty$, we recover the Hausdorff distance. A convergence notion and more precisely a metrizable topology is naturally attached to the ρ -Hausdorff distance. When \mathbb{H} is finite dimensional, the convergence with respect to the ρ -Hausdorff distances is nothing but the classical Painlevé-Kuratowski set-convergence.

To study stability of minimization problems, the authors in [39] introduced the ρ -Hausdorff distance between functions by identifying them with their epigraphs.

Definition 2.5.1. For $\rho \geq 0$, the ρ -Hausdorff (epi-)distance between two functions $f, g : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\}$ is

$$\text{haus}_\rho(f, g) \stackrel{\text{def}}{=} \text{haus}_\rho(\text{epi } f, \text{epi } g),$$

where the unit ball is the box norm on $\mathbb{H} \times \mathbb{R}$.

This device was extended in [36] to set-valued operators by identifying them with their graphs.

Definition 2.5.2. For $\rho \geq 0$, the ρ -Hausdorff distance between two operators $A, B : H \rightrightarrows H$ is

$$\text{haus}_\rho(A, B) \stackrel{\text{def}}{=} \text{haus}_\rho(\text{gph } f, \text{gph } g),$$

where the unit ball is that of $\mathbb{H} \times \mathbb{H}$ equipped with the box norm.

2.6 Other deterministic technical results

In this section we present some technical deterministic results we will need throughout this thesis, we will omit the details of lemmas that are straightforward to prove.

Lemma 2.6.1. *Let $a, b \in \mathbb{R}$ and $x, y \in \mathbb{H}$, then*

$$\|ax - by\| \leq \max\{|a|, |b|\}\|x - y\| + |a - b| \max\{\|x\|, \|y\|\}.$$

Lemma 2.6.2. *Let $t_0 \geq 0$ and $a, b : [t_0, +\infty[\rightarrow \mathbb{R}_+$. If $\lim_{t \rightarrow +\infty} a(t)$ exists, $b \notin L^1([t_0, +\infty[)$ and $\int_{t_0}^{\infty} a(s)b(s)ds < +\infty$, then $\lim_{t \rightarrow +\infty} a(t) = 0$.*

Lemma 2.6.3. *Let $q : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be a non-decreasing differentiable function, if $q \notin L^1([t_0, +\infty[)$, then $\frac{q'}{q} \notin L^1([t_0, +\infty[)$*

Proof. By definition,

$$\int_{t_0}^{+\infty} \frac{q'(s)}{q(s)} ds = \lim_{t \rightarrow +\infty} \ln(q(t)) - \ln(q(t_0)) = +\infty.$$

□

Lemma 2.6.4. *Take $t_0 > 0$, and let $f \in L^1([t_0, +\infty[)$ be continuous. Consider a non-decreasing function $\varphi : [t_0, +\infty[\rightarrow \mathbb{R}_+$ such that $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$. Then $\lim_{t \rightarrow +\infty} \frac{1}{\varphi(t)} \int_{t_0}^t \varphi(s)f(s)ds = 0$.*

Proof. See the proof in [33, Lemma A.5]

□

Lemma 2.6.5. *Let $t_0 > 0$, and $a, b : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be two functions such that $a \notin L^1([t_0, +\infty[)$, $\lim_{u \rightarrow +\infty} b(u) = 0$, and define $A(t) \stackrel{\text{def}}{=} \int_{t_0}^t a(u)du$ and $B(t) \stackrel{\text{def}}{=} e^{-A(t)} \int_{t_0}^t a(u)e^{A(u)}b(u)du$. Then $\lim_{t \rightarrow +\infty} B(t) = 0$.*

Proof. Let $\varepsilon > 0$ arbitrary, let us take T_ε such that $t_0 < T_\varepsilon$ and $b(u) \leq \varepsilon$ for $u \geq T_\varepsilon$. For $t > T_\varepsilon$, let us write

$$\begin{aligned} B(t) &= e^{-A(t)} \int_{t_0}^{T_\varepsilon} a(u)e^{A(u)}b(u)du + e^{-A(t)} \int_{T_\varepsilon}^t a(u)e^{A(u)}b(u)du \\ &\leq e^{-A(t)} \int_{t_0}^{T_\varepsilon} a(u)e^{A(u)}b(u)du + \varepsilon. \end{aligned}$$

Since $a \notin L^1([t_0, +\infty[)$, then $\lim_{t \rightarrow +\infty} e^{-A(t)} = 0$, we get

$$\limsup_{t \rightarrow +\infty} B(t) \leq \varepsilon.$$

This being true for any $\varepsilon > 0$, we infer that $\lim_{t \rightarrow +\infty} B(t) = 0$, which gives the claim. □

The next result is an adaptation of [143, Proposition 2.3] to our specific context but under slightly less stringent assumptions.

Lemma 2.6.6 (Comparison Lemma). *Let $t_0 \geq 0$ and $T > t_0$. Assume that $h : [t_0, +\infty[\rightarrow \mathbb{R}_+$ is measurable with $h \in L^1([t_0, T])$, that $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and non-decreasing, $\varphi_0 > 0$ and the Cauchy problem*

$$\begin{cases} \varphi'(t) = -\psi(\varphi(t)) + h(t) & \text{for almost all } t \in [t_0, T] \\ \varphi(t_0) = \varphi_0 \end{cases}$$

has an absolutely continuous solution $\varphi : [t_0, T] \rightarrow \mathbb{R}_+$. If a lower semicontinuous function $\omega : [t_0, T] \rightarrow \mathbb{R}_+$ is bounded from below and satisfies

$$\omega(t) \leq \omega(s) - \int_s^t \psi(\omega(\tau))d\tau + \int_s^t h(\tau)d\tau$$

for $t_0 \leq s < t \leq T$ and $\omega(t_0) = \varphi_0$, then

$$\omega(t) \leq \varphi(t) \quad \text{for } t \in [t_0, T].$$

Lemma 2.6.7. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $\liminf_{t \rightarrow +\infty} f(t) \neq \limsup_{t \rightarrow +\infty} f(t)$. Then, for every α satisfying $\liminf_{t \rightarrow +\infty} f(t) < \alpha < \limsup_{t \rightarrow +\infty} f(t)$, and for every $\beta > 0$, we can define a sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that*

$$f(t_k) > \alpha, \quad t_{k+1} > t_k + \beta, \quad \forall k \in \mathbb{N}.$$

Proof. Since $\liminf_{t \rightarrow +\infty} f(t)$ and $\limsup_{t \rightarrow +\infty} f(t)$ are different real numbers, α in the lemma obviously exists. Moreover, by definition of \limsup , there exists a sequence $(t_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$ and $f(t_k) > \alpha$. Let $\beta > 0$ and $n_0 = 0$, let us define recursively for $j \geq 1$, $n_j = \min\{n > n_{j-1} : t_n - t_{n_{j-1}} > \beta\}$. Let $j' \in \mathbb{N}$ be the first natural number such that $n_{j'} = +\infty$. This implies that for every $n > n_{j'-1}$, $t_n \leq \beta + t_{n_{j'-1}} < +\infty$, a contradiction since $\lim_{n \rightarrow +\infty} t_n = +\infty$, then for every $j \in \mathbb{N}$, $n_j < +\infty$. Thus, we can define $(t_{n_j})_{j \in \mathbb{N}}$ a subsequence of $(t_k)_{k \in \mathbb{N}}$ such that $\lim_{j \rightarrow +\infty} t_{n_j} = +\infty$ and for every $j \in \mathbb{N}$, $t_{n_{j+1}} - t_{n_j} > \beta$. \square

Lemma 2.6.8. *For $a, x > 0$, let us define the upper incomplete Gamma function as:*

$$\Gamma_{inc}(a; x) = \int_x^{+\infty} s^{a-1} e^{-s} ds.$$

Then, the following holds:

- (i) $x^{1-a} e^x \Gamma_{inc}(a; x) \leq 1$ for $0 < a \leq 1$.
- (ii) $x^{1-a} e^x \Gamma_{inc}(a; x) \geq 1$ for $a \geq 1$.
- (iii) $\lim_{x \rightarrow +\infty} x^{1-a} e^x \Gamma_{inc}(a; x) = 1$

Proof. See [160, Section 8]. \square

Lemma 2.6.9. *Let us define $p > 0$ and $I_p(t) \stackrel{\text{def}}{=} \int_0^1 e^{-tu} (1-u)^p du$. Then*

- (i) $I_p(t) \leq t^{-1}$ for every $t > 0$.
- (ii) $I_p(t) \sim t^{-1}$ as $t \rightarrow +\infty$.

Proof. The first result comes from bounding the term $(1-u)^p$ by 1 in the integral, then we can notice directly that $I_p(t) \leq t^{-1}$ for every $t > 0$. The second result is an application of Watson's Lemma (see [195]). \square

2.7 Stochastic and measure-theoretic results

Let us recall some elements of stochastic analysis; for a more complete account, we refer to [159, 167, 141, 101]. Throughout the thesis, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{\mathcal{F}_t | t \geq 0\}$ is a filtration of the σ -algebra \mathcal{F} . Given $\mathcal{C} \subseteq \Omega$, we will denote $\sigma(\mathcal{C})$ the σ -algebra generated by \mathcal{C} . We denote $\mathcal{F}_{+\infty} \stackrel{\text{def}}{=} \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right) \in \mathcal{F}$.

The expectation of an \mathbb{H} -valued random variable $\xi : \Omega \rightarrow \mathbb{H}$ is denoted by

$$\mathbb{E}(\xi) \stackrel{\text{def}}{=} \int_{\Omega} \xi(\omega) d\mathbb{P}(\omega).$$

As said in Section 2.1, for $1 \leq p \leq +\infty$, $L^p(\Omega; \mathbb{H})$ is the space of \mathbb{H} -valued random variables ξ such that $\mathbb{E}(\|\xi\|^p) < +\infty$, with the usual adaptation when $p = +\infty$.

An event $E \in \mathcal{F}$ happens almost surely if $\mathbb{P}(E) = 1$, and it will be denoted as " E , \mathbb{P} -a.s." or simply " E , a.s.". The characteristic function of an event $E \in \mathcal{F}$ is denoted by

$$\mathbb{1}_E(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $t_0 \geq 0$. An \mathbb{H} -valued stochastic process is a function $X : \Omega \times [t_0, +\infty[\rightarrow \mathbb{H}$. It is said to be continuous if $X(\omega, \cdot) \in C([t_0, +\infty[; \mathbb{H})$ for almost all $\omega \in \Omega$. We will denote $X(t) \stackrel{\text{def}}{=} X(\cdot, t)$.

We are going to study stochastic differential equations, and in order to ensure the uniqueness of a solution, we introduce a relation over stochastic processes. Let $T > t_0$, two stochastic processes $X, Y : \Omega \times [t_0, T] \rightarrow \mathbb{H}$ are said to be equivalent if $X(t) = Y(t), \forall t \in [t_0, T], \mathbb{P}$ -a.s.. This leads us to define the equivalence relation \mathcal{R} , which associates the equivalent stochastic processes in the same class.

Furthermore, we will need some properties about the measurability of these processes. A stochastic process $X : \Omega \times [t_0, +\infty[\rightarrow \mathbb{H}$ is progressively measurable if for every $t \geq t_0$, the map $\Omega \times [t_0, t] \rightarrow \mathbb{H}$ defined by $(\omega, s) \rightarrow X(\omega, s)$ is $\mathcal{F}_t \otimes \mathcal{B}([t_0, t])$ -measurable, where \otimes is the product σ -algebra and \mathcal{B} is the Borel σ -algebra. On the other hand, X is \mathcal{F}_t -adapted if $X(t)$ is \mathcal{F}_t -measurable for every $t \geq t_0$. It is a direct consequence of the definition that if X is progressively measurable, then X is \mathcal{F}_t -adapted.

For $T > t_0$, let us define the quotient space:

$$S_{\mathbb{H}}^0[t_0, T] \stackrel{\text{def}}{=} \{X : \Omega \times [t_0, T] \rightarrow \mathbb{H} : X \text{ is a prog. measurable cont. stochastic process}\} / \mathcal{R}.$$

We set $S_{\mathbb{H}}^0[t_0] \stackrel{\text{def}}{=} \bigcap_{T \geq t_0} S_{\mathbb{H}}^0[t_0, T]$, and $S_{\mathbb{H}}^0 \stackrel{\text{def}}{=} S_{\mathbb{H}}^0[0]$. Furthermore, for $\nu > 0$, we define $S_{\mathbb{H}}^\nu[t_0, T]$ as the subset of processes $X(t)$ in $S_{\mathbb{H}}^0[t_0, T]$ such that

$$S_{\mathbb{H}}^\nu[t_0, T] \stackrel{\text{def}}{=} \{X \in S_{\mathbb{H}}^0[t_0, T] : \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t)\|^\nu \right) < +\infty\}.$$

We define $S_{\mathbb{H}}^\nu[t_0] \stackrel{\text{def}}{=} \bigcap_{T \geq t_0} S_{\mathbb{H}}^\nu[t_0, T]$, and $S_{\mathbb{H}}^\nu \stackrel{\text{def}}{=} S_{\mathbb{H}}^\nu[0]$.

Let $I \subseteq \mathbb{N}$ be a numerable set such that $\{e_i\}_{i \in I}$ is an orthonormal basis of \mathbb{K} , and $\{w_i(t)\}_{i \in I, t \geq t_0}$ be a sequence of independent Brownian motions defined on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. The process

$$W(t) = \sum_{i \in I} w_i(t) e_i$$

is well-defined (independent from the election of $\{e_i\}_{i \in I}$) and is called a \mathbb{K} -valued Brownian motion. Besides, let $G : \Omega \times [t_0, +\infty[\rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ be a measurable and \mathcal{F}_t -adapted process, then we can define $\int_{t_0}^t G(s) dW(s)$ which is the stochastic integral of G , and we have that $G \rightarrow \int_{t_0}^\cdot G(s) dW(s)$ is an isometry between the measurable and \mathcal{F}_t -adapted $\mathcal{L}_2(\mathbb{K}; \mathbb{H})$ -valued processes and the space of \mathbb{H} -valued continuous square-integrable martingales (see [101, Theorem 2.3]).

Theorem 2.7.1 (Egorov's Theorem). [182, Chapter 3, Exercise 16] *Let (X, \mathcal{A}, μ) be a measure space with $\mu(X) < +\infty$, and $(f_t)_{t \in \mathbb{R}_+}$ is a family of real measurable functions such that for μ -almost all $x \in X$:*

1. $\lim_{t \rightarrow +\infty} f_t(x) = f(x)$, and
2. $t \mapsto f_t(x)$ is continuous.

Then, for every $\delta > 0$, there exists a measurable set $E_\delta \subset X$, with $\mu(X \setminus E_\delta) < \delta$, such that $(f_t)_{t \in \mathbb{R}_+}$ converges uniformly on E_δ .

Lemma 2.7.2. *Let $\delta > 0, \Omega_\delta \in \mathcal{F}$ such that $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$ and $h : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ a stochastic process such that $\sup_{t \geq 0} \mathbb{E}[h(t)^2] < +\infty$. Then, there exists a constant $C_h > 0$ (independent of δ) such that*

$$\mathbb{E}[h(t) \mathbb{1}_{\Omega \setminus \Omega_\delta}] \leq C_h \sqrt{\delta}.$$

Proof. Note that $\mathbb{P}(\Omega \setminus \Omega_\delta) \leq \delta$ and thus Cauchy-Schwarz inequality gives:

$$\mathbb{E}[h(t) \mathbb{1}_{\Omega \setminus \Omega_\delta}] = \int_{\Omega} h(\omega, t) \mathbb{1}_{\Omega \setminus \Omega_\delta}(\omega) d\mathbb{P}(\omega) \leq \sqrt{\delta} \sqrt{\mathbb{E}[h(t)^2]} \leq \underbrace{\sqrt{\delta} \sqrt{\sup_{t \geq 0} \mathbb{E}[h(t)^2]}}_{\stackrel{\text{def}}{=} C_h}.$$

□

Let us consider $t_0 \geq 0, \nu \geq 2$ and the SDE with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ which is \mathcal{F}_{t_0} -measurable:

$$\begin{cases} dX(t) = F(t, X(t))dt + G(t, X(t))dW(t), & t \geq 0, \\ X(t_0) = X_0, \end{cases} \quad (2.14)$$

where $F : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathbb{H}$, $G : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ are measurable functions and W is an \mathcal{F}_t -adapted \mathbb{K} -valued Brownian Motion.

Definition 2.7.3. A solution of (2.14) is a process $X \in S_{\mathbb{H}}^0[t_0]$ such that:

$$X(t) = X_0 + \int_{t_0}^t F(s, X(s))ds + \int_{t_0}^t G(s, X(s))dW(s), \quad a.s., \forall t \geq t_0.$$

Theorem 2.7.4. (See [101, Theorem 3.3 and Theorem 3.5]) Let $F : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathbb{H}$ and $G : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ be measurable functions satisfying

$$\sup_{t \geq t_0} (\|F(t, 0)\| + \|G(t, 0)\|_{\text{HS}}) < +\infty, \quad (2.15)$$

and for every $T > t_0$ and some constant $C_1 \geq 0$,

$$\|F(t, x) - F(t, y)\| + \|G(t, x) - G(t, y)\|_{\text{HS}} \leq C_1 \|x - y\|, \quad \forall x, y \in \mathbb{H}, \forall t \in [t_0, T]. \quad (2.16)$$

Then (2.14) has a unique solution $X \in S_{\mathbb{H}}^\nu[t_0]$.

Let us now present Itô's formula which plays a central role in the theory of stochastic differential equations.

Theorem 2.7.5. [101, Theorem 2.9] (Itô's Formula) Let $X(t)$ be a solution of (2.14) and assume that $\phi : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathbb{R}$ is such that ϕ is continuous and its Fréchet partial derivatives $\frac{\partial \phi}{\partial t}, \nabla \phi, \nabla^2 \phi$ are continuous and bounded on bounded sets of $[t_0, +\infty[\times \mathbb{H}$. Then, the following Itô's formula holds:

$$\begin{aligned} \phi(t, X(t)) &= \phi(t_0, X_0) + \int_{t_0}^t \frac{\partial \phi}{\partial t}(s, X(s))ds \\ &+ \int_{t_0}^t \langle \nabla \phi(s, X(s)), F(s, X(s)) \rangle ds + \int_{t_0}^t \langle \nabla \phi(s, X(s)), G(s) dW(s) \rangle \\ &+ \frac{1}{2} \int_{t_0}^t \text{tr}[\nabla^2 \phi(s, X(s))G(s, X(s))G^*(s, X(s))]ds, \quad a.s., \quad \forall t \geq t_0. \end{aligned}$$

The C^2 assumption on $\phi(t, \cdot)$ in Itô's formula is crucial. However, this can be weakened in certain cases leading to the following inequality that will be useful in our context.

Proposition 2.7.6. Consider \mathbb{H} a finite-dimensional space and X a solution of (2.14), $\phi_1 \in C^1([t_0, +\infty[)$, $\phi_2 \in C_L^{1,1}(\mathbb{H})$ and $\phi(t, x) = \phi_1(t)\phi_2(x)$. Then the process

$$Y(t) = \phi(t, X(t)) = \phi_1(t)\phi_2(X(t)),$$

is an Itô Process such that

$$\begin{aligned} Y(t) &\leq Y(t_0) + \int_{t_0}^t \phi_1'(s)\phi_2(X(s))ds + \int_{t_0}^t \phi_1(s) \langle \nabla \phi_2(X(s)), F(s, X(s)) \rangle ds \\ &+ \int_{t_0}^t \langle \sigma^*(s, X(s))\phi_1(s)\nabla \phi_2(X(s)), dW(s) \rangle + \frac{L}{2} \int_{t_0}^t \phi_1(s)\text{tr}(G(s, X(s))G^*(s, X(s))) ds. \end{aligned} \quad (2.17)$$

Moreover, if $\mathbb{E}[Y(t_0)] < +\infty$, and if for all $T > t_0$

$$\mathbb{E} \left(\int_{t_0}^T \|G^*(s, X(s))\phi_1(s)\nabla \phi_2(X(s))\|^2 ds \right) < +\infty,$$

$$\begin{aligned} \mathbb{E}[Y(t)] &\leq \mathbb{E}[Y(t_0)] + \mathbb{E} \left(\int_{t_0}^t \phi_1'(s) \phi_2(X(s)) ds \right) + \mathbb{E} \left(\int_{t_0}^t \phi_1(s) \langle \nabla \phi_2(X(s)), F(s, X(s)) \rangle ds \right) \\ &\quad + \frac{L}{2} \mathbb{E} \left(\int_{t_0}^t \phi_1(s) \text{tr} (G(s, X(s)) G^*(s, X(s))) ds \right). \end{aligned} \quad (2.18)$$

Proof. Analogous to the proof of [149, Proposition C.2] using Rademacher's theorem instead of Alexandrov's. \square

2.8 On martingales

Considering \mathbb{H} , a real separable Hilbert space, the following theorem extends Doob's martingale convergence theorem, originally stated for \mathbb{R} -valued martingales and usually generalized to \mathbb{R}^d -valued martingales, to \mathbb{H} -valued martingales. Since we could not find a direct proof of this generalization, we propose an independent proof inspired by the ideas presented in [103].

Theorem 2.8.1. *Let \mathbb{H} be a real separable Hilbert space and $(M_t)_{t \geq 0} : \Omega \rightarrow \mathbb{H}$ be a continuous martingale such that $\sup_{t \geq 0} \mathbb{E}(\|M_t\|^2) < +\infty$. Then there exists a \mathbb{H} -valued random variable $M_\infty \in L^2(\Omega; \mathbb{H})$ such that $\lim_{t \rightarrow +\infty} \|M_t - M_\infty\| = 0$ a.s..*

Proof. Consider $(M_k)_{k \in \mathbb{N}}$ to be the embedded discrete parameter martingale. Since $\sup_{k \in \mathbb{N}} \mathbb{E}\|M_k\|^2 < +\infty$, then $(M_k)_{k \in \mathbb{N}}$ is uniformly integrable and by [183, Theorem 3], there exists a measurable \mathbb{H} -valued random variable $M_\infty \in L^2(\Omega; \mathbb{H})$ such that $\lim_{k \rightarrow +\infty} \|M_k - M_\infty\| = 0$ a.s.. In turn, using the dominated convergence theorem (see [182, Theorem 1.34]), we also have

$$\lim_{k \rightarrow +\infty} \mathbb{E}(\|M_k - M_\infty\|^2) = 0. \quad (2.19)$$

The rest of the proof is inspired by the arguments in the proof of [103, Theorem 2.2].

We consider an arbitrary $k \in \mathbb{N}^*$ and $\delta > 0$. Since $(M_{t+k} - M_k)_{t \geq 0}$ is also a \mathbb{H} -valued martingale, we can use Doob's maximal inequalities for \mathbb{H} -valued martingales shown in [101, Theorem 2.2], which gives us

$$\delta^2 \mathbb{P} \left(\sup_{s \in [0, t]} \|M_{s+k} - M_k\| > \delta \right) \leq \mathbb{E}(\|M_{t+k} - M_k\|^2). \quad (2.20)$$

Let $n \in \mathbb{N}^*$ be arbitrary. We have

$$\mathbb{P} \left(\sup_{s \in \mathbb{Q} \cap [0, n]} \|M_{s+k} - M_\infty\| > \delta \right) \leq \mathbb{P} \left(\sup_{s \in \mathbb{Q} \cap [0, n]} \|M_{s+k} - M_k\| > \frac{\delta}{2} \right) + \mathbb{P} \left(\|M_k - M_\infty\| > \frac{\delta}{2} \right).$$

Using (2.20) and Markov's inequality, we get the bound

$$\begin{aligned} \delta^2 \mathbb{P} \left(\sup_{s \in \mathbb{Q} \cap [0, n]} \|M_{s+k} - M_\infty\| > \delta \right) &\leq 4\mathbb{E}(\|M_{n+k} - M_k\|^2) + 4\mathbb{E}(\|M_k - M_\infty\|^2) \\ &\leq 8\mathbb{E}(\|M_{n+k} - M_\infty\|^2) + 12\mathbb{E}(\|M_k - M_\infty\|^2). \end{aligned} \quad (2.21)$$

In turn, we get

$$\begin{aligned} \delta^2 \mathbb{P} \left(\sup_{s \in \mathbb{Q}, s \geq k} \|M_s - M_\infty\| > \delta \right) &\leq \delta^2 \mathbb{P} \left(\bigcup_{n \in \mathbb{N}^*} \left\{ \sup_{s \in \mathbb{Q} \cap [0, n]} \|M_{s+k} - M_\infty\| > \delta \right\} \right) \\ &\leq \delta^2 \liminf_{n \rightarrow +\infty} \mathbb{P} \left(\sup_{s \in \mathbb{Q} \cap [0, n]} \|M_{s+k} - M_\infty\| > \delta \right) \\ &\leq 12\mathbb{E}(\|M_k - M_\infty\|^2), \end{aligned}$$

where we have used (2.21) and in the last inequality, that $\lim_{n \rightarrow +\infty} \mathbb{E}(\|M_{n+k} - M_\infty\|^2) = 0$ by (2.19). Taking $k \rightarrow +\infty$, and using again (2.19), we conclude that for all $\delta > 0$

$$\lim_{k \rightarrow +\infty} \mathbb{P} \left(\sup_{s \in \mathbb{Q}, s \geq k} \|M_s - M_\infty\| > \delta \right) = 0.$$

For $k \in \mathbb{N}^*$, we define $A_k \stackrel{\text{def}}{=} \{\omega \in \Omega : \sup_{s \in \mathbb{Q}, s \geq k} \|M_s(\omega) - M_\infty(\omega)\| > \delta\}$, since $(A_k)_{k \in \mathbb{N}^*}$ is a non-increasing sequence of sets:

$$0 = \lim_{k \rightarrow +\infty} \mathbb{P}(A_k) = \mathbb{P} \left(\bigcap_{k \in \mathbb{N}^*} A_k \right).$$

Defining for $l \geq 0$, $D_l = \{\omega \in \Omega : \|M_l(\omega) - M_\infty(\omega)\| > \delta\}$, it is direct to check that $\bigcup_{l \geq k, l \in \mathbb{Q}} D_l \subseteq A_k$ for every $k \in \mathbb{N}^*$. Therefore, we obtain that

$$\mathbb{P} \left(\bigcap_{k \in \mathbb{N}^*} \bigcup_{l \geq k, l \in \mathbb{Q}} D_l \right) = 0,$$

which is equivalent to $\lim_{s \rightarrow +\infty, s \in \mathbb{Q}} \|M_t - M_\infty\| = 0$ a.s.. The result follows from classical arguments of continuity of the martingale. \square

Theorem 2.8.2. [141, Theorem 1.3.9] *Let $\{A_t\}_{t \geq 0}$ and $\{U_t\}_{t \geq 0}$ be two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s.. Let $\{M_t\}_{t \geq 0}$ be a real valued continuous local martingale with $M_0 = 0$ a.s.. Let ξ be a nonnegative \mathcal{F}_0 -measurable random variable. Define*

$$X_t = \xi + A_t - U_t + M_t \quad \text{for } t \geq 0.$$

If X_t is nonnegative and $\lim_{t \rightarrow +\infty} A_t < +\infty$ a.s., then a.s. $\lim_{t \rightarrow +\infty} X_t$ exists and is finite, and $\lim_{t \rightarrow +\infty} U_t < +\infty$.

Proposition 2.8.3. (see [71] and [178, Section 1.2]) (Burkholder-Davis-Gundy Inequality) *Let $p > 0$, W be a \mathbb{K} -valued Brownian motion defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and $g : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{K}$ a progressively measurable process (with our usual notation $g(t) \stackrel{\text{def}}{=} g(\cdot, t)$) such that*

$$\mathbb{E} \left[\left(\int_0^T \|g(s)\|^2 ds \right)^{\frac{p}{2}} \right] < +\infty, \quad \forall T > 0.$$

Then, there exists $C_p > 0$ (only depending on p) for every $T > 0$ such that:

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_0^t \langle g(s), dW(s) \rangle \right|^p \right] \leq C_p \mathbb{E} \left[\left(\int_0^T \|g(s)\|^2 ds \right)^{\frac{p}{2}} \right].$$

Part I

First-Order Systems

Chapter 3

A First-Order SDE Perspective on Stochastic Convex Optimization

In this chapter, we analyze the global and local behavior of gradient-like flows under stochastic errors towards the aim of solving convex optimization problems with noisy gradient input. We first study the unconstrained differentiable convex case, using a stochastic differential equation where the drift term is minus the gradient of the objective function and the diffusion term is either bounded or square-integrable. In this context, under Lipschitz continuity of the gradient, our first main result shows almost sure weak convergence of the trajectory process towards a minimizer of the objective function. We also provide a comprehensive complexity analysis by establishing several new pointwise and ergodic convergence rates in expectation for the convex, strongly convex, and (local) Łojasiewicz case. The latter, which involves local analysis, is challenging and requires non-trivial arguments from measure theory. Then, we extend our study to the constrained case and more generally to certain non-smooth situations. We show that several of our results have natural extensions obtained by replacing the gradient of the objective function by a cocoercive monotone operator. This makes it possible to obtain similar convergence results for optimization problems with an additively “smooth + non-smooth” convex structure. Finally, we consider another extension of our results to non-smooth optimization which is based on the Moreau envelope.

Main contributions of this chapter

- ▶ Almost sure weak convergence of the trajectory generated by the SDE to the set of minimizers (Theorem 3.2.3).
- ▶ Global convergence rates in expectation and in almost sure sense of the SDE under convexity and strong convexity of the objective (Theorem 3.2.5, Proposition 3.2.6).
- ▶ Local convergence rates of the values in expectation under Łojasiewicz Inequality (Theorem 3.3.3).
- ▶ Extension of the previous results to the maximal monotone case (Subsection 3.4.1).

The content of this chapter was revised in [144].

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3.1 Introduction

We aim to solve convex minimization problems by means of stochastic differential equations (SDE) whose drift term is driven by the gradient of the objective function. This allows for noisy (inaccurate) gradient input to be taken into account. Consider \mathbb{H} a real separable Hilbert space and the minimization problem

$$\min_{x \in \mathbb{H}} f(x), \quad (\mathcal{P}_1)$$

where the objective f satisfies the following standing assumptions:

$$\begin{cases} f \text{ is convex and continuously differentiable with } L\text{-Lipschitz continuous gradient;} \\ \mathcal{S} \stackrel{\text{def}}{=} \operatorname{argmin}(f) \neq \emptyset. \end{cases} \quad (\mathbf{H}_f)$$

We will also later deal with the constrained case, and more generally with additively structured “smooth + non-smooth” convex optimization.

To solve (\mathcal{P}_1) , a fundamental dynamic to consider is the gradient flow of f , *i.e.* the gradient descent dynamic with initial condition $x_0 \in \mathbb{H}$:

$$\begin{cases} \dot{x}(t) = -\nabla f(x(t)), & t > 0; \\ x(0) = x_0. \end{cases} \quad (\mathbf{GF})$$

It is well known since the founding papers of Brezis, Baillon, Bruck in the 1970s that, if the solution set $\operatorname{argmin}(f)$ of (\mathcal{P}_1) is non-empty, then each solution trajectory of (\mathbf{GF}) converges, and its limit belongs to $\operatorname{argmin}(f)$. In fact, this result is true in a more general setting, simply assuming that the objective function f is convex, lower semicontinuous (lsc) and proper, in which case we must consider the differential inclusion obtained by replacing in (\mathbf{GF}) the gradient of f by the sub-differential ∂f (detailed in Chapter 4).

Consider \mathbb{K} a real separable Hilbert space. To solve (\mathcal{P}_1) , we will refer to Section 1.2 for the approach that leads to the dynamic that we will study in this chapter, specifically the stochastic counterpart of (\mathbf{GF}) , which is the following SDE:

$$\begin{cases} dX(t) = -\nabla f(X(t))dt + \sigma(t, X(t))dW(t), & t > 0; \\ X(0) = X_0, \end{cases} \quad (\mathbf{SDE})$$

defined over a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where the diffusion (volatility) term $\sigma : \mathbb{R}_+ \times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ (see Section 2.1 for notation) is a measurable function that satisfies (\mathbf{H}_σ) , W is a \mathcal{F}_t -adapted \mathbb{K} -valued Brownian motion (see Section 2.7 for a precise definition), and the initial data X_0 is an \mathcal{F}_0 -measurable \mathbb{H} -valued random variable.

3.2 Convergence properties for convex differentiable functions

We consider f (called the potential) and study the dynamic (SDE) under hypotheses (\mathbf{H}_f) (i.e. $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$) and (\mathbf{H}_σ) . Recall the definitions of σ_* and $\sigma_\infty(t)$ from (1.4). Throughout the chapter, we will use the shorthand notation

$$\Sigma(t, x) \stackrel{\text{def}}{=} \sigma(t, x)\sigma(t, x)^*.$$

For the necessary notation and preliminaries on stochastic processes, see Section 2.7.

We emphasize that Theorem 2.7.4 provides us with sufficient conditions to ensure the existence and uniqueness of the solution to (SDE). These conditions are met in our case under assumptions (\mathbf{H}_f) and (\mathbf{H}_σ) .

Let us now present Itô's formula (see Theorem 2.7.5) in the specific case of (SDE).

Proposition 3.2.1. *Consider X a solution of (SDE), $\phi : \mathbb{R}_+ \times \mathbb{H} \rightarrow \mathbb{R}$ be such that $\phi(\cdot, x) \in C^1(\mathbb{R}_+)$ for every $x \in \mathbb{H}$ and $\phi(t, \cdot) \in C^2(\mathbb{H})$ for every $t \geq 0$. Then the process*

$$Y(t) = \phi(t, X(t)),$$

is an Itô Process such that for all $t \geq 0$:

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t \frac{\partial \phi}{\partial t}(s, X(s)) ds - \int_0^t \langle \nabla \phi(s, X(s)), \nabla f(X(s)) \rangle ds \\ &\quad + \int_0^t \langle \sigma^*(s, X(s)) \nabla \phi(s, X(s)), dW(s) \rangle + \frac{1}{2} \int_0^t \text{tr}(\sigma(s, X(s)) \sigma^*(s, X(s)) \nabla^2 \phi(s, X(s))) ds. \end{aligned} \quad (3.1)$$

Moreover, if $\mathbb{E}[Y(0)] < +\infty$, and if for all $T > 0$:

$$\mathbb{E} \left(\int_0^T \|\sigma^*(s, X(s)) \nabla \phi(s, X(s))\|^2 ds \right) < +\infty,$$

then $\int_0^t \langle \sigma^*(s, X(s)) \nabla \phi(s, X(s)), dW(s) \rangle$ is a square-integrable continuous martingale and

$$\begin{aligned} \mathbb{E}[Y(t)] &= \mathbb{E}[Y(0)] + \mathbb{E} \left(\int_0^t \frac{\partial \phi}{\partial t}(s, X(s)) ds \right) - \mathbb{E} \left(\int_0^t \langle \nabla \phi(s, X(s)), \nabla f(X(s)) \rangle ds \right) \\ &\quad + \frac{1}{2} \mathbb{E} \left(\int_0^t \text{tr}(\sigma(s, X(s)) \sigma^*(s, X(s)) \nabla^2 \phi(s, X(s))) ds \right). \end{aligned} \quad (3.2)$$

The C^2 assumption on $\phi(t, \cdot)$ in Itô's formula is crucial. However, this can be weakened in certain cases as shown in Proposition 2.7.6.

3.2.1 Almost sure weak convergence of the trajectory

Recall that our focus in this thesis is on an optimization perspective, and as we argued in the introduction, we will study the long time behavior of our SDE as the diffusion term vanishes when $t \rightarrow +\infty$. Therefore, we recall that we assume that the diffusion (volatility) term σ satisfies:

$$\begin{cases} \sup_{t \geq 0, x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} < +\infty, \\ \|\sigma(t, x') - \sigma(t, x)\|_{\text{HS}} \leq l_0 \|x' - x\|, \end{cases} \quad (\mathbf{H}_\sigma)$$

for some $l_0 > 0$ and for all $t \geq 0, x, x' \in \mathbb{H}$. The Lipschitz continuity assumption is mild and classical and will be required to ensure the well-posedness of (SDE). Let us also recall that we define $\sigma_\infty : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as

$$\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}}.$$

Remark 3.2.2. Under the hypothesis (\mathbf{H}_σ) we have that there exists $\sigma_* > 0$ such that:

$$\|\sigma(t, x)\|_{\text{HS}}^2 = \text{tr}[\Sigma(t, x)] \leq \sigma_*^2,$$

for all $t \geq 0, x \in \mathbb{H}$, where $\Sigma \stackrel{\text{def}}{=} \sigma \sigma^*$.

Our first main result establishes almost sure weak convergence of $X(t)$ to an \mathcal{S} -valued random variable as $t \rightarrow +\infty$.

Theorem 3.2.3. *Consider the dynamic (SDE), where f and σ satisfy the assumptions (\mathbf{H}_f) and (\mathbf{H}_σ) , respectively. Additionally, let $\nu \geq 2$, $X_0 \in L^\nu(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Then, there exists a unique solution $X \in S_{\mathbb{H}}^\nu$ of (SDE). Moreover, if $\sigma_\infty \in L^2(\mathbb{R}_+)$, then the following holds:*

(i) $\mathbb{E}[\sup_{t \geq 0} \|X(t)\|^\nu] < +\infty$.

(ii) $\forall x^* \in \mathcal{S}$, $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s. and $\sup_{t \geq 0} \|X(t)\| < +\infty$ a.s..

(iii) $\lim_{t \rightarrow +\infty} \|\nabla f(X(t))\| = 0$ a.s.. As a result, $\lim_{t \rightarrow +\infty} f(X(t)) = \min f$ a.s..

(iv) In addition to (iii), there exists an \mathcal{S} -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s..

Remark 3.2.4. (i) In finite dimension, the assumptions on the noise variance are compatible with the theory of asymptotic pseudotrajectories (APT) [50] and their weak version (WAPT) [51]. As we have already discussed in the Introduction, the theory of APT can indeed be used to study convergence properties of $X(t)$. For instance, in finite dimension using [50, Proposition 7.4] one can show easily that assuming $\sup_{x \in \mathbb{R}^d} \|\sigma(t, x)\|_F = o(1/\sqrt{\log(t)})$ (where $\|\cdot\|_F$ is the Frobenius norm) and that $X(t)$ is bounded almost surely, one has almost sure subsequential convergence of $X(t)$ to points in \mathcal{S} . Our work leverages convexity, does not need any boundedness assumption and shows almost sure global convergence of the process (not just subsequentially).

(ii) In finite dimension, ergodic properties of $X(t)$ can be derived from the theory of WAPT as developed in [51] under the weaker assumptions that $\|\sigma(x, t)\| \leq \alpha(t)$, for some decreasing function α such that $\alpha(t) \rightarrow 0$ as $t \rightarrow +\infty$. For instance, an immediate consequence of [51, Proposition 1 and Corollary 1] and Theorem 3.2.3(i) is that the fraction of time spent by X in an arbitrary neighborhood of \mathcal{S} goes to one with probability one. We also have by combining [51, Corollary 2] and Theorem 3.2.3(i), and since \mathcal{S} is convex in our case, that the average process $\frac{1}{t} \int_0^t X(s) ds$ converges almost surely to a point in \mathcal{S} .

Proof. The existence and uniqueness of a solution follows directly from the fact that the conditions of Theorem 2.7.4 are satisfied under (\mathbf{H}_f) and (\mathbf{H}_σ) . The architecture of the proof of Theorem 3.2.3 consists of three steps that we briefly describe:

- The first step is based on Itô's formula (Proposition 3.2.1) and Burkholder-Davis-Gundy inequality (Proposition 2.8.3) that let us prove a uniform bound (on time) for the ν -moment of $X(t)$.
- The second step is also based on Itô's formula. Instead of the previous step, we use Theorem 2.8.2 that allows us to conclude that for every $x^* \in \mathcal{S}$, $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s.. Then, a separability argument is used to conclude that almost surely, for all $x^* \in \mathcal{S}$, $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists.
- The third step consists in using another conclusion of Theorem 2.8.2 to conclude that $\|\nabla f(X(\cdot))\|^2 \in L^1(\mathbb{R}_+)$ a.s.. After proving that this function is eventually uniformly continuous, we proceed according to Barbalat's Lemma (see [95]) to conclude that $\lim_{t \rightarrow +\infty} \|\nabla f(X(t))\| = 0$ a.s.. As a consequence of the convexity of f we deduce that $\lim_{t \rightarrow +\infty} f(X(t)) = \min f$ a.s..

- Finally, the fourth step consists in using Opial's Lemma to conclude that there exists an \mathcal{S} -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s..
- (i) Let x^* be taken arbitrarily in \mathcal{S} . Let us define the corresponding anchor function $\phi(x) = \frac{\|x - x^*\|^2}{2}$. Using Itô's formula we obtain

$$\begin{aligned} \phi(X(t)) &= \underbrace{\frac{\|X_0 - x^*\|^2}{2}}_{\xi = \phi(X_0)} + \underbrace{\frac{1}{2} \int_0^t \text{tr}(\Sigma(s, X(s))) ds}_{A_t} - \underbrace{\int_0^t \langle \nabla f(X(s)), X(s) - x^* \rangle ds}_{U_t} \\ &\quad + \underbrace{\int_0^t \langle \sigma^*(s, X(s))(X(s) - x^*), dW(s) \rangle}_{M_t}. \end{aligned} \quad (3.3)$$

Let us now embark from (3.3) and use that

$$0 \leq \text{tr}(\Sigma(s, X(s))) \leq \sigma_\infty^2(s) \quad \text{and} \quad \langle \nabla f(X(s)), X(s) - x^* \rangle \geq 0,$$

where the second inequality is due to the convexity of f , to get

$$\phi(X(t)) \leq \phi(X_0) + \frac{1}{2} \int_0^{+\infty} \sigma_\infty^2(s) ds + M_t.$$

Taking power $\frac{\nu}{2}$ at both sides and using that $(a + b + c)^{\frac{\nu}{2}} \leq 3^{\frac{\nu-2}{2}} (a^{\frac{\nu}{2}} + b^{\frac{\nu}{2}} + c^{\frac{\nu}{2}})$, we have that

$$\|X(t) - x^*\|^\nu \leq 3^{\frac{\nu-2}{2}} \left[\|X_0 - x^*\|^\nu + \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}} + 2^{\frac{\nu}{2}} |M_t|^{\frac{\nu}{2}} \right].$$

Let $T > 0$, applying the supremum over $t \in [0, T]$ and then taking expectation, we obtain

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^\nu \right) \leq 3^{\frac{\nu-2}{2}} \left[\mathbb{E}(\|X_0 - x^*\|^\nu) + \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}} + 2^{\frac{\nu}{2}} \mathbb{E} \left(\sup_{t \in [0, T]} |M_t|^{\frac{\nu}{2}} \right) \right].$$

Letting $g(s) = \sigma^*(s, X(s))(X(s) - x^*)$ and $p = \frac{\nu}{2}$ in Proposition 2.8.3, we get

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^\nu \right) &\leq 3^{\frac{\nu-2}{2}} \left[\mathbb{E}(\|X_0 - x^*\|^\nu) + \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}} \right] \\ &\quad + 3^{\frac{\nu-2}{2}} 2^{\frac{\nu}{2}} C_{\frac{\nu}{2}} \mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^{\frac{\nu}{2}} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{4}} \right). \end{aligned}$$

Using that $ab \leq \frac{a^2}{2K} + \frac{Kb^2}{2}$ for every $K > 0$,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^\nu \right) &\leq 3^{\frac{\nu-2}{2}} \left[\mathbb{E}(\|X_0 - x^*\|^\nu) + \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}} \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^\nu \right) + 6^{\frac{\nu-2}{2}} C_{\frac{\nu}{2}} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}}. \end{aligned}$$

And we end up with

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|X(t) - x^*\|^\nu \right) \leq 3^{\frac{\nu-2}{2}} 2 \left[\mathbb{E}(\|X_0 - x^*\|^\nu) + (1 + 2^{\frac{\nu-2}{2}} C_{\frac{\nu}{2}}) \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^{\frac{\nu}{2}} \right].$$

Since the right-hand side is independent of T , we take $\liminf_{T \rightarrow +\infty}$ on the previous expression and apply Fatou's Lemma to show the first claim.

(ii) Observe that, since $\nu \geq 2$, we have that $\mathbb{E}(\sup_{t \geq 0} \|X(t)\|^2) < +\infty$. Moreover $\sigma_\infty \in L^2(\mathbb{R}_+)$, and therefore

$$\mathbb{E} \left(\int_0^{+\infty} \|\sigma^\star(s, X(s)) (X(s) - x^\star)\|^2 ds \right) \leq \mathbb{E} \left(\sup_{t \geq 0} \|X(t) - x^\star\|^2 \right) \int_0^{+\infty} \sigma_\infty^2(s) ds < +\infty.$$

Therefore M_t is a square-integrable continuous martingale. It is also a continuous local martingale (see [141, Theorem 1.3.3]), which implies that $\mathbb{E}(M_t) = 0$.

On the other hand, A_t and U_t defined as in (3.3) are two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s.. Since $\phi(X(t))$ is nonnegative and $\sup_{x \in \mathbb{H}} \|\sigma(\cdot, x)\|_{\text{HS}} \in L^2(\mathbb{R}_+)$, we deduce that $\lim_{t \rightarrow +\infty} A_t < +\infty$. Then, we can use Theorem 2.8.2 to conclude that

$$\int_0^{+\infty} \langle \nabla f(X(s)), X(s) - x^\star \rangle ds < +\infty \quad a.s. \quad (3.4)$$

and

$$\forall x^\star \in \mathcal{S}, \exists \Omega_{x^\star} \in \mathcal{F}, \text{ such that } \mathbb{P}(\Omega_{x^\star}) = 1 \text{ and } \lim_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| \text{ exists } \forall \omega \in \Omega_{x^\star}. \quad (3.5)$$

Since \mathbb{H} is separable, there exists a countable set $\mathcal{Z} \subseteq \mathcal{S}$, such that $\text{cl}(\mathcal{Z}) = \mathcal{S}$ (where cl stands for the closure of the set). Let $\tilde{\Omega} = \bigcap_{z \in \mathcal{Z}} \Omega_z$. Since \mathcal{Z} is countable, a union bound shows

$$\mathbb{P}(\tilde{\Omega}) = 1 - \mathbb{P} \left(\bigcup_{z \in \mathcal{Z}} \Omega_z^c \right) \geq 1 - \sum_{z \in \mathcal{Z}} \mathbb{P}(\Omega_z^c) = 1.$$

For arbitrary $x^\star \in \mathcal{S}$, there exists a sequence $(z_k)_{k \in \mathbb{N}} \subseteq \mathcal{Z}$ such that $\lim_{k \rightarrow +\infty} z_k = x^\star$. In view of (3.5), for every $k \in \mathbb{N}$ there exists $\tau_k : \Omega_{z_k} \rightarrow \mathbb{R}_+$ such that

$$\lim_{t \rightarrow +\infty} \|X(\omega, t) - z_k\| = \tau_k(\omega), \quad \forall \omega \in \Omega_{z_k}. \quad (3.6)$$

Now, let $\omega \in \tilde{\Omega}$. Since $\tilde{\Omega} \subset \Omega_{z_k}$ for any $k \in \mathbb{N}$, and using the triangle inequality and (3.6), we obtain that

$$\tau_k(\omega) - \|z_k - x^\star\| \leq \liminf_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| \leq \limsup_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| \leq \tau_k(\omega) + \|z_k - x^\star\|.$$

Now, passing to $k \rightarrow +\infty$, we deduce

$$\limsup_{k \rightarrow +\infty} \tau_k(\omega) \leq \liminf_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| \leq \limsup_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| \leq \liminf_{k \rightarrow +\infty} \tau_k(\omega),$$

whence we deduce that $\lim_{k \rightarrow +\infty} \tau_k(\omega)$ exists on the set $\tilde{\Omega}$ of probability 1, and in turn

$$\lim_{t \rightarrow +\infty} \|X(\omega, t) - x^\star\| = \lim_{k \rightarrow +\infty} \tau_k(\omega).$$

Let us recall that there exists $\Omega_{\text{cont}} \in \mathcal{F}$ such that $\mathbb{P}(\Omega_{\text{cont}}) = 1$ and $X(\omega, \cdot)$ is continuous for every $\omega \in \Omega_{\text{cont}}$. Now let $x^\star \in \mathcal{S}$ arbitrary, since the limit exists, for every $\omega \in \tilde{\Omega} \cap \Omega_{\text{cont}}$ there exists $T(\omega)$ such that $\|X(\omega, t) - x^\star\| \leq 1 + \lim_{k \rightarrow +\infty} \tau_k(\omega)$ for every $t \geq T(\omega)$. Besides, since $X(\omega, \cdot)$ is continuous, by Bolzano's theorem

$$\sup_{t \in [0, T(\omega)]} \|X(\omega, t)\| = \max_{t \in [0, T(\omega)]} \|X(\omega, t)\| \stackrel{\text{def}}{=} h(\omega) < +\infty.$$

Therefore, $\sup_{t \geq 0} \|X(\omega, t)\| \leq \max\{h(\omega), 1 + \lim_{k \rightarrow +\infty} \tau_k(\omega) + \|x^\star\|\} < +\infty$.

(iii) Let $N_t = \int_0^t \sigma(s, X(s)) dW(s)$. This is a continuous martingale (w.r.t. the filtration \mathcal{F}_t), which verifies

$$\mathbb{E}(\|N_t\|^2) = \mathbb{E} \left(\int_0^t \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \right) \leq \mathbb{E} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right) < +\infty, \forall t \geq 0.$$

According to Theorem 2.8.1, we deduce that there exists a \mathbb{H} -valued random variable N_∞ w.r.t. \mathcal{F}_∞ , and which verifies: $\mathbb{E}(\|N_\infty\|^2) < +\infty$, and there exists $\Omega_N \in \mathcal{F}$ such that $\mathbb{P}(\Omega_N) = 1$ and

$$\lim_{t \rightarrow +\infty} N_t(\omega) = N_\infty(\omega) \text{ for every } \omega \in \Omega_N.$$

Besides, by convexity of f and (3.4), we have that there exists $\Omega_f \in \mathcal{F}$ such that $\mathbb{P}(\Omega_f) = 1$ and $(f(X(\omega, \cdot)) - \min f) \in L^1(\mathbb{R}_+)$ for every $\omega \in \Omega_f$. By Corollary 2.7, we obtain that $\|\nabla f(X(\omega, \cdot))\| \in L^2(\mathbb{R}_+)$ for every $\omega \in \Omega_f$.

Let $\Omega_{\text{conv}} \stackrel{\text{def}}{=} \tilde{\Omega} \cap \Omega_{\text{cont}} \cap \Omega_f \cap \Omega_N$, hence $\mathbb{P}(\Omega_{\text{conv}}) = 1$. Let $\omega \in \Omega_{\text{conv}} \subseteq \Omega_f$ arbitrary, then $\liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = 0$. If $\limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = 0$ then we conclude. Suppose by contradiction that there exists $\omega_0 \in \Omega_{\text{conv}}$ such that $\limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t))\| > 0$. Then, by Lemma 2.6.7, there exists $\delta(\omega_0) > 0$ satisfying

$$0 = \liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t))\| < \delta(\omega_0) < \limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t))\|,$$

and there exists $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$,

$$\|\nabla f(X(\omega_0, t_k))\| > \delta \text{ and } t_{k+1} - t_k > 1, \quad \forall k \in \mathbb{N}.$$

We allow ourselves the abuse of notation $X(t) \stackrel{\text{def}}{=} X(\omega_0, t)$ and $\delta \stackrel{\text{def}}{=} \delta(\omega_0)$ during the rest of the proof from this point.

Let $\varepsilon \in]0, \min\{\frac{\delta^2}{4L^2}, 1\}[$. Note that this choice entails that the intervals $([t_k, t_k + \frac{\varepsilon}{2}])_{k \in \mathbb{N}}$ are disjoint. On the other hand, according to the convergence property of N_t and the fact that $\|\nabla f(X(\cdot))\| \in L^2(\mathbb{R}_+)$, there exists $k' > 0$ such that for every $k \geq k'$

$$\sup_{t \geq t_k} \|N_t - N_{t_k}\|^2 < \frac{\varepsilon}{4} \text{ and } \int_{t_k}^{+\infty} \|\nabla f(X(s))\|^2 ds \leq \frac{1}{2}.$$

Besides, for every $k \geq k'$, $t \in [t_k, t_k + \frac{\varepsilon}{2}]$

$$\|X(t) - X(t_k)\|^2 \leq 2(t - t_k) \int_{t_k}^t \|\nabla f(X(s))\|^2 ds + 2\|N_t - N_{t_k}\|^2 \leq (t - t_k) + \frac{\varepsilon}{2} \leq \varepsilon.$$

Since $f \in C_L^{1,1}(\mathbb{H})$ and $L^2\varepsilon \leq (\frac{\delta}{2})^2$ by assumption on ε , we have that for every $k \geq k'$ and $t \in [t_k, t_k + \frac{\varepsilon}{2}]$

$$\|\nabla f(X(t)) - \nabla f(X(t_k))\|^2 \leq L^2 \|X(t) - X(t_k)\|^2 \leq \left(\frac{\delta}{2}\right)^2.$$

Therefore, for every $k \geq k'$, $t \in [t_k, t_k + \frac{\varepsilon}{2}]$

$$\|\nabla f(X(t))\| \geq \|\nabla f(X(t_k))\| - \underbrace{\|\nabla f(X(t)) - \nabla f(X(t_k))\|}_{\leq \frac{\delta}{2}} \geq \frac{\delta}{2}.$$

Finally,

$$\int_0^{+\infty} \|\nabla f(X(s))\|^2 ds \geq \sum_{k \geq k'} \int_{t_k}^{t_k + \frac{\varepsilon}{2}} \|\nabla f(X(s))\|^2 ds \geq \sum_{k \geq k'} \frac{\delta^2 \varepsilon}{8} = +\infty,$$

which contradicts $\|\nabla f(X(\cdot))\| \in L^2(\mathbb{R}_+)$. So,

$$\limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = \liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = \lim_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = 0, \quad \forall \omega \in \Omega_{\text{conv}}.$$

Let $x^* \in \mathcal{S}$ and $\omega \in \Omega_{\text{conv}}$ taken arbitrary. By convexity and Cauchy-Schwarz inequality:

$$0 \leq f(X(\omega, t)) - \min f \leq \|\nabla f(X(\omega, t))\| \|X(\omega, t) - x^*\|.$$

The claim then follows because we have already obtained that $\lim_{t \rightarrow +\infty} \|X(\omega, t) - x^*\|$ exists, and $\lim_{t \rightarrow +\infty} \|\nabla f(X(\omega, t))\| = 0$.

(iv) Let $\omega \in \Omega_{\text{conv}}$ and $\tilde{X}(\omega)$ be a weak sequential limit point of $X(\omega, t)$. Equivalently, there exists an increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$ and

$$\text{w-lim}_{k \rightarrow +\infty} X(\omega, t_k) = \tilde{X}(\omega).$$

Since $\lim_{t \rightarrow +\infty} f(X(\omega, t)) = \min f$ and the fact that f is weakly lower semicontinuous (since it is convex and continuous), we obtain directly that $\tilde{X}(\omega) \in \mathcal{S}$. Finally, by Opial's Lemma (see [162]) we conclude that there exists $X^*(\omega) \in \mathcal{S}$ such that $\text{w-lim}_{t \rightarrow +\infty} X(\omega, t) = X^*(\omega)$. In other words, since $\omega \in \Omega_{\text{conv}}$ was arbitrary, there exists an \mathcal{S} -valued random variable X^* such that $\text{w-lim}_{t \rightarrow +\infty} X(t) = X^*$ a.s.. □

3.2.2 Convergence rates of the objective

Our second main result, stated below, summarizes the global convergence rates in expectation satisfied by the trajectories of (SDE).

Theorem 3.2.5. *Consider the dynamic (SDE) where f and σ satisfy the assumptions (\mathbf{H}_f) and (\mathbf{H}_σ) . Additionally, $X_0 \in L^2(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. The following statements are satisfied by the solution trajectory $X \in \mathcal{S}_{\mathbb{H}}^2$ of (SDE):*

(i) Let $\overline{f \circ X}(t) \stackrel{\text{def}}{=} t^{-1} \int_0^t f(X(s)) ds$ and $\overline{X}(t) = t^{-1} \int_0^t X(s) ds$. Then

$$\mathbb{E}(f(\overline{X}(t)) - \min f) \leq \mathbb{E}(\overline{f \circ X}(t) - \min f) \leq \frac{\mathbb{E}(\text{dist}(X_0, \mathcal{S})^2)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t > 0. \quad (3.7)$$

Besides, if σ_∞ is $L^2(\mathbb{R}_+)$, then

$$\mathbb{E}(f(\overline{X}(t)) - \min f) \leq \mathbb{E}(\overline{f \circ X}(t) - \min f) = \mathcal{O}\left(\frac{1}{t}\right). \quad (3.8)$$

(ii) Moreover, if $f \in \Gamma_\mu(\mathbb{H})$ with $\mu > 0$, then $\mathcal{S} = \{x^*\}$ and

(a)

$$\mathbb{E}(\|X(t) - x^*\|^2) \leq \mathbb{E}(\|X_0 - x^*\|^2) e^{-\mu t} + \frac{\sigma_*^2}{\mu}, \quad \forall t \geq 0. \quad (3.9)$$

Besides, if σ_∞ is nonincreasing and vanishes at infinity, then for every $\lambda \in]0, 1[$:

$$\mathbb{E}(\|X(t) - x^*\|^2) \leq \mathbb{E}(\|X_0 - x^*\|^2) e^{-\mu t} + \frac{\sigma_*^2}{\mu} e^{-\mu(1-\lambda)t} + \sigma_\infty^2(\lambda), \quad \forall t \geq 0. \quad (3.10)$$

(b) Furthermore, if \mathbb{H} is finite-dimensional or $f \in C^2(\mathbb{H})$:

$$\mathbb{E}(f(X(t)) - \min f) \leq \mathbb{E}(f(X_0) - \min f) e^{-2\mu t} + \frac{L\sigma_*^2}{4\mu}, \quad \forall t \geq 0. \quad (3.11)$$

Besides, if σ_∞ is nonincreasing and vanishes at infinity, then for every $\lambda \in]0, 1[$:

$$\mathbb{E}(f(X(t)) - \min f) \leq \mathbb{E}(f(X_0) - \min f) e^{-2\mu t} + \frac{L\sigma_*^2}{4\mu} e^{-2\mu(1-\lambda)t} + \frac{L}{2} \sigma_\infty^2(\lambda), \quad \forall t \geq 0. \quad (3.12)$$

Proof. (i) Let $x^* \in \mathcal{S}$. Let $g(t) = \phi(X(t)) = \frac{\|X(t) - x^*\|^2}{2}$ and $G(t) = \mathbb{E}(g(t))$. By applying Proposition 3.2.1 with ϕ , and using the convexity of f , we obtain

$$\begin{aligned} G(t) - G(0) &= \mathbb{E}\left(\int_0^t \langle \nabla f(X(s)), x^* - X(s) \rangle ds\right) + \frac{1}{2} \mathbb{E}\left(\int_0^t \text{tr}[\Sigma(s, X(s))] ds\right) \\ &\leq -\mathbb{E}\left(\int_0^t (f(X(s)) - \min f) ds\right) + \frac{1}{2} \mathbb{E}\left(\int_0^t \text{tr}[\Sigma(s, X(s))] ds\right) \\ &\leq -\mathbb{E}\left(\int_0^t (f(X(s)) - \min f) ds\right) + \frac{\sigma_*^2}{2} t. \end{aligned} \quad (3.13)$$

Then rearranging the terms in (3.13), using $G(t) \geq 0$, and dividing by $t > 0$, we obtain

$$\frac{1}{t} \mathbb{E} \left(\int_0^t (f(X(s)) - \min f) ds \right) \leq \frac{\mathbb{E} \left(\|X_0 - x^*\|^2 \right)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t > 0. \quad (3.14)$$

Since x^* is arbitrary, by taking the infimum with respect to $x^* \in \mathcal{S}$ in (3.14), we obtain

$$\frac{1}{t} \mathbb{E} \left(\int_0^t (f(X(s)) - \min f) ds \right) \leq \frac{\mathbb{E} \left(\text{dist}(X_0, \mathcal{S})^2 \right)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t > 0. \quad (3.15)$$

Moreover, if $\sigma_\infty \in L^2(\mathbb{R}_+)$, then using inequality (3.13), we have

$$G(t) - G(0) \leq -\mathbb{E} \left(\int_0^t (f(X(s)) - \min f) ds \right) + \frac{1}{2} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right).$$

Rearranging as before, we conclude that

$$\frac{1}{t} \mathbb{E} \left(\int_0^t (f(X(s)) - \min f) ds \right) \leq \frac{\mathbb{E} \left(\text{dist}(X_0, \mathcal{S})^2 \right)}{2t} + \frac{1}{2t} \int_0^{+\infty} \sigma_\infty^2(s) ds, \quad \forall t > 0. \quad (3.16)$$

Then complete the result with the inequality

$$\mathbb{E} \left(f(\bar{X}(t)) - \min f \right) \leq \mathbb{E} \left(f \circ \bar{X}(t) - \min f \right)$$

which follows from convexity of f and Jensen's inequality.

(ii) (a) Let $g(t) = \phi(X(t)) = \frac{\|X(t) - x^*\|^2}{2}$, $G(t) = \mathbb{E}(g(t))$. By Proposition 3.2.1 with ϕ , we obtain

$$G(t) - G(0) = \mathbb{E} \left(\int_0^t \langle -\nabla f(X(s)), X(s) - x^* \rangle ds \right) + \frac{1}{2} \mathbb{E} \left(\int_0^t \text{tr}[\Sigma(s, X(s))] ds \right). \quad (3.17)$$

Using that $f \in \Gamma_\mu(\mathbb{H})$, we deduce that

$$G(t) \leq G(0) - \mu \int_0^t G(s) ds + \int_0^t \frac{\sigma_*^2}{2} ds, \quad \forall t \geq 0. \quad (3.18)$$

In order to invoke Lemma 2.6.6, we solve the ODE

$$\begin{cases} y'(t) &= -\mu y(t) + \frac{\sigma_*^2}{2}, \quad \forall t > 0, \\ y(0) &= \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right). \end{cases}$$

Solving it by the integrating factor method, we conclude that

$$G(t) \leq \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right) e^{-\mu t} + \frac{\sigma_*^2}{2\mu}, \quad \forall t \geq 0.$$

Suppose now that σ_∞ is nonincreasing and vanishes at infinity. We can bound the trace term by σ_∞^2 in (3.17). To use Lemma 2.6.6, we need to solve

$$\begin{cases} y'(t) &= -\mu y(t) + \frac{\sigma_\infty^2(t)}{2}, \quad \forall t > 0, \\ y(0) &= \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right). \end{cases}$$

Let $\lambda \in]0, 1[$, using the integrating factor method, we get

$$\begin{aligned} y(t) &\leq y(0)e^{-\mu t} + e^{-\mu t} \int_0^t \frac{\sigma_\infty^2(s)}{2} e^{\mu s} ds \\ &\leq y(0)e^{-\mu t} + e^{-\mu t} \left(\int_0^{\lambda t} \frac{\sigma_\infty^2(s)}{2} e^{\mu s} ds + \int_{\lambda t}^t \frac{\sigma_\infty^2(s)}{2} e^{\mu s} ds \right) \\ &\leq y(0)e^{-\mu t} + e^{-\mu t} \left(\frac{\sigma_*^2}{2} \int_0^{\lambda t} e^{\mu s} ds + \frac{\sigma_\infty^2(\lambda t)}{2} \int_{\lambda t}^t e^{\mu s} ds \right) \\ &\leq y(0)e^{-\mu t} + e^{-\mu t} \left(\frac{\sigma_*^2}{2\mu} e^{\mu \lambda t} + \frac{\sigma_\infty^2(\lambda t)}{2} e^{\mu t} \right), \quad \forall t \geq 0. \end{aligned}$$

According to Lemma 2.6.6, we deduce that

$$G(t) \leq \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right) e^{-\mu t} + \frac{\sigma_\infty^2}{2\mu} e^{-\mu(1-\lambda)t} + \frac{\sigma_\infty^2(\lambda t)}{2}, \quad \forall t \geq 0,$$

which is our claim (3.10).

- (b) Since $f \in \Gamma_\mu(\mathbb{H})$, it is well known that f satisfies the Polyak-Łojasiewicz inequality 2.2.3, *i.e.*

$$2\mu(f(x) - \min f) \leq \|\nabla f(x)\|^2, \quad \forall x \in \mathbb{H},$$

(see Section 2.4 for an explanation of this inequality). Besides, since $f \in \Gamma_0(\mathbb{H}) \cap C_L^{1,1}(\mathbb{H})$ and $X_0 \in L^2(\Omega; \mathbb{H})$, we have that $\mathbb{E}(f(X_0) - \min f) < +\infty$.

We take the function $\widehat{\phi}(x) = f(x) - \min f$ and apply Proposition 3.2.1. Then, defining $\widehat{g}(t) = f(X(t)) - \min f$ and $\widehat{G}(t) = \mathbb{E}(g(t))$, we obtain

$$\widehat{G}(t) - \widehat{G}(0) \leq -\mathbb{E} \left(\int_0^t \|\nabla f(X(s))\|^2 ds \right) + \frac{L}{2} \int_0^t \sigma_\infty^2(s) ds.$$

Using the Polyak-Łojasiewicz inequality, we end up having

$$\widehat{G}(t) - \widehat{G}(0) \leq -2\mu \left(\int_0^t \widehat{G}(s) ds \right) + \frac{L}{2} \int_0^t \sigma_\infty^2(s) ds. \quad (3.19)$$

And we conclude by continuing the analysis as in the previous item after arriving to (3.18). \square

We will rephrase assumption (\mathbf{H}_f) on the objective f to:

$$\left\{ \begin{array}{l} f \text{ is convex and continuously differentiable with } L\text{-Lipschitz continuous gradient;} \\ f \in C^2(\mathbb{H}) \text{ or } \mathbb{H} \text{ is finite-dimensional;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{array} \right. \quad (\mathbf{H}_f^*)$$

(\mathbf{H}_f^*) coincides with (\mathbf{H}_f) in the infinite-dimensional case, but is weaker than (\mathbf{H}_f) when \mathbb{H} is finite-dimensional.

Under the previous assumption (\mathbf{H}_f^*) and a stronger assumption on σ_∞ , we also have the following pointwise sublinear convergence rate in expectation.

Proposition 3.2.6. *Let $\nu \geq 2$ and consider the dynamic (SDE) where f and σ satisfy the assumptions (\mathbf{H}_f^*) and (\mathbf{H}_σ) , respectively. With initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Assume that there exists $K \geq 0, \beta \in [0, 1[$ such that*

$$\int_0^t (s+1)\sigma_\infty^2(s) ds \leq Kt^\beta, \quad \forall t \geq 0. \quad (3.20)$$

Then the solution trajectory $X \in S_{\mathbb{H}}^\nu$ of (SDE) satisfies

$$\mathbb{E}(f(X(t)) - \min f) = \mathcal{O}(t^{\beta-1}).$$

Remark 3.2.7. The case $\beta = 0$ in the previous Proposition is equivalent to the condition $t \mapsto t\sigma_\infty^2(t) \in L^1(\mathbb{R}_+)$, in this case we could conclude that:

$$\mathbb{E}(f(X(t)) - \min f) = \mathcal{O}(t^{-1}).$$

Proof. Given $x^* \in \mathcal{S}$, let us apply Proposition 2.7.6 successively with $V_1(t, x) = t(f(x) - \min f)$, then with $V_2(x) = \frac{1}{2} \|x - x^*\|^2$. Taking the expectation and adding the two results, we get

$$\begin{aligned} \mathbb{E}(V_1(t, X(t)) + V_2(X(t))) &\leq \mathbb{E}\left(\frac{\|X_0 - x^*\|^2}{2}\right) + \frac{L}{2} \int_0^t s \sigma_\infty^2(s) ds + \frac{1}{2} \int_0^t \sigma_\infty^2(s) ds \\ &\leq \mathbb{E}\left(\frac{\|X_0 - x^*\|^2}{2}\right) + \frac{\max\{1, L\}}{2} \left(\int_0^t (s+1) \sigma_\infty^2(s) ds\right), \end{aligned}$$

where we have used the convexity of f in the first inequality. Then we conclude that

$$\mathbb{E}(f(X(t)) - \min f) \leq \frac{\mathbb{E}\left(\|X_0 - x^*\|^2\right)}{2t} + \frac{K \max\{1, L\}}{2} t^{\beta-1} = \mathcal{O}(t^{\beta-1}).$$

□

When f is also C^2 and the first order moment of σ_∞^2 is bounded, we get an improved $o(t^{-1})$ global convergence rate on the objective in almost sure sense.

Theorem 3.2.8. *Consider the dynamic (SDE). Assume that $f \in C^2(\mathbb{H})$ and σ satisfy the assumptions (H_f) and (H_σ) , respectively. Additionally, $X_0 \in L^2(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable, and that $t \mapsto t\sigma_\infty^2(t) \in L^1(\mathbb{R}_+)$. Then, the solution trajectory $X \in S_{\mathbb{H}}^2$ of (SDE) obeys:*

- (i) $t \mapsto t \|\nabla f(X(t))\|^2 \in L^1(\mathbb{R}_+)$ a.s..
- (ii) $f(X(t)) - \min f = o(t^{-1})$ a.s..

Proof. By applying Itô's formula in Proposition 3.2.1 with $\phi(t, x) = t(f(x) - \min f)$ we get

$$\begin{aligned} t(f(X(t)) - \min f) &= \int_0^t (f(X(s)) - \min f) ds + \frac{1}{2} \int_0^t \text{tr}[\Sigma(s, X(s)) \nabla^2 f(X(s))] ds \\ &\quad - \int_0^t s \|\nabla f(X(s))\|^2 ds + \int_0^t \langle s \sigma^*(s, X(s)) \nabla f(X(s)), dW(s) \rangle. \end{aligned}$$

By (3.4) and convexity of f , we deduce that $f(X(\cdot)) - \min f \in L^1(\mathbb{R}_+)$ a.s.. Moreover,

$$\int_0^{+\infty} \text{str}[\Sigma(s, X(s)) \nabla^2 f(X(s))] ds \leq L \int_0^{+\infty} s \sigma_\infty^2(s) ds < +\infty.$$

Then by Theorem 2.8.2, we have that $\lim_{t \rightarrow +\infty} t(f(X(t)) - \min f)$ exists a.s. and

$$\int_0^{+\infty} t \|\nabla f(X(t))\|^2 dt < +\infty, \quad \text{a.s..}$$

Finally, since $\int_0^{+\infty} \frac{dt}{t} = +\infty$, by Lemma 2.6.2 we conclude that $\lim_{t \rightarrow +\infty} t(f(X(t)) - \min f) = 0$ a.s.. □

3.3 Convergence rates under the Łojasiewicz inequality

As presented in Section 2.4, the local convergence rate of the first-order descent methods can be understood using the Łojasiewicz property, the associated Łojasiewicz exponent, and the corresponding Error Bound Inequality.

In this line, we now state the following ergodic local convergence rate.

Proposition 3.3.1. *Consider the hypotheses of Theorem 3.2.5 and let $\varepsilon > 0$. If $f \in \text{EB}^p(\mathcal{S})$ and this property holds on $[f \leq r_\varepsilon]$ for $r_\varepsilon > \min f + \frac{\sigma_*^2}{2} + \varepsilon$, then $\exists t_\varepsilon > 0$ such that*

$$\text{dist}(\mathbb{E}(\bar{X}(t)), \mathcal{S}) = \mathcal{O}(t^{-\frac{1}{p}}) + \mathcal{O}\left(\sigma_*^{\frac{2}{p}}\right), \quad \forall t \geq t_\varepsilon.$$

Proof. There exists $t_\varepsilon > 0$ such that for all $t \geq t_\varepsilon$, $\frac{\mathbb{E}(\text{dist}(X_0, \mathcal{S})^\varepsilon)}{2t} < \varepsilon$. Thus, from (3.7) and Jensen's inequality, we have

$$f(\mathbb{E}[\bar{X}(t)]) \leq \mathbb{E}[f(\bar{X}(t))] \leq \min f + \frac{\sigma_*^2}{2} + \varepsilon \leq r_\varepsilon, \quad \forall t \geq t_\varepsilon.$$

This reflects the fact that, $\mathbb{E}[\bar{X}(t)] \in [f \leq r_\varepsilon]$ for $t \geq t_\varepsilon$. Using Theorem 3.2.5 and that $f \in \text{EB}^p(\mathcal{S})$ is valid on $([f \leq r_\varepsilon])$, letting $\gamma > 0$ the coefficient of the error bound, we have

$$\gamma \text{dist}(\mathbb{E}[\bar{X}(t)], \mathcal{S})^p \leq f(\mathbb{E}[\bar{X}(t)]) - \min f \leq \frac{\mathbb{E}(\text{dist}(X_0, \mathcal{S})^2)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t \geq t_\varepsilon.$$

Dividing by $\gamma > 0$, then taking the power $\frac{1}{p}$ on both sides of the previous inequality, and finally using the subadditivity of the power function $(\cdot)^{1/p}$ on $[0, +\infty[$ (recall $p \geq 1$), we obtain

$$\text{dist}(\mathbb{E}[\bar{X}(t)], \mathcal{S}) \leq \left(\frac{\mathbb{E}(\text{dist}(X_0, \mathcal{S})^2)}{2\gamma} \right)^{\frac{1}{p}} t^{-\frac{1}{p}} + \left(\frac{\sigma_*^2}{2\gamma} \right)^{\frac{1}{p}}, \quad \forall t \geq t_\varepsilon.$$

□

3.3.1 Discussion on the localization of the process

Let us take a moment to elaborate on the localization of the process $X(t)$ generated by (SDE) when $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$ and $\sigma_\infty \in L^2(\mathbb{R}_+)$. This discussion is essential to understand the challenges underlying the analysis of the local convergence properties and rates in a stochastic setting under (local) error bounds. First, observe that the hypothesis of Lipschitz continuity of the gradient is incompatible with a global hypothesis of error bound or Łojasiewicz inequality unless the exponent is $p = 2$ or $q = \frac{1}{2}$, respectively. Therefore, we can only ask for these inequalities to be locally satisfied. Even though, thanks to convexity, we could introduce a global desingularizing function (see [59, Theorem 3]), this function would not be concave nor convex, a fundamental property usually at the heart of the local analysis. In recent literature on stochastic processes and local properties, it is usual to find hypotheses about the almost sure localization of the process or that it is essentially bounded. Nevertheless, these assumptions are unrealistic or outright false due to the behavior of the Brownian Motion. Hence, we will avoid making these kinds of assumptions.

What we will do is to consider that by Theorem 3.2.3 we have that $\lim_{t \rightarrow +\infty} f(X(t)) = \min f$ a.s., which means that there exists $\Omega_{\text{conv}} \in \mathcal{F}$ such that $\mathbb{P}(\Omega_{\text{conv}}) = 1$, and $(\forall r > \min f, \forall \omega \in \Omega_{\text{conv}})$, $(\exists t_r(\omega) > 0)$ such that $(\forall t > t_r(\omega))$, $X(\omega, t) \in [f \leq r]$. However, one should not infer from this that $X(t) \in [f \leq r]$ a.s. for t large enough. Indeed, t_r is a random variable which cannot be in general bounded uniformly on Ω_{conv} . Rather, in this work, we will invoke measure theoretic arguments to pass from a.s. convergence to almost uniform convergence thanks to Egorov's theorem (see Theorem 2.7.1). More precisely, we will show that

$$(\forall \delta > 0, \forall r > \min f), (\exists \Omega_\delta \in \mathcal{F} \text{ s.t. } \mathbb{P}(\Omega_\delta) \geq 1 - \delta \text{ and } \exists \hat{t}_{r,\delta} > 0), (\forall \omega \in \Omega_\delta, \forall t > \hat{t}_{r,\delta}), \\ X(\omega, t) \in [f \leq r].$$

Hence, this property will allow us to localize $X(t)$ in the sublevel set of f at r for t large enough with probability at least $1 - \delta$. In turn, we will be able to invoke the error bound (or Łojasiewicz) inequality.

3.3.2 Convergence rates under Łojasiewicz Inequality

We start with a useful Corollary of Lemma 2.7.2.

Corollary 3.3.2. *Let $\delta > 0, \Omega_\delta \in \mathcal{F}$ such that $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$. Consider (SDE) where f and σ satisfy the assumptions (\mathbf{H}_f^*) and (\mathbf{H}_σ) , respectively. Assume that $X_0 \in L^4(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Moreover, suppose that $\sigma_\infty \in L^2(\mathbb{R}_+)$. Let $X \in S_{\mathbb{H}}^4$ be the unique solution of (SDE), then there exists $C_d, C_f > 0$ (independent of δ) such that:*

$$\mathbb{E} \left[\frac{\text{dist}(X(t), \mathcal{S})^2}{2} \right] - \mathbb{E} \left[\frac{\text{dist}(X(t), \mathcal{S})^2}{2} \mathbb{1}_{\Omega_\delta} \right] \leq C_d \sqrt{\delta},$$

and

$$\mathbb{E} [f(X(t)) - \min f] - \mathbb{E} [(f(X(t)) - \min f) \mathbb{1}_{\Omega_\delta}] \leq C_f \sqrt{\delta}.$$

Proof. Let $x^* \in \mathcal{S}$. Using Proposition 2.7.6 with $\hat{\phi}(x) = \frac{\text{dist}(x, \mathcal{S})^2}{2}$, squaring the obtained inequality and taking expectation, we obtain

$$\begin{aligned} \mathbb{E} \left[\frac{\text{dist}(X(t), \mathcal{S})^4}{4} \right] &\leq \frac{3}{4} \mathbb{E}(\text{dist}(X_0, \mathcal{S})^4) + \frac{3}{4} \left(\int_0^t \sigma_\infty^2(s) ds \right)^2 \\ &\quad + 3 \mathbb{E} \left[\left(\int_0^t \langle \sigma^*(s, X(s))(X(s) - P_{\mathcal{S}}(X(s))), dW(s) \rangle \right)^2 \right] \\ &\leq \frac{3}{4} \mathbb{E}(\text{dist}(X_0, \mathcal{S})^4) + \frac{3}{4} \left(\int_0^t \sigma_\infty^2(s) ds \right)^2 + 3 \sup_{t \geq 0} \mathbb{E}[\|X(t) - x^*\|^2] \left[\int_0^t \sigma_\infty^2(s) ds \right]. \end{aligned}$$

Taking the supremum over $t \geq 0$, we obtain

$$\begin{aligned} \sup_{t \geq 0} \mathbb{E} \left[\left(\frac{\text{dist}(X(t), \mathcal{S})^2}{2} \right)^2 \right] &\leq \frac{3}{4} \mathbb{E}(\text{dist}(X_0, \mathcal{S})^4) + \frac{3}{4} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^2 \\ &\quad + 3 \sup_{t \geq 0} \mathbb{E}[\|X(t) - x^*\|^2] \left[\int_0^{+\infty} \sigma_\infty^2(s) ds \right] \stackrel{\text{def}}{=} C_d < +\infty. \end{aligned}$$

In the above estimation we used that $\sigma_\infty \in L^2(\mathbb{R}_+)$ and $\sup_{t \geq 0} \mathbb{E}[\|X(t) - x^*\|^2] < +\infty$ by Theorem 3.2.3(i).

On the other hand, since $f \in \Gamma_0(\mathbb{H}) \cap C_L^{1,1}(\mathbb{H})$ and $X_0 \in L^4(\Omega; \mathbb{H})$, we have that

$$\mathbb{E}([f(X_0) - \min f]^2) \leq \frac{1}{2} \mathbb{E}(\|\nabla f(X_0) - \nabla f(x^*)\|^4) + \frac{1}{2} \mathbb{E}(\|X_0 - x^*\|^4) < \frac{L^4 + 1}{2} \mathbb{E}(\|X_0 - x^*\|^4) < +\infty.$$

Then using Proposition 2.7.6 with $\tilde{\phi}(x) = f(x) - \min f$, squaring it, and taking expectation, we obtain

$$\begin{aligned} \mathbb{E} [[f(X(t)) - \min f]^2] &\leq 3 \mathbb{E}([f(X_0) - \min f]^2) + \frac{3L^2}{4} \left(\int_0^t \sigma_\infty^2(s) ds \right)^2 \\ &\quad + 3 \mathbb{E} \left[\left(\int_0^t \langle \sigma^*(s, X(s))(\nabla f(X(s))), dW(s) \rangle \right)^2 \right] \\ &\leq 3 \mathbb{E}([f(X_0) - \min f]^2) + \frac{3L^2}{4} \left(\int_0^t \sigma_\infty^2(s) ds \right)^2 \\ &\quad + 3L^2 \sup_{t \geq 0} \mathbb{E}[\|X(t) - x^*\|^2] \left[\int_0^t \sigma_\infty^2(s) ds \right]. \end{aligned}$$

Taking the supremum over $t \geq 0$, we obtain

$$\begin{aligned} \sup_{t \geq 0} \mathbb{E} [[f(X(t)) - \min f]^2] &\leq 3 \mathbb{E}([f(X_0) - \min f]^2) + \frac{3L^2}{4} \left(\int_0^{+\infty} \sigma_\infty^2(s) ds \right)^2 \\ &\quad + 3L^2 \sup_{t \geq 0} \mathbb{E}[\|X(t) - x^*\|^2] \left[\int_0^{+\infty} \sigma_\infty^2(s) ds \right] \stackrel{\text{def}}{=} C_f < +\infty. \end{aligned}$$

And we have proved the hypothesis of Lemma 2.7.2 in both cases, applying this lemma, we conclude the proof. \square

Now we consider $\sigma_\infty \in L^2(\mathbb{R}_+)$, $L > 0$, $\delta > 0$, $\beta \in [0, 1[$ and some positive constants C_l, C_k . We also consider the functions $l_\delta, k_\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by:

$$l_\delta(t) = \frac{L}{2}\sigma_\infty^2(t) + C_l\sqrt{\delta} \frac{\sigma_\infty^2(t)}{2\sqrt{\int_{\hat{t}_\delta}^t \sigma_\infty^2(u)du}}, \quad (3.21)$$

$$k_\delta(t) = \frac{L}{2}\sigma_\infty^2(t) + C_k\sqrt{\delta} \frac{\sigma_\infty^2(t)t^{\beta-1}}{2\sqrt{\int_{\hat{t}_\delta}^t \sigma_\infty^2(u)u^{\beta-1}du}}. \quad (3.22)$$

We are now ready to state our main local convergence result.

Theorem 3.3.3. *Consider (SDE) where f and σ satisfy the assumptions (\mathbf{H}_f^*) and (\mathbf{H}_σ) , respectively. Additionally, $X_0 \in L^4(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Let $X \in S_{\mathbb{H}}^4$ the unique solution trajectory of (SDE). Suppose also that $\sigma_\infty \in L^2(\mathbb{R}_+)$ ($C_\infty \stackrel{\text{def}}{=} \|\sigma_\infty\|_{L^2(\mathbb{R}_+)}$). Let $p \geq 2$ and $q \stackrel{\text{def}}{=} 1 - \frac{1}{p} \in [\frac{1}{2}, 1[$, and assume that $f \in L^q(\mathcal{S})$. Consider also the positive constants C_*, C_l, C_k, C_d, C_f (detailed in the proof). Then, for all $\delta > 0$, there exists a measurable set Ω_δ such that $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$ and $\hat{t}_\delta > 0$ such that the following statements hold.*

(i) *If $q = \frac{1}{2}$ and σ_∞ is nonincreasing, then σ_∞ vanishes at infinity and there exists $\mu > 0$ such that for every $\lambda \in]0, 1[$,*

$$\begin{aligned} \mathbb{E}(f(X(t)) - \min f) &\leq e^{-\mu^2(t-\hat{t}_\delta)}\mathbb{E}(f(X(\hat{t}_\delta)) - \min f) \\ &+ e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) \\ &+ \frac{l_\delta(\hat{t}_\delta + \lambda(t - \hat{t}_\delta))}{\mu^2} + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned} \quad (3.23)$$

Moreover, if (3.20) holds, then

$$\begin{aligned} \mathbb{E}(f(X(t)) - \min f) &\leq e^{-\mu^2(t-\hat{t}_\delta)}\mathbb{E}(f(X(\hat{t}_\delta)) - \min f) \\ &+ e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_k C_\infty \sqrt{\hat{t}_\delta^{\beta-1}} \sqrt{\delta} \right) \\ &+ \frac{k_\delta(\hat{t}_\delta + \lambda(t - \hat{t}_\delta))}{\mu^2} + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned} \quad (3.24)$$

(ii) *If $q > \frac{1}{2}$, there exists $\mu > 0$ such that:*

$$\mathbb{E}[f(X(t)) - \min f] \leq w_\delta^*(t) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta, \quad (3.25)$$

where w_δ^* is the solution of the Cauchy problem

$$(C.1) \quad \begin{cases} y'(t) &= -\mu^2 y(t)^{2q} + l_\delta(t), \quad t > \hat{t}_\delta \\ y(\hat{t}_\delta) &= \mathbb{E}([f(X(\hat{t}_\delta)) - \min f] \mathbb{1}_{\Omega_\delta}). \end{cases}$$

Moreover, if (3.20) holds, then

$$\mathbb{E}[f(X(t)) - \min f] \leq z_\delta^*(t) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta, \quad (3.26)$$

where z_δ^* is the solution of the Cauchy problem

$$(C.2) \quad \begin{cases} y'(t) &= -\mu^2 y(t)^{2q} + k_\delta(t), \quad \forall t > \hat{t}_\delta \\ y(\hat{t}_\delta) &= \mathbb{E}([f(X(\hat{t}_\delta)) - \min f] \mathbb{1}_{\Omega_\delta}). \end{cases}$$

Before proceeding with the proof, a few remarks are in order.

Remark 3.3.4. The hypothesis that f has a Lipschitz continuous gradient restricts the Łojasiewicz exponent q to be in $[\frac{1}{2}, 1[$.

Remark 3.3.5. If we have a *global* error bound (or Łojasiewicz inequality), then as noted in the discussion of Section 3.3.1, one necessarily has $q = \frac{1}{2}$. In this case, the statements (i) of Theorem 3.3.3 will hold if we replace $\sigma_\infty \in L^2(\mathbb{R}_+)$ by σ_∞ being nonincreasing and vanishing at infinity, δ by 0 and \hat{t}_δ by 0. Clearly, one recovers (3.10).

Remark 3.3.6. It is important to highlight the trade-off in the selection of δ . Although δ can be arbitrarily small, the time from which the inequalities are satisfied, \hat{t}_δ , surely increases when δ approaches 0^+ . Besides, let $q_{\delta, \hat{t}_\delta} : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a decreasing function. Our convergence rates in Theorem 3.3.3 are of the form $\mathbb{E}[f(X(t)) - \min f] \leq q_{\delta, \hat{t}_\delta}(t) + C\sqrt{\delta}$, $\forall t > t_\delta$. Let $\varepsilon \in]0, 2C[$ and $\delta^* = \frac{\varepsilon^2}{4C^2}$. Then one gets an ε -optimal solution for $t > \max\{q^*(\varepsilon), \hat{t}_{\delta^*}\}$.

Remark 3.3.7. Referring again to the discussion of Section 3.3.1, we have that there exists $\delta > 0$ and $\Omega_\delta \in \mathcal{F}$ with $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$ over which we have uniform convergence of the objective. If δ could be 0 (a.s. uniform convergence), there would be a $\hat{t} > 0$ such that $X(t) \in [f \leq r], \forall t > \hat{t}$ a.s.. Thus, the statements in Theorem 3.3.3 would hold if we replace δ by 0 and \hat{t}_δ by \hat{t} . The proof is far easier in this case. It is however not easy to ensure the existence of such \hat{t} in general.

Remark 3.3.8. In order to find explicit convergence rates in Theorem 3.3.3 we have to solve or bound the solution of the Cauchy problems (C.1) and (C.2). We can generalize these problems as follows: Let $a > 0, b > 1, \hat{t}_\delta > 0, \delta > 0, y_0(\hat{t}_\delta, \delta) > 0$ and p_δ a nonnegative integrable function. Consider

$$(C.0) \quad \begin{cases} y'(t) &= -ay^b(t) + p_\delta(t), \quad \forall t > \hat{t}_\delta \\ y(\hat{t}_\delta) &= y_0(\hat{t}_\delta, \delta). \end{cases}$$

Although one could give an explicit ad-hoc p_δ in order to find a particular solution of (C.0), the dependence of this function on \hat{t}_δ is unavoidable, which is a problem, since p_δ is explicitly related to σ_∞ , and this in turn is the one that defines \hat{t}_δ in the first place.

To the best of our knowledge, there is no way to arithmetically solve this non linear ODE, not even a sharp bound of the solution.

Nevertheless, if $y(t) = \mathcal{O}\left((t+1)^{-\frac{1}{b-1}}\right)$, then $p_\delta(t) = \mathcal{O}\left((t+1)^{-\frac{b}{b-1}}\right)$. Which leads us to make the following conjecture:

Conjecture 3.3.9. *If $p_\delta = \mathcal{O}(\sigma_\infty^2)$ and $\sigma_\infty^2(t) = \mathcal{O}\left((t+1)^{-\frac{b}{b-1}}\right)$ (for constants independent of δ and \hat{t}_δ), then $y(t) = \mathcal{O}\left((t+1)^{-\frac{1}{b-1}}\right)$.*

Proof. Given that $\sigma_\infty \in L^2(\mathbb{R}_+)$, if it is nonincreasing, we have immediately that it vanishes at infinity. Let $x^* \in \mathcal{S}$. Let us recall that by claim (i) of Theorem 3.2.3, there exists $C^* > 0$ such that

$$\sup_{t \geq 0} \mathbb{E}(f(X(t)) - \min f) \leq \frac{1}{2} \mathbb{E}(\|\nabla f(X(t)) - \nabla f(x^*)\|^2) + \frac{1}{2} \mathbb{E}(\|X(t) - x^*\|^2) \leq \frac{L^2 + 1}{2} C^* < +\infty.$$

On the other hand, by Theorem 3.2.3(iii), there exists a set $\Omega_{\text{conv}} \in \mathcal{F}$ such that $\mathbb{P}(\Omega_{\text{conv}}) = 1$ where, for all $\omega \in \Omega_{\text{conv}}$: $\lim_{t \rightarrow +\infty} f(X(\omega, t)) = \min f$, $t \mapsto f(X(\omega, t))$ is continuous. Then, by Theorem 2.7.1 for every $\delta > 0$ there exists $\Omega_\delta \in \mathcal{F}$ such that $\Omega_\delta \subset \Omega_{\text{conv}}$, $\mathbb{P}(\Omega_\delta) > 1 - \delta$ and $f(X(\cdot, t))$ converges uniformly to $\min f$ on Ω_δ . This means that given $r \geq \min f$, and for every $\delta > 0$, there exist $\hat{t}_\delta > 0$

and $\Omega_\delta \in \mathcal{F}$ with $\mathbb{P}(\Omega_\delta) > 1 - \delta$ such that $X(\omega, t) \in [f \leq r]$ for all $t \geq \hat{t}_\delta$ and $\omega \in \Omega_\delta$. On the other hand, since $f \in \mathcal{L}^q(\mathcal{S})$, by Proposition 2.4.6, there exists $r > \min f$ such that f verifies the p -Hölderian error bound inequality (2.11) on $[\min f < f < r]$. Consequently, for any $\delta > 0$, there exists $t \geq \hat{t}_\delta$ large enough such that the p -Hölderian error bound inequality holds at $X(\omega, t)$ for all $t \geq \hat{t}_\delta$ and $\omega \in \Omega_\delta$.

We are now ready to start. Let $x^* \in \mathcal{S}$, $\delta > 0$, and $t \geq \hat{t}_\delta$.

(i) $q = \frac{1}{2}$: Denote $\tilde{g}(t) = \tilde{\phi}(X(t)) = f(X(t)) - \min f$ and $\tilde{G}(t) = \mathbb{E}(\mathbb{1}_{\Omega_\delta} \tilde{g}(t))$. By Proposition 2.7.6

$$\begin{aligned} \tilde{g}(t) &\leq \tilde{g}(\hat{t}_\delta) - \int_{\hat{t}_\delta}^t \left\langle \nabla f(X(s)), \nabla \tilde{\phi}(X(s)) \right\rangle ds + \frac{L}{2} \int_{\hat{t}_\delta}^t \text{tr}[\Sigma(s, X(s))] ds \\ &\quad + \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) \nabla f(X(s)), dW(s) \rangle. \end{aligned} \quad (3.27)$$

Multiplying both sides by $\mathbb{1}_{\Omega_\delta}$ and taking expectation we obtain

$$\begin{aligned} \tilde{G}(t) - \tilde{G}(\hat{t}_\delta) &\leq -\mathbb{E} \left[\int_{\hat{t}_\delta}^t \|\nabla f(X(s))\|^2 \mathbb{1}_{\Omega_\delta} ds \right] + \frac{L}{2} \mathbb{E} \left[\int_{\hat{t}_\delta}^t \text{tr}[\Sigma(s, X(s))] ds \right] \\ &\quad + \mathbb{E} \left[\mathbb{1}_{\Omega_\delta} \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) \nabla f(X(s)), dW(s) \rangle \right]. \end{aligned} \quad (3.28)$$

On the other hand, we have

$$\begin{aligned} \mathbb{E} \left(\int_0^T \|\sigma^*(s, X(s)) \nabla f(X(s))\|^2 ds \right) &\leq L^2 \mathbb{E} \left(\int_0^T \sigma_\infty^2(s) \|X(s) - x^*\|^2 ds \right) \\ &\leq L^2 C^* \int_0^{+\infty} \sigma_\infty^2(s) ds < +\infty, \quad \forall T > 0. \end{aligned}$$

Since $\mathbb{E} \left[\int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) \nabla f(X(s)), dW(s) \rangle \right] = 0$, we have

$$\begin{aligned} &\mathbb{E} \left[\mathbb{1}_{\Omega_\delta} \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) (\nabla f(X(s))), dW(s) \rangle \right] = \\ &\quad - \mathbb{E} \left[\mathbb{1}_{\Omega_{\text{conv}} \setminus \Omega_\delta} \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) (\nabla f(X(s))), dW(s) \rangle \right]. \end{aligned}$$

The last term can be bounded as

$$\begin{aligned} &\left| \mathbb{E} \left[\mathbb{1}_{\Omega_{\text{conv}} \setminus \Omega_\delta} \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) (\nabla f(X(s))), dW(s) \rangle \right] \right| \\ &\leq \sqrt{\mathbb{E}(\mathbb{1}_{\Omega_{\text{conv}} \setminus \Omega_\delta})} \sqrt{\mathbb{E} \left[\left(\int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) (\nabla f(X(s))), dW(s) \rangle \right)^2 \right]} \\ &\leq L\sqrt{\delta} \sqrt{\mathbb{E} \left[\int_{\hat{t}_\delta}^t \sigma_\infty^2(s) \|X(s) - x^*\|^2 ds \right]} \\ &\leq L\sqrt{C^*} \sqrt{\delta} \sqrt{\int_{\hat{t}_\delta}^t \sigma_\infty^2(s) ds} = L\sqrt{C^*} \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s)}{2\sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) du}} ds, \end{aligned}$$

where we have used the fundamental theorem of calculus to arrive at the last equality. Let us notice that if (3.20) holds, then Proposition 3.2.6 tells us that $\mathbb{E}(f(X(t)) - \min f) \leq K't^{\beta-1}$ with $\beta \in [0, 1]$, and for some $K' > 0$. In this case, Cauchy-Schwarz inequality and Corollary 2.7 yield

$$\left| \mathbb{E} \left[\mathbb{1}_{\Omega_{\text{conv}} \setminus \Omega_\delta} \int_{\hat{t}_\delta}^t \langle \sigma^*(s, X(s)) (\nabla f(X(s))), dW(s) \rangle \right] \right| \leq \sqrt{2LK'} \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s) s^{\beta-1}}{2\sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) u^{\beta-1} du}} ds.$$

Injecting this into (3.28), we have for all $t > \hat{t}_\delta$

$$\begin{aligned} \tilde{G}(t) \leq \tilde{G}(\hat{t}_\delta) - \mathbb{E} \left[\int_{\hat{t}_\delta}^t \|\nabla f(X(s))\|^2 \mathbb{1}_{\Omega_\delta} ds \right] + \frac{L}{2} \int_{\hat{t}_\delta}^t \sigma_\infty^2(s) ds \\ + \begin{cases} C_k \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s) s^{\beta-1}}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) u^{\beta-1} du}} ds, & \forall t > \hat{t}_\delta \text{ if (3.20) holds,} \\ C_l \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s)}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) du}} ds & \text{otherwise,} \end{cases} \end{aligned} \quad (3.29)$$

where $C_l = L\sqrt{C^*}$, $C_k = \sqrt{2LK'}$ and recall that $C_\infty = \sqrt{\int_0^{+\infty} \sigma_\infty^2(s) ds}$. Recalling $l_\delta(t)$ and $k_\delta(t)$ from (3.21)-(3.22), and by Fubini's theorem, (3.29) becomes

$$\tilde{G}(t) \leq \tilde{G}(\hat{t}_\delta) - \int_{\hat{t}_\delta}^t \mathbb{E} \left[\|\nabla f(X(s))\|^2 \mathbb{1}_{\Omega_\delta} \right] ds + \begin{cases} \int_{\hat{t}_\delta}^t k_\delta(s) ds & \text{if (3.20) holds,} \\ \int_{\hat{t}_\delta}^t l_\delta(s) ds & \text{otherwise.} \end{cases} \quad (3.30)$$

Since $f \in \mathbb{L}^{1/2}(\mathcal{S})$, there exists $\mu > 0$ such that

$$\tilde{G}(t) \leq \tilde{G}(\hat{t}_\delta) - \mu^2 \int_{\hat{t}_\delta}^t \tilde{G}(s) ds + \begin{cases} \int_{\hat{t}_\delta}^t k_\delta(s) ds & \text{if (3.20) holds,} \\ \int_{\hat{t}_\delta}^t l_\delta(s) ds & \text{otherwise.} \end{cases} \quad (3.31)$$

To get an explicit bound in (3.31), we use Lemma 2.6.6, which involves solving

$$(E.2) \quad \begin{cases} y'(t) &= -\mu^2 y(t) + l_\delta(t), \quad \forall t > \hat{t}_\delta \\ y(\hat{t}_\delta) &= \tilde{G}(\hat{t}_\delta) \end{cases}$$

$$(E.3) \quad \begin{cases} y'(t) &= -\mu^2 y(t) + k_\delta(t), \quad \forall t > \hat{t}_\delta \\ y(\hat{t}_\delta) &= \tilde{G}(\hat{t}_\delta) \end{cases}$$

Let $\lambda \in]0, 1[$. Using the integrating factor method as in (i), we get for (E.2)

$$\begin{aligned} y(t) &\leq e^{-\mu^2(t-\hat{t}_\delta)} \mathbb{E}(\tilde{g}(\hat{t}_\delta)) \\ &+ \begin{cases} e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) + \frac{l_\delta(\hat{t}_\delta + \lambda(t-\hat{t}_\delta))}{\mu^2} & \text{for (E.2)} \\ e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_k C_\infty \sqrt{\hat{t}_\delta^{\beta-1} \sqrt{\delta}} \right) + \frac{k_\delta(\hat{t}_\delta + \lambda(t-\hat{t}_\delta))}{\mu^2} & \text{for (E.3).} \end{cases} \end{aligned}$$

Using Lemma 2.6.6 and then Corollary 3.3.2, we obtain

$$\begin{aligned} \mathbb{E} [f(X(t)) - \min f] &\leq y(t) + C_f \sqrt{\delta} \\ &\leq e^{-\mu^2(t-\hat{t}_\delta)} \mathbb{E} [f(X(\hat{t}_\delta)) - \min f] + C_f \sqrt{\delta} \\ &+ \begin{cases} e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) + \frac{l_\delta(\hat{t}_\delta + \lambda(t-\hat{t}_\delta))}{\mu^2} & \text{for (E.2)} \\ e^{-\mu^2(1-\lambda)(t-\hat{t}_\delta)} \left(\frac{LC_\infty^2}{2} + C_k C_\infty \sqrt{\hat{t}_\delta^{\beta-1} \sqrt{\delta}} \right) + \frac{k_\delta(\hat{t}_\delta + \lambda(t-\hat{t}_\delta))}{\mu^2} & \text{for (E.3).} \end{cases} \end{aligned}$$

(ii) $q > \frac{1}{2}$:

We use that $f \in \mathbb{L}^q(\mathcal{S})$ with $q > \frac{1}{2}$ and the computations of (i). We embark from inequality (3.29) to get

$$\begin{aligned} \tilde{G}(t) \leq \tilde{G}(\hat{t}_\delta) - \mathbb{E} \left[\int_{\hat{t}_\delta}^t \|\nabla f(X(s))\|^2 \mathbb{1}_{\Omega_\delta} ds \right] + \frac{L}{2} \int_{\hat{t}_\delta}^t \sigma_\infty^2(s) ds \\ + \begin{cases} C_k \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s) s^{\beta-1}}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) u^{\beta-1} du}} ds, & \forall t > \hat{t}_\delta \text{ if (3.20) holds,} \\ C_l \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s)}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) du}} ds & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} \tilde{G}(t) \leq \tilde{G}(\hat{t}_\delta) - \mu^2 \mathbb{E} \left[\int_{\hat{t}_\delta}^t \tilde{G}^{2q}(s) ds \right] + \frac{L}{2} \int_{\hat{t}_\delta}^t \sigma_\infty^2(s) ds \\ + \begin{cases} C_k \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s) s^{\beta-1}}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) u^{\beta-1} du}} ds, & \forall t > \hat{t}_\delta \text{ if (3.20) holds,} \\ C_l \sqrt{\delta} \int_{\hat{t}_\delta}^t \frac{\sigma_\infty^2(s)}{2 \sqrt{\int_{\hat{t}_\delta}^s \sigma_\infty^2(u) du}} ds & \text{otherwise,} \end{cases} \end{aligned}$$

In the last inequality, we used that $q > \frac{1}{2}$ and Jensen's inequality.

The idea is again to use the comparison lemma (Lemma 2.6.6), which will now involve solving the Cauchy problem (C.1), and finally invoke Corollary 3.3.2. \square

3.4 Non-smooth structured convex optimization

In this section, we turn to the composite convex minimization problem with additive structure

$$\min_{x \in \mathbb{H}} F(x) \stackrel{\text{def}}{=} f(x) + g(x), \quad (\mathcal{P}_{\text{comp}})$$

where the objective F satisfies the following standing assumptions:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ g : \mathbb{H} \rightarrow \mathbb{R} \text{ is proper, lsc and convex;} \\ \mathcal{S}_F \stackrel{\text{def}}{=} \text{argmin}(F) \neq \emptyset. \end{cases} \quad (\mathbf{H}_F)$$

The importance of this class of problems comes from its wide spectrum of applications ranging from data processing, to machine learning and statistics to name a few.

We consider two different approaches leading to different SDE's. The first is based on a fixed point argument and the use of the notion of cocoercive monotone operator. The second approach is based on a regularization/smoothing argument, for instance the Moreau envelope.

3.4.1 Fixed point approach via cocoercive monotone operators

Let us start with some classical definitions concerning monotone operators.

Definition 3.4.1. The graph of an operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is the set:

$$\text{gph}(A) = \{(x, u) \in \mathbb{H} \times \mathbb{H} : u \in A(x)\}.$$

Definition 3.4.2. An operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is monotone if

$$\langle u - v, x - y \rangle \geq 0, \quad \forall (x, u) \in \text{gph}(A), (y, v) \in \text{gph}(A).$$

It is maximally monotone if there exists no monotone operator whose graph properly contains $\text{gph}(A)$. Moreover, A is α -strongly monotone with modulus $\alpha > 0$ if

$$\langle u - v, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall (x, u) \in \text{gph}(A), (y, v) \in \text{gph}(A).$$

Remark 3.4.3. If A is maximally monotone and strongly monotone, then

$$A^{-1}(0) \stackrel{\text{def}}{=} \{x \in \mathbb{H} : A(x) = 0\}$$

is non-empty and reduced to a singleton.

Remark 3.4.4. The subdifferential operator ∂g of $g \in \Gamma_0(\mathbb{H})$ is maximally monotone.

We recall that cocoercivity was defined in Section 2.3.

Remark 3.4.5. If $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$, then the operator ∇f is L^{-1} -cocoercive.

Our interest now is to solve the structured monotone inclusion problem

$$0 \in A(x) + B(x),$$

where A is maximally monotone, and B is cocoercive with $(A + B)^{-1}(0) \neq \emptyset$. This is of course a generalization of $(\mathcal{P}_{\text{comp}})$ by taking $A = \partial g$ and $B = \nabla f$.

A favorable situation occurs when one can compute the resolvent operator of A

$$J_{\mu A} = (I + \mu A)^{-1}, \quad \mu > 0.$$

In this case, we can develop a strategy parallel to the one which consists in replacing a maximally monotone operator by its Yosida approximation. Indeed, given $\mu > 0$, we have

$$(A + B)(x) \ni 0 \iff x - J_{\mu A}(x - \mu B(x)) = 0 \iff M_{A,B,\mu}(x) = 0, \quad (3.32)$$

where $M_{A,B,\mu} : \mathbb{H} \rightarrow \mathbb{H}$ is the single-valued operator defined by

$$M_{A,B,\mu}(x) = \frac{1}{\mu} (x - J_{\mu A}(x - \mu B(x))). \quad (3.33)$$

$M_{A,B,\mu}$ is closely tied to the well-known forward-backward fixed point operator. Moreover, when $B = 0$, $M_{A,B,\mu} = \frac{1}{\mu} (I - J_{\mu A})$ which is nothing but the Yosida regularization of A with index μ . As a remarkable property, for the μ parameter properly set, the operator $M_{A,B,\mu}$ is cocoercive. This is made precise in the following result.

Proposition 3.4.6. [19, Lemma B.1] *Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a general maximally monotone operator, and let $B : \mathbb{H} \rightarrow \mathbb{H}$ be a monotone operator which is λ -cocoercive. Assume that $\mu \in]0, 2\lambda[$. Then, $M_{A,B,\mu}$ is ρ -cocoercive with*

$$\rho = \mu \left(1 - \frac{\mu}{4\lambda}\right).$$

We first focus on finding the zeros of M , where

$$M : \mathbb{H} \rightarrow \mathbb{H} \text{ is cocoercive and } M^{-1}(0) \neq \emptyset. \quad (\mathbf{H}_M)$$

We will then specialize our results to the case of a structured operator of the form $M_{A,B,\mu}$.

Our goal is to handle the situation where M can be evaluated up to a stochastic error. We therefore consider the following SDE with an \mathcal{F}_0 -measurable initial data X_0 :

$$\begin{cases} dX(t) = -M(X(t))dt + \sigma(t, X(t))dW(t), & t \geq 0; \\ X(0) = X_0. \end{cases} \quad (\text{SDE}^M)$$

As in Section 3.1, we will assume that the volatility matrix $\sigma : \mathbb{R}_+ \times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ satisfies (\mathbf{H}_σ) , W is a \mathcal{F}_t -adapted \mathbb{K} -valued Brownian motion.

Remark 3.4.7. The motivation of (SDE^M) comes again from the Robbins-Monro stochastic approximation algorithm where the martingale difference noise/error is induced by randomly approximating the action of the whole fixed point operator $M_{A,B,\mu}$. This allows for instance for inexact computation of the resolvent of A with random noise. However, the situation is more intricate when the noise is solely on B (*i.e.* inside the resolvent), as it is standard in many applications (think of $B = \nabla f$ and the latter is accessible only some unbiased stochastic estimator). In this case, to justify moving the noise outside of the resolvent, one has to modify (SDE^M) to a limiting continuous-time process of a forward-backward scheme, which would take us to the land of *stochastic differential inclusions* (SDI). SDI's were only introduced in the early 80's by [121, 122], where the notion of solutions with path-wise

uniqueness of a solution for a certain class of maximal monotone operators A was introduced. Theory of SDI's has subsequently received much attention with general applications including beyond the maximal monotone case; see *e.g.* [120]. We point out in particular the results of [173] who was the first to show existence and uniqueness of a solution to SDI's with maximal monotone A and Lipschitz continuous B using the Yosida approximation of A , hence extending known results of Brézis [66] in the deterministic case. This is yet another justification behind our second approach using Moreau-Yosida regularization, though restricted to functions. However, handling SDI's properly necessitates much more care and many new techniques and notions. This will be the whole content of Chapter 4.

Let us now state the natural extensions of our main results to this situation.

Theorem 3.4.8. *Let $M : \mathbb{H} \rightarrow \mathbb{H}$ be a cocoercive operator. Consider the stochastic differential equation (SDE^M), with M and σ under the hypotheses (H_M) and (H_σ), respectively. Additionally, let $\nu \geq 2$, $X_0 \in L^\nu(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Then, there exists a unique solution $X \in S_{\mathbb{H}}^\nu$. Moreover, if $\sigma_\infty \in L^2(\mathbb{R}_+)$, then:*

$$(i) \mathbb{E} [\sup_{t \geq 0} \|X(t)\|^\nu] < +\infty.$$

$$(ii) \forall x^* \in M^{-1}(0), \lim_{t \rightarrow +\infty} \|X(t) - x^*\| \text{ exists a.s. and } \sup_{t \geq 0} \|X(t)\| < +\infty \text{ a.s..}$$

$$(iii) \lim_{t \rightarrow +\infty} \|M(X(t))\| = 0 \text{ a.s..}$$

$$(iv) \text{ There exists an } M^{-1}(0)\text{-valued random variable } X^* \text{ such that } w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^* \text{ a.s..}$$

Proof. Existence and uniqueness follow from Theorem 2.7.4 since M is Lipschitz continuous and σ verifies (H_σ). The proof of the first three items remains the same as for Theorem 3.2.3, where we use the cocoercivity of M instead of the convexity of f in the third item to prove that $\lim_{t \rightarrow +\infty} \|M(X(t))\| = 0$ a.s.. For the last item, it suffices to use that the operator M is demiclosed (since it is maximal monotone) to conclude with Opial's Lemma. \square

Theorem 3.4.9. *Consider the dynamic (SDE^M) where M and σ satisfy the assumptions (H_M) and (H_σ). Moreover, let M be a ρ -cocoercive operator. Additionally, $X_0 \in L^2(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Let $X \in S_{\mathbb{H}}^2$ be the unique solution of (SDE^M), then the following properties are satisfied:*

$$(i) \text{ Let } \overline{M \circ X}(t) \stackrel{\text{def}}{=} t^{-1} \int_0^t M(X(s)) ds \text{ and } \overline{\|M(X(t))\|^2} \stackrel{\text{def}}{=} t^{-1} \int_0^t \|M(X(s))\|^2 ds. \text{ We have}$$

$$\mathbb{E} \left[\|\overline{M \circ X}(t)\|^2 \right] \leq \mathbb{E} \left[\overline{\|M(X(t))\|^2} \right] \leq \frac{\mathbb{E} (\text{dist}(X_0, M^{-1}(0))^2)}{2\rho t} + \frac{\sigma_*^2}{2\rho}, \quad \forall t > 0. \quad (3.34)$$

Besides, if σ_∞ is $L^2(\mathbb{R}_+)$, then

$$\mathbb{E} \left[\|\overline{M \circ X}(t)\|^2 \right] \leq \mathbb{E} \left[\overline{\|M(X(t))\|^2} \right] = \mathcal{O} \left(\frac{1}{t} \right), \quad \forall t > 0. \quad (3.35)$$

(ii) If M is α -strongly monotone, then $M^{-1}(0) = \{x^*\}$ and

$$\mathbb{E} \left(\|X(t) - x^*\|^2 \right) \leq \mathbb{E} \left(\|X_0 - x^*\|^2 \right) e^{-\alpha t} + \frac{\sigma_*^2}{\alpha}, \quad \forall t \geq 0. \quad (3.36)$$

If, moreover, σ_∞ is nonincreasing and vanishes at infinity, then for every $\lambda \in]0, 1[$

$$\mathbb{E} \left(\|X(t) - x^*\|^2 \right) \leq \mathbb{E} \left(\|X_0 - x^*\|^2 \right) e^{-\gamma t} + \frac{\sigma_*^2}{\alpha} e^{-\alpha t(1-\lambda)} + \sigma_\infty^2(\lambda t), \quad \forall t > 0. \quad (3.37)$$

Proof. Analogous to Theorem 3.2.5. \square

As an immediate consequence of the above Theorem, by considering the cocoercive operator $M_{A,B,\mu}$ defined in (3.33), we obtain the following result.

Corollary 3.4.10. *Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximally monotone operator and $B : \mathbb{H} \rightarrow \mathbb{H}$ be a λ -cocoercive operator, $\lambda > 0$. Let $M_{A,B,\mu}$ be the operator defined in (3.33). Assume that $\mu \in]0, 2\lambda[$ and $(A + B)^{-1}(0) \neq \emptyset$. Then, the operator $M_{A,B,\mu}$ is ρ -cocoercive with $\rho = \mu \left(1 - \frac{\mu}{4\lambda}\right)$, letting $\nu \geq 2$ and considering the SDE with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ which is \mathcal{F}_0 -measurable:*

$$\begin{cases} dX(t) = -M_{A,B,\mu}(X(t))dt + \sigma(t, X(t))dW(t), & t \geq 0; \\ X(0) = X_0, \end{cases} \quad (\text{SDE}^{M_{A,B,\mu}})$$

we can conclude the same results as in Theorem 3.4.8 and Theorem 3.4.9. In particular, if $\sigma_\infty \in L^2(\mathbb{R}_+)$, there exists an $(A + B)^{-1}(0)$ -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s..

This result naturally applies to problem $(\mathcal{P}_{\text{comp}})$ when $\mathcal{S}_F = \text{argmin}(f + g) \neq \emptyset$ by taking $A = \partial g$ and $B = \nabla f$. In this case, one has that $X(t)$ converges a.s. to an \mathcal{S}_F -valued random variable. Moreover, using standard inequalities, see e.g. [47], one can show that

$$\mathbb{E} \left[F \left(t^{-1} \int_0^t (\text{prox}_{\mu g}(X(s) - \mu \nabla f(X(s)))) ds \right) - \min(F) \right] = \mathcal{O} \left(\sqrt{\mathbb{E} \left[\|M_{\partial g, \nabla f, \mu}(X(t))\|^2 \right]} \right),$$

where $\text{prox}_{\mu g} = (I + \mu \partial g)^{-1}$ is the proximal mapping of g . From this, one can deduce an $\mathcal{O}(t^{-1/2})$ rate thanks to (3.34) and (3.35).

Remark 3.4.11. In the finite dimensional setting (*i.e.* $\mathbb{H} = \mathbb{R}^d$), we could turn to the local convergence properties of (SDE^M) by extending the results presented in Theorem 3.3.3. To this end, we would need an extension of the Hölderian error bound inequality (or Łojasiewicz inequality) to the operator setting. For convex functions, it is known that error bound inequalities are closely related to metric subregularity of the subdifferential [9, 124, 123]. This leads to the following definition.

Definition 3.4.12. Let $M : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a single-valued operator. We say that M satisfies the Hölder metric subregularity property with exponent $p \geq 2$ at $x^* \in M^{-1}(0)$ if there exists $\varrho > 0$ and a neighbourhood \mathcal{V}_{x^*} such that

$$\|M(x)\|^2 \geq \varrho \text{dist}(x, M^{-1}(0))^p, \quad \forall x \in \mathcal{V}_{x^*}. \quad (3.38)$$

If this inequality holds for any $x^* \in M^{-1}(0)$ with the same ϱ , we write $M \in \text{HMS}^p(\mathbb{H})$.

3.4.2 Approach via Moreau-Yosida regularization

The previous approach, though it is able to deal with more general setting (that of monotone inclusions), took us out of the framework of convex optimization by considering instead a dynamic governed by a cocoercive operator. In particular, the perturbation/noise is considered on the whole operator evaluation and not on a part of it (*i.e.* B) as it is standard in many applications. Moreover this approach led to a pessimistic convergence rate estimate when specialized to convex function minimization. By contrast, the following approach will operate directly on problem $(\mathcal{P}_{\text{comp}})$ and is based on a standard smoothing approach, replacing the non-smooth part g by its Moreau envelope [48].

3.4.2.1 Moreau envelope

Let us start by recalling some basic facts concerning the Moreau envelope.

Definition 3.4.13. Let $g \in \Gamma_0(\mathbb{H})$. Given $\theta > 0$, the Moreau envelope of g of parameter θ is the function

$$g_\theta(x) \stackrel{\text{def}}{=} \inf_{y \in \mathbb{H}} \left(g(y) + \frac{1}{2\theta} \|x - y\|^2 \right) = \left(g \square \frac{1}{\theta} q \right) (x)$$

where \square is the infimal convolution operator and $q(x) = \frac{1}{2} \|x\|^2$.

The Moreau envelope has remarkable approximation and regularization properties, as summarized in the following statement.

Proposition 3.4.14. *Let $g \in \Gamma_0(\mathbb{H})$.*

- (i) $g_\theta(x) \downarrow \inf g(\mathbb{H})$ as $\theta \uparrow +\infty$.
- (ii) $g_\theta(x) \uparrow g(x)$ as $\theta \downarrow 0$.
- (iii) $g_\theta(x) \leq g(x)$ for any $\theta > 0$ and $x \in \mathbb{H}$,
- (iv) $\operatorname{argmin}(g_\theta) = \operatorname{argmin}(g)$ for any $\theta > 0$,
- (v) $g(x) - g_\theta(x) \leq \frac{\theta}{2} \|\partial^0 g(x)\|^2$ for any $\theta > 0$ and $x \in \operatorname{dom}(\partial g)$,
- (vi) $g_\theta \in C_{\frac{1}{\theta}}^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$ for any $\theta > 0$.

We use the following notation in the rest of the section: $F \stackrel{\text{def}}{=} f + g$, $\mathcal{S} \stackrel{\text{def}}{=} \operatorname{argmin}(f)$, $F_\theta \stackrel{\text{def}}{=} f + g_\theta$ and $\mathcal{S}_\theta \stackrel{\text{def}}{=} \operatorname{argmin}(f_\theta)$.

Note that $F_\theta \in C_{L+\frac{1}{\theta}}^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H})$. Thus we will use F_θ as the potential driving in (SDE), that is

$$\begin{cases} dX(t) = -\nabla F_\theta(X(t))dt + \sigma(t, X(t))dW(t), & t \geq 0; \\ X(0) = X_0. \end{cases} \quad (\text{SDE}_\theta)$$

Throughout this section, we assume that $X_0 \in L^2(\Omega; \mathbb{H})$ and is \mathcal{F}_0 -measurable. Under (\mathbf{H}_F) and (\mathbf{H}_σ) , we will show almost sure weak convergence of the trajectory and corresponding convergence rates.

Remark 3.4.15. Though we focus here on the Moreau envelope, our convergence results, in particular, Proposition 3.4.18, still hold with infimal-convolution based smoothing using more general smooth kernels beyond the norm squared; see [48, Section 4.4].

3.4.2.2 Convergence of the trajectory

Applying Theorem 3.2.3 to F_θ , we have the following result.

Proposition 3.4.16. *For any $\theta > 0$, let $X_0 \in L^2(\Omega; \mathbb{H})$ and $X_\theta \in S_{\mathbb{H}}^2$ be the unique solution of the dynamic (SDE $_\theta$) governed by the potential F_θ , and make assumptions (\mathbf{H}_F) , $\mathcal{S}_\theta \neq \emptyset$, (\mathbf{H}_σ) and $\sigma_\infty \in L^2(\mathbb{R}_+)$. Then there exists an \mathcal{S}_θ -valued random variable X_θ^* such that*

$$\text{w-lim}_{t \rightarrow +\infty} X_\theta(t) = X_\theta^*, \quad \text{a.s.}$$

If $f = 0$, then $\mathcal{S}_\theta = \mathcal{S}$ (see Proposition 3.4.14(iv)), and Proposition 3.4.16 provides almost sure weak convergence to a solution of ($\mathcal{P}_{\text{comp}}$). On the other hand for $f \neq 0$, $\mathcal{S} \neq \mathcal{S}_\theta$ in general and we only obtain an ‘‘approximate’’ solution of ($\mathcal{P}_{\text{comp}}$); see Proposition 3.4.17(ii) for a quantitative estimate of this approximation when f is strongly convex. To obtain a true solution of the initial problem, a common device consists in using a diagonalization process that combines the dynamic with the approximation. Specifically, one considers

$$\begin{cases} dX(t) = -\nabla F_{\theta(t)}(X(t))dt + \sigma(t, X(t))dW(t), & t \geq 0; \\ X(0) = X_0, \end{cases} \quad (\text{SDE}_{\theta(t)})$$

where $\theta(t) \downarrow 0$ as $t \rightarrow +\infty$. In the deterministic case, an abundant literature has been devoted to the convergence of this type of systems. Note that unlike the cocoercive approach, we are now faced with a non-autonomous stochastic differential equation, making this a difficult problem, a subject for further research (see also Remark 3.4.7).

3.4.2.3 Convergence rates

We start with the following uniform bound on \mathcal{S}_θ which holds under slightly reinforced, but reasonable assumptions on f and g .

Proposition 3.4.17. *Consider f, g where f and g are proper lsc and convex, and g is also L_0 -Lipschitz continuous.*

(i) *Assume that $F = f + g$ is coercive. Then, there exists $C > 0$, such that for any $\theta \in [0, 1]$*

$$\sup_{z \in \mathcal{S}_\theta} \|z\| \leq C. \quad (3.39)$$

(ii) *Assume that $f \in \Gamma_\mu(\mathbb{H})$ for $\mu > 0$. Then (3.39) holds for every $\theta \in [0, 1]$. Moreover, $\mathcal{S} = \{x^*\}$, $\mathcal{S}_\theta = \{x_\theta^*\}$ and*

$$\|x_\theta^* - x^*\|^2 \leq \frac{L_0^2}{\mu} \theta. \quad (3.40)$$

Proof. (i) Since F is coercive, so is F_θ . Thus both \mathcal{S} and \mathcal{S}_θ are non-empty compact sets. Let $x_\theta^* \in \mathcal{S}_\theta$ and $x^* \in \mathcal{S}$. By Proposition 3.4.14(v) and Lipschitz continuity of g , we obtain

$$F(x_\theta^*) - F_\theta(x_\theta^*) = g(x_\theta^*) - g_\theta(x_\theta^*) \leq \frac{L_0^2}{2} \theta.$$

Moreover,

$$F_\theta(x_\theta^*) + \frac{L_0^2}{2} \theta \leq F_\theta(x^*) + \frac{L_0^2}{2} \theta \leq F(x^*) + \frac{L_0^2}{2} \theta \leq \min(F) + \frac{L_0^2}{2} \theta \stackrel{\text{def}}{=} \tilde{C},$$

where the second inequality is given by Proposition 3.4.14(iv). On the other hand, the coercivity of F implies that there exists $a > 0, b \in \mathbb{R}$ such that for any $x \in \mathbb{H}$

$$a \|x\| + b \leq F(x).$$

Therefore, collecting the above inequalities yields

$$a \|x_\theta^*\| + b \leq F(x_\theta^*) \leq \tilde{C}.$$

Using that x_θ^* is arbitrary in \mathcal{S}_θ , and defining $C \stackrel{\text{def}}{=} \frac{\tilde{C}-b}{a} \geq 0$, we obtain (3.39), or equivalently that the set of approximate minimizers is bounded independently of θ .

(ii) Since f is μ -strongly convex, so are F and F_θ . In turn, F is coercive and thus (3.39) holds by claim (i). Strong convexity implies uniqueness of minimizers of F and F_θ . Moreover,

$$\frac{\mu}{2} \|x_\theta^* - x^*\|^2 \leq F_\theta(x^*) - F_\theta(x_\theta^*). \quad (3.41)$$

From Proposition 3.4.14(iii)-(v) and Lipschitz continuity of g , we infer that

$$F_\theta(x^*) - F_\theta(x_\theta^*) \leq F(x^*) - F_\theta(x_\theta^*) \leq F(x_\theta^*) - F_\theta(x_\theta^*) \leq \frac{L_0^2}{2} \theta. \quad (3.42)$$

Combining (3.41) and (3.42), we get the claimed bound. \square

We are now ready to establish complexity results.

Proposition 3.4.18. *Suppose that in addition to (\mathbf{H}_F) and (\mathbf{H}_σ) , $F = f + g$ is coercive and g is L_0 -Lipschitz continuous. Let $X_0 \in \mathbb{L}^2(\Omega; \mathbb{H})$ and $X_\theta \in \mathcal{S}_{\mathbb{H}}^2$ be the unique solution of (SDE_θ) governed by F_θ with $\theta \in [0, 1]$. Let $C_0 = \mathbb{E}[(\|X_0\| + C)^2]$, where C is the constant defined in (3.39). Then the following statements hold for any $t > 0$.*

(i) Let $\overline{X}_\theta(t) = t^{-1} \int_0^t X_\theta(s) ds$, then

$$\mathbb{E} (F(\overline{X}_\theta(t)) - \min F) \leq \frac{C_0}{2t} + \frac{\sigma_*^2}{2} + \theta \frac{L_0^2}{2}.$$

Besides, if $\sigma_\infty \in L^2(\mathbb{R}_+)$, then

$$\mathbb{E} (F(\overline{X}_\theta(t)) - \min F) = \frac{C_0 + \int_0^{+\infty} \sigma_\infty^2(s) ds}{2t} + \theta \frac{L_0^2}{2}.$$

(ii) If σ_∞ verifies (3.20) with $\beta \in [0, 1[$, and $\theta \in]0, 1]$, then

$$\mathbb{E} (F(X(t)) - \min F) = \frac{C_0}{2t} + \frac{K(1+L)}{2t^{1-\beta\theta}} + \theta \frac{L_0^2}{2}.$$

(iii) If, in addition, $f \in \Gamma_\mu(\mathbb{H})$ for some $\mu > 0$, then $\mathcal{S} = \{x^*\}$ and

$$\mathbb{E} (\|X_\theta(t) - x^*\|^2) \leq 2C_0 e^{-\mu t} + \frac{2\sigma_*^2}{\mu} + 2 \frac{L_0^2}{\mu} \theta.$$

Besides, if σ_∞ is nonincreasing and vanishes at infinity, then $\forall \lambda \in]0, 1[$:

$$\mathbb{E} (\|X_\theta(t) - x^*\|^2) \leq 2C_0 e^{-\mu t} + \frac{2\sigma_*^2}{\mu} e^{-\mu(1-\lambda)t} + 2\sigma_\infty^2(\lambda t) + 2 \frac{L_0^2}{\mu} \theta.$$

Remark 3.4.19. Observe that when $f = 0$, then $\mathcal{S}_\theta = \mathcal{S} = \{x^*\}$. Therefore in Proposition 3.4.18, the last term in θ can be dropped.

Proof. (i) Combine Theorem 3.2.5(i) applied to F_θ , Proposition 3.4.14(iii) and (v), and Proposition 3.4.17(i).

(ii) Argue as in claim (i) using Proposition 3.2.6 instead of Theorem 3.2.5(i), and use the fact that ∇F_θ is Lipschitz continuous with constant

$$L + \frac{1}{\theta} \leq \frac{L+1}{\theta} \quad \text{for } \theta \in]0, 1].$$

(iii) Combine Theorem 3.2.5(ii) applied to F_θ , Proposition 3.4.17(ii) and Jensen's inequality. □

Chapter 4

Tikhonov Regularization for Stochastic Non-Smooth Convex Optimization

To solve convex optimization problems with a noisy gradient input, we analyze the global behavior of subgradient-like flows under stochastic errors. The objective function is composite, being equal to the sum of two convex functions, one being differentiable and the other potentially non-smooth. We then use stochastic differential inclusions (SDIs) where the drift term is minus the subdifferential of the objective function, and the diffusion term is either bounded or square-integrable. In this context, under Lipschitz's continuity of the differentiable term and a growth condition of the non-smooth term, our first main result shows almost sure weak convergence of the trajectory process towards a minimizer of the objective function. Then, using Tikhonov regularization with a properly tuned vanishing parameter, we can obtain almost sure strong convergence of the trajectory towards the minimum norm solution. We find an explicit tuning of this parameter when our objective function satisfies a local error-bound inequality. We also provide a comprehensive complexity analysis by establishing several new pointwise and ergodic convergence rates in expectation for the convex and strongly convex case.

Main contributions of this chapter

- ▶ Almost sure weak convergence of the trajectory generated by the SDI to the set of minimizers (Theorem 4.3.2).
- ▶ Global convergence rates in expectation of the SDI under convexity and strong convexity of the objective (Theorem 4.3.4).
- ▶ Almost sure strong convergence of the trajectory generated by the Tikhonov regularized SDI, to the minimal norm minimizer (Theorem 4.4.1).
- ▶ Practical situations where the Tikhonov parameter can be exhibited (Theorem 4.4.5).

The content of this chapter was submitted in [145].

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4.1 Introduction

4.1.1 Problem statement

We aim to solve convex minimization problems by means of stochastic differential inclusions (SDI), showing the existence, uniqueness, and properties of the solution. Then, we work with Tikhonov regularization, specifically when the drift term is the sum of the (sub-)gradient of the objective function and of a Tikhonov regularization term with a vanishing coefficient. This makes it possible to take into account a noisy (imprecise) gradient input and obtain convergence a.s. to the minimal norm solution.

Let us consider the minimization problem

$$\min_{x \in \mathbb{H}} F(x) \stackrel{\text{def}}{=} f(x) + g(x), \quad (\mathcal{P}_2)$$

where \mathbb{H} is a separable real Hilbert space, and the objective F satisfies the following standing assumptions:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ g : \mathbb{H} \rightarrow \mathbb{R} \text{ is proper, lsc and convex;} \\ \mathcal{S}_F \stackrel{\text{def}}{=} \operatorname{argmin}(F) \neq \emptyset. \end{cases} \quad (\mathbf{H}_F)$$

To solve (\mathcal{P}_2) , a fundamental dynamic to consider is the subgradient flow, which is the following differential inclusion (DI) starting in $t_0 \geq 0$ with initial condition $x_0 \in \mathbb{H}$:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)), & t > t_0; \\ x(t_0) = x_0. \end{cases} \quad (\mathbf{DI})$$

It is well known since the founding articles of Brezis, Baillon, Bruck in the 1970s that, when the initial data x_0 is in the domain of F , (more generally when it is in its closure), there exists a unique strong global solution of (DI). Moreover, if the solution set $\operatorname{argmin}(F)$ of (\mathcal{P}_2) is nonempty then each solution trajectory of (DI) converges weakly, and its limit belongs to $\operatorname{argmin}(F)$.

Consider \mathbb{K} a real separable Hilbert space. To solve (\mathcal{P}_2) , we will refer to Section 1.2 for the approach that leads to the dynamic that we will study in this chapter, specifically the stochastic counterpart of (DI), which is the following SDI:

$$\begin{cases} dX(t) \in -\partial F(X(t)) + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0, \end{cases} \quad (\text{SDI})$$

where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ (see notation in Section 2.1) is a measurable function that satisfies (\mathbf{H}_σ) , and W is a \mathbb{K} -valued Brownian motion (see Section 2.7 for a precise definition), and the initial data X_0 is an \mathcal{F}_{t_0} -measurable \mathbb{H} -valued random variable. This dynamic can be viewed as a stochastic dissipative system that aims to minimize F if the diffusion term vanishes sufficiently fast. Also, it is the natural extension to the non-smooth setting of the work done in Chapter 3.

An important aspect of our work concerns the Tikhonov regularization of (DI) and (SDI). Given $t_0 > 0$, and a regularization parameter $\varepsilon : [t_0, +\infty[\rightarrow \mathbb{R}_+$, which is a measurable function that vanishes asymptotically in a controlled way, the Tikhonov regularization of (DI) is written:

$$\begin{cases} \dot{x}(t) \in -\partial F(x(t)) - \varepsilon(t)x(t), & t > t_0; \\ x(t_0) = x_0. \end{cases} \quad (\text{DI-TA})$$

The stochastic counterpart of (DI-TA) (which is the Tikhonov regularization of (SDI)), is the following stochastic differential inclusion with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$):

$$\begin{cases} dX(t) \in -\partial F(X(t)) - \varepsilon(t)X(t) + \sigma(t, X(t))dW(t), & t > t_0; \\ X(t_0) = X_0. \end{cases} \quad (\text{SDI-TA})$$

The impact of the Tikhonov term has been studied in depth in the deterministic case (DI-TA) (see [84]). The fact that the Tikhonov regularization parameter $\varepsilon(t)$ tends to zero not too fast as $t \rightarrow +\infty$ induces a hierarchical minimization property: the limit of any trajectory no longer depends on the initial data, it is precisely the minimum norm solution. We propose to extend these results to the stochastic case (SDI-TA) based on the work presented in Chapter 3.

4.1.2 Deterministic subgradient flow with Tikhonov regularization

Let us first recall some basic facts about the deterministic case. To solve (\mathcal{P}_2) , a fundamental dynamic to consider is the subgradient flow of F , *i.e.* the following differential inclusion:

$$\dot{x}(t) \in -\partial F(x(t)). \quad (\text{DI})$$

It is well known since the founding papers of Brezis, Baillon, and Bruck in the 1970s that, if the solution set $\text{argmin}(F)$ of (\mathcal{P}_2) is non-empty and F is convex, lower semicontinuous (lsc) and proper, then each solution trajectory of (DI) converges weakly, and its weak limit belongs to $\text{argmin}(F)$.

In general, the limit solution depends on the initial data and is a priori difficult to specify when one has a set of solutions not reduced to only one element. To remedy this difficulty we consider the differential inclusion with vanishing Tikhonov regularization, $\varepsilon(t) \rightarrow 0$ (denoted (DI-TA)) which gives

$$\dot{x}(t) + \partial F(x(t)) + \varepsilon(t)x(t) \ni 0. \quad (\text{DI-TA})$$

Let \mathcal{H} be a real Hilbert space. To analyze the convergence properties of this dynamic, let us recall basic facts concerning the Tikhonov approximation (1963). It consists in approximating the convex minimization problem (possibly ill-posed)

$$\min_{x \in \mathcal{H}} F(x), \quad (\mathcal{P})$$

by the strongly convex minimization problem ($\varepsilon > 0$):

$$\min_{x \in \mathcal{H}} F(x) + \frac{\varepsilon}{2} \|x\|^2, \quad (\mathcal{P}_\varepsilon)$$

whose unique solution is denoted by x_ε . The following result was first obtained by Browder in 1966 [67, 68].

Theorem 4.1.1. (*Hierarchical minimization*). *Suppose that $\mathcal{S}_F = \operatorname{argmin}(F) \neq \emptyset$. Then,*

$$(i) \lim_{\varepsilon \rightarrow 0} \|x_\varepsilon - x^*\| = 0 \quad \text{where } x^* = P_{\mathcal{S}_F}(0).$$

$$(ii) \|x_\varepsilon\| \leq \|x^*\| \text{ for all } \varepsilon > 0.$$

The system **(DI – TA)** is a special case of the general dynamic model

$$\dot{x}(t) + \nabla F(x(t)) + \varepsilon(t)\nabla\Psi(x(t)) \ni 0 \quad (4.1)$$

which involves two functions F and Ψ intervening with different time scale. When $\varepsilon(\cdot)$ tends to zero moderately slowly, it was shown in [29] that the trajectories of (4.1) converge asymptotically to equilibria that are solutions of the following hierarchical problem: they minimize the function Ψ on the set of minimizers of F . The continuous and discrete-time versions of these systems have a natural connection to the best response dynamics for potential games, domain decomposition for PDE's, optimal transport, and coupled wave equations. In the case of the Tikhonov approximation, a natural choice is to take $\Psi(x) = \|x - x_d\|^2$ where x_d is a desired state. By doing so, we obtain asymptotically the closest possible solution to x_d . By translation, we can immediately reduce ourselves to the case $x_d = 0$, as considered in this article.

The following theorem establishes the convergence of the trajectories of **(DI – TA)** towards the minimum norm solution under minimal assumptions on the parameter $\varepsilon(t)$.

Theorem 4.1.2. (Cominetti-Peypouquet-Sorin, [84])

Suppose that $\varepsilon : [t_0, +\infty[\rightarrow \mathbb{R}_+$ is a measurable function that satisfies:

$$(i) \varepsilon(t) \rightarrow 0 \text{ as } t \rightarrow +\infty;$$

$$(ii) \int_{t_0}^{+\infty} \varepsilon(t) dt = +\infty.$$

*Let $x(\cdot)$ be a solution trajectory of the continuous dynamic **(DI – TA)**, and $x^* \stackrel{\text{def}}{=} P_{\mathcal{S}_F}(0)$. Then, $\text{s-lim}_{t \rightarrow +\infty} x(t) = x^*$.*

Proof. Set $F_\varepsilon(x) \stackrel{\text{def}}{=} F(x) + \frac{\varepsilon}{2}\|x\|^2$. Then **(DI – TA)** can be written equivalently in a denser form as

$$\dot{x}(t) + \partial F_{\varepsilon(t)}(x(t)) \ni 0.$$

Set $h(t) \stackrel{\text{def}}{=} \frac{1}{2}\|x(t) - x^*\|^2$ where $x^* = P_{\mathcal{S}_F}(0)$. Derivation of f and constitutive equation **(DI – TA)** give

$$\dot{h}(t) + \langle -\dot{x}(t), x(t) - x^* \rangle = 0, \quad (4.2)$$

where $-\dot{x}(t) \in \partial F_{\varepsilon(t)}(x(t))$. By strong convexity of $F_{\varepsilon(t)}$, we get

$$F_{\varepsilon(t)}(x^*) \geq F_{\varepsilon(t)}(x(t)) + \langle y(t), x^* - x(t) \rangle + \frac{\varepsilon(t)}{2}\|x(t) - x^*\|^2,$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$.

Using that $F_{\varepsilon(t)}(x(t)) \geq F_{\varepsilon(t)}(x_{\varepsilon(t)})$, we get

$$F(x^*) + \frac{\varepsilon(t)}{2}\|x^*\|^2 \geq F(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2}\|x_{\varepsilon(t)}\|^2 + \langle y(t), x^* - x(t) \rangle + \frac{\varepsilon(t)}{2}\|x(t) - x^*\|^2,$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$.

From $F(x^*) \leq F(x_{\varepsilon(t)})$ we deduce

$$\langle y(t), x(t) - x^* \rangle \geq \varepsilon(t)h(t) + \frac{\varepsilon(t)}{2} (\|x_{\varepsilon(t)}\|^2 - \|x^*\|^2), \quad (4.3)$$

for every $y(t) \in \partial F_{\varepsilon(t)}(x(t))$.

Combining (4.2) with (4.3) we obtain

$$\dot{h}(t) + \varepsilon(t)h(t) \leq \frac{1}{2}\varepsilon(t) (\|x^*\|^2 - \|x_{\varepsilon(t)}\|^2).$$

Integrate the above inequality from t_0 to t . With $m(t) \stackrel{\text{def}}{=} \exp \int_{t_0}^t \varepsilon(s) ds$ we get

$$h(t) \leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^t m'(s) (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds. \quad (4.4)$$

According to hypothesis (i) and the classical property of the Tikhonov approximation we have $x_{\varepsilon(t)} \rightarrow x^*$, and hence $\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2 \rightarrow 0$. To pass to the limit on (4.4) we use hypothesis (ii) which tells us that $m(t) \rightarrow +\infty$. Let us complete the argument by using that convergence implies ergodic convergence. Precisely, given $\delta > 0$, let $t_\delta > t_0$ such that $|\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2| \leq \delta$ for $s \geq t_\delta$. Then split the integral as follows

$$h(t) \leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^{t_\delta} m'(s) (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds + \delta \frac{1}{2m(t)} \int_{t_\delta}^t m'(s) ds \quad (4.5)$$

$$\leq \frac{h(t_0)}{m(t)} + \frac{1}{2m(t)} \int_{t_0}^{t_\delta} m'(s) (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds + \frac{\delta}{2}. \quad (4.6)$$

Then let t tend to infinity, to get $\limsup_{t \rightarrow +\infty} h(t) \leq \frac{\delta}{2}$. This being true for any $\delta > 0$ gives the result. \square

4.2 Stochastic differential inclusions

In this section, we will work with stochastic differential inclusions. For the history of this concept, we refer the reader to [120, Preface]. We will start by showing a general version of the (SDI) dynamic, formally describing what it means to be a solution of that dynamic, and then we will move on to show the conditions under which you can have the existence and uniqueness of a solution. Existence is due to [173] and uniqueness is proven here. Then we will focus on (SDI) and study the conditions on the diffusion term in order to ensure the almost sure weak convergence of the trajectory towards the set of minimizers. Finally, we will show some convergence rates of the objective under the hypothesis of convexity or strong convexity.

4.2.1 Existence and uniqueness of solution

Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$, $b : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathbb{H}$ and $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$. Let $t_0 \geq 0$ and consider the general stochastic differential inclusion:

$$\begin{cases} dX(t) \in b(t, X(t))dt - A(X(t))dt + \sigma(t, X(t))dW(t), & t > 0; \\ X(t_0) = X_0, \end{cases} \quad (\text{SDI}_0)$$

defined over a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function; W is a \mathcal{F}_t -adapted \mathbb{K} -valued Brownian motion; and the initial data X_0 is an \mathcal{F}_{t_0} -measurable \mathbb{H} -valued random variable.

Definition 4.2.1. A solution of (SDI₀) is a couple (X, ϑ) of \mathcal{F}_t -adapted processes such that almost surely:

- (i) X is continuous with sample paths in the domain of A ;
- (ii) ϑ is absolutely continuous, such that $\vartheta(t_0) = 0$, and $\forall T > t_0$, $\vartheta' \in L^2([t_0, T]; \mathbb{H})$, $\vartheta'(t) \in A(X(t))$ for almost all $t \geq t_0$;

(iii) For $t > t_0$,

$$\begin{cases} X(t) &= X_0 + \int_{t_0}^t b(s, X(s))ds - \vartheta(t) + \int_{t_0}^t \sigma(s, X(s))dW(s); \\ X(t_0) &= X_0. \end{cases} \quad (4.7)$$

For brevity, we sometimes omit the process ϑ and say that X is a solution of (SDI_0) , meaning that, there exists a process ϑ such that (X, ϑ) satisfies the previous definition.

The definition of uniqueness for the process X is presented in Section 2.7.

Throughout the chapter it will be assumed:

$$\begin{cases} A \text{ is a maximal monotone operator with closed domain;} \\ \mathcal{S} \stackrel{\text{def}}{=} A^{-1}(0) \neq \emptyset. \end{cases} \quad (\text{H}_A)$$

$$\begin{cases} \exists L > 0, \|b(t, x) - b(t, y)\| \vee \|\sigma(t, x) - \sigma(t, y)\|_{\text{HS}} \leq L\|x - y\|, \forall t \geq t_0, \forall x, y \in \mathbb{H}; \\ \sup_{t \geq t_0} (\|b(t, 0)\| \vee \|\sigma(t, 0)\|_{\text{HS}}) < +\infty. \end{cases} \quad (\text{H}_{b,\sigma})$$

The Lipschitz continuity assumption is mild and required to ensure the well-posedness of (SDI_0) .

We are interested in ensuring the existence and uniqueness of a solution for (SDI_0) . Although there are several works that deal with the subject of stochastic differential inclusions (see [120, 53, 40, 177, 173, 106]), those of [173, 106] are the closest to our setting and define a solution in the sense of Definition 4.2.1, thus generalizing the work of Brézis [66] in the deterministic case to the stochastic setting. In this work, we consider the sequence of solutions $\{X_\lambda\}_{\lambda > 0}$ of the stochastic differential equations

$$\begin{cases} dX_\lambda(t) = b(t, X_\lambda(t))dt - A_\lambda(X_\lambda(t))dt + \sigma(t, X_\lambda(t))dW(t), & t > t_0; \\ X_\lambda(t_0) = X_0, \end{cases} \quad (\text{SDE}_\lambda)$$

where $A_\lambda = (I - (I + \lambda A)^{-1})/\lambda$ is the Yosida approximation of A with parameter $\lambda > 0$. Under the integrability condition

$$\limsup_{\lambda \downarrow 0} \int_{t_0}^T \mathbb{E}(\|A_\lambda(X_\lambda(t))\|^2)dt < +\infty, \quad (\text{H}_\lambda)$$

it was shown in [173] that there exists a couple (X, ϑ) of stochastic processes such that for every $T > t_0$,

$$\lim_{\lambda \downarrow 0} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X_\lambda(t) - X(t)\|^2 \right) = 0, \quad \lim_{\lambda \downarrow 0} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|\vartheta_\lambda - \vartheta\|^2 \right) = 0,$$

where $\vartheta_\lambda(t) = \int_{t_0}^t A_\lambda(X_\lambda(s))ds$, and that (X, ϑ) is a solution of (SDI_0) in the sense of Definition 4.2.1. Moreover, one can even have a.s. convergence of the process X_λ when the diffusion term is state-independent; see [173, Proposition 6.3].

Remark 4.2.2. Condition (H_λ) is satisfied under different conditions, some examples are mentioned in [173]. One case where this condition holds is when A is full domain and there exists $C_0 > 0$ such that $\|A^0(x)\| \leq C_0(1 + \|x\|)$ for $x \in \mathbb{H}$, where $A^0(x) = \operatorname{argmin}_{y \in A(x)} \|y\|$.

Let us present our extension of Itô's formula for a multi-valued drift, which plays a central role in the study of SDI's.

Proposition 4.2.3. *Consider (SDI_0) under the assumptions of Theorem (4.2.4). Let $(X, \vartheta) \in S_{\mathbb{H}}^{\nu}[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$ be the unique solution of (SDI_0) , and let $\phi : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathbb{R}$ be such that $\phi(\cdot, x) \in C^1([t_0, +\infty[)$ for every $x \in \mathbb{H}$ and $\phi(t, \cdot) \in C^2(\mathbb{H})$ for every $t \geq t_0$. Then the process*

$$Y(t) = \phi(t, X(t)),$$

is an Itô Process such that for all $t \geq 0$

$$\begin{aligned} Y(t) &= Y(t_0) + \int_{t_0}^t \frac{\partial \phi}{\partial t}(s, X(s)) ds + \int_{t_0}^t \langle \nabla \phi(s, X(s)), b(s, X(s)) - \vartheta'(s) \rangle ds \\ &\quad + \int_{t_0}^t \langle \sigma^*(s, X(s)) \nabla \phi(s, X(s)), dW(s) \rangle + \frac{1}{2} \int_{t_0}^t \text{tr}(\sigma(s, X(s)) \sigma^*(s, X(s)) \nabla^2 \phi(s, X(s))) ds, \end{aligned} \quad (4.8)$$

where $\vartheta'(t) \in A(X(t))$ a.s. for almost all $t \geq t_0$. Moreover, if $\mathbb{E}[Y(t_0)] < +\infty$, and if for all $T > t_0$

$$\mathbb{E} \left(\int_{t_0}^T \|\sigma^*(s, X(s)) \nabla \phi(s, X(s))\|^2 ds \right) < +\infty,$$

then $\int_{t_0}^t \langle \sigma^*(s, X(s)) \nabla \phi(s, X(s)), dW(s) \rangle$ is a square-integrable continuous martingale and

$$\begin{aligned} \mathbb{E}[Y(t)] &= \mathbb{E}[Y(t_0)] + \mathbb{E} \left(\int_{t_0}^t \frac{\partial \phi}{\partial t}(s, X(s)) ds \right) + \mathbb{E} \left(\int_{t_0}^t \langle \nabla \phi(s, X(s)), b(s, X(s)) - \vartheta'(s) \rangle ds \right) \\ &\quad + \frac{1}{2} \mathbb{E} \left(\int_{t_0}^t \text{tr}(G(s, X(s)) G^*(s, X(s)) \nabla^2 \phi(s, X(s))) ds \right). \end{aligned} \quad (4.9)$$

Proof. The unique solution $(X, \vartheta) \in S_{\mathbb{H}}^{\nu}[t_0] \times C^1([t_0, +\infty[; \mathbb{H}])$ of (SDI_0) satisfies (by definition) the following equation:

$$\begin{cases} X(t) &= X_0 + \int_{t_0}^t [b(s, X(s)) - \vartheta'(s)] ds + \int_{t_0}^t \sigma(s, X(s)) dW(s), \quad t > t_0; \\ X(t_0) &= X_0. \end{cases} \quad (4.10)$$

and $\vartheta'(s) \in A(X(s))$ for almost all $t \geq t_0$ a.s.. Then, (4.10) is an Itô process with drift $s \mapsto b(s, X(s)) - \vartheta'(s)$ and diffusion $s \mapsto \sigma(s, X(s))$. Consequently, we can apply the classical Itô's formula (see Theorem 2.7.5) to obtain the desired result. \square

Now we are ready to state the existence and uniqueness of a solution of (SDI_0) .

Theorem 4.2.4. Consider (SDI_0) , where A and (b, σ) satisfy the assumption (H_A) and $(\text{H}_{b, \sigma})$, respectively. Additionally, suppose that A satisfy (H_{λ}) and let $\nu \geq 2$ such that $X_0 \in L^{\nu}(\Omega; \mathbb{H})$ and is \mathcal{F}_{t_0} -measurable. Then, there exists a unique solution $(X, \vartheta) \in S_{\mathbb{H}}^{\nu}[t_0] \times C^1([t_0, +\infty[; \mathbb{H}])$ of (SDI_0) .

Proof. The existence of a solution (X, ϑ) in the sense of Definition 4.2.1 comes from [173, Theorem 3.5]. We now turn to uniqueness. Let (X_1, ϑ_1) and (X_2, ϑ_2) be two solutions of (SDI_0) . By Itô's formula (see Theorem 4.2.3), we have

$$\begin{aligned} \|X_1(t) - X_2(t)\|^2 &= 2 \int_{t_0}^t \langle b(s, X_1(s)) - b(s, X_2(s)), X_1(s) - X_2(s) \rangle ds \\ &\quad - 2 \int_{t_0}^t \langle \vartheta_1'(s) - \vartheta_2'(s), X_1(s) - X_2(s) \rangle ds + \int_{t_0}^t \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|_{\text{HS}}^2 ds \\ &\quad + \int_{t_0}^t \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] dW(s) \rangle. \end{aligned}$$

Since for almost all $t \geq 0$, $\vartheta_i'(t) \in A(X_i(t))$, $i = \{1, 2\}$, by monotonicity of A , we have that for almost all $t \geq t_0$,

$$\langle \vartheta_1'(t) - \vartheta_2'(t), X_1(t) - X_2(t) \rangle \geq 0,$$

and thus the second term on the right-hand side is non-positive. Now, let $n \in \mathbb{N}$ arbitrary and consider the stopping time $\tau_n = \inf\{t \geq t_0 : \|X_1(t) - X_2(t)\| \geq n\}$ and evaluate the previous equation at $t \wedge \tau_n$,

denoting $X_i^n(t) = X_i(t \wedge \tau_n)$ ($i = \{1, 2\}$), we have

$$\begin{aligned}
\|X_1^n(t) - X_2^n(t)\|^2 &\leq 2 \int_{t_0}^{t \wedge \tau_n} \langle b(s, X_1(s)) - b(s, X_2(s)), X_1(s) - X_2(s) \rangle ds \\
&\quad + \int_{t_0}^{t \wedge \tau_n} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|_{\text{HS}}^2 ds \\
&\quad + \int_{t_0}^{t \wedge \tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] dW(s) \rangle \\
&\leq L(L+2) \int_{t_0}^{t \wedge \tau_n} \|X_1(s) - X_2(s)\|^2 ds \\
&\quad + \int_{t_0}^{t \wedge \tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] dW(s) \rangle \\
&\leq L(L+2) \int_{t_0}^t \|X_1^n(s) - X_2^n(s)\|^2 ds \\
&\quad + \int_{t_0}^{t \wedge \tau_n} \langle X_1(s) - X_2(s), [\sigma(s, X_1(s)) - \sigma(s, X_2(s))] dW(s) \rangle.
\end{aligned}$$

Note that we have used Cauchy-Schwarz inequality and the Lipschitz assumption on (b, σ) in the second inequality. Taking expectation of both sides and using the properties of Itô's integral we obtain

$$\mathbb{E}(\|X_1^n(t) - X_2^n(t)\|^2) \leq L(L+2) \int_{t_0}^t \mathbb{E}(\|X_1^n(s) - X_2^n(s)\|^2) ds.$$

By Grönwall's inequality, we obtain that

$$\mathbb{E}(\|X_1^n(t) - X_2^n(t)\|^2) = 0, \forall t \geq t_0, \forall n \in \mathbb{N}.$$

On the other hand, we have that $\lim_{n \rightarrow +\infty} t \wedge \tau_n = t$. Therefore, taking $\liminf_{n \rightarrow +\infty}$ in the previous expression, using Fatou's Lemma and the fact that X_1, X_2 are a.s. continuous processes, we conclude that $\mathbb{E}(\|X_1(t) - X_2(t)\|^2) = 0$, consequently

$$\mathbb{P}(X_1(t) = X_2(t), \forall t \in [t_0, T]) = 1, \quad \text{for every } T > t_0.$$

Let $T > t_0$ arbitrary, let us prove that $\mathbb{E}(\sup_{t \in [t_0, T]} \|X(t)\|^\nu) < +\infty$. Using Itô's formula (see Theorem 4.2.3) with the solution process X and the anchor function $\phi(x) = \|x - x^*\|^2$ for $x^* \in A^{-1}(0)$, we obtain for every $t \in [t_0, T]$:

$$\begin{aligned}
\|X(t) - x^*\|^2 &= \|X_0 - x^*\|^2 + 2 \int_{t_0}^t \langle b(s, X(s)), X(s) - x^* \rangle ds - 2 \int_{t_0}^t \langle \vartheta'(s), X(s) - x^* \rangle ds \\
&\quad + \int_{t_0}^t \|\sigma(s, X(s))\|_{\text{HS}}^2 ds + 2 \int_{t_0}^t \langle X(s) - x^*, \sigma(s, X(s)) dW(s) \rangle.
\end{aligned}$$

Since $\vartheta'(t) \in A(X(t))$ for almost all $t \geq 0$, and $0 \in A(x^*)$, by monotonicity of A we have that for every $t \in [t_0, T]$,

$$\langle \vartheta'(t), X(t) - x^* \rangle \geq 0, \quad \text{for almost all } t \geq 0.$$

Thus the second integral is nonnegative, which implies

$$\begin{aligned}
\|X(t) - x^*\|^2 &\leq \|X_0 - x^*\|^2 + 2 \int_{t_0}^t \langle b(s, X(s)), X(s) - x^* \rangle ds + \int_{t_0}^t \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \\
&\quad + 2 \int_{t_0}^t \langle X(s) - x^*, \sigma(s, X(s)) dW(s) \rangle.
\end{aligned} \tag{4.11}$$

Moreover, we have

$$2\langle b(t, x), x - x^* \rangle + \|\sigma(t, x)\|_{\text{HS}}^2 \leq 2\|b(t, x)\| \|x - x^*\| + \|\sigma(t, x)\|_{\text{HS}}^2 \leq C(1 + \|x - x^*\|^2), \quad \forall t \geq t_0, \forall x \in \mathbb{H}.$$

We now proceed as in the proof of [112, Lemma 3.2] to conclude that $X \in S_{\mathbb{H}}^{\nu}[t_0]$. In fact, we take power $\frac{\nu}{2}$ at both sides of (4.11), then using that $(a + b + c)^{\frac{\nu}{2}} \leq 3^{\frac{\nu-2}{2}}(a^{\frac{\nu}{2}} + b^{\frac{\nu}{2}} + c^{\frac{\nu}{2}})$ we have

$$\begin{aligned} \|X(t) - x^*\|^{\nu} &\leq 3^{\frac{\nu-2}{2}} \left(\|X_0 - x^*\|^{\nu} + C^{\frac{\nu}{2}} \left(\int_{t_0}^t 1 + \|X(s) - x^*\|^2 ds \right)^{\frac{\nu}{2}} \right) \\ &\quad + 3^{\frac{\nu-2}{2}} 2^{\frac{\nu}{2}} \left(\int_{t_0}^t \langle X(s) - x^*, \sigma(s, X(s)) dW(s) \rangle \right)^{\frac{\nu}{2}}. \end{aligned}$$

Now taking supremum $t \in [t_0, T]$ and then expectation at both sides, we have that there exists $K = K(\nu, T)$ such that:

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) &\leq K \left(1 + \mathbb{E}(\|X_0 - x^*\|^{\nu}) + \int_{t_0}^T \mathbb{E}(\|X(s) - x^*\|^{\nu}) ds \right) \\ &\quad + K \mathbb{E} \left(\sup_{t \in [t_0, T]} \left| \int_{t_0}^t \langle X(s) - x^*, \sigma(s, X(s)) dW(s) \rangle \right|^{\frac{\nu}{2}} \right). \end{aligned}$$

By Proposition 2.8.3, we get that, for a redefined $K = K(\nu, T)$,

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) &\leq K \left(1 + \mathbb{E}(\|X_0 - x^*\|^{\nu}) + \int_{t_0}^T \mathbb{E}(\|X(s) - x^*\|^{\nu}) ds \right) \\ &\quad + K \mathbb{E} \left(\left| \int_{t_0}^T \|X(s) - x^*\|^2 \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \right|^{\frac{\nu}{4}} \right). \end{aligned} \tag{4.12}$$

Note that by Cauchy-Schwarz and Young's inequality,

$$\begin{aligned} &\mathbb{E} \left(\left| \int_{t_0}^T \|X(s) - x^*\|^2 \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \right|^{\frac{\nu}{4}} \right) \\ &\leq \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\frac{\nu}{2}} \left(\int_{t_0}^T \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \right)^{\frac{\nu}{4}} \right) \\ &\leq \frac{1}{2K} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) + \frac{K}{2} \mathbb{E} \left[\left(\int_{t_0}^T \|\sigma(s, X(s))\|_{\text{HS}}^2 ds \right)^{\frac{\nu}{2}} \right] \\ &\leq \frac{1}{2K} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) + \frac{KC^{\frac{\nu}{2}}}{2} \mathbb{E} \left[\left(\int_{t_0}^T 1 + \|X(s) - x^*\|^2 ds \right)^{\frac{\nu}{2}} \right] \\ &\leq \frac{1}{2K} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) + \frac{KC^{\frac{\nu}{2}}}{2} T^{\frac{\nu-2}{2}} \mathbb{E} \left[\left(\int_{t_0}^T (1 + \|X(s) - x^*\|^2)^{\frac{\nu}{2}} ds \right) \right]. \end{aligned}$$

Substituting this into (4.12), we have, for a possibly different $K = K(\nu, T)$,

$$\mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) \leq K \left(1 + \mathbb{E}(\|X_0 - x^*\|^{\nu}) + \int_{t_0}^T \mathbb{E} \left(\sup_{t \in [0, s]} \|X(t) - x^*\|^{\nu} \right) ds \right).$$

By Grönwall's inequality, we obtain

$$\mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^{\nu} \right) \leq K (1 + \mathbb{E}(\|X_0 - x^*\|^{\nu})) e^{KT} < +\infty.$$

Since $T > t_0$ is arbitrary, we conclude that $X \in S_{\mathbb{H}}^{\nu}[t_0]$. \square

Corollary 4.2.5. *Consider (SDE $_{\lambda}$), where A and (b, σ) satisfy the assumption (\mathbf{H}_A) and $(\mathbf{H}_{b, \sigma})$, respectively. Additionally, let us consider that A satisfy (\mathbf{H}_{λ}) and let $\nu \geq 2$ such that $X_0 \in L^{\nu}(\Omega; \mathbb{H})$ and is \mathcal{F}_{t_0} -measurable. Then,*

$$\sup_{\lambda > 0} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X_{\lambda}(t)\|^{\nu} \right) < +\infty.$$

Proof. Since $A^{-1}(0) = A_\lambda^{-1}(0)$ and A_λ is monotone, we replace ϑ' by $A_\lambda(X_\lambda)$ in the proof of Theorem 4.2.4, then we realize that the constant that bounds $\mathbb{E} \left(\sup_{t \in [t_0, T]} \|X_\lambda(t)\|^\nu \right)$ is independent from λ to conclude. \square

4.3 Convergence properties for convex functions

Recall that our focus in this thesis is on an optimization perspective, and as we argued in the introduction, we will study the long time behavior of our SDE as the diffusion term vanishes when $t \rightarrow +\infty$. Therefore, we recall that we assume that the diffusion (volatility) term σ satisfies:

$$\begin{cases} \sup_{t \geq t_0, x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} < +\infty, \\ \|\sigma(t, x') - \sigma(t, x)\|_{\text{HS}} \leq l_0 \|x' - x\|, \end{cases} \quad (\mathbf{H}_\sigma)$$

for some $l_0 > 0$ and for all $t \geq t_0, x, x' \in \mathbb{H}$. The Lipschitz continuity assumption is mild and classical and will be required to ensure the well-posedness of (SDI). Let us also recall that we define $\sigma_\infty : [t_0, +\infty[\rightarrow \mathbb{R}_+$ as

$$\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}}.$$

Remark 4.3.1. (\mathbf{H}_σ) implies the existence of $\sigma_* > 0$ such that:

$$\|\sigma(t, x)\|_{\text{HS}}^2 = \text{tr}[\Sigma(t, x)] \leq \sigma_*^2,$$

for all $t \geq t_0, x \in \mathbb{H}$, where $\Sigma \stackrel{\text{def}}{=} \sigma\sigma^*$.

We consider $f + g$ (called the potential) and study the dynamic (SDI) under the hypotheses (\mathbf{H}_F) (i.e. $f \in C_L^{1,1}(\mathbb{H}) \cap \Gamma_0(\mathbb{H}), g \in \Gamma_0(\mathbb{H})$) and (\mathbf{H}_σ) . Recall the definitions of σ_* and $\sigma_\infty(t)$ from (1.5). And we also recall the notation

$$\begin{aligned} F(x) &\stackrel{\text{def}}{=} f(x) + g(x) \\ \Sigma(t, x) &\stackrel{\text{def}}{=} \sigma(t, x)\sigma(t, x)^* \\ \mathcal{S}_F &\stackrel{\text{def}}{=} \text{argmin}(F). \end{aligned}$$

4.3.1 Almost sure weak convergence of the trajectory

Our first main result establish almost sure weak convergence of $X(t)$ to a point that belongs in \mathcal{S}_F . It is based on Itô's formula, and on Barbalat's and Opial's Lemma. It follows the same ideas as in Theorem 3.2.3.

Theorem 4.3.2. *Consider $F = f + g$ and σ satisfying (\mathbf{H}_F) and (\mathbf{H}_σ) respectively. Suppose further that ∂g verifies (\mathbf{H}_λ) . Let $\nu \geq 2, t_0 \geq 0$, and consider the dynamic (SDI) with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$, i.e.:*

$$\begin{aligned} dX(t) &\in -\partial F(X(t))dt + \sigma(t, X(t))dW(t); \\ X(t_0) &= X_0, \end{aligned} \quad (4.13)$$

where W is a \mathbb{K} -valued Brownian motion. Then, there exists a unique solution (in the sense of Theorem 4.2.4) $(X, \vartheta) \in \mathcal{S}_{\mathbb{H}}^\nu[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$.

Moreover, if $\sigma_\infty \in L^2([t_0, +\infty[)$, then the following holds:

(i) $\mathbb{E}[\sup_{t \geq t_0} \|X(t)\|^\nu] < +\infty$.

(ii) $\forall x^* \in \mathcal{S}_F, \lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s. and $\sup_{t \geq t_0} \|X(t)\| < +\infty$ a.s..

(iii) If g is continuous, then ∇f is constant on \mathcal{S}_F , $\text{s-lim}_{t \rightarrow +\infty} \nabla f(X(t)) = \nabla f(x^*)$ a.s. for any $x^* \in \mathcal{S}_F$, and

$$\int_{t_0}^{+\infty} [F(X(t)) - \min F] dt < +\infty.$$

(iv) If (iii) holds, then there exists an \mathcal{S}_F -valued random variable X^* such that $\text{w-lim}_{t \rightarrow +\infty} X(t) = X^*$.

Remark 4.3.3. By classical properties of the Yosida approximation

$$(\partial g(x))_\lambda = \nabla g_\lambda(x) = \frac{1}{\lambda}(x - \text{prox}_{\lambda g}(x)),$$

where g_λ is the Moreau envelope of g with parameter $\lambda > 0$. If there exists $C > 0$, such that

$$\|x - \text{prox}_{\lambda g}(x)\| \leq \lambda C,$$

the assumption (\mathbf{H}_λ) is satisfied by ∂g . As mentioned in Remark 4.2.2, if g is continuous and

$$\|\partial^0 g(x)\| \leq C_0(1 + \|x\|),$$

for some $C_0 > 0$, then ∂g also satisfies (\mathbf{H}_λ) .

Proof. (i) Directly from Theorem 4.2.4.

(ii) Since F is convex, we first notice that $\mathcal{S}_F = (\partial F)^{-1}(0)$.

Now let us consider $(X, \vartheta) \in \mathcal{S}_{\mathbb{H}}^\nu[t_0] \times C^1([t_0, +\infty[; \mathbb{H})$ be the unique solution of (\mathbf{SDI}_0) given by Theorem 4.2.4, and $\phi(x) = \frac{\|x - x^*\|^2}{2}$, where $x^* \in \mathcal{S}_F$. Then by Itô's formula

$$\begin{aligned} \phi(X(t)) &= \underbrace{\frac{\|X_0 - x^*\|^2}{2}}_{\xi = \phi(X_0)} + \underbrace{\frac{1}{2} \int_{t_0}^t \text{tr}(\Sigma(s, X(s))) ds}_{A_t} - \underbrace{\int_{t_0}^t \langle \vartheta'(s) + \nabla f(X(s)), X(s) - x^* \rangle ds}_{U_t} \\ &\quad + \underbrace{\int_{t_0}^t \langle \sigma^*(s, X(s)) (X(s) - x^*), dW(s) \rangle}_{M_t}. \end{aligned} \quad (4.14)$$

Let us observe that, since $\nu \geq 2$, we have that $\mathbb{E}(\sup_{t \geq t_0} \|X(t)\|^2) < +\infty$. Moreover, since $\sigma_\infty \in \mathbf{L}^2([t_0, +\infty[)$ we have

$$\mathbb{E} \left(\int_{t_0}^{+\infty} \|\sigma^*(s, X(s)) (X(s) - x^*)\|^2 ds \right) \leq \mathbb{E} \left(\sup_{t \geq t_0} \|X(t) - x^*\|^2 \right) \int_{t_0}^{+\infty} \sigma_\infty^2(s) ds < +\infty.$$

Therefore M_t is a square-integrable continuous martingale. It is also a continuous local martingale (see [141, Theorem 1.3.3]), which implies that $\mathbb{E}(M_t) = 0$.

Moreover, since F is a convex function, then ∂F is a monotone operator. On the other hand $\vartheta'(t) \in \partial g(X(t))$ a.s. for almost all $t \geq t_0$, so

$$\langle \vartheta'(t) + \nabla f(X(t)), X(t) - x^* \rangle \geq 0, \text{ a.s. for almost all } t \geq t_0.$$

We have that A_t and U_t defined as in (4.14) are two continuously adapted increasing processes with $A_0 = U_0 = 0$ a.s.. Since $\phi(X(t))$ is nonnegative and $\sup_{x \in \mathbb{H}} \|\sigma(\cdot, x)\|_{\text{HS}} \in \mathbf{L}^2([t_0, +\infty[)$, we deduce that $\lim_{t \rightarrow +\infty} A_t < +\infty$. Then, we can use Theorem 2.8.2 to conclude that

$$\int_{t_0}^{+\infty} \langle \vartheta'(t) + \nabla f(X(t)), X(t) - x^* \rangle dt < +\infty \quad \text{a.s.} \quad (4.15)$$

and

$$\forall x^* \in \mathcal{S}_F, \exists \Omega_{x^*} \in \mathcal{F}, \text{ such that } \mathbb{P}(\Omega_{x^*}) = 1 \text{ and } \lim_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| \text{ exists } \forall \omega \in \Omega_{x^*}. \quad (4.16)$$

Since \mathbb{H} is separable, there exists a countable set $\mathcal{Z} \subseteq \mathcal{S}_F$, such that $\text{cl}(\mathcal{Z}) = \mathcal{S}_F$ (where cl stands for the closure of the set). Let $\tilde{\Omega} = \bigcap_{z \in \mathcal{Z}} \Omega_z$. Since \mathcal{Z} is countable, a union bound shows

$$\mathbb{P}(\tilde{\Omega}) = 1 - \mathbb{P}\left(\bigcup_{z \in \mathcal{Z}} \Omega_z^c\right) \geq 1 - \sum_{z \in \mathcal{Z}} \mathbb{P}(\Omega_z^c) = 1.$$

For arbitrary $x^* \in \mathcal{S}_F$, there exists a sequence $(z_k)_{k \in \mathbb{N}} \subseteq \mathcal{Z}$ such that $\lim_{k \rightarrow +\infty} z_k = x^*$. In view of (4.16), for every $k \in \mathbb{N}$ there exists $\tau_k : \Omega_{z_k} \rightarrow \mathbb{R}_+$ such that

$$\lim_{t \rightarrow +\infty} \|X(\omega, t) - z_k\| = \tau_k(\omega), \quad \forall \omega \in \Omega_{z_k}. \quad (4.17)$$

Now, let $\omega \in \tilde{\Omega}$. Since $\tilde{\Omega} \subset \Omega_{z_k}$ for any $k \in \mathbb{N}$, and using the triangle inequality and (4.17), we obtain that

$$\tau_k(\omega) - \|z_k - x^*\| \leq \liminf_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| \leq \limsup_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| \leq \tau_k(\omega) + \|z_k - x^*\|.$$

Now, passing to $k \rightarrow +\infty$, we deduce

$$\limsup_{k \rightarrow +\infty} \tau_k(\omega) \leq \liminf_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| \leq \limsup_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| \leq \liminf_{k \rightarrow +\infty} \tau_k(\omega),$$

whence we deduce that $\lim_{k \rightarrow +\infty} \tau_k(\omega)$ exists on the set $\tilde{\Omega}$ of probability 1, and in turn

$$\lim_{t \rightarrow +\infty} \|X(\omega, t) - x^*\| = \lim_{k \rightarrow +\infty} \tau_k(\omega).$$

Let us recall that there exists $\Omega_{\text{cont}} \in \mathcal{F}$ such that $\mathbb{P}(\Omega_{\text{cont}}) = 1$ and $X(\omega, \cdot)$ is continuous for every $\omega \in \Omega_{\text{cont}}$. Now let $x^* \in \mathcal{S}_F$ arbitrary, since the limit exists, for every $\omega \in \tilde{\Omega} \cap \Omega_{\text{cont}}$ there exists $T(\omega)$ such that $\|X(\omega, t) - x^*\| \leq 1 + \lim_{k \rightarrow +\infty} \tau_k(\omega)$ for every $t \geq T(\omega)$. Besides, since $X(\omega, \cdot)$ is continuous, by Bolzano's theorem

$$\sup_{t \in [0, T(\omega)]} \|X(\omega, t)\| = \max_{t \in [0, T(\omega)]} \|X(\omega, t)\| \stackrel{\text{def}}{=} h(\omega) < +\infty.$$

Therefore, $\sup_{t \geq t_0} \|X(\omega, t)\| \leq \max\{h(\omega), 1 + \lim_{k \rightarrow +\infty} \tau_k(\omega) + \|x^*\|\} < +\infty$.

(iii) Let $N_t = \int_{t_0}^t \sigma(s, X(s)) dW(s)$. This is a continuous martingale (w.r.t. the filtration \mathcal{F}_t), which verifies

$$\mathbb{E}(\|N_t\|^2) = \mathbb{E}\left(\int_{t_0}^t \|\sigma(s, X(s))\|_{\text{HS}}^2 ds\right) \leq \mathbb{E}\left(\int_{t_0}^{+\infty} \sigma_\infty^2(s) ds\right) < +\infty, \forall t \geq t_0.$$

According to Theorem 2.8.1, we deduce that there exists a \mathbb{H} -valued random variable N_∞ w.r.t. \mathcal{F}_∞ , and which verifies: $\mathbb{E}(\|N_\infty\|^2) < +\infty$, and there exists $\Omega_N \in \mathcal{F}$ such that $\mathbb{P}(\Omega_N) = 1$ and

$$\lim_{t \rightarrow +\infty} N_t(\omega) = N_\infty(\omega) \text{ for every } \omega \in \Omega_N.$$

On the other hand, since $x^* \in (\partial F)^{-1}(0) = (\nabla f + \partial g)^{-1}(0)$, then $-\nabla f(x^*) \in \partial g(x^*)$. Let $T > t_0$ such that $\vartheta'(t) \in \partial g(X(t))$ a.s., consequently,

$$\begin{aligned} \langle \vartheta'(t) + \nabla f(X(t)), X(t) - x^* \rangle &= \underbrace{\langle \vartheta'(t) - (-\nabla f(x^*)), X(t) - x^* \rangle}_{\geq 0} \\ &\quad + \langle \nabla f(X(t)) - \nabla f(x^*), X(t) - x^* \rangle \\ &\geq \langle \nabla f(X(t)) - \nabla f(x^*), X(t) - x^* \rangle \\ &\geq \frac{1}{L} \|\nabla f(X(t)) - \nabla f(x^*)\|^2, \end{aligned}$$

where $\langle \vartheta'(t) - (-\nabla f(x^*)), X(t) - x^* \rangle \geq 0$ by monotonicity of ∂g . Then by (4.15) we obtain

$$\int_{t_0}^{+\infty} \|\nabla f(X(t)) - \nabla f(x^*)\|^2 dt < +\infty \quad \text{a.s.} \quad (4.18)$$

Let $\Omega_{\text{HS}} \in \mathcal{F}$ be the event where (4.15) (and consequently (4.18)) is satisfied ($\mathbb{P}(\Omega_{\text{HS}}) = 1$). Let $\Omega_{\vartheta} \in \mathcal{F}$ be the event where $\vartheta'(t) \in \partial g(X(t))$ for almost all $T > t_0$ ($\mathbb{P}(\Omega_{\vartheta}) = 1$). Finally, let $\Omega_{\text{conv}} \stackrel{\text{def}}{=} \tilde{\Omega} \cap \Omega_{\text{cont}} \cap \Omega_{\text{HS}} \cap \Omega_M \cap \Omega_{\vartheta}$, hence $\mathbb{P}(\Omega_{\text{conv}}) = 1$. Let also $\omega \in \Omega_{\text{conv}} \subseteq \Omega_{\text{HS}}$ arbitrary, then

$$\liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^*)\| = 0.$$

If also

$$\limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^*)\| = 0$$

then we conclude with the proof. Suppose by contradiction that there exists $\omega_0 \in \Omega_{\text{conv}}$ such that

$$\limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^*)\| > 0.$$

Then, by Lemma 2.6.7, there exists $\delta(\omega_0) > 0$ satisfying

$$0 = \liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^*)\| < \delta(\omega_0) < \limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega_0, t)) - \nabla f(x^*)\|,$$

and there exists $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$,

$$\|\nabla f(X(\omega_0, t_k)) - \nabla f(x^*)\| > \delta(\omega_0) \text{ and } t_{k+1} - t_k > 1, \quad \forall k \in \mathbb{N}.$$

Additionally, consider $\vartheta'(\omega_0, t) \in \partial g(X(\omega_0, t))$ for almost all $T > t_0$. Since $\sup_{t \geq t_0} \|X(\omega_0, t)\| < +\infty$, ∂g is full domain, and the fact that ∂g maps bounded sets onto bounded sets, we have that there exists $C_{\vartheta}(\omega_0) \geq 0$ such that $\|\vartheta'(\omega_0, t)\|^2 \leq C_{\vartheta}(\omega_0)$ for almost all $T > t_0$.

We allow ourselves the abuse of notation $X(t) \stackrel{\text{def}}{=} X(\omega_0, t)$, $\vartheta'(t) \stackrel{\text{def}}{=} \vartheta'(\omega_0, t)$, $C_{\vartheta} \stackrel{\text{def}}{=} C_{\vartheta}(\omega_0)$ and $\delta \stackrel{\text{def}}{=} \delta(\omega_0)$ during the rest of the proof from this point.

Let

- $C_0 \stackrel{\text{def}}{=} C_{\vartheta} + \|\nabla f(x^*)\|^2$;
- $C_1 \stackrel{\text{def}}{=} \frac{(2C_0+1)^2-1}{C_0} > 0$;
- $\varepsilon \in]0, \min\{\frac{\delta^2}{4L^2}, C_1\}[$;
- and $C(\varepsilon) \stackrel{\text{def}}{=} \frac{\sqrt{C_0\varepsilon+1}-1}{4C_0} \in]0, \frac{1}{2}]$.

Note that this choice entails that the intervals $([t_k, t_k + C(\varepsilon)])_{k \in \mathbb{N}}$ are disjoint. On the other hand, according to the convergence of N_t and the fact that $\|\nabla f(X(t)) - \nabla f(x^*)\| \in L^2([t_0, +\infty[)$, there exists $k' > 0$ such that for every $k \geq k'$,

$$\sup_{t \geq t_k} \|N_t - N_{t_k}\|^2 < \frac{\varepsilon}{4} \text{ and } \int_{t_k}^{+\infty} \|\nabla f(X(t)) - \nabla f(x^*)\|^2 dt < 1.$$

Also, we compute

$$\begin{aligned} \int_{t_k}^t \|\vartheta'(s) + \nabla f(X(s))\|^2 ds &\leq 2 \int_{t_k}^t \|\nabla f(X(s)) - \nabla f(x^*)\|^2 ds + 2 \int_{t_k}^t \|\vartheta'(s) + \nabla f(x^*)\|^2 ds \\ &\leq 2 + 4C_0(t - t_k). \end{aligned}$$

Furthermore, $C(\varepsilon)$ was chosen such that $C(\varepsilon) + 2C_0C(\varepsilon)^2 \leq \frac{\varepsilon}{8}$. Besides for every $k \geq k'$, $t \in [t_k, t_k + C(\varepsilon)]$,

$$\begin{aligned} \|X(t) - X(t_k)\|^2 &\leq 2(t - t_k) \int_{t_k}^t \|\vartheta'(s) + \nabla f(X(s))\|^2 ds + 2\|N_t - N_{t_k}\|^2 \\ &\leq 4(t - t_k) + 8C_0(t - t_k)^2 + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

Since ∇f is L -Lipschitz and $L^2\varepsilon \leq (\frac{\delta}{2})^2$ by assumption on ε , we have that for every $k \geq k'$ and $t \in [t_k, t_k + C(\varepsilon)]$

$$\|\nabla f(X(t)) - \nabla f(X(t_k))\|^2 \leq L^2 \|X(t) - X(t_k)\|^2 \leq \left(\frac{\delta}{2}\right)^2.$$

Therefore, for every $k \geq k'$, $t \in [t_k, t_k + C(\varepsilon)]$

$$\|\nabla f(X(t)) - \nabla f(x^*)\| \geq \|\nabla f(X(t_k)) - \nabla f(x^*)\| - \underbrace{\|\nabla f(X(t)) - \nabla f(X(t_k))\|}_{\leq \frac{\delta}{2}} \geq \frac{\delta}{2}.$$

Finally,

$$\begin{aligned} \int_{t_0}^{+\infty} \|\nabla f(X(s)) - \nabla f(x^*)\|^2 ds &\geq \sum_{k \geq k'} \int_{t_k}^{t_k + C(\varepsilon)} \|\nabla f(X(s)) - \nabla f(x^*)\|^2 ds \\ &\geq \sum_{k \geq k'} \frac{\delta^2 C(\varepsilon)}{4} = +\infty, \end{aligned}$$

which contradicts $\|\nabla f(X(\cdot)) - \nabla f(x^*)\| \in L^2([t_0, +\infty[)$. So, for every $\omega \in \Omega_{\text{conv}}$,

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^*)\| &= \liminf_{t \rightarrow +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^*)\| \\ &= \lim_{t \rightarrow +\infty} \|\nabla f(X(\omega, t)) - \nabla f(x^*)\| = 0. \end{aligned}$$

On the other hand, since F is convex, by (4.15), we obtain

$$\int_{t_0}^{+\infty} F(X(t)) - \min F dt < +\infty, \quad a.s.. \quad (4.19)$$

Since $\sup_{t \geq t_0} \|X(t)\| < +\infty$ a.s., and (∂F) maps bounded sets onto bounded sets (since g is convex and continuous), we can show that there exists $\tilde{L} > 0$ such that

$$|F(X(t_1)) - F(X(t_2))| \leq \tilde{L} \|X(t_1) - X(t_2)\|, \quad \forall t_1, t_2 \geq t_0, a.s..$$

Using the same technique as before, we can conclude that $\lim_{t \rightarrow +\infty} F(X(t)) = \min F$ a.s..

- (iv) Let $\omega \in \Omega_{\text{conv}}$ and $\tilde{X}(\omega)$ be a weak sequential limit point of $X(\omega, t)$. Equivalently, there exists an increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$ and

$$\text{w-lim}_{k \rightarrow +\infty} X(\omega, t_k) = \tilde{X}(\omega).$$

Since $\lim_{t \rightarrow +\infty} F(X(\omega, t)) = \min F$ and the fact that f is weakly lower semicontinuous (since it is convex and continuous), we obtain directly that $\tilde{X}(\omega) \in \mathcal{S}_F$. Finally, by Opial's Lemma (see [162]) we conclude that there exists $X^*(\omega) \in \mathcal{S}_F$ such that

$$\text{w-lim}_{t \rightarrow +\infty} X(\omega, t) = X^*(\omega).$$

In other words, since $\omega \in \Omega_{\text{conv}}$ was arbitrary, there exists an \mathcal{S}_F -valued random variable X^* such that $\text{w-lim}_{t \rightarrow +\infty} X(t) = X^*$ a.s.. □

4.3.2 Convergence rates of the objective

Our first result, stated below, summarizes the global convergence rates in expectation satisfied by the trajectories of (SDI).

Theorem 4.3.4. *Consider the dynamic (SDI) where $F = f + g$ and σ satisfy the assumptions (\mathbf{H}_F) and (\mathbf{H}_σ) , furthermore assume that ∂g satisfies (\mathbf{H}_λ) . Additionally, $X_0 \in L^2(\Omega; \mathbb{H})$ and is \mathcal{F}_{t_0} -measurable. The following statements are satisfied by the unique solution trajectory $X \in S_{\mathbb{H}}^2[t_0]$ of (SDI):*

(i) Let $\overline{F \circ X}(t) \stackrel{\text{def}}{=} t^{-1} \int_{t_0}^t F(X(s)) ds$ and $\overline{X}(t) = t^{-1} \int_{t_0}^t X(s) ds$. Then

$$\mathbb{E} (F(\overline{X}(t)) - \min F) \leq \mathbb{E} (\overline{F \circ X}(t) - \min F) \leq \frac{\mathbb{E} (\text{dist}(X_0, \mathcal{S}_F)^2)}{2t} + \frac{\sigma_*^2}{2}, \quad \forall t > t_0. \quad (4.20)$$

Besides, if σ_∞ is $L^2([t_0, +\infty[)$, then

$$\mathbb{E} (F(\overline{X}(t)) - \min F) \leq \mathbb{E} (\overline{F \circ X}(t) - \min F) = \mathcal{O} \left(\frac{1}{t} \right). \quad (4.21)$$

(ii) Moreover, if $F \in \Gamma_\mu(\mathbb{H})$ with $\mu > 0$, then $\mathcal{S}_F = \{x^*\}$ and

$$\mathbb{E} (\|X(t) - x^*\|^2) \leq \mathbb{E} (\|X_0 - x^*\|^2) e^{-\mu t} + \frac{\sigma_*^2}{\mu}, \quad \forall t > t_0. \quad (4.22)$$

Besides, if σ_∞ is nonincreasing and vanishes at infinity, then:

$$\mathbb{E} (\|X(t) - x^*\|^2) \leq \mathbb{E} (\|X_0 - x^*\|^2) e^{-\mu t} + \frac{\sigma_*^2}{\mu} e^{\frac{\mu t_0}{2}} e^{-\frac{\mu t}{2}} + \sigma_\infty^2 \left(\frac{t_0 + t}{2} \right), \quad \forall t > t_0. \quad (4.23)$$

Proof. Using that $F(x) - \min F \leq \langle y, x - x^* \rangle$ for every $y \in \partial F(x)$, $x^* \in \mathcal{S}_F$, and Itô's formula with the anchor function $\phi(x) = \frac{\|x - x^*\|^2}{2}$ (for $x^* \in \mathcal{S}_F$), the proof is analogous to the one in Theorem 3.2.5. \square

4.4 Tikhonov regularization: Convergence properties for convex functions

It is important to provide insight into the technique of Tikhonov regularization. This allows us to pass from the almost sure weak convergence towards the set of minimizers of the trajectory generated by (SDI₀) to achieving almost sure strong convergence of the trajectory generated by (SDI – TA), not only towards the set of minimizers but to the minimal norm solution. The price to pay in order to achieve this is the proper tuning of the Tikhonov parameter that depends on a local constant that could be hard to compute, besides that, we obtain slower convergence rates of the objective, passing from $\mathcal{O}(t^{-1})$ to $\mathcal{O}(t^{-r} + R(t))$, where $r < 1$ and $R(t) \rightarrow 0$ (defined below in (4.33)).

4.4.1 Almost sure convergence of the trajectory to the minimal norm solution

Our second main result establish almost sure convergence of $X(t)$ to $x^* = P_{\mathcal{S}_F}(0)$ as $t \rightarrow +\infty$. It is based on a subtle tuning of the Tikhonov parameter $\varepsilon(t)$ formulated as conditions (T₁), (T₂), and (T₃) below. We know that $\|x^*\|^2 - \|x_{\varepsilon(t)}\|^2$ tends to zero as $t \rightarrow +\infty$. We shall see that the conditions (T₁), (T₂), and (T₃) are compatible for tame functions, *i.e.* which satisfy a Kurdyka-Łojasiewicz property (see Section 2.4).

Theorem 4.4.1. *Consider the dynamic (SDI – TA) where $F = f + g$ and σ satisfy the assumptions (H_F) and (H_σ), respectively, furthermore assume that ∂g satisfy (H_λ). Let $\nu \geq 2$, and its initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$. Then, there exists a unique solution $X \in S_{\mathbb{H}}^\nu[t_0]$ of (SDI – TA). Let $x^* \stackrel{\text{def}}{=} P_{\mathcal{S}_F}(0)$ be the minimum norm solution, and for $\varepsilon > 0$ let x_ε be the unique minimizer of $F_\varepsilon(x) \stackrel{\text{def}}{=} F(x) + \frac{\varepsilon}{2} \|x\|^2$. Suppose that $\sigma_\infty \in L^2(\mathbb{R}_+)$, and that $\varepsilon : [t_0, +\infty[\rightarrow \mathbb{R}_+$ satisfies the conditions:*

(T₁) $\varepsilon(t) \rightarrow 0$ as $t \rightarrow +\infty$;

(T₂) $\int_{t_0}^{+\infty} \varepsilon(t) dt = +\infty$;

(T₃) $\int_{t_0}^{+\infty} \varepsilon(t) (\|x^*\|^2 - \|x_{\varepsilon(t)}\|^2) dt < +\infty$.

Then the following holds:

- (i) $\int_{t_0}^{+\infty} \varepsilon(t) \mathbb{E}[\|X(t) - x^*\|^2] dt < +\infty.$
- (ii) $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s. and $\sup_{t \geq t_0} \|X(t)\| < +\infty$ a.s..
- (iii) $\int_{t_0}^{+\infty} \varepsilon(t) \|X(t) - x^*\|^2 dt < +\infty$ a.s..
- (iv) $s\text{-}\lim_{t \rightarrow +\infty} X(t) = x^*$ a.s.

Proof. The existence and uniqueness of a solution $X \in S_{\mathbb{H}}^{\mathcal{U}}[t_0]$ follow directly from the fact that the conditions of Theorem 4.2.4 are satisfied under (\mathbf{H}_F) and (\mathbf{H}_σ) . The only subtlety to check is that $\sup_{t \geq t_0} |\varepsilon(t)| < +\infty$, but this can be assumed without loss of generality since $\varepsilon(t) \rightarrow 0$ as $t \rightarrow +\infty$ (it might be necessary a redefinition of t_0).

Our stochastic dynamic **(SDI – TA)** can be written equivalently as follows

$$\begin{cases} dX(t) \in -\partial F_{\varepsilon(t)}(X(t))dt + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0, \end{cases} \quad (\text{SDIT})$$

- (i) Let us define the anchor function $\phi(x) = \frac{\|x - x^*\|^2}{2}$. Since ∂g satisfy (\mathbf{H}_λ) , there exists a stochastic process $\tilde{\vartheta} : \Omega \times [t_0, +\infty[\rightarrow \mathbb{H}$ such that $\tilde{\vartheta}(t) \in \partial F_{\varepsilon(t)}(X(t))$ a.s. for almost all $t \geq t_0$. Using Itô's formula we obtain

$$\begin{aligned} \phi(X(t)) &= \underbrace{\frac{\|X_0 - x^*\|^2}{2}}_{\xi} + \underbrace{\frac{1}{2} \int_{t_0}^t \text{tr}(\Sigma(s, X(s))) ds}_{A_t} - \underbrace{\int_{t_0}^t \langle \tilde{\vartheta}(s), X(s) - x^* \rangle ds}_{U_t} \\ &\quad + \underbrace{\int_{t_0}^t \langle \sigma^*(s, X(s)) (X(s) - x^*), dW(s) \rangle}_{M_t}. \end{aligned} \quad (4.24)$$

Since $X \in S_{\mathbb{H}}^2[t_0]$ by Proposition 4.2.3, we have for every $T > t_0$, that

$$\mathbb{E} \left(\int_{t_0}^T \|\sigma^*(s, X(s)) (X(s) - x^*)\|^2 ds \right) \leq \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^2 \right) \int_{t_0}^{+\infty} \sigma_\infty^2(s) ds < +\infty.$$

Therefore M_t is a square-integrable continuous martingale. It is also a continuous local martingale, which implies that $\mathbb{E}(M_t) = 0$.

Let us now take the expectation of (4.24). Using that

$$0 \leq \text{tr}(\Sigma(s, X(s))) \leq \sigma_\infty^2(s)$$

and (4.3) that we recall below

$$\langle y(t), X(t) - x^* \rangle \geq \varepsilon(t) \phi(X(t)) + \frac{\varepsilon(t)}{2} (\|x_{\varepsilon(t)}\|^2 - \|x^*\|^2), \quad (4.25)$$

where $y : \Omega \times [t_0, +\infty[\rightarrow \mathbb{H}$ is such that $y(t) \in \partial F_{\varepsilon(t)}(X(t))$ a.s., we obtain that

$$\begin{aligned} &\mathbb{E}(\phi(X(t))) + \int_{t_0}^t \varepsilon(s) \mathbb{E}(\phi(X(s))) ds \\ &\leq \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right) + \frac{1}{2} \int_{t_0}^t \sigma_\infty^2(s) ds + \frac{1}{2} \int_{t_0}^t \varepsilon(s) (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds. \end{aligned}$$

According to our assumptions, we can write briefly the above relation as

$$\mathbb{E}(\phi(X(t))) + \int_{t_0}^t \varepsilon(s) \mathbb{E}(\phi(X(s))) ds \leq g(t), \quad (4.26)$$

with g a nonnegative function defined by

$$g(t) \stackrel{\text{def}}{=} \mathbb{E} \left(\frac{\|X_0 - x^*\|^2}{2} \right) + \frac{1}{2} \int_{t_0}^t \sigma_\infty^2(s) ds + \frac{1}{2} \int_{t_0}^t \varepsilon(s) (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds$$

which satisfies $\lim_{t \rightarrow +\infty} g(t) = g_\infty < +\infty$.

Let us integrate the above relation (4.26). We set

$$\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \mathbb{E}(\phi(X(s))) ds.$$

We have $\dot{\theta}(t) = \mathbb{E}(\phi(X(t)))$ and (4.26) is written equivalently as

$$\dot{\theta}(t) + \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds \leq g(t). \quad (4.27)$$

Equivalently

$$\frac{1}{\varepsilon(t)} \frac{d}{dt} \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds + \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds \leq g(t), \quad (4.28)$$

that is

$$\frac{d}{dt} \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds + \varepsilon(t) \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds \leq \varepsilon(t) g(t). \quad (4.29)$$

With $m(t) \stackrel{\text{def}}{=} \exp \int_{t_0}^t \varepsilon(s) ds$ we get

$$\frac{d}{dt} \left(m(t) \int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds \right) \leq \varepsilon(t) m(t) g(t). \quad (4.30)$$

After integration we get

$$\int_{t_0}^t \varepsilon(s) \dot{\theta}(s) ds \leq \frac{1}{m(t)} \int_{t_0}^t m'(s) g(s) ds. \quad (4.31)$$

Since g is bounded by assumption (T₂), we get

$$\sup_{t \geq t_0} \mathbb{E} \left[\int_{t_0}^t \varepsilon(s) \|X(s) - x^*\|^2 ds \right] < +\infty.$$

Equivalently

$$\int_{t_0}^{+\infty} \mathbb{E} \left[\|X(t) - x^*\|^2 \right] \varepsilon(t) dt < +\infty.$$

The assumption (T₂) guarantees that the above inequality forces $\mathbb{E} \left[\|X(t) - x^*\|^2 \right]$ to tend to zero.

(ii) Consider (4.24), we define

$$\tilde{A}_t \stackrel{\text{def}}{=} A_t + \int_{t_0}^t \frac{\varepsilon(s)}{2} (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds, \quad \text{and} \quad \tilde{U}_t \stackrel{\text{def}}{=} U_t + \int_{t_0}^t \frac{\varepsilon(s)}{2} (\|x^*\|^2 - \|x_{\varepsilon(s)}\|^2) ds.$$

By (4.25) we have that $\tilde{U}_t \geq \int_{t_0}^t \varepsilon(s) \phi(X(s)) ds \geq 0$. We can rewrite (4.24) as

$$\phi(X(t)) = \xi + \tilde{A}_t - \tilde{U}_t + M_t.$$

Since $\sigma_\infty \in L^2([t_0, +\infty[)$ and (T₃), then $\lim_{t \rightarrow +\infty} \tilde{A}_t < +\infty$. Let us observe that, since $X \in S_{\mathbb{H}}^2[t_0]$ by Proposition 4.2.3, we have for every $T > t_0$ that

$$\mathbb{E} \left(\int_{t_0}^T \|\sigma^*(s, X(s)) (X(s) - x^*)\|^2 ds \right) \leq \mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t) - x^*\|^2 \right) \int_{t_0}^{+\infty} \sigma_\infty^2(s) ds < +\infty.$$

Therefore M_t is a square-integrable continuous martingale. It is also a continuous local martingale (see [141, Theorem 1.3.3]), which implies that $\mathbb{E}(M_t) = 0$.

By Theorem 2.8.2, we get that $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s. and that $\lim_{t \rightarrow +\infty} \tilde{U}_t < +\infty$ a.s..

(iii) Using the lower bound we had on \tilde{U}_t , we obtain

$$\int_{t_0}^{+\infty} \varepsilon(t) \|X(t) - x^*\|^2 dt < +\infty.$$

(iv) By the previous item, (T_2) , and Lemma 2.6.2 we conclude that $\lim_{t \rightarrow +\infty} X(t) = x^*$ a.s..

This completes the proof. \square

4.4.2 Practical situations

We will consider situations where the three conditions (T_1) , (T_2) and (T_3) are satisfied simultaneously. These are properties of the viscosity curve that we will now study. The difficulty comes from (T_2) and (T_3) which are a priori not compatible. Indeed, (T_2) requires the parameter $\varepsilon(t)$ to converge slowly towards zero for the Tikhonov regularization to be effective. On the other hand in (T_3) the parameter $\varepsilon(t)$ must converge sufficiently quickly towards zero so that the term $(\|x^*\|^2 - \|x_{\varepsilon(t)}\|^2)$ converges to zero fairly quickly, and thus corrects the infinite value of the integral of $\varepsilon(t)$.

4.4.2.1 Quantitative stability of variational systems

Our first objective is to evaluate the rate of convergence towards zero of $(\|x^*\|^2 - \|x_{\varepsilon}\|^2)$ as $\varepsilon \rightarrow 0$. Using the differentiability properties of the viscosity curve is not a good idea, because the viscosity curve can be of infinite length in the case of a general differentiable convex function, see [192]. To overcome this difficulty, we assume that $F = f + g$ satisfies the Łojasiewicz property (see (2.10)). This basic property has its roots in algebraic geometry, and it essentially describes a relationship between the objective value and its gradient (or subgradient). Once this is assumed, we will need tools from variational analysis to conclude. More precisely, we will need the following two results that have been obtained in [39, 36]. Recall from Section 2.5 the different notations and the notion of ρ -Hausdorff distance for functions and operators.

Lemma 4.4.2 ([36, Proposition 1.2]). *Let $A, B : \mathbb{H} \rightrightarrows \mathcal{H}$ be two maximal monotone operators, then*

$$\|(I + A)^{-1}(0) - \|(I + B)^{-1}(0)\| \leq 3\text{haus}_{\|(I+A)^{-1}(0)\|}(A, B).$$

In fact, in [36], it is shown that convergence of sequences of maximal monotone operators in the bounded Hausdorff topology is equivalent the uniform convergence on bounded sets of the resolvents.

Lemma 4.4.3 ([39, Theorem 5.2]). *Let f and g be proper lsc convex functions on \mathbb{H} . To any*

$$\rho > \max(\text{dist}(0, \text{epi } f), \text{dist}(0, \text{epi } g))$$

there correspond some constants κ and ρ_0 (that depend on ρ) such that

$$\text{haus}_{\rho}(\partial f, \partial g) \leq \kappa[\text{haus}_{\rho_0}(f, g)]^{\frac{1}{2}}.$$

The following proposition is new and is a consequence of the previous two results, since this is not obvious, we are going to present the whole proof.

Proposition 4.4.4. *Let $f \in \Gamma_0(\mathbb{H})$ be a function such that $\mathcal{S} \neq \emptyset$, and that $f \in \text{EB}^p(\mathcal{S})$. Let also $x^* = \text{P}_{\mathcal{S}}(0)$ and for $\varepsilon > 0$, let x_{ε} be the unique minimizer of $f_{\varepsilon}(x) = f(x) + \frac{\varepsilon}{2}\|x\|^2$. Then there exists $C_0, \varepsilon^* > 0$ such that*

$$\|x_{\varepsilon} - x^*\| \leq C_0 \varepsilon^{\frac{1}{2p}}, \quad \forall \varepsilon \in]0, \varepsilon^*].$$

Consequently, there exists $C > 0$ such that

$$\|x^*\|^2 - \|x_{\varepsilon}\|^2 \leq C \varepsilon^{\frac{1}{2p}}, \quad \forall \varepsilon \in]0, \varepsilon^*].$$

Proof. We have that

$$x_\varepsilon + \frac{1}{\varepsilon} \partial f(x_\varepsilon) \ni 0,$$

that is

$$x_\varepsilon = (I + \partial \varphi_\varepsilon)^{-1}(0),$$

where

$$\varphi_\varepsilon \stackrel{\text{def}}{=} \frac{1}{\varepsilon} (f - \min f).$$

We have that φ_ε increases to $\iota_{\mathcal{S}}$ as ε decreases to zero, and

$$x^* = \mathbf{P}_{\mathcal{S}}(0) = (I + N_{\mathcal{S}})^{-1}(0).$$

Therefore

$$\|x_\varepsilon - x^*\| = \|(I + \partial \varphi_\varepsilon)^{-1}(0) - (I + N_{\mathcal{S}})^{-1}(0)\|.$$

By Lemma 4.4.2 with $A = \partial \varphi_\varepsilon$, and $B = \partial \iota_{\mathcal{S}} = N_{\mathcal{S}}$, we have that

$$\|x_\varepsilon - x^*\| \leq 3 \text{haus}_\rho(\partial \varphi_\varepsilon, N_{\mathcal{S}}),$$

for $\rho > \|x^*\|$, where we used that haus_ρ is increasing with ρ . Now, since

$$\max(\text{dist}(0, \text{epi } \varphi_\varepsilon), \text{dist}(0, \text{epi } \iota_{\mathcal{S}})) \leq \|x^*\|,$$

we let $\rho > \|x^*\|$, and in turn Lemma 4.4.3 entails that there exist constants $\kappa, \rho_0 > 0$ (depending on ρ) such that

$$\|x_\varepsilon - x^*\| \leq 3\kappa [\text{haus}_{\rho_0}(\varphi_\varepsilon, \iota_{\mathcal{S}})]^{\frac{1}{2}}.$$

To complete our proof we just need to bound the right hand side. To that end, we use the $\text{EB}^p(\mathcal{S})$ assumption. Since $f \in \text{EB}^p(\mathcal{S})$, there exists $\varrho > 0, r > \min f$ such that

$$f(x) - \min f \geq \varrho \text{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f \leq f \leq r].$$

Since $\iota_{\mathcal{S}} \geq \varphi_\varepsilon$ we just need to compute the excess function of $(\text{epi } \varphi_\varepsilon)_{\rho_0}$ on $\text{epi } \iota_{\mathcal{S}} = \mathcal{S} \times \mathbb{R}_+$. We have

$$\iota_{\mathcal{S}}(x) \geq \varphi_\varepsilon(x) \geq \frac{\varrho}{\varepsilon} \text{dist}(x, \mathcal{S})^p, \quad \forall x \in [\min f \leq f \leq r].$$

By Definition 2.5.1, we have

$$\text{haus}_{\rho_0}(\varphi_\varepsilon, \iota_{\mathcal{S}}) = e((\text{epi } \varphi_\varepsilon)_{\rho_0}, \text{epi } \iota_{\mathcal{S}}) = \max_{x_1 \in \text{epi } \varphi_\varepsilon \cap \rho_0 \mathbb{B}, |r_1| \leq \rho_0} \min_{(x_2, r_2) \in \mathcal{S} \times \mathbb{R}_+} \max(\|x_1 - x_2\|, |r_1 - r_2|).$$

Besides, in inner minimization problem is bounded above by taking $r_1 = r_2$. We then have

$$\text{haus}_{\rho_0}(\varphi_\varepsilon, \iota_{\mathcal{S}}) \leq \max_{x_1 \in \text{epi } \varphi_\varepsilon \cap \rho_0 \mathbb{B}} \min_{x_2 \in \mathcal{S}} \|x_1 - x_2\| = \max_{x_1 \in \text{epi } \varphi_\varepsilon \cap \rho_0 \mathbb{B}} \text{dist}(x_1, \mathcal{S}).$$

Now let $\varepsilon_0 \stackrel{\text{def}}{=} \frac{r - \min f}{\rho_0} > 0$ and consider $\varepsilon \in]0, \varepsilon_0]$, then

$$\max_{x_1 \in \text{epi } \varphi_\varepsilon \cap \rho_0 \mathbb{B}} \text{dist}(x_1, \mathcal{S}) \leq \sup_{x_1 \in [f \leq \rho_0 \varepsilon + \min f]} \text{dist}(x_1, \mathcal{S}) \leq \left(\frac{\rho_0}{\varrho}\right)^{\frac{1}{p}} \varepsilon^{\frac{1}{p}},$$

where we have used that $\rho_0 \varepsilon + \min f \leq r$ and the hypothesis $\text{EB}^p(\mathcal{S})$ in the last inequality.

Let $C_0 \stackrel{\text{def}}{=} 3\kappa \left(\frac{\rho_0}{\varrho}\right)^{\frac{1}{2p}}$, then for $\varepsilon \in]0, \varepsilon_0]$,

$$\|x_\varepsilon - x^*\| \leq C_0 \varepsilon^{\frac{1}{2p}}.$$

On the other hand, since $\lim_{\varepsilon \rightarrow 0^+} x_\varepsilon = x^*$, then there exists $\varepsilon_1 > 0$ such that $\|x_\varepsilon\| \leq 1 + \|x^*\|$, for every $\varepsilon \leq \varepsilon_1$. Let $C_1 \stackrel{\text{def}}{=} 1 + 2\|x^*\|$, and we get

$$\|x^*\|^2 - \|x_\varepsilon\|^2 \leq C_1 \|x_\varepsilon - x^*\|, \quad \forall \varepsilon \in]0, \varepsilon_1].$$

Now letting $\varepsilon^\star \stackrel{\text{def}}{=} \min(\varepsilon_0, \varepsilon_1)$, and $C \stackrel{\text{def}}{=} C_0 C_1$, then we get

$$\|x^\star\|^2 - \|x_\varepsilon\|^2 \leq C\varepsilon^{\frac{1}{2p}}, \forall \varepsilon \in]0, \varepsilon^\star].$$

□

Theorem 4.4.5. *Consider the setting of Theorem 4.4.1 and suppose that $F = f + g \in \text{EB}^p(\mathcal{S}_F)$. Then taking the Tikhonov parameter $\varepsilon(t) = \frac{1}{t^r}$ with*

$$1 \geq r > \frac{2p}{2p+1},$$

then the three conditions (T₁), (T₂), and (T₃) of Theorem 4.3.2 are satisfied simultaneously. In particular, the solution $X \in S_{\mathbb{H}}^\nu[t_0]$ of (SDI – TA) is unique and $X(t)$ converges almost surely (as $t \rightarrow +\infty$) in the strong topology to the minimal norm solution $x^\star = P_{\mathcal{S}_F}(0)$.

Proof. It is direct to check (T₁) and (T₂). In order to check (T₃), let $\varepsilon^\star > 0$ from Proposition 4.4.4 and $T^\star = \max[t_0, (\frac{1}{\varepsilon^\star})^{\frac{1}{r}}]$, then we have

$$\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2 \leq C \frac{1}{t^{\frac{r}{2p}}}, \quad \forall t \geq T^\star.$$

So

$$\int_{t_0}^{+\infty} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt = \underbrace{\int_{t_0}^{T^\star} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt}_{I_1} + \underbrace{\int_{T^\star}^{+\infty} \frac{\|x^\star\|^2 - \|x_{\varepsilon(t)}\|^2}{t^r} dt}_{I_2}.$$

It is clear that I_1 is bounded (by $T^\star t_0^{-r} \|x^\star\|^2$ for instance), hence (T₃) holds under the condition that

$$\int_{T^\star}^{+\infty} \frac{1}{t^r} \frac{C}{t^{\frac{r}{2p}}} dt < +\infty$$

which is true when $r + \frac{r}{2p} > 1$. This in turn gives the condition $1 \geq r > \frac{2p}{2p+1}$. □

4.4.3 Convergence rates of the objective in the smooth case

We are going to show global convergence rates in expectation in the smooth convex case, in order to do that, it is worth citing a result from [27], where they deal with the deterministic case.

Take $\varepsilon(t) = \frac{1}{t^r}$, $0 < r < 1$, $t_0 > 0$. The convergence rate of the values and the strong convergence to the minimum norm solution is described below, see Attouch, Chbani, Riahi [27, Theorem 5].

Theorem 4.4.6. *Take $\varepsilon(t) = \frac{1}{t^r}$ and $0 < r < 1$. Let us consider (DI – TA) in the case where $g \equiv 0$, i.e.*

$$\dot{x}(t) + \nabla f(x(t)) + \frac{1}{t^r} x(t) = 0. \quad (4.32)$$

Let $x : [t_0, +\infty[\rightarrow \mathcal{H}$ be a solution trajectory of (DI – TA). For $\varepsilon > 0$ define $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) + \frac{\varepsilon}{2} \|x\|^2$, let x_ε be the unique minimizer of f_ε , and consider the Lyapunov function

$$E(t) \stackrel{\text{def}}{=} f_{\varepsilon(t)}(x(t)) - f_{\varepsilon(t)}(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2} \|x(t) - x_{\varepsilon(t)}\|^2.$$

Then, we have

$$(i) \quad E(t) = \mathcal{O}\left(\frac{1}{t}\right) \text{ as } t \rightarrow +\infty;$$

$$(ii) \quad f(x(t)) - \min(f) = \mathcal{O}\left(\frac{1}{t^r}\right) \text{ as } t \rightarrow +\infty;$$

$$(iii) \|x(t) - x_{\varepsilon(t)}\|^2 = \mathcal{O}\left(\frac{1}{t^{1-r}}\right) \text{ as } t \rightarrow +\infty.$$

In addition, we have strong convergence of $x(t)$ to the minimum norm solution, named $x^* = P_{\mathcal{S}}(0)$. Moreover, if $f \in \text{EB}^p(\mathcal{S})$

$$(iv) \|x(t) - x^*\|^2 = \begin{cases} \mathcal{O}\left(\frac{1}{t^p}\right), & \text{if } r \in \left]0, \frac{p}{p+1}\right[; \\ \mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } r \in \left[\frac{p}{p+1}, 1\right[\end{cases} \text{ as } t \rightarrow +\infty.$$

Remark 4.4.7. The last item of this already known Theorem is new and direct from our Proposition 4.4.4.

Now we have the necessary tools in order to show our first result in this sense, this one summarizes the global convergence rates in expectation satisfied by the trajectories of (SDI – TA) in the case where $g \equiv 0$.

Let us consider the assumption:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ f \in C^2(\mathbb{H}) \text{ or } \mathbb{H} \text{ is finite-dimensional;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{cases} \quad (\mathbf{H}_f^*)$$

Theorem 4.4.8. Let f satisfying (\mathbf{H}_f^*) , and also $f \in \text{EB}^p(\mathcal{S})$, σ satisfying (\mathbf{H}_σ) , and $\sigma_\infty \in L^2([t_0, +\infty[)$ and is nonincreasing. Let $\nu \geq 2$, and let us consider $\varepsilon(t) = \frac{1}{t^r}$ where $0 < r < 1$, then we evaluate (SDI – TA) in the case where $g \equiv 0$, and with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$, i.e.

$$\begin{cases} dX(t) = -\nabla f(X(t))dt - \frac{1}{t^r}X(t)dt + \sigma(t, X(t))dW(t), & t \geq t_0; \\ X(t_0) = X_0. \end{cases} \quad (\text{SDE} - \text{TA})$$

For $\varepsilon > 0$, let us define $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) + \frac{\varepsilon}{2}\|x\|^2$, x_ε be the unique minimizer of f_ε , consider the Lyapunov function

$$E(t, x) \stackrel{\text{def}}{=} f_{\varepsilon(t)}(x) - f_{\varepsilon(t)}(x_{\varepsilon(t)}) + \frac{\varepsilon(t)}{2}\|x - x_{\varepsilon(t)}\|^2,$$

and for $t_1 > t_0$,

$$R(t) \stackrel{\text{def}}{=} e^{-\frac{t^{1-r}}{1-r}} \int_{t_1}^t e^{\frac{s^{1-r}}{1-r}} \sigma_\infty^2(s) ds. \quad (4.33)$$

Consider $x^* \stackrel{\text{def}}{=} P_{\mathcal{S}}(0)$. Then, the solution trajectory $X \in S_{\mathbb{H}}^\nu[t_0]$ is unique, and we have that:

(i) $R(t) \rightarrow 0$ as $t \rightarrow +\infty$;

(ii) Furthermore, we can obtain a convergence rate for R ,

$$R(t) = \mathcal{O}\left(\exp(-t^r(1-2^{-r})) + t^r \sigma_\infty^2\left(\frac{t_1+t}{2}\right)\right).$$

Moreover, if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > 1$, then $R(t) = \mathcal{O}(t^{r-\alpha})$.

(iii) $\mathbb{E}[E(t, X(t))] = \mathcal{O}\left(\frac{1}{t} + R(t)\right)$

(iv) $\mathbb{E}[f(X(t)) - \min(f)] = \mathcal{O}\left(\frac{1}{t^r} + R(t)\right)$. Moreover if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > 1$, then

$$\mathbb{E}[f(X(t)) - \min(f)] = \begin{cases} \mathcal{O}\left(\frac{1}{t^{\alpha-r}}\right), & \text{if } \alpha \in]1, 2r[; \\ \mathcal{O}\left(\frac{1}{t^r}\right), & \text{if } \alpha \geq 2r; \end{cases}$$

(v) $\mathbb{E}[\|X(t) - x_{\varepsilon(t)}\|^2] = \mathcal{O}\left(\frac{1}{t^{1-r}} + t^r R(t)\right)$, which goes to 0 as $t \rightarrow +\infty$ if $r \in]0, \frac{1}{2}]$. Moreover, if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > \max\{2r, 1\}$, then

$$\mathbb{E}[\|X(t) - x_{\varepsilon(t)}\|^2] = \begin{cases} \mathcal{O}\left(\frac{1}{t^{\alpha-2r}}\right), & \text{if } \alpha \in]\max\{1, 2r\}, r+1[; \\ \mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } \alpha \geq r+1. \end{cases}$$

(vi) $\mathbb{E}[\|X(t) - x^*\|^2] = \mathcal{O}\left(\frac{1}{t^{1-r}} + \frac{1}{t^{\frac{r}{p}}} + t^r R(t)\right)$, which goes to 0 as $t \rightarrow +\infty$ if $r \in]0, \frac{1}{2}]$. Moreover, if $\sigma_\infty^2(t) = \mathcal{O}(t^{-\alpha})$ for $\alpha > \max\{2r, 1\}$, then

$$\mathbb{E}[\|X(t) - x^*\|^2] = \mathcal{O}\left(\frac{1}{t^{1-r}} + \frac{1}{t^{\frac{r}{p}}} + \frac{1}{t^{\alpha-2r}}\right).$$

In particular,

$$\mathbb{E}[\|X(t) - x^*\|^2] = \begin{cases} \mathcal{O}\left(\frac{1}{t^{1-r}}\right), & \text{if } r \in]\frac{p}{p+1}, 1[, \alpha > r+1; \\ \mathcal{O}\left(\frac{1}{t^{\frac{r}{p}}}\right), & \text{if } r \in]0, \frac{p}{p+1}[, \alpha > \max\{1, \frac{r(2p+1)}{p}\}; \\ \mathcal{O}\left(\frac{1}{t^{\alpha-2r}}\right), & \text{if } r \in]\frac{p}{2p+1}, 1[, \alpha \in \left(\max\{2r, 1\}, \min\{r+1, \frac{r(2p+1)}{p}\}\right). \end{cases}$$

Remark 4.4.9. The expression in (ii) goes to 0 as $t \rightarrow +\infty$ since $\lim_{t \rightarrow +\infty} t\sigma_\infty^2(t) = 0$ and $r < 1$.

Proof. The existence and uniqueness of a solution was already stated in Theorem 4.2.4.

The first item is a direct consequence of Lemma 2.6.4, for the second one we recall that $\sigma_\infty \in L^2([t_0, +\infty[)$ and is nonincreasing, and we proceed as follows:

$$\begin{aligned} R(t) &= e^{-\frac{t^{1-r}}{1-r}} \int_{t_1}^{\frac{t_1+t}{2}} e^{\frac{s^{1-r}}{1-r}} \sigma_\infty^2(s) ds + e^{-\frac{t^{1-r}}{1-r}} \int_{\frac{t_1+t}{2}}^t e^{\frac{s^{1-r}}{1-r}} \sigma_\infty^2(s) ds \\ &\leq e^{(\frac{t_0}{2})^r} e^{-t^r(1-2^{-r})} \int_{t_1}^{+\infty} \sigma_\infty^2(s) ds + \sigma_\infty^2\left(\frac{t_1+t}{2}\right) D_{\frac{1}{1-r}, 1-r}(t), \end{aligned}$$

where

$$D_{a,b}(t) = e^{-at^b} \int_0^t e^{as^b} ds.$$

As a corollary of an upper bound of the Dawson integral shown in [160, Section 7.8], we have that

$$D_{a,b}(t) \leq \frac{2}{ab} t^{1-b}, \quad 0 < b \leq 2, a > 0, t > 0,$$

thus we obtain

$$R(t) = \mathcal{O}\left(\exp(-t^r(1-2^{-r})) + t^r \sigma_\infty^2\left(\frac{t_1+t}{2}\right)\right),$$

and $\lim_{t \rightarrow +\infty} t^r \sigma_\infty^2(t) = 0$, to show this, we recall that σ_∞^2 is nonincreasing, then

$$0 \leq t\sigma_\infty^2(t) \leq 2 \int_{\frac{t}{2}}^t \sigma_\infty^2(u) du,$$

and the right hand side goes to 0 as $t \rightarrow +\infty$ since $\sigma_\infty \in L^2([t_0, +\infty[)$, thus we obtain that $\lim_{t \rightarrow +\infty} t\sigma_\infty^2(t) = 0$ (as mentioned in Remark 4.4.9), and this directly implies the desired.

The rest of the proof follows by using Itô's formula with $\phi(t, x) = e^{\int_{t_1}^t \frac{ds}{s^\gamma}} E(t, x)$, where either $f \in C^2(\mathbb{H})$ or \mathbb{H} is finite-dimensional, in the latter case we use Proposition 2.7.6. Then, taking expectation and following the calculus done in [27, Theorem 5], we obtain the same results as in Theorem 4.4.6 (or [27, Theorem 5]) up to the term $R(t)$, therefore, this result could be seen as the stochastic counterpart of the mentioned Theorem. \square

Remark 4.4.10. Tikhonov regularization implies strong convergence of the trajectory to the minimal norm solution, therefore, in the stochastic case you have to be careful in the tuning of the noise in order to not break this convergence. In the almost sure sense, you can tune appropriately the Tikhonov parameter without assuming more than $\sigma_\infty \in L^2([t_0, +\infty[)$, nevertheless, in expectation, you may require a stronger assumption in general on the noise σ_∞^2 in order to obtain a useful convergence rate, this is reflected in items (v) and (vi) of Theorem 4.4.8.

Part II

Second-Order Systems

Chapter 5

From First to Second Order Methods via Time Scaling and Averaging

In this chapter, we aim to use the theory we developed in Chapter 4 and Chapter 5 for stochastic convex minimization problems through the lens of stochastic inertial differential inclusions that are driven by the subgradient of a convex objective function. This will provide a general mathematical framework for analyzing the convergence properties of stochastic second-order inertial continuous-time dynamics involving vanishing viscous damping and measurable stochastic subgradient selections. Our chief goal in this chapter is to develop a systematic and unified way that transfers the properties studied for first-order stochastic differential equations (Chapters 4-5) to second-order inertial ones driven even by subgradients in lieu of gradients. This program will rely on two tenets: time scaling and averaging, following an approach recently developed in the literature by one of the co-authors in the deterministic case. Under a mild integrability assumption involving the diffusion term and the viscous damping, our first main result shows that almost surely, there is weak convergence of the trajectory towards a minimizer of the objective function and fast convergence of the values and gradients. We also provide a comprehensive complexity analysis by establishing several new pointwise and ergodic convergence rates in expectation for the convex, strongly convex, and (local) Polyak-Łojasiewicz case. Finally, using Tikhonov regularization with a properly tuned vanishing parameter, we can obtain almost sure strong convergence of the trajectory towards the minimum norm solution.

Main contributions of this chapter

- ▶ Almost sure weak convergence of the trajectory generated by the second-order inertial SDI with a particular geometric damping, to the set of minimizers (Theorem 5.2.2).
- ▶ Global convergence rates of the values in expectation under smoothness of the objective for the convex (Theorem 5.2.3), and Polyak-Łojasiewicz inequality case (Theorem 5.2.6).
- ▶ Fast convergence of the values in almost sure sense and in expectation under $\alpha > 3$, $\gamma(t) = \frac{\alpha}{t}$ and $\beta = \frac{t}{\alpha-1}$ (Corollary 5.2.5).
- ▶ Almost sure strong convergence of the trajectory generated by the second-order inertial SDI regularized by a Tikhonov parameter, to the minimal norm minimizer (Theorem 5.3.1).

The content of this chapter is submitted in [147].

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5.1 Introduction

5.1.1 Problem Statement

Let us consider again the minimization problem

$$\min_{x \in \mathbb{H}} F(x) \stackrel{\text{def}}{=} f(x) + g(x), \quad (\mathcal{P}_2)$$

where \mathbb{H} is a separable real Hilbert space, and the objective F satisfies the following standing assumptions:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ g : \mathbb{H} \rightarrow \mathbb{R} \cup \{\pm\infty\} \text{ is proper, lower semi-continuous (lsc) and convex;} \\ \mathcal{S}_F \stackrel{\text{def}}{=} \operatorname{argmin}(F) \neq \emptyset. \end{cases} \quad (\mathbf{H}_F)$$

Consider \mathbb{K} a real separable Hilbert space. To solve (\mathcal{P}_2) , we will refer to Section 1.2 for the approach that leads to the dynamic that we will study in this chapter, specifically the following stochastic differential inclusion (SDI), which is the stochastic counterpart of $(\text{ISIHD}_{\text{NS}})$:

$$\begin{cases} dX(t) = V(t)dt; \\ dV(t) \in -\gamma(t)V(t)dt - \partial F(X(t) + \beta(t)V(t))dt + \sigma(t, X(t) + \beta(t)V(t))dW(t); \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{S} - \text{ISIHD}_{\text{NS}})$$

This SDI is defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ (for some $\nu \geq 2$) are the initial data; the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function that satisfies (\mathbf{H}_σ) ; and W is a \mathbb{K} -valued Brownian motion (see definition in Section 2.7). When $g \equiv 0$, we recover the stochastic counterpart of (ISIHD) as the following SDE:

$$\begin{cases} dX(t) = V(t)dt; \\ dV(t) = -\gamma(t)V(t)dt - \nabla f(X(t) + \beta(t)V(t))dt + \sigma(t, X(t) + \beta(t)V(t))dW(t); \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{S} - \text{ISIHD})$$

In this work, our goal is to provide a general mathematical framework for analyzing the convergence properties of $(\text{S} - \text{ISIHD}_{\text{NS}})$. In this context, considering inertial dynamics with a time-dependent vanishing viscosity coefficient γ is a key ingredient to obtain fast convergent methods. We will develop

a systematic and unified way that transfers the properties of stochastic first-order dynamics studied in Chapters 3-4 to second-order ones. Our program will then rely on two pillars: time scaling and averaging, following the methodology recently developed by Attouch, Bot, and Nguyen in [17] in the deterministic gradient case.

5.1.2 Other assumptions

Recall that our focus in this thesis is on an optimization perspective, and as we argued in the introduction, we will study the long time behavior of our SDE's and SDI's (in particular (S – ISIHD) and (S – ISIHD_{NS})) as the diffusion term vanishes when $t \rightarrow +\infty$. Therefore, we recall that we assume that the diffusion (volatility) term σ satisfies:

$$\begin{cases} \sup_{t \geq t_0, x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} < +\infty, \\ \|\sigma(t, x') - \sigma(t, x)\|_{\text{HS}} \leq l_0 \|x' - x\|, \end{cases} \quad (\mathbf{H}_\sigma)$$

for some $l_0 > 0$ and for all $t \geq t_0, x, x' \in \mathbb{H}$. The Lipschitz continuity assumption is mild and classical and will be required to ensure the well-posedness of (S – ISIHD) and (S – ISIHD_{NS}). Let us also define $\sigma_\infty : [t_0, +\infty[\rightarrow \mathbb{R}_+$ as

$$\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}}.$$

Remark 5.1.1. (\mathbf{H}_σ) implies the existence of $\sigma_* > 0$ such that:

$$\|\sigma(t, x)\|_{\text{HS}}^2 = \text{tr}[\Sigma(t, x)] \leq \sigma_*^2,$$

for all $t \geq t_0, x \in \mathbb{H}$, where $\Sigma \stackrel{\text{def}}{=} \sigma \sigma^*$.

For $t_0 > 0$, let $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be a viscous damping and denote:

$$p(t) \stackrel{\text{def}}{=} \exp\left(\int_{t_0}^t \gamma(s) ds\right).$$

If

$$\begin{cases} \gamma \text{ is upper bounded by a non-increasing function for every } t \geq t_0; \\ \int_{t_0}^{+\infty} \frac{ds}{p(s)} < +\infty. \end{cases} \quad (\mathbf{H}_\gamma)$$

We define $\Gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ by

$$\Gamma(t) \stackrel{\text{def}}{=} p(t) \int_t^{+\infty} \frac{ds}{p(s)}. \quad (5.1)$$

Remark 5.1.2. Let us notice that Γ satisfies the relation $\Gamma' = \gamma\Gamma - 1$.

We denote

$$I[h](t) \stackrel{\text{def}}{=} \exp\left(-\int_{t_0}^t \frac{du}{\Gamma(u)}\right) \int_{t_0}^t h(u) \frac{\exp\left(\int_{t_0}^u \frac{ds}{\Gamma(s)}\right)}{\Gamma(u)} du.$$

The stochastic version of (GF) where f is smooth has been well studied and documented in Chapter 3. Since we are going to show results in the smooth case, we rewrite (\mathbf{H}_F) when $g \equiv 0$,

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{cases} \quad (\mathbf{H}_f)$$

5.2 From first-order to second-order systems

5.2.1 Time scaling and averaging

We apply a time scaling and then an averaging technique to the system (SDI) to derive an insightful reparametrization of a particular case of our second-order system (S – ISIH_{NS}), specifically, the case when $\beta \equiv \Gamma$. The main advantage of this method is that the results of (SDI) directly carry over to obtain results on the convergence behavior of (S – ISIH_{NS}) without passing through a dedicated Lyapunov analysis.

Let $\nu \geq 2$, $s_0 \geq 0$. We consider the potential $F = f + g$ where g satisfies (H _{λ}). Let σ_1 be a diffusion term in the time parametrization by s . We will study the dynamic (SDI) in s , starting at s_0 , with diffusion term σ_1 under hypotheses (H _{F}) and (H _{σ}). Let $\sigma_{1*} > 0$ be such that

$$\|\sigma_1(s, x)\|_{\text{HS}} \leq \sigma_{1*}^2, \quad \forall s \geq s_0, \forall x \in \mathbb{H},$$

and $\sigma_{1\infty}(s) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma_1(s, x)\|_{\text{HS}}$. We rewrite (SDI) adapted to our case,

$$\begin{cases} dZ(s) \in -\partial F(Z(s))ds + \sigma_1(s, Z(s))dW(s), & s > s_0; \\ Z(s_0) = Z_0, \end{cases} \quad (5.2)$$

where $Z_0 \in L^\nu([s_0, +\infty[; \mathbb{H})$.

Let us make the change of time $s = \tau(t)$ in the dynamic (5.2), where τ is an increasing function from $[t_0, +\infty[$ to $[s_0, +\infty[$, which is twice differentiable, and which satisfies $\lim_{t \rightarrow +\infty} \tau(t) = +\infty$. Denote $Y(t) \stackrel{\text{def}}{=} Z(s)$ and t_0 be such that $s_0 = \tau(t_0)$. By the chain rule and [159, Theorem 8.5.7], we have

$$\begin{cases} dY(t) \in -\tau'(t)\partial F(Y(t))dt + \sqrt{\tau'(t)}\sigma_1(\tau(t), Y(t))dW(t), & t > t_0; \\ Y(t_0) = Z_0. \end{cases} \quad (5.3)$$

Consider the smooth case, *i.e.* when $g \equiv 0$ and the hypotheses of Theorem 3.2.8 ($f \in C_L^2(\mathbb{H})$ and $s \mapsto s\sigma_{1\infty}^2(s) \in L^1([s_0, +\infty[)$), then we can conclude the following convergence rates for (5.3) (when $g \equiv 0$):

$$f(Y(t)) - \min f = o\left(\frac{1}{\tau(t)}\right) \text{ a.s.}, \quad (5.4)$$

and by Proposition 3.2.6 in the case $\beta = 0$,

$$\mathbb{E}(f(Y(t)) - \min f) = \mathcal{O}\left(\frac{1}{\tau(t)}\right) \text{ a.s.}, \quad (5.5)$$

By introducing a function τ that grows faster than the identity ($\tau(t) \geq t$), we have accelerated the dynamic, passing from the asymptotic convergence rate $1/s$ for (5.2) to $1/\tau(t)$ for (5.3). The price to pay is that the drift term in (5.3) is non-autonomous, furthermore, when the coefficient in front of the gradient tends to infinity as $t \rightarrow +\infty$, it will preclude the use of an explicit discretization in time. To overcome this, we adapt from [17] the following approach, which is called averaging.

Consider (5.3) and let $X, V : \Omega \times [t_0, +\infty[\rightarrow \mathbb{H}$ be two stochastic processes such that:

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ Y(t) = X(t) + \tau'(t)V(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0, \end{cases} \quad (5.6)$$

where $Y(t)$ is the process in (5.3), and $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ is the initial data that is \mathcal{F}_{t_0} -measurable. This leads us to set $Z_0 \stackrel{\text{def}}{=} X_0 + \tau'(t_0)V_0$ in order for the equations to fit. According to the averaging, the differential form of $Y(t)$ is

$$dY(t) = dX(t) + \tau''(t)V(t)dt + \tau'(t)dV(t).$$

Combining the previous equation with (5.3), we have that

$$-\tau'(t)\partial F(Y(t))dt + \sqrt{\tau'(t)}\sigma_1(\tau(t), Y(t))dW(t) \ni dX(t) + \tau''(t)V(t)dt + \tau'(t)dV(t).$$

Using that $dX(t) = V(t)dt$ and dividing by τ' , we then have

$$-\partial F(X(t) + \tau'(t)V(t))dt + \frac{1}{\sqrt{\tau'(t)}}\sigma_1(\tau(t), X(t) + \tau'(t)V(t))dW(t) \ni \frac{1 + \tau''(t)}{\tau'(t)}V(t)dt + dV(t).$$

Therefore, after the time scaling and averaging, we obtain the following dynamic:

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) \in -\frac{1+\tau''(t)}{\tau'(t)}V(t)dt - \partial F(X(t) + \tau'(t)V(t))dt \\ \quad + \frac{1}{\sqrt{\tau'(t)}}\sigma_1(\tau(t), X(t) + \tau'(t)V(t))dW(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{ISIHD-S.1})$$

Let $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ satisfying (\mathbf{H}_γ) . We are going to determine τ in order to obtain a viscous damping coefficient equal to γ , *i.e.*,

$$\frac{1 + \tau''(t)}{\tau'(t)} = \gamma(t).$$

Clearly, τ' solves the following ODE in ζ

$$\zeta' = \gamma\zeta - 1.$$

As observed in Remark 5.1.2, the function Γ also satisfies the same ODE, and thus we can adjust the initial condition of τ' to obtain

$$\tau'(t) = \Gamma(t) = p(t) \int_t^{+\infty} \frac{du}{p(u)} \quad \forall t \geq t_0.$$

We then integrate and take $\tau(t) = s_0 + \int_{t_0}^t \Gamma(u)du$ to get $\tau(t_0) = s_0$ as required. This is a valid selection of τ since $t \mapsto s_0 + \int_{t_0}^t \Gamma(u)du$ is an increasing function from $[t_0, +\infty[$ to $[s_0, +\infty[$, twice differentiable and $\Gamma \notin L^1([t_0, +\infty[)$ because Γ is lower bounded by a non-decreasing function since γ is upper bounded by a non-increasing function (see [18, Proposition 2.2]) by (\mathbf{H}_γ) . For this particular selection of τ , and defining $\tilde{\sigma}_1(t, \cdot) \stackrel{\text{def}}{=} \frac{\sigma_1(\tau(t), \cdot)}{\sqrt{\Gamma(t)}}$, we have that (ISIHD-S.1) is equivalent to

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) \in -\gamma(t)V(t)dt - \partial F(X(t) + \Gamma(t)V(t))dt + \tilde{\sigma}_1(t, X(t) + \Gamma(t)V(t))dW(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{ISIHD-S.2})$$

Clearly, (ISIHD-S.2) is nothing but $(\mathbf{S} - \text{ISIHD}_{\text{NS}})$ when $\beta \equiv \Gamma$ and $\sigma \equiv \tilde{\sigma}_1$.

In order to be able to transfer the convergence results on Z in (5.2) (via (5.3)) to X in (ISIHD-S.2), it remains to express X in terms of Y only. For this, let

$$a(t) \stackrel{\text{def}}{=} \frac{1}{\tau'(t)}, \quad A(t) \stackrel{\text{def}}{=} \int_{t_0}^t a(u)du.$$

Recalling the averaging in (5.6), we need to integrate the following equation

$$V(t) + a(t)X(t) = a(t)Y(t). \quad (5.7)$$

Multiplying both sides by $e^{A(t)}$ and using (5.6), we get equivalently

$$d\left(e^{A(t)}X(t)\right) = a(t)e^{A(t)}Y(t)dt. \quad (5.8)$$

Integrating and using again (5.6), we obtain

$$\begin{aligned} X(t) &= e^{-A(t)}X(t_0) + e^{-A(t)} \int_{t_0}^t a(u)e^{A(u)}Y(u)du \\ &= e^{-A(t)}Y(t_0) + e^{-A(t)} \int_{t_0}^t a(u)e^{A(u)}Y(u)du - e^{-A(t)}\tau'(t_0)V(t_0). \end{aligned}$$

Then we can write

$$X(t) = \int_{t_0}^t Y(u)d\mu_t(u) + \xi(t), \quad (5.9)$$

where μ_t is the probability measure on $[t_0, t]$ defined by

$$\mu_t \stackrel{\text{def}}{=} e^{-A(t)}\delta_{t_0} + a(u)e^{A(u)-A(t)}du, \quad (5.10)$$

where δ_{t_0} is the Dirac measure at t_0 , $a(u)e^{A(u)-A(t)}du$ is the measure with density $a(\cdot)e^{A(\cdot)-A(t)}$ with respect to the Lebesgue measure on $[t_0, t]$, and $\xi(t)$ is a random process since V_0 is a random variable, *i.e.*,

$$\xi(t) \stackrel{\text{def}}{=} \xi(\omega, t) = -e^{-A(t)}\tau'(t_0)V_0(\omega) \quad \forall \omega \in \Omega. \quad (5.11)$$

5.2.2 Almost sure weak convergence of the trajectory and convergence rates under general γ , and $\beta \equiv \Gamma$

We here state the main results of this section. We start with a useful lemma and then show almost sure convergence of the trajectory of (S – ISIHD_{NS}) to a random variable taking values in the set of minimizers of F . When $g \equiv 0$, we also provide convergence rates.

Lemma 5.2.1. *Under hypothesis (H $_\gamma$), then*

$$\int_{t_0}^{\infty} \frac{ds}{\Gamma(s)} = +\infty.$$

Proof. Let $q(t) \stackrel{\text{def}}{=} \int_t^{\infty} \frac{ds}{p(s)}$, since $\int_{t_0}^{\infty} \frac{ds}{p(s)} < +\infty$, then $\lim_{t \rightarrow \infty} q(t) = 0$ and $q'(t) = -\frac{1}{p(t)}$. On the other hand

$$\int_{t_0}^{\infty} \frac{ds}{\Gamma(s)} = - \int_{t_0}^{\infty} \frac{q'(t)}{q(t)} = \ln(q(t_0)) - \lim_{t \rightarrow \infty} \ln(q(t)) = +\infty. \quad \square$$

Theorem 5.2.2. *Let $\nu \geq 2$ and consider the dynamic (S – ISIHD_{NS}) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable, where $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ satisfies (H $_\gamma$), and $\beta \equiv \Gamma$. Besides, $F = f + g$ and σ satisfy Assumptions (H $_F$) and (H $_\sigma$). Moreover, suppose that g satisfies (H $_\lambda$). Then, there exists a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of (S – ISIHD_{NS}). Additionally, if $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$, then there exists an \mathcal{S}_F -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s. and $w\text{-}\lim_{t \rightarrow +\infty} \Gamma(t)V(t) = 0$. a.s..*

Proof. Let $\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \Gamma(u)du$, $\tilde{\sigma}(s, \cdot) \stackrel{\text{def}}{=} \sigma(\theta^{-1}(s), \cdot)\sqrt{\Gamma(\theta^{-1}(s))}$, and $\tilde{\sigma}_\infty(s) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\tilde{\sigma}(s, x)\|_{\text{HS}}$. Then $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$ is equivalent to $\tilde{\sigma}_\infty \in L^2(\mathbb{R}_+)$. Consider the dynamic:

$$\begin{cases} dZ(s) & \in -\partial F(Z(s)) + \tilde{\sigma}(s, Z(s))dW(s), \quad s > 0; \\ Z(0) & = X_0 + \Gamma(t_0)V_0. \end{cases} \quad (5.12)$$

By Theorem 4.3.2, we have that there exists a unique solution $(Z, \vartheta) \in S_{\mathbb{H}}^\nu \times C^1(\mathbb{R}_+; \mathbb{H})$ of (5.12), and an \mathcal{S}_F -valued random variable X^* such that $w\text{-}\lim_{s \rightarrow +\infty} Z(s) = X^*$ a.s.. Moreover, using the time scaling $\tau \equiv \theta$ and the averaging described in this section, we end up with the dynamic (S – ISIHD_{NS}) in the case where $\beta \equiv \Gamma$.

It is direct to check that the time scaling and averaging preserves the uniqueness of a solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^0[t_0]$. Now let us validate $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$. Since

$$\mathbb{E} \left(\sup_{s \in [0, T]} \|Z(s)\|^\nu \right) < +\infty, \quad \forall T > 0,$$

we directly obtain

$$\mathbb{E} \left(\sup_{t \in [t_0, T]} \|Y(t)\|^\nu \right) < +\infty, \quad \forall T > t_0.$$

Thanks to the relation (5.9), the following holds

$$\begin{aligned} \|X(t)\|^\nu &\leq \nu \left(\left\| X(t) - \int_{t_0}^t Y(u) d\mu_t(u) \right\|^\nu + \left\| \int_{t_0}^t Y(u) d\mu_t(u) \right\|^\nu \right) \\ &\leq \nu \left(\|\xi(t)\|^\nu + (t - t_0)^{\nu-1} \int_{t_0}^t \|Y(u)\|^\nu d\mu_t(u) \right). \end{aligned}$$

Let $T > t_0$ be arbitrary. Taking supremum over $[t_0, T]$ and then expectation at both sides, we obtain that

$$\mathbb{E} \left(\sup_{t \in [t_0, T]} \|X(t)\|^\nu \right) \leq \nu \left(\mathbb{E}(\|V_0\|^\nu) \|\Gamma(t_0)\|^\nu + (T - t_0)^{\nu-1} \mathbb{E} \left(\sup_{t \in [t_0, T]} \|Y(t)\|^\nu \right) \right) < +\infty.$$

Since $V(t) = \frac{Y(t) - X(t)}{\Gamma(t)}$, we have

$$\|V(t)\|^\nu \leq \frac{\nu}{\Gamma^\nu(t)} (\|Y(t)\|^\nu + \|X(t)\|^\nu).$$

Similarly as before, we let $T > t_0$ be arbitrary, and take the supremum over $[t_0, T]$ and then expectation at both sides to obtain

$$\mathbb{E} \left(\sup_{t \in [t_0, T]} \|V(t)\|^\nu \right) \leq \nu \sup_{t \in [t_0, T]} \frac{1}{\Gamma^\nu(t)} \left(\mathbb{E} \left(\sup_{t \in [t_0, T]} (\|Y(t)\|^\nu + \|X(t)\|^\nu) \right) \right).$$

Since Γ is a continuous positive function, by the extreme value theorem, we have that there exists $t_T \in [t_0, T]$ such that $\sup_{t \in [t_0, T]} \frac{1}{\Gamma^\nu(t)} = \frac{1}{\Gamma^\nu(t_T)} < +\infty$, and we conclude that $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$.

Now we prove that there exists an \mathcal{S}_F -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s.. By virtue of Theorem 4.3.2, there exists an \mathcal{S}_F -valued random variable X^* such that $w\text{-}\lim_{s \rightarrow +\infty} Z(s) = X^*$ a.s.. We also notice that we have directly $w\text{-}\lim_{t \rightarrow +\infty} Y(t) = X^*$ a.s.. Let $h \in \mathbb{H}$ be arbitrary and use the relation (5.9) as follows:

$$\begin{aligned} |\langle X(t) - X^*, h \rangle| &\leq \left| \left\langle X(t) - \int_{t_0}^t Y(u) d\mu_t(u), h \right\rangle \right| + \left| \left\langle \int_{t_0}^t Y(u) d\mu_t(u) - X^*, h \right\rangle \right| \\ &= |\langle \xi(t), h \rangle| + \left| \left\langle \int_{t_0}^t (Y(u) - X^*) d\mu_t(u), h \right\rangle \right| \\ &= |\langle \xi(t), h \rangle| + \left| \int_{t_0}^t \langle Y(u) - X^*, h \rangle d\mu_t(u) \right| \\ &\leq \|\xi(t)\| \|h\| + \int_{t_0}^t |\langle Y(u) - X^*, h \rangle| d\mu_t(u), \end{aligned}$$

where the second equality comes from the dominated convergence theorem, since $\sup_{s > t_0} \|Y(s)\| < +\infty$ a.s. (by (ii) of Theorem 4.3.2).

Now let $a(t) = \frac{1}{\Gamma(t)}$ and $A(t) = \int_{t_0}^t \frac{du}{\Gamma(u)}$. By Lemma 5.2.1, we have that $\lim_{t \rightarrow +\infty} \|\xi(t)\| = 0$ a.s.. On the other hand, it holds that

$$\int_{t_0}^t |\langle Y(u) - X^*, h \rangle| d\mu_t(u) \leq e^{-A(t)} |\langle Y(t_0) - X^*, h \rangle| + e^{-A(t)} \int_{t_0}^t a(u) e^{A(u)} |\langle Y(u) - X^*, h \rangle| du.$$

Now let $b(u) = |\langle Y(u) - X^*, h \rangle|$. Since we already proved that $\lim_{u \rightarrow +\infty} b(u) = 0$ a.s., and we have that $a \notin L^1([t_0, +\infty[)$ by Lemma 5.2.1, we utilize Lemma 2.6.5 with our respective a, b functions. This let us conclude that

$$\lim_{t \rightarrow +\infty} |\langle X(t) - X^*, h \rangle| = 0 \quad a.s..$$

Thus, $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s.. Finally, since

$$Y(t) = X(t) + \Gamma(t)V(t),$$

and the fact that X and Y have (a.s.) the same limit, we conclude that

$$w\text{-}\lim_{t \rightarrow +\infty} \Gamma(t)V(t) = 0 \quad a.s..$$

□

In the smooth case, we also have convergence rates on the objective value and the gradient.

Theorem 5.2.3. *Let $\nu \geq 2$ and consider the dynamic (S – ISIHD) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} –measurable, and such that f and σ satisfy (H_f) and (H_σ) , and in the case where γ satisfies (H_γ) , $\beta \equiv \Gamma$. Moreover, suppose that either \mathbb{H} is finite dimensional or $f \in C^2(\mathbb{H})$, and*

$$t \mapsto \sqrt{\theta(t)}\Gamma(t)\sigma_\infty \in L^2([t_0, +\infty[),$$

where $\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \Gamma(u)du$. Then the solution trajectory $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ is unique and satisfies:

$$\mathbb{E}[f(X(t)) - \min f] = \mathcal{O} \left(\max \left\{ e^{-A(t)}, I \left[\frac{1}{\theta} \right] (t) \right\} \right), \quad \forall t > t_0,$$

where $A(t) \stackrel{\text{def}}{=} \int_{t_0}^t \frac{du}{\Gamma(u)}$ and we recall that $I \left[\frac{1}{\theta} \right] (t) = e^{-A(t)} \int_{t_0}^t \frac{1}{\theta(u)} \frac{e^{A(u)}}{\Gamma(u)} du$.

From hypothesis (H_γ) we have that $\lim_{t \rightarrow +\infty} e^{-A(t)} = 0$, and since $\Gamma \notin L^1([t_0, +\infty[)$, we can use Lemma 2.6.5 to check that $\lim_{t \rightarrow +\infty} I \left[\frac{1}{\theta} \right] (t) = 0$.

Proof. We will utilize the averaging technique used in Theorem 5.2.2 and Jensen’s inequality. First, we have

$$\mathbb{E}(f(X(t)) - \min f) = \mathbb{E} \left(f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) \right) + \mathbb{E} \left(f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) - \min f \right).$$

Let us recall that $\mathbb{E}(\sup_{s \geq 0} \|Z(s)\|) < +\infty$, which implies that $\mathbb{E}(\sup_{t \geq t_0} \|X(t)\|) < +\infty$. We bound the first term using the gradient convexity inequality on f to get

$$\begin{aligned} f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) &\leq \|\nabla f(X(t))\| \|\xi(t)\| \\ &\leq \|\xi(t)\| (L \|X(t)\| + \|\nabla f(0)\|) \\ &\leq \|\xi(t)\| \left(L \sup_{t \geq t_0} \|X(t)\| + \|\nabla f(0)\| \right), \end{aligned}$$

and we conclude that

$$\mathbb{E} \left(f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) \right) = \mathcal{O}(e^{-A(t)}).$$

For the second term, we use Jensen’s inequality to obtain

$$\begin{aligned} \mathbb{E} \left(f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) - \min f \right) &\leq \int_{t_0}^t \mathbb{E}[f(Y(u)) - \min f] d\mu_t(u) \\ &\leq e^{-A(t)} \mathbb{E}[f(Y(t_0)) - \min f] + e^{-A(t)} \int_{t_0}^t \frac{e^{A(u)}}{\Gamma(u)} \mathbb{E}(f(Y(u)) - \min f) du. \end{aligned}$$

Since $\sqrt{\theta}\Gamma\sigma_\infty \in L^2([t_0, +\infty[))$ is equivalent to $s \mapsto s\tilde{\sigma}_\infty^2(s) \in L^1(\mathbb{R}_+)$, by Proposition 3.2.6 with $\beta = 0$, we have that there exists $C > 0$ such that $\mathbb{E}(f(Z(s)) - \min f) \leq \frac{C}{s}$. Then, we have

$$\mathbb{E}(f(Y(t)) - \min f) \leq \frac{C}{\theta(t)}.$$

Hence, there exists $C_0 > 0$ such that

$$\mathbb{E}(f(X(t)) - \min f) \leq C_0 e^{-A(t)} + CI \left[\frac{1}{\theta} \right] (t).$$

□

Theorem 5.2.4. *Let $\nu \geq 2$ and consider the dynamic (S – ISIHD) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable, and such that f and σ satisfy (\mathbf{H}_f) and (\mathbf{H}_σ) , and in the case where γ satisfies (\mathbf{H}_γ) , $\beta \equiv \Gamma$. Moreover, suppose that $f \in C^2(\mathbb{H})$ and*

$$t \mapsto \theta(t)\Gamma^2(t)\sigma_\infty^2(t) \in L^1([t_0, +\infty[),$$

where $\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \Gamma(u)du$. Then the solution trajectory $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ is unique and satisfies

$$\int_{t_0}^{+\infty} \theta(u)\Gamma(u)\|\nabla f(X(u) + \Gamma(u)V(u))\|^2 du < +\infty \quad \text{a.s.} \quad (5.13)$$

Proof. Consider (5.12) and the technique used in Theorem 5.2.2. We have that $t \mapsto \theta(t)\Gamma^2(t)\sigma_\infty^2(t) \in L^1([t_0, +\infty[)$ is equivalent to $s \mapsto s\tilde{\sigma}_\infty^2(s) \in L^1(\mathbb{R}_+)$. Therefore, we can use Theorem 3.2.8 to state that

$$\int_0^{+\infty} s\|\nabla f(Z(s))\|^2 ds < +\infty \quad \text{a.s.}$$

Using the time scaling $\tau \equiv \theta$ and making the change of variable $\theta(t) = s$ in the previous integral, we obtain

$$\int_{t_0}^{+\infty} \theta(t)\Gamma(t)\|\nabla f(Y(t))\|^2 dt < +\infty \quad \text{a.s.}$$

Recalling that in the averaging we impose that $Y = X + \Gamma V$, we conclude. □

5.2.3 Fast convergence under $\alpha > 3, \gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \frac{t}{\alpha-1}$

In the following, we show fast convergence results in expectation.

Corollary 5.2.5 (Case $\frac{\alpha}{t}$). *Let $\nu \geq 2, \alpha > 3$ and consider the dynamic (S – ISIHD) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable, in the case where $\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \frac{t}{\alpha-1}$. Besides, consider that f and σ satisfy (\mathbf{H}_f) and (\mathbf{H}_σ) . Moreover, let $f \in C^2(\mathbb{H})$ and $t \mapsto t^2\sigma_\infty(t) \in L^2([t_0, +\infty[)$. Then the solution trajectory $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ is unique and satisfies:*

(i) $f(X(t)) - \min f = o(t^{-2})$ a.s..

(ii) $\mathbb{E}[f(X(t)) - \min f] = \mathcal{O}(t^{-2})$.

(iii)

$$\int_{t_0}^{+\infty} t^3 \left\| \nabla f \left(X(t) + \frac{t}{\alpha-1} V(t) \right) \right\|^2 dt < +\infty \quad \text{a.s.}$$

Proof. Consider (5.12) with $\Gamma(t) = \frac{t}{\alpha-1}$ and $\theta(t) = \frac{t^2 - t_0^2}{2(\alpha-1)}$. Let $\tilde{\sigma}(s, \cdot) = \sigma(\theta^{-1}(s), \cdot)\sqrt{\Gamma(\theta^{-1}(s))}$. Notice that $t \mapsto t^2\sigma_\infty(t) \in L^2([t_0, +\infty[)$ is equivalent to $s \mapsto s\tilde{\sigma}_\infty^2(s) \in L^1(\mathbb{R}_+)$. We apply Theorem 3.2.8 to deduce that

$$f(Z(s)) - \min f = o(s^{-1}) \quad \text{a.s.}$$

Using the time scaling $\tau \equiv \theta$ and then the averaging technique as in the proof of Theorem 5.2.2, we have that

$$f(Y(t)) - \min f = o(t^{-2}) \text{ a.s.}$$

Moreover, it holds that

$$X(t) = \int_{t_0}^t Y(u) d\mu_t(u) + \xi(t).$$

(i) Now we prove the first point in the following way:

$$t^2(f(X(t)) - \min f) = t^2 \left(f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) \right) + t^2 \left(f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) - \min f \right).$$

Let us bound the first term using the convexity of f :

$$\begin{aligned} t^2 \left(f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) \right) &\leq t^2 \|\nabla f(X(t))\| \|\xi(t)\| \\ &\leq t^2 \|\xi(t)\| (L \|X(t)\| + \|\nabla f(0)\|) \\ &\leq t^2 \|\xi(t)\| \left(L \sup_{t \geq t_0} \|X(t)\| + \|\nabla f(0)\| \right). \end{aligned}$$

Let us recall that $\sup_{s \geq 0} \|Z(s)\| < +\infty$ a.s.. Due to the time scaling and averaging, it is direct to check that $\sup_{t \geq t_0} \|X(t)\| < +\infty$ a.s.. On the other hand, $\|\xi(t)\| = \mathcal{O}(t^{1-\alpha})$ a.s.. Therefore, we have

$$t^2 \left(f(X(t)) - f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) \right) = \mathcal{O}(t^{3-\alpha}) \text{ a.s.} \quad (5.14)$$

Now let us bound the second term using Jensen's inequality,

$$\begin{aligned} t^2 \left(f \left(\int_{t_0}^t Y(u) d\mu_t(u) \right) - \min f \right) &\leq t^2 \left(\int_{t_0}^t [f(Y(u)) - \min f] d\mu_t(u) \right) \\ &= \frac{t_0^{\alpha-1}}{t^{\alpha-3}} [f(Y(t_0)) - \min f] \\ &\quad + \frac{\alpha-1}{t^{\alpha-3}} \int_{t_0}^t u^{\alpha-4} (u^2 (f(Y(u)) - \min f)) du. \end{aligned}$$

In order to calculate the limit of this second term, let $a(t) = \frac{\alpha-1}{t}$, $b(u) = u^2 (f(Y(u)) - \min f)$, by Lemma 2.6.5 we have that

$$\lim_{t \rightarrow +\infty} \frac{\alpha-1}{t^{\alpha-1}} \int_{t_0}^t u^{\alpha-2} b(u) du = 0 \text{ a.s.}$$

Since $\alpha > 3$, we also have that

$$\lim_{t \rightarrow +\infty} \frac{\alpha-3}{t^{\alpha-3}} \int_{t_0}^t u^{\alpha-4} b(u) du = 0 \text{ a.s.} \quad (5.15)$$

Therefore, we conclude that

$$\lim_{t \rightarrow +\infty} t^2 (f(X(t)) - \min f) = 0 \text{ a.s.}$$

(ii) By Theorem 5.2.3 in the case $\gamma(t) = \frac{\alpha}{t}$, we have that $e^{-A(t)} = t_0^{\alpha-1} t^{1-\alpha}$ and $\theta(t) = \frac{t^2 - t_0^2}{2(\alpha-1)}$. On the other hand

$$I \left[\frac{1}{\theta} \right] (t) = 2(\alpha-1)^2 t^{1-\alpha} \int_{t_0}^t \frac{u^{\alpha-2}}{u^2 - t_0^2} = \mathcal{O}(t^{1-\alpha} + t^{-2}).$$

Since $\alpha > 3$, we have that $\mathcal{O}(t^{1-\alpha})$ is also $\mathcal{O}(t^{-2})$, and we conclude that

$$\mathbb{E}(f(X(t)) - \min f) = \mathcal{O}(t^{-2}).$$

(iii) This point follows directly from Theorem 5.2.4 in the case $\gamma(t) = \frac{\alpha}{t}$.

□

5.2.4 Convergence rate under Polyak-Łojasiewicz inequality

In this subsection, we show a local convergence rate under Polyak-Łojasiewicz inequality.

Theorem 5.2.6. *Let $\nu \geq 2$ and consider the dynamic (S – ISIH) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable, where f satisfies (H_f), and σ satisfies (H_σ). Besides, $f \in \mathcal{P}L_\mu(\mathcal{S})$ and suppose that either \mathbb{H} is finite dimensional or $f \in C^2(\mathbb{H})$. Let also, $\gamma \equiv \sqrt{2\mu}$, $\beta \equiv \Gamma \equiv \frac{1}{\sqrt{2\mu}}$, and such that $\sigma_\infty \in L^2([t_0, +\infty[)$.*

Then the solution trajectory $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ is unique. Moreover, letting $\delta > 0$, then there exists $\hat{t}_\delta > t_0, K_{\mu, \delta}, C_l, C_f > 0$ such that:

$$\mathbb{E}(f(X(t)) - \min f) \leq K_{\mu, \delta} e^{-\frac{\mu}{2}(t - \hat{t}_\delta)} + \frac{1}{\mu} l_\delta \left(\frac{t + 3\hat{t}_\delta - 4t_0}{4\mu} \right) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta, \quad (5.16)$$

where

$$l_\delta(s) = \frac{L}{2} \sigma_\infty^2(s) + C_l \sqrt{\delta} \frac{\sigma_\infty^2(s)}{2 \sqrt{\int_{\hat{s}_\delta}^s \sigma_\infty^2(u) du}}.$$

Besides, if $f \in \mathcal{P}L_\mu(\mathcal{S})$ holds on the entire space (i.e. $r = +\infty$), then we have that there exists $K_\mu > 0$ such that:

$$\mathbb{E}(f(X(t)) - \min f) \leq K_\mu e^{-\frac{\mu}{2}(t - t_0)} + \frac{L}{2\mu} \sigma_\infty^2 \left(\frac{t - t_0}{4\mu} \right), \quad \forall t > t_0, \quad (5.17)$$

Proof. Consider the dynamic (S – ISIH) with $\gamma \equiv c, \beta \equiv \Gamma \equiv \frac{1}{c}$, where $c > 0$ is a constant that will be fixed later.

Let us also define $\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \Gamma(u) du = \frac{t - t_0}{c}$ and $\tilde{\sigma}(s, \cdot) \stackrel{\text{def}}{=} \sigma(\theta^{-1}(s), \cdot) \sqrt{\Gamma(\theta^{-1}(s))}$. Then $\sigma_\infty \in L^2([t_0, +\infty[)$ is equivalent to $\tilde{\sigma}_\infty \in L^2(\mathbb{R}_+)$. Now consider the dynamic:

$$\begin{cases} dZ(s) &= -\nabla f(Z(s)) + \tilde{\sigma}(s, Z(s)) dW(s), \quad s > 0; \\ Z(0) &= X_0 + \Gamma(t_0) V_0. \end{cases} \quad (5.18)$$

Let $\delta > 0$ and apply the result of Theorem 3.3.3, item (i) (with coefficient $\sqrt{2\mu}$), that is, there exists $\hat{s}_\delta > 0$ such that for every $\lambda \in]0, 1[$,

$$\begin{aligned} \mathbb{E}(f(Z(s)) - \min f) &\leq e^{-2\mu(s - \hat{s}_\delta)} \mathbb{E}(f(Z(\hat{s}_\delta)) - \min f) \\ &+ e^{-2\mu(1-\lambda)(s - \hat{s}_\delta)} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) \\ &+ \frac{l_\delta(\hat{s}_\delta + \lambda(s - \hat{s}_\delta))}{2\mu} + C_f \sqrt{\delta}, \quad \forall s > \hat{s}_\delta, \end{aligned} \quad (5.19)$$

where $C_\infty, C_l, C_f > 0$ and the establishment of l_δ are detailed in Section 3.3.

Consider the time scaling $\tau \equiv \theta, Y(t) = Z(\theta(t))$ and $\hat{t}_\delta > t_0$ such that $\theta(\hat{t}_\delta) = \hat{s}_\delta$ (i.e. $\hat{t}_\delta = c\hat{s}_\delta + t_0$), we have that:

$$\begin{aligned} \mathbb{E}(f(Y(t)) - \min f) &\leq e^{-2\mu(\theta(t) - \hat{s}_\delta)} \mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) \\ &+ e^{-2\mu(1-\lambda)(\theta(t) - \hat{s}_\delta)} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) \\ &+ \frac{l_\delta(\hat{s}_\delta + \lambda(\theta(t) - \hat{s}_\delta))}{2\mu} + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned} \quad (5.20)$$

Let $a(t) = c$ and $A(t) = c(t - \hat{t}_\delta)$. Now, we consider the averaging as in (5.8) but change the initial condition to \hat{t}_δ . Thus, we have

$$X(t) = \int_{\hat{t}_\delta}^t Y(u) d\tilde{\mu}_t(u) + \tilde{\xi}(t), \quad (5.21)$$

where $\tilde{\mu}_t$ is the probability measure on $[\hat{t}_\delta, t]$ defined by

$$\tilde{\mu}_t = e^{-c(t-\hat{t}_\delta)}\delta_{\hat{t}_\delta} + ce^{c(u-t)}du, \quad (5.22)$$

where $\delta_{\hat{t}_\delta}$ is the Dirac measure at \hat{t}_δ and

$$\tilde{\xi}(t) \stackrel{\text{def}}{=} -\frac{1}{c}e^{-c(t-\hat{t}_\delta)}V(\hat{t}_\delta). \quad (5.23)$$

Then

$$\mathbb{E}(f(X(t)) - \min f) = \mathbb{E}\left(f(X(t)) - f\left(\int_{\hat{t}_\delta}^t Y(u)d\mu_t(u)\right)\right) + \mathbb{E}\left(f\left(\int_{\hat{t}_\delta}^t Y(u)d\mu_t(u)\right) - \min f\right).$$

We can bound the first term using convexity and Cauchy-Schwarz inequality in the following way

$$\begin{aligned} \mathbb{E}\left(f(X(t)) - f\left(\int_{\hat{t}_\delta}^t Y(u)d\tilde{\mu}_t(u)\right)\right) &\leq \sqrt{\mathbb{E}(\|\nabla f(X(t))\|^2)}\sqrt{\mathbb{E}(\|\tilde{\xi}(t)\|^2)} \\ &\leq \frac{\sqrt{\mathbb{E}(\|V(\hat{t}_\delta)\|^2)}}{c}\sqrt{2\|\nabla f(0)\|^2 + 2L^2\mathbb{E}\left(\sup_{t \geq t_0} \|X(t)\|^2\right)}e^{-c(t-\hat{t}_\delta)}, \end{aligned}$$

where $\mathbb{E}(\sup_{t \geq t_0} \|X(t)\|^2) < +\infty$ as mentioned in Corollary 5.2.5.

On the other hand, we can bound the second term using Jensen's inequality and then (5.20)

$$\begin{aligned} \mathbb{E}\left(f\left(\int_{\hat{t}_\delta}^t Y(u)d\tilde{\mu}_t(u)\right) - \min f\right) &\leq \int_{\hat{t}_\delta}^t \mathbb{E}(f(Y(u) - \min f))d\tilde{\mu}_t(u) \\ &\leq \int_{\hat{t}_\delta}^t e^{-2\mu(\theta(u)-\hat{s}_\delta)}\mathbb{E}(f(Y(\hat{t}_\delta)) - \min f)d\tilde{\mu}_t(u) \\ &\quad + \int_{\hat{t}_\delta}^t e^{-2\mu(1-\lambda)(\theta(u)-\hat{s}_\delta)}\left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta}\right)d\tilde{\mu}_t(u) \\ &\quad + \int_{\hat{t}_\delta}^t \frac{l_\delta(\hat{s}_\delta + \lambda(\theta(u) - \hat{s}_\delta))}{2\mu}d\tilde{\mu}_t(u) + C_f \sqrt{\delta} \\ &= \left(\mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) + \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta}\right) + \frac{l_\delta(\hat{s}_\delta)}{2\mu}\right)e^{-c(t-\hat{t}_\delta)} \\ &\quad + c\mathbb{E}(f(Y(\hat{t}_\delta)) - \min f)e^{-ct} \int_{\hat{t}_\delta}^t e^{-2\mu(\theta(u)-\hat{s}_\delta)}e^{cu}du \\ &\quad + c\left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta}\right)e^{-ct} \int_{\hat{t}_\delta}^t e^{-2\mu(1-\lambda)(\theta(u)-\hat{s}_\delta)}e^{cu}du \\ &\quad + \frac{c}{2\mu}e^{-ct} \int_{\hat{t}_\delta}^t l_\delta(\hat{s}_\delta + \lambda(\theta(u) - \hat{s}_\delta))e^{cu}du + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned}$$

We bound the first integral as follows:

$$e^{-ct} \int_{\hat{t}_\delta}^t e^{-2\mu(\theta(u)-\hat{s}_\delta)}e^{cu}du \leq e^{2\mu(\frac{t_0}{c} + \hat{s}_\delta)}e^{-\frac{2\mu}{c}t}.$$

And the second integral in the same way

$$e^{-ct} \int_{\hat{t}_\delta}^t e^{-2\mu(1-\lambda)(\theta(u)-\hat{s}_\delta)}e^{cu}du \leq e^{2\mu(1-\lambda)(\frac{t_0}{c} + \hat{s}_\delta)}e^{-\frac{2\mu(1-\lambda)}{c}t}.$$

To treat the third integral we are going to split the integral in two in order to find a useful convergence rate. Let us recall that $l_\delta \in L^1([\hat{s}_\delta, +\infty[)$ and that l_δ is decreasing. Let us define

$$\varphi_{\lambda,c,\delta}(t) \stackrel{\text{def}}{=} \hat{s}_\delta + \lambda\left(\frac{t + \hat{t}_\delta - 2t_0}{2c} - \hat{s}_\delta\right),$$

then

$$\begin{aligned} e^{-ct} \int_{\hat{t}_\delta}^t l_\delta(\hat{s}_\delta + \lambda(\theta(u) - \hat{s}_\delta)) e^{cu} du &= e^{-ct} \int_{\hat{t}_\delta}^{\frac{\hat{t}_\delta+t}{2}} l_\delta(\hat{s}_\delta + \lambda(\theta(u) - \hat{s}_\delta)) e^{cu} du \\ &\quad + e^{-ct} \int_{\frac{\hat{t}_\delta+t}{2}}^t l_\delta(\hat{s}_\delta + \lambda(\theta(u) - \hat{s}_\delta)) e^{cu} du \\ &\leq \frac{c}{\lambda} e^{-\frac{c\hat{t}_\delta}{2}} C_\infty e^{-\frac{ct}{2}} + l_\delta(\varphi_{\lambda,c,\delta}(t)). \end{aligned}$$

Now that we have bounded all the terms, we have the following bound

$$\begin{aligned} \mathbb{E}(f(X(t)) - \min f) &\leq \frac{\sqrt{\mathbb{E}(\|V(\hat{t}_\delta)\|^2)}}{c} \sqrt{2\|\nabla f(0)\|^2 + 2L^2\mathbb{E}\left(\sup_{t \geq t_0} \|X(t)\|^2\right)} e^{-c(t-\hat{t}_\delta)} \\ &\quad + \left(\mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) + \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) + \frac{l_\delta(\hat{s}_\delta)}{2\mu} \right) e^{-c(t-\hat{t}_\delta)} \\ &\quad + c \mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) e^{2\mu\left(\frac{t_0+\hat{t}_\delta}{c} + \hat{s}_\delta\right)} e^{-\frac{2\mu}{c}(t-\hat{t}_\delta)} \\ &\quad + c \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) e^{2\mu(1-\lambda)\left(\frac{t_0+\hat{t}_\delta}{c} + \hat{s}_\delta\right)} e^{-\frac{2\mu(1-\lambda)}{c}(t-\hat{t}_\delta)} \\ &\quad + \frac{c}{2\mu} \left(\frac{c}{\lambda} C_\infty e^{-\frac{c(t-\hat{t}_\delta)}{2}} + l_\delta(\varphi_{\lambda,c,\delta}(t)) \right) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned}$$

Letting $\lambda = \frac{1}{2}$ and $c = \sqrt{2\mu}$ we obtain

$$\begin{aligned} \mathbb{E}(f(X(t)) - \min f) &\leq \frac{\sqrt{\mathbb{E}(\|V(\hat{t}_\delta)\|^2)}}{\sqrt{2\mu}} \sqrt{2\|\nabla f(0)\|^2 + 2L^2\mathbb{E}\left(\sup_{t \geq t_0} \|X(t)\|^2\right)} e^{-\sqrt{2\mu}(t-\hat{t}_\delta)} \\ &\quad + \left(\mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) + \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) + \frac{l_\delta(\hat{s}_\delta)}{2\mu} \right) e^{-\sqrt{2\mu}(t-\hat{t}_\delta)} \\ &\quad + \sqrt{2\mu} \mathbb{E}(f(Y(\hat{t}_\delta)) - \min f) e^{2\mu\left(\frac{t_0+\hat{t}_\delta}{\sqrt{2\mu}} + \hat{s}_\delta\right)} e^{-\sqrt{2\mu}(t-\hat{t}_\delta)} \\ &\quad + \sqrt{2\mu} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) e^{\mu\left(\frac{t_0+\hat{t}_\delta}{\sqrt{2\mu}} + \hat{s}_\delta\right)} e^{-\frac{\sqrt{2\mu}}{2}(t-\hat{t}_\delta)} \\ &\quad + 2C_\infty e^{-\frac{\sqrt{2\mu}(t-\hat{t}_\delta)}{2}} + \frac{1}{\sqrt{2\mu}} l_\delta(\varphi_{\frac{1}{2},\sqrt{2\mu},\delta}(t)) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \end{aligned}$$

Letting $K_{\mu,\delta} \stackrel{\text{def}}{=} \sqrt{2\mu} \left(\frac{LC_\infty^2}{2} + C_l C_\infty \sqrt{\delta} \right) e^{\mu\left(\frac{t_0+\hat{t}_\delta}{\sqrt{2\mu}} + \hat{s}_\delta\right)} + 2C_\infty$, we conclude that

$$\mathbb{E}(f(X(t)) - \min f) \leq K_{\mu,\delta} e^{-\frac{\sqrt{2\mu}}{2}(t-\hat{t}_\delta)} + \frac{1}{\sqrt{2\mu}} l_\delta(\varphi_{\frac{1}{2},\sqrt{2\mu},\delta}(t)) + C_f \sqrt{\delta}, \quad \forall t > \hat{t}_\delta. \quad (5.24)$$

□

5.3 From weak to strong convergence under general γ and $\beta \equiv \Gamma$

5.3.1 General result

We consider the Tikhonov regularization of the dynamic (**S – ISIH_{NS}**), *i.e.*, for $t > 0$,

$$\begin{cases} dX(t) &= V(t)dt; \\ dV(t) &\in -\gamma(t)V(t)dt - \partial F(X(t) + \beta(t)V(t))dt - \varepsilon(t)(X(t) + \beta(t)V(t))dt \\ &\quad + \sigma(t, X(t) + \beta(t)V(t))dW(t); \\ X(t_0) &= X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{S – ISIH}_{\text{NS}} - \text{TA})$$

We show some conditions (on $\gamma, \beta, \varepsilon, \sigma$) under which we can obtain strong convergence of the trajectory.

Theorem 5.3.1. Consider that $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ satisfies (\mathbf{H}_γ) . Besides, $F = f + g$ and σ satisfy assumptions (\mathbf{H}_F) and (\mathbf{H}_σ) . Moreover, suppose that g satisfies (\mathbf{H}_λ) and let $\nu \geq 2$. Consider $(\mathbf{S} - \mathbf{ISIHD}_{\text{NS}} - \mathbf{TA})$ with $\beta \equiv \Gamma$ and initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable.

Then, there exists a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of $(\mathbf{S} - \mathbf{ISIHD}_{\text{NS}} - \mathbf{TA})$. Additionally, let $x^* \stackrel{\text{def}}{=} P_{S_F}(0)$ be the minimum norm solution, and for $\varepsilon > 0$ let x_ε be the unique minimizer of $F_\varepsilon(x) \stackrel{\text{def}}{=} F(x) + \frac{\varepsilon}{2}\|x\|^2$. If we suppose that $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$, and that $\varepsilon : [t_0, +\infty[\rightarrow \mathbb{R}_+$ satisfies the conditions:

$$(T'_1) \quad \varepsilon(t) \rightarrow 0 \text{ as } t \rightarrow +\infty;$$

$$(T'_2) \quad \int_{t_0}^{+\infty} \varepsilon(t)\Gamma(t)dt = +\infty;$$

$$(T'_3) \quad \int_{t_0}^{+\infty} \varepsilon(t)\Gamma(t) (\|x^*\|^2 - \|x_{\varepsilon(t)}\|^2) dt < +\infty.$$

Then $\text{s-lim}_{t \rightarrow +\infty} X(t) = x^*$ a.s., and $\|V(t)\| = o\left(\frac{1}{\Gamma(t)}\right)$ a.s..

Proof. Let $s_0 > 0$, $\theta(t) \stackrel{\text{def}}{=} s_0 + \int_{t_0}^t \Gamma(u)du$; $\tilde{\varepsilon}(t) = \varepsilon(\theta^{-1}(t))$; and $\tilde{\sigma}(s, \cdot) \stackrel{\text{def}}{=} \sigma(\theta^{-1}(s), \cdot)\sqrt{\Gamma(\theta^{-1}(s))}$. Then ε satisfying $(T'_1), (T'_2)$, and (T'_3) is equivalent to $\tilde{\varepsilon}$ satisfying $(T_1), (T_2)$, and (T_3) . Besides, $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$ is equivalent to $\tilde{\sigma}_\infty \in L^2(\mathbb{R}_+)$. Consider the dynamic:

$$\begin{cases} dZ(s) & \in -\partial F(Z(s)) - \tilde{\varepsilon}(s)Z(s) + \tilde{\sigma}(s, Z(s))dW(s), \quad s > s_0; \\ Z(s_0) & = X_0 + \Gamma(t_0)V_0. \end{cases} \quad (5.25)$$

By Theorem 4.4.1, we have that there exists a unique solution $Z \in S_{\mathbb{H}}^\nu[s_0]$, and that $\lim_{s \rightarrow +\infty} Z(s) = x^*$ a.s. (Recall that $x^* \stackrel{\text{def}}{=} P_{S_F}(0)$). Using the time scaling $\tau \equiv \theta$ and the averaging described at the beginning of this section, we end up with the dynamic $(\mathbf{S} - \mathbf{ISIHD}_{\text{NS}} - \mathbf{TA})$ in the case where $\beta \equiv \Gamma$. The existence and uniqueness of solution, and the fact that $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ goes analogously as in the proof of Theorem 5.2.2.

Now we prove the claim, since $\lim_{s \rightarrow +\infty} Z(s) = x^*$ a.s., this implies directly that $\lim_{t \rightarrow +\infty} Y(t) = x^*$ a.s.. Besides, we have the relation (5.9), i.e.

$$X(t) = \int_{t_0}^t Y(u)d\mu_t(u) + \xi(t),$$

where μ_t and ξ are defined in (5.10) and (5.11), respectively. Consequently, we have

$$\begin{aligned} \|X(t) - x^*\| & \leq \left\| X(t) - \int_{t_0}^t Y(u)d\mu_t(u) \right\| + \left\| \int_{t_0}^t Y(u)d\mu_t(u) - x^* \right\| \\ & \leq \|\xi(t)\| + \left\| \int_{t_0}^t Y(u)d\mu_t(u) - x^* \right\|. \end{aligned}$$

Let $a(t) = \frac{1}{\Gamma(t)}$ and $A(t) = \int_{t_0}^t \frac{du}{\Gamma(u)}$. By Lemma 5.2.1, we have that $\lim_{t \rightarrow +\infty} \|\xi(t)\| = 0$. On the other hand

$$\begin{aligned} \left\| \int_{t_0}^t Y(u)d\mu_t(u) - x^* \right\| & = \left\| \int_{t_0}^t (Y(u) - x^*)d\mu_t(u) \right\| \\ & \leq \int_{t_0}^t \|Y(u) - x^*\|d\mu_t(u) \\ & = e^{-A(t)}\|Y(t_0) - x^*\| + e^{-A(t)} \int_{t_0}^t a(u)e^{A(u)}\|Y(u) - x^*\|du. \end{aligned}$$

Let $b(u) = \|Y(u) - x^*\|$. Since we already proved that $\lim_{u \rightarrow +\infty} b(u) = 0$ a.s., and we have that $a \notin L^1([t_0, +\infty[)$ by Lemma 5.2.1, we utilize Lemma 2.6.5 with our respective a, b functions. This let

us conclude that

$$\lim_{t \rightarrow +\infty} \left\| \int_{t_0}^t Y(u) d\mu_t(u) - x^* \right\| = 0 \quad \text{a.s..}$$

Thus, $\lim_{t \rightarrow +\infty} X(t) = x^*$ a.s.. Finally, since

$$Y(t) = X(t) + \Gamma(t)V(t),$$

and the fact that X and Y have (a.s.) the same limit, we conclude that

$$\lim_{t \rightarrow +\infty} \Gamma(t)V(t) = 0 \quad \text{a.s..}$$

□

5.3.2 Practical situations

We give some conditions when (T'_1) , (T'_2) , and (T'_3) of Theorem 5.3.1 are satisfied simultaneously.

Theorem 5.3.2. *Consider the setting of Theorem 5.3.1 and suppose that $F = f + g \in \text{EB}^p(\mathcal{S}_F)$ (recall Definition 2.11). Let $s_0 > 0$ and denote $\theta(t) \stackrel{\text{def}}{=} s_0 + \int_{t_0}^t \Gamma(s) ds$, then taking the Tikhonov parameter $\varepsilon(t) = \frac{1}{\theta^r(t)}$ with*

$$1 \geq r > \frac{2p}{2p+1},$$

then the three conditions (T'_1) , (T'_2) , and (T'_3) of Theorem 5.3.1 are satisfied simultaneously. In particular, for any solution $(X, V) \in \mathcal{S}_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of (5.3.1), we get almost sure (strong) convergence of $X(t)$ to the minimal norm solution named $x^* = P_{\mathcal{S}_F}(0)$ and that $\|V(t)\| = o\left(\frac{1}{\Gamma(t)}\right)$.

Proof. We proceed as in the proof of Theorem 5.3.1 and arrive to the dynamic (5.25), since

$$\tilde{\varepsilon}(t) = \varepsilon(\theta^{-1}(t)) = \frac{1}{t^r},$$

the proof goes as in Theorem 4.4.5. □

Let us consider the assumption:

$$\begin{cases} f : \mathbb{H} \rightarrow \mathbb{R} \text{ is continuously differentiable and convex with } L\text{-Lipschitz continuous gradient;} \\ f \in C^2(\mathbb{H}) \text{ or } \mathbb{H} \text{ is finite-dimensional;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{cases}$$

(H $_f^*$)

Theorem 5.3.3. *Let f satisfying (H_f^*) , and also $f \in \text{EB}^p(\mathcal{S})$, σ satisfying (H_σ) , and $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$ and is nonincreasing. Let $\nu \geq 2$, and let us consider $\varepsilon(t) = \frac{1}{t^r}$ where $0 < r < 1$, then we evaluate $(\mathbf{S} - \text{ISIHD}_{\text{NS}} - \text{TA})$ in the case where γ satisfies (H_γ) , $g \equiv 0$, $\beta \equiv \Gamma$, and with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable, i.e., for $t > t_0$,*

$$\begin{cases} dX(t) = V(t)dt; \\ dV(t) = -\gamma(t)V(t)dt - \nabla f(X(t) + \Gamma(t)V(t))dt - \varepsilon(t)(X(t) + \Gamma(t)V(t)) \\ \quad + \sigma(t, X(t) + \Gamma(t)V(t))dW(t); \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (5.26)$$

For $\varepsilon > 0$, let us define $f_\varepsilon(x) \stackrel{\text{def}}{=} f(x) + \frac{\varepsilon}{2}\|x\|^2$, and let x_ε be the unique minimizer of f_ε . Moreover, let $s_0 > 0$ and for $s_1 > s_0$ consider,

$$R(s) \stackrel{\text{def}}{=} e^{-\frac{s^{1-r}}{1-r}} \int_{s_1}^s e^{\frac{u^{1-r}}{1-r}} \sigma_\infty^2(\theta^{-1}(u)) \Gamma(\theta^{-1}(u)) du, \quad (5.27)$$

where $\theta(t) \stackrel{\text{def}}{=} s_0 + \int_{t_0}^t \Gamma(u)du$. Let also $x^* \stackrel{\text{def}}{=} P_S(0)$, $A(s) \stackrel{\text{def}}{=} \int_{s_1}^s \frac{du}{\Gamma(u)}$, and $t_1 \stackrel{\text{def}}{=} \theta^{-1}(s_1)$. Then, the solution trajectory $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ is unique, and we have that:

(i) $R(\theta(t)) \rightarrow 0$ as $t \rightarrow +\infty$.

(ii) Let $\bar{\sigma}(t) = \Gamma(t)\sigma_\infty^2(t)$, then

$$R(\theta(t)) = \mathcal{O} \left(\exp(-\theta^r(t)(1 - 2^{-r})) + \theta^r(t)\bar{\sigma} \left(\frac{s_1 + \theta(t)}{2} \right) \right).$$

Moreover, if $\bar{\sigma}(t) = \mathcal{O}(\theta^{-\Delta}(t))$ for $\Delta > 1$, then $R(\theta(t)) = \mathcal{O}(\theta^{r-\Delta}(t))$.

Besides, we have the following convergence rate in expectation:

(iii) For the values, we have:

$$\mathbb{E}[f(X(t)) - \min(f)] = \mathcal{O} \left(\max\{e^{-A(t)}, I[h_1](t)\} \right),$$

where $h_1(t) = \frac{1}{\theta^r(t)} + R(\theta(t))$.

(iv) And for the trajectory, we obtain:

$$\mathbb{E}[\|X(t) - x^*\|^2] = \mathcal{O} \left(\max\{e^{-A(t)}, I[h_2](t)\} \right),$$

where $h_2(t) = \theta^{r-1}(t) + \theta^{-\frac{r}{p}}(t) + \theta^r(t)R(\theta(t))$.

Proof. We proceed as in the proof of Theorem 5.2.2 and define analogously $\tilde{\sigma}$, $\tilde{\varepsilon}$, we also consider the dynamic (5.12). By Theorem 4.4.8 we obtain that

$$R(s) = e^{-\frac{s^{1-r}}{1-r}} \int_{s_1}^s e^{\frac{u^{1-r}}{1-r}} \tilde{\sigma}_\infty^2(u) du,$$

where $\tilde{\sigma}_\infty^2 \in L^2([s_0, +\infty[)$, satisfies the following:

- $R(s) \rightarrow 0$ as $t \rightarrow +\infty$.
- $R(s) = \mathcal{O} \left(\exp(-s^r(1 - 2^{-r})) + s^r \tilde{\sigma}_\infty^2 \left(\frac{s_1+s}{2} \right) \right)$. Moreover if $\tilde{\sigma}_\infty^2(s) = \mathcal{O}(s^{-\Delta})$ for $\Delta > 1$, then $R(s) = \mathcal{O}(s^{r-\Delta})$.

And evaluating at $s = \theta(t)$ we obtain the first two items of the theorem. For the third and fourth items we used that

- $\mathbb{E}[f(Z(s)) - \min(f)] = \mathcal{O} \left(\frac{1}{s^r} + R(s) \right)$.
- $\mathbb{E}[\|Z(s) - x^*\|^2] = \mathcal{O} \left(\frac{1}{s^{1-r}} + \frac{1}{s^{\frac{r}{p}}} + s^r R(s) \right)$.

Then, proceeding as in the proof of Theorem 5.2.3, we obtain the desired results. \square

Corollary 5.3.4. Consider Theorem 5.3.2 in the case where $\gamma(t) = \frac{\alpha}{t}$ for $\alpha > 1$, $\beta(t) = \frac{t}{\alpha-1}$ then we have that:

1. If $\sigma_\infty^2(t) = \mathcal{O}(t^{-2(\Delta+1)})$ for $\Delta > 1$, and $\alpha \neq \{1 + 2r, 1 + 2(\Delta - r)\}$, then

$$\mathbb{E}[f(X(t)) - \min(f)] = \mathcal{O} \left(\max\{t^{-(\alpha-1)}, t^{-2r}, t^{-2(\Delta-r)}\} \right).$$

In particular, if $\alpha > 3$

2. If $\sigma_\infty^2(t) = \mathcal{O}(t^{-2(\Delta+1)})$ for $\Delta > \max\{1, 2r\}$, and $\alpha \neq \{3 - 2r, 1 + \frac{2r}{p}, 1 + 2(2r - \Delta)\}$, then

$$\mathbb{E}[\|X(t) - x^*\|^2] = \mathcal{O} \left(\max\{t^{-(\alpha-1)}, t^{-2(1-r)}, t^{-\frac{2r}{p}}, t^{-2(2r-\Delta)}\} \right).$$

Chapter 6

An SDE Perspective on Stochastic Inertial Gradient Dynamics with Time-Dependent Viscosity

Our goal in this chapter is to solve convex minimization problems by means of stochastic inertial differential equations which are driven by the gradient of the objective function. This will provide a general mathematical framework for analyzing fast optimization algorithms with stochastic gradient input. Our study is a natural extension of our previous work devoted to the first-order in time stochastic steepest descent in Chapter 3 and those of Chapter 5 where we used time scaling and averaging. We will then develop these results further by considering second-order stochastic differential equations in time, incorporating a viscous time-dependent damping and a Hessian-driven damping with general coefficients. To develop this program, we rely on stochastic Lyapunov analysis. Assuming a square-integrability condition on the diffusion term times a function dependant on the viscous damping, and that the Hessian-driven damping is a positive constant, our first main result shows that almost surely, there is convergence of the values, and states fast convergence of the values in expectation. Besides, in the case where the Hessian-driven damping is zero, we conclude with the fast convergence of the values in expectation and in almost sure sense, we also managed to prove almost sure weak convergence of the trajectory. We provide a comprehensive complexity analysis by establishing several new convergence rates in expectation and in almost sure sense for the convex and strongly convex case.

Main contributions of this chapter

- ▶ Almost sure convergence of the gradient evaluated at the trajectory generated by the second-order inertial SDE with a general viscous and geometric damping, to zero (Theorem 6.2.7).
- ▶ Global convergence rates of the second-order inertial SDE in expectation under convexity (Theorem 6.2.8), and strong convexity (Theorem 6.2.13).
- ▶ Fast convergence of the values in expectation under $\alpha > 3$, $\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \gamma_0 + \frac{\beta_0}{t}$ (Corollary 6.2.11).
- ▶ Tighter global convergence rates of the values in almost sure sense and in expectation when the geometric damping is null, including the fast convergence case (Theorem 6.3.5).
- ▶ Almost sure weak convergence of the trajectory generated by the second-order inertial SDE with a non-increasing viscous damping and a null geometric damping, to the set of minimizers (Theorem 6.3.7).

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6.1 Introduction

Let us fix the framework of our study. We consider again the minimization problem

$$\min_{x \in \mathbb{H}} f(x), \quad (\mathcal{P}_1)$$

where \mathbb{H} is a real separable Hilbert space and the objective function $f : \mathbb{H} \rightarrow \mathbb{R}$ satisfies the following standing assumptions:

$$\begin{cases} f \text{ is convex and continuously twice differentiable with } L\text{-Lipschitz continuous gradient;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{cases} \quad (\mathbf{H}_f)$$

Consider \mathbb{K} a real separable Hilbert space. To solve (\mathcal{P}_1) , we will refer to Section 1.2 for the approach that leads to the dynamic that we will study in this chapter, specifically the following SDE:

$$\begin{cases} dX(t) &= V(t)dt; \\ dV(t) &= -\gamma(t)V(t)dt - \nabla f(X(t) + \beta(t)V(t))dt + \sigma(t, X(t) + \beta(t)V(t))dW(t); \\ X(t_0) &= X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{S – ISIHD})$$

This SDE is defined over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, where the diffusion (volatility) term $\sigma : [t_0, +\infty[\times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K}; \mathbb{H})$ is a measurable function that satisfies (\mathbf{H}_σ) , and W is a \mathbb{K} -valued Brownian motion. Where $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ for some $\nu \geq 2$ are given initial data that is \mathcal{F}_{t_0} -measurable. Besides, γ and β are parameters called viscous damping and geometric damping, respectively. They are explained and discussed in more detail below.

Our goal is to provide a general mathematical framework for analyzing fast gradient-based optimization algorithms with stochastic gradient input. For this, we will study second-order stochastic differential equations in time, *i.e.*, also involving acceleration, and whose drift term is the gradient of the function to be minimized. In this context, considering inertial dynamics with a time-dependent viscosity coefficient is a key property to obtain fast convergent methods. Our study is related to two recent works:

- On the one hand, it is a natural extension of Chapter 3 devoted to the first-order in time stochastic steepest descent. Moreover, it extends Chapter 5, which studies second-order in time stochastic systems with a particular geometric damping, to a focus on general geometric damping.
- On the other hand, we will rely on the Lyapunov analysis for the dynamic (ISIHD) done by Attouch, Fadili, and Kungurtsev in [33]; and the one for (IGS $_\gamma$) done by Attouch and Cabot in [18].

Our objectives are largely motivated by recent analysis in the deterministic setting. In fact, the dynamic (S – ISIHD) comes naturally as a stochastic version of (ISIHD). (ISIHD) is one of the most recent developments regarding the use of gradient-based dynamic systems for optimization. We refer to the Introduction for a deep insight on second order systems.

On stochastic differential equations. For the necessary notation and preliminaries on stochastic processes, see [145, Section A.2]. Moreover, the existence and uniqueness of a solution of (S – ISIHD) is discussed in Proposition 6.2.4.

Let us now present Itô's formula (Theorem 2.7.5) in the particular case of (S – ISIHD):

Proposition 6.1.1. [101, Section 2.3] Consider (X, V) a solution of (S – ISIHD) and W a \mathbb{K} -valued Brownian motion, let $\phi : [t_0, +\infty[\times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ be such that $\phi(\cdot, x, v) \in C^1([t_0, +\infty[)$ for every $x, v \in \mathbb{H}$, $\phi(t, \cdot, \cdot) \in C^2(\mathbb{H} \times \mathbb{H})$ for every $t \geq t_0$. Then the process

$$Y(t) = \phi(t, X(t), V(t)),$$

is an Itô Process, such that for all $t \geq t_0$

$$\begin{aligned} Y(t) &= Y(t_0) + \int_{t_0}^t \frac{\partial \phi}{\partial t}(s, X(s), V(s)) ds + \int_{t_0}^t \langle \nabla_x \phi(s, X(s), V(s)), V(s) \rangle ds \\ &\quad - \int_{t_0}^t \langle \nabla_v \phi(s, X(s), V(s)), \gamma(s)V(s) + \nabla f(X(s) + \beta(s)V(s)) \rangle ds \\ &\quad + \int_{t_0}^t \langle \sigma^*(s, X(s) + \beta(s)V(s)) \nabla_v \phi(s, X(s), V(s)), dW(s) \rangle \\ &\quad + \frac{1}{2} \int_{t_0}^t \text{tr}[\sigma(s, X(s) + \beta(s)V(s)) \sigma^*(s, X(s) + \beta(s)V(s)) \nabla_v^2 \phi(s, X(s), V(s))] ds, \end{aligned} \tag{6.1}$$

where ∇_v^2 is the Hessian with respect to the double differentiation of v and σ^* is the adjoint operator of σ . Moreover, if for all $T > t_0$

$$\mathbb{E} \left(\int_{t_0}^T \|\sigma^*(s, X(s) + \beta(s)V(s)) \nabla_v \phi(s, X(s), V(s))\|^2 ds \right) < +\infty,$$

then $\int_{t_0}^t \langle \sigma^*(s, X(s) + \beta(s)V(s)) \nabla_v \phi(s, X(s), V(s)), dW(s) \rangle$ is a square-integrable continuous martingale and

$$\mathbb{E} \left(\int_{t_0}^t \langle \sigma^*(s, X(s) + \beta(s)V(s)) \nabla_v \phi(s, X(s), V(s)), dW(s) \rangle \right) = 0 \tag{6.2}$$

6.2 (S – ISIHD) with general γ and β

In this section, we will develop a Lyapunov analysis based on [33] to study almost sure, and in expectation properties of the dynamic (S – ISIHD), when the parameters γ and β are general functions. This will allow to go much further and consider parameters not covered in Chapter 5 which exploits the relationship between first-order and second-order systems. We will also apply our results to two special cases: (i) γ is a differentiable, decreasing and vanishing function, with vanishing derivative, and β is a positive constant; and (ii) $\gamma(t) = \frac{\alpha}{t}$, and $\beta(t) = \gamma_0 + \frac{\beta}{t}$ (with $\gamma_0, \beta > 0$). These cases are again not covered by results in Chapter 5.

Recall that our focus in this thesis is on an optimization perspective, and as we argued in the introduction, we will study the long time behavior of (S – ISIHD) as the diffusion term vanishes when $t \rightarrow +\infty$. Therefore, we recall that we assume that the diffusion (volatility) term σ satisfies:

$$\begin{cases} \sup_{t \geq t_0, x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}} < +\infty, \\ \|\sigma(t, x') - \sigma(t, x)\|_{\text{HS}} \leq l_0 \|x' - x\|, \end{cases} \tag{H_\sigma}$$

for some $l_0 > 0$ and for all $t \geq t_0, x, x' \in \mathbb{H}$. The Lipschitz continuity assumption is mild and required to ensure the well-posedness of (S – ISIHD).

Remark 6.2.1. Under the hypothesis (H_σ) we have that there exists $\sigma_*^2 > 0$ such that

$$\|\sigma(t, x)\|_{\text{HS}}^2 = \text{tr}[\Sigma(t, x)] \leq \sigma_*^2, \quad \forall t \geq t_0, \forall x \in \mathbb{H},$$

where $\Sigma \stackrel{\text{def}}{=} \sigma\sigma^*$. Let us also define $\sigma_\infty : [t_0, +\infty[\rightarrow \mathbb{R}_+$ as: $\sigma_\infty(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{H}} \|\sigma(t, x)\|_{\text{HS}}$.

Now, we follow with the hypotheses we will require over the damping parameters.

For $t_0 > 0$, let $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be a viscous damping, we denote

$$p(t) \stackrel{\text{def}}{=} \exp\left(\int_{t_0}^t \gamma(s) ds\right). \quad (6.3)$$

Besides, if

$$\int_{t_0}^{+\infty} \frac{ds}{p(s)} < +\infty, \quad (H_\gamma)$$

we define $\Gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ by

$$\Gamma(t) \stackrel{\text{def}}{=} p(t) \int_t^{+\infty} \frac{ds}{p(s)}. \quad (6.4)$$

Remark 6.2.2. Let us notice that Γ satisfies the relation

$$\Gamma' = \gamma\Gamma - 1.$$

Besides, there are some results that will require the additional following assumption (introduced in [18]):

$$\text{there exists } t_2 \geq t_0 \text{ and } m < \frac{3}{2} \text{ such that } \gamma(t)\Gamma(t) \leq m \text{ for every } t \geq t_2. \quad (H'_\gamma)$$

For $t_0 > 0$, let $\beta : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be a geometric damping that we will assume to be a differentiable function. We will occasionally need to impose the additional assumption that there exists $c_1, c_2 > 0$, and $t_1 > t_0$ such that for every $t \geq t_1$:

$$\begin{cases} \beta(t) & \leq c_1; \\ \left| \frac{\beta'(t) - \gamma(t)\beta(t) + 1}{\beta(t)} \right| & \leq c_2. \end{cases} \quad (H_\beta)$$

We present a Lemma that will be useful in this chapter.

Lemma 6.2.3. *Let us consider the viscous damping function $\gamma : [t_0, +\infty[\rightarrow \mathbb{R}_+$ defined by $\gamma(t) = \frac{\alpha}{t^r}$ with $r \in]0, 1[$ and $\alpha \geq 1 - r$, then:*

(i) γ satisfies (H_γ) .

(ii) $\Gamma(t) = \mathcal{O}(t^r)$.

(iii) γ satisfies (H'_γ) .

Proof. (i) Let $c \stackrel{\text{def}}{=} \frac{\alpha}{1-r} \geq 1$, we first notice that after the change of variable $u = cs^{1-r}$, we get

$$\int_0^\infty \exp(-cs^{1-r}) ds = \frac{1}{\alpha c^{\frac{r}{1-r}}} \int_0^\infty u^{\frac{1}{1-r}-1} e^{-u} du < +\infty,$$

since the last integral is the classical Gamma function (see *e.g.* [160, Section 5]) evaluated at $\frac{1}{1-r}$, and this function is well defined for positive arguments, then (H_γ) is satisfied.

(ii) Besides, by definition $\Gamma(t) = \exp(ct^{1-r}) \int_t^\infty \exp(-cs^{1-r}) ds$. Using the same change of variable as before, we obtain that

$$\Gamma(t) = \frac{\exp(ct^{1-r})}{\alpha c^{\frac{r}{1-r}}} \Gamma_{inc} \left(\frac{1}{1-r}, ct^{1-r} \right). \quad (6.5)$$

By (iii) of Lemma 2.6.8 with $a = \frac{1}{1-r} > 1$ and $x = ct^{1-r}$, for every $\varepsilon > 0$, there exists $t_1 > t_0$ such that for every $t > t_1$:

$$\Gamma(t) \leq \frac{1 + \varepsilon}{\alpha c^{\frac{r}{1-r}}} t^r.$$

(iii) Moreover, if we restrict $\varepsilon \in]0, \frac{1}{2}[$, there exists $t_1 > t_0$ such that for every $t > t_1$:

$$\gamma(t)\Gamma(t) \leq \frac{1 + \varepsilon}{c^{\frac{1}{1-r}}} \leq 1 + \varepsilon.$$

Defining m as $1 + \varepsilon$, we have that $m < \frac{3}{2}$, and we conclude. \square

6.2.1 Reformulation of (S – ISIHD)

The formulation of the dynamic (S – ISIHD) is known as the Hamiltonian formulation. However, it is not the only one. In the deterministic case, an alternative equivalent and more flexible first-order reformulation of (ISIHD) was proposed in [33]. The motivation there was that this equivalent reformulation can handle the case where f is non-smooth. Although we will not consider the non-smooth case here, we will still extend and use that equivalent reformulation to the stochastic case.

Consider the dynamic (S – ISIHD), and let us define the auxiliary variable

$$Y(t) = X(t) + \beta(t)V(t), \quad t > t_0.$$

We have that

$$\begin{aligned} dY(t) &= dX(t) + \beta'(t)V(t) + \beta(t)dV(t) \\ &= -\beta(t)\nabla f(Y(t))dt - (\beta'(t) - \gamma(t)\beta(t) + 1) \left(\frac{X(t) - Y(t)}{\beta(t)} \right) dt + \beta(t)\sigma(t, Y(t))dW(t). \end{aligned}$$

So we can reformulate (S – ISIHD) in terms of X, Y in the following way:

$$\begin{cases} dX(t) &= - \left(\frac{X(t) - Y(t)}{\beta(t)} \right) dt, \quad t > t_0; \\ dY(t) &= -\beta(t)\nabla f(Y(t))dt - (\beta'(t) - \gamma(t)\beta(t) + 1) \left(\frac{X(t) - Y(t)}{\beta(t)} \right) dt + \beta(t)\sigma(t, Y(t))dW(t), \quad t > t_0; \\ X(t_0) &= X_0; \quad Y(t_0) = X_0 + \beta(t_0)V_0, \end{cases} \quad (\text{ISIHD} - \text{S}_R)$$

where the subscript 'R' indicates that this is a reformulation. Moreover, we can reformulate (ISIHD – S_R) in the product space $\mathbb{H} \times \mathbb{H}$ by setting $Z(t) = (X(t), Y(t)) \in \mathbb{H} \times \mathbb{H}$, and thus (ISIHD – S_R) can be equivalently written as

$$\begin{cases} dZ(t) &= -\beta(t)\nabla \mathcal{G}(Z(t))dt - \mathcal{D}(t, Z(t))dt + \hat{\sigma}(t, Z(t))dW(t), \quad t > t_0; \\ Z(t_0) &= (X_0, X_0 + \beta(t_0)V_0), \end{cases} \quad (6.6)$$

where $\mathcal{G} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ is the convex function defined as $\mathcal{G}(Z) = f(Y)$, and the time-dependent operator $\mathcal{D} : [t_0, +\infty[\times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ is given by

$$\mathcal{D}(t, Z) = \left(\frac{1}{\beta(t)}(X - Y), \frac{\beta'(t) - \gamma(t)\beta(t) + 1}{\beta(t)}(X - Y) \right), \quad (6.7)$$

and the stochastic noise $\hat{\sigma} \in \mathcal{M}_{2 \times 2}(\mathcal{L}_2(\mathbb{K}; \mathbb{H}))$ defined by

$$\hat{\sigma}(t, Z) = \begin{pmatrix} 0 & 0 \\ 0 & \beta(t)\sigma(t, Y) \end{pmatrix},$$

and $W(t) = (W_1(t), W_2(t))$, where W_1, W_2 are two independent \mathbb{K} -valued Brownian motions.

6.2.2 Existence and uniqueness of a solution

Proposition 6.2.4. *Consider $\nu \geq 2$, $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$, f and σ satisfying (\mathbf{H}_f) and (\mathbf{H}_σ) , respectively. Consider also γ satisfying (\mathbf{H}_γ) , and β satisfying (\mathbf{H}_β) . Then (S – ISIHD) has a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$.*

Remark 6.2.5. Hypothesis (\mathbf{H}_β) does not allow us to consider the case $\beta \equiv 0$, nevertheless, this case is well studied in Section 6.3.

Proof. We rewrite (S – ISIHD) as in the reformulation (ISIHD – S_R), we recall (6.6):

$$\begin{cases} dZ(t) &= -\beta(t)\nabla\mathcal{G}(Z(t))dt - \mathcal{D}(t, Z(t))dt + \hat{\sigma}(t, Z(t))dW(t), & t > t_0, \\ Z(t_0) &= (X_0, X_0 + \beta(t_0)V_0), \end{cases}$$

Since $\beta(t) \leq c_1$, we have that $-\beta(t)\nabla\mathcal{G}(Z(t))$ is Lipschitz. Besides, since

$$\left| \frac{\beta'(t) - \gamma(t)\beta(t) + 1}{\beta(t)} \right| \leq c_2,$$

we have that \mathcal{D} is a Lipschitz operator. Then, using the hypotheses on σ we can use Theorem 2.7.4 and conclude the existence and uniqueness of a process $Z \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$, this, in turn, implies the existence and uniqueness of a solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of (S – ISIHD). \square

6.2.3 Fast convergence properties: convex case

To obtain properties in almost sure sense and in expectation of (S – ISIHD), we are going to adapt the Lyapunov analysis shown on [33] for the dynamic (ISIHD).

To that purpose, let us consider $t_1 > t_0$, $\gamma, \beta : [t_0, +\infty[\rightarrow \mathbb{R}_+$ be fixed functions and let $a, b, c, d : [t_0, +\infty[\rightarrow \mathbb{R}$ be differentiable functions (on $]t_0, +\infty[$) satisfying the following system for all $t > t_1$:

$$\begin{cases} a'(t) - b(t)c(t) & \leq 0, \\ -a(t)\beta(t) & \leq 0, \\ -a(t)\gamma(t)\beta(t) + a(t)\beta'(t) + a(t) - c(t)^2 + b(t)c(t)\beta(t) & = 0, \\ b'(t)b(t) + \frac{d'(t)}{2} & \leq 0, \\ b'(t)c(t) + b(t)(b(t) + c'(t) - c(t)\gamma(t)) + d(t) & = 0, \\ c(t)(b(t) + c'(t) - c(t)\gamma(t)) & \leq 0. \end{cases} \quad (\mathbf{S}_{a,b,c,d})$$

Given $x^* \in \mathcal{S}$, we consider

$$\mathcal{E}(t, x, v) = a(t)(f(x + \beta(t)v) - \min(f)) + \frac{1}{2}\|b(t)(x - x^*) + c(t)v\|^2 + \frac{d(t)}{2}\|x - x^*\|^2. \quad (6.8)$$

Remark 6.2.6. It was shown in [3, Section 3.1] and [33, Lemma 1] that energy function \mathcal{E} with a, b, c, d satisfying the system $(\mathbf{S}_{a,b,c,d})$ is a Lyapunov function for (ISIHD) when $\gamma(t) = \frac{\alpha}{t}$ (with $\alpha > 3$) and $\beta(t) = \gamma_0 + \frac{\beta}{t}$ (with $\gamma_0, \beta \geq 0$), hence, useful to obtain convergence guarantees of that dynamic. We will see that the same system $(\mathbf{S}_{a,b,c,d})$ also covers the case of general coefficients γ and β , hence providing insights on the convergence properties of (S – ISIHD) when one can find the corresponding functions a, b, c, d .

In the following proposition, we state different abstract integral bounds and almost sure properties for (S – ISIHD), finally concluding with the almost sure convergence of the gradient towards zero.

Proposition 6.2.7. *Consider that f, σ satisfy (H_f) and (H_σ) , respectively. Let $\nu \geq 2$, and consider the dynamic (S – ISIHD) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} –measurable. Consider also γ, β from (S – ISIHD) satisfying (H_γ) and (H_β) , respectively, and a, b, c, d satisfying $(S_{a,b,c,d})$. Finally, we consider \mathcal{E} the energy function defined in (6.8).*

Then, there exists a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of (S – ISIHD). Moreover, if $t \mapsto m(t)\sigma_\infty^2(t) \in L^1([t_0, +\infty[)$, where $m(t) \stackrel{\text{def}}{=} \max\{1, a(t), c^2(t)\}$, then the following properties are satisfied:

(i)

$$\lim_{t \rightarrow +\infty} \mathcal{E}(t, X(t), V(t)) \text{ exists a.s..}$$

(ii) $\int_{t_0}^\infty (b(s)c(s) - a'(s))(f(X(s) + \beta(s)V(s)) - \min f) ds < +\infty$, a.s..

(iii) $\int_{t_0}^\infty a(s)\beta(s)\|\nabla f(X(s) + \beta(s)V(s))\|^2 ds < +\infty$, a.s..

(iv) $\int_{t_0}^\infty \left(b(s)b'(s) + \frac{d'(s)}{2}\right) \|X(s) - x^*\|^2 ds < +\infty$, a.s..

(v) $\int_{t_0}^\infty c(s)(\gamma(s)c(s) - c'(s) - b(s))\|V(s)\|^2 ds < +\infty$, a.s..

(vi) *If $b(t)c(t) - a'(t) = \mathcal{O}(c(t)(\gamma(t)c(t) - c'(t) - b(t)))$, then*

$$\int_{t_0}^\infty (b(s)c(s) - a'(s))(f(X(s)) - \min f) ds < +\infty \quad \text{a.s..}$$

(vii) *If there exists $\eta > 0, \hat{t} > t_0$ such that*

$$\eta \leq c(t)(\gamma(t)c(t) - c'(t) - b(t)), \quad \eta \leq a(t)\beta(t), \quad \gamma(t) \leq \eta, \quad \forall t > \hat{t},$$

then $\lim_{t \rightarrow \infty} \|V(t)\| = 0$ a.s., $\lim_{t \rightarrow \infty} \|\nabla f(X(t) + \beta(t)V(t))\| = 0$ a.s., and $\lim_{t \rightarrow \infty} \|\nabla f(X(t))\| = 0$ a.s..

Proof. The existence and uniqueness of a solution is a direct consequence of Corollary 6.2.4. Moreover, applying Proposition 6.1.1 with \mathcal{E} , we can obtain

$$\begin{aligned} \mathcal{E}(t, X(t), V(t)) &\leq \mathcal{E}(t_0, X_0, V_0) - \int_{t_0}^t (b(s)c(s) - a'(s))(f(X(s) + \beta(s)V(s)) - \min f) ds \\ &\quad - \int_{t_0}^t a(s)\beta(s)\|\nabla f(X(s) + \beta(s)V(s))\|^2 ds - \int_{t_0}^t \left(b(s)b'(s) + \frac{d'(s)}{2}\right) \|X(s) - x^*\|^2 ds \\ &\quad - \int_{t_0}^t c(s)(b(s) + c'(s) - c(s)\gamma(s))\|V(s)\|^2 ds + \int_{t_0}^t (La(s)\beta^2(s) + c^2(s))\sigma_\infty^2(s) ds + M_t, \end{aligned} \tag{6.9}$$

where

$$M_t \stackrel{\text{def}}{=} \int_{t_0}^t \langle \sigma^*(s, X(s) + \beta(s)V(s))(a(s)\beta(s)\nabla f(X(s) + \beta(s)V(s)) + c(s)[b(s)(X(s) - x^*) + c(s)V(s)], dW(s) \rangle.$$

Since $\sup_{t \in [t_0, T]} \mathbb{E}(\|X(t)\|^2) < +\infty, \sup_{t \in [t_0, T]} \mathbb{E}(\|V(t)\|^2) < +\infty$ for every $T > t_0$, and a, b, c, β are continuous functions, we have that M_t is a continuous martingale, on the other hand, we have that

$$\int_{t_0}^\infty (La(s)\beta^2(s) + c^2(s))\sigma_\infty^2(s) ds < +\infty.$$

Then, we can apply Theorem 2.8.2 and conclude that $\lim_{t \rightarrow \infty} \mathcal{E}(t, X(t), V(t))$ exists a.s. and

- $\int_{t_0}^\infty (b(s)c(s) - a'(s))(f(X(s) + \beta(s)V(s)) - \min f) ds < +\infty$ a.s..
- $\int_{t_0}^\infty a(s)\beta(s)\|\nabla f(X(s) + \beta(s)V(s))\|^2 ds < +\infty$ a.s..
- $\int_{t_0}^\infty \left(b(s)b'(s) + \frac{d'(s)}{2}\right) \|X(s) - x^*\|^2 ds < +\infty$ a.s..

- $\int_{t_0}^{\infty} c(s)(\gamma(s)c(s) - c'(s) - b(s))\|V(s)\|^2 ds < +\infty$ a.s..

This let us conclude with items (i) to (v).

Let $\tilde{b}(t) = b(t)c(t) - a'(t)$, and

$$\begin{aligned} I_f &\stackrel{\text{def}}{=} \int_{t_0}^{\infty} \tilde{b}(s)(f(X(s)) - \min f) ds \\ &\leq \int_{t_0}^{\infty} \tilde{b}(s)(f(X(s)) - f(X(s) + \beta(s)V(s))) ds + \int_{t_0}^{\infty} \tilde{b}(s)(f(X(s) + \beta(s)V(s)) - \min f) ds \end{aligned}$$

Using Descent Lemma, Cauchy Schwarz Inequality and Corollary 2.7:

$$\begin{aligned} I_f &\leq \sqrt{2L}\beta_0 \left(\int_{t_0}^{\infty} \tilde{b}(s)(f(X(s) + \beta(s)V(s)) - \min f) ds \right)^{\frac{1}{2}} \left(\int_{t_0}^{\infty} \tilde{b}(s)\|V(s)\|^2 ds \right)^{\frac{1}{2}} \\ &\quad + \frac{L\beta_0^2}{2} \int_{t_0}^{\infty} \tilde{b}(s)\|V(s)\|^2 ds + \int_{t_0}^{\infty} \tilde{b}(s)(f(X(s) + \beta(s)V(s)) - \min f) ds. \end{aligned}$$

If $\tilde{b}(t) = b(t)c(t) - a'(t) = \mathcal{O}(c(t)(\gamma(t)c(t) - c'(t) - b(t)))$, we have that

$$\int_{t_0}^{\infty} \tilde{b}(s)\|V(s)\|^2 ds < +\infty \quad \text{a.s..}$$

And we conclude with item (vi).

To prove (vii), in particular that $\lim_{t \rightarrow \infty} \|V(t)\| = 0$, we consider that if there exists $\eta > 0, \hat{t} > t_0$ such that $\eta \leq c(t)(\gamma(t)c(t) - c'(t) - b(t)), \forall t > \hat{t}$, then there exists $\Omega_v \in \mathcal{F}$ such that $\mathbb{P}(\Omega_v) = 1$ and

$$\int_{t_0}^{\infty} \|V(\omega, s)\|^2 ds < +\infty, \quad \forall \omega \in \Omega_v.$$

Then, we have $\liminf_{t \rightarrow \infty} \|V(\omega, t)\| = 0, \forall \omega \in \Omega_v$. Let us suppose that for every $\omega \in \Omega_v$, $\limsup_{t \rightarrow \infty} \|V(\omega, t)\| > 0$, then by Lemma 2.6.7, there exists $\delta > 0$ satisfying

$$0 = \liminf_{t \rightarrow \infty} \|V(\omega, t)\| < \delta < \limsup_{t \rightarrow \infty} \|V(\omega, t)\|, \quad \forall \omega \in \Omega_v.$$

And there exists $(t_k)_{k \in \mathbb{N}} \subset [t_0, +\infty[$ such that $\lim_{k \rightarrow \infty} t_k = +\infty$,

$$\|V(\omega, t_k)\| > \delta, \quad \forall \omega \in \Omega_v \text{ and } t_{k+1} - t_k > 1, \quad \forall k \in \mathbb{N}.$$

Let $N_t \stackrel{\text{def}}{=} \int_{t_0}^t \sigma(s, X(s) + \beta(s)V(s)) dW(s)$. This is a continuous martingale (w.r.t. the filtration \mathcal{F}_t), which verifies

$$\mathbb{E}(\|N_t\|^2) = \mathbb{E} \left(\int_{t_0}^t \|\sigma(s, X(s) + \beta(s)V(s))\|_{\text{HS}}^2 ds \right) \leq \mathbb{E} \left(\int_{t_0}^{\infty} \sigma_{\infty}^2(s) ds \right) < +\infty, \forall t \geq t_0.$$

According to Theorem 2.8.1, we deduce that there exists a \mathbb{H} -valued random variable N_{∞} w.r.t. \mathcal{F}_{∞} , and which verifies: $\mathbb{E}(\|N_{\infty}\|^2) < +\infty$, and there exists $\Omega_N \in \mathcal{F}$ such that $\mathbb{P}(\Omega_N) = 1$ and

$$\lim_{t \rightarrow +\infty} N_t(\omega) = N_{\infty}(\omega) \text{ for every } \omega \in \Omega_N.$$

Let $\omega_0 \in \Omega_{nv} \stackrel{\text{def}}{=} \Omega_N \cap \Omega_v$ ($\mathbb{P}(\Omega_{nv}) = 1$) and the notation $V(t) \stackrel{\text{def}}{=} V(\omega_0, t)$, $\varepsilon \in \left(0, \min\{1, \frac{\delta^2}{4}\}\right)$ arbitrary and recall that $\eta \leq a(t)\beta(t), \gamma(t) \leq \eta$ for every $t > \hat{t}$. Let $k' \in \mathbb{N}$ be such that $t_{k'} > \hat{t}$, $k > k'$ and

$t \in [t_k, t_k + \varepsilon]$, then

$$\begin{aligned}
\|V(t) - V(t_k)\|^2 &\leq 3(t - t_k) \int_{t_k}^t \gamma^2(s) \|V(s)\|^2 ds + 3(t - t_k) \int_{t_k}^t \|\nabla f(X(s) + \beta(s)V(s))\|^2 ds \\
&\quad + 3\|N_t - N_{t_k}\|^2 \\
&\leq 3\eta^2(t - t_k) \int_{t_k}^t \|V(s)\|^2 ds + \frac{3}{\eta}(t - t_k) \int_{t_k}^t a(s)\beta(s) \|\nabla f(X(s) + \beta(s)V(s))\|^2 ds \\
&\quad + 3\|N_t - N_{t_k}\|^2 \\
&\leq 3\eta^2\varepsilon \int_{t_k}^t \|V(s)\|^2 ds + \frac{3}{\eta} \int_{t_k}^t a(s)\beta(s) \|\nabla f(X(s) + \beta(s)V(s))\|^2 ds \\
&\quad + 3\|N_t - N_{t_k}\|^2.
\end{aligned}$$

Now let $k'' \in \mathbb{N}$ be such that for every $k > k''$,

$$\int_{t_k}^{\infty} \|V(s)\|^2 ds < \frac{1}{9\eta^2}, \int_{t_k}^{\infty} a(s)\beta(s) \|\nabla f(X(s) + \beta(s)V(s))\|^2 ds < \frac{\eta}{9}, \sup_{t > t_k} \|N_t - N_{t_k}\|^2 < \frac{\varepsilon}{9}.$$

Then, we have that

$$\|V(t) - V(t_k)\|^2 \leq \varepsilon \leq \frac{\delta^2}{4}, \forall t \in [t_k, t_k + \varepsilon], k > \max\{k', k''\}.$$

For such t , we bound using the triangular inequality and obtain

$$\|V(t)\| \geq \|V(t_k)\| - \|V(t) - V(t_k)\| > \frac{\delta}{2}.$$

Now we consider

$$\int_{t_0}^{\infty} \|V(s)\|^2 ds \geq \sum_{k > \max\{k', k''\}} \int_{t_k}^{t_k + \varepsilon} \|V(s)\|^2 ds \geq \sum_{k > \max\{k', k''\}} \frac{\varepsilon \delta^2}{4} = +\infty.$$

Which is a contradiction, then we conclude that $\liminf_{t \rightarrow \infty} \|V(t)\| = \limsup_{t \rightarrow \infty} \|V(t)\| = 0$, a.s..

To prove the second part of (vii), i.e. that $\lim_{t \rightarrow \infty} \|\nabla f(X(t) + \beta(t)V(t))\| = 0$, we recall that there exists $\eta > 0, \hat{t} > t_0$ such that $\eta \leq a(t)\beta(t)$, then there exists $\Omega_y \in \mathcal{F}$ such that $\mathbb{P}(\Omega_y) = 1$ and

$$\int_{t_0}^{\infty} \|\nabla f(X(\omega, s) + \beta(s)V(\omega, s))\|^2 ds < +\infty \quad \forall \omega \in \Omega_y$$

So we have that

$$\liminf_{t \rightarrow \infty} \|\nabla f(X(\omega, t) + \beta(t)V(\omega, t))\| = 0, \quad \forall \omega \in \Omega_y.$$

Moreover, if we suppose that

$$\limsup_{t \rightarrow \infty} \|\nabla f(X(\omega, t) + \beta(t)V(\omega, t))\| > 0, \quad \forall \omega \in \Omega_y,$$

by Lemma 2.6.7, there exists $\delta > 0, (t_k)_{k \in \mathbb{N}} \subset [t_0, +\infty[$ such that $\lim_{k \rightarrow \infty} t_k = +\infty$,

$$\|\nabla f(X(\omega, t_k) + \beta(t_k)V(\omega, t_k))\| > \delta \quad \forall \omega \in \Omega_y \text{ and } t_{k+1} - t_k > 1, \quad \forall k \in \mathbb{N}.$$

Recall that by (H $_{\beta}$), there exists β_0 such that $\beta(t) \leq \beta_0$. Let $\varepsilon \in \left(0, \min\{1, \frac{\delta^2}{4L^2}\}\right)$ arbitrary and consider $Y(t) = X(t) + \beta(t)V(t)$, let also $k \in \mathbb{N}$ arbitrary and $t \in [t_k, t_k + \varepsilon]$. Then, using Lemma 2.6.1

and Jensen's inequality we can bound as follows:

$$\begin{aligned}
\|Y(t) - Y(t_k)\|^2 &\leq 2\|X(t) - X(t_k)\|^2 + 2\|\beta(t)V(t) - \beta(t_k)V(t_k)\|^2 \\
&\leq 2\left\|\int_{t_k}^t V(s)ds\right\|^2 + 2(\beta_0\|V(t) - V(t_k)\| + |\beta(t) - \beta(t_k)| \max\{\|V(t)\|, \|V(t_k)\|\})^2 \\
&\leq 2(t - t_k) \int_{t_k}^{\infty} \|V(s)\|^2 ds \\
&\quad + 2(\beta_0\|V(t) - V(t_k)\| + |\beta(t) - \beta(t_k)| \max\{\|V(t)\|, \|V(t_k)\|\})^2 \\
&\leq 2(t - t_k) \int_{t_k}^{\infty} \|V(s)\|^2 ds \\
&\quad + 4\beta_0^2\|V(t) - V(t_k)\|^2 + 4|\beta(t) - \beta(t_k)|^2 \max\{\|V(t)\|, \|V(t_k)\|\}^2.
\end{aligned}$$

By the previous point, we have that there exists $\Omega_v \in \mathcal{F}$ such that $\mathbb{P}(\Omega_v) = 1$ such that

$$\int_{t_0}^{\infty} \|V(\omega, s)\|^2 ds < +\infty \quad \forall \omega \in \Omega_v,$$

and $k' \in \mathbb{N}$ such that for every $k > k'$, for all $t \in [t_k, t_k + \varepsilon]$:

$$\int_{t_k}^{\infty} \|V(\omega, s)\|^2 ds < \frac{1}{6}, \quad \max\{\|V(\omega, t)\|, \|V(\omega, t_k)\|\} < 1, \quad \|V(\omega, t) - V(\omega, t_k)\|^2 \leq \frac{\varepsilon}{12\beta_0^2}.$$

We consider an arbitrary $\omega_0 \in \Omega_y \cap \Omega_v$ ($\mathbb{P}(\Omega_y \cap \Omega_v) = 1$), and we let us use the abuse of notation $X(t) = X(\omega_0, t)$, $V(t) = V(\omega_0, t)$, and $Y(t) = Y(\omega_0, t)$ for the rest of this proof.

On the other hand, β is continuous, so there exists $\tilde{\delta} > 0$ such that, if $|t - t_k| < \tilde{\delta}$, then $|\beta(t) - \beta(t_k)| < \frac{\sqrt{\varepsilon}}{2\sqrt{3}}$.

Therefore, letting $\varepsilon' = \min\{\varepsilon, \tilde{\delta}\}$, we have that

$$\|\nabla f(Y(t)) - \nabla f(Y(t_k))\|^2 \leq L^2\|Y(t) - Y(t_k)\|^2 \leq L^2\varepsilon \leq \frac{\delta^2}{4}, \quad \forall k > k', \forall t \in [t_k, t_k + \varepsilon'].$$

Then, we obtain

$$\|\nabla f(Y(t))\| \geq \|\nabla f(Y(t_k))\| - \|\nabla f(Y(t)) - \nabla f(Y(t_k))\| \geq \frac{\delta}{2}, \quad \forall k > k', \forall t \in [t_k, t_k + \varepsilon'].$$

This implies that

$$\int_{t_0}^{\infty} \|\nabla f(Y(s))\|^2 ds \geq \sum_{k > k'} \int_{t_k}^{t_k + \varepsilon'} \|\nabla f(Y(s))\|^2 ds \geq \sum_{k > k'} \int_{t_k}^{t_k + \varepsilon'} \frac{\delta^2}{4} = \sum_{k > k'} \frac{\varepsilon' \delta^2}{4} = +\infty.$$

Which is a contradiction, then we conclude that

$$\liminf_{t \rightarrow \infty} \|\nabla f(X(t) + \beta(t)V(t))\| = \limsup_{t \rightarrow \infty} \|\nabla f(X(t) + \beta(t)V(t))\| = 0, \quad a.s..$$

To prove the last part of (vii), we consider that $\beta(t) \leq \beta_0$, then

$$\begin{aligned}
\|\nabla f(X(t))\| &\leq \|\nabla f(X(t) + \beta(t)V(t))\| + \|\nabla f(X(t)) - \nabla f(X(t) + \beta(t)V(t))\| \\
&\leq \|\nabla f(X(t) + \beta(t)V(t))\| + L\beta_0\|V(t)\|.
\end{aligned}$$

With this bound, we can conclude that $\lim_{t \rightarrow \infty} \|\nabla f(X(t))\| = 0$ a.s.. □

The following proposition states abstract bounds in expectation of (S – ISIHD).

Proposition 6.2.8. *Consider the setting of Proposition 6.2.7, then we have that:*

$$(i) \quad \mathbb{E}(f(X(t) + \beta(t)V(t)) - \min f) = \mathcal{O}\left(\frac{1}{a(t)}\right).$$

Moreover, if there exists $D > 0, \tilde{t} > t_0$ such that $d(t) \geq D$ for $t > \tilde{t}$, then :

$$(ii) \sup_{t > t_0} \mathbb{E}(\|X(t) - x^*\|^2) < +\infty.$$

$$(iii) \mathbb{E}(\|V(t)\|^2) = \mathcal{O}\left(\frac{1 + b^2(t)}{c^2(t)}\right).$$

$$(iv) \mathbb{E}(f(X(t)) - \min f) = \mathcal{O}\left(\max\left\{\frac{1}{a(t)}, \frac{\beta(t)\sqrt{1+b^2(t)}}{\sqrt{a(t)}c(t)}, \frac{\beta^2(t)(1+b^2(t))}{c^2(t)}\right\}\right).$$

Proof. To prove this proposition we are going to take expectation in (6.9). First, we are going to bound the negative terms by 0, denoting $E_0 \stackrel{\text{def}}{=} \mathcal{E}(t_0) + \max\{1, L\} \int_{t_0}^{\infty} (a(s)\beta^2(s) + c^2(s))\sigma_{\infty}^2(s)ds$, we obtain that

$$\mathbb{E}(\mathcal{E}(t, X(t), V(t))) \leq E_0.$$

This implies that

- $\mathbb{E}(f(X(t) + \beta(t)V(t)) - \min f) \leq \frac{E_0}{a(t)}$.
- $\mathbb{E}(\|b(t)(X(t) - x^*) + c(t)V(t)\|^2) \leq 2E_0$.

If there exists $D > 0, \tilde{t} > t_0$ such that $d(t) \geq D$ for $t > \tilde{t}$, then for $t > \tilde{t}$:

- $\mathbb{E}(\|X(t) - x^*\|^2) \leq \frac{2E_0}{D}$.
- And also,

$$\begin{aligned} \mathbb{E}(\|V(t)\|^2) &\leq \frac{2}{c^2(t)} [\mathbb{E}(\|b(t)(X(t) - x^*) + c(t)V(t)\|^2) + b^2(t)\mathbb{E}(\|X(t) - x^*\|^2)] \\ &\leq \frac{2}{c^2(t)} \left(2E_0 + \frac{2E_0b^2(t)}{D}\right) = \frac{4E_0}{c^2(t)} \left(1 + \frac{b^2(t)}{D}\right). \end{aligned}$$

- We bound the following term using the Descent Lemma

$$\begin{aligned} \mathbb{E}(f(X(t)) - f(X(t) + \beta(t)V(t))) &\leq \beta(t)\sqrt{\mathbb{E}(\|\nabla f(X(t) + \beta(t)V(t))\|^2)}\sqrt{\mathbb{E}(\|V(t)\|^2)} \\ &\quad + \frac{L}{2}\beta^2(t)\mathbb{E}(\|V(t)\|^2). \end{aligned}$$

Using Corollary 2.7, we have

$$\begin{aligned} \mathbb{E}(f(X(t)) - f(X(t) + \beta(t)V(t))) &\leq \beta(t)\sqrt{2L\mathbb{E}(f(X(t) + \beta(t)V(t)) - \min f)}\sqrt{\mathbb{E}(\|V(t)\|^2)} \\ &\quad + \frac{L}{2}\beta^2(t)\mathbb{E}(\|V(t)\|^2) \\ &\leq 2E_0\sqrt{2L}\frac{\beta(t)}{\sqrt{a(t)}}\frac{\sqrt{1+\frac{b^2(t)}{D}}}{c(t)} + 2LE_0\frac{\beta^2(t)\left(1+\frac{b^2(t)}{D}\right)}{c^2(t)} \\ &= \mathcal{O}\left(\max\left\{\frac{\beta(t)}{\sqrt{a(t)}}\frac{\sqrt{1+b^2(t)}}{c(t)}, \frac{\beta^2(t)(1+b^2(t))}{c^2(t)}\right\}\right) \end{aligned}$$

Then, we notice that

$$\begin{aligned} \mathbb{E}(f(X(t)) - \min f) &= \mathbb{E}[f(X(t)) - f(X(t) + \beta(t)V(t))] + \mathbb{E}[f(X(t) + \beta(t)V(t)) - \min f] \\ &= \mathcal{O}\left(\max\left\{\frac{\beta(t)}{\sqrt{a(t)}}\frac{\sqrt{1+b^2(t)}}{c(t)}, \frac{\beta^2(t)(1+b^2(t))}{c^2(t)}, \frac{1}{a(t)}\right\}\right). \end{aligned}$$

□

The previous two propositions generalizes the results proved in [33] to the stochastic setting, and is not restricted to the case $\gamma(t) = \frac{\alpha}{t}$. However, they lack practical use if we cannot exhibit a, b, c, d functions that satisfy (S_{a,b,c,d}). Although we are not able to solve this system in general, in Corollaries 6.2.9 and 6.2.11 we will specify some particular cases for γ and β where such functions a, b, c, d can be exhibited to satisfy the system (S_{a,b,c,d}).

The following corollary provides a specific case where a solution to the system $(S_{a,b,c,d})$ can be exhibited, which was not discussed in [3, 33]. Moreover, we show the implications it has on the stochastic setting.

Corollary 6.2.9 (Decreasing and vanishing γ , with vanishing γ' and positive constant β). Consider the context of Proposition 6.2.7 in the case where $\beta(t) \equiv \beta > 0$, γ satisfying (H_γ) , such that it is a differentiable, decreasing, and vanishing function, with $\lim_{t \rightarrow +\infty} \gamma'(t) = 0$, and satisfying (H'_γ) . Let $b \in]2(m-1), 1[$, then choosing

$$\begin{aligned} a(t) &= \frac{\Gamma(t)(\Gamma(t) - \beta b)}{1 - \beta\gamma(t)}, \\ b(t) &= b, \\ c(t) &= \Gamma(t), \\ d(t) &= b(1 - b), \end{aligned}$$

there exists $\hat{t} > t_0$ such that the system $(S_{a,b,c,d})$ is satisfied for every $t \geq \hat{t}$.

Given $x^* \in \mathcal{S}$ and σ_∞ be such that $t \mapsto \Gamma(t)\sigma_\infty(t) \in L^2([t_0, +\infty[)$, then the following statements hold:

- (i) $\int_{t_0}^{+\infty} \Gamma(s) (f(X(s)) - \min f + \|V(s)\|^2) ds < +\infty$ a.s..
- (ii) $\lim_{t \rightarrow +\infty} \|\nabla f(X(t))\| + \|V(t)\| = 0$ a.s..
- (iii) $\mathbb{E}(f(X(t)) - \min f + \|V(t)\|^2) = \mathcal{O}\left(\frac{1}{\Gamma^2(t)}\right)$.

Remark 6.2.10. When $\gamma(t) = \frac{\alpha}{t}$ with $\alpha > 3$ and $t\sigma_\infty(t) \in L^2([t_0, +\infty[)$, the previous corollary ensures fast convergence of the values, i.e., $\mathcal{O}(t^{-2})$. Besides, by Lemma 6.2.3, when $\gamma(t) = \frac{\alpha}{t^r}$ with $r \in]0, 1[$, $\alpha \geq 1 - r$, and $t^r\sigma_\infty(t) \in L^2([t_0, +\infty[)$, the previous corollary ensures convergence of the objective at a rate $\mathcal{O}(t^{-2r})$. The latter choice indicates that one can require a weaker integrability condition on the noise, compared to the case $\gamma(t) = \frac{\alpha}{t}$ ($\alpha > 3$), but at the price of a slower convergence rate.

Proof. We start by noticing that since γ is decreasing, by [18, Corollary 2.3] we have that $\Gamma(t)$ is increasing and $\gamma(t)\Gamma(t) \geq 1$, for every $t \geq t_0$. Also, it is direct that with a fixed $\beta > 0$ we satisfy (H_β) .

Letting $b \in]2(m-1), 1[$ and $t_1 > t_0$ such that $\beta \leq \frac{1}{\gamma(t_1)}$, this t_1 exists since $t \mapsto \frac{1}{\gamma(t)}$ is an increasing function. We choose $c(t) = \Gamma(t)$, by the fifth equation of $(S_{a,b,c,d})$, we get that $d = b(1 - b)$, and the fourth equation is trivial. The third equation implies that $a(t) = \frac{\Gamma(t)(\Gamma(t) - \beta b)}{1 - \beta\gamma(t)}$ and the choice of β implies that the second equation is satisfied for $t \geq t_1$, since $\beta \leq \frac{1}{\gamma(t_1)} \leq \frac{1}{\gamma(t)} \leq \Gamma(t)$ for every $t > t_1$. By the definition of $c(t)$ and the fact that $b < 1$, we directly have that the sixth equation also holds. We just need to check the first equation, to do that we can see that this equation is equivalent to

$$\frac{\Gamma'(t)(2\Gamma(t) - \beta b)(1 - \beta\gamma(t)) + \beta\Gamma(t)(\Gamma(t) - \beta b)\gamma'(t)}{(1 - \beta\gamma(t))^2} \leq b\Gamma(t),$$

which in turn is equivalent to the following:

$$\begin{aligned} 2\Gamma(t)\Gamma'(t) - \beta b\Gamma'(t) - 2\beta\gamma(t)\Gamma(t)\Gamma'(t) + b\beta^2\gamma(t)\Gamma'(t) + \beta\Gamma^2(t)\gamma'(t) - b\beta^2\Gamma(t)\gamma'(t) \\ \leq b\Gamma(t) - 2b\beta\gamma(t)\Gamma(t) + b\beta^2\gamma^2\Gamma(t). \end{aligned} \quad (6.10)$$

By (H'_γ) , there exists $t_2 > t_0$ such that $\gamma(t)\Gamma(t) \leq m$ and $\Gamma'(t) \leq m - 1$ for every $t \geq t_2$. Note that the terms $b\beta^2\gamma(t)\Gamma'(t)$, $-2b\beta\gamma(t)\Gamma(t)$ are upper and lower bounded by constants. Since the terms

$$-2\beta\gamma(t)\Gamma(t)\Gamma'(t), \quad -\beta b\Gamma'(t), \quad \beta\Gamma^2(t)\gamma'(t),$$

are negative, and $b\beta^2\gamma^2\Gamma(t)$ is positive, if we could prove that there exists $t_3 \geq \max\{t_0, t_1, t_2\}$ such that

$$\begin{aligned} -b\beta^2\gamma'(t) \leq b - 2\Gamma'(t) \\ - 110 - \end{aligned}$$

for $t \geq t_3$, this would imply that there exists $\hat{t} \geq t_3$ such that (6.10) holds for every $t \geq \hat{t}$. In fact, we see that the previous inequality holds for t large enough (*i.e.* there exists such a t_3) since $\lim_{t \rightarrow +\infty} -\gamma'(t) = 0$ and the fact that $2(m-1) < b$ implies that $b - 2\Gamma'(t) > 0$. Thus, we have checked that the proposed a, b, c, d satisfy the system $(S_{a,b,c,d})$ for $t > \hat{t}$.

The rest of the proof is direct from replacing the specified $a, b, c, d, \gamma, \beta$ functions in Propositions 6.2.7-6.2.8, and the fact that for t large enough, $b\Gamma(t) - a'(t) \geq (b - 2(m-1))\Gamma(t) - C_b$ for some $C_b > 0$, that $\lim_{t \rightarrow +\infty} \Gamma(t) = +\infty$, and also $a(t) \geq \Gamma^2(t)$. \square

The following result gives us another case in which we can satisfy the system $(S_{a,b,c,d})$. This generalizes to the stochastic setting the results presented in [3, Section 3.1] and [33, Lemma 1]. Besides, it ensures fast convergence of the values whenever $t \mapsto t\sigma_\infty(t) \in L^2([t_0, +\infty[)$.

Corollary 6.2.11 ($\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \gamma_0 + \frac{\beta_0}{t}$). *Consider the context of Proposition 6.2.7 in the case where $\gamma(t) = \frac{\alpha}{t}$ and $\beta(t) = \gamma_0 + \frac{\beta_0}{t}$, where $\alpha > 3, \gamma_0 > 0, \beta_0 \geq 0$. Then choosing*

$$\begin{aligned} a(t) &= t^2 \left(1 + \frac{(\alpha - b)\gamma_0 t - \beta_0(\alpha + 1 - b)}{t^2 - \alpha\gamma_0 t - \beta_0(\alpha + 1)} \right), \\ b(t) &= b \in (2, \alpha - 1), \\ c(t) &= t, \\ d(t) &= b(\alpha - 1 - b), \end{aligned}$$

the system $(S_{a,b,c,d})$ is satisfied.

Given $x^* \in \mathcal{S}$ and σ_∞ be such that $t \mapsto t\sigma_\infty(t) \in L^2([t_0, +\infty[)$, then the following statements hold:

- (i) $\int_{t_0}^{+\infty} s (f(X(s)) - \min f + \|V(s)\|^2) ds < +\infty$ a.s..
- (ii) $\lim_{t \rightarrow +\infty} \|\nabla f(X(t))\| + \|V(t)\| = 0$ a.s..
- (iii) $\mathbb{E}(f(X(t)) - \min f + \|V(t)\|^2) = \mathcal{O}\left(\frac{1}{t^2}\right)$.

Proof. Direct from replacing the specified $a, b, c, d, \gamma, \beta$ functions in Propositions 6.2.7-6.2.8, and the fact that for t large enough $bt - a'(t) \geq \frac{(\alpha-3)t}{2}$, and also $a(t) \geq t^2, 0 < \gamma_0 < \beta(t) \leq \gamma_0 + \frac{\beta_0}{t_0}$. \square

Remark 6.2.12. We can use the choices for a, b, c, d presented in Corollaries 6.2.9 and 6.2.11 in Propositions 6.2.7 and 6.2.8 to obtain additional integral bounds, almost sure and in expectation properties of (S – ISIHD). We leave this to the reader.

6.2.4 Strongly convex case

In the following theorem, we consider the case where the objective function is strongly convex and we present a choice of parameters γ and β to obtain a fast linear convergence to a noise dominated region.

Theorem 6.2.13. *Assume that $f : \mathbb{H} \rightarrow \mathbb{R}$ satisfies (H_f) , and is μ -strongly convex, $\mu > 0$, and denote x^* its unique minimizer. Suppose also that σ obeys (H_σ) . Let $\nu \geq 2$, consider the dynamic (S – ISIHD) with initial data $X_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable. Consider also $\gamma \equiv 2\sqrt{\mu}$, and a constant β such that $0 \leq \beta \leq \frac{1}{2\sqrt{\mu}}$. Moreover, suppose that σ_∞ is a nonincreasing function such that $\sigma_\infty \in L^2([t_0, +\infty[)$. Define the function $\mathcal{E} : [t_0, +\infty[\times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}_+$ as*

$$\mathcal{E}(t, x, v) \stackrel{\text{def}}{=} f(x + \beta v) - \min f + \frac{1}{2} \|\sqrt{\mu}(x - x^*) + v\|^2.$$

Then, (S – ISIHD) has q unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$. In addition, there exists positive constants M_1, M_2 such that

$$\mathbb{E}[\mathcal{E}(t, X(t), V(t))] \leq \mathcal{E}(t_0, X_0, V_0) e^{-\frac{\sqrt{\mu}}{2}(t-t_0)} + M_1 e^{-\frac{\sqrt{\mu}}{4}(t-t_0)} + M_2 \sigma_\infty \left(\frac{t_0 + t}{2} \right), \quad \forall t > t_0.$$

Let $\Theta : [t_0, +\infty[\rightarrow \mathbb{R}_+$ defined as $\Theta(t) \stackrel{\text{def}}{=} \max\{e^{-\frac{\sqrt{\mu}}{4}(t-t_0)}, \sigma_\infty\left(\frac{t+t_0}{2}\right)\}$. Consequently,

$$\begin{aligned}\mathbb{E}(f(X(t)) - \min f) &= \mathcal{O}(\Theta(t)), \\ \mathbb{E}(\|X(t) - x^*\|^2) &= \mathcal{O}(\Theta(t)), \\ \mathbb{E}(\|V(t)\|^2) &= \mathcal{O}(\Theta(t)), \\ \mathbb{E}(\|\nabla f(X(t))\|^2) &= \mathcal{O}(\Theta(t)).\end{aligned}$$

Proof. Using Itô's formula with \mathcal{E} , taking expectation and denoting $E(t) \stackrel{\text{def}}{=} \mathbb{E}(\mathcal{E}(t, X(t), V(t)))$, we have

$$E(t) \leq E(t_0) - \int_{t_0}^t \frac{\sqrt{\mu}}{2} E(s) ds - \int_{t_0}^t C(s) ds + (L\beta^2 + 1) \int_{t_0}^t \sigma_\infty^2(s) ds,$$

where

$$\begin{aligned}C(t) &\stackrel{\text{def}}{=} \beta \|\nabla f(X(t) + \beta V(t))\|^2 + \beta \sqrt{\mu} \langle \nabla f(X(t) + \beta V(t)), V(t) \rangle + \frac{\sqrt{\mu}}{2} (\beta^2 \mu + 1) \|V(t)\|^2 \\ &\quad + \beta \mu \sqrt{\mu} \langle X(t) - x^*, V(t) \rangle + \frac{\mu \sqrt{\mu}}{4} \|X(t) - x^* + \beta V(t)\|^2.\end{aligned}$$

It was proved in [33, Theorem 4.2] that under the condition $0 \leq \beta \leq \frac{1}{2\sqrt{\mu}}$ we obtain that $C(t)$ is a nonnegative function. Therefore, we can write the following

$$E(t) \leq E(t_0) - \int_{t_0}^t \frac{\sqrt{\mu}}{2} E(s) ds + (L\beta^2 + 1) \int_{t_0}^t \sigma_\infty^2(s) ds.$$

We continue by using Lemma 2.6.6, to do this, we need to solve the following Cauchy problem:

$$\begin{cases} Y'(t) &= -\frac{\sqrt{\mu}}{2} Y(t) + (L\beta^2 + 1) \sigma_\infty^2(t), \\ Y(t_0) &= \mathcal{E}(t_0, X_0, V_0). \end{cases}$$

Using the integrating factor method, we deduce that for all $t \geq t_0$:

$$\begin{aligned}Y(t) &= Y(t_0) e^{\frac{\sqrt{\mu}}{2}(t_0-t)} + (L\beta^2 + 1) e^{-\frac{\sqrt{\mu}}{2}t} \int_{t_0}^t e^{\frac{\sqrt{\mu}}{2}s} \sigma_\infty^2(s) ds \\ &\leq Y(t_0) e^{\frac{\sqrt{\mu}}{2}(t_0-t)} + (L\beta^2 + 1) e^{-\frac{\sqrt{\mu}}{2}t} \left(\int_{t_0}^{\frac{t_0+t}{2}} e^{\frac{\sqrt{\mu}}{2}s} \sigma_\infty^2(s) ds + \int_{\frac{t_0+t}{2}}^t e^{\frac{\sqrt{\mu}}{2}s} \sigma_\infty^2(s) ds \right) \\ &\leq Y(t_0) e^{\frac{\sqrt{\mu}}{2}(t_0-t)} + (L\beta^2 + 1) \sigma_\infty^2\left(\frac{t_0+t}{2}\right) + (L\beta^2 + 1) \sigma_\infty^2(t_0) e^{\frac{\sqrt{\mu}}{4}(t_0-t)} \\ &= \mathcal{O}(\Theta(t)).\end{aligned}$$

By Lemma 2.6.6, we conclude that $E(t) = \mathcal{O}(\Theta(t))$, immediately we observe that

$$\begin{aligned}\mathbb{E}(f(X(t) + \beta V(t)) - \min f) &= \mathcal{O}(\Theta(t)) \\ \mathbb{E}(\|\sqrt{\mu}(X(t) - x^*) + V(t)\|^2) &= \mathcal{O}(\Theta(t))\end{aligned}$$

By the strong convexity of f , we have that $\mathbb{E}(\|X(t) - x^* + \beta V(t)\|^2) = \mathcal{O}(\Theta(t))$, since $\beta \neq \frac{1}{\sqrt{\mu}}$ ($\beta \leq \frac{1}{2\sqrt{\mu}}$), then $\mathbb{E}(\|X(t) - x^*\|^2) = \mathbb{E}(\|V(t)\|^2) = \mathcal{O}(\Theta(t))$, on the other hand, using Lemma 2.3.2 and Lemma 2.7, $\mathbb{E}(f(X(t)) - f(X(t) + \beta V(t))) = \mathcal{O}(\Theta(t))$, thus,

$$\mathbb{E}(f(X(t)) - \min f) = \mathbb{E}(\|\nabla f(X(t))\|^2) = \mathcal{O}(\Theta(t)).$$

□

6.3 (S – ISIHD) with general γ and $\beta \equiv 0$

In this section we are going to study properties of the dynamic (S – ISIHD) in expectation and in almost sure sense, when the parameter γ is a general function and $\beta \equiv 0$. The noiseless case and under deterministic noise is well documented in [18].

Consider the dynamic (S – ISHD) when $\beta \equiv 0$. This dynamic will be a stochastic version of the Hamiltonian formulation of (IGS $_\gamma$) and it will be described by:

$$\begin{cases} dX(t) = V(t)dt, & t > t_0; \\ dV(t) = -\gamma(t)V(t)dt - \nabla f(X(t))dt + \sigma(t, X(t))dW(t), & t > t_0; \\ X(t_0) = X_0; \quad V(t_0) = V_0. \end{cases} \quad (\text{IGS}_\gamma - \text{S})$$

The main motivation for a separate analysis is that, in Section 6.2 we consider hypothesis (\mathbf{H}_β) to establish the existence and uniqueness of a solution, from which, the rest of the results follow. This hypothesis is incompatible with the case $\beta \equiv 0$.

We will demonstrate almost sure convergence of the velocity to zero and of the objective to its minimum value, under assumptions that are satisfied for $\gamma(t) = \frac{\alpha}{t^r}$, with $r \in [0, 1]$, $\alpha \geq 1 - r$. Additionally, we will show that for this particular choice of β , we can obtain almost sure (weak) convergence of the trajectory.

6.3.1 Minimization properties

Let γ satisfying (\mathbf{H}_γ), and let us define for $c > 0$,

$$\lambda_c(t) = \frac{p(t)}{c + \int_{t_0}^t p(s)ds}.$$

Remark 6.3.1. We can deduce that $\lambda'_c + \lambda_c^2 - \gamma\lambda_c = 0$. Besides, since $p \notin L^1([t_0, +\infty[)$, then $\lambda_c \notin L^1([t_0, +\infty[)$.

Theorem 6.3.2. Assume that f and σ satisfy assumptions (\mathbf{H}_f) and (\mathbf{H}_σ), respectively. Let $\nu \geq 2$, and consider the dynamic (IGS $_\gamma - \text{S}$) with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable. Then, there exists a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of (IGS $_\gamma - \text{S}$). Additionally, if $\sigma_\infty \in L^2([t_0, +\infty[)$, then

$$\int_{t_0}^{+\infty} \gamma(s) \|V(s)\|^2 ds < +\infty \quad a.s..$$

Moreover, suppose that

$$\text{there exists } \hat{t} \geq t_0, \text{ and } c > 0 \text{ such that } \gamma(t) \leq \lambda_c(t) \quad \forall t \geq \hat{t}, \quad (\mathbf{H}_a)$$

and

$$\int_{t_0}^{+\infty} \lambda_c(s) \|V(s)\|^2 ds < +\infty \quad a.s.. \quad (\mathbf{H}_b)$$

Then the following properties are satisfied:

- (i) $\int_{t_0}^{+\infty} \lambda_c(s) (f(X(s)) - \min f) ds < +\infty$ a.s..
- (ii) $\lim_{t \rightarrow +\infty} \|V(t)\| = 0$ a.s. and $\lim_{t \rightarrow +\infty} f(X(t)) - \min f = 0$ a.s..

Proof. If we define $F_0 : [t_0, +\infty[\times \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$, $G_0 : [t_0, +\infty[\times \mathbb{H} \times \mathbb{H} \rightarrow \mathcal{L}_2(\mathbb{K} \times \mathbb{K}; \mathbb{H} \times \mathbb{H})$ defined by

$$F_0(t, x, v) = \begin{pmatrix} v \\ -\gamma(t)v - \nabla f(x) \end{pmatrix}, \quad G_0(t, x, v) = \begin{pmatrix} 0_{\mathbb{K}; \mathbb{H}} & 0_{\mathbb{K}; \mathbb{H}} \\ 0_{\mathbb{K}; \mathbb{H}} & \sigma(t, x) \end{pmatrix}.$$

Let W be a $\mathbb{K} \times \mathbb{K}$ -valued Brownian Motion and $Z(t) = (X(t), Y(t))$, then (IGS $_\gamma - \text{S}$) can be equivalently be written as

$$\begin{cases} dZ(t) = F_0(t, Z(t))dt + G_0(t, Z(t))dW(t); \\ Z(t_0) = (X_0, V_0). \end{cases}$$

Then the existence and uniqueness of a solution of (IGS $_{\gamma}$ – S) is a direct consequence of Theorem 2.7.4 in the product space $\mathbb{H} \times \mathbb{H}$.

Let $x^* \in \mathcal{S}$ and $\phi_0 : (x, v) \mapsto \mathbb{R}$ defined by $\phi_0(x, v) = f(x) - \min f + \frac{\|v\|^2}{2}$, by Itô's formula, Theorem 2.8.2 and the fact that $\sigma_{\infty}^2 \in L^2([t_0, +\infty[)$ we obtain that $\int_{t_0}^{+\infty} \gamma(s) \|V(s)\|^2 ds < +\infty$ a.s., and that

$$\lim_{t \rightarrow +\infty} f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \text{ exists a.s.} \quad (6.11)$$

Moreover, if we assume the hypotheses (H $_a$) and (H $_b$), then:

- (i) Let $x^* \in \mathcal{S}$ and $\phi : (t, x, v) \mapsto \mathbb{R}$ defined by $\phi(t, x, v) = \frac{\|\lambda_c(t)(x-x^*)+v\|^2}{2} + (f(x) - \min f)$.
LConsider \hat{t} the one defined in the statement, by Itô's formula from \hat{t} to t , we have

$$\begin{aligned} f(X(t)) - \min f + \frac{\|\lambda_c(t)(X(t) - x^*) + V(t)\|^2}{2} &= f(X(\hat{t})) - \min f + \frac{\|\lambda_c(t)(X(\hat{t}) - x^*) + V(\hat{t})\|^2}{2} \\ &+ \int_{\hat{t}}^t \lambda_c(s) \lambda'_c(s) \|X(s) - x^*\|^2 - \gamma(t) \|V(s)\|^2 - \lambda_c(t) \langle \nabla f(X(s)), X(s) - x^* \rangle ds \\ &+ \int_{\hat{t}}^t \lambda_c(t) \|V(s)\|^2 + \text{tr}[\Sigma(s, X(s))] ds + \underbrace{\int_{\hat{t}}^t \langle [\lambda_c(s)(X(s) - x^*) + V(s)] \sigma^*(s, X(s)), dW(s) \rangle}_{M_t}. \end{aligned}$$

By the hypotheses, we have that

$$\int_{\hat{t}}^{+\infty} (\lambda_c(s) \|V(s)\|^2 + \text{tr}[\Sigma(s, X(s))]) ds \leq \int_{\hat{t}}^{+\infty} (\lambda_c(s) \|V(s)\|^2 + \sigma_{\infty}^2(s)) ds < +\infty \text{ a.s.}$$

Besides $(M_t)_{t \geq \hat{t}}$ is a continuous martingale. Moreover, by convexity of f and the fact that $\lambda'_c(t) \leq 0, \forall t \geq \hat{t}$, we have

$$\begin{aligned} &\int_{\hat{t}}^t \lambda_c(s) \lambda'_c(s) \|X(s) - x^*\|^2 - \gamma(t) \|V(s)\|^2 - \lambda_c(t) \langle \nabla f(X(s)), X(s) - x^* \rangle ds \\ &\leq - \int_{\hat{t}}^t \lambda_c(s) (f(X(s)) - \min f) ds. \end{aligned}$$

Then, by Theorem 2.8.2,

$$\int_{\hat{t}}^{+\infty} \lambda_c(s) \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) ds < +\infty \text{ a.s.}, \quad (6.12)$$

and $\lim_{t \rightarrow +\infty} f(X(t)) - \min f + \frac{\|\lambda_c(t)(X(t) - x^*) + V(t)\|^2}{2}$ exists a.s..

- (ii) Recalling that $\lambda_c \notin L^1([t_0, +\infty[)$, by Lemma 2.6.2, (6.11) and (6.12), we conclude that

$$\lim_{t \rightarrow +\infty} f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} = 0, \text{ a.s.}$$

□

Corollary 6.3.3. Consider the context of Theorem 6.3.2 with $\gamma(t) = \frac{\alpha}{t^r}$, where $r \in [0, 1]$ and $\alpha > 1 - r$. Then (H $_a$) and (H $_b$) are satisfied and thus the conclusions of Theorem 6.3.2 hold.

Proof. • We will prove the case $r = 1$ first, since it is direct, in such case, letting $c = \frac{t_0}{\alpha+1}$ we have that $\lambda_c(t) = \frac{\alpha+1}{t}$, which satisfies (H $_a$), moreover, $\lambda(t) = \frac{\alpha+1}{\alpha} \gamma(t)$, so (H $_b$) is also satisfied.

- Let $r \in]0, 1[$, $c = \frac{\int_0^{t_0} e^{\alpha s^{1-r}} ds}{e^{\alpha t_0^{1-r}}}$. Instead of proving $\gamma(t) \leq \lambda_c(t)$, we will prove the equivalent inequality $\frac{1}{t \lambda_c(t)} \leq \frac{1}{t \gamma(t)}$. In fact, by a change of variable we have that (see notation of I_p in Lemma 2.6.9):

$$\frac{1}{\lambda_c(t)t} = \frac{(1-r)^{\frac{r}{1-r}}}{\alpha^{\frac{1}{1-r}}} I_{\frac{r}{1-r}} \left(\frac{\alpha}{1-r} t^{1-r} \right),$$

Moreover, by the first result of Lemma 2.6.9 we have that

$$\frac{1}{t\lambda_c(t)} \leq \left(\frac{1-r}{\alpha}\right)^{\frac{1}{1-r}} \frac{t^{r-1}}{\alpha} \leq \frac{1}{t\gamma(t)},$$

where the last inequality comes from the fact that $1-r \leq \alpha$. Moreover, by the second result of Lemma 2.6.9, we obtain that:

$$\left(\frac{\alpha}{1-r}\right)^{\frac{1}{1-r}} \frac{1}{t\lambda_c(t)} \sim \frac{1}{t\gamma(t)}, \quad \text{as } t \rightarrow +\infty.$$

This implies that for every $\varepsilon \in]0, 1[$ there exists $\hat{t} > t_0, \Lambda_\varepsilon \geq 1$ such that $\lambda_c(t) \leq \Lambda_\varepsilon \gamma(t)$ for every $t > \hat{t}$ $\left(\Lambda_\varepsilon = \left(\frac{\alpha}{1-r}\right)^{\frac{1}{1-r}} \frac{1}{(1-\varepsilon)}\right)$, this implies (H_b) . □

Remark 6.3.4. Finding all (or at least a larger class of) continuous functions γ that satisfy (H_a) and for which one can prove (H_b) in general is an open problem.

6.3.2 Tighter convergence rates of the values

In order to illustrate the context of the following result, it is useful to mention that if $\gamma(t) = \frac{\alpha}{t}$, then Theorem 6.3.2 gives us minimization properties in the case $\alpha > 0$. However, it is widely known in the continuous deterministic setting (IGS_γ) with such γ and $\alpha > 3$, then the values converge at the rate $o(1/t^2)$ (see [18, 37]). Based on [18], we will depict that effect for a general γ in the continuous stochastic setting.

We will rephrase assumption (H_f) on the objective f to:

$$\begin{cases} f \text{ is convex and continuously differentiable with } L\text{-Lipschitz continuous gradient;} \\ f \in C^2(\mathbb{H}) \text{ or } \mathbb{H} \text{ is finite-dimensional;} \\ \mathcal{S} \stackrel{\text{def}}{=} \text{argmin}(f) \neq \emptyset. \end{cases} \quad (H_f^*)$$

(H_f^*) coincides with (H_f) in the infinite-dimensional case, but is weaker than (H_f) when \mathbb{H} is finite-dimensional.

Theorem 6.3.5. *Assume that f, σ and γ satisfy assumptions $(H_f^*), (H_\sigma)$ and (H_γ) - (H'_γ) , respectively. Let $\nu \geq 2$, and consider the dynamic $(IGS_\gamma - S)$ with initial data $X_0, V_0 \in L^\nu(\Omega; \mathbb{H})$ that is \mathcal{F}_{t_0} -measurable. Then, there exists a unique solution $(X, V) \in S_{\mathbb{H} \times \mathbb{H}}^\nu[t_0]$ of $(IGS_\gamma - S)$, for every $\nu \geq 2$. Additionally, if $\Gamma\sigma_\infty \in L^2([t_0, +\infty[)$, then:*

$$(i) \int_{t_2}^{+\infty} \Gamma(t)(f(X(t)) - \min f + \|V(t)\|^2)dt < +\infty \text{ a.s..}$$

$$(ii) f(X(t)) - \min f + \|V(t)\|^2 = o\left(\frac{1}{\Gamma^2(t)}\right) \text{ a.s..}$$

$$(iii) \mathbb{E}(f(X(t)) - \min f + \|V(t)\|^2) = \mathcal{O}\left(\frac{1}{\Gamma^2(t)}\right).$$

Moreover, assume that $\Gamma \notin L^1([t_0, +\infty[)$, and let $\theta(t) \stackrel{\text{def}}{=} \int_{t_0}^t \Gamma(s)ds$. If also $\theta\sigma_\infty^2 \in L^1([t_0, +\infty[)$, then:

$$(iv) \int_{t_2}^{+\infty} \gamma(t)\theta(t)\|V(t)\|^2 dt < +\infty \text{ a.s..}$$

$$(v) f(X(t)) - \min f + \|V(t)\|^2 = o\left(\frac{1}{\theta(t)}\right) \text{ a.s..}$$

$$(vi) \mathbb{E}(f(X(t)) - \min f + \|V(t)\|^2) = \mathcal{O}\left(\min\left\{\frac{\int_{t_0}^t \frac{ds}{\Gamma(s)}}{\theta(t)}, \frac{1}{\Gamma^2(t)}\right\}\right).$$

Remark 6.3.6. The claim (ii) is new even in the deterministic case (i.e. $\sigma(\cdot, \cdot) = 0_{\mathbb{K}; \mathbb{H}}$). According to the first three items of the previous theorem, the conclusions of Remark 6.2.10 are also valid, this is that when $\gamma(t) = \frac{\alpha}{t}$ with $\alpha > 3$ and $t\sigma_\infty(t) \in L^2([t_0, +\infty[)$, the previous theorem ensures fast

convergence of the values, *i.e.*, $\mathcal{O}(t^{-2})$ in expectation and $o(t^{-2})$ in almost sure sense. Besides, by Corollary 6.2.3, when $\gamma(t) = \frac{\alpha}{t^r}$ with $r \in]0, 1[$, $\alpha \geq 1 - r$, and $t^r \sigma_\infty(t) \in L^2([t_0, +\infty[)$, the previous theorem ensures convergence of the objective at a rate $\mathcal{O}(t^{-2r})$ in expectation and $o(t^{-2r})$ in almost sure sense, moreover by (v) the convergence rate is actually $o(t^{-(r+1)})$ in almost sure sense, which is faster than $o(t^{-2r})$. Regarding (v), this can be seen as the extension of [18, Theorem 3.6] to the stochastic setting.

Proof. (i) Let $m < \frac{3}{2}$ and t_2 defined in (\mathbf{H}'_γ) , let also $b \in]2(m-1), 1[$ and $x^* \in \mathcal{S}$. Based on $(\mathbf{S}_{a,b,c,d})$ with $\beta \equiv 0$, we introduce $\phi_1 : (t, x, v) \mapsto \mathbb{R}$ defined by

$$\phi_1(t, x, v) = \Gamma^2(t)(f(x) - \min f) + \frac{\|b(x - x^*) + \Gamma(t)v\|^2}{2} + \frac{b(1-b)}{2}\|x - x^*\|^2.$$

Since $f \in C^2(\mathbb{H})$, we use Itô's formula from t_2 to t to get

$$\begin{aligned} & \phi_1(t, X(t), V(t)) - \phi_1(t_2, X(t_2), V(t_2)) = \\ & \int_{t_2}^t \Gamma(s)[2\Gamma'(s)(f(X(s)) - \min f) - b\langle \nabla f(X(s)), X(s) - x^* \rangle] ds \\ & + (b-1) \int_{t_2}^t \Gamma(s)\|V(s)\|^2 ds + \int_{t_2}^t \Gamma^2(s)\text{tr}[\Sigma(s, X(s))] ds \\ & + \underbrace{\int_{t_2}^t \langle [\Gamma^2(s)V(s) + b\Gamma(s)(X(s) - x^*)] \sigma^*(s, X(s)), dW(s) \rangle}_{M_t}. \end{aligned} \quad (6.13)$$

When \mathbb{H} is finite-dimensional but f is not $C^2(\mathbb{H})$, we can use mollifiers as in [149, Proposition C.2], and get (6.13) as an inequality in this case.

Besides, we have that

$$\int_{t_2}^{+\infty} \Gamma^2(s)\text{tr}[\Sigma(s, X(s))] ds \leq \int_{t_2}^{+\infty} \Gamma^2(s)\sigma_\infty^2(s) ds < +\infty.$$

Besides $(M_t)_{t \geq t_2}$ is a continuous martingale. Moreover, by convexity of f , we have that

$$\begin{aligned} & \int_{t_2}^t \Gamma(s)[2\Gamma'(s)(f(X(s)) - \min f) - b\langle \nabla f(X(s)), X(s) - x^* \rangle] ds \\ & \leq \int_{t_2}^t \Gamma(s)(2\Gamma'(s) - b)(f(X(s)) - \min f) ds. \end{aligned}$$

Since $b-1 < 0$, and

$$2\Gamma'(t) - b = 2\gamma(t)\Gamma(t) - 2 - b \leq 2(m-1) - b < 0, \quad \forall t > t_2.$$

By Theorem 2.8.2,

$$\int_{t_2}^{+\infty} \Gamma(s)(f(X(s)) - \min f + \|V(s)\|^2) ds < +\infty \quad a.s., \quad (6.14)$$

and

$$\lim_{t \rightarrow +\infty} \Gamma^2(t)(f(X(t)) - \min f) + \frac{\|b(X(t) - x^*) + \Gamma(t)V(t)\|^2}{2} + \frac{b(1-b)}{2}\|X(t) - x^*\|^2 \text{ exists a.s.}$$

(ii) On the other hand, let $\phi_2 : (t, x, v) \mapsto \mathbb{R}$ defined by $\phi_2(t, x, v) = \Gamma^2(t) \left(f(x) - \min f + \frac{\|v\|^2}{2} \right)$.

Recalling the discussion for ϕ_1 , we get that by Itô's formula from t_2 to t , we have

$$\begin{aligned} \phi_2(t, X(t), V(t)) &= \phi_2(t_2, X(t_2), V(t_2)) + \int_{t_2}^t 2\Gamma(s)\Gamma'(s)(f(X(s)) - \min f)ds \\ &\quad - \int_{t_2}^t \Gamma(s)\|V(s)\|^2 ds + \int_{t_2}^t \Gamma^2(s)\text{tr}[\Sigma(s, X(s))]ds \\ &\quad + \underbrace{\int_{t_2}^t \Gamma^2(s)\langle V(s)\sigma^*(s, X(s)), dW(s) \rangle}_{M_t}. \end{aligned} \quad (6.15)$$

And also, that

$$\begin{aligned} &\int_{t_2}^{+\infty} 2\Gamma(s)\Gamma'(s)(f(X(s)) - \min f) + \Gamma^2(s)\text{tr}[\Sigma(s, X(s))]ds \\ &\leq \int_{t_2}^{+\infty} \Gamma(s)(f(X(s)) - \min f) + \Gamma^2(s)\sigma_\infty^2(s)ds < +\infty \quad a.s.. \end{aligned}$$

Besides $(M_t)_{t \geq t_2}$ is a continuous martingale. By Theorem 2.8.2, we get again that

$$\int_{t_2}^{+\infty} \Gamma(s)\|V(s)\|^2 ds < +\infty \quad a.s.,$$

and that

$$\lim_{t \rightarrow +\infty} \Gamma^2(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) \text{ exists a.s.} \quad (6.16)$$

Let us recall that $\frac{1}{\Gamma} \notin L^1([t_0, +\infty[)$ by Lemma 5.2.1. Therefore, by (6.14) and (6.16), we can use Lemma 2.6.2 to obtain that

$$\lim_{t \rightarrow +\infty} \Gamma^2(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) = 0 \quad a.s..$$

(iii) Taking expectation on (6.13) and denoting

$$\begin{aligned} K_1 &\stackrel{\text{def}}{=} \Gamma^2(t_2)\mathbb{E}(f(X(t_2)) - \min f) + \frac{1}{2}\mathbb{E}(\|b(X(t_2)) - x^* + \Gamma(t_2)V(t_2)\|^2) + \frac{b(1-b)}{2}\|X(t_2) - x^*\|^2, \\ K_\Gamma &\stackrel{\text{def}}{=} \int_{t_2}^{+\infty} \Gamma^2(s)\sigma_\infty^2(s)ds, \end{aligned}$$

we obtain directly that

$$\mathbb{E} \left(\Gamma^2(t)(f(X(t)) - \min f) + \frac{\|b(X(t)) - x^* + \Gamma(t)V(t)\|^2}{2} + \frac{b(1-b)}{2}\|X(t) - x^*\|^2 \right) \leq K_1 + K_\Gamma.$$

From this, is direct that $\sup_{t \geq t_2} \mathbb{E}(\|X(t) - x^*\|^2) < +\infty$, and this in turn imply

$$\mathbb{E} \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) = \mathcal{O} \left(\frac{1}{\Gamma^2(t)} \right).$$

(iv) Moreover, assume that $\Gamma \notin L^1([t_0, +\infty[)$, and let $\theta(t) = \int_{t_0}^t \Gamma(s)ds$. If also $\theta\sigma_\infty^2 \in L^1([t_0, +\infty[)$, then we consider $\phi_3(t, x, v) = \theta(t) \left(f(x) - \min f + \frac{\|v\|^2}{2} \right)$, by Itô's formula from t_2 to t , we get

$$\begin{aligned} \phi_3(t, X(t), V(t)) &= \phi_3(t_2, X(t_2), V(t_2)) + \int_{t_2}^t \Gamma(s) \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) ds \\ &\quad - \int_{t_2}^t \gamma(s)\theta(s)\|V(s)\|^2 ds + \frac{1}{2} \int_{t_2}^t \theta(s)\text{tr}[\Sigma(s, X(s))]ds \\ &\quad + \underbrace{\int_{t_2}^t \theta(s)\langle V(s)\sigma^*(s, X(s)), dW(s) \rangle}_{M_t}. \end{aligned} \quad (6.17)$$

Also, by the first item and the new hypothesis on the diffusion term, we get that

$$\begin{aligned} & \int_{t_2}^t \Gamma(s) \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) + \theta(s) \text{tr}[\Sigma(s, X(s))] ds \\ & \leq \int_{t_2}^{+\infty} \Gamma(s) \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) + \theta(s) \sigma_\infty^2(s) ds < +\infty. \end{aligned} \quad (6.18)$$

Besides $(M_t)_{t \geq t_2}$ is a continuous martingale. By Theorem 2.8.2, we get that

$$\int_{t_2}^{+\infty} \gamma(s) \theta(s) \|V(s)\|^2 ds < +\infty \quad a.s..$$

(v) By the previous, we also get that

$$\lim_{t \rightarrow +\infty} \theta(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) \text{ exists a.s.}, \quad (6.19)$$

Using Lemma 2.6.3 with $q(t) = \theta(t)$, we get that $\frac{\Gamma}{\theta} \notin L^1([t_2, +\infty[)$. Besides, recalling that

$$\int_{t_2}^{+\infty} \Gamma(s) \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) < +\infty, \quad a.s.,$$

we invoke Lemma 2.6.2 to conclude that $\lim_{t \rightarrow +\infty} \theta(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) = 0$ a.s..

(vi) Taking expectation in (6.17) and upper bounding we get

$$\begin{aligned} \mathbb{E}(\phi_3(t, X(t), V(t))) & \leq \mathbb{E}(\phi_3(t_2, X(t_2), V(t_2))) + \int_{t_2}^t \Gamma(s) \mathbb{E} \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) ds \\ & \quad + \frac{1}{2} \int_{t_2}^{+\infty} \theta(s) \sigma_\infty^2(s) ds. \end{aligned} \quad (6.20)$$

By the third item, we have that $\mathbb{E} \left(f(X(s)) - \min f + \frac{\|V(s)\|^2}{2} \right) = \mathcal{O} \left(\frac{1}{\Gamma^2(s)} \right)$, so we conclude that

$$\mathbb{E} \left(\theta(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) \right) = \mathcal{O} \left(\int_{t_2}^t \frac{ds}{\Gamma(s)} \right). \quad (6.21)$$

Thus,

$$\mathbb{E} \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right) = \mathcal{O} \left(\min \left\{ \frac{\int_{t_2}^t \frac{ds}{\Gamma(s)}}{\theta(t)}, \frac{1}{\Gamma^2(t)} \right\} \right).$$

□

6.3.3 Almost sure weak convergence of trajectories

In the deterministic setting with $\alpha > 3$, it is also well-known that one can obtain weak convergence of the trajectory. Our aim in this section is to show this claim for a general γ in the stochastic setting.

Theorem 6.3.7. *Consider the setting of Theorem 6.3.5. Then, if $\Gamma \sigma_\infty \in L^2([t_0, +\infty[)$ we have that:*

(i) $\mathbb{E} [\sup_{t \geq t_2} \|X(t)\|^\nu] < +\infty$.

(ii) $\forall x^* \in \mathcal{S}$, $\lim_{t \rightarrow +\infty} \|X(t) - x^*\|$ exists a.s..

(iii) If γ is nonincreasing, there exists an \mathcal{S} -valued random variable X^* such that $w\text{-}\lim_{t \rightarrow +\infty} X(t) = X^*$ a.s..

Proof. (i) Analogous to the proof of the first point of Theorem 3.2.3.

(ii) Recalling the proof of Theorem 6.3.5, we combine the fact that both

$$\lim_{t \rightarrow +\infty} \Gamma^2(t) (f(X(t)) - \min f) + \frac{\|b(X(t) - x^*) + \Gamma(t)V(t)\|^2}{2} + \frac{b(1-b)}{2} \|X(t) - x^*\|^2,$$

and

$$\lim_{t \rightarrow +\infty} \Gamma^2(t) \left(f(X(t)) - \min f + \frac{\|V(t)\|^2}{2} \right)$$

exist a.s.. We can subtract both quantities to obtain that

$$\lim_{t \rightarrow +\infty} \frac{\|X(t) - x^*\|^2}{2} + \Gamma(t) \langle X(t) - x^*, V(t) \rangle \text{ exists a.s..}$$

Thus, for every $x^* \in \mathcal{S}$ there exists $\Omega_{x^*} \in \mathcal{F}$ with $\mathbb{P}(\Omega_{x^*}) = 1$ and $\exists \ell : \Omega_{x^*} \mapsto \mathbb{R}$ such that

$$\lim_{t \rightarrow +\infty} \frac{\|X(\omega, t) - x^*\|^2}{2} + \Gamma(t) \langle V(\omega, t), X(\omega, t) - x^* \rangle = \ell(\omega).$$

Let $Z(\omega, t) = \frac{\|X(\omega, t) - x^*\|^2}{2} - \ell(\omega)$ and $\varepsilon > 0$ arbitrary. There exists $T(\omega) \geq t_0$ such that $\forall t \geq T(\omega)$

$$\left\| Z(\omega, t) + \Gamma(t) \langle V(\omega, t), X(\omega, t) - x^* \rangle \right\| < \varepsilon.$$

Let $g(t) \stackrel{\text{def}}{=} \exp\left(\int_{t_2}^t \frac{ds}{\Gamma(s)}\right)$, multiplying the previous inequality by $\frac{g(t)}{\Gamma(t)}$, there exists $T(\omega) \geq t_0$ such that for every $t \geq T(\omega)$:

$$\left\| \frac{g(t)}{\Gamma(t)} Z(t) + g(t) \langle V(\omega, t), X(\omega, t) - x^* \rangle \right\| < \frac{\varepsilon}{\Gamma(t)} g(t).$$

On the other hand, $dZ(t) = \langle V(t), X(t) - x^* \rangle dt$ and

$$d\left(\frac{g(t)}{\Gamma(t)} Z(t)\right) = \left(\frac{g(t)}{\Gamma(t)} Z(t) + g(t) \langle V(t), X(t) - x^* \rangle\right) dt.$$

Thus,

$$\begin{aligned} \|g(t)Z(t) - g(T)z(T)\| &= \left\| \int_T^t d(g(s)Z(s)) \right\| = \left\| \int_T^t \left(\frac{g(s)}{\Gamma(s)} Z(s) + g(s) \langle V(s), X(s) - x^* \rangle \right) ds \right\| \\ &\leq \varepsilon \int_T^t \frac{g(s)}{\Gamma(s)} ds = \varepsilon(g(t) - g(T)). \end{aligned}$$

So,

$$\|Z(t)\| \leq \frac{g(T)}{g(t)} \|z(T)\| + \varepsilon.$$

By Lemma 5.2.1, we obtain that $\lim_{t \rightarrow +\infty} g(t) = +\infty$. Hence, $\limsup_{t \rightarrow +\infty} \|Z(t)\| \leq \varepsilon$. And we conclude that for every $x^* \in \mathcal{S}$, $\lim_{t \rightarrow +\infty} \frac{\|X(t) - x^*\|}{2}$ exists a.s.. By a separability argument (see proof of Theorem 3.2.3 or Theorem 4.3.2) there exists $\tilde{\Omega} \in \mathcal{F}$ (independent of x^*) such that $\mathbb{P}(\tilde{\Omega}) = 1$ and $\lim_{t \rightarrow +\infty} \frac{\|X(\omega, t) - x^*\|}{2}$ exists for every $\omega \in \tilde{\Omega}, x^* \in \mathcal{S}$.

(iii) If γ is nonincreasing, then Γ is nondecreasing (see [18, Corollary 2.3]). Then, by item (ii) of Theorem 6.3.5, we have that:

$$\lim_{t \rightarrow +\infty} f(X(t)) = \min f \quad \text{a.s..}$$

Let $\Omega_f \in \mathcal{F}$ be the set of events on which this limit is satisfied. Thus $\mathbb{P}(\Omega_f) = 1$. Set $\Omega_{\text{conv}} \stackrel{\text{def}}{=} \Omega_f \cap \tilde{\Omega}$. We have $\mathbb{P}(\Omega_{\text{conv}}) = 1$. Now, let $\omega \in \Omega_{\text{conv}}$ and $\tilde{X}(\omega)$ be a weak sequential cluster point of $X(\omega, t)$ (which exists by boundedness). Equivalently, there exists an increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow +\infty} t_k = +\infty$ and

$$\text{w-lim}_{k \rightarrow +\infty} X(\omega, t_k) = \tilde{X}(\omega).$$

Since $\lim_{t \rightarrow +\infty} f(X(\omega, t)) = \min f$ and the fact that f is weakly lower semicontinuous (since it is convex and continuous), we obtain directly that $\tilde{X}(\omega) \in \mathcal{S}$. Finally by Opial's Lemma (see [162]) we conclude that there exists $X^*(\omega) \in \mathcal{S}$ such that $\text{w-lim}_{t \rightarrow +\infty} X(\omega, t) = X^*(\omega)$. In other words, since $\omega \in \Omega_{\text{conv}}$ was arbitrary, there exists an \mathcal{S} -valued random variable X^* such that $\text{w-lim}_{t \rightarrow +\infty} X(t) = X^*$ a.s..

□

Chapter 7

Second-Order Systems with Hessian-driven Damping for Non-Convex Optimization in Finite Dimension

In this chapter, we aim to study non-convex minimization problems via second-order (in-time) dynamics, including a non-vanishing viscous damping and a geometric Hessian-driven damping. Second-order systems that only rely on a viscous damping may suffer from oscillation problems towards the minima, while the inclusion of a Hessian-driven damping term is known to reduce this effect without explicit construction of the Hessian in practice. There are essentially two ways to introduce the Hessian-driven damping term: explicitly or implicitly. For each setting, we provide conditions on the damping coefficients to ensure convergence of the gradient towards zero. Moreover, if the objective function is definable, we show global convergence of the trajectory towards a critical point as well as convergence rates. Besides, in the autonomous case, if the objective function is Morse, we conclude that the trajectory converges to a local minimum of the objective for almost all initializations. We also study algorithmic schemes for both dynamics and prove all the previous properties in the discrete setting under proper choice of the stepsize.

Main contributions of this chapter

- ▶ Global convergence of the trajectory generated by (ISEHD) and (SIHD) with a non-vanishing viscous damping and constant geometric damping to a critical point under KL-type conditions over the objective (Theorem 7.2.1-7.3.1).
- ▶ Moreover, global convergence of the trajectory to a local minimum for Morse-type functions (Theorem 7.2.3-7.3.2).
- ▶ Global convergence rates of the trajectory under KL-type conditions over the Lyapunov function (Theorem 7.2.4-7.3.3).
- ▶ Analogous results for the proposed algorithmic schemes (ISEHD-Disc), (SIHD-Disc) (Subsection 7.2.2-7.3.2).

The content of this chapter is submitted in [148].

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7.1 Introduction

7.1.1 Problem statement

Let us consider the minimization problem

$$\min_{x \in \mathbb{R}^d} f(x), \quad (\mathcal{P}_3)$$

where the objective function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the following standing assumptions:

$$\begin{cases} f \in C^2(\mathbb{R}^d); \\ \inf f > -\infty. \end{cases} \quad (\mathbf{H}_0)$$

Since the objective function is potentially non-convex, the problem (\mathcal{P}_3) is NP-Hard. We refer to the introduction for a general discussion about second-order systems and to Section 1.2.5 for a detailed discussion on this problem. As discussed there, an important dynamic to consider in this regard is the Inertial System with Explicit Hessian Damping:

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \beta(t)\nabla^2 f(x(t))\dot{x}(t) + \nabla f(x(t)) = 0, & t > t_0; \\ x(t_0) = x_0; \quad \dot{x}(t_0) = v_0. \end{cases} \quad (\text{ISEHD})$$

Where $\gamma, \beta : [t_0, +\infty[\rightarrow \mathbb{R}_+$ are the viscous and geometric damping parameters, respectively.

And the Inertial System with Implicit Hessian Damping:

$$\begin{cases} \ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla f(x(t)) + \beta(t)\dot{x}(t) = 0, & t > t_0; \\ x(t_0) = x_0; \quad \dot{x}(t_0) = v_0. \end{cases} \quad (\text{ISIHD})$$

It was discussed in the introduction that following the physical interpretation of these two ODEs, we call the non-negative parameters γ and β the viscous and geometric damping coefficients, respectively. The two ODEs (ISEHD) and (ISIHD) were found to have a smoothing effect on the energy error and oscillations [3, 152, 23]. Moreover, in [33] they obtain fast convergence rates for the values for the two ODEs when $\gamma(t) = \frac{\alpha}{t}$ ($\alpha > 3$) and $\beta(t) = \beta > 0$.

However, the previous results (including the ones discussed in the introduction) are exclusive to the convex case. Nevertheless, in [76], the authors analyzes (ISEHD) with constant viscous and geometric damping in a non-convex setting, concluding that the bounded solution trajectories converges to

a critical point of the objective under Łojasiewicz inequality and giving sublinear convergence rates. Moreover, in [74] the author shows that the previous dynamic avoids strict saddle points when the objective is Morse. In these two works they propose a discretization called INNA, where the conditions on the stepsize are more stringent than ours but let them consider arbitrary values for the geometric damping. To the best of our knowledge, there is no further analysis of (ISEHD) and (SIHD) when the objective is non-convex and definable. In this work, our goal is to fill this gap.

In this chapter we will establish convergence of the trajectory to a critical point for (ISEHD) and (SIHD) with variable, non-vanishing viscous damping and constant geometric damping, as well as for the proposed algorithms ((ISEHD-Disc), (SIHD-Disc)). Moreover, in the autonomous case we will conclude with the avoidance of strict saddle points. We would like to point out that the proof for the continuous-time case will necessitate (as in [74]) a more stringent assumption on the class of functions, for instance, that f is Morse, while this is not necessary for the discrete algorithms.

7.1.2 Preliminaries

The following lemma is essential to some arguments presented in this work:

Lemma 7.1.1. [188, Theorem 3] Consider $A, B, C, D \in \mathbb{R}^{d \times d}$ and such that $AC = CA$. Then

$$\det \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \det(AD - CB).$$

And the following definitions will be important throughout the chapter:

Definition 7.1.2 (Local extrema and saddle points). Consider a function $f \in C^2(\mathbb{R}^d)$. We will say that \hat{x} is a local minimum (resp. maximum) of f if $\hat{x} \in \text{crit}(f)$, $\nabla^2 f(\hat{x})$ is positive (resp. negative) definite. If \hat{x} is a critical point that is neither a local minimum nor a local maximum, we will say that \hat{x} is a saddle point of f .

Definition 7.1.3 (Strict saddle point). Consider a function $f \in C^2(\mathbb{R}^d)$, we will say that \hat{x} is a strict saddle point of f if $\hat{x} \in \text{crit}(f)$ and $\lambda_{\min}(\nabla^2 f(\hat{x})) < 0$.

Remark 7.1.4. According to this definition, local maximum points are strict saddles.

Definition 7.1.5 (Strict saddle property). A function $f \in C^2(\mathbb{R}^d)$ will satisfy the strict saddle property if every critical point is either a local minimum or a strict saddle.

This property is a reasonable assumption for smooth minimization. In practice, it holds for specific problems of interest, such as low-rank matrix recovery and phase retrieval. Moreover, as a consequence of Sard's theorem, for a full measure set of linear perturbations of a function f , the linearly perturbed function satisfies the strict saddle property. Consequently, in this sense, the strict saddle property holds generically in smooth optimization. We also refer to the discussion in [128, Conclusion].

Definition 7.1.6 (Morse function). A function $f \in C^2(\mathbb{R}^d)$ will be Morse if it satisfies the following conditions:

- (i) For each critical point \hat{x} , $\nabla^2 f(\hat{x})$ is non-singular.
- (ii) There exists a nonempty set $I \subseteq \mathbb{N}$ and $(\hat{x}_k)_{k \in I}$ such that $\text{crit}(f) = \bigcup_{k \in I} \{\hat{x}_k\}$.

Remark 7.1.7. By definition, a Morse function satisfies the strict saddle property.

Remark 7.1.8. Morse functions can be shown to be generic in the Baire sense in the space of C^2 functions; see [41].

We refer to Subsection 2.4 for the introduction of KL and Łojasiewicz inequality.

Also, we recall Remark 2.4.2, *i.e.*, the concavity of the desingularizing functions is only required in the discrete setting (Subsection 7.2.2-7.3.2).

Remark 7.1.9. Morse functions verify the Łojasiewicz inequality with exponent $q = 1/2$; see [15].

7.2 Inertial System with Explicit Hessian Damping

Throughout the chapter we will consider (ISEHD) and (SIHD) with $t_0 = 0$. Also we will assume that the viscous damping $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous and is such that $\exists c, C > 0$, $c \leq C$, and

$$c \leq \gamma(t) \leq C, \quad \forall t \geq 0. \quad (\mathbf{H}_\gamma)$$

Moreover, throughout this work, we consider a constant geometric damping, *i.e.* $\beta(t) \equiv \beta > 0$.

7.2.1 Continuous-time dynamics

Let us consider (ISEHD), as in [33], we will say that $x : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a solution trajectory of (ISEHD) with initial conditions $x(0) = x_0, \dot{x}(0) = v_0$, if and only if, $x \in C^2(\mathbb{R}_+; \mathbb{R}^d)$ and there exists $y \in C^1(\mathbb{R}_+; \mathbb{R}^d)$ such that (x, y) satisfies:

$$\begin{cases} \dot{x}(t) + \beta \nabla f(x(t)) - \left(\frac{1}{\beta} - \gamma(t)\right) x(t) + \frac{1}{\beta} y(t) = 0, \\ \dot{y}(t) - \left(\frac{1}{\beta} - \gamma(t) - \beta \gamma'(t)\right) x(t) + \frac{1}{\beta} y(t) = 0, \end{cases} \quad (7.1)$$

with initial conditions $x(0) = x_0, y(0) = y_0 \stackrel{\text{def}}{=} -\beta(v_0 + \beta \nabla f(x_0)) + (1 - \beta \gamma(0))x_0$.

7.2.1.1 Global convergence of the trajectory

Our first main result is the following theorem.

Theorem 7.2.1. *Assume that $0 < \beta < \frac{2c}{C^2}$, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies (\mathbf{H}_0) , and $\gamma \in C^1(\mathbb{R}_+; \mathbb{R}_+)$ obeys (\mathbf{H}_γ) .*

Consider (ISEHD) in this setting, then the following holds:

(i) There exists a global solution trajectory $x : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ of (ISEHD).

(ii) We have that $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) If we suppose that the solution trajectory x is bounded over \mathbb{R}_+ , then

$$\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = \lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0,$$

and $\lim_{t \rightarrow +\infty} f(x(t))$ exists.

(iv) In addition to (iii), if we also assume that f is definable, then $\dot{x} \in L^1(\mathbb{R}_+; \mathbb{R}^d)$ and $x(t)$ converges (as $t \rightarrow +\infty$) to a critical point of f .

Remark 7.2.2. The boundedness assumption in assertion (ii) can be dropped if ∇f is supposed to be globally Lipschitz continuous.

Proof. (i) We will start by showing the existence of a solution. Setting $Z = (x, y)$, then (7.1) can be equivalently written as

$$\dot{Z}(t) + \nabla \mathcal{G}(Z(t)) + \mathcal{D}(t, Z(t)) = 0, \quad Z(0) = (x_0, y_0), \quad (7.2)$$

where $\mathcal{G}(Z) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the function defined by $\mathcal{G}(Z) = \beta f(x)$ and the time-dependent operator $\mathcal{D} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is given by:

$$\mathcal{D}(t, Z) \stackrel{\text{def}}{=} \left(- \left(\frac{1}{\beta} - \gamma(t) \right) x + \frac{1}{\beta} y, - \left(\frac{1}{\beta} - \gamma(t) - \beta \gamma'(t) \right) x + \frac{1}{\beta} y \right).$$

Since the map $(t, Z) \mapsto \nabla \mathcal{G}(Z) + \mathcal{D}(t, Z)$ is continuous in the first variable and locally Lipschitz in the second (by hypothesis **(H₀)** and the assumptions on γ), by the classical Cauchy-Lipschitz theorem, we have that there exists $T_{\max} > 0$ and a unique maximal solution of (7.2) denoted $Z \in C^1([0, T_{\max}[; \mathbb{R}^d \times \mathbb{R}^d)$. Consequently, there exists a unique maximal solution of **(ISEHD)** $x \in C^2([0, T_{\max}[; \mathbb{R}^d)$.

Let us consider the energy function $V : [0, T_{\max}[\rightarrow \mathbb{R}$ defined by

$$V(t) = f(x(t)) + \frac{1}{2} \|\dot{x}(t) + \beta \nabla f(x(t))\|^2.$$

We will prove that it is indeed a Lyapunov function for **(ISEHD)**. We see that

$$\begin{aligned} V'(t) &= \langle \nabla f(x(t)), \dot{x}(t) \rangle + \langle \ddot{x}(t) + \beta \frac{d}{dt} \nabla f(x(t)), \dot{x}(t) + \beta \nabla f(x(t)) \rangle \\ &= \langle \nabla f(x(t)), \dot{x}(t) \rangle + \langle -\gamma(t) \dot{x}(t) - \nabla f(x(t)), \dot{x}(t) + \beta \nabla f(x(t)) \rangle \\ &= -\langle \gamma(t) \dot{x}(t), \dot{x}(t) \rangle - \beta \langle \gamma(t) \dot{x}(t), \nabla f(x(t)) \rangle - \beta \|\nabla f(x(t))\|^2 \\ &\leq -c \|\dot{x}(t)\|^2 + \frac{\beta^2 \|\gamma(t) \dot{x}(t)\|^2}{2\varepsilon} + \frac{\varepsilon \|\nabla f(x(t))\|^2}{2} \\ &\leq -c \|\dot{x}(t)\|^2 + \frac{\beta^2 C^2 \|\dot{x}(t)\|^2}{2\varepsilon} + \frac{\varepsilon \|\nabla f(x(t))\|^2}{2}, \end{aligned}$$

where the last bound is due to Young's inequality with $\varepsilon > 0$. Now let $\varepsilon = \frac{\beta^2 C^2}{c}$, then

$$V'(t) \leq -\frac{c}{2} \|\dot{x}(t)\|^2 - \beta \left(1 - \frac{\beta C^2}{2c} \right) \|\nabla f(x(t))\|^2 - \delta_1 (\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2). \quad (7.3)$$

For $\delta_1 \stackrel{\text{def}}{=} \min \left(\frac{c}{2}, \beta \left(1 - \frac{\beta C^2}{2c} \right) \right) > 0$.

We will now show that the maximal solution Z of (7.2) is actually global. For this, we use a standard argument and argue by contradiction assuming that $T_{\max} < +\infty$. It is sufficient to prove that x and y have a limit as $t \rightarrow T_{\max}$, and local existence will contradict the maximality of T_{\max} . Integrating inequality (7.3), we obtain $\dot{x} \in L^2([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ x \in L^2([0, T_{\max}[; \mathbb{R}^d)$, hence implying that $\dot{x} \in L^1([0, T_{\max}[; \mathbb{R}^d)$ and $\nabla f \circ x \in L^1([0, T_{\max}[; \mathbb{R}^d)$, which in turn entails that $(x(t))_{t \in [0, T_{\max}[}$ satisfies the Cauchy property whence we get that $\lim_{t \rightarrow T_{\max}} x(t)$ exists. Besides, by the first equation of (7.1), we have that $\lim_{t \rightarrow T_{\max}} y(t)$ exists if $\lim_{t \rightarrow T_{\max}} x(t)$, $\lim_{t \rightarrow T_{\max}} \nabla f(x(t))$ and $\lim_{t \rightarrow T_{\max}} \dot{x}(t)$ exist. We have already that the first two limits exist by continuity of ∇f , and thus we just have to check that $\lim_{t \rightarrow T_{\max}} \dot{x}(t)$ exists. A sufficient condition would be to prove that $\ddot{x} \in L^1([0, T_{\max}[; \mathbb{R}^d)$. By **(ISEHD)** this will hold if \dot{x} , $\nabla f \circ x$, $(\nabla^2 f \circ x) \dot{x}$ are in $L^1([0, T_{\max}[; \mathbb{R}^d)$. We have already checked that the first two terms are in $L^1([0, T_{\max}[; \mathbb{R}^d)$. To conclude, it remains to check that $\nabla^2 f \circ x \in L^\infty([0, T_{\max}[; \mathbb{R}^d)$ and this is true since $\nabla^2 f$ is continuous, x is continuous on $[0, T_{\max}]$, and the latter is compact. Consequently, the solution Z of (7.2) is global, thus the solution x of **(ISEHD)** is also global.

- (ii) Integrating (7.3) and using that V is well-defined for every $t > 0$ and is bounded from below, we deduce that $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$.
- (iii) We recall that we are assuming that $\sup_{t \geq 0} \|x(t)\| < +\infty$ and $f \in C^2(\mathbb{R}^d)$, hence

$$\sup_{t \geq 0} \|\nabla^2 f(x(t))\| < +\infty.$$

In turn, ∇f is Lipschitz continuous on bounded sets. Moreover, as $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ and is continuous, then $\dot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$. The last two facts imply that $t \mapsto \nabla f(x(t))$ is uniformly continuous. In fact, for every $t, s \geq 0$, we have

$$\|\nabla f(x(t)) - \nabla f(x(s))\| \leq \sup_{\tau \geq 0} \|\nabla^2 f(x(\tau))\| \|\dot{x}(\tau)\| |t - s|.$$

This combined with $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ yields

$$\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = 0.$$

We also have that $\frac{d}{dt} \nabla f(x(t)) = \nabla^2 f(x(t)) \dot{x}(t)$, and thus $(\nabla^2 f \circ x) \dot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$. We also have $\nabla f \circ x \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$ by continuity of ∇f and boundedness of x . It then follows from (ISEHD) that $\ddot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$. This implies that

$$\|\dot{x}(t) - \dot{x}(s)\| \leq \sup_{\tau \geq 0} \|\ddot{x}(\tau)\| |t - s|,$$

meaning that $t \mapsto \dot{x}(t)$ is uniformly continuous. Recalling that $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ gives that

$$\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0.$$

We have from (7.3) that V is non-increasing. Since it is bounded from below, it has a limit, *i.e.* $\lim_{t \rightarrow +\infty} V(t)$ exists and we will denote this limit by \tilde{L} . Recall from the definition of V that

$$f(x(t)) = V(t) - \frac{1}{2} \|\dot{x}(t) + \beta \nabla f(x(t))\|^2.$$

Using the above three limits we get

$$\lim_{t \rightarrow +\infty} f(x(t)) = \lim_{t \rightarrow +\infty} V(t) = \tilde{L}.$$

- (iv) From boundedness of $(x(t))_{t \geq 0}$, by a Lyapunov argument (see *e.g.* [34, Proposition 4.1], [109]), the set of its cluster points $\mathfrak{C}(x(\cdot))$ satisfies:

$$\begin{cases} \mathfrak{C}(x(\cdot)) \subseteq \text{crit}(f); \\ \mathfrak{C}(x(\cdot)) \text{ is non-empty, compact and connected}; \\ f \text{ is constant on } \mathfrak{C}(x(\cdot)). \end{cases} \quad (7.4)$$

We consider the function

$$E : (x, v, w) \in \mathbb{R}^{3d} \mapsto f(x) + \frac{1}{2} \|v + w\|^2. \quad (7.5)$$

Since f is definable, so is E as the sum of a definable function and an algebraic one. Therefore, E satisfies the KL inequality [125]. Let $\mathfrak{C}_1 = \mathfrak{C}(x(\cdot)) \times \{0_d\} \times \{0_d\}$. Observe that E takes the constant value \tilde{L} on \mathfrak{C}_1 and $\mathfrak{C}_1 \subset \text{crit}(E)$. It then follows from the uniformized KL property [60, Lemma 6] that $\exists r, \eta > 0$ and $\exists \psi \in \kappa(0, \eta)$ such that for all $(x, v, w) \in \mathbb{R}^{3d}$ verifying $x \in \mathfrak{C}(x(\cdot)) + B_r, v \in B_r, w \in B_r$ (where B_r is the \mathbb{R}^d -ball centered at 0_d with radius r) and $0 < E(x, v, w) - \tilde{L} < \eta$, one has

$$\psi'(E(x, v, w) - \tilde{L}) \|\nabla E(x, v, w)\| \geq 1. \quad (7.6)$$

It is clear that $V(t) = E(x(t), \dot{x}(t), \beta \nabla f(x(t)))$, and that $x^* \in \text{crit}(f)$ if and only if $(x^*, 0, 0) \in \text{crit}(E)$.

Let us define the translated Lyapunov function $\tilde{V}(t) = V(t) - \tilde{L}$. By the properties of V proved above, we have $\lim_{t \rightarrow +\infty} \tilde{V}(t) = 0$ and \tilde{V} is non-increasing, and we can conclude that $\tilde{V}(t) \geq 0$ for every $t > 0$. Without loss of generality, we may assume that $\tilde{V}(t) > 0$ for every $t > 0$ (since

otherwise $\tilde{V}(t)$ is eventually zero and thus $\dot{x}(t)$ is eventually zero in view of (7.3), meaning that $x(\cdot)$ has finite length). This in turn implies that $\lim_{t \rightarrow +\infty} \psi(\tilde{V}(t)) = 0$. Define the constants $\delta_2 = \max(4, 1 + 4\beta^2)$ and $\delta_3 = \frac{\delta_1}{\sqrt{\delta_2}}$. We have from (7.4) that $\lim_{t \rightarrow +\infty} \text{dist}(x(t), \mathfrak{C}(x(\cdot))) = 0$. This together with the convergence claims on \dot{x} , $\nabla f(x)$ and \tilde{V} imply that there exists $T > 0$ large enough such that for all $t \geq T$:

$$\begin{cases} x(t) \in \mathfrak{C}(x(\cdot)) + B_r, \\ \|\dot{x}(t)\| < r, \\ \beta \|\nabla f(x(t))\| < r, \\ 0 < \tilde{V}(t) < \eta, \\ \frac{1}{\delta_3} \psi(\tilde{V}(t)) < \frac{r}{2\sqrt{2}}. \end{cases} \quad (7.7)$$

We are now in position to apply (7.6) to obtain

$$\psi'(\tilde{V}(t)) \|\nabla E(x(t), \dot{x}(t), \beta \nabla f(x(t)))\| \geq 1, \quad \forall t \geq T. \quad (7.8)$$

On the other hand, for every $t \geq T$:

$$-\frac{d}{dt} \psi(\tilde{V}(t)) = \psi'(\tilde{V}(t))(-\tilde{V}'(t)) \geq -\frac{\tilde{V}'(t)}{\|\nabla E(x(t), \dot{x}(t), \beta \nabla f(x(t)))\|}. \quad (7.9)$$

Additionally, for every $t > 0$ we have the bounds

$$\begin{aligned} -\tilde{V}'(t) &\geq \delta_1 (\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2), \\ \|\nabla E(x(t), \dot{x}(t), \beta \nabla f(x(t)))\|^2 &\leq \delta_2 (\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2). \end{aligned} \quad (7.10)$$

Combining the two previous bounds, then for every $t > 0$:

$$\|\nabla E(x(t), \dot{x}(t), \beta \nabla f(x(t)))\| \leq \sqrt{\frac{\delta_2}{\delta_1}} \sqrt{-\tilde{V}'(t)}. \quad (7.11)$$

By (7.9), for every $t \in [T, +\infty[$

$$-\frac{d}{dt} \psi(\tilde{V}(t)) \geq \sqrt{\frac{\delta_1}{\delta_2}} \frac{-\tilde{V}'(t)}{\sqrt{-\tilde{V}'(t)}} = \sqrt{\frac{\delta_1}{\delta_2}} \sqrt{-\tilde{V}'(t)} \geq \delta_3 \sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2}. \quad (7.12)$$

Integrating from T to $+\infty$, we obtain

$$\int_T^{+\infty} \sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2} dt \leq \frac{1}{\delta_3} \psi(V(T)) < \frac{r}{2\sqrt{2}}. \quad (7.13)$$

Thus

$$\int_T^{+\infty} \|\dot{x}(t)\| dt \leq \int_T^{+\infty} \sqrt{\|\dot{x}(t)\|^2 + \|\nabla f(x(t))\|^2} dt < \frac{r}{2\sqrt{2}},$$

this implies that $\dot{x} \in L^1(\mathbb{R}_+; \mathbb{R}^d)$. Therefore $x(t)$ has the Cauchy property and this in turn implies that $\lim_{t \rightarrow +\infty} x(t)$ exists, and is a critical point of f since $\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = 0$. \square

7.2.1.2 Trap avoidance

In the previous section, we have seen the convergence of the trajectory to a critical point of the objective, which includes strict saddle points. We will call trap avoidance the effect of avoiding such points at the limit. If the objective function satisfies the strict saddle property (recall Definition 7.1.5) as is the case for a Morse function, this would imply convergence to a local minimum of the objective. The following theorem gives conditions to obtain such an effect.

Theorem 7.2.3. *Let $c > 0$, assume that $0 < \beta < \frac{2}{c}$ and take $\gamma \equiv c$. Suppose that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies (\mathbf{H}_0) and is a Morse function. Consider (ISEHD) in this setting. If the solution trajectory x is bounded over \mathbb{R}_+ , then the conclusions of Theorem 7.2.1 hold. If, moreover, $\beta \neq \frac{1}{c}$, then for Lebesgue almost all initial conditions $x_0, v_0 \in \mathbb{R}^d$, $x(t)$ converges (as $t \rightarrow +\infty$) to a local minimum of f .*

Proof. Since Morse functions are C^2 and satisfy the KL inequality (see Remark 7.1.9), then all the claims of Theorem 7.2.1, and in particular (iv)¹, hold.

As in [105, Theorem 4], we will use the global stable manifold theorem [172, page 223] to get the last claim. We recall that (ISEHD) is equivalent to (7.1), and that we are in the case $\gamma(t) = c$ for all t , i. e.,

$$\begin{cases} \dot{x}(t) + \beta \nabla f(x(t)) - \left(\frac{1}{\beta} - c\right) x(t) + \frac{1}{\beta} y(t) & = 0, \\ \dot{y}(t) - \left(\frac{1}{\beta} - c\right) x(t) + \frac{1}{\beta} y(t) & = 0, \end{cases} \quad (7.14)$$

with initial conditions $x(0) = x_0, y(0) = y_0 \stackrel{\text{def}}{=} -\beta(v_0 + \beta \nabla f(x_0)) + (1 - \beta c)x_0$. Let us consider $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$F(x, y) = \left(-\beta \nabla f(x) + \left(\frac{1}{\beta} - c\right) x - \frac{1}{\beta} y, \left(\frac{1}{\beta} - c\right) x - \frac{1}{\beta} y \right).$$

Defining $z(t) = (x(t), y(t))$ and $z_0 = (x_0, v_0) \in \mathbb{R}^{2d}$, then (7.14) is equivalent to the Cauchy problem

$$\begin{cases} \dot{z}(t) & = F(z(t)), \\ z(0) & = z_0. \end{cases} \quad (7.15)$$

We stated that when $0 < \beta < \frac{2}{c}$ and f is definable (see the first claim above), then the solution trajectory $z(t)$ converges (as $t \rightarrow +\infty$) to an equilibrium point of F . Let us denote $\Phi(z_0, t)$, the value at t of the solution (7.15) with initial condition z_0 . Assume that \hat{z} is a hyperbolic equilibrium point of F (to be shown below), meaning that $F(\hat{z}) = 0$ and that no eigenvalue of $J_F(\hat{z})$ has zero real part. Consider the invariant set

$$W^s(\hat{z}) = \{z_0 \in \mathbb{R}^{2d} : \lim_{t \rightarrow +\infty} \Phi(z_0, t) = \hat{z}\}.$$

The global stable manifold theorem [172, page 223] asserts that $W^s(\hat{z})$ is an immersed submanifold of \mathbb{R}^{2d} , whose dimension equals the number of eigenvalues of $J_F(\hat{z})$ with negative real part.

First, we will prove that each equilibrium point of F is hyperbolic. We notice that the set of equilibrium points of F is $\{(\hat{x}, (1 - \beta c)\hat{x}) : \hat{x} \in \text{crit}(f)\}$. On the other hand, we compute

$$J_F(x, y) = \begin{pmatrix} -\beta \nabla^2 f(x) + \left(\frac{1}{\beta} - c\right) I_d & -\frac{1}{\beta} I_d \\ \left(\frac{1}{\beta} - c\right) I_d & -\frac{1}{\beta} I_d \end{pmatrix}.$$

Let $\hat{z} = (\hat{x}, (1 - \beta c)\hat{x})$, where $\hat{x} \in \text{crit}(f)$. Then the eigenvalues of $J_F(\hat{z})$ are characterized by the roots in $\lambda \in \mathbb{C}$ of

$$\det \left(\begin{pmatrix} -\beta \nabla^2 f(\hat{x}) + \left(\frac{1}{\beta} - c - \lambda\right) I_d & -\frac{1}{\beta} I_d \\ \left(\frac{1}{\beta} - c\right) I_d & -\left(\lambda + \frac{1}{\beta}\right) I_d \end{pmatrix} \right) = 0. \quad (7.16)$$

By Lemma 7.1.1, we have that (7.16) is equivalent to

$$\det((1 + \lambda\beta)\nabla^2 f(\hat{x}) + (\lambda^2 + \lambda c)I_d) = 0. \quad (7.17)$$

¹In fact, the proof is even more straightforward since the set of cluster points $\mathfrak{C}(x(\cdot))$ satisfies (7.4) and the critical points are isolated; see the proof of [34, Theorem 4.1].

If $\lambda = -\frac{1}{\beta}$, then by (7.17), $\beta c = 1$, which is excluded by hypothesis. Therefore, $-\frac{1}{\beta}$ cannot be an eigenvalue, *i.e.* $\lambda \neq -\frac{1}{\beta}$. We then obtain that (7.17) is equivalent to

$$\det \left(\nabla^2 f(\hat{x}) + \frac{\lambda^2 + \lambda c}{(1 + \lambda\beta)} I_d \right) = 0. \quad (7.18)$$

It follows that λ satisfies (7.18) if and only if

$$\frac{\lambda^2 + \lambda c}{(1 + \lambda\beta)} = -\eta$$

where $\eta \in \mathbb{R}$ is an eigenvalue of $\nabla^2 f(\hat{x})$. Equivalently,

$$\lambda^2 + (c + \eta\beta)\lambda + \eta = 0. \quad (7.19)$$

Let $\Delta_\lambda \stackrel{\text{def}}{=} (c + \eta\beta)^2 - 4\eta$. We distinguish two cases.

- $\Delta_\lambda \geq 0$: then the roots of (7.19) are real and we rewrite (7.19) as

$$\lambda(\lambda + (c + \eta\beta)) = -\eta,$$

since $\eta \neq 0$ (because f is a Morse function), then $\lambda \neq 0$.

- $\Delta_\lambda < 0$: then (7.19) has a pair of complex conjugate roots whose real part is $-\frac{c+\eta\beta}{2}$. Besides, $\Delta_\lambda = c^2 + 2\beta\eta c + \eta^2\beta^2 - 4\eta$ can be seen as a quadratic on c whose discriminant is given by $\Delta_c = 16\eta$. The fact that $\Delta_\lambda < 0$ implies $\Delta_c > 0$, and thus $\eta > 0$, therefore $-\frac{c+\eta\beta}{2} < 0$.

Overall, this shows that every equilibrium point of F is hyperbolic.

Let us recall that $\text{crit}(f) = \bigcup_{k \in I} \{\hat{x}_k\}$. Thus, the set of equilibria of F is also finite and each one takes the form $\hat{z}_k = (\hat{x}_k, (1 - \beta c)\hat{x}_k)$. Since we have already shown that each solution trajectory x of (ISEHD) converges towards some \hat{x}_k , the following partition then holds

$$\mathbb{R}^d \times \mathbb{R}^d = \bigcup_{k \in I} W^s(\hat{z}_k).$$

Let

$$I^- = \{k \in I : \text{each eigenvalue of } J_F(\hat{z}_k) \text{ has negative real part.}\},$$

and $J \stackrel{\text{def}}{=} I \setminus I^-$. Now, the global stable manifold theorem [172, page 223] allows to claim that $W^s(\hat{z}_k)$ is an immersed submanifold of \mathbb{R}^{2d} whose dimension is $2d$ when $k \in I^-$ and at most $2d - 1$ when $k \in J$.

Let $k \in I^-$, we claim that $\nabla^2 f(\hat{x}_k)$ has only positive eigenvalues. By contradiction, let us assume that $\eta_0 < 0$ is an eigenvalue of $\nabla^2 f(\hat{x}_k)$ ($\eta_0 = 0$ is not possible due to the Morse hypothesis). Each solution λ of (7.19) is an eigenvalue of $J_F(\hat{z}_k)$ and one of these solutions is

$$\frac{-(c + \eta\beta) + \sqrt{(c + \eta\beta)^2 - 4\eta_0}}{2}$$

which is positive since $\eta_0 < 0$. We then have

$$\frac{-(c + \eta_0\beta) + \sqrt{(c + \eta_0\beta)^2 - 4\eta_0}}{2} > \frac{-(c + \eta_0\beta) + |c + \eta_0\beta|}{2} \geq 0,$$

hence contradicting the assumption that $k \in I^-$. In conclusion, the set of initial conditions z_0 such that $\Phi(z_0, t)$ converges to $(x_b, (1 - \beta c)x_b)$ (as $t \rightarrow +\infty$), where x_b is not a local minimum of f is $\bigcup_{k \in J} W^s(\hat{z}_k)$ which has Lebesgue measure zero. Therefore, due to the equivalence between (7.1) and (ISEHD) and that Morse functions satisfy the strict saddle property (see Remark 7.1.7), we indeed have that for almost all initial conditions $x_0, v_0 \in \mathbb{R}^d$, the solution trajectory of (ISEHD) will converge to a local minimum of f . \square

7.2.1.3 Convergence rate

When the objective function is definable, we now provide the convergence rate on the Lyapunov function E in (7.5), hence f , and on the solution trajectory x .

Theorem 7.2.4. *Consider the setting of Theorem 7.2.1 with f being also definable and the solution trajectory x is bounded. Recall the function E from (7.5), which is also definable, and denote ψ its desingularizing function and Ψ any primitive of $-\psi'^2$. Then, $x(t)$ converges (as $t \rightarrow +\infty$) to $x_\infty \in \text{crit}(f)$. Denote $\tilde{V}(t) \stackrel{\text{def}}{=} E(x(t), \dot{x}(t), \beta \nabla f(x(t))) - f(x_\infty)$. The following rates of convergence hold:*

- If $\lim_{t \rightarrow 0} \Psi(t) \in \mathbb{R}$, we have $E(x(t), \dot{x}(t), \beta \nabla f(x(t)))$ converges to $f(x_\infty)$ in finite time.
- If $\lim_{t \rightarrow 0} \Psi(t) = +\infty$, there exists some $t_1 \geq 0$ such that

$$\tilde{V}(t) = \mathcal{O}(\Psi^{-1}(t - t_1)). \quad (7.20)$$

Moreover,

$$\|x(t) - x_\infty\| = \mathcal{O}(\psi \circ \Psi^{-1}(t - t_1)). \quad (7.21)$$

Proof. This proof is a generalization of [81, Theorem 2.7] to the dynamics (ISEHD). Let $\delta_0 \stackrel{\text{def}}{=} \frac{\delta_2}{\delta_1}$, $\delta_3 > 0$ and $T > 0$ for $\delta_1, \delta_2, \delta_3, T$ defined in the proof of Theorem 7.2.1. Using (7.10) then (7.9), we have for $t > T$

$$\begin{aligned} \frac{d}{dt} \Psi(\tilde{V}(t)) &= \Psi'(\tilde{V}(t)) \tilde{V}'(t) \\ &= -\psi'^2(\tilde{V}(t)) \tilde{V}'(t) \\ &\geq \delta_0 \psi'^2(\tilde{V}(t)) \|\nabla E(x(t), \dot{x}(t), \beta \nabla f(x(t)))\|^2 \\ &\geq \delta_0. \end{aligned} \quad (7.22)$$

Integrating on both sides from T to t we obtain that for every $t > T$

$$\Psi(\tilde{V}(t)) \geq \delta_0(t - T) + \Psi(\tilde{V}(T)).$$

Following the arguments shown in [100, Theorem 3.1.12], if $\lim_{t \rightarrow 0} \Psi(t) \in \mathbb{R}$, then $\tilde{V}(t)$ converges to 0 in finite time. Otherwise, we take the inverse of Ψ , which is non-increasing, on both sides of (7.22) to obtain the desired bound. Finally, using (7.13) we also have for every $t > T$

$$\|x(t) - x_\infty\| \leq \int_t^{+\infty} \|\dot{x}(s)\| ds \leq \frac{1}{\delta_3} \psi(\tilde{V}(t)) \leq \frac{1}{\delta_3} \psi \circ \Psi^{-1}(\delta_0(t - T) + \Psi(\tilde{V}(T))). \quad (7.23)$$

□

Remark 7.2.5. Observe that the convergence rate (7.20) holds also on $f(x(t)) - f(x_\infty)$ and $\|\dot{x}(t) + \nabla f(x(t))\|^2$.

We now specialize this to the Łojasiewicz case.

Corollary 7.2.6. *Consider the setting of Theorem 7.2.4 where now f satisfies the Łojasiewicz inequality with desingularizing function $\psi_f(s) = c_f s^{1-q}$, $q \in [0, 1[$, $c_f > 0$. Then there exists some $t_1 > 0$ such that the following convergence rates hold:*

- If $q \in [0, \frac{1}{2}]$, then

$$\tilde{V}(t) = \mathcal{O}(\exp(-(t - t_1))) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left(\exp\left(\frac{t - t_1}{2}\right)\right). \quad (7.24)$$

- If $q \in]\frac{1}{2}, 1[$, then

$$\tilde{V}(t) = \mathcal{O}\left((t - t_1)^{\frac{-1}{2q-1}}\right) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left((t - t_1)^{-\frac{1-q}{2q-1}}\right). \quad (7.25)$$

Proof. E is a separable quadratic perturbation of f . But a quadratic function is Łojasiewicz with exponent $1/2$. It then follows from the Łojasiewicz exponent calculus rule in [130, Theorem 3.3] that the desingularizing function of E is $\psi_E(s) = c_E s^{1-q_E}$ for some $c_E > 0$ and $q_E = \max(q, \frac{1}{2})$. Then,

- If $q \in [0, \frac{1}{2}]$ then $q_E = \frac{1}{2}$ and $\Psi(s) = \frac{c_1^2}{4} \ln(\frac{1}{s})$. This implies that $\Psi^{-1}(s) = \frac{4}{c_1^2} \exp(-s)$.
- If $q \in]\frac{1}{2}, 1[$ then $q_E = q$ and $\Psi(s) = \frac{c_1^2}{4(2q-1)} s^{1-2q}$. This implies that $\Psi^{-1}(s) = \frac{4(2q-1)}{c_1^2} s^{\frac{-1}{2q-1}}$.

We conclude in both cases by using Theorem 7.2.4. \square

7.2.2 Algorithmic scheme

Now we will consider the following finite differences explicit discretization of (ISEHD) with stepsize $h > 0$ and for $k \geq 1$:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \gamma(kh) \frac{x_{k+1} - x_k}{h} + \beta \frac{\nabla f(x_k) - \nabla f(x_{k-1})}{h} + \nabla f(x_k) = 0. \quad (7.26)$$

Rearranging, this equivalently reads

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) - \beta_k(\nabla f(x_k) - \nabla f(x_{k-1})), \\ x_{k+1} &= y_k - s_k \nabla f(x_k), \end{cases} \quad (\text{ISEHD-Disc})$$

with initial conditions $x_0, x_1 \in \mathbb{R}^d$, where $\gamma_k \stackrel{\text{def}}{=} \gamma(kh)$, $\alpha_k \stackrel{\text{def}}{=} \frac{1}{1+\gamma_k h}$, $\beta_k \stackrel{\text{def}}{=} \beta h \alpha_k$, $s_k \stackrel{\text{def}}{=} h^2 \alpha_k$.

7.2.2.1 Global convergence and trap avoidance

The following theorem summarizes our main results on the behavior of (ISEHD-Disc). Observe that as the discretization is explicit, we will need ∇f to be globally Lipschitz continuous.

Theorem 7.2.7. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be satisfying (H₀) with ∇f being globally L -Lipschitz-continuous. Consider the scheme (ISEHD-Disc) with $h > 0$, $\beta \geq 0$ and $c \leq \gamma_k \leq C$ for some $c, C > 0$ and all $k \in \mathbb{N}$. Then the following holds:*

- (i) *If $\beta + \frac{h}{2} < \frac{c}{L}$, then $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, in particular*

$$\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0.$$

- (ii) *Moreover, if $(x_k)_{k \in \mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \rightarrow +\infty$) to a critical point of f .*

- (iii) *Furthermore, if $\gamma_k \equiv c > 0$, $0 < \beta < \frac{c}{L}$, $\beta \neq \frac{1}{c}$, and $h < \min(2(\frac{c}{L} - \beta), \frac{1}{L\beta})$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a local minimum of f .*

Remark 7.2.8. When $\beta = 0$, we recover the HBF method and the condition $h < \min(2(\frac{c}{L} - \beta), \frac{1}{L\beta})$ becomes $h < \frac{2c}{L}$.

Proof. (i) By definition of x_{k+1} in (ISEHD-Disc), for $k \in \mathbb{N}^*$

$$x_{k+1} = \operatorname{argmin}_{x \in \mathbb{R}^d} \frac{1}{2} \|x - (y_k - s_k \nabla f(x_k))\|^2. \quad (7.27)$$

1-strong convexity of $x \mapsto \frac{1}{2} \|x - (y_k - s_k \nabla f(x_k))\|^2$ then yields

$$\frac{1}{2} \|x_{k+1} - (y_k - s_k \nabla f(x_k))\|^2 \leq \frac{1}{2} \|x_k - (y_k - s_k \nabla f(x_k))\|^2 - \frac{1}{2} \|x_{k+1} - x_k\|^2. \quad (7.28)$$

Let $\underline{\alpha} = \frac{1}{1+Ch}$, $\bar{\alpha} = \frac{1}{1+ch}$, $\underline{s} = h^2\underline{\alpha}$, $\bar{s} = h^2\bar{\alpha}$, and thus for every $k \in \mathbb{N}$, $\underline{\alpha} \leq \alpha_k \leq \bar{\alpha}$ and $\underline{s} \leq s_k \leq \bar{s}$. Let also $v_k \stackrel{\text{def}}{=} x_k - x_{k-1}$, $z_k \stackrel{\text{def}}{=} \alpha_k v_k - \beta_k (\nabla f(x_k) - \nabla f(x_{k-1}))$, then $y_k = x_k + z_k$. After expanding the terms of (7.28) we have that

$$\begin{aligned} \langle \nabla f(x_k), v_{k+1} \rangle &\leq -\frac{\|v_{k+1}\|^2}{s_k} + \frac{1}{s_k} \langle v_{k+1}, z_k \rangle \\ &\leq -\frac{\|v_{k+1}\|^2}{\bar{s}} + \frac{1}{h^2} \langle v_{k+1}, v_k \rangle - \frac{\beta}{h} \langle v_{k+1}, \nabla f(x_k) - \nabla f(x_{k-1}) \rangle. \end{aligned} \quad (7.29)$$

By the descent lemma for L -smooth functions, we obtain

$$f(x_{k+1}) \leq f(x_k) + \langle \nabla f(x_k), v_{k+1} \rangle + \frac{L}{2} \|v_{k+1}\|^2. \quad (7.30)$$

Using the bound in (7.29), we get

$$f(x_{k+1}) \leq f(x_k) + \frac{1}{h^2} \langle v_{k+1}, v_k \rangle - \frac{\beta}{h} \langle v_{k+1}, \nabla f(x_k) - \nabla f(x_{k-1}) \rangle - \left(\frac{1}{\bar{s}} - \frac{L}{2} \right) \|v_{k+1}\|^2. \quad (7.31)$$

According to our hypothesis $h < \frac{2c}{L}$, so $h < \frac{c + \sqrt{c^2 + 2L}}{L}$ and this implies that $\bar{s} < \frac{2}{L}$. Using Young's inequality twice, for $\varepsilon, \varepsilon' > 0$, the fact that ∇f is L -Lipschitz, and adding $\frac{\varepsilon + \varepsilon'}{2} \|v_{k+1}\|^2$ at both sides, then

$$\begin{aligned} f(x_{k+1}) + \frac{\varepsilon + \varepsilon'}{2} \|v_{k+1}\|^2 &\leq f(x_k) + \frac{\varepsilon + \varepsilon'}{2} \|v_k\|^2 \\ &\quad + \left(\frac{1}{2} \left[\frac{1}{h^4 \varepsilon} + \varepsilon + \frac{\beta^2 L^2}{h^2 \varepsilon'} + \varepsilon' \right] - \left(\frac{1}{\bar{s}} - \frac{L}{2} \right) \right) \|v_{k+1}\|^2. \end{aligned} \quad (7.32)$$

In order to make the last term negative, we want to impose

$$\frac{1}{2} \left[\frac{1}{h^4 \varepsilon} + \varepsilon + \frac{\beta^2 L^2}{h^2 \varepsilon'} + \varepsilon' \right] < \left(\frac{1}{\bar{s}} - \frac{L}{2} \right). \quad (7.33)$$

Minimizing the left-hand side with respect to $\varepsilon, \varepsilon' > 0$ we get $\varepsilon = \frac{1}{h^2}$, $\varepsilon' = \frac{\beta L}{h}$, and one can check that in this case, the condition (7.33) becomes equivalent to $\beta + \frac{h}{2} < \frac{c}{L}$ which is assumed in the hypothesis.

Setting $C_1 \stackrel{\text{def}}{=} \frac{1}{h^2} + \frac{\beta L}{h}$, $\delta \stackrel{\text{def}}{=} \frac{1}{\bar{s}} - \frac{L}{2} - \frac{1}{h^2} - \frac{\beta L}{h} > 0$, and defining $V_k \stackrel{\text{def}}{=} f(x_k) + \frac{C_1}{2} \|v_k\|^2$, we have for any $k \in \mathbb{N}^*$

$$V_{k+1} \leq V_k - \delta \|v_{k+1}\|^2. \quad (7.34)$$

Clearly, V_k is non-increasing and bounded from below, hence $\lim_{k \rightarrow +\infty} V_k$ exists (say \tilde{L}). Summing this inequality over k , we have that $(\|v_{k+1}\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$ entailing that $\lim_{k \rightarrow +\infty} \|v_k\| = 0$. In turn, we have that $\lim_{k \rightarrow +\infty} f(x_k) = \tilde{L}$. Embarking again from the update in (ISEHD-Disc), we have

$$\begin{aligned} s_k \|\nabla f(x_k)\| &= \|x_{k+1} - y_k\| \leq \|x_{k+1} - x_k\| + \|x_k - y_k\| \\ &\leq \|v_{k+1}\| + (\alpha_k + \beta_k L) \|v_k\| \\ &\leq \|v_{k+1}\| + \|v_k\|, \end{aligned}$$

since $\bar{\alpha}(1 + \beta h L) < 1$ by hypothesis. Therefore

$$\|\nabla f(x_k)\|^2 \leq \delta_2 (\|v_{k+1}\|^2 + \|v_k\|^2),$$

where $\delta_2 = \frac{2}{\bar{s}^2}$. Consequently $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, which implies that $\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0$.

(ii) If, moreover, $(x_k)_{k \in \mathbb{N}}$ is bounded, then the set of its cluster points $\mathfrak{C}((x_k)_{k \in \mathbb{N}})$ satisfies (see *e.g.* [60, Lemma 5]):

$$\begin{cases} \mathfrak{C}((x_k)_{k \in \mathbb{N}}) \subseteq \text{crit}(f); \\ \mathfrak{C}((x_k)_{k \in \mathbb{N}}) \text{ is non-empty, compact and connected}; \\ f \text{ is constant on } \mathfrak{C}((x_k)_{k \in \mathbb{N}}). \end{cases} \quad (7.35)$$

Define

$$E : (x, v) \in \mathbb{R}^{2d} \mapsto f(x) + \frac{C_1}{2} \|v\|^2. \quad (7.36)$$

Since f is definable, so is E as the sum of a definable function and an algebraic one, whence E satisfies the KL inequality. Let $\mathfrak{C}_1 = \mathfrak{C}((x_k)_{k \in \mathbb{N}}) \times \{0_d\}$. Since $E|_{\mathfrak{C}_1} = \tilde{L}$, $\nabla E|_{\mathfrak{C}_1} = 0$, $\exists r, \eta > 0, \exists \psi \in \kappa(0, \eta)$ such that for every (x, v) such that $x \in \mathfrak{C}((x_k)_{k \in \mathbb{N}}) + B_r, v \in B_r$, (where B_r is the \mathbb{R}^d -ball centred at 0_d with radius r) and $0 < E(x, v) - \tilde{L} < \eta$, one has

$$\psi'(E(x, v) - \tilde{L}) \|\nabla E(x, v)\| \geq 1. \quad (7.37)$$

Let us define $\tilde{V}_k = V_k - \tilde{L}$, or equivalently $\tilde{V}_k = E(x_k, v_k) - \tilde{L}$. From (7.34), $(\tilde{V}_k)_{k \in \mathbb{N}}$ is a non-increasing sequence and its limit is 0 by definition of \tilde{L} . This implies that $\tilde{V}_k \geq 0$ for all $k \in \mathbb{N}^*$. We may assume without loss of generality that $\tilde{V}_k > 0$. Indeed, suppose there exists $K \in \mathbb{N}$ such that $\tilde{V}_K = 0$, then the decreasing property (7.34) implies that $\tilde{V}_k = 0$ holds for all $k \geq K$. Thus $v_{k+1} = 0$, or equivalently $x_k = x_K$, for all $k \geq K$, hence $(x_k)_{k \in \mathbb{N}}$ has finite length. Since $\lim_{k \rightarrow +\infty} \text{dist}(x_k, \mathfrak{C}((x_k)_{k \in \mathbb{N}})) = 0, \lim_{k \rightarrow +\infty} \|v_k\| = 0$, and $\lim_{k \rightarrow +\infty} V_k = 0$, there exists $\tilde{K} \in \mathbb{N}$ such that for all $k \geq \tilde{K}$,

$$\begin{cases} x_k \in \mathfrak{C}((x_k)_{k \in \mathbb{N}}) + B_r; \\ \|v_k\| < r; \\ 0 < \tilde{V}_k < \eta. \end{cases}$$

Then, by (7.37), we have

$$\psi'(\tilde{V}_k) \|\nabla E(x_k, v_k)\| \geq 1, \quad \forall k \geq \tilde{K}. \quad (7.38)$$

By concavity of ψ and (7.38), we have

$$\begin{aligned} \psi(\tilde{V}_k) - \psi(\tilde{V}_{k+1}) &\geq -\psi'(\tilde{V}_k)(\tilde{V}_{k+1} - \tilde{V}_k) \\ &\geq \delta \psi'(\tilde{V}_k) \|v_{k+1}\|^2 \\ &\geq \delta \frac{\|v_{k+1}\|^2}{\|\nabla E(x_k, v_k)\|}. \end{aligned}$$

On the other hand,

$$\|\nabla E(x_k, v_k)\| \leq \delta_3 (\|v_{k+1}\| + \|v_k\|) \quad (7.39)$$

where $\delta_3 = \sqrt{C_1^2 + \delta_2}$. Let us define for $k \in \mathbb{N}^*$, $(\Delta\psi)_k \stackrel{\text{def}}{=} \psi(\tilde{V}_k) - \psi(\tilde{V}_{k+1})$ and $\delta_4 = \frac{\delta_3}{\delta}$. We then have for all $k \geq \tilde{K}$,

$$\|v_{k+1}\|^2 \leq \delta_4 (\Delta\psi)_k (\|v_k\| + \|v_{k+1}\|).$$

Using Young's inequality and concavity of $\sqrt{\cdot}$, this implies that for every $\varepsilon > 0$

$$\|v_{k+1}\| \leq \delta_4 \frac{(\Delta\psi)_k}{\sqrt{2\varepsilon}} + \varepsilon \frac{\|v_k\| + \|v_{k+1}\|}{\sqrt{2}}.$$

Rearranging the terms and imposing $0 < \varepsilon < \sqrt{2}$ gives

$$\left(1 - \frac{\varepsilon}{\sqrt{2}}\right) \|v_{k+1}\| \leq \delta_4 \frac{(\Delta\psi)_k}{\sqrt{2\varepsilon}} + \varepsilon \frac{\|v_k\|}{\sqrt{2}}.$$

Dividing by $\left(1 - \frac{\varepsilon}{\sqrt{2}}\right)$ on both sides, we get

$$\|v_{k+1}\| \leq \delta_4 \frac{(\Delta\psi)_k}{\varepsilon(\sqrt{2} - \varepsilon)} + \varepsilon \frac{\|v_k\|}{\sqrt{2} - \varepsilon}. \quad (7.40)$$

Choosing now ε such that $0 < \varepsilon < \frac{\sqrt{2}}{2}$, we get that $0 < \frac{\varepsilon}{\sqrt{2} - \varepsilon} < 1$. Since $(\Delta\psi)_k \in \ell^1(\mathbb{N}^*)$ as a telescopic sum, we conclude that $(\|v_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N}^*)$. This means that $(x_k)_{k \in \mathbb{N}}$ has finite length, hence is a Cauchy sequence, entailing that x_k has a limit (as $k \rightarrow +\infty$) denoted x_∞ which is a critical point of f since $\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0$.

- (iii) If $\gamma_k \equiv c$, we denote $\alpha_k \equiv \alpha = \frac{1}{1+ch}$, $\beta_k \equiv \tilde{\beta} = \beta h \alpha$, $s_k \equiv s = h^2 \alpha$. Let $z_k = (x_k, x_{k-1})$ for $k \geq 1$, and $g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$g : (x_+, x_-) \mapsto [(1 + \alpha)x_+ - \alpha x_- - (\tilde{\beta} + s)\nabla f(x_+) + \tilde{\beta}\nabla f(x_-), x_+].$$

(ISEHD-Disc) is then equivalent to

$$z_{k+1} = g(z_k). \quad (7.41)$$

To complete the proof, we will capitalize on [128, Corollary 1] which builds on the center stable manifold theorem [187, Theorem III.7]. For this, one needs to check two conditions:

(a) $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$.

(b) Let $\mathcal{A}_g^* \stackrel{\text{def}}{=} \{(x, x) \in \mathbb{R}^{2d} : x \in \text{crit}(f), \max_i |\lambda_i(J_g(x, x))| > 1\}$, \mathcal{X}^* be the set of strict saddle points of f , and $\hat{\mathcal{X}} \stackrel{\text{def}}{=} \{(x, x) \in \mathbb{R}^{2d} : x \in \mathcal{X}^*\}$. Check that $\hat{\mathcal{X}} \subset \mathcal{A}_g^*$.

Notice that \mathcal{A}_g^* is the set of unstable fixed points. Indeed, the fixed points of g are of the form (x^*, x^*) where $x^* \in \text{crit}(f)$.

Let us check (a). We first compute $J_g(x_+, x_-)$, given by

$$\begin{pmatrix} (1 + \alpha)I_d - (\tilde{\beta} + s)\nabla^2 f(x_+) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x_-) \\ I_d & 0_{d \times d} \end{pmatrix} \quad (7.42)$$

This is a block matrix that comes in a form amenable to applying Lemma 7.1.1. We then have

$$\det(J_g(x_+, x_-)) = \det(\alpha I_d - \tilde{\beta}\nabla^2 f(x_-)).$$

Since the eigenvalues of $\nabla^2 f(x_-)$ are contained in $[-L, L]$, if $\alpha > L\tilde{\beta}$, then $\alpha - \tilde{\beta}\eta \neq 0$ for every eigenvalue $\eta \in \mathbb{R}$ of $\nabla^2 f(x_-)$. This implies that the first condition is satisfied, *i.e.* $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$. The condition $\alpha > L\tilde{\beta}$ in terms of h reads $h < \frac{1}{L\tilde{\beta}}$, since we already needed $h < 2(\frac{c}{L} - \beta)$, we just ask h to be less than the minimum of the two quantities.

To check (b), let us take x a strict saddle point of f , *i.e.* $x \in \text{crit}(f)$ and $\lambda_{\min}(\nabla^2 f(x)) = -\eta < 0$. To compute the eigenvalues of $J_g(x, x)$ we consider

$$\det \left(\begin{pmatrix} (1 + \alpha - \lambda)I_d - (\tilde{\beta} + s)\nabla^2 f(x) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x) \\ I_d & -\lambda I_d \end{pmatrix} \right) = 0.$$

Again by Lemma 7.1.1, we get that

$$\begin{aligned} & \det \left(\begin{pmatrix} (1 + \alpha - \lambda)I_d - (\tilde{\beta} + s)\nabla^2 f(x) & -\alpha I_d + \tilde{\beta}\nabla^2 f(x) \\ I_d & -\lambda I_d \end{pmatrix} \right) = \\ & \det[(-\lambda(1 + \alpha) + \lambda^2)I_d + \lambda(\tilde{\beta} + s)\nabla^2 f(x) + \alpha I_d - \tilde{\beta}\nabla^2 f(x)] = \\ & \det[(\lambda(\tilde{\beta} + s) - \tilde{\beta})\nabla^2 f(x) + (\lambda^2 - \lambda(1 + \alpha) + \alpha)I_d]. \end{aligned}$$

We then need to solve for λ

$$\det[(\lambda(\tilde{\beta} + s) - \tilde{\beta})\nabla^2 f(x) + (\lambda^2 - \lambda(1 + \alpha) + \alpha)I_d] = 0. \quad (7.43)$$

If $\lambda = \frac{\tilde{\beta}}{\tilde{\beta} + s}$, then (7.43) becomes

$$\left(\frac{\tilde{\beta}}{\tilde{\beta} + s} \right)^2 - \left(\frac{\tilde{\beta}}{\tilde{\beta} + s} \right) (1 + \alpha) + \alpha = 0.$$

This implies that $\alpha = \frac{\tilde{\beta}}{\tilde{\beta} + s}$, which in terms of β and c is equivalent to $\beta c = 1$. But this case is excluded by hypothesis. Now we can focus on the case where $\lambda \neq \frac{\tilde{\beta}}{\tilde{\beta} + s}$ and we can rewrite (7.43)

as

$$\det \left(\nabla^2 f(x) - \frac{\lambda^2 - \lambda(1 + \alpha) + \alpha}{\tilde{\beta} - \lambda(\tilde{\beta} + s)} I_d \right) = 0. \quad (7.44)$$

Therefore, as argued for the time-continuous dynamic, for every eigenvalue $\eta' \in \mathbb{R}$ of $\nabla^2 f(x)$, λ satisfies (7.44) if and only if

$$\frac{\lambda^2 - \lambda(1 + \alpha) + \alpha}{\tilde{\beta} - \lambda(\tilde{\beta} + s)} = \eta'.$$

where $\eta' \in \mathbb{R}$ is an eigenvalue of $\nabla^2 f(\hat{x})$. Thus if $\eta' = -\eta$ is negative, we have

$$\lambda^2 - \lambda((1 + \alpha) + \eta(\tilde{\beta} + s)) + \alpha + \eta\tilde{\beta} = 0. \quad (7.45)$$

We analyze its discriminant $\Delta_\lambda = ((1 + \alpha) + \eta(\tilde{\beta} + s))^2 - 4(\alpha + \eta\tilde{\beta})$. After developing the terms we get that

$$\Delta_\lambda = \alpha^2 + 2\alpha(\eta(\tilde{\beta} + s) - 1) + (\eta(\tilde{\beta} + s) + 1)^2 - 4\eta\tilde{\beta},$$

which can be seen as a quadratic equation on α . We get that its discriminant Δ_α is $-16\eta s$, which is negative (since η, s are positive), thus the quadratic equation on α does not have real roots, implying that $\Delta_\lambda > 0$. We can write the solutions of (7.45),

$$\lambda = \frac{((1 + \alpha) + \eta(\tilde{\beta} + s)) \pm \sqrt{\Delta_\lambda}}{2}.$$

Let us consider the biggest solution (the one with the plus sign) and let us see that $\lambda > 1$, this is equivalent to

$$\frac{((1 + \alpha) + \eta(\tilde{\beta} + s)) + \sqrt{\Delta_\lambda}}{2} > 1,$$

which in turn is equivalent to

$$\sqrt{\Delta_\lambda} > 2 - (1 + \alpha) - \eta(\tilde{\beta} + s).$$

Squaring both sides of this inequality, we have

$$\begin{aligned} [(1 - \alpha) - \eta(\tilde{\beta} + s)]^2 &< \Delta_\lambda \\ &= [(1 + \alpha) + \eta(\tilde{\beta} + s)]^2 - 4(\alpha + \eta\tilde{\beta}). \end{aligned}$$

After expanding the terms, we see that the inequality is equivalent to $0 < 4\eta s$, which is always true as $\eta > 0$. Consequently, $\lambda > 1$ and in turn $\hat{\mathcal{X}} \subset \mathcal{A}_g^*$.

We have then checked the two conditions (a)-(b) above. This entails that the invariant set $\{z_1 \in \mathbb{R}^{2d} : \lim_{k \rightarrow +\infty} g^k(z_1) \in \hat{\mathcal{X}}\}$ has Lebesgue measure zero. Equivalently, the set of initializations $x_0, x_1 \in \mathbb{R}^d$ for which x_k converges to a strict saddle point of f has Lebesgue measure zero. □

7.2.2.2 Convergence rate

The following result provides the convergence rates for algorithm (ISEHD-Disc) in the case where f has the Łojasiewicz property. The original idea of proof for descent-like algorithms can be found in [14, Theorem 5].

Theorem 7.2.9. *Consider the setting of Theorem 7.2.7, where f also satisfies the Łojasiewicz property with exponent $q \in [0, 1[$. Then $x_k \rightarrow x_\infty \in \text{crit}(f)$ as $k \rightarrow +\infty$ at the rates:*

- If $q \in [0, \frac{1}{2}]$ then there exists $\rho \in]0, 1[$ such that

$$\|x_k - x_\infty\| = \mathcal{O}(\rho^k). \quad (7.46)$$

- If $q \in]\frac{1}{2}, 1[$ then

$$\|x_k - x_\infty\| = \mathcal{O}\left(k^{-\frac{1-q}{2q-1}}\right). \quad (7.47)$$

Proof. Recall the function E from (7.36). Since f satisfies the Łojasiewicz property with exponent $q \in [0, 1[$, and E is a separable quadratic perturbation of f , it follows from [130, Theorem 3.3] that E has the Łojasiewicz property with exponent $q_E = \max(q, 1/2) \in [1/2, 1[$, *i.e.* there exists $c_E > 0$ such that the desingularizing function of E is $\psi_E(s) = c_E s^{1-q_E}$.

Let $v_k = x_k - x_{k-1}$ and $\Delta_k = \sum_{p=k}^{+\infty} \|v_{p+1}\|$. The triangle inequality yields $\Delta_k \geq \|x_k - x_\infty\|$ so it suffices to analyze the behavior of Δ_k to obtain convergence rates for the trajectory. Recall the constants $\delta, \delta_3, \delta_4 > 0$ and the sequences $((\Delta\psi)_k)_{k \in \mathbb{N}}$ and $(\tilde{V}_k)_{k \in \mathbb{N}}$ defined in the proof of Theorem 7.2.7.

Denote $\lambda = \frac{\varepsilon}{\sqrt{2}-\varepsilon} \in (0, 1)$ and $M = \frac{\delta_4}{\varepsilon(\sqrt{2}-\varepsilon)}$ for $0 < \varepsilon < \frac{\sqrt{2}}{2}$. Using (7.40), we have that there exists $\tilde{K} \in \mathbb{N}$ large enough such that for all $k \geq \tilde{K}$

$$\|v_{k+1}\| \leq \lambda \|v_k\| + M(\Delta\psi)_k.$$

Recall that $q_E \in [1/2, 1[$ (so $\frac{1-q_E}{q_E} \leq 1$) and that $\lim_{k \rightarrow +\infty} \tilde{V}_k = 0$. We obtain by induction that for all $k \geq \tilde{K}$

$$\sum_{p=k}^{+\infty} \|v_{p+1}\| \leq \frac{\lambda}{1-\lambda} \|v_k\| + \frac{M c_E}{1-\lambda} \tilde{V}_k^{1-q_E}. \quad (7.48)$$

Or equivalently,

$$\Delta_k \leq \frac{\lambda}{1-\lambda} (\Delta_{k-1} - \Delta_k) + \frac{M c_E}{1-\lambda} \tilde{V}_k^{1-q_E}. \quad (7.49)$$

Denoting $c_2 = (c_E(1-q_E))^{\frac{1-q_E}{q_E}}$, then by (7.38) and (7.39)

$$\begin{aligned} \tilde{V}_k^{1-q_E} &\leq c_2 \|\nabla E(x_k, v_k)\|^{\frac{1-q_E}{q_E}} \\ &\leq c_2 \delta_3^{\frac{1-q_E}{q_E}} (\|v_k\| + \|v_{k+1}\|)^{\frac{1-q_E}{q_E}} \\ &\leq c_2 \delta_3^{\frac{1-q_E}{q_E}} (\Delta_{k-1} - \Delta_k + \Delta_k - \Delta_{k+1})^{\frac{1-q_E}{q_E}} \\ &= c_2 \delta_3^{\frac{1-q_E}{q_E}} (\Delta_{k-1} - \Delta_{k+1})^{\frac{1-q_E}{q_E}}. \end{aligned}$$

Plugging this into (7.49), and using that $\Delta_k \rightarrow 0$ and $\frac{1-q_E}{q_E} \leq 1$, then there exists an integer $\tilde{K}_1 \geq \tilde{K}$ such that for every $k \geq \tilde{K}_1$

$$\Delta_k \leq \frac{\lambda}{1-\lambda} (\Delta_{k-1} - \Delta_k)^{\frac{1-q_E}{q_E}} + \frac{M_1}{1-\lambda} (\Delta_{k-1} - \Delta_{k+1})^{\frac{1-q_E}{q_E}},$$

where $M_1 = c_E c_2 \delta_3^{\frac{1-q_E}{q_E}} M$. Taking the power $\frac{q_E}{1-q_E} \geq 1$ on both sides and using the fact that

$$\Delta_{k+1} \leq \Delta_k,$$

we have for every $k \geq \tilde{K}_1$:

$$\Delta_k^{\frac{q_E}{1-q_E}} \leq M_2 (\Delta_{k-1} - \Delta_{k+1}), \quad (7.50)$$

where we set $M_2 = (1-\lambda)^{-\frac{q_E}{1-q_E}} \max(\lambda, M_1)^{\frac{q_E}{1-q_E}}$. We now distinguish two cases:

- $q \in [0, 1/2]$, hence $q_E = \frac{1}{2}$: (7.50) then becomes

$$\Delta_k \leq M_2 (\Delta_{k-1} - \Delta_{k+1}),$$

with $M_2 = (1-\lambda)^{-1} \max(\lambda, M_1)$ and $M_1 = \frac{c_E^2 \delta_3^2}{\delta}$. Using again that $\Delta_{k+1} \leq \Delta_k$, we obtain that for $k \geq \tilde{K}_1$

$$\Delta_k \leq \frac{M_2}{1+M_2} \Delta_{k-2},$$

which implies

$$\Delta_k \leq \left(\frac{M_2}{1+M_2} \right)^{\frac{k-\tilde{K}_1}{2}} \Delta_{\tilde{K}_1} = \mathcal{O}(\rho^k),$$

for $\rho \stackrel{\text{def}}{=} \left(\frac{M_2}{1+M_2} \right)^{\frac{1}{2}} \in]0, 1[$.

- $q \in]\frac{1}{2}, 1[$, hence $q_E = q$: we define the function $h : \mathbb{R}_+^* \rightarrow \mathbb{R}$ by $h(s) = s^{-\frac{q}{1-q}}$. Let $R > 1$. Assume first that $h(\Delta_k) \leq Rh(\Delta_{k-1})$. Then from (7.50), we get

$$\begin{aligned} 1 &\leq M_2(\Delta_{k-1} - \Delta_{k+1})h(\Delta_k) \\ &\leq RM_2(\Delta_{k-1} - \Delta_{k+1})h(\Delta_{k-1}) \\ &\leq RM_2 \int_{\Delta_{k+1}}^{\Delta_{k-1}} h(s) ds \\ &\leq RM_2 \frac{1-q}{1-2q} \left(\Delta_{k-1}^{\frac{1-2q}{1-q}} - \Delta_{k+1}^{\frac{1-2q}{1-q}} \right). \end{aligned}$$

Setting $\nu = \frac{2q-1}{1-q} > 0$ and $M_3 = \frac{\nu}{RM_2} > 0$, one obtains

$$0 < M_3 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}. \quad (7.51)$$

Now assume that $h(\Delta_k) > Rh(\Delta_{k-1})$. Since h is decreasing and $\Delta_{k+1} \leq \Delta_k$, then

$$h(\Delta_{k+1}) > Rh(\Delta_{k-1}).$$

Set $q = R^{\frac{2q-1}{q}} > 1$, we directly have that

$$\Delta_{k+1}^{-\nu} > q\Delta_{k-1}^{-\nu}.$$

Since $q - 1 > 0$ and $\Delta_k^{-\nu} \rightarrow +\infty$ as $k \rightarrow +\infty$, there exists $M_4 > 0$ and a large enough integer $\tilde{K}_2 \geq \tilde{K}_1$ such that for every $k \geq \tilde{K}_2$ that satisfies our assumption ($h(\Delta_k) > Rh(\Delta_{k-1})$), we have

$$0 < M_4 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}. \quad (7.52)$$

Taking $M_5 = \min(M_3, M_4)$, (7.51) and (7.52) show that for all $k \geq \tilde{K}_2$

$$0 < M_5 \leq \Delta_{k+1}^{-\nu} - \Delta_{k-1}^{-\nu}.$$

Summing both sides from \tilde{K}_2 up to $K - 1 \geq \tilde{K}_2$, we obtain

$$M_5(K - \tilde{K}_2) \leq \Delta_K^{-\nu} - \Delta_{\tilde{K}_2}^{-\nu} + \Delta_{K-1}^{-\nu} - \Delta_{\tilde{K}_2-1}^{-\nu} \leq 2(\Delta_K^{-\nu} - \Delta_{\tilde{K}_2-1}^{-\nu}).$$

Therefore

$$\Delta_K^{-\nu} \geq \Delta_{\tilde{K}_2-1}^{-\nu} + \frac{M_5}{2}(K - \tilde{K}_2). \quad (7.53)$$

Inverting, we get

$$\Delta_K \leq \left[\Delta_{\tilde{K}_2-1}^{-\nu} + \frac{M_5}{2}(K - (\tilde{K}_2 + 1)) \right]^{-\frac{1}{\nu}} = \mathcal{O}(K^{-\frac{1}{\nu}}). \quad (7.54)$$

□

7.2.2.3 General coefficients

The discrete scheme (**ISEHD-Disc**) opens the question of whether we can consider α_k, β_k, s_k to be independent. Though this would omit the fact they arise from a discretization of the continuous-time dynamic (**ISEHD**), hence ignoring its physical interpretation, it will give us a more flexible choice of these parameters while preserving the desired convergence behavior.

Theorem 7.2.10. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be satisfying (\mathbf{H}_0) with ∇f being globally L -Lipschitz continuous. Consider $(\alpha_k)_{k \in \mathbb{N}}, (\beta_k)_{k \in \mathbb{N}}, (s_k)_{k \in \mathbb{N}}$ to be three positive sequences, and the following algorithm with $x_0, x_1 \in \mathbb{R}^d$:*

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}) - \beta_k(\nabla f(x_k) - \nabla f(x_{k-1})), \\ x_{k+1} &= y_k - s_k \nabla f(x_k). \end{cases} \quad (7.55)$$

If there exists $\bar{s} > 0$ such that:

- $0 < \inf_{k \in \mathbb{N}} s_k \leq \sup_{k \in \mathbb{N}} s_k \leq \bar{s} < \frac{2}{L}$;
- $\sup_{k \in \mathbb{N}} \left(\frac{\alpha_k + \beta_k L}{s_k} \right) < \frac{1}{\bar{s}} - \frac{L}{2}$.

Then the following holds:

(i) $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and thus

$$\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k \in \mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \rightarrow +\infty$) to a critical point of f .

(iii) Furthermore, if $\alpha_k \equiv \alpha, \beta_k \equiv \beta, s_k \equiv s$, then the previous conditions reduce to

$$\alpha + \beta L + \frac{sL}{2} < 1.$$

If, in addition, $\alpha \neq \frac{\beta}{\beta + s}$, and $\alpha > \beta L$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a local minimum of f .

Remark 7.2.11. If α_k, β_k, s_k are given as in $(\mathbf{ISEHD-Disc})$, i.e. $\alpha_k = \frac{1}{1 + \gamma_k h}, \beta_k = \beta h \alpha_k, s_k = h^2 \alpha_k$, then the requirements of Theorem 7.2.10 reduce to $\beta + \frac{h}{2} < \frac{c}{L}$ (recall that c is such that $c \leq \gamma_k$).

Proof. Adjusting equation (7.29) to this setting, i.e. not using the dependent explicit forms of α_k, β_k, s_k , we get an analogous proof to the one of Theorem 7.2.7. We omit the details for the sake of brevity. \square

7.3 Inertial System with Implicit Hessian Damping

7.3.1 Continuous-time dynamics

We now turn to the second-order system with implicit Hessian damping as stated in (\mathbf{ISIH}) , where we consider a constant geometric damping, i.e. $\beta(t) \equiv \beta > 0$. We will use the following equivalent reformulation of (\mathbf{ISIH}) proposed in [33]. We will say that x is a solution trajectory of (\mathbf{ISIH}) with initial conditions $x(0) = x_0, \dot{x}(0) = v_0$, if and only if, $x \in C^2(\mathbb{R}_+; \mathbb{R}^d)$ and there exists $y \in C^1(\mathbb{R}_+; \mathbb{R}^d)$ such that (x, y) satisfies:

$$\begin{cases} \dot{x}(t) + \frac{x(t) - y(t)}{\beta} &= 0, \\ \dot{y}(t) + \beta \nabla f(y(t)) + \left(\frac{1}{\beta} - \gamma(t) \right) (x(t) - y(t)) &= 0, \end{cases} \quad (7.56)$$

with initial conditions $x(0) = x_0, y(0) = y_0 \stackrel{\text{def}}{=} x_0 + \beta v_0$.

7.3.1.1 Global convergence of the trajectory

Our next main result is the following theorem, which is the implicit counterpart of Theorem 7.2.1.

Theorem 7.3.1. *Let $0 < \beta < \frac{2c}{C^2}$, $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (\mathbf{H}_0) , γ is continuous and satisfies (\mathbf{H}_γ) .*

Consider (\mathbf{ISIH}) in this setting, then the following holds:

- (i) *There exists a global solution trajectory $x : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ of (\mathbf{ISIH}) .*
- (ii) *We have that $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\nabla f \circ (x + \beta\dot{x}) \in L^2(\mathbb{R}_+; \mathbb{R}^d)$.*
- (iii) *If we suppose that the solution trajectory x is bounded over \mathbb{R}_+ , then $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$,*

$$\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = \lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0,$$

and $\lim_{t \rightarrow +\infty} f(x(t))$ exists.

- (iv) *In addition to (iii), if we also assume that f is definable, then $\dot{x} \in L^1(\mathbb{R}_+; \mathbb{R}^d)$ and $x(t)$ converges (as $t \rightarrow +\infty$) to a critical point of f .*

Proof. (i) We will start by showing the existence of a solution. Setting $Z = (x, y)$, (7.56) can be equivalently written as:

$$\dot{Z}(t) + \nabla \mathcal{G}(Z(t)) + \mathcal{D}(t, Z(t)) = 0, \quad Z(0) = (x_0, y_0), \quad (7.57)$$

where $\mathcal{G}(Z) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the function defined by $\mathcal{G}(Z) = \beta f(y)$ and the time-dependent operator $\mathcal{D} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is given by:

$$\mathcal{D}(t, Z) = \left(\frac{x - y}{\beta}, \left(\frac{1}{\beta} - \gamma(t) \right) (x - y) \right).$$

Since the map $(t, Z) \mapsto \nabla \mathcal{G}(Z) + \mathcal{D}(t, Z)$ is continuous in the first variable and locally Lipschitz in the second (by (\mathbf{H}_0) and the assumptions on γ), we get from Cauchy-Lipschitz theorem that there exists $T_{\max} > 0$ and a unique maximal solution of (7.2) denoted $Z \in C^1([0, T_{\max}]; \mathbb{R}^d \times \mathbb{R}^d)$. Consequently, there exists a unique maximal solution of (\mathbf{ISIH}) $x \in C^2([0, T_{\max}]; \mathbb{R}^d)$.

Let us consider the energy function $V : [0, T_{\max}[\rightarrow \mathbb{R}$ defined by

$$V(t) = f(x(t) + \beta\dot{x}(t)) + \frac{1}{2} \|\dot{x}(t)\|^2.$$

Proceeding as in the proof of Theorem 7.2.1, we prove it is indeed a Lyapunov function for (\mathbf{ISIH}) . Denoting $\delta_1 \stackrel{\text{def}}{=} \min\left(\frac{c}{2}, \beta\left(1 - \frac{\beta C^2}{2c}\right)\right) > 0$, we have

$$V'(t) \leq -\delta_1 (\|\dot{x}(t)\|^2 + \|\nabla f(x(t) + \beta\dot{x}(t))\|^2). \quad (7.58)$$

We will now show that the maximal solution Z of (7.57) is actually global. For this, we argue by contradiction and assume that $T_{\max} < +\infty$. It is sufficient to prove that x and y have a limit as $t \rightarrow T_{\max}$, and local existence will contradict the maximality of T_{\max} . Integrating (7.58), we obtain $\dot{x} \in L^2([0, T_{\max}]; \mathbb{R}^d)$ and $\nabla f \circ (x + \beta\dot{x}) \in L^2([0, T_{\max}]; \mathbb{R}^d)$, which entails that $\dot{x} \in L^1([0, T_{\max}]; \mathbb{R}^d)$ and $\nabla f \circ (x + \beta\dot{x}) \in L^1([0, T_{\max}]; \mathbb{R}^d)$, and in turn $(x(t))_{t \in [0, T_{\max}[}$ satisfies the Cauchy property and $\lim_{t \rightarrow T_{\max}} x(t)$ exists. Besides, by the first equation of (7.56), we will have that $\lim_{t \rightarrow T_{\max}} y(t)$ will exist if both $\lim_{t \rightarrow T_{\max}} x(t)$ and $\lim_{t \rightarrow T_{\max}} \dot{x}(t)$ exist. So we just have to check the existence of the second limit. A sufficient condition would be to prove that $\ddot{x} \in L^1([0, T_{\max}]; \mathbb{R}^d)$. By (\mathbf{ISIH}) this will hold if $\dot{x}, \nabla f \circ (x + \beta\dot{x})$ are in $L^1([0, T_{\max}]; \mathbb{R}^d)$. But we have already shown these claims. Consequently, the solution Z of (7.57) is global, and thus the solution x of (\mathbf{ISIH}) is also global.

- (ii) Integrating (7.58), using that V is well-defined and bounded from below, we get that $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, and $\nabla f(x(t) + \beta\dot{x}(t)) \in L^2(\mathbb{R}_+; \mathbb{R}^d)$.

(iii) By assumption, $\sup_{t>0} \|x(t)\| < +\infty$. Moreover, since $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ and continuous, $\dot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$ and then using that ∇f is locally Lipschitz, we have

$$\begin{aligned} \int_0^{+\infty} \|\nabla f(x(t))\|^2 dt &\leq 2 \int_0^{+\infty} \|\nabla f(x(t) + \beta \dot{x}(t)) - \nabla f(x(t))\|^2 dt \\ &\quad + 2 \int_0^{+\infty} \|\nabla f(x(t) + \beta \dot{x}(t))\|^2 dt \\ &\leq 2\beta^2 L_0^2 \int_0^{+\infty} \|\dot{x}(t)\|^2 dt + 2 \int_0^{+\infty} \|\nabla f(x(t) + \beta \dot{x}(t))\|^2 dt < +\infty, \end{aligned}$$

where L_0 is the Lipschitz constant of ∇f on the centered ball of radius

$$\sup_{t>0} \|x(t)\| + \beta \sup_{t>0} \|\dot{x}(t)\| < +\infty.$$

Moreover, for every $t, s \geq 0$,

$$\|\nabla f(x(t)) - \nabla f(x(s))\| \leq L_0 \sup_{\tau \geq 0} \|\dot{x}(\tau)\| |t - s|.$$

This combined with $\nabla f \circ x \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ yields

$$\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = 0.$$

We also have that

$$\begin{aligned} \sup_{t>0} \|\nabla f(x(t) + \beta \dot{x}(t))\| &\leq \sup_{t>0} (\|\nabla f(x(t) + \beta \dot{x}(t)) - \nabla f(0)\|) + \|\nabla f(0)\| \\ &\leq L_0 \sup_{t>0} \|x(t)\| + L_0 \beta \sup_{t>0} \|\dot{x}(t)\| + \|\nabla f(0)\| < +\infty. \end{aligned}$$

Therefore, in view of (ISIH), we get that $\ddot{x} \in L^\infty(\mathbb{R}_+; \mathbb{R}^d)$. This implies that

$$\|\dot{x}(t) - \dot{x}(s)\| \leq \sup_{\tau \geq 0} \|\ddot{x}(\tau)\| |t - s|.$$

Combining this with $\dot{x} \in L^2(\mathbb{R}_+; \mathbb{R}^d)$ gives that $\lim_{t \rightarrow +\infty} \|\dot{x}(t)\| = 0$.

From (7.58), V is non-increasing, and since it is bounded from below, $V(t)$ has a limit, say \tilde{L} . Passing to the limit in the definition of $V(t)$, using that the velocity vanishes, gives

$$\lim_{t \rightarrow +\infty} f(x(t) + \beta \dot{x}(t)) = \tilde{L}.$$

On the other hand, we have

$$\begin{aligned} |f(x(t) + \beta \dot{x}(t)) - f(x(t))| &= \beta \left| \int_0^1 \langle \nabla f(x(t) + s\beta \dot{x}(t)), \dot{x}(t) \rangle ds \right| \\ &\leq \beta \left(\int_0^1 \|\nabla f(x(t) + s\beta \dot{x}(t))\| ds \right) \|\dot{x}(t)\|. \end{aligned}$$

Passing to the limit as $t \rightarrow +\infty$, the right hand side goes to 0 from the above limits on $\nabla f(x(t))$ and $\dot{x}(t)$. We deduce that $\lim_{t \rightarrow +\infty} f(x(t)) = \tilde{L}$.

(iv) As in the proof for (ISEH), since $(x(t))_{t \geq 0}$ is bounded, then (7.4) holds. Besides, consider the function

$$E : (x, v, w) \in \mathbb{R}^{3d} \mapsto f(x + v) + \frac{1}{2} \|w\|^2. \quad (7.59)$$

Since f is definable, so is E . In turn, E satisfies has the KL property. Let

$$\mathfrak{C}_1 \stackrel{\text{def}}{=} \mathfrak{C}(x(\cdot)) \times \{0_d\} \times \{0_d\}.$$

Since $E|_{\mathfrak{C}_1} = \tilde{L}$, $\nabla E|_{\mathfrak{C}_1} = 0$, $\exists r, \eta > 0$, $\exists \psi \in \kappa(0, \eta)$ such that for every $(x, v, w) \in \mathbb{R}^{3d}$ such that $x \in \mathfrak{C}(x(\cdot)) + B_r, v \in B_r, w \in B_r$ and $0 < E(x, v, w) - \tilde{L} < \eta$, we have

$$\psi'(E(x, v, w) - \tilde{L}) \|\nabla E(x, v, w)\| \geq 1 \quad (7.60)$$

By definition, we have $V(t) = E(x(t), \beta\dot{x}(t), \dot{x}(t))$. We also define $\tilde{V}(t) = V(t) - \tilde{L}$. By the properties of V above, we have $\lim_{t \rightarrow +\infty} \tilde{V}(t) = 0$ and \tilde{V} is a non-increasing function. Thus $\tilde{V}(t) \geq 0$ for every $t > 0$. Without loss of generality, we may assume that $\tilde{V}(t) > 0$ for every $t > 0$ (since otherwise $\tilde{V}(t)$ is eventually zero entailing that $\dot{x}(t)$ is eventually zero in view of (7.58), meaning that $x(\cdot)$ has finite length).

Define the constants $\delta_2 = 2, \delta_3 = \frac{\delta_1}{\sqrt{2}}$. In view of the convergence claims on \dot{x} and \tilde{V} above, there exists $T > 0$, such that for any $t > T$

$$\begin{cases} x(t) \in \mathfrak{C}(x(\cdot)) + B_r, \\ 0 < \tilde{V}(t) < \eta, \\ \max(\beta, 1) \|\dot{x}(t)\| < r, \\ \frac{1}{\delta_3} \psi(\tilde{V}(t)) < \frac{r}{2\sqrt{2}}. \end{cases} \quad (7.61)$$

The rest of the proof is analogous to the one of Theorem 7.2.1. Since

$$\|\nabla E(x(t), \beta\dot{x}(t), \dot{x}(t))\|^2 \leq \delta_2(\|\dot{x}(t)\|^2 + \|\nabla f(x(t) + \beta\dot{x}(t))\|^2), \quad (7.62)$$

and

$$\psi'(\tilde{V}(t)) \|\nabla E(x(t), \beta\dot{x}(t), \dot{x}(t))\| \geq 1, \quad \forall t \geq T. \quad (7.63)$$

We can lower bound the term $-\frac{d}{dt}\psi(\tilde{V}(t))$ for $t \geq T$ (as in (7.12)) and conclude that $\dot{x} \in L^1(\mathbb{R}_+; \mathbb{R}^d)$, and that this implies that $x(t)$ has finite length and thus has a limit as $t \rightarrow +\infty$. This limit is necessarily a critical point of f since $\lim_{t \rightarrow +\infty} \|\nabla f(x(t))\| = 0$. \square

7.3.1.2 Trap avoidance

We now show that (ISIH) provably avoids strict saddle points, hence implying convergence to a local minimum if the objective function is Morse.

Theorem 7.3.2. *Let $c > 0$, $0 < \beta < \frac{2}{c}$ and $\gamma \equiv c$. Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies (\mathbf{H}_0) and is a Morse function. Consider (ISIH) in this setting. If the solution trajectory x is bounded over \mathbb{R}_+ , then the conclusions of Theorem 7.3.1 hold. If, moreover, $\beta \neq \frac{1}{c}$, then for almost all $x_0, v_0 \in \mathbb{R}^d$ initial conditions, $x(t)$ converges (as $t \rightarrow +\infty$) to a local minimum of f .*

Proof. Since Morse functions are C^2 and satisfy the KL inequality, and x is assumed bounded, then all the claims of Theorem 7.3.1 hold.

As in the proof of Theorem 7.2.3, we will use again the global stable manifold theorem to prove the last point. Since, $\gamma(t) = c$ for all t , introducing the velocity variable $v = \dot{x}$, we have the equivalent phase-space formulation of (ISIH)

$$\begin{cases} \dot{x}(t) &= v(t), \\ \dot{v}(t) &= -cv(t) - \nabla f(x(t) + \beta v(t)), \end{cases} \quad (7.64)$$

with initial conditions $x(0) = x_0, v(0) = v_0$. Let us consider $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$F(x, v) = (v, -cv - \nabla f(x + \beta v)).$$

Defining $z(t) = (x(t), v(t))$ and $z_0 = (x_0, v_0) \in \mathbb{R}^{2d}$, then (7.64) is equivalent to

$$\begin{cases} \dot{z}(t) &= F(z(t)), \\ z(0) &= z_0. \end{cases} \quad (7.65)$$

We know from above that under our conditions, the solution trajectory $z(t)$ converges (as $t \rightarrow +\infty$) to an equilibrium point of F , and the set of equilibria is $\{(\hat{x}, 0) : \hat{x} \in \text{crit}(f)\}$. Following the same ideas

as in the proof of Theorem 7.2.3, first, we will prove that each equilibrium point of F is hyperbolic. We first compute the Jacobian

$$J_F(x, y) = \begin{pmatrix} 0_{d \times d} & I_d \\ -\nabla^2 f(x + \beta v) & -cI_d - \beta \nabla^2 f(x + \beta v) \end{pmatrix}.$$

Let $\hat{z} = (\hat{x}, 0)$, where $\hat{x} \in \text{crit}(f)$. Then the eigenvalues of $J_F(\hat{z})$ are characterized by the solutions on $\lambda \in \mathbb{C}$ of

$$\det \left(\begin{pmatrix} -\lambda I_d & I_d \\ -\nabla^2 f(\hat{x}) & -(\lambda + c)I_d - \beta \nabla^2 f(\hat{x}) \end{pmatrix} \right) = 0. \quad (7.66)$$

By Lemma 7.1.1, (7.66) is equivalent to

$$\det((1 + \lambda\beta)\nabla^2 f(\hat{x}) + (\lambda^2 + \lambda c)I_d) = 0. \quad (7.67)$$

This is the exact same equation as (7.17). Thus the rest of the analysis goes as in the proof of Theorem 7.2.3. \square

7.3.1.3 Convergence rate

We now give asymptotic convergence rates on the objective and trajectory.

Theorem 7.3.3. *Consider the setting of Theorem 7.3.1 with f being also definable. Recall the function E from (7.59), which is also definable, and denote ψ its desingularizing function and Ψ any primitive of $-\psi'^2$. Then, $x(t)$ converges (as $t \rightarrow +\infty$) to $x_\infty \in \text{crit}(f)$. Denote $\tilde{V}(t) \stackrel{\text{def}}{=} E(x(t), \beta \dot{x}(t), \dot{x}(t)) - f(x_\infty)$. Then, the following rates of convergence hold:*

- If $\lim_{t \rightarrow 0} \Psi(t) \in \mathbb{R}$, we have $E(x(t), \dot{x}(t), \beta \nabla f(x(t)))$ converges to $f(x_\infty)$ in finite time.
- If $\lim_{t \rightarrow 0} \Psi(t) = +\infty$, there exists some $t_1 \geq 0$ such that

$$\tilde{V}(t) = \mathcal{O}(\Psi^{-1}(t - t_1)) \quad (7.68)$$

Moreover,

$$\|x(t) - x_\infty\| = \mathcal{O}(\psi \circ \Psi^{-1}(t - t_1)) \quad (7.69)$$

Proof. Analogous to Theorem 7.2.4. \square

When f has the Łojasiewicz property, we get the following corollary of Theorem 7.3.3.

Corollary 7.3.4. *Consider the setting of Theorem 7.3.3 where now f satisfies the Łojasiewicz inequality with desingularizing function $\psi_f(s) = c_f s^{1-q}$, $q \in [0, 1[$, $c_f > 0$. Then there exists some $t_1 > 0$ such that:*

- If $q \in [0, \frac{1}{2}]$, then

$$\tilde{V}(t) = \mathcal{O}(\exp(-(t - t_1))) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}\left(\exp\left(\frac{t - t_1}{2}\right)\right) \quad (7.70)$$

- If $q \in]\frac{1}{2}, 1[$, then

$$\tilde{V}(t) = \mathcal{O}((t - t_1)^{\frac{-1}{2q-1}}) \quad \text{and} \quad \|x(t) - x_\infty\| = \mathcal{O}((t - t_1)^{-\frac{1-q}{2q-1}}) \quad (7.71)$$

Proof. Analogous to Corollary 7.2.6. \square

7.3.2 Algorithmic scheme

In this section, we will study the properties of an algorithmic scheme derived from the following explicit discretization discretization of (ISIH) with stepsize $h > 0$ and for $k \geq 1$:

$$\frac{x_{k+1} - 2x_k + x_{k-1}}{h^2} + \gamma(kh) \frac{x_{k+1} - x_k}{h} + \nabla f \left(x_k + \beta \frac{x_k - x_{k-1}}{h} \right) = 0. \quad (7.72)$$

This is equivalently written as

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} &= y_k - s_k \nabla f(x_k + \beta'(x_k - x_{k-1})), \end{cases} \quad (\text{ISIH-Disc})$$

with initial conditions $x_0, x_1 \in \mathbb{R}^d$, where $\gamma_k \stackrel{\text{def}}{=} \gamma(kh)$, $\alpha_k \stackrel{\text{def}}{=} \frac{1}{1+\gamma_k h}$, $s_k \stackrel{\text{def}}{=} h^2 \alpha_k$ and $\beta' \stackrel{\text{def}}{=} \frac{\beta}{h}$.

7.3.2.1 Global convergence and trap avoidance

We have the following result which characterizes the asymptotic behavior of algorithm (ISIH-Disc), which shows that the latter enjoys the same guarantees as (ISEHD-Disc) given in Theorem 7.2.7. We will again require that ∇f is globally Lipschitz-continuous.

Theorem 7.3.5. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying (H₀) with ∇f being globally L -Lipschitz-continuous. Consider algorithm (ISIH-Disc) with $h > 0$, $\beta \geq 0$ and $c \leq \gamma_k \leq C$ for some $c, C > 0$ and for every $k \in \mathbb{N}$. Then the following holds:*

(i) *If $\beta + \frac{h}{2} < \frac{c}{L}$, then $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, in particular*

$$\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0.$$

(ii) *Moreover, if $(x_k)_{k \in \mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \rightarrow +\infty$) to a critical point of f .*

(iii) *Furthermore, if $\gamma_k \equiv c > 0$, $0 < \beta < \frac{c}{L}$, $\beta \neq \frac{1}{c}$, and $h < \min\left(2\left(\frac{c}{L} - \beta\right), \frac{1}{L\beta}\right)$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a local minimum of f .*

Proof. (i) Let $v_k \stackrel{\text{def}}{=} x_k - x_{k-1}$, $\bar{\alpha} \stackrel{\text{def}}{=} \frac{1}{1+ch}$, $\alpha \stackrel{\text{def}}{=} \frac{1}{1+Ch}$, $\bar{s} = h^2 \bar{\alpha}$, $s = h^2 \alpha$, so $\alpha \leq \alpha_k \leq \bar{\alpha}$ and $s \leq s_k \leq \bar{s}$ for every $k \in \mathbb{N}$. Proceeding as in the proof of Theorem 7.2.7, we have by definition that for $k \in \mathbb{N}^*$

$$x_{k+1} = \operatorname{argmin}_{x \in \mathbb{R}^d} \frac{1}{2} \|x - (y_k - s_k \nabla f(x_k + \beta' v_k))\|^2, \quad (7.73)$$

and 1-strong convexity of $x \mapsto \frac{1}{2} \|x - (y_k - s_k \nabla f(x_k + \beta' v_k))\|^2$ then gives

$$\frac{1}{2} \|x_{k+1} - (y_k - s_k \nabla f(x_k + \beta' v_k))\|^2 \leq \frac{1}{2} \|x_k - (y_k - s_k \nabla f(x_k + \beta' v_k))\|^2 - \frac{1}{2} \|x_{k+1} - x_k\|^2. \quad (7.74)$$

Expanding and rearranging, we obtain

$$\langle \nabla f(x_k + \beta' v_k), v_{k+1} \rangle \leq -\frac{\|v_{k+1}\|^2}{s_k} + \frac{1}{h^2} \langle v_k, v_{k+1} \rangle. \quad (7.75)$$

Combining this with the descent lemma of L -smooth functions applied to f , we arrive at

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), v_{k+1} \rangle + \frac{L}{2} \|v_{k+1}\|^2 \\ &= f(x_k) + \langle \nabla f(x_k) - \nabla f(x_k + \beta' v_k), v_{k+1} \rangle + \langle \nabla f(x_k + \beta' v_k), v_{k+1} \rangle + \frac{L}{2} \|v_{k+1}\|^2 \\ &\leq f(x_k) + \left(\beta' L + \frac{1}{h^2} \right) \|v_k\| \|v_{k+1}\| - \left(\frac{1}{\bar{s}} - \frac{L}{2} \right) \|v_{k+1}\|^2. \end{aligned}$$

Where we have used that the gradient of f is L -Lipschitz and Cauchy-Schwarz inequality in the last bound. Denote $\tilde{\alpha} = \beta' L + \frac{1}{h^2}$. We can check that since $h < \frac{2c}{L}$, then $0 < \bar{s} < \frac{2}{L}$ and the last term of the inequality is negative. Using Young's inequality we have that for $\varepsilon > 0$:

$$f(x_{k+1}) \leq f(x_k) + \frac{\tilde{\alpha}^2}{2\varepsilon} \|v_k\|^2 + \varepsilon \frac{\|v_{k+1}\|^2}{2} - \left(\frac{1}{\bar{s}} - \frac{L}{2} \right) \|v_{k+1}\|^2.$$

Or equivalently,

$$f(x_{k+1}) + \frac{\tilde{\alpha}^2}{2\varepsilon} \|v_{k+1}\|^2 \leq f(x_k) + \frac{\tilde{\alpha}^2}{2\varepsilon} \|v_k\|^2 + \left[\frac{\varepsilon}{2} + \frac{\tilde{\alpha}^2}{2\varepsilon} - \left(\frac{1}{\bar{s}} - \frac{L}{2} \right) \right] \|v_{k+1}\|^2.$$

In order to make the last term negative, we impose

$$\frac{\varepsilon}{2} + \frac{\tilde{\alpha}^2}{2\varepsilon} < \frac{1}{\bar{s}} - \frac{L}{2}.$$

Minimizing for ε at the left-hand side we obtain $\varepsilon = \tilde{\alpha}$ and the condition to satisfy is

$$\bar{s} < \frac{2}{2\tilde{\alpha} + L}. \quad (7.76)$$

Recalling the definitions of $\bar{s}, \tilde{\alpha}, \beta'$, this is equivalent to

$$\frac{h^2}{1 + ch} < \frac{2}{2\left(L\frac{\beta}{h} + \frac{1}{h^2}\right) + L} \iff 2L\beta h + 2 + Lh^2 < 2 + 2ch.$$

Simplifying, this reads

$$\beta + \frac{h}{2} < \frac{c}{L},$$

which is precisely what we have assumed. Let $\delta = \left(\frac{1}{\bar{s}} - \frac{L}{2}\right) - \tilde{\alpha} > 0$, then

$$f(x_{k+1}) + \frac{\tilde{\alpha}}{2} \|v_{k+1}\|^2 \leq f(x_k) + \frac{\tilde{\alpha}}{2} \|v_k\|^2 - \delta \|v_{k+1}\|^2. \quad (7.77)$$

Toward our Lyapunov analysis, define now $V_k = f(x_k) + \frac{\tilde{\alpha}}{2} \|v_k\|^2$ for $k \in \mathbb{N}^*$. In view of (7.77), V_k obeys

$$V_{k+1} \leq V_k - \delta \|v_{k+1}\|^2. \quad (7.78)$$

and thus V_k is non-increasing. Since it is also bounded from below, V_k converges to a limit, say \tilde{L} . Summing (7.77) over $k \in \mathbb{N}^*$, we get that $(\|v_{k+1}\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, hence $\lim_{k \rightarrow +\infty} \|v_k\| = 0$. Besides, since $\bar{\alpha} < 1$

$$\begin{aligned} \|\nabla f(x_k + \beta' v_k)\| &= \frac{1}{s_k} \|x_{k+1} - y_k\| \leq \frac{1}{\underline{s}} (\|x_{k+1} - x_k\| + \|x_k - y_k\|) \\ &\leq \frac{1}{\underline{s}} (\|v_{k+1}\| + \bar{\alpha} \|v_k\|) \\ &\leq \frac{1}{\underline{s}} (\|v_{k+1}\| + \|v_k\|), \end{aligned}$$

which implies

$$\|\nabla f(x_k + \beta' v_k)\|^2 \leq \delta_2 (\|v_{k+1}\|^2 + \|v_k\|^2),$$

where $\delta_2 = \frac{2}{\underline{s}^2}$. Consequently $(\|\nabla f(x_k + \beta' v_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and

$$\begin{aligned} \|\nabla f(x_k)\|^2 &= 2(\|\nabla f(x_k) - \nabla f(x_k + \beta' v_k)\|^2 + \|\nabla f(x_k + \beta' v_k)\|^2) \\ &\leq 2(L^2 \beta'^2 \|v_k\|^2 + \|\nabla f(x_k + \beta' v_k)\|^2). \end{aligned}$$

Thus, $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, hence $\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0$.

- (ii) When $(x_k)_{k \in \mathbb{N}}$ is bounded and f is definable, we proceed analogously as in the proof of Theorem 7.2.7 to conclude that $(\|v_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N}^*)$, so $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence which implies that it has a limit (as $k \rightarrow +\infty$) denoted x_∞ , which is a critical point of f since

$$\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0.$$

- (iii) When $\gamma_k \equiv c$, we let $\alpha_k \equiv \alpha = \frac{1}{1+ch}$, $s_k \equiv s = h^2\alpha$. Let $z_k = (x_k, x_{k-1})$, and $g : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ defined by

$$g : (x_+, x_-) \mapsto [(1 + \alpha)x_+ - \alpha x_- - s\nabla f(x_+ + \beta'(x_+ - x_-)), x_+].$$

(**ISIHD-Disc**) is then equivalent to

$$z_{k+1} = g(z_k). \quad (7.79)$$

To conclude, we will again use [128, Corollary 1], similarly to what we did in the proof of Theorem 7.2.7, by checking that:

- (a) $\det(J_g(x_+, x_-)) \neq 0$ for every $x_+, x_- \in \mathbb{R}^d$.
 (b) $\hat{\mathcal{X}} \subset \mathcal{A}_g^*$, where $\mathcal{A}_g^* \stackrel{\text{def}}{=} \{(x, x) \in \mathbb{R}^{2d} : x \in \text{crit}(f), \max_i |\lambda_i(J_g(x, x))| > 1\}$ and $\hat{\mathcal{X}} \stackrel{\text{def}}{=} \{(x, x) \in \mathbb{R}^{2d} : x \in \mathcal{X}^*\}$, with \mathcal{X}^* the set of strict saddle points of f .

Let us check condition (a). The Jacobian $J_g(x_+, x_-)$ reads

$$\begin{pmatrix} (1 + \alpha)I_d - s(1 + \beta')\nabla^2 f(x_+ + \beta'(x_+ - x_-)) & -\alpha I_d + s\beta'\nabla^2 f(x_+ + \beta'(x_+ - x_-)) \\ I_d & 0_{d \times d} \end{pmatrix}. \quad (7.80)$$

This is a block matrix, where the bottom-left matrix commutes with the upper-left matrix (since is the identity matrix), then by Lemma 7.1.1:

$$\det(J_g(x_+, x_-)) = \det(\alpha I_d - \beta' s \nabla^2 f(x_+ + \beta'(x_+ - x_-))).$$

Since the eigenvalues of $\nabla^2 f(x_+ + \beta'(x_+ - x_-))$ are contained in $[-L, L]$. It is then sufficient that $\alpha > \beta' L s$ to have that $\eta - \frac{\alpha}{\beta' s} \neq 0$ for every eigenvalue $\eta \neq 0$ of $\nabla^2 f(x_+ + \beta'(x_+ - x_-))$. This means that under $\alpha > \beta' L s$, condition (a) is in force. Requiring $\alpha > \beta' L s$ is equivalent to $h < \frac{1}{L\beta}$, and since we already need $h < 2(\frac{c}{L} - \beta)$, we just ask h to be less than the minimum of the two quantities.

Let us check condition (b). Let x be a strict saddle point of f , *i.e.* $x \in \text{crit}(f)$ and

$$\lambda_{\min}(\nabla^2 f(x)) = -\eta < 0.$$

To characterize the eigenvalues of $J_g(x, x)$ we could use Lemma 7.1.1 as before, however, we will present an equivalent argument. Let $\eta_i \in \mathbb{R}$, $i = 1, \dots, d$, be the eigenvalues of $\nabla^2 f(x)$. By symmetry of the Hessian, it is easy to see that the $2d$ eigenvalues of $J_g(x, x)$ coincide with the eigenvalues of the 2×2 matrices

$$\begin{pmatrix} (1 + \alpha) - s(1 + \beta')\eta_i & -\alpha + s\beta'\eta_i \\ 1 & 0 \end{pmatrix}.$$

These eigenvalues are therefore the (complex) roots of

$$\lambda^2 - \lambda((1 + \alpha) - s(1 + \beta')\eta_i) + \alpha - s\beta'\eta_i = 0. \quad (7.81)$$

If $\lambda = \frac{\beta'}{\beta'+1}$, then (7.81) becomes

$$\left(\frac{\beta'}{\beta'+1}\right)^2 - \left(\frac{\beta'}{\beta'+1}\right)(1 + \alpha) + \alpha = 0.$$

This implies that $\alpha = \frac{\beta'}{\beta'+1}$, or equivalently $\frac{1}{1+ch} = \frac{\beta}{\beta+h}$. But this contradicts our assumption that $\beta c \neq 1$, and thus this case cannot occur. Let us now solve (7.81) for $\eta_i = -\eta$. Its discriminant is

$$\Delta_\lambda = \alpha^2 + 2\alpha(\eta s(1 + \beta') - 1) + (\eta s(1 + \beta') + 1)^2 - 4\eta s\beta',$$

which can be seen as a quadratic equation in α whose discriminant $\Delta_\alpha = -16\eta s$. Since $\Delta_\alpha < 0$ (recall that $\eta, s > 0$). Therefore the quadratic equation on α does not have real roots implying

that $\Delta_\lambda > 0$. We can then write the solutions of (7.81),

$$\lambda = \frac{((1 + \alpha) + \eta s(1 + \beta')) \pm \sqrt{\Delta_\lambda}}{2}.$$

Let us examine the largest solution (the one with the plus sign) and show that actually $\lambda > 1$. Simple algebra shows that this is equivalent to verifying that

$$\Delta_\lambda > (2 - (1 + \alpha) - \eta s(1 + \beta'))^2.$$

or, equivalently,

$$\begin{aligned} (1 - \alpha)^2 + \eta^2 s^2 (1 + \beta')^2 - 2(1 - \alpha)\eta s(1 + \beta') &< \Delta_\lambda \\ &= ((1 + \alpha)^2 + \eta(s - \beta'))^2 - 4(\alpha - \eta\beta'). \end{aligned}$$

Simple algebra again shows that this inequality is equivalent to $0 < 4\eta s$, which is always true as $\eta > 0$. We have thus shown that $\hat{\mathcal{X}} \subset \mathcal{A}_g^*$.

Overall, we have checked the two conditions (a)-(b) above. Therefore the invariant set $\{z_1 \in \mathbb{R}^{2d} : \lim_{k \rightarrow +\infty} g^k(z_1) \in \hat{\mathcal{X}}\}$ has Lebesgue measure zero. This means that the set of initializations $x_0, x_1 \in \mathbb{R}^d$ for which x_k converges to a strict saddle point of f has Lebesgue measure zero. \square

7.3.2.2 Convergence rate

The asymptotic convergence rate of algorithm (ISIHD-Disc) for Łojasiewicz functions is given in the following theorem. This shows that (ISIHD-Disc) enjoys the same asymptotic convergence rates as (ISEHD-Disc).

Theorem 7.3.6. *Consider the setting of Theorem 7.3.5, where f also satisfies the Łojasiewicz property with exponent $q \in [0, 1[$. Then $x_k \rightarrow x_\infty \in \text{crit}(f)$ as $k \rightarrow +\infty$ at the rates:*

- If $q \in [0, \frac{1}{2}]$ then there exists $\rho \in]0, 1[$ such that

$$\|x_k - x_\infty\| = \mathcal{O}(\rho^k). \quad (7.82)$$

- If $q \in]\frac{1}{2}, 1[$ then

$$\|x_k - x_\infty\| = \mathcal{O}\left(k^{-\frac{1-q}{2q-1}}\right). \quad (7.83)$$

Proof. Since the Lyapunov analysis of (ISIHD-Disc) is analogous to that of (ISEHD-Disc) (though the Lyapunov functions are different), the proof of this theorem is similar to the one of Theorem 7.2.9. \square

7.3.2.3 General coefficients

As discussed for the explicit case, the discrete scheme (ISIHD-Disc) rises from a discretization of the ODE (ISIHD). However, the parameters α_k, s_k are linked to each other. We now consider (ISIHD-Disc) where α_k, s_k are independent. Though this would hide somehow the physical interpretation of these parameters, it allows for some flexibility in their choice while preserving the convergence behavior.

Theorem 7.3.7. *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be satisfying (H₀) with ∇f being globally L -Lipschitz-continuous. Consider $(\alpha_k)_{k \in \mathbb{N}}, (\beta_k)_{k \in \mathbb{N}}, (s_k)_{k \in \mathbb{N}}$ to be three positive sequences, and the following algorithm with $x_0, x_1 \in \mathbb{R}^d$:*

$$\begin{cases} y_k &= x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} &= y_k - s_k \nabla f(x_k + \beta_k(x_k - x_{k-1})). \end{cases} \quad (7.84)$$

If there exists $\bar{s} > 0$ such that:

- $0 < \inf_{k \in \mathbb{N}} s_k \leq \sup_{k \in \mathbb{N}} s_k \leq \bar{s} < \frac{2}{L}$;

$$\bullet \sup_{k \in \mathbb{N}} \left(\beta_k L + \frac{\alpha_k}{s_k} \right) < \frac{1}{s} - \frac{L}{2}.$$

Then the following holds:

(i) $(\|\nabla f(x_k)\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, and $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$, hence

$$\lim_{k \rightarrow +\infty} \|\nabla f(x_k)\| = 0.$$

(ii) Moreover, if $(x_k)_{k \in \mathbb{N}}$ is bounded and f is definable, then $(\|x_{k+1} - x_k\|)_{k \in \mathbb{N}} \in \ell^1(\mathbb{N})$ and x_k converges (as $k \rightarrow +\infty$) to a critical point of f .

(iii) Furthermore, if $\alpha_k \equiv \alpha, \beta_k \equiv \beta, s_k \equiv s$, then the previous conditions reduce to

$$\alpha + sL \left(\beta + \frac{1}{2} \right) < 1.$$

If, in addition, $\alpha \neq \frac{\beta}{\beta+1}$, $\alpha > \beta L s$, then for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a critical point of f that is not a strict saddle. Consequently, if f satisfies the strict saddle property, for almost all $x_0, x_1 \in \mathbb{R}^d$, x_k converges (as $k \rightarrow +\infty$) to a local minimum of f .

Proof. Adjusting equation (7.75) to our setting (*i.e.* not using the dependency of α_k, s_k) we get an analogous proof to the one of Theorem 7.3.5. We omit the details. \square

7.4 Numerical experiments

Before describing the numerical experiments, let us start with a few observations on the computational complexity and memory storage requirement of (ISEHD-Disc) and (ISIHD-Disc). The number of gradient access per iteration is the same for Gradient Descent (GD), discrete HBF, (ISEHD-Disc) and (ISIHD-Disc) is the same (one per iteration). However, the faster convergence (in practice) of inertial methods comes at the cost of storing previous information. For the memory storage requirement per iteration, GD stores only the previous iterate, the discrete HBF and (ISIHD-Disc) store the two previous iterates, while (ISEHD-Disc) additionally stores the previous gradient iterate as well. This has to be kept in mind when comparing these algorithms especially in very high dimensional settings.

We will illustrate our findings with two numerical experiments. The first one is the optimization of the Rosenbrock function in \mathbb{R}^2 , while the second one is on image deblurring. We will apply the proposed discrete schemes (ISEHD-Disc) and (ISIHD-Disc) and compare them with gradient descent and the (discrete) HBF. We will call $\|\nabla f(x_k)\|$ the residual.

7.4.1 Rosenbrock function

We will minimize the classical Rosenbrock function, *i.e.*,

$$f : (x, y) \in \mathbb{R}^2 \mapsto (1 - x)^2 + 100(y - x^2)^2,$$

with global minimum at $(x^*, y^*) = (1, 1)$.

We notice that its global minimum is the only critical point. Therefore this function is Morse and thus satisfies the Łojasiewicz inequality with exponent $\frac{1}{2}$ (see Remark 7.1.9). Consider (ISEHD) and (ISIHD) with the Rosenbrock function as the objective, $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ continuous and satisfying (H_γ) (*i.e.* $0 < c \leq \gamma(t) \leq C < +\infty$), and $0 < \beta < \frac{2c}{C^2}$. By Theorems 7.2.1-7.2.9 for (ISEHD), and Theorems 7.3.1-7.3.6 for (ISIHD), we get that the solution trajectories of these dynamics will converge to the global minimum eventually at a linear rate. Due to the low dimensionality of this problem, we could use an ODE solver to show numerically these results. However, we will just the iterates generated by our proposed algorithmic schemes (ISEHD-Disc) and (ISIHD-Disc). Although the gradient of the objective is not globally Lipschitz continuous, our proposed algorithmic schemes worked very well for h

small enough. This suggests that we may relax this hypothesis in future work, as proposed in [165, 118] for GD.

We applied (**ISEHD-Disc**) and (**ISIHD-Disc**) with $\beta \in \{0.02, 0.04\}$, $\gamma(t) \equiv \gamma_0 = 3$, $h = 10^{-3}$ and initial conditions $x_0 = (-1.5, 0), x_1 = x_0$. We compared our algorithms with GD and HBF (with the same initial conditions) after $2 * 10^4$ iterations:

$$x_{k+1} = x_k - \frac{h^2}{1 + \gamma_0 h} \nabla f(x_k), \quad (\text{GD})$$

and

$$\begin{cases} y_k &= x_k + \frac{1}{1 + \gamma_0 h} (x_k - x_{k-1}), \\ x_{k+1} &= y_k - \frac{h^2}{1 + \gamma_0 h} \nabla f(x_k). \end{cases} \quad (\text{HBF})$$

The behavior of all algorithms is depicted in Figures 7.1 and 7.2.

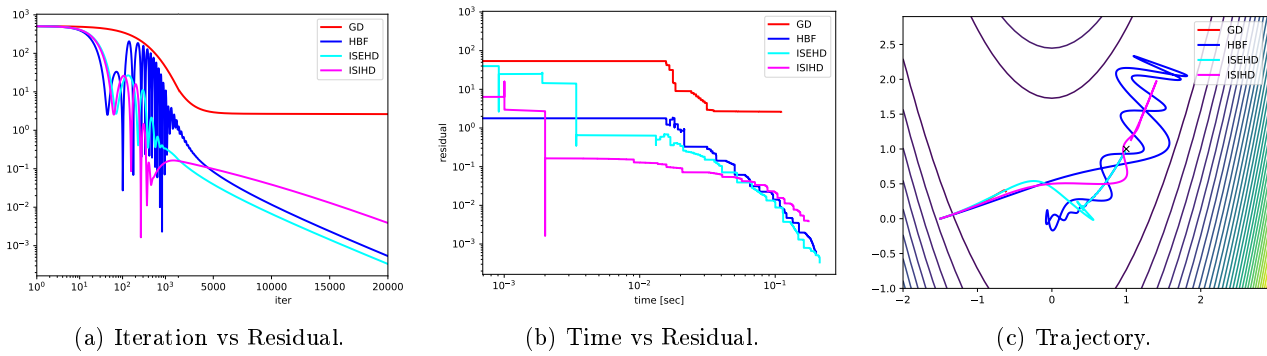


Figure 7.1: Results on the Rosenbrock function with $\beta = 0.02$.

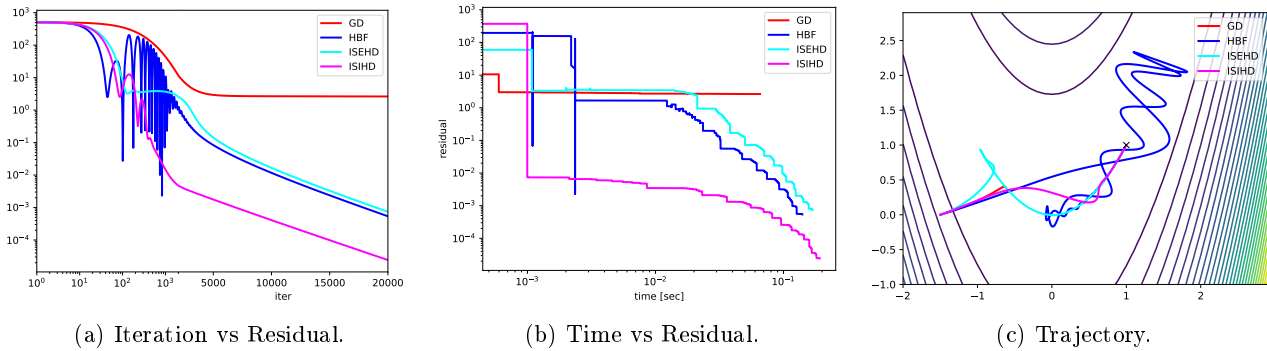


Figure 7.2: Results on the Rosenbrock function with $\beta = 0.04$.

We can notice that the iterates generated by (**ISEHD-Disc**) and (**ISIHD-Disc**) oscillate much less towards the minimum than (**HBF**), and this damping effect is more notorious as β gets larger. In the case $\beta = 0.02$, we observe there are still some oscillations, which benefit the dynamic generated by (**ISEHD-Disc**) more than the one generated by (**ISIHD-Disc**). However, we have the opposite effect in the case $\beta = 0.04$, where the oscillations are more damped. These three methods ((**ISEHD-Disc**), (**ISIHD-Disc**), (**HBF**)) share a similar asymptotic convergence rate, which is linear as predicted (recall f is Łojasiewicz with exponent $1/2$), and they are significantly faster than (**GD**).

7.4.2 Image deblurring

In the task of image deblurring, we are given a blurry and noisy (gray-scale) image $b \in \mathbb{R}^{n_x \times n_y}$ of size $n_x \times n_y$. The blur corresponds to a convolution with a known low-pass kernel. Let $A : \mathbb{R}^{n_x \times n_y} \rightarrow \mathbb{R}^{n_x \times n_y}$

be the blur linear operator. We aim to solve the (linear) inverse problem of reconstructing $u^* \in \mathbb{R}^{n_x \times n_y}$ from the relation $b = A\bar{u} + \xi$, where ξ is the noise, that is additive pixel-wise, has 0-mean and is Gaussian. Through this experiment, we used $n_x = n_y = 256$.

In order to reduce noise amplification when inverting the operator A , we solve a regularized optimization problem to recover u^* as accurately as possible. As natural images can be assumed to be smooth except for a (small) edge-set between objects in the image, we use a non-convex logarithmic regularization term that penalizes finite forward differences in horizontal and vertical directions of the image, implemented as linear operators $K_x, K_y : \mathbb{R}^{n_x \times n_y} \rightarrow \mathbb{R}^{n_x \times n_y}$ with Neumann boundary conditions. In summary, we aim to solve the following:

$$\min_{u \in \mathbb{R}^{n_x \times n_y}} f(u), \quad f(u) \stackrel{\text{def}}{=} \frac{1}{2} \|Au - b\|^2 + \frac{\mu}{2} \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \log(\rho + (K_x u)_{i,j}^2 + (K_y u)_{i,j}^2),$$

where μ, ρ are positive constants for regularization and numerical stability set to $5 \cdot 10^{-5}$ and 10^{-3} , respectively. Also f is definable as the sum of compositions of definable mappings, and ∇f is Lipschitz continuous.

To solve the above optimization problem, we have used (**ISEHD-Disc**) and (**ISIHD-Disc**) with parameters $\beta = 1.3$, $\gamma_k \equiv 0.25$, $h = 0.5$, and initial conditions $x_0 = x_1 = 0_{n_x \times n_y}$. We compared both algorithms with the baseline algorithms (**GD**), (**HBF**) (with the same initial condition). All algorithms were run for 250 iterations. The results are shown in Figure 7.3 and Figure 7.4.

In Figure 7.3, the original image \bar{u} is shown on the left. In the middle, we display the blurry and noise image b . Finally, the image recovered by (**ISEHD-Disc**) is shown on the right.

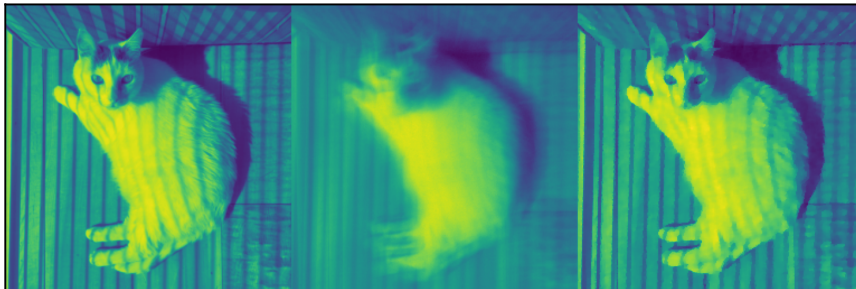
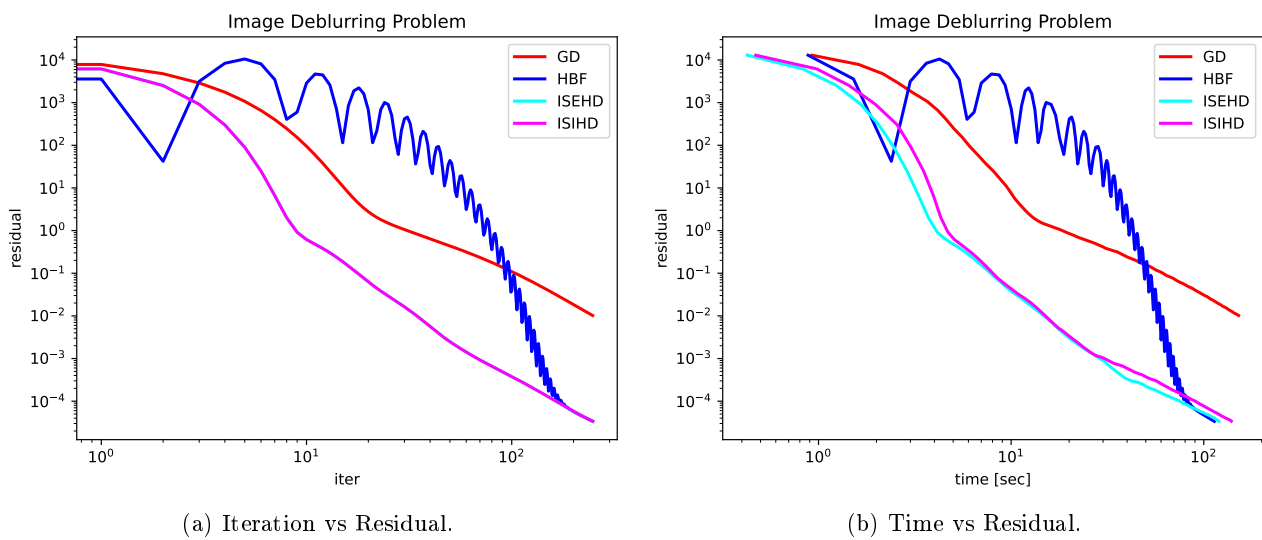


Figure 7.3: Results of the discretization of ISEHD.

In Figure 7.4, see that the residual plots of (**ISEHD-Disc**) and (**ISIHD-Disc**) overlap. Again, as expected, the trajectory of (**ISEHD-Disc**) and (**ISIHD-Disc**) has much less oscillation than (**HBF**) which is a very desirable feature in practice. At the same time, (**ISEHD-Disc**) and (**ISIHD-Disc**) exhibit faster convergence, though (**HBF**) eventually attains a similar convergence rate. Again, (**GD**) is the slowest. Overall, (**ISEHD-Disc**) and (**ISIHD-Disc**) seem to take the best of both worlds: small oscillations and a faster asymptotic convergence rate.



(a) Iteration vs Residual.

(b) Time vs Residual.

Figure 7.4: Results on the image deblurring problem.

Chapter 8

Conclusion and Perspectives

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8.1 Summary

We have exploited the close link between continuous-time dissipative dynamical systems and optimization algorithms to provide a systematic analysis of the global and local behavior of several first- and second-order systems, focusing on convex, stochastic, and infinite-dimensional settings on the one hand, and non-convex, deterministic, and finite-dimensional settings on the other hand. We have proposed new analysis tools and results, including convergence of the trajectories, in both continuous and discrete time, in expectation and almost surely, as well as global and local rates. We have also described trap avoidance properties for some of these dynamics. We have made our arguments with a high level of generality such that they apply to infinite-dimensional problems. Our assumptions have been outlined in such a way that future researchers can easily understand in which way they contribute to the arguments we make.

We summarize the main conclusions to be drawn from our work:

- (i) The works of Chapters 3-6 was intended to uncover the global and local convergence properties of trajectories of (sub)gradient-based continuous-time dynamics with stochastic errors under the umbrella of stochastic differential equations and inclusions. The aim is to solve convex optimization problems with noisy gradient input with vanishing variance. We have shed light on these properties and provided a comprehensive local and global complexity analysis both in the smooth and non-smooth case. This continuous-time perspective offers a deep insight and unveils the key properties of the dynamic of algorithms, without being tied to a specific discretization. This in turn enlightens the behavior of the sequence generated by some specific algorithm such as SGD. In turn, studying the continuous-time diffusion will allow to predict the convergence behavior of algorithmic schemes and other stochastic algorithms.
- (ii) First-order stochastic diffusion for stochastic convex optimization: Chapter 3 consider first-order stochastic differential equations and an extension to maximal monotone operators through smoothing, while Chapter 4 studies stochastic differential inclusions in a non-smooth convex setting together with a Tikhonov regularized version. Convergence properties of the trajectory and convergence rates are derived in almost sure sense and in expectation.
- (iii) Second-order stochastic diffusion for stochastic convex optimization: Chapters 5-6 are devoted to second-order stochastic inertial dynamics. Chapter 5 capitalizes on the results obtained in

the previous chapters and a technique called “time scaling and averaging” to obtain convergence properties of a second-order SDE with specific coefficients without having to go through a specific Lyapunov analysis. The case with coefficients is at the heart of Chapter 6 where one has to go through intricate Lyapunov analysis.

(iv) Second-order systems and algorithms for non-convex optimization: our main take-away messages are:

- Under definability conditions on the objective and suitable choice of the parameters, the trajectories of the proposed dynamics converge globally. Besides, in the autonomous setting, and under a Morse condition on the objective, the trajectory provably and generically converges to a local minimum of the objective.
- The same properties hold for respective proposed algorithmic schemes.
- The inclusion of the Hessian driven damping indeed helps to reduce oscillations towards critical points, and, when chosen appropriately, without harming the speed of convergence.
- The selection of Hessian driven damping parameter is important. If it is chosen too close to zero, it may not significantly reduce oscillations. Conversely, if it is chosen too large (even within the theoretical bounds), the trade-off for reduced oscillations might be a worse practical convergence speed.

8.2 Future Work

Several extensions and research avenues are possible in the context of future work. Some of the most promising ones in our opinion are the following.

Itô formula in general separable Hilbert spaces. Extension of Itô formula to $C_L^{1,1}(\mathbb{H})$ functions where \mathbb{H} is a general separable Hilbert space is rather challenging. More precisely, the goal is to extend Proposition 2.7.6 to such spaces. However, the argument in finite dimension relies on mollifiers which does not extend to infinite dimension. In the latter, a promising direction should rather go through general approximation theory of functions (see [150]).

Beyond the convex case in the stochastic setting. We think for instance to the quasi-convex case, and we refer to the recent work of [82] which offers us some perspective concerning the extension of our work to the non-convex KL setting. The second-order stochastic dynamics we proposed were not related provably to a stochastic algorithmic scheme in the small stepsize regime. Whether these dynamics can approximate some stochastic algorithm, at which rate, and how this could be used to prove trap avoidance of such algorithms is an important challenge.

Assumption (H_λ) . This assumption appears somewhat stringent though it is verified in some special cases (see Remark 4.2.2). Investigating other conditions to ensure existence of a strong solution to an SDI, extending the theory of [173], is an important question.

Maximal monotone operators. A challenging question is the extension of our general theory for non-smooth optimization problems to the case of maximal monotone operators, *i.e.*, replace ∇f by A (a maximal monotone operator) and ∂g by B (a set-valued maximal monotone operator) and investigate the convergence guarantees of the corresponding stochastic dynamics for finding the zeros of $A + B$; see the recent paper [64].

Linearly constrained convex minimization. Inspired by [22], we could consider a structured convex minimization problem under a linear constraint and use a stochastic version of the Temporally Rescaled Inertial Augmented Lagrangian System (TRIALS) to solve it.

Non-euclidian geometry. In our results in Chapter 7, we focus on the Euclidian case. Extending our results to (non-Euclidean) Bregman-type geometry, *i.e.* those where the objective is not globally Lipschitz-smooth but only relatively so, appears worth investigating in a future work.

List of Publications

Revised journal paper

- (1) R. Maulen-Soto, J. Fadili, and H. Attouch. An SDE perspective on stochastic convex optimization. *Mathematics of Operations Research* (revised), 2022. arXiv:2207.02750.

Submitted preprints

- (2) R. Maulen-Soto, J. Fadili, and H. Attouch. Tikhonov regularization for stochastic non-smooth convex optimization in Hilbert spaces. *Open Journal of Mathematical Optimization* (submitted), 2024. arXiv:2403.06708.
- (3) R. Maulen-Soto, J. Fadili, H. Attouch, and P. Ochs. An SDE perspective on stochastic inertial gradient dynamics with time-dependent viscosity and geometric damping. *SIAM J. Optim.* (submitted), 2024. arXiv:2407.04562.
- (4) R. Maulen-Soto, J. Fadili, H. Attouch, and P. Ochs. Stochastic inertial dynamics via time scaling and averaging. *Stochastic Systems* (submitted), 2024. arXiv:2403.16775.
- (5) R. Maulen-Soto, J. Fadili, and P. Ochs. Inertial methods with viscous and Hessian driven damping for non-convex optimization. *Mathematical Programming* (submitted), 2024. arXiv:2407.12518.

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