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Innovative multichannel models for pricing and inventory decisions considering service level

Minh Tam Tran

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**Modèles multicanaux innovants pour les décisions de
tarification et de stocks en considérant le niveau de service**

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MASSONNET	Guillaume	Associate Professor	IMT Atlantique	Examiner
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Abstract

The retail landscape has undergone a profound transformation in recent years, driven by technological advancements, consumer preferences, and the proliferation of digital channels. In this dynamic environment, retailers face many challenges, from optimizing pricing strategies and inventory management to navigating the complexities of multichannel retailing. This thesis explores critical opportunities and challenges in modern retail management across three domains: multichannel retailing, dual-channel pricing, and data-driven inventory management.

Firstly, we delve into the complexities of multichannel retailing, where the convergence of physical and online channels presents both opportunities and challenges for inventory management. We uncover the strategic imperatives and operational considerations that shape modern retailing by examining customer behavior, channel preferences, and market dynamics.

Secondly, building upon our exploration of multichannel retailing, we investigate the realm of dual-channel pricing, where the strategic balancing act between profitability and customer satisfaction takes center stage. Drawing on insights from market-share models and demand theory, we develop a novel pricing model that captures the complex interactions between channels and sheds light on the elusive quest for pricing decisions in a dual-channel environment.

Thirdly, to further investigate the importance of service level in retailing, we focus on data-driven inventory management, where historical data and advanced methodologies converge to inform decision-making under uncertainty. Leveraging the power of Kernel Density Estimation (KDE) and nonlinear programming (NLP), we tackle the formidable challenge of integrating service-level constraints into inventory optimization models, offering a glimpse into the future of data-driven decision-making in retail.

Our journey through multichannel retailing, dual-channel pricing, and data-driven inventory management has yielded several key insights. We have highlighted the importance of adopting a customer-centric approach, leveraging data-driven methodologies, and embracing continuous adaptation to navigate the complexities of modern retail environments.

The contributions of this thesis extend beyond theoretical insights to practical methodologies and findings. We have developed a comprehensive conceptual framework for understanding multichannel retailing, advanced dual-channel pricing optimization methodologies, and novel data-driven inventory management approaches. Through numerical experiments, we have uncovered valuable insights into the effectiveness of different retail management strategies and the potential of data-driven methodologies to drive business success.

Looking to the future, several avenues for further research beckon. Advanced demand modeling techniques, dynamic pricing strategies, supply chain resilience considerations, and sustainability and ethics in retail management represent promising areas for future exploration and innovation.

In conclusion, this thesis offers a comprehensive exploration of key challenges and opportunities in modern retail management, providing valuable insights and practical methodologies for retailers to handle the complexities of the digital age and drive business growth in an increasingly dynamic and interconnected world.

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Nomenclature

Acronyms / Abbreviations

3PL Third-Party Logistics

APICS American Production and Inventory Control Society

ASCM Association for Supply Chain Management

CSCMP Council of Supply Chain Management Professionals

EOQ Economic Order Quantity

IMCs Intermodal Marketing Companies

IoT Internet of Things

IT Information Technology

OM Operations Management

KDE Kernel Density Estimation

MCI Multiplicative Competitive Interaction

MILP Mixed-Integer Linear Programming

MNL Multinomial Logit

NLP Nonlinear Programming

SCM Supply Chain Management

SL Service Level

Chapter 1

Getting started

1.1 Problem statement

The retail business has profoundly evolved in an era of unprecedented technological advancement and rapid digital transformation. The rise of e-commerce, the proliferation of digital channels, and the growing influence of data analytics have revolutionized how consumers shop and interact with brands. In this dynamic and ever-changing environment, retailers face many challenges, from optimizing pricing strategies and inventory management to navigating the complexities of multichannel retailing.

The modern retailing is complex, uncertain, and intense competition. Retailers must grapple with shifting consumer preferences, fluctuating market trends, and disruptive technological innovations while striving to deliver seamless, personalized experiences across multiple channels. In this environment, success depends on adapting quickly, making informed decisions, and staying ahead of the curve.

The intricate interplay between pricing, inventory management, and channel strategy lies at the heart of these challenges. Retailers must carefully balance the competing demands of maximizing profitability, minimizing stockouts, and satisfying customer expectations while navigating the nuances of multichannel retailing. From the strategic allocation of inventory to the tactical execution of pricing strategies, every decision has implications for the bottom line and the business's overall success.

Against this backdrop, the need for rigorous analysis, innovative methodologies, and data-driven decision-making has never been greater. By leveraging advanced analytics, optimization techniques, and empirical insights, retailers can better understand consumer behavior, market dynamics, and competitive pressures, empowering them to make smarter, more strategic decisions.

Supply chain management in the digital age

To fulfill its promises to customers, a company must effectively utilize its resources to balance supply and demand. Supply chain management is vital in aligning the company's manufacturing and service resources with suppliers, internal partners, distribution channels, and customers. In today's business environment, information is a critical component that significantly influences supply chains and their management. With the increasing availability of information, one of the challenges businesses face today is handling and leveraging this information to create more value in the supply chain. In a customer-driven world where consumers have a stronger voice through social media, greater reach, higher expectations, and more choices, supply chain management must adapt.

The development of the internet and advancements in information technology, such as cloud storage and telecommunications capabilities, has provided supply chain managers access to vast amounts of data in real time. The emergence of the "Internet of Things" (IoT) has further accelerated this development by integrating technologies and communication solutions, enabling the interaction between physical and virtual worlds. This interaction has led to the multichannel concept, with conditions on the channel interaction and customers' expectations.

This thesis aims to develop innovative frameworks for flow management that blur the boundaries between the physical and digital supply chain and facilitate an integrated and optimized multichannel configuration.

While the operations management community has examined various aspects of online retailing, such as the costs and benefits of "buy online and pick up in store" policies, information sharing, and multichannel price optimization, there is still a research gap when it comes to optimizing end-to-end flow management in a stochastic decision-making environment.

To contextualize this thesis within the broader research outlook, the following section offers an introduction to three key research areas: operations management, supply chain management, and inventory management.

1.2 Overview on the landscape of research

In this section, we give an overview of the related base knowledge for the thesis, from the most general operations management to supply chain management and inventory management.

Operations management

The most general background that the thesis falls into is operation management. Operations management (OM) is a subject of multifaceted inquiry, marked by many issues and diverse research methodologies. Although some degree of coherence exists between the subject matter and chosen research methods, the discipline still exhibits notable, occasionally

intricate, variability. A fundamental concept within OM pertains to operations as a transformation process, wherein resources (including machinery and human labor) and inputs (encompassing materials, personnel, and information) are transmuted into concrete outputs in the form of goods and services. This foundational notion underscores the ubiquity of operations in our daily existence, extending its purview beyond manufacturing to encompass sales, services, administrative processes, and numerous other domains.

The scope of OM is extensive and hinges on the operational aspect, which has several vital facets. First and foremost is the strategic perspective, which delves into the function and objectives of operations, particularly the pivotal role it plays within an organization's business model. A key point is how operations contribute to business model innovation and development. This, in turn, informs the strategic direction for operations and how they enable a firm to remain competitive in the marketplace. Operations systems are inherently intertwined with the products and services intended to generate. The design of these systems encompasses not only internal processes but also external elements such as procurement and distribution systems. This comprehensive design process includes the planning of information and material flows, layout decisions, and the selection of appropriate process technologies for transformation activities. Operations design extends to shaping the organization, its processes, structures, and the allocation of human resources. Moreover, it entails the critical selection and development of capacity, inventory, and transformation activity planning and control mechanisms within both internal and external production and service systems. The design process also extends to the creation of supportive systems for quality assurance, system maintenance, and continuous improvement.

The field of OM exhibits distinctive characteristics that shape its approach and methodology. It is a practical discipline with a managerial orientation, addressing real-world issues and challenges. OM is inherently interdisciplinary, drawing from economics, finance, accounting, organizational behavior, marketing, mathematics, etc. Within each of these domains, research investigates many issues using diverse research methods, often informed by the perspectives of different disciplines. The research methodologies include surveys employing questionnaires and interviews, single and multiple case studies, longitudinal field studies, action research, as well as modeling and simulation techniques Karlsson (2016).

The advent of the Internet has brought about significant transformations in the operational and functional aspects of businesses. It has reshaped how companies procure resources and meet customer expectations. This shift is driven by various factors, including advances in information technology (IT), the globalization of markets, the decentralization of operations, and a growing awareness of environmental concerns. As a result, industries are compelled to

reevaluate their strategies and techniques related to productivity and quality, including their overall approach to OM.

Meredith (2001) and Zhao et al. (2009) have highlighted several significant trends in the field of OM, including the following:

- **Emphasis on resource-based theory and transaction cost theory:** These theories have gained importance in understanding and shaping OM practices.
- **Integration of operations and supply chain strategies:** The interplay between operations and supply chain strategies is explored through contingency and configuration theories.
- **Operations choice from theoretical perspectives:** Decisions regarding operations are made while drawing insights from institutional, contingency, and configuration theories.
- **Supply chain management and management issues:** The supply chain is a focal point, with an emphasis on aspects like trust, commitment, and configuration.

Our thesis aims to advance, in particular, the knowledge of supply chain management.

Supply chain management

Supply chain management (SCM) is described as the systematic and strategic alignment of conventional business operations and strategies across these operations within a specific organization, as well as among businesses throughout the supply chain, with the goal of enhancing the sustained performance of both individual companies and the overall supply chain (Mentzer et al., 2001).

The list of topics and their definitions in supply chain management are as follows. In cases where feasible, the definitions have been sourced from the Council of Supply Chain Management Professionals (CSCMP) and the American Production and Inventory Control Society (APICS) (or Association for Supply Chain Management (ASCM)).

- **Contracts and coordination:** This area centers on topics related to legal agreements and documentation. A contract is a legal arrangement between two or more parties to provide specific products or services. Coordination relates to the processes and activities within various business functions, such as marketing, sales, product design, finance, and information technology (Council of Supply Chain Management Professionals, 2013).
- **Customer service strategy:** This addresses activities that enhance or facilitate the sale or use of a seller's products or services through interactions between the buyer and seller (Council of Supply Chain Management Professionals, 2013).

- Demand management and forecasting: This field encompasses the management of all demand for goods and services to support the marketplace (APICS The Association for Operations Management, 2011).
- Human resource development: This process involves developing and maximizing human expertise through organizational development and personnel training, ultimately aimed at enhancing performance (Swanson, 2022).
- Humanitarian operations, logistics, and supply chains: This encompasses planning, implementing, and controlling the cost flow's efficiency and effectiveness and storage of goods, materials, and related information from the point of origin to the point of consumption. The primary objective is to alleviate the suffering of vulnerable populations (Swanson et al., 2018).
- Information technology and data analytics: This area involves the use of technology for the development, maintenance, and utilization of computer systems, software, and networks to process and distribute data. It also encompasses the examination, cleansing, transformation, and modeling of data to discover valuable insights, draw conclusions, and support decision-making.
- Inventory: Inventory refers to components, raw materials, work in process, finished goods, and supplies needed for the creation of goods and services. It can also denote the quantity and value of goods held in stock by a company (Council of Supply Chain Management Professionals, 2013).
- Manufacturing: Manufacturing is the process of converting raw materials, components, or parts into completed goods that meet customer expectations or specifications. Typically, manufacturing operations involve a combination of human labor and machinery, with the division of labor in large-scale production.
- Network analysis (optimization): Network analysis, or optimization, is a process or methodology used to maximize the efficiency, functionality, and effectiveness of a network. It may employ mathematical techniques to identify the best solutions (Council of Supply Chain Management Professionals, 2013).
- Performance measurement and metrics: Performance measurement involves assessing the work performed and the outcomes achieved in an activity, process, or organizational unit. These measures encompass both non-financial and financial indicators and facilitate periodic comparisons and benchmarking (Council of Supply Chain Management Professionals, 2013).

- **Product and process innovation:** Product innovation involves the introduction of new or significantly enhanced goods or services, while process innovation relates to the implementation of novel or markedly improved production or delivery methods.
- **Purchasing and supply management (planning):** This process involves identifying, prioritizing, and aggregating all sources of supply that add value to the management of a product or service at the desired level, horizon, and interval (Council of Supply Chain Management Professionals, 2013).
- **Quality:** Quality refers to the extent to which the defined characteristics of a product or service meet known requirements. It is a perceptual and somewhat subjective attribute, with its meaning contingent on specific functions and objects (Council of Supply Chain Management Professionals, 2013).
- **Relationships and collaboration:** This refers to cooperative efforts and communication among individuals and systems, including business partners, suppliers, and customers, to achieve common business objectives. It also involves a strategy for cost reduction through the consolidation of shipments from multiple sources under a mutual agreement between shippers and carriers. This approach may include collaborating with competing shippers (Council of Supply Chain Management Professionals, 2013).
- **Reverse logistics:** Reverse logistics denotes the process of shifting goods from their final destination for the sake of capturing value or ensuring proper disposal, which may involve activities such as remanufacturing and refurbishing (Reverse Logistics Association, 2022).
- **Risk management:** Risk management encompasses the identification and evaluation of risks, followed by the coordination and cost-effective allocation of resources to mitigate, monitor, and control the likelihood and impact of unfavorable events (Council of Supply Chain Management Professionals, 2013).
- **Strategy:** Strategy involves specific actions taken to achieve an objective (Council of Supply Chain Management Professionals, 2013).
- **Supply chain agility:** This refers to the ability to swiftly and cost-effectively adapt to market changes without significant negative effects on quality or reliability (Council of Supply Chain Management Professionals, 2013).
- **Supply chain integration:** Supply chain integration measures the extent to which a manufacturer strategically collaborates with its supply chain partners and manages processes both within and between organizations (Flynn et al., 2010).

- **Supply chain resilience:** Supply chain resilience denotes the extent to which the supply chain is fortified against disasters (Council of Supply Chain Management Professionals, 2013).
- **Sustainability:** Sustainability involves the strategic and transparent integration and attainment of an organization's social, environmental, and economic objectives in the systematic coordination of crucial inter-organizational business practices, all aimed at enhancing the long-term economic performance of the individual enterprise and its supply chains (Carter and Rogers, 2008).
- **Third-party logistics and outsourcing:** This entails outsourcing some or all of a company's logistics operations to a specialized company. Initially, the term "3PL" was used to identify intermodal marketing companies (IMCs) in transportation contracts. These companies, acting as intermediaries, accept shipments from shippers and tender them to rail carriers. As such, they became the third party to the contract, hence "3PL". The definition has since expanded to include any company offering logistics services for hire, often integrating various services such as transportation, warehousing, cross-docking, inventory management, packaging, and freight forwarding (Council of Supply Chain Management Professionals, 2013).
- **Transportation and logistics:** This involves the process of planning, implementing, and controlling procedures to efficiently and effectively transport and store goods, including related information and services, from the point of origin to the point of utilization in line with customer requirements. This definition encompasses inbound, outbound, internal, and external movements (Council of Supply Chain Management Professionals, 2013).

Over the last twenty years, there has been a notable emphasis on advancing theoretical research within the field of SCM, and scholars have been responsive to this call. Nevertheless, it is evident that certain subjects have received greater attention in research compared to others, implying that these areas may be deemed more "theoretical" and are progressing at a swifter pace. For a comprehensive analysis of the research on these topics, we refer to the work of Swanson et al. (2018).

Our thesis focuses on inventory management in particular, taking into account customer behaviors in a multichannel business with advancing data analytics.

Inventory management

Inventory management is a crucial aspect of operations for organizations across various industries. It involves the careful control and tracking of materials, parts, and finished goods

that a company possesses. These inventories are maintained for future use in production processes or for eventual sale to customers. Effective inventory management is important because it directly impacts a company's operational efficiency and financial performance.

The challenges associated with inventory management are multifaceted. On one hand, not having enough inventory can be costly. Shortages of essential materials and parts can disrupt production processes, leading to delays in product delivery and even stockouts of finished goods. Such stockouts can result in lost sales opportunities and harm a company's reputation. On the other hand, carrying excess inventory also incurs costs. This includes the opportunity cost of capital tied up in inventory, storage expenses, insurance premiums, taxes, and costs related to spoilage and obsolescence. Therefore, striking the right balance in inventory levels is of paramount importance for any organization.

Inventory theory is a field of study that provides analytical models and solution techniques to assist organizations in managing their inventories optimally. Its primary objective is to help organizations meet their service requirements while minimizing the total expected costs associated with ordering, holding inventory, and addressing shortages. These models take into account various tradeoffs, including economies of scale, lead time (the time taken to receive ordered items), and uncertainties in supply and demand. A fundamental outcome of inventory theory is the development of inventory-control policies. These policies dictate when to reorder items (referred to as the reorder point) and how much to order (referred to as the order quantity). The choice of an appropriate inventory-control policy depends on several factors, including the nature of demand (deterministic or stochastic, stationary or nonstationary), product characteristics (perishability), and the handling of shortages (lost sales or backlogging).

Inventory models can be categorized based on several characteristics:

- **Cost structure:** The cost structure can include fixed ordering costs, among other factors, which influence the economic order quantity and optimal inventory-control policies.
- **Demand nature:** The nature of demand can be deterministic or stochastic, stationary or nonstationary, and the demand distribution may be known or unknown.
- **Inventory monitoring:** Models can involve discrete-time or continuous-time inventory monitoring, affecting how inventory levels are tracked.
- **Planning period:** Inventory models can be single or multiple planning periods, depending on whether they consider a short-term or long-term perspective.
- **Product and stage:** Inventory models may apply to single or multiple products, as well as single or multi-stage (or location) scenarios within the supply chain.

- Supply system: The supply system can vary in terms of single- or dual-source, exogenous or endogenous supply, and finite or infinite capacity.

There are several research-based textbooks and handbooks that offer in-depth coverage and references in this field, including works by Arrow et al. (1958); Axsäter (2015); de Kok and Graves (2003); Graves et al. (1993); Hadley and Whitin (1963); Julien and David (1997); Nahmias (2011); Porteus (2002); Silver et al. (1998); Snyder and Shen (2019); Song (2023); Zipkin (2000).

Recently, inventory models have been categorized into three primary classes (Petropoulos et al., 2023):

1. Optimal inventory-control policies: This class focuses on characterizing optimal inventory control policies within specific supply and demand contexts and cost structures. A common approach involves formulating multi-period inventory decision problems as dynamic programs, simplifying the original formulations through state reduction. Researchers then identify the structural attributes of the cost function for a single period to establish the optimal policy form for individual periods. It is essential to demonstrate that these properties remain valid over time. Calculating optimal policy parameters may pose challenges, motivating the development of efficient algorithms for their computation.
2. Performance evaluation tools: This class of models is dedicated to devising effective tools for assessing the performance of specific inventory policies. This is particularly valuable when state reduction is impractical, given the exponential growth of system state dimensions over periods, often termed “the curse of dimensionality”. These models analyze continuous-review systems characterized by stochastic demand patterns and yield performance metrics for various policies, including average inventory levels, average backorders, stockout rates, and long-term average costs. Subsequently, optimization tools are employed to identify policy parameters that minimize long-term average costs.
3. Asymptotic analysis: The third class of models conducts asymptotic analysis to establish the asymptotic optimality of simple-structured policies for complex inventory systems where exact optimal policies are challenging to ascertain.

In inventory literature, there are some classic models in which optimal policies are demonstrated to exhibit uncomplicated structures: the economic order quantity (EOQ) model, the newsvendor model, and dynamic backlogging models. Our thesis utilizes the newsvendor model as a foundation for mathematical formulations.

The newsvendor model, also known as the newsboy or single-period model, is a pivotal mathematical construct within operations management and applied economics. It serves as a robust framework for optimizing inventory levels in scenarios characterized by fixed pricing structures and unpredictable demand dynamics, particularly applicable to perishable products. Historically, the origins of this mathematical conundrum can be traced back to 1888, when Edgeworth invoked the central limit theorem to ascertain the optimal cash reserves required to address random withdrawals from depositors (Edgeworth, 1888). According to Chen et al. (2016), the term “newsboy” gained prominence as it was introduced in Morse and Kimball’s book. However, the modern conceptualization of the newsvendor model is largely attributed to a paper published in *Econometrica* by Arrow et al. (1951). In this paper, Arrow et al. (1951) examined the newsvendor-type inventory planning model and derived the critical fractile solution. The newsvendor model addresses the challenge of managing inventory levels amidst demand uncertainty, particularly for seasonal products. This model operates as a single-period model, allowing for just one ordering opportunity before the selling season commences. It assumes an estimated demand distribution and negligibly low fixed order costs. Following the arrival of the ordered quantity, the selling season begins, and demand is realized. At the season’s end, there may be either unsold units (overage) or unmet demand (underage). The unit overage cost (o) corresponds to the purchasing cost minus the salvage value, while the unit underage cost (u) relates to lost profit. The optimal newsvendor order quantity is determined by the demand distribution’s critical ratio, expressed as $u/(u + o)$. This model can be extended to incorporate various complexities, including random yield, diverse cost structures, pricing considerations, distribution-free bounds (Gallego and Moon, 1993; Petruzzi and Dada, 1999; Qin et al., 2011), and multi-location scenarios featuring risk-pooling effects (Bimpikis and Markakis, 2016; Eppen, 1979). Our thesis focuses on the extension of pricing for the newsvendor problem in multichannel settings and the data-driven newsvendor problem.

While extensive research exists on inventory control utilizing the newsvendor model, there remains a significant gap in understanding inventory and pricing decisions within multichannel environments. Given the evolving landscape of shopping behaviors facilitated by technological advancements, managers need to collaboratively address pricing and inventory policies while integrating customer behaviors into their decision-making processes. Consequently, further research is warranted to develop practical models for inventory decision-making that account for customer criteria and behaviors, leveraging advanced analytics to guide inventory managers in formulating optimal strategies. For instance, how should retailers determine optimal pricing and inventory policies while considering factors such as customer service levels, demand uncertainty, and interactions between different selling

channels? Moreover, how can the application of advanced techniques, including data-driven learning models, enhance the derivation of informed decisions in this regard? These inquiries underscore the complexity of the issue and highlight the potential for valuable insights to be gained through future research endeavors.

In summary, inventory management is a multifaceted field that plays a pivotal role in the efficient functioning of organizations. By leveraging inventory theory and various modeling techniques, companies can make informed decisions regarding when and how much to order, ultimately enhancing their operational efficiency and financial performance. The ongoing research and development in this field continue to yield insights and strategies that help organizations navigate the complexities of inventory management in an ever-developing business scene.

1.3 Research questions

To address this research gap in the multichannel supply chain, we embark on a journey to explore the following key questions:

- How can supply chain flow management enhance the multichannel shopping experience with a focus on customer service level satisfaction?
- What are the unique challenges introduced by multichannel supply chain management, and how have cross-channel interactions reshaped the flow management? How can traditional flow management policies be adapted to address these challenges effectively?
- To what extent do data-driven approaches and advanced decision-making analytics contribute to the advancement of the shopping experience in multichannel retailing?

The aim of this thesis is to investigate the above research questions. Thus, each research question corresponds to one chapter of the thesis.

1.4 Scope of the thesis

This thesis comprehensively investigates the challenges and opportunities presented by integrating digital and physical supply chains within a multichannel environment. It explores various factors, including demand modeling, pricing optimization, inventory control, and customer behavior analysis. The research is geared towards developing practical solutions that can be applied by businesses seeking to thrive in this dynamic environment.

In particular, the scope of this thesis is given as follows.

- Explore the strategies and practices that supply chain flow management can employ to enhance the shopping experience across multiple channels. It aims to investigate how factors such as inventory availability, order fulfillment efficiency, and customer service quality contribute to customer satisfaction and influence purchasing behavior.
- Identify and analyze the unique challenges multichannel supply chain management introduces. It seeks to understand how the integration of various channels complicates traditional flow management practices and impacts inventory management and order fulfillment. Additionally, it aims to propose innovative solutions and adaptations to traditional policies to effectively address these challenges by incorporating customer behavior and retailer offers of promotion to prevent stockout.
- Explore the role of data analytics and advanced analytical techniques in optimizing supply chain flow management and enhancing the multichannel shopping experience. It aims to investigate how retailers can leverage data insights to improve inventory management, order quantity optimization, and personalized customer experiences. Additionally, it seeks to assess the impact of data-driven approaches on operational efficiency, cost reduction, and revenue growth in multichannel retail environments.

1.5 Structure of the thesis

The thesis unfolds in five parts, each dedicated to addressing critical aspects of operations management, supply chain management in general, and inventory management in particular:

- **Part I (Chapter 1: Getting started).** Chapter 1 introduces the research topic, emphasizing the shifting panorama of retailing amidst technological advancements and centering consumer behavior. We highlight the importance of robust analysis, innovative methodologies, and data-driven decision-making to optimize supply chain flow management. The chapter outlines research questions and associated objectives while providing an overview of recent advances in inventory, operations, and supply chain management. This sets the stage, underlining the impact of technology, data availability, and changing consumer expectations, urging innovative approaches to adapt to multichannel retail complexities and elevate customer experiences.
- **Part II (Chapter 2: Optimal pricing for a multichannel stochastic demand with service levels).** Chapter 2 delves into the complexities of multichannel retailing, where multiple physical and online sales channels coexist. Within this dynamic setting, we confront the challenge of simultaneously optimizing prices and inventory levels while

adhering to service-level requirements. The problem involves determining optimal prices across channels to maximize expected profit while meeting service levels. This challenge is compounded by the fact that customer demand is influenced by the price of the corresponding channel and the prices set in other channels. Our research thus explores innovative pricing strategies that consider the stochastic nature of global demand and the non-linear relationships between prices and channel demand to address this intricate problem.

- **Part III (Chapter 3: Optimal pricing for dual-channel retailing with stochastic attraction demand model):** Chapter 3 extends the investigation into dual-channel retailing, where companies operate a physical store and an online sales platform. This chapter grapples with optimizing prices when customer behavior is pivotal in channel choice. The problem entails determining optimal pricing strategies that maximize expected profit across both online and physical channels while factoring in the uncertainty of potential market demand and the influence of prices on channel selection. Further complexity arises from customers' diverse reactions when physical stores experience stockouts, with some opting to purchase online in response to promotions.
- **Part IV (Chapter 4: Data-driven nonlinear optimization for the chance-constrained newsvendor problem):** Chapter 4 outlines our innovative solution to address the data-driven newsvendor problem. Our methodology utilizes Kernel Density Estimation (KDE) to represent service level constraints and investigates a nonlinear programming (NLP) formulation. Importantly, our approach guarantees that the achieved service level exceeds the target, affirming its efficacy in improving inventory decision-making. This study advances inventory management practices by harnessing data-driven methodologies and novel modeling techniques, furnishing businesses with more resilient and dependable strategies for navigating dynamic and uncertain conditions.
- **Part V (Chapter 5: Conclusion and perspectives):** In this concluding chapter, we offer a thorough overview of the thesis, emphasizing its narrative, the acquired insights, contributions to inventory, operations, and supply chain management, and prospects for further research. We delve into the exploration of multichannel retail, dual-channel pricing, and data-centric inventory management, weaving connections across chapters and underlining the importance of our discoveries.

Each of the three chapters, Chapter 2, Chapter 3, and Chapter 4, starts with the general abstract, introduction, literature review, research problems and solutions analysis, and conclusion, corresponding to the three research questions.

1.6 Related publications

Here, we represent all the publications linked to the thesis.

- Tran, M. T., Rekik, Y., & Hadj-Hamou, K. (2024). Optimal pricing for dual-channel retailing with stochastic attraction demand model. *International Journal of Production Economics*, 268, 109127.
- Tran, M. T., Rekik, Y., & Hadj-Hamou, K. (2021, August). Pricing Decisions for an Omnichannel Retailing Under Service Level Considerations. In *IFIP International Conference on Advances in Production Management Systems* (pp. 175-185). Cham: Springer International Publishing.
- Tran, M. T., Rekik, Y., & Hadj-Hamou, K. Optimal pricing for a multi-channel stochastic demand with service levels. Under revision in *OR Spectrum*, 2024.
- Tran, M. T., Rekik, Y., & Hadj-Hamou, K. Data-driven nonlinear optimization for chance-constrained newsvendor problem. Finalized and targeted to *International Journal of Production Economics*, 2024.

Chapter 2 of this thesis constitutes the revised manuscript submitted to the *OR Spectrum* journal and the proceeding (Tran et al., 2021). Chapter 3 represents the paper published in the *International Journal of Production Economics* (Tran et al., 2024). Chapter 4 involves the manuscript preparing for submission to the *International Journal of Production Economics*.

1.7 Summary for the first chapter

In this chapter, we introduce the research topic, shedding light on the evolving domain of retailing amidst technological advancements and consumer behaviors. Emphasizing the significance of rigorous analysis, innovative methodologies, and data-driven decision-making, we explore the opportunities and challenges faced in optimizing supply chain flow management.

The chapter outlines the research questions that guide the study and the objectives associated with each research question. In particular, we aim to investigate how customer behaviors impact supply chain management practices, explore the implications of consumer expectations and requirements on operations management strategies, and assess the role of innovative approaches in adapting to the complexities of multichannel retailing.

Furthermore, the chapter provides an overview of recent advances and topics in inventory management, operations management, and supply chain management. This discussion

underscores the necessity for inventive solutions to adapt to the developing retail practice, driven by technological advancements, data accessibility, and evolving consumer demands.

Overall, this chapter serves as a comprehensive introduction to the research topic, laying the groundwork for subsequent discussions on theoretical frameworks, methodology, and findings of the study.

Chapter 2

Optimal pricing for a multichannel stochastic demand with service levels

In practice, the service level (SL) is used to measure service quality and to set the inventory policy. As retail enterprises integrate physical and online stores, one question arises regarding multichannel retail under SL requirements. To investigate this issue, we propose a multichannel inventory-pricing model, price-based stochastic demand, and SL-based order quantity. Motivated by market-share models, we adopt a demand model involving attraction between channels. In contrast to previous works, we consider that the global demand is stochastic and channel demand is a non-linear function of prices across channels. The chapter delivers two contributions. First, by attraction model structure, the profit function is not jointly concave with its variables; thus, we propose a novel approach to prove that in the case of target SLs, the objective function is still reasonably well-behaved. In particular, we show the existence of a unique solution to the first-order conditions with non-linear demand and objective functions under mild assumptions. The second contribution is related to the applicability of the model in practice. In particular, we characterize the behavior of the optimal prices and analyze the performance gap with different SL and price settings. We find that common knowledge on service levels in inventory control cannot apply in the same way under a multichannel configuration.

2.1 Introduction

Multichannel retail is the practice of selling merchandise and services to consumers on more than one sales channel (Levy et al. (2012)). Multichannel retailers are companies that participate in multichannel retailing as their principal source of revenue. Today, many leading

retailers adopt a multichannel business model and modern innovations to integrate their multiple selling channels and to provide clients with a comprehensive buying experience. These channels frequently aim to captivate several consumer segments and enable retailers to cater to different purchasing habits and preferences (Kireyev et al. (2017)).

Multichannel is an inventory management study cluster with various difficulties concerning demand modeling, pricing optimization, and inventory decision-making (Tsay and Agrawal (2004)). The inventory of each channel in a multichannel setting affects how efficiently the orders from customers are filled. Sharing the demand through these channels depends moreover on the associated prices, which raises another challenge. Despite recent investigations on multichannel retailing, few studies take into account the influence of cross-channel interaction and channel integration on store operations, and how retailers should respond (Mou et al. (2018)).

Furthermore, profitability in retailer pricing has become a paramount concern. While attempting to maximize profitability, retailers must pay attention to every marketing mix component. Pricing has always been a crucial strategic aspect for them to “get right” (Kireyev et al. (2017)).

In this context, this research presents a cross-channel price interaction-based framework that structures the customers’ demand and decision process through which the retailer can develop and implement a multichannel pricing strategy. In particular, an attraction demand model is employed to quantify the cross-channel interaction. This model is drawn from Harsha et al. (2019) and used to represent the price-dependent and integrated decision under multichannel configurations. Here, from a more general perspective than that of Harsha et al. (2019), the potential market demand is stochastic and has a known distribution.

To shed light on the multichannel pricing problem, we consider a pricing and inventory decision model for a retail company selling a single product to consumers through several channels. We concentrate on merchandise rather than service retailing since product management and delivery issues are significantly more straightforward for service retailing (Zhang et al. (2010)). The model incorporates the effects of service level on the retailer’s pricing and ordering decisions. In particular, the retailer is interested in determining the prices corresponding to its selling channels to maximize its expected profit when the order quantity satisfies the constraints at a service level. In our settings, each channel’s demand depends not only on its related price but also on the other channels’ prices. Given the potential market stochastic demand, each channel’s demand is determined through an attractive model (a function of all the selling prices across channels). The price-setting single-period newsvendor model then determines the profit of each channel. Thus, the proposed model applies to

seasonal products as well as products with short life cycles. It is also applicable to perishable products when units carried on the shelf are all of the same age.

The optimization problem aims to maximize the retailer's total expected profit while respecting a target service level for each channel. We focus on the relationship between expected profit and the service level for a price-setting multichannel model since the service level (SL) is frequently used by inventory managers as a metric to measure business quality and set the inventory policy, rather than using the shortage cost to represent the economic consequence of a stock out (Abad (2014); Wei et al. (2021)). Thus, the equality constraints are set initially on the α -type service level - the probability that the demand does not exceed the order quantity. Consequently, it is a non-linear chance-constrained problem, and given the equality constraints, there are numerous mathematical difficulties. Most of these are related to structure, the stability of transforming chance constraints into deterministic constraints, and the existence and uniqueness of the optimal solution.

This might raise a question concerning the convergence of numerical methods to determine the solution. Consider a particular form of the attractive model with deterministic demand, the multinomial logit model Hanson and Martin (1996) construct an example in which the profit function is not jointly concave (and also, not jointly quasi-concave) on its decision variables (prices across channels and related order quantities). Thus, in our setting, when demand is stochastic, without the service level constraints, the profit function is not jointly concave (and quasi-concave) in both prices and order quantities. Therefore, determining the optimal solution for the optimization problem without constraints may require a complex and difficult search procedure.

Despite these challenges, we prove that the optimization problem under the service level requirements behaves well mathematically. In particular, we transform the initial pricing problem (when the decision variables are both prices and order quantities and the objective function is not jointly quasi-concave) to an equivalent unconstrained problem in which the decision variables are only the prices. We show the existence and uniqueness of a vector of prices that satisfies the first-order conditions. This is the optimal solution to the equivalent unconstrained problem with a non-linear objective function and stochastic non-linear attraction channel demand. It implies that a single vector of prices and order quantities exists that solves the initial multichannel pricing problem under the inventory service level. Thus, a simple search procedure should be sufficient to find the optimal solution.

The main contributions of this chapter could be summarized as follows:

- **Multichannel demand modeling:** We introduce a multichannel stochastic demand model on which a series of advanced multichannel retail studies and decisions can be built. While considering that the potential market demand faces uncertainty, our

model employs the attraction choice model to quantify the proportion of each channel's demand and the cross-channel interactions. This framework enables us to simulate multichannel demand for big retail chains that launch many stores (or channels).

- **Multichannel price optimization with stochastic demand:** We study a single-period product's multichannel pricing problem (MCPS) when the potential market demand meets uncertainty with known distribution. The MCPS objective is to maximize total expected profit across all channels while satisfying service-level performance constraints. Although the objective function of MCPS is not jointly concave on its decision variables (prices across channels), we show that there exists a unique solution to the first-order conditions under mild assumptions, and that this is the global solution to our MCPS problem. The solution procedure should be computationally efficient in calculating the optimal price.
- **Management value assessment:** We perform a management value assessment based on a numerical analysis for different SL settings. We analyze the variation of optimal prices, order quantities, market share, and expected profit with SL. The case study illustrates that when the service level is high enough, there is always a trade-off between the total expected profit and the service level. This suggests that inequality-constrained and equality-constrained problems are equivalent if the minimal service level (and target service level) is high enough.

2.2 Literature review

2.2.1 Service level and inventory management

One of the most fundamental and difficult core problems in inventory management is to increase profits over time while offering high-quality customer service in the presence of demand uncertainty. Many inventory systems can be classified as either a backorder, a lost-sales system, or a combination of the two.

In the inventory control literature, backorder and lost-sales inventory models have been widely examined in the so-called canonical cost-based model, where there is no specific targeted service level guarantee, and the goal is commonly to minimize the predicted total holding, ordering, and stock-out costs (Zipkin (2000)). A widespread misconception is that optimizing a lost-sales inventory system is considerably more difficult than optimizing a backorder inventory system when there is positive lead time (i.e., the time lag between when a purchase order is placed to replenish products and when the order is received in the

warehouse). While the backorder system typically has an ideal order-up-to (or base-stock) policy (e.g., Zipkin (2000)), the lost-sales system is complicated to analyze, and the structure of its optimal policy appears to be little understood. As an exact solution is unavailable (for a literature overview, see Bijvank and Vis (2011)), asymptotic analysis has been provided by many academics.

While much of the inventory literature has previously concentrated on the cost-based model due to analytical tractability rather than proper characterization of reality (Bertsimas and Paschalidis (2001)), there are also works in the literature that study inventory policy using a so-called service-based model. This is because the cost of unfulfilled demand is frequently challenging to quantify in practice (e.g., Chen and Krass (2001)). Service level is therefore generally employed as a more direct criterion for evaluating the performance of inventory replenishment procedures (Ghiani et al. (2004)).

In practice, the notion of service levels is actually frequently used. Service levels are one of the key performance indicators used by many businesses, including Walmart (Xin et al. (2017)). Service level agreements are severely enforced in some industries (such as semiconductors) to ensure product delivery (Katok et al. (2008)). When determining reorder points or order-up-to levels, the enterprise resource planning system SAP Retail (a commercial version of ECC 6.04) uses only service level (Rawe (2008)). Research by Jing and Lewis (2011), Craig et al. (2016) shows that inventory service levels and stockout rates have a significant influence on demand.

There are several ways to determine a service level in inventory literature. The event-oriented α -service level (or “ready rate”) is defined as the probability of no stock-out. The quantity-based β -service level (or “fill rate”) is defined as the proportion of total demand instantaneously satisfied without time lag, capturing the stock-out event and the amount of stock-out. The time-and-quantity-related γ -service level is defined to represent the number of backorders and the time it takes for the backorder requests to be fulfilled. Among these three, the α -service level is one of the most commonly utilized service level criteria in both the literature and practice of inventory management (e.g., Rawe (2008), Snyder and Shen (2019), Jiang et al. (2019)). Our research adopts the α -type service level as a performance constraint for our model.

In inventory management, one line of study is to investigate the inventory models under service level constraints in a multi-period setting. Bitran and Yanasse (1984) propose a deterministic approximation to stochastic production problems. Chen and Krass (2001) study an inventory model with periodic review, constant lead time, and i.i.d. demands under minimal service level constraints. Wei et al. (2021) investigate a deterministic approximation of inventory systems given the constraints on sequential α -service level. Sachs et al. (2022)

analyze a manufacturer's ordering decisions from the perspective of a multi-product, multi-period problem with an aggregate service-level constraint in which the problem can be broken down on a daily basis and considered as a single-period problem.

Another line of research is investigating the effect of service level on the single-period models. In the inventory world, the single-period problem is often referred to as a newsvendor problem.

The newsvendor problem is one of the main issues in operations management. The fundamental model considers a decision-maker who must choose how many perishable products to order to maximize expected profit in the presence of stochastic demand. Its natural extension, the problem of price-setting or a monopolistic newsvendor, has been studied extensively. Many model variations and extensions have been developed in the literature (Choi (2012)). A review of these models is provided by Petruzzi and Dada (1999).

The service level is useful for a newsvendor-based problem. In particular, when demand faces uncertainty, the newsvendor must make an implicit tradeoff between profit and the potential loss of goodwill and future profits related to a stock-out situation. The difficulty is reduced using the shortage cost technique to either maximize implicit profit or minimize implicit cost. However, it can be challenging to estimate the cost of a shortage. Practitioners interpret the service level and cost as conflicting goals and the inventory control as a dual-criterion issue. It follows that they will prefer the service level approach when it is available (Abad (2014)).

In general, it is noteworthy that the α -service level is helpful for single-period problems. In particular, solving a single-period problem with underage (u) and overage (h) unit cost configuration is equivalent to solving it by setting the probability of not stocking out, i.e., the service level equal to the ratio $\frac{u}{u+h}$. Even if it is not explicitly mentioned as a metric in solving the single-period problem, the service level is used. Results deduced from the single-period model could moreover be applicable in a multi-period setting under some conditions where the latter is a series of independent single-period problems. This is the case of myopic policies. Veinott Jr (1965a,b) established a set of conditions for optimizing the myopic base stock policy for different dynamic inventory settings. Heyman and Sobel (1984) provide some practical examples where solving the dynamic multi-period problem is as easy as solving an equivalent static single-period problem. This is the case, for instance, when a retailer uses debt financing, ensuring that the preference is to avoid storing unsold items and ordering each selling period (Heyman and Sobel (1984)). Perishable products which should be disposed of at the end of each period could fit with a myopic policy.

Existing inventory papers that consider α -service level constraints in a single-period setting include Jammerneegg and Kischka (2013), Abad (2014), and Jiang et al. (2019).

Jammerneegg and Kischka (2013) study the optimal ordering decisions under the condition that the service level must be greater than the critical service level, and the probability of actual profit being negative must be lower than the predetermined limit. Abad (2014) investigates a price-setting single-channel newsvendor model under service level with both additive and multiplicative demand uncertainty, and studies its optimal strategy. Jiang et al. (2019) study a model that incorporates the service-level constraints and the penalty cost for each unsatisfied demand. Both of them consider a single-channel scenario.

Compared to the existing models in the service level literature, our model shares similarities in the adoption of α -service level as the constraint for a single-period model. However, unlike the existing literature, we focus on studying the optimal decisions for a price-setting newsvendor-based model in a multichannel configuration.

2.2.2 Multichannel and pricing decisions

In business management, multichannel and its particular case of a dual-channel supply chain are the clusters for inventory management research with several challenges of demand modeling, price optimization, and inventory decision-making (Tsay and Agrawal (2004)). In the multichannel literature, the studies focus on channel selection (Matsui (2018); Zhang et al. (2017)), channel coordination (Atan et al. (2018); Gallino et al. (2017); Haitao Cui et al. (2007)), channel competition (Bernstein et al. (2009); Lan et al. (2018); Ouardighi et al. (2013)), and pricing with decision optimization (He et al. (2020)). Despite recent investigations on multichannel retailing, few studies consider the impact of channel integration on store operations and how retailers should respond (Mou et al. (2018)).

There are several options for choosing the demand model, depending on the features it captures and the relevant estimation parameters using historical sales data. Attraction demand models are one of the most often employed demand functions to characterize buyer behavior in the marketing, economics, and revenue management literature, both in empirical studies and theoretical models (see Leeflang and Wittink (2000)). They generalize the widespread multinomial logit (MNL) and multiplicative competitive interaction (MCI) demand models, which are based on the random utility theory of economics (Harsha et al. (2019)). According to Luce (1959), the attraction demand model can be axiomatically constructed from basic presumptions about customer behavior. The attraction demand model, as detailed in Anderson et al. (1992) and Lariviere (1999), has been used successfully in econometric research to estimate demand and is becoming more widely recognized in marketing (Besanko et al. (1998)). For its uses in the operations management community, see So (2000), Bernstein and Federgruen (2004), Gallego et al. (2006), Harsha et al. (2019), and references therein.

In the multichannel field, we employ these attraction demand functions to simulate consumer demand and the channel choice of a consumer. The multichannel demand model that comes closest to ours is the one in Harsha et al. (2019), which analyzes a joint price and fulfillment planning problem. However, three critical features differentiate their work from ours. First, in a price-setting newsvendor framework, we consider the service-level measure “non-stockout probability”, which results in a more sophisticated objective function. Second, Harsha et al. (2019) essentially assumes deterministic demand functions. In comparison, we assume stochastic demands. Third, we consider a general multichannel setting without requiring that the prices be the same for some channels. Thus, our work complements theirs in certain dimensions, to set it closer to practical considerations in retailing.

Our research is related to the existing literature on pricing decisions of a multichannel supply chain. In marketing and operations management, pricing and inventory decisions of multichannel retailing have received considerable attention (Li and Mizuno (2022)). The multichannel supply chain’s pricing and inventory control problem has been studied in some works in a multi-period setting, with the assumption that the demand in each channel is deterministic. These works have endeavored to bring managerial insights and perspectives to increase long-term company profitability (He et al. (2020); Li et al. (2021); Moon et al. (2010)). Several works have studied the determination of optimal prices and inventory/order quantities for a multichannel supply chain with uncertain demand in a single-period environment. Dumrongsiri et al. (2008) have developed a model involving a manufacturer and a retailer over a consumer choice process with a price and service quality-sensitive demand. Huang et al. (2021) examine the optimal pricing decisions of both retailer and manufacturer with the stochastic demand in a Stackelberg game framework. Lan et al. (2018) analyze competition and coordination in a three-tier supply chain with two competing suppliers. Modak and Kelle (2019) investigate inventory decisions of a multichannel supply chain when stochastic consumer demand depends on both price and delivery time. Qiu et al. (2021) formulate a Stackelberg game model for a multichannel supply chain under different order fulfillment policies and study pricing, ordering, and order fulfillment decisions with a distribution-free approach.

To the best of our knowledge, there is no research that explicitly analyzes retailers’ pricing and ordering problems in a multichannel environment when the service level acts as a performance constraint. Thus, in line with this research stream, we study the multichannel optimal pricing and ordering decisions, given the service level constraints.

2.3 Proposed multichannel stochastic demand and optimization problem

2.3.1 Multichannel stochastic demand model

In this section, we present the design of a stochastic multichannel demand framework to quantify cross-channel demand interactions on which a series of advanced retailing decisions can be built.

Let us consider a multichannel retailer selling a single perishable product to customers through n channels $I = \{1, 2, \dots, n\}$. Let r_i be the price for the product sold via channel $i \in I$, and $\mathbf{r} = (r_1, r_2, \dots, r_n)$ be the corresponding vector of prices in all the channels. Let $\mathbf{D} = (D_1, D_2, \dots, D_n)$ be the vector of demand in all the channels. In order to simulate customer preferences across channels in a multichannel environment, we assume that the demand for the product in a specific channel depends on the attributes of all the channels.

There are several options for selecting the demand model, depending on the attributes it captures as well as the practical estimation parameters using past sales data. In the marketing, economics, and revenue management literature, attraction demand models are one of the most widely used demand functions to describe consumer choice. They are based on economics' random utility theory and generalize the well-known multinomial logit (MNL) and multiplicative competitive interaction (MCI) demand models (Harsha et al. (2019)). In the multichannel research field, we employ these attraction demand functions to simulate consumer demand and the channel choice of a consumer. In particular, we assume that the demand D_i of channel i has the following form:

$$D_i(\mathbf{r}) = \text{Potential Market Size} * \text{Market Share of channel } i \quad (2.1)$$

$$= \xi \frac{g_i(r_i)}{g_0 + \sum_{i \in I} g_i(r_i)}, \quad (2.2)$$

where ξ is the market size, $g_i(r_i)$ is the attraction function of buyers to channel i , g_0 is the attractiveness of the no-purchase option, assumed to be positive.

This demand modeling framework for the multichannel environment accurately quantifies cross-channel demand interactions. The potential market size shows the upper bound of the number of people who are interested in the product, whereas the market share, also known as the purchase/choice probability, describes how people choose amongst multiple options, including not buying. Here, the potential market size, ξ , is assumed to be stochastic. It is treated as a continuous random variable, characterized by its probability density functions (PDF) $f_\xi(x)$ and cumulative distribution functions (CDF) $F_\xi(x)$. The model is thus referred to

as the stochastic multichannel demand model. It is similar to the Luce selection model (Luce (1959)). In the literature, there are some commonly used attraction models: linear attraction model: $g_i(r_i) = a_i - b_i r_i$, with $a_i > 0, b_i > 0, \min a_i - b_i r_i > 0$; multinomial logit (MNL) model: $g_i(r_i) = \exp(a_i - b_i r_i)$ with $a_i > 0, b_i > 0$; multiplicative competitive interaction (MCI) model: $g_i(r_i) = a_i r_i^{-b_i}$ with $a_i > 0, b_i > 1$ (Gallego et al. (2006)). In general, a_i, b_i are constants that ensure the negative price elasticity of demand. We assume that the attractive function satisfies the following standard conditions.

Assumption 1. Consider an attraction function $g : \mathbb{R} \rightarrow \mathbb{R}^+$. It then satisfies:

- i. $g(\cdot)$ is strictly decreasing and is twice differentiable on \mathbb{R} ,
- ii. $\lim_{x \rightarrow \infty} g(x) = 0$, and $\lim_{x \rightarrow \infty} xg(x) = 0$.

Assumption 1 states that the attractiveness of a channel is decreasing in its corresponding price. This assumption implies that $\frac{g_i(r_i)}{g_0 + \sum_{i \in I} g_i(r_i)}$ is decreasing in r_i and increasing in $r_{i'}$ for $i' \neq i$, which can be interpreted as follows: the expected demand for channel i is decreasing in its own retailing price and increasing in the retailing prices of the other channels. It also ensures that a channel's contribution (channel's demand) becomes zero as its price becomes arbitrarily large.

Our formulation includes information about the no-purchase part. It refers to the instances when consumers find all the available purchase choices less attractive. Mathematically, it refers to the component $\xi \frac{g_0}{g_0 + \sum_{i \in I} g_i(r_i)}$. In practice, g_0 is often normalized by setting $g_0 = 1$ (Harsha et al. (2019); Subramanian and Harsha (2021)).

The next section investigates a pricing problem based on the stochastic multichannel demand model.

2.3.2 Multichannel price optimization with stochastic demand and service level

In this section, we propose a multichannel pricing optimization framework for seasonal perishables as well as products with short life cycles. The purpose is to find the best prices across all channels, taking into account a variety of retailer goals and actual business standards. We focus on the relationship between the retail chain's expected profit and the service level. This link is essential since practitioners prefer to investigate the economic consequences of a stock out using the service level measure rather than the shortage cost (Abad (2014); Wei et al. (2021)).

In our proposed stochastic multichannel pricing framework, the newsvendor model is used to determine the order quantity and the selling price for every single channel. The newsvendor approach is used for inventory management of seasonal products, products with a short life cycle, or perishable products like clothing, electronics, and so on. It can also be used to support long-term strategic capacity investment (Van Mieghem (2003)). Thus, our stochastic multichannel pricing model is generally valid for any type of product, such that the newsvendor strategy is applicable. From a mathematical perspective, it allows the integrated pricing problem across the retail chain to be seen as a single-period pricing problem.

Using the notation introduced in Section 2.3.1, for each channel $i \in I$, let c_i denote the unit item retailing cost (including purchase cost, operating cost, and shipping cost if necessary) ($c_i \leq r_i$); s_i ($0 \leq s_i \leq c_i$) the unit salvage value; and Q_i ($Q_i \geq 0$) the order quantity. Let \mathbf{Q} be the vector of order quantity, \mathbf{c} be the vector of retailing cost per unit, and \mathbf{s} be the vector of salvage values for unsold units left at the end of the period. If $Q_i \geq D_i$, then $Q_i - D_i$ units are left over at the end of the period and are salvaged for a per unit value of s_i ; and if $Q_i < D_i$, then $D_i - Q_i$ units cost the retailer zero per unit. At the end of the selling season, the actual profit for the retailer from channel $i \in I$ is

$$\Pi_i(D_i, Q_i, r_i) = r_i \min(D_i, Q_i) + s_i(Q_i - D_i)^+ - c_i Q_i. \quad (2.3)$$

The retailer's total actual profit at the end of the selling period is

$$\mathbf{\Pi} = \sum_{i \in I} \Pi_i(D_i, Q_i, r_i). \quad (2.4)$$

The retailer wants to optimize the profit. However, the retailer is unable to determine the actual end-of-selling period profit because the demand was not realized at the start of the selling season. As a result, the classical method to solve the optimization problem is to assume a risk-neutral retailer who makes the best pricing decision at the start of the sales season in order to maximize the overall expected profit.

In this research, the α -service level is considered as the performance constraint since it is useful for retailers to evaluate the quality of their selling process (Abad (2014); Wei et al. (2021)). Recall that the service level related to channel $i \in I$ is the probability that the demand does not exceed the order quantity $\mathbb{P}(D_i \leq Q_i)$. Let $SL_i \in [0, 1]$ be the target service level related to channel i , specified by the retailer. Thus, the equality constraint on SL for channel i is

$$\mathbb{P}(D_i \leq Q_i) = SL_i. \quad (2.5)$$

The general non-linear multichannel price optimization problem with stochastic demand and target service level denoted by MCPS-I is as follows.

$$\begin{aligned}
& \max_{Q_i, r_i} \sum_{i \in I} \mathbb{E}[\Pi_i(D_i, Q_i, r_i)] \\
& \text{s.t. } \mathbb{P}(D_i \leq Q_i) = SL_i, \forall i \in I \\
& \quad r_i \geq c_i, \forall i \in I \\
& \quad Q_i \geq 0, \forall i \in I.
\end{aligned} \tag{MCPS-I}$$

The decision variables in the above formulation are the prices and order quantities in all channels, and the objective is to maximize the total expected profit of the retailer across the retail chain.

For all $i \in I$, denote

$$G_i(\mathbf{r}) = \frac{g_i(r_i)}{g_0 + \sum_{i \in I} g_i(r_i)}. \tag{2.6}$$

It follows that

$$D_i(\mathbf{r}) = \xi G_i(\mathbf{r}). \tag{2.7}$$

Given Assumption 1, it is obvious that $0 < G_i < 1$ and $\sum_{i \in I} G_i < 1$. Hence, $G_i(\mathbf{r})$ can be seen as the proportion of market sharing of channel i .

Thus, the expected profit for channel $i \in I$,

$$\mathbb{E}[\Pi_i(D_i, Q_i, r_i)] = \mathbb{E}[\Pi_i(\xi G_i(\mathbf{r}), Q_i, r_i)] := \Pi_i(Q_i, \mathbf{r}), \tag{2.8}$$

is a function of the corresponding channel order quantity and all the retailing prices. The explicit formula is given in the Remark 1 as follows.

Remark 1. *The expected profit for each channel $i \in I$ is as follows.*

$$\begin{aligned}
\Pi_i(Q_i, \mathbf{r}) &= (r_i - s_i)G_i(\mathbf{r})\mathbb{E}[\xi] - (c_i - s_i)Q_i \\
&\quad - (r_i - s_i)G_i(\mathbf{r}) \int_{\frac{Q_i}{G_i(\mathbf{r})}}^{\infty} \left(x - \frac{Q_i}{G_i(\mathbf{r})}\right) f_{\xi}(x) dx.
\end{aligned} \tag{2.9}$$

Proof. Sketch of the proof. We embed the Eq. 2.7 into the objective function of Problem MCPS-I and proceed with a process of algebraic transformation and simplification. See Appendix A.3.1 for the detail of the proof. \square

However, it is not possible to verify if $\sum_{i \in I} \Pi_i(Q_i, \mathbf{r})$ is jointly concave on all its variables (all $(Q_i)_{i \in I}$ and $(r_i)_{i \in I}$). Thus, the (unconstrained) problem of maximizing $\sum_{i \in I} \Pi_i(Q_i, \mathbf{r})$ includes difficulties (algorithmic, computational approaches) in finding global solutions.

From the service level constraint given in Eq. 2.5, we have

$$SL_i = \mathbb{P}(D_i \leq Q_i) = \mathbb{P}(\xi G_i(\mathbf{r}) \leq Q_i) = \mathbb{P}\left(\xi \leq \frac{Q_i}{G_i(\mathbf{r})}\right) = F_\xi\left(\frac{Q_i}{G_i(\mathbf{r})}\right) \quad (2.10)$$

$$\Leftrightarrow F_\xi^{-1}(SL_i) = F_\xi^{-1}\left(F_\xi\left(\frac{Q_i}{G_i(\mathbf{r})}\right)\right) = \frac{Q_i}{G_i(\mathbf{r})}. \quad (2.11)$$

Or equivalently,

$$Q_i = G_i(\mathbf{r})F_\xi^{-1}(SL_i). \quad (2.12)$$

Therefore, we can derive the relationship between the order quantity of each channel with respect to the retailing prices across channels, which is given in Proposition 1 as follows.

Proposition 1. Consider a non-negative random variable ξ , if $SL_i > 0$, then

$$i. \frac{\partial Q_i}{\partial r_i}(\mathbf{r}) < 0.$$

$$ii. \frac{\partial Q_i}{\partial r_j}(\mathbf{r}) > 0.$$

Proof. Note that by Assumption 1, we have $g_i(r_i) > 0$, $g'_i(r_i) < 0$, $0 < G_i(\mathbf{r}) < 1$ for all $i \in I$. Moreover, $F_\xi^{-1}(SL_i) > 0$, since

$$F_\xi^{-1}(SL_i) > 0 \Leftrightarrow F_\xi(F_\xi^{-1}(SL_i)) > F_\xi(0) \Leftrightarrow SL_i > F_\xi(0) \Leftrightarrow SL_i > \mathbb{P}(\xi < 0) \Leftrightarrow SL_i > 0. \quad (2.13)$$

Thus, the results are deduced from the representations as follows.

$$\frac{\partial Q_i}{\partial r_i}(\mathbf{r}) = \frac{\partial G_i(\mathbf{r})F_\xi^{-1}(SL_i)}{\partial r_i} = \frac{g'_i(r_i)}{g_i(r_i)}G_i(\mathbf{r})(1 - G_i(\mathbf{r}))F_\xi^{-1}(SL_i) < 0, \quad (2.14)$$

$$\frac{\partial Q_i}{\partial r_j}(\mathbf{r}) = \frac{\partial G_i(\mathbf{r})F_\xi^{-1}(SL_i)}{\partial r_j} = -\frac{g'_j(r_j)}{g_j(r_j)}G_i(\mathbf{r})G_j(\mathbf{r})F_\xi^{-1}(SL_i) > 0. \quad (2.15)$$

□

Proposition 1 states that the order quantity of each channel decreases with its corresponding retailing price and increases with other channels' retailing prices. Or, equivalently, the order quantity has the property of price elasticity.

Given the service level requirements, Eq. 2.12 gives us a relation between order quantity and retailing price so that the decision variables of the optimal problem MCPS-I are only the

prices across channels. The expected profit for channel $i \in I$,

$$\mathbb{E}[\Pi_i(D_i, Q_i, r_i)] = \mathbb{E} \left[\Pi_i \left(\xi G_i(\mathbf{r}), G_i(\mathbf{r}) F_\xi^{-1}(SL_i), r_i \right) \right] := \Pi_i(\mathbf{r}), \quad (2.16)$$

is a function of only the retailing prices. The explicit formula is given in the Remark 2 as follows.

Remark 2. *Given the constraints on service level, the expected profit for channel $i \in I$ is represented as follows*

$$\begin{aligned} \Pi_i(\mathbf{r}) = G_i(\mathbf{r}) \left[(r_i - c_i) F_\xi^{-1}(SL_i) - (r_i - s_i) SL_i F_\xi^{-1}(SL_i) \right. \\ \left. + (r_i - s_i) \int_{-\infty}^{F_\xi^{-1}(SL_i)} x f_\xi(x) dx \right]. \end{aligned} \quad (2.17)$$

Proof. Sketch of the proof. We embed the Eq. 2.12 into the expected profit function given in Remark 1 and proceed with a process of algebraic transformation and simplification. See Appendix A.3.2 for detail of the proof. \square

Consider a non-negative potential market demand ξ . Since $SL_i \geq 0$, then $F_\xi^{-1}(SL_i) \geq 0$. Thus, $Q_i \geq 0$. Let $\Omega = [c_1, \infty) \times [c_2, \infty) \times \dots \times [c_n, \infty)$ be the feasible domain for \mathbf{r} . The multichannel price optimization problem with stochastic demand and target service level MCPS-I becomes

$$\max_{\mathbf{r} \in \Omega} \sum_{i \in I} \Pi_i(\mathbf{r}) \quad (\text{MCPS-II})$$

The optimization problem MCPS-II is a non-constrained optimization problem and equivalent to the initial multichannel pricing problem MCPS-I. The decision variables for Problem MCPS-II are only prices across channels.

In the next section, we prove that the objective function of Problem MCPS-II behaves well mathematically. Although it is difficult to examine the joint concavity, we show that there exists a unique solution to the first-order conditions, and it is the global solution for Problem MCPS-II.

2.4 Solution of integrated multichannel pricing problem

In general, the difficulty in dealing with stochastic multichannel pricing problem MCPS-I is that the corresponding objective function is not jointly concave in all the decision variables (prices and order quantities). Given the constraints on service level, we can transform the

Problem MCPS-I into an equivalent problem, which is the Problem MCPS-II. In this section, we show that the objective function of Problem MCPS-II is mathematically well-behaved under a set of assumptions. In particular, there exists a unique solution to the first-order conditions, which is the unique solution of Problem MCPS-II. Consequently, there is a unique vector of prices and order quantities solving the integrated stochastic multichannel pricing Problem MCPS-I, derived via the optimal solution of Problem MCPS-II.

Recall that a \mathcal{C}^2 multi-variable function is jointly concave at a point if its associated Hessian matrix is negative semi-definite Boyd et al. (2004). To examine the existence and uniqueness of the solution of Problem MCPS-II, the general approach of checking the negative semi-definiteness of the Hessian matrix at all the points in the domain is not applicable. We therefore use a novel approach based on a property where a function has a unique stationary point if: (i) there exists at least one point such that the gradient of the function at this point is zero and (ii) at all the stationary points, it follows that the corresponding Hessian matrix is a diagonal matrix with strictly negative elements on its diagonal. Note that statement (ii) is a sufficient condition for strict concavity at a point. Thus, it can be seen that if we have a good structure (a particular form for associated Hessian matrices) for all the zero-gradient points, the objective function also has a good structure (uniqueness of the stationary point). The property is represented in two lemmas for the uni-dimensional and multi-dimensional cases, Lemma 1 and Lemma 2, which are shown later in this section.

Consider Problem MCPS-II. It is challenging to analyze the joint concavity of the objective function of Problem MCPS-II since it is impossible to check if its associated Hessian matrix is negative semi-definite at every point in Ω . Despite the difficulties, we claim that the function

$$\Pi(\mathbf{r}) := \sum_{i \in I} \Pi_i(\mathbf{r}) \quad (2.18)$$

is well-behaved, given a set of assumptions as follows.

Assumption 2. For all $i \in I$, let $\zeta_i(r_i) := \frac{g_i'(r_i)}{g_i(r_i)}$. Consider the potential market ξ . It satisfies

i. $\xi \geq 0$, $\mathbb{E}[\xi] < \infty$

ii. $\lim_{r_i \rightarrow \infty} \zeta_i(r_i) > -\infty$, $\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} < 0$

iii. $\exists \mathbf{c}^* \in \Omega : \nabla \Pi(\mathbf{c}^*) > \mathbf{0}$.

Assumption 2.i guarantees that the demand is non-negative and has a finite expectation. If the support of ξ contains a negative part, its truncated version can be used to restrict

the domain. When Assumption 2.i is satisfied, then $F_{\xi}^{-1}(SL_i) \geq 0$. Thus, Q_i is obviously non-negative for all $i \in I$.

Assumption 2.ii is rather technical and not very restrictive. For instance, the assumption is satisfied for linear, MCI, and MNL attraction functions. By Assumption 1, we deduce that $\zeta_i(r_i) < 0$ for all r_i .

From a management interpretation point of view, Assumption 2.iii states that there is a vector of prices, \mathbf{c}^* , in the domain so that the related expected profit increases when we increase a small value of prices. For simplicity, one can consider that \mathbf{c}^* equal to \mathbf{c} , the lower bound of the prices. Mathematically, Assumption 2.iii is to secure the existence of a point \mathbf{r}^* satisfying $\frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = 0$ for all $i \in \{1, 2, \dots, n\}$ (solution to the first-order conditions). Moreover, to our best understanding through experiments, the solution to the first-order conditions exists without Assumption 2.iii, but no analytical proof is available.

Assumptions 1 and 2 ensure the existence and uniqueness of the stationary point of $\Pi(\mathbf{r})$. To provide the structural results on the stochastic multichannel pricing problem MCPS-I, we now introduce two key lemmas mentioned at the beginning of this section.

Lemma 1. *Given a real number a , let $\psi : [a, \infty) \rightarrow \mathbb{R}$ be a \mathcal{C}^2 function of the single variable u . Suppose that*

- i. There exists at least one point $u^* \in [a, \infty)$ such that $\psi'(u^*) = 0$,*
- ii. At any u^* that satisfies $\psi'(u^*) = 0$, it follows that $\psi''(u^*) < 0$.*

Then, there exists a unique u^ that satisfies $\psi'(u^*) = 0$, and u^* is the argument of the maxima of $\psi(\cdot)$.*

Proof. From the first part of the lemma, there exists at least one point u such that $\psi'(u) = 0$. We need to prove its uniqueness.

Let $u^* = \min\{u \mid \psi'(u) = 0\}$. By definition, we have $\psi'(u^*) = 0$ and from condition (ii), $\psi''(u^*) < 0$. Thus, u^* is a local maximizer of $\psi(\cdot)$. Moreover, $\psi''(u)$ is a continuous function. Therefore, by the fundamental theorem of integral calculus, there exists an $\varepsilon > 0$ such that $\psi'(u) < 0$ for $u \in (u^*, u^* + \varepsilon)$.

Now, assuming that there exists at least one point u such that $u > u^*$ and $\psi'(u) = 0$. Let $u^{**} = \min\{u \mid u > u^*, \psi'(u) = 0\}$. By the definition of u^{**} and the continuity of $\psi'(u)$, $\psi'(u)$ does not change its sign between two zero points u^* and u^{**} . As $\psi'(u) < 0$ for $u \in (u^*, u^* + \varepsilon)$, we deduce that $\psi'(u) < 0$ for $u \in (u^*, u^{**})$.

Besides, by the condition (ii) of the lemma, we have $\psi''(u^{**}) < 0$, or u^{**} is also a local maximizer of $\psi(\cdot)$. Thus, there exists an $\eta > 0$ such that $\psi'(u) > 0$ for $u \in (u^{**} - \eta, u^{**})$,

which is a contradiction to the above property that $\psi'(u) < 0$ for $u \in (u^*, u^{**})$. Thus, no such a u^{**} exists. \square

Lemma 2 extends the result of Lemma 1 in the multi-dimensional case as follows.

Lemma 2. *Given a real value vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, let $\Psi : W_1 \times W_2 \times \dots \times W_n \rightarrow \mathbb{R}$, $W_i = [a_i, \infty) \subset \mathbb{R}$, be a \mathcal{C}^2 function of $\mathbf{u} = (u_1, u_2, \dots, u_n) \in W_1 \times W_2 \times \dots \times W_n$. For all $i \neq j$, and $i, j \in I = \{1, 2, \dots, n\}$, suppose that*

- i. *There exists at least one point \mathbf{u}^* such that $\nabla \Psi(\mathbf{u}^*) = \mathbf{0}$,*
- ii. *At all the points \mathbf{u}^* mentioned above, $\frac{\partial^2 \Psi}{\partial u_i^2}(\mathbf{u}^*) < 0$ and $\frac{\partial^2 \Psi}{\partial u_i \partial u_j}(\mathbf{u}^*) = 0$ ($j \neq i$) hold true.*

Then, there exists a unique vector \mathbf{u}^ that satisfies $\nabla \Psi(\mathbf{u}^*) = \mathbf{0}$, and \mathbf{u}^* maximizes $\Psi(\cdot)$.*

Proof. Sketch of the proof. We prove this by induction on the number of dimensions of space. The base case holds true from Lemma 1. The induction step is proved by using the induction hypothesis and envelope theorem. See Appendix A.1.1 for the detail of the proof. \square

From now, to simplify and shorten the formulations, for all $i \in I$, let us denote

- i. $\alpha_i := F_\xi^{-1}(SL_i)$,
- ii. $A_i := \int_{-\infty}^{\alpha_i} x f_\xi(x) dx$,
- iii. $U_i := \alpha_i(1 - SL_i) + A_i$,
- iv. $V_i(r_i) := \alpha_i[(r_i - c_i) - (r_i - s_i)SL_i] + (r_i - s_i)A_i$.

Given SL_i , α_i , A_i , and U_i are constants. $V_i(\cdot)$ is a function of r_i only. Furthermore, $\Pi_i(\mathbf{r}) = G_i(\mathbf{r})V_i(r_i)$ and $\Pi(\mathbf{r}) = \sum_{i \in I} G_i(\mathbf{r})V_i(r_i)$.

Remark 3. *Assume that $SL_i > 0$ for all $i \in I$. We then have*

- i. $\alpha_i > 0$
- ii. $A_i > 0$
- iii. $U_i > 0$
- iv. $V_i(c_i) < 0$
- v. $U_i < \alpha_i$

vi. U_i is an increasing function in α_i and SL_i .

Proof. i. $\alpha_i = F_\xi^{-1}(SL_i)$. If ξ is a non-negative random variable and $SL_i > 0$ then $\alpha_i > 0$.

ii. $A_i = \int_{-\infty}^{\alpha_i} x f_\xi(x) dx$. If ξ is a non-negative random variable and $SL_i > 0$, then $\alpha_i > 0$, thus $A_i = \int_0^{\alpha_i} x f_\xi(x) dx > 0$.

iii. $U_i = \alpha_i(1 - SL_i) + A_i$. If ξ is a non-negative random variable and $SL_i > 0$, then $\alpha_i > 0$ and $A_i > 0$, thus $U_i \geq 0$.

iv. Sketch of the proof. We have $V_i(c_i) = (c_i - s_i)H(\alpha_i)$, where $H(t) := -tF_\xi(t) + \int_{-\infty}^t x f_\xi(x) dx$. Then the result follows from $H(\alpha_i) \leq 0$. See Appendix A.3.3 for the detail of the proof.

v. Note that $H(\alpha_i) = U_i - \alpha_i$ and $H(\alpha_i) \leq 0$, the result is followed.

vi. Sketch of the proof. Calculate the first-order derivative of U_i with respect to SL_i , and the result is followed. See Appendix A.3.3 for the detail of the proof. \square

The proposition below shows the relationship between second-order derivatives and cross derivatives with first-order derivatives of the objective function of problem MCPS-II. It represents key equations to prove the uniqueness of the stationary point.

Proposition 2. For all $i, j \in \{1, 2, \dots, n\}, i \neq j$, we have

$$i. \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) = \left[\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} \right] G_i(\mathbf{r}) U_i + \left[\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i) (1 - 2G_i(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_i}(\mathbf{r}),$$

$$ii. \frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}) = - \left\{ \zeta_j(r_j) G_j(\mathbf{r}) \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) + \zeta_i(r_i) G_i(\mathbf{r}) \frac{\partial \Pi}{\partial r_j}(\mathbf{r}) \right\}.$$

Proof. Sketch of the proof. The results are obtained by transformation by combining the results of Lemma 6, Lemma 5, and Lemma 7 in the appendix. See Appendix A.2.1 for the detail of the proof. \square

Remark 4. From Proposition 2 and Remark 3, we deduce that

$$i. \text{ If there exists a point } \mathbf{r}^* \text{ satisfying } \frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = 0, \text{ it follows that } \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}^*) < 0.$$

$$ii. \text{ If there exists a point } \mathbf{r}^* \text{ satisfying } \frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = \frac{\partial \Pi}{\partial r_j}(\mathbf{r}^*) = 0, \text{ it follows that } \frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}^*) = 0.$$

$$iii. \text{ If there exists a point } \mathbf{r}^* \text{ satisfying } \nabla \Pi(\mathbf{r}^*) = \mathbf{0}, \text{ it follows that } \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}^*) < 0 \text{ and}$$

$$\frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}^*) = 0, \text{ for all } i, j \in \{1, 2, \dots, n\}, i \neq j.$$

Proof. i. If $\frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = 0$, by Proposition 2, it follows that $\frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}^*) = \left[\zeta_i(r_i^*) - \frac{\zeta_i'(r_i^*)}{\zeta_i(r_i^*)} \right] G_i(\mathbf{r}^*) U_i$.

By Assumption 2 and Remark 3, we have $\zeta_i(r_i^*) - \frac{\zeta_i'(r_i^*)}{\zeta_i(r_i^*)} < 0$ and $U_i > 0$. Note that

$0 < G_i(\mathbf{r}^*) < 1$. Thus, $\frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}^*) < 0$.

ii. If $\frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = \frac{\partial \Pi}{\partial r_j}(\mathbf{r}^*) = 0$, it follows directly from Proposition 2 that $\frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}^*) = 0$.

iii. The result is the consequence of the results in i and ii. □

From Lemma 2 and Proposition 2, we obtain the structural results on the multichannel pricing problem. They are represented in Proposition 3 as follows.

Proposition 3. *Suppose that Assumptions 1, 2 hold. Then, there exists a unique point \mathbf{r}^* satisfying $\nabla \Pi(\mathbf{r}^*) = \mathbf{0}$, and it is the global solution for Problem MCPS-II.*

Furthermore, let $\mathbf{Q}^ = (G_1(\mathbf{r}^*)F_\xi^{-1}(SL_1), G_2(\mathbf{r}^*)F_\xi^{-1}(SL_2), \dots, G_n(\mathbf{r}^*)F_\xi^{-1}(SL_n))$. Then, $(\mathbf{Q}^*, \mathbf{r}^*)$ is the unique global solution to the maximization Problem MCPS-I.*

Proof. We claim that there exists a point \mathbf{r}^* such that $\nabla \Pi(\mathbf{r}^*) = \mathbf{0}$. By Assumption 2, we have $\frac{\partial \Pi}{\partial r_i}(\mathbf{c}^*) > 0$ for all $i \in \{1, 2, \dots, n\}$. Thus, to prove the claim, we need to show that $\lim_{r_i \rightarrow \infty} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) < 0$. We show that $\lim_{r_i \rightarrow \infty} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = 0^-$. Here, the upper-script minus sign means that $\lim_{r_i \rightarrow \infty} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = 0$ and $\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) < 0$ when r_i is big enough.

Our analysis uses big \mathcal{O} notation (Landau's symbol) to represent the speed at which a function grows or declines Cormen et al. (2022). For the formal definition, suppose $f(x)$ and $g(x)$ are two functions defined on some subset of the real numbers. We write $f(x) = \mathcal{O}(g(x))$ (or $f(x) = \mathcal{O}(g(x))$ for $x \rightarrow \infty$ to be more precise) if and only if there exist constants N and C such that $|f(x)| < C|g(x)|$ for all $x > N$. Intuitively, this means that f does not grow faster than g .

From Lemma 7, we have

$$\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r}) V_k(r_k)) \right] \right\} \quad (2.19)$$

$$= G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - G_i(\mathbf{r}) V_i(r_i) - \sum_{k \neq i} (G_k(\mathbf{r}) V_k(r_k)) \right] \right\}. \quad (2.20)$$

Note that $U_i > 0$ is a constant with respect to \mathbf{r} , and $\frac{\partial V_i(r_i)}{\partial r_i} = U_i$. Thus $\lim_{r_i \rightarrow \infty} V_i(r_i) = \infty$ and $V_i(r_i) = \mathcal{O}(r_i)$. From Assumption 1, we have $\lim_{r_i \rightarrow \infty} G_i(\mathbf{r}) = 0$ and $\lim_{r_i \rightarrow \infty} r_i G_i(\mathbf{r}) = 0$, for all $i \in \{1, 2, \dots, n\}$. Therefore $\lim_{r_i \rightarrow \infty} G_i(\mathbf{r}) V_i(r_i) = 0$.

Since $\zeta_i(r_i) < 0$ and $\lim_{r_i \rightarrow \infty} \zeta_i(r_i) > -\infty$, we have

$$U_i + \zeta_i(r_i) V_i(r_i) < 0 \text{ for } r_i \text{ big enough, and } U_i + \zeta_i(r_i) V_i(r_i) = \mathcal{O}(r_i). \quad (2.21)$$

Therefore,

$$\lim_{r_i \rightarrow \infty} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = \lim_{r_i \rightarrow \infty} G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - G_i(\mathbf{r}) V_i(r_i) - \sum_{k \neq i} (G_k(\mathbf{r}) V_k(r_k)) \right] \right\} \quad (2.22)$$

$$= \lim_{r_i \rightarrow \infty} G_i(\mathbf{r}) (U_i + \zeta_i(r_i) V_i(r_i)) = 0^-. \quad (2.23)$$

Together with Assumption 2, we deduce the existence of such a point \mathbf{r}^* satisfying $\frac{\partial \Pi}{\partial r_i}(\mathbf{r}^*) = 0$ for all $i \in \{1, 2, \dots, n\}$. Together with the results given in Remark 4 and Lemma 2, it completes the proof for Proposition 3. \square

Remark 5.

From the proof of Proposition 3, $\lim_{r_i \rightarrow \infty} \frac{\partial \Pi}{\partial r_i} = 0^-$. Thus, the Assumption 2 is to technically provide the existence of a zero point of $\nabla \Pi(\mathbf{r})$. Through numerical experiments, we believe that a zero-gradient point exists without Assumption 2, but no analytical proof can be provided.

Lemma 2 above assures that if there exists a point satisfying the first-order conditions, then it is unique. In practice, even if the Assumption 2 is not satisfied, if the zero-point of the first-order derivatives can be found numerically, then it is not only unique but also the optimal solution to the proposed multichannel pricing model.

To summarize, in this section, we study a stochastic multichannel optimization problem under equality constraints on SL. We prove that the objective function of the problem is well-behaved, given a set of assumptions. In particular, there exists a unique solution derived from first-order conditions, and a simple search procedure should be sufficient to find the optimal solution.

Next, we discuss the behavior of the order quantities, market share, and expected profit at optimal prices by varying the value of SL.

2.5 Numerical analysis

The previous section proves that our objective function is mathematically well-behaved under certain assumptions. As in the Remark 5, if there exists a point satisfying the first-order conditions, it is unique. In practice, even if the Assumption 2 is violated, if the point satisfying the first-order conditions can be found numerically, then it secures the existence and uniqueness of the optimal solution to the pricing problem.

In this section, we aim to derive some managerial insights on using the SL constraint in a multichannel configuration. We provide an analysis of the model given a particular set of parameters, and we particularly focus on the following questions:

- Does the common knowledge on service levels in inventory control apply similarly under a multichannel configuration?
- If there is a lower bound for SL, how should the target service level be set for each channel?
- Is there any trade-off between a higher target SL and the total expected profit?

From a managerial perspective, it would be useful to consider a lower bound at service level. However, equality in the service level constraint is motivated by the mathematical complexity of the model. It is worthwhile to notice that the resulting profit function in the developed framework is generally non-concave and non-linear. Thus, standard non-linear search procedures may terminate at a local optimum, which is far from the global optimum. Because of this, finding a global solution may require a sophisticated search procedure Hanson and Martin (1996); Harsha et al. (2019). According to Hanson and Martin (1996), “as the logit model approaches the integer programming model at the limit, and the integer programming optimization problem is NP-hard, it is logical to conclude that optimizing problems involving logit probabilities are also difficult”. Considering equality constraints, we are successful in deriving the optimal conditions in a tractable way for attraction profit functions. In practice, studying optimization problems with inequality (or lower bounds) on service-level constraints can be seen as the study of several optimization problems with equality constraints at a service level. In particular, if we assume that the minimal service level requirement for a single channel is 80%, we can investigate and compare the results of optimization problems with equality constraints set by 80%, 81%, 82%, . . . , etc., and decide the pricing strategy. Therefore, in the numerical analysis, we perform a sensitivity study by varying the target SL_i for each channel and deriving managerial insights on the linkage between these fixed targets and the associated profit expectation/market sharing for each channel. We do think that the tractable mathematical optimization followed by the sensitivity

Table 2.1 Optimal prices and related values for given service levels extracted from extended tables in the appendix

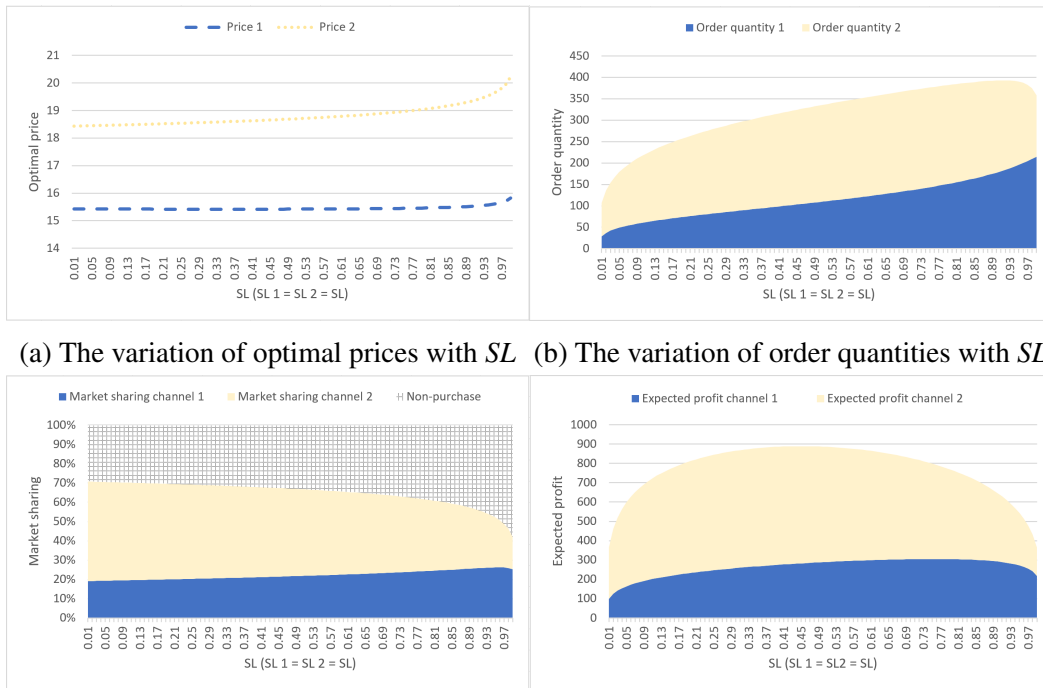
SL_1	SL_2	r_1^*	r_2^*	Q_1^*	Q_2^*	G_1^*	G_2^*	G_0^*	Π_1	Π_2	$\Pi(Q^*, r^*)$
0.1	0.1	15.423	18.471	60	156	0.195	0.506	0.298	196.9	510.4	707.3
0.1	0.2	15.678	18.43	48	202	0.155	0.540	0.305	168.1	615.8	783.9
0.1	0.3	15.793	18.447	44	230	0.142	0.545	0.313	158.7	659.8	818.5
0.1	0.4	15.831	18.489	43	248	0.140	0.537	0.322	158.7	671.1	829.8
0.1	0.5	15.81	18.551	45	260	0.148	0.520	0.332	166.1	657.5	823.5
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
0.9	0.5	15.932	18.605	95	258	0.137	0.516	0.347	183.3	664.1	847.4
0.9	0.6	15.889	18.69	103	264	0.148	0.491	0.360	195.4	630.7	826.1
0.9	0.7	15.813	18.81	116	264	0.167	0.456	0.377	214.2	574.4	788.7
0.9	0.8	15.698	18.994	138	252	0.199	0.402	0.399	243.4	488.3	731.7
0.9	0.9	15.523	19.332	178	215	0.257	0.311	0.433	292.1	353.6	645.7

analysis permits us to make the equality assumption on SL constraint less critical from a managerial perspective.

For simplicity, let us consider the case with two channels, a particular case of multichannel ($n = 2$): a pricing problem with stochastic logit demand in which the online (resp. offline) channel demand is characterized by $a_1 = 15$ and $b_1 = 1$ (resp. $a_2 = 19$ and $b_2 = 1$). We assume the non-purchased option is normalized, $g_0 = 1$. With the higher parameter a_2 , the offline channel has more selling potential. The unit purchase costs are set by $c_1 = 12$ and $c_2 = 15$. Here, the smaller value of c_1 compared to c_2 explains that the total purchase cost and operation cost for the online channel is lower than those of the offline channel. The unit salvage costs are $s_1 = 9$ and $s_2 = 10$. We consider a (truncated) normal distribution for the global market demand $\xi \sim \mathcal{N}(500, 150^2)$. All the numbers mentioned after this paragraph are already rounded off.

To produce the following analysis, we first fix a couple of target service levels SL_1, SL_2 . We then derive the optimal price r_1^*, r_2^* corresponding to these target SLs. The resulting order quantity of channel 1, order quantity of channel 2, total order quantity, market sharing of channel 1, market sharing of channel 2, total market sharing in two channels, expected profit of channel 1, expected profit of channel 2, and total expected profit are then deduced given the target SLs and related optimal price r_1^*, r_2^* . The SL_1, SL_2 are varied to investigate their impact on related indicators (optimal price, order quantity, market sharing, and expected profit).

Table 2.1 represents the optimal prices and related resulting indicators for a part of the results when we vary SL_1 in $[0.1, 0.2, 0.3, \dots, 0.9]$ and SL_2 in $[0.1, 0.2, 0.3, \dots, 0.9]$. The full table of numerical results for this setting is in Appendix Table A.1. Figure 2.1 represents the visualizations of results related to the scheme that $SL_1 = SL_2$ with a step size of 0.01



(a) The variation of optimal prices with SL (b) The variation of order quantities with SL
 (c) The variation of market share with SL (d) The variation of expected profit with SL
 (a), (b), (c), (d) evaluated with $SL_1 = SL_2 = SL$, and (b), (c), (d) evaluated at the optimal prices.

Figure 2.1 The variation of optimal prices, order quantities, market share, and expected profit given equal SL .

(see Appendix Table A.2 for the full numerical results). In this scheme, the optimal price of both the channels, order quantity of channel 2, total order quantity, market sharing of channel 1, market sharing of channel 2, total market sharing of two channels, expected profit for channel 1, expected profit for channel 2, and total expected profit curves are concave with respect to SL . The order quantity of the first channel increases with SL . The maximal order quantity of channel 1 is 214, attained when $SL = 0.99$; the maximal order quantity of channel 2 is 233, attained when $SL = 0.71$; the maximal total order quantity is 393, attained when $SL = 0.91$. The maximal market sharing of channel 1 is 26.2%, attained when $SL = 0.96$; the maximal market sharing of channel 2 is 51.6%, attained when $SL = 0.01$; the maximal market sharing of two channels is 70.7%, attained when $SL = 0.01$. The maximal expected profit of channel 1 is 304.9, attained when $SL = 0.76$; the maximal expected profit of channel 2 is 611.5, attained when $SL = 0.37$; the maximal expected profit of two channels is 888.8, attained when $SL = 0.45$.

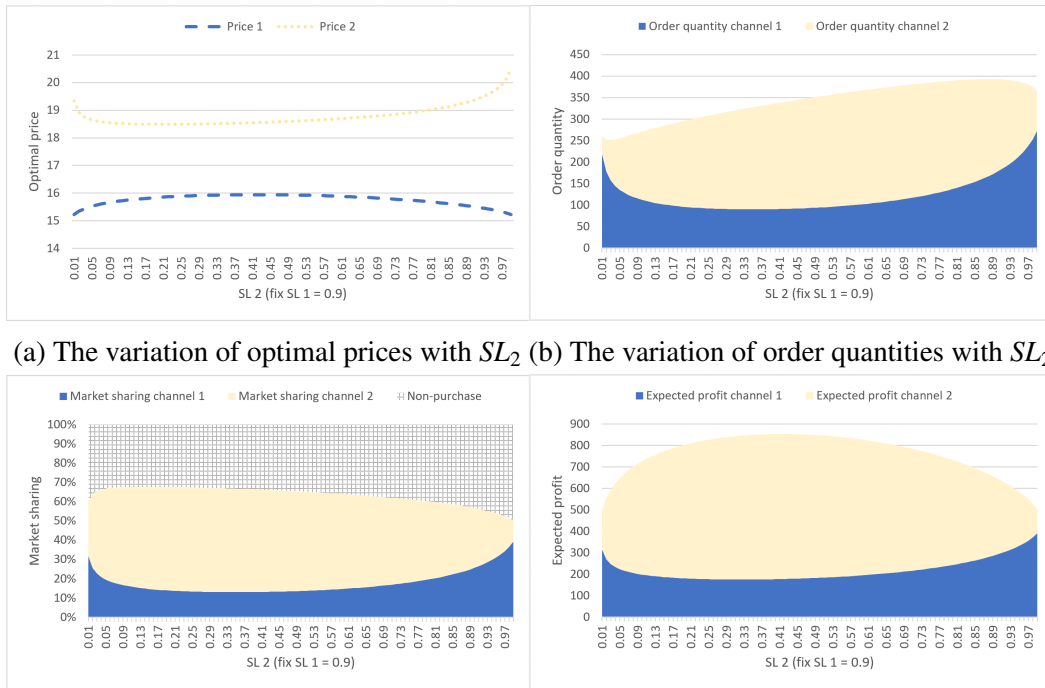
This first set of results shows that some of the established and known results in inventory control no longer apply in our multichannel configuration. It is well known in the classical inventory literature that the service level is strictly increasing with the ordering quantity.

Under a multichannel configuration with cross-interactions between channels' demands, this common knowledge result does not apply. As illustrated in Figure 2.1, the ordering quantity for channel 2 decreases with SL for high target SLs. Despite the fact that they both have the same demand elasticity ($b_1 = b_2 = 1$), channel 2's optimal price considerably increases by setting a higher target service level, whereas channel 1's ordering quantity increases when the target service level is set higher.

It is moreover a common practice in retailing to target high service levels. Setting a high target service for both channels leads to an increase in the non-purchase option with a low market share for each channel. This is due to the high pricing "filter" applied by both channels to achieve these high target SLs. As illustrated in Figure 2.1, the expected profit for both channels could be negatively impacted by such a strategy. This result is intuitive: for a higher service level, the retailer has to sacrifice the expected profit. It also suggests that when the same lower bound on service levels is set high enough, the optimal pricing decision should be taken at this boundary to maximize the profit. These results imply that the retailer should take into account the interaction between channels and the trade-off between profit and service level in order to provide reasonable pricing decisions.

Let us now investigate the scenario where the target service level for each channel is different. In particular, we focus on a case study where there is a "strict" requirement for the service level of a single channel and a "flexible" option for the other. For this purpose, we first assume that channel 1 is "strict" in the choice of the target service level by fixing its own SL equal to 90%, and we vary the target service level for the second channel with a step size of 0.01. The full table of numerical results for this setting is in the Appendix Table A.3. Figure 2.2 illustrates the optimal solution (ordering, pricing, market share, and expected profit) under this scenario.

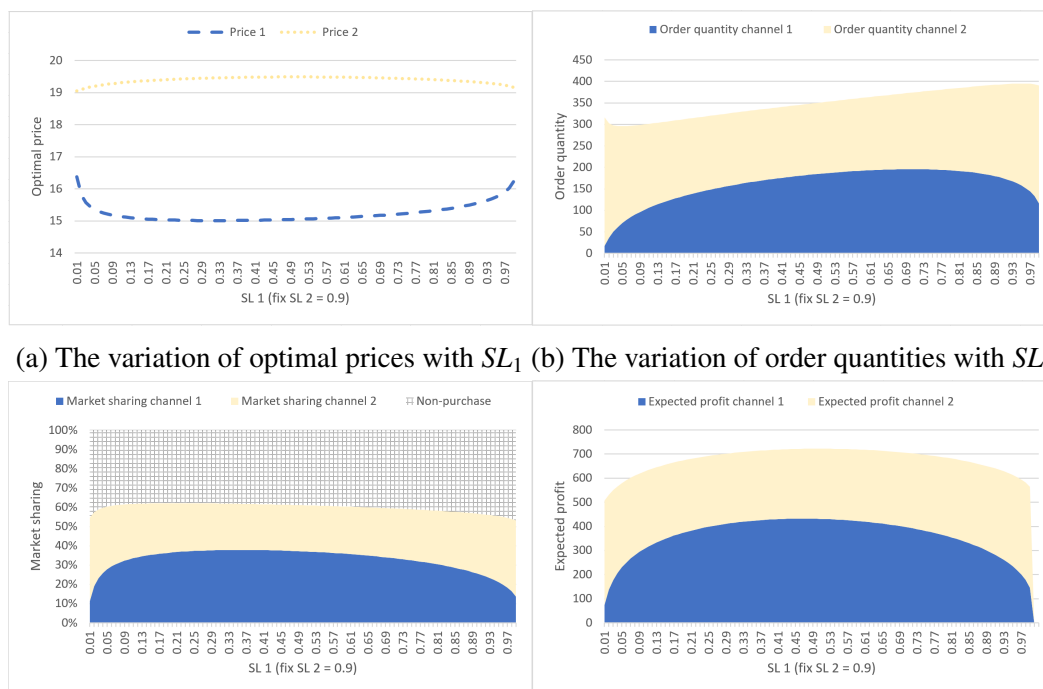
When $SL_1 = 0.9$: The maximal order quantity of channel 1 is 272, attained if the retailer sets channel 2's target service level equal to $SL_2 = 0.99$. This also corresponds to the highest expected profit channel 1 could achieve (equal to 391.5). In other words, channel 1 is better off if the retailer aims to avoid second channel's shortage by setting a very high target service level. Channel 1 can profit from such a situation by, in contrast, lowering its selling price and reaching the highest market share. Concerning the retailer's decision, if the condition $SL_1 = 0.9$ target level is set for channel 1, the retailer should set channel 2's target service level $SL_2 = 0.4$ to achieve channel 2's highest profit (676.5) and should set it equal to $SL_2 = 0.42$ in order to achieve the highest profit for both channels (853.2 as total expected profit). In practice, if the minimal service level for channel 2 is higher than 42%, the trade-off between service level and total expected profit can be seen.



(a) The variation of optimal prices with SL_2 (b) The variation of order quantities with SL_2
 (c) The variation of market share with SL_2 (d) The variation of expected profit with SL_2
 (a), (b), (c), (d) evaluated with $SL_1 = 90\%$, and (b), (c), (d) evaluated at the optimal prices.

Figure 2.2 The variation of optimal prices, order quantities, market share, and expected profit with respect to the SL of channel 2 at a fixed SL level of channel 1: $SL_1 = 90\%$.

Conversely, we now assume that channel 2 is “strict” in the choice of the service level by setting its target level SL_2 equal to 90%. We vary SL_1 with a step size of 0.01 to analyze what the best pricing strategy would be if we had a boundary condition for SL_1 . The related numerical results for this settings is in the Appendix Table A.4. Figure 2.3 illustrates the optimal solution (pricing, ordering quantity, market share, and expected profit). It is worth noticing that the shape of plots is now completely different from those illustrated in Figure 2.2. For instance, the maximal expected profit of channel 2 is 431.4, attained when $SL_1 = 0.01$. In other words, channel 2 is better off totally excluding channel 1. If the retailer has to decide on channel 1’s target service level to derive the highest expected profit from channel 1, based on channel 2’s ‘strict’ condition, he should set it to $SL_1 = 0.46$, which is close to the SL_1 level needed to achieve the highest profit for the two-channels. The maximal expected profit of the two channels is 722.1, attained when $SL_1 = 0.5$. The latter total profit for the two channels is lower than the one achieved in the previous case where channel 1 is under the “strict” condition of the target service level. These results also suggest a trade-off between the profit and service level of channel 1 when the latter is higher than 0.5. In particular, maximizing total profit under the minimal service level of channel 1 is equivalent to maximizing total

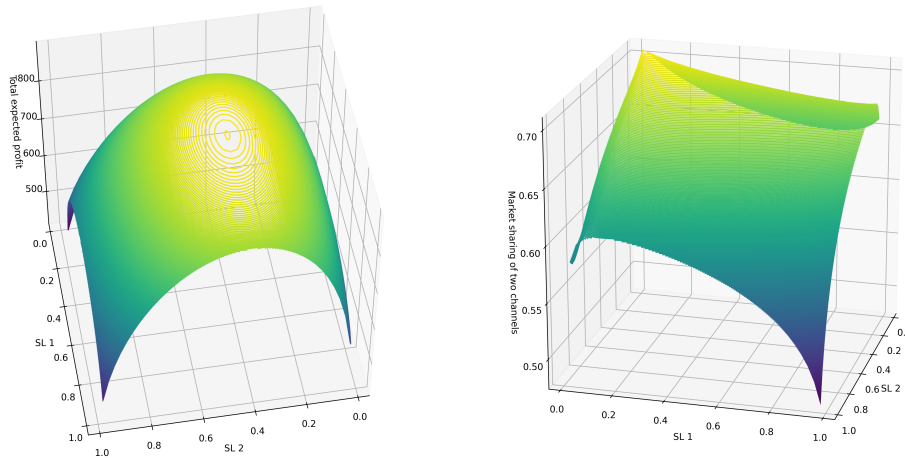


(a) The variation of optimal prices with SL_1 (b) The variation of order quantities with SL_1
 (c) The variation of market share with SL_1 (d) The variation of expected profit with SL_1
 (a), (b), (c), (d) evaluated with $SL_2 = 90\%$, and (b), (c), (d) evaluated at the optimal prices.

Figure 2.3 The variation of optimal prices, order quantities, market share, and expected profit with respect to the SL of channel 1 at a fixed SL level of channel 2: $SL_2 = 90\%$.

profit under the equality constraint on the minimal service level if the minimal service level is no less than a certain value.

From a macro vision, the impacts of an increase in service level on the optimal selling price, order size, market share, and expected profit are indeterminate in the multichannel attractive demand configurations. However, through the case study, when the service level is high enough, there is always a trade-off between the total expected profit and service level. Figure 2.4 illustrating total profit and total market share for the two channels can help decision-makers set these levels by trading-off economic function and customer satisfaction. It suggests that inequality-constrained and equality-constrained problems are equivalent if the minimal service level (and target service level) is high enough.



(a) The variation of total expected profit (b) The variation of total market share
(a), (b) evaluated at the optimal prices.

Figure 2.4 The variation of total expected profit and total market share with SL_1 and SL_2 .

2.6 Conclusion and future research

This chapter investigates a stochastic multichannel pricing problem under service level effects on retailers' pricing and ordering decisions. In particular, each channel's demand is a stochastic function of prices. Each channel's order quantity satisfies the retailer's service level constraint. A multiplicative attractive model is employed to quantify the cross-channel dependencies. Under the constraints on service level and some regular conditions, we reduce the number of decision variables and prove that the objective function of the multichannel pricing problem is mathematically well-behaved. In particular, there is a unique global solution: a vector of prices and order quantities, which corresponds to the zero point of the gradient of the objective function. The methodology for verifying the existence and uniqueness of the optimal solution is highlighted when the usual approach of checking the negative semi-definiteness of the objective function's associated Hessian matrix at all the points in the domain is not applicable. This chapter represents a novel approach by looking at the structure of zero-gradient points and their associated Hessian matrices, which secures the stationary point's existence and uniqueness, resulting in a global solution. This method can also be applied to other objective functions in case they are not jointly concave. Thus, it enables researchers and retailers to examine more complex profit functions.

We provide a management value assessment based on numerical analysis for various SL scenarios. We study how the order quantities, market share, and expected profit change with SL, given the optimal prices in a particular multichannel case study: a dual-channel

environment. We discovered that some conventional knowledge about inventory control with the service level does not hold in a dual-channel environment in particular or a multichannel environment in general. For example, the service level does not always increase when the number of orders increases. Furthermore, channel coordination is required to trade off expected profit for different stores, as well as the service levels attained and losses in the market share. A negative impact for all the parts could be caused by decisions on target SLs. It also allows an inequality-constrained problem to be seen as an equality-constrained problem.

For future research, one line of study would be to analyze the optimization problem for a more extensive collection of demand models (for example, additive demand) and another service measure (such as fill rate). It would also be interesting to improve the multichannel model, which is embedded in the objective function, with multiple choice of ordering and dynamic pricing. Last but not least, it would be meaningful to investigate the case of a decentralized supply chain and its effects on the objective function and the optimal solutions.

Chapter 3

Optimal pricing for dual-channel retailing with stochastic attraction demand model

In dual-channel supply chains, where retailers sell their goods both online and in physical stores, determining the optimal pricing strategy while considering customer behavior is a critical challenge. This study introduces and investigates a dual-channel pricing model that accounts for customer channel choice behavior. Drawing inspiration from market-share models, we incorporate a demand model that reflects the attraction between online and physical stores. Our approach includes stochastic assumptions for potential market demand and price-based interactions between the two channels. In particular, we model the channel's stochastic demand as a non-linear function of prices and we allow for different customer reactions when the physical store runs out of stock. This chapter makes two key contributions. First, we highlight the analytical complexity involved in verifying the joint concavity of the retailer's expected profit function with respect to selling prices. To address this challenge, we introduce a novel approach to establish the existence of optimal global prices in the context of non-linear demand and a non-linear, non-concave objective function. Secondly, our study offers practical insights by applying the model to various operational scenarios. We provide guidance on the best pricing strategy when physical store capacity is limited. Depending on customer channel preferences, prioritizing the showroom may lead to higher profits. However, optimizing for profit could result in a reduced market share. In a showroom configuration, the retailer's choice may shift between exclusive physical and exclusive online retailing to maximize profit.

3.1 Introduction

The practice of selling products and services on more than one sales channel is known as Multichannel Retail (Levy et al., 2012). Multichannel retailers are businesses that derive their primary source of income from multichannel retailing. Today's leading retailers frequently use a multichannel business strategy, which involves selling their items both offline and online. These channels typically try to attract various consumer categories and allow retailers to accommodate diverse buying interests and habits (Kireyev et al., 2017). According to Neslin and Shankar (2009), research on multichannel management provides important insights on topics like channel choice (Montoya-Weiss et al., 2003), channel migration (Ansari et al., 2008), and the value of multichannel versus single channel customers (Kushwaha and Shankar, 2008). However, there are many remaining managerial and research-related issues to explore (Liu et al., 2018).

Multichannel and its particular case of a dual-channel supply chain are the clusters for inventory management study, with various issues concerning demand modeling, inventory decision-making, and pricing optimization (Tsay and Agrawal, 2004). The inventory of each channel in a multichannel setting affects how efficiently the orders from the customers are filled. Sharing the demand through these channels depends moreover on the associated prices, which raises another challenge. Despite recent investigations on multichannel retailing, few studies take into account the influence of cross-channel interaction and channel integration on store operations and how retailers should respond (Mou et al., 2018).

Profitability in retailer pricing has become a paramount concern. While attempting to maximize profitability, retailers must pay attention to every marketing mix component. Pricing has always been a crucial strategic aspect for them to "get right" (Kireyev et al., 2017). A typical supermarket today has thousands of items, is bigger than ever, and is a part of a much larger retail chain due to mergers and acquisitions (Bolton et al., 2010). Retailers are therefore challenged to develop a coherent and profitable pricing strategy (Bolton and Shankar, 2018).

This chapter presents a cross-channel price interaction-based framework that structures the customers' demand and decision process, so that the retailer can develop and implement a dual-channel pricing strategy. We concentrate on product rather than service retailing since product management and delivery issues are significantly more straightforward for service retailing (Zhang et al., 2010). We consider the problem of optimal pricing for a retail company selling a single product to consumers through two channels: online and physical stores. Our model acknowledges the inherent limitations of physical store inventory capacity, which primarily constitutes a long-term tactical decision. Therefore, in our single-period problem formulation, we do not treat it as an operational decision. The retailer faces the

problem of determining pricing policies in two channels where the channel's demand depends not only on its related price but also on the price of the other channel. Given the potential market stochastic demand, each channel's demand is determined through a channel choice model as a function of all the selling prices across channels. In particular, the demand settings are inspired from the attraction demand model, (Gallego et al., 2006; Harsha et al., 2019) in order to capture the price-dependent and cross-channel interaction in dual-channel configurations. Here, more general than Harsha et al. (2019), the potential market demand is stochastic and has a known distribution. Furthermore, using the exogenous model for the online channel, we consider different reactions of the retailer when the physical store is stock-out: customers can buy online or receive promotions from the retailer. Therefore, key questions involve the existence, uniqueness, and stability of the optimal solution when the retailer jointly decides the prices across channels to maximize total profits.

Consider the multinomial logit model, a particular form of the attraction demand model with deterministic setting. Hanson and Martin (1996) construct an example in which the profit function is not jointly concave on its decision variables. Thus, determining the optimal solution may require a complex and challenging search procedure. Generally, the objective function is not jointly concave on its decision variables (prices across channels). In our attraction demand-based model, it is impossible to analytically provide a set of conditions based on the Hessian matrix's negative semi-definiteness to secure the joint concavity. We introduce a novel approach to derive a property in which the existence of a local optimizer will result in its uniqueness. Thus, the existence of a local optimizer is a sufficient condition to guarantee the uniqueness of the optimal global solution to the proposed dual-channel pricing problem with a non-linear objective function and stochastic non-linear attraction channel demand. It follows that a simple search procedure is sufficient to find the approximation of the optimal solution.

Our research makes significant contributions to the dual-channel management literature:

- It offers an approach for researchers and practitioners to develop and estimate an optimal pricing strategy.
- It incorporates cross-channel effects based on channels' prices and the attraction demand model.
- It produces several interesting substantive insights regarding cross-channel effects.

The rest of the chapter is structured as follows. First, we conduct a literature review. We then introduce and investigate the retailer's profit optimization problem. We consider the effect of the potential market, channel choice, and attraction model on the retailer's

pricing decision. Following that, we represent the structural results of the retailer's profit maximization problem. We provide numerical studies, analyze their consequences, and end with research objectives for the future.

3.2 Literature review

In this section, we start by introducing the attraction demand models and their utilization in Economics and Management; and we then embed our research within this stream of study. Thereafter, we show our contribution to the dual-channel pricing problem literature by reviewing the research gap of demand modeling under uncertainty with attraction models and its related pricing problem in dual-channel retailing.

Attraction demand models are among the most often employed demand functions to characterize buyer behavior in the Marketing, Economics, and Revenue Management literature (Feng et al., 2022; Gallego et al., 2006; Mahajan and van Ryzin, 1999). They generalize the widespread multinomial logit (MNL) and multiplicative competitive interaction (MCI) demand models, which are based on the random utility theory of economics (Harsha et al., 2019). According to Luce (1959), the attraction demand model can be axiomatically constructed from basic presumptions about customer behavior. The attraction demand model, as detailed in Anderson et al. (1992) and Lariviere (1999), has been used successfully in econometric research to estimate demand and is becoming more widely recognized in marketing (Besanko et al., 1998). Furthermore, empirical studies have also embraced the attraction model. Among them, Besanko et al. (1998) theoretically and empirically found the Nash equilibrium prices for multiple manufacturers and retailers using the MNL model, which is one of the attraction models. For other MNL models in the empirical studies, the reader is referred to Khan and Jain (2005) and Nevo (2000). Few researchers (Nakanishi and Cooper, 1982; Parks, 1969) conducted empirical research employing MCI models, which also belong to the category of price attraction models. In empirical studies, the MNL and MCI models are transformed into linear models to estimate the model parameters (Cooper and Nakanishi, 1988; Nakanishi and Cooper, 1982). Recently, the attraction demand model has been recommended to study the profit optimization problem (Harsha et al., 2019; Subramanian and Harsha, 2021) and consumer choice models and estimation (Feng et al., 2022). It is operationally convenient because of its parsimony in the number of coefficients to be estimated (see, e.g., Harsha et al. (2019)). For its uses in the operations management community, see So (2000), Bernstein and Federgruen (2004), Gallego et al. (2006), Feng et al. (2022) and references therein.

As a first step toward analyzing channel strategies, understanding and modeling consumer preferences are essential in a multichannel world (Neslin et al., 2006). Some recent publi-

cations in the marketing literature have investigated customer dynamics in a multichannel environment, namely consumer migration across various channels like the web and catalogs (Ansari et al., 2008) or online and brick-and-mortar channels (Chintagunta et al., 2012). Between sales made online and in physical stores, Goolsbee (2001) discovers significant cross-price elasticity. He draws the conclusion that channels cannot be viewed as different marketplaces because the decision to purchase online is based on the costs in-store. These articles use logit models to adjust the consumer substitution behavior between two channels. We likewise employ discrete choice models, the attraction demand models, to estimate consumer channel preferences.

Considering the demand function under uncertainty, several research papers treat demand in each channel as a random variable in a dual-channel environment (Huang et al., 2013). Many assume that online and physical store demand are uncorrelated and follow a standard or well-known probability distribution (Ozbilge et al., 2022). Some research considers the price-dependent and cross-channel interaction in dual-channel settings, but the demand is linearly dependent on the prices across channels Cai et al. (2009); Dumrongsiri et al. (2008). Our proposed dual-channel formulation based on the attraction demand model differs from these models by the non-linear structure for the dependence of demand and prices across channels. Huang et al. (2013) highlight that both linear and attraction models yield analytically tractable results for price competition scenarios. However, the attraction models offer an advantage by accommodating non-linear effects that often arise in competitive settings. This advantage has led many researchers to prefer attraction models when exploring price competition among firms.

In addition, the stochastic demand is commonly specified as an additive, a multiplicative, or an additive–multiplicative model (Young, 1978), and is frequently used for pricing newsvendor-type problems (Petruzzi and Dada, 1999). Although these assumptions may sound relatively simple, they allow researchers to handle novel dual-channel phenomena, such as demand spillovers from the competing channel in the case of stock-outs (Boyaci, 2005; Geng and Mallik, 2007; Yang et al., 2017). Similar to the consideration in the above papers, we assume that the potential market demand is stochastic with a known probability distribution function. However, since almost all the stochastic formulations for the demand are based on a deterministic linear demand model, our model differs from the others insofar as it proposes a non-linear function of prices across channels.

In marketing and operations management, studies on dual-channel pricing have primarily targeted the scenarios in which a dual-channel system would benefit the manufacturer (e.g., Cai (2010); David and Adida (2015); Zhou et al. (2019)) or the retailer (e.g., Kireyev et al.

(2017); Nie et al. (2019); Sun et al. (2022)). In line with the stream of dual-channel pricing problems, our research targets retailers' decision problems.

From an operational perspective, many academic studies concentrate on single-product pricing problems (Chen and Simchi-Levi, 2012). To our knowledge, the focus has been on single-channel or multichannel pricing with independent channel operation settings. Commercially available pricing solutions employed by retailers today do not incorporate cross-channel interactions and often price the multiple channels separately (Harsha et al., 2019). In contrast, this chapter focuses on an integrated dual-channel pricing problem in the presence of cross-channel demand interactions and critical operational considerations.

Several publications have examined various parametric and non-parametric approaches from the perspective of pricing optimization using customer choice models Harsha et al. (2019). (Hanson and Martin, 1996) demonstrate that the profit as a function of prices is not quasi-concave for the multinomial logit (MNL) demand model. Further investigation of this issue, by Akçay et al. (2010), revealed that the resulting profit function is unimodal in the pricing space. To show that the objective function is jointly concave in the space of the market share variables, Dong et al. (2009) presented a market share variable transformation. Later, Schön (2010) and Keller et al. (2014) extended this transformation concept for MNL demand models to a generic class of attraction models.

Our research on single-product pricing is connected to the price optimization problem utilizing a combination of the attraction demand models. It is acknowledged by (Keller et al., 2014) that this is an open problem, and they create a local optimal heuristic solution by using an approximate demand model and supposed convexity. Similarly, assuming that the cost function with attraction demand is convex, Gallego et al. (2006) showed the existence of a unique Nash equilibrium for the price competition game. We propose a novel approach and a set of conditions in which there is only one candidate for the optimal global solution to the general non-concave, non-linear objective function of the dual-channel pricing problem with attraction demand models.

In short, motivated by the unexplored dual-channel retailing problems under the stochastic attraction demand consideration, our research proposes a dual-channel pricing model under the impacts of customer behavior on retailers' pricing decisions. We contribute to the dual-channel literature by considering the different offers of the retailer when the physical store is stock-out, which is a practical problem faced by numerous firms in reality. Moreover, we propose a novel approach to go beyond the difficulty of the non-concave structure of the objective function under attraction models. Our research also contributes to the pricing problem literature by demonstrating its significant role in affecting optimal price decisions in dual distribution channels.

3.3 Dual-channel price optimization with stochastic attraction demand model

Model practical context:

We formulate a pricing problem based on the stochastic dual-channel attraction demand model, considering the case where the stock related to the physical channel is limited compared to an unlimited stock capacity for the online channel. This framework could model the practical case where a retailer uses a capacitated local store/showroom for the physical distribution and is able to fulfill online orders by using a central uncapacitated warehouse. Customers who need to test the product or who would like an instantaneous fulfillment of their orders could be satisfied by the physical store and online customers are satisfied by the central warehouse. Given the limited capacity of the physical store, out-of-stock demand could be partially converted into an online demand and customers are offered a voucher in such cases (this voucher could be a free shipping delivery for instance).

To solve the problem described above, we propose a dual-channel pricing optimization framework for products in order to find the best prices across online and physical stores, taking into account a variety of retailer goals and actual business standards. We focus on the relationship between the retail chain's expected profit and customer behavior in case of stockout in the physical channel. This link is essential since practitioners prefer to investigate the economic consequences of a stock out (Abad, 2014).

Our proposed framework employs the newsvendor model to represent the connection between the central warehouse and the local store. The latter with a fixed capacity of K should be replenished in a regular selling period (daily or weekly for instance). Prices for the online and physical channels are decided for the period leading to two demand distributions resulting from the attraction model. The pricing decisions are a trade-off between the overage and underage risks in the physical store. In case of a shortage in the physical store, a margin loss might occur, but some of the lost demand could be converted into online sales. We also assume a voucher code is provided to customers in such a case. In case of overstock in the physical store, a unit holding cost is applied to model either the necessity to return the unsold items to the central warehouse or to transfer them to the forthcoming selling period. Even if the product does not belong to single-period type products (perishable, fashion, ...), the connection between the central warehouse and the local store could be modeled as a newsvendor framework since it targets trade-offs between overage and underage risks.

It is worth noting that the developed framework can model retail situations involving anticipatory shipping. In this scenario, the decision maker proactively delivers a certain quantity to a local location, like a pickup store, to meet the immediate needs of customers.

The physical store capacity K is not a decision variable in our model, but the numerical section will vary it in a sensitivity analysis to investigate the effect of this capacity on the pricing strategy. Hence, the single-period problem that we develop in this chapter could be extended in a straightforward way to a multi-period setting since the myopic policy is by design optimal in our model. That is, the optimization of the multi-period problem is a sequence of optimization of the single-period problem.

Model description:

There is a retailer who sells a product through two channels, physical and online channels. We denote the set of the channels by $I = \{p, o\}$: channel p represents the physical store, and channel o represents the online store.

The main stock of the retailer is located in a central warehouse. As the inventory policy of the central warehouse is not within the scope of the chapter, we assume that the online stock at the central warehouse is uncapacitated. The physical store's capacity is however limited and fixed equal to K . In each selling period, the stock of the physical store is replenished, and pricing decisions for the two channels are decided. In the physical store, the retailer faces a price-setting newsvendor problem: there is a pricing decision r_p to be made before random demand is realized, so there may eventually be unmet demand or leftovers in the physical store. Let c_p be the unit cost of inventory and h_p be the value of leftover units at the end of the selling period. The cost c_p models the unit cost of the product after being transferred to the local cost (including purchasing cost, delivery, and operational cost at the physical store). The unit cost h_p models the overage penalty paid for each unsold unit which could cover the transfer of the product to the forthcoming period, or the reverse logistics cost if it needs to be shipped back to the central warehouse.

The online channel is modeled exogenously: each unit of online demand results in a net profit margin of $r_o - c_o$ for the retailer. Here, r_o and c_o are respectively the unit retailing price and the per unit cost for the online channel. The cost c_o models costs incurred for a product in the central warehouse (including purchasing cost and operational costs in the warehouse).

3.3.1 Dual-channel stochastic attraction demand model

In this part, we outline the construction of a stochastic dual-channel attraction demand framework that could quantify cross-channel demand interactions and serve as the foundation for several sophisticated retailing decisions. In the dual-channel world, we employ attraction models to simulate the channel choice of a consumer.

Let $\mathbf{D} = (D_p, D_o)$ be the vector of demands pertaining to the two stores. To simulate customer preferences across channels in a dual-channel environment, we assume that the

demand for the product in a particular store depends on the attributes of the two selling channels. In particular, we assume that the demand D_i of channel i ($i \in I$) has the following form:

$$D_i(\mathbf{r}) = \text{Potential Market Size} * \text{Market Share of Channel } i \quad (3.1)$$

$$= \xi \frac{g_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)}, \quad (3.2)$$

where

- ξ is the potential market size,
- $g_i(r_i)$ is the attraction function of customers to channel i .

This framework quantifies the cross-channel demand interactions in particular. The potential market size depicts the total number of consumers interested in the goods. In contrast, the market share, also known as the purchase/choice probability, illustrates how consumers choose between physical and online stores, including choosing not to buy the product. Here, the market size, ξ , is assumed to be stochastic. It is considered as a continuous random variable, characterized by its cumulative distribution function (CDF) $F_\xi(x)$ and probability density function (PDF) $f_\xi(x)$. As a result, the model is referred to as the stochastic dual-channel attraction demand model.

Our research considers that ξ is a bounded random variable, $|\xi| < M$ (big M refers to a large number). It follows that $\mathbb{E}[|\xi|] < M$. In practice, if the support of ξ is not bounded, we can use an equivalent truncated distribution for ξ . It is intuitive since demand realization is non-negative and bounded from above.

Frequently used attraction models in the literature are:

- Linear attraction model: $g_i(r_i) = a_i - b_i r_i$, with $a_i > 0, b_i > 0, \min a_i - b_i r_i > 0$;
- Multinomial logit (MNL) model: $g_i(r_i) = \exp(a_i - b_i r_i)$ with $a_i > 0, b_i > 0$;
- Multiplicative competitive interaction (MCI) model: $g_i(r_i) = a_i r_i^{-b_i}$ with $a_i > 0, b_i > 1$.

In general, a_i and b_i are constants that guarantee the negative price elasticity of demand (Gallego et al., 2006). All these parameters can be estimated by fitting the real-world data with the desired attraction model.

Our research considers a more extensive set of attraction models. The theoretical results later hold not only for the linear, MNL, and MCI models but also for all the attraction models that satisfy all the following assumptions. Similar to Harsha et al. (2019), the first assumption below shows standard conditions for attraction functions.

Assumption 3. Consider an attraction function $g : \mathbb{R} \rightarrow \mathbb{R}^+$. It then satisfies

- i. $g(\cdot)$ is strictly decreasing and is twice differentiable on \mathbb{R} ,
- ii. $\lim_{x \rightarrow \infty} g(x) = 0$, and $\lim_{x \rightarrow \infty} xg(x) = 0$.

According to assumption 3, a channel's attractiveness reduces as its corresponding price increases, which implies that $\frac{g_p(r_p)}{1 + \sum_{i \in I} g_i(r_i)}$ is increasing in r_o and decreasing in r_p . This property can be interpreted as the expected demand for the physical store increasing in the retailing price of the online store and decreasing in the retailing price of the physical store, and vice versa for the demand for the online store. Moreover, this assumption guarantees that a channel's demand becomes zero as its price becomes arbitrarily large.

The second assumption represents a sufficient condition to prove the theoretical result given in Proposition 4. It naturally comes from our model settings and is defined as follows.

Assumption 4. For all $i \in I$, let $\zeta_i(r_i) := \frac{g'_i(r_i)}{g_i(r_i)}$. It satisfies

- i. $\lim_{r_i \rightarrow \infty} \zeta_i(r_i) > -\infty$
- ii. $\zeta_i(r_i) - \frac{\zeta'_i(r_i)}{\zeta_i(r_i)} < 0$

Assumption 4 is rather technical and not very restrictive. For instance, the assumption is satisfied for the linear, MNL, and MCI models. By assumption 3, we deduce that $\zeta_i(r_i) < 0$ for all r_i , which is also needed for the mathematical analysis later.

Significantly, the two assumptions above create a set of attraction models. This set contains all the linear, MNL, and MCI models.

In our model, the component $\frac{1}{1 + \sum_{i \in I} g_i(r_i)}$ quantifies the probability of the no-purchase option, capturing the situation when customers may still be interested in the item, but do not decide to buy it (Harsha et al., 2019; Subramanian and Harsha, 2021). It follows that even the “potential market demand” ξ remains fixed by its distribution function, the “total real demand” - i.e. the total number of items that are generated after pricing decisions of the retailer, $\xi \frac{g_p(r_p) + g_o(r_o)}{1 + \sum_{i \in I} g_i(r_i)}$ - is not fixed and price-dependent. Given the retailing prices of online and physical stores, the law of “total realized demand” is determined by the law of “potential market demand”.

3.3.2 Dual-channel price optimization with stochastic attraction demand

To shorten the formulation, for all $i \in I$, we denote

$$G_i(\mathbf{r}) = \frac{g_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)}. \quad (3.3)$$

Therefore, $D_i(\mathbf{r}) = \xi G_i(\mathbf{r})$ for all $i \in I$.

We now consider the retailer's decision problem, assuming that the retailer knows the potential market size distribution. If a customer goes to the online store, all his demands will be fulfilled. If he goes to the physical store, he may encounter two possible outcomes:

- If the store has inventory, then he can make a purchase on the spot.
- If the store is out of stock, the customer receives a promotion offer (denoted by ν) from the physical store's staff to encourage him to order the item online. If he does not accept the offer, he will be considered as a lost customer. We assume that customers' probability of accepting the offer (denoted by β) is known. It is worth noting that ν and β are supposed to be independent.

In addition, since customers tend to make an additional purchase (other products) when they come to the physical store (Cavallo, 2017; UPS, 2015), there is an extra profit α from every demand fulfilled in this channel. The retailer's expected profit under the attraction demand model is then as follows.

$$\begin{aligned} \Pi &= r_p \mathbb{E}[\min(D_p, K)] + h_p \mathbb{E}[(K - D_p)^+] - c_p K + \alpha \mathbb{E}[D_p] \\ &\quad + \beta((1 - \nu)r_o - c_o) \mathbb{E}[(D_p - K)^+] + (r_o - c_o) \mathbb{E}[D_o] \end{aligned} \quad (3.4)$$

$$\begin{aligned} &= r_p \mathbb{E}[\min(G_p(\mathbf{r})\xi, K)] + h_p \mathbb{E}[(K - G_p(\mathbf{r})\xi)^+] - c_p K + \alpha \mathbb{E}[G_p(\mathbf{r})\xi] \\ &\quad + \beta((1 - \nu)r_o - c_o) \mathbb{E}[(G_p(\mathbf{r})\xi - K)^+] + (r_o - c_o) \mathbb{E}[G_o(\mathbf{r})\xi]. \end{aligned} \quad (3.5)$$

Given the store inventory capacity of K , the retailers' expected profit from selling the product in the store channel is shown in the first three terms above. The fourth term represents the additional profit for the retailer obtained when customers visiting the physical store buy other products. The fifth describes the profit from customers who encounter stockouts in the physical channel and who are converted into online buyers, and the last represents the profit generated by customers who shop online directly.

Remark 6. *The explicit formulation of expected profit for the retailer is as follows:*

$$\begin{aligned} \Pi(r_p, r_o) &= (r_p - h_p + \alpha)G_p(\mathbf{r})\mathbb{E}[\xi] - (c_p - h_p)K + (r_o - c_o)G_o(\mathbf{r})\mathbb{E}[\xi] \\ &\quad - [(r_p - h_p) - \beta((1 - \nu)r_o - c_o)]G_p(\mathbf{r}) \int_{\frac{K}{G_p(\mathbf{r})}}^{\infty} \left(x - \frac{K}{G_p(\mathbf{r})}\right) f(x)dx. \end{aligned} \quad (3.6)$$

Proof. We have $\min(D_p, K) = D_p - (D_p - K)^+$ and $(K - D_p)^+ = (D_p - K)^+ - (D_p - K)$. Thus

$$\begin{aligned} \Pi &= r_p\mathbb{E}[\min(D_p, K)] + h_p\mathbb{E}[(K - D_p)^+] - c_pK + \alpha\mathbb{E}[D_p] \\ &\quad + \beta((1 - \nu)r_o - c_o)\mathbb{E}[(D_p - K)^+] + (r_o - c_o)\mathbb{E}[D_o] \end{aligned} \quad (3.7)$$

$$\begin{aligned} &= r_p\mathbb{E}[D_p - (D_p - K)^+] + h_p\mathbb{E}[(D_p - K)^+ - (D_p - K)] - c_pK + \alpha\mathbb{E}[D_p] \\ &\quad + \beta((1 - \nu)r_o - c_o)\mathbb{E}[(D_p - K)^+] + (r_o - c_o)\mathbb{E}[D_o] \end{aligned} \quad (3.8)$$

$$\begin{aligned} &= (r_p - h_p + \alpha)\mathbb{E}[D_p] - (c_p - h_p)K - (r_p - h_p)\mathbb{E}[(D_p - K)^+] \\ &\quad + \beta((1 - \nu)r_o - c_o)\mathbb{E}[(D_p - K)^+] + (r_o - c_o)\mathbb{E}[D_o] \end{aligned} \quad (3.9)$$

$$\begin{aligned} &= (r_p - h_p + \alpha)\mathbb{E}[D_p] - (c_p - h_p)K + (r_o - c_o)\mathbb{E}[D_o] \\ &\quad - [(r_p - h_p) - \beta((1 - \nu)r_o - c_o)]\mathbb{E}[(D_p - K)^+] \end{aligned} \quad (3.10)$$

$$\begin{aligned} &= (r_p - h_p + \alpha)\mathbb{E}[G_p\xi] - (c_p - h_p)K + (r_o - c_o)\mathbb{E}[G_o\xi] \\ &\quad - [(r_p - h_p) - \beta((1 - \nu)r_o - c_o)]\mathbb{E}[(G_p\xi - K)^+] \end{aligned} \quad (3.11)$$

$$\begin{aligned} &= (r_p - h_p + \alpha)G_p\mathbb{E}[\xi] - (c_p - h_p)K + (r_o - c_o)G_o\mathbb{E}[\xi] \\ &\quad - [(r_p - h_p) - \beta((1 - \nu)r_o - c_o)]G_p\mathbb{E}\left[\left(\xi - \frac{K}{G_p}\right)^+\right] \end{aligned} \quad (3.12)$$

$$\begin{aligned} &= (r_p - h_p + \alpha)G_p(\mathbf{r})\mathbb{E}[\xi] - (c_p - h_p)K + (r_o - c_o)G_o(\mathbf{r})\mathbb{E}[\xi] \\ &\quad - [(r_p - h_p) - \beta((1 - \nu)r_o - c_o)]G_p(\mathbf{r}) \int_{\frac{K}{G_p(\mathbf{r})}}^{\infty} \left(x - \frac{K}{G_p(\mathbf{r})}\right) f(x)dx. \end{aligned} \quad (3.13)$$

□

The dual-channel pricing (DCP) problem with stochastic attraction demand is as follows:

$$\begin{aligned} \max_{r_p, r_o} \quad & \Pi(r_p, r_o) \\ \text{s.t.} \quad & r_i \geq c_i, i \in I. \end{aligned} \quad (\text{DCP})$$

3.4 Solution to the dual-channel pricing problem

In the previous section, we present a general mathematical formulation for optimization problems DCP. The decision problem is to find the optimal r_p, r_o to maximize the expected profit function Eq. 3.6. In this section, we study the existence of an optimal global solution to the problem DCP.

Note that a \mathcal{C}^2 multi-variable function is jointly concave at a point if its associated Hessian matrix is negative semi-definite (Boyd et al., 2004). A function is jointly concave in the whole domain if it is jointly concave at every point in the domain. This is a sufficient condition for the uniqueness of an optimal global solution. However, the mathematical complexity of all the derivatives of the DCP objective function makes it impossible to verify its joint concavity property in the whole domain. In fact, it is difficult to analyze the negative semi-definiteness of the associated Hessian matrix at every point in the unbounded domain ($[c_p, \infty) \times [c_o, \infty)$). In general, this function is not jointly concave in its domain Hanson and Martin (1996). Although we have explicit derivative formulations, searching algorithms (gradient descent, stochastic gradient descent, etc.) may terminate at a local optima.

In this situation, we represent a mathematical property of a \mathcal{C}^2 function in which the existence of a stationary point (the point where the associated gradient is zero) results in the existence and uniqueness of the global solution. In particular, a \mathcal{C}^2 function has a unique stationary point if:

- there exists at least one point such that the gradient of the function at this point is zero (stationary point);
- at every stationary point, the corresponding Hessian matrix is diagonal with strictly negative elements on its diagonal.

Note that the second condition is a sufficient condition for strict concavity at a point. Thus, we can see that if we have a good structure (a particular form for associated Hessian matrices) for all the zero-gradient points, the objective function also has a good structure (uniqueness of the stationary point). The property is represented by Lemma 3 and 4 as follows:

Lemma 3. *Given a real number a , let $\psi : [a, \infty) \rightarrow \mathbb{R}$ be a \mathcal{C}^2 function of the single variable u . Suppose that*

- i. *There exists at least one point $u^* \in [a, \infty)$ such that $\psi'(u^*) = 0$,*
- ii. *At any u^* that satisfies $\psi'(u^*) = 0$, it follows that $\psi''(u^*) < 0$.*

Then, there exists only one u^* that satisfies $\psi'(u^*) = 0$, and u^* is the argument of the maxima of $\psi(\cdot)$.

Proof. See B.1.1 for the detail of the proof. \square

Lemma 4 extends the result of Lemma 3 in the multi-dimensional case as follows.

Lemma 4. Given a real value vector $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, let $\Psi : W_1 \times W_2 \times \dots \times W_n \rightarrow \mathbb{R}$, $W_i = [a_i, \infty) \subset \mathbb{R}$, be a \mathcal{C}^2 function of $\mathbf{u} = (u_1, u_2, \dots, u_n) \in W_1 \times W_2 \times \dots \times W_n$. For all $i \neq j$, and $i, j \in I = \{1, 2, \dots, n\}$, suppose that

i. There exists at least one point \mathbf{u}^* such that $\nabla\Psi(\mathbf{u}^*) = \mathbf{0}$,

ii. At all the points \mathbf{u}^* mentioned above, $\frac{\partial^2\Psi}{\partial u_i^2}(\mathbf{u}^*) < 0$ and $\frac{\partial^2\Psi}{\partial u_i \partial u_j}(\mathbf{u}^*) = 0$ ($j \neq i$) hold true.

Then, there exists only one vector \mathbf{u}^* that satisfies $\nabla\Psi(\mathbf{u}^*) = \mathbf{0}$, and \mathbf{u}^* maximizes $\Psi(\cdot)$.

Proof. Sketch of the proof. We prove this by induction on the number of dimensions of space. The base case holds true from Lemma 3. The induction step is proved by using the induction hypothesis and envelope theorem. See B.1.2 for the detail of the proof. \square

The two lemmas above represent the main idea of showing the uniqueness of an optimal global solution to the optimization problem DCP. This method is recommended when the uniqueness of the optimal global solution cannot be analyzed by checking the negative semi-definiteness property of the Hessian matrix at all the points in the domain. Thus, it enables researchers to study the existence and uniqueness of an optimal global solution for attractive profit functions and other complex objective functions.

Based on this idea, Proposition 4 below gives us a structural result on problem DCP.

Proposition 4. Given a non-negative bounded potential market demand distribution ξ , $\xi < M$, let Ω be a subset of the domain such that $G_p(r_p, r_o) \leq \frac{K}{M}$, $\forall (r_p, r_o) \in \Omega \subset [c_p, \infty) \times [c_o, \infty)$. The objective function $\Pi(r_p, r_o)$ is then mathematically well-behaved over Ω . In particular, if a stationary point exists in Ω , then it is unique and represents the global solution to the optimal problem DCP.

Proof. The proof for Proposition 4 is based on the idea represented above. The detail of the proof is given in B.2.1. \square

In Proposition 4, we show that in a part of the domain $\Omega = \left\{ (r_p, r_o) \mid G_p(r_p, r_o) \leq \frac{K}{M} \right\}$, if a stationary point exists, then it is unique and represents the optimal global solution. Thus, the existence of a local optimizer implies the existence and uniqueness of a global optimizer. In short, from the above results, we deduce that considering the attraction demand model, it is impossible to analyze the joint concavity property, existence, and uniqueness of a global solution with a traditional approach by verifying the negative semi-definiteness of the associated Hessian matrix for all the points in the domain. In such a difficult situation, we prove that there is a part of the domain in which the objective function has good mathematical behavior. In particular, if a local solution exists, then it is unique and represents the global solution.

The methodology for verifying the uniqueness of the optimal global solution is highlighted when the usual approach of checking the negative semi-definiteness of the objective function's associated Hessian matrix at all the points in the domain is not applicable. The idea for the proof of Proposition 4 represents a novel approach by looking at the structure of zero-gradient points and their associated Hessian matrices. This structure secures the stationary point's uniqueness, resulting in a global solution. This method can also be applied to other objective functions in case they are not jointly concave. Thus, it enables researchers and retailers to investigate complex profit functions.

Proposition 4 also suggests that when the objective function is complex and generally not jointly concave, a simple search procedure is enough to find the optimal global solution if the domain can be divided into some parts so that it is not difficult to find the global solution numerically for every single part. The optimal global solution in the whole domain is then delivered based on optimization results from all the parts of the domain. In the case that it is challenging to find such a partition, then a set of boundary conditions should be applied.

Proposition 5. *If we assume that $\frac{g_p(c_p)}{1 + g_p(c_p)} \leq \frac{K}{M}$, then the objective function $\Pi(r_p, r_o)$ of problem DCP is mathematically well-behaved over its domain. In particular, if a stationary point exists, then it is unique and represents the optimal global solution to the problem DCP.*

Proof. Note that $G_p(r_p, r_o)$ is monotonically decreasing with r_p and $G_p(r_p, r_o)$ is monotonically increasing with r_o . By assumption 3, recall that $g_i(\cdot)$ is a non-negative decreasing function. Thus, for all $(r_p, r_o) \in [c_p, \infty) \times [c_o, \infty)$, we have

$$G_p(r_p, r_o) = \frac{g_p(r_p)}{1 + g_p(r_p) + g_o(r_o)} \leq \frac{g_p(c_p)}{1 + g_p(c_p) + g_o(r_o)} \quad (3.14)$$

$$\leq \frac{g_p(c_p)}{1 + g_p(c_p)} \leq \frac{K}{M}. \quad (3.15)$$

Therefore, for all $(r_p, r_o) \in [c_p, \infty) \times [c_o, \infty)$, we have $G_p(r_p, r_o) \leq \frac{K}{M}$. Proposition 2 is then deduced from the result of Proposition 1. \square

Remark 7. *Proposition 5 implies that if a simple search procedure (gradient descent, stochastic gradient descent, etc.) converges in $[c_p, \infty) \times [c_o, \infty)$, it converges to the unique global solution.*

In practice, assumption $\frac{g_p(c_p)}{1 + g_p(c_p)} \leq \frac{K}{M}$ may be violated. In such a case, because of the mathematical complexity, it is challenging to find a partition of $[c_p, \infty) \times [c_o, \infty)$ such that it will be simple to find the global solution to the problem on every single subset. We therefore need to have boundary conditions $r_p \leq r_p^{max}$ and $r_o \leq r_o^{max}$ for searching the domain to ensure that a simple search procedure converges to the optimal global solution. If the domain is bounded in \mathbb{R}^2 , then it is a compact set. Therefore, the optimal global solution always exists (see, e.g., Rudin et al. (1976), Theorem 4.16). In a real-world problem, the prices are normally lower than a certain value. Moreover, we only need to provide an approximation for the optimal prices. Then, with an acceptance rate of error, we can find the optimal pricing solution by studying all the profit function values on the grid values of r_p and r_o over $[c_p, r_p^{max}] \times [c_o, r_o^{max}]$. The step size is small enough to keep the error rate acceptable.

In summary, this section represents a novel approach to prove the existence of an optimal global solution to problem DCP. This global optimizer can be found by a simple search procedure. This method allows us to go beyond the difficulty when the classical approach by verifying the joint concavity property of the objective function is not valid.

3.5 Numerical analysis

This section aims to derive some managerial insights from the model. As demonstrated in the previous section, a bounded domain should be considered to secure the existence of a global solution to the optimization problem DCP when the condition $\frac{g_p(c_p)}{1 + g_p(c_p)} \leq \frac{K}{M}$ is violated. We delve into and investigate in this section the numerical results for a demand involving linear attraction. Results for MNL and MCI demand models, which have similar messages are put in B.3.1 and B.3.2, respectively. In particular, we consider $g_i = a_i - b_i r_i$, for all $i \in I$. This type of demand results in a unit selling price r_p upper-bounded by $\frac{a_p}{b_p}$ for the physical channel, and an upper bound $\frac{a_o}{b_o}$ for the online channel price r_o .

We provide an analysis of the model given a particular set of parameters, and we particularly focus on the following managerial insights:

- What would be the best capacity for the physical store, when a showroom configuration should be preferred, and what are the trade-offs involved in the choice of the best capacity value?
- What is the impact of customer "behavioral" type parameters on the pricing strategy? The conversion of stock-outs in the physical store into possible sales in the online channel with a promotion voucher (such as free online shipping for instance) is compared to the added profit enabled thanks to the physical presence of the customers in the store.
- What is the effect of the online operational cost on the pricing strategy? How might the dual-channel configuration provide more flexibility to the retailer to enable a smooth transfer of clients between the two channels.

With regards to the potential market size, we consider a uniform distribution with an average equal to 500 and we assume two variability configurations: $\xi \sim \mathcal{U}(100, 900)$ for the high variability and $\xi \sim \mathcal{U}(400, 600)$ for the low variability.

Table 3.1 represents optimization results with different level of K . All the fixed parameters are as follows.

- Attraction demand coefficients: $a_p = 20, b_p = 1, a_o = 18, b_o = 1$
- Cost coefficients: $c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, v = 0.1$

The physical store capacity K is not an operational decision, but rather a more long-term, tactical one. However, changing it in a sensitivity analysis allows us to derive interesting managerial results. One can imagine that the retailer's operating history as an online business is afforded the opportunity to sell products through a physical selling point with a limited capacity of K . Changing K from zero to the upper bound of the demand allows for the conditions under which the retailer has to use the dual channel structure: using the physical store as a showroom with no stock or with limited stock, or completely relying on it to fulfill the demand. On the other hand, it could help for a long-term strategic decision incorporating the future market demand and increase/reduce the store capacity dedicated to the desired item.

With the problem parameters of Table 3.1, the two channels are similar in terms of demand sensitivity to price and they have close demand intercept (a_p is slightly higher than a_o).

Given that the operational cost of the physical channel is twice that of the online store, the retailer has a real interest in using the physical channel as a showroom by setting $K = 0$

(a selling point for demonstration purposes and sales are fulfilled online for the fraction β). Because of the closeness of demand parameters and the high conversion of physical stock-outs to online sales ($\beta = 0.8$), the optimal prices for the two channels are close and the showroom scenario promotes the channel with the highest unit margin.

Table 3.1 Optimal prices and related values for given inventory levels when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$

K	$\xi \sim \mathcal{U}(100, 900)$						$\xi \sim \mathcal{U}(400, 600)$					
	r_p^*	r_o^*	$G_p(\mathbf{r}^*)$	$G_o(\mathbf{r}^*)$	$G_p + G_o$	$\Pi(\mathbf{r}^*)$	r_p^*	r_o^*	$G_p(\mathbf{r}^*)$	$G_o(\mathbf{r}^*)$	$G_p + G_o$	$\Pi(\mathbf{r}^*)$
0	6.00	18.00	0.933	0.000	0.933	5160.9	16.62	16.87	0.613	0.206	0.819	6366.5
100	18.44	15.40	0.302	0.504	0.806	4800.0	16.62	16.87	0.613	0.206	0.819	5866.5
200	17.79	15.67	0.399	0.420	0.819	4799.7	16.59	16.92	0.620	0.198	0.818	5366.1
300	17.34	15.95	0.466	0.359	0.825	4609.5	16.96	16.40	0.539	0.283	0.822	4820.1
400	17.00	16.34	0.530	0.293	0.823	4282.6	16.62	16.87	0.613	0.206	0.819	4366.5
500	16.74	16.69	0.585	0.236	0.821	3857.6	16.62	16.87	0.613	0.206	0.819	3866.5
600	16.62	16.87	0.613	0.206	0.819	3366.5	16.62	16.87	0.613	0.206	0.819	3366.5
700	16.62	16.87	0.613	0.206	0.819	2866.5	16.62	16.87	0.613	0.206	0.819	2866.5
800	16.62	16.87	0.613	0.206	0.819	2366.5	16.62	16.87	0.613	0.206	0.819	2366.5
900	16.62	16.87	0.613	0.206	0.819	1866.5	16.62	16.87	0.613	0.206	0.819	1866.5

The result is completely different in Table 3.2 where all the parameters are the same as the settings of Table 1, except for the demand intercept of the online channel which is decreased to $a_o = 12$ (the other parameters are $a_p = 20, b_p = 1, b_o = 1$ and $c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$). The showroom option is no longer the best one for the demand with high variability ($\xi \sim \mathcal{U}(100, 900)$). In fact, the retailer would be better off with a capacity of $K = 300$ for the physical store.

The price difference between the two channels is now more important since the maximum marginal unit profit for the physical store $\left(\frac{a_p}{b_p} - c_p\right)$ is sufficiently higher than the marginal unit profit for the online store $\left(\frac{a_o}{b_o} - c_o\right)$.

Table 3.2 Optimal prices and related values for given inventory levels when $a_p = 20, b_p = 1, a_o = 12, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$

K	$\xi \sim \mathcal{U}(100, 900)$						$\xi \sim \mathcal{U}(400, 600)$					
	r_p^*	r_o^*	$G_p(\mathbf{r}^*)$	$G_o(\mathbf{r}^*)$	$G_p + G_o$	$\Pi(\mathbf{r}^*)$	r_p^*	r_o^*	$G_p(\mathbf{r}^*)$	$G_o(\mathbf{r}^*)$	$G_p + G_o$	$\Pi(\mathbf{r}^*)$
0	6.00	12.00	0.933	0.000	0.933	3145.0	16.47	12.00	0.779	0.000	0.779	6221.9
100	17.58	10.67	0.509	0.280	0.789	3143.3	16.47	12.00	0.779	0.000	0.779	5721.9
200	17.51	10.85	0.537	0.248	0.785	3501.9	16.47	12.00	0.779	0.000	0.779	5221.9
300	17.19	11.21	0.611	0.171	0.782	3631.3	17.20	11.39	0.635	0.138	0.773	4113.4
400	17.04	11.62	0.683	0.087	0.770	3595.7	16.88	12.00	0.757	0.000	0.757	4108.0
500	16.89	12.00	0.757	0.000	0.757	3439.9	16.47	12.00	0.779	0.000	0.779	3721.9
600	16.71	12.00	0.767	0.000	0.767	3152.7	16.47	12.00	0.779	0.000	0.779	3221.9
700	16.48	12.00	0.779	0.000	0.779	2721.9	16.47	12.00	0.779	0.000	0.779	2721.9
800	16.47	12.00	0.779	0.000	0.779	2221.9	16.47	12.00	0.779	0.000	0.779	2221.9
900	16.47	12.00	0.779	0.000	0.779	1721.9	16.47	12.00	0.779	0.000	0.779	1721.9

The results in the two tables above show that the use of the physical store either as a regular store with a given capacity or as a showroom may result in a profit loss or gain, depending on the demand parameters. It suggests that when the maximum marginal unit profit for the physical store $\left(\frac{a_p}{b_p} - c_p\right)$ is sufficiently higher than the marginal unit profit for the online store $\left(\frac{a_o}{b_o} - c_o\right)$, then the physical store should be used as a regular store and its capacity could be optimized.

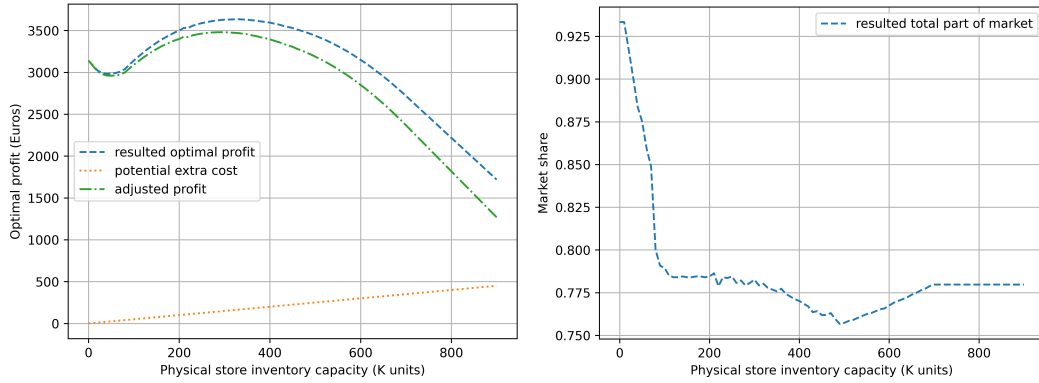
For a more detailed illustration, we keep the same parameters as above and we vary K from 0 to 900 with a step size of 10. We thus derive some figures concerning the relationship between K and optimization results.

Assuming that the capacity of the physical store is not free, we introduce the term “potential extra cost” since each unit of inventory in the physical store can generate an extra cost for the retailer (unit renting space for instance). For simplicity of presentation, we consider that this potential extra cost has a linear relationship with K . In particular, we consider *potential extra cost* = $0.5K$. The adjusted profit is then equal to the resulting optimal profit decreased by the potential extra cost.

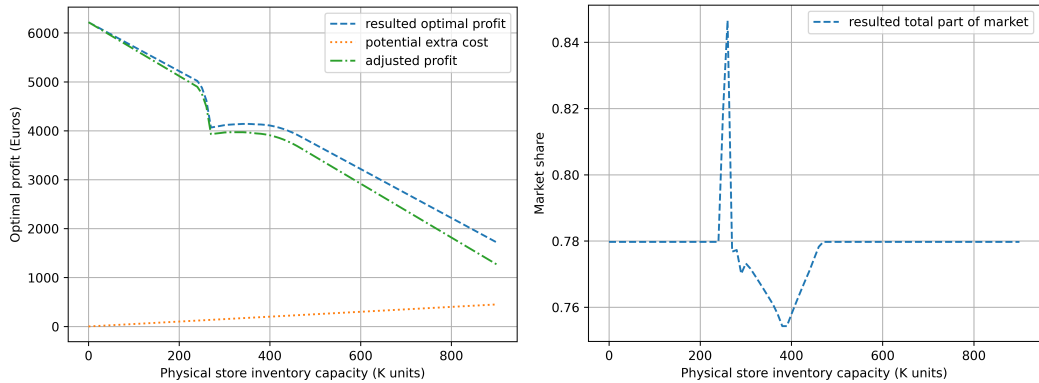
Figure 3.1 confirms that a sequential optimization is possible to determine the best physical store capacity. This best capacity results from a complex trade-off between:

- The demand attraction and the interaction between the two channels’ price elasticities.
- The margin difference between the physical and online channels.
- The rate of conversion of stock-outs in the physical channel, to online sales.

- The loss of margin incurred in case of sales with a voucher (promotion).
- The extra potential benefit is enabled by the physical presence of customers in the store (modeled with the parameter α).



(a) Optimal profit versus $K, \xi \sim \mathcal{U}(100, 900)$ (b) Market share versus $K, \xi \sim \mathcal{U}(100, 900)$

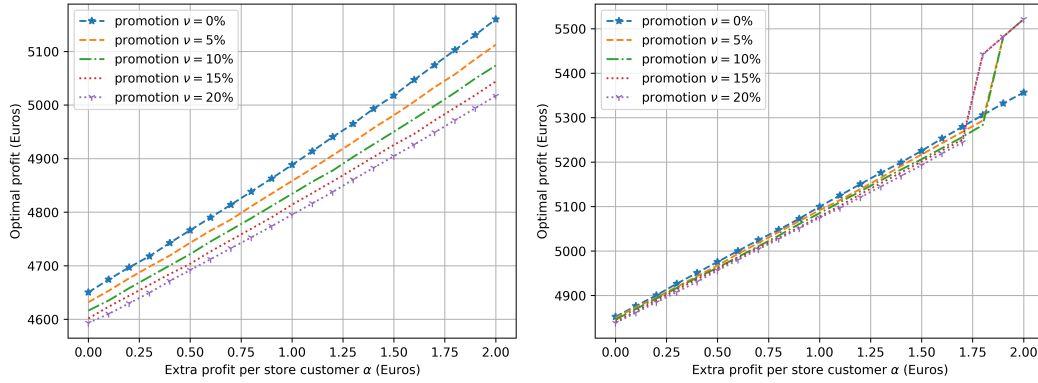


(c) Optimal profit versus $K, \xi \sim \mathcal{U}(400, 600)$ (d) Market share versus $K, \xi \sim \mathcal{U}(400, 600)$

Figure 3.1 The variation of resulted optimal profit and market share at the optimal prices when $a_p = 20, b_p = 1, a_o = 12, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, v = 0.1$

Remember that the presence of customers in the physical store is beneficial because it could generate extra sales of other products. But it is also risky, for in cases of shortage there is no guarantee of conversion to online sales, and even with conversion, the customer may have paid with a promotion voucher. To investigate margin loss due to promotion vouchers in case of stock-out and extra margin enabled by the presence of customers in the physical store, we consider a new instance of parameters. We set $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 250$. Figure 3.2 represents the evolution of the optimal profit with α (step size 0.1) for different promotion values $v = 0\%, 5\%, 10\%, 15\%, 20\%$ when $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$. Figure 3.2 shows that in both demand settings,

a higher α results in higher profits, as one would intuitively expect. Moreover, different combinations of α and v can imply the same optimal profit. However and counter-intuitively, giving a higher promotion offer with the same α will not always result in a lower profit (cf. Figure 3.2. b).



(a) Optimal profit versus α , $\xi \sim \mathcal{U}(100, 900)$ (b) Optimal profit versus α , $\xi \sim \mathcal{U}(400, 600)$

Figure 3.2 The variation of resulted optimal profit at the optimal prices with respect to α and v when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 250$.

Delivery cost is known to be very important for online businesses. This cost is embedded in c_o in our model. We now consider an instance by fixing $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, \alpha = 0.5, \beta = 0.8, K = 250$ or $K = 0$. Figure 3.3 represents the implication on optimal prices when increasing c_o (step size 0.5) given $v = 0\%, 10\%, 20\%$ when $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$. In the case of a regular physical store with $K = 250$, it is worth noting that the price increase for the online channel does not follow the same slope increase of the cost c_o and, more interestingly, the price in the physical store is decreasing to allow the transfer of sales opportunities between the two channels. Under the low demand uncertainty, Figure 3.3. b) shows the threshold $c_o = 6$, above which the online price is set to its maximum value $\left(\frac{a_o}{b_o}\right)$, pushing the system to rely exclusively on the physical store to fulfill the demand. In the showroom configuration, the same result applies, but a switch between exclusively physical and exclusively online could be a "binary" choice for the highly variable market, as shown in Figure 3.3. c).

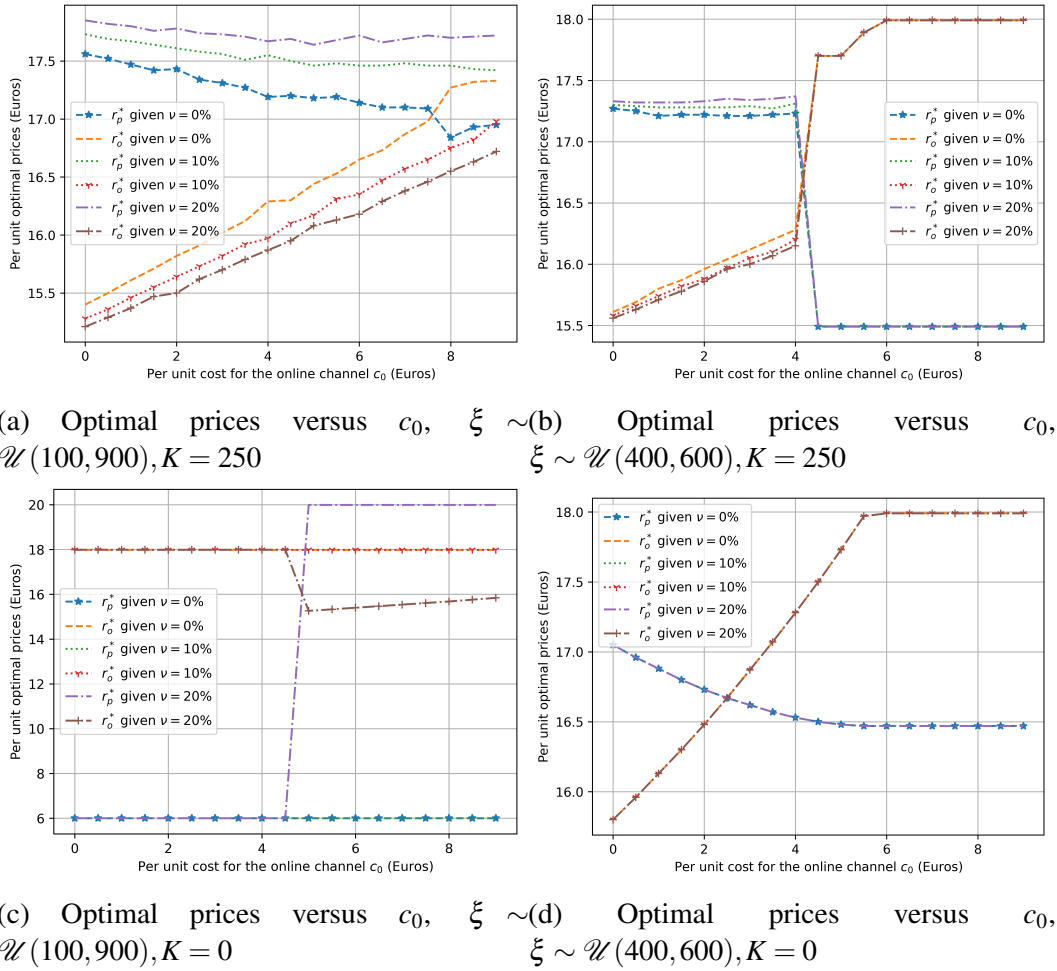


Figure 3.3 The variation of resulted optimal prices with respect to c_0 and different v when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, \alpha = 0.5, \beta = 0.8, K = 250$ and $K = 0$.

Until now, we have assumed a high value for the conversion of stockouts in the physical store to online sales. Given the importance of the rate β , we now consider an instance by fixing $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, c_o = 3, h_p = 1, \alpha = 0.5$. Figure 3.4 represents the implication on optimal prices and profit when increasing β (step size 0.05) given $K = 0, 250$ when $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$. Unsurprisingly, for $K = 250$, the profit increases with β . However, the optimal prices could increase or decrease with β , depending on the market size variability. The online and physical prices behave oppositely for each market variability, as illustrated in 3.4. b). Under the case $K = 0$ and $\xi \sim \mathcal{U}(100, 900)$, there is a threshold of β above which the retailer moves from online only to a physical only channel. The total profit increases considerably after this threshold. For the other case ($K = 0$ and $\xi \sim \mathcal{U}(400, 600)$), the optimal profit is counter-intuitively insensitive to β . This could be explained by the closeness of the optimal prices.

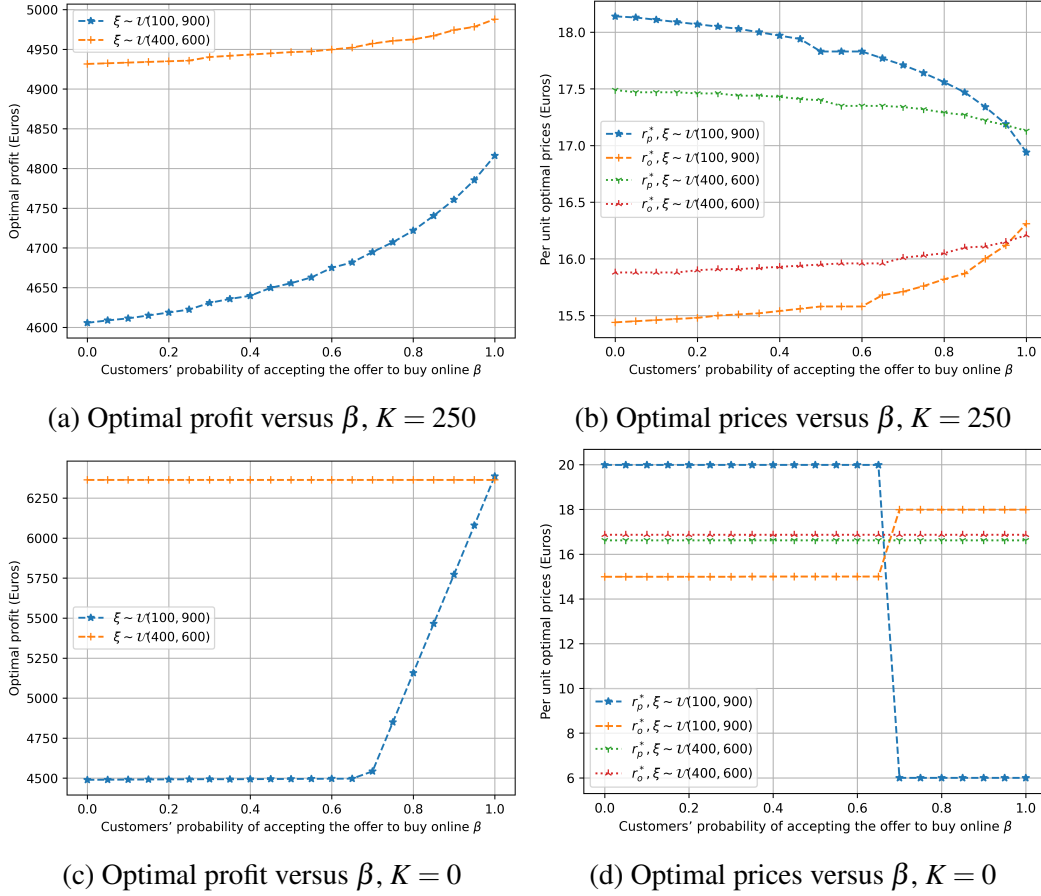


Figure 3.4 The variation of resulted optimal profit and prices with respect to β when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, c_o = 3, h_p = 1, \alpha = 0.5, K = 250$ and $K = 0$, $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$

In brief, this section offers a numerical analysis-based management value assessment for different parameter configurations. Using the optimal prices, we examine how the market share and anticipated profit change with variations in inventory level, extra profit, promotion, the proportion of customers who opt to buy online, and the operational cost of the online channel. Furthermore, our numerical analysis indicates that determining the optimal capacity for the physical store is complex. Depending on the attraction model parameters or, in other words, customers' channel preference, a showroom may be more advantageous for profit. A better capacity value for profit might result in a lower market share, which is the number of items sold divided by the potential market demand. We also contrast the conversion of stock-outs in the physical store into potential sales in the online channel with a promotion voucher, against the added profit enabled by the physical presence of customers in the store. Moreover, the pricing decisions are influenced by the online channel's operational cost. The

prices in both channels can increase or decrease to enable the transfer of sales opportunities between them. In a showroom configuration, the retailer's "binary" choice could be to switch between exclusively physical and exclusively online sales.

3.6 Conclusion and future research

This chapter proposes a retail model and investigates the corresponding optimal pricing strategy. The model employs attraction models as the framework to quantify the cross-channel effects of prices. Numerical analysis then provides a set of managerial insights from the pricing decisions.

In particular, this chapter investigates a stochastic dual-channel pricing problem and how consumer behavior affects retailers' pricing decisions. We develop a stylized model that captures the situation where the stock of the physical channel may be depleted. In this scheme, customers receive offers from the retailer to instead shop online. We employ attraction models to quantify the cross-channel dependencies and the customers' channel choice. Each channel's demand is thus a non-concave, non-linear function of prices multiplied by the stochastic potential market demand. The total market demand is generated from the pricing decision of the retailer.

The attraction model commonly used in marketing and economics literature is investigated less in the inventory control literature. This could be related to the non-linear, non-convex profit function derived from the attraction model. Our research focuses on an integrated dual-channel pricing problem in the presence of cross-channel demand interactions and critical operational considerations. We prove that the objective function of the dual-channel pricing problem is mathematically well-behaved. In particular, the existence of a local solution in a subset of the domain results in the existence and uniqueness of the optimal global solution over it. This implies that if a simple search algorithm converges in this subset, it converges to the unique optimal global solution.

This chapter also highlights the methodology for verifying the uniqueness of the optimal solution over its domain when the usual approach of checking the negative semi-definiteness of the objective function's associated Hessian matrix at all the points in the domain is not applicable. In particular, by looking at the structure of zero-gradient points and their associated Hessian matrices, this research represents a novel approach to ensure that a local optimizer is unique, resulting in a global solution. This method can be applied when the objective function is not jointly concave. Thus, it enables researchers and retailers to examine more complex profit functions.

We also provide a management value assessment based on numerical analysis for various parameter settings. Given the optimal prices, we study how the market share and expected profit vary with the inventory level K , the extra profit α , the promotion v , the proportion of customers who agree to buy online β , and the operational cost of the online channel c_o . The numerical study shows that it is complicated to figure out the best capacity for the physical store. Depending on the attraction model parameters, or equivalently, the channel preference of customers, a showroom can be given priority for profit. Better capacity value for the sake of profit may result in a lower market share, i.e., the number of sold items divided by the potential market demand. Second, the conversion of stock-outs in the physical store into possible sales in the online channel with a promotion voucher (like free online shipping, for instance) is contrasted to the added profit enabled thanks to the physical presence of the customers in the store. Last but not least, the online channel's operational cost impacts the pricing decisions. The price in the two channels can increase or decrease to permit the transfer of sales opportunities between the two channels. In the showroom configuration, a switch between exclusively physical and exclusively online could be the retailer's "binary" choice.

Beyond the scope of our current analysis, one line of study would be to analyze the optimization problem for a more extensive collection of demand models (for example, addictive demand). It would also be interesting to improve the multichannel model, which is embedded in the objective function, with multiple choice of ordering and dynamic pricing. To simplify the analysis, in our model, we have imposed an assumption that the online channel is exogenous and always in stock. Thus, another line of research would be to consider the extension of the model when the retailer has limited inventory in both the store and online channels. Furthermore, our model acknowledges the inherent limitations of physical store inventory capacity, which primarily constitute a long-term tactical decision. Therefore, another stream of research would be to theoretically investigate the optimal physical store capacity as a long-term decision. Our research also acknowledges the assumption that the probability of customers choosing to shop online during stockout in physical stores is independent of prices and promotion. This independence assumption between prices, promotional activities, and customer acceptance is a limitation in our work, prompting avenues for future research to explore the potential dependency of this probability on prices and promotion. Thanks to the applicability of our model on different attraction models, we have detailed the managerial insights for the linear attraction and showed in the appendix that these insights are not contradicted for the MNL and MCI models. A more comprehensive comparative study on different attraction models would be an interesting direction for this

research. Last but not least, it would be meaningful to investigate the case of a decentralized supply chain and its effects on the objective function and the optimal solutions.

Chapter 4

Data-driven nonlinear optimization for the chance-constrained newsvendor problem

The newsvendor model integrates considerations of inventory costs, customer service levels, and ordering decisions within the framework of known demand distribution. Yet in practical settings, decision-makers often face challenges accessing true demand distributions and their associated parameters. To tackle this issue, empirical approaches leveraging datasets have gained prominence, providing more precise insights and reducing the risk of uninformed decisions. The proliferation of high-quality datasets, advancements in optimization algorithms, and increased computational capabilities have collectively fueled the emergence of data-driven methodologies. However, the data-driven inventory decision in existing literature, which is constructed thanks to mixed integer linear programming (MILP) will not meet the constraint of the service-level based newsvendor model, resulting in the non-validity of solution when the constraint on service level is strict. Thus, this research aims to propose a different approach to solve the data-driven newsvendor problem. We use the Kernel Density Estimation (KDE) to model the constraint on service level and investigate a nonlinear programming (NLP) formulation of the problem. The numerical results show that on average, our nonlinear programming approach is better than the mixed integer linear programming approach. Our proposed methodology provides a smaller gap between the achieved and target service levels. The achieved service level of our method is also more stationary given the same size of the data compared to the MILP formulation. Additionally, the achieved service level of our model is higher than the target service level, giving the validity for the proposed methodology.

4.1 Introduction

The newsvendor problem stands as a classical problem within operations management, focusing on a decision-maker tasked with determining the optimal quantity of perishable or seasonal products to order, aiming to maximize expected profit amidst uncertain demand. This problem considers a retailer who plans to sell a good that has a stochastic demand with a known distribution. Various model variations and extensions have been explored in the academic literature (Choi, 2012).

In practice, managers cannot access the actual demand distribution before making inventory decisions. In particular, although decision-makers typically rely on historical data related to demands and various influencing factors such as price, promotions, quality, and seasonality, they cannot access the exact demand distribution (Erkip, 2022). The difficulty in determining actual demand distributions has led to a recent surge in the adoption of data-driven approaches in the newsvendor problem. Notably, Levi et al. (2015), Ban and Rudin (2019), Oroojlooyjadid et al. (2020), and Bertsimas and Koduri (2022) have explored data-driven newsvendor problems, considering historical demands and feature data.

The concept of service level, especially α -type service level (or “non-stockout probability”), proves invaluable in addressing newsvendor-based problems. When available, practitioners tend to favor the service-level approach over other methods (Abad, 2014). While there are two formulations for the newsvendor model, the cost-based model and the service level-based model, most existing research focuses on the former model. There exist only two research of Beutel and Minner (2012) and van der Laan et al. (2022) considering data-driven newsvendor problems under service level formulation. However, based on mixed-integer linear programming (MILP), their method has a drawback because the average achieved service level is lower than the target service level.

Within this observation, our research investigates the data-driven newsvendor problem incorporating features and employing the widely-used α -type service level. Our objective is to develop novel approaches to attain the target service level. Similar to the idea represented in Beutel and Minner (2012) and van der Laan et al. (2022), the motivation behind our approaches stems from recognizing that the newsvendor problem with a constraint in the non-stockout probability can be considered as a chance constraint optimization problem. Chance constraint optimization problems have been extensively studied in the stochastic programming literature (Birge and Louveaux, 2011; Shapiro et al., 2021).

Building upon the fact that the inventory decision taken as the result of the MILP formulation cannot achieve the required (minimum) service level, we introduce a new formulation for the approximation of the service level constraint considering the data-driven newsvendor problem as a chance-constrained program and solve it as a nonlinear programming (NLP)

problem, leveraging earlier theoretical findings from relevant literature. In particular, the proposed approach and corresponding mathematical optimization model are derived from the Kernel Density Estimation (KDE) methodology. Notably, the application and empirical evaluation of KDE in the context of the data-driven newsvendor problem have yet to be previously explored.

Consequently, our primary contributions revolve around demonstrating the efficacy of the Kernel Density Estimation method in handling service constraints and empirically substantiating its viability for addressing the data-driven newsvendor problem. To validate our claims, we conducted an extensive numerical study encompassing structured experiments based on simulation.

The detailed contributions of our study can be summarized as follows.

- **New data-driven nonparametric approach:** We propose a methodology to address the data-driven newsvendor problem with features under a service-level constraint. To that end, we use the KDE approach to approximate the constraint of the chance-constrained optimization problems. Our approaches come with asymptotic optimality guarantees.
- **Numerical experiments based on simulation:** We conduct a set of numerical studies, including comparative analyses against the MILP method, to assess the effectiveness of the proposed approach. Through these evaluations, we aim to provide a thorough and reliable assessment of the performance and applicability of our approach.

The subsequent sections of this chapter are structured as follows. In Section 2, we present a formal depiction of the data-driven newsvendor problem, outlining its key characteristics and considerations. Subsequently, we conduct a comprehensive literature review focusing on previous studies addressing the problem and highlighting pertinent theoretical findings originating from the realm of chance-constrained stochastic programming and an in-depth analysis of the Kernel Density Estimation (KDE) approach, delving into its fundamental principles and methodologies. Building upon this understanding, in Section 3, we introduce novel data-driven approaches based on KDE for chance-constrained optimization, presenting the underlying framework and mathematical models. To evaluate the performance of our proposed approaches, Section 4 encompasses extensive numerical experiments that examine and compare their efficacy. Finally, in Section 5, we conclude our findings, provide insights into potential avenues for future research, and summarize the contributions of this study.

4.2 Problem description and literature review

4.2.1 Newsvendor problem

The newsvendor problem holds significant importance in operations management, focusing on a decision-maker task of determining the optimal quantity of perishable products to order, aiming to maximize expected profit in the face of stochastic demand. The classical formulation of the newsvendor problem involves a decision-maker who must determine the optimal order quantity, denoted as Q , before observing the uncertain demand, represented as D , for a product with a limited shelf life. This core model has been extensively studied, with its natural extension involving pricing strategies or monopolistic scenarios receiving considerable attention in research. Many model variations and extensions have been explored in academic literature (Choi, 2012; Qin et al., 2011).

The concept of service level proves invaluable in addressing newsvendor-based problems, especially when demand is stochastic. In such situations, the newsvendor must implicitly balance profit considerations against the potential loss of goodwill and future earnings due to stockouts. The challenge is mitigated by employing the shortage cost technique to either maximize implicit profit or minimize implicit cost. However, accurately estimating the cost associated with shortages poses a notable challenge. Practitioners often view service level and cost as conflicting objectives, framing inventory control as a dual-criterion issue. Consequently, they tend to favor the service level approach whenever feasible (Abad, 2014).

In the newsvendor as a service-level constrained problem, the objective is to identify an inventory level that ensures a certain prescribed probability of meeting the demand while minimizing the expected excess inventory. This formulation aims to strike a balance between satisfying customer demand and minimizing the potential waste of unsold products. Mathematically, the service-level based newsvendor problem can be formulated as follows:

$$\begin{aligned} \min_{Q \geq 0} \quad & \mathbb{E}[(Q - D)^+] \\ \text{s.t.} \quad & \mathbb{P}(Q \geq D) \geq \alpha. \end{aligned} \tag{4.1}$$

Here, we define the parameter α , which lies within the interval $[0, 1]$, as a lower limit on the probability of non-stockout (α -type service level). Equivalently, $1 - \alpha$ is an upper limit on the probability of stockout. Furthermore, the notation a^+ is used to denote the maximum value between a and 0, i.e., $a^+ := \max\{a, 0\}$.

The formulation of the newsvendor problem described above is considered well-defined if the distribution of the demand is known. If this is the case, then the optimal solution of

problem 4.1 is $Q_{\text{service level}}^* = F^{-1}(\alpha)$, where $F(\cdot)$ is the cumulative distribution function of the demand distribution and $F^{-1}(\cdot)$ is its inverse.

It is noteworthy that the α -service level is useful in addressing newsvendor or single-period problems. Specifically, solving a single-period problem with configured unit costs for underages (u) and overages (o) is equivalent to determining the probability of avoiding stockouts, i.e., setting the service level as the ratio $\frac{u}{u+o}$. Even if not explicitly stated as a metric in solving single-period problems, the service level remains implicitly employed.

In practical scenarios, it is unlikely that the actual demand distribution is known, rendering the representation for optimal solutions infeasible. When the demand distribution is unknown, it becomes impossible to directly evaluate the optimal order quantity. Consequently, we are left with the possibility of obtaining only an approximate solution for problem 4.1 (Levi et al., 2015).

We target to solve problem 4.1. In the situation of unknown demand distribution, we propose a new numerical approach based on historical information to handle the newsvendor problem under the service level constraint.

4.2.2 Data-driven newsvendor problem

There has been a growing trend among companies to gather extensive data and utilize data science methodologies to enhance their decision-making processes (Huber et al., 2019). This shift towards data-driven approaches has sparked significant interest in the realm of inventory management, particularly concerning the newsvendor problem. In the context of the data-driven newsvendor problem, decision-makers have access to historical demand data, as well as potentially influential features that can impact demand. These features may encompass various factors such as price, promotions, weather conditions, and seasonal variations, among others.

Within this context, two primary categories of methods have emerged for addressing the challenge of unknown demand distribution in inventory decision-making: parametric and nonparametric approaches. The parametric approach assumes that the demand distribution belongs to a specific parametric family, albeit with unknown parameter values (Huh et al., 2011). The Bayesian approach, as one of the earliest solutions within this group, is well-documented in (Scarf, 1959). Another parametric approach, known as Operational Statistics, has been developed to simultaneously estimate demand and optimize inventory, and its details can be found in (Liyanage and Shanthikumar, 2005).

In contrast, the nonparametric methods belong to a family of techniques that require no specific assumptions about the demand distribution or its parameters. These approaches, often referred to as data-driven in the literature, rely on empirical information rather than

assumptions to derive solutions (Cheung and Simchi-Levi, 2019). They can be implemented as single-stage methods or as separate steps for parameter estimation and optimization. Moreover, these methods have the flexibility to incorporate contextual information, commonly referred to as data features, to enhance predictive analytics models.

Other methodologies employed for data-driven inventory control problems include Artificial Intelligence approaches, learning-based approaches, and hybrid combinations of different methodologies (Erkip, 2022). For a comprehensive review of data-driven approaches to inventory problems, readers are encouraged to refer to (Erkip, 2022), which offers a structured overview of the field.

Interestingly, a noteworthy observation is that, except for two studies, the data-driven newsvendor problem predominantly focuses on minimizing costs as the primary objective. The two exceptions are the investigations conducted by Beutel and Minner (2012) and (van der Laan et al., 2022), which tackle the more challenging and arguably more practical service-level constrained problem. In particular, Beutel and Minner (2012) propose a linear programming formulation to address the newsvendor problem with features data. The key implication is that the conventional percentile-based solution for the newsvendor problem becomes a function of the features, making it dependent on these characteristics. The identification of features is problem-specific and may vary accordingly. The study considers service level benchmarks to compare various models addressing the same problem. The linear programming problem resolves the order quantity decision of the demand model and regression coefficients as dependent on the features data. However, Beutel and Minner (2012)'s approach is prone to overfitting, posing a risk of generating infeasible solutions. van der Laan et al. (2022) demonstrate that Beutel and Minner (2012)' method can be translated into an application of the sample average approximation - hindsight approach, providing theoretical and practical insights into its ineffectiveness. The hindsight approach may suffer from poor estimation of the solution.

There is a recent surge in interest in developing data-driven solutions, highlighting the novelty and early stages of this topic. However, it could not yet be concluded to what extent the data-driven approaches are more accurate and applicable than their model-based counterparts and in what scenarios the single-stage solutions outperform the two-step methods (Ban and Rudin, 2019; Huber et al., 2019).

Our study extends the knowledge in solving data-driven newsvendor problems as a service-level constrained problem. We propose a new approach to approximate the probability constraint and numerically solve the approximated problem. The data-driven newsvendor problem is also referred to as the feature-based newsvendor problem, as follows.

Feature-based newsvendor formulation

In this research, we assume that the historical data consists of N demand observations $D_1, \dots, D_N \in \mathbb{R}$, and corresponding d -dimensional feature information $\mathbf{x}_1, \dots, \mathbf{x}_N \in \mathcal{X} \subset \mathbb{R}^d$. The decision maker's problem in the subsequent decision period is to choose an inventory level Q upon observing the new feature vector \mathbf{x}_{N+1} . The structure of data is given in the Table 4.1. To that end, the decision maker should use a decision rule of the form function $Q(\cdot) : \mathcal{X} \rightarrow \mathbb{R}$.

Table 4.1 Features and demand observations

Observation	\mathcal{X}			D
1	$\mathbf{x}_1 = (x_1^1$	\cdots	$x_1^d)$	D_1
2	$\mathbf{x}_2 = (x_2^1$	\cdots	$x_2^d)$	D_2
\vdots	\vdots		\vdots	\vdots
N	$\mathbf{x}_N = (x_N^1$	\cdots	$x_N^d)$	D_N
$N+1$	$\mathbf{x}_{N+1} = (x_{N+1}^1 \cdots x_{N+1}^d)$			N/A

The decision space, denoted as \mathcal{Q} , within which the function $Q(\cdot)$ operates, offers various options for selection. A commonly employed approach, as demonstrated in prior studies (Ban and Rudin, 2019; Beutel and Minner, 2012; van der Laan et al., 2022), involves utilizing a linear relationship. For technical convenience, let us assume that the first element of the feature vector is 1 for each observation. Thus, we have $\mathbf{x}_i = (1, \tilde{\mathbf{x}}_i)$. The decision rule can be formulated as follows:

$$\mathcal{Q} = \left\{ Q : \mathcal{X} \rightarrow \mathbb{R} : Q(\mathbf{x}_i) = \mathbf{q}^\top \mathbf{x}_i = \sum_{j=1}^d q_j x_i^j \right\}, \quad (4.2)$$

where $\mathbf{x}_i = (1, \tilde{\mathbf{x}}_i) = (1, x_i^2, \dots, x_i^d) \in \mathbb{R}^d$, and $\mathbf{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$. In this formulation, the coefficient q_1 represents the ‘‘intercept’’ and (q_2, \dots, q_d) represents the ‘‘slopes’’ related to the features information.

4.2.3 Chance constrained optimization

In order to formulate our approach, we adopt the interpretation of the service-level constraint as a chance constraint. By considering it as such, we can leverage relevant findings from the field of chance-constrained programming, which will be examined and discussed in the following.

Suppose that ξ is an m -dimensional random vector defined on some probability space $\Omega, \mathcal{A}, \mathbb{P}$. Consider then the chance-constrained optimization problem

$$\begin{aligned} \min_{q \in \mathcal{Q}} \quad & J(q, \xi) \\ \text{s.t.} \quad & \mathbb{P}(G(q, \xi) \geq 0) \geq \alpha, \end{aligned} \tag{4.3}$$

where

- $\mathcal{Q} \subset \mathbb{R}^d$ is the admissible set for the decision variables q
- $J: \mathcal{Q} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is the objective function
- $G: \mathcal{Q} \times \mathbb{R}^m \rightarrow \mathbb{R}$ defines an inequality constraint
- $\alpha \in [0, 1]$ is a probability threshold

The treatment of this type of problem dates back to the 1950s (Charnes et al., 1958). In 1959, Charnes and Cooper (1959) presented chance-constrained programming as a method for solving optimization problems in the face of uncertainty. They handled the problem by proposing a methodology for ensuring that a model's decision resulted in a certain probability of constraint compliance. Notably, a comprehensive theory was then established by Prekopa (Prškopa, 1970; Prekopa, 1973), who introduced the concept of convexity theory based on log concavity. Further advancements in the log concavity theory in stochastic programming can be found in related works by Dentcheva and Ruszczyński (2003); Prekopa (1995); Ruszczyński and Shapiro (2003); Shapiro et al. (2021). Numerous results have been derived concerning the regularity of the constraint function and the error between approximate solutions of chance-constrained optimization problems. Two fundamental theorems concerning the continuity and convexity of the constraint function were proven by Prekopa (1973) and Raik (1971). In situations where the probability distribution of ξ is unknown and replaced by an estimator, Henrion and Römisch (2004) established hypotheses that enable the estimation of the difference between the solution of the original problem and the one where the estimator is employed.

Chance-constrained optimal control problems have also been addressed using alternative techniques, including the scenario approach (Calafiore and Campi, 2006; Campi and Garatti, 2011; Carś et al., 2015; van der Laan et al., 2022) and Monte Carlo approach (Shapiro et al., 2021). The scenario approach offers the advantage of providing a priori certificates for the chance constraint. However, when extending this approach to incorporate an optimal control context, a challenge arises in that the problem may become infeasible for certain samples due to a lack of controllability. In contrast, this issue does not arise when employing a Kernel

Density Estimation-based approach (Caillaud et al., 2018). In the case of the Monte Carlo approach, Monte Carlo algorithms involve repeatedly sampling variables and parameters of a problem to obtain numerical results, treating them as random quantities. This approach can be particularly advantageous when dealing with problems that have a high number of dimensions, numerous degrees of freedom, or unknown probability distributions. The core step in a Monte Carlo method is selecting a probability distribution for the inputs and generating random input values within the specified domain. The mathematical theory supporting these methods depends on the specific type chosen; however, the fundamental result on which Monte Carlo methods are based is the strong law of large numbers.

The methodology we introduce in the subsequent section for numerically solving the data-driven newsvendor problem as a chance-constrained control problem based on Kernel Density Estimation (KDE). This technique involves estimating the probability density function (PDF) of an unknown distribution random variable based on a provided sample. Unlike Monte Carlo methods, which typically necessitate a substantial number of simulations, KDE can provide reliable density approximations with a limited number of samples in practical applications.

4.2.4 Kernel Density Estimation

Kernel Density Estimation is a numerical solution approach for solving chance-constrained control problems. This technique involves the approximation of the probability density function (PDF) of a random variable that possesses an unknown distribution using a provided sample.

Consider U , a random variable with an unknown distribution f that requires estimation, and let U_1, U_2, \dots, U_N be a sample of size N drawn from U . A Kernel Density Estimator for the probability density function (PDF) f can be defined as follows.

$$\hat{f}_{N,h}(u) := \frac{1}{Nh} \sum_{i=1}^N K\left(\frac{u - U_i}{h}\right), \quad (4.4)$$

where the function K is called kernel, and the smoothing parameter h is called bandwidth.

The origins of the Kernel Density Estimation method in its current form can be traced back to the early 1950s, as evidenced by the works of Rosenblatt (1956) and Parzen (1962). This method is sometimes referred to as the Parzen-Rosenblatt window. Silverman's book (Silverman, 1986) is considered a foundational text, providing a comprehensive understanding of KDE. Additionally, Terrell and Scott (1992) offers a detailed analysis of the various properties associated with this technique.

A crucial milestone in the development of KDE is the consistency result established by Nadaraya (1965). Variations of this theorem have been explored by Silverman (1978) and Devroye (1985), while an earlier but less general version of this result can also be found in the original work of Parzen (1962).

Theorem 6. (Nadaraya, 1965). *Assume that the kernel $K : \mathbb{R} \rightarrow \mathbb{R}^+$ is a function of bounded variation and $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a uniformly continuous density function. If h satisfies $\sum_{N=1}^{+\infty} e^{-\zeta N h^2} < +\infty$, then*

$$\mathbb{P}\left[\lim_{N \rightarrow +\infty} \sup_u |\hat{f}_{N,h}(u) - f(u)| = 0\right] = 1. \quad (4.5)$$

The approximation error between f and $\hat{f}_{n,h}$ depends on the choice of both kernel K and bandwidth h . The kernel K is assumed to be satisfying the following standard conditions.

$$\int_{-\infty}^{+\infty} K(u) du = 1, \quad \int_{-\infty}^{+\infty} u K(u) du = 0, \quad \text{and} \quad \int_{-\infty}^{+\infty} u^2 K(u) du > 0. \quad (4.6)$$

The application of KDE extends beyond its use in the field of statistics. It has found relevance in diverse disciplines such as archaeology, banking, climatology, economics, genetics, hydrology, and physiology. For a more extensive list of references, Sheather's work (Sheather, 2004) can be consulted. However, to the best of our knowledge, there is no application of the KDE in inventory control in general and data-driven newsvendor problems in particular. Thus, our research offers an initial framework to fill this gap.

4.3 Application of KDE approach to data-driven newsvendor problem

Consider the following service-level constrained newsvendor problem:

$$\begin{aligned} \min_{Q \geq 0} \quad & \mathbb{E}[(Q - D)^+] \\ \text{s.t.} \quad & \mathbb{P}(Q \geq D) \geq \alpha \end{aligned} \quad (4.7)$$

To gain an understanding of the connection between chance-constrained programs and a service-level constraint in the context of the data-driven newsvendor problem, consider a scenario where the decision rule Q associates an observed feature vector \mathbf{x} with a desired inventory level. Similar to the discussion of feature-based newsvendor formulation above, we suppose that this decision rule is linear in \mathbf{x} , expressed as $Q = Q(\mathbf{x}) = \mathbf{q}^\top \mathbf{x}$.

The chance-constrained programming for newsvendor problem is thus:

$$\begin{aligned} \min_{\mathbf{q} \in \mathcal{Q}} \quad & \mathbb{E}[(\mathbf{q}^\top \mathbf{x} - D)^+] \\ \text{s.t.} \quad & \mathbb{P}(\mathbf{q}^\top \mathbf{x} - D \geq 0) \geq \alpha \end{aligned} \quad (4.8)$$

which becomes a special case of (4.3) if we define $\xi = (\mathbf{x}, D)$ and $G(\mathbf{q}, \xi) = \mathbf{q}^\top \mathbf{x} - D$.

Chance constraints are typically challenging to handle due to their inherent computational complexity Nemirovski and Shapiro (2007). This is because even for a fixed vector \mathbf{q} , evaluating the probability $\mathbb{P}[G(\mathbf{q}, \xi) \geq 0]$ requires the computation of a multidimensional integral. Moreover, the feasible region defined by (4.3), i.e., $\mathbf{q} \in \mathcal{Q} : \mathbb{P}(G(\mathbf{q}, \xi) \geq 0) \geq \alpha$, is generally non-convex, even if the set \mathcal{Q} itself is convex. Consequently, optimizing under the framework of (4.3) becomes highly challenging, particularly when the distribution of ξ is not known. To tackle these difficulties, this study proposes the use of Kernel Density Estimation to approximate chance constraints.

Through the use of Kernel Density Estimation, we can construct an estimate of the probability density function that defines the chance constraint. This enables us to replace the probability calculation with the integral of the estimated PDF, thereby transforming the stochastic optimization problem into a deterministic one. For a given \mathbf{q} within the set \mathcal{Q} , let $f_{\mathbf{q}}$ represent the true PDF of the random variable $\mathbf{q}^\top \mathbf{x} - D$, where \mathbf{q} serves as a parameter. Additionally, let $\hat{f}_{\mathbf{q}}$ denote the approximation of the PDF. To maintain clarity in the notation, we omit the subscripts denoting the number of samples, kernel, and bandwidth, as used in (4.4). Nevertheless, it is important to note that the approximated PDF, $\hat{f}_{\mathbf{q}}$, is dependent on the sample size, choice of kernel, and bandwidth. We have

$$\mathbb{P}(\mathbf{q}^\top \mathbf{x} - D \geq 0) = \int_0^{+\infty} f_{\mathbf{q}}(u) du, \quad (4.9)$$

where $F_{\mathbf{q}}$ denotes the cumulative distribution function of the random variable $\mathbf{q}^\top \mathbf{x} - D$ for a given $\mathbf{q} \in \mathcal{Q}$. We then build the estimator $\hat{f}_{\mathbf{q}}$ of $f_{\mathbf{q}}$ via KDE.

We can write an approximation of the chance-constrained programming for the newsvendor problem (4.8) in the form

$$\begin{aligned} \min_{\mathbf{q} \in \mathcal{Q}} \quad & \frac{1}{N} \sum_{i=1}^N (\mathbf{q}^\top \mathbf{x}_i - D_i)^+ \\ \text{s.t.} \quad & \int_0^{+\infty} \hat{f}_{\mathbf{q}}(u) du \geq \alpha \end{aligned} \quad (4.10)$$

Let \mathbf{q}^* and $\hat{\mathbf{q}}^*$ denote the solutions to problems (4.8) and (4.10), respectively. While an explicit estimation of the error between \mathbf{q}^* and $\hat{\mathbf{q}}^*$ in terms of the difference between $\hat{f}_{\hat{\mathbf{q}}^*}$ and $f_{\mathbf{q}^*}$ may not be available, we can still rely on the law of large numbers to validate our results retrospectively.

Remark 8. *Note that*

$$\mathbb{P}(\mathbf{q}^\top \mathbf{x} - D \geq 0) = 1 - \mathbb{P}(\mathbf{q}^\top \mathbf{x} - D < 0) = 1 - \int_{-\infty}^0 f_{\mathbf{q}}(u) du = 1 - F_{\mathbf{q}}(0), \quad (4.11)$$

where $F_{\mathbf{q}}$ denotes the cumulative distribution function of the random variable $\mathbf{q}^\top \mathbf{x} - D$ for a given $\mathbf{q} \in \mathcal{Q}$. Thus, by defining

$$\hat{F}_{\mathbf{q}}(s) := \int_{-\infty}^s \hat{f}_{\mathbf{q}}(u) du, \quad (4.12)$$

we can write an approximation of our chance-constrained optimization problem (4.8) in the form

$$\min_{\mathbf{q} \in \mathcal{Q}} \frac{1}{N} \sum_{i=1}^N (\mathbf{q}^\top \mathbf{x}_i - D_i)^+ \quad (4.13)$$

$$\text{s.t.} \quad \hat{F}_{\mathbf{q}}(0) \leq 1 - \alpha \quad (4.14)$$

The numerical solution of the chance-constrained programming for the newsvendor problem involves the following steps.

Data preparation.

Obtain a sample $\xi_1, \xi_2, \dots, \xi_N$ of size N from historical data of the random vector $\xi = (\mathbf{x}, D)$. This operation is performed only once at the beginning of the optimization procedure because the realizations of ξ depend solely on its historical data and are independent of the decision variable \mathbf{q} . For any given value of \mathbf{q} , the data contains N realizations $\mathbf{q}^\top \mathbf{x}_1 - D_1, \dots, \mathbf{q}^\top \mathbf{x}_N - D_N$ of the random variable $\mathbf{q}^\top \mathbf{x} - D$.

Compute the constraint function.

We need to determine the value of $\hat{f}_{\mathbf{q}}(u)$ for any given \mathbf{q} . For each element ξ_i in the sample, we evaluate the constraint function $\mathbf{q}^\top \mathbf{x} - D$. This allows us to compute an approximation of the probability density function (PDF) based on the chosen kernel and bandwidth, as described in equation (4.4).

To estimate $\int_0^{+\infty} \hat{f}_{\mathbf{q}}(u) du$, we employ a quadrature rule to approximate the integral of the estimated density function. This involves a two-level approximation scheme for the

chance constraint, where Kernel Density Estimation is used for PDF estimation, followed by quadrature to approximate the cumulative distribution function of the estimated density.

For the quadrature, we employ the composite Simpson's rule. It involves dividing the interval $[a, b]$ into an even number M of subintervals and applying the corresponding formula to compute the integral of the function v :

$$\int_a^b v(u)du \approx \frac{1}{3} \frac{b-a}{M} \left(v(a) + 2 \sum_{i=1}^{M/2-1} v(u_{2i}) + 4 \sum_{i=1}^{M/2} v(u_{2i-1}) + v(b) \right), \quad (4.15)$$

where $u_i := a + \frac{b-a}{M}i, i \in \{0, 1, \dots, M-1\}$. We utilize Simpson's rule due to its favorable trade-off between code implementation simplicity and precision. This quadrature formula ensures a bounded error, which can be expressed as follows:

$$\left(\frac{b-a}{M} \right)^4 (b-a) \max_{u \in [a,b]} |v^{(4)}(u)| \quad (4.16)$$

Further information regarding this formula can be explored in the study conducted by Young and Gregory (1988).

In our data-driven newsvendor problem, we can choose zero for the lower bound of the integration. Consider the upper bound of the integration. If we assume that the random variable $\mathbf{q}^\top \mathbf{x} - D$ is normally distributed, i.e., $G(\mathbf{q}, \xi) \sim \mathcal{N}(\mu, \sigma^2)$ with both μ and σ unknown and its N realization are $\{G_1, \dots, G_N\} = \{\mathbf{q}^\top \mathbf{x}_1 - D_1, \dots, \mathbf{q}^\top \mathbf{x}_N - D_N\}$, then we can choose the upper bound for the integration equal to $\bar{G}_N + T_p s_N \sqrt{1 + (1/N)}$, where \bar{G}_N is the sample mean and T_p is the $100(1-p)^{th}$ percentile of Student's t-distribution with $N-1$ degrees of freedom. The upper bound is thus chosen with the confidence level of $100(1-p)\%$, i.e., $\mathbb{P}(G_{N+1} \leq \bar{G}_N + T_p s_N \sqrt{1 + (1/N)}) = 1-p$. In general, if we don't have an assumption on the distribution of G , the nonparametric approach gives us the upper bound, which is the maximum value of N realizations, G_1, \dots, G_N . Since all observations have an equal probability of being the maximum, the probability that G_{N+1} is the maximum is $\frac{1}{N+1}$. Therefore, $\mathbb{P}(G_{N+1} \leq \max\{G_i, i = 1, \dots, N\}) = \frac{N}{N+1}$. When $N = 20, 50, 100$, this probability is approximately 95%, 98%, 99%, respectively. For more advanced technical methods to find the upper bound, we refer readers to the conformal prediction and the work of (Shafer and Vovk, 2008), (Barber et al., 2023), and (Angelopoulos et al., 2023). In practice, the upper bound can be considered big enough but not very big to avoid computational complexity.

Solve the approximated problem.

Having established the approximation for the constraint, we are now able to treat the problem (4.10) as a conventional deterministic optimization problem and proceed with its solution. It is a nonlinear constrained optimization, and we use Artelys Knitro solver on a hosted local Jupyter Notebook on a personal computer with CPU RAM 8G to solve it.

The nonlinear programming (NLP) for solving the data-driven newsvendor problem is given as follows.

$$\min_{\mathbf{q}, \boldsymbol{\varphi}} \frac{1}{N} \sum_{i=1}^N \varphi_i \quad (4.17)$$

$$\text{s.t. } \varphi_i \geq \mathbf{q}^\top \mathbf{x}_i - D_i, i = 1, \dots, N \quad (4.18)$$

$$\varphi_i \geq 0, i = 1, \dots, N \quad (4.19)$$

$$\frac{1}{3} \frac{b-a}{M} \left(\hat{f}(a) + 2 \sum_{j=1}^{M/2-1} \hat{f}(u_{2j}) + 4 \sum_{j=1}^{M/2} \hat{f}(u_{2j-1}) + \hat{f}(b) \right) \geq \alpha \quad (4.20)$$

where

- $$\hat{f}(u) := \frac{1}{Nh} \sum_{i=1}^N K \left(\frac{u - (\mathbf{q}^\top \mathbf{x}_i - D_i)}{h} \right), \quad (4.21)$$

- $K(z)$ is the Kernel function,
- h is the optimal bandwidth for the Kernel function $K(z)$,
- a and b are the estimations for the upper and lower bounds of the integration. As discussion in the previous step, we can choose $a = 0$ and $b = \max(\mathbf{q}^\top \mathbf{x}_i - D_i, i = 1, \dots, N)$.
- $u_j = a + j \frac{b-a}{M}$.
- $\mathbf{x}_i = (1, \tilde{x}_i)$ and $D_i, i = 1, \dots, N$, are the observed features and corresponding demand data.
- $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N)$ is the surplus inventory.
- $\mathbf{q} = (q_1, \dots, q_d)$ is the decision rule need to be determined.

In the above formulation, equations (18-20) represent the objective function, and equation (21) describes the service level constraint. When we establish the solution process in the previous settings, the optimal bandwidth and estimated boundaries of the integration depend

on the decision variable \mathbf{q} , which causes the model to be very complex to solve. Indeed, through experiments, we observe that the Jupyter Notebook hosting the solver runs out of memory and then crashes even if N is small. Therefore, we can not solve it due to a memory crash. Thus, a way to go beyond this difficulty is to provide a posterior estimation of the bandwidth and boundaries for the integration.

The mixed integer linear programming (MILP) now works as a tool to estimate the bandwidth and the boundaries of the integration. Similar to Beutel and Minner (2012), the MILP for data-driven newsvendor problem is as follows:

$$\min_{\mathbf{q}, \boldsymbol{\varphi}, \boldsymbol{\zeta}} \frac{1}{N} \sum_{i=1}^N \varphi_i \quad (4.22)$$

$$\text{s.t. } \varphi_i \geq \mathbf{q}^\top \mathbf{x}_i - D_i, i = 1, \dots, N \quad (4.23)$$

$$\varphi_i \geq 0, i = 1, \dots, N \quad (4.24)$$

$$\mathbf{q}^\top \mathbf{x}_i + \zeta_i C \geq D_i, i = 1, \dots, N \quad (4.25)$$

$$\sum_{i=1}^N \zeta_i \leq (1 - \alpha)N \quad (4.26)$$

$$\zeta_i \in \{0, 1\}, i = 1, \dots, N \quad (4.27)$$

where C is a large positive constant. In this MILP formulation, the decision rule produced by the optimum \mathbf{q} minimizes the overall inventory level while achieving a service level of at least α in the past. To illustrate this, note that unless $\zeta_i = 1$, equation (26) requires that the demand be satisfied in the i^{th} period. As per equation (27), the maximum number of periods during which the market demand is unsatisfied is $(1 - \alpha)N$, meaning that the demand is satisfied in at least αN of N times. Additionally, equations (23), (24), and (25) are the same as equations (18), (19), and (20) of the NLP formulation. Thus, the goal of the MILP is also to reduce the overall amount of excess inventory with a different approach for approximating the service level constraint. Note that the solution of the mixed integer linear programming is not validated since, on average, the achieved service level is always lower than the target one Beutel and Minner (2012).

Let us denote the solution of the MILP by \mathbf{q}^{MILP} . Then, the estimation for the bandwidth and boundaries of the integration is obtained by considering the set of values $\{(\mathbf{q}^{MILP})^\top \mathbf{x}_i - D_i, i = 1, \dots, N\}$. In literature, given N observations $\{(\mathbf{q}^{MILP})^\top \mathbf{x}_i - D_i, i = 1, \dots, N\}$, the related optimal bandwidth can be obtained by several methods, including cross-validation and plug-in methods (Chiu, 1991). These methods are not valid to apply directly in our NLP formulation because of the dependency of the optimal bandwidth h and the decision variable \mathbf{q} , which is the target solution of the NLP. Thus, with MILP solution \mathbf{q}^{MILP} , we are not only

available to provide an estimation for the boundaries of the integration but also an estimation for the bandwidth by applying complex methods like cross-validation and plug-in methods to the set $\{(\mathbf{q}^{MILP})^\top \mathbf{x}_i - D_i, i = 1, \dots, N\}$.

In the numerical application for the methodology introduced in this chapter, for the sake of simplicity, we consider only the Gaussian kernel, i.e., $K(z) := \frac{e^{-z^2/2}}{\sqrt{2\pi}}$. Thus, Silverman's rule of thumb is applied to provide the optimal bandwidth, i.e., $h = 1.06 \min\left(s_N, \frac{IQR}{1.34}\right) N^{-1/5}$, where s_N is the standard deviation and IQR represents an approximation of the interquartile range of the sample (Wilcox, 2011). In essence, IQR provides an estimation of the difference between the third and the first quantiles of the sample $\{(\mathbf{q}^{MILP})^\top \mathbf{x}_i - D_i, i = 1, \dots, N\}$. Although the numerical application in this chapter only considers the Gaussian kernel and Silverman's rule of thumb, it is noteworthy that other kernel functions and their estimation for the optimal bandwidth through the MILP decision rule are also eligible to be considered in a real-world problem.

Remark 9. *Given the above discussion, the general procedure to solve the data-driven service-level-based newsvendor problem is given as follows.*

- i. **First step.** *Solve the MILP to obtain \mathbf{q}^{MILP} , which will be used as a foundation to provide estimation for the bandwidth and the boundaries of the integration.*
- ii. **Second step.** *Given \mathbf{q}^{MILP} , calculate and investigate the set of values $\{(\mathbf{q}^{MILP})^\top \mathbf{x}_i - D_i, i = 1, \dots, N\}$ to obtain an estimation for the boundaries of the integration and the bandwidth. We can generally consider any kernel function and its related optimal bandwidth.*
- iii. **Third step.** *Solve the NLP with the kernel function and its estimated optimal bandwidth to obtain \mathbf{q}^{NLP} .*

The next section will provide numerical results of the proposed NLP approach considering the Gaussian kernel with simulated sample data.

4.4 Numerical analysis

In a controlled simulation study, we assess the efficacy of the proposed Nonlinear Programming (NLP) approach against the conventional Mixed Integer Linear Programming (MILP) method. Our experiment involves the generation of datasets, varied sample sizes, and the application of the proposed methodology for comprehensive evaluation.

We assume here that, in addition to demand (apart from the constant term), the historical data also includes one more attribute. In particular, we consider the case when the demand is a linear function of the sales price \tilde{x} plus an error component e that is normally distributed, with a mean of zero and a variance of σ^2 . Mathematically, the demand is defined as $D = \kappa - \omega\tilde{x} + e$. For an overview of price-dependent demand, we refer the readers to the work of Petruzzi and Dada (1999). In practical situations, the parameters κ , ω , and σ^2 must be estimated since they are unknown. The pricing range is normalized to $[0, 1]$ for each randomly generated problem instance, and then we draw the known parameters κ , ω , and σ^2 in the following manner:

- Market size is uniformly distributed $\kappa \sim \mathcal{U}(100, 200)$,
- Slope is uniformly distributed $\omega \sim \mathcal{U}(30, 50)$,
- The volatility of demand is generated in such a manner that the coefficient of variation (CV) at the mean price \tilde{x} is set to 0.3, expressed as $CV = \frac{\sigma}{\mu} = 0.3$.

While κ and ω are unknown to the decision maker, they are fixed for every instance of the problem. For a given instance (a fixed value of κ and ω), the feature \tilde{x} is sampled from the uniform distribution $\mathcal{U}(0, 1)$. The demand is truncated at zero to prevent negative values. An example with 200 demand observations taken for an instance with $\kappa = 189.356013$ and $\omega = 37.614526$ is shown in Fig. 4.1.

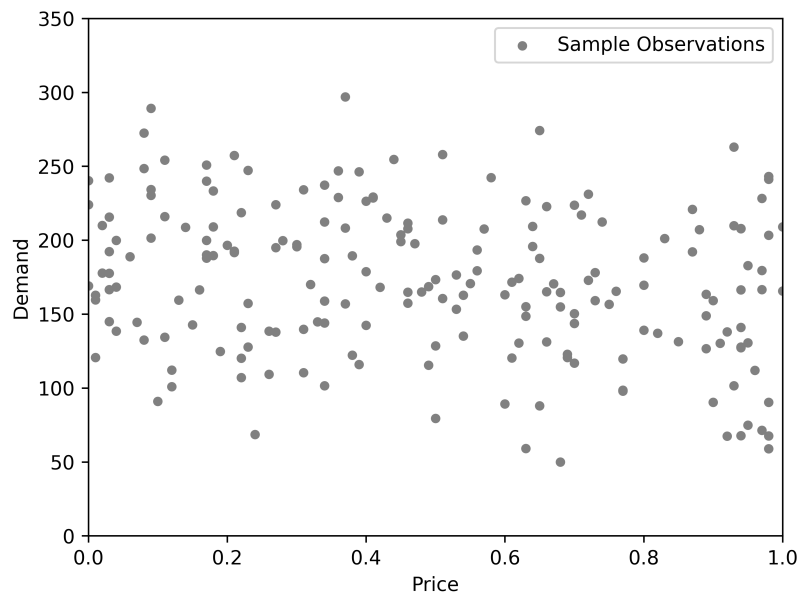


Figure 4.1 Example of demand observations for one instance

Within the same instance, from all N observations, the models are estimated, and order quantities are set such that the constraint on the target service level (the probability of non-stockout) is met. Let $\mathbf{q}^{MILP} = (q_1^{MILP}, q_2^{MILP})$ be the MILP decision rule, $\mathbf{q}^{NLP} = (q_1^{NLP}, q_2^{NLP})$ be the NLP decision rule. The following table gives us the results of the MILP decision rule, bandwidth estimation (\hat{h}), the NLP decision rule, the achieved service level of the MILP decision rule (SL_{MILP}), and the achieved service level of the NLP decision rule (SL_{NLP}) for out of sample testing data (100000 sample points) when the minimum service level is set at 90%. The size of data used for training is increasing from 10 to 200.

Table 4.2 Example of decision rule and achieved service level for one instance

N	q_1^{MILP}	q_2^{MILP}	\hat{h}	SL_{MILP}	q_1^{NLP}	q_2^{NLP}	SL_{NLP}
10	266.2175	-118.9785	33.53863	0.73626	235.7603	-15.59856	0.86605
20	217.0735	0	27.13921	0.81217	235.0427	-6.465675	0.8794
30	248.8846	-47.43279	27.845	0.85529	274.175	-56.79324	0.92763
40	245.0613	-30.51236	25.58473	0.87614	270.096	-42.94082	0.93562
50	247.4383	-42.76741	23.7493	0.85936	265.5149	-48.13231	0.91627
60	247.0716	-41.58426	24.14976	0.86053	258.6216	-33.12421	0.91785
70	247.0716	-41.58426	22.40278	0.86053	255.4594	-32.01039	0.91009
80	245.0613	-30.51236	21.5157	0.87614	252.1073	-21.35275	0.91525
90	244.7195	-28.50158	21.13481	0.87867	251.8907	-17.54084	0.91986
100	259.8818	-52.56872	21.36075	0.89026	252.7805	-13.70197	0.9271
110	259.8818	-52.56872	20.80239	0.89026	253.6881	-17.25707	0.92529
120	249.5863	-36.22668	20.0828	0.8829	252.2686	-18.29202	0.91996
130	249.5863	-36.22668	19.78155	0.8829	254.0026	-22.4929	0.91947
140	251.6159	-39.44825	19.52841	0.88483	257.8692	-28.43501	0.92239
150	249.5863	-36.22668	19.11758	0.8829	258.6068	-34.09133	0.9165
160	251.6159	-39.44825	18.77785	0.88483	257.7052	-32.4755	0.91601
170	244.4777	-27.07959	18.60586	0.88029	258.0031	-35.48973	0.91265
180	259.8818	-52.56872	18.43569	0.89026	258.1032	-33.36557	0.91597
190	251.6159	-39.44825	18.14028	0.88483	257.9045	-35.79798	0.91187
200	244.4777	-27.07959	18.10078	0.88029	256.7589	-35.57276	0.9085

From Table 4.2, it can be seen that the achieved SL of MILP is always lower than the target SL of 0.9, making this method not validated if the constraint on SL is strictly imposed. The achieved SL from NLP is always higher than the MILP case, and for this instance, the achieved SL for NLP is higher than the target SL of 0.9 when N is at least 30.

The actual optimal order quantity at a fixed price, knowing the dependency of demand and price, is given as follows (Beutel and Minner, 2012):

$$Q_{true} = \mu + SI, \quad (4.28)$$

where μ is the mean demand and SI is the safety inventory. As we consider normally distributed errors e , letting Φ denote the PDF of the standard normal distribution $Z \sim \mathcal{N}(0, 1)$,

i.e.,

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\left\{-\frac{u^2}{2}\right\} du,$$

then the required safety factors under service level constraint become:

$$SI = \sigma\Phi^{-1}(\alpha). \quad (4.29)$$

The true decision rule for a new observation $x = (1, \tilde{x})$, where \tilde{x} is the feature value (price in our case study), is given as follows.

$$\mathbf{q}_{true} = (\kappa + \sigma\Phi^{-1}(\alpha), -\omega), \quad (4.30)$$

and the order quantity decision at a given price \tilde{x} is $\kappa + \sigma\Phi^{-1}(\alpha) - \omega\tilde{x}$.

Given $N = 200$, Figure 4.2 represents the dependency of order quantity and the price in three cases: known demand (actual), MILP, and NLP decisions for the above instance.

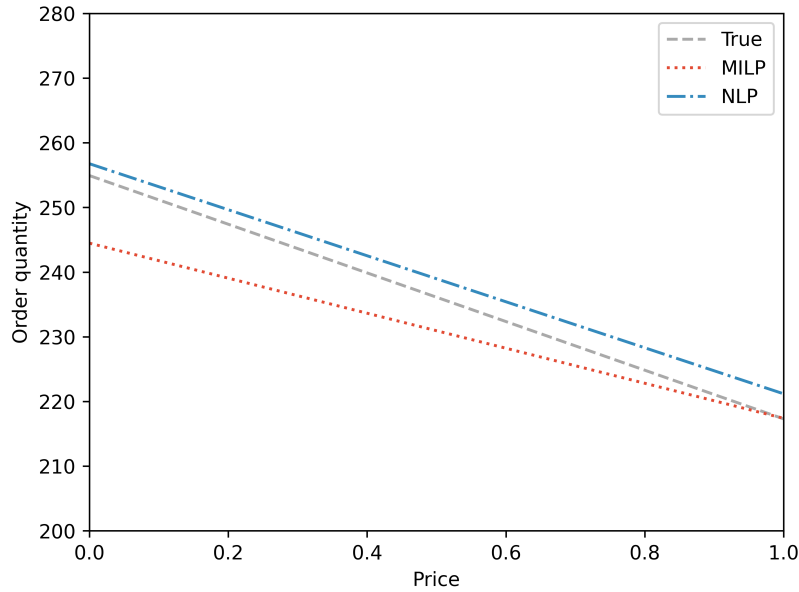


Figure 4.2 Example of order quantity decision for one instance

The results observed can vary for different instances. To evaluate the performance of the proposed NLP method compared to MILP and known demand, we solve the problem for 100 instances and calculate the average and the variance of the results, including the average achieved SL and its variance. The results of analyzing 100 instances are given in Table 4.3 and Figure 4.3.

Table 4.3 Mean and variance of the achieved service level for 100 instances

N	Mean SL_{True}	Var SL_{True}	Mean SL_{MILP}	Var SL_{MILP}	Mean SL_{NLP}	Var SL_{NLP}
10	0.899972	1e-06	0.748454	0.016859	0.884024	0.007667
20	0.899972	1e-06	0.817137	0.006175	0.906279	0.002917
30	0.899972	1e-06	0.836877	0.004067	0.912276	0.001744
40	0.899972	1e-06	0.852812	0.00277	0.915041	0.001396
50	0.899972	1e-06	0.856726	0.002544	0.913277	0.001205
60	0.899972	1e-06	0.8631	0.002038	0.913142	0.00102
70	0.899972	1e-06	0.866044	0.001626	0.913583	0.000928
80	0.899972	1e-06	0.870114	0.001431	0.914869	0.00078
90	0.899972	1e-06	0.873076	0.001156	0.913853	0.000608
100	0.899972	1e-06	0.873288	0.000785	0.91221	0.000566
110	0.899972	1e-06	0.873547	0.00074	0.911296	0.000466
120	0.899972	1e-06	0.873426	0.000768	0.911753	0.000465
130	0.899972	1e-06	0.87575	0.000706	0.911001	0.000461
140	0.899972	1e-06	0.876258	0.000687	0.911139	0.000429
150	0.899972	1e-06	0.877423	0.000685	0.910843	0.000419
160	0.899972	1e-06	0.878364	0.000664	0.910789	0.000412
170	0.899972	1e-06	0.878108	0.00063	0.910303	0.000375
180	0.899972	1e-06	0.879006	0.000566	0.910147	0.00035
190	0.899972	1e-06	0.878747	0.000577	0.909412	0.000343
200	0.899972	1e-06	0.881085	0.000571	0.909408	0.000344

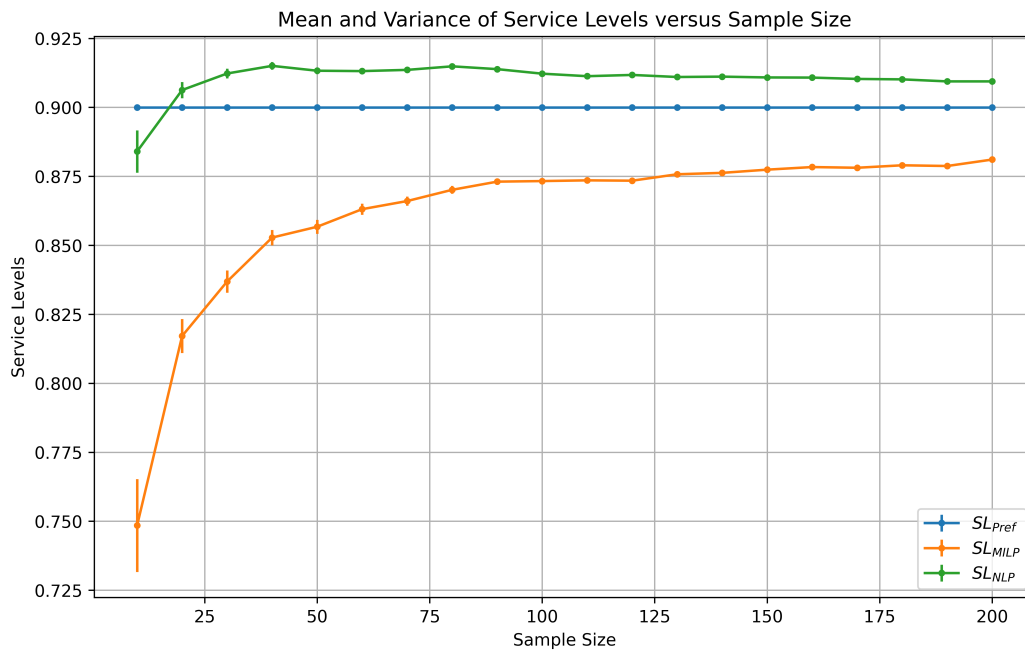


Figure 4.3 The average achieved service level and its variance

In Table 4.3, the achieved SLs when we know the actual distribution (Mean SL_{True}) are calculated through the decision rule q_{true} based on 100000 out of sample. Thus, they can deviate slightly from the actual value of 0.9. It can be seen that the average achieved SL results of NLP (Mean SL_{NLP}) are always closer to the target SL. The variance of the SL achieved by our proposed method is smaller than the MILP's, meaning our NLP formulation performs better than the MILP formulation. Moreover, while the average achieved service level of MILP is always lower than the target level of 0.9, the average achieved service level of NLP is higher than the target level when the sample size is bigger than 20. Therefore, when the constraint on service level is strictly applied, the MILP decision rule is invalid, and our proposed method is more appropriate.

To test the robustness of our proposed NLP model, we consider different values of the service level requirement $\alpha \in \{0.7, 0.8, 0.9, 0.95, 0.99\}$ and sample size $N \in \{20, 50, 100, 200\}$. We solve the MILP and NLP for 100 instances. The results are given in Table 4.4. Table 4.4

Table 4.4 Mean and variance of the achieved service level with different values of target service level α and sample size N

α	N	Mean SL_{True}	Var SL_{True}	Mean SL_{MILP}	Var SL_{MILP}	Mean SL_{NLP}	Var SL_{NLP}
0.7	20	0.700086	1.97e-06	0.591069	0.009973	0.703479	0.008148
0.7	50	0.700086	1.97e-06	0.648405	0.004926	0.714194	0.003734
0.7	100	0.700086	1.97e-06	0.666462	0.002185	0.710584	0.001497
0.7	200	0.700086	1.97e-06	0.675676	0.001212	0.708506	0.000889
0.8	20	0.800199	1.37e-06	0.691106	0.010177	0.808442	0.005953
0.8	50	0.800199	1.37e-06	0.755963	0.00378	0.818228	0.002594
0.8	100	0.800199	1.37e-06	0.765878	0.001634	0.814267	0.001074
0.8	200	0.800199	1.37e-06	0.779018	0.000872	0.810759	0.000628
0.9	20	0.899972	7.36e-07	0.817137	0.006175	0.906279	0.002917
0.9	50	0.899972	7.36e-07	0.856727	0.002544	0.913277	0.001205
0.9	100	0.899972	7.36e-07	0.873288	0.000785	0.912211	0.000566
0.9	200	0.899972	7.36e-07	0.881085	0.000571	0.909409	0.000344
0.95	20	0.94997	3.73e-07	0.863662	0.005522	0.952375	0.0014
0.95	50	0.94997	3.73e-07	0.917545	0.00138	0.956313	0.000561
0.95	100	0.94997	3.73e-07	0.922787	0.00067	0.957018	0.00029
0.95	200	0.94997	3.73e-07	0.933415	0.000318	0.955745	0.000172
0.99	20	0.990003	7.99e-08	0.931528	0.002804	0.984756	0.000313
0.99	50	0.990003	7.99e-08	0.968635	0.000699	0.989335	9.67e-05
0.99	100	0.990003	7.99e-08	0.96925	0.000382	0.99085	4.86e-05
0.99	200	0.990003	7.99e-08	0.977818	0.000114	0.991018	2.21e-05

confirms the same behavior observed in Table 4.3, i.e., the NLP is better than the MILP in the solution performance and the validation of the service level requirement.

With the same considerations for $\alpha \in \{0.7, 0.8, 0.9, 0.95, 0.99\}$ and $N \in \{20, 50, 100, 200\}$, Table 4.5 represents the mean time in seconds used for solving MILP and NLP models. The first four columns illustrate the wall time and CPU time used to solve the MILP and the NLP

models, while the last two columns display the “aggregate” time consumption to solve the NLP model. This aggregate time is equal to the sum of the time consumed to solve the MILP and NLP models, as the MILP model is used to provide an estimation for the bandwidth for the NLP model. Table 4.5 confirms that with not too much additional calculation time, the NLP model provides a better solution compared to the MILP model.

Table 4.5 The average time (seconds) used to solve the MILP and NLP problem

α	N	Wall time MILP	CPU time MILP	Wall time NLP	CPU time NLP	Aggregate wall time NLP	Aggregate CPU time NLP
0.7	20	0.17	0.43	1.78	4.82	1.95	5.25
0.7	50	1.74	4.31	5.96	15.63	7.70	19.95
0.7	100	8.96	24.20	16.13	42.57	25.09	66.77
0.7	200	186.45	591.62	35.47	93.38	221.92	685.00
0.8	20	0.26	0.49	2.02	5.43	2.28	5.92
0.8	50	2.50	5.20	6.33	16.59	8.83	21.79
0.8	100	13.67	33.55	20.58	53.62	34.25	87.17
0.8	200	184.40	542.74	42.17	110.67	226.57	653.41
0.9	20	0.27	0.35	2.19	5.92	2.47	6.26
0.9	50	2.40	4.93	7.29	19.13	9.69	24.05
0.9	100	12.32	28.01	17.93	47.21	30.26	75.22
0.9	200	95.50	283.93	46.19	122.56	141.69	406.49
0.95	20	0.17	0.18	2.35	6.34	2.52	6.52
0.95	50	1.02	1.37	7.25	19.10	8.27	20.47
0.95	100	11.23	29.05	18.18	47.48	29.41	76.53
0.95	200	40.22	108.43	43.38	114.74	83.60	223.16
0.99	20	0.01	0.02	2.43	6.57	2.44	6.59
0.99	50	0.02	0.02	7.56	19.71	7.57	19.73
0.99	100	0.33	0.37	16.95	44.87	17.29	45.24
0.99	200	7.14	22.44	41.49	109.79	48.63	132.24

Note that in the numerical example, we use a simple model for NLP with the Gaussian kernel, and an approximation of integration by considering the number of sub-intervals $M = 1000$. In practice, given more computational capacity, the approximation for the constraint can be better thanks to increasing the number of sub-intervals and another method to estimate optimal bandwidth. As the first research on applying the nonlinear approach to solving the data-driven newsvendor problem, we consider the Gaussian kernel and Silverman’s rule of thumb for the optimal bandwidth for numerical study in this chapter. We keep the comparative study for different kernel functions and their estimated optimal bandwidth of the NLP for the data-driven newsvendor problem for future research.

4.5 Extension for multichannel setting

In this section, we introduce the data-driven nonlinear optimization model for the service-level constrained inventory control problem in a multichannel setting as a natural extension of the single-channel data-driven newsvendor represented earlier in this chapter. Because of the computational complexity, we only represent the formulation of the multichannel setting. We keep the numerical analysis and comparative study for future research.

Consider a retailer who manages n channels of sales. We denote each channel in the set of n channels by $k \in \{1, 2, \dots, n\}$. The objective of the multichannel inventory model is to minimize the waste of unsold products while satisfying a minimum service level for each channel. In practice, each channel can have a different priority. Thus, to model the importance of each channel, we associate the unsold products in each channel with a “weight” coefficient. The multichannel inventory control problem under service level constraints is given as follows.

$$\begin{aligned} \min_{Q^1, Q^2, \dots, Q^n \geq 0} \quad & \sum_{k=1}^n \beta_k \mathbb{E}[(Q^k - D^k)^+] \\ \text{s.t.} \quad & \mathbb{P}(Q^k \geq D^k) \geq \alpha_k, \forall k \in \{1, 2, \dots, n\} \end{aligned} \quad (4.31)$$

In the above formulation, for each channel $k \in \{1, 2, \dots, n\}$, β_k represents the weight coefficient, Q^k, D^k stand for the inventory decisions and demand, and α_k is the minimum service level for this channel.

We assume that there is a correlation between the two channels through the channels’ feature values. For example, the price, seasonality, promotion, ..., of each channel have an impact on the demand of the other channel. Thus, the feature vector \mathbf{x} contains all the features of the two channels. Similar to the single channel setting, we assume that the inventory decision for the two channels is linear depending on the feature values. In particular, $Q^k = \mathbf{q}^{k, \top} \mathbf{x}$, where $\mathbf{q}^{k, \top} = (\mathbf{q}^k)^\top$ for $k \in \{1, 2, \dots, n\}$. The chance-constrained programming for the multichannel retailing problem is given as follows.

$$\begin{aligned} \min_{\mathbf{q}^1, \mathbf{q}^2, \dots, \mathbf{q}^n \in \mathcal{Q}} \quad & \sum_{k=1}^n \beta_k \mathbb{E}[(\mathbf{q}^{k, \top} \mathbf{x} - D^k)^+] \\ \text{s.t.} \quad & \mathbb{P}(\mathbf{q}^{k, \top} \mathbf{x} - D^k \geq 0) \geq \alpha_k, \forall k \in \{1, 2, \dots, n\} \end{aligned} \quad (4.32)$$

For $k \in \{1, 2, \dots, n\}$, let

$$\hat{f}_k(u) := \frac{1}{Nh_k} \sum_{i=1}^N K_k \left(\frac{u - (\mathbf{q}^{k, \top} \mathbf{x}_i - D_i^k)}{h_k} \right). \quad (4.33)$$

The nonlinear programming (NLP) for solving the data-driven multichannel retailing problem is given as follows.

$$\min_{\mathbf{q}^1, \mathbf{q}^2, \dots, \mathbf{q}^n, \boldsymbol{\varphi}^1, \boldsymbol{\varphi}^2, \dots, \boldsymbol{\varphi}^n} \sum_{k=1}^n \beta_k \frac{1}{N} \sum_{i=1}^N \varphi_{k,i} \quad (4.34)$$

$$\text{s.t. } \varphi_{k,i} \geq \mathbf{q}^{k,\top} \mathbf{x}_i - D_i^k, i = 1, \dots, N, \forall k \in \{1, 2, \dots, n\} \quad (4.35)$$

$$\varphi_{k,i} \geq 0, i = 1, \dots, N, \forall k \in \{1, 2, \dots, n\} \quad (4.36)$$

$$\frac{1}{3} \frac{b_k - a_k}{M} \left(\hat{f}_k(a_k) + 2 \sum_{j=1}^{M_k/2-1} \hat{f}_k(u_{k,2j}) + 4 \sum_{j=1}^{M_k/2} \hat{f}_k(u_{k,2j-1}) + \hat{f}_k(b_k) \right) \geq \alpha_k, \quad (4.37)$$

$$\forall k \in \{1, 2, \dots, n\}$$

where for each channel k , we have

- $K_k(z)$ is the Kernel function,
- h_k is the optimal bandwidth for the Kernel function $K_k(z)$,
- a_k and b_k are the estimations for the upper and lower bounds of the integration estimation for service level constraint.
- $u_{k,j} = a_k + j \frac{b_k - a_k}{M_k}$.
- $\mathbf{x}_i = (1, \tilde{x}_i)$ and D_i^k , $i = 1, \dots, N$, are the observed features and corresponding demand data.
- $\boldsymbol{\varphi}^k = (\varphi_{k,1}, \dots, \varphi_{k,N})$ is the surplus inventory.
- $\mathbf{q}^k = (q_1^k, \dots, q_d^k)$ is the decision rule need to be determined.

The above formulation is challenging to solve because of computational complexity. Compared to the single-channel setting, the difficulty of the multichannel configuration additionally comes from the possibility of different kernel choices for each channel. Moreover, the number of decision variables is significantly higher, even in the simplest case of two channels. In particular, if we consider that $n = 2$ and only one feature of price as the single channel case, i.e., assuming that $D_1 = \kappa_1 - \omega_1 \tilde{x}_1 + \nu_1 \tilde{x}_2 + e_1$ and $D_2 = \kappa_2 - \omega_2 \tilde{x}_2 + \nu_2 \tilde{x}_1 + e_2$, the decision rules are $\mathbf{q}^1 = (q_1^1, q_2^1, q_3^1)$ and $\mathbf{q}^2 = (q_1^2, q_2^2, q_3^2)$. This study also needs a comprehensive comparative study of different kernel choices for each channel and an initial estimation of the bandwidth and boundary of the integration with MILP. Therefore, we will continue with the numerical analysis and comparative investigation of the multichannel

setting in future research once we have the feedback of the research community on the single channel analysis.

4.6 Conclusion and future research

In this study, we have addressed the data-driven newsvendor problem, a significant challenge in inventory management, by introducing novel methodologies leveraging Kernel Density Estimation (KDE) within the framework of chance-constrained optimization. Our contributions have advanced the understanding and resolution of this problem in several key aspects. Firstly, we proposed a data-driven nonparametric approach that integrates features and service-level constraints, using KDE to approximate chance constraints. This methodology offers asymptotic optimality guarantees, providing decision-makers with robust tools to manage inventory under uncertainty while meeting stringent service quality requirements. Secondly, through numerical experiments and comparative analyses against the MILP method, we have demonstrated the effectiveness and applicability of our approach. Our results show superior performance in achieving target service levels, underlying the practical value of adopting data-driven methodologies in inventory control decision-making.

There are several avenues for further research and development followed. Firstly, exploring the application of advanced machine learning techniques, such as deep learning and reinforcement learning, could enhance the predictive capabilities of our models, particularly in capturing complex demand patterns and dynamic market conditions. Additionally, investigating the scalability and computational efficiency of our methodologies for large-scale inventory management problems remains a crucial area of interest. This entails devising optimization algorithms and heuristics capable of handling massive datasets and real-time decision-making scenarios, thereby facilitating seamless integration into industrial applications. In our model, we consider the process involves evaluating the PDF within the quadrature through KDE; this step becomes the primary bottleneck of the proposed approach. Thus, one promising direction for future research is to study how to model and evaluate the constraint effectively. Furthermore, extending our research to consider additional constraints and objectives, such as sustainability goals and supply chain resilience, would enrich the applicability and relevance of our methodologies in contemporary inventory management practices.

Overall, our study contributes to bridging the gap between theory and practice in inventory management by offering innovative solutions based on data-driven approaches, which can be considered as a novel framework for enhancing operational efficiency in an increasingly dynamic and uncertain business environment.

Chapter 5

Conclusion and perspectives

In this concluding chapter, we offer a comprehensive summary of the thesis, highlighting the story it tells, the insights it reveals, the contributions it delivers, and avenues for future research. Our reflection on the adventure undertaken, spanning multichannel retail, dual-channel pricing, and data-driven inventory management, seeks to draw connections across chapters and emphasize the significance of our findings.

Overview of the thesis

Our exploration through inventory management, operations management, and supply chain management to multichannel retailing, dual-channel pricing, and data-driven inventory management has been a rich tapestry of exploration, discovery, and innovation. Each chapter has added depth and nuance to our understanding of modern retail management, from the complexities of consumer behavior to the intricacies of pricing optimization. Our adventure begins amidst the bustling world of multichannel retail, where the convergence of physical and online channels presents opportunities and challenges for retail managers.

Chapter 1: We begin our research by providing an overview of recent advances and topics in operations management, supply chain management, and inventory management. This discussion highlights the necessity for inventive solutions to adapt to the evolving retail business, driven by technological advancements, data accessibility, and uncertain consumer demands.

Chapter 2: We examine the complexities of multichannel retailing, exploring the challenges of inventory allocation and pricing optimization across several physical and online channels. Through comprehensive modeling of customer behavior, channel preferences, market dynamics, and service level requirements, we uncover the strategic imperatives and operational considerations that shape the modern retail landscape.

In particular, in this chapter, we address a multichannel pricing problem that considers the effects of service level on retailers' pricing and ordering decisions. Our approach involves

modeling each channel's demand as a stochastic function of prices, with order quantities subject to service level constraints. Utilizing attraction demand models, we quantify cross-channel dependencies. We establish the mathematical well-behavedness of the objective function under service level and regular conditions, identifying a unique global solution corresponding to the zero point of the gradient. Our methodology ensures the existence and uniqueness of the optimal solution, even when conventional approaches are not applicable, by examining zero-gradient points and associated Hessian matrices. This novel method extends applicability to non-concave objective functions, offering insights for researchers and retailers into complex profit functions.

Through numerical analysis, we provide managerial insights for various service-level scenarios. We study implications in order quantities, market share, and expected profit with varying service levels. Coordination between channels becomes crucial, necessitating trade-offs between expected profit, service levels, and market share losses.

For future research, we propose extending the analysis to consider a broader range of demand models and service measures, such as addictive demand and fill rate. Enhancing the multichannel model with multiple ordering choices and dynamic pricing presents an exciting avenue. Additionally, investigating decentralized supply chains' impact on the objective function and optimal solutions promises valuable insights.

Chapter 3: Building upon our exploration of multichannel retailing, we delve into dual-channel pricing, where the strategic balancing act between profitability and customer satisfaction takes center stage. Drawing on insights from market-share models and demand theory, we develop a novel pricing model that captures the complex interactions between channels, retailers, and customers and sheds light on the elusive quest for pricing decisions in a dual-channel environment.

Specifically, our study delves into a stochastic dual-channel pricing problem and its impact on retailer pricing decisions amidst consumer behavior dynamics. We develop a stylized model capturing scenarios where physical store stock depletion enables online shopping offers to customers. We quantify cross-channel dependencies and customer channel choices by employing attraction models, resulting in non-concave, non-linear demand and objective functions. We establish the well-behavedness of the objective function, demonstrating the existence and uniqueness of the optimal global solution within its domain. We continue to use the methodology mentioned in Chapter 2 to verify the optimal solution's uniqueness and confirm a novel approach based on zero-gradient points and associated Hessian matrices. This method extends applicability to non-concave objective functions, facilitating the examination of complex profit functions.

Our numerical analysis provides managerial insights across various parameter settings, highlighting complexities in determining optimal physical store capacity and addressing stock-out conversions to online sales with promotions. Additionally, we identify the impact of online channel operational costs on pricing decisions, indicating potential adjustments to channel prices for sales transfer between channels.

Future research directions include analyzing optimization problems with broader demand models, improving retailing models with dynamic pricing, and extending the model to consider limited inventory in physical and online channels. Investigating optimal physical store capacity as a long-term decision and exploring dependencies between prices, promotions, and customer acceptance during stock-outs also warrant attention. In addition, comparative studies on different attraction models are promising avenues for further research.

Chapter 4: Finally, we focus on data-driven inventory management, where historical data and advanced methodologies converge to inform decision-making under uncertainty. Leveraging the power of Kernel Density Estimation (KDE) and nonlinear programming (NLP), we tackle the formidable challenge of integrating service-level constraints into inventory optimization models, offering a glimpse into the future of data-driven decision-making in retail.

In particular, this chapter tackles the data-driven newsvendor problem in inventory management by introducing novel methodologies leveraging Kernel Density Estimation (KDE) within the framework of chance-constrained optimization. Our contributions enhance understanding and resolution of this problem in several key aspects. Firstly, we propose a data-driven nonparametric approach integrating features and service-level constraints, using KDE to approximate chance constraints. This methodology offers asymptotic optimality guarantees, providing decision-makers with robust tools to manage inventory under uncertainty while meeting stringent service quality requirements. Secondly, through numerical experiments and comparative analyses against the mixed-integer linear programming (MILP) method, we demonstrate the effectiveness and applicability of our approach. Our results show superior performance in achieving target service levels, highlighting the practical value of adopting data-driven methodologies in inventory control decision-making.

Several avenues for further research and development are identified. Exploring advanced machine learning techniques like deep and reinforcement learning could enhance predictive capabilities, particularly in capturing complex demand patterns and dynamic market conditions. Investigating scalability and computational efficiency for large-scale inventory management problems is crucial, requiring optimization algorithms and heuristics capable of handling massive datasets and real-time decision-making scenarios. Additionally, modeling and evaluating constraints effectively, particularly the PDF evaluation within the quadrature

through KDE, presents promising future research directions. Extending our research to consider additional constraints and objectives, such as sustainability goals and supply chain resilience, would enrich the applicability and relevance of our methodologies in contemporary inventory management practices.

Overall, our study contributes to bridging the gap between theory and practice in inventory management by offering innovative solutions based on data-driven approaches. These methodologies represent a novel framework for enhancing operational efficiency in an increasingly dynamic and uncertain business environment.

Chapter 5: We conclude our study by summarizing the story throughout the thesis, encapsulating its essence, and reaffirming the significance of our contributions to retail management.

Insights

Throughout the thesis, several key insights emerge:

- **Customer-centric approach:** Success in retail hinges on a customer-centric approach, where pricing strategies, inventory management decisions, and service-level considerations are tailored to meet the needs and preferences of customers across channels.
- **Multichannel retailing:** Incorporating service-level constraints introduces complex interactions between physical and online channels, necessitating sophisticated pricing and inventory management strategies to optimize performance across channels.
- **Dual-channel retailing:** Understanding customer behavior is crucial for effective pricing decisions, especially in environments where customers may exhibit varying preferences and purchase behaviors, depending on the promotion offer and intention to switch between channels when stockout occurs.
- **Data-driven decision making:** Leveraging data-driven approaches enables more informed inventory management decisions, allowing retailers to adapt to dynamic market conditions and uncertainty more effectively.

Contributions

Our exploration of multichannel, dual-channel, and data-driven inventory management has made several notable contributions to the field:

- **Conceptual framework:** We have developed a comprehensive conceptual framework for understanding and addressing the unique challenges of multichannel retailing, providing a roadmap for retailers to navigate the complexities of modern retail environments.

- **Methodological advances:** Our exploration has led to the development of novel methodologies and analytical tools for pricing optimization, inventory management, and data-driven decision-making, pushing the boundaries of theoretical and practical approaches in retail management.
- **Managerial insights:** Through numerical experiments, we have uncovered valuable insights into the effectiveness of different inventory management strategies, the impact of customer behavior on pricing decisions, and the potential of data-driven methodologies to drive business success.

Future research

Results obtained in this dissertation provide interesting aspects of retail management and stimulate the development of further research. Among research perspectives, the following ones are of special interest.

- **Advanced modeling:** Further exploration of advanced modeling techniques, especially for chapter 4, such as machine learning algorithms and predictive analytics, could enhance the accuracy and predictive capabilities of inventory management models, enabling retailers to anticipate better and respond to uncertain market conditions.
- **Dynamic pricing strategies:** All the models considered in this thesis are single-period. Thus, investigating dynamic pricing strategies, personalized pricing algorithms, and real-time pricing adjustments could unlock new opportunities for revenue optimization and customer targeting in long-term multichannel retail environments.
- **Supply chain resilience:** Exploring the integration of supply chain resilience considerations into inventory management models, including risk mitigation strategies, supply chain diversification, and adaptive logistics, could help retailers build more resilient and adaptable supply chains.
- **Sustainability and ethics:** Examining the intersection of retail management with sustainability and ethical considerations, including environmentally friendly practices, fair labor standards, and ethical sourcing, could pave the way for more responsible and socially conscious retail practices.

Conclusion

In conclusion, this thesis has addressed the multifaceted challenges facing retailers in the era of digital transformation and technological advancement. The evolving retail landscape, characterized by the rise of e-commerce, the proliferation of digital channels, and the growing

influence of data analytics, has presented retailers with complex and dynamic scenarios that demand innovative solutions.

Throughout this study, we have highlighted the critical role of supply chain management in navigating the complexities of modern retailing. Recognizing the importance of effectively balancing supply and demand, we have emphasized the need for retailers to leverage advanced analytics and optimization techniques to optimize flow management across channels. Our exploration of innovative frameworks for flow management has underscored the significance of blurring the boundaries between the physical and digital supply chain. By facilitating an integrated and optimized multichannel configuration, these frameworks offer retailers the opportunity to enhance customer experiences and drive competitive advantage.

Furthermore, our analysis has shed light on the transformative impact of information technology on supply chain management. The emergence of the Internet of Things (IoT) and advancements in information technology have provided supply chain managers with access to vast amounts of real-time data, enabling more informed decision-making and excellent responsiveness to customer demands. However, despite the strides made in understanding and addressing the challenges of modern retailing, there remain areas that need to be explored for further research and exploration. The optimization of end-to-end flow management in stochastic decision-making environments represents a notable research gap that warrants attention from the operations management community.

In summary, this thesis has contributed to advancing our understanding of retailing in the digital age. It has provided valuable insights and frameworks for retailers seeking to thrive in an increasingly complex and competitive environment. By embracing innovation, leveraging data-driven approaches, and prioritizing customer-centric strategies, retailers can position themselves for success in the dynamic world of modern retailing.

Final note

This thesis presents a comprehensive investigation into the complexities of retail management in the digital age, focusing on multichannel retailing, dual-channel pricing, and data-driven inventory management. Through a series of interconnected chapters, we explore the strategic imperatives and operational considerations shaping the modern retail landscape, offering insights and methodologies to address the multifaceted challenges faced by retailers.

Chapter 1 provides an overview of recent topics and advancements in operations management, supply chain management, and inventory management, emphasizing the need for inventive solutions to adapt to evolving retail dynamics driven by technological developments and uncertain consumer demands.

Chapter 2 delves into the intricacies of multichannel retailing, examining inventory allocation and pricing optimization across physical and online channels. We address a

multichannel pricing problem, considering the effects of service level on pricing and ordering decisions, and propose a methodology to ensure the existence and uniqueness of the optimal solution, even for non-concave objective functions. Managerial insights gleaned from numerical analysis highlight the importance of channel coordination and the impact of service levels on market share and profitability.

Building upon our exploration of multichannel retailing, Chapter 3 investigates dual-channel pricing, focusing on the strategic balancing act between profitability and customer satisfaction. We develop a novel pricing model capturing complex interactions between channels, retailers, and customers. We establish the well-behavedness of the objective function, ensuring the existence and uniqueness of the optimal global solution. Managerial insights underscore the importance of determining optimal physical store capacity and addressing stock-out conversions to online sales with promotions.

Chapter 4 introduces data-driven inventory management methodologies leveraging Kernel Density Estimation (KDE) within chance-constrained optimization frameworks. Our approach offers robust tools for managing inventory under uncertainty while meeting stringent service quality requirements, demonstrating superior performance in achieving target service levels compared to traditional methods. Future research directions include exploring advanced machine learning techniques and extending the model to consider sustainability and supply chain resilience.

In conclusion, this thesis promotes our understanding of retail management in the digital age, offering valuable insights and methodologies for navigating the complexities of modern retailing. By embracing innovation, leveraging data-driven approaches, and prioritizing customer-centric strategies, retailers can position themselves for success in an increasingly dynamic and competitive environment.

Appendix A

Appendix of Chapter 2

A.1 Proofs of Lemmas

In this section, we first give the proof for Lemma 2, then introduce Lemma 6, Lemma 5, and Lemma 7 with their detailed proofs.

A.1.1 Proof of Lemma 2

The first condition (i) of the lemma secures the existence of a point \mathbf{u}^* such that $\nabla\Psi(\mathbf{u}^*) = \mathbf{0}$. Thus, if \mathbf{u}^* is an argument of the maxima of $\Psi(\mathbf{u})$, then $\frac{\partial\Psi}{\partial u_i}(\mathbf{u}^*) = 0$ for all $i \in \{1, 2, \dots, n\}$. In order to complete the proof, we need to show the uniqueness. We prove by induction on n , the number of decision variables.

When $n = 1$, the result follows from Lemma 1. Assuming that the lemma holds when $n = k - 1$, we need to show that it is true when $n = k$.

At a given u_k , $\Psi(\mathbf{u})$ can be seen as a function of $k - 1$ variables. By the induction assumption, given u_k , there exists unique vector of decision variables $\mathbf{u}^*(u_k) = (u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k))$ that satisfies

$$\frac{\partial\Psi}{\partial u_i}(u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k), u_k) = 0, \quad (\text{A.1})$$

for all $i \in \{1, 2, \dots, k - 1\}$. Moreover, $\mathbf{u}^*(u_k)$ is the maximizer of $\Psi(\mathbf{u})$ for this given value of u_k .

Now, instead of maximizing $\Psi(\mathbf{u})$ over $\mathbf{u} = (u_1, u_2, \dots, u_k)$, we can maximize the function $\Psi^*(u_k) := \Psi(\mathbf{u}^*(u_k), u_k)$ over u_k . Applying the envelope theorem, the first-order derivative of $\Psi^*(u_k)$ is given as follows:

$$\frac{\partial\Psi^*}{\partial u_k}(u_k) = \frac{\partial\Psi}{\partial u_k}(\mathbf{u}^*(u_k), u_k). \quad (\text{A.2})$$

Thus, the second-order derivative of $\Psi^*(u_k)$ is given as follows

$$\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) = \frac{\partial^2 \Psi}{\partial u_k^2}(\mathbf{u}^*(u_k), u_k) + \sum_{i=1}^{k-1} \frac{\partial^2 \Psi}{\partial u_k \partial u_i}(\mathbf{u}^*(u_k), u_k) \frac{\partial u_i^*(u_k)}{\partial u_k}. \quad (\text{A.3})$$

Now, we will prove that at all the point u_k such that $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$, then $\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) < 0$.

Indeed, when $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$, from Eq. A.1, Eq. A.2, for all $i \in \{1, 2, 3, \dots, k\}$, we have

$$\frac{\partial \Psi}{\partial u_i}(u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k), u_k) = 0. \quad (\text{A.4})$$

Then, by assumption (ii) of the lemma, we have

$$\frac{\partial^2 \Psi}{\partial u_k^2}(\mathbf{u}^*(u_k), u_k) < 0, \quad \frac{\partial^2 \Psi}{\partial u_k \partial u_i}(\mathbf{u}^*(u_k), u_k) = 0, \quad i \in \{1, 2, \dots, k-1\}. \quad (\text{A.5})$$

Embedding these properties in Eq. A.3, we deduce that

$$\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) < 0, \quad (\text{A.6})$$

given that $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$.

The lemma holds for $n = 1$ (confirm by Lemma 1), we deduce that there exists a unique u_k^* that maximizes $\Psi^*(u_k)$. Therefore, there exists a unique vector $\mathbf{u}^* = (u_1^*(u_k^*), u_2^*(u_k^*), \dots, u_{k-1}^*(u_k^*), u_k^*)$ that satisfies $\frac{\partial \Psi}{\partial u_i}(\mathbf{u}^*) = 0$ for $i \in \{1, 2, 3, \dots, k\}$ and \mathbf{u}^* maximizes $\Psi(\mathbf{u})$.

A.2 Proofs of Propositions

In this section, we provide all the complement proofs for the propositions in the main body of the paper.

A.2.1 Proof of Proposition 2

With the above notations, to prove this proposition, we need to represent explicit formulations for the part of the market function $G_i(\mathbf{r})$; channel's profit function $\Pi_i(\mathbf{r})$; total profit function $\Pi(\mathbf{r})$; their first-order derivatives, second-order derivatives, and cross derivatives via Lemma 5, Lemma 6, and Lemma 7 as follows.

Lemma 5. For all $i, j, k, l \in \{1, 2, \dots, n\}, i \neq j, i \neq l, j \neq l$, we have

a.
$$\frac{\partial G_i}{\partial r_i}(\mathbf{r}) = \zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) < 0 \quad (\text{A.7})$$

b.
$$\frac{\partial G_j}{\partial r_i}(\mathbf{r}) = -\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r}) > 0 \quad (\text{A.8})$$

c.
$$\frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) = [\zeta_i'(r_i) + (1 - 2G_i(\mathbf{r}))\zeta_i^2(r_i)]G_i(\mathbf{r})(1 - G_i(\mathbf{r})) \quad (\text{A.9})$$

d.
$$\frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r}) = -[\zeta_i'(r_i) + (1 - 2G_i(\mathbf{r}))\zeta_i^2(r_i)]G_i(\mathbf{r})G_j(\mathbf{r}) \quad (\text{A.10})$$

e.
$$\frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r}) = -\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})(1 - 2G_i(\mathbf{r})) \quad (\text{A.11})$$

f.
$$\frac{\partial^2 G_l}{\partial r_i \partial r_j}(\mathbf{r}) = 2\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})G_l(\mathbf{r}) \quad (\text{A.12})$$

Proof. Note that we have $\zeta_i(r_i) < 0, 0 < G_i(\mathbf{r}) < 1$.

a.

$$\frac{\partial G_i}{\partial r_i}(\mathbf{r}) \quad (\text{A.13})$$

$$= \frac{\partial \frac{g_i(r_i)}{g_0 + \sum_{k=1}^n g_k(r_k)}}{\partial r_i} \quad (\text{A.14})$$

$$= \frac{1}{(g_0 + \sum_{k=1}^n g_k(r_k))^2} \left[g_i'(r_i) \left(g_0 + \sum_{k=1}^n g_k(r_k) \right) - g_i(r_i) \frac{\partial (g_0 + \sum_{k=1}^n g_k(r_k))}{\partial r_i} \right] \quad (\text{A.15})$$

$$= \frac{g_i'(r_i)}{g_0 + \sum_{k=1}^n g_k(r_k)} - \frac{g_i(r_i)g_i'(r_i)}{(g_0 + \sum_{k=1}^n g_k(r_k))^2} \quad (\text{A.16})$$

$$= \frac{g_i'(r_i)}{g_i(r_i)} \frac{g_i(r_i)}{g_0 + \sum_{k=1}^n g_k(r_k)} - \frac{g_i'(r_i)}{g_i(r_i)} \frac{g_i^2(r_i)}{(g_0 + \sum_{k=1}^n g_k(r_k))^2} \quad (\text{A.17})$$

$$= \zeta_i(r_i)G_i(\mathbf{r}) - \zeta_i(r_i)G_i^2(\mathbf{r}) \quad (\text{A.18})$$

$$= \zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) < 0. \quad (\text{A.19})$$

b.

$$\frac{\partial G_j}{\partial r_i}(\mathbf{r}) = \frac{\partial \frac{g_j(r_j)}{g_0 + \sum_{k=1}^n g_k(r_k)}}{\partial r_i} \quad (\text{A.20})$$

$$= \frac{1}{(g_0 + \sum_{k=1}^n g_k(r_k))^2} \left[-g_j(r_j) \frac{\partial (g_0 + \sum_{k=1}^n g_k(r_k))}{\partial r_i} \right] \quad (\text{A.21})$$

$$= -\frac{g_j(r_j) g'_i(r_i)}{(g_0 + \sum_{k=1}^n g_k(r_k))^2} \quad (\text{A.22})$$

$$= -\frac{g'_i(r_i)}{g_i(r_i)} \frac{g_i(r_i)}{g_0 + \sum_{k=1}^n g_k(r_k)} \frac{g_j(r_j)}{g_0 + \sum_{k=1}^n g_k(r_k)} \quad (\text{A.23})$$

$$= -\zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) > 0. \quad (\text{A.24})$$

c.

$$\frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) = \frac{\partial}{\partial r_i} \left(\frac{\partial G_i}{\partial r_i} \right) (\mathbf{r}) \quad (\text{A.25})$$

$$= \frac{\partial}{\partial r_i} [\zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r}))] \quad (\text{A.26})$$

$$= \zeta'_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left(\frac{\partial G_i}{\partial r_i}(\mathbf{r}) (1 - G_i(\mathbf{r})) - G_i(\mathbf{r}) \frac{\partial G_i}{\partial r_i}(\mathbf{r}) \right) \quad (\text{A.27})$$

$$= \zeta'_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left(\frac{\partial G_i}{\partial r_i}(\mathbf{r}) (1 - 2G_i(\mathbf{r})) \right) \quad (\text{A.28})$$

$$= \zeta'_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left([\zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r}))] (1 - 2G_i(\mathbf{r})) \right) \quad (\text{A.29})$$

$$= \zeta'_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) + \zeta_i^2(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) (1 - 2G_i(\mathbf{r})) \quad (\text{A.30})$$

$$= G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) [\zeta'_i(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))]. \quad (\text{A.31})$$

d.

$$\frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r}) \quad (\text{A.32})$$

$$= \frac{\partial}{\partial r_i} \left(\frac{\partial G_j}{\partial r_i} \right) (\mathbf{r}) \quad (\text{A.33})$$

$$= \frac{\partial}{\partial r_i} [-\zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r})] \quad (\text{A.34})$$

$$= - \left[\zeta'_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) + \zeta_i(r_i) \frac{\partial G_i}{\partial r_i}(\mathbf{r}) G_j(\mathbf{r}) + \zeta_i(r_i) G_i(\mathbf{r}) \frac{\partial G_j}{\partial r_i}(\mathbf{r}) \right] \quad (\text{A.35})$$

$$= - \left\{ \zeta'_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) + \zeta_i(r_i) [\zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r}))] G_j(\mathbf{r}) \right. \\ \left. + \zeta_i(r_i) G_i(\mathbf{r}) [-\zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r})] \right\} \quad (\text{A.36})$$

$$= - [\zeta'_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) + \zeta_i^2(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) G_j(\mathbf{r}) - \zeta_i^2(r_i) G_i^2(\mathbf{r}) G_j(\mathbf{r})] \quad (\text{A.37})$$

$$= - G_i(\mathbf{r}) G_j(\mathbf{r}) [\zeta'_i(r_i) + \zeta_i^2(r_i) (1 - G_i(\mathbf{r})) - \zeta_i^2(r_i) G_i(\mathbf{r})] \quad (\text{A.38})$$

$$= - G_i(\mathbf{r}) G_j(\mathbf{r}) [\zeta'_i(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))]. \quad (\text{A.39})$$

e.

$$\frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial}{\partial r_j} \left(\frac{\partial G_i}{\partial r_i} \right) (\mathbf{r}) \quad (\text{A.40})$$

$$= \frac{\partial}{\partial r_j} [\zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r}))] \quad (\text{A.41})$$

$$= \zeta_i(r_i) \left[\frac{\partial G_i}{\partial r_j}(\mathbf{r}) (1 - G_i(\mathbf{r})) - G_i(\mathbf{r}) \frac{\partial G_i}{\partial r_j}(\mathbf{r}) \right] \quad (\text{A.42})$$

$$= \zeta_i(r_i) \frac{\partial G_i}{\partial r_j}(\mathbf{r}) (1 - 2G_i(\mathbf{r})) \quad (\text{A.43})$$

$$= \zeta_i(r_i) [-\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r})] (1 - 2G_i(\mathbf{r})) \quad (\text{A.44})$$

$$= -\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) (1 - 2G_i(\mathbf{r})). \quad (\text{A.45})$$

f.

$$\frac{\partial^2 G_l}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial}{\partial r_j} \left(\frac{\partial G_l}{\partial r_i} \right) (\mathbf{r}) \quad (\text{A.46})$$

$$= \frac{\partial}{\partial r_j} [-\zeta_i(r_i) G_i(\mathbf{r}) G_l(\mathbf{r})] \quad (\text{A.47})$$

$$= -\zeta_i(r_i) \left[\frac{\partial G_i}{\partial r_j}(\mathbf{r}) G_l(\mathbf{r}) + G_i(\mathbf{r}) \frac{\partial G_l}{\partial r_j}(\mathbf{r}) \right] \quad (\text{A.48})$$

$$= -\zeta_i(r_i) \left\{ [-\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r})] G_l(\mathbf{r}) + G_i(\mathbf{r}) [-\zeta_j(r_j) G_j(\mathbf{r}) G_l(\mathbf{r})] \right\} \quad (\text{A.49})$$

$$= -\zeta_i(r_i) [-2\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_l(\mathbf{r})] \quad (\text{A.50})$$

$$= 2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_l(\mathbf{r}). \quad (\text{A.51})$$

□

Lemma 6 and Lemma 7 represent profit functions and their derivatives as follows.

Lemma 6. For all $i, j, l \in \{1, 2, \dots, n\}, i \neq j, i \neq l, j \neq l$, we have

a.

$$\Pi(\mathbf{r}) = \sum_{k=1}^n G_k(\mathbf{r})V_k(r_k) \quad (\text{A.52})$$

b.

$$\frac{\partial \Pi_i}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r})U_i + \frac{\partial G_i}{\partial r_i}(\mathbf{r})V_i(r_i) \quad (\text{A.53})$$

c.

$$\frac{\partial \Pi_j}{\partial r_i}(\mathbf{r}) = \frac{\partial G_j}{\partial r_i}(\mathbf{r})V_j(r_j) \quad (\text{A.54})$$

d.

$$\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r})U_i + \sum_{k=1}^n \left(\frac{\partial G_k}{\partial r_i}(\mathbf{r})V_k(r_k) \right) \quad (\text{A.55})$$

e.

$$\frac{\partial^2 \Pi_i}{\partial r_i^2}(\mathbf{r}) = 2 \frac{\partial G_i}{\partial r_i}(\mathbf{r})U_i + \frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r})V_i(r_i) \quad (\text{A.56})$$

f.

$$\frac{\partial^2 \Pi_j}{\partial r_i^2}(\mathbf{r}) = \frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r})V_j(r_j) \quad (\text{A.57})$$

g.

$$\frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) = 2 \frac{\partial G_i}{\partial r_i}(\mathbf{r})U_i + \sum_{k=1}^n \left(\frac{\partial^2 G_k}{\partial r_i^2}(\mathbf{r})V_k(r_k) \right) \quad (\text{A.58})$$

h.

$$\frac{\partial^2 \Pi_i}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial G_i}{\partial r_j}(\mathbf{r})U_i + \frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r})V_i(r_i) \quad (\text{A.59})$$

i.

$$\frac{\partial^2 \Pi_l}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial^2 G_l}{\partial r_i \partial r_j}(\mathbf{r})V_l(r_l) \quad (\text{A.60})$$

j.

$$\frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial G_i}{\partial r_j}(\mathbf{r})U_i + \frac{\partial G_j}{\partial r_i}(\mathbf{r})U_j + \sum_{k=1}^n \left(\frac{\partial^2 G_k}{\partial r_i \partial r_j}(\mathbf{r})V_k(r_k) \right) \quad (\text{A.61})$$

Proof. a. Deduce from the notation introduced in the paper.

b. Note that $\frac{\partial V_i(r_i)}{\partial r_i} = U_i$, then take the derivative w.r.t r_i with $\Pi_i(\mathbf{r}) = G_i(\mathbf{r})V_i(r_i)$.

- c. Similar to the proof of part b, note that $\frac{\partial V_j(r_j)}{\partial r_i} = 0$.
- d. Take the sum over all derivatives calculated in part a and part b.
- e., ..., j. Take the derivative with respect to r_i, r_j both sides of part b, and part c and take the sum over all indexes. \square

Lemma 7. For all $i, j, k, l \in \{1, 2, \dots, n\}, i \neq j, i \neq l, j \neq l$, we have

a.
$$\frac{\partial \Pi_i}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r})[U_i + \zeta_i(r_i)(1 - G_i(\mathbf{r}))V_i(r_i)] \quad (\text{A.62})$$

b.
$$\frac{\partial \Pi_j}{\partial r_i}(\mathbf{r}) = -\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r})V_j(r_j) \quad (\text{A.63})$$

c.
$$\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\} \quad (\text{A.64})$$

d.
$$\frac{\partial^2 \Pi_i}{\partial r_i^2}(\mathbf{r}) = G_i(\mathbf{r})(1 - G_i(\mathbf{r})) \left\{ 2\zeta_i(r_i)U_i + [\zeta_i'(r_i) + \zeta_i^2(r_i)(1 - 2G_i(\mathbf{r}))]V_i(r_i) \right\} \quad (\text{A.65})$$

e.
$$\frac{\partial^2 \Pi_j}{\partial r_i^2}(\mathbf{r}) = -G_i(\mathbf{r})G_j(\mathbf{r})[\zeta_i'(r_i) + \zeta_i^2(r_i)(1 - 2G_i(\mathbf{r}))]V_j(r_j) \quad (\text{A.66})$$

f.
$$\begin{aligned} \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) = G_i(\mathbf{r}) \left\{ 2\zeta_i(r_i)(1 - G_i(\mathbf{r}))U_i \right. \\ \left. + [\zeta_i'(r_i) + \zeta_i^2(r_i)(1 - 2G_i(\mathbf{r}))] \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\}, \end{aligned} \quad (\text{A.67})$$

g.
$$\frac{\partial^2 \Pi_i}{\partial r_i \partial r_j}(\mathbf{r}) = -\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})[U_i + \zeta_i(r_i)(1 - 2G_i(\mathbf{r}))V_i(r_i)] \quad (\text{A.68})$$

h.
$$\frac{\partial^2 \Pi_l}{\partial r_i \partial r_j}(\mathbf{r}) = 2\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})G_l(\mathbf{r})V_l(r_l) \quad (\text{A.69})$$

i.

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}) = & -G_i(\mathbf{r})G_j(\mathbf{r}) \left\{ \zeta_j(r_j) \left[U_i + \zeta_i(r_i) \left(V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right) \right] \right. \\ & \left. + \zeta_i(r_i) \left[U_j + \zeta_j(r_j) \left(V_j(r_j) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right) \right] \right\}. \end{aligned} \quad (\text{A.70})$$

Proof. a.

$$\frac{\partial \Pi_i}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r})U_i + \frac{\partial G_i}{\partial r_i}(\mathbf{r})V_i(r_i) \quad (\text{A.71})$$

$$= G_i(\mathbf{r})U_i + \zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))V_i(r_i) \quad (\text{A.72})$$

$$= G_i(\mathbf{r})[U_i + \zeta_i(r_i)(1 - G_i(\mathbf{r}))V_i(r_i)]. \quad (\text{A.73})$$

b.

$$\frac{\partial \Pi_j}{\partial r_i}(\mathbf{r}) = \frac{\partial G_j}{\partial r_i}(\mathbf{r})V_i(r_i) \quad (\text{A.74})$$

$$= -\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r})V_i(r_i). \quad (\text{A.75})$$

c.

$$\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r})U_i + \sum_{k=1}^n \left(\frac{\partial G_k}{\partial r_i}(\mathbf{r})V_k(r_k) \right) \quad (\text{A.76})$$

$$= G_i(\mathbf{r})U_i + \frac{\partial G_i}{\partial r_i}(\mathbf{r})V_i(r_i) + \sum_{k \neq i} \left(\frac{\partial G_k}{\partial r_i}(\mathbf{r})V_k(r_k) \right) \quad (\text{A.77})$$

$$= G_i(\mathbf{r})U_i + \zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))V_i(r_i) + \sum_{k \neq i} \left(-\zeta_i(r_i)G_i(\mathbf{r})G_k(\mathbf{r})V_k(r_k) \right) \quad (\text{A.78})$$

$$= G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i)(1 - G_i(\mathbf{r}))V_i(r_i) - \zeta_i(r_i) \sum_{k \neq i} (G_k(\mathbf{r})V_k(r_k)) \right\} \quad (\text{A.79})$$

$$= G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[(1 - G_i(\mathbf{r}))V_i(r_i) - \sum_{k \neq i} (G_k(\mathbf{r})V_k(r_k)) \right] \right\} \quad (\text{A.80})$$

$$= G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\}. \quad (\text{A.81})$$

d.

$$\frac{\partial^2 \Pi_i}{\partial r_i^2}(\mathbf{r}) \quad (\text{A.82})$$

$$= 2 \frac{\partial G_i}{\partial r_i}(\mathbf{r}) U_i + \frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) V_i(r_i) \quad (\text{A.83})$$

$$= 2 \zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) U_i + G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] V_i(r_i) \quad (\text{A.84})$$

$$= G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) \left\{ 2 \zeta_i(r_i) U_i + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] V_i(r_i) \right\}. \quad (\text{A.85})$$

e.

$$\frac{\partial^2 \Pi_j}{\partial r_i^2}(\mathbf{r}) = \frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r}) V_j(r_j) \quad (\text{A.86})$$

$$= -G_i(\mathbf{r}) G_j(\mathbf{r}) [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] V_j(r_j). \quad (\text{A.87})$$

f.

$$\frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) \quad (\text{A.88})$$

$$= 2 \frac{\partial G_i}{\partial r_i}(\mathbf{r}) U_i + \sum_{k=1}^n \left(\frac{\partial^2 G_k}{\partial r_i^2}(\mathbf{r}) V_k(r_k) \right) \quad (\text{A.89})$$

$$= 2 \frac{\partial G_i}{\partial r_i}(\mathbf{r}) U_i + \frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) V_i(r_i) + \sum_{k \neq i} \left(\frac{\partial^2 G_k}{\partial r_i^2}(\mathbf{r}) V_k(r_k) \right) \quad (\text{A.90})$$

$$= 2 \zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) U_i + G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] V_i(r_i) + \sum_{k \neq i} (-G_i(\mathbf{r}) G_k(\mathbf{r}) [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] V_k(r_k)) \quad (\text{A.91})$$

$$= G_i(\mathbf{r}) \left\{ 2 \zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] (1 - G_i(\mathbf{r})) V_i(r_i) + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \sum_{k \neq i} (-G_k(\mathbf{r}) V_k(r_k)) \right\} \quad (\text{A.92})$$

$$= G_i(\mathbf{r}) \left\{ 2 \zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \left[V_i(r_i) - G_i(\mathbf{r}) V_i(r_i) - \sum_{k \neq i} (G_k(\mathbf{r}) V_k(r_k)) \right] \right\} \quad (\text{A.93})$$

$$\begin{aligned}
&= G_i(\mathbf{r}) \left\{ 2\zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i \right. \\
&\quad \left. + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r}) V_k(r_k)) \right] \right\}. \tag{A.94}
\end{aligned}$$

g.

$$\frac{\partial^2 \Pi_i}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial G_i}{\partial r_j}(\mathbf{r}) U_i + \frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r}) V_i(r_i) \tag{A.95}$$

$$= -\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) U_i - \zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) (1 - 2G_i(\mathbf{r})) V_i(r_i) \tag{A.96}$$

$$= -\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_i + \zeta_i(r_i) (1 - 2G_i(\mathbf{r})) V_i(r_i)]. \tag{A.97}$$

h.

$$\frac{\partial^2 \Pi_l}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial^2 G_l}{\partial r_i \partial r_j}(\mathbf{r}) V_l(r_l) \tag{A.98}$$

$$= 2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_l(\mathbf{r}) V_l(r_l). \tag{A.99}$$

i.

$$\frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}) \tag{A.100}$$

$$= \frac{\partial G_i}{\partial r_j}(\mathbf{r}) U_i + \frac{\partial G_j}{\partial r_i}(\mathbf{r}) U_j + \sum_{k=1}^n \left(\frac{\partial^2 G_k}{\partial r_i \partial r_j}(\mathbf{r}) V_k(r_k) \right) \tag{A.101}$$

$$= \frac{\partial^2 \Pi_i}{\partial r_i \partial r_j}(\mathbf{r}) + \frac{\partial^2 \Pi_j}{\partial r_i \partial r_j}(\mathbf{r}) + \sum_{k \neq i, k \neq j} \left(\frac{\partial^2 \Pi_k}{\partial r_i \partial r_j}(\mathbf{r}) \right) \tag{A.102}$$

$$\begin{aligned}
&= -\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_i + \zeta_i(r_i) (1 - 2G_i(\mathbf{r})) V_i(r_i)] \\
&\quad - \zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_j + \zeta_j(r_j) (1 - 2G_j(\mathbf{r})) V_j(r_j)] \\
&\quad + \sum_{k \neq i, k \neq j} (2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_k(\mathbf{r}) V_k(r_k)) \tag{A.103}
\end{aligned}$$

$$\begin{aligned}
&= -\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_i + \zeta_i(r_i) V_i(r_i) - 2\zeta_i(r_i) G_i(\mathbf{r}) V_i(r_i)] \\
&\quad - \zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_j + \zeta_j(r_j) V_j(r_j) - 2\zeta_j(r_j) G_j(\mathbf{r}) V_j(r_j)] \\
&\quad + 2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) \sum_{k \neq i, k \neq j} (G_k(\mathbf{r}) V_k(r_k)) \tag{A.104}
\end{aligned}$$

$$\begin{aligned}
&= -\zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_i + \zeta_i(r_i) V_i(r_i)] + 2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_i(\mathbf{r}) V_i(r_i) \\
&\quad - \zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) [U_j + \zeta_j(r_j) V_j(r_j)] + 2\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) G_j(\mathbf{r}) V_j(r_j)
\end{aligned}$$

$$+ 2\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r}) \sum_{k \neq i, k \neq j} (G_k(\mathbf{r})V_k(r_k)) \quad (\text{A.105})$$

$$= -\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})[U_i + \zeta_i(r_i)V_i(r_i)] - \zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r})[U_j + \zeta_j(r_j)V_j(r_j)] \\ + 2\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r}) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \quad (\text{A.106})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) \left\{ [\zeta_j(r_j)U_i + \zeta_i(r_i)\zeta_j(r_j)V_i(r_i)] + [\zeta_i(r_i)U_j + \zeta_i(r_i)\zeta_j(r_j)V_j(r_j)] \right. \\ \left. - 2\zeta_i(r_i)\zeta_j(r_j) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right\} \quad (\text{A.107})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) \left\{ \left[\zeta_j(r_j)U_i + \zeta_i(r_i)\zeta_j(r_j)V_i(r_i) - \zeta_i(r_i)\zeta_j(r_j) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right. \\ \left. + \left[\zeta_i(r_i)U_j + \zeta_i(r_i)\zeta_j(r_j)V_j(r_j) - \zeta_i(r_i)\zeta_j(r_j) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\} \quad (\text{A.108})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) \left\{ \zeta_j(r_j) \left[U_i + \zeta_i(r_i)V_i(r_i) - \zeta_i(r_i) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right. \\ \left. + \zeta_i(r_i) \left[U_j + \zeta_j(r_j)V_j(r_j) - \zeta_j(r_j) \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\} \quad (\text{A.109})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) \left\{ \zeta_j(r_j) \left[U_i + \zeta_i(r_i) \left(V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right) \right] \right. \\ \left. + \zeta_i(r_i) \left[U_j + \zeta_j(r_j) \left(V_j(r_j) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right) \right] \right\}. \quad (\text{A.110})$$

□

Let us return to Proposition 2. From the formulations given in Lemma 5, Lemma 6, Lemma 7, we have

$$\frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = G_i(\mathbf{r}) \left\{ U_i + \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \right\} \quad (\text{A.111})$$

$$\Leftrightarrow \frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) = U_i + \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \quad (\text{A.112})$$

$$\Leftrightarrow \frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) - U_i = \zeta_i(r_i) \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)) \right] \quad (\text{A.113})$$

$$\Leftrightarrow \frac{1}{\zeta_i(r_i)} \left(\frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) - U_i \right) = V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r})V_k(r_k)). \quad (\text{A.114})$$

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) &= G_i(\mathbf{r}) \left\{ 2\zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i \right. \\ &\quad \left. + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \left[V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r}) V_k(r_k)) \right] \right\} \end{aligned} \quad (\text{A.115})$$

$$\begin{aligned} &= G_i(\mathbf{r}) \left\{ 2\zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i \right. \\ &\quad \left. + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \left[\frac{1}{\zeta_i(r_i)} \left(\frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) - U_i \right) \right] \right\} \end{aligned} \quad (\text{A.116})$$

$$\begin{aligned} &= G_i(\mathbf{r}) \left\{ 2\zeta_i(r_i) (1 - G_i(\mathbf{r})) U_i - [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \frac{1}{\zeta_i(r_i)} U_i \right. \\ &\quad \left. + [\zeta_i'(r_i) + \zeta_i^2(r_i) (1 - 2G_i(\mathbf{r}))] \frac{1}{\zeta_i(r_i)} \frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) \right\} \end{aligned} \quad (\text{A.117})$$

$$\begin{aligned} &= G_i(\mathbf{r}) \left\{ U_i \left[\left(2\zeta_i(r_i) - 2\zeta_i(r_i) G_i(\mathbf{r}) \right) - \left(\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i) - 2\zeta_i(r_i) G_i(\mathbf{r}) \right) \right] \right\} \\ &\quad + \left[\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i) (1 - 2G_i(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) \end{aligned} \quad (\text{A.118})$$

$$= G_i(\mathbf{r}) U_i \left[\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} \right] + \left[\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i) (1 - 2G_i(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_i}(\mathbf{r}). \quad (\text{A.119})$$

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial r_i \partial r_j}(\mathbf{r}) &= -G_i(\mathbf{r}) G_j(\mathbf{r}) \left\{ \zeta_j(r_j) \left[U_i + \zeta_i(r_i) \left(V_i(r_i) - \sum_{k=1}^n (G_k(\mathbf{r}) V_k(r_k)) \right) \right] \right. \\ &\quad \left. + \zeta_i(r_i) \left[U_j + \zeta_j(r_j) \left(V_j(r_j) - \sum_{k=1}^n (G_k(\mathbf{r}) V_k(r_k)) \right) \right] \right\} \end{aligned} \quad (\text{A.120})$$

$$= -G_i(\mathbf{r}) G_j(\mathbf{r}) \left\{ \zeta_j(r_j) \frac{1}{G_i(\mathbf{r})} \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) + \zeta_i(r_i) \frac{1}{G_j(\mathbf{r})} \frac{\partial \Pi}{\partial r_j}(\mathbf{r}) \right\} \quad (\text{A.121})$$

$$= - \left\{ \zeta_j(r_j) G_j(\mathbf{r}) \frac{\partial \Pi}{\partial r_i}(\mathbf{r}) + \zeta_i(r_i) G_i(\mathbf{r}) \frac{\partial \Pi}{\partial r_j}(\mathbf{r}) \right\}. \quad (\text{A.122})$$

Thus, Proposition 2 is proved.

A.3 Proofs of Remarks

In this section, we provide complementary proof of the remarks that are mentioned in the paper.

A.3.1 Proof of Remark 1

The actual end-of-period profit for the retailer is

$$\Pi_i(D_i, Q_i, r_i) = r_i \min(D_i, Q_i) + s_i(Q_i - D_i)^+ - c_i Q_i. \quad (\text{A.123})$$

We claim that it can also be represented as follows

$$\Pi_i(D_i, Q_i, r_i) = (r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)(D_i - Q_i)^+. \quad (\text{A.124})$$

Indeed, when $D_i > Q_i$, then

$$r_i \min(D_i, Q_i) + s_i(Q_i - D_i)^+ - c_i Q_i = r_i Q_i - c_i Q_i = (r_i - c_i)Q_i, \quad (\text{A.125})$$

and

$$\begin{aligned} (r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)(D_i - Q_i)^+ \\ = (r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)(D_i - Q_i) \end{aligned} \quad (\text{A.126})$$

$$= (r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)D_i + (r_i - s_i)Q_i \quad (\text{A.127})$$

$$= [(r_i - s_i) - (c_i - s_i)]Q_i = (r_i - c_i)Q_i. \quad (\text{A.128})$$

Similarly, when $D_i \leq Q_i$, we have

$$r_i \min(D_i, Q_i) + s_i(Q_i - D_i)^+ - c_i Q_i = r_i D_i + s_i(Q_i - D_i) - c_i Q_i \quad (\text{A.129})$$

$$= (r_i - s_i)D_i - (c_i - s_i)Q_i, \quad (\text{A.130})$$

and

$$(r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)(D_i - Q_i)^+ = (r_i - s_i)D_i - (c_i - s_i)Q_i. \quad (\text{A.131})$$

Thus, the expected profit for channel i is

$$\mathbb{E}[\Pi_i(D_i, Q_i, r_i)] = \mathbb{E}[(r_i - s_i)D_i - (c_i - s_i)Q_i - (r_i - s_i)(D_i - Q_i)^+]. \quad (\text{A.132})$$

Since $D_i = \xi G_i(\mathbf{r})$, we deduce that

$$\mathbb{E}[\Pi_i(D_i, Q_i, r_i)] = \mathbb{E}[(r_i - s_i)\xi G_i(\mathbf{r}) - (c_i - s_i)Q_i - (r_i - s_i)(\xi G_i(\mathbf{r}) - Q_i)^+]. \quad (\text{A.133})$$

Moreover, $G_i(\mathbf{r}) > 0$, it follows that

$$\Pi_i(Q_i, \mathbf{r}) = \mathbb{E}[\Pi_i(D_i, Q_i, r_i)] \quad (\text{A.134})$$

$$\begin{aligned} &= (r_i - s_i)G_i(\mathbf{r})\mathbb{E}[\xi] - (c_i - s_i)Q_i \\ &\quad - (r_i - s_i)G_i(\mathbf{r}) \int_{\frac{Q_i}{G_i(\mathbf{r})}}^{\infty} \left(x - \frac{Q_i}{G_i(\mathbf{r})}\right) f_{\xi}(x) dx. \end{aligned} \quad (\text{A.135})$$

Thus, Remark 1 holds true.

A.3.2 Proof of Remark 2

For each index i , $Q_i = G_i(\mathbf{r})F_{\xi}^{-1}(SL_i)$. Thus

$$\begin{aligned} &\int_{\frac{Q_i}{G_i(\mathbf{r})}}^{\infty} \left(x - \frac{Q_i}{G_i(\mathbf{r})}\right) f_{\xi}(x) dx \\ &= \int_{F_{\xi}^{-1}(SL_i)}^{\infty} \left(x - F_{\xi}^{-1}(SL_i)\right) f_{\xi}(x) dx \end{aligned} \quad (\text{A.136})$$

$$= \int_{F_{\xi}^{-1}(SL_i)}^{\infty} x f_{\xi}(x) dx - F_{\xi}^{-1}(SL_i) \int_{F_{\xi}^{-1}(SL_i)}^{\infty} f_{\xi}(x) dx \quad (\text{A.137})$$

$$= E[\xi] - \int_{-\infty}^{F_{\xi}^{-1}(SL_i)} x f_{\xi}(x) dx - F_{\xi}^{-1}(SL_i) \left(1 - \int_{-\infty}^{F_{\xi}^{-1}(SL_i)} f_{\xi}(x) dx\right) \quad (\text{A.138})$$

$$= E[\xi] - \int_{-\infty}^{F_{\xi}^{-1}(SL_i)} x f_{\xi}(x) dx - F_{\xi}^{-1}(SL_i) \left(1 - F_{\xi} \left(F_{\xi}^{-1}(SL_i)\right)\right) \quad (\text{A.139})$$

$$= E[\xi] - \int_{-\infty}^{F_{\xi}^{-1}(SL_i)} x f_{\xi}(x) dx - F_{\xi}^{-1}(SL_i)(1 - SL_i). \quad (\text{A.140})$$

Therefore

$$\Pi_i(\mathbf{r}) = \Pi_i(G_i(\mathbf{r})F_{\xi}^{-1}(SL_i), \mathbf{r}) \quad (\text{A.141})$$

$$\begin{aligned} &= (r_i - s_i)G_i(\mathbf{r})E[\xi] - (c_i - s_i)G_i(\mathbf{r})F_{\xi}^{-1}(SL_i) \\ &\quad - (r_i - s_i)G_i(\mathbf{r}) \int_{F_{\xi}^{-1}(SL_i)}^{\infty} \left(x - F_{\xi}^{-1}(SL_i)\right) f_{\xi}(x) dx \end{aligned} \quad (\text{A.142})$$

$$\begin{aligned} &= (r_i - s_i)G_i(\mathbf{r})E[\xi] - (c_i - s_i)G_i(\mathbf{r})F_{\xi}^{-1}(SL_i) \\ &\quad - (r_i - s_i)G_i(\mathbf{r}) \left(E[\xi] - \int_{-\infty}^{F_{\xi}^{-1}(SL_i)} x f_{\xi}(x) dx - F_{\xi}^{-1}(SL_i)(1 - SL_i)\right) \end{aligned} \quad (\text{A.143})$$

$$= -(c_i - s_i)G_i(\mathbf{r})F_{\xi}^{-1}(SL_i)$$

$$\begin{aligned}
& + (r_i - s_i)G_i(\mathbf{r}) \int_{-\infty}^{F_\xi^{-1}(SL_i)} x f_\xi(x) dx \\
& + (r_i - s_i)G_i(\mathbf{r})F_\xi^{-1}(SL_i) - (r_i - s_i)G_i(\mathbf{r})F_\xi^{-1}(SL_i)SL_i
\end{aligned} \tag{A.144}$$

$$\begin{aligned}
& = (r_i - c_i)G_i(\mathbf{r})F_\xi^{-1}(SL_i) - (r_i - s_i)G_i(\mathbf{r})F_\xi^{-1}(SL_i)SL_i \\
& + (r_i - s_i)G_i(\mathbf{r}) \int_{-\infty}^{F_\xi^{-1}(SL_i)} x f_\xi(x) dx
\end{aligned} \tag{A.145}$$

$$\begin{aligned}
& = G_i(\mathbf{r}) \left[(r_i - c_i)F_\xi^{-1}(SL_i) - (r_i - s_i)SL_i F_\xi^{-1}(SL_i) \right. \\
& \left. + (r_i - s_i) \int_{-\infty}^{F_\xi^{-1}(SL_i)} x f_\xi(x) dx \right].
\end{aligned} \tag{A.146}$$

Thus, Remark 2 is proved.

A.3.3 Proof of Remark 3

iv. We have

$$V_i(c_i) = \alpha_i [(c_i - c_i) - (c_i - s_i)SL_i] + (c_i - s_i)A_i \tag{A.147}$$

$$= (c_i - s_i)(-\alpha_i SL_i + A_i) \tag{A.148}$$

$$= (c_i - s_i) \left(-\alpha_i F_\xi(F_\xi^{-1}(SL_i)) + \int_{-\infty}^{\alpha_i} x f_\xi(x) dx \right) \tag{A.149}$$

$$= (c_i - s_i) \left(-\alpha_i F_\xi(\alpha_i) + \int_{-\infty}^{\alpha_i} x f_\xi(x) dx \right). \tag{A.150}$$

Denote $H(t) := -tF_\xi(t) + \int_{-\infty}^t x f_\xi(x) dx$, we have

$$\frac{\partial H}{\partial t}(t) = -F_\xi(t) - t f_\xi(t) + t f_\xi(t) = -F_\xi(t) \leq 0. \tag{A.151}$$

Thus, when $t > 0$, $H(t) \leq H(0) = \int_{-\infty}^0 x f_\xi(x) dx$. If ξ is a non-negative random variable, then $H(0) = 0$. We deduce that $H(t) \leq 0$ for all $t \geq 0$. As $\alpha_i \geq 0$, we have $H(\alpha_i) \leq 0$. Then $V_i(c_i) = (c_i - s_i)H(\alpha_i) \leq 0$.

vi. We have

$$U_i = \alpha_i(1 - SL_i) + A_i \tag{A.152}$$

$$= \alpha_i - \alpha_i F_\xi(F_\xi^{-1}(SL_i)) + \int_{-\infty}^{\alpha_i} x f_\xi(x) dx \tag{A.153}$$

$$= \alpha_i - \alpha_i F_\xi(\alpha_i) + \int_{-\infty}^{\alpha_i} x f_\xi(x) dx := K(\alpha_i). \quad (\text{A.154})$$

Let $K(t) := t - tF_\xi(t) + \int_{-\infty}^t x f_\xi(x) dx$, we have

$$\frac{\partial K}{\partial t}(t) = 1 - F_\xi(t) - t f_\xi(t) + t f_\xi(t) = 1 - F_\xi(t) \geq 0. \quad (\text{A.155})$$

We also have

$$F_\xi(\alpha_i) = SL_i \quad (\text{A.156})$$

$$\Rightarrow \frac{\partial F_\xi(\alpha_i)}{\partial SL_i} = 1 \quad (\text{A.157})$$

$$\Rightarrow \frac{\partial F_\xi(\alpha_i)}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial SL_i} = 1 \quad (\text{A.158})$$

$$\Rightarrow f_\xi(\alpha_i) \frac{\partial \alpha_i}{\partial SL_i} = 1 \quad (\text{A.159})$$

$$\Rightarrow \frac{\partial F_\xi^{-1}(SL_i)}{\partial SL_i} = \frac{\partial \alpha_i}{\partial SL_i} = \frac{1}{f_\xi(\alpha_i)} = \frac{1}{f_\xi(F_\xi^{-1}(SL_i))} > 0. \quad (\text{A.160})$$

Therefore,

$$U_i = \alpha_i(1 - SL_i) + A_i \quad (\text{A.161})$$

$$= F_\xi^{-1}(SL_i)(1 - SL_i) + \int_{-\infty}^{F_\xi^{-1}(SL_i)} x f_\xi(x) dx \quad (\text{A.162})$$

$$\Rightarrow \frac{\partial U_i}{\partial SL_i} = \frac{1}{f_\xi(F_\xi^{-1}(SL_i))} (1 - SL_i) - F_\xi^{-1}(SL_i) \quad (\text{A.163})$$

$$+ \frac{1}{f_\xi(F_\xi^{-1}(SL_i))} F_\xi^{-1}(SL_i) f(F_\xi^{-1}(SL_i)) \quad (\text{A.164})$$

$$= \frac{1}{f_\xi(F_\xi^{-1}(SL_i))} (1 - SL_i) > 0. \quad (\text{A.165})$$

A.4 Extended tables

Table A.1 shows the full table of numerical results of pricing problem varying CSL_1 and CSL_2 with step size 0.1.

SL_1	SL_2	r_1^*	r_2^*	Q_1^*	Q_2^*	G_1^*	G_2^*	G_0^*	Π_1	Π_2	Π
0.1	0.1	15.423	18.471	60	156	0.195	0.506	0.298	196.9	510.4	707.3

0.1	0.2	15.678	18.43	48	202	0.155	0.540	0.305	168.1	615.8	783.9
0.1	0.3	15.793	18.447	44	230	0.142	0.545	0.313	158.7	659.8	818.5
0.1	0.4	15.831	18.489	43	248	0.140	0.537	0.322	158.7	671.1	829.8
0.1	0.5	15.81	18.551	45	260	0.148	0.520	0.332	166.1	657.5	823.5
0.1	0.6	15.737	18.636	50	265	0.164	0.493	0.343	180.8	620.9	801.7
0.1	0.7	15.612	18.757	59	262	0.192	0.453	0.355	204.9	559.0	763.9
0.1	0.8	15.425	18.944	74	244	0.241	0.390	0.369	243.2	464.6	707.7
0.1	0.9	15.153	19.292	101	198	0.329	0.287	0.384	305.3	320.8	626.0
0.2	0.1	15.219	18.586	91	140	0.242	0.456	0.302	266.2	475.7	741.9
0.2	0.2	15.421	18.515	75	185	0.200	0.495	0.305	234.3	579.8	814.2
0.2	0.3	15.515	18.521	70	212	0.186	0.503	0.311	224.1	623.4	847.5
0.2	0.4	15.546	18.558	69	229	0.185	0.496	0.319	224.6	634.2	858.8
0.2	0.5	15.532	18.619	72	240	0.193	0.480	0.328	233.1	620.6	853.7
0.2	0.6	15.477	18.707	78	244	0.210	0.453	0.338	249.7	584.5	834.2
0.2	0.7	15.382	18.834	89	239	0.238	0.412	0.349	275.8	524.3	800.1
0.2	0.8	15.24	19.03	107	220	0.285	0.352	0.363	315.7	433.8	749.5
0.2	0.9	15.036	19.395	137	177	0.366	0.255	0.379	377.9	298.7	676.6
0.3	0.1	15.157	18.651	110	133	0.261	0.433	0.306	301.5	460.2	761.7
0.3	0.2	15.336	18.565	92	177	0.219	0.474	0.307	268.5	563.6	832.1
0.3	0.3	15.42	18.565	86	203	0.205	0.482	0.312	258.1	606.7	864.8
0.3	0.4	15.449	18.6	86	220	0.204	0.477	0.319	258.8	617.5	876.2
0.3	0.5	15.438	18.66	89	230	0.212	0.461	0.328	267.6	604.2	871.8
0.3	0.6	15.391	18.749	96	233	0.228	0.434	0.338	284.6	568.7	853.3
0.3	0.7	15.308	18.878	108	228	0.257	0.394	0.349	311.4	509.6	821.0
0.3	0.8	15.186	19.079	127	210	0.301	0.335	0.363	351.4	421.5	772.9
0.3	0.9	15.009	19.449	159	168	0.377	0.243	0.380	413.2	290.4	703.6
0.4	0.1	15.148	18.687	123	130	0.267	0.423	0.310	318.2	454.3	772.5
0.4	0.2	15.315	18.593	104	174	0.226	0.465	0.309	284.9	557.3	842.2
0.4	0.3	15.392	18.591	98	199	0.212	0.473	0.314	274.8	599.9	874.7
0.4	0.4	15.42	18.624	97	216	0.211	0.468	0.321	275.5	610.7	886.3
0.4	0.5	15.41	18.684	101	226	0.219	0.452	0.329	284.5	597.6	882.1
0.4	0.6	15.367	18.773	109	229	0.235	0.426	0.339	301.6	562.5	864.2
0.4	0.7	15.292	18.903	121	224	0.262	0.387	0.351	328.2	504.4	832.6
0.4	0.8	15.18	19.105	141	206	0.305	0.329	0.366	367.9	417.6	785.5
0.4	0.9	15.018	19.478	174	165	0.377	0.238	0.384	429.2	288.4	717.6
0.5	0.1	15.172	18.7	132	130	0.264	0.423	0.313	320.9	455.5	776.4

0.5	0.2	15.33	18.605	112	173	0.224	0.463	0.312	288.7	557.6	846.3
0.5	0.3	15.405	18.601	106	199	0.211	0.472	0.317	278.7	600.2	878.9
0.5	0.4	15.431	18.634	105	215	0.210	0.466	0.323	279.7	610.9	890.5
0.5	0.5	15.422	18.694	109	225	0.218	0.451	0.332	288.7	597.8	886.5
0.5	0.6	15.381	18.783	117	228	0.234	0.425	0.342	305.6	563.1	868.7
0.5	0.7	15.31	18.913	130	224	0.260	0.386	0.354	331.7	505.5	837.2
0.5	0.8	15.203	19.115	151	206	0.301	0.329	0.369	370.9	419.3	790.2
0.5	0.9	15.049	19.487	185	166	0.371	0.239	0.390	431.3	290.9	722.1
0.6	0.1	15.223	18.693	136	132	0.253	0.430	0.317	311.9	462.4	774.3
0.6	0.2	15.378	18.602	116	175	0.216	0.469	0.315	281.2	563.9	845.1
0.6	0.3	15.45	18.599	110	201	0.204	0.477	0.319	271.9	606.1	878.1
0.6	0.4	15.476	18.632	109	218	0.203	0.471	0.326	273.0	616.8	889.8
0.6	0.5	15.467	18.692	113	228	0.210	0.455	0.335	281.8	603.9	885.7
0.6	0.6	15.427	18.781	121	231	0.225	0.430	0.345	298.3	569.3	867.7
0.6	0.7	15.358	18.91	135	227	0.250	0.392	0.358	323.6	512.1	835.7
0.6	0.8	15.253	19.11	156	210	0.291	0.335	0.374	361.8	426.2	788.0
0.6	0.9	15.101	19.48	193	170	0.358	0.245	0.396	421.3	297.2	718.4
0.7	0.1	15.308	18.667	136	137	0.235	0.446	0.319	290.6	475.6	766.3
0.7	0.2	15.461	18.584	116	180	0.200	0.482	0.318	262.6	576.1	838.7
0.7	0.3	15.532	18.584	110	206	0.189	0.488	0.322	254.3	617.9	872.2
0.7	0.4	15.557	18.618	109	223	0.189	0.482	0.329	255.6	628.4	884.0
0.7	0.5	15.548	18.678	113	233	0.195	0.466	0.338	264.0	615.6	879.7
0.7	0.6	15.509	18.766	121	237	0.210	0.441	0.349	279.6	581.5	861.1
0.7	0.7	15.439	18.894	135	233	0.234	0.403	0.363	304.0	524.2	828.2
0.7	0.8	15.334	19.091	158	217	0.272	0.347	0.380	340.5	438.3	778.8
0.7	0.9	15.18	19.455	196	178	0.338	0.257	0.405	398.3	308.1	706.3
0.8	0.1	15.442	18.617	129	145	0.207	0.472	0.322	255.2	496.2	751.4
0.8	0.2	15.598	18.549	110	188	0.176	0.503	0.321	230.9	595.4	826.3
0.8	0.3	15.669	18.555	104	214	0.167	0.508	0.325	224.1	636.6	860.7
0.8	0.4	15.693	18.591	104	231	0.166	0.501	0.333	225.6	647.0	872.6
0.8	0.5	15.683	18.651	108	243	0.173	0.485	0.342	233.5	634.3	867.8
0.8	0.6	15.643	18.738	117	247	0.186	0.460	0.354	247.8	600.5	848.2
0.8	0.7	15.571	18.863	130	245	0.208	0.423	0.369	270.2	543.5	813.6
0.8	0.8	15.463	19.055	153	230	0.244	0.367	0.388	304.1	457.3	761.4
0.8	0.9	15.303	19.41	193	191	0.307	0.276	0.416	358.9	325.2	684.0
0.9	0.1	15.687	18.535	113	158	0.163	0.514	0.323	198.2	528.4	726.6

0.9	0.2	15.847	18.491	96	201	0.139	0.538	0.323	180.0	625.5	805.4
0.9	0.3	15.92	18.505	91	227	0.131	0.540	0.329	175.0	666.1	841.0
0.9	0.4	15.944	18.544	91	246	0.131	0.532	0.337	176.5	676.5	853.0
0.9	0.5	15.932	18.605	95	258	0.137	0.516	0.347	183.3	664.1	847.4
0.9	0.6	15.889	18.69	103	264	0.148	0.491	0.360	195.4	630.7	826.1
0.9	0.7	15.813	18.81	116	264	0.167	0.456	0.377	214.2	574.4	788.7
0.9	0.8	15.698	18.994	138	252	0.199	0.402	0.399	243.4	488.3	731.7
0.9	0.9	15.523	19.332	178	215	0.257	0.311	0.433	292.1	353.6	645.7

Table A.1 Numerical results when varying SL_1 and SL_2 with step size 0.1

Table A.2 shows the full table of numerical results of pricing problem when $SL_1 = SL_2 = SL$. We vary SL_1 with step size 0.01.

SL_1	SL_2	r_1^*	r_2^*	Q_1^*	Q_2^*	G_1^*	G_2^*	G_0^*	Π_1	Π_2	Π
0.01	0.01	15.426	18.433	29	78	0.191	0.516	0.293	98.3	265.4	363.7
0.02	0.02	15.426	18.437	37	99	0.192	0.515	0.293	124.6	335.0	459.6
0.03	0.03	15.425	18.442	42	112	0.192	0.514	0.294	141.3	377.7	519.0
0.04	0.04	15.425	18.446	46	122	0.193	0.513	0.295	153.6	409.0	562.6
0.05	0.05	15.425	18.45	49	130	0.193	0.512	0.295	163.6	433.6	597.1
0.06	0.06	15.425	18.454	52	136	0.193	0.511	0.296	171.9	453.9	625.8
0.07	0.07	15.424	18.458	54	142	0.194	0.510	0.296	179.3	471.0	650.3
0.08	0.08	15.424	18.462	56	147	0.194	0.509	0.297	185.7	485.9	671.6
0.09	0.09	15.424	18.466	58	152	0.195	0.508	0.298	191.5	499.0	690.5
0.1	0.1	15.423	18.471	60	156	0.195	0.506	0.298	196.9	510.4	707.3
0.11	0.11	15.423	18.475	62	160	0.196	0.505	0.299	201.8	520.8	722.6
0.12	0.12	15.423	18.479	64	163	0.196	0.504	0.300	206.3	530.1	736.4
0.13	0.13	15.423	18.483	65	167	0.197	0.503	0.300	210.4	538.6	749.0
0.14	0.14	15.422	18.488	67	170	0.197	0.502	0.301	214.6	546.1	760.6
0.15	0.15	15.422	18.492	68	173	0.198	0.501	0.301	218.2	553.1	771.3
0.16	0.16	15.422	18.497	70	175	0.198	0.500	0.302	221.8	559.4	781.2
0.17	0.17	15.422	18.501	71	178	0.199	0.499	0.303	225.1	565.3	790.4
0.18	0.18	15.422	18.506	72	180	0.199	0.497	0.304	228.3	570.6	798.9
0.19	0.19	15.421	18.51	74	183	0.200	0.496	0.304	231.4	575.4	806.8
0.2	0.2	15.421	18.515	75	185	0.200	0.495	0.305	234.3	579.8	814.2
0.21	0.21	15.421	18.52	76	187	0.201	0.494	0.306	237.1	583.9	821.0
0.22	0.22	15.421	18.524	77	189	0.201	0.493	0.306	239.7	587.7	827.4

0.23	0.23	15.421	18.529	78	191	0.201	0.492	0.307	242.3	591.1	833.3
0.24	0.24	15.421	18.534	80	193	0.202	0.490	0.308	244.8	594.1	838.9
0.25	0.25	15.42	18.539	81	195	0.203	0.489	0.308	247.3	596.8	844.0
0.26	0.26	15.42	18.544	82	197	0.203	0.488	0.309	249.6	599.3	848.8
0.27	0.27	15.42	18.549	83	199	0.204	0.486	0.310	251.8	601.5	853.3
0.28	0.28	15.42	18.555	84	200	0.204	0.485	0.311	254.0	603.4	857.4
0.29	0.29	15.42	18.56	85	202	0.205	0.484	0.312	256.1	605.2	861.3
0.3	0.3	15.42	18.565	86	203	0.205	0.482	0.312	258.1	606.7	864.8
0.31	0.31	15.42	18.571	88	205	0.206	0.481	0.313	260.1	608.0	868.1
0.32	0.32	15.42	18.576	89	206	0.206	0.480	0.314	261.9	609.1	871.1
0.33	0.33	15.42	18.582	90	208	0.207	0.478	0.315	263.9	609.9	873.8
0.34	0.34	15.42	18.588	91	209	0.207	0.477	0.316	265.7	610.6	876.3
0.35	0.35	15.42	18.593	92	210	0.208	0.476	0.317	267.3	611.1	878.5
0.36	0.36	15.42	18.599	93	212	0.209	0.474	0.317	269.1	611.4	880.5
0.37	0.37	15.42	18.605	94	213	0.209	0.473	0.318	270.7	611.5	882.3
0.38	0.38	15.42	18.611	95	214	0.210	0.471	0.319	272.3	611.5	883.8
0.39	0.39	15.42	18.617	96	215	0.210	0.470	0.320	273.9	611.2	885.1
0.4	0.4	15.42	18.624	97	216	0.211	0.468	0.321	275.5	610.7	886.3
0.41	0.41	15.42	18.63	99	217	0.212	0.466	0.322	277.0	610.2	887.2
0.42	0.42	15.42	18.637	100	218	0.212	0.465	0.323	278.5	609.4	887.9
0.43	0.43	15.42	18.643	101	219	0.213	0.463	0.324	279.8	608.6	888.4
0.44	0.44	15.42	18.65	102	220	0.214	0.461	0.325	281.2	607.5	888.7
0.45	0.45	15.421	18.657	103	221	0.214	0.460	0.326	282.5	606.3	888.8
0.46	0.46	15.421	18.664	104	222	0.215	0.458	0.327	283.8	604.9	888.7
0.47	0.47	15.421	18.671	105	223	0.215	0.456	0.328	285.1	603.4	888.5
0.48	0.48	15.421	18.679	107	224	0.216	0.454	0.330	286.4	601.6	888.0
0.49	0.49	15.422	18.686	108	225	0.217	0.453	0.331	287.4	599.9	887.3
0.5	0.5	15.422	18.694	109	225	0.218	0.451	0.332	288.7	597.8	886.5
0.51	0.51	15.422	18.702	110	226	0.218	0.449	0.333	289.9	595.6	885.5
0.52	0.52	15.423	18.71	111	227	0.219	0.447	0.334	290.9	593.4	884.3
0.53	0.53	15.423	18.718	112	227	0.220	0.445	0.335	291.9	590.9	882.8
0.54	0.54	15.424	18.726	114	228	0.220	0.443	0.337	292.8	588.4	881.3
0.55	0.55	15.424	18.735	115	229	0.221	0.441	0.338	293.9	585.5	879.5
0.56	0.56	15.425	18.743	116	229	0.222	0.439	0.339	294.7	582.8	877.5
0.57	0.57	15.425	18.752	117	230	0.223	0.437	0.341	295.7	579.6	875.3
0.58	0.58	15.426	18.762	119	230	0.224	0.434	0.342	296.6	576.3	873.0

0.59	0.59	15.427	18.771	120	231	0.224	0.432	0.344	297.4	573.0	870.4
0.6	0.6	15.427	18.781	121	231	0.225	0.430	0.345	298.3	569.3	867.7
0.61	0.61	15.428	18.79	122	232	0.226	0.428	0.347	298.9	565.8	864.7
0.62	0.62	15.429	18.801	124	232	0.227	0.425	0.348	299.8	561.8	861.5
0.63	0.63	15.43	18.811	125	232	0.228	0.423	0.350	300.4	557.8	858.2
0.64	0.64	15.431	18.822	127	233	0.228	0.420	0.352	301.0	553.5	854.6
0.65	0.65	15.432	18.833	128	233	0.229	0.417	0.353	301.6	549.1	850.8
0.66	0.66	15.433	18.844	129	233	0.230	0.415	0.355	302.2	544.6	846.7
0.67	0.67	15.435	18.856	131	233	0.231	0.412	0.357	302.6	539.8	842.5
0.68	0.68	15.436	18.868	132	233	0.232	0.409	0.359	303.1	534.8	838.0
0.69	0.69	15.437	18.881	134	233	0.233	0.406	0.361	303.6	529.6	833.2
0.7	0.7	15.439	18.894	135	233	0.234	0.403	0.363	304.0	524.2	828.2
0.71	0.71	15.441	18.907	137	233	0.235	0.400	0.365	304.2	518.7	822.9
0.72	0.72	15.442	18.921	139	233	0.236	0.397	0.367	304.5	512.8	817.4
0.73	0.73	15.444	18.936	140	233	0.237	0.394	0.369	304.8	506.7	811.6
0.74	0.74	15.447	18.951	142	233	0.238	0.390	0.372	304.8	500.6	805.4
0.75	0.75	15.449	18.966	144	233	0.239	0.387	0.374	304.9	494.1	799.0
0.76	0.76	15.451	18.982	145	232	0.240	0.383	0.377	304.9	487.3	792.2
0.77	0.77	15.454	18.999	147	232	0.241	0.380	0.379	304.8	480.3	785.1
0.78	0.78	15.457	19.017	149	231	0.242	0.376	0.382	304.6	473.0	777.6
0.79	0.79	15.46	19.036	151	231	0.243	0.372	0.385	304.5	465.3	769.7
0.8	0.8	15.463	19.055	153	230	0.244	0.367	0.388	304.1	457.3	761.4
0.81	0.81	15.467	19.076	155	229	0.245	0.363	0.392	303.7	449.0	752.7
0.82	0.82	15.471	19.097	157	228	0.247	0.358	0.395	303.0	440.4	743.5
0.83	0.83	15.475	19.12	159	227	0.248	0.354	0.399	302.4	431.3	733.7
0.84	0.84	15.48	19.144	162	226	0.249	0.348	0.402	301.5	421.9	723.4
0.85	0.85	15.486	19.17	164	225	0.250	0.343	0.407	300.4	412.0	712.5
0.86	0.86	15.492	19.198	166	223	0.251	0.337	0.411	299.3	401.6	700.9
0.87	0.87	15.498	19.227	169	222	0.253	0.331	0.416	297.9	390.6	688.5
0.88	0.88	15.505	19.259	172	220	0.254	0.325	0.421	296.3	379.0	675.2
0.89	0.89	15.514	19.294	175	218	0.255	0.318	0.427	294.3	366.7	661.0
0.9	0.9	15.523	19.332	178	215	0.257	0.311	0.433	292.1	353.6	645.7
0.91	0.91	15.534	19.374	181	212	0.258	0.303	0.440	289.4	339.6	629.1
0.92	0.92	15.547	19.421	184	209	0.259	0.294	0.447	286.3	324.6	610.9
0.93	0.93	15.562	19.474	188	205	0.260	0.284	0.456	282.5	308.4	590.8
0.94	0.94	15.579	19.535	191	200	0.261	0.273	0.466	278.0	290.5	568.4

0.95	0.95	15.601	19.607	196	194	0.262	0.260	0.478	272.3	270.7	543.0
0.96	0.96	15.63	19.695	200	187	0.262	0.246	0.492	265.1	248.4	513.4
0.97	0.97	15.668	19.807	205	178	0.262	0.228	0.510	255.5	222.3	477.7
0.98	0.98	15.725	19.964	210	165	0.260	0.204	0.536	241.7	190.2	431.9
0.99	0.99	15.829	20.228	214	144	0.252	0.169	0.578	218.1	146.3	364.4

Table A.2 Numerical results when $SL_1 = SL_2 = SL$, we vary SL with step size 0.01

Table A.3 shows the full table of numerical results of pricing problem when $SL_1 = 0.9$ and SL_2 is varied with step size 0.01.

SL_1	SL_2	r_1^*	r_2^*	Q_1^*	Q_2^*	G_1^*	G_2^*	G_0^*	Π_1	Π_2	Π
0.9	0.01	15.228	19.338	220	43	0.317	0.284	0.399	315.1	184.9	500.0
0.9	0.02	15.339	18.936	178	74	0.256	0.384	0.360	268.7	286.1	554.8
0.9	0.03	15.417	18.784	157	93	0.227	0.428	0.345	246.8	346.3	593.1
0.9	0.04	15.477	18.702	145	108	0.209	0.454	0.337	233.3	389.4	622.7
0.9	0.05	15.526	18.649	136	119	0.196	0.472	0.332	223.7	423.2	646.9
0.9	0.06	15.567	18.612	129	129	0.187	0.485	0.329	216.4	450.9	667.3
0.9	0.07	15.603	18.584	124	138	0.179	0.495	0.326	210.4	474.5	684.9
0.9	0.08	15.634	18.564	119	145	0.172	0.503	0.325	205.7	494.7	700.4
0.9	0.09	15.662	18.548	116	152	0.167	0.509	0.324	201.7	512.5	714.2
0.9	0.1	15.687	18.535	113	158	0.163	0.514	0.323	198.2	528.4	726.6
0.9	0.11	15.71	18.525	110	164	0.159	0.519	0.323	195.2	542.7	737.9
0.9	0.12	15.731	18.516	107	169	0.155	0.523	0.322	192.5	555.6	748.1
0.9	0.13	15.75	18.51	105	174	0.152	0.526	0.322	190.2	567.2	757.4
0.9	0.14	15.767	18.504	103	179	0.149	0.529	0.322	188.2	577.9	766.0
0.9	0.15	15.783	18.5	102	183	0.147	0.531	0.322	186.4	587.6	774.0
0.9	0.16	15.798	18.497	100	187	0.145	0.533	0.322	184.8	596.5	781.3
0.9	0.17	15.812	18.494	99	191	0.143	0.535	0.322	183.3	604.7	788.0
0.9	0.18	15.825	18.492	98	194	0.141	0.536	0.323	182.0	612.3	794.3
0.9	0.19	15.836	18.491	97	198	0.140	0.537	0.323	180.9	619.1	800.1
0.9	0.2	15.847	18.491	96	201	0.139	0.538	0.323	180.0	625.5	805.4
0.9	0.21	15.857	18.49	95	204	0.137	0.539	0.324	179.0	631.4	810.4
0.9	0.22	15.867	18.491	94	207	0.136	0.539	0.324	178.2	636.8	815.0
0.9	0.23	15.875	18.491	94	210	0.135	0.540	0.325	177.5	641.7	819.3
0.9	0.24	15.883	18.492	93	213	0.134	0.540	0.325	176.9	646.3	823.2
0.9	0.25	15.891	18.494	93	216	0.134	0.540	0.326	176.4	650.4	826.9

0.9	0.26	15.898	18.495	92	218	0.133	0.541	0.326	175.9	654.3	830.2
0.9	0.27	15.904	18.497	92	221	0.132	0.541	0.327	175.6	657.7	833.3
0.9	0.28	15.91	18.499	91	223	0.132	0.541	0.328	175.3	660.9	836.1
0.9	0.29	15.915	18.502	91	225	0.131	0.540	0.328	175.1	663.6	838.7
0.9	0.3	15.92	18.505	91	227	0.131	0.540	0.329	175.0	666.1	841.0
0.9	0.31	15.924	18.508	91	230	0.131	0.539	0.330	174.9	668.3	843.2
0.9	0.32	15.928	18.511	90	232	0.131	0.539	0.330	174.8	670.2	845.0
0.9	0.33	15.931	18.514	90	234	0.131	0.538	0.331	174.8	671.9	846.7
0.9	0.34	15.934	18.518	90	236	0.130	0.538	0.332	175.0	673.2	848.2
0.9	0.35	15.937	18.522	90	237	0.130	0.537	0.333	175.1	674.4	849.5
0.9	0.36	15.939	18.526	90	239	0.130	0.536	0.334	175.3	675.3	850.5
0.9	0.37	15.941	18.53	90	241	0.131	0.535	0.334	175.5	676.0	851.4
0.9	0.38	15.942	18.535	91	242	0.131	0.534	0.335	175.8	676.3	852.1
0.9	0.39	15.943	18.54	91	244	0.131	0.533	0.336	176.2	676.4	852.6
0.9	0.4	15.944	18.544	91	246	0.131	0.532	0.337	176.5	676.5	853.0
0.9	0.41	15.944	18.55	91	247	0.132	0.530	0.338	177.1	676.1	853.2
0.9	0.42	15.944	18.555	91	249	0.132	0.529	0.339	177.5	675.6	853.2
0.9	0.43	15.944	18.56	92	250	0.132	0.528	0.340	178.0	675.0	853.0
0.9	0.44	15.943	18.566	92	251	0.133	0.526	0.341	178.7	674.0	852.7
0.9	0.45	15.942	18.572	92	252	0.133	0.525	0.342	179.3	672.9	852.2
0.9	0.46	15.941	18.578	93	254	0.134	0.523	0.343	180.0	671.6	851.6
0.9	0.47	15.939	18.585	93	255	0.135	0.521	0.344	180.8	669.9	850.7
0.9	0.48	15.937	18.591	94	256	0.135	0.520	0.345	181.6	668.2	849.8
0.9	0.49	15.935	18.598	94	257	0.136	0.518	0.346	182.4	666.3	848.7
0.9	0.5	15.932	18.605	95	258	0.137	0.516	0.347	183.3	664.1	847.4
0.9	0.51	15.93	18.612	95	259	0.138	0.514	0.349	184.2	661.8	846.0
0.9	0.52	15.926	18.62	96	260	0.139	0.512	0.350	185.3	659.1	844.4
0.9	0.53	15.923	18.628	97	260	0.140	0.509	0.351	186.3	656.3	842.6
0.9	0.54	15.919	18.636	97	261	0.141	0.507	0.352	187.5	653.3	840.7
0.9	0.55	15.915	18.644	98	262	0.142	0.505	0.354	188.6	650.1	838.7
0.9	0.56	15.91	18.652	99	263	0.143	0.502	0.355	189.8	646.7	836.5
0.9	0.57	15.906	18.661	100	263	0.144	0.500	0.356	191.0	643.1	834.1
0.9	0.58	15.9	18.67	101	264	0.145	0.497	0.357	192.5	639.2	831.6
0.9	0.59	15.895	18.68	102	264	0.147	0.494	0.359	193.9	635.1	829.0
0.9	0.6	15.889	18.69	103	264	0.148	0.491	0.360	195.4	630.7	826.1
0.9	0.61	15.883	18.7	104	265	0.150	0.488	0.362	196.9	626.2	823.1

0.9	0.62	15.877	18.71	105	265	0.151	0.486	0.363	198.5	621.5	820.0
0.9	0.63	15.87	18.721	106	265	0.153	0.482	0.365	200.2	616.5	816.7
0.9	0.64	15.863	18.733	107	265	0.155	0.479	0.367	202.0	611.2	813.2
0.9	0.65	15.856	18.744	108	265	0.156	0.476	0.368	203.7	605.8	809.5
0.9	0.66	15.848	18.756	110	265	0.158	0.472	0.370	205.6	600.1	805.7
0.9	0.67	15.84	18.769	111	265	0.160	0.468	0.372	207.7	594.1	801.7
0.9	0.68	15.831	18.782	113	265	0.163	0.464	0.373	209.8	587.8	797.6
0.9	0.69	15.823	18.796	114	264	0.165	0.460	0.375	211.9	581.3	793.2
0.9	0.7	15.813	18.81	116	264	0.167	0.456	0.377	214.2	574.4	788.7
0.9	0.71	15.804	18.825	117	263	0.170	0.451	0.379	216.6	567.4	783.9
0.9	0.72	15.794	18.84	119	263	0.172	0.447	0.381	219.0	560.0	779.0
0.9	0.73	15.783	18.856	121	262	0.175	0.442	0.383	221.6	552.2	773.9
0.9	0.74	15.772	18.873	123	261	0.178	0.437	0.385	224.4	544.2	768.5
0.9	0.75	15.761	18.891	125	260	0.181	0.432	0.387	227.2	535.8	763.0
0.9	0.76	15.749	18.909	127	258	0.184	0.426	0.389	230.1	527.1	757.2
0.9	0.77	15.737	18.929	130	257	0.188	0.421	0.392	233.3	517.9	751.2
0.9	0.78	15.725	18.949	132	255	0.191	0.415	0.394	236.4	508.6	745.0
0.9	0.79	15.711	18.971	135	254	0.195	0.408	0.397	239.9	498.6	738.5
0.9	0.8	15.698	18.994	138	252	0.199	0.402	0.399	243.4	488.3	731.7
0.9	0.81	15.683	19.018	141	249	0.203	0.395	0.402	247.2	477.5	724.7
0.9	0.82	15.669	19.043	144	247	0.207	0.388	0.405	251.0	466.3	717.3
0.9	0.83	15.653	19.071	147	244	0.212	0.380	0.408	255.3	454.4	709.7
0.9	0.84	15.637	19.1	150	241	0.217	0.372	0.411	259.6	442.1	701.8
0.9	0.85	15.62	19.131	154	238	0.223	0.363	0.414	264.2	429.2	693.5
0.9	0.86	15.603	19.165	158	234	0.228	0.354	0.418	269.1	415.7	684.8
0.9	0.87	15.584	19.201	163	230	0.235	0.344	0.421	274.3	401.4	675.7
0.9	0.88	15.565	19.241	167	226	0.241	0.334	0.425	279.9	386.4	666.2
0.9	0.89	15.545	19.284	172	221	0.249	0.323	0.429	285.7	370.5	656.2
0.9	0.9	15.523	19.332	178	215	0.257	0.311	0.433	292.1	353.6	645.7
0.9	0.91	15.501	19.385	183	209	0.265	0.298	0.437	298.8	335.8	634.6
0.9	0.92	15.477	19.445	190	201	0.274	0.283	0.442	306.2	316.6	622.8
0.9	0.93	15.451	19.513	197	193	0.285	0.268	0.447	314.3	296.0	610.3
0.9	0.94	15.424	19.592	205	184	0.296	0.251	0.453	323.0	273.8	596.9
0.9	0.95	15.395	19.686	214	173	0.309	0.231	0.459	332.7	249.6	582.3
0.9	0.96	15.362	19.801	225	160	0.325	0.209	0.466	343.8	222.6	566.4
0.9	0.97	15.326	19.949	237	144	0.342	0.184	0.474	356.4	192.1	548.5

0.9	0.98	15.284	20.157	252	123	0.364	0.152	0.484	371.7	156.2	527.9
0.9	0.99	15.232	20.503	272	94	0.393	0.110	0.496	391.5	110.5	502.0

Table A.3 Numerical results when $SL_1 = 0.9$, we vary SL_2 with step size 0.01

Table A.4 shows the full table of numerical results of pricing problem when $SL_2 = 0.9$ and SL_1 is varied with step size 0.01.

SL_1	SL_2	r_1^*	r_2^*	Q_1^*	Q_2^*	G_1^*	G_2^*	G_0^*	Π_1	Π_2	Π
0.01	0.9	16.376	19.05	17	299	0.115	0.432	0.454	75.3	431.4	506.8
0.02	0.9	15.802	19.1	37	266	0.191	0.385	0.425	137.6	393.8	531.4
0.03	0.9	15.572	19.139	51	247	0.232	0.357	0.411	177.8	373.0	550.7
0.04	0.9	15.442	19.171	61	235	0.259	0.339	0.402	207.3	359.2	566.4
0.05	0.9	15.355	19.198	70	225	0.278	0.325	0.397	230.7	348.9	579.7
0.06	0.9	15.293	19.221	78	218	0.293	0.315	0.393	250.1	341.1	591.2
0.07	0.9	15.245	19.242	85	212	0.305	0.306	0.389	266.8	334.5	601.3
0.08	0.9	15.208	19.26	91	207	0.314	0.298	0.387	281.1	329.2	610.4
0.09	0.9	15.178	19.277	96	202	0.323	0.292	0.385	293.9	324.7	618.6
0.1	0.9	15.153	19.292	101	198	0.329	0.287	0.384	305.3	320.8	626.0
0.11	0.9	15.132	19.306	106	195	0.335	0.282	0.383	315.6	317.3	632.9
0.12	0.9	15.114	19.319	110	192	0.341	0.278	0.382	325.0	314.2	639.2
0.13	0.9	15.099	19.331	114	189	0.345	0.274	0.381	333.6	311.5	645.1
0.14	0.9	15.086	19.342	118	187	0.349	0.270	0.381	341.4	309.1	650.5
0.15	0.9	15.074	19.352	122	185	0.353	0.267	0.380	348.7	306.9	655.6
0.16	0.9	15.065	19.362	125	183	0.356	0.264	0.380	355.4	305.0	660.3
0.17	0.9	15.056	19.371	128	181	0.359	0.262	0.379	361.7	303.1	664.8
0.18	0.9	15.048	19.379	131	180	0.361	0.260	0.379	367.5	301.5	669.0
0.19	0.9	15.042	19.387	134	178	0.363	0.257	0.379	372.8	300.1	672.9
0.2	0.9	15.036	19.395	137	177	0.366	0.255	0.379	377.9	298.7	676.6
0.21	0.9	15.031	19.402	139	176	0.367	0.254	0.379	382.6	297.4	680.1
0.22	0.9	15.026	19.408	142	174	0.369	0.252	0.379	387.0	296.3	683.3
0.23	0.9	15.023	19.415	144	173	0.371	0.250	0.379	391.1	295.3	686.4
0.24	0.9	15.019	19.421	147	172	0.372	0.249	0.379	395.1	294.3	689.3
0.25	0.9	15.017	19.426	149	171	0.373	0.248	0.379	398.5	293.6	692.1
0.26	0.9	15.014	19.431	151	171	0.374	0.247	0.379	401.9	292.7	694.7
0.27	0.9	15.013	19.436	153	170	0.375	0.246	0.380	405.0	292.1	697.1
0.28	0.9	15.011	19.441	155	169	0.376	0.244	0.380	408.0	291.4	699.4

0.29	0.9	15.01	19.445	157	169	0.376	0.244	0.380	410.7	290.9	701.5
0.3	0.9	15.009	19.449	159	168	0.377	0.243	0.380	413.2	290.4	703.6
0.31	0.9	15.009	19.453	161	168	0.377	0.242	0.381	415.5	290.0	705.5
0.32	0.9	15.009	19.457	162	167	0.378	0.241	0.381	417.7	289.6	707.3
0.33	0.9	15.009	19.46	164	167	0.378	0.241	0.381	419.6	289.3	708.9
0.34	0.9	15.009	19.463	166	166	0.378	0.240	0.382	421.5	289.0	710.5
0.35	0.9	15.01	19.466	167	166	0.378	0.240	0.382	423.1	288.8	711.9
0.36	0.9	15.011	19.469	169	166	0.378	0.239	0.382	424.7	288.6	713.2
0.37	0.9	15.012	19.472	170	165	0.378	0.239	0.383	426.1	288.4	714.5
0.38	0.9	15.014	19.474	172	165	0.378	0.239	0.383	427.2	288.4	715.6
0.39	0.9	15.016	19.476	173	165	0.378	0.238	0.384	428.2	288.4	716.7
0.4	0.9	15.018	19.478	174	165	0.377	0.238	0.384	429.2	288.4	717.6
0.41	0.9	15.02	19.48	176	165	0.377	0.238	0.385	430.0	288.4	718.4
0.42	0.9	15.022	19.481	177	165	0.377	0.238	0.385	430.6	288.6	719.2
0.43	0.9	15.025	19.483	178	165	0.376	0.238	0.386	431.2	288.7	719.9
0.44	0.9	15.027	19.484	179	165	0.376	0.238	0.386	431.7	288.8	720.4
0.45	0.9	15.03	19.485	181	165	0.375	0.238	0.387	431.9	289.0	720.9
0.46	0.9	15.034	19.486	182	165	0.374	0.238	0.387	432.0	289.3	721.3
0.47	0.9	15.037	19.486	183	165	0.374	0.239	0.388	432.0	289.7	721.7
0.48	0.9	15.041	19.487	184	165	0.373	0.239	0.388	431.9	290.0	721.9
0.49	0.9	15.044	19.487	185	165	0.372	0.239	0.389	431.7	290.3	722.1
0.5	0.9	15.049	19.487	185	166	0.371	0.239	0.390	431.3	290.9	722.1
0.51	0.9	15.053	19.487	186	166	0.370	0.240	0.390	430.8	291.3	722.1
0.52	0.9	15.057	19.487	187	166	0.369	0.240	0.391	430.3	291.7	722.0
0.53	0.9	15.062	19.487	188	167	0.368	0.241	0.391	429.6	292.3	721.9
0.54	0.9	15.067	19.486	189	167	0.367	0.241	0.392	428.7	292.9	721.6
0.55	0.9	15.072	19.485	190	167	0.365	0.242	0.393	427.7	293.5	721.3
0.56	0.9	15.077	19.485	190	168	0.364	0.242	0.393	426.8	294.1	720.9
0.57	0.9	15.083	19.484	191	168	0.363	0.243	0.394	425.6	294.8	720.4
0.58	0.9	15.089	19.482	192	169	0.361	0.244	0.395	424.1	295.7	719.8
0.59	0.9	15.095	19.481	192	169	0.360	0.245	0.396	422.7	296.4	719.2
0.6	0.9	15.101	19.48	193	170	0.358	0.245	0.396	421.3	297.2	718.4
0.61	0.9	15.108	19.478	193	170	0.357	0.246	0.397	419.5	298.1	717.6
0.62	0.9	15.114	19.476	194	171	0.355	0.247	0.398	417.7	299.0	716.7
0.63	0.9	15.122	19.474	194	172	0.353	0.248	0.399	415.7	300.0	715.7
0.64	0.9	15.129	19.472	194	173	0.351	0.249	0.400	413.7	301.0	714.6

0.65	0.9	15.137	19.47	195	173	0.349	0.250	0.400	411.5	302.0	713.5
0.66	0.9	15.145	19.467	195	174	0.347	0.252	0.401	409.1	303.2	712.2
0.67	0.9	15.153	19.464	195	175	0.345	0.253	0.402	406.6	304.3	710.9
0.68	0.9	15.162	19.461	195	176	0.343	0.254	0.403	403.9	305.6	709.5
0.69	0.9	15.171	19.458	196	177	0.340	0.256	0.404	401.1	306.8	707.9
0.7	0.9	15.18	19.455	196	178	0.338	0.257	0.405	398.3	308.1	706.3
0.71	0.9	15.19	19.452	196	179	0.336	0.258	0.406	395.2	309.4	704.6
0.72	0.9	15.201	19.448	196	180	0.333	0.260	0.407	391.8	311.0	702.8
0.73	0.9	15.211	19.444	196	181	0.330	0.262	0.408	388.4	312.4	700.8
0.74	0.9	15.223	19.44	195	182	0.327	0.264	0.409	384.8	314.0	698.8
0.75	0.9	15.235	19.435	195	184	0.324	0.266	0.410	380.9	315.8	696.7
0.76	0.9	15.247	19.431	195	185	0.321	0.267	0.411	377.0	317.4	694.4
0.77	0.9	15.26	19.426	194	187	0.318	0.269	0.413	372.8	319.2	692.0
0.78	0.9	15.274	19.421	194	188	0.315	0.272	0.414	368.3	321.2	689.5
0.79	0.9	15.288	19.416	193	190	0.311	0.274	0.415	363.8	323.1	686.8
0.8	0.9	15.303	19.41	193	191	0.307	0.276	0.416	358.9	325.2	684.0
0.81	0.9	15.319	19.404	192	193	0.304	0.279	0.418	353.7	327.4	681.1
0.82	0.9	15.336	19.398	191	195	0.299	0.281	0.419	348.4	329.6	678.0
0.83	0.9	15.354	19.391	190	197	0.295	0.284	0.420	342.6	332.1	674.7
0.84	0.9	15.373	19.384	189	199	0.291	0.287	0.422	336.7	334.6	671.3
0.85	0.9	15.394	19.377	187	201	0.286	0.291	0.424	330.3	337.3	667.6
0.86	0.9	15.416	19.369	186	204	0.281	0.294	0.425	323.6	340.2	663.8
0.87	0.9	15.439	19.36	184	206	0.275	0.298	0.427	316.4	343.2	659.7
0.88	0.9	15.465	19.352	182	209	0.269	0.302	0.429	308.9	346.4	655.3
0.89	0.9	15.493	19.342	180	212	0.263	0.306	0.431	300.7	349.9	650.7
0.9	0.9	15.523	19.332	178	215	0.257	0.311	0.433	292.1	353.6	645.7
0.91	0.9	15.557	19.321	175	218	0.249	0.316	0.435	282.7	357.7	640.4
0.92	0.9	15.595	19.31	172	222	0.241	0.321	0.438	272.6	362.0	634.6
0.93	0.9	15.637	19.297	168	226	0.233	0.327	0.440	261.6	366.7	628.3
0.94	0.9	15.686	19.283	164	231	0.223	0.334	0.443	249.3	372.0	621.4
0.95	0.9	15.744	19.267	158	237	0.212	0.342	0.446	235.5	378.1	613.6
0.96	0.9	15.813	19.249	152	243	0.200	0.351	0.450	219.9	384.9	604.8
0.97	0.9	15.902	19.228	144	250	0.184	0.362	0.454	201.3	393.1	594.5
0.98	0.9	16.025	19.202	133	260	0.165	0.376	0.460	178.2	403.5	581.7
0.99	0.9	16.228	19.166	116	274	0.137	0.396	0.467	145.6	418.3	563.8

Table A.4 Numerical results when $SL_2 = 0.9$, we vary SL_1 with step size 0.01

Appendix B

Appendix of Chapter 3

B.1 Proofs of Lemmas

B.1.1 Proof of Lemma 1

From the first part of the lemma, there exists at least one point u such that $\psi'(u) = 0$. We need to prove the uniqueness of such point. Let $u^* = \min\{u \mid \psi'(u) = 0\}$. By definition, we have $\psi'(u^*) = 0$ and from condition (ii), $\psi''(u^*) < 0$. Thus, u^* is a local maximizer of $\psi(\cdot)$. Moreover, $\psi''(u)$ is a continuous function. Therefore, by the fundamental theorem of integral calculus, there exists an $\varepsilon > 0$ such that $\psi'(u) < 0$ for $u \in (u^*, u^* + \varepsilon)$.

Now, assuming that there exists at least one point u such that $u > u^*$ and $\psi'(u) = 0$. Let $u^{**} = \min\{u \mid u > u^*, \psi'(u) = 0\}$. By the definition of u^{**} and the continuity of $\psi'(u)$, $\psi'(u)$ doesn't change its sign between two zero points u^* and u^{**} . As $\psi'(u) < 0$ for $u \in (u^*, u^* + \varepsilon)$, we deduce that $\psi'(u) < 0$ for $u \in (u^*, u^{**})$.

Besides, by the condition (ii) of the lemma, we have $\psi''(u^{**}) < 0$, or u^{**} is also a local maximizer of $\psi(\cdot)$. Thus, there exists an $\eta > 0$ such that $\psi'(u) > 0$ for $u \in (u^{**} - \eta, u^{**})$, which is a contradiction to the above property that $\psi'(u) < 0$ for $u \in (u^*, u^{**})$. Thus, no such a u^{**} exists.

B.1.2 Proof of Lemma 2

The first condition (i) of the lemma secures the existence of a point \mathbf{u}^* such that $\nabla\Psi(\mathbf{u}^*) = \mathbf{0}$. Thus, if \mathbf{u}^* is an argument of the maxima of $\Psi(\mathbf{u})$, then $\frac{\partial\Psi}{\partial u_i}(\mathbf{u}^*) = 0$ for all $i \in \{1, 2, \dots, n\}$. In order to complete the proof, we need to show uniqueness. We prove by induction on n , the number of decision variables.

When $n = 1$, the result follows from Lemma 3. Assuming that the lemma holds when $n = k - 1$, we need to show that it is true when $n = k$.

At a given u_k , $\Psi(\mathbf{u})$ can be seen as a function of $k - 1$ variables. By the induction assumption, given u_k , there exists unique vector of decision variables $\mathbf{u}^*(u_k) = (u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k))$ that satisfies

$$\frac{\partial \Psi}{\partial u_i}(u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k), u_k) = 0, \quad (\text{B.1})$$

for all $i \in \{1, 2, \dots, k - 1\}$. Moreover, $\mathbf{u}^*(u_k)$ is the maximizer of $\Psi(\mathbf{u})$ for this given value of u_k .

Now, instead of maximizing $\Psi(\mathbf{u})$ over $\mathbf{u} = (u_1, u_2, \dots, u_k)$, we can maximize the function $\Psi^*(u_k) := \Psi(\mathbf{u}^*(u_k), u_k)$ over u_k . Applying the envelope theorem, the first-order derivative of $\Psi^*(u_k)$ is given as follows:

$$\frac{\partial \Psi^*}{\partial u_k}(u_k) = \frac{\partial \Psi}{\partial u_k}(\mathbf{u}^*(u_k), u_k). \quad (\text{B.2})$$

Thus, the second-order derivative of $\Psi^*(u_k)$ is given as follows

$$\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) = \frac{\partial^2 \Psi}{\partial u_k^2}(\mathbf{u}^*(u_k), u_k) + \sum_{i=1}^{k-1} \frac{\partial^2 \Psi}{\partial u_k \partial u_i}(\mathbf{u}^*(u_k), u_k) \frac{\partial u_i^*(u_k)}{\partial u_k}. \quad (\text{B.3})$$

Now, we will prove that at all the point u_k such that $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$, then $\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) < 0$.

Indeed, when $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$, from Eq. B.1, Eq. B.2, for all $i \in \{1, 2, 3, \dots, k\}$, we have

$$\frac{\partial \Psi}{\partial u_i}(u_1^*(u_k), u_2^*(u_k), \dots, u_{k-1}^*(u_k), u_k) = 0. \quad (\text{B.4})$$

Then, by assumption (ii) of the lemma, we have

$$\frac{\partial^2 \Psi}{\partial u_k^2}(\mathbf{u}^*(u_k), u_k) < 0, \frac{\partial^2 \Psi}{\partial u_k \partial u_i}(\mathbf{u}^*(u_k), u_k) = 0, i \in \{1, 2, \dots, k - 1\}. \quad (\text{B.5})$$

Embedding these properties in Eq. B.3, we deduce that

$$\frac{\partial^2 \Psi^*}{\partial u_k^2}(u_k) < 0, \quad (\text{B.6})$$

given that $\frac{\partial \Psi^*}{\partial u_k}(u_k) = 0$.

The lemma holds for $n = 1$ (confirm by Lemma 3), we deduce that there exists a unique u_k^* that maximizes $\Psi^*(u_k)$. Therefore, there exists a unique vector $\mathbf{u}^* = (u_1^*(u_k^*), u_2^*(u_k^*), \dots, u_{k-1}^*(u_k^*), u_k^*)$ that satisfies $\frac{\partial \Psi}{\partial u_i}(\mathbf{u}^*) = 0$ for $i \in \{1, 2, 3, \dots, k\}$ and \mathbf{u}^* maximizes $\Psi(\mathbf{u})$.

B.2 Proofs of Propositions

B.2.1 Proof of Proposition 1

To prove this proposition, we introduce two lemmas concerning the detailed formulation of the (first and second order) derivatives for the part of the market and profit functions.

We represent explicit formulations for the part of the market function $G_i(\mathbf{r})$ and its first-order derivatives, second-order derivatives, and cross derivatives via Lemma 8 as follows.

Lemma 8. For all $i, j \in I, i \neq j$, we have

$$a. \quad \frac{\partial G_i}{\partial r_i}(\mathbf{r}) = \zeta_i(r_i) G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) < 0 \quad (\text{B.7})$$

$$b. \quad \frac{\partial G_j}{\partial r_i}(\mathbf{r}) = -\zeta_i(r_i) G_i(\mathbf{r}) G_j(\mathbf{r}) > 0 \quad (\text{B.8})$$

$$c. \quad \frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) = [\zeta_i'(r_i) + (1 - 2G_i(\mathbf{r})) \zeta_i^2(r_i)] G_i(\mathbf{r}) (1 - G_i(\mathbf{r})) \quad (\text{B.9})$$

$$d. \quad \frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r}) = -[\zeta_i'(r_i) + (1 - 2G_i(\mathbf{r})) \zeta_i^2(r_i)] G_i(\mathbf{r}) G_j(\mathbf{r}) \quad (\text{B.10})$$

$$e. \quad \frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r}) = -\zeta_i(r_i) \zeta_j(r_j) G_i(\mathbf{r}) G_j(\mathbf{r}) (1 - 2G_i(\mathbf{r})) \quad (\text{B.11})$$

Proof. Note that we have $\zeta_i(r_i) < 0, 0 < G_i(\mathbf{r}) < 1$.

$$\frac{\partial G_i}{\partial r_i}(\mathbf{r}) = \frac{\partial \frac{g_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)}}{\partial r_i} \quad (\text{B.12})$$

$$= \frac{1}{(1 + \sum_{i \in I} g_i(r_i))^2} \left[g_i'(r_i) \left(1 + \sum_{i \in I} g_i(r_i) \right) - g_i(r_i) \frac{\partial (1 + \sum_{i \in I} g_i(r_i))}{\partial r_i} \right] \quad (\text{B.13})$$

$$= \frac{g'_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)} - \frac{g_i(r_i)g'_i(r_i)}{(1 + \sum_{i \in I} g_i(r_i))^2} \quad (\text{B.14})$$

$$= \frac{g'_i(r_i)}{g_i(r_i)} \frac{g_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)} - \frac{g'_i(r_i)}{g_i(r_i)} \frac{g_i^2(r_i)}{(1 + \sum_{i \in I} g_i(r_i))^2} \quad (\text{B.15})$$

$$= \zeta_i(r_i)G_i(\mathbf{r}) - \zeta_i(r_i)G_i^2(\mathbf{r}) \quad (\text{B.16})$$

$$= \zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) < 0. \quad (\text{B.17})$$

$$\frac{\partial G_j}{\partial r_i}(\mathbf{r}) = \frac{\partial \frac{g_j(r_j)}{1 + \sum_{i \in I} g_i(r_i)}}{\partial r_i} \quad (\text{B.18})$$

$$= \frac{1}{(1 + \sum_{i \in I} g_i(r_i))^2} \left[-g_j(r_j) \frac{\partial (1 + \sum_{i \in I} g_i(r_i))}{\partial r_i} \right] \quad (\text{B.19})$$

$$= -\frac{g_j(r_j)g'_i(r_i)}{(1 + \sum_{i \in I} g_i(r_i))^2} \quad (\text{B.20})$$

$$= -\frac{g'_i(r_i)}{g_i(r_i)} \frac{g_i(r_i)}{1 + \sum_{i \in I} g_i(r_i)} \frac{g_j(r_j)}{1 + \sum_{i \in I} g_i(r_i)} \quad (\text{B.21})$$

$$= -\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r}) > 0. \quad (\text{B.22})$$

$$\frac{\partial^2 G_i}{\partial r_i^2}(\mathbf{r}) = \frac{\partial}{\partial r_i} \left(\frac{\partial G_i}{\partial r_i} \right) (\mathbf{r}) \quad (\text{B.23})$$

$$= \frac{\partial}{\partial r_i} [\zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))] \quad (\text{B.24})$$

$$= \zeta'_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left(\frac{\partial G_i}{\partial r_i}(\mathbf{r})(1 - G_i(\mathbf{r})) - G_i(\mathbf{r}) \frac{\partial G_i}{\partial r_i}(\mathbf{r}) \right) \quad (\text{B.25})$$

$$= \zeta'_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left(\frac{\partial G_i}{\partial r_i}(\mathbf{r})(1 - 2G_i(\mathbf{r})) \right) \quad (\text{B.26})$$

$$= \zeta'_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) + \zeta_i(r_i) \left([\zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))](1 - 2G_i(\mathbf{r})) \right) \quad (\text{B.27})$$

$$= \zeta'_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r})) + \zeta_i^2(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))(1 - 2G_i(\mathbf{r})) \quad (\text{B.28})$$

$$= G_i(\mathbf{r})(1 - G_i(\mathbf{r})) [\zeta'_i(r_i) + \zeta_i^2(r_i)(1 - 2G_i(\mathbf{r}))]. \quad (\text{B.29})$$

$$\frac{\partial^2 G_j}{\partial r_i^2}(\mathbf{r}) = \frac{\partial}{\partial r_i} \left(\frac{\partial G_j}{\partial r_i} \right) (\mathbf{r}) \quad (\text{B.30})$$

$$= \frac{\partial}{\partial r_i} [-\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r})] \quad (\text{B.31})$$

$$= - \left[\zeta_i'(r_i)G_i(\mathbf{r})G_j(\mathbf{r}) + \zeta_i(r_i)\frac{\partial G_i}{\partial r_i}(\mathbf{r})G_j(\mathbf{r}) + \zeta_i(r_i)G_i(\mathbf{r})\frac{\partial G_j}{\partial r_i}(\mathbf{r}) \right] \quad (\text{B.32})$$

$$= - \left\{ \zeta_i'(r_i)G_i(\mathbf{r})G_j(\mathbf{r}) + \zeta_i(r_i) [\zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))] G_j(\mathbf{r}) \right. \\ \left. + \zeta_i(r_i)G_i(\mathbf{r}) [-\zeta_i(r_i)G_i(\mathbf{r})G_j(\mathbf{r})] \right\} \quad (\text{B.33})$$

$$= - [\zeta_i'(r_i)G_i(\mathbf{r})G_j(\mathbf{r}) + \zeta_i^2(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))G_j(\mathbf{r}) - \zeta_i^2(r_i)G_i^2(\mathbf{r})G_j(\mathbf{r})] \quad (\text{B.34})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) [\zeta_i'(r_i) + \zeta_i^2(r_i)(1 - G_i(\mathbf{r})) - \zeta_i^2(r_i)G_i(\mathbf{r})] \quad (\text{B.35})$$

$$= -G_i(\mathbf{r})G_j(\mathbf{r}) [\zeta_i'(r_i) + \zeta_i^2(r_i)(1 - 2G_i(\mathbf{r}))]. \quad (\text{B.36})$$

$$\frac{\partial^2 G_i}{\partial r_i \partial r_j}(\mathbf{r}) = \frac{\partial}{\partial r_j} \left(\frac{\partial G_i}{\partial r_i} \right) (\mathbf{r}) \quad (\text{B.37})$$

$$= \frac{\partial}{\partial r_j} [\zeta_i(r_i)G_i(\mathbf{r})(1 - G_i(\mathbf{r}))] \quad (\text{B.38})$$

$$= \zeta_i(r_i) \left[\frac{\partial G_i}{\partial r_j}(\mathbf{r})(1 - G_i(\mathbf{r})) - G_i(\mathbf{r})\frac{\partial G_i}{\partial r_j}(\mathbf{r}) \right] \quad (\text{B.39})$$

$$= \zeta_i(r_i)\frac{\partial G_i}{\partial r_j}(\mathbf{r})(1 - 2G_i(\mathbf{r})) \quad (\text{B.40})$$

$$= \zeta_i(r_i) [-\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})] (1 - 2G_i(\mathbf{r})) \quad (\text{B.41})$$

$$= -\zeta_i(r_i)\zeta_j(r_j)G_i(\mathbf{r})G_j(\mathbf{r})(1 - 2G_i(\mathbf{r})). \quad (\text{B.42})$$

□

The lemma below shows the relationship between second-order derivatives and cross derivatives with first-order derivatives of the objective function of problem DCP when $(r_p, r_o) \in \Omega$. It represents key equations to prove Proposition 4.

Lemma 9. For all $(r_p, r_o) \in \Omega$, for all $i \in I$, we have

$$i. \frac{\partial^2 \Pi}{\partial r_i^2}(\mathbf{r}) = \mathbb{E}[\xi] G_i(\mathbf{r}) \left[\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} \right] + \left[\frac{\zeta_i'(r_i)}{\zeta_i(r_i)} + \zeta_i(r_i)(1 - 2G_i(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_i}(\mathbf{r})$$

$$ii. \frac{\partial^2 \Pi}{\partial r_p \partial r_o}(\mathbf{r}) = - \left\{ \zeta_o(r_o)G_o(\mathbf{r})\frac{\partial \Pi}{\partial r_p}(\mathbf{r}) + \zeta_p(r_p)G_p(\mathbf{r})\frac{\partial \Pi}{\partial r_o}(\mathbf{r}) \right\}$$

From now on, to shorten the formula representations, we denote $\zeta_i(r_i)$ by ζ_i , $G_i(\mathbf{r})$ by G_i , $\frac{\partial G_i}{\partial r_i}(\mathbf{r})$ by $\frac{\partial G_i}{\partial r_i}$, $\frac{\partial G_i}{\partial r_j}(\mathbf{r})$ by $\frac{\partial G_i}{\partial r_j}$, $\frac{\partial \Pi}{\partial r_i}(\mathbf{r})$ by $\frac{\partial \Pi}{\partial r_i}$. We denote by μ the expectation of ξ , $\mu = \mathbb{E}[\xi]$.

Proof. When $(r_p, r_o) \in \Omega$, we have $G_p(r_p, r_o) < K/M$. Thus, $\frac{K}{G_p(r_p, r_o)} > M$. Note that ξ is bounded by M . Therefore, $\int_{\frac{K}{G_p(\mathbf{r})}}^{\infty} \left(x - \frac{K}{G_p(\mathbf{r})}\right) f(x) dx = 0$. It follows that

$$\Pi(r_p, r_o) = \mu(r_p - h_p + \alpha)G_p - (c_p - h_p)K + \mu(r_o - c_o)G_o \quad (\text{B.43})$$

$$\frac{\partial \Pi}{\partial r_p} = \mu G_p + \mu(r_p - h_p + \alpha) \frac{\partial G_p}{\partial r_p} + \mu(r_o - c_o) \frac{\partial G_o}{\partial r_p} \quad (\text{B.44})$$

$$= \mu G_p + \mu(r_p - h_p + \alpha) [\delta_p G_p (1 - G_p)] + \mu(r_o - c_o) [-\delta_p G_p G_o] \quad (\text{B.45})$$

$$= \mu G_p + \mu(r_p - h_p + \alpha) \delta_p G_p (1 - G_p) - \mu(r_o - c_o) \delta_p G_p G_o \quad (\text{B.46})$$

$$\frac{\partial \Pi}{\partial r_o} = \mu(r_p - h_p + \alpha) \frac{\partial G_p}{\partial r_o} + \mu(r_o - c_o) \frac{\partial G_o}{\partial r_o} + \mu G_o \quad (\text{B.47})$$

$$= \mu(r_p - h_p + \alpha) [-\delta_o G_p G_o] + \mu(r_o - c_o) [\delta_o G_o (1 - G_o)] + \mu G_o \quad (\text{B.48})$$

$$= -\mu(r_p - h_p + \alpha) \delta_o G_p G_o + \mu(r_o - c_o) \delta_o G_o (1 - G_o) + \mu G_o \quad (\text{B.49})$$

$$\frac{\partial^2 \Pi}{\partial r_p^2} = \mu \frac{\partial G_p}{\partial r_p} + \mu \frac{\partial G_p}{\partial r_p} + \mu(r_p - h_p + \alpha) \frac{\partial^2 G_p}{\partial r_p^2} + \mu(r_o - c_o) \frac{\partial^2 G_o}{\partial r_p^2} \quad (\text{B.50})$$

$$= 2\mu [\delta_p G_p (1 - G_p)] + \mu(r_p - h_p + \alpha) [G_p (1 - G_p) (\delta'_p + \delta_p^2 (1 - 2G_p))] + \mu(r_o - c_o) [-G_p G_o (\delta'_p + \delta_p^2 (1 - 2G_p))] \quad (\text{B.51})$$

$$= 2\mu \delta_p G_p (1 - G_p) + \mu(r_p - h_p + \alpha) G_p (1 - G_p) (\delta'_p + \delta_p^2 (1 - 2G_p)) - \mu(r_o - c_o) G_p G_o (\delta'_p + \delta_p^2 (1 - 2G_p)) \quad (\text{B.52})$$

$$\frac{\partial^2 \Pi}{\partial r_o^2} = \mu(r_p - h_p + \alpha) \frac{\partial^2 G_p}{\partial r_o^2} + \mu \frac{\partial G_o}{\partial r_o} + \mu(r_o - c_o) \frac{\partial^2 G_o}{\partial r_o^2} + \mu \frac{\partial G_o}{\partial r_o} \quad (\text{B.53})$$

$$= 2\mu [\delta_o G_o (1 - G_o)] + \mu(r_p - h_p + \alpha) [-G_p G_o (\delta'_o + \delta_o^2 (1 - 2G_o))] + \mu(r_o - c_o) [G_o (1 - G_o) (\delta'_o + \delta_o^2 (1 - 2G_o))] \quad (\text{B.54})$$

$$= 2\mu \delta_o G_o (1 - G_o) - \mu(r_p - h_p + \alpha) G_p G_o (\delta'_o + \delta_o^2 (1 - 2G_o)) + \mu(r_o - c_o) G_o (1 - G_o) (\delta'_o + \delta_o^2 (1 - 2G_o)) \quad (\text{B.55})$$

$$\frac{\partial^2 \Pi}{\partial r_p \partial r_o} = \mu \frac{\partial G_p}{\partial r_o} + \mu \frac{\partial G_o}{\partial r_p} + \mu(r_p - h_p + \alpha) \frac{\partial^2 G_p}{\partial r_p \partial r_o} + \mu(r_o - c_o) \frac{\partial^2 G_o}{\partial r_p \partial r_o} \quad (\text{B.56})$$

$$= \mu[-\delta_o G_p G_o] + \mu(r_p - h_p + \alpha)[-\delta_p \delta_o G_p G_o(1 - 2G_p)] + \mu[-\delta_p G_p G_o] + \mu(r_o - c_o)[-\delta_p \delta_o G_p G_o(1 - 2G_o)] \quad (\text{B.57})$$

$$= -\mu \delta_o G_p G_o - \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p G_o(1 - 2G_p) - \mu \delta_p G_p G_o - \mu(r_o - c_o) \delta_p \delta_o G_p G_o(1 - 2G_o) \quad (\text{B.58})$$

For the first part of the lemma, we have

$$\left[\frac{\zeta'_p}{\zeta_p} + \zeta_p(1 - 2G_p) \right] \frac{\partial \Pi}{\partial r_p} = \left[\frac{\zeta'_p}{\zeta_p} + \zeta_p(1 - 2G_p) \right] \times [\mu G_p + \mu(r_p - h_p + \alpha) \delta_p G_p(1 - G_p) - \mu(r_o - c_o) \delta_p G_p G_o] \quad (\text{B.59})$$

$$= \left[\frac{\zeta'_p}{\zeta_p} + \zeta_p(1 - 2G_p) \right] \mu G_p + \mu(r_p - h_p + \alpha) G_p(1 - G_p)(\delta'_p + \delta_p^2(1 - 2G_p)) - \mu(r_o - c_o) G_p G_o(\delta'_p + \delta_p^2(1 - 2G_p)) \quad (\text{B.60})$$

$$= \left[\frac{\zeta'_p}{\zeta_p} + \zeta_p(1 - 2G_p) \right] \mu G_p - 2\mu \delta_p G_p(1 - G_p) + \frac{\partial^2 \Pi}{\partial r_p^2} \quad (\text{B.61})$$

$$= \mu G_p \left[\frac{\zeta'_p}{\zeta_p} + \zeta_p(1 - 2G_p) - 2\delta_p(1 - G_p) \right] + \frac{\partial^2 \Pi}{\partial r_p^2} \quad (\text{B.62})$$

$$= \mu G_p \left[\frac{\zeta'_p}{\zeta_p} - \zeta_p \right] + \frac{\partial^2 \Pi}{\partial r_p^2} \quad (\text{B.63})$$

Thus,

$$\frac{\partial^2 \Pi}{\partial r_p^2}(\mathbf{r}) = \mu G_p(\mathbf{r}) \left[\zeta_p(r_p) - \frac{\zeta'_p(r_p)}{\zeta_p(r_p)} \right] + \left[\frac{\zeta'_p(r_p)}{\zeta_p(r_p)} + \zeta_p(r_p)(1 - 2G_p(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_p}(\mathbf{r}) \quad (\text{B.64})$$

Similarly,

$$\left[\frac{\zeta'_o}{\zeta_o} + \zeta_o(1 - 2G_o) \right] \frac{\partial \Pi}{\partial r_o} = \left[\frac{\zeta'_o}{\zeta_o} + \zeta_o(1 - 2G_o) \right] \times [\mu G_o - \mu(r_p - h_p + \alpha) \delta_o G_p G_o + \mu(r_o - c_o) \delta_o G_o(1 - G_o)] \quad (\text{B.65})$$

$$= \left[\frac{\zeta'_o}{\zeta_o} + \zeta_o(1 - 2G_o) \right] \mu G_o$$

$$\begin{aligned}
& -\mu(r_p - h_p + \alpha)G_p G_o(\delta'_o + \delta_o^2(1 - 2G_o)) \\
& + \mu(r_o - c_o)G_o(1 - G_o)(\delta'_o + \delta_o^2(1 - 2G_o))
\end{aligned} \tag{B.66}$$

$$= \left[\frac{\zeta'_o}{\zeta_o} + \zeta_o(1 - 2G_o) \right] \mu G_o - 2\mu \delta_o G_o(1 - G_o) + \frac{\partial^2 \Pi}{\partial r_o^2} \tag{B.67}$$

$$= \mu G_o \left[\frac{\zeta'_o}{\zeta_o} + \zeta_o(1 - 2G_o) - 2\delta_o(1 - G_o) \right] + \frac{\partial^2 \Pi}{\partial r_o^2} \tag{B.68}$$

$$= \mu G_o \left[\frac{\zeta'_o}{\zeta_o} - \zeta_o \right] + \frac{\partial^2 \Pi}{\partial r_o^2} \tag{B.69}$$

Thus,

$$\frac{\partial^2 \Pi}{\partial r_o^2}(\mathbf{r}) = \mu G_o(\mathbf{r}) \left[\zeta_o(r_o) - \frac{\zeta'_o(r_o)}{\zeta_o(r_o)} \right] + \left[\frac{\zeta'_o(r_o)}{\zeta_o(r_o)} + \zeta_o(r_o)(1 - 2G_o(\mathbf{r})) \right] \frac{\partial \Pi}{\partial r_o}(\mathbf{r}) \tag{B.70}$$

Therefore, the first part of the lemma holds true. For the second part, we have

$$\delta_o G_o \frac{\partial \Pi}{\partial r_p} = \delta_o G_o [\mu G_p + \mu(r_p - h_p + \alpha) \delta_p G_p(1 - G_p) - \mu(r_o - c_o) \delta_p G_p G_o] \tag{B.71}$$

$$= \mu \delta_o G_p G_o + \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p G_o(1 - G_p) - \mu(r_o - c_o) \delta_p \delta_o G_p G_o^2 \tag{B.72}$$

$$\delta_p G_p \frac{\partial \Pi}{\partial r_o} = \delta_p G_p [-\mu(r_p - h_p + \alpha) \delta_o G_p G_o + \mu(r_o - c_o) \delta_o G_o(1 - G_o) + \mu G_o] \tag{B.73}$$

$$= \mu \delta_p G_p G_o - \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p^2 G_o + \mu(r_o - c_o) \delta_p \delta_o G_p G_o(1 - G_o) \tag{B.74}$$

Thus,

$$\begin{aligned}
\delta_o G_o \frac{\partial \Pi}{\partial r_p} + \delta_p G_p \frac{\partial \Pi}{\partial r_o} &= \mu \delta_o G_p G_o + \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p G_o(1 - G_p) - \mu(r_o - c_o) \delta_p \delta_o G_p G_o^2 \\
&\quad + \mu \delta_p G_p G_o - \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p^2 G_o + \mu(r_o - c_o) \delta_p \delta_o G_p G_o(1 - G_o)
\end{aligned} \tag{B.75}$$

$$\begin{aligned}
&= \mu \delta_o G_p G_o + \mu(r_p - h_p + \alpha) \delta_p \delta_o G_p G_o(1 - 2G_p) \\
&\quad + \mu \delta_p G_p G_o + \mu(r_o - c_o) \delta_p \delta_o G_p G_o(1 - 2G_o)
\end{aligned} \tag{B.76}$$

$$= -\frac{\partial^2 \Pi}{\partial r_p \partial r_o} \tag{B.77}$$

Therefore,

$$\frac{\partial^2 \Pi}{\partial r_p \partial r_o}(\mathbf{r}) = - \left\{ \zeta_o(r_o) G_o(\mathbf{r}) \frac{\partial \Pi}{\partial r_p}(\mathbf{r}) + \zeta_p(r_p) G_p(\mathbf{r}) \frac{\partial \Pi}{\partial r_o}(\mathbf{r}) \right\} \quad (\text{B.78})$$

The lemma is proved. \square

We now have enough materials to prove Proposition 4. For all the values of \mathbf{r} , we have $0 < G_i(\mathbf{r}) < 1$ and $\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} < 0$. Note that the later inequality condition is given in Assumption 4, and is satisfied by the linear attraction demand model, multinomial logit demand model, and multiplicative competitive interaction demand model. We then deduce that $\mu G_i \left[\zeta_i(r_i) - \frac{\zeta_i'(r_i)}{\zeta_i(r_i)} \right] < 0$. Lemma 9 above then implies that at all the point \mathbf{r}^* such that $\nabla \Pi(\mathbf{r}^*) = 0$, we then have $\frac{\partial^2 \Pi}{\partial r_p \partial r_o} = 0$ and $\frac{\partial^2 \Pi}{\partial r_i^2} < 0$ for all $i \in I$. Apply the result of Lemma 4 with dimension two ($n = 2$), and the result of Proposition 4 holds true.

B.3 Numerical results for MNL and MCI demand models

B.3.1 Numerical results for MNL demand model

In this part, we provide a set of numerical results considering the MNL model, i.e.,

$$g_i(r_i) = \exp(a_i - b_i r_i),$$

with $a_i > 0, b_i > 0$. Note that, if the condition

$$\frac{g_p(c_p)}{1 + g_p(c_p)} \leq \frac{K}{M} \quad (\text{B.79})$$

is not met, our objective function does not exhibit joint concavity. As a result, finding an optimal global solution becomes challenging. In practical scenarios, even when our decision variables are bounded, algorithms may converge to optimal local solutions. Therefore, it is necessary to establish upper bounds for prices to facilitate grid search for locating the optimal global solution. In the following discussion, we will explore how to pragmatically establish these boundaries.

Our demand can be described as a bounded random variable with $|\xi| < M$. When the market segment corresponding to a particular channel is relatively small, the number of items

sold in that channel approaches zero, indicating channel saturation. Consequently, in practice, the upper price bounds should be determined when the respective channels are saturated.

Mathematically, a physical channel is considered saturated when the integer part of ξG_p is 0, or equivalently,

$$\xi G_p < 0.5. \quad (\text{B.80})$$

We can establish the following inequality:

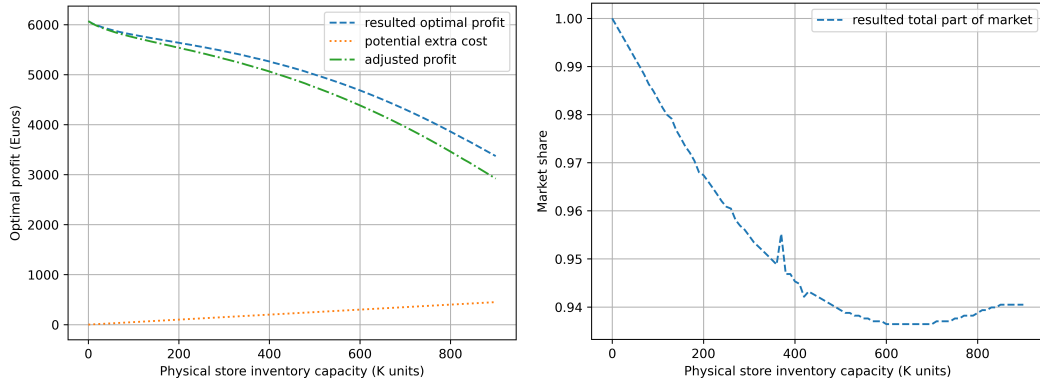
$$\xi G_p < M G_p = M \frac{e^{a_p - b_p r_p}}{1 + e^{a_p - b_p r_p} + e^{a_o - b_o r_o}} < M e^{a_p - b_p r_p}. \quad (\text{B.81})$$

If $r_p > \frac{a_p + \ln(M) + \ln(2)}{b_p}$, then $M e^{a_p - b_p r_p} < 0.5$, which implies $\xi G_p < 0.5$, or the physical channel is saturated. Consequently, we can define the upper bound for r_p as $r_p^{\max} := \frac{a_p + \ln(M) + \ln(2)}{b_p}$.

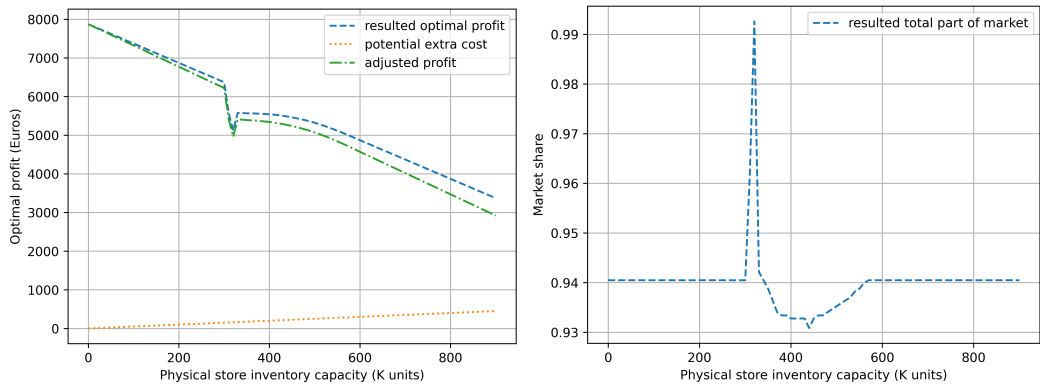
Similarly, the upper bound for r_o can be set as $r_o^{\max} := \frac{a_o + \ln(M) + \ln(2)}{b_o}$. We acknowledge that when the online channel is saturated, the demand transferred from the physical store due to accepting a promotional offer will be met from the warehouse with the same parameters (price, cost, etc.) as the online channel.

With regards to the potential market size we consider a uniform distribution with an average equal to 500 and we assume two variability configurations: $\xi \sim \mathcal{U}(100, 900)$ for the high variability and $\xi \sim \mathcal{U}(400, 600)$ for the low one. We use the same notations and definitions (for example, *potential extra cost*, *adjusted profit*, etc.) as given in Section 3.5.

Consider *potential extra cost* = $0.5K$, attraction demand coefficients: $a_p = 20, b_p = 1, a_o = 12, b_o = 1$, cost coefficients: $c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$ figure B.1 represents optimization results with different level of physical store inventory capacity K . In both two cases $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 27.5)$ and $(c_o, 19.5)$, respectively.



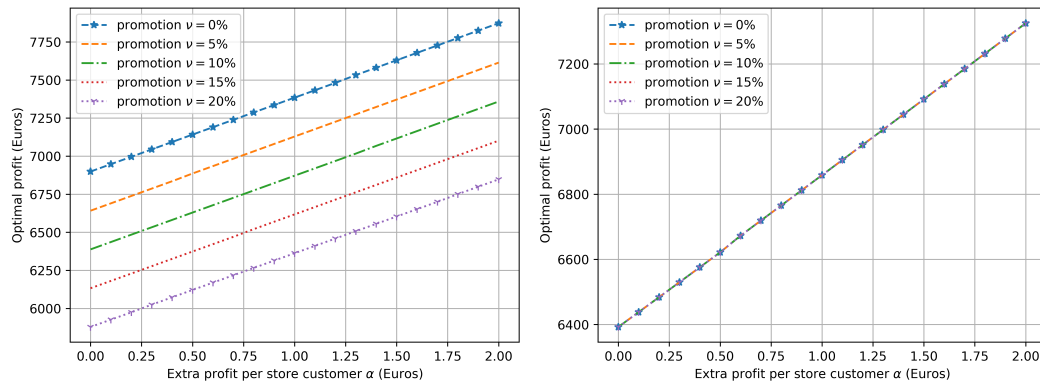
(a) Optimal profit versus $K, \xi \sim \mathcal{U}(100, 900)$ (b) Market share versus $K, \xi \sim \mathcal{U}(100, 900)$



(c) Optimal profit versus $K, \xi \sim \mathcal{U}(400, 600)$ (d) Market share versus $K, \xi \sim \mathcal{U}(400, 600)$

Figure B.1 The variation of resulted optimal profit and market share at the optimal prices when $a_p = 20, b_p = 1, a_o = 12, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$

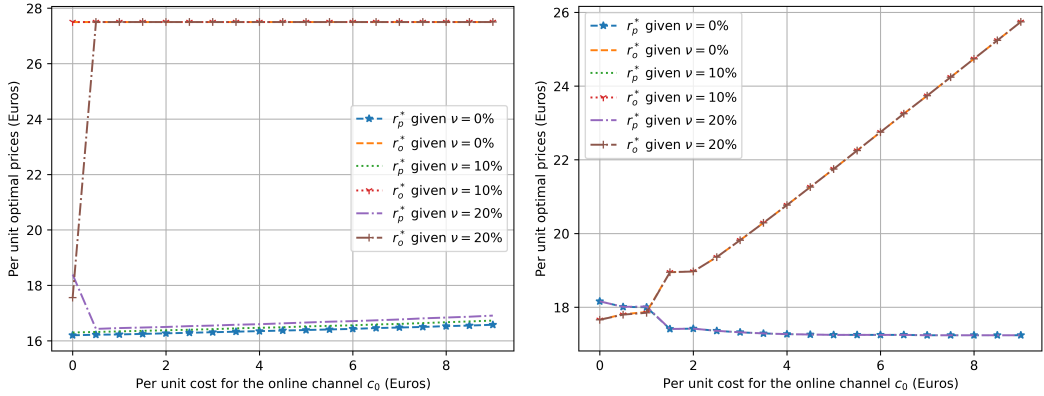
The variation of resulted optimal profit at the optimal prices with respect to α and ν when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 250$ is represented in Figure B.2 as follows. In both two cases $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 27.5)$ and $(c_o, 25.5)$, respectively.



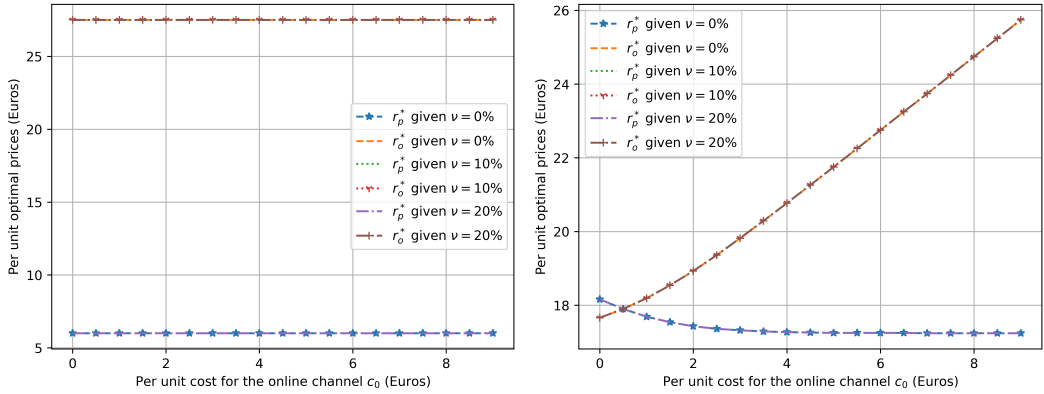
(a) Optimal profit versus α , $\xi \sim \mathcal{U}(100, 900)$ (b) Optimal profit versus α , $\xi \sim \mathcal{U}(400, 600)$

Figure B.2 The variation of resulted optimal profit at the optimal prices with respect to α and ν when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 250$.

Figure B.3 represents the variation of resulted optimal prices with respect to c_o and different ν when $a_p = 20, b_p = 1, a_o = 20, b_o = 1, c_p = 6, h_p = 1, \alpha = 0.5, \beta = 0.8, K = 250$ and $K = 0$. In both two cases $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 27.5)$ and $(c_o, 27.5)$, respectively.



(a) Optimal prices versus c_0 , $\xi \sim \mathcal{U}(100, 900), K = 250$ (b) Optimal prices versus c_0 , $\xi \sim \mathcal{U}(400, 600), K = 250$



(c) Optimal prices versus c_0 , $\xi \sim \mathcal{U}(100, 900), K = 0$ (d) Optimal prices versus c_0 , $\xi \sim \mathcal{U}(400, 600), K = 0$

Figure B.3 The variation of resulted optimal prices with respect to c_0 and different v when $a_p = 20, b_p = 1, a_o = 20, b_o = 1, c_p = 6, h_p = 1, \alpha = 0.5, \beta = 0.8, K = 250$ and $K = 0$.

The variation of resulted optimal profit and prices with respect to β when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, c_o = 3, h_p = 1, \alpha = 0.5, K = 250$ and $K = 0, \xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$ is represented in Figure B.4 as follows. In both two cases $\xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 27.5)$ and $(c_o, 25.5)$, respectively.

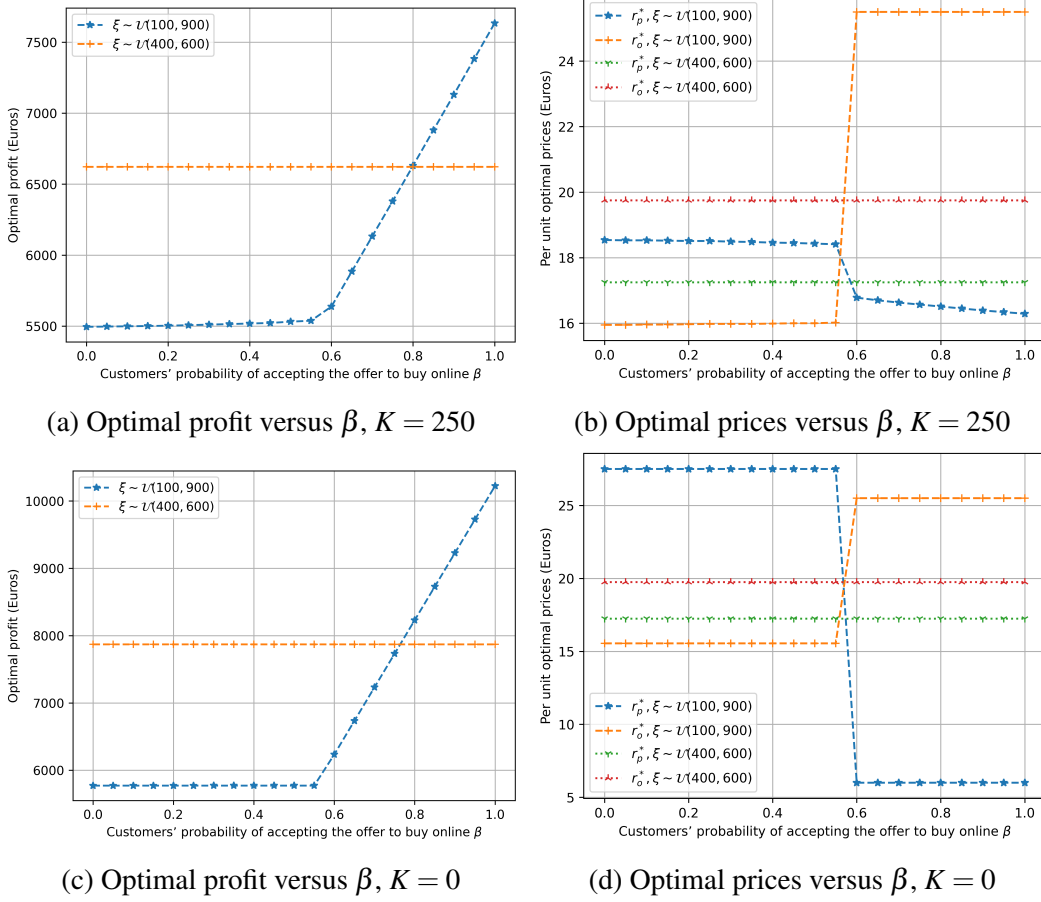


Figure B.4 The variation of resulted optimal profit and prices with respect to β when $a_p = 20, b_p = 1, a_o = 18, b_o = 1, c_p = 6, c_o = 3, h_p = 1, \alpha = 0.5, K = 250$ and $K = 0, \xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$

B.3.2 Numerical results for MCI demand model

In this part, we provide a set of numerical results considering the MCI model, i.e.,

$$g_i(r_i) = a_i r_i^{-b_i},$$

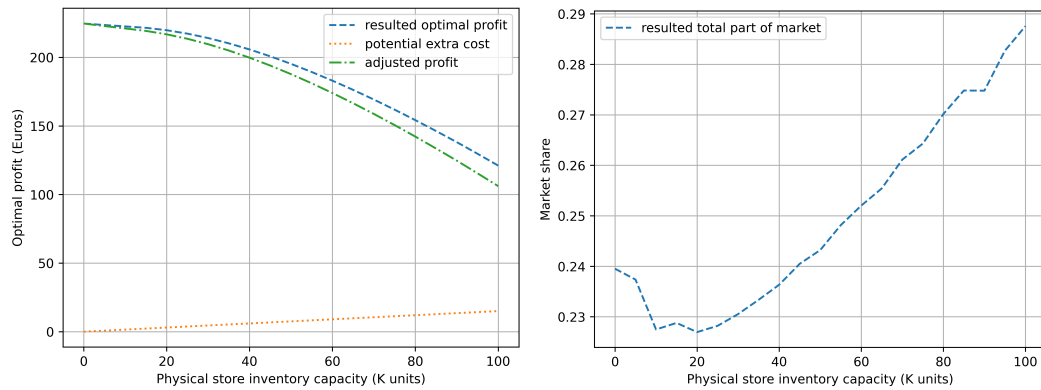
with $a_i > 0, b_i > 1$. With the same idea as the MNL model, we can establish the following inequality:

$$\xi G_p < M G_p = M \frac{a_p r_p^{-b_p}}{1 + a_p r_p^{-b_p} + a_o r_o^{-b_o}} < M a_p r_p^{-b_p}. \quad (\text{B.82})$$

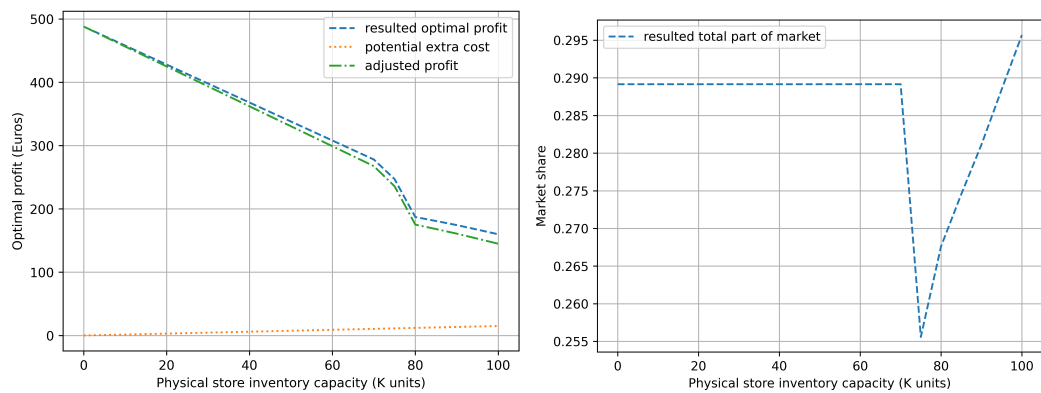
If $r_p > \sqrt[b_p]{2Ma_p}$, then $Ma_p r_p^{-b_p} < 0.5$, which implies $\xi G_p < 0.5$, or the physical channel is saturated. Consequently, we can define the upper bound for r_p as $r_p^{\max} := \sqrt[b_p]{2Ma_p}$. Similarly, the upper bound for r_o can be set as $r_o^{\max} := \sqrt[b_o]{2Ma_o}$. We acknowledge that when the online channel is saturated, the demand transferred from the physical store due to accepting a promotional offer will be met from the warehouse with the same parameters (price, cost, etc.) as the online channel.

With regards to the potential market size we consider a uniform distribution with an average equal to 500 and we assume two variability configurations: $\xi \sim \mathcal{U}(100, 900)$ for the high variability and $\xi \sim \mathcal{U}(400, 600)$ for the low one. We use the same notations and definitions (for example, *potential extra cost*, *adjusted profit*, etc.) as given in Section 3.5.

Consider *potential extra cost* = 0.15K, attraction demand coefficients: $a_p = 20, b_p = 3, a_o = 20, b_o = 3$, cost coefficients: $c_p = 4, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$, figure B.5 represents optimization results with different level of physical store inventory capacity K . When $\xi \sim \mathcal{U}(100, 900)$, the boundaries for prices are $(c_p, 33.1)$ and $(c_o, 33.1)$; when $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 28.9)$ and $(c_o, 28.9)$, respectively.



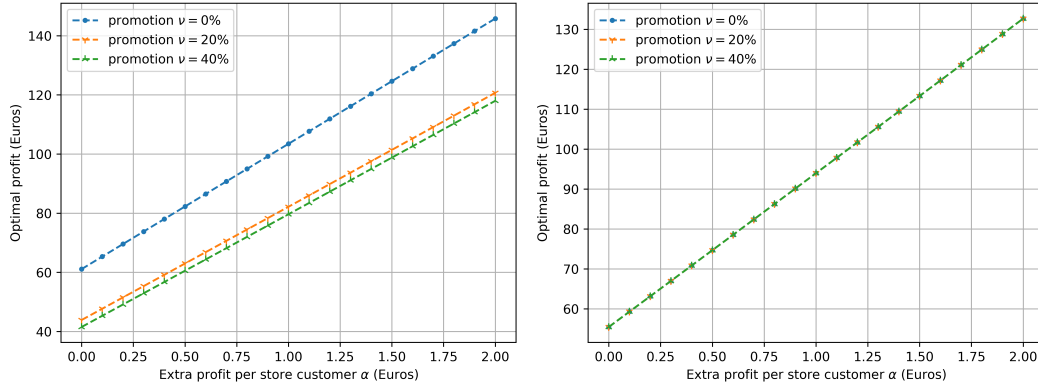
(a) Optimal profit versus K , $\xi \sim \mathcal{U}(100, 900)$ (b) Market share versus K , $\xi \sim \mathcal{U}(100, 900)$



(c) Optimal profit versus K , $\xi \sim \mathcal{U}(400, 600)$ (d) Market share versus K , $\xi \sim \mathcal{U}(400, 600)$

Figure B.5 The variation of resulted optimal profit and market share at the optimal prices when $a_p = 20, b_p = 3, a_o = 20, b_o = 3, c_p = 4, h_p = 1, c_o = 3, \alpha = 0.5, \beta = 0.8, \nu = 0.1$

The variation of resulted optimal profit at the optimal prices with respect to α and ν when $a_p = 20, b_p = 3, a_o = 18, b_o = 3, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 50$ is represented in Figure B.6 as follows. When $\xi \sim \mathcal{U}(100, 900)$, the boundaries for prices are $(c_p, 33.1)$ and $(c_o, 31.9)$; when $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 28.9)$ and $(c_o, 27.9)$, respectively.



(a) Optimal profit versus α , $\xi \sim \mathcal{U}(100, 900)$ (b) Optimal profit versus α , $\xi \sim \mathcal{U}(400, 600)$

Figure B.6 The variation of resulted optimal profit at the optimal prices with respect to α and v when $a_p = 20, b_p = 3, a_o = 18, b_o = 3, c_p = 6, h_p = 1, c_o = 3, \beta = 0.8, K = 50$.

Figure B.7 represents the variation of resulted optimal prices with respect to c_o and different v when $a_p = 20, b_p = 3, a_o = 20, b_o = 3, c_p = 4, h_p = 1, \alpha = 0.5, \beta = 0.1, K = 30$ and $K = 0$. When $\xi \sim \mathcal{U}(100, 900)$, the boundaries for prices are $(c_p, 33.1)$ and $(c_o, 33.1)$; when $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 28.9)$ and $(c_o, 28.9)$, respectively.

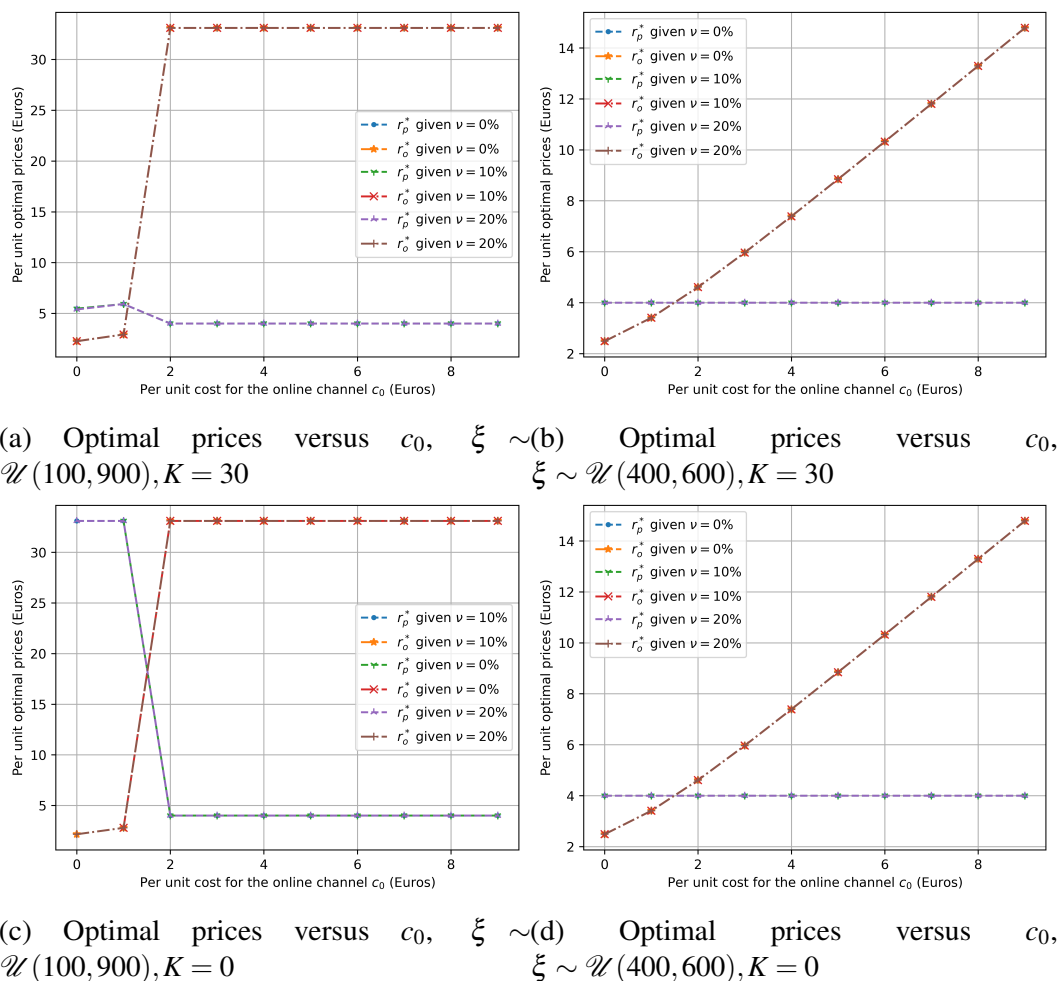


Figure B.7 The variation of resulted optimal prices with respect to c_o and different v when $a_p = 20, b_p = 3, a_o = 20, b_o = 3, c_p = 4, h_p = 1, \alpha = 0.5, \beta = 0.1, K = 30$ and $K = 0$.

The variation of resulted optimal profit and prices with respect to β when $a_p = 20, b_p = 3, a_o = 20, b_o = 3, c_p = 4, c_o = 3, h_p = 1, \alpha = 0.5, K = 30$ and $K = 0, \xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$ is represented in Figure B.8 as follows. When $\xi \sim \mathcal{U}(100, 900)$, the boundaries for prices are $(c_p, 33.1)$ and $(c_o, 33.1)$; when $\xi \sim \mathcal{U}(400, 600)$, the boundaries for prices are $(c_p, 28.9)$ and $(c_o, 28.9)$, respectively.

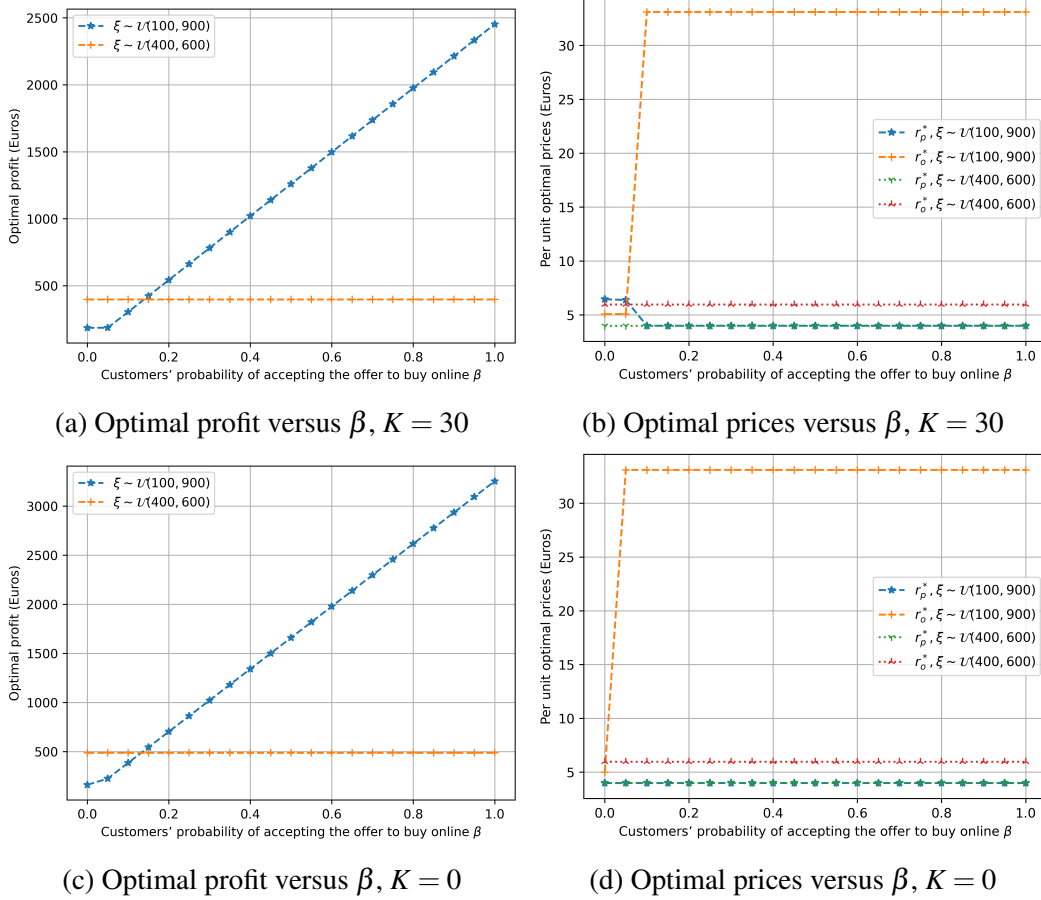


Figure B.8 The variation of resulted optimal profit and prices with respect to β when $a_p = 20, b_p = 3, a_o = 20, b_o = 3, c_p = 4, c_o = 3, h_p = 1, \alpha = 0.5, K = 30$ and $K = 0, \xi \sim \mathcal{U}(100, 900)$ and $\xi \sim \mathcal{U}(400, 600)$

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FOLIO ADMINISTRATIF

THESE DE L'INSA LYON, MEMBRE DE L'UNIVERSITE DE LYON

NOM : **TRAN**
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DATE de SOUTENANCE : **28/06/2024**

Prénoms : **Minh Tam**

TITRE : **Innovative multichannel models for pricing and inventory decisions considering service level**

NATURE : **Doctorat**

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École Doctorale : **Informatique et Mathématiques de Lyon (ED 512 InfoMaths)**

Spécialité : **Génie industriel**

RÉSUMÉ :

The thesis investigates contemporary challenges in retail management amidst the digital revolution, with a focus on multichannel retailing, dual-channel pricing, and data-driven inventory management. This thesis first begins with an overview of evolving retail dynamics driven by technological advancements and shifting consumer demands, emphasizing the necessity for inventive solutions to navigate these complexities. Second, by exploring multichannel retailing in-depth, the study examines inventory allocation and pricing optimization across physical and online channels. It addresses a multichannel pricing problem, proposing a methodology to ensure optimal solutions and highlighting the importance of channel coordination and service levels on market share and profitability. Thirdly, further delving into dual-channel pricing, the thesis presents a novel pricing model capturing intricate interactions between channels, retailers, and customers. It emphasizes the significance of determining optimal physical store capacity and managing stock-out conversions to online sales with promotions. Fourth, introducing data-driven inventory management methodologies, the study leverages Kernel Density Estimation (KDE) within chance-constrained optimization frameworks. By demonstrating superior performance in achieving target service levels compared to traditional methods, the thesis emphasizes the importance of managing inventory under uncertainty while maintaining service quality. Last but not least, the thesis concludes by promoting a deeper understanding of retail management in the digital age, offering valuable insights and methodologies to navigate modern retailing complexities. By embracing innovation, data-driven approaches, and customer-centric strategies, retailers can position themselves for success in an increasingly dynamic and competitive environment. Future research directions include exploring advanced machine learning techniques and extending the model to consider sustainability and supply chain resilience.

MOTS-CLÉS : **inventory, data-driven, multichannel, pricing, service level**

Laboratoire(s) de recherche : **Décision et Information pour les Systèmes de Production (DISP)**

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Composition du Jury :

Ou TANG (rapporteur), Zied JEMAI (rapporteur), Gülgün ALPAN (examinatrice), Guillaume MASSONNET (examineur), Khaled HADJ-HAMOU (directeur de thèse), Yacine REKIK (co-directeur de thèse)