Mathematical modelling, observation and identification of epidemiological models with reinfection

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Colophon

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Mots clés : modèles compartimentaux en épidémiologie ; réinfection ; structuration en âge ; multistabilité ; observation ; identification ; observateur asymptotique ; observateur adaptatif

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À la mémoire de mon grand-père

献给我的家人

Mais quoi, l'Histoire n'est amère qu'à ceux qui l'attendent sucrée.

Sandor Krasna

Mathematical modelling, observation and identification of epidemiological models with reinfection

Abstract

The subject of this work is the mathematical modelling of infectious diseases with reinfections and the analysis of the corresponding models. First, we introduce a general class of compartmental models counting the number of reinfections, consisting of an infinite number of ordinary differential equations, and study its well-posedness. The proposed class also allows for the modelling of heterogeneous transmission, whose characteristics depend upon the number of past reinfections. In the homogeneous situation (where the past reinfections do not affect disease transmission), the global behavior obeys a usual compartmental model. Asymptotic results are established, and formulas giving the mean number of reinfections at the endemic equilibrium are provided. Next, we investigate a two-stage reinfection model intended for the modelling of diseases for which the subsequent reinfections behave differently from the primary infection. We describe thoroughly the steady states of the model, which may contain up to three endemic equilibriums, and study the disease persistence. Employing Li and Muldowney theory, we prove asymptotic convergence of every trajectory, in a particular case that may exhibit multiple endemic equilibriums. Using semigroup theory, we then establish the well-posedness for a class of models structured both in age and number of reinfections. The latter are constituted by an infinite number of partial differential equations. This allows the computation of several interesting quantities at endemic equilibrium, such as the average age in each compartment or the mean number of reinfections at each age. Finally, we investigate some issues of Control theory, more precisely whether the use of additional reinfection data may improve parameter and state estimation. To this end, we study identifiability and observability of a SIS model, based on the measure of the number of infected and of primary infected, and propose an asymptotic observer and an adaptive observer respectively for the state, and for the joint state and parameter, estimation.

Keywords: compartmental models in epidemiology; reinfection; age structure; multistability; observation; identification; asymptotic observer; adaptative observer

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Résumé

Le sujet de ce travail est la modélisation mathématique des maladies infectieuses avec réinfections et l'analyse des modèles correspondants. Nous introduisons d'abord une classe générale de modèles compartimentaux comptant les réinfections, constitués d'un nombre infini d'équations différentielles ordinaires, et nous étudions leur caractère bien posé. La classe proposée permet également de modéliser une transmission hétérogène, dont les caractéristiques dépendent du nombre de réinfections antérieures. Dans le cas homogène (où les réinfections passées n'affectent pas la transmission de la maladie), le comportement global obéit à un modèle compartimental usuel. Des résultats asymptotiques sont établis et des formules donnant le nombre moyen de réinfections à l'équilibre endémique sont données. Nous étudions ensuite un modèle de réinfection à deux étages destiné à la modélisation de maladies pour lesquelles les réinfections ultérieures se comportent différemment de l'infection primaire. Nous décrivons en détail les équilibres du modèle, qui peut contenir jusqu'à trois équilibres endémiques, et étudions la persistance de la maladie. Grâce à la théorie de Li et Muldowney, nous prouvons la convergence asymptotique de chaque trajectoire dans un cas particulier qui peut présenter plusieurs équilibres endémiques. En utilisant la théorie des semigroupes, nous établissons ensuite le caractère bien-posé d'une classe de modèles structurés à la fois en âge et en nombre de réinfections. Ces derniers sont constitués d'un nombre infini d'équations aux dérivées partielles. Ceci permet de calculer plusieurs quantités intéressantes à l'équilibre endémique, telles que l'âge moyen dans chaque compartiment ou le nombre moyen de réinfections à chaque âge. Nous examinons enfin des questions de théorie du contrôle, plus précisément si l'utilisation de données supplémentaires sur les réinfections peut améliorer l'estimation des paramètres et de l'état. À cette fin, nous étudions l'identifiabilité et l'observabilité d'un modèle SIS, basées sur la mesure du nombre d'infectés et de primo-infectés, et proposons un observateur asymptotique et un observateur adaptatif respectivement pour l'estimation de l'état, et pour l'estimation conjointe de l'état et des paramètres.

Mots clés : modèles compartimentaux en épidémiologie ; réinfection ; structuration en âge ; multistabilité ; observation ; identification ; observateur asymptotique ; observateur adaptatif

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И мы знаем, что так было всегда,
Что Судьбою больше любим,
Кто живет по законам другим
И кому умирать молодым.
Он не помнит слово «да» и слово «нет»,
Он не помнит ни чинов, ни имен.
И способен дотянуться до звезд,
Не считая, что это сон,
И упасть, опаленным Зведой по имени Солнце.

Виктор Р. Цой

Introduction

Mathematical epidemiology and compartmental models

For those of us living in Western society that has long since embraced hygienics ideas and standards [103], we tend to forget how communicable diseases affect human societies, and to what extent human life is built around this natural reality. The recent COVID-19 pandemic has served as a reminder of the potential consequences of these diseases. Throughout history, infectious diseases have indeed constantly influenced social relationships and humans societies, and there are countless examples of historical events partially shaped by them. To cite a few, the Antonine plague, which may have been caused by increasing urban concentration and trade due to the economic growth during the Pax Romana, contributed to the end of the golden age of the Roman Empire [36] in the second century. The Justinian plague in the sixth century, which is credited as the earliest pandemic due to the pathogen *yersinia pestis*, halted the expansion of the Byzantine Empire and significantly weakened the Sassanian army, ultimately leading to the rise of Islam [104]. Another well-known example of plague caused by *yersinia pestis* is the Black Death in the fourteenth century, whose spread followed the expansion of the Mongol Empire (see Figure 1). The plague killed roughly one third to one half of the total European population between 1347 and 1351. The devastating effects of the Black Death had a significant impact on the collective consciousness of the late Middle Ages and the Renaissance (see Figure 2). Moreover, the resulting massive reduction of the workforce and the abundance of land changed the power dynamics between landowners and the peasantry, which in response marked the decline of feudalism in Western Europe and a renewal of serfdom in the Eastern counterpart [98].

Even in the present day, infectious diseases remain a serious challenge for humanity, with millions of people dying of measles, malaria, tuberculosis, diarrhea, and other diseases that may be treated with modern medicine [116]. Several of these diseases are indeed endemic in many parts of the world [93], although they are considered eradicated or declining in developed countries. This unequal burden between global north and south countries with regard to infectious diseases accentuates the inequality and further hinders development efforts.

Despite the omnipresence of communicable diseases in human society, the progress in the biological understanding of their transmission is relatively recent. For a long time, people thought that disease came from invisible things called “bad air” or “miasma”. It wasn’t until the 19th century that the germ theory of disease was developed and accepted by the scientific community, highlighting the role of microorganisms as the cause of infectious disease [95]. Nowadays, the transmission mechanism of most of infectious diseases is well-known, and can be classified into three main classes, according to their mode of transmission. First, it may be transmitted directly by viral agents, which is the case for influenza, measles, HIV/AIDS or SARS, or transmitted by bacteria, for example in the case of tuberculosis, cholera, pneumonia, or meningitis. Finally, the diseases may also be transmitted indirectly through vectors (usually insects), in the case of dengue, chikungunya or malaria.

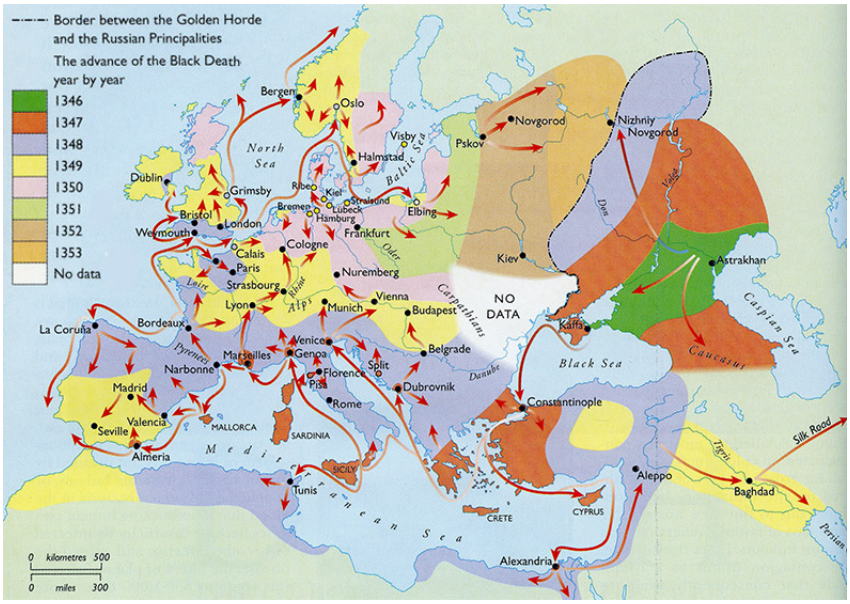


Figure 1: Spread of the Black Death in Europe. Source : *The Black Death, 1346-1353: The Complete History*, O.J. Benedictow, Boydell Press, 2004, [13].

Figure 2: *The Triumph of Death*, Pieter Bruegel the Elder, c. 1562.

The idea of applying mathematics to advance the understanding of disease transmission can be traced back to the works of Daniel Bernoulli in 1760 on the benefits of inoculation against smallpox [14]. Compartmental models, which are now massively employed in mathematical epidemiology, have been for their part introduced at the beginning of the 20th century through

the works of W.O. Kermack and A.G. McKendrick [68].

Generally speaking, a compartmental model simulates the interaction between individuals or objects distributed across multiple classes, called *compartments*. Each compartment represents then a group of entities (such as population, molecules, health status, capital, individuals...) that are assumed to be homogeneous with respect to the processes being studied. Due to their versatility, compartmental models are often used in many fields, such as pharmacology, ecology, systems biology or economics. In mathematical epidemiology, they are employed for the modelling of host and vector population, classified according to their health status with regard to the disease. For instance, the well-known SIR model introduced by Kermack & McKendrick [68] contains three compartments, which are

1. The compartment S, which includes individuals who are *susceptible* to the disease.

2. The compartment I, including *infected* individuals, able to transmit the disease.

3. The compartment R contains the populations *removed* from the spreading process, either due to immunity acquired, quarantine, or death.

Schematically, the SIR model is illustrated by the flowchart in Figure 3.

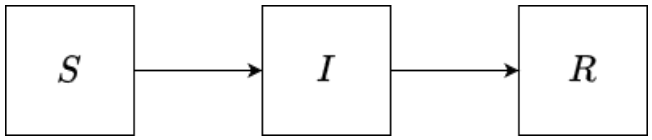


Figure 3: Flowchart of the SIR model

When the demographic effects may be considered much slower than the epidemic dynamics, these effects may be neglected and the SIR model is written as follows:

$$\begin{aligned} \dot{S} &= -\beta SI, \\ \dot{I} &= \beta SI - \gamma I, \\ \dot{R} &= \gamma I. \end{aligned} \tag{1}$$

The model takes then the following assumption.

1. The total population $N := S + I + R$ is constant.

2. βI is called the *force of infection* of the model and represents the number of contacts with infectious hosts that susceptible makes per unit of time. As it is proportional to the number of infected in the population, the transmission is said to respect the law of *pseudo mass action*. If instead the force of infection is modelled by the density dependent term $\beta \frac{I}{N}$, the transmission is said to obey the law of *mass action*. In the case where the total population N is constant, it is easy to see with a normalization that both laws are equivalent.

3. γ represents the recruitment rate of recovered from the infected, which is equivalent to say that γ^{-1} represents the average infectious period of the disease.

Although the SIR model is surprisingly simple, it captures already several major features of an epidemic, which makes it a powerful tool for the modelling of disease transmission. Notably,

the solution of the system (1) gives a bell-shaped curve for the infected, which is typical in the outbreak of a disease (see Figure 4). On the other hand, let us rewrite the equation of $\dot{I}$ as

$$\dot{I} = (\beta S - \gamma)I.$$

It is thus clear that, if the initial condition is such that $S(0) < \frac{\beta}{\gamma}$, I will remain non-increasing thus excluding the possibility of an outbreak. This fact is known in mathematical epidemiology as the *threshold phenomenon*, that prevents the possibility of outbreaks if the population does not contain enough susceptible. Alternatively, if we consider the population as *almost* entirely susceptible initially and approximate $S \approx 1$, the threshold condition became

$$1 < \frac{\beta}{\gamma},$$

which gives then a condition on the intrinsic parameters of the disease for its spreading in a disease-free population. The quantity

$$\mathcal{R}_0 := \frac{\beta}{\gamma},$$

called the *basic reproduction number* plays a major role. It is defined as "the average number of secondary cases arising from an average primary case in an entirely susceptible population" [33].

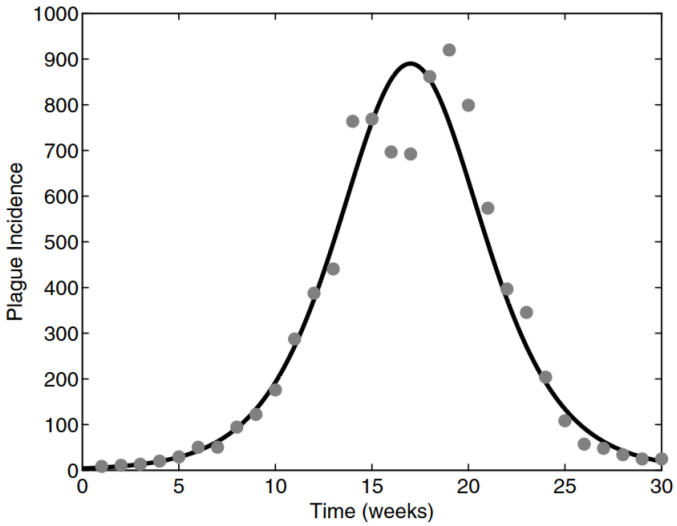


Figure 4: Weakly deaths from plague in Bombay from December 17, 1905 to July 21, 1906 (filled circles) compared to the approximate solution from Kermack and McKendrick's model (solid line). Source : *Modeling Infectious Diseases in humans and animals*, M. J. Keeling and P. Rohani, 2008 [67].

Reinfection

Since the SIR model of Kermack and McKendrick, the field of mathematical epidemiology has evolved considerably, incorporating increasingly complex factors such as quarantine, vaccine,

spatial, hosts heterogeneity, variants, spatial dynamics or stochastic effects [83]. One of these factors is the phenomenon of *reinfection*.

Biologically speaking, for various diseases, reinfection refers to the ability of the host to be infected multiple times by the pathogen during its lifetime. This is either due to the lack of immune memory against specific diseases, or because the immunity granted is incomplete or only lasts a certain amount of time, which concerns numerous viral diseases. In fact, it is more reasonable to expect the immunity response to be generally imperfect, and in most situations the presence of reinfections is the norm rather than the exception. However, this biological reality is not always reflected in the field of mathematical epidemiology, which is often concerned only with mathematical models assuming lifelong immunity [51]. Indeed, the assumption of permanent immunity of this class of model corresponds only to a limited set of diseases, for instance measles and rubella [3].

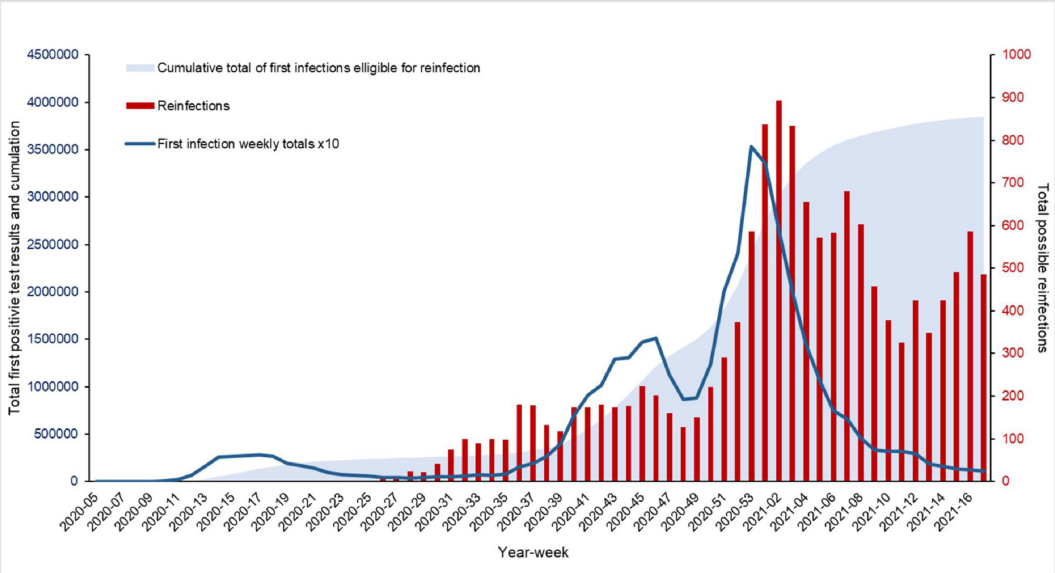


Figure 5: Weekly cumulative total of SARS-CoV-2 infection eligible for reinfection (within 90 days interval), weekly number of possible SARS-CoV-2 reinfection and primary infection (times 10) in all ages, England, January 2020 April 2021. Source: *Disease severity during SARS-COV-2 reinfection: a nationwide study*, A.A. Mensah et al., 2022, [89].

On the contrary, there are countless examples of diseases including reinfection. To cite a few, tuberculosis [50] is well-known to be able to reoccur in the infected's later life, due to either reactivation of the same strain of *mycobacterium tuberculosis*, i.e., relapse, or reinfection with a new strain. Another sadly famous example is that of COVID-19, which is caused by the strain of coronavirus *SARS-CoV-2*. The reinfection in this case may be jointly explained by the protection offered by the immune system, which is only temporary, and the appearance of new variants of the virus [25]. In addition, a major class of diseases are those offering no protection to subsequent infection. This is for example the case for several sexually transmitted diseases such as syphilis or chlamydia, or for the influenza.

On the subject of modeling, the simplest compartmental model for disease with reinfections

is the SIS model, which is given by the equations:

$$\begin{aligned} \dot{S} &= -\beta SI + \gamma I, \\ \dot{I} &= \beta SI - \gamma I. \end{aligned} \tag{2}$$

This model assumes that the infected became immediately susceptible, thus does not include an

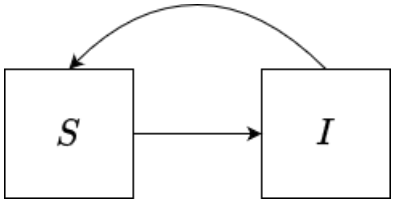


Figure 6: Flowchart of the SIS model

immune compartment. It is therefore adapted to the modelling of sexually transmitted diseases.

Another reinfection model that we are centrally interested in this thesis is the SEIRS model, which can be defined by the following system of differential equations:

$$\begin{aligned} \dot{S} &= bN - \beta SI + \omega R - \mu S, \\ \dot{E} &= \beta SI - (\sigma + \mu)E, \\ \dot{I} &= \sigma E - (\gamma + \mu + \nu)I, \\ \dot{R} &= \gamma I - (\omega + \mu)R. \end{aligned} \tag{3}$$

In this case, the system contains additional demographic coefficients b and μ, which are respectively the birth and death rates. The presence of demographic terms is motivated by the fact that, for reinfection models, we are often interested in the behavior of the disease transmission over a longer timeframe. Moreover, this model includes a period of incubation σ^{-1} for the individuals *exposed* to the disease to become infectious, which is represented by the addition of a compartment E. Furthermore, the recovered may lose their immunity and be recruited again as susceptible with rate ω, enabling reinfection. The SEIRS reinfection model is particularly fitted for numerous infectious diseases such as influenza or COVID-19 [15], and will be further investigated in Chapter 1.

In the last decades, the need for a more detailed understanding of the above-mentioned diseases has driven the interest in studying numerous compartmental reinfection models [1, 8, 51]. However, several points of interest remain to be explored. First, most of reinfected models presented in the literature do not include information about the number of times the hosts have been infected, which may be an interesting data to evaluate. Another closely related issue is the reinfection related heterogeneity in the disease, and most of existing models do not account for the fact that the behavior of the disease may change depending on how many times the hosts have been reinfected. Finally, a third underexplored topic is the one of identification/observation of the epidemiological models, i.e., the state/parameter estimation and whether these data are even retrievable from measurements. This is crucial, for example, for the monitoring and control of epidemics. From the perspective of control theory, this issue remains insufficiently considered [56]. In this regard, one of our goals in this thesis is to investigate whether the use of reinfection data (typically the breakdown of the prevalence or the incidence according to primary infections and reinfections, see Figure 5) can enrich the knowledge and improve the control of epidemics.

Outline of the thesis and list of publications

Considering the above observations, in Chapter 1 we investigate an infinite reinfection model that is endowed with a counting of the number of times the hosts have been infected, which can be infinitely many. Moreover, the parameters of the model may vary according to the number of reinfections, which accounts for reinfection induced heterogeneity. For the corresponding infinite system of ordinary differential equations (ODEs), the well-posedness is proved in a Banach space of sequences that describe the number of hosts according to their epidemiological status (e.g. susceptible, exposed, infected, recovered) and the number of reinfections they already went through. Moreover, the case where parameters are constant with regard to reinfection number, i.e., no reinfection-induced heterogeneity, is investigated in detail. Considering the basic reproduction number $\mathcal{R}_0$ of the model, the global stability of the disease-free and endemic equilibriums is described for $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively. In addition, thanks to the description of the endemic equilibrium, we can also retrieve some valuable information, such as the average number of reinfections in the population at the endemic state of the model. Finally, we explore the model with geometric progression of the coefficients according to the number of reinfections, and derive existence and uniqueness result on the endemic equilibrium of the system.

In Chapter 2, we turn our attention to a class of reinfection models that distinguish primary infection from the subsequent reinfections. This class of model takes the assumption that the primary infection behaves in a significantly different way from the remaining reinfections, which is a reasonable hypothesis. Concurrently with our research, this class of models has been employed for the modelling of COVID-19 epidemics [18, 63]. Description of the equilibrium states of the corresponding system is delivered in detail, and in particular, the system is shown to exhibit simultaneously up to 2 and 3 endemic equilibriums in the case $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ respectively. The result along with numerical simulations for the case $\mathcal{R}_0 > 1$ show the existence of a reinfection threshold in certain configurations. The multistability of the system makes it difficult to analyze the global asymptotic behavior in a classical Lyapunov theory framework. We apply instead the geometrical approach of Li and Muldowney [75], better known in the control community as the theory of *k-contraction* [117]. With this approach, we demonstrate the asymptotic convergence toward equilibrium for a specific case of the general model, where the heterogeneity is made up of partial immunity after the first infection that allows the emergence of multistability.

Continuing on the topic of reinfections counting, in Chapter 3 we add an age-structure on top of the infinite system in Chapter 1. Indeed, this is motivated by the fact that epidemic dynamics are highly dependent on the age structure of the host population, which influences for instance the transmission or the severity of the disease. Similarly to Chapter 1, we prove the well-posed of the corresponding infinite system of partial differential equations (PDEs) employing perturbed semigroup theory [31, 96, 102] in a Banach space setting. Then, the attention is again given to the simplest setting of the model, where all parameters are assumed to be non-varying, now with regard both to the age and reinfection number. This allows us to once again derive global stability properties for the disease-free and endemic steady states when $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$ respectively. In addition, the description of the endemic state enables the computation of supplementary values of interest that characterizes the endemic equilibrium, such as the average age in each compartment of the model or the mean number of reinfections according to the age of the hosts.

Finally, in Chapter 4, we turn to some control theory issues. More precisely, we explore observation and identification issues for a compartmental SIS model with reinfection. From the state-space representation of the model, we analyze the identifiability and observability of the model with a fixed but unknown portion of the infected as measured output. We find out that the model is identifiable/observable if in addition the same portion of primary infections is also

measured. This motivates in the sequel the design of an observer for the SIS model measuring both infections and primary infections. An asymptotic Luenberger-like observer is proposed, after an appropriate change of coordinates that transforms the system into a Persidskii form [88, 97]. The stability of the observer is derived using copositive Lyapunov functions or of Lur'e form and involves the testing of linear matrix inequalities (LMIs). Last, an adaptive observer is designed for the case where the parameters of the model are also unknown. The strategy to its design is similar to the approach of *Immersion & Invariance* [9] and its stability is analyzed under the framework of Input-to-Output stability.

Several findings in the thesis are the subject of scientific publications, either already published or in progress.

1. The article [40], based on the contents of Chapter 1 and Chapter 3, is currently in preparation.

2. The article [42], based on the contents of Chapter 2, has been submitted for peer-review.

3. The contents of Chapter 4 have been the subject of several conference papers [41, 43, 44]. An additional article [45] has been submitted for peer-review.

Chapter 1

An infinite SEIRS model tracking the number of reinfections

1.1 Introduction

Compartmental models have witnessed a rich development in mathematical epidemiology since their introduction by Kermack and McKendrik in 1927 [68]. In the last decades, driven by the needs to understand several diseases such as tuberculosis, measles, pertussis or recently Covid-19, increasing attention has been paid to the modelling and analysis of compartmental models with reinfection, which generally consist of variants of the classical SIS, SIRS or SEIRS models. To cite a few works, reinfection by multiple strains has been studied in [1, 4], deriving among others threshold conditions for the invasion by a new strain or the disease persistence. In [8], the authors analyze the relation between vaccination and reinfection using a SVIRS model, exhibiting the possibility of bistability which complicates the prediction of the epidemics and the efficiency of vaccination campaign. Similarly, driven by the need to understand the effectiveness of vaccination, Gomes et al. [51] delivered a comprehensive analysis of SIRS models with vaccination and imperfect immunity.

Completing the work in [51], a threshold condition for endemicity of reinfection models is derived by Katriel [66], alongside attack rate and mean number of reinfections for models without demography rates. At the end of the paper, the author made several modifications to the SIRS model to obtain a remarkable model tracking the number of reinfections, which was then briefly analyzed. Continuing in the same direction, a general SEIRS model counting reinfection is investigated in this chapter. The novelty and at the same time the difficulty of this kind of system lie in the infinite number of differential equations describing its dynamics. In this sense, they are similar to Becker-Döring model [10, 34, 71], which describes, for example, the coagulation and fragmentation of droplets in a condensing vapor.

The chapter is organized as follows. In the Section 1.2, we briefly present the classical SEIRS model for infectious diseases. Subsequently, we introduce the SEIRS model that tracks the number of reinfections, which is described by a system of infinite number of ordinary differential equations. Then, in Section 1.3, the general model is proved to be well-posed, under quite general conditions. The well-posedness constitutes the main and most important result of the general model. Afterward, the attention shifts to the study of two particular cases. In Section 1.4, we start looking into the most simple situation, where the coefficients are invariant regarding the number of reinfections. In this case, the evolution of each class of hosts in its entirety,

regardless of the number of reinfections, is determined by the classical SEIRS model of finite dimension. Therefore, the coupling of the classical SEIRS system and the infinite-dimensional system tracking reinfections present a triangular structure already noticed in [66], which renders the analysis much easier and making it possible to obtain significant results such as the mean number of reinfections at the asymptotic limit. Finally, in Section 1.5, the case where the coefficients progress geometrically at each stage of reinfections is considered, and a threshold condition is derived for the appearance of endemic equilibrium and a sufficient condition for its uniqueness.

1.2 General SEIRS model tracking the number of reinfections

Usually, reinfection processes are represented in compartmental models by a transfer from the recovered compartments to either the susceptible class, as in the SIS, SIRS, or SEIRS model [67], or the infected class as in the SIRI model [94]. For example, consider a common SEIRS (Susceptible-Exposed-Infected-Recovered-Susceptible) model for infectious diseases with demography and disease induced mortality, which has been employed for modelling infectious diseases with an incubation period, such as SARS-CoV-2 or Ebola [15, 83].

$$\begin{aligned} \dot{S} &= bN - \beta I\frac{S}{N} + \omega R - \mu S, \\ \dot{E} &= \beta I\frac{S}{N} - (\sigma+\mu)E, \\ \dot{I} &= \sigma E - (\gamma+\mu+\nu)I, \\ \dot{R} &= \gamma I - (\omega+\mu)R, \end{aligned} \tag{1.1}$$

where the quantity N is the total number of hosts in the system

$$N := S + E + I + R,$$

As illustrated in Figure 1.1, the model assumes that the healthy subject in the susceptible compartment S goes through an incubation phase in the exposed compartment E after the contact with the disease and before becoming infectious and being transferred to the infected class I. After recovery, the subject moves to the compartment R and acquires a non-permanent immunity, whose loss may result in further infections. The infection is characterized by the term

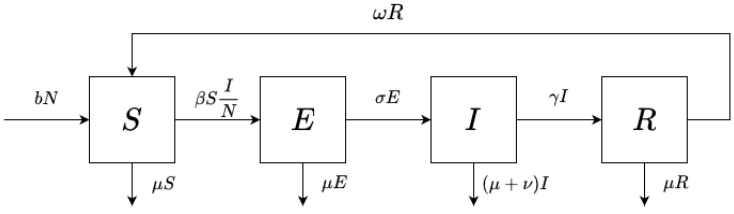


Figure 1.1: Flowchart of system (1.1)

$\beta\frac{S}{N}I$, which follows the principle of mass action law [57]. Moreover, the coefficients b and μ are respectively the natural birth and death rate, ν the disease induced death rate, β represents the

infectious contact rate, ω the loss of immunity, and σ^{-1} and γ^{-1} are respectively latency and infectious period.

It should be noted that classical reinfection models enable the hosts to be repeatedly infected. For the SEIRS model introduced above, this is represented by the loop in Figure 1.1, and the hosts are here assumed to repeat the same infection process. However, the disease dynamics can also be considered with regard to the number of reinfections of the hosts. Indeed, it may be interesting to investigate how the first, second, third, and so on infections behave, with possible heterogeneity according to the number of reinfections. To study these questions, we introduce a general compartmental reinfection model, with the ability of considering the infection history of the hosts.

To make our approach clearer, let us consider a simpler version of the SEIRS model [1]

$$\begin{aligned} \dot{S} &= \mu N - \beta I \frac{S}{N} + \omega R - \mu S, \\ \dot{E} &= \beta I \frac{S}{N} - (\sigma + \mu) E, \\ \dot{I} &= \sigma E - (\gamma + \mu) I, \\ \dot{R} &= \gamma I - (\omega + \mu) R. \end{aligned} \tag{1.2}$$

which is obtained by dropping the disease-induced mortality ν in (1.1) and fixing $b = \mu$ in order to have a constant population (as a matter of fact, one checks that for any trajectory of (1.2), one has $\dot{N} = \dot{S} + \dot{E} + \dot{I} + \dot{R} = \mu(N - S - E - I - R) \equiv 0$). After normalizing, the system (1.2) is equivalent to

$$\begin{aligned} \dot{S} &= \mu - \beta SI + \omega R - \mu S, \\ \dot{E} &= \beta SI - (\sigma + \mu) E, \\ \dot{I} &= \sigma E - (\gamma + \mu) I, \\ \dot{R} &= \gamma I - (\omega + \mu) R. \end{aligned} \tag{1.3}$$

As we are interested in a system modelling the reinfection structure of (1.3), a natural way to proceed is to divide the compartments according to the number of reinfections. This results in the following SEIRS ODE model with infinite equations tracking the number of reinfections.

$$\begin{aligned} \dot{S}_1 &= \mu - \beta S_1 \textstyle\sum_{j \geqslant 1} I_j - \mu S_1, \\ \dot{S}_i &= \omega R_{i-1} - \beta S_i \textstyle\sum_{j \geqslant 1} I_j - \mu S_i, && i > 1, \\ \dot{E}_i &= \beta S_i \textstyle\sum_{j \geqslant 1} I_j - (\sigma + \mu) E_i, && i \geqslant 1, \\ \dot{I}_i &= \sigma E_i - (\gamma + \mu) I_i, && i \geqslant 1, \\ \dot{R}_i &= \gamma I_i - (\omega + \mu) R_i, && i \geqslant 1, \end{aligned} \tag{1.4}$$

It is quite relevant to generalize the model (1.4) with the introduction of coefficients depending on the number of reinfections. This leads finally to the model:

$$\begin{aligned} \dot{S}_i &= \omega_{i-1} R_{i-1} - S_i \textstyle\sum_{j \geqslant 1} \beta_{i,j} I_j - \mu S_i, && i \geqslant 1, \\ \dot{E}_i &= S_i \textstyle\sum_{j \geqslant 1} \beta_{i,j} I_j - (\sigma_i + \mu) E_i, && i \geqslant 1, \\ \dot{I}_i &= \sigma_i E_i - (\gamma_i + \mu) I_i, && i \geqslant 1, \\ \dot{R}_i &= \gamma_i I_i - (\omega_i + \mu) R_i, && i \geqslant 1. \end{aligned} \tag{1.5}$$

[1] We come back to consider the model (1.1) in Section 1.4.

Here, R_0 is defined by $R_0 \equiv \frac{\mu}{\omega_0}$ to simplify the writing of the equations. Similarly to the system (1.3), the coefficient μ is the natural birth and death rate and $\beta_{i,j}$, $i, j \geqslant 1$, are infectious contact rates, depending both on the number of reinfections of the infectious individual i and the infected j. On the other hand, the constants ω_i represent loss of immunity, σ_i^{-1} and γ_i^{-1} are respectively incubation and infectious periods, which all depend on the number of reinfection i in the general model. Finally, notice that the normalized system (1.5) admits a corresponding denormalized system, which is

$$\begin{aligned}
\dot{S}_i &= \omega_{i-1}R_{i-1} - \frac{S_i}{N}\textstyle\sum_{j\geqslant 1}\beta_{i,j}I_j - \mu S_i, \quad i \geqslant 1,\\
\dot{E}_i &= \frac{S_i}{N}\textstyle\sum_{j\geqslant 1}\beta_{i,j}I_j - (\sigma_i+\mu)E_i, \qquad i \geqslant 1,\\
\dot{I}_i &= \sigma_i E_i - (\gamma_i+\mu)I_i, \qquad i \geqslant 1,\\
\dot{R}_i &= \gamma_i I_i - (\omega_i+\mu)R_i, \qquad i \geqslant 1,
\end{aligned} \tag{1.6}$$

where $R_0 \equiv \frac{\mu N}{\omega_0}$, and

$$N(t) := \sum_{i\geqslant 1}(S_i(t) + E_i(t) + I_i(t) + R_i(t)),$$

is the total number of hosts in the system (1.6). The flowchart of (1.6) is shown in Figure 1.2 as an illustration.

For the sake of simplicity, we ignored the supplementary death rate induced by the disease ν in the general model (1.5)-(1.6). However, in Section 1.4, the infinite model tracking reinfections of the SEIRS system (1.2) with disease-induced mortality and varying total population will be investigated. Before that, we now verify in the following section the well-posedness of the infinite systems (1.5) and (1.6) that we have obtained.

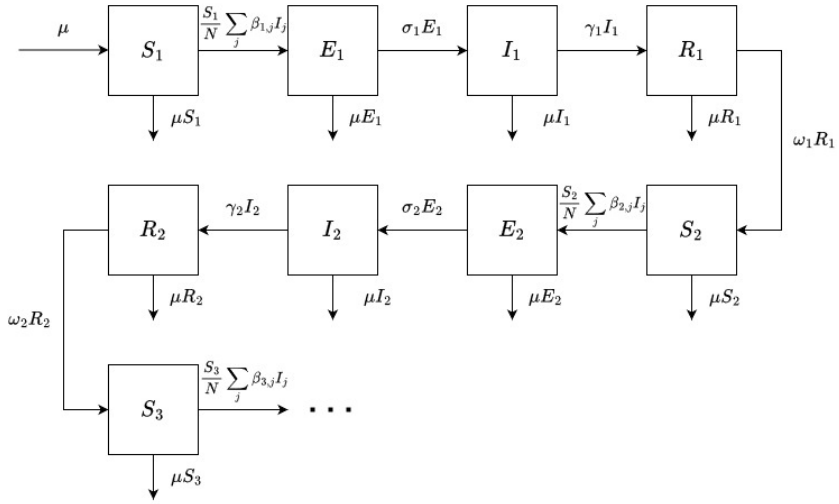


Figure 1.2: Flowchart of system (1.6)

1.3 Well-posedness of the general system

As the systems of ordinary differential equations (1.5) and (1.6) have an infinite number of equations, the question of their well-posedness is non-trivial and will be investigated in the present section. For the analysis, we consider the system in a Banach space setting. For this, we define for all $n \in \mathbb{N}$ the Banach spaces

$$X^n := \underbrace{\ell^1 \times \cdots \times \ell^1}_{n \text{ times}}, \qquad X^n_+ := \underbrace{\ell^1_+ \times \cdots \times \ell^1_+}_{n \text{ times}}, \qquad X^n_{++} := \underbrace{\ell^1_{++} \times \cdots \times \ell^1_{++}}_{n \text{ times}}, \tag{1.7}$$

where, by definition, ℓ^1 is the Banach space of summable sequences of, $\ell^1_+ \subset \ell^1$ the subspace of sequences with nonnegative values, and $\ell^1_{++} \subset \ell^1_+$ for sequences with positive values. The space X^n is endowed with the induced norm $\|\cdot\|_{X^n}$ such that for any $x := (x_1, x_2, \ldots, x_n) \in X^n$, $\|x\|_{X^n} = \sum_{i=1}^n \|x_i\|_{\ell^1}$.

Let us begin the analysis with a definition of the solution of system (1.5).

Definition 1.3.1. *Lets $0 < T \leqslant \infty$, and $x_0 := (S_{i,0}, E_{i,0}, I_{i,0}, R_{i,0})_{i\geqslant 0} \in X^4_+$. We call solution of (1.5) on $[0,T)$ any function $x : [0,T) \to X^4_+, t \mapsto x(t) := (S_i(t), E_i(t), I_i(t), R_i(t))_{i\geqslant 1}$, such that:*

1. *Every component $(S_i, E_i, I_i, R_i) : [0,T) \to \mathbb{R}^4_+$, $i \geqslant 1$ is continuous and $\sup_{t\in[0,T)} \|x(t)\|_{X^4} < \infty$.*

2. *For all $t \in [0,T)$,*

$$\begin{aligned}
S_i(t) &= S_{i,0} + \int_0^t \Big(\omega_{i-1}R_{i-1}(s) - S_i(s)\textstyle\sum_j \beta_{i,j}I_j(s) - \mu S_i(s)\Big)\, ds, \qquad i \geqslant 1,\\
E_i(t) &= E_{i,0} + \int_0^t \Big(S_i(s)\textstyle\sum_j \beta_{i,j}I_j(s) - (\sigma_i+\mu)E_i(s)\Big)\, ds, \qquad i \geqslant 1,\\
I_i(t) &= I_{i,0} + \int_0^t (\sigma_i E_i(s) - (\gamma_i+\mu)I_i(s))\, ds, \qquad i \geqslant 1,\\
R_i(t) &= R_{i,0} + \int_0^t (\gamma_i I_i(s) - (\omega_i+\mu)R_i(s))\, ds, \qquad i \geqslant 1.
\end{aligned} \tag{1.8}$$

Under this definition, the next result proves indeed that the system (1.5) is well-posed. In addition, it suggests that the quantities S, E, I, R do not admit, in general, a closed form expression governing their dynamics.

Theorem 1.3.1 (Existence and uniqueness)**.** *Suppose that $\mu > 0$, and $\sigma_i, \gamma_i, \omega_i$ and $\beta_{i,j}$ are positive and uniformly bounded with regard to $i, j \geqslant 1$. Then for any initial condition $x_0 \in X^4_+$ and any $T \in (0, +\infty)$, there is a unique solution $x = (S_i, E_i, I_i, R_i)_{i\geqslant 1}$ of (1.5) on $[0,T)$. Moreover, let us denote*

$$S := \sum_{i\geqslant 1} S_i, \quad E := \sum_{i\geqslant 1} E_i, \quad I := \sum_{i\geqslant 1} I_i, \quad R := \sum_{i\geqslant 1} R_i, \quad N = S + E + I + R,$$

then every component of the solution x and additionally S, E, I, R, N are continuously differen-

tiable, and the following differential equations are satisfied.

$$\dot{S} = \mu + \sum_{i\geqslant 1}\omega_i R_i - \sum_{i,j\geqslant 1}\beta_{i,j}S_iI_j - \mu S, \quad \dot{E} = \sum_{i,j\geqslant 1}\beta_{i,j}S_iI_j - \sum_{i\geqslant 1}\sigma_i E_i - \mu E,$$
$$\dot{I} = \sum_{i\geqslant 1}\sigma_i E_i - \sum_{i\geqslant 1}\gamma_i I_i - \mu I, \quad \dot{R} = \sum_{i\geqslant 1}\gamma_i I_i - \sum_{i\geqslant 1}\omega_i R_i - \mu R,$$
$$\dot{N} = \mu(1-N).$$

Proof of Theorem 1.3.1.

Existence. Let T be a positive real number, $n \in \mathbb{N}^*$ and denote $y^n := (S_i^n, E_i^n, I_i^n, R_i^n)_{1\leqslant i\leqslant n}$, the unique continuously differentiable solution on $[0,T)$ of the ODE:

$$\begin{aligned}
\dot{S}_i^n &= \omega_{i-1}R_{i-1}^n - S_i^n\textstyle\sum_{j=1}^n \beta_{i,j}I_j^n - \mu S_i^n, && S_i^n(0) = S_{i,0}, \quad 1\leqslant i\leqslant n,\\
\dot{E}_i^n &= S_i^n\textstyle\sum_{j=1}^n \beta_{i,j}I_j^n - (\sigma_i+\mu)E_i^n, && E_i^n(0) = E_{i,0}, \quad 1\leqslant i\leqslant n,\\
\dot{I}_i^n &= \sigma_i E_i^n - (\gamma_i+\mu)I_i^n, && I_i^n(0) = I_{i,0}, \quad 1\leqslant i\leqslant n,\\
\dot{R}_i^n &= \gamma_i I_i^n - (\omega_i+\mu)R_i^n, && R_i^n(0) = R_{i,0}, \quad 1\leqslant i\leqslant n-1,\\
\dot{R}_n^n &= \gamma_n I_n^n - \mu R_n^n, && R_n^n(0) = R_{n,0}.
\end{aligned} \tag{1.9}$$

Similarly to (1.5), $\omega_0 R_0^n \equiv \mu$. The $4n$ dimensional differential system (1.9) may be considered as a truncation of (1.5) obtained by setting $\omega_n = 0$ hence allowing at most n reinfections.

Remark that the boundary of $\mathbb{R}_+^{4n}$ is characterized by elements $y \in \mathbb{R}_+^{4n}$ such that at least one component of y is zero. Then, from the equations (1.9), at the boundary of $\mathbb{R}_+^{4n}$ the derivatives along trajectories of (1.9) of the zero-valued components are always nonnegative, hence $\mathbb{R}_+^{4n}$ is a positively invariant set for (1.9) and $y^n(t) \in \mathbb{R}_+^{4n}$ for all $t \in [0,T)$ if $y^n(0) \in \mathbb{R}_+^{4n}$.

Let us define

$$S^n := \sum_{i=1}^n S_i^n, \quad E^n := \sum_{i=1}^n E_i^n, \quad I^n := \sum_{i=1}^n I_i^n, \quad R^n := \sum_{i=1}^n R_i^n,$$
$$N^n := S^n + E^n + I^n + R^n.$$

Then, we sum up the $4n$ differential equations to obtain

$$\dot{N}^n = \sum_{1\leqslant 1\leqslant n} \dot{S}_i^n(t) + \dot{E}_i^n(t) + \dot{I}_i^n(t) + \dot{R}_i^n(t) = \mu(1-N^n),$$

As $N^n(0) \leqslant N(0) = \|x_0\|_{X^4}$, from the last equation it is easy to see that $N^n(t) \leqslant \max\{\|x_0\|_{X^4}, 1\}$. As all components of $y^n(t)$ are nonnegative, one deduces that $S_i^n, E_i^n, I_i^n, R_i^n$, S^n, E^n, I^n, R^n are upper bounded as well by $c_1 := \max\{\|x_0\|_{X^4}, 1\}$. We complete the solution $y^n \in \mathbb{R}_+^{4n}$ with zero to form the function $x^n = (S_i^n, E_i^n, I_i^n, R_i^n)_{i\geqslant 1}$ taking values in X_+^4 with $S_i^n, E_i^n, I_i^n, R_i^n \equiv 0$ for $i \geqslant n+1$. Then, by the fact that the coefficients $(\omega_i)_{i\geqslant 1}, (\sigma_i)_{i\geqslant 1}, (\gamma_i)_{i\geqslant 1}, (\beta_{i,j})_{i,j\geqslant 1}$ are uniformly bounded, there exists $c_2 \in \mathbb{R}_+$ upper bounding the coefficients, therefore

$$|\dot{A}_i^n(t)| \leqslant c_2c_1(1+c_1), \quad n\geqslant 0, \quad i = 1,\dots,n, \quad t\geqslant 0, \quad A \in \{S,E,I,R\}.$$

As the derivatives are uniformly bounded, thus $S_i^n, E_i^n, I_i^n, R_i^n$ are equicontinuous on $[0,T)$. Applying the Arzelà-Ascoli theorem [21], we extract a subsequence of index n_k such that for all $A \in \{S,E,I,R\}$, the sequence of function $A_i^{n_k}$ converges uniformly to a continuous function

$A_i : [0,\infty) \to \mathbb{R}^+$ on every compact of $[0,\infty)$ when $k \to \infty$. Given that $N^n(t) \leqslant c_1$, then for all $A \in \{S, E, I, R\}$ and n_k, A^{n_k} is upper bounded by c_1, which implies, taking $A = I$ and passing to the limit, that the series $I = \sum_{i \geqslant 1} I_i(t)$ is also upper bounded, thus integrable on any compact of $[0,\infty)$. Finally, after an integration of equations (1.9), for any i and $t \in [0,\infty)$

$$
\begin{aligned}
S_i^{n_k}(t) &= S_i(0) + \int_0^t (\omega_{i-1} R_{i-1}^{n_k} - S_i^{n_k} \sum_j \beta_{i,j} I_j^{n_k} - \mu S_i^{n_k}) ds,\\
E_i^{n_k}(t) &= E_i(0) + \int_0^t (S_i^{n_k}(s) \sum_j \beta_{i,j} I_j^{n_k}(s) - (\sigma_i + \mu) E_i^{n_k}(s)) ds,\\
I_i^{n_k}(t) &= I_i(0) + \int_0^t (\sigma_i E_i^{n_k}(s) - (\gamma_i + \mu) I_i^{n_k}(s)) ds,\\
R_i^{n_k}(t) &= R_i(0) + \int_0^t (\gamma_i I_i^{n_k}(s) - (\omega_i + \mu) R_i^{n_k}(s)) ds.
\end{aligned}
$$

Hence it is enough to consider the limit $k \to \infty$ and the uniform convergence allows the interchanging of limit and integral. Therefore, the functions S_i, E_i, I_i, R_i obtained are indeed a solution of (1.5).

Uniqueness. Let $x := (S_i, E_i, I_i, R_i)_{1 \leqslant i}$ and $x' := (S_i', E_i', I_i', R_i')_{1 \leqslant i}$ be two functions solutions of (1.5) sharing a same initial condition. Denote

$$
y_i^S := S_i - S_i', \quad y_i^E := E_i - E_i', \quad y_i^I := I_i - I_i', \quad y_i^R := R_i - R_i',
$$

hence,

$$
\begin{aligned}
\frac{d|y_i^S|}{dt} &= \operatorname{sgn}(y_i^S)\, \omega_{i-1} y_{i-1}^R - \operatorname{sgn}(y_i^S) \sum_j \beta_{i,j} (S_i I_j - S_i' I_j') - \mu |y_i^S|,\\
\frac{d|y_i^E|}{dt} &= \operatorname{sgn}(y_i^E) \sum_j \beta_{i,j} (S_i I_j - S_i' I_j') - (\sigma_i + \mu) |y_i^E|,\\
\frac{d|y_i^I|}{dt} &= \operatorname{sgn}(y_i^I)\, \sigma_i y_i^E - (\gamma_i + \mu) |y_i^I|,\\
\frac{d|y_i^R|}{dt} &= \operatorname{sgn}(y_i^R)\, \gamma_i y_i^I - (\omega_i + \mu) |y_i^R|.
\end{aligned}
$$

Moreover,

$$
\sum_j \beta_{i,j} (S_i I_j - S_i' I_j') = S_i \sum_j \beta_{i,j} y_j^I + y_i^S \sum_j \beta_{i,j} I_j'.
$$

Hence,

$$
\begin{aligned}
\sum_{A \in \{S,E,I,R\}} \frac{d|y_i^A|}{dt} \quad = \quad & \operatorname{sgn}(y_i^S)\, \omega_{i-1} y_{i-1}^R - \omega_i |y_i^R| + \gamma_i (\operatorname{sgn}(y_i^R y_i^I) - 1) |y_i^I| + \sigma_i (\operatorname{sgn}(y_i^E y_i^I) - 1) |y_i^E| \\
& + (\operatorname{sgn}(y_i^E) - \operatorname{sgn}(y_i^S)) \sum_j \beta_{i,j} (S_i I_j - S_i' I_j') - \mu (|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \operatorname{sgn}(y_i^S)\,\omega_{i-1}y_{i-1}^R - \omega_i|y_i^R| - \mu(|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|)\\
&\quad +(\operatorname{sgn}(y_i^E) - \operatorname{sgn}(y_i^S))(S_i\sum_j \beta_{i,j}y_j^I + y_i^S\sum_j \beta_{i,j}I_j')\\
&\leqslant \operatorname{sgn}(y_i^S)\,\omega_{i-1}y_{i-1}^R - \omega_i|y_i^R| - \mu(|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|)\\
&\quad +(\operatorname{sgn}(y_i^E y_i^S) - 1)|y_i^S|\sum_j \beta_{i,j}I_j' + (\operatorname{sgn}(y_i^E) - \operatorname{sgn}(y_i^S))S_i\sum_j \beta_{i,j}y_j^I.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^n \sum_{A\in\{S,E,I,R\}} \frac{d|y_i^A|}{dt} \leqslant\ & -\mu\sum_{i=1}^n(|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|) + \sum_{i=1}^n(\operatorname{sgn}(y_i^S)\,\omega_{i-1}y_{i-1}^R - \omega_i|y_i^R|)\\
& +2\sum_{i=1}^n\sum_j \beta_{i,j}S_i|y_j^I|.
\end{aligned}$$

By the fact that the coefficients $(\beta_{i,j})_{i,j\geqslant 1}$ are uniformly bounded with respect to i,j and that, by definition $\sup_{t\in[0,T)}\|x(t)\|_{X^4} < \infty$, there exists $c_3 \in \mathbb{R}_+$ such that $2\sum_i S_i\beta_{i,j} < c_3$ for all $j \geqslant 0$. Hence,

$$\begin{aligned}
\sum_{i=1}^n \sum_{A\in\{S,E,I,R\}} \frac{d|y_i^A|}{dt} &\leqslant -\mu\sum_{i=1}^n(|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|) + \sum_{i=1}^n \operatorname{sgn}(y_i^S y_{i-1}^R - 1)\omega_{i-1}|y_{i-1}^R|\\
&\quad - \omega_n|y_n^R| + \omega_0|y_0^R| + c_3\sum_{j=1}^n |y_j^I|\\
&\leqslant (c_3 - \mu)\sum_{i=1}^n(|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|).
\end{aligned}$$

The equality $\sum_{i\geqslant 1}|y_i^S(0)| + |y_i^E(0)| + |y_i^I(0)| + |y_i^R(0)| = 0$ leads to

$$\left(\sum_{i=1}^n |y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|\right)(t) \leqslant (c_3 - \mu)\int_0^t \left(\sum_{i=1}^n |y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|\right) ds.$$

Taking n to the limit $n \to \infty$ and interchanging sum and integral (which is possible as $\sum_{i\geqslant 1}|y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|$ is upper bounded), we obtain

$$\left(\sum_{i\geqslant 1} |y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|\right)(t) \leqslant c_3\int_0^t \left(\sum_{i\geqslant 1} |y_i^S| + |y_i^E| + |y_i^I| + |y_i^R|\right) ds.$$

Finally the Gronwall's lemma implies

$$\sum_{i\geqslant 1} |y_i^S| + |y_i^E| + |y_i^I| + |y_i^R| \equiv 0.$$

and the uniqueness of the solution of (1.5). Finally, in order to prove the smoothness of the sums S, E, I, R, the following Lemma is first required.

Lemma 1.3.2. *With the hypothesis and notations of Theorem 1.3.1, the functions* $\phi_i := \sum_j \beta_{i,j}I_j$

are well-defined and continuous on $[0,T)$ *for all* $i \geqslant 1$.

Proof of Lemma 1.3.2. As the coefficients $\beta_{i,j}$, σ_i, γ_i are positive and uniformly bounded with respect to i, j and moreover the series of positive terms $\sum_j I_j$, $\sum_j E_j$ are upper bounded by $\|x(0)\|_{X^4}$, hence the partial sums $\sum_{j=1}^n \dot{I}_j$ are uniformly bounded by a constant, leading to the equicontinuity of the family of functions $\{\sum_{j=1}^n \beta_{i,j} I_j\}_{n\geqslant 1}$, for any $i \geqslant 1$. Then, by Arzelà-Ascoli theorem and the pointwise convergence of partial sums $\sum_{j=1}^n \beta_{i,j} I_j$, the convergence is uniform, and the limit ϕ_i is a continuous function on any bounded interval. □

Smoothness. As the solution x is continuous, it is not difficult to see that the integrands in the integral formulas (1.8) are continuous. Hence, any component S_i, E_i, I_i, R_i of x is continuously derivable. Moreover, the differential equations of the partial sums are given by

$$\begin{aligned}
\sum_{1\leqslant i\leqslant n} \dot{S}_i &= \mu N + \sum_{2\leqslant i\leqslant n-1} \omega_i R_i - \sum_{1\leqslant i\leqslant n} S_i\phi_i - \mu \sum_{1\leqslant i\leqslant n} S_i,\\
\sum_{1\leqslant i\leqslant n} \dot{E}_i &= \sum_{1\leqslant i\leqslant n} S_i\phi_i - \sum_{1\leqslant i\leqslant n} (\sigma_i+\mu)E_i,\\
\sum_{1\leqslant i\leqslant n} \dot{I}_i &= \sum_{1\leqslant i\leqslant n} \sigma_i E_i - \sum_{1\leqslant i\leqslant n} (\gamma_i+\mu)I_i,\\
\sum_{1\leqslant i\leqslant n} \dot{R}_i &= \sum_{1\leqslant i\leqslant n} \gamma_i I_i - \sum_{1\leqslant i\leqslant n} (\omega_i+\mu)R_i.
\end{aligned}$$

By a similar argument as before and an application of Dini's Theorem [105], every partial sum in the derivatives above converges uniformly. Hence, the derivatives of the partial sums also converge uniformly. Thus, it is allowed to interchange derivation and summation, which proves the final point of the theorem. Finally, the integrands in formulas (1.8) are continuous. Then, the solutions are continuously differentiable, and their derivatives are given by (1.5). □

To conclude the section, with a slight change in the integral formulas, we define the solution of (1.6) similarly to Definition 1.3.1 for the normalized system (1.5). Then, the wellposed-ness of the normalized system (1.5) implies the same for the non-normalized system (1.6).

Theorem 1.3.3. *Under the same hypotheses as in Theorem 1.3.1. Then for any initial condition* $x_0 \in X_+^4$ *and any* $T \in (0,+\infty)$*, there is a unique solution* $x = (S_i, E_i, I_i, R_i)_{i\geqslant 1} \in X_+^4$ *of* (1.6) *on* $[0,T)$*, whose components are continuously differentiable.*

Proof. Assume that $x := (S_i, E_i, I_i, R_i)_{i\geqslant 1}$ is a nonnegative solution of (1.6) on $[0,T)$. Let us prove first that the total number of hosts of (1.6) is constant. By Dini's Theorem, the partial sum $\sum_{i=1}^n (S_i + E_i + I_i + R_i)$ converges uniformly to N when $n \to +\infty$ (and consequently R_n to 0 also) on any compact subset $[0, T-\varepsilon]$ of $[0,T)$, for $\varepsilon > 0$. Hence, the same convergence is true for the partial sum

$$\sum_{i=1}^n (\dot{S}_i + \dot{E}_i + \dot{I}_i + \dot{R}_i) = \mu(N - \sum_{i=1}^n (S_i + E_i + I_i + R_i)) - \omega_n R_n.$$

Therefore, taking $n \to +\infty$, $\dot{N}$ is defined and is equal to 0 on $[0, T-\varepsilon]$. Taking $\varepsilon \to 0$, this proves that any solution of (1.6) has a constant total population N on $[0,T)$. Dividing the system (1.6) by N, we retrieve the normalized system (1.5) and Theorem 1.3.1 implies the wellposed-ness of (1.6). □

1.4 Underlying reinfection structure of a common SEIRS model

This section studies an infinite model, slightly different from (1.6), with a birth rate $b \neq \mu$, an additional disease-induced mortality rate $\nu > 0$ and coefficients $\beta_{i,j}, \sigma_i, \gamma_i, \omega_i$ that do not vary with respect to the number of reinfections. In subsection 1.4.1 the infinite model with the addition of disease related mortality ν and a birth rate $b \neq \mu$ is presented. An easier and explicit proof of well-posedness for non-varying parameters, based on the existence of a closed form expression for the dynamics of S, E, I, R, may be given, as we shall see in subsection 1.4.2. In subsection 1.4.3 results on the asymptotic behavior and stability of the steady states are derived. In fact, the asymptotic behavior of the model can be well-described even with the additional parameters ν and b. Moreover, from the description of the steady state, we derive the mean numbers of reinfection at the endemic equilibrium of the model in subsection 1.4.4. Finally, numerical simulations are computed in subsection 1.4.5 and to make the reading easier, the proofs of several results of this section are reported in subsection 1.4.6.

1.4.1 A SEIRS model counting reinfections with varying population

As already discussed in Section 1.2, the hosts in the SEIRS model are allowed to be infected multiple times. Hence, the SEIRS model (1.2) presents a hidden reinfection structure that we intend to unravel in the present section. While maintaining the disease induced mortality and keeping the coefficients invariant, we proceed by dividing the compartments in a similar fashion as in Section 1.2 to obtain the following system tracking reinfection numbers.

$$\begin{aligned}
\dot{S}_i &= \omega R_{i-1} - \beta S_i \frac{I}{N} - \mu S_i, \quad i \geqslant 1,\\
\dot{E}_i &= \beta S_i \frac{I}{N} - (\sigma + \mu) E_i, \qquad i \geqslant 1,\\
\dot{I}_i &= \sigma E_i - (\gamma + \mu + \nu) I_i, \qquad i \geqslant 1,\\
\dot{R}_i &= \gamma I_i - (\omega + \mu) R_i, \qquad i \geqslant 1,
\end{aligned} \tag{1.10}$$

with, as in (1.5), $R_0(t) = \frac{b}{\omega} N(t)$, and

$$S := \sum_{i \geqslant 1} S_i, \quad E := \sum_{i \geqslant 1} E_i, \quad I := \sum_{i \geqslant 1} I_i, \quad R := \sum_{i \geqslant 1} R_i, \quad N := S + E + I + R.$$

Moreover, the system (1.10) is endowed with initial condition

$$(S_i(0), E_i(0), I_i(0), R_i(0))_{i \geqslant 1} = (S_{i,0}, E_{i,0}, I_{i,0}, R_{i,0})_{i \geqslant 1}.$$

Since the system (1.1) concerns the total population at each health status irrespective of their disease history, it can be considered as a *macroscopic* system. Therefore, by contrast, (1.10) is considered as the *microscopic* system.

1.4.2 Well-posedness

Because system (1.10) differs slightly from (1.5) due to supplementary deaths from the disease and the varying total population, it is needed to check again the well-posedness. To this end, we borrow the same spaces (1.7) and the definition of solution similar to Definition 1.3.1 used

in Section 1.3. The solutions of (1.10) are again assumed to be evolving in the space X^4, and the initial conditions $(S_{i,0}, E_{i,0}, I_{i,0}, R_{i,0})_{i\geqslant 1}$ are included in X^4_+. In this setting, the next result proves that the problem is still well-posed. In addition, the solution $x(t)$ is proved to be positive component-wise for all $t > 0$ and, more interestingly, (S, E, I, R) constitutes, in fact, a solution of the macroscopic system (1.5). Thus, in the simple case with constant coefficients, the macroscopic quantities possess a closed form expression for their dynamics.

Theorem 1.4.1 (Existence and uniqueness)**.** *For all $x_0 \in X^4_+$ and any $T > 0$, there exists a unique function $x : [0, t) \to X^4_+, t \mapsto (S_i, E_i, I_i, R_i)_{i\geqslant 1}$ such that:*

1. *Every component $(S_i, E_i, I_i, R_i) : [0, T) \to \mathbb{R}^4_+, i \geqslant 1$ is continuous and $\sup_{t\in[0,T)} \|x(t)\|_{X^4} < +\infty$.*

2. *For all $t \in [0, T)$, x is a solution of the integral equations*

$$\begin{aligned}
S_i(t) &= S_i(0) + \int_0^t (\omega R_{i-1}(s) - \beta S_i(s)\frac{I(s)}{N(s)} - \mu S_i(s))ds, \qquad i \geqslant 1,\\
E_i(t) &= E_i(0) + \int_0^t (\beta S_i(s)\frac{I(s)}{N(s)} - (\sigma+\mu)E_i(s))ds, \qquad i \geqslant 1,\\
I_i(t) &= I_i(0) + \int_0^t (\sigma E_i(s) - (\mu+\gamma+\nu)I_i(s))ds, \qquad i \geqslant 1,\\
R_i(t) &= S_i(0) + \int_0^t (\gamma I_i(s) - (\omega+\mu)R_i(s))ds, \qquad i \geqslant 1.
\end{aligned} \tag{1.11}$$

Moreover,

1. *every component is continuously differentiable and x is the solution of the system of differential equations* (1.10) *for $t \in [0, T)$.*

2. *If*

$$\|E^0\|_{\ell^1} + \|I^0\|_{\ell^1} = \sum_{i\geqslant 1} E_{i,0} + I_{i,0} > 0 \tag{1.12}$$

   is verified, then $x(t) \in X^4_{++}$ for all $t > 0$.

3. *For $t \in [0, T)$, (S, E, I, R) is a solution of the macroscopic system* (1.1)*.*

The well-posedness of system (1.10) can be proved, with some little adjustments, as in Theorem 1.3.3 in Section 1.3 through a normalization. However, we will deliver an alternative demonstration of Theorem 1.4.1 with the construction of an explicit solution, on the basis that the quantities S, E, I, R of system (1.10) will be proved to admit the closed form expression (1.2) governing their dynamics. Thus, the system presents a special *triangular* structure. The proof is transferred to the subsection 1.4.6.1 to make the reading less cumbersome.

1.4.3 Steady states, stability and asymptotic behavior

In this subsection, we examine the steady states and the asymptotic behavior of the system (1.10). As in the preceding subsection 1.4.2 the proofs of the results are reported to subsection 1.4.6 to facilitate the reading.

Let us start by noticing that the total population of (1.10) may diverge or converge to zero. To overcome these situations, let us study the evolution of the following normalized quantities :

$$\bar{S}_i(t) = \frac{S_i(t)}{N(t)}, \quad \bar{E}_i(t) = \frac{E_i(t)}{N(t)}, \quad \bar{I}_i(t) = \frac{I_i(t)}{N(t)}, \quad \bar{R}_i(t) = \frac{R_i(t)}{N(t)}, \quad i \geqslant 1,$$

$$\bar{S}(t) = \frac{S(t)}{N(t)}, \quad \bar{E}(t) = \frac{E(t)}{N(t)}, \quad \bar{I}(t) = \frac{I(t)}{N(t)}, \quad \bar{R}(t) = \frac{R(t)}{N(t)}.$$

The normalized counterpart of system (1.10) is:

$$\begin{aligned}
\dot{\bar{S}}_i &= \frac{\dot{S}_i}{N} - \frac{S_i}{N^2}\dot{N} = \omega \bar{R}_{i-1} - (\beta - \nu)\bar{S}_i\bar{I} - b\bar{S}_i, \\
\dot{\bar{E}}_i &= \frac{\dot{E}_i}{N} - \frac{E_i}{N^2}\dot{N} = \beta \bar{S}_i\bar{I} - (\sigma + b)\bar{E}_i + \nu\bar{I}\bar{E}_i, \\
\dot{\bar{I}}_i &= \frac{\dot{I}_i}{N} - \frac{I_i}{N^2}\dot{N} = \sigma \bar{E}_i - (\gamma + b + \nu)\bar{I}_i + \nu\bar{I}\bar{I}_i, \\
\dot{\bar{R}}_i &= \frac{\dot{R}_i}{N} - \frac{R_i}{N^2}\dot{N} = \gamma \bar{I}_i - (\omega + b)\bar{R}_i + \nu\bar{I}\bar{R}_i,
\end{aligned} \tag{1.13}$$

and the counterpart of system (1.1) is

$$\begin{aligned}
\dot{\bar{S}} &= \frac{\dot{S}}{N} - \frac{S}{N^2}\dot{N} = b - (\beta - \nu)\bar{S}\bar{I} + \omega\bar{R} - b\bar{S}, \\
\dot{\bar{E}} &= \frac{\dot{E}}{N} - \frac{E}{N^2}\dot{N} = \beta \bar{S}\bar{I} - (\sigma + b)\bar{E} + \nu\bar{I}\bar{E}, \\
\dot{\bar{I}} &= \frac{\dot{I}}{N} - \frac{I}{N^2}\dot{N} = \sigma \bar{E} - (\gamma + b + \nu)\bar{I} + \nu\bar{I}^2, \\
\dot{\bar{R}} &= \frac{\dot{R}}{N} - \frac{R}{N^2}\dot{N} = \gamma \bar{I} - (\omega + b)\bar{R} + \nu\bar{I}\bar{R},
\end{aligned} \tag{1.14}$$

where $\omega\bar{R}_0 = b$. Because the non-normalized system is already shown to be well-posed in Theorem 1.4.1, it is easy to see that this implies in particular the well-posedness of the normalized system (1.14). Again, in order to simplify reading, the proofs of all the results present in this section are transferred to subsection 1.4.6.

Remark 1.4.2. *The natural death rate μ does not appear in the normalized systems* (1.13) *and* (1.14).

For the normalized system (1.13), it is natural to study the trajectories $\bar{x} := (\bar{S}_i, \bar{E}_i, \bar{I}_i, \bar{R}_i)_{i\geqslant 1}$ of (1.13) only in the invariant set

$$\Gamma := \{\bar{x} \in X_+^4 : \|\bar{x}\|_{X^4} = 1\}. \tag{1.15}$$

Lemma 1.4.3. *For every $\bar{x}_0 \in \Gamma$, the solution $\bar{x}$ of* (1.13) *is such that*

$$\|\bar{x}(t)\|_{X^4} = 1 \quad \textit{for all} \quad t \geqslant 0. \tag{1.16}$$

Proof. The proof follows easily from the equation

$$\dot{\bar{S}} + \dot{\bar{E}} + \dot{\bar{I}} + \dot{\bar{R}} = b - b(\bar{S} + \bar{E} + \bar{I} + \bar{R}).$$

□

Within the invariant set Γ, the steady states of (1.13), composed of disease-free and endemic equilibriums, are described in the following proposition.

Theorem 1.4.4 (Steady states)**.** *System* (1.13) *admits a unique disease-free equilibrium (DFE)*

$$\bar{x}^{DFE} := (\delta_i^1, 0, 0, 0)_{i\geqslant 1},$$

where δ_i^1 *is the Kronecker delta. Moreover, define the basic reproduction number*

$$\mathcal{R}_0 := \frac{\sigma}{\sigma+b}\frac{\beta}{\gamma+\nu+b}. \tag{1.17}$$

For $\mathcal{R}_0 > 1$*, the system* (1.13) *admits a unique endemic equilibrium (EE)*

$$\bar{x}^{EE} := (\alpha_\nu^{i-1}\bar{S}_1^{EE}, \alpha_\nu^{i-1}\bar{E}_1^{EE}, \alpha_\nu^{i-1}\bar{I}_1^{EE}, \alpha_\nu^{i-1}\bar{R}_1^{EE})_{i\geqslant 1},$$

where the coefficients $\alpha_\nu, \bar{S}_1^{EE}, \bar{E}_1^{EE}, \bar{I}_1^{EE}, \bar{R}_1^{EE} \in (0,1)$ *are defined by*

$$\alpha_\nu := \frac{\omega}{(\beta-\nu)\bar{I}^{EE}+b}\frac{\gamma}{\omega+b-\nu\bar{I}^{EE}}\mathcal{S}_\nu\bar{I}^{EE}, \quad \mathcal{S}_\nu := \frac{\beta}{\sigma+b-\nu\bar{I}^{EE}}\frac{\sigma}{\gamma+b+\nu-\nu\bar{I}^{EE}}, \tag{1.18a}$$

$$\bar{S}_1^{EE} = \frac{b}{(\beta-\nu)\bar{I}^{EE}+b}, \quad \bar{E}_1^{EE} = \frac{\beta\bar{I}^{EE}}{\sigma+b-\nu\bar{I}^{EE}}\frac{b}{(\beta-\nu)\bar{I}^{EE}+b}, \tag{1.18b}$$

$$\bar{I}_1^{EE} = \frac{b}{(\beta-\nu)\bar{I}^{EE}+b}\mathcal{S}_\nu\bar{I}^{EE}, \quad \bar{R}_1^{EE} = \frac{\gamma}{\omega+b-\nu\bar{I}^{EE}}\frac{b}{(\beta-\nu)\bar{I}^{EE}+b}\mathcal{S}_\nu\bar{I}^{EE}, \tag{1.18c}$$

and $(\bar{S}^{EE}, \bar{E}^{EE}, \bar{I}^{EE}, \bar{R}^{EE})$ *is the unique EE of system* (1.14) *in the interior of* $\{\bar{x} \in \mathbb{R}_+^4 : \|\bar{x}\|_{\mathbb{R}^4} = 1\}$*, satisfying the equations*

$$\mathcal{R}_0\left(1+\frac{\gamma}{b}\frac{\omega}{\omega+b-\nu\bar{I}^{EE}}\right) = \left(\frac{\beta-\nu}{b}\bar{I}^{EE}+1\right)\left(1-\frac{\nu\bar{I}^{EE}}{\sigma+b}\right)\left(1-\frac{\nu\bar{I}^{EE}}{\gamma+\nu+b}\right),$$

$$\bar{S}^{EE} = \frac{1}{\mathcal{S}_\nu}, \quad \bar{E}^{EE} = \frac{\gamma+\nu+b-\nu\bar{I}^{EE}}{\sigma}\bar{I}^{EE}, \quad \bar{R}^{EE} = \frac{\gamma}{\omega+b-\nu\bar{I}^{EE}}\bar{I}^{EE}.$$

The proof of Theorem 1.4.4 is given in subsection 1.4.6.2. The result characterizes the EE of (1.13) using the value $\bar{I}^{EE}$ of the EE of system (1.14) that has already been described in [80]. In the particular case where the disease-induced mortality is zero, there is an explicit expression for the value of $\bar{I}^{EE}$.

Corollary 1.4.5 ($\nu = 0$)**.** *Assume* $\nu = 0$ *and* $\mathcal{R}_0 > 1$*, then*

$$\mathcal{R}_0 = \mathcal{S}_0 = \frac{\beta}{\gamma+b}\frac{\sigma}{\sigma+b}, \qquad \alpha_0 = \frac{\gamma\omega}{\beta(\omega+b)-\gamma\omega}(\mathcal{R}_0-1),$$

and $\bar{I}^{EE}$ *has an explicit expression :*

$$\bar{I}^{EE} = \frac{\mathcal{R}_0-1}{\mathcal{R}_0}\frac{1}{\zeta}, \qquad \textit{where} \quad \zeta = \frac{(\gamma+b)(\sigma+b)(\omega+b)-\omega\gamma\sigma}{\sigma b(\omega+b)} > 1.$$

Remark 1.4.6. *The coefficient* ζ *is defined in the literature as the critical stability number [23].*

Finally, the asymptotic behavior of system (1.13) is well-described, as shown in the following Theorem.

Theorem 1.4.7 (Global stability)**.** *For any $\bar{x}_0 \in \Gamma$ (1.15), let $\bar{x}(t) = (\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t))_{i\geqslant 1}$ be the solution of the system* (1.14) *with initial condition $\bar{x}(0) = \bar{x}_0$. The following convergence results hold:*

1. *if $\mathcal{R}_0 < 1$,*

$$\lim_{t\to+\infty} \bar{x}(t) = \bar{x}^{DFE},$$

   and the disease-free equilibrium is globally asymptotically stable in Γ.

2. *if $\mathcal{R}_0 > 1$ and* (1.12) *holds,*

$$\lim_{t\to+\infty} \bar{x}(t) = \bar{x}^{EE},$$

   and the endemic equilibrium is globally asymptotically stable in $\{\bar{x} \in X_+^4 : \|\bar{x}\|_{X^4} = 1$ and $\sum_{i\geqslant 1}(\bar{E}_i + \bar{I}_i) > 0\}$.

Theorem 1.4.7 shows that the asymptotic behavior of (1.10) is qualitatively simple. In fact, every non-trivial solution of the system (1.10) is either converging to the unique DFE when $\mathcal{R}_0 < 1$ or to the unique EE if $\mathcal{R}_0 > 1$ and the initial condition fulfills (1.12). In particular, this implies the non-existence of EE for $\mathcal{R}_0 < 1$.

Remark 1.4.8. *The proof of the convergence of the microscopic system* (1.13) *relies on the global stability result of the macroscopic system* (1.1)*, as seen in [53] for $\mathcal{R}_0 < 1$ and [80] for $\mathcal{R}_0 > 1$.*

As a side note, the asymptotic convergence of the normalized system (1.14) also enables a complete description of the asymptotic behavior of N :

Theorem 1.4.9 (Asymptotic behavior of the total population)**.** *Let $N(t)$ be the total number of the population at time t of the system* (1.1) *corresponding to a trajectory with initial conditions fulfilling* (1.12)*, then*

1. *If $b < \mu$, $N(t)$ converges to 0 while $t \to +\infty$.*
2. *If $b = \mu$,*
   (a) *If $\nu = 0$, then $N(t) = N(0)$ for all $t \geqslant 0$.*
   (b) *If $\nu > 0$,*
      i. *$N(t)$ converges to a positive limit if $\mathcal{R}_0 < 1$.*
      ii. *$N(t)$ converges to 0 while $t \to \infty$ if $\mathcal{R}_0 > 1$.*
3. *If $b > \mu$,*
   (a) *If $b - \mu > \bar{I}^{EE}$, then $N(t)$ diverges to $+\infty$,*
   (b) *If $\mathcal{R}_0 > 1$ and $b - \mu < \bar{I}^{EE}$, then $N(t)$ converges to 0.*

Remark 1.4.10. *Again, the proof of the Theorem can be found in subsection 1.4.6.4. As $\bar{I}^{EE} \in (0, 1)$ and does not depend on μ, the case* (3a) *is realized for example if $b < \mu + 1$, and the case* (3b) *is realized if $b - \bar{I}^{EE} < \mu < b$.*

-

1.4.4 Mean numbers of reinfection

Thanks to Theorem 1.4.7, the system converges asymptotically towards a unique EE for $\mathcal{R}_0 > 1$ in the presence of the disease, which makes the epidemiological prediction easier for this model. From the description of the EE presented in Theorem 1.4.4, it is moreover possible to obtain some valuable insight about the final state of the epidemics according to the reinfection structure. This kind of additional information may, for example, be used for parameters identification purpose for epidemic modelling. The next theorem delivers such a result about the mean numbers of reinfection in each category of the population.

Theorem 1.4.11 (Mean numbers of reinfection)**.** *Let $\mathcal{R}_0 > 1$, and α_ν, $\mathcal{S}_\nu$ be defined as in Theorem 1.4.4. Then,*

$$\frac{\sum_{i\geqslant 1}(i-1)\bar{S}_i^{EE}}{\sum_{i\geqslant 1}\bar{S}_i^{EE}} = \frac{\alpha_\nu}{1-\alpha_\nu}, \tag{1.19a}$$

$$\frac{\sum_{i\geqslant 1} i\bar{E}_i^{EE}}{\sum_{i\geqslant 1}\bar{E}_i^{EE}} = \frac{\sum_{i\geqslant 1} i\bar{I}_i^{EE}}{\sum_{i\geqslant 1}\bar{I}_i^{EE}} = \frac{\sum_{i\geqslant 1} i\bar{R}_i^{EE}}{\sum_{i\geqslant 1}\bar{R}_i^{EE}} = \frac{1}{1-\alpha_\nu}, \tag{1.19b}$$

$$\frac{\sum_{i\geqslant 1}((i-1)\bar{S}_i^{EE} + i(\bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE})}{\sum_{i\geqslant 1}(\bar{S}_i^{EE} + \bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE})} = \frac{1}{(1-\alpha_\nu)^2}\frac{b - b\alpha_\nu - \nu\bar{I}_1^{EE}}{b-\nu\bar{I}^{EE}} - \frac{1}{\mathcal{S}_\nu}. \tag{1.19c}$$

To provide some insights, the equation (1.19a) gives an expression of mean numbers of reinfections among the susceptible at the endemic equilibrium, and (1.19b) refers to the same values but among the exposed, infected or recovered. Finally, (1.19c) gives the mean number of reinfections in the total population at the same EE.

Proof. Assume $\mathcal{R}_0 > 1$, by Theorem 1.4.4, the sequences $(\bar{S}_i^{EE})_{i\geqslant 1}$, $(\bar{E}_i^{EE})_{i\geqslant 1}$, $(\bar{I}_i^{EE})_{i\geqslant 1}$ and $(\bar{R}_i^{EE})_{i\geqslant 1}$ are geometric with ratio α_ν such that $0 < \alpha_\nu < 1$. Therefore, as $|\alpha_\nu| < 1$, we apply the formula $\sum_{i\geqslant 1} i\alpha_\nu^{i-1} = \frac{1}{(1-\alpha_\nu)^2}$ to deduce:

$$\frac{\sum_{i\geqslant 1} i\bar{S}_i^{EE}}{\sum_{i\geqslant 1}\bar{S}_i^{EE}} = \frac{\sum_{i\geqslant 1} i\bar{E}_i^{EE}}{\sum_{i\geqslant 1}\bar{E}_i^{EE}} = \frac{\sum_{i\geqslant 1} i\bar{I}_i^{EE}}{\sum_{i\geqslant 1}\bar{I}_i^{EE}} = \frac{\sum_{i\geqslant 1} i\bar{R}_i^{EE}}{\sum_{i\geqslant 1}\bar{R}_i^{EE}} = \frac{1}{1-\alpha_\nu},$$

hence the formulas (1.19a) and (1.19b). Moreover,

$$\sum_{i\geqslant 1} i(\bar{S}_i^{EE} + \bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE}) = (\bar{S}_1^{EE} + \bar{E}_1^{EE} + \bar{I}_1^{EE} + \bar{R}_1^{EE})\sum_{i\geqslant 1} i\alpha_\nu^{i-1} = \bar{N}_1^{EE}\sum_{i\geqslant 1} i\alpha_\nu^{i-1},$$

where we denote,

$$\bar{N}_i := \bar{S}_i + \bar{E}_i + \bar{I}_i + \bar{R}_i,$$

and summing up (1.13), we obtain

$$\dot{\bar{N}}_i = \omega\bar{R}_{i-1} - b\bar{N}_i + \nu\bar{I}\bar{N}_i - \omega\bar{R}_i - \nu\bar{I}_i.$$

Finally, at endemic equilibrium we have

$$\bar{N}_1^{EE} = \frac{b - \omega\bar{R}_1^{EE} - \nu\bar{I}_1^{EE}}{b-\nu\bar{I}^{EE}} = \frac{b - b\alpha_\nu - \nu\bar{I}_1^{EE}}{b-\nu\bar{I}^{EE}},$$

$$\bar{N}_i^{EE} = \frac{\omega(\bar{R}_{i-1}^{EE} - \bar{R}_i^{EE}) - \nu \bar{I}_i^{EE}}{b - \nu \bar{I}^{EE}}, \quad i \geqslant 2.$$

Thus

$$\sum_{i \geqslant 1} i(\bar{S}_i^{EE} + \bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE}) = \frac{b - b\alpha_\nu - \nu \bar{I}_1^{EE}}{b - \nu \bar{I}^{EE}} \frac{1}{(1-\alpha_\nu)^2},$$

and because $\sum_{i \geqslant 1} \bar{S}_i^{EE} = \bar{S}^{EE} = \frac{1}{\mathcal{S}_\nu}$, (1.19c) is verified. □

In the simple case without disease-induced mortality ($\nu = 0$), we recall that there exists an explicit expression for the value of $\bar{I}^{EE}$ and therefore also for the mean numbers of reinfections.

Proposition 1.4.12. *Assume $\nu = 0$ and $\mathcal{R}_0 > 1$, then:*

$$\frac{\sum_{i \geqslant 1}(i-1)\bar{S}_i^{EE}}{\sum_{i \geqslant 1} \bar{S}_i^{EE}} = \frac{\gamma\omega(\mathcal{R}_0 - 1)}{\beta(\omega + b) - \gamma\omega\mathcal{R}_0}, \tag{1.20a}$$

$$\frac{\sum_{i \geqslant 1} i\bar{E}_i^{EE}}{\sum_{i \geqslant 1} \bar{E}_i^{EE}} = \frac{\sum_{i \geqslant 1} i\bar{I}_i^{EE}}{\sum_{i \geqslant 1} \bar{I}_i^{EE}} = \frac{\sum_{i \geqslant 1} i\bar{R}_i^{EE}}{\sum_{i \geqslant 1} \bar{R}_i^{EE}} = \frac{\beta(\omega + b) - \gamma\omega}{\beta(\omega + b) - \gamma\omega\mathcal{R}_0}, \tag{1.20b}$$

$$\frac{\sum_{i \geqslant 1}((i-1)\bar{S}_i^{EE} + i(\bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE})}{\sum_{i \geqslant 1}(\bar{S}_i^{EE} + \bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE})} = \frac{\beta(\omega + b)(\mathcal{R}_0 - 1)}{(\beta(\omega + b) - \gamma\omega\mathcal{R}_0)\mathcal{R}_0}. \tag{1.20c}$$

Remark 1.4.13. *The right-hand side of the equations above are positive thanks to $\mathcal{R}_0 > 1$.*

Proof. Assuming that $\nu = 0$ and $\mathcal{R}_0 > 1$, by Corollary 1.4.5:

$$\frac{1}{1 - \alpha_0} = \frac{1}{1 - \frac{\gamma\omega}{\beta(\omega+b) - \gamma\omega}(\mathcal{R}_0 - 1)} = \frac{\beta(\omega + b) - \gamma\omega}{\beta(\omega + b) - \gamma\omega\mathcal{R}_0},$$

hence,

$$\frac{\alpha_0}{1 - \alpha_0} = \frac{\gamma\omega(\mathcal{R}_0 - 1)}{\beta(\omega + b) - \gamma\omega\mathcal{R}_0}.$$

We retrieve (1.20a) and (1.20b) by Theorem 1.4.11. Moreover, due to (1.19c) and $\nu = 0$:

$$\sum_{i \geqslant 1}((i-1)\bar{S}_i^{EE} + i(\bar{E}_i^{EE} + \bar{I}_i^{EE} + \bar{R}_i^{EE}) = \frac{1}{(1-\alpha_0)^2}\frac{b - b\alpha_0}{b} - \frac{1}{\mathcal{S}_0} = \frac{1}{1-\alpha_0} - \frac{1}{\mathcal{R}_0},$$

hence (1.20c). □

1.4.5 Numerical simulations

With the aim of illustrating the previous results, numerical simulations of the underlying reinfection dynamics (1.13) are computed along the trajectories of the macroscopic system (1.14). The parameter values are borrowed from [15]:

$$\gamma^{-1} = 14 \text{ days}, \quad \sigma^{-1} = 7 \text{ days},$$
$$\omega^{-1} = 1 \text{ year}, \quad b^{-1} = \mu^{-1} = 76 \text{ years}, \quad \beta = 0.21 \text{ days}^{-1}, \text{ and } \nu = 0.$$

Moreover, the initial condition is chosen as $\bar{I}_0 = \bar{I}_{1,0} = 10^{-3}, S_0 = S_{1,0} = 1 - I_0$ and the simulations are computed with the solver ODEINT in Python.

Figure 1.3 shows a trajectory of (1.13) and (1.14) and its evolution in each compartment up to the index $n = 5$. The graphs suggest that the convergence of the macroscopic components is slower than the convergence of microscopic ones, although the time frames during which these components begin and stop evolving are delayed as the numbers of reinfections progress. Notice also that the figures highlight how the oscillations in macroscopic solution are, in fact, produced by successive waves in the different compartments of the microscopic system.

For $b = \mu$ and $\nu = 0$, the system (1.13) is equivalent to (1.10) with N constant equals to 1. An easy computation gives

$$\frac{\bar{S}_1^{EE}}{\bar{S}^{EE}} = \frac{b\sigma\beta}{(\beta\bar{I}^{EE} + b)(\gamma + b)(\sigma + b)} \simeq 0.02.$$

This shows that, at the endemic equilibrium, only a tiny portion of susceptible hosts have never been infected in their lifetime. Moreover, the Jacobian matrix for the system (1.14) describing the evolution of the macroscopic components is

$$\begin{pmatrix} -\beta\bar{I}^{EE} - b & 0 & -\beta\bar{S}^{EE} & \omega \\ \beta\bar{I}^{EE} & -(\sigma + b) & \beta\bar{S}^{EE} + \bar{E}^{EE} & 0 \\ 0 & \sigma & -(\gamma + b) & 0 \\ 0 & 0 & \gamma & -(\omega + b) \end{pmatrix},$$

while all diagonal blocks of the block-triangular Jacobian matrix of the system (1.13), corresponding to the evolution of a finite number of microscopic modes, are expressed by

$$\begin{pmatrix} -\beta\bar{I}^{EE} - b & 0 & 0 & 0 \\ \beta\bar{I}^{EE} & -(\sigma + b) & 0 & 0 \\ 0 & \sigma & -(\gamma + b) & 0 \\ 0 & 0 & \gamma & -(\omega + b) \end{pmatrix}.$$

The latter matrix is diagonal, its spectrum is real and numerically approximated to $\{-1.01, -1.87, -26.08, -52.17\}$. On the other hand, the numerical computation of the spectrum of the former matrix, which appears to be complex, gives $\{-1.28 \times 10^{-2}, -68.62, -6.24 \pm 3.16i\}$. Hence, both matrices are Hurwitz, and the largest real part of the eigenvalues is associated to the macroscopic evolution, as foreseen.

1.4.6 Proofs

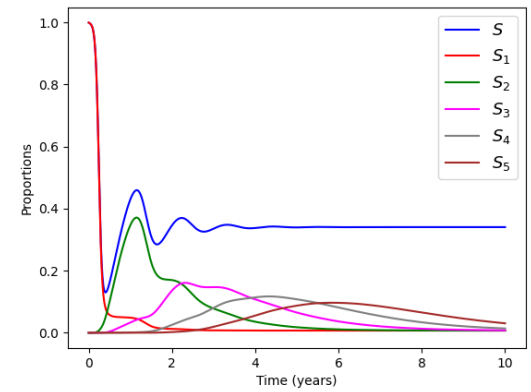
We collect in this subsection the proofs of several results stated earlier in the chapter.

1.4.6.1 Proof of Theorem 1.4.1

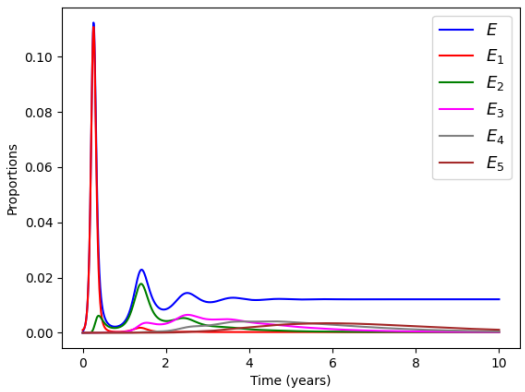
The proof of Theorem 1.4.1 requires the following Lemmas.

Lemma 1.4.14. *For all $T \in (0, +\infty)$, there exists a continuously differentiable solution $x = (S_i, E_i, I_i, R_i)_{i \geqslant 1}$ for the system (1.10) defined on $[0, T)$ taking values in X_+^4. Moreover, (S, E, I, R) is solution of (1.1). Finally, if (1.12) is verified, $S(t), E(t), I(t), R(t) > 0$ and $x(t) \in X_{++}^4$ for all $t \in (0, T)$.*

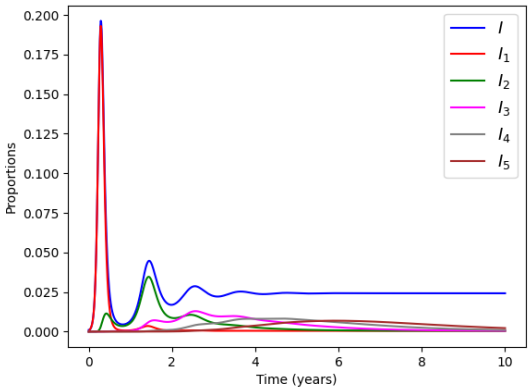
Lemma 1.4.15. *Any continuous solution of the integral equations (1.11) on $[0, T)$ is continuously differentiable.*



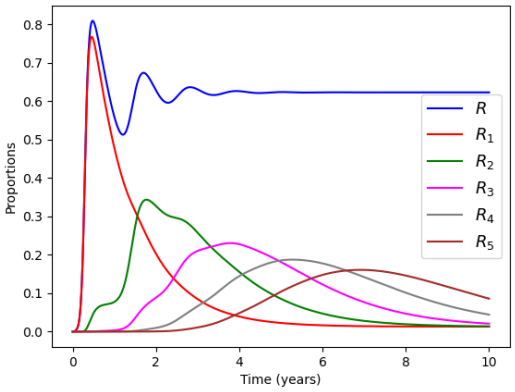
(a) Susceptible compartments



(b) Exposed compartments



(c) Infected compartments



(d) Recovered compartments

Figure 1.3: Trajectories of solutions of (1.13) and (1.14)

Lemma 1.4.16. *If $x = (S_i, E_i, I_i, R_i)_{i\geqslant 1}$, $x' = (S_i', E_i', I_i', R_i')_{i\geqslant 1}$ are two continuously differentiable solutions of* (1.10) *defined on* $[0,T)$*, taking values in* X_+^4 *and fulfilling* $x(0) = x'(0)$*. Then* $x = x'$*.*

The construction of an explicit (continuously differentiable) solution is included in Lemma 1.4.14. Lemma 1.4.15 and 1.4.16 guarantee the uniqueness of the solution. This implies in particular $x(t) \in X_{++}^4$ when (1.12) is fulfilled.

Proof du Lemma 1.4.14.
• By Picard-Lindelöf Theorem, there is a unique solution which is maximal for the system (1.1) evolving in $\mathbb{R}_+^4$ for any initial condition $(S_0, E_0, I_0, R_0) \in \mathbb{R}_+^4$. This solution is global, as it does not blow up in finite time due to the inequality

$$-(\mu+\nu)N \leqslant \dot{N} \leqslant bN, \tag{1.21}$$

which implies,

$$0 \leqslant N(0)e^{-(\mu+\nu)t} \leqslant N(t) \leqslant N(0)e^{bt} < +\infty \quad \text{for all } t \geqslant 0.$$

Moreover, for $N(0) > 0$, one has $N(t) > 0$ for all $t > 0$. Let $(s, e, \imath, r)$ be the unique continuously differentiable solution on $[0,T)$, $0 < T \leqslant \infty$, of the system (1.1) with the initial condition $(s(0), e(0), \imath(0), r(0)) \in \mathbb{R}_+^4$ such that :

$$s(0) = \|S_{i,0}\|_{l^1}, \quad e(0) = \|E_{i,0}\|_{l^1}, \quad \imath(0) = \|I_{i,0}\|_{l^1}, \quad r(0) = \|R_{i,0}\|_{l^1}$$

and fulfilling (1.12). The equation can be rewritten in the matrix form :

$$\begin{pmatrix} \dot{s} \\ \dot{e} \\ \dot{\imath} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} b - \beta\frac{\imath(t)}{n(t)} - \mu & b & b & b+\omega \\ \beta\frac{\imath(t)}{n(t)} & -(\sigma+\mu) & 0 & 0 \\ 0 & \sigma & -(\gamma+\mu+\nu) & 0 \\ 0 & 0 & \gamma & -(\omega+\mu) \end{pmatrix} \begin{pmatrix} s \\ e \\ \imath \\ r \end{pmatrix},$$

with

$$n(t) = s(t) + e(t) + \imath(t) + r(t).$$

An integration gives :

$$\begin{aligned} s(t) &= s(0)e^{-\int_0^t (\beta\frac{\imath(\tau)}{n(\tau)}+\mu)d\tau} + \int_0^t (\omega r(s) + bn(s))e^{-\int_s^t (\beta\frac{\imath(\tau)}{n(\tau)}+\mu)d\tau} ds, \\ e(t) &= e(0)e^{-(\sigma+\mu)t} + \int_0^t \beta\imath(\tau)\frac{s(\tau)}{n(\tau)}e^{-(\sigma+\mu)(t-\tau)}d\tau, \\ \imath(t) &= \imath(0)e^{-(\gamma+\mu+\nu)t} + \int_0^t \sigma e(s)e^{-(\gamma+\mu+\nu)(t-s)}ds, \\ r(t) &= r(0)e^{-(\omega+\mu)t} + \int_0^t \gamma\imath(s)e^{-(\omega+\mu)(t-s)}ds. \end{aligned} \tag{1.22}$$

For nonnegative initial conditions verifying either $e(0) > 0$ or $\imath(0) > 0$, it is easy to verify that the integral formulas (1.22) imply the existence of $\varepsilon > 0$ such that $s, e, \imath, r$ are positive for $t \in (0, \varepsilon)$. On the other hand, let us assume that

$$t_1 = \sup\{t > 0 \mid s > 0,\ e > 0,\ \imath > 0,\ r > 0\} < +\infty.$$

Then the same formulas indicate that

$$s(t_1) > 0, \quad e(t_1) > 0, \quad \imath(t_1) > 0, \quad r(t_1) > 0,$$

which contradicts the definition of t_1 as the continuity of solution implies there exists $\varepsilon_1 > 0$ such that the same inequality remains true for $t < t_1 + \varepsilon_1$. Hence, $t_1 = +\infty$ and

$$\forall t > 0, \qquad s(t) > 0, \quad e(t) > 0, \quad \imath(t) > 0, \quad r(t) > 0.$$

Remark moreover that from (1.21) we deduce that $N(t) \leqslant N(0)e^{bt}$ for all $t \geqslant 0$ and $s, e, \imath, r$ are always bounded.

• We construct now recursively the components (S_i, E_i, I_i, R_i), $i \geqslant 1$ as solutions on $[0, \infty)$ of the system :

$$\begin{pmatrix} \dot{S}_i \\ \dot{E}_i \\ \dot{I}_i \\ \dot{R}_i \end{pmatrix} = \begin{pmatrix} -(\beta \frac{\imath}{n} + \mu) & 0 & 0 & 0 \\ \beta \frac{\imath}{n} & -(\sigma + \mu) & 0 & 0 \\ 0 & \sigma & -(\gamma + \mu + \nu) & 0 \\ 0 & 0 & \gamma & -(\omega + \mu) \end{pmatrix} \begin{pmatrix} S_i \\ E_i \\ I_i \\ R_i \end{pmatrix} + \begin{pmatrix} \omega R_{i-1} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.23}$$

For $i = 1$, thanks to $\omega R_0 = bN$, the property

$$\forall t > 0, \quad S_1(t) > 0, \quad E_1(t) > 0, \quad I_1(t) > 0, \quad R_1(t) > 0.$$

can be verified with the same arguments employed for $s, e, \imath, r$. Now, assume that the property is verified for $i - 1$, integrating the equations above, we derive :

$$\begin{aligned} S_i(t) &= S_i(0)e^{-\int_0^t (\beta \frac{\imath(\tau)}{n(\tau)} + \mu)d\tau} + \int_0^t \omega R_{i-1} e^{-\int_s^t (\beta \frac{\imath(\tau)}{n(\tau)} + \mu)d\tau} ds, \\ E_i(t) &= E_i(0)e^{-(\sigma+\mu)t} + \int_0^t \beta \imath(\tau) \frac{S_i(\tau)}{n(\tau)} e^{-(\sigma+\mu)(t-\tau)} d\tau, \\ I_1(t) &= I_1(0)e^{-(\gamma+\mu+\nu)t} + \int_0^t \sigma E_i(s) e^{-(\gamma+\mu+\nu)(t-s)} ds, \\ R_i(t) &= R_1(0)e^{-(\omega+\mu)t} + \int_0^t \gamma I_i(s) e^{-(\omega+\mu)(t-s)} ds. \end{aligned}$$

Assume the initial condition is nonnegative and verifies either $e(0) > 0$ or $\imath(0) > 0$. As $\imath(t) > 0, R_{i-1}(t) > 0$ for $t > 0$, the integrals above imply again the existence of $\varepsilon > 0$ such that $S_i, E_i, I_i, R_i > 0$ on $(0, \varepsilon)$. Then, by a reasoning similar to the one employed for $s, e, \imath, r$, the integral formulas indicate that the components cannot be canceled for $t > 0$, hence

$$S_i(t) > 0, \quad E_i(t) > 0, \quad I_i(t) > 0, \quad R_i(t) > 0, \quad \forall t > 0, \quad i \geqslant 1. \tag{1.24}$$

• Let us denote

$$\begin{aligned} 0 \leqslant s_n(t) &:= \sum_{i=1}^n S_i(t) \leqslant s(t), \quad 0 \leqslant e_n(t) := \sum_{i=1}^n E_i(t) \leqslant e(t), \\ 0 \leqslant \imath_n(t) &:= \sum_{i=1}^n I_i(t) \leqslant \imath(t), \quad 0 \leqslant r_n(t) := \sum_{i=1}^n R_i(t) \leqslant r(t), \end{aligned} \tag{1.25}$$

then $(s_n, e_n, \imath_n, r_n)$ is solution of:

$$\begin{pmatrix} \dot{s}_n \\ \dot{e}_n \\ \dot{\imath}_n \\ \dot{r}_n \end{pmatrix} = \begin{pmatrix} -(\beta \frac{\imath(t)}{n(t)} + \mu) & 0 & 0 & \omega \\ \beta \frac{\imath(t)}{n(t)} & -(\sigma+\mu) & 0 & 0 \\ 0 & \sigma & -(\gamma+\mu+\nu) & 0 \\ 0 & 0 & \gamma & -(\omega+\mu) \end{pmatrix} \begin{pmatrix} s_n \\ e_n \\ \imath_n \\ r_n \end{pmatrix} + \omega \begin{pmatrix} R_0 - R_n & 0 & 0 & 0 \end{pmatrix}^\intercal . \tag{1.26}$$

The sequences of functions $(s_n)_{n\geqslant 1}, (e_n)_{n\geqslant 1}, (\imath_n)_{n\geqslant 1}$ and $(r_n)_{n\geqslant 1}$ are non-decreasing and bounded. Therefore, they converge pointwise on $[0,\infty)$ to the limits S, E, I, R.

Let us verify then that $(S, E, I, R) = (s, e, \imath, r)$. By (1.26), $(\dot{s}_n)_{n\geqslant 1}, (\dot{e}_n)_{n\geqslant 1}, (i_n)_{n\geqslant 1}, (\dot{r}_n)_{n\geqslant 1}$ are uniformly bounded with respect to n on any compact of $[0,\infty)$, thus the families of functions $\{s_n, n \in \mathbb{N}^*\}$, $\{e_n, n \in \mathbb{N}^*\}$, $\{\imath_n, n \in \mathbb{N}^*\}$, $\{r_n, n \in \mathbb{N}^*\}$ are equicontinuous. Then the Ascoli-Arzelà theorem states that we can extract subsequences of $(s_n)_{n\geqslant 1}, (e_n)_{n\geqslant 1}, (\imath_n)_{n\geqslant 1}$ and $(r_n)_{n\geqslant 1}$ that converge uniformly on any compact to the continuous functions S, E, I, R. Applying Dini's Theorem, the convergence of $(s_n)_{n\geqslant 1}, (e_n)_{n\geqslant 1}, (\imath_n)_{n\geqslant 1}, (r_n)_{n\geqslant 1}$ to S, E, I, R is uniform on any compact of $[0,\infty)$, therefore, from the relations (1.26), it also implies the uniform convergence of $(\dot{s}_n)_{n\geqslant 1}, (\dot{e}_n)_{n\geqslant 1}, (i_n)_{n\geqslant 1}, (\dot{r}_n)_{n\geqslant 1}$ to the limits that we denote $\dot{S}, \dot{E}, \dot{I}, \dot{R}$. Interchanging the sum and the derivatives, we deduce that (S, E, I, R) is also a solution of the same Cauchy problem as $(s, e, \imath, r)$, then, by uniqueness of the solution, they are indeed equal. □

Proof of Lemma 1.4.15. Let $T \in (0,\infty)$, and $x = (S_i, E_i, I_i, R_i)_{i\geqslant 1}$ be a continuous solution of (1.10) on $[0,T)$ and the sequence $(s_n, e_n, \imath_n, r_n)_{n\geqslant 1}$ defined as in (1.25). Then, for any $n \geqslant 1$, $(s_n, e_n, \imath_n, r_n)$ is solution of the differential equations:

$$\begin{pmatrix} \dot{s}_n \\ \dot{e}_n \\ \dot{\imath}_n \\ \dot{r}_n \end{pmatrix} = \begin{pmatrix} -(\beta \frac{I(t)}{N(t)} + \mu) & 0 & 0 & \omega \\ \beta \frac{I(t)}{N(t)} & -(\sigma+\mu) & 0 & 0 \\ 0 & \sigma & -(\gamma+\mu+\nu) & 0 \\ 0 & 0 & \gamma & -(\omega+\mu) \end{pmatrix} \begin{pmatrix} s_n \\ e_n \\ \imath_n \\ r_n \end{pmatrix} + \omega \begin{pmatrix} R_0 - R_n & 0 & 0 & 0 \end{pmatrix}^\intercal . \tag{1.27}$$

As

$$\sup_{t\in[0,T)} \|x(t)\|_{X^4} < \infty,$$

and $s_n, e_n, \imath_n, r_n \leqslant \|x\|_{X^4}$, taking $n \to +\infty$, $s_n, e_n, \imath_n$ and r_n converge to limits that we denote S, E, I, R, moreover the uniform bound implies that the derivatives $\dot{s}_n, \dot{e}_n, i_n, \dot{r}_n$ are also uniformly bounded by (1.27), hence the equicontinuity of $(s_n)_{n\geqslant 1}, (e_n)_{n\geqslant 1}, (\imath_n)_{n\geqslant 1}, (r_n)_{n\geqslant 1}$.

Applying once again the Ascoli-Arzelà Theorem, the limits S, E, I, R are continuous, and Dini's theorem states that the convergence of $s_n, e_n, \imath_n, r_n$ is uniform on any compact subset $[0,T)$, thus the same for $\dot{s}_n, \dot{e}_n, i_n, \dot{r}_n$ by (1.27). As the previous derivatives are continuous, the uniform convergence also implies that $\dot{S}, \dot{E}, \dot{I}, \dot{R}$ are defined and continuous. Then, the integrands in (1.11) are continuous, hence every component of the solution x is continuously differentiable. □

Proof of Lemma 1.4.16. The proof of Lemma 1.4.15 above implies in particular that any continuously differentiable solution $x = (S_i, E_i, I_i, R_i)_{i\geqslant 1}$ of (1.10) is such that $S := \sum_{i\geqslant 1} S_i$, $E := \sum_{i\geqslant 1} E_i$, $I := \sum_{i\geqslant 1} I_i$, $R := \sum_{i\geqslant 1} R_i$ fulfil the system (1.1), which admits a unique solution. Once we retrieved the unique macroscopic solution $(s, e, \imath, r)$, $(S_i, E_i, R_i, I_i)_{i\geqslant 1}$, is de-

rived uniquely as solution of system (1.23) for $i \geqslant 1$. Therefore, we obtain the uniqueness of continuously differentiable solution and Lemma 1.4.16. □

1.4.6.2 Proof of Theorem 1.4.4

Let $(\bar{S}_{i,0}, \bar{E}_{i,0}, \bar{I}_{i,0}, \bar{R}_{i,0})_{i\geqslant 1}$ be a DFE of (1.13). By definition of DFE, $E_{i,0}, I_{i,0} = 0$ for $i \geqslant 1$. The uniqueness and value of the DFE follow then easily from the steady state equations

$$0 = \omega\bar{R}_{i-1,0} - b\bar{S}_{i,0}, \quad 0 = -(\sigma+b)\bar{E}_{i,0}, \quad 0 = -(\gamma+b+\nu)\bar{I}_{i,0}, \quad 0 = -(\omega+b)\bar{R}_{i,0}, \quad i \geqslant 1.$$

Concerning the characterization of the basic reproduction $\mathcal{R}_0$ of (1.13), notice that the system is none other than an unfolded version of system (1.14) with regard to the number of reinfections. Therefore, both systems will share the same asymptotic behavior regarding the threshold $\mathcal{R}_0$. Let us compute then the $\mathcal{R}_0$ of (1.14) using the next-generation matrix method [32, 35]. Let $y = (\bar{E}, \bar{I})$, thus from (1.14) the differential equations for y can be written as

$$\dot{y} = \mathcal{F} - \mathcal{V}$$

with

$$\mathcal{F} = \begin{pmatrix} \beta\bar{S}\bar{I} \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} (\sigma+b-\nu\bar{I})\bar{E} \\ (\gamma+b+\nu)\bar{I} - \sigma\bar{E} - \nu\bar{I}^2 \end{pmatrix}.$$

Let F, V be the Jacobian of $\mathcal{F}$, $\mathcal{V}$ at the DFE:

$$F = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \sigma+b & 0 \\ -\sigma & \gamma+b+\nu \end{pmatrix}.$$

The inverse of V is

$$V^{-1} = \frac{1}{(\sigma+b)(\gamma+b+\nu)} \begin{pmatrix} \gamma+b+\nu & 0 \\ \sigma & \sigma+b \end{pmatrix}.$$

Hence, the next-generation matrix is

$$FV^{-1} = \frac{1}{(\sigma+b)(\gamma+\nu+b)} \begin{pmatrix} \beta\sigma & \beta(\sigma+b) \\ 0 & 0 \end{pmatrix}.$$

By definition, $\mathcal{R}_0$ is then the spectral radius of the next-generation matrix, which is given by (1.17).

Assume now $(\bar{S}_i^{EE}, \bar{E}_i^{EE}, \bar{I}_i^{EE}, \bar{R}_i^{EE})_{i\geqslant 1}$ is an EE for (1.13), it is then easy to see from Theorem 1.4.1 that $(\bar{S}^{EE}, \bar{E}^{EE}, \bar{I}^{EE}, \bar{R}^{EE})$, where $\bar{S}^{EE} = \sum_{i\geqslant 1} \bar{S}_i^{EE}, \bar{E}^{EE} = \sum_{i\geqslant 1} \bar{E}_i^{EE}, \bar{I}^{EE} = \sum_{i\geqslant 1} \bar{I}_i^{EE}, \bar{R}^{EE} = \sum_{i\geqslant 1} \bar{R}_i^{EE}$, is an EE for system (1.14). For the EE of (1.14), the steady state equations

$$\begin{aligned} 0 &= b - (\beta-\nu)\bar{S}^{EE}\bar{I}^{EE} + \omega\bar{R}^{EE} - b\bar{S}^{EE}, \\ 0 &= \beta\bar{S}^{EE}\bar{I}^{EE} - (\sigma+b)\bar{E}^{EE} + \nu\bar{I}^{EE}\bar{E}^{EE}, \\ 0 &= \sigma\bar{E}^{EE} - (\gamma+b+\nu)\bar{I}^{EE} + \nu(\bar{I}^{EE})^2, \\ 0 &= \gamma\bar{I}^{EE} - (\omega+b)\bar{R}^{EE} + \nu\bar{I}^{EE}\bar{R}^{EE}, \end{aligned} \tag{1.28}$$

imply the relations

$$\begin{aligned} \bar{R}^{EE} &= \frac{\gamma}{\omega+b-\nu\bar{I}^{EE}}\bar{I}^{EE}, \qquad \bar{E}^{EE} = \frac{\gamma+b+\nu-\nu\bar{I}^{EE}}{\sigma}\bar{I}^{EE}, \\ \beta\bar{S}^{EE}\bar{I}^{EE} &= (\sigma+b-\nu\bar{I}^{EE})\bar{E}^{EE}, \qquad b(1-\bar{S}^{EE}) = (\beta-\nu)\bar{S}^{EE}\bar{I}^{EE} + \omega\bar{R}^{EE}. \end{aligned} \tag{1.29}$$

Summing up the four identities in (1.28), we deduce that any solution of (1.28) fulfills $\bar{S}^{EE} + \bar{E}^{EE} + \bar{I}^{EE} + \bar{R}^{EE} = 1$. In addition, equations (1.29) can be rewritten as follows

$$\bar{R}^{EE} = \frac{\gamma \bar{I}^{EE}}{\omega + b - \nu \bar{I}^{EE}}, \quad \bar{E}^{EE} = \frac{(\gamma + b + \nu - \nu \bar{I}^{EE})\bar{I}^{EE}}{\sigma},$$
$$\bar{S}^{EE} = \frac{\sigma + b - \nu \bar{I}^{EE}}{\sigma} \frac{\gamma + b + \nu - \nu \bar{I}^{EE}}{\beta},$$

and

$$b(1 - \frac{\sigma + b - \nu \bar{I}^{EE}}{\sigma} \frac{\gamma + b + \nu - \nu \bar{I}^{EE}}{\beta}) = (\beta - \nu) \frac{\sigma + b - \nu \bar{I}^{EE}}{\sigma} \frac{\gamma + b + \nu - \nu \bar{I}^{EE}}{\beta} \bar{I}^{EE} - \omega \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} \bar{I}^{EE}.$$

The last equality can be rewritten as

$$1 = (\frac{\beta - \nu}{b} \bar{I}^{EE} + 1) \frac{\sigma + b - \nu \bar{I}^{EE}}{\sigma} \frac{\gamma + \nu + b - \nu \bar{I}^{EE}}{\beta} - \frac{\gamma}{b} \frac{\omega}{\omega + b - \nu \bar{I}^{EE}}. \tag{1.30}$$

Because $\bar{S}^{EE}, \bar{E}^{EE}, \bar{I}^{EE}, \bar{R}^{EE} > 0$, the quantity $\bar{I}^{EE}$ has to be a solution of a 4th degree polynomial equation taking value in $(0, \max\{1, \frac{\omega+b}{\nu}, \frac{\sigma+b}{\nu}\})$. It is proved in [80] that, under the condition $\mathcal{R}_0 > 1$, such a solution exists and is indeed unique in this interval. Finally, let $(\bar{S}^{EE}, \bar{E}^{EE}, \bar{I}^{EE}, \bar{R}^{EE})$ be the unique EE of (1.14), the endemic equilibrium equations of (1.13) are

$$\begin{aligned} 0 &= \omega \bar{R}_{i-1}^{EE} - (\beta - \nu) \bar{S}_i^{EE} \bar{I}^{EE} - b \bar{S}_i^{EE} \\ 0 &= \beta \bar{S}_i^{EE} \bar{I}^{EE} - (\sigma + b) \bar{E}_i^{EE} + \nu \bar{I}^{EE} \bar{E}_i^{EE} \\ 0 &= \sigma \bar{E}_i^{EE} - (\gamma + \nu + b) \bar{I}_i^{EE} + \nu \bar{I}^{EE} \bar{I}_i^{EE} \\ 0 &= \gamma \bar{I}_i^{EE} - (\omega + b) \bar{R}_i^{EE} + \nu \bar{I}^{EE} \bar{R}_i^{EE}. \end{aligned} \tag{1.31}$$

We rewrite the equations

$$\bar{R}_i^{EE} = \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} \bar{I}_i^{EE}, \qquad \bar{E}_i^{EE} = \frac{\gamma + \nu + b - \nu \bar{I}^{EE}}{\sigma} \bar{I}_i^{EE},$$
$$\bar{S}_i^{EE} = \frac{1}{\beta \bar{I}^{EE}} (\sigma + b - \nu \bar{I}^{EE}) \bar{E}_i^{EE} = \frac{\sigma + b - \nu \bar{I}^{EE}}{\beta} \frac{\gamma + \nu + b - \nu \bar{I}^{EE}}{\sigma} \frac{\bar{I}_i^{EE}}{\bar{I}^{EE}},$$
$$\bar{R}_{i-1}^{EE} = \frac{(\beta - \nu)\bar{I}^{EE} + b}{\omega} \bar{S}_i^{EE} = \frac{(\beta - \nu)\bar{I}^{EE} + b}{\omega} \frac{\sigma + b - \nu \bar{I}^{EE}}{\beta} \frac{\gamma + \nu + b - \nu \bar{I}^{EE}}{\sigma} \frac{\omega + b - \nu \bar{I}^{EE}}{\gamma} \frac{\bar{R}_i^{EE}}{\bar{I}^{EE}}.$$

Thus,

$$\bar{R}_i^{EE} = \frac{\omega}{(\beta - \nu)\bar{I}^{EE} + b} \frac{\beta}{\sigma + b - \nu \bar{I}^{EE}} \frac{\sigma}{\gamma + \nu + b - \nu \bar{I}^{EE}} \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} \bar{I}^{EE} \bar{R}_{i-1}^{EE}. \tag{1.32}$$

Notice, the relation above is still valid if we replace $(\bar{R}_i^{EE}, \bar{R}_{i-1}^{EE})$ by either $(\bar{I}_i^{EE}, \bar{I}_{i-1}^{EE})$, $(\bar{E}_{i-1}^{EE}, \bar{E}_{i-1}^{EE})$ or $(\bar{S}_i^{EE}, \bar{S}_{i-1}^{EE})$. Therefore, the coefficient

$$\alpha_\nu = \frac{\omega}{(\beta - \nu)\bar{I}^{EE} + b} \frac{\beta}{\sigma + b - \nu \bar{I}^{EE}} \frac{\sigma}{\gamma + \nu + b - \nu \bar{I}^{EE}} \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} \bar{I}^{EE},$$

governs the progression of endemic equilibrium values between the i-th and $i-1$th stages of the model, for $i \geqslant 2$. Denoting

$$S_\nu := \frac{\beta}{\sigma + b - \nu \bar{I}^{EE}} \frac{\sigma}{\gamma + b + \nu - \nu \bar{I}^{EE}},$$

then

$$\alpha_\nu = \frac{\omega}{(\beta - \nu)\bar{I}^{EE} + b} \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} S_\nu \bar{I}^{EE},$$

and the first line of (1.18) is proved. Moreover, if $\mathcal{R}_0 > 1$, the solution $\bar{I}^{EE}$ fulfills by definition

$$\begin{aligned} (\beta - \nu)\bar{I}^{EE} + b > 0, \quad (\sigma + b) - \nu \bar{I}^{EE} > 0, \\ (\gamma + \nu + b) - \nu \bar{I}^{EE} > 0, \quad (\omega + b) - \nu \bar{I}^{EE} > 0. \end{aligned} \tag{1.33}$$

Hence, $\alpha_\nu > 0$. Moreover, multiplying $\alpha_\nu \frac{b}{\gamma} \frac{\omega + b - \nu \bar{I}^{EE}}{\omega}$ before adding α_ν in both sides of the equation (1.30) leads to

$$\alpha_\nu (1 + \frac{b}{\gamma} \frac{\omega + b - \nu \bar{I}^{EE}}{\omega}) = \bar{I}^{EE}.$$

In particular, $\alpha_\nu < 1$. Hence, the series

$$\begin{aligned} \sum_{i=1}^{n} \bar{S}_i^{EE} = \bar{S}_1^{EE} \sum_{i \geqslant 1} \alpha_\nu^{i-1}, \quad \sum_{i=1}^{n} \bar{E}_i^{EE} = \bar{E}_1^{EE} \sum_{i \geqslant 1} \alpha_\nu^{i-1}, \\ \sum_{i=1}^{n} \bar{I}_i^{EE} = \bar{I}_1^{EE} \sum_{i \geqslant 1} \alpha_\nu^{i-1}, \quad \sum_{i=1}^{n} \bar{R}_i^{EE} = \bar{R}_1^{EE} \sum_{i \geqslant 1} \alpha_\nu^{i-1}, \end{aligned}$$

converge absolutely, and this allows to sum up (1.31) for $i \geqslant 1$, from which we retrieve as expected the equations (1.28), hence

$$\bar{S}^{EE} = \sum_{i \geqslant 1} \bar{S}_i^{EE}, \quad \bar{E}^{EE} = \sum_{i \geqslant 1} \bar{E}_i^{EE}, \quad \bar{I}^{EE} = \sum_{i \geqslant 1} \bar{I}_i^{EE}, \quad \bar{R}^{EE} = \sum_{i \geqslant 1} \bar{R}_i^{EE}.$$

Finally, consider (1.31) with $i = 1$ and the equality $\omega \bar{R}_0^{EE} = \mu$, we recover the expression for $(S_1^{EE}, E_1^{EE}, I_1^{EE}, R_1^{EE})$:

$$\bar{S}_1^{EE} = \frac{b}{(\beta - \nu)\bar{I}^{EE} + b}, \qquad \bar{E}_1^{EE} = \frac{\beta \bar{I}^{EE}}{\sigma + b - \nu \bar{I}^{EE}} \bar{S}_1^{EE}, \tag{1.34a}$$

$$\bar{I}_1^{EE} = \frac{\sigma}{\gamma + \nu + b - \nu \bar{I}^{EE}} \bar{E}_1^{EE}, \qquad \bar{R}_1^{EE} = \frac{\gamma}{\omega + b - \nu \bar{I}^{EE}} \bar{I}_1^{EE}. \tag{1.34b}$$

It is easy to verify that (1.34) is equivalent to the expression for $(\bar{S}_1^{EE}, \bar{E}_1^{EE}, \bar{I}_1^{EE}, \bar{R}_1^{EE})$ in (1.18). Moreover, we can see that the value of EE of (1.13) is uniquely determined by the unique EE of (1.14), hence the uniqueness.

1.4.6.3 Proof of Corollary 1.4.5

In the case where $\mathcal{R}_0 > 1$ and $\nu = 0$, adding the first two equalities of (1.28), initially

computed in [23], we recover

$$\bar{S}^{EE} = 1 + \frac{\omega}{b}\bar{R}^{EE} - \frac{\sigma+b}{b}\bar{E}^{EE} = 1 - \frac{(\gamma+b)(\sigma+b)(\omega+b) - \omega\gamma\sigma}{\sigma b(\omega+b)}\bar{I}^{EE} = 1 - \zeta\bar{I}^{EE}.$$

As $\bar{I}^{EE} > 0$ by assumption, replacing $\bar{S}^{EE}$ by the expression above, the second equation of (1.28) leads to

$$(1 - \zeta\bar{I}^{EE}) = \bar{S}^{EE} = \frac{\sigma+b}{\beta}\frac{\bar{E}^{EE}}{\bar{I}^{EE}} = \frac{1}{\mathcal{R}_0},$$

which is the same equation for $\bar{I}^{EE}$ as presented in the Corollary. Moreover,

$$\begin{aligned}
\alpha_0 &= \frac{\omega}{\beta\bar{I}^{EE}+b}\frac{\gamma}{\omega+b}\mathcal{R}_0\bar{I}^{EE} \\
&= \frac{\gamma}{\frac{b(\omega+b)(\beta\sigma-(\sigma+b)(\gamma+b))}{(\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega}+b}\frac{\sigma b(\omega+b)}{(\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega}\frac{\omega}{\omega+b}(\mathcal{R}_0-1) \\
&= \frac{\gamma}{\frac{b(\omega+b)(\beta\sigma-(\sigma+b)(\gamma+b))+b((\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega)}{(\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega}}\frac{\sigma b(\omega+b)}{(\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega}\frac{\omega}{\omega+b}(\mathcal{R}_0-1) \\
&= \frac{\gamma((\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega)}{b(\omega+b)\beta\sigma - b\sigma\gamma\omega}\frac{\sigma b(\omega+b)}{(\sigma+b)(\omega+b)(\gamma+b)-\sigma\gamma\omega}\frac{\omega}{\omega+b}(\mathcal{R}_0-1) \\
&= \frac{\gamma\omega}{\beta(\omega+b)-\gamma\omega}(\mathcal{R}_0-1),
\end{aligned}$$

we retrieved the identity for α_0 presented in the Corollary. Finally, let us verify that $\zeta > 1$. In fact, developing the numerator above,

$$\begin{aligned}
\zeta &= \frac{b(\sigma+b)(\omega+b)+\gamma(\sigma+b)(\omega+b)-\omega\gamma\sigma}{\sigma b(\omega+b)} = \frac{b(\sigma+b)(\omega+b)+\gamma b(\omega+b)+\gamma\sigma b}{\sigma b(\omega+b)} \\
&> \frac{b(\sigma+b)(\omega+b)}{\sigma b(\omega+b)} > 1,
\end{aligned}$$

which concludes the proof of Corollary 1.4.5.

1.4.6.4 Proof of Theorem 1.4.7

The case $\mathcal{R}_0 < 1$.

• For the macroscopic system, it was proven in [53] that the DFE of (1.14) is globally asymptotically stable for $\mathcal{R}_0 < 1$.

• For the microscopic system, let us prove the asymptotic behavior of each component $\bar{x}_i := (\bar{S}_i, \bar{E}_i, \bar{I}_i, \bar{R}_i)$. As

$$\lim_{t\to\infty}\bar{I}(t) = \lim_{t\to\infty}\bar{E}(t) = \lim_{t\to\infty}\bar{R}(t) = 0,$$

and

$$\bar{I}_i(t) \leqslant \bar{I}(t),\ \bar{E}_i(t) \leqslant \bar{E}(t),\ \bar{R}_i(t) \leqslant \bar{R}(t),$$

for all $i \geqslant 1$ and $t \geqslant 0$, thus

$$\lim_{t\to\infty}\bar{I}_i(t) = \lim_{t\to\infty}\bar{E}_i = \lim_{t\to\infty}\bar{R}_i = 0, \quad \text{for all} \quad i \geqslant 1.$$

Additionally, for any $\varepsilon > 0$, by definition there is a $T_\varepsilon > 0$ such that, for all $t \geqslant T_\varepsilon$,

$$t \geqslant T_\varepsilon \quad \Rightarrow \quad \bar{I}(t) \in [0, \varepsilon), \quad \bar{R}_i(t) \in [0, \varepsilon) \quad i \geqslant 1.$$

Let

$$A_\varepsilon := \begin{pmatrix} -b + \nu\varepsilon & 0 & 0 & 0 \\ \varepsilon\beta & -(\sigma + b) + \varepsilon\nu & 0 & 0 \\ 0 & \sigma & -(b + \gamma) + \varepsilon\nu & 0 \\ 0 & 0 & \gamma & -(\omega + b) + \varepsilon\nu \end{pmatrix},$$

then

$$\dot{\bar{x}}_{i+1} \leqslant A_\varepsilon \bar{x}_{i+1} + (\omega\varepsilon, 0, 0, 0)^\intercal, \quad i \geqslant 2, \quad t \geqslant T_\varepsilon.$$

For sufficiently small ε, the matrix A_ε is Metzler, and we may use the comparison result from [26] to deduce that, for y solution of

$$\dot{y} = A_\varepsilon y + (\omega\varepsilon, 0, 0, 0)^\intercal, \quad y(T_\varepsilon) = \bar{x}_{i+1}(T_\varepsilon), \tag{1.35}$$

we have

$$\forall t \geqslant T_\varepsilon, \quad \bar{x}_{i+1}(t) \leqslant y(t).$$

Moreover, for any ε, the matrix A_ε is Hurwitz, then any solution y of (1.35) converges to $-\omega\epsilon A_\varepsilon^{-1}(1, 0, 0, 0)^\intercal$ and therefore:

$$\limsup_{t \to +\infty} \bar{x}_i(t) \leqslant \limsup_{t \to +\infty} y(t) = -\omega\varepsilon A_\varepsilon^{-1}(1, 0, 0, 0)^\intercal, \quad i \geqslant 2.$$

As ε may be arbitrarily small, taking to the limits we deduce that

$$\limsup_{t \to +\infty} \bar{x}_i(t) \leqslant 0.$$

Consequently

$$\lim_{t \to \infty} \bar{S}_i(t) = 0, \quad i \geqslant 2.$$

Finally, remark that the following inequality is verified for $i = 1$ and $t \geqslant 0$:

$$\dot{\bar{x}_1} \geqslant A_0 \bar{x}_1 + b(1, 0, 0, 0)^\intercal,$$

hence

$$\liminf_{t \to +\infty} \bar{x}_1(t) \geqslant -bA_0^{-1}(1, 0, 0, 0)^\intercal,$$

but

$$-bA_0^{-1}(1, 0, 0, 0)^\intercal = \frac{-b}{b(\sigma + b)(b + \gamma)(\omega + b)}(-(\sigma + b)(\gamma + b)(\omega + b), 0, 0, 0)^\intercal = (1, 0, 0, 0)^\intercal,$$

and S_1 is upper bounded by 1, thus $\lim_{t \to \infty} \bar{S}_1(t) = 1$ and the result for $\mathcal{R}_0 < 1$.

The case $\mathcal{R}_0 > 1$.

• For the asymptotic behavior of the macroscopic system, the global stability of the unique endemic equilibrium $(\bar{S}^{EE}, \bar{E}^{EE}, \bar{I}^{EE}, \bar{R}^{EE})$ of (1.14) in $\mathbb{R}^4_+ \setminus \{(\bar{S}, \bar{E}, \bar{I}, \bar{R}) \in \mathbb{R}^4 : E + I = 0\}$ has been proven in [80] using the same geometrical approach of Li and Muldowney [75].

• For solutions starting in the set $\Gamma \setminus \{\bar{x} \in X_+^4 : \sum_{i\geqslant 1}(\bar{E}_i + \bar{I}_i) = 0\}$, let us prove the asymptotic convergence to the endemic equilibrium of (1.13) recursively using the asymptotic result of the macroscopic counterpart. For $i = 1$, we have

$$\begin{aligned}
\dot{\bar{S}}_1 &= b - (\beta - \nu)\bar{I}\bar{S}_1 - b\bar{S}_1,\\
\dot{\bar{E}}_1 &= \beta\bar{I}\bar{S}_1 - (\sigma + b)\bar{E}_1 + \nu\bar{I}\bar{E}_1,\\
\dot{\bar{I}}_1 &= \sigma\bar{E}_1 - (\gamma + \nu + b)\bar{I}_1 + \nu\bar{I}\bar{I}_1,\\
\dot{\bar{R}}_1 &= \gamma\bar{I}_1 - (\omega + b)\bar{R}_1 + +\nu\bar{I}\bar{R}_1.
\end{aligned} \tag{1.36}$$

Let ε be any positive number and $T_\varepsilon > 0$ such that

$$t \geqslant T_\varepsilon \qquad \Rightarrow \qquad \bar{I}(t) - \bar{I}^{EE} \in (-\varepsilon, \varepsilon).$$

For all $t \geqslant T_\varepsilon$, let

$$B_\varepsilon := \begin{pmatrix}
-\beta(\bar{I}^{EE}-\varepsilon)+\nu(\bar{I}^{EE}+\varepsilon)-b & 0 & 0 & 0\\
\beta(\bar{I}^{EE}+\varepsilon) & -(\sigma+b)+\nu(\bar{I}^{EE}+\varepsilon) & 0 & 0\\
0 & \sigma & -(\gamma+\nu+b)+\nu(\bar{I}^{EE}+\varepsilon) & 0\\
0 & 0 & \gamma & -(\omega+b)+\nu(\bar{I}^{EE}+\varepsilon)
\end{pmatrix},$$

then,

$$\begin{pmatrix}\dot{\bar{S}}_1 & \dot{\bar{E}}_1 & \dot{\bar{I}}_1 & \dot{\bar{R}}_1\end{pmatrix}^\intercal \leqslant B_\varepsilon \begin{pmatrix}\bar{S}_1 & \bar{E}_1 & \bar{I}_1 & \bar{R}_1\end{pmatrix}^\intercal + \begin{pmatrix}b & 0 & 0 & 0\end{pmatrix}^\intercal.$$

For all $\varepsilon > 0$, the matrix B_ε is Metzler. Let $\bar{x}_1$ be a solution of system (1.36) and y be the solution of

$$\dot{y} = B_\varepsilon y + b(1, 0, 0, 0)^\intercal, \qquad y(T_\varepsilon) = \bar{x}_1(T_\varepsilon). \tag{1.38}$$

the comparison principle [26] gives,

$$\forall t \geqslant T_\varepsilon, \qquad \bar{x}_1(t) \leqslant y(t),$$

where the vector inequality is interpreted component-wise. Moreover, by (1.33), for sufficiently small $\varepsilon > 0$ the matrix B_ε is Hurwitz and every solution y of (1.38) converges to $-bB_\varepsilon^{-1}(1, 0, 0, 0)^\intercal$. Therefore:

$$\limsup_{t\to+\infty} \bar{x}_1(t) \leqslant \limsup_{t\to+\infty} y(t) = -bB_0^{-1}(1, 0, 0, 0)^\intercal.$$

then, replacing ε by $-\varepsilon$, the same reasoning may be used to obtain :

$$\liminf_{t\to+\infty} \bar{x}_1(t) \geqslant -bB_0^{-1}(1, 0, 0, 0)^\intercal.$$

Hence finally:

$$\liminf_{t\to+\infty} \bar{x}_1(t) = \limsup_{t\to+\infty} \bar{x}_1(t) = \lim_{t\to+\infty} \bar{x}_1(t) = -bB_0^{-1}(1, 0, 0, 0)^\intercal.$$

Thus $\bar{x}_1(t)$ converges to $-bB_0^{-1}(1,0,0,0)^\intercal$, that is,

$$-bB_0^{-1}(1,0,0,0)^\intercal = \begin{pmatrix} \frac{b}{(\beta-\nu)\bar{I}^{EE}+b} \\ \frac{b\beta\bar{I}^{EE}}{(\sigma+b-\nu\bar{I}^{EE})((\beta-\nu)\bar{I}^{EE}+b)} \\ \frac{b\beta\bar{I}^{EE}\sigma}{(\sigma+b-\nu\bar{I}^{EE})(\gamma+\nu+b-\nu\bar{I}^{EE})((\beta-\nu)\bar{I}^{EE}+b)} \\ \frac{b\sigma\gamma\beta\bar{I}^{EE}}{(\sigma+b-\nu\bar{I}^{EE})(\gamma+\nu+b-\nu\bar{I}^{EE})(\omega+b-\nu\bar{I}^{EE})((\beta-\nu)\bar{I}^{EE}+b)} \end{pmatrix}.$$

By (1.34) the convergence is verified for $i = 1$.

Assuming now that the convergence is true for $i \geqslant 1$ and let us prove that it implies the same for $i+1$. Let $\bar{x}_{i+1}(t) = (\bar{S}_{i+1}(t), \bar{E}_{i+1}(t), \bar{I}_{i+1}(t), \bar{R}_{i+1}(t))$, ε be any positive number and $T_\varepsilon > 0$ such that

$$t \geqslant T_\varepsilon \qquad \Rightarrow \qquad \bar{I}(t) - \bar{I}^{EE} \in (-\varepsilon, \varepsilon), \quad \bar{R}_i - \bar{R}_i^{EE} \in (-\varepsilon, \varepsilon).$$

Then, for all $t \geqslant T_\varepsilon$,

$$\dot{\bar{x}}_{i+1}(t) \leqslant B_\varepsilon \bar{x}_{i+1}(t) + (\omega(\bar{R}_i^{EE} + \epsilon), 0, 0, 0)^\intercal.$$

Similarly to case $i = 1$, we have:

$$\limsup_{t\to+\infty} \bar{x}_{i+1}(t) \leqslant -bB_0^{-1}(\omega\bar{R}_i^{EE}, 0, 0, 0)^\intercal,$$

and

$$\liminf_{t\to+\infty} \bar{x}_{i+1}(t) \geqslant -bB_0^{-1}(\omega\bar{R}_i^{EE}, 0, 0, 0)^\intercal.$$

Thus:

$$\liminf_{t\to+\infty} \bar{x}_{i+1}(t) = \limsup_{t\to+\infty} \bar{x}_{i+1}(t) = \lim_{t\to+\infty} \bar{x}_{i+1}(t) = -bB_0^{-1}(\omega\bar{R}_i^{EE}, 0, 0, 0)^\intercal$$

consequently $\bar{x}_{i+1}(t)$ converges to $-bB_0^{-1}(\omega\bar{R}_i^{EE}, 0, 0, 0)^\intercal$, which equals to

$$-bB_0^{-1}(\omega\bar{R}_i^{EE}, 0, 0, 0)^\intercal = K_i \begin{pmatrix} (\gamma+\nu+b-\nu\bar{I}^{EE})(\sigma+b-\nu\bar{I}^{EE})(\omega+b-\nu\bar{I}^{EE}) \\ \beta\bar{I}^{EE}(\gamma+\nu+b-\nu\bar{I}^{EE})(\omega+b-\nu\bar{I}^{EE}) \\ \beta\bar{I}^{EE}\sigma(\omega+b-\nu\bar{I}^{EE}) \\ \sigma\gamma\beta\bar{I}^{EE} \end{pmatrix},$$

where

$$K_i = \frac{\omega\bar{R}_i^{EE}}{(\sigma+b-\nu\bar{I}^{EE})(\gamma+\nu+b-\nu\bar{I}^{EE})(\omega+b-\nu\bar{I}^{EE})(\beta\bar{I}^{EE}+b-\nu\bar{I}^{EE})}.$$

Finally, thanks to (1.32), the convergence is verified for $i+1$. Hence, the convergence is demonstrated.

Global asymptotic stability properties. We proved that the solutions of (1.10) converge component-wise to either the DFE or the EE depending on the value of $\mathcal{R}_0$. As the solutions are moreover bounded, evolving in the invariant set Γ, this implies the convergence in X^4. In this setting, the DFE (resp. EE) is moreover stable for $\mathcal{R}_0 < 1$ (resp. $\mathcal{R}_0 > 1$) thanks to the stability result proved in [55, Theorem 3.3.2] for semigroups.

Indeed, let X be a complete metric space and $T(t) : X \to X$ be a C^r-semigroup, $r \geqslant 0$. If the semigroup $T(t)$ is moreover *asymptotically smooth*, i.e., for any nonempty, closed, bounded

set $B \subset X$ such that $T(t)B \subset B$, there is a compact set $C \subset B$ such that C attracts B. Then, in particular, [55, Theorem 3.3.2] states that any compact invariant set J attracting points locally is stable if and only if there is a bounded neighborhood W of J such that $T(t)W \subset W, t \geqslant 0$, and J attracts compact sets of W.

Taking $X = \Gamma$, which is complete, the well-posedness of the system (1.10) implies that it generates a C^0-semigroup $T(t)$. The asymptotic smoothness can be easily derived, as any trajectory of (1.10) is either attracted to the DFE or EE.

- For $\mathcal{R}_0 < 1$, it is sufficient to take Γ as the positively invariant set corresponding to W in the Theorem and the DFE as the invariant attracting set J. As the DFE attracts globally in Γ, then it attracts a fortiori every compact subset of Γ. Therefore, the DFE is stable for $\mathcal{R}_0 < 1$.

- For $\mathcal{R}_0 > 1$, $\{\bar{x} \in X_+^4 : \|\bar{x}\|_{X^4} = 1 \text{ and } \sum_{i\geqslant 1}(\bar{E}_i + \bar{I}_i) > 0\}$ is a suitable candidate for the positively invariant set W, and as the EE attracts globally W, it is stable.

To conclude, the DFE (resp. EE) is globally asymptotically stable in Γ for $\mathcal{R}_0 < 1$ (resp. in $\{\bar{x} \in X_+^4 : \|\bar{x}\|_{X^4} = 1 \text{ and } \sum_{i\geqslant 1}(\bar{E}_i + \bar{I}_i) > 0\}$ for $\mathcal{R}_0 > 1$).

1.4.6.5 Proof of Theorem 1.4.9

• The assertions 1. and 2.(a) are evident due to

$$\dot{N}(t) = (b - \mu)N(t) - \nu I(t).$$

• If $b = \mu$ and $\mathcal{R}_0 > 1$, the Theorem 1.4.7 establishes that $\bar{I}(t)$ converges to a positive limit $\bar{I}^{EE}$. Rewriting the differential equation in this case, we have

$$\dot{N} = -\nu I(t) = -\nu \bar{I}(t)N(t).$$

Thus, let ε be a positive number, for sufficiently large t, $\dot{N}$ verifies

$$\dot{N} \leqslant -\nu(\bar{I}^{EE} - \varepsilon)N(t).$$

It is then sufficient to take ε small enough to have $\bar{I}^{EE} - \varepsilon > 0$ and 2.$(b).ii$.
• For the case 2.$(b).i$ (i.e., $b = \mu$ and $\mathcal{R}_0 < 1$), $\bar{I}$ converges exponentially to 0 by the proof of Theorem 1.4.7. Hence, taking t to the limit $t \to +\infty$, there are $c_1, c_2 > 0$ such that

$$-\nu c_1 e^{-c_2 t} N \leqslant \dot{N} \leqslant 0,$$

applying then the logarithm, we obtain

$$\frac{d}{dt} \log N \geqslant \nu \frac{c_1}{c_2} \frac{d}{dt} e^{-c_2 t},$$

thus

$$\log N(t) \geqslant \log N(0) + \nu \frac{c_1}{c_2}(e^{-c_2 t} - 1) \geqslant \log N(0) - \nu \frac{c_1}{c_2},$$

and finally :

$$N(t) \geqslant N(0) e^{-\nu \frac{c_1}{c_2}} > 0.$$

$N(t)$ is therefore non-increasing and bounded from below by a positive quantity, and 2.(b).i is proved.

• If $b > \mu$ and $b - \mu > \bar{I}^{EE}$, then for all $\varepsilon > 0$ and $t > 0$ large enough:

$$\left(b - \mu - \nu(\bar{I}^{EE} + \varepsilon)\right) N \leqslant \dot{N},$$

therefore, we obtain $b - \mu - \nu(\bar{I}^{EE} + \varepsilon) > 0$ and 3.(a) for ε sufficiently small.

• Finally, for the case $b - \mu < \bar{I}^{EE}$, we consider $-\varepsilon$ instead of ε in the computation above to derive 3.(b).

1.5 Coefficients with geometric progression

We consider in this section the system (1.5) with coefficients following the geometric progression

$$\beta_{i,j} = \beta\rho_S^i\rho_I^j, \quad \omega_i = \omega\rho_R^i, \quad \sigma_i = \sigma, \quad \gamma_i = \gamma,$$

and $\beta, \sigma, \gamma, \omega > 0$, $0 \leqslant \rho_R, \rho_S, \rho_I \leqslant 1$. In other words, there is a geometrical decrease in susceptibility in a proportion of ρ_S, infectivity in a proportion of ρ_I and an increase in immunity in a proportion of ρ_R at every infection. The quite natural assumption on the model, which motivates the study, is that every reinfection strengthens the infected host's immunity and hence decreases its susceptibility and infectivity with regard to the disease, and slows down the immunity waning. Moreover, it is assumed that all these decays are exponential with respect to the number of reinfections. Last, we assume here that the incubation rate σ and the recovery rate ω do not depend upon the number of reinfections (but such hypotheses may be considered if needed). This leads to the system :

$$\begin{aligned}
\dot{S}_i &= \omega\rho_R^{i-1}R_{i-1} - \beta\rho_S^i S_i \textstyle\sum_j \rho_I^j I_j - \mu S_i, \qquad i \geqslant 1,\\
\dot{E}_i &= \beta\rho_S^i S_i \textstyle\sum_j \rho_I^j I_j - (\sigma + \mu)E_i, \qquad i \geqslant 1,\\
\dot{I}_i &= \sigma E_i - (\gamma + \mu)I_i, \qquad i \geqslant 1,\\
\dot{R}_i &= \gamma I_i - (\omega\rho_R^i + \mu)R_i, \qquad i \geqslant 1,
\end{aligned} \tag{1.40}$$

where we have the equality $R_0 = \frac{\mu}{\omega}$. The macroscopic quantities are then solutions of the differential system

$$\begin{aligned}
\dot{S} &= \mu - \beta \sum_i \rho_S^i S_i \sum_j \rho_I^j I_j + \omega \sum_i \rho_R^i R_i - \mu S,\\
\dot{E} &= \beta \sum_i \rho_S^i S_i \sum_j \rho_I^j I_j - (\sigma + \mu)E,\\
\dot{I} &= \sigma E - (\gamma + \mu)I,\\
\dot{R} &= \gamma I - \omega \sum_i \rho_R^i R_i - \mu R.
\end{aligned} \tag{1.41}$$

The equations above show that there is apparently no closed form formula for the macroscopic behavior. Therefore, the case where the coefficients follow the geometric progression is more intricate for the analysis. However, we were able to derive a threshold for the appearance of endemic equilibrium, which is the content of the subsection 1.5.1 below.

1.5.1 Equilibrium, basic reproduction number

Theorem 1.5.1. *Define* $\mathcal{R}_0 := \frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\rho_I\rho_S$ *and* $\delta := \frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\rho_I}{\rho_S}$. *System* (1.40) *has a unique disease-free equilibrium* $x^{DFE} := (\delta_{1i}, 0, 0, 0)_{i\geqslant 1}$. *When* $\mathcal{R}_0 < 1$, *there is no endemic equilibrium. The existence of endemic equilibrium is guaranteed for* $\mathcal{R}_0 > 1$ *and moreover its uniqueness for* $\mathcal{R}_0 > (\frac{\delta}{1-\delta})^2$.

Proof. **Existence.**

System (1.40) possesses only one DFE which is $(\delta_i^1, 0, 0, 0)_{i\geqslant 1}$. At any endemic equilibrium $(S_i^{EE}, E_i^{EE}, I_i^{EE}, R_i^{EE})_{i\geqslant 1}$, the derivatives cancel out, and the equilibrium point fulfills the equations below

$$\begin{aligned}
0 &= \omega\rho_R^{i-1}R_{i-1}^{EE} - \beta\rho_S^i S_i^{EE}\sum_j \rho_I^j I_j^{EE} - \mu S_i^{EE}, \qquad i \geqslant 1,\\
0 &= \beta\rho_S^i S_i^{EE}\sum_j \rho_I^j I_j^{EE} - (\sigma+\mu)E_i^{EE}, \qquad i \geqslant 1,\\
0 &= \sigma E_i^{EE} - (\gamma+\mu)I_i^{EE}, \qquad i \geqslant 1,\\
0 &= \gamma I_i^{EE} - (\omega\rho_R^i + \mu)R_i^{EE}, \qquad i \geqslant 1.
\end{aligned} \tag{1.42}$$

Denoting $I^{EE} := \sum_i \rho_I^i I_i^{EE}$, then I^{EE} must be < 1 in order to be an equilibrium value that is realistic. From the equations (1.42) we deduce the relation

$$R_i^{EE} = \frac{\gamma}{\omega\rho_R^i+\mu}I_i^{EE}, \quad I_i^{EE} = \frac{\sigma}{\gamma+\mu}E_i^{EE}, E_i^{EE} = \frac{\beta\rho_S^i I^{EE}}{\sigma+\mu}S_i^{EE}, \quad S_i^{EE} = \frac{\omega\rho_R^{i-1}}{\beta\rho_S^i I^{EE}+\mu}R_{i-1}^{EE},$$

hence, for $i \geqslant 2$,

$$R_i^{EE} = \frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\beta\rho_S^j I^{EE}}{\beta\rho_S^j I^{EE}+\mu}\frac{\omega\rho_R^{j-1}}{\omega\rho_R^j+\mu}R_{i-1}^{EE} = \alpha_i R_{i-1}^{EE}. \tag{1.43}$$

Furthermore, as $R_0 = \frac{\mu}{\omega}$,

$$R_i^{EE} = \frac{\mu}{\omega}\prod_{j=1}^{i}\left(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\beta\rho_S^j I^{EE}}{\beta\rho_S^j I^{EE}+\mu}\frac{\omega\rho_R^{j-1}}{\omega\rho_R^j+\mu}\right), \qquad i \geqslant 1.$$

Define

$$\Phi_i(I^{EE}) := \prod_{j=1}^{i}\left(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\beta\rho_S^j I^{EE}}{\beta\rho_S^j I^{EE}+\mu}\frac{\omega\rho_R^{j-1}}{\omega\rho_R^j+\mu}\right).$$

Hence for all $i \geqslant 1$,

$$\begin{aligned}
R_i^{EE} &= \frac{\mu}{\omega}\Phi_i(I^{EE}), \qquad I_i^{EE} = \frac{\mu+\omega\rho_R^i}{\gamma}R_i^{EE} = \frac{\mu+\omega\rho_R^i}{\gamma}\frac{\mu}{\omega}\Phi_i(I^{EE}),\\
I^{EE} &= \sum_{i\geqslant 1}\rho_I^i I_i^{EE} = \frac{\mu}{\omega}\sum_{i\geqslant 1}\rho_I^i\frac{\mu+\omega\rho_R^i}{\gamma}\Phi_i(I^{EE}) := \Phi(I^{EE}),
\end{aligned}$$

with the expression of $\Phi(x)$ defined by

$$\begin{aligned}\Phi(x) &= \frac{\mu}{\omega}\sum_{i\geqslant 1}(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu})^i\rho_I^i(\prod_{j=1}^{i}\frac{\beta\rho_S^j x}{\beta\rho_S^j x+\mu})(\prod_{j=1}^{i}\frac{\omega\rho_R^{j-1}}{\omega\rho_R^j+\mu})\frac{\mu+\omega\rho_R^i}{\gamma}\\ &= \frac{\mu}{\gamma}\sum_{i\geqslant 1}\rho_I^i(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu})^i(\prod_{j=1}^{i}\frac{\beta\rho_S^j x}{\beta\rho_S^j x+\mu})(\prod_{j=1}^{i-1}\frac{\omega\rho_R^j}{\omega\rho_R^j+\mu}).\end{aligned}$$

Let us fix $x=0$, thus $\Phi(0)=0$ by the previous formula. For $x=1$, we have

$$\begin{aligned}\Phi(1) &= \sum_{i\geqslant 1}\frac{\mu}{\gamma}\rho_I^i(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu})^i(\prod_{j=1}^{i-1}\frac{\omega\rho_R^j}{\omega\rho_R^j+\mu})(\prod_{j=1}^{i}\frac{\beta\rho_S^j}{\beta\rho_S^j+\mu})\\ &\leqslant \frac{\mu}{\gamma}\frac{\omega\rho_R+\mu}{\omega\rho_R}\sum_{i\geqslant 1}(\rho_I\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\omega\rho_R}{\omega\rho_R+\mu}\frac{\beta\rho_S}{\beta\rho_S+\mu})^i\\ &\leqslant \frac{\mu}{\gamma}\frac{\omega\rho_R+\mu}{\omega\rho_R}\frac{\rho_I\gamma\sigma\omega\rho_R\beta\rho_S}{(\gamma+\mu)(\sigma+\mu)(\omega\rho_R+\mu)(\beta\rho_S+\mu)-\rho_I\gamma\sigma\omega\rho_R\beta\rho_S}\\ &\leqslant \frac{\mu(\omega\rho_R+\mu)\rho_I\sigma\beta\rho_S}{(\gamma+\mu)(\sigma+\mu)(\omega\rho_R+\mu)(\beta\rho_S+\mu)-\rho_I\gamma\sigma\omega\rho_R\beta\rho_S}\\ &\leqslant \frac{\mu(\omega\rho_R+\mu)\sigma\beta\rho_S}{(\gamma+\mu)(\sigma+\mu)(\omega\rho_R+\mu)(\beta\rho_S+\mu)-\gamma\sigma\omega\rho_R\beta\rho_S}<1.\end{aligned}$$

Additionally,

$$\Phi'(0)=a_1\frac{\rho_S\beta}{\mu}=\frac{\mu}{\gamma}\rho_I\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\frac{\rho_S\beta}{\mu}=\frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\rho_I\rho_S.$$

Thus for $\mathcal{R}_0:=\frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\rho_I\rho_S>1$, $\Phi(x)>x$ on the right of the point $x=0$. Therefore, by the fact that $\Phi(1)<1$, $\Phi(x)$ has at least a fixed point in $[0,1]$, hence the existence of at least one equilibrium for the system (1.40) in the case when $\mathcal{R}_0>1$.

Let us consider the case $\frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}\rho_I\rho_S<1$. Let θ be $\theta:=\rho_I(\frac{\gamma}{\gamma+\mu}\frac{\sigma}{\sigma+\mu})$ such that

$$\Phi(x)=\frac{\mu}{\gamma}\sum_{i\geqslant 1}\theta^i\prod_{j=1}^{i}\left(\frac{\beta\rho_S^j x}{\beta\rho_S^j x+\mu}\right)\prod_{j=1}^{i-1}\left(\frac{\omega\rho_R^j}{\omega\rho_R^j+\mu}\right),$$

therefore

$$\Phi(x)\leqslant\frac{\mu}{\gamma}\sum_{i\geqslant 1}\theta^i(\frac{\beta\rho_S x}{\beta\rho_S x+\mu})^i\leqslant\frac{\mu}{\gamma}\frac{\theta\beta\rho_S x}{\beta\rho_S x+\mu-\theta\beta\rho_S x}.$$

Assuming then $0<x<1$ and $\Phi(x)=x$. Divide the inequality by x on both sides

$$1\leqslant\frac{\mu}{\gamma}\frac{\theta\beta\rho_S}{(1-\theta)\beta\rho_S x+\mu},\text{ hence }(1-\theta)\beta\rho_S x\leqslant\mu(\frac{\theta\beta\rho_S}{\gamma}-1),$$

however we have

$$\frac{\theta\beta\rho_S}{\gamma}-1=\frac{\beta}{\gamma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I\rho_S-1=\mathcal{R}_0-1.$$

Hence the previous inequality could not be fulfilled if $\mathcal{R}_0 < 1$. As a consequence, there is no endemic equilibrium in this case.

Uniqueness.

• **Bounding Φ'**

Defining

$$\varphi(x) := \frac{\beta x}{\beta x + \mu},$$

it is then easy to prove that, for all $x > 0$,

$$0 < \varphi(x) < \min\left\{\frac{\beta}{\mu}x, 1\right\}.$$

The function φ is positive, increasing, and concave.

On the other hand the function Φ can be rewritten as

$$\Phi(x) = \frac{\mu}{\gamma}\sum_{i\geqslant 1}\left(\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I\right)^i\left(\prod_{j=1}^{i-1}\frac{\omega\rho_R^j}{\omega\rho_R^j+\mu}\right)\left(\prod_{j=1}^{i}\varphi(\rho_S^j x)\right). \tag{1.44}$$

For all $a > 0$, denote $\varphi_a : x \to \varphi(ax)$, thus for all $i \geqslant 1$, one has

$$\frac{d}{dx}\left(\prod_{j=1}^{i}\varphi(\rho_S^j x)\right) = \sum_{j=1}^{i}\varphi(\rho_S x)\dots\varphi(\rho_S^{j-1}x)\varphi'_{\rho_S^j}(x)\varphi(\rho_S^{j-1}x)\dots\varphi(\rho_S^i x). \tag{1.45}$$

and the derivation of ϕ_a led to

$$\varphi'_a(x) = \frac{\beta a\mu}{(\beta a x+\mu)^2} = \frac{a}{b}\frac{(\beta b x+\mu)^2}{(\beta a x+\mu)^2}\varphi'_b(x) = \frac{b}{a}\left(\frac{1+\frac{\mu}{\beta b x}}{1+\frac{\mu}{\beta a x}}\right)^2\varphi'_b(x).$$

In particular, while φ and its derivative are positive, if $0 < a \leqslant b$, then for all $x \geqslant 0$:

$$\varphi'_a(x) \leqslant \frac{b}{a}\varphi'_b(x).$$

Thus, from (1.45) and the fact that $\rho_S^j \leqslant \rho_S$ (because $\rho_S \leqslant 1$):

$$\begin{aligned}\frac{d}{dx}\left(\prod_{j=1}^{i}\varphi(\rho_S^j x)\right) &\leqslant \varphi(\rho_S x)^{i-1}\sum_{j=1}^{i}\varphi'_{\rho_S^j}(x) \leqslant \varphi(\rho_S x)^{i-1}\sum_{j=1}^{i}\rho_S^{1-j}\varphi'_{\rho_S}(x)\\ &\leqslant i\varphi(\rho_S x)^{i-1}\varphi'_{\rho_S}(x)\rho_S^{-(i-1)} = \frac{d}{dx}(\varphi^i_{\rho_S}(x))\rho_S\rho_S^{-i}.\end{aligned}$$

Then the derivative of (1.44) is, for all $x > 0$

$$\frac{d\Phi(x)}{dx} < \frac{\mu}{\gamma}\sum_{i\geqslant 1}\left(\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I\right)^i\frac{d}{dx}\left(\prod_{j=1}^{i}\varphi_{\rho_S^j}(x)\right) \leqslant \frac{\mu\rho_S}{\gamma}\frac{d}{dx}\left(\sum_{i\geqslant 1}\left(\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\frac{\rho_I}{\rho_S}\varphi_{\rho_S}(x)\right)^i\right).$$

The computations of the series and its derivative lead to

Lemma 1.5.2. *Assuming $\delta\varphi(\rho_S) \in (0,1)$ with δ defined as*

$$\delta := \frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\frac{\rho_I}{\rho_S}. \tag{1.46}$$

For all $x > 0$,

$$\frac{d\Phi(x)}{dx} < \frac{\mu\rho_S}{\gamma}\frac{d}{dx}\left(\frac{\delta\varphi_{\rho_S}(x)}{1-\delta\varphi_{\rho_S}(x)}\right) = \frac{\mu\rho_S}{\gamma}\frac{d}{dx}\left(\frac{\delta\beta\rho_S x}{(1-\delta)\beta\rho_S x+\mu}\right) = \frac{\mu\rho_S}{\gamma}\frac{\delta\beta\rho_S\mu}{((1-\delta)\beta\rho_S x+\mu)^2}.$$

Remark that the upper bound above is *exact* for $x = 0$. Indeed,

$$\Phi'(0) = \frac{\delta\beta\rho_S^2}{\gamma} = \frac{\sigma}{\sigma+\mu}\frac{\mu}{\gamma+\mu}\beta\rho_I\rho_S =: \mathcal{R}_0.$$

• **Bounding Φ**

Thanks to (1.44) and the fact that φ is increasing, one has for all $x > 0$

$$\Phi(x) < \frac{\mu}{\gamma}\sum_{i\geqslant 1}\left(\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I\right)^i\left(\prod_{j=1}^{i}\varphi(\rho_S^j x)\right) \leqslant \frac{\mu}{\gamma}\sum_{i\geqslant 1}\left(\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I\varphi(\rho_S x)\right)^i = \frac{\mu}{\gamma}\frac{\delta'\varphi(\rho_S x)}{1-\delta'\varphi(\rho_S x)}$$
$$= \frac{\mu}{\gamma}\frac{\delta'\beta\rho_S x}{(1-\delta')\beta\rho_S x+\mu}, \quad \text{where } \delta' := \frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}\rho_I = \rho_S\delta.$$

On the other side, we derive a lower bound for Φ by retaining only the first term of the series. The next lemma summarizes this result :

Lemma 1.5.3. *For all $x > 0$,*

$$\frac{\mu}{\gamma}\frac{\delta'\beta\rho_S x}{\beta\rho_S x+\mu} < \Phi(x) < \frac{\mu}{\gamma}\frac{\delta'\beta\rho_S x}{(1-\delta')\beta\rho_S x+\mu}.$$

Considering now the rational function below

$$\psi(x) := \frac{\mu}{\gamma}\frac{\delta'\beta\rho_S x}{\beta\rho_S x+\mu}.$$

As well as φ studied previously, the function ψ is positive, increasing, and concave. Moreover, the result of lemma 1.5.2 and 1.5.3 can be expressed as following :

$$\psi(x) < \Phi(x) < \frac{1}{1-\delta'}\psi((1-\delta')x), \quad \Phi'(x) < \psi'((1-\delta)x), \qquad x > 0.$$

• **Uniqueness of the fixed point**

The following result below is required.

Lemma 1.5.4. *Assume,*

$$\mathcal{R}_0 = \Phi'(0) = \frac{\delta'\beta\rho_S}{\gamma} > 1.$$

Let x^{EE} be the unique positive value such that $\psi(x^{EE}) = x^{EE}$. Then,

$$\psi'((1-\delta)x^{EE}) \leqslant 1 \qquad \Rightarrow \qquad \exists!\ x^{EE} > 0,\ \Phi(x^{EE}) = x^{EE}. \tag{1.47}$$

Proof of Lemma 1.47. Indeed, at any point $x \in (0, x^{EE})$, one has $x < \psi(x) < \Phi(x)$. Every fixed point of the function Φ belongs then to the interval $(x^{EE}, +\infty)$. As ψ is concave, then for every point x of this interval

$$1 \geqslant \psi'((1-\delta)x^{EE}) > \psi'((1-\delta)x) > \Phi'(x) > 0.$$

The fixed point of Φ is then unique (and belongs to $(x^{EE}, +\infty)$) if the condition (1.47) is fulfilled (One may also obtain another estimation using Lemma 1.5.3). □

By definition, x^{EE} fulfills

$$\frac{\mu}{\gamma}\frac{\delta'\beta\rho_S}{\beta\rho_S x^{EE}+\mu} = 1,$$

then,

$$x^{EE} = \frac{\mu}{\beta\rho_S}\left(\frac{\delta'\beta\rho_S}{\gamma} - 1\right).$$

and

$$\psi'((1-\delta)x^{EE}) = \frac{\mu}{\gamma}\frac{\delta'\beta\rho_S\mu}{((1-\delta)\beta\rho_S x^{EE}+\mu)^2} = \frac{\mathcal{R}_0}{((1-\delta)(\mathcal{R}_0-1)+1)^2} = \frac{\mathcal{R}_0}{(1-\delta)^2\mathcal{R}_0^2 + 2\delta(1-\delta)\mathcal{R}_0 + \delta^2}.$$

the condition (1.47) is thus equivalent to

$$\frac{\mathcal{R}_0}{((1-\delta)\mathcal{R}_0+\delta)^2} < 1,$$

Therefore the uniqueness of the fixed point and consequently uniqueness of the endemic equilibrium are guaranteed for $\mathcal{R}_0 > \max\left\{1, \left(\frac{\delta}{1-\delta}\right)^2\right\}$. □

Chapter 2

Distinguishing primary and secondary infections: a two-stage reinfection model

2.1 Introduction

This chapter is intended for the study of reinfection models where all reinfections occur along identical characteristics, which are different from the characteristics of the primary infection. The motivation for this study comes from the observation that, in most of the existing models for reinfection, the infection and reinfection processes are assumed to behave essentially the same, and there is no comprehensive investigation into the possible consequences of heterogeneity between the primary infections and reinfections. This is because in the standard reinfection models [15, 67], the infected often belong to the same compartment regardless of their history of infections, and these types of models are usually not suitable for considering disease with different reinfection behaviors. Nevertheless, several works have considered reinfection models with partial immunity, which is a simple example of the kind of heterogeneity that we are concerned about. Notably, a SIRI model with partial immunity has been thoroughly analyzed in [51]. It is demonstrated therein, among other things, the existence of a threshold on the reinfection parameter by which the system transits from low-infectivity to high-infectivity equilibrium. On the other hand, a simple SIRI reinfection model with partial immunity and without demography effects has been studied in [94], exhibiting bistability for $\mathcal{R}_0 < 1$.

To gain further insight into the effects of heterogeneity between infection and reinfection, we introduce in this chapter a *two-stage* SEIRS model, illustrated by the flowchart in Figure 2.1. As in the standard SEIRS model, the classes of compartments *S, E, I, R* represent in this case respectively the susceptible, exposed, infectious and recovered. The SEIRS models take the assumption that there exists a latent period for hosts infected by the disease before becoming infectious [15]. The specificity of the model that we have introduced is that it separates primary infections from the subsequent infections. This is rendered through the subscripts 1 and r (as 'remaining'). In terms of differential equations, we obtain the following system for the two-stage

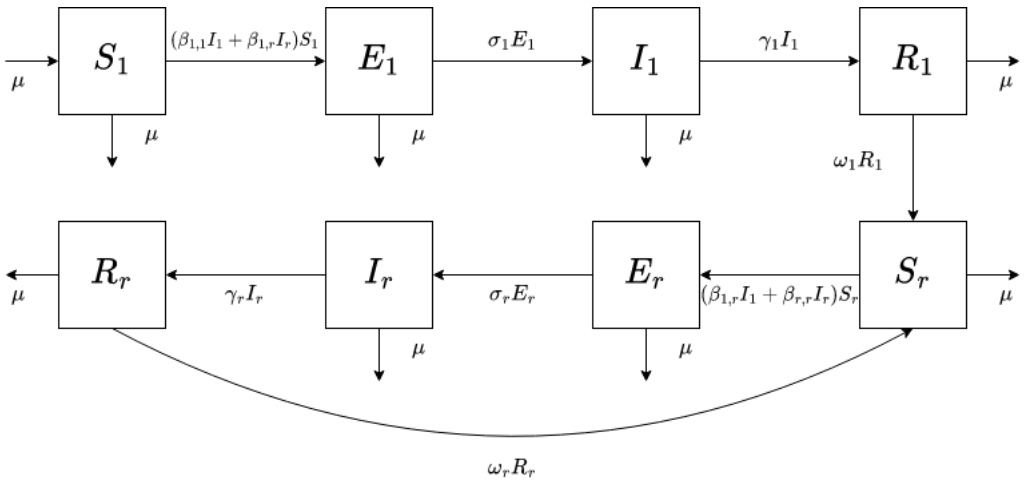


Figure 2.1: Flowchart of the two-stage SEIRS model

SEIRS model:

$$\begin{aligned}
\dot{S}_1 &= \mu - (\beta_{1,1}I_1 + \beta_{1,r}I_r + \mu)S_1,\\
\dot{E}_1 &= (\beta_{1,1}I_1 + \beta_{1,r}I_r)S_1 - (\sigma_1 + \mu)E_1,\\
\dot{I}_1 &= \sigma_1 E_1 - (\gamma_1 + \mu)I_1,\\
\dot{R}_1 &= \gamma_1 I_1 - (\omega_1 + \mu)R_1,\\
\dot{S}_r &= \omega_1 R_1 + \omega_r R_r - (\beta_{r,1}I_1 + \beta_{r,r}I_r + \mu)S_r,\\
\dot{E}_r &= (\beta_{r,1}I_1 + \beta_{r,r}I_r)S_r - (\sigma_r + \mu)E_r,\\
\dot{I}_r &= \sigma_r E_r - (\gamma_r + \mu)I_r,\\
\dot{R}_r &= \gamma_r I_r - (\omega_r + \mu)R_r,
\end{aligned} \tag{2.1}$$

where the parameter μ is the constant natural birth rate, hence the model considers a constant total population, which can be assumed to be equal to one. Next, $\beta_{1,1}, \beta_{1,r}, \beta_{r,1}, \beta_{r,r}$ are transmission rates, respectively for primary infections acquired from primary infected and from reinfected; and for reinfections transmitted by primary infected and by reinfected. The parameters σ_1, σ_r are respectively the rates at which the primary and secondary exposed hosts become infective. Moreover, γ_1, γ_r are rates of recovery and ω_1, ω_r represent the loss of immunity, depending again on whether the infection is primary or secondary.

In the spirit of Chapter 1, the two-stage reinfection models are initially derived from the study of the infinite reinfection models tracking the number of reinfection. In fact, the present model can be considered as the macroscopic model corresponding to the general infinite model (1.5), in the case where the epidemiological parameters differ only between the primary infection and the reinfections. Therefore, we may expect to find in the same way results somehow similar to the ones in Section 1.4 of Chapter 1, taking advantage again of the existence of a macroscopic counterpart to the infinite model. For instance, the endemic equilibriums of the model and the mean number of reinfections at equilibrium of the infinite model are again topics worth investigating. However, these results depend strongly on the properties of the macroscopic two-stage model, which will be explored in the present chapter.

It is noteworthy to mention that, concurrently with our study and driven by the need for

more complex models during the COVID-19 pandemic, two-stage reinfection models have been independently proposed in several papers of mathematical epidemiology. In fact, an age-specific two-stage SEIRS model has been utilized for statistical inference of infection levels in Covid-19 epidemics [18]. Moreover, in [63] Kaklamanos et al. investigated the backward bifurcations and the different time scales regarding the dynamics of a two-stage SIRS model, which appears to be very similar to our own albeit with a lower level of complexity. Furthermore, two-stage models have also found their application for the modelling of animal infections, such as bovine tuberculosis (BTB) or Aujesky's disease (pseudorabies virus) in pigs [7, 54]. These considerations underscore the significance of the two-stage model for diseases that permit reinfections.

In the present chapter, we are mainly interested in describing the steady states of the model, how the parameters affect the reinfection and whether there is a reinfection threshold similar to the one underlined by Gomes et al. [50]. In a second time, the persistence of the system is analyzed in the mathematical framework developed in [49]. Persistence is a well-known topic in mathematical biology [108, 112], and particularly in epidemiology, as it is concerned with the issue of disease eradication [50, 82]. Furthermore, the persistence result may in some instances serve as the initial step in establishing asymptotic behavior results, in case where the direct Lyapunov method fails [74, 75]. Finally, the question of asymptotic behavior of the epidemiological models is also studied. As we shall see, the two-stage models often exhibit multistability, and this makes the analysis of asymptotic behavior a non-trivial problem.

The analysis begins in Section 2.2 with the description of the disease-free equilibrium and the computation of the basic reproduction number of the model. Next, a comprehensive result on the existence and number of endemic equilibriums is established in Section 2.3. In fact, a quite precise description is provided that characterizes the number of endemic equilibriums, according to the parameter values. It is shown that there may exist up to three distinct endemic equilibriums when the basic reproduction number is larger than one, and up to two such equilibriums when it is smaller or equal than one. The result and the simulations displayed later in the chapter allow us to discuss the existence of a reinfection threshold for the two-stage model. In particular, we argue that our findings shed some lights on the controversy that opposed Gomes et al. to Breban and Blower [20] about the exact nature of the threshold. Disease persistence of the model is then considered in Section 2.4, and an extensive set of numerical simulations of endemic equilibriums and trajectories are computed in Section 2.5 for each possible situation. In Section 2.6, the same analysis is repeated for a simpler two-stage SIS model of reduced complexity, for which one checks that the same results persist in this most simplified reinfection model with two stages. Finally, in Section 2.7, we introduce and discuss a number of tools for the analysis of asymptotic behavior within a multistable setting, and we deliver a result on the convergence to equilibrium of compartmental models presenting partial immunity, which can be considered as a particular case of the general two-stage model. The proof of this result is based on Li-Muldowney theory [75].

2.2 Disease-free equilibrium and basic reproduction number $\mathcal{R}_0$

We begin our investigation with an easy computation showing that the system (2.1) admits a unique disease-free equilibrium, i.e., an equilibrium in which $E_1 = I_1 = E_r = I_r = 0$. The value of the equilibrium is given by

$$E_0 := (1, 0, 0, 0, 0, 0, 0, 0).$$

Additionally, we may characterize the basic reproduction number $\mathcal{R}_0$ by the threshold condition for local stability of the DFE. Remarkably, from the expression below, $\mathcal{R}_0$ depends solely on coefficients intervening in the primary infection.

Proposition 2.2.1. *System* (2.1) *admits the unique disease-free equilibrium* $(1,0,0,0,0,0,0,0)$. *Moreover, denoting*

$$\mathcal{R}_0 := \frac{\beta_{1,1}}{\gamma_1+\mu}\frac{\sigma_1}{\sigma_1+\mu}, \tag{2.2}$$

the DFE is locally asymptotically stable for $\mathcal{R}_0 < 1$ *and unstable when* $\mathcal{R}_0 > 1$.

Proof. By definition, $E_1 = I_1 = E_r = I_r = 0$ at disease-free equilibrium, which means that any DFE E_0 is of the form

$$E_0 = (S_1^0, 0, 0, R_1^0, S_r^0, 0, 0, R_1^0).$$

Moreover, the remaining values $S_1^0, R_1^0, S_r^0, R_1^0$ of the steady state fulfill the equations

$$\mu - \mu S_1^* = 0, \quad -(\omega_1+\mu)R_1^* = 0, \quad \omega_1 R_1^* + \omega_r R_r^* - \mu S_r^* = 0, \quad -(\omega_r+\mu)R_r^* = 0.$$

Hence, $S_1^0 = 1$ and $R_1^0 = S_r^0 = R_1^0 = 0$. The Jacobian of the system (2.1) at the DFE is

$$A := \begin{pmatrix} -\mu & 0 & -\beta_{1,1} & 0 & 0 & 0 & -\beta_{1,r} & 0 \\ 0 & -(\sigma_1+\mu) & \beta_{1,1} & 0 & 0 & 0 & \beta_{1,r} & 0 \\ 0 & \sigma_1 & -(\gamma_1+\mu) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_1 & -(\omega_1+\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_1 & -\mu & 0 & 0 & \omega_r \\ 0 & 0 & 0 & 0 & 0 & -(\sigma_r+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_r & -(\gamma_r+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_r & -(\omega_r+\mu) \end{pmatrix}.$$

Let us compute the characteristic polynomial associated to the Jacobian matrix.

$$\begin{aligned}\det(A-\lambda\mathbf{I}_8) =& (-\mu-\lambda)^2(-\omega_r-\mu-\lambda)(-\gamma_r-\mu-\lambda)(-\sigma_r-\mu-\lambda)(-\omega_1-\mu-\lambda) \\ &\times \begin{vmatrix} -(\sigma_1+\mu+\lambda) & \beta_{1,1} \\ \sigma_1 & -(\gamma_1+\mu+\lambda) \end{vmatrix}.\end{aligned}$$

The factorization shows that at least 6 eigenvalues counting multiplicity exhibited in the factor above are negative, and the remaining two are the eigenvalues of the matrix

$$\begin{pmatrix} -(\sigma_1+\mu) & \beta_{1,1} \\ \sigma_1 & -(\gamma_1+\mu) \end{pmatrix}.$$

As the trace of this matrix is negative, it is necessary and sufficient that the determinant of this matrix is positive to have two negative eigenvalues, assumption which is equivalent to

$$(\sigma_1+\mu)(\gamma_1+\mu) - \beta_{1,1}\sigma_1 > 0 \Leftrightarrow \mathcal{R}_0 < 1,$$

where $\mathcal{R}_0$ is defined in (2.2).

Additionally, one may verify, using the method of next generation matrix [35], that $\mathcal{R}_0$ is indeed the basic reproduction number of the model. The system (2.1) has 4 infected compartments E_1, I_1, E_r, I_r, which will be alternatively denoted x_1, x_2, x_3, x_4 respectively for simplicity. On the other hand, the remaining compartments S_1, R_1, S_r, R_r are denoted respectively x_5, x_6, x_7, x_8.

We write now the dynamics of $x := (x_i)_{1\leqslant i\leqslant 8}$ in the form

$$\dot{x}_i = F_i(x) - V_i(x).$$

Here, F_i is the rate of appearance of new infections in the compartment x_i and $V_i = V_i^- - V_i^+$, where V_i^+ (resp. V_i^-) is the rate of transition into by all others means (resp. out of) the compartment x_i. Denoting $F(x) := (F_i(x))_{1\leqslant i\leqslant 8}$ and $V(x) := (V_i)_{1\leqslant i\leqslant 8}$, and the Jacobians of F and V at the disease-free equilibrium are

$$DF(E_0) = \begin{pmatrix} \mathbb{F} & \mathbf{0}_{4\times 4} \\ \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} \end{pmatrix}, \text{ where } \mathbb{F} = \begin{pmatrix} 0 & \beta_{1,1} & 0 & \beta_{1,r} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$DV(E_0) = \begin{pmatrix} \mathbb{V} & \mathbf{0}_{4\times 4} \\ J_1 & J_2 \end{pmatrix}, \text{ where } \mathbb{V} = \begin{pmatrix} \sigma_1+\mu & 0 & 0 & 0 \\ -\sigma_1 & \gamma_1+\mu & 0 & 0 \\ 0 & 0 & \sigma_r+\mu & 0 \\ 0 & 0 & -\sigma_r & \gamma_r+\mu \end{pmatrix}.$$

In addition,

$$\mathbb{V}^{-1} = \begin{pmatrix} (\sigma_1+\mu)^{-1} & 0 & 0 & 0 \\ \sigma_1(\sigma_1+\mu)^{-1}(\gamma_1+\mu)^{-1} & (\gamma_1+\mu)^{-1} & 0 & 0 \\ 0 & 0 & (\sigma_r+\mu)^{-1} & 0 \\ 0 & 0 & \sigma_r(\sigma_r+\mu)^{-1}(\gamma_r+\mu)^{-1} & (\gamma_r+\mu)^{-1} \end{pmatrix}.$$

The basic reproduction number of the model (2.1) is then given by the spectral value of $\mathbb{F}\mathbb{V}^{-1}$, which is equal to $\frac{\beta_{1,1}}{\gamma_1+\mu}\frac{\sigma_1}{\sigma_1+\mu}$. □

2.3 Endemic equilibriums

In this section, which forms the main part of the chapter, we investigate the endemic equilibriums of (2.1). First, in subsection 2.3.1 the equilibrium equations are transformed into a polynomial equation which is suitable for analysis. Moreover, we introduce the coefficient a_r as the bifurcation parameter for the study of the polynomial. In subsection 2.3.2, we state that the polynomial equation may imply the possibility for the system (2.1) to have multiple endemic equilibriums. In this case, there exists a total ordering between these points. The main results characterizing the number of endemic equilibriums are contained in subsections 2.3.3, 2.3.4 and 2.3.5 respectively for the case $\mathcal{R}_0 < 1$, $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 > 1$ (respectively in Theorems 2.3.7, 2.3.8, 2.3.9). Finally, in subsection 2.3.6 we describe some particular situations where the uniqueness of endemic equilibrium is guaranteed.

2.3.1 Preparation for the bifurcation analysis

Let $(S_1^*, E_1^*, I_1^*, R_1^*, S_r^*, E_r^*, I_r^*, R_r^*)$ be an endemic equilibrium of system (2.1), then the following equations hold.

$$\begin{aligned} 0 &= \mu - (\beta_{1,1}I_1^* + \beta_{1,r}I_r^* + \mu)S_1^*, \\ 0 &= (\beta_{1,1}I_1^* + \beta_{1,r}I_r^*)S_1^* - (\sigma_1+\mu)E_1^*, \\ 0 &= \sigma_1 E_1^* - (\gamma_1+\mu)I_1^*, \end{aligned}$$

$$
\begin{aligned}
0 &= \gamma_1 I_1^* - (\omega_1 + \mu) R_1^*, \\
0 &= \omega_1 R_1^* + \omega_r R_r^* - (\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu) S_r^*, \\
0 &= (\beta_{r,1} I_1^* + \beta_{r,r} I_r^*) S_r^* - (\sigma_r + \mu) E_r^*, \\
0 &= \sigma_r E_r^* - (\gamma_r + \mu) I_r^*, \\
0 &= \gamma_r I_r^* - (\omega_r + \mu) R_r^*.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
R_r^* = \frac{\gamma_r}{\omega_r + \mu} I_r^*, \quad E_r^* &= \frac{\gamma_r + \mu}{\sigma_r} I_r^*, \\
E_r^* = \frac{\beta_{r,1} I_1^* + \beta_{r,r} I_r^*}{\sigma_r + \mu} S_r^*, \quad S_r^* &= \frac{\omega_1 R_1^* + \omega_r R_r^*}{\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu}, \\
R_1^* = \frac{\gamma_1}{\omega_1 + \mu} I_1^*, \quad E_1^* &= \frac{\gamma_1 + \mu}{\sigma_1} I_1^*, \\
E_1^* = \frac{\beta_{1,1} I_1^* + \beta_{1,r} I_r^*}{\sigma_1 + \mu} S_1^*, \quad S_1^* &= \frac{\mu}{\beta_{1,1} I_1^* + \beta_{1,r} I_r^* + \mu},
\end{aligned}
\tag{2.3}
$$

After elimination of $S_1^*, E_1^*, R_1^*, S_r^*, E_r^*, R_r^*$, the remaining quantities I_1^* and I_r^* are related by

$$
\begin{aligned}
I_1^* &= \frac{\sigma_1}{\sigma_1 + \mu} \frac{\mu}{\gamma_1 + \mu} \frac{\beta_{1,1} I_1^* + \beta_{1,r} I_r^*}{\beta_{1,1} I_1^* + \beta_{1,r} I_r^* + \mu}, \\
I_r^* &= \frac{\beta_{r,1} I_1^* + \beta_{r,r} I_r^*}{\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu} \frac{\sigma_r}{\sigma_r + \mu} \frac{\gamma_r}{\gamma_r + \mu} \frac{\omega_r}{\omega_r + \mu} \left(\frac{\omega_1}{\omega_r} \frac{\omega_r + \mu}{\omega_1 + \mu} \frac{\gamma_1}{\gamma_r} I_1^* + I_r^* \right).
\end{aligned}
\tag{2.4}
$$

All the coefficients being positive, any nonnegative nonzero solution (I_1^*, I_r^*) of (2.4) yields a nonnegative nonzero equilibrium value $(S_1^*, E_1^*, I_1^*, R_1^*, S_r^*, E_r^*, I_r^*, R_r^*)$ such that $S_1^* + E_1^* + I_1^* + R_1^* + S_r^* + E_r^* + I_r^* + R_r^* = 1$. The endemic equilibriums are therefore in one-to-one correspondence with the nonnegative nonzero solutions of system (2.4).

In order to solve this system of equations, we rewrite (2.4) in the form:

$$
I_1^* = A_1 \frac{I_1^* + B_1 I_r^*}{I_1^* + B_1 I_r^* + C_1}, \quad I_r^* = A_r \frac{B_r I_1^* + I_r^*}{B_r I_1^* + I_r^* + C_r} (D_r I_1^* + I_r^*),
$$

with coefficients defined by

$$
\begin{aligned}
A_1 = \frac{\mu}{\gamma_1 + \mu} \frac{\sigma_1}{\sigma_1 + \mu}, \; B_1 = \frac{\beta_{1,r}}{\beta_{1,1}}, \; C_1 = \frac{\mu}{\beta_{1,1}}, \\
A_r = \frac{\sigma_r}{\sigma_r + \mu} \frac{\gamma_r}{\gamma_r + \mu} \frac{\omega_r}{\omega_r + \mu}, \; B_r = \frac{\beta_{r,1}}{\beta_{r,r}}, \; C_r = \frac{\mu}{\beta_{r,r}}, \; D_r = \frac{\omega_1}{\omega_r} \frac{\omega_r + \mu}{\omega_1 + \mu} \frac{\gamma_1}{\gamma_r}.
\end{aligned}
\tag{2.5}
$$

Clearly, all coefficients are positive, and $A_1, A_r < 1$. For simplicity, we denote in addition

$$
x := I_1^*, \quad y := I_r^*,
$$

and the equations above are equivalent to

$$
x^2 + (b_1 y - a_1) x - c_1 y = 0, \quad y^2 + (a_r - b_r x) y - c_r x^2 = 0,
\tag{2.6}
$$

where the coefficients are

$$\begin{gathered} a_1 = A_1 - C_1, \quad b_1 = B_1 > 0, \quad c_1 = A_1 B_1 > 0, \\ a_r = \frac{C_r}{1 - A_r} > 0,\ b_r = \frac{1}{1 - A_r}(A_r(B_r + D_r) - B_r),\ c_r = \frac{A_r B_r D_r}{1 - A_r} > 0. \end{gathered} \tag{2.7}$$

Notice that the signs of a_1 and b_r are unknown in general, moreover

$$a_1 > 0 \Leftrightarrow \mathcal{R}_0 > 1, \quad a_1 < 0 \Leftrightarrow \mathcal{R}_0 < 1, \tag{2.8}$$

and in any case $a_1 b_1 < c_1$ is verified. Moreover, for any $y > 0$,

$$y = \Psi_1(x) := \frac{1}{2}\left(b_r x - a_r + \sqrt{(b_r x - a_r)^2 + 4c_r x^2}\right), \tag{2.9}$$

is the unique solution of the second equation (the other root is negative). On the other hand, the expression of y given by the first identity in (2.6) must be

$$y = \Psi_r(x) := x\frac{x - a_1}{c_1 - b_1 x}. \tag{2.10}$$

Notice that,

$$\max\{0, a_1\} < \frac{c_1}{b_1} < 1.$$

Therefore, $\Psi_r(x) > 0$ with $x > 0$ if and only if

$$x \in (0, 1) \cap (a_1, \frac{c_1}{b_1}), \tag{2.11}$$

and in fact we proved the following result.

Lemma 2.3.1. *The endemic equilibriums of system* (2.1) *are in one-to-one correspondence with the roots of the equation*

$$\Psi_1(x) = \Psi_r(x), \quad x \in (0, 1) \cap (a_1, \frac{c_1}{b_1}),$$

for Ψ_1, Ψ_r *given in* (2.9)-(2.10).

The problem may be further transformed by isolating the square root in $\Psi_1(x)$ and squaring both sides of the equality, hence the equality above yields

$$2x\frac{x - a_1}{c_1 - b_1 x} - (b_r x - a_r) = \sqrt{(b_r x - a_r)^2 + 4c_r x^2}, \tag{2.12}$$

thus

$$\left(2x\frac{x - a_1}{c_1 - b_1 x} - (b_r x - a_r)\right)^2 = (b_r x - a_r)^2 + 4c_r x^2,$$

that is

$$4x^2(\frac{x - a_1}{c_1 - b_1 x})^2 - 4x\frac{x - a_1}{c_1 - b_1 x}(b_r x - a_r) = 4c_r x^2.$$

The following relation is therefore verified for any solution of (2.13) fulfilling (2.11),

$$x\frac{x-a_1}{c_1-b_1x}\left(x\frac{x-a_1}{c_1-b_1x}-(b_rx-a_r)\right)=c_rx^2>0. \tag{2.13}$$

As $x\frac{x-a_1}{c_1-b_1x}>0$ in the interval given in (2.11), the previous inequality implies

$$b_rx-a_r<x\frac{x-a_1}{c_1-b_1x}<2x\frac{x-a_1}{c_1-b_1x},$$

and the left-hand side of (2.12) is indeed positive for any solution of (2.13) in the interval (2.11). Therefore, the problem is reduced to finding the solutions of polynomial equation (2.13) instead of (2.12). Last, we multiply by $\frac{(c_1-b_1x)^2}{x}$ on both sides of (2.13) to obtain a cubic polynomial equation and deduce the next Lemma.

Lemma 2.3.2. *The endemic equilibriums of system* (2.1) *are in one-to-one correspondence with the roots of the equation* $\mathcal{P}(x)=0$ *in* $(0,1)\cap(a_1,\frac{c_1}{b_1})$*, where*

$$\mathcal{P}(x):=x(x-a_1)^2-(x-a_1)(c_1-b_1x)(b_rx-a_r)-c_r(c_1-b_1x)^2x.$$

Remark 2.3.3. *One shows easily that*

$$\mathcal{P}(a_1)=-c_ra_1(c_1-b_1a_1)^2,\quad \mathcal{P}(0)=-a_1c_1a_r,\quad \mathcal{P}\left(\frac{c_1}{b_1}\right)=\frac{c_1}{b_1}\left(\frac{c_1}{b_1}-a_1\right)^2>0.$$

The values $\mathcal{P}(a_1),\mathcal{P}(0)$ *are thus both positive* iff $a_1<0$*, that is* $\mathcal{R}_0<1$*; and both negative* iff $a_1>0$*, that is* $\mathcal{R}_0>1$*.*

For the subsequent analysis, we consider $a_r>0$ as a bifurcation parameter. In fact, writing separately the terms depending on a_r from the rest of the polynomial $\mathcal{P}$ gives

$$\mathcal{P}(x)=xP(x)+a_rQ(x),$$
$$P(x):=(x-a_1)^2-b_r(x-a_1)(c_1-b_1x)-c_r(c_1-b_1x)^2,\quad Q(x):=(x-a_1)(c_1-b_1x).$$

As the roots of Q are well known, this decomposition helps to visualize the roots of $\mathcal{P}$ geometrically and motivates the choice of taking a_r as the bifurcation parameter. Indeed,

$$P(a_1)=-c_r(c_1-b_1a_1)^2<0,\quad P\left(\frac{c_1}{b_1}\right)=\left(\frac{c_1}{b_1}-a_1\right)^2>0, \tag{2.14a}$$

$$Q(a_1)=Q\left(\frac{c_1}{b_1}\right)=0,\quad Q>0 \text{ on } \left(a_1,\frac{c_1}{b_1}\right). \tag{2.14b}$$

On the other hand, there is no prescribed sign for the two following quantities

$$P(0)=a_1^2+a_1c_1b_r-c_rc_1^2,\quad \lim_{x\to+\infty}\frac{P(x)}{x^2}=1+b_rb_1-c_rb_1^2. \tag{2.14c}$$

In order to study the roots of $\mathcal{P}$, it is necessary to recall the following result on cubic polynomials, which can be found in [61].

Definition 2.3.1. *For a given cubic polynomial* ax^3+bx^2+cx+d *in the variable* x*, the discriminant is defined by* $b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd$*.*

The number of roots of a 3rd-degree polynomial depends upon its discriminant in the following way : if the latter is positive, then the polynomial has 3 distinct real roots; if it is null, then the polynomial has a real root which is double or triple; last, if it is negative, then it has a unique real root, which is simple. For future use, we denote by Δ the discriminant of the cubic polynomial $\mathcal{P}(x)$, **considered as a function of the variable** a_r. It is easy to see that Δ is a polynomial of degree up to 4 in the variable a_r.

Remark 2.3.4. *Notice that the parameter a_r is independent of the value of $\mathcal{R}_0$, which is expressed only as a function of the parameters intervening in the first stage of the infection dynamic (see the formula* (2.2) *that provides its value). Keeping the coefficients $\mu, \beta_{1,1}, \beta_{1,r}, \sigma_1, \sigma_r, \gamma_1, \gamma_r, \omega_1, \omega_r$ fixed and considering* (2.5) *and* (2.7)*, we see that a variation in $\beta_{r,1}$ and $\beta_{r,r}$ of the same proportion amounts to modify a_r in an inversely proportional way.* **Therefore, an increase in the parameter a_r can be interpreted as a decrease in the susceptibility to reinfections.**

2.3.2 Number and ordering of the endemic equilibriums

Prior to the actual polynomial analysis, let us notice that $\mathcal{P}$ introduced in Lemma 2.3.2 is of third degree, which permits to state an interesting fact concerning the number of endemic equilibriums of system (2.1).

Corollary 2.3.5. *The number of endemic equilibriums of system* (2.1) *is 0, 1, 2 or 3.*

The actual number of equilibriums depends upon the values of the model parameters. Before deciphering this dependence and showing in subsections 2.3.3 to 2.3.5 that all four cases depicted in Corollary 2.3.5 are actually realizable, we state a last general result, related to ordering. As a matter of fact, in the case of multiple equilibriums, the question of their possible ordering arises. The following result solves completely this issue and establishes the existence of a total order relation between all equilibrium points.

Theorem 2.3.6. *Let $(S_1^*, E_1^*, I_1^*, R_1^*, S_r^*, E_r^*, I_r^*, R_r^*)$ and $(S_1^{**}, E_1^{**}, I_1^{**}, R_1^{**}, S_r^{**}, E_r^{**}, I_r^{**}, R_r^{**})$ be two distinct endemic equilibriums of system* (2.1)*. Assume S_1^* is larger (resp. smaller) than S_1^{**}. Then every remaining component of $(S_1^*, E_1^*, I_1^*, R_1^*, S_r^*, E_r^*, I_r^*, R_r^*)$ is smaller (resp. larger) than its analogue in $(S_1^{**}, E_1^{**}, I_1^{**}, R_1^{**}, S_r^{**}, E_r^{**}, I_r^{**}, R_r^{**})$.*

Proof. Assume that S_1^* is smaller than S_1^{**}, let us check first that I_1^* is larger than I_1^{**}. Differentiating first Ψ_r from (2.10) :

$$\begin{aligned}\Psi_r'(x) &= \frac{x-a_1}{c_1-b_1x} + x\frac{c_1-b_1x+b_1(x-a_1)}{(c_1-b_1x)^2},\\ &= \frac{(x-a_1)(c_1-b_1x)+x(c_1-b_1x+b_1(x-a_1))}{(c_1-b_1x)^2} > 0, \quad \forall x \in (a_1, \frac{c_1}{b_1}).\end{aligned} \tag{2.15}$$

Thus the endemic value I_r^* increases along with I_1^*. We deduce that

$$\frac{\mu}{\beta_{1,1}x + \beta_{1,r}\Psi_r(x) + \mu}$$

is a decreasing function in $(a_1, \frac{c_1}{b_1})$. Therefore, thanks to (2.3) and $S_1^* \leqslant S_1^{**}$, we conclude that $I_1^{**} \leqslant I_1^*$ and (also) $I_r^{**} \leqslant I_r^*$. The inequalities for $E_1^*, R_1^*, E_r^*, R_r^*$ are easily deduced from (2.3) using $I_r^{**} \leqslant I_r^*$. It remains to check S_r^*. One has the following equality, derived from (2.3)

$$S_r^* = \frac{\sigma_r+\mu}{\beta_{r,1}I_1^* + \beta_{r,r}I_r^*}E_r^* = \frac{\sigma_r+\mu}{\beta_{r,1}I_1^* + \beta_{r,r}I_r^*}\frac{\gamma_r+\mu}{\sigma_r}I_r^* = \frac{\sigma_r+\mu}{\sigma_r}\frac{\gamma_r+\mu}{\beta_{r,r}}\frac{I_r^*}{B_rI_1^* + I_r^*}$$

with B_r defined in (2.5). Let us define then the function ξ as

$$\xi(x) := \frac{\sigma_r + \mu}{\sigma_r} \frac{\gamma_r + \mu}{\beta_{r,r}} \frac{\Psi_r(x)}{B_r x + \Psi_r(x)}.$$

Its derivative is

$$\begin{aligned}\xi'(x) &= \frac{\sigma_r + \mu}{\sigma_r} \frac{\gamma_r + \mu}{\beta_{r,r}} \frac{\Psi_r'(x)(B_r x + \Psi_r(x)) - \Psi_r(x)(B_r + \Psi_r'(x))}{(B_r x + \Psi_r(x))^2} \\ &= \frac{\sigma_r + \mu}{\sigma_r} \frac{\gamma_r + \mu}{\beta_{r,r}} \frac{\Psi_r'(x) B_r x - B_r \Psi_r(x)}{(B_r x + \Psi_r(x))^2} = \frac{\sigma_r + \mu}{\sigma_r} \frac{\gamma_r + \mu}{\beta_{r,r}} \frac{B_r(\Psi_r'(x) x - \Psi_r(x))}{(B_r x + \Psi_r(x))^2}.\end{aligned}$$

Notice that, thanks to (2.15)

$$x\Psi_r'(x) = \Psi_r(x) + x^2 \frac{c_1 - a_1 b_1}{(c_1 - b_1 x)^2} > \Psi_r(x).$$

We conclude that $\xi'(x) > 0,\ \forall x \in (a_1, \frac{c1}{b1})$, and as $I_1^* \geqslant I_1^{**}$, then $S_r^* \geqslant S_r^{**}$. □

2.3.3 The case $\mathcal{R}_0 < 1$

We proceed now to the description of the endemic equilibriums in case where $\mathcal{R}_0 < 1$. Recall that this is equivalent to $a_1 < 0$, see formula (2.8). In this case, the result below describes comprehensively the steady states according to the parameters of the system. In particular, it shows the possibility for the system to exhibit up to two endemic equilibriums, even if the threshold condition for outbreaks is not reached.

Theorem 2.3.7. *Assume $\mathcal{R}_0 < 1$ (that is $a_1 < 0$). Then,*

1. *if $a_1^2 + a_1 c_1 b_r - c_r c_1^2 \geqslant 0$, then system (2.1) admits no endemic equilibrium;*
2. *if $a_1^2 + a_1 c_1 b_r - c_r c_1^2 < 0$, then there exists at least a value at which Δ vanishes, and the roots of Δ are necessarily positive. Moreover,*
   (a) *if $a_r > 0$ is smaller than the smallest zero of Δ, then system (2.1) admits two endemic equilibriums;*
   (b) *if a_r is equal to the smallest zero of Δ, then system (2.1) admits a unique endemic equilibrium;*
   (c) *if a_r is larger the smallest zero of Δ, then system (2.1) admits no endemic equilibrium.*

Notice that the value $a_1^2 + a_1 c_1 b_r - c_r c_1^2$ does not depend upon the bifurcation parameter a_r. To provide some insights, Theorem 2.3.7 states that, for $\mathcal{R}_0 < 1$ and $a_1 < 0$, the system exhibits endemic equilibriums under the conditions $a_1^2 + a_1 c_1 b_r - c_r c_1^2 < 0$ and a_r less than or equal to the smallest positive zero of Δ. Even though the first condition appears to be difficult to interpret, the second condition indicates that the system produces endemic equilibriums only in the interval between 0 and the smallest root of Δ. From the Remark 2.3.4, this means that endemic equilibriums appear when susceptibility to reinfections is sufficiently high.

Proof. For $\mathcal{R}_0 < 1$, notice first that $a_1 < 0$, and the endemic equilibriums are in one-to-one correspondence with the zeros of $\mathcal{P}$ in $\left(0, \frac{c_1}{b_1}\right)$. First, let us consider the zeroes of the polynomial P. From (2.14a), we deduce that P has a zero in $\left(a_1, \frac{c_1}{b_1}\right)$ and there can be no other zero in this

interval, as P is of degree 1 or 2. If it exists, the second zero of P is then either in $(-\infty, a_1)$ (and necessarily P is convex) or in $\left(\frac{c_1}{b_1}, +\infty\right)$ (and necessarily P is concave). Moreover, the zero of P located in $\left(a_1, \frac{c_1}{b_1}\right)$ may be either in $(a_1, 0)$ (in which case necessarily $P(0) = a_1^2 + a_1 c_1 b_r - c_r c_1^2 > 0$), in $\left(0, \frac{c_1}{b_1}\right)$ (in which case $P(0) < 0$) or at 0 (in which case $P(0) = 0$).

Consider now these three cases separately. Notice that the derivative of $xP(x)$ at $x = 0$ is equal to $P(0)$. Therefore,

- if $P(0) = a_1^2 + a_1 c_1 b_r - c_r c_1^2 > 0$, then $xP(x)$ is strictly increasing at $x = 0$ and in addition $P(x) > 0$ in $\left(0, \frac{c_1}{b_1}\right)$. As $Q(x) > 0$ in this interval, one concludes that, for any $a_r > 0$, $\mathcal{P}(x) = xP(x) + a_r Q(x) > 0$ in $\left(0, \frac{c_1}{b_1}\right)$, and there is no endemic equilibrium. This demonstrates point 1 in the case of a strict inequality.

- if $P(0) = a_1^2 + a_1 c_1 b_r - c_r c_1^2 = 0$, 0 is a root of the polynomial P. Thus, due to (2.14a), $P(x) > 0$ in $\left(0, \frac{c_1}{b_1}\right)$. The same argument shows that, for any $a_r > 0$, $\mathcal{P}(x) > 0$ in $\left(0, \frac{c_1}{b_1}\right)$: no endemic equilibrium exists. This, together with the previous point, fully demonstrates case 1.

- if $P(0) = a_1^2 + a_1 c_1 b_r - c_r c_1^2 < 0$ (case 2), consider first the situation where $1 + b_r b_1 - c_r b_1^2 \neq 0$ and P is a 2nd-degree polynomial. In this case, $xP(x)$, $\mathcal{P}(x)$ are 3rd-degree polynomials and, for some $x^* \in \left(0, \frac{c_1}{b_1}\right)$, one has

$$P(x) < 0 \text{ in } (0, x^*), \quad P(x^*) = 0, \quad P(x) > 0 \text{ in } \left(x^*, \frac{c_1}{b_1}\right).$$

  Therefore, the 3rd-order polynomial $xP(x)$ vanishes at $x = 0$ and $x = x^*$, is negative in $(0, x^*)$ and positive in $\left(x^*, \frac{c_1}{b_1}\right)$. As $Q > 0$ in $[0, x^*] \subset \left(a_1, \frac{c_1}{b_1}\right)$, then we can already notice that, for any sufficiently small positive value of a_r, the polynomial $\mathcal{P}(x) = xP(x) + a_r Q(x)$ admits two distinct roots in $(0, x^*)$.

  Moreover, $xP(x) > 0$ at $x = a_1 < 0$ and at $x = \frac{c_1}{b_1}$ thanks to (2.14a). The polynomial $xP(x)$ being of 3rd degree, it goes to infinity when $|x| \to +\infty$, with different signs when $x \to +\infty$ and $x \to -\infty$. Therefore, one of the two limits is negative, and the third (real) zero of $xP(x)$ is either in $(-\infty, a_1)$, or in $\left(\frac{c_1}{b_1}, +\infty\right)$. Let us study now these two cases in more details.

  - If $xP(x)$ possesses a zero in $(-\infty, a_1)$, then the latter is unique and $xP(x)$ diverges to $-\infty$ when $x \to -\infty$, while taking positive values in $\left(\frac{c_1}{b_1}, +\infty\right)$. Recall in addition that the polynomial Q is of degree 2, vanishes at $x = a_1$ and is negative in $(-\infty, a_1)$, so for any positive value a_r, $\mathcal{P}(x) = xP(x) + a_r Q(x)$ has a unique zero in $(-\infty, a_1)$. For $a_r > 0$ close to zero, it is already described that $\mathcal{P}$ possesses two roots in $(0, x^*) \subset (0, \frac{c_1}{b_1})$. When the parameter a_r increases, the same situation holds, until the two roots located in $(0, x^*)$ collapse and disappear, as $a_r Q(x)$ (which is positive in $((0, \frac{c_1}{b_1}))$) became not cancellable by the term $xP(x)$. This happens when a_r reaches a critical value a_r^*, where a_r^* is such that the associated polynomial $\mathcal{P}(x)$ admits a *double root* in the interval $\left(0, \frac{c_1}{b_1}\right)$, that is $\Delta = 0$ for $a_r = a_r^*$.

Notice that a pair of zeros of $\mathcal{P}$ appears in $\left(\frac{c_1}{b_1}, +\infty\right)$ for large enough values of a_r. One has to wonder whether this happens for a value of a_r *smaller* than a_r^*. For such a value, $\mathcal{P}$ would have one root in $(-\infty, a_1)$, two distinct roots in $\left(0, \frac{c_1}{b_1}\right)$ and (at least) one in $\left(\frac{c_1}{b_1}, +\infty\right)$. Such a configuration with 4 roots is impossible for a 3rd-degree polynomial, and a_r^* is thus the *smallest* $a_r > 0$ for which Δ vanishes.

- On the contrary, if $xP(x)$ possesses a zero on $\left(\frac{c_1}{b_1}, +\infty\right)$, then $xP(x)$ diverges to $-\infty$ when $x \to +\infty$ and a similar argument applies to deduce that $\mathcal{P}$ has at least one zero in $(\frac{c_1}{b_1}, +\infty)$. Therefore, $\mathcal{P}$ cannot have a third zero in $\left(0, \frac{c_1}{b_1}\right) \subset \left(a_1, \frac{c_1}{b_1}\right)$. Again, a critical value a_r^* exists at which the corresponding Δ vanishes. For a_r greater than a_r^*, $\mathcal{P}$ has no root in $\left(a_1, \frac{c_1}{b_1}\right)$. In parallel, a pair of zeros of $\mathcal{P}$ appears in $(-\infty, a_1)$ for large enough values of a_r, but one shows with the same argument on the polynomial degree, that this occurs only after the disappearance of the two roots in $\left(0, \frac{c_1}{b_1}\right)$. Again, the critical value a_r^* is the smallest $a_r > 0$ for which $\Delta = 0$.
- Last, let us consider the degenerate case where $P(0) = a_1^2 + a_1 c_1 b_r - c_r c_1^2 < 0$, but $1 + b_r b_1 - c_r b_1^2 = 0$ and the degree of P is 1; The polynomial P is an increasing linear function that cuts the x-axis at a point $x^* \in (0, \frac{c_1}{b_1})$. The function $xP(x)$ is positive in $(-\infty, 0)$ and $(x^*, +\infty)$, negative in $(0, x^*)$. Thus, for $a_r = 0$, $\mathcal{P}$ admits a root at $x = 0$ and $x = x^* \in (0, \frac{c_1}{b_1})$ and no roots elsewhere as its degree 2 precludes the appearance of new roots. Moreover, the polynomial Q is positive on $(a_1, \frac{c_1}{b_1})$. Therefore, for $a_r > 0$ close to 0, an increase in a_r leads to the displacement of the first root of $\mathcal{P}$ to the right and the second to the left. This goes on until the roots merge and disappear at a critical value a_r^*, beyond which no root exists in this interval. Again, these are precisely the situations depicted in the cases 2a, 2b and 2c.

Finally, let us prove that any root of Δ must be positive in this case. Assuming that $a_r < 0$, as $xP(x)$ cancels out at least once in $[0, \frac{c_1}{b_1}) \subset (a_1, \frac{c_1}{b_1})$, is positive at the extremities of $(a_1, \frac{c_1}{b_1})$ and moreover $a_r Q(x)$ is negative in $(a_1, \frac{c_1}{b_1})$ with its roots situated at both extremities of the same interval, this implies that $xP(x) + a_r Q(x)$ is positive at $x = a_1$ and $x = \frac{c_1}{b_1}$, and negative at $x = 0$, thus it vanishes twice in $(a_1, \frac{c_1}{b_1})$. Additionally, $\mathcal{P}$ has a third root in $(-\infty, a_1) \cup (\frac{c_1}{b_1}, +\infty)$. This means that $\mathcal{P}$ has always three real roots and Δ does not cancel out when $a_r < 0$ and $a_1^2 + a_1 c_1 b_r - c_r c_1^2 < 0$.

The previous considerations thus prove that when $P(0) < 0$, the behavior always obeys the analysis provided by the three cases 2a to 2c.

This completes the proof of Theorem 2.3.7. □

2.3.4 The case $\mathcal{R}_0 = 1$

For the intermediate case $\mathcal{R}_0 = 1$, that is $a_1 = 0$ (see (2.8)), it is again possible to have up to 2 endemic equilibriums.

Theorem 2.3.8. *Assume $\mathcal{R}_0 = 1$. Then, there exists a value at which Δ vanishes, and the roots of Δ are necessarily positive. Moreover,*

1. *if a_r is smaller than the smallest zero of Δ, system (2.1) admits a unique endemic equilibrium.*

2. *if* $1 + b_r b_1 - c_r b_1^2 \leqslant 0$,

   (a) *if* a_r *is equal to or larger than the smallest zero of* Δ, *system* (2.1) *admits no endemic equilibrium.*

3. *if* $1 + b_r b_1 - c_r b_1^2 > 0$,

   (a) *if* $3b_1c_r - b_r \geqslant 0$, *system* (2.1) *admits no endemic equilibrium for* a_r *equal to or larger than this zero of* Δ.

   (b) *if* $3b_1c_r - b_r < 0$ *system* (2.1) *admits at least a second zero of* Δ. *Furthermore,*

      i. *if* a_r *is between the smallest and the second-smallest zero of* Δ, *system* (2.1) *admits two endemic equilibriums.*

      ii. *if* a_r *is equal to the second-smallest zero of* Δ, *system* (2.1) *admits a unique endemic equilibrium.*

      iii. *if* a_r *is larger than the second-smallest zero of* Δ, *system* (2.1) *admits no endemic equilibrium.*

Notice that, similarly to Theorem 2.3.7, the value of the quantities $1+b_r b_1-c_r b_1^2$ and $3b_1c_r-b_r$ do not depend upon the bifurcation parameter a_r.

Proof. When $\mathcal{R}_0 = 1$, $a_1 = 0$, and the endemic equilibriums are in one-to-one correspondence with the zeros of $\mathcal{P}$ in $\left(0, \frac{c_1}{b_1}\right)$.

The polynomial P has a zero in $\left(0, \frac{c_1}{b_1}\right)$ due to (2.14a). Additionally, as P is of degree 1 or 2, there cannot be another zero in this interval. If it exists, the second zero of P is thus either in $(-\infty, 0)$ (and necessarily P is convex) or in $\left(\frac{c_1}{b_1}, +\infty\right)$ (and necessarily P is concave). Hence, $xP(x)$ has a root at $x = 0$, another one in $(0, \frac{c_1}{b_1})$ and the last one is either in $(-\infty, 0)$ or $(\frac{c_1}{b_1}, +\infty)$. Anyway, because $Q(0) = Q(a_1) = 0$ the polynomial $\mathcal{P}$ has always a zero at $x = 0$ when $\mathcal{R}_0 = 1$. The analysis further divides into two cases :

- If P is strictly concave (which is equivalent to $\frac{P(x)}{x^2} = 1 + b_r b_1 - c_r b_1^2 < 0$), the second root of P is in $(\frac{c_1}{b_1}, +\infty)$, which leads $xP(x)$ and thus $\mathcal{P}$ to diverge to $-\infty$ when $x \to +\infty$. As $\mathcal{P}(\frac{c_1}{b_1}) > 0$, there is always a root in $(\frac{c_1}{b_1}, +\infty)$ and consequently the polynomial $\mathcal{P}$ has always three real roots for every value of $a_r \geqslant 0$. Now, assume that $a_r = 0$, the third root of $\mathcal{P}$ is in $(0, \frac{c_1}{b_1})$ and is equal to the root of P in the same interval. Recalling again that Q is positive in $(0, \frac{c_1}{b_1})$ and negative outside of it, then, as a_r is increasing, $\mathcal{P}$ increases in $(0, \frac{c_1}{b_1})$ and thus the third root of $\mathcal{P}$ moves to the left and leaves the interval $(0, \frac{c_1}{b_1})$ when reaching $x = 0$. It happens when a_r reaches the smallest positive root of Δ.

- If P is a linear function (i.e., $1 + b_r b_1 - c_r b_1^2 = 0$), it is thus an increasing function as its root is in $(0, \frac{c_1}{b_1})$ and that $P(\frac{c_1}{b_1}) > 0$. $\mathcal{P}$ is then a convex function with roots at $x = 0$ and in $(0, \frac{c_1}{b_1})$ when $a_r = 0$. As a_r is increasing, the later root decreases and leaves the interval $(0, \frac{c_1}{b_1})$ by forming a double root at $x = 0$ when a_r reaches the first positive root of Δ. Thus, endemic equilibrium of the system disappears when $a_r \geqslant a_r^*$.

- If P is strictly convex (i.e., $1+b_r b_1-c_r b_1^2 > 0$) the second root is in $(-\infty, 0)$, $\mathcal{P}(x)$ vanishes now once respectively at $x = 0$, in $(-\infty, 0)$ and in $(0, \frac{c_1}{b_1})$ when $a_r = 0$. The function $xP(x)$ is positive between the first zero and $x = 0$, negative between $x = 0$ and the third zero, and again positive between the third zero and $+\infty$. Moreover, an increase of $a_r > 0$ leads the function $\mathcal{P}$ to decrease in $(-\infty, a_1)$ and increase in $(a_1, \frac{c_1}{b_1})$. Thus, the first root of $\mathcal{P}$

is displacing to the right and the third is decreasing as a_r increases before merging and vanish. As the two roots have opposite sign for $a_r = 0$, this implies that one of them must reach the third root $x = 0$ in order to merge and form a double root with each other. This happens when a_r reaches the smallest positive root a_r^* of Δ. Therefore, we have three different scenarios:

- If it is the positive root that reaches $x = 0$ before becoming negative, then the system loses its unique endemic equilibrium when $a_r > a_r^*$.
- It the two roots converge exactly at the point $x = 0$, then they also vanish at the same point and, as previously, the system loses its endemic equilibrium when $a_r > a_r^*$.
- If it is the negative root that passes through $x = 0$ and becomes positive, then the system gains a second endemic equilibrium before that the two come together and then vanish when a_r reaches the second positive root of Δ.

We may further determine which root is reaching zero at the first critical value a_r^* by computing the value of $\mathcal{P}''(0)$ when $a_r = a_r^*$.

- If $\mathcal{P}''(0) > 0$, $\mathcal{P}$ has a double root at $x = 0$ and $\mathcal{P}$ is positive in a neighborhood of $x = 0$, then thanks to $\lim_{x\to-\infty} \mathcal{P}(x) = -\infty$ (due to the convexity of P), the third root is in $(-\infty, 0)$ and it is the initially positive root that is passing through zero.
- If $\mathcal{P}''(0) < 0$, $\mathcal{P}$ has a double root at $x = 0$ and is negative in a neighborhood of $x = 0$, by the same argument we have $\lim_{x\to+\infty} \mathcal{P}(x) = +\infty$ and then the third root is positive. Hence, it is the initially negative root that is passing through zero.
- If $\mathcal{P}''(0) = 0$, $\mathcal{P}$ has a triple root at $x = 0$, then both the positive and negative roots are reaching zero together.

Let us compute $\mathcal{P}''(0)$:

$$\begin{aligned}
\mathcal{P}' &= P(x) + xP'(x) + a_r^* Q'(x),\\
\mathcal{P}'' &= 2P'(x) + xP''(x) + a_r^* Q''(x),\\
\mathcal{P}''(0) &= 2P'(0) + a_r^* Q''(0)\\
&= 2(-2a_1 - b_r c_1 - a_1 b_1 b_r + 2c_r b_1 c_1) - 2b_1 a_r^* = 2(-b_r c_1 + 2c_r b_1 c_1) - 2b_1 a_r^*.
\end{aligned}$$

Thanks to the fact that $0 = \mathcal{P}'(0) = P(0) + a_r^* Q'(0)$, one has

$$a_r^* = -\frac{P(0)}{Q'(0)} = \frac{-c_r c_1^2}{c_1 + a_1 b_1} = -c_r c_1.$$

Thus,

$$\mathcal{P}''(0) = 2(3c_r b_1 c_1 - b_r c_1) = 2c_1(3c_r b_1 - b_r),$$

and which scenario occurs ultimately depends on the sign of $3c_r b_1 - b_r$.

Finally, let us show that any root of Δ must be positive. Notice in any case that, for a negative value of a_r, $a_r Q(x)$ is negative in $(0, \frac{c_1}{b_1})$ and its roots are located at the extremities of the interval. On the other hand, as $P(0) < 0$, $xP(x)$ is negative on the right side of $x = 0$ and positive on the left side of $x = 0$ and at $x = \frac{c_1}{b_1}$ (see (2.14a)). Hence, $\mathcal{P}$ is negative on the right side of $x = 0$ and positive at $x = \frac{c_1}{b_1}$, and $\mathcal{P}$ has at least two distinct roots, one at the point $x = 0$ and the other one in $(0, \frac{c_1}{b_1})$. Moreover, if $\mathcal{P}$ is of degree 3, there must be a third root in the set $(-\infty, 0) \cup (\frac{c_1}{b_1}, +\infty)$, as $\mathcal{P} > 0$ at $x = \frac{c_1}{b_1}$ and on the

right side of the point $x = 0$. Therefore, the corresponding Δ of $\mathcal{P}$ cannot vanish for a non-positive value of a_r.

All cases being examined, this concludes the proof of Theorem 2.3.8. □

2.3.5 The case $\mathcal{R}_0 > 1$

In case where $\mathcal{R}_0 > 1$, the situation is different, and the system may have up to 3 endemic equilibriums under specific conditions described below.

Theorem 2.3.9. *Assume $\mathcal{R}_0 > 1$. Then,*

1. *if $1 + b_r b_1 - c_r b_1^2 \leqslant 0$, then system (2.1) admits a unique endemic equilibrium;*
2. *if $1 + b_r b_1 - c_r b_1^2 > 0$, then Δ vanishes at two, three, or four distinct values, which are necessarily positive. Moreover,*
   (a) *if Δ has two distinct zeroes, then system (2.1) admits a unique endemic equilibrium;*
   (b) *if Δ has three distinct zeroes, then system (2.1) admits two endemic equilibriums at the intermediate zero and otherwise a unique endemic equilibrium;*
   (c) *if Δ has four distinct zeroes, then system (2.1) admits two endemic equilibriums at the two intermediate zeroes, three endemic equilibriums between the two intermediate zeroes and otherwise a unique endemic equilibrium.*

Notice that, here also, the value of $1 + b_r b_1 - c_r b_1^2$ is independent of the bifurcation parameter a_r. We see that, for $\mathcal{R}_0 > 1$, the condition $1 + b_r b_1 - c_r b_1^2 \leqslant 0$ guarantees the uniqueness of endemic equilibrium. If this condition is violated, additional steady states may arise as the number of roots of Δ increases. However, the question of how to interpret biologically these conditions is difficult to answer. Nonetheless, it can be observed that multiple endemic equilibriums typically arise for values of a_r between two intermediate roots of Δ, hence it represents a transition between a mode with low susceptibility to reinfections to a higher one. To facilitate the visualization of the phenomenon, one can refer to the simulation in figures 2.11a and 2.12a in subsection 2.5.5.2 below.

Particularly, the remark above and the corresponding numerical simulations show indeed that in some situations there exists a swift transition from a low-infectivity endemic equilibrium to a high-infectivity equilibrium as the susceptibility to reinfection increases. This suggests the existence of a reinfection threshold, already underscored by Gomes et al. [51] for simpler reinfection SIRI models. The fact that in the present system the transition may be, in some instances, accompanied by a bifurcation sheds some additional light on the controversy that opposes Gomes et al. to Breban and Blower [20], the latter arguing that the reinfection threshold suggested in [51] does not exist as it is not a bifurcation point. However, the existence of a reinfection threshold is not always guaranteed, as illustrated numerically by the simulation in figure 2.9a, which does not include abrupt change between two modes.

The proof of the Theorem employs the same techniques as the proof of Theorem 2.3.7, in a somewhat more involved manner.

Proof. When $\mathcal{R}_0 > 1$, then $a_1 > 0$, and the endemic equilibriums are in one-to-one correspondence with the zeros of $\mathcal{P}$ in $\left(a_1, \frac{c_1}{b_1}\right)$.

As in the proof of Theorem 2.3.7, the polynomial P has a zero in $\left(a_1, \frac{c_1}{b_1}\right)$, and due to (2.14a) and the fact that it is of degree 1 or 2, there can be no other zero in this interval. If the degree

of P is 2, then the second zero of P is either in $(-\infty, a_1)$ (and necessarily P is convex) or in $\left(\frac{c_1}{b_1}, +\infty\right)$ (and necessarily P is concave).

Let us study the behavior of $xP(x)$, and afterward of $\mathcal{P}(x) = xP(x) + a_r Q(x)$.

- If $\lim_{x\to+\infty} \frac{P(x)}{x^2} = 1 + b_r b_1 - c_r b_1^2 < 0$, then P is concave, and its second zero is in $\left(\frac{c_1}{b_1}, +\infty\right)$. The 3rd-degree polynomial $xP(x)$ possesses one zero at $x = 0 \in (-\infty, a_1)$, one in $\left(a_1, \frac{c_1}{b_1}\right)$, and one in $\left(\frac{c_1}{b_1}, +\infty\right)$. Moreover, it converges to $+\infty$ (resp. $-\infty$) when $x \to -\infty$ (resp. $+\infty$). The polynomial Q being negative in each interval $(-\infty, a_1)$ and $\left(\frac{c_1}{b_1}, +\infty\right)$, positive in $\left(a_1, \frac{c_1}{b_1}\right)$ and null at a_1 and $\frac{c_1}{b_1}$, the polynomial $\mathcal{P}(x)$ thus possesses, for any $a_r > 0$, (at least) one zero in each of these three intervals. Applying Lemma 2.3.2, one deduces existence of a unique endemic equilibrium of the system (2.1).

- If $\lim_{x\to+\infty} \frac{P(x)}{x^2} = 1 + b_r b_1 - c_r b_1^2 = 0$. In this case, P is a polynomial of degree 1, which is increasing due to (2.14a). The 2nd-degree polynomial $xP(x)$ then possesses two zeroes, one at $x = 0 \in (-\infty, a_1)$ and one in $\left(a_1, \frac{c_1}{b_1}\right)$, and is negative between the two roots, and positive outside. Arguing as in the previous case, one deduces that $\mathcal{P}(x)$ possesses, for any $a_r > 0$, (at least) one zero in $\left(a_1, \frac{c_1}{b_1}\right)$. This zero is indeed unique, as otherwise the 2nd-degree polynomial $\mathcal{P}(x)$ would have three zeroes. This, together with the previous considerations, establishes the case 1 of the statement.

- If $\lim_{x\to+\infty} \frac{P(x)}{x^2} = 1 + b_r b_1 - c_r b_1^2 > 0$ (case 2), then P is convex, and its second zero is inside $(-\infty, a_1)$: for some $x^* < a_1$ and $x^{**} \in \left(a_1, \frac{c_1}{b_1}\right)$, one has

$$P(x) < 0 \text{ in } (x^*, x^{**}), \quad P(x^*) = P(x^{**}) = 0, \quad P(x) > 0 \text{ in } \left(x^{**}, \frac{c_1}{b_1}\right),$$

and similarly for $xP(x)$ except at $x = 0$ where $xP(x) = 0$. Recall in addition that $Q(x) > 0$ in $\left(a_1, \frac{c_1}{b_1}\right)$ and $Q(x) < 0$ in $\mathbb{R}\setminus\left[a_1, \frac{c_1}{b_1}\right]$. Therefore, for $a_r = 0$ or a sufficiently small positive value of $a_r > 0$, $\mathcal{P}(x) = xP(x) + a_r Q(x)$ possesses, similarly to $xP(x)$, two zeroes in $(-\infty, a_1)$, and exactly one in $\left(a_1, \frac{c_1}{b_1}\right)$ (recall $\mathcal{P}$ is a 3rd-degree polynomial). This situation goes on while a_r increases, until the two roots located in $(-\infty, a_1)$ collapse and disappear, at a point where Δ necessarily vanishes.

On the other hand, for sufficiently large values of $a_r > 0$, we see that $\mathcal{P}(x)$ possesses this time two zeroes in $\left(\frac{c_1}{b_1}, +\infty\right)$, as $\mathcal{P}(\frac{c_1}{b_1}) > 0$, $\lim_{x\to+\infty} \mathcal{P}(x) > 0$ and $Q(x) < 0$ for $x > \frac{c_1}{b_1}$. Invoking again the fact that $\mathcal{P}$ cannot have more than 3 roots, one sees that this occurs through the appearance of a double root of $\mathcal{P}$ as the parameter a_r increases and reaches a new zero of the equation $\Delta = 0$, and afterward splits into two distinct zeroes that subsequently persist for arbitrary large values of $a_r > 0$ greater than this zero of Δ.

Moreover, the polynomial Δ being of degree 4, it may still have, on top of the two previous real zeroes exhibit, up to two distinct zeroes.

  - First, if there is no supplementary real zero of Δ, then from the observation above the polynomial $\mathcal{P}$ has always a unique zero in $\left(a_1, \frac{c_1}{b_1}\right)$ for any $a_r > 0$, and by Lemma 2.3.2 there exists a unique endemic equilibrium. This situation is covered by the case 2a in the statement.

- Second, assume two distinct real zeroes exist, these zeroes cannot be larger than the second critical value exhibited. As they are also larger than the first one, they are thus sandwiched between the two critical values, indicating the apparition of two supplementary roots of $\mathcal{P}$ for any value of a_r between these two zeroes of Δ (case 2c). Fixing x and varying a_r, the quantity $\mathcal{P}(x)$ is an increasing function of a_r when $x \in (-\infty, a_1)$ and decreasing one when $x \in (\frac{c_1}{b_1}, +\infty)$. Thus $\mathcal{P}(x)$ is positive for $x \in (-\infty, a_1)$ and a_r bigger than the smallest positive root of Δ and negative for $x \in (\frac{c_1}{b_1}, +\infty)$ and a_r smaller than the greatest root of Δ. Consequently, the two supplementary roots of $\mathcal{P}$ may only appear in the interval $(a_1, \frac{c_1}{b_1})$, which implies the apparition of two supplementary endemic equilibriums. Moreover, in the degenerate cases where a_r is equal to one of these two roots (case 2b), only a single supplementary endemic equilibrium appears.
- Third, if a unique real zero exists, it is the degenerate case where the two roots of the previous case are equal. In this case, two endemic equilibriums exist at this critical point (case 2b), and one otherwise (case 2a).

To conclude, let us show that the roots of Δ must be positive in the present case. For negative values of a_r, due to $Q(x) < 0$ for $x < a_1$, $Q(a_1) = 0$, $\lim_{x\to-\infty} xP(x) < 0$, $a_1P(a_1) < 0$, then $\mathcal{P}(x) = xP(x) + a_rQ(x)$ is negative at $x = a_1$, positive at $x = 0$ and $\lim_{x\to-\infty} \mathcal{P} = -\infty$. Thus $\mathcal{P}$ has always three real roots when, $a_r < 0$ and Δ does not vanish in this setting.

All cases are shown to enter one of the categories depicted in the statement. This completes the proof of Theorem 2.3.9. □

Theorem 2.3.9 states necessary and sufficient conditions for the existence of three endemic equilibriums points is $1 + b_1b_r - c_rb_1^2 > 0$. The following proposition provides simpler necessary conditions on a_r for the existence of three endemic equilibriums.

Proposition 2.3.10. *Assume that the system* (2.1) *presents three endemic equilibriums, then*

- $2a_1b_r + 2c_1c_r + \frac{a_1}{b_1} - 3a_1b_1c_r - b_r\frac{c_1}{b_1} < a_r < 3\frac{c_1}{b_1^2} + b_r\frac{c_1}{b_1} - \frac{a_1}{b_1}$,
- $3a_1b_1c_r - a_1b_r - c_rc_1 < a_r < 3\frac{c_1}{b_1^2} + 2b_r\frac{c_1}{b_1} - c_1c_r - 2\frac{a_1}{b_1} - a_1b_r$.

Proof. Let us apply Descartes' rule of sign [29] to the polynomials $P_1 := \mathcal{P}(x + a_1)$ and $P_2 := \mathcal{P}(-x + \frac{c_1}{b_1})$ in order to check respectively the maximal number of roots of $\mathcal{P}$ on the right of $x = a_1$ and left of $x = \frac{c_1}{b_1}$. Writing the previous polynomials in the expanded form $P_i(x) = a_1^i x^3 + a_2^i x^2 + a_3^i x + a_4^i$ for $i = \{1, 2\}$, the coefficients are given by :

$$\begin{cases} a_1^1 = 1 + b_1b_r - c_rb_1^2 > 0 \\ a_2^1 = a_1(-c_rb_1^2 + b_rb_1 + 1) - b_r(c_1 - a_1b_1) - a_rb_1 + 2b_1c_r(c_1 - a_1b_1) \\ a_3^1 = a_r(c_1 - a_1b_1) - c_r(c_1 - a_1b_1)^2 - a_1(b_r(c_1 - a_1b_1) - 2b_1c_r(c_1 - a_1b_1)) \\ a_4^1 = -a_1c_r(c_1 - a_1b_1)^2 < 0 \end{cases}$$

$$\begin{cases} a_1^2 = c_rb_1^2 - b_1b_r - 1 < 0 \\ a_2^2 = 2(\frac{c_1}{b_1} - a_1) + b_1b_r(\frac{c_1}{b_1} - a_1) + \frac{c_1}{b_1}(-c_rb_1^2 + b_rb_1 + 1) - a_rb_1 \\ a_3^2 = \frac{c_1}{b_1}(2(a_1 - \frac{c_1}{b_1}) + b_1b_r(a_1 - \frac{c_1}{b1})) - a_rb_1(a_1 - \frac{c_1}{b_1}) - (a_1 - \frac{c_1}{b_1})^2 \\ a_4^2 = \frac{c_1}{b_1}(a_1 - \frac{c_1}{b_1})^2 > 0 \end{cases}$$

Then by Descartes' rule of sign [29], the number of roots of $\mathcal{P}$ (counting multiplicity) on the right of $x = a_1$ (respectively on the left of $x = \frac{c_1}{b_1}$) is at most the number of sign changes in the sequence of polynomial's coefficients of P_1 (respectively P_2). As the three endemic equilibriums state implies that $1 + b_1 b_r - c_r b_1^2 > 0$ and moreover that $\frac{c_1}{b_1} > a_1$, the sign of first and last coefficient of both sequences are known, we conclude then that $a_2^1,\ a_3^2 < 0$ and $a_3^1,\ a_2^2 > 0$ as a necessary condition for the existence of three equilibriums. From $a_2^1,\ a_3^2 < 0$ we deduce

$$\frac{a_1}{b_1}(1 + b_r b_1 - c_r b_1^2) + (2b_1 c_r - b_r)(\frac{c_1}{b_1} - a_1) < a_r < \frac{c_1}{b_1^2}(3 + b_1 b_r - a_1 \frac{b_1}{c_1}),$$

and from $a_3^1,\ a_2^2 > 0$

$$2a_1 b_1 c_r - a_1 b_r - c_r(c_1 - a_1 b_1) < a_r < (\frac{2}{b_1} + b_r)(\frac{c_1}{b_1} - a_1) + \frac{c_1}{b_1^2}(1 + b_r b_1 - c_r b_1^2)$$

and we conclude by expanding the previous inequalities. □

We deduce from Proposition 2.3.10 the following necessary conditions for the existence of three endemic equilibriums, *independent of* a_r. This is essential, as this parameter characterizes the relative increase in susceptibility of the individuals having recovered from an infection.

Corollary 2.3.11. *Assume that the system* (2.1) *presents three endemic equilibriums, then the following inequalities are fulfilled*

- $a_1 b_r + c_1 c_r + \frac{a_1}{b_1} < \frac{c_1}{b_1^2} + b_r \frac{c_1}{b_1} + a_1 b_1 c_r$,
- $3a_1 b_1 c_r + \frac{a_1}{b_1} < 3\frac{c_1}{b_1^2} + b_r \frac{c_1}{b_1} + a_1 b_r + c_r c_1$,
- $3a_1 b_1 c_r + 2\frac{a_1}{b_1} < 3\frac{c_1}{b_1^2} + 2b_r \frac{c_1}{b_1}$.

2.3.6 Particular cases with unique endemic equilibrium for $\mathcal{R}_0 > 1$

In this subsection, we look at some cases that may come up naturally in practice. First, we investigate the case $\beta_{1,1} = \beta_{1,r},\ \beta_{r,1} = \beta_{r,r}$, that is, when the contact rates depend entirely on the history of the susceptible in the transmission. Next, we check the case $\beta_{1,1} = \beta_{r,1},\ \beta_{1,r} = \beta_{r,r}$. Finally, the case $\beta_{1,1} = \beta,\ \beta_{1,r} = \beta_{r,1} = \rho\beta,\ \beta_{r,r} = \rho^2\beta$, i.e., the contact rate decreases geometrically w.r.t. the history of the infectious and infected in the transmission, is considered. The results show that, for "well-balanced" contact rates, the system exhibits a unique endemic equilibrium in the case where $\mathcal{R}_0 > 1$.

2.3.6.1 Reinfection induced change of susceptibility only

Proposition 2.3.12. *Assume that* $\mathcal{R}_0 > 1$ *and* $\beta_{1,1} = \beta_{1,r},\ \beta_{r,1} = \beta_{r,r}$. *Then system* (2.1) *admits a unique endemic equilibrium. This remains true for small enough perturbations of every parameter choice fulfilling the previous identities.*

Proof. For $\beta_{1,1} = \beta_{1,r},\ \beta_{r,1} = \beta_{r,r},\ B_1 = B_r = 1$, the following equality is verified

$$1 + b_r b_1 - c_r b_1^2 = 1 + b_r - c_r = 1 - B_r + \frac{A_r D_r}{1 - A_r} - \frac{A_r D_r}{1 - A_r} = 0.$$

According to Theorem 2.3.9, the system has a unique endemic equilibrium.

Let us show now that this property remains true in a neighborhood of every set of parameters verifying the assumption. As the 3rd degree leading coefficient of the polynomial $\mathcal{P}$ vanishes, $\mathcal{P}$ is now the quadratic polynomial written below :

$$\mathcal{P}(x) = (2b_1c_1c_r - a_rb_1 - b_rc_1 - a_1b_1b_r - 2a_1)x^2 + (a_1^2 + b_ra_1c_1 + a_rb_1a_1 - c_rc_1^2 + a_rc_1)x - a_1a_rc_1.$$

As $\mathcal{P}(x)$ has a root in $(a_1, \frac{c_1}{b_1})$, $\mathcal{P}(a_1) < 0$ and $\mathcal{P}(\frac{c_1}{b_1}) > 0$, the position of the second root is uniquely determined by the sign of the coefficient of the second degree monomial. If $a_r < \frac{2b_1c_1c_r - b_rc_1 - a_1b_1b_r - 2a_1}{b_1}$, the second root is on the left side of a_1, and otherwise on the right side of $\frac{c_1}{b_1}$ if $a_r > \frac{2b_1c_1c_r - b_rc_1 - a_1b_1b_r - 2a_1}{b_1}$ and vanishes when there is equality. To conclude, we can see that the discriminant Δ, which remains a quartic polynomial w.r.t. the parameter a_r, has a double root at $a_r = \frac{2b_1c_1c_r - b_rc_1 - a_1b_1b_r - 2a_1}{b_1}$ and is strictly positive otherwise. Thanks to the continuity of the roots w.r.t. to polynomial parameters, Δ has still two imaginary roots if we perturb the parameters, thus according to the Theorem 2.3.9 the endemic equilibrium is still unique. □

2.3.6.2 Reinfection induced change of infectivity only

Proposition 2.3.13. *Assume that $\mathcal{R}_0 > 1$ and $\beta_{1,1} = \beta_{r,1}$, $\beta_{1,r} = \beta_{r,r}$. Then system (2.1) admits a unique endemic equilibrium. This remains true for small enough perturbations of every parameter choice fulfilling the previous identities.*

Proof. Let ξ be $\xi := \frac{\beta_{1,r}}{\beta_{1,1}}$, we deduce that $B_1 = \xi$ and $B_r = \frac{1}{\xi}$. Then,

$$1 + b_1b_r - c_rb_1^2 = 1 + b_r\xi - c_r\xi^2 = 1 - B_r\xi + \frac{A_rD_r}{1 - A_r}\xi - \frac{A_rD_r}{1 - A_r}B_r\xi^2 = 0.$$

The rest of the proof is the same as in Proposition 2.3.12. □

2.3.6.3 Reinfection induced geometrical dependence of infectivity and susceptibility

Proposition 2.3.14. *Assume that $\mathcal{R}_0 > 1$ and there is a $\beta, \rho > 0$ such that $\beta_{1,1} = \beta$, $\beta_{1,r} = \beta_{r,1} = \rho\beta$, $\beta_{r,r} = \rho^2\beta$. Then system (2.1) admits a unique endemic equilibrium. This remains true for small perturbation of the parameters.*

Proof. If $\beta_{1,1} = \beta$, $\beta_{1,r} = \beta_{r,1} = \rho\beta$, $\beta_{r,r} = \rho^2\beta$, we have now $B_1 = \rho$ and $B_r = \frac{1}{\rho}$. Thus,

$$1 + b_1b_r - c_rb_1^2 = 1 + b_r\rho - c_r\rho^2 = 1 - B_r\rho + \frac{A_rD_r}{1 - A_r}\rho - \frac{A_rD_r}{1 - A_r}B_r\rho^2 = 0.$$

We conclude in the same way as in Propositions 2.3.12, 2.3.13. □

2.4 Persistence analysis for $\mathcal{R}_0 > 1$

An interesting topic in epidemiological models is the question of disease persistence, i.e., if the disease eventually dies out after a certain time. We saw that the two-stage SEIRS model presents quite complicated dynamics, with possibly up to three endemic equilibriums and one disease-free equilibrium. This situation of multistability makes global analysis of the system difficult and motivates instead the interest to study persistence. For this purpose, we introduce the following definitions from [49].

Definition 2.4.1. *Let X be a metric space with metric d, and ϕ be a continuous flow defined on X. Let E be a closed subset of X with boundary ∂E and non-empty interior $\mathring{E}$ nonempty. Assume moreover that E is positively invariant under the flow ϕ. Denoting $\phi_t(x)$ as the flow map, the flow ϕ is called* uniformly persistent *(regarding the set E) if there exists $\varepsilon > 0$ such that for all $x \in \mathring{E}$*

$$\liminf_{t\to\infty} d(\phi_t(x), \partial E) > \varepsilon.$$

Consider then the set $X := \mathbb{R}^8_+$, and the invariant biologically feasible set Σ defined by:

$$\Sigma := \{(S_1, E_1, I_1, R_1, S_r, E_r, I_r, R_r) \in X | S_1 + E_1 + I_1 + R_1 + S_r + E_r + I_r + R_r = 1\}. \quad (2.16)$$

We denote moreover $\partial\Sigma$ as the boundary of the set Σ, and introduce finally the set

$$\partial\Sigma_{\max} := \{(S_1, E_1, I_1, R_1, S_r, E_r, I_r, R_r) \in \Sigma | E_1 = 0, I_1 = 0, E_r = 0, I_r = 0\}, \quad (2.17)$$

one shows easily that $\partial\Sigma_{\max}$ is invariant from the equations (2.1). Indeed, $\partial\Sigma_{\max}$ is the maximal invariant set in the boundary $\partial\Sigma$, which will be proven later.

For the uniform persistence of the system (2.1) in the invariant set Σ, we established the following result, which is the contribution of the present section. In particular, this indicates that the disease persists for the SEIRS system (2.1) when $\mathcal{R}_0 > 1$. Moreover, the proportion of the population bearing the disease is asymptotically bounded from below by a positive constant, uniformly regardless of the initial conditions.

Theorem 2.4.1. *System* (2.1) *is uniformly persistent in the interior $\mathring{\Sigma}$ of Σ defined in* (2.16) *if $\mathcal{R}_0 > 1$ and non-persistent if $\mathcal{R}_0 < 1$.*

To prove the Theorem, additional definitions (from [49]) and lemmas are required.

Definition 2.4.2. *A nonempty subset $M \subset X$ is called an* isolated set *if there exists $\varepsilon > 0$ such that for any invariant set N contained entirely in $\{x : x \in X, d(x, M) \leqslant \varepsilon\}$, we have $N \subset M$.*

Definition 2.4.3. *The flow ϕ is* point dissipative over a nonempty set $M \subset X$ *if there exists a compact set $N \subset X$ such that for any $y \in M$, there exists $t(y) > 0$ such that for any $t \geqslant t(y)$, $\phi_t(y) \in \mathring{N}$.*

Lemma 2.4.2. *If the initial condition of system* (2.1) *fulfills $E_1(0) + I_1(0) + E_r(0) + I_r(0) > 0$, then $E_1(t) > 0, I_1(t) > 0, E_r(t) >, I_r(t) > 0$ for any $t > 0$.*

Proof. Let us begin by observing that every state $g \in \{S_1, E_1, I_1, R_1, S_r, E_r, I_r, R_r\}$ of the system (2.1) fulfills the inequality $\dot{g} \geqslant -a_g g$ where a_g is a non-negative constant. Hence, if any state of the system (2.1) is positive at a time $T \geqslant 0$, the positivity is preserved for any $t > T$ thanks to a straightforward application of Grönwall Lemma. Therefore, the Lemma is verified if the initial state vector is positive element-wise, and it remains to check that the Lemma is still valid if the vector of initial condition has some zero components.

If the initial condition fulfills $S_1(0) = 0$, $\dot{S}_1$ is strictly positive and $S_1(t)$ for $t > 0$. Thus, S_1 is positive for any $t > 0$ in any case. Adding $\dot{I}_1$ to $\dot{I}_r$, we deduce from (2.1) that

$$\dot{I}_1 + \dot{I}_r > \min(\sigma_1, \sigma_r)(E_1 + E_r) - (\max(\gamma_1, \gamma_r) + \mu)(I_1 + I_r).$$

Thus, the condition $E_1(0) + I_1(0) + E_r(0) + I_r(0) > 0$ implies either $I_1(0) + I_r(0) > 0$ or $I_1(0) + I_r(0) = 0$ and $E_1(0) + E_r(0) > 0$. In the second case, thanks to the inequality above, we see that $\dot{I}_1 + \dot{I}_r > 0$ on the right side of $t = 0$. In any case, one deduces $I_1(t) + I_r(t) > 0$ for $t > 0$. This and the fact that $S_1(t) > 0$ for $t > 0$ force $\dot{E}_1$ to be positive on the right side of $t = 0$

in the hypothetical situation where $E_1(0) = 0$. Therefore, $E_1(t) > 0$ for any $t > 0$ also. Finally, the repeated application of the same reasoning leads successively to $I_1(t) > 0, R_1(t) > 0, S_r(t) > 0, E_r(t) > 0, I_r(t) > 0, R_r(t) > 0$ for $t > 0$ which completes the proof. □

Lemma 2.4.3. *Assume that*

$$\lim_{t\to\infty} E_1 = 0, \qquad \lim_{t\to\infty} I_1 = 0, \qquad \lim_{t\to\infty} E_r = 0, \qquad \lim_{t\to\infty} I_r = 0,$$

then,

$$\lim_{t\to\infty} S_1 = 1, \qquad \lim_{t\to\infty} R_r = 0, \qquad \lim_{t\to\infty} S_r = 0, \qquad \lim_{t\to\infty} R_r = 0.$$

Proof. For $\varepsilon > 0$, there is $T > 0$ such that for $t > T$

$$E_1(t) < \varepsilon, \qquad I_1(t) < \varepsilon, \qquad E_r(t) < \varepsilon, \qquad I_r(t) < \varepsilon.$$

Then, we have the following differential inequalities for $t > T$:

$$\begin{gathered}
\mu - \mu S_1 - (\beta_{1,1} + \beta_{1,r})\varepsilon \leqslant \dot{S}_1 \leqslant \mu - \mu S_1, \\
-(\omega_1 + \mu)R_1 \leqslant \dot{R}_1 \leqslant \gamma_1\varepsilon - (\omega_1 + \mu)R_1, \\
-\mu S_r \leqslant \dot{S}_r \leqslant (\gamma_1 + \gamma_r)\varepsilon - \mu S_r, \\
-(\omega_r + \mu)R_r \leqslant \dot{R}_r \leqslant \gamma_r\varepsilon - (\omega_r + \mu)R_r,
\end{gathered}$$

which implies, for $t > T$,

$$\begin{gathered}
S_1(T)e^{-\mu(t-T)} + \int_T^t [\mu - \varepsilon(\beta_{1,1} + \beta_{1,r})]e^{-\mu(t-s)}ds \leqslant S_1(t) \leqslant S_1(T)e^{-\mu(t-T)} + \int_T^t \mu e^{-\mu(t-s)}ds, \\
E_1(T)e^{-(\omega_1+\mu)(t-T)} \leqslant E_1(t) \leqslant E_1(T)e^{-(\omega_1+\mu)(t-T)} + \int_T^t \gamma_1\varepsilon e^{-(\omega_1+\mu)(t-s)}ds, \\
S_r(T)e^{-\mu(t-T)} \leqslant S_r(t) \leqslant S_r(T)e^{-\mu(t-T)} + \int_T^t (\gamma_1 + \gamma_r)\varepsilon e^{-\mu(t-s)}ds, \\
E_r(T)e^{-(\omega_r+\mu)(t-T)} \leqslant E_r(t) \leqslant E_r(T)e^{-(\omega_r+\mu)(t-T)} + \int_T^t \gamma_r\varepsilon e^{-(\omega_r+\mu)(t-s)}ds,
\end{gathered}$$

As $\lim_{t\to+\infty} \int_T^t e^{-\lambda(t-s)}ds = \lambda^{-1}$ for $\lambda \in \mathbb{R}$, the inequalities lead to

$$\begin{gathered}
1 - \mu^{-1}\varepsilon(\beta_{1,1} + \beta_{1,r}) \leqslant \liminf_{t\to\infty} S_1(t) \leqslant \limsup_{t\to\infty} S_1(t) \leqslant 1, \\
0 \leqslant \liminf_{t\to\infty} E_1(t) \leqslant \limsup_{t\to\infty} E_1(t) \leqslant (\omega_1 + \mu)^{-1}\gamma_1\varepsilon, \\
0 \leqslant \liminf_{t\to\infty} S_r(t) \leqslant \limsup_{t\to\infty} S_r(t) \leqslant \mu^{-1}(\gamma_1 + \gamma_r)\varepsilon, \\
0 \leqslant \liminf_{t\to\infty} E_r(t) \leqslant \limsup_{t\to\infty} E_r(t) \leqslant (\omega_r + \mu)^{-1}\gamma_r\varepsilon.
\end{gathered}$$

As the previous inequalities are valid for all $\varepsilon > 0$, taking $\varepsilon \to 0$ we obtain

$$\lim_{t\to\infty} S_1(t) = 1, \lim_{t\to\infty} E_1(t) = 0, \lim_{t\to\infty} S_r(t) = 0, \lim_{t\to\infty} E_r(t) = 0.$$

This completes the proof. □

Lemma 2.4.4. *For $\mathcal{R}_0 > 1$, no solution of system (2.1) converges to the DFE except those starting on the invariant set $\partial\Sigma_{\max}$ (2.17).*

Proof. Assume that the contrary is true, i.e., that for $\mathcal{R}_0 > 1$ there is a solution x converging to the DFE and such that $x(0) \notin \partial\Sigma_{max}$ (note that Lemma 2.4.2 implies then $x(t) \notin \partial\Sigma_{\max}$ for any $t > 0$). Let L be $L := E_1 + \frac{\sigma_1+\mu}{\sigma_1} I_1$, the derivative $\dot{L}$ along trajectories of system (2.1) is

$$\dot{L} = (\beta_{1,1} I_1 + \beta_{1,r} I_r) S_1 - \frac{(\sigma_1 + \mu)(\gamma_1 + \mu)}{\sigma_1} I_1.$$

Let $\varepsilon_1 > 0$ such that $(1-\varepsilon_1)\beta_{1,1} > (\gamma_1 + \mu)\frac{\sigma_1+\mu}{\sigma_1}$ (again this is possible because $\mathcal{R}_0 > 1$). As the solution x converges toward the DFE, there is a $T > 0$ such that $1 - \varepsilon_1 < S_1(t) < 1$ for $t > T$. Thus for $t > T$, we have the differential inequality :

$$\dot{L} > [(1-\varepsilon_1)\beta_{1,1} - \frac{(\sigma_1 + \mu)(\gamma_1 + \mu)}{\sigma_1}] I_1 > 0,$$

and the quantity $L(t) = E_1(t) + \frac{\sigma_1+\mu}{\sigma_1} I_1(t)$ is strictly increasing for $t > T$, which contradicts the convergence to the DFE. Finally, the convergence towards the DFE in the invariant set $\partial\Sigma_{\max}$ is self-evident, as the dynamics (2.1) are reduced to the following system in the set $\partial\Sigma_{\max}$

$$\begin{aligned}
\dot{S}_1 &= \mu - \mu S_1, \\
\dot{R}_1 &= -(\omega_1 + \mu) R_1, \\
\dot{S}_r &= -\mu S_r, \\
\dot{R}_r &= -(\omega_r + \mu) R_r,
\end{aligned}$$

whose solutions converge to the point $(1, 0, 0, 0)$. □

The previous lemmas and Theorem 4.3 in [49] allow us to derive that the system (2.1) is uniformly persistent.

Proof of Theorem 2.4.1. The non persistence is deduced from the fact that the DFE is locally asymptotically stable when $\mathcal{R}_0 < 1$.

The uniform persistence follows from an application of Theorem 4.3 in [49]. More precisely, let X be the space $\mathbb{R}^8_+$ and $\Sigma, \partial\Sigma_{\max}$ defined as in (2.16) and (2.17). We show that $\partial\Sigma_{\max}$ is the maximal invariant set in $\partial\Sigma$. Indeed, taking $x_0 \in \partial\Sigma$ but $x_0 \notin \partial\Sigma_{\max}$, then $E_1(0) + I_1(0) + E_r(0) + I_r(0) > 0$ and Lemma 2.4.2 implies that $\phi_t(x_0) \in \mathring{\Sigma}$ for $t > 0$, therefore we have proved that $\partial\Sigma_{\max}$ is the maximal invariant set in $\partial\Sigma$.

Moreover, notice that the hypothesis $(\mathcal{H})$ that ensures in the aforementioned theorem by Freedman the nonexistence of cycle in $\partial\Sigma_{\max}$ is fulfilled, as the maximal invariant set $\partial\Sigma_{\max}$ is closed and connected. Indeed, the $(\mathcal{H})$ hypothesis requires that $\partial\Sigma_{\max}$ is a closed invariant set and there exists a cover $\{N_\alpha\}_{\alpha\in A}$ of $\partial\Sigma_{\max}$, where A is a nonempty index set, $N_\alpha \subset \partial E, \partial\Sigma_{\max} \subset \cup_{\alpha\in A} N_\alpha$ and $N_\alpha (\alpha \in A)$ are pairwise disjoint closed invariant sets. Furthermore, it is assumed that

- All N_α are isolated invariant sets of the flow ϕ.
- $\{N_\alpha\}_{\alpha\in A}$ is acyclic, that is, any finite subset of $\{N_\alpha\}_{\alpha\in A}$ does not form a cycle (see [22]).
- Any compact subset of $\partial\Sigma$ contains, at most, finitely many sets of $\{N_\alpha\}_{\alpha\in A}$.

As the set $\partial\Sigma_{\max}$ is already closed and connected, we take simply the cover $\{N_\alpha\}_{\alpha\in A} = \{\partial\Sigma_{\max}\}$ and the three points above can be verified. Furthermore, the system is naturally point-dissipative over the invariant compact set Σ, thus Theorem 4.3 in [49] can be applied. According to this Theorem and Lemma 2.4.2, uniform persistence of the system is equivalent to the non-existence of trajectories starting in $\mathring{\Sigma}$ and approaching asymptotically the set $\partial\Sigma_{\max}$. Now, due to Lemma 2.4.3, the convergence towards the set $\partial\Sigma_{\max}$ implies, in fact, the convergence towards the DFE, which is indeed excluded when $\mathcal{R}_0 > 1$ due to Lemma 2.4.4. □

2.5 Numerical simulations

In this section, we consider various sets of parameters and show that the corresponding numbers of endemic equilibriums coincide with the ones predicted by Theorems 2.3.7, 2.3.8, and 2.3.9. In subsection 2.5.1 we investigate first whether, given a set of polynomial parameters $a_1, b_1, c_1, a_r, b_r, c_r$ for simulations, a corresponding set of physical parameters $\beta_{i,j}, \sigma_i, \omega_i, \gamma_i,\ i,j \in \{1,r\}$, is guaranteed to exist. Subsection 2.5.2 presents the contents, the principles, and methodology of the simulations. Next, the case $\mathcal{R}_0 < 1$, $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 > 1$ are respectively considered for simulations in subsections 2.5.3, 2.5.4 and 2.5.5. In particular, the simulations show that the most significant scenarios with 3 endemic equilibriums for the case $\mathcal{R}_0 > 1$ and 2 endemic equilibriums for the case $\mathcal{R}_0 < 1$ are indeed realized.

2.5.1 Correspondence between physical and polynomial $\mathcal{P}$ parameters

It is essential to check if the transformed problem (2.3.2), in the context which the previous theorems are obtained and the simulations will be performed, has always at least one equivalent in the original problem (2.4). The following results ensure that the existence is guaranteed under certain necessary constraints.

Lemma 2.5.1. *For any $\mu > 0$, let the function Φ_1 be defined by*

$$\begin{cases} \Phi_1 : \Omega \to \mathbb{R}^7 \\ \zeta = (\gamma_1, \sigma_1, \omega_1, \beta_{1,1}\beta_{1,r}, \gamma_r, \sigma_r, \omega_r, \beta_{r,1}, \beta_{r,r}) \mapsto \Phi_1(\zeta) = (A_1, B_1, C_1, A_r, B_r, C_r, D_r) \end{cases}$$

where $\Omega := \mathbb{R}_+^{*10}$ and $A_1, B_1, C_1, A_r, B_r, C_r, D_r$ are defined by the identities (2.5)*. Then*

- $\operatorname{Im}\Phi_1 = \Omega'$ *with, denoting* $\zeta' := (A_1, B_1, C_1, A_r, B_r, C_r, D_r)$,

$$\Omega' := \left\{ \zeta' \in \mathbb{R}_+^{*7} \ :\ A_1, A_r < 1 \text{ and } D_r < \frac{1-A_1}{A_1}\frac{1-A_r}{A_r} \right\}.$$

- *The map $\Phi_1^g : \Omega' \to \Omega$ defined by*

$$\Phi_1^g(\zeta') = \begin{pmatrix} \gamma_1 & \sigma_1 & \omega_1 & \beta_{1,1} & \beta_{1,r} & \gamma_r & \sigma_r & \omega_r & \beta_{r,1} & \beta_{r,r} \end{pmatrix},$$

where for $\Theta := D_r \frac{A_1}{1-A_1}\frac{A_r}{1-A_r}$,

$$\beta_{1,1} = \frac{\mu}{C_1}, \quad \beta_{1,r} = \mu\frac{B_1}{C_1}, \quad \beta_{r,r} = \frac{\mu}{C_r}, \quad \beta_{r,1} = \mu\frac{B_r}{C_r},$$
$$\omega_1 = \mu\frac{\Theta^{\frac{1}{4}}}{1-\Theta^{\frac{1}{4}}}, \quad \sigma_1 = \mu\frac{\Theta^{\frac{1}{4}}(1-A_1)+A_1}{(1-\Theta^{\frac{1}{4}})(1-A_1)}, \quad \omega_r = \mu\frac{A_r}{(1-\Theta^{\frac{1}{4}})(1-A_r)},$$
$$\sigma_r = \frac{\mu}{(1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}})} - \mu, \quad \gamma_r = \mu\frac{1-(1-A_r)\Theta^{\frac{1}{4}}}{(1-A_r)\Theta^{\frac{1}{2}}}, \quad \gamma_1 = \mu\frac{\Theta^{\frac{1}{4}}(1-A_1)}{A_1},$$

is a right inverse of Φ_1*, that is :*

$$\forall \zeta' \in \Omega', \quad \Phi_1 \circ \Phi_1^g(\zeta') = \zeta'.$$

Lemma 2.5.2. *Let* Ω' *be defined as previously, and the function* Φ_2 *be*

$$\begin{cases} \Phi_2 : \Omega' \to \mathbb{R}^6 \\ \Phi_2(\zeta') = \begin{pmatrix} a_1 & b_1 & c_1 & a_r & b_r & c_r \end{pmatrix}, \end{cases}$$

where $(a_1, b_1, c_1, a_r, b_r, c_r, d_r)$ *are defined by the identities* (2.7)*. Then*

- $\operatorname{Im}\Phi_2 = \Omega''$ *with, denoting* $\zeta'' := (a_1, b_1, c_1, a_r, b_r, c_r)$

$$\Omega'' := \left\{ \zeta'' \in \Omega' \to \mathbb{R} \times \mathbb{R}_+^{*3} \times \mathbb{R} \times \mathbb{R}_+^* \ : \ a_1 < \frac{c_1}{b_1} < 1, \frac{c_1}{b_1} < \frac{2}{2 + b_r + \sqrt{b_r^2 + 4c_r}} \right\}.$$

- *The map* $\Phi_2^g : \Omega'' \to \Omega'$ *defined by*

$$\Phi_2^g(\zeta'') = \begin{pmatrix} A_1 & B_1 & C_1 & A_r & B_r & C_r & D_r \end{pmatrix},$$

where

$$A_1 = \frac{c_1}{b_1}, \quad B_1 = b_1, \quad C_1 = \frac{c_1}{b_1} - a_1,$$
$$A_r = \frac{\sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}{1 + \sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}, \quad B_r = \frac{1}{2}(-b_r + \sqrt{b_r^2 + 4c_r}),$$
$$C_r = \frac{a_r}{1 + \sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}, \quad D_r = \sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})},$$

is a right inverse of Φ_2*, that is :*

$$\forall \zeta'' \in \Omega'', \Phi_2 \circ \Phi_2^g(\zeta'') = \zeta''.$$

The theorem below is a direct consequence of the previous lemmas.

Proposition 2.5.3. *Let* $\mu \in \mathbb{R}_+^*$ *and the functions* $\Phi_1, \Phi_2, \Phi_1^g, \Phi_2^g$ *be defined as in Lemmas 2.5.1 and 2.5.2. Then* $\operatorname{Im}\Phi_2 \circ \Phi_1 = \Omega''$ *and* $\Phi_1^g \circ \Phi_2^g$ *is a right inverse of the function* $\Phi_2 \circ \Phi_1$ *defined on* Ω''*.*

In particular, for every $\zeta'' := (a_1, b_1, c_1, a_r, b_r, c_r) \in \Omega''$*, let*

$$\zeta := \begin{pmatrix} \gamma_1 & \sigma_1 & \omega_1 & \beta_{1,1} & \beta_{1,r} & \gamma_r & \sigma_r & \omega_r & \beta_{r,1} & \beta_{r,r} \end{pmatrix} = \Phi_1^g \circ \Phi_2^g(\zeta'').$$

Then $\zeta \in \Omega$ *and is related to* ζ'' *by the identities* (2.5), (2.7).

Proof of Lemma 2.5.1. Let us first show that $\operatorname{Im} \Phi_1 \in \Omega'$. Let $\zeta \in \Omega$, the fact that $A_1, A_r < 1$ is obvious from (2.5). For simplicity, for any $x \in \mathbb{R}_+$ we write

$$\bar{x} = \frac{x}{x + \mu}.$$

The formulas (2.5) then simplify as

$$A_1 = \bar{\sigma}_1(1 - \bar{\gamma}_1), \quad A_r = \bar{\sigma}_r \bar{\gamma}_r \bar{\omega}_r, \quad D_r = \frac{\bar{\omega}_1}{\bar{\omega}_r} \frac{\gamma_1}{\gamma_r}.$$

Then, considering the equality

$$D_r = \frac{\bar{\omega}_1}{\bar{\omega}_r} \frac{\gamma_1}{1 - \bar{\gamma}_1} \frac{1 - \bar{\gamma}_r}{\gamma_r},$$

and that

$$\bar{\gamma}_1 = 1 - \frac{A_1}{\bar{\sigma}_1} = \frac{\bar{\sigma}_1 - A_1}{\bar{\sigma}_1}, \quad \bar{\gamma}_r = \frac{A_r}{\bar{\sigma}_r \bar{\omega}_r}, \tag{2.20}$$

we obtain

$$D_r = \frac{\bar{\omega}_1}{\bar{\omega}_r} \frac{\bar{\sigma}_1 - A_1}{A_1} \frac{\bar{\sigma}_r \bar{\omega}_r - A_r}{A_r}, \tag{2.21}$$

as $\bar{x} < 1$ for any $x \in \mathbb{R}_+$, thus

$$D_r < \frac{\bar{\sigma}_1 - A_1}{A_1} \frac{\bar{\sigma}_r \bar{\omega}_r - A_r}{A_r \bar{\omega}_r} < \frac{1 - A_1}{A_1} \frac{1 - A_r}{A_r},$$

which concludes $\operatorname{Im} \Phi_1 \subset \Omega'$.

Let us now establish that $\Omega' \in \operatorname{Im} \Phi_1$, by showing that $\Phi_1^g : \Omega' \to \Omega$ defined in the statement is a right inverse of Φ_1. Let $\zeta' = (A_1, B_1, C_1, A_r, B_r, C_r, D_r) \in \Omega'$. Let us verify that $\Phi_1^g(\zeta')$ is an element of Ω fulfilling the identities (2.5). Let $\Theta > 0$ be such that

$$\Theta = \frac{A_1}{1 - A_1} \frac{A_r}{1 - A_r} D_r. \tag{2.22}$$

Then $\Theta \in (0, 1)$ due to the inequality $D_r < \frac{1 - A_1}{A_1} \frac{1 - A_r}{A_r}$. Now, using the identities 2.21, 2.22 yields

$$\begin{aligned} \Theta &= \frac{A_1}{1 - A_1} \frac{A_r}{1 - A_r} \frac{\bar{\omega}_1}{\bar{\omega}_r} \frac{\bar{\sigma}_1 - A_1}{A_1} \frac{\bar{\sigma}_r \bar{\omega}_r - A_r}{A_r} \\ &= \bar{\omega}_1 \left(\frac{\bar{\sigma}_1 - A_1}{A_1} \frac{A_1}{1 - A_1} \right) \left(\frac{\bar{\sigma}_r \bar{\omega}_r - A_r}{A_r} \frac{A_r}{\bar{\omega}_r - A_r} \right) \left(\frac{\bar{\omega}_r - A_r}{A_r \bar{\omega}_r} \frac{A_r}{1 - A_r} \right). \end{aligned}$$

We take each of the four factors in the previous equation equal to $\Theta^{\frac{1}{4}} < 1$ (that is possible

because each factor is inferior to 1), which implies

$$\bar{\omega}_1 = \Theta^{\frac{1}{4}} \in (0,1), \quad \bar{\sigma}_1 = \Theta^{\frac{1}{4}}(1-A_1) + A_1 \in (\Theta^{\frac{1}{4}}, 1),$$
$$\bar{\omega}_r - A_r = \bar{\omega}_r(1-A_r)\Theta^{\frac{1}{4}}, \quad \text{so that} \quad \bar{\omega}_r = \frac{A_r}{1-(1-A_r)\Theta^{\frac{1}{4}}} \in (A_r, 1),$$

and

$$\bar{\sigma}_r\bar{\omega}_r = (\bar{\omega}_r - A_r)\Theta^{\frac{1}{4}} + A_r = \frac{A_r}{1-(1-A_r)\Theta^{\frac{1}{4}}}\left[(1-1+(1-A_r)\Theta^{\frac{1}{4}})\Theta^{\frac{1}{4}} + 1 - (1-A_r)\Theta^{\frac{1}{4}}\right],$$

that is :

$$\bar{\sigma}_r = 1 - (1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}}) \in (0,1).$$

As the value of μ is known, then also the values of ω_1, σ_1, ω_r, σ_r thanks to the relation $x = \mu\frac{\bar{x}}{1-\bar{x}}$:

$$\omega_1 = \mu\frac{\Theta^{\frac{1}{4}}}{1-\Theta^{\frac{1}{4}}} > 0, \quad \omega_r = \mu\frac{A_r}{(1-\Theta^{\frac{1}{4}})(1-A_r)} > 0,$$
$$\sigma_1 = \mu\frac{\Theta^{\frac{1}{4}}(1-A_1)+A_1}{1-\Theta^{\frac{1}{4}}(1-A_1)-A_1} = \mu\frac{\Theta^{\frac{1}{4}}(1-A_1)+A_1}{(1-\Theta^{\frac{1}{4}})(1-A_1)} > 0,$$
$$\sigma_r = \mu\frac{1-(1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}})}{(1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}})} = \frac{\mu}{(1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}})} - \mu > 0.$$

Finally we deduce (see (2.20))

$$\bar{\gamma}_r = \frac{A_r}{\bar{\sigma}_r\bar{\omega}_r} = \frac{1-(1-A_r)\Theta^{\frac{1}{4}}}{1-(1-A_r)\Theta^{\frac{1}{4}}(1-\Theta^{\frac{1}{4}})} \in (0,1),$$

$$\bar{\gamma}_1 = 1 - \frac{A_1}{\bar{\sigma}_1} = \frac{\Theta^{\frac{1}{4}}(1-A_1)}{\Theta^{\frac{1}{4}}(1-A_1)+A_1} \in (0, 1-A_1) \subset (0,1),$$
$$\gamma_r = \mu\frac{1-(1-A_r)\Theta^{\frac{1}{4}}}{(1-A_r)\Theta^{\frac{1}{2}}} > 0, \quad \gamma_1 = \mu\frac{\Theta^{\frac{1}{4}}(1-A_1)}{A_1} > 0.$$

Last, identities (2.5) imply that

$$\beta_{1,1} = \frac{\mu}{C_1}, \quad \beta_{1,r} = \beta_{1,1}B_1 = \mu\frac{B_1}{C_1}, \quad \beta_{r,r} = \frac{\mu}{C_r}, \quad \beta_{r,1} = \beta_{r,r}B_r = \mu\frac{B_r}{C_r}.$$

We have shown that $\Phi_1^g(\zeta')$ is an element of Ω that fulfills the identities (2.5) for a given $\zeta' \in \Omega'$. Hence, Φ_1^g is, in fact, a right inverse of Φ_1, as $\Phi_1 \circ \Phi_1^g(\zeta') = \zeta'$. Moreover, $\Omega' \in \operatorname{Im}\Phi_1$, and finally $\Omega' = \operatorname{Im}\Phi_1$. This concludes the proof of Lemma 2.5.1. □

Proof of Lemma 2.5.2. Let us show that $\operatorname{Im}\Phi_2 \in \Omega''$. Let $\zeta' \in \Omega'$, the fact that $a_1 < \frac{c_1}{b_1} < 1$ is obvious from (2.7), and (2.7) also gives the equations

$$b_r = -B_r + \frac{A_r}{1-A_r}D_r, \quad c_r = \frac{A_r}{1-A_r}D_rB_r.$$

Denoting $X := B_r, Y := \frac{A_r}{1-A_r} D_r$, by (2.7) one has

$$\begin{cases} -X + Y = b_r \\ XY = c_r \end{cases} \Leftrightarrow \begin{cases} Y = X + b_r \\ X(X + b_r) = c_r \end{cases}.$$

We obtain the equation

$$X^2 + b_r X - c_r = 0,$$

whose discriminant is positive. As $X > 0$, then it is the positive root of the quadratic equation, thus :

$$X := B_r = \frac{1}{2}(-b_r + \sqrt{b_r^2 + 4c_r}), \quad Y := \frac{A_r}{1 - A_r} D_r = \frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}). \tag{2.23}$$

The second identity can be further rewritten as

$$D_r = \frac{1 - A_r}{A_r} \frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}).$$

Due to $A_1 = \frac{c_1}{b_1}$ and the inequality $D_r < \frac{1-A_r}{A_r} \frac{1-A_1}{A_1}$ from Lemma 2.5.1, one has

$$\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}) < \frac{b_1}{c_1} - 1,$$

so that :

$$1 + \frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}) < \frac{b_1}{c_1},$$

and finally,

$$\frac{c_1}{b_1} < \frac{2}{2 + b_r + \sqrt{b_r^2 + 4c_r}},$$

which concludes that $\operatorname{Im} \Phi_2 \subset \Omega''$.

Conversely, let us show that $\Phi_2^g : \Omega'' \to \Omega'$ is a right inverse of Φ_2 and that $\Phi'' \in \operatorname{Im} \Phi_2$. The following equations for Φ_2^g are deduced directly from (2.7)

$$A_1 = \frac{c_1}{b_1} > 0, \quad B_1 = b_1 > 0, \quad C_1 = \frac{c_1}{b_1} - a_1 > 0,$$

We have also from (2.23) and (2.7)

$$B_r = \frac{1}{2}(-b_r + \sqrt{b_r^2 + 4c_r}), \quad \frac{A_r}{1 - A_r} D_r = \frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}), \quad C_r = \frac{a_r}{\sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}.$$

Now we take

$$\frac{A_r}{1 - A_r} = D_r := \sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})},$$

then

$$A_r = \frac{\sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}{1 + \sqrt{\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r})}}.$$

Therefore thanks to the definition of Ω'', $A_1 < 1$ and obviously $A_r < 1$. The same equation also implies

$$D_r = \frac{1 - A_r}{A_r} \frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}).$$

The following inequality is deduced from the definition of Ω'' (which has been already shown)

$$\frac{1}{2}(b_r + \sqrt{b_r^2 + 4c_r}) < \frac{b_1}{c_1} - 1 = \frac{b_1 - c_1}{b_1} \frac{b_1}{c_1} = \frac{1 - A_1}{A_1}.$$

Thus $D_r < \frac{1-A_1}{A_1} \frac{1-A_r}{A_r}$, $\Phi_2^g(\zeta) \in \Omega'$ and Φ_2^g is a right inverse of Φ_2, which concludes the proof of Lemma 2.5.2. □

2.5.2 Guidelines of the simulations

The coefficient μ in the simulations below is fixed beforehand as $\mu = \frac{1}{75}$ years^{-1} for each simulation shown below. Then we proceed as follows for the simulations:

1. Choice of adequate values of the normalized parameters a_1, b_1, c_1, b_r, c_r, deduction of the physical parameters $\gamma_1, \sigma_1, \omega_1, \beta_{1,1}, \beta_{1,r}, \gamma_r, \sigma_r, \omega_r$ through Proposition 2.5.3.
2. Plot of the branches of curves that give the roots of $\mathcal{P}$ as a function of the last parameter a_r. The horizontal lines correspond to the values $\max 0, a_1$ and $\frac{c_1}{b_1}$ which define the extremities of the interval, inside which lie the roots that correspond to endemic equilibriums (see Lemma 2.3.2).
3. Plots of $\mathcal{P}$ for several pertinent values of a_r to illustrate situations described in Theorems 2.3.7 to 2.3.9. This choice is done according to the previous plot.
4. Numerical computation of the spectrum of the Jacobian $J_{\mathbb{E}}$ at every equilibrium point of the system as a function of function a_r, each value of a_r assesses corresponding value to the two missing physical quantities $\beta_{r,1}, \beta_{r,r}$, and drawing of the bifurcation diagram showing the stability or instability of equilibrium points. The spectrum is computed numerically in MATLAB environment by applying the eigenvalue function "eigs" to the Jacobian.
5. Last, for a representative sampling of situations, numerical simulations and plot of trajectories depending on the vicinity of the steady states, in order to illustrate and check their local stability properties.

2.5.3 The case $\mathcal{R}_0 < 1$

When $\mathcal{R}_0 < 1$, Theorem 2.3.7 applies. We study in this subsection two situations where case 1 and 2 of Theorem 2.3.7 are fulfilled.

2.5.3.1 Theorem 2.3.7, case 1 — 0 endemic equilibrium

Set

$$a_1 = -0.001,\ b_1 = 0.81,\ c_1 = 0.004,\ b_r = -10,\ c_r = 1,$$

Then

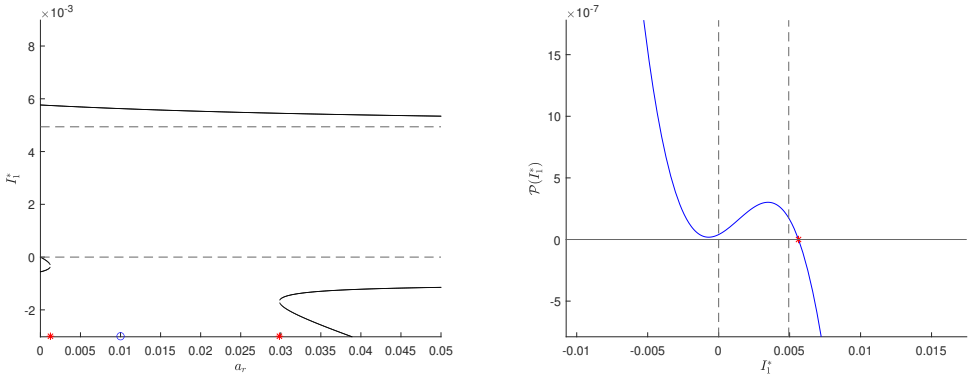
$$a_1^2 + a_1 c_1 b_r - c_r c_1^2 = 2,5 \times 10^{-5} > 0,$$

$\beta_{1,1}$	$\beta_{1,r}$	$\beta_{r,1}$	$\beta_{r,r}$	ω_1	$\mathcal{R}_0$
6.15×10^{-3}	4.98×10^{-3}	4.85×10^{-2}	4.80×10^{-3}	6.39×10^{-6}	0.83
ω_r	σ_1	σ_r	γ_1	γ_r	
1.35×10^{-5}	6.60×10^{-6}	3.42×10^{-4}	1.10×10^{-3}	1.92×10^{-3}	

Table 2.1: Physical parameters for the example in Figure 2.2

and we are in the case 1 of Theorem 2.3.7 and the system has no endemic equilibrium.

Figure 2.2a shows the roots of $\mathcal{P}$ as functions of a_r. The two critical values are shown with red stars. Fig. 2.2b shows the curve $\mathcal{P}(I_1^*)$ corresponding to the value $a_r = 0.0100$ marked in Fig. 2.2a by the blue circle.



(a) Roots of $\mathcal{P}$ depending on a_r

(b) Case 1 $a_r = 0.0100$ - 0 EE

Figure 2.2: Illustration of Theorem 2.3.7 case 1

2.5.3.2 Theorem 2.3.7 case 2 — 0, 1 or 2 endemic equilibriums

For the values:

$$a_1 = -0.001,\ b_1 = 0.81,\ c_1 = 0.0004,\ b_r = 15,\ c_r = 15,$$

we have

$$a_1^2 + a_1 c_1 b_r - c_r c_1^2 = -0.0003 < 0.$$

Thus it corresponds to the case 2 of Theorem 2.3.7 and examples with zero, one or two endemic equilibriums will be exhibited. Figure 2.3a shows the roots of $\mathcal{P}$ depending on a_r. The critical values and plotted values of a_r are also respectively illustrated by red stars and blue circles. Figures 2.3b, 2.3c, 2.3d-2.3f are examples of numerical plots of $\mathcal{P}$ in cases with zero, one or two endemic equilibriums. Next, the bifurcation diagram computed in Figure 2.4a exhibits that, in case where $\mathcal{R}_0 < 1$ and the system presents two endemic equilibriums, the greatest is locally stable while the smallest is unstable. This is also illustrated in the trajectories computed in Figures 2.4b and 2.4c which show that trajectories starting near the greatest steady state converge towards the latter while trajectories starting near the second endemic equilibrium either converge slowly towards the first endemic equilibrium or the disease-free equilibrium (shown in green).

Finally, Tables 2.2 and 2.4 expose physical parameters corresponding to each example calculated using Proposition 2.5.3. Notice that only the parameter a_r is varying throughout the

examples. Then, thanks to Lemma 2.5.2, only the intermediary parameter C_r depends on a_r, and finally by Lemma 2.5.1 only the physical parameters $\beta_{r,r}$ and $\beta_{r,1}$ are varying.

Fig.	a_r	$\beta_{r,1}$	$\beta_{r,r}$
2.3b	0.0275	6.23×10^{-3}	6.62×10^{-3}
2.3c	0.0551	3.12×10^{-3}	3.31×10^{-3}
2.3d	0.0826	2.08×10^{-3}	2.21×10^{-3}
2.3e	0.1157	1.48×10^{-3}	1.58×10^{-3}
2.3f	0.1736	9.89×10^{-4}	1.05×10^{-3}

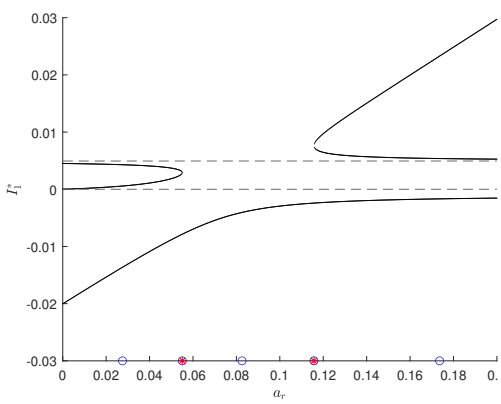
Table 2.2: Physical parameters for examples in Figure 2.3

Fig.	Values of I_1^*	$\max_{\lambda \in \mathrm{spec}(J_{\mathbb{E}})}\{\Re(\lambda)\}$
2.3b	$(4.28, 0.57) \times 10^{-3}$	$(-3.65, 0.92) \times 10^{-5}$
2.3c	2.91×10^{-3}	0

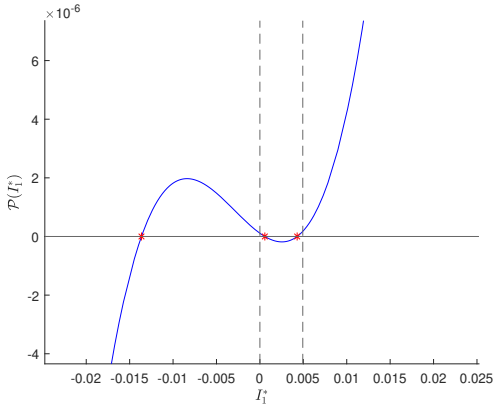
Table 2.3: Values of I_1^* and spectral abscissa of the Jacobian matrices at each equilibrium points

$\beta_{1,1}$	$\beta_{1,r}$	ω_1	ω_r	$\mathcal{R}_0$
6.15×10^{-3}	4.98×10^{-3}	4.13×10^{-5}	3.11×10^{-4}	0.83
σ_1	σ_r	γ_1	γ_r	
4.16×10^{-5}	6.96×10^{-4}	3.90×10^{-3}	5.80×10^{-4}	

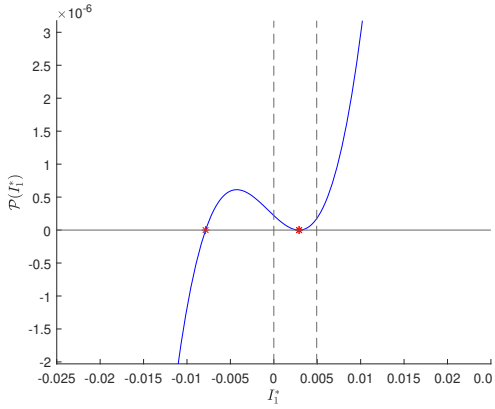
Table 2.4: Physical parameters for examples in Figure 2.3



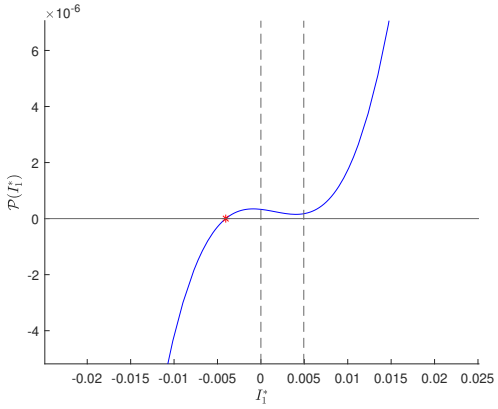
(a) Roots of $\mathcal{P}$ depending on a_r



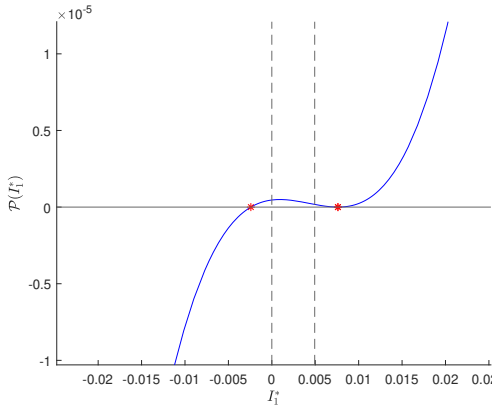
(b) Case 2a - $a_r = 0.0275$ – 2 EE



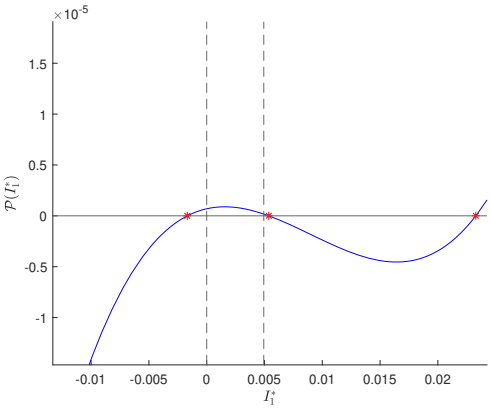
(c) Case 2b - $a_r = 0.0551$ – 1 EE



(d) Case 2c - $a_r = 0.0826$ – 0 EE

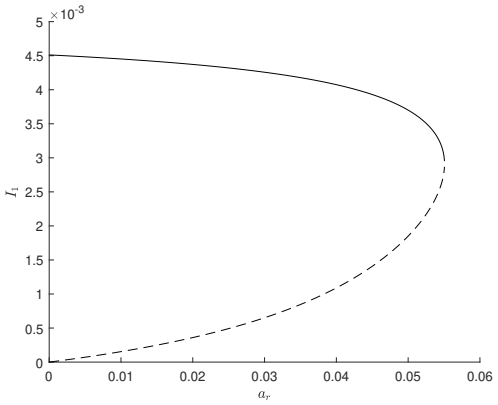


(e) Case 2c - $a_r = 0.1157$ – 0 EE

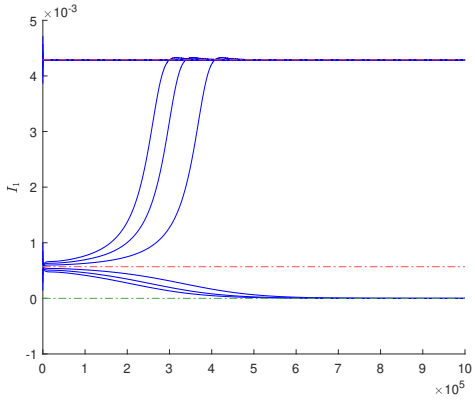


(f) Case 2c - $a_r = 0.1736$ – 0 EE

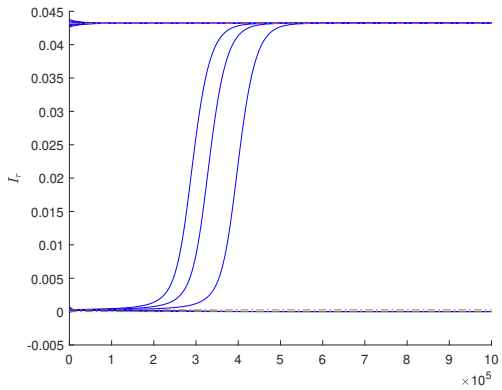
Figure 2.3: Illustration of Theorem 2.3.7 case 2



(a) Bifurcation diagram



(b) I_1 trajectories — $a_r = 0.0275$



(c) I_r trajectories — $a_r = 0.0275$

Figure 2.4: Trajectories in the case 2 of Theorem 2.3.7

2.5.4 The case $\mathcal{R}_0 = 1$

When $\mathcal{R}_0 = 1$, Theorem 2.3.8 applies. Case 2a and 3b of Theorem 2.3.8 are studied in this subsection.

2.5.4.1 Theorem 2.3.8 case 2a — 0 or 1 endemic equilibrium

Let

$$a_1 = 0, b_1 = 0.50, c_1 = 0.001, b_r = -30, c_r = 10, \tag{2.26}$$

Then

$$1 + b_r b_1 - c_r b_1^2 = -16.5 < 0,$$

and this corresponds to the case 2a of Theorem 2.3.8 allowing only the presence of one or zero endemic equilibrium. The graph of the roots of $\mathcal{P}$ as functions of a_r as well as critical and plotted values of a_r are shown in Figure 2.5a. Plots of $\mathcal{P}$ are shown in Figure 2.5b in an example with one endemic equilibrium and Figures 2.5c and 2.5d in examples without endemic equilibrium. The physical parameters of the examples are shown in Table 2.7, 2.5 and the numerical value of the root in Table 2.6. The bifurcation diagram 2.6a computed for this case shows that the unique endemic equilibrium present in the case 2a is stable, which is further demonstrated by the trajectories simulated in Figures 2.6b and 2.6c.

Fig.	a_r	$\beta_{r,1}$	$\beta_{r,r}$
2.5b	0.0050	3.49×10^{-1}	1.15×10^{-2}
2.5c	0.0100	1.74×10^{-1}	5.75×10^{-3}
2.5d	0.0150	1.16×10^{-1}	3.83×10^{-3}

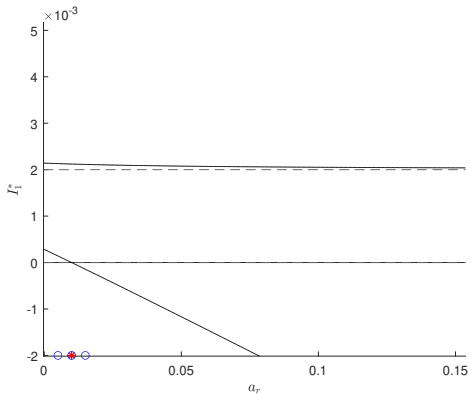
Table 2.5: Physical parameters for examples in Figure 2.5

Fig.	Values of I_1^*	$\max_{\lambda \in \mathrm{spec}(J_{\mathbb{E}})}\{\Re(\lambda)\}$
2.5b	1.42×10^{-4}	-1.47×10^{-6}

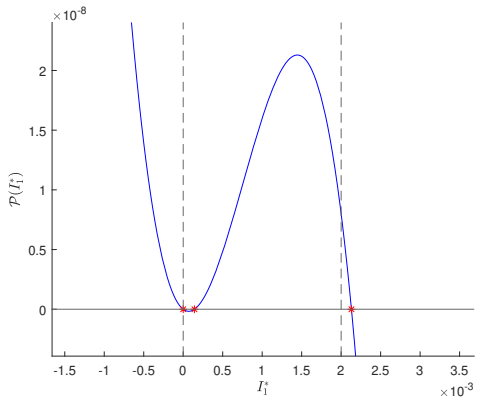
Table 2.6: Value of I_1^* and spectral abscissa of the Jacobian matrix at equilibrium point

$\beta_{1,1}$	$\beta_{1,r}$	ω_1	ω_r	$\mathcal{R}_0$
1.83×10^{-2}	9.13×10^{-3}	6.98×10^{-6}	2.50×10^{-5}	1
σ_1	σ_r	γ_1	γ_r	
7.06×10^{-6}	3.91×10^{-4}	2.92×10^{-3}	2.01×10^{-3}	

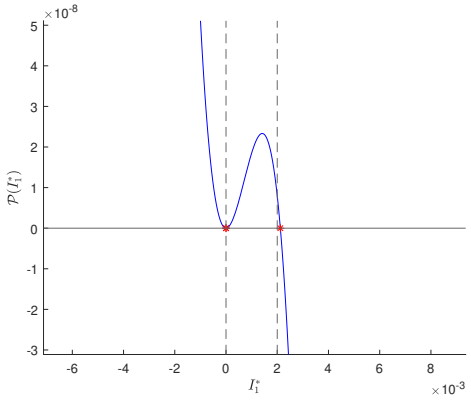
Table 2.7: Physical parameters for examples in Figure 2.5



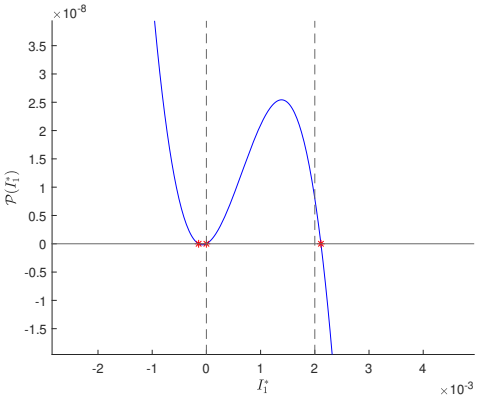
(a) Roots of $\mathcal{P}$ depending on a_r



(b) Case 1 - $a_r = 0.0050$ - 1 EE

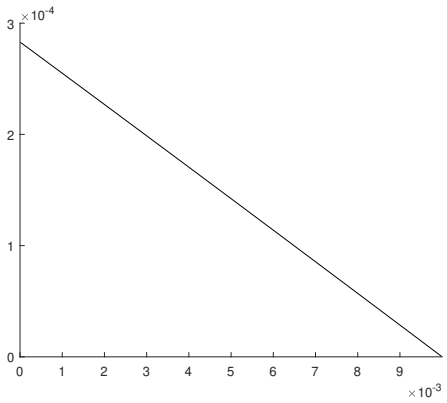


(c) Case 2a - $a_r = 0.0100$ - 0 EE

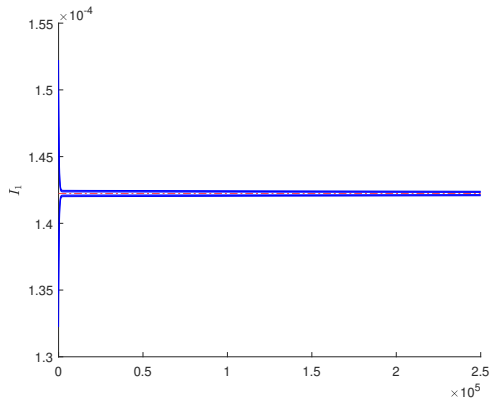


(d) Case 2a - $a_r = 0.0150$ - 0 EE

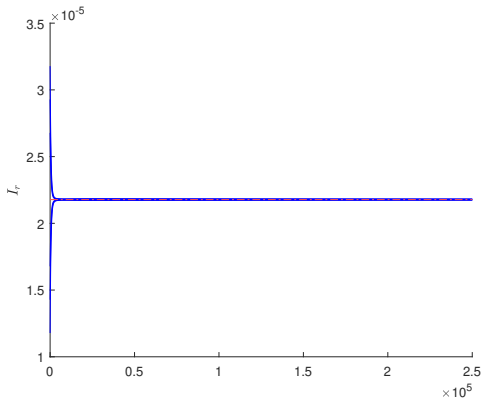
Figure 2.5: Polynomial illustrations of Theorem 2.3.8 case 2a



(a) Bifurcation diagram



(b) I_1 trajectories - $a_r = 0.0050$ - 1 EE



(c) I_r trajectories - $a_r = 0.0050$ - 1 EE

Figure 2.6: Trajectories in the case 2a of Theorem 2.3.8

2.5.4.2 Theorem 2.3.8 case 3b — 0, 1 or 2 endemic equilibriums

Consider

$$a_1 = 0,\ b_1 = 0.50,\ c_1 = 0.001,\ b_r = 30,\ c_r = 10.$$

As

$$1 + b_1 b_r - c_r b_1^2 = 13.5 > 0, \quad 3b_1 c_1 c_r - b_r c_1 = -0.015 < 0,$$

this case corresponds to the case 3b of Theorem 2.3.8 allowing the possibility of having up to 2 endemic equilibriums. Figure 2.7a shows the critical values (in red star) and plotted values (in blue circle) alongside the graph of roots of $\mathcal{P}$ depending on the parameter a_r. Figure 2.7b-2.7c, 2.7d, 2.7e-2.7f show respectively examples of plots of the function $\mathcal{P}$ in case with zero, one and two endemic equilibriums. Tables 2.8, 2.10 show the physical parameters of the examples computed using Proposition 2.5.3. As previously, only the physical parameter $\beta_{r,1}$ and $\beta_{r,r}$ vary throughout the simulations. In the case when $\mathcal{R}_0 = 1$ and the system presents two endemic equilibriums, the bifurcation diagram 2.8a shows that the greatest endemic equilibrium is locally stable while the smallest is unstable. Figure 2.8d, 2.8e illustrate that trajectories starting near the unstable equilibrium either converge towards the stable endemic equilibrium or towards the disease-free equilibrium.

Fig.	a_r	$\beta_{r,1}$	$\beta_{r,r}$
2.7b	0.0033	2.35×10^{-2}	7.13×10^{-2}
2.7c	0.0100	7.84×10^{-3}	2.38×10^{-2}
2.7d	0.0243	3.22×10^{-3}	9.78×10^{-3}
2.7e	0.0680	1.15×10^{-3}	3.50×10^{-3}
2.7f	0.0146	5.36×10^{-4}	1.63×10^{-3}

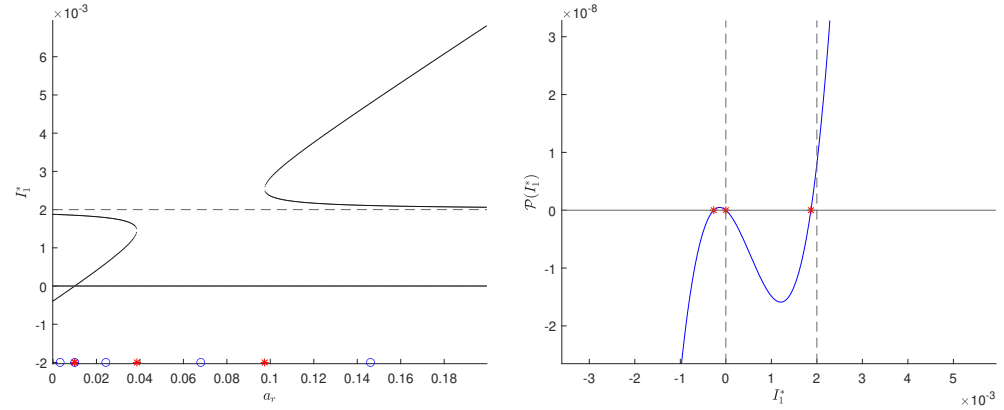
Table 2.8: Physical parameters for examples in Figure 2.7

Fig.	I_1^*	$\max_{\lambda \in \mathrm{spec}(J_{\mathbb{E}})}\{\Re(\lambda)\}$
2.7b	1.87×10^{-3}	-3.65×10^{-5}
2.7c	1.85×10^{-3}	-3.65×10^{-5}
2.7d	$(1.79, 0.59) \times 10^{-3}$	$(-3.65, 1.21) \times 10^{-5}$

Table 2.9: Values of I_1^* and spectral abscissa of the Jacobian matrix at each equilibrium point

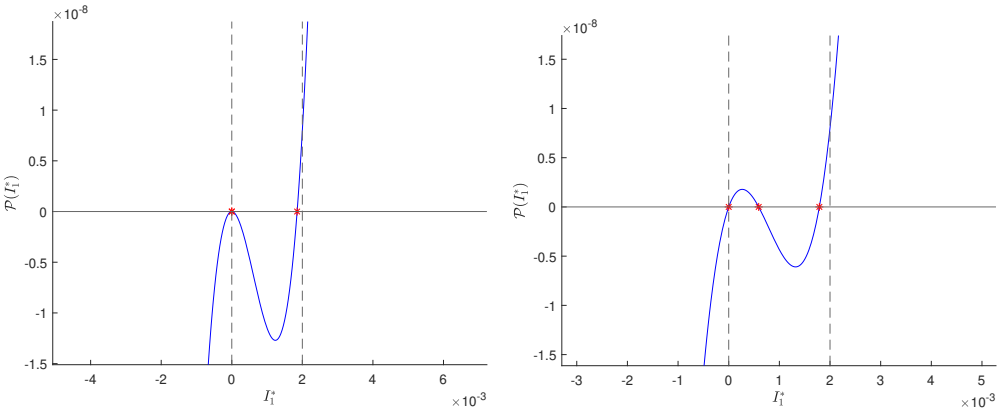
$\beta_{1,1}$	$\beta_{1,r}$	ω_1	ω_r	$\mathcal{R}_0$
1.83×10^{-2}	9.13×10^{-3}	3.60×10^{-5}	4.00×10^{-4}	1
σ_1	σ_r	γ_1	γ_r	
3.62×10^{-5}	9.14×10^{-4}	9.05×10^{-3}	8.91×10^{-4}	

Table 2.10: Physical parameters for examples in Figure 2.7



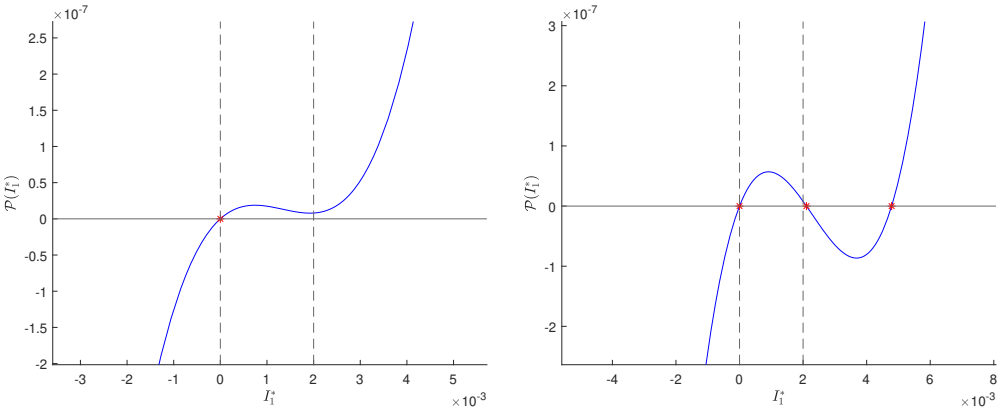
(a) Roots of $\mathcal{P}$ depending on a_r

(b) Case 3 - $a_r = 0.0033$ - 1 EE



(c) Case 3 - $a_r = 0.0100$ - 1 EE

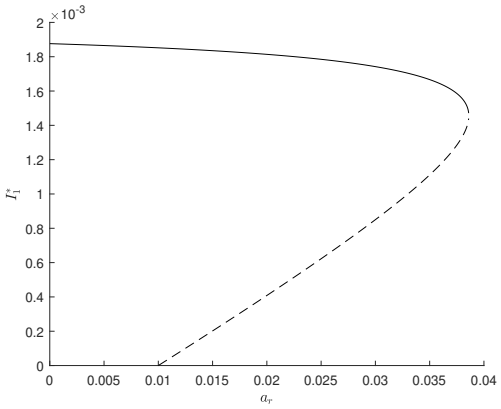
(d) Case 3(b)i - $a_r = 0.0243$ - 2 EE



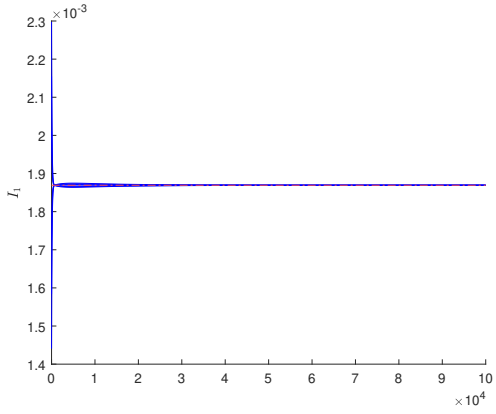
(e) Case 3(b)iii - $a_r = 0.0680$ - 0 EE

(f) Case 3(b)iii - $a_r = 0.1460$ - 0 EE

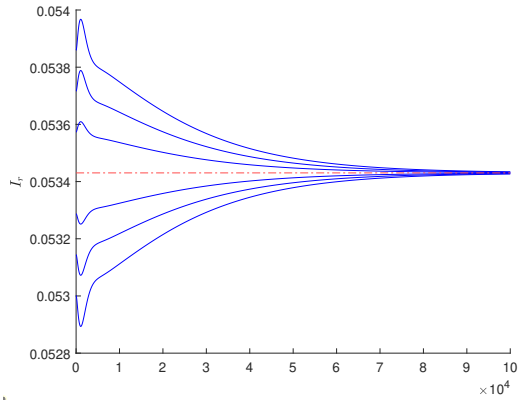
Figure 2.7: Polynomial illustrations of Theorem 2.3.8 case 3b



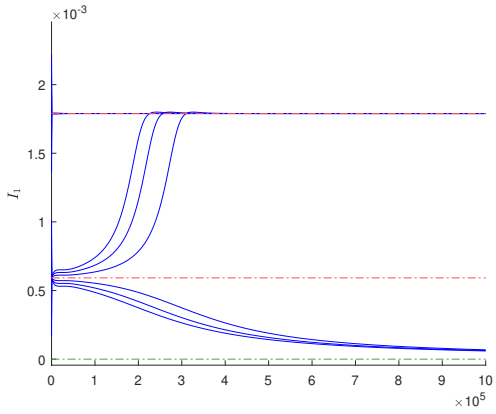
(a) Bifurcation diagram



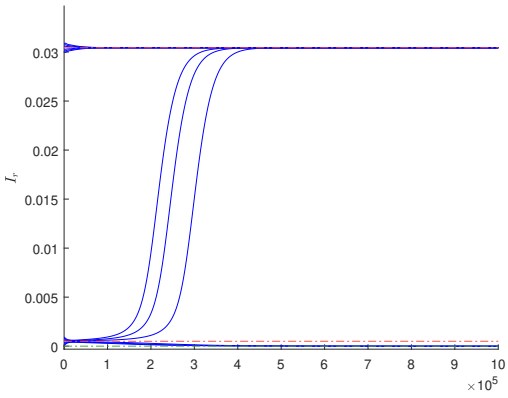
(b) I_1 trajectories – $a_r = 0.0033$ - 1 EE



(c) I_r trajectories – $a_r = 0.0033$ - 1 EE



(d) I_1 trajectories – $a_r = 0.0243$ - 2 EE



(e) I_r trajectories – $a_r = 0.0243$ - 2 EE

Figure 2.8: Trajectories in the case 3b of Theorem 2.3.8

2.5.5 The case $\mathcal{R}_0 > 1$

When $\mathcal{R}_0 > 1$, Theorem 2.3.9 applies. In this subsection, we consider two examples of the cases 1 and 2c of Theorem 2.3.9.

2.5.5.1 Theorem 2.3.9 case 1 — 1 endemic equilibrium

Set

$$a_1 = 0.0009, b_1 = 1.5, c_1 = 0.004, b_r = 0.01, c_r = 150.$$

Hence,

$$1 + b_r b_1 - c_r b_1^2 = -336,49 < 0,$$

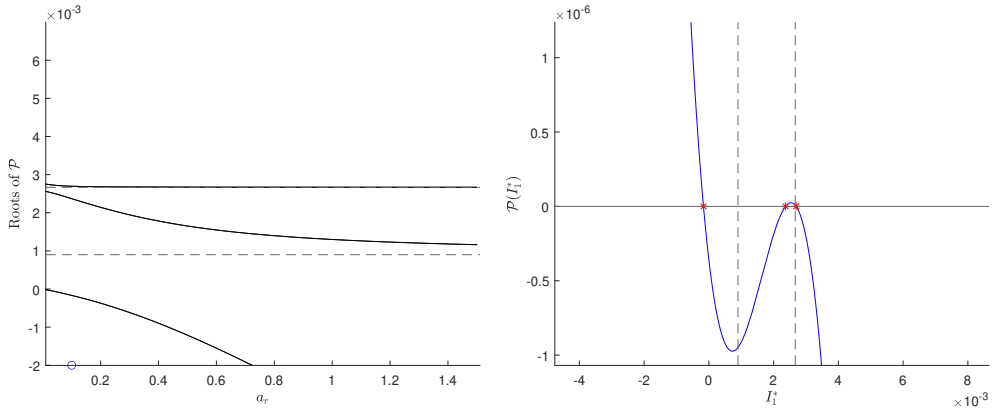
which implies that this case corresponds to (1) of Theorem 2.3.9 and has a unique endemic equilibrium. Figure 2.9a illustrates the distribution of real roots of $\mathcal{P}$, the critical values, and the plotted values of a_r. An example of the plot is presented in Figure 2.9b and illustrates the uniqueness of endemic equilibrium in this case. Finally, Figures 2.10a to 2.10c show that the unique endemic equilibrium in the present case is locally stable.

Fig.	Values of I_1^*	$\max_{\lambda \in \text{spec}(J_{\mathbb{E}})}\{\Re(\lambda)\}$
2.9b	2.56×10^{-3}	-3.65×10^{-5}

Table 2.11: Value of I_1^* and spectral abscissa of the Jacobian matrix at the equilibrium point

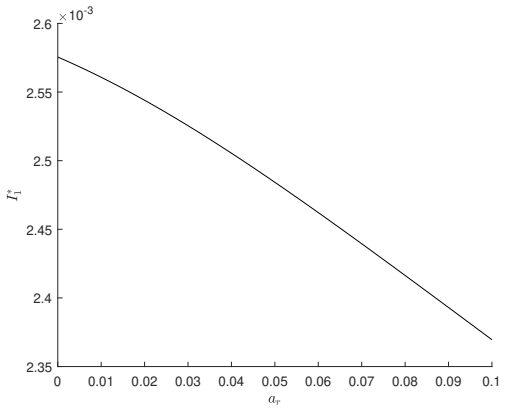
$\beta_{1,1}$	$\beta_{1,r}$	$\beta_{r,1}$	$\beta_{r,r}$	ω_1	$\mathcal{R}_0$
2.07×10^{-2}	3.10×10^{-2}	2.01×10^{-2}	1.64×10^{-3}	2.71×10^{-5}	1.51
ω_r	σ_1	σ_r	γ_1	γ_r	a_r
2.23×10^{-4}	2.72×10^{-5}	6.36×10^{-4}	5.58×10^{-3}	8.22×10^{-4}	0.1

Table 2.12: Physical parameters for the example in Figure 2.9

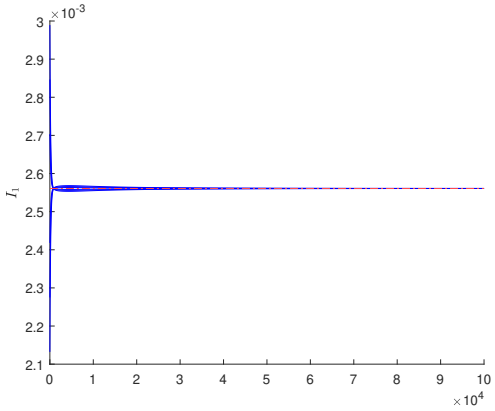


(a) Roots of $\mathcal{P}$ depending on a_r (b) Case 1 - $a_r = 0.1$ - 1 EE

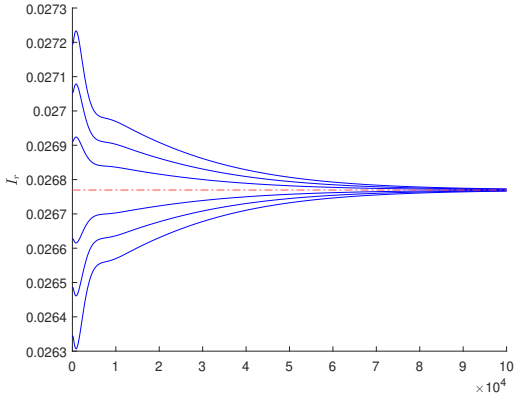
Figure 2.9: Polynomial illustrations of Theorem 2.3.9 case 1



(a) Bifurcation diagram



(b) I_1 trajectories with $a_r = 0.1000$



(c) I_r trajectories with $a_r = 0.1000$

Figure 2.10: Trajectories in the case 1 of Theorem 2.3.9

2.5.5.2 Theorem 2.3.9 case 2 — 1, 2 or 3 endemic equilibriums

The coefficients in this example are:

$$a_1 = 0.00025,\ b_1 = 0.81,\ c_1 = 0.0004,\ b_r = 150,\ c_r = 30.$$

As

$$1 + b_r b_1 - c_r b_1^2 = 102.82 > 0,$$

and that $\Delta > 0$ has four distinct positives roots as shown in Figure 2.11a, this corresponds to the case 2c of Theorem 2.3.9 where the polynomial Δ has four distinct roots, thus exhibiting cases with three endemic equilibriums. Figure 2.11a shows the distribution of real roots of $\mathcal{P}$ depending on the values of the parameter a_r. The critical values of a_r are indicated with red stars, and the values taken for simulations are indicated with blue circles. Figures 2.11b-2.11f illustrate some cases with one, two and three endemic equilibriums. In particular, the case with three endemic equilibriums is exhibited by Figure 2.11e as announced in the Theorem 2.3.9. Tables 2.13, 2.15 show values of physical parameters obtained according to Proposition 2.5.3. In the case where $\mathcal{R}_0 > 1$ and the system presents three endemic equilibriums, the bifurcation diagram 2.12a and simulations of trajectories in Figures 2.12d and 2.12e show that both the greatest and smallest equilibrium are locally stable while the endemic equilibrium located in between is unstable. Furthermore, the diagram in Figure 2.12a depicts indeed a transitional phase during which the system swiftly transitions from a high infection level to a low infection level when a_r increases, thus exhibiting a reinfection threshold.

Fig.	a_r	$\beta_{r,1}$	$\beta_{r,r}$
2.11b	0.0051	1.91×10^{-2}	9.58×10^{-2}
2.11c	0.0321	3.01×10^{-3}	1.50×10^{-2}
2.11d	0.0643	1.50×10^{-3}	7.53×10^{-3}
2.11e	0.0658	1.47×10^{-3}	7.36×10^{-3}
2.11f	0.0912	1.06×10^{-3}	5.31×10^{-3}

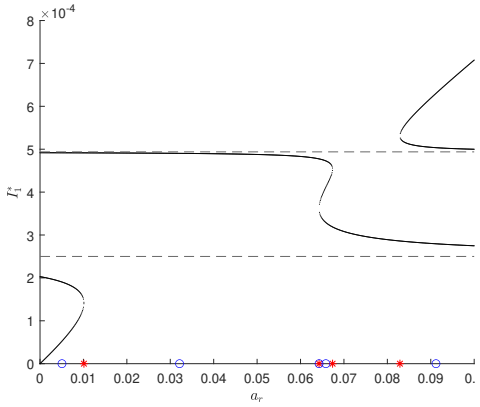
Table 2.13: Physical parameters for the example in Figure 2.11

Fig.	Value of I_1^*	$\max_{\lambda \in \mathrm{spec}(J_{\mathbb{E}})}\{\Re(\lambda)\}$
2.11b	4.91×10^{-4}	-3.65×10^{-5}
2.11c	4.90×10^{-4}	-3.65×10^{-5}
2.11d	$(4.77, 3.62) \times 10^{-4}$	-3.21×10^{-5}, 0
2.11e	$(4.73, 4.12, 3.29) \times 10^{-4}$	$(-2.21, 0.73, -1.17) \times 10^{-5}$
2.11f	2.80×10^{-4}	-3.65×10^{-5}

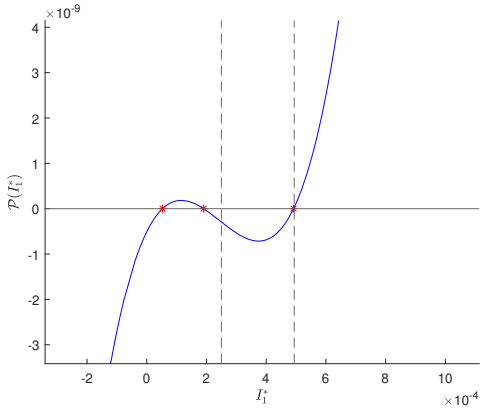
Table 2.14: Values of I_1^* and spectral abscissa of the Jacobian matrix at each equilibrium point

$\beta_{1,1}$	$\beta_{1,r}$	ω_1	ω_r	$\mathcal{R}_0$
1.50×10^{-1}	1.21×10^{-1}	3.99×10^{-5}	9.36×10^{-4}	2.03
σ_1	σ_r	γ_1	γ_r	
3.99×10^{-5}	1.90×10^{-3}	3.86×10^{-2}	1.71×10^{-3}	

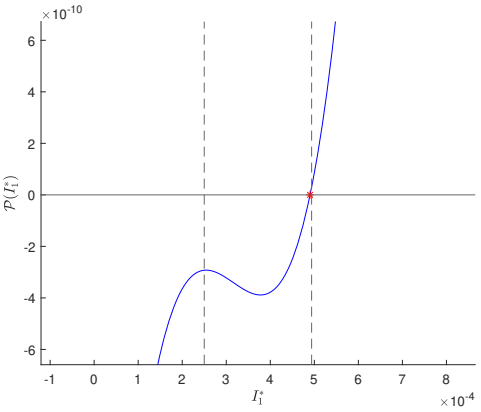
Table 2.15: Physical parameters for the example in Figure 2.11



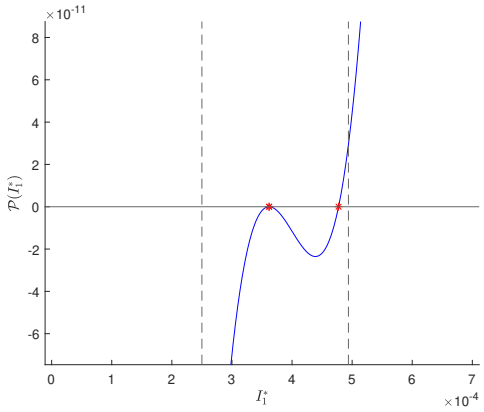
(a) Roots of $\mathcal{P}$ depending on a_r



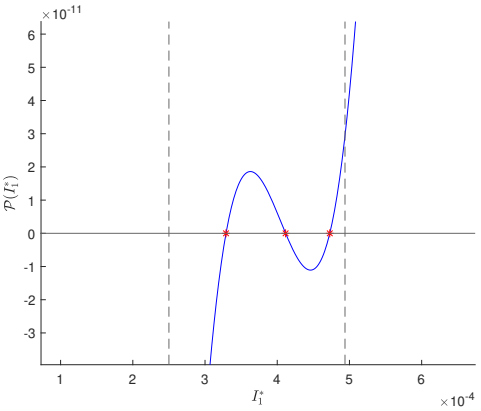
(b) Case 2c - $a_r = 0.0051$ - 1 EE



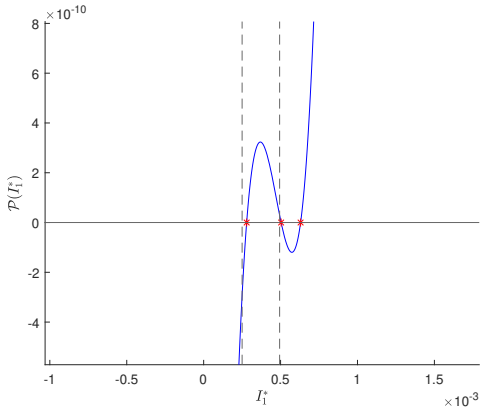
(c) Case 2c - $a_r = 0.0321$ - 1 EE



(d) Case 2c - $a_r = 0.0643$ - 2 EE

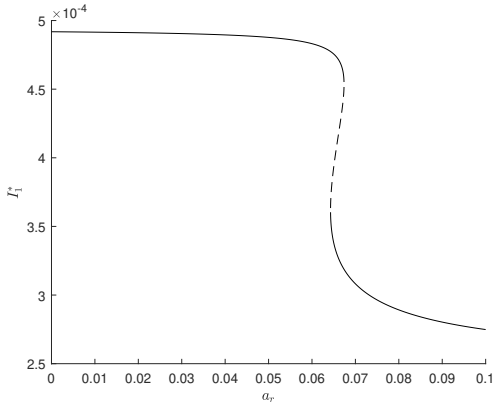


(e) Case 2c - $a_r = 0.0658$ - 3 EE

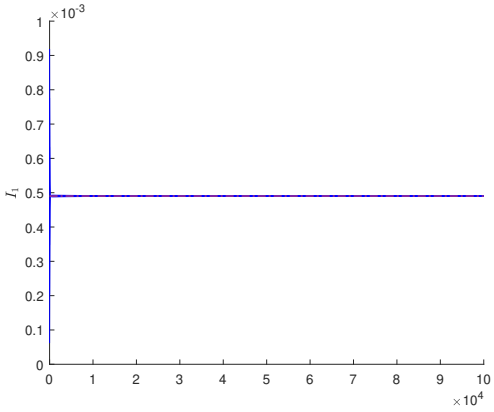


(f) Case 2c - $a_r = 0.0912$ - 1 EE

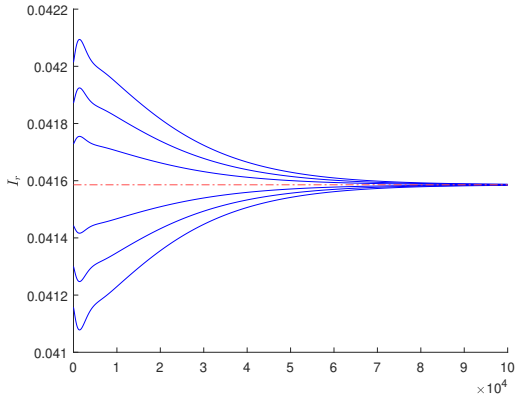
Figure 2.11: Polynomial Illustrations of Theorem 2.3.9 case 2



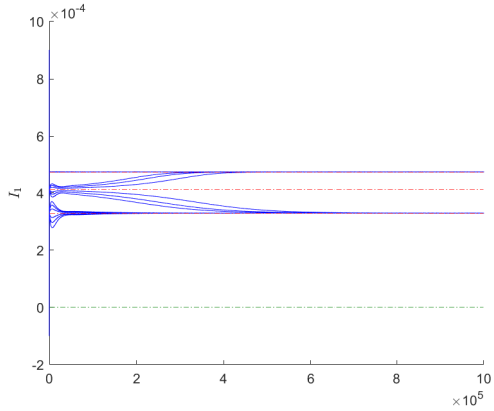
(a) Bifurcation diagram



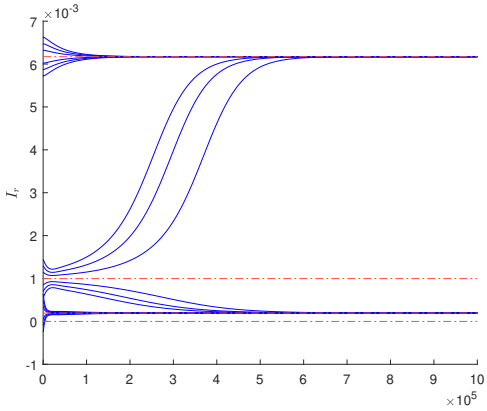
(b) I_1 trajectories - $a_r = 0.0321$ - 1 EE



(c) I_r trajectories - $a_r = 0.0321$ - 1 EE



(d) I_1 trajectories - $a_r = 0.0658$ - 3 EE



(e) I_r trajectories - $a_r = 0.0658$ - 3 EE

Figure 2.12: Trajectories in the case 2 of Theorem 2.3.9

2.6 Two-stage SIS model

In this section, we review a simple two-stage SIS reinfection model, which is,

$$\begin{aligned}
\dot{S}_1 &= \mu - (\beta_{1,1}I_1 + \beta_{1,r}I_r + \mu)S_1,\\
\dot{I}_1 &= (\beta_{1,1}I_1 + \beta_{1,r}I_r)S_1 - (\gamma_1 + \mu)I_1,\\
\dot{S}_r &= \gamma_1 I_1 + \gamma_r I_r - (\beta_{r,1}I_1 + \beta_{r,r}I_r + \mu)S_r,\\
\dot{I}_r &= (\beta_{r,1}I_1 + \beta_{r,r}I_r)S_r - (\gamma_r + \mu)I_r.
\end{aligned} \tag{2.27}$$

The motivation for the study of this SIS model is to examine whether the findings presented previously exhibiting complex behaviors of the steady states are still valid even in the most simple SIS configuration. Its reduced dimension allows for the SIS system allows for the examination of the asymptotic behavior under the Li and Muldowney theory framework [75], which is computationally difficult for the SEIRS model. The analysis is done in Section 2.7 below.

2.6.1 Equilibrium analysis

The invariant and biologically feasible set of the system is

$$\Sigma := \{(S_1, I_1, S_r, I_r) \in \mathbb{R}^4_+ | S_1 + I_1 + S_r + I_r = 1\}. \tag{2.28}$$

Again the system has a unique DFE $(1, 0, 0, 0)$, and a result similar to Proposition 2.2.1 can be deduced on local stability of the DFE.

Proposition 2.6.1. *The system* (2.27) *has a unique disease-free equilibrium*

$$E_0 = (1, 0, 0, 0).$$

Moreover, denoting,

$$\mathcal{R}_0 := \frac{\beta_{1,1}}{\gamma_1 + \mu},$$

the DFE is locally asymptotically stable when $\mathcal{R}_0 < 1$ *and unstable when* $\mathcal{R}_0 > 1$.

Proof. As disease-free equilibrium $E_0 = (S_1^0, I_1^0, S_r^0, I_r^0)$, first $I_1^0 = I_r^0 = 0$ by definition and S_1^0, S_r^0 fulfill

$$\mu - \mu S_1^0 = 0, \qquad -\mu S_r^0 = 0.$$

Hence, $S_1^0 = 1$ and $S_r^0 = 0$. The Jacobian of system (2.27) at the DFE is

$$A := \begin{pmatrix} -\mu & -\beta_{1,1} & 0 & -\beta_{1,r} \\ 0 & \beta_{1,1} - (\gamma_1 + \mu) & 0 & \beta_{1,r} \\ 0 & \gamma_1 & -\mu & \gamma_r \\ 0 & 0 & 0 & -(\gamma_r + \mu) \end{pmatrix}.$$

Expanding the first and third columns while computing the determinant, we can easily see that μ is an eigenvalue with multiplicity two and the remaining are eigenvalues of the matrix :

$$\begin{pmatrix} \beta_{1,1} - (\gamma_1 + \mu) & \beta_{1,r} \\ 0 & -(\gamma_r + \mu) \end{pmatrix},$$

which are the coefficients in the diagonal entries. Thus, we can conclude that the DFE is locally asymptotically stable if $\beta_{1,1} < \gamma_1 + \mu$, i.e., if $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$. □

Each endemic equilibrium $(S_1^*, I_1^*, S_r^*, I_r^*)$ fulfills now the following relationships

$$\begin{aligned}
0 &= \mu - (\beta_{1,1} I_1^* + \beta_{1,r} I_r^* + \mu) S_1^*, \\
0 &= (\beta_{1,1} I_1^* + \beta_{1,r} I_r^*) S_1^* - (\gamma_1 + \mu) I_1^*, \\
0 &= \gamma_1 I_1^* + \gamma_r I_r^* - (\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu) S_r^*, \\
0 &= (\beta_{r,1} I_1^* + \beta_{r,r} I_r^*) S_r^* - (\gamma_r + \mu) I_r^*,
\end{aligned}$$

Thus,

$$\begin{aligned}
S_1^* &= \frac{\mu}{\beta_{1,1} I_1^* + \beta_{1,r} I_r^* + \mu}, \quad I_1^* = \frac{(\beta_{1,1} I_1^* + \beta_{1,r} I_r^*) S_1^*}{\gamma_1 + \mu}, \\
S_r^* &= \frac{\gamma_1 I_1^* + \gamma_r I_r^*}{\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu}, \quad I_r^* = \frac{(\beta_{r,1} I_1^* + \beta_{r,r} I_r^*) S_r^*}{\gamma_r + \mu}.
\end{aligned}$$

Again, by elimination of S_1^*, S_r^*, we deduced that I_1^*, I_r^* must fulfil the equations underneath

$$\begin{aligned}
I_1^* &= \frac{\mu}{\gamma_1 + \mu} \frac{\beta_{1,1} I_1^* + \beta_{1,r} I_r^*}{\beta_{1,1} I_1^* + \beta_{1,r} I_r^* + \mu}, \\
I_r^* &= \frac{\gamma_r}{\gamma_r + \mu} \left(\frac{\gamma_1}{\gamma_r} I_1^* + I_r^*\right) \frac{\beta_{r,1} I_1^* + \beta_{r,r} I_r^*}{\beta_{r,1} I_1^* + \beta_{r,r} I_r^* + \mu}.
\end{aligned}$$

This could be written with a similar set of coefficients as those introduced for the analysis of SEIRS model.

$$I_1^* = A_1 \frac{I_1^* + B_1 I_r^*}{I_1^* + B_1 I_r^* + C_1}, \qquad I_r^* = A_r \frac{B_r I_1^* + I_r^*}{B_r I_1^* + I_r^* + C_r} (D_r I_1^* + I_r^*).$$

where the coefficients are

$$\begin{aligned}
A_1 &= \frac{\mu}{\gamma_1 + \mu}, \; B_1 = \frac{\beta_{1,r}}{\beta_{1,1}}, \; C_1 = \frac{\mu}{\beta_{1,1}}, \\
A_r &= \frac{\gamma_r}{\gamma_r + \mu}, \; B_r = \frac{\beta_{r,1}}{\beta_{r,r}}, \; C_r = \frac{\mu}{\beta_{r,r}}, \; D_r = \frac{\gamma_1}{\gamma_r}.
\end{aligned}$$

Thus, introducing the coefficients $a_1, b_1, c_r, a_r, b_r, c_r$ defined in the same way as before, we can see in a straightforward manner that the relationship

$$a_1 > 0 \Leftrightarrow \mathcal{R}_0 > 1, \quad a_1 < 0 \Leftrightarrow \mathcal{R}_0 < 1,$$

and all the results on existence and number of endemic equilibriums of the SEIRS model still hold in the SIS case. Moreover, in the SIS case, we can derive simple expression of these coefficients depending on the parameters of the system :

$$\begin{aligned}
a_1 &= \frac{\mu}{\beta_{1,1}} (\mathcal{R}_0 - 1), \quad b_1 = \frac{\beta_{1,r}}{\beta_{1,1}}, \quad c_1 = \frac{\beta_{1,r}}{\beta_{1,1}} \frac{\mu}{\gamma_1 + \mu}, \\
a_r &= \frac{\gamma_r + \mu}{\beta_{r,r}}, \quad b_r = -\frac{\beta_{r,1}}{\beta_{r,r}} + \frac{\gamma_1}{\mu}, \quad c_r = \frac{\gamma_1}{\mu} \frac{\beta_{r,1}}{\beta_{r,r}}.
\end{aligned} \tag{2.29}$$

As in Proposition 2.5.3 of the previous SEIRS model, it is required to check if there is a

specific set of physical parameters which corresponds to a given choice of $a_1,\ b_1,\ c_1,\ a_r,\ b_r,\ c_r$.

Proposition 2.6.2. *Let $\mu \in \mathbb{R}_+^*$, let the function Φ be defined by,*

$$\begin{cases} \Phi : \mathbb{R}_+^{*6} \to \mathbb{R}^6 \\ \zeta = \frac{1}{\mu}(\gamma_1, \beta_{1,1}\beta_{1,r}, \gamma_r, \beta_{r,1}, \beta_{r,r}) \mapsto \Phi(\zeta) = (a_1, b_1, c_1, a_r, b_r, c_r) \end{cases}$$

where the parameters $(a_1,\ b_1,\ c_1,\ a_r,\ b_r,\ c_r)$ are defined by the equations (2.29)*. Then*

- $\operatorname{Im}\Phi = \Omega'$ *with, denoting* $\zeta' := (a_1, b_1, c_1, a_r, b_r, c_r)$,

$$\Omega' := \Big\{ \zeta' \in \mathbb{R}_+^{*6} : b_1,\ c_1,\ a_r,\ c_r > 0,\ a_1 < \frac{c_1}{b_1} < 1,\ \frac{b_1}{c_1} - 1 - b_r > 0, \\ c_r = \frac{b_1 - c_1}{c_1}\Big(\frac{b_1 - c_1}{c_1} - b_r\Big) \Big\}.$$

- *For a fixed $\zeta' \in \Omega'$, the inverse image $\Phi^{-1}(\zeta')$ is defined by*

$$\Phi^{-1}(\zeta') = \{\frac{1}{\mu}(\gamma_1, \beta_{1,1}\beta_{1,r}, \gamma_r, \beta_{r,1}, \beta_{r,r}) \in \Omega : \zeta \textit{ fulfils } (2.30)\}$$

$$\begin{aligned} &\frac{\gamma_1}{\mu} = \frac{b_1}{c_1} - 1, \quad \frac{\beta_{1,1}}{\mu} = (\frac{c_1}{b_1} - a_1)^{-1}, \quad \frac{\beta_{1,r}}{\mu} = b_1(\frac{c_1}{b_1} - a_1)^{-1}, \\ &\beta_{r,r} > \frac{\mu}{a_r}, \quad \frac{\beta_{r,1}}{\mu} = \frac{\beta_{r,r}}{\mu}(\frac{b_1}{c_1} - 1)^{-1}c_r, \quad \frac{\gamma_r}{\mu} = a_r\frac{\beta_{r,r}}{\mu} - 1. \end{aligned} \tag{2.30}$$

Proof. Positivity of $b_1,\ c_1,\ a_r,\ c_r$ is easily derived and $\frac{c_1}{b_1} < 1$ from (2.29). From equations (2.29),

$$\frac{\mu}{\gamma_1 + \mu}\frac{\beta_{1,1} - \gamma_1 - \mu}{\beta_{1,1}} = a_1,\ \frac{\mu}{\gamma_1 + \mu} = \frac{c_1}{b_1},\ \frac{\mu}{\beta_{1,1}} = \frac{c_1}{b_1} - a_1,\ \frac{\beta_{1,r}}{\mu} = c_1\frac{\beta_{1,1}}{\mu}\frac{\gamma_1 + \mu}{\mu} = b_1(\frac{c_1}{b_1} - a_1)^{-1},$$

$$\frac{\beta_{r,1}}{\beta_{r,r}} = \frac{\gamma_1}{\mu} - b_r = \frac{b_1}{c_1} - 1 - b_r = (\frac{b_1}{c_1} - 1)^{-1}c_r, \quad \frac{\gamma_r + \mu}{\beta_{r,r}} = a_r.$$

The first equality shows that $a_1 < \frac{c_1}{b_1}$, the fifth equality implies the second and third condition of Ω', and the equations (2.30) can be easily deduced. □

This result shows that in the SIS model, the set of admissible parameters $(a_1, b_1, c_1, a_r, b_r, c_r)$ for polynomial analysis of endemic equilibrium has one additional constrain $c_r = \frac{b_1 - c_1}{c_1}(\frac{b_1 - c_1}{c_1} - b_r)$. Then, even if previous theorems on the numbers of equilibrium are still technically correct, it is necessary to check again if each case described in these theorems may still realize under this supplementary constraint. To this end, we rewrite the coefficients a, c of the polynomial $P = ax^2 + bx + c$ and $d = 3b_1c_r - b_r$ that intervene in Theorems 2.3.7, 2.3.8, 2.3.9 and eliminate c_r to obtain:

$$\begin{aligned} a &= 1 + b_rb_1 - c_rb_1^2 = 1 + b_rb_1 - (\frac{b_1}{c_1} - 1)(\frac{b_1}{c_1} - b_r - 1)b_1^2, \\ c &= a_1^2 + b_ra_1c_1 - c_rc_1^2 = a_1^2 + b_ra_1c_1 - (b_1 - c_1)(b_1 - c_1b_r - c_1), \\ d &= 3b_1c_r - b_r = 3b_1(\frac{b_1}{c_1} - 1)(\frac{b_1}{c_1} - b_r - 1) - b_r. \end{aligned}$$

It is easy to see that coefficients above may still be both positive or negative by tuning a_1, b_1 and b_r.

- For the case $\mathcal{R}_0 < 1$ ($a_1 < 0$), the number of endemic equilibrium depends solely on c. Then, for parameters besides a_1 fixed, c can be both positive (when a_1 tends to infinity) and negative (a_1 close to zero). Both cases enunciated in Theorem 2.3.7 are then still possible.

- For $\mathcal{R}_0 = 1$ ($a_1 = 0$), the situation depends on a and d. For each value of b_r, a can be both positive with b_1 close to zero and negative when b_1 is arbitrarily large.

- Last, when $\mathcal{R}_0 > 1$, ($a_1 > 0$), the number of endemic equilibrium depends on the coefficient a and the number of roots of Δ.The coefficient a can be tuned as in the case $\mathcal{R}_0 = 1$ to obtain the desired sign and the existence of the case 3 endemic equilibriums, i.e., of a setting where the polynomial Δ has 4 roots is verified numerically with the following parameters :

$$a_1 = 0.001, \quad b_1 = 0.1, \quad c_1 = 1, \quad b_r = 3.$$

2.6.2 Persistence analysis for $\mathcal{R}_0 > 1$

For persistence analysis, the techniques developed in Section 2.4 for the SEIRS system can be adapted for SIS system. Borrowing the same notations, we set instead X as $X := \mathbb{R}^4_+$, the invariant and biologically feasible set Σ as in (2.28) with boundary $\partial\Sigma$ and interior $\mathring{\Sigma}$. We also define the invariant set $\partial\Sigma_{\max}$ as

$$\partial\Sigma_{\max} := \{(S_1, I_1, S_r, I_r) \in \Sigma | I_1 = 0, I_r = 0\},$$

and the proof of it being the maximal invariant set in $\partial\Sigma$ will again be brought later on. The SIS model version of the intermediary Lemmas 2.4.2, 2.4.3, 2.4.4 are

Lemma 2.6.3. *If the initial condition fulfills $I_1(0) + I_r(0) > 0$, then $I_1(t) > 0, I_r(t) > 0$ for each $t > 0$.*

Proof. Again, let us begin by noticing that every state $g \in \{S_1, I_1, S_r, I_r\}$ of the system (2.27) fulfills the inequality $\dot{g} \geqslant -a_g g$ where a_g is a (non-negative) constant. Thus, as observed in Lemma 2.4.2, the positivity of any state g is preserved for $t > T$ if it is positive at a time T and the Lemma is verified if the initial state vector is positive element wise. Therefore, once again, it remains to check the case where the initial state vector has some zero valued components. First, if we have either $S_1(0) = 0$ or $S_r(0) = 0$, then due to equations (2.27) and $I_1(0) + I_r(0) > 0$, the derivatives $\dot{S}_1(0)$, $\dot{S}_r(0)$ are strictly positive and therefore, thanks to the observation above, $S_1(t), S_r(t) > 0$ for $t > 0$. It remains to check the situations where $I_1(0) = 0$ or $I_r(0) = 0$ to prove the Lemma. As we have the inequality,

$$\dot{I}_1 + \dot{I}_r \geqslant (\beta_{1,1} I_1 + \beta_{1,r} I_r) S_1 + (\beta_{r,1} I_1 + \beta_{r,r} I_r) S_r - (\max\{\gamma_1, \gamma_r\} + \mu)(I_1 + I_r),$$

we deduce that $I_1(t) + I_r(t) > 0$ for any $t \geqslant 0$ as $I_1(0) + I_r(0) > 0$. This fact, and $S_1(t) > 0, S_r(t) > 0$ for $t > 0$ imply that the derivatives $\dot{I}_1(t)$, $\dot{I}_r(t)$ are positive on the right side of $t = 0$ in the hypothetical cases where $I_1(0) = 0$ or $I_r(0) = 0$. Thus, in any case, we have $I_1(t), I_r(t) > 0$ for $t > 0$, which concludes the proof. □

Lemma 2.6.4. *Assume that $\lim_{t\to\infty} I_1 = 0, \lim_{t\to\infty} I_r = 0$, then $\lim_{t\to\infty} S_1 = 1, \lim_{t\to\infty} S_r = 0$.*

Proof. Assume that $\lim_{t\to\infty} I_1 = 0, \lim_{t\to\infty} I_r = 0$, then for each $\varepsilon > 0$, there is $T > 0$ such that $I_1(t) < \varepsilon, I_r(t) < \varepsilon$ for $t > T$. Therefore, the differential inequalities below are verified for $t > T$:

$$\mu - \mu S_1 - (\beta_{1,1} + \beta_{1,r})\varepsilon \leqslant \dot{S}_1 \leqslant \mu - \mu S_1,$$
$$-\mu S_r \leqslant \dot{S}_r \leqslant (\gamma_1 + \gamma_r)\varepsilon - \mu S_r.$$

Which imply, for $t > T$,

$$S_1(T)e^{-\mu(t-T)} + \int_T^t [\mu - \varepsilon(\beta_{1,1} + \beta_{1,r})]e^{-\mu(t-s)}ds \leqslant S_1(t) \leqslant S_1(T)e^{-\mu(t-T)} + \int_T^t \mu e^{-\mu(t-s)}ds,$$
$$S_r(T)e^{-\mu(t-T)} \leqslant S_r(t) \leqslant S_r(T)e^{-\mu(t-T)} + \int_T^t (\gamma_1 + \gamma_r)\varepsilon e^{-\mu(t-s)}ds,$$

As $\lim_{t\to+\infty} \int_T^t e^{-\mu(t-s)}ds = \mu^{-1}$, the inequalities imply

$$1 - \mu^{-1}\varepsilon(\beta_{1,1} + \beta_{1,r}) \leqslant \liminf_{t\to\infty} S_1(t) \leqslant \limsup_{t\to\infty} S_1(t) \leqslant 1,$$
$$0 \leqslant \liminf_{t\to\infty} S_r(t) \leqslant \limsup_{t\to\infty} S_r(t) \leqslant \mu^{-1}(\gamma_1 + \gamma_r)\varepsilon.$$

Because the previous inequalities are valid for all $\varepsilon > 0$, by taking $\varepsilon \to 0$, we obtain

$$\lim_{t\to\infty} S_1(t) = 1, \lim_{t\to\infty} S_r(t) = 0.$$

□

Lemma 2.6.5. *For $\mathcal{R}_0 > 1$, no solution of system (2.27) converges to the DFE except those starting on the invariant set $\partial\Sigma_{\max}$.*

Proof. Assume that the contrary is true, i.e., for $\mathcal{R}_0 > 1$ there is a solution x converging to the DFE and such that $x(0) \notin \partial\Sigma_{\max}$ which is equivalent to $I_1(0) + I_r(0) > 0$. According to Lemma 2.6.3 the solution cannot reach the invariant set $\partial\Sigma_{\max}$ in finite time, thus $x(t) \notin \partial\Sigma_{\max}$ for $t \geqslant 0$. Setting now $\varepsilon_1 > 0$ such that $(1 - \varepsilon_1)\beta_{1,1} > \gamma_1 + \mu$ (which is possible as $\mathcal{R}_0 > 1$), the convergence of the solution implies that there is $T > 0$ such that $1 - \varepsilon_1 < S_1(t) < 1$ for $t > T$. Thus, for $t > T$, we have the following differential inequality

$$\dot{I}_1 > [(1 - \varepsilon_1)\beta_{1,1} - \gamma_1 - \mu]I_1 > 0,$$

which implies that I_1 moves exponentially away from zero asymptotically, which is absurd as $\lim_{t\to\infty} I_1 = 0$ because the solution is assumed to converge to the DFE.

Finally, the convergence towards the DFE on the invariant set $\partial\Sigma$ (invariance which can be deduced easily from the equations of the system) is self-evident as system 2.27 is reduced to the following equations in the set $\partial\Sigma_{\max}$

$$\dot{S}_1 = \mu - \mu S_1,$$
$$\dot{S}_r = -\mu S_r.$$

□

Employing the Lemma 2.6.3, 2.6.4, 2.6.5 demonstrated above, the following Theorem, which is the SIS version of Theorem 2.4.1 is now proved using the same arguments as in the proof of the Theorem aforementioned.

Theorem 2.6.6. *System* (2.27) *is uniformly persistent in the interior* $\mathring{\Sigma}$ *of the set* Σ *defined in* (2.28) *if* $\mathcal{R}_0 > 1$ *and non-persistent if* $\mathcal{R}_0 < 1$.

Proof. The non persistence is deduced from the fact that the DFE is locally asymptotically stable when $\mathcal{R}_0 < 1$. Moreover, the uniform persistence follows from an application of Theorem 4.3 in [49].

Thanks to the Lemma 2.6.3, for the biological feasible and invariant region Σ (2.28), the maximal invariant set on the boundary $\partial\Sigma$ is the set $\partial\Sigma_{\max}$. As the set $\partial\Sigma_{\max}$ is closed and connected, we take simply, as in the proof of Theorem 2.4.1, the cover $\{N_\alpha\}_{\alpha\in A} = \{\partial\Sigma_{\max}\}$ and the three points of the $(\mathcal{H})$ hypothesis of [49] are again verified. Furthermore, the system is naturally point-dissipative over the invariant compact set Σ, thus Theorem 4.3 in [49] can be applied. According to this Theorem and Lemma 2.6.3, uniform persistence of the system is equivalent to the non-existence of trajectories starting in the interior $\mathring{\Sigma}$ and approaching asymptotically the set $\partial\Sigma_{\max}$. Now, due to Lemma 2.6.4, the convergence towards the set $\partial\Sigma_{\max}$ implies, in fact, the convergence towards the DFE, which is indeed excluded when $\mathcal{R}_0 > 1$ due to Lemma 2.6.5. □

2.7 Global analysis of reinfection models with partial immunity

2.7.1 Elements of multistability analysis and contraction theory

As shown in Section 2.6, the two-stage models exhibit multistability. In fact, the models present up to 2 endemic equilibriums for $\mathcal{R}_0 < 1$, 3 endemic equilibriums for $\mathcal{R}_0 > 1$ alongside a disease-free equilibrium which always exists. In this section, we intend to discuss the asymptotic behavior of the two stage model.

For non-planar systems exhibiting multistability, the asymptotic behavior analysis is often difficult. Nevertheless, several approaches have been proposed in the literature for multistability analysis. First, the asymptotic convergence towards a steady state may be shown if the system presents some monotonicity properties [107], and whether a system is monotone regarding a particular cone can be easily tested. Nevertheless, it is often very difficult to find a priori a cone leading to monotonicity [5], but one may refer to [70] for an algorithmic approach that has been recently proposed on this issue.

Another way to deduce asymptotic convergence is to implement a generalization of Lyapunov's direct method to systems with multiple invariant sets [37]. The method requires finding a Lyapunov-like function proving that the invariant sets are attracting, which is again difficult in practice. Similarly, the theory of Input-to-State stability (ISS) has also been generalized to systems with multiple invariant sets [6], allowing the consideration of the stability of interconnected systems with multistability. Indeed, an analysis of ISS property of cascade interconnections of ISS subsystems is provided in [48]. However, it is not clear to what extent the generalized ISS property can be conserved for more complex interconnections, such as the ones in (2.27).

Finally, it is also possible to analyze multistability through contraction theory [78]. In short, for a non-linear system

$$\dot{x} = f(x), \tag{2.32}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function, contraction is a differential property on the corresponding

variational equation

$$\dot{\delta x} = \frac{df(x)}{dx}\delta x, \tag{2.33}$$

that implies the exponential convergence of any virtual displacement δx in the tangent space to zero. This in turn implies the convergence of all trajectories with respect to one another, which is a form of stability that is called *incremental*. In this case, every solution of (2.32) converges towards a unique equilibrium.

Recently, contraction theory has been generalized to include the contraction of k-dimensional parallelotope in the tangent space [117], with the classical contraction property corresponding to the case where $k = 1$. In fact, the motivation to study the generalized k-contraction comes from the work of Li and Muldowney who proved that any solution of a 2-contracting system converges towards an equilibrium, which is not necessarily unique [75]. For planar systems, the autonomous convergence theorem demonstrated by Li and Muldowney can be considered as a generalization of the well-known Bendixson-Dulac criterion [72]. This convergence result has been since applied to several compartmental models presenting one unique endemic equilibrium. In these models, the 2-contraction property is combined with a persistence result to guarantee the global stability of the unique endemic equilibrium in the interior of the invariant biologically feasible set when $\mathcal{R}_0 > 1$. As the 2-contraction property does not necessarily require the uniqueness of the steady states, this offers also an interesting framework for the asymptotic behavior analysis of multistable systems.

We present now a quick review of Li and Muldowney autonomous convergence result. Let $\delta x_1(t), \delta x_2(t), \ldots, \delta x_k(t) \in \mathbb{R}^n$ be k time-varying vectors representing virtual displacements which are solutions of (2.33). We may represent the k-dimensional parallelotope generated by the vectors $\delta x_1(t), \delta x_2(t), \ldots, \delta x_k(t)$ as an exterior product $z(t) := \wedge^i_{1\leqslant i\leqslant k}\delta x_i(t)$ evolving in $\mathbb{R}^{\binom{n}{k}}$, where $\binom{n}{k}$ is the binomial coefficient. With this definition, $z(t)$ is a solution of the *k-th compound equation* below associated to (2.33) (see e.g. [76, Section 2]).

$$\dot{z} = \frac{df}{dx}^{[k]}(x)z, \tag{2.34}$$

where, for any matrix A, we denote $A^{[k]}$ as the *k-th additive compound matrix* of A. For an introduction to exterior product and compound matrices, one refers to [76] and [91, 90]. With these definitions, we introduce the convergence result proved by Li and Muldowney in [75] and reformulated by Ballyk et al. [11] who relaxed the need for a unique equilibrium. The statement of the following theorem is slightly adapted from [11, Theorem 6.2].

Theorem 2.7.1. *Let $k = 2$, D be a simply connected open subset of $\mathbb{R}^n$ such that solutions of equation (2.32) with $x(0) = x_0 \in D$ remain in D for all finite time. Let $\|\cdot\|$ be a norm on $\mathbb{R}^{\binom{n}{2}}$ and let Q be an $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function on D such that $\|Q^{-1}\|$ is bounded on D. If there exist real numbers $T, g > 0$ such that for all solutions of (2.32) in D and all $t \geqslant T$, any solution $z(t)$ of (2.34) satisfies*

$$\|Q(x(t))z(t)\| \leqslant \|Q(x(0))z(0)\|e^{-gt}, \tag{2.35}$$

then every solution of (2.32) in D bounded away from the boundary converges to an equilibrium.

In other words, if the origin of the 2-nd compound equation (2.34) is exponentially asymptotically stable uniformly w.r.t. a state dependent norm $\|Q(x)z\|$ for any trajectory of (2.32) in the set D, then the ω-limit set of any solution of (2.32) in D not approaching its boundary is

simple, i.e., is reduced to an equilibrium point.

To give a brief idea of the proof of Li and Muldowney's result in [75], the condition (2.35) is first proved to be a *Bendixson criterion* [72] that precludes the existence of periodic solution for system (2.32). Indeed, let $D = \mathbb{R}^n$ for simplicity, the quantity $\|Q(x)z\|$ gives a measure of the volume of a $k-$parallelotope z in $\mathbb{R}^n$. Hence, the condition (2.35) for $k = 2$ translates the fact that the area of any surface in $\mathbb{R}^n$ shrinks exponentially under the flow of (2.32), and this rules out periodic solutions. In order to obtain the convergence to an equilibrium, the criterion is moreover demonstrated to be robust under C^1 *perturbations of the vector field* f. The Pugh's closing Lemma [101] is finally invoked to deduce the non-existence of non-constant non-wandering points, hence any non-wandering point is an equilibrium.

Practically, the Bendixson criterion (2.35) is demonstrated through the use of Lyapunov-like function. In fact, a carefully selected vector norm $|\cdot|$ on $\mathbb{R}^{\binom{n}{2}}$ is employed as Lyapunov function to prove the property (2.35). In this case, it is interesting to introduce the *Lozinskii measure* $\mu(E)$ of a matrix E with respect to the norm $|\cdot|$, defined by

$$\mu(E) = \lim_{h\to 0^+} \frac{|\mathbf{I} + hE| - 1}{h},$$

which is used to estimate the rate of expansion $-g$ in (2.35). Indeed, it is possible to verify that (see [86])

$$D_+|z(t)| \leqslant \mu\left(\frac{df}{dx}^{[2]}(x(t))\right)|z(t)|,$$

where D_+ is the right-hand derivative. More generally, for $V(x, z) := |Q(x)z|$, where $Q(\cdot)$ is a C^1 non-singular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function

$$D_+|Q(x(t))z(t)| \leqslant \mu\left(Q_f(x(t))Q^{-1}(x(t)) + Q(x(t))\frac{df}{dx}^{[2]}(x(t))Q^{-1}(x(t))\right)|Q(x(t))z(t)|,$$

where Q_f is the directional derivative of the matrix Q in the direction of the vector field f. For more details about Lozinskii measure, one may refer to [26].

Denoting $B(x) := Q_f(x)Q^{-1}(x) + Q(x)\frac{df}{dx}^{[2]}(x)Q^{-1}(x)$, if

$$\sup_{x\in D} \mu(B(x)) < 0, \tag{2.36}$$

or more generally,

$$\limsup_{t\to\infty} \sup_{x_0\in D} \frac{1}{t}\int_0^t \mu(B(x(s, x_0))ds < 0, \tag{2.37}$$

then the property (2.35) is fulfilled. In fact, the conditions (2.36) and (2.37) are the original Bendixson criterions proposed by Li and Muldowney [77, 75]. Insofar as they are used in the proof of results in Section 2.7, it is also worth noting that the Bendixson criterion (2.37) can be further relaxed, see for example [79, Theorem 2.4] or [80, Theorems 2.2, 2.3 and 2.4].

As epidemiological models are often considered in invariant and biologically feasible sets of co-dimension 1 (generated by the fact that the total population is constant along the time), a useful extension of the convergence result is available for differential equations evolving on invariant manifolds [76].

Indeed, if the system (2.32) possesses a simply connected invariant manifold $\Gamma := \{x \in$

$\mathbb{R}^n_+ | g(x) = 0\}$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is C^2 and $\dim(\frac{dg}{dx}) = m$, then there is a function $N : \mathbb{R}^n \to \mathbb{R}^m$ [76] such that

$$\frac{\partial g}{\partial x}(x) f(x) = N(x) g(x),$$

and the convergence result of Li and Muldowney holds as well if we replace the 2-nd compound equation (2.34) by the $(m+2)$-th compound equation (2.38) [76]:

$$\dot{z} = (\frac{df}{dx}^{[m+2]}(x) - \nu(x)\mathbf{I})z, \tag{2.38}$$

where

$$\nu(x) := \mathrm{Tr}(N(x)), \tag{2.39}$$

is the trace of $N(x)$. By the same arguments then as in [11, Theorem 6.2], we deduce a result similar to Theorem 2.7.1 in the context of systems evolving in an invariant manifold.

Theorem 2.7.2. *D be a simply connected open subset of Γ such that solutions of equation (2.32) with $x(0) = x_0 \in D$ remain in D for all finite time. Let $\|\cdot\|$ be a norm on $\mathbb{R}^{\binom{n}{m+2}}$ and let Q be a $\binom{n}{m+2} \times \binom{n}{m+2}$ matrix-valued function on D such that $\|Q^{-1}\|$ is bounded on D. If there exist reals $T, g > 0$ such that for all solutions of (2.32) in D and all $t \geqslant T$, any solution $z(t)$ of (2.34) satisfies*

$$\|Q(x(t))z(t)\| \leqslant \|Q(x(0))z(0)\| e^{-gt}, \tag{2.40}$$

then every solution of (2.32) in D bounded away from the boundary converges to an equilibrium.

The interest of studying the associated compound (2.38) instead of (2.34) lies in the fact that the $(m+2)$-th additive compound matrix may be in some cases smaller in size than the 2-nd additive compound matrix. This is essential because compound matrices explode rapidly in size as the dimension n increases.

In addition, it is noteworthy that, if the same assumptions in Theorems 2.7.1 and 2.7.2 hold in a compact invariant set Σ instead, then [76, Theorem 6.1] states that any trajectory in Σ converges to an equilibrium.

Coming back to contraction theory, the convergence criterion (2.35) proposed is in fact the definition of *k-contraction*, (see e.g. [120, Definition 1]), with $k = 2$ in the context of Li and Muldowney theory. To demonstrate the k-contraction, another interesting method which involves the numerical testing of linear-matrix inequalities (LMIs) has been recently proposed [120]. In fact, we may verify simply the contraction property with the following result, which is related to the existence of a quadratic Lyapunov function $V(z) := zQz$ for proving the asymptotic stability of (2.34).

Theorem 2.7.3. *Let $\Sigma \subset \mathbb{R}^n$ be a forward invariant set of (2.32) and suppose there exists a real number $\eta > 0$ and a symmetric positive definite matrix $Q \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{k}}$ such that*

$$Q\left(\frac{\partial f}{\partial x}(x)^{[k]}\right) + \left(\frac{\partial f}{\partial x}(x)^{[k]}\right)^{\top} Q \preceq -\eta\mathbf{I}, \quad \forall x \in \Sigma.$$

Then, the system (2.32) is k-contractive on Σ.

Another criterion has also been proposed in [120] for k-contraction without the involvement of compound matrix, which may increase rapidly in size for larger dimension n of the system

(2.32), however it is not useful for the present case as $n = 4$ remains small. Additionally, the notion of k-contraction was compared in [120] to the closely related property of p-dominance [47]. Defining the inertia of a matrix P as $In(P) = (p, 0, n-p)$ in the case where P has p negative eigenvalues, and $n-p$ positive eigenvalues, then the property of p-dominance is defined as follows.

Definition 2.7.1 (Forni and Sepulchre [47])**.** *System* (2.32) *is said to be* p-dominant *on* $\Sigma \subset \mathbb{R}^n$ *if there exist real numbers* $\mu, \varepsilon \geqslant 0$ *and a symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *with inertia* $In(P) = (p, 0, n-p)$ *such that*

$$P\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \prec -2\mu P - \varepsilon \mathbf{I}.$$

The property is strict *if* $\varepsilon > 0$.

Similarly to the property of k-contraction, the notion of p-dominance implies also, for a small number p, that the system is asymptotically well-behaved.

Proposition 2.7.4 (Forni and Sepulchre [47])**.** *For a strictly* p*-dominant system, every bounded solution asymptotically converges to*

1. *a unique fixed point if* $p = 0$*;*
2. *a fixed point if* $p = 1$*;*
3. *a simple attractor if* $p = 2$*, that is, a fixed point, a set of fixed points and connecting arcs, or a limit cycle.*

Notice that the conditions above need to be solved for all $x \in \Sigma$, which is a complex problem. To overcome this difficulty, the LMIs for k-contraction and p-dominance can be first transformed into tractable conditions by convex relaxation, as suggested in [47, 120].

2.7.2 SIR model with partial immunity and demography

We examine here the endemic equilibriums and asymptotic behavior of a SIRI model, which takes the assumption that the primary infection leads to a partial immunity to subsequent reinfections. This was initially studied in [51]. The asymptotic behavior and stability of the model without demography has been analyzed in [94]. Taking a step further, we investigate the asymptotic behavior and stability of the SIRI model *including demography.*

The SIRI model with demography introduced by [51] can be considered, in fact, as a particular case of system (2.27). To see this, let us begin by fixing the coefficients in (2.27) such that $\beta_{1,1} = \beta_{1,r} = \beta$, $\beta_{r,1} = \beta_{r,r} = \psi\beta$, $\gamma_1 = \gamma_r = \gamma$. In this situation, the equations of the system (2.27) are

$$\begin{aligned}
\dot{S}_1 &= \mu - \beta(I_1 + I_r)S_1 - \mu S_1,\\
\dot{I}_1 &= \beta(I_1 + I_r)S_1 - (\gamma + \mu)I_1,\\
\dot{S}_r &= \gamma(I_1 + I_r) - \psi\beta(I_1 + I_r)S_r - \mu S_r,\\
\dot{I}_r &= \psi\beta(I_1 + I_r)S_r - (\gamma + \mu)I_r.
\end{aligned} \tag{2.41}$$

Introducing then the state variables

$$I := I_1 + I_r, \quad S := S_1, \quad R := S_r,$$

we obtain with these definitions the SIRI system with partial immunity and demography [50].

$$\begin{aligned}\dot{S} &= \mu - \beta IS - \mu S,\\ \dot{I} &= \beta IS + \psi\beta IR - (\gamma+\mu)I,\\ \dot{R} &= \gamma I - \psi\beta IR - \mu R.\end{aligned} \tag{2.42}$$

We describe in the following theorem the endemic equilibriums of the systems (2.41) and (2.42), we apply Theorem 2.3.9 and Theorem 2.3.7 respectively to the case $\mathcal{R}_0 > 1$ and $\mathcal{R}_0 < 1$. For $\mathcal{R}_0 > 1$, this implies the uniqueness of endemic equilibrium. On the other hand, the matter is more intricate for $\mathcal{R}_0 < 1$.

Theorem 2.7.5. *The systems* (2.41) *and* (2.42) *admit*

1. *a unique endemic equilibrium if* $\mathcal{R}_0 > 1$,

2. *no endemic equilibrium if* $\mathcal{R}_0 < 1$ *and* $\beta \leqslant \mu$.

Moroever, if $\mathcal{R}_0 < 1$ *and* $\beta > \mu$, *there exists a threshold such that the systems* (2.41) *and* (2.42) *admit*

1. *no endemic equilibrium if* $\psi < \psi^*$,

2. *a unique endemic equilibrium if* $\psi = \psi^*$,

3. *two endemic equilibrium if* $\psi > \psi^*$.

Remark 2.7.6. *For a disease ensuring partial immunity after infection, one has* $\psi < 1$. *Consequently, in such case, systems* (2.41) *and* (2.42) *do not exhibit endemic equilibriums when* $\mathcal{R}_0 < 1$.

Proof. As $\beta_{1,1} = \beta_{1,r} = \beta$, $\beta_{r,1} = \beta_{r,r} = \psi\beta$, $\gamma_1 = \gamma_r = \gamma$, by formulas in (2.29), $b_1 = 1$ and

$$1 + b_r b_1 - c_r b_1^2 = 1 + b_r - c_r = 1 + \frac{\gamma}{\mu} - 1 - \frac{\gamma}{\mu} = 0.$$

We conclude the uniqueness of endemic equilibrium for $\mathcal{R}_0 > 1$ by applying point 1 of Theorem 2.3.9. This implies the same for system (2.42).

Let us demonstrate now the statement of the Theorem for system (2.41) and the case $\mathcal{R}_0 \leqslant 1$. Clearly, from the Theorem 2.3.7, a sufficient condition for the non-existence of endemic equilibrium is

$$c = a_1^2 + a_1 c_1 b_r - c_r c_1^2 \geqslant 0. \tag{2.43}$$

Thanks to (2.29), the quantity c can also be expressed in terms of epidemiological parameters

$$\begin{aligned}c &= (\frac{\mu}{\beta}\frac{\beta-\gamma-\mu}{\gamma+\mu})^2 + \frac{\mu}{\beta}\frac{\beta-\gamma-\mu}{\gamma+\mu}\frac{\mu}{\gamma+\mu}(\frac{\gamma}{\mu}-1) - \frac{\gamma}{\gamma+\mu}\frac{\mu}{\gamma+\mu}\\ &= \frac{\mu}{\beta}\frac{\beta-\gamma-\mu}{(\gamma+\mu)^2}(\mu\frac{\beta-\gamma-\mu}{\beta} + \gamma - \mu - \gamma\frac{\beta}{\beta-\gamma-\mu}).\end{aligned}$$

As $\mathcal{R}_0 < 1$, thus $\beta - \gamma - \mu < 0$ and the condition (2.43) is equivalent to

$$\begin{aligned}\mu\frac{\beta-\gamma-\mu}{\beta} + \gamma &\leqslant \gamma\frac{\beta}{\beta-\gamma-\mu} + \mu,\\ \frac{(\beta-\gamma-\mu)\mu+\gamma\beta}{\beta} &\leqslant \frac{\gamma\beta+\mu(\beta-\gamma-\mu)}{\beta-\gamma-\mu},\\ (\gamma+\mu)\frac{\beta-\mu}{\beta} &\leqslant (\gamma+\mu)\frac{\beta-\mu}{\beta-\gamma-\mu}.\end{aligned}$$

Now, if $\beta > \mu$, the last inequality above is the same as

$$\frac{\beta-\gamma-\mu}{\beta} \geqslant 1,$$

which is always false. On the other hand, if $\beta < \mu$ then the same condition is now equivalent to

$$\frac{\beta-\gamma-\mu}{\beta} \leqslant 1,$$

which is always verified. Finally, if $\beta = \mu$, the same condition is trivially verified. To sum up, the condition (2.43) is the same as

$$\beta \leqslant \mu.$$

Therefore, if $\beta \leqslant \mu$ the system does not admit endemic equilibrium.

For the contrary case, Theorem 2.3.7 states that the smallest zero of Δ acts as a threshold on a_r through which the system passes from a two endemic equilibrium regime to a zero endemic equilibrium regime. As $a_r = \frac{\gamma+\mu}{\psi\beta}$ is inversely proportional to ψ, this is equivalent to say that there exists a threshold ψ^* on ψ for the appearance of two endemic equilibriums.

Finally, let $\psi = 1$ and denote $I := I_1 + I_r$, $R := S_1 + S_r$, then the system (2.41) can be reduced to the SIS system:

$$\begin{aligned}\dot{S} &= \mu - \beta IS - \mu S + \gamma I,\\ \dot{I} &= \beta IS - (\gamma+\mu)I.\end{aligned}$$

It is easy to verify that this SIS system does not admit endemic equilibrium for $\mathcal{R}_0 < 1$. In particular, this forces ψ^* to be greater than 1.

We have shown that the system (2.41) exhibits one endemic and one disease-free equilibrium for $\mathcal{R}_0 > 1$, and up to 2 endemic equilibriums and one disease-free equilibrium for $\mathcal{R}_0 < 1$. In addition, the endemic equilibriums of the two-stage SIS system are endowed with a total ordering in the same way as in Theorem 2.3.6 for the SEIRS system. Hence, Theorem 2.7.5 still holds for the reduced SIRI model (2.42), as the multiple endemic equilibriums that may arise in system (2.41) cannot be confounded in the reduced model (2.42) due to the ordering. □

We have shown that the system (2.41) exhibits one endemic and one disease-free equilibrium for $\mathcal{R}_0 > 1$, and up to 2 endemic equilibriums and one disease-free equilibrium for $\mathcal{R}_0 < 1$. In addition, it is easy to see that the endemic equilibriums of the two-stage SIS system are endowed with a total ordering in the same way as in Theorem 2.3.6. Therefore, the same Theorem 2.7.5 still holds for the reduced SIRI model (2.42) as the multiple endemic equilibriums that may arise in system (2.41) cannot be confounded in the reduced model (2.42) due to the ordering.

Subsequently, we turn our attention to the asymptotic convergence of the solutions. In the final result of this subsection, we prove first that any trajectory converges to an equilibrium for

the reduced system (2.42), assuming that $\mathcal{R}_0 \neq 1$. In particular, this implies the same for the two-stage SIS system (2.41).

Theorem 2.7.7. *Assume $\mathcal{R}_0 \neq 1$, any solution $x(t)$ of system (2.41) (resp. system (2.42)) with initial condition $x(0) \in \{x \in \mathbb{R}^4_+ | S_1 + I_1 + S_r + I_r = 1\}$ (resp. $x(0) \in \{x \in \mathbb{R}^3_+ | S + I + R = 1\}$) converges to an equilibrium.*

Notice that, due to the persistence result of Theorem 2.6.6, Theorem 2.7.7 implies in particular that, for $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable in the interior of the respective biologically feasible and invariant sets of both systems (2.42) and (2.41). In the alternative case where $\mathcal{R}_0 < 1$, the result guarantees the convergence to an equilibrium of every solution even in the case where multiple endemic equilibriums coexist.

Proof. For $\mathcal{R}_0 \neq 1$, let us show that any trajectory of (2.42) converges to an equilibrium using the Li and Muldowney theory summarized in Section 2.7.1. The invariant and biologically feasible set Γ is defined in this case by $\Gamma = \{x \in \mathbb{R}^3_+ | g(x) = 0\}$, where $g : \mathbb{R}^3 \to \mathbb{R}^m$, $g(x) = S + I + R - 1$. In the rest of the proof, we may consider only the trajectories evolving in the interior $\mathring{\Gamma}$ of Γ. This is justified as trajectories starting in Γ either converge to the disease-free equilibrium if $I(0) = 0$, or fulfill $S(t), I(t), R(t) > 0$ for $t > 0$ thanks to the differential inequalities:

$$\dot{S} \geqslant \mu(1 - S), \quad \dot{I} \geqslant -(\gamma + \mu)I, \quad \dot{R} \geqslant -\mu R,$$

and the fact that $\dot{S}(0) > 0$ (resp. $\dot{R}(0) > 0$) if $S(0) = 0$ (resp. $R(0) = 0$). In addition, notice that

$$\frac{\partial g}{\partial x}(x) f(x) = \mu(1 - S - I - R) = -\mu g(x),$$

thus the function ν introduced in (2.39) of the Section 2.7.1 above corresponds to the constant $-\mu$ and $m = 1$ for the system (2.42). The associated 3-rd additive compound matrix of $\frac{df}{dx}(x)$ associated to the equation is (see [90])

$$\dot{z} = (\frac{df}{dx}^{[3]}(x) + \mu I)z.$$

As

$$\frac{\partial f}{\partial x}(x) = \begin{pmatrix} -\beta I - \mu & -\beta S & 0 \\ \beta I & \beta(S + \psi R) - \gamma - \mu & \psi\beta I \\ 0 & \gamma - \psi\beta R & -\psi\beta I - \mu \end{pmatrix}, \tag{2.44}$$

is a 3×3 matrix, its 3-rd additive compound is simply its trace $\frac{df}{dx}^{[3]} = \mathrm{Tr}(\frac{df}{dx})$. Therefore, the compound equation (2.38) is reduced to :

$$\dot{z} = [\beta(S + \psi R) - \gamma - \beta(1 + \psi)I - 2\mu]z.$$

Moreover,

$$\beta(S + \psi R) - \gamma - \beta I(1 + \psi) - 2\mu = \frac{\dot{I}}{I} - \beta I(1 + \psi) - \mu.$$

Then, after integration of the compound equation, we obtain

$$z(t) = z(0)\frac{I(t)}{I(0)} e^{-\int_0^t (\beta(1+\psi)I(s) + \mu) ds}.$$

As $\lim_{t\to+\infty}\int_0^t(\beta(1+\psi)I(s)+\mu)ds=\infty$, thus $\lim_{t\to+\infty}z(t)=0$ exponentially, and the system (2.42) satisfies a *Bendixson condition* (see [75] or Section 2.7.1) in $\mathring{\Gamma}$. Therefore, Theorem 2.7.2 holds with $D=\mathring{\Gamma}$, $Q=\mathbf{I}$ and any solution $x(t)$ of (2.42) either converges to an equilibrium or approaches the boundary of $\mathring{\Gamma}$. The later case is precluded for $\mathcal{R}_0>1$ due to uniform persistence. On the other hand, for $\mathcal{R}_0<1$, it can be observed that the solutions may only approach the component $\{x\in\mathbb{R}^3_+|S+R=1,I=0\}$ of the boundary. Indeed, we have $\dot{S}>0$ if S is close to 0 and the same for $\dot{R}$ if $I>0$ and R close to 0. In addition, from the expression of the Jacobian (2.44), the DFE is locally asymptotically stable, thus its basin of attraction is an open subset of Γ [24]. As the DFE attracts $\{x\in\mathbb{R}^3_+|S+R=1,I=0\}$, hence it also attracts a close neighborhood of the latter and the solution converges to the DFE in this case. □

2.7.3 SEIR model with partial immunity and demography

We investigate now a more sophisticated two-stage reinfection model compared to (2.41), with an additional class E modelling the disease incubation period. Again, the model can be considered as a special case of the general two-stage model. Indeed, setting

$$\beta_{1,1}=\beta_{1,r}=\beta,\quad \beta_{r,1}=\beta_{r,r}=\psi\beta,\quad \sigma_1=\sigma_r=\sigma,\quad \gamma_1=\gamma_r=\gamma,\quad \omega_1=\omega_r=+\infty,$$

we obtain the following two-stage SEIS system:

$$\begin{aligned}
\dot{S}_1&=\mu-(\beta(I_1+I_r)+\mu)S_1,\\
\dot{E}_1&=(\beta(I_1+I_r)S_1-(\sigma+\mu)E_1,\\
\dot{I}_1&=\sigma E_1-(\gamma+\mu)I_1,\\
\dot{S}_r&=\gamma(I_1+I_r)-(\psi\beta(I_1+I_r)+\mu)S_r,\\
\dot{E}_r&=\psi\beta(I_1+I_r)S_r-(\sigma+\mu)E_r,\\
\dot{I}_r&=\sigma E_r-(\gamma+\mu)I_r.
\end{aligned}\tag{2.45}$$

Now, let us introduce the state variables

$$S:=S_1,\quad E:=E_1+E_R,\quad I:=I_1+I_r,\quad R:=S_r.$$

This leads to a reduced SEIRE system with partial immunity and demography:

$$\begin{aligned}
\dot{S}&=\mu-\beta SI-\mu S,\\
\dot{E}&=\beta SI+\psi\beta IR-(\sigma+\mu)E,\\
\dot{I}&=\sigma E-(\gamma+\mu)I,\\
\dot{R}&=\gamma I-\psi\beta IR-\mu R.
\end{aligned}\tag{2.46}$$

It can be checked straightforwardly that the results of Theorems 2.3.6 to 2.3.9 and Theorem 2.4.1 still hold for system (2.45), albeit with $A_r=\frac{\sigma}{\sigma+\mu}\frac{\gamma}{\gamma+\mu}$ and $D_r=1$ instead of the definition in (2.5). Moreover in this case,

$$1-A_r=1-\frac{\sigma\gamma}{\sigma\gamma+\mu(\sigma+\gamma)+\mu^2}=\frac{\mu(\sigma+\gamma+\mu)}{\sigma\gamma+\mu(\sigma+\gamma+\mu)},$$

the polynomial coefficients $a_1, b_1, c_1, a_r, b_r, c_r$ are

$$a_1 = \mu\frac{\beta\sigma - (\gamma+\mu)(\sigma+\mu)}{\beta(\gamma+\mu)(\sigma+\mu)}, \quad b_1 = 1, \quad c_1 = \frac{\mu}{\gamma+\mu}\frac{\sigma}{\sigma+\mu},$$
$$a_r = \frac{\sigma\gamma + \mu(\sigma+\gamma+\mu)}{\psi\beta(\sigma+\gamma+\mu)}, \quad b_r = \frac{\sigma\gamma - (\sigma+\gamma+\mu)\mu}{\mu(\sigma+\gamma+\mu)}, \quad c_r = \frac{\sigma\gamma}{\mu(\sigma+\gamma+\mu)},$$

and the basic reproduction number is expressed by

$$\mathcal{R}_0 = \frac{\beta}{\gamma+\mu}\frac{\sigma}{\sigma+\mu}.$$

Similarly to systems (2.41) and (2.42), the endemic equilibrium is unique for $\mathcal{R}_0 > 1$ and the description is more complicated for $\mathcal{R}_0 < 1$.

Theorem 2.7.8. *The systems* (2.45) *and* (2.46) *admit*

1. *a unique endemic equilibrium if* $\mathcal{R}_0 > 1$,
2. *no endemic equilibrium if* $\mathcal{R}_0 < 1$ *and*

$$\mu(\sigma+\gamma+\mu)(\beta\sigma - (\gamma+\mu)(\sigma+\mu)) + \gamma\sigma^2\beta \leqslant 0. \tag{2.47}$$

Moreover, if $\mathcal{R}_0 < 1$ *and* (2.47) *is not fulfilled, there exists a threshold* $\psi^* > 1$ *and systems* (2.45) *and* (2.46) *admit*

1. *no endemic equilibrium if* $\psi < \psi^*$,
2. *a unique endemic equilibrium if* $\psi = \psi^*$,
3. *two endemic equilibriums if* $\psi > \psi^*$.

Remark 2.7.9. *Again, for disease ensuring partial immunity after infection, systems* (2.45) *and* (2.46) *cannot exhibit endemic equilibriums when* $\mathcal{R}_0 < 1$ *(see Remark 2.7.6).*

Proof. The computation is mostly the same as in the proof of Theorem 2.7.5 for the case $\mathcal{R}_0 > 1$.

On the other hand, for $\mathcal{R}_0 \leqslant 1$, we prove first the statement for the system (2.45). The quantity c in this setting is equal to

$$\begin{aligned} c =& a_1^2 + a_1 c_1 b_r - c_r c_1^2 \\ =& \sigma\frac{\beta\sigma\mu - \mu(\gamma+\mu)(\sigma+\mu)}{\beta(\gamma+\mu)^2(\sigma+\mu)^2(\sigma+\gamma+\mu)}\Bigg(\mu(\sigma+\gamma+\mu)\frac{\beta\sigma-(\gamma+\mu)(\sigma+\mu)}{\beta\sigma} + \gamma\sigma - (\sigma+\gamma+\mu)\mu \\ & - \gamma\sigma\frac{\beta\sigma}{\beta\sigma-(\gamma+\mu)(\sigma+\mu)}\Bigg). \end{aligned}$$

For $\mathcal{R}_0 < 1$, $\beta\sigma < (\gamma+\mu)(\sigma+\mu)$ and the condition $c \geqslant 0$ is equivalent to

$$\frac{\mu(\sigma+\gamma+\mu)(\beta\sigma-(\gamma+\mu)(\sigma+\mu)) + \gamma\sigma^2\beta}{\beta\sigma} \leqslant \frac{\mu(\sigma+\gamma+\mu)(\beta\sigma-(\gamma+\mu)(\sigma+\mu)) + \gamma\sigma^2\beta}{\beta\sigma-(\gamma+\mu)(\sigma+\mu)}.$$

If $\mu(\sigma+\gamma+\mu)(\beta\sigma-(\gamma+\mu)(\sigma+\mu)) + \gamma\sigma^2\beta > 0$, the condition $c \geqslant 0$ is the equivalent to

$$\frac{\beta\sigma-(\gamma+\mu)(\sigma+\mu)}{\beta\sigma} \geqslant 1,$$

which is always false. On the contrary, if $\mu(\sigma+\gamma+\mu)(\beta\sigma-(\gamma+\mu)(\sigma+\mu))+\gamma\sigma^2\beta \leqslant 0$, the same condition is equivalent to

$$\frac{\beta\sigma-(\gamma+\mu)(\sigma+\mu)}{\beta\sigma} \leqslant 1,$$

which is verified. To sum up, the condition $c \geqslant 0$ is equivalent to

$$\mu(\sigma+\gamma+\mu)(\beta\sigma-(\gamma+\mu)(\sigma+\mu))+\gamma\sigma^2\beta \leqslant 0.$$

Similarly to the proof of Theorem 2.7.5, the existence of the threshold ψ^* is then deduced using Theorem 2.3.7. Moreover, for $\psi = 1$, we denote $S := S_1 + S_r$, $I := I_1 + I_r$, $E := E_1 + E_r$, then the system (2.45) can be again reduced to the SEIS system below.

$$\begin{aligned}
\dot{S} &= \mu - \beta SI - \mu S + \gamma I,\\
\dot{E} &= \beta SI - (\sigma+\mu)E,\\
\dot{I} &= \sigma E - (\gamma+\mu)I.
\end{aligned}$$

It is known that this system does not admit endemic equilibrium for $\mathcal{R}_0 < 1$ [39]. In particular, this implies that $\psi^* > 1$.

Notice finally that the result still holds for the reduced system (2.46), thanks to an adaptation of Theorem 2.3.6 which gives an ordering of the equilibriums. □

Notice in addition that the equilibrium results above still hold for the reduced system (2.46), thanks to an adaptation of Theorem 2.3.6 which gives an ordering of the equilibriums.

We also present a persistence result, that will be needed for later, and whose proof can be easily adapted from the one of Theorem 2.4.1.

Theorem 2.7.10. *System* (2.45) *is uniformly persistent in the interior of* $\{x \in \mathbb{R}^6_+ | S_1 + E_1 + I_1 + S_r + E_r + I_r = 1\}$ *if* $\mathcal{R}_0 > 1$ *and non-persistent if* $\mathcal{R}_0 < 1$.

We have shown that systems (2.45) and (2.46) present again one unique endemic equilibrium when $\mathcal{R}_0 > 1$ and up to two endemic equilibriums when $\mathcal{R}_0 < 1$. Excluding the critical case $\mathcal{R}_0 = 1$, the next result shows that any solution of (2.46) converges either to the disease-free equilibrium or to an endemic equilibrium.

Theorem 2.7.11. *Assume* $\mathcal{R}_0 \neq 1$*, any solution* $x(t)$ *of* (2.45) *(resp. of* (2.46)*) with initial condition* $x(0) \in \{x \in \mathbb{R}^6_+ | S_1+E_1+I_1+S_r+E_r+I_r = 1\}$ *(resp.* $x(0) \in \{x \in \mathbb{R}^4_+ | S+E+I+R = 1\}$*) converges to an equilibrium.*

Notice that, combined with the persistence result of Theorem 2.7.10, the unique endemic equilibrium is again globally asymptotically stable in the interior of the invariant set when $\mathcal{R}_0 > 1$.

The proof of Theorem 2.7.11 requires the following Lemma.

Lemma 2.7.12. *Let* $x(t) = (S(t), E(t), I(t), R(t))$ *be any solution of* (2.46) *in the invariant set* $\{x \in \mathbb{R}^4_+ | S + E + I + R = 1\}$*. Then ultimately the following inequality is verified*

$$\gamma \geqslant \psi\beta R.$$

Proof of Lemma 2.7.12. The first inequality is deduced from the inequality

$$\dot{S} \geqslant \mu - (\beta+\mu)S.$$

Notice that the derivative of R, solution of (2.46), is non-positive on the set $\{x \in \mathbb{R}^+_4 | \gamma = \psi\beta R\}$, hence $\{x \in \mathbb{R}^+_4 | \gamma \geqslant \psi\beta R\}$ is an invariant set of (2.46).

For trajectories such that $I(0) + E(0) = 0$, it is easy to see that $I(t) = 0$ for $t \geqslant 0$ and $R(t)$ converges also to 0 in this case, hence $\gamma \geqslant \psi\beta R$ ultimately.

On the other hand, assume $I(0) + E(0) > 0$, then $I(t), R(t) > 0$ for $t > 0$. This is because, if $I(0) = 0$, $E(0) > 0$ and $\dot{I} > 0$ and I is positive in a positive neighborhood of zero. Thus, we may assume with no loss of generality that $I(0) > 0$. In this case, the differential inequalities

$$\dot{I}(t) \geqslant -(\gamma + \mu)I, \quad \dot{R} \geqslant -(\psi\beta I + \mu)R,$$

and the fact that $\dot{R}(0) > 0$ if $R(0) = 0$ imply that $I(t), R(t) > 0$ for $t > 0$ anyway. Therefore, the fact that $I(t), R(t) > 0$ for all $t > 0$ implies $\dot{R}(t) < 0$ if $\gamma \leqslant \psi\beta R(t)$ and again $\gamma \geqslant \psi\beta R$ ultimately. □

Proof of Theorem 2.7.11. In a similar fashion as in the proof of Theorem 2.7.7 for the SIR system above, notice first that only trajectories evolving in the interior of $\Gamma := \{x \in \mathbb{R}^4_+ | S+E+I+R = 1\}$ matter, as any solution in Γ either fulfills $S(t), E(t), I(t), R(t) > 0$ for $t > 0$ or converges to the disease-free equilibrium when $E(0) + I(0) = 0$. We apply again the geometrical approach of Li and Muldowney [76]. The Jacobian matrix of (2.46) is

$$\frac{df}{dx} = \begin{pmatrix} -(\beta I + \mu) & 0 & -\beta S & 0 \\ \beta I & -(\sigma + \mu) & \beta S + \psi\beta R & \psi\beta I \\ 0 & \sigma & -(\gamma + \mu) & 0 \\ 0 & 0 & -\psi\beta R + \gamma & -(\psi\beta I + \mu) \end{pmatrix}.$$

and its 3-rd additive compound is defined by (see the Appendix of [90])

$$\frac{df}{dx}^{[3]} = \begin{pmatrix} k_1 & 0 & -\psi\beta I & 0 \\ -\psi\beta R + \gamma & k_2 & \beta S + \psi\beta R & \beta S \\ 0 & \sigma & k_3 & 0 \\ 0 & 0 & \beta I & k_4 \end{pmatrix},$$

where

$$k_1 = -\beta I - \sigma - \gamma - 3\mu, \quad k_2 = -(1 + \psi)\beta I - \sigma - 3\mu,$$
$$k_3 = -(1 + \psi)\beta I - \gamma - 3\mu, \quad k_4 = -\sigma - \gamma - \psi\beta I - 3\mu.$$

Hence, the associated compound equation (2.38) is in this case defined as

$$\dot{z} = \left(\frac{df}{dx}^{[3]} + \mu \mathbf{I} \right) z.$$

Let us introduce now

$$A := \mathrm{diag}(R, I, E, S),$$

then

$$A^{-1} = \mathrm{diag}(R^{-1}, I^{-1}, E^{-1}, S^{-1}) \text{ and } A_f A^{-1} = \mathrm{diag}(\frac{\dot{R}}{R}, \frac{\dot{I}}{I}, \frac{\dot{E}}{E}, \frac{\dot{S}}{S}),$$

where A_f stands for the directional derivative of the matrix A in the direction of the vector field

f. One has in addition,

$$A\frac{df}{dx}^{[3]}A^{-1} = \begin{pmatrix} k_1 & 0 & \psi\beta I\frac{R}{E} & 0 \\ (\gamma-\psi\beta R)\frac{I}{R} & k_2 & (\beta S+\psi\beta R)\frac{I}{E} & \beta I \\ 0 & \sigma\frac{E}{I} & k_3 & 0 \\ 0 & 0 & \beta I\frac{S}{E} & k_4 \end{pmatrix}.$$

Thus,

$$B := A_f A^{-1} + A\frac{df}{dx}^{[3]}A^{-1} + \mu I = \begin{pmatrix} k_5 & 0 & \psi\beta I\frac{R}{E} & 0 \\ (\gamma-\psi\beta R)\frac{I}{R} & k_6 & (\beta S+\psi\beta R)\frac{I}{E} & \beta I \\ 0 & \sigma\frac{E}{I} & k_7 & 0 \\ 0 & 0 & \beta I\frac{S}{E} & k_8 \end{pmatrix},$$

where

$$k_5 = \frac{\dot R}{R} - \beta I - \sigma - \gamma - 2\mu, \quad k_6 = \frac{\dot I}{I} - (1+\psi)\beta I - \sigma - 2\mu,$$
$$k_7 = \frac{\dot E}{E} - (1+\psi)\beta I - \gamma - 2\mu, \quad k_8 = \frac{\dot S}{S} - \sigma - \gamma - \psi\beta I - 2\mu.$$

Thanks to Lemma 2.7.12, we may assume directly that $\gamma - \psi\beta R \geqslant 0$ in the sequel. Defining $y(t) := A(x(t))z(t)$, y is determined by the differential equation

$$\dot y = By.$$

Let us prove (2.35) with $Q(x) = A(x)$ and considering the L^∞ norm on the vector space $\mathbb{R}^4$. For a given matrix $M = (M_{i,j})_{1\leqslant i,j\leqslant 4}$ and denoting

$$l_i := M_{i,i} + \sum_{j\neq i}|M_{i,j}|,$$

the Lozinskii measure μ_∞ associated to the L^∞ norm is defined by (see for example [91])

$$\mu_\infty(M) = \max_{1\leqslant i\leqslant 4}\{l_i\}.$$

We provide the following formulas, which are required for the computation of l_i.

$$\begin{aligned} \frac{\dot S}{S} &= \mu S^{-1} - \beta I - \mu, \quad \frac{\dot E}{E} = \beta S\frac{I}{E} + \psi\beta\frac{I}{E}R - (\sigma+\mu), \\ \frac{\dot I}{I} &= \sigma\frac{E}{I} - (\gamma+\mu), \quad \frac{\dot R}{R} = \gamma\frac{I}{R} - \psi\beta I - \mu. \end{aligned} \tag{2.48}$$

Let us compute now the value of l_i for the matrix B.

$$\begin{aligned}
l_1 &= \frac{\dot{R}}{R} - \beta I - \sigma - \gamma - 2\mu + \psi\beta I\frac{R}{E} \\
&= \frac{\dot{E}}{E} + \frac{\dot{R}}{R} - \beta I(1 + \frac{I}{E}) - \gamma - \mu, \\
l_2 &= \frac{\dot{I}}{I} - (1+\psi)\beta I - \sigma - 2\mu + (\gamma - \psi\beta R)\frac{I}{R} + (\beta S + \psi\beta R)\frac{I}{E} + \beta I \\
&= \frac{\dot{E}}{E} + \frac{\dot{I}}{I} + \frac{\dot{R}}{R} - \psi\beta I, \\
l_3 &= \frac{\dot{E}}{E} - (1+\psi)\beta I - \gamma - 2\mu + \sigma\frac{E}{I} \\
&= \frac{\dot{E}}{E} - (1+\psi)\beta I - \mu, \\
l_4 &= \frac{\dot{S}}{S} - \sigma - \gamma - \psi\beta I - 2\mu + \beta I\frac{S}{E} \\
&= \frac{\dot{S}}{S} + \frac{\dot{E}}{E} - \gamma - \psi\beta I(1 + \frac{R}{E}) - \mu.
\end{aligned}$$

Therefore, an easy adaptation of [80, Theorem 2.2] to the norm L^∞ makes it possible to say that it is sufficient, for all $1 \leqslant i \leqslant 4$, to have $\bar{g}_i$ such that, for any trajectory of (2.46) in a given invariant set Σ,

$$\limsup_{t\to+\infty} \frac{1}{t}\int_0^t l_i(s)ds \leqslant \bar{g}_i < 0, \tag{2.49}$$

in order to guarantee (2.40) and for the system (2.46) to satisfy a Bendixson condition in this invariant set. For l_1, l_3, l_4, the property (2.49) is verified for any trajectory in the interior of Γ. For example, one has

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{1}{t}\int_0^t l_1(s)ds &\leqslant \lim_{t\to+\infty} \frac{1}{t}\int_0^t (\frac{\dot{E}}{E} + \frac{\dot{R}}{R} - \gamma - \mu)ds \\
&= \lim_{t\to+\infty} \frac{1}{t}\left(\log(\frac{E(t)}{E(0)}) + \log(\frac{R(t)}{R(0)})\right) - \gamma - \mu = -(\gamma+\mu).
\end{aligned}$$

However, for l_2, the property is not always verified, as we shall see.

$$\begin{aligned}
\frac{1}{t}\int_0^t l_2(s)ds &= \frac{1}{t}[\log(\frac{E(t)}{E(0)}) + \log(\frac{I(t)}{I(0)}) + \log(\frac{R(t)}{R(0)})] - \psi\beta\frac{1}{t}\int_0^t I(s)ds, \\
\limsup_{t\to+\infty} \frac{1}{t}\int_0^t l_2(s)ds &= -\psi\beta \liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds.
\end{aligned}$$

For $\mathcal{R}_0 > 1$, the persistence result in Theorem 2.7.10 guarantees the existence of $\varepsilon > 0$ such that

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds > \varepsilon,$$

and the property (2.49) is indeed verified for any trajectory in the interior $\mathring{\Gamma}$ of Γ. The uniform

persistence gives also a compact absorbing subset (see e.g. [75] for a definition) $\mathcal{K}$ in $\mathring{\Gamma}$, where there exists $\varepsilon_{\mathcal{K}} > 0$ such that $S, E, I, R \geqslant \varepsilon_{\mathcal{K}}$ and $\|A^{-1}\|$ is uniformly bounded in $\mathcal{K}$. Therefore, the result in [80, Theorem 2.4] adapted to the L^∞ norm allows us to deduce that any trajectory starting in the interior of Γ converges to the unique endemic equilibrium when $\mathcal{R}_0 > 1$.

On the other hand, for $\mathcal{R}_0 < 1$, no persistence result is available, and we need to first examine under which circumstances the solution of (2.46) may fulfill

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds = 0. \tag{2.50}$$

An integration of the equation for $\dot{I}$ in (2.46) gives

$$0 = \limsup_{t\to+\infty} \frac{1}{t}[I(t) - I(0)] = \sigma \limsup_{t\to+\infty} \frac{1}{t}\int_0^t E(s)ds - (\gamma+\mu)\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds.$$

Assume $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds = 0$, the equation above implies $\limsup_{t\to+\infty} \frac{1}{t}\int_0^t E(s)ds = 0$. As $E(t) \geqslant 0$, hence $\lim_{t\to+\infty} \frac{1}{t}\int_0^t E(s)ds$ is then defined and equal to zero. Integrating again the same differential equation for $\dot{I}$, we obtain $\lim_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds = 0$. Furthermore, the differential inequality

$$\dot{R} \leqslant \gamma I - \mu R,$$

gives

$$0 \leqslant \gamma \lim_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds - \mu \lim_{t\to+\infty} \frac{1}{t}\int_0^t R(s)ds.$$

As $\lim_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds = 0$ and $R \geqslant 0$, thus $\lim_{t\to+\infty} \frac{1}{t}\int_0^t R(s)ds = 0$. Finally, thanks to $S + E + I + R = 1$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t S(s)ds = 1$. In this scenario, the solution must visit any neighborhood containing points arbitrarily close to the disease-free equilibrium. Otherwise, there would be $\varepsilon > 0$ such that $S(t) < 1 - \varepsilon$ for $t \geqslant 0$, which contradicts $\lim_{t\to+\infty} \frac{1}{t}\int_0^t S(s)ds = 1$. Hence, for all $\varepsilon > 0$, there must be a $t \geqslant 0$ such that $S(t) \geqslant 1 - \varepsilon$. As $E(t), I(t), R(t) \geqslant 0$ and $S(t)+E(t)+I(t)+R(t) = 1$, then $I(t), E(t), R(t) \leqslant \varepsilon$. As ε can be made arbitrarily small, indeed this is equivalent to saying that the solution visits any neighborhood of the DFE. Moreover, the Jacobian matrix at the disease-free equilibrium

$$\begin{pmatrix} -\mu & 0 & -\beta & 0 \\ 0 & -(\sigma+\mu) & \beta & 0 \\ 0 & \sigma & -(\gamma+\mu) & 0 \\ 0 & 0 & \gamma & -\mu \end{pmatrix}$$

is Hurwitz for $\mathcal{R}_0 < 1$. Indeed, $-\mu$ is an eigenvalue of the matrix of algebraic multiplicity 2 and the remaining eigenvalues are the same as those of the matrix

$$\begin{pmatrix} -(\sigma+\mu) & \beta \\ \sigma & -(\gamma+\mu) \end{pmatrix}.$$

As the 2×2 matrix has a negative trace and moreover its determinant $(\sigma+\mu)(\sigma+\mu)-\beta\sigma$ is positive when $\mathcal{R}_0 < 1$, this yields the local stability of the DFE. Therefore, if $\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s)ds = 0$, then the trajectory must converge to the disease-free equilibrium, as it is locally asymptotically stable for $\mathcal{R}_0 < 1$. Inversely, if the solution converges to the DFE, naturally (2.50) is verified. Hence, the solutions verifying (2.50) are those starting in Γ and in the basin of attraction of the

DFE, which we refer to as $\Lambda \subset \mathbb{R}^4$.

For $x_0 \in \mathring{\Gamma}$, we denote now $x(t, x_0) = (S(t, x_0), E(t, x_0), I(t, x_0), R(t, x_0))$ the solution of (2.46) with initial condition x_0. Notice then that

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t I(s,x)ds = \sup_{t\geqslant 0}\inf_{s\geqslant t}\frac{1}{s}\int_0^s I(\tau,x)d\tau,$$

is lower semi-continuous with regard to the variable x. Moreover, $\Gamma \setminus \Lambda$ is closed and compact (Λ is open as the basin of attraction of the locally asymptotically stable DFE [24]). Therefore, there exists $\varepsilon > 0$ such that

$$\inf_{x_0\in\Gamma\setminus\Lambda}\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(s,x_0)ds = \min_{x_0\in\Gamma\setminus\Lambda}\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(s,x_0)ds > \varepsilon.$$

Otherwise there exists $x_0 \in \Gamma \setminus \Lambda$ such that $\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(s,x_0)ds = 0$ and the solution $x(t, x_0)$ converges to the DFE, which contradicts the definition of $\Gamma \setminus \Lambda$.

Hence, the property (2.49) is satisfied in $\Gamma \setminus \Lambda$. However, it is not guaranteed that $\|A^{-1}\|$ is bounded in $\Gamma \setminus \Lambda$, and we have to show that there is an absorbing subset Σ of $\Gamma \setminus \Lambda$ where the inverse is defined and bounded.

Thanks to Lemma 2.7.12, an ultimate lower bound

$$\liminf_{t\to+\infty} S(t) \geqslant \varepsilon_S := \frac{\mu}{\beta+\mu} > 0$$

is already derived for any trajectory in Γ. It remains to derive similar bounds for E, I and R. Notice first that $\partial\Gamma_1 := \{x \in \Gamma | E + I = 0\}$ is a compact subset of Λ. As Λ is open, for each point of $x_\Lambda \in \partial\Gamma_1$, there exists ε_{x_Λ} such that $\mathcal{C}_{x_\Lambda} := \{x \in \mathbb{R}^4 | \|x - x_\Lambda\| < \varepsilon_{x_\Lambda}\} \subset \Lambda$. Thus, $\{\mathcal{C}_x\}_{x\in\partial\Gamma_1}$ constitutes a cover of $\partial\Gamma_1$ by open subsets of Λ. From the compactness of $\partial\Gamma_1$, we extract a finite subcover from $\{\mathcal{C}_x\}_{x\in\partial\Gamma_1}$, and this allows to deduce the existence of $\varepsilon_{E+I} > 0$ such that

$$\min_{(S,E,I,R)\in\Gamma\setminus\Lambda} E + I \geqslant \varepsilon_{E+I}.$$

This lower bound and equations (2.46) imply the differential inequality in the invariant set $\Gamma \setminus \Lambda$:

$$\dot{I} \geqslant \sigma\varepsilon_{E+I} - (\gamma + \sigma + \mu)I.$$

Hence, any trajectory in $\Gamma \setminus \Lambda$ fulfills ultimately

$$\liminf_{t\to+\infty} I(t) \geqslant \varepsilon_I := \frac{\sigma\varepsilon_{E+I}}{\gamma+\sigma+\mu} > 0.$$

Again from (2.46), this implies that ultimately, the differential inequalities below are verified for any solution in $\Gamma \setminus \Lambda$.

$$\liminf_{t\to+\infty}\dot{E}(t) \geqslant \beta\varepsilon_S\varepsilon_I - (\sigma+\mu)E, \quad \liminf_{t\to+\infty}\dot{R}(t) \geqslant \gamma\varepsilon_I - \mu R.$$

Thus, there exists also $\varepsilon_E, \varepsilon_R > 0$ such that, for any trajectory in $\Gamma \setminus \Lambda$, the inequalities

$$\liminf_{t\to+\infty} E(t) \geqslant \varepsilon_E, \quad \liminf_{t\to+\infty} R(t) \geqslant \varepsilon_R,$$

are verified. The compact $\Gamma \setminus \Lambda \cap \{(S, E, I, R) \in \Gamma | S \geqslant \varepsilon_S, E \geqslant \varepsilon_E, I \geqslant \varepsilon_I, R \geqslant \varepsilon_R\}$ is then an

absorbing subset of $\Gamma \setminus \Lambda$ where $\|A^{-1}\|$ is uniformly bounded and the Bendixson criterion (2.49) is satisfied. An application of [76, Theorem 6.1] concludes that any trajectory in the invariant set $\Gamma \setminus \Lambda$ converges to an equilibrium. This completes the proof of Theorem 2.7.11. □

2.7.4 Commentary on the general system

Unfortunately, the implementation of Li-Muldowney theory in the general case is more intricate, and no convergence result has yet been discovered. In order to show the complication, let us compute the 3-rd additive compound matrix of $\frac{df}{dx}$ for the general SIS system. Indeed, the system (2.27) has the following Jacobian matrix

$$\frac{\partial f}{\partial x}(x) = \begin{pmatrix} -(\beta_{1,1}I_1 + \beta_{1,r}I_r + \mu) & -\beta_{1,1}S_1 & 0 & -\beta_{1,r}S_1 \\ \beta_{1,1}I_1 + \beta_{1,r}I_r & \beta_{1,1}S_1 - \gamma_1 - \mu & 0 & \beta_{1,r}S_1 \\ 0 & \gamma_1 - \beta_{r,1}S_r & -(\beta_{r,1}I_1 + \beta_{r,r}I_r + \mu) & \gamma_r - \beta_{r,r}S_r \\ 0 & \beta_{r,1}S_r & \beta_{r,1}I_1 + \beta_{r,r}I_r & \beta_{r,r}S_r - \gamma_r \end{pmatrix},$$

therefore its 3-additive compound matrix is

$$\frac{\partial f^{[3]}}{\partial x}(x) = \begin{pmatrix} k_1 & \gamma_r - \beta_{r,r}S_r & -\beta_{1,r}S_1 & -\beta_{1,r}S_1 \\ \beta_{r,1}I_1 + \beta_{r,r}I_r & k_2 & 0 & 0 \\ -\beta_{r,1}S_r & \gamma_1 - \beta_{r,1}S_r & k_3 & -\beta_{1,1}S_1 \\ 0 & 0 & \beta_{1,1}I_1 + \beta_{1,r}I_r & k_4 \end{pmatrix},$$

where k_1, k_2, k_3, k_4 are defined by

$$\begin{aligned} k_1 &= \beta_{1,1}S_1 - (\beta_{1,1} + \beta_{r,1})I_1 - (\beta_{1,r} + \beta_{r,r})I_r - \gamma_1 - 3\mu, \\ k_2 &= \beta_{1,1}S_1 + \beta_{r,r}S_r - \beta_{1,1}I_1 - \beta_{1,r}I_r - \gamma_1 - \gamma_r - 3\mu, \\ k_3 &= \beta_{r,r}S_r - (\beta_{1,1} + \beta_{r,1})I_1 - (\beta_{1,r} + \beta_{r,r})I_r - \gamma_r - 3\mu, \\ k_4 &= \beta_{1,1}S_1 + \beta_{r,r}S_r - \beta_{r,1}I_1 - \beta_{r,r}I_r - \gamma_1 - \gamma_r - 3\mu. \end{aligned}$$

To simplify the problem, we introduce a matrix-valued function $A(x)$ so that the matrix

$$A_f A^{-1} + A\frac{df}{dx}^{[3]} A^{-1} + \mu\mathbf{I},$$

is in the simplest possible form.

Defining first the transformation matrix

$$P_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$B := P_1 \frac{\partial f^{[3]}}{\partial x}(x) P_1^{-1} = \begin{pmatrix} k_1 & \gamma_r - \beta_{r,r}S_r & -\beta_{1,r}S_1 & 0 \\ \beta_{r,1}I_1 + \beta_{r,r}I_r & k_2 & 0 & 0 \\ -\beta_{r,1}S_r & \gamma_1 - \beta_{r,1}S_r & k_5 & -\gamma_1 \\ 0 & 0 & \beta_{1,1}I_1 + \beta_{1,r}I_r & k_6 \end{pmatrix},$$

where

$$k_5 = \beta_{r,r}S_r - \beta_{r,1}I_1 - \beta_{r,r}I_r - \gamma_r - 3\mu,$$
$$k_6 = \beta_{1,1}S_1 + \beta_{r,r}S_r - (\beta_{1,1} + \beta_{r,1})I_1 - (\beta_{1,r} + \beta_{r,r})I_r - \gamma_1 - \gamma_r - 3\mu.$$

In addition, we introduce

$$P_2 := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$C := P_2 B P_2^{-1} = \begin{pmatrix} k_7 & 0 & -\beta_{1,r}S_1 & 0 \\ \beta_{r,1}I_1 + \beta_{r,r}I_r & k_8 & 0 & 0 \\ -\beta_{r,1}S_r & \gamma_1 & k_5 & -\gamma_1 \\ 0 & 0 & \beta_{1,1}I_1 + \beta_{1,r}I_r & k_6 \end{pmatrix},$$

where

$$k_7 = \beta_{1,1}S_1 - \beta_{1,1}I_1 - \beta_{1,r}I_r - \gamma_1 - 3\mu,$$
$$k_8 = \beta_{1,1}S_1 + \beta_{r,r}S_r - (\beta_{1,1} + \beta_{r,1})I_1 - (\beta_{1,r} + \beta_{r,r})I_r - \gamma_1 - \gamma_r - 3\mu.$$

Furthermore, define

$$P_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$D := P_3 C P_3^{-1} = \begin{pmatrix} k_7 & -\beta_{1,r}S_1 & 0 & 0 \\ -\beta_{r,1}S_r & k_5 & \gamma_1 & -\gamma_1 \\ \beta_{r,1}I_1 + \beta_{r,r}I_r & 0 & k_8 & 0 \\ 0 & \beta_{1,1}I_1 + \beta_{1,r}I_r & 0 & k_8 \end{pmatrix}.$$

Finally, with

$$P_4 = \begin{pmatrix} \frac{I_r}{S_r} & 0 & 0 & 0 \\ 0 & \frac{I_1}{S_r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we derive the matrix

$$E = P_4 D P_4^{-1} = \begin{pmatrix} k_7 & -\beta_{1,r}S_1\frac{I_r}{I_1} & 0 & 0 \\ -\beta_{r,1}S_r\frac{I_1}{I_r} & k_5 & \gamma_1\frac{I_1}{S_r} & -\gamma_1\frac{I_1}{S_r} \\ (\beta_{r,1}I_1 + \beta_{r,r}I_r)\frac{S_r}{I_r} & 0 & k_8 & 0 \\ 0 & (\beta_{1,1}I_1 + \beta_{1,r}I_r)\frac{S_r}{I_1} & 0 & k_8 \end{pmatrix}.$$

The last transformation is intended to obtain non-diagonal coefficients which can be cancelled as far as possible by diagonal's ones, which is related to the computation of the Lozinskii measure below. Define finally the matrix A as $A := P_4 P_3 P_2 P_1$, it is easy to verify $A_f A^{-1} = \mathrm{diag}(\frac{\dot{I}_r}{I_r} -$

$\frac{\dot{S}_r}{S_r}, \frac{\dot{I}_1}{I_1} - \frac{\dot{S}_r}{S_r}, 0, 0)$. We derived finally the matrix

$$F := A_f A^{-1} + E + \mu \mathbf{I} = \begin{pmatrix} k_8 & -\beta_{1,r} S_1 \frac{I_r}{I_1} & 0 & 0 \\ -\beta_{r,1} S_r \frac{I_1}{I_r} & k_9 & \gamma_1 \frac{I_1}{S_r} & -\gamma_1 \frac{I_1}{S_r} \\ (\beta_{r,1} I_1 + \beta_{r,r} I_r) \frac{S_r}{I_r} & 0 & k_{10} & 0 \\ 0 & (\beta_{1,1} I_1 + \beta_{1,r} I_r) \frac{S_r}{I_1} & 0 & k_{10} \end{pmatrix}, \quad (2.51)$$

where the diagonal entries are

$$\begin{aligned} k_8 &= \frac{\dot{I}_r}{I_r} - \frac{\dot{S}_r}{S_r} + \beta_{1,1} S_1 - \beta_{1,1} I_1 - \beta_{1,r} I_r - \gamma_1 - 2\mu, \\ k_9 &= \frac{\dot{I}_1}{I_1} - \frac{\dot{S}_r}{S_r} + \beta_{r,r} S_r - \beta_{r,1} I_1 - \beta_{r,r} I_r - \gamma_r - 2\mu, \\ k_{10} &= \beta_{1,1} S_1 + \beta_{r,r} S_r - (\beta_{1,1} + \beta_{r,1}) I_1 - (\beta_{1,r} + \beta_{r,r}) I_r - \gamma_1 - \gamma_r - 2\mu, \end{aligned}$$

which can be alternatively expressed by

$$\begin{aligned} k_8 &= \frac{\dot{I}_r}{I_r} - \frac{\dot{S}_r}{S_r} + \frac{\dot{I}_1}{I_1} - \beta_{1,r} \frac{I_r}{I_1} S_1 - \beta_{1,1} I_1 - \beta_{1,r} I_r - \mu, \\ k_9 &= \frac{\dot{I}_1}{I_1} - \frac{\dot{S}_r}{S_r} + \frac{\dot{I}_r}{I_r} - \beta_{r,1} \frac{I_1}{I_r} S_r - \beta_{r,1} I_1 - \beta_{r,r} I_r - \mu, \\ k_{10} &= \frac{\dot{I}_1}{I_1} + \frac{\dot{I}_r}{I_r} - \beta_{1,r} \frac{I_r}{I_1} S_1 - \beta_{r,1} \frac{I_1}{I_r} S_r - (\beta_{1,1} + \beta_{r,1}) I_1 - (\beta_{1,r} + \beta_{r,r}) I_r. \end{aligned}$$

To obtain 2-contraction, let us define a vector norm $\|\cdot\|$ such that the associated Lozinskii measure $\mu(A_f A^{-1} + E + \mu \mathbf{I})$ fulfills

$$\limsup_{t \to +\infty} \lim_{x_0 \in \Sigma} \frac{1}{t} \int_0^t \mu(A_f A^{-1}(s) + E(s) + \mu \mathbf{I}) ds < 0. \quad (2.52)$$

Usually, the vector norms employed are the L^1 and L^∞ norms, or a hybrid combination of both types (for example the norm $\|(x_1, x_2, x_3, x_4)\| := \max\{|x_1| + |x_4|, |x_2|, |x_3|\}$ is employed in [80]). This is because the computation of Lozinskii measure associated with these norms is rather easy, and can be typically expressed as a sum of diagonal entries and the absolute value of non-diagonal entries. However, this method does not yield positive results for the present problem, as the matrix defined in (2.51) presents non-diagonal terms that cannot be compensated by the diagonal's ones, which is required in order to ensure (2.52). To see this, let us denote $F = (F_{i,j})_{1 \leqslant i,j \leqslant 4}$ and take for example the norm $\|(x_1, x_2, x_3, x_4)\| := \max\{|x_1|, |x_2|, |x_3| + |x_4|\}$. As described in [84] and similarly to [80], the associated Lozinskii measure of F can be estimated by $\mu(F) \leqslant \max\{l_1, l_2, l_3\}$, where

$$\begin{aligned} l_1 &:= F_{1,1} + |F_{1,2}| + \max\{|F_{1,3}|, |F_{1,4}|\}, \\ l_2 &:= F_{2,2} + |F_{2,1}| + \max\{|F_{2,3}|, |F_{2,4}|\}, \\ l_3 &:= \max\{F_{3,3} + |F_{4,3}|, F_{4,4} + |F_{3,4}|\} + |F_{3,1}| + |F_{3,2}| + |F_{4,1}| + |F_{4,2}|. \end{aligned}$$

The issue in the present attempt is the quantity l_3, which is in fact equal to

$$l_3 = \frac{\dot{I}_1}{I_1} + \frac{\dot{I}_r}{I_r} - \beta_{1,r}\frac{I_r}{I_1}S_1 - (\beta_{1,1} + \beta_{r,1})I_1 - (\beta_{1,r} + \beta_{r,r})I_r + \beta_{r,r}S_r + (\beta_{1,1}I_1 + \beta_{1,r}I_r)\frac{S_r}{I_1},$$

but the remaining positive terms above cannot be further compensated in order to have (2.52). Several similar attempts have been performed, leading to the same situation. Moreover, with the attempt exposed previously, it can be seen that the failure of the application of Li and Muldowney method comes from the complex structure of the recruitment rates $(\beta_{1,1}I_1+\beta_{1,r}I_r)S_1$ and $(\beta_{r,1}I_1 + \beta_{r,r}I_r)S_r$, caused by the heterogeneities in the model, while the method were valid for the simpler contact rate βSI in (2.42).

We also tried to prove numerically 2-contraction and 1-dominance for the general system (2.27) by solving, after a convex relaxation, the LMIs conditions included in Theorem 2.7.3 and Definition 2.7.1. We employed the semidefinite programming solvers SeDuMi and SDPT3 in MATLAB, which failed to deliver a solution to the problems. This suggests in particular that we may not expect a state-independent quadratic Lyapunov function $V(z) := zQz$ to prove the 2-contraction of the general system. On the other hand, the problem of finding state-dependent quadratic storage function for k-contraction and p-dominance is difficult and constitutes a domain of active research [47, 120].

Chapter 3

Joint age and reinfection structured model and associated mean values

3.1 Introduction

Following the infinite system of ODE tracking the number of reinfection in Chapter 1, we discuss in this chapter a class of epidemiological models numbering the reinfection with additional age dependency. In fact, it is often noticed that the characteristics of the actual diseases, for example the severity or the rate of transmission, are highly dependent on the host population structure [28, 81, 106]. Hence, a single disease may exhibit significant variations in the resulting epidemic dynamics owing to the disparities in the age distribution of the population, which vary, for example, from country to country. It is therefore often necessary to include age dependencies in the epidemiological models in order to improve decision-making [18] or formulate recommendations for targeted public health policies such as vaccination or quarantine [52, 114].

The age dependency is considered mathematically by so-called *age-structured* models. For epidemiological models, this can be implemented with the ODE-based compartmental models with discrete age groups [18, 119] or with a system of PDE assuming the age as a coordinate in a continuous domain (for examples of age-structured epidemiological models, one can refer to [60]). In the present chapter, we are moreover interested in the *joint structure of age and reinfection*. Particularly, we noticed that the reinfection-structured microscopic ODE model presented in Chapter 1 also admit a "hidden" structure, this time according to the age, in the same way as the former is itself an "unfolding" according to the number of reinfections of macroscopic models. Hence, we intend to investigate the underlying age structuration of reinfection models of Chapter 1.

To this end, an age-structured SEIRS system with an infinite number of partial differential equations taking also into account the number of reinfections is introduced and its well-posedness deduced. The general framework within which this system will be handled is that of abstract differential equation and semigroup theory [96]. The chapter is organized as follows. Section 3.2 introduces the age and reinfection structured model as an infinite system of evolution equations. In Section 3.3 well-posedness results, elaborated by Desch et al.[31] and adapted by Pugliese and Tonetto [102], on a class of abstract Cauchy problems are recalled as a preliminary for the well-posedness analysis of the model. These results are then employed in Section 3.4 where the

age and reinfection structured model is adapted into an abstract Cauchy problem setting and proved to be well-posed. Finally, in Section 3.5 the system, in its simplest setting with non-varying parameters with regard to the age and reinfection number, is studied in detail. Similarly to the infinite system studied in Section 1.4 of Chapter 1, the system in Section 3.5 can also be considered can also be considered as revealing an underlying structure of the same SEIRS model (1.1), now with regard to both the age and the reinfection number, with the additional assumptions that the disease is non-lethal, and the total population is non-varying. We call this model, which possesses two "microscopic dimensions", a *nanoscopic model*. The asymptotic stability of the disease-free and endemic steady states are studied, and the age-structure allows the computation of supplementary interesting quantities on the endemic state of the SEIRS model, namely the mean numbers of reinfection at each age and the average age values in each compartment.

3.2 An infinite age-structured SEIRS reinfection model

As announced, we investigate in this chapter the joint age and reinfection structure in the context of SEIRS models. Introducing the variables $b_j(t,a)$, $b \in \{s,e,\imath,r\}$, $j \geqslant 1$, which denote the age density of hosts with health status b (which can be either s, e, $\imath$, r, respectively for 'Susceptible', 'Exposed', 'Infected', 'Recovered') that have been infected exactly j times ($j-1$ times in the case of variables s_j, $j \geqslant 1$). The variables take two arguments (a,t), which are the age of the hosts a and time t. With these definitions, the dynamics of b_j can be captured by the following model consisting of evolution equations, which is similar to the ODE model (1.5) albeit with an additional age structure.

$$\begin{aligned}
&\frac{\partial s_1}{\partial t} + \frac{\partial s_1}{\partial a} = -\lambda_1(t,a)s_1 - \mu(a)s_1, \\
&\frac{\partial s_i}{\partial t} + \frac{\partial s_i}{\partial a} = \omega_{i-1}(a)r_{i-1} - \lambda_i(t,a)s_i - \mu(a)s_i, \quad i > 1, \\
&\frac{\partial e_i}{\partial t} + \frac{\partial e_i}{\partial a} = \lambda_i(t,a)s_i - (\sigma_i(a) + \mu(a))e_i, \qquad i \geqslant 1, \\
&\frac{\partial \imath_i}{\partial t} + \frac{\partial \imath_i}{\partial a} = \sigma_i(a)e_i - (\gamma_i(a) + \mu(a))\imath_i, \qquad i \geqslant 1, \\
&\frac{\partial r_i}{\partial t} + \frac{\partial r_i}{\partial a} = \gamma_i(a)I_i - (\omega_i(a) + \mu(a))r_i, \qquad i \geqslant 1,
\end{aligned} \tag{3.1}$$

where the forces of infection λ_i defined by

$$\lambda_i(t,a) = \frac{\sum_j \int_0^{+\infty} \beta_{i,j}(a,a')\imath_j(t,a')da'}{N(t)}, \tag{3.2}$$

represent the rate at which susceptible hosts, of age a that have already been infected $n-1$ times, acquire the infectious disease at time t, which follows the *true mass-action law*. The total population of the system, denoted $N(t)$, is defined by :

$$N(t) := \sum_i \int_0^{+\infty} \Big[s_i(t,a) + e_i(t,a) + \imath_i(t,a) + r_i(t,a) \Big] da. \tag{3.3}$$

In the equations (3.1), the left-hand side includes components of transport equations modelling the aging process. On the right-hand side, the nonlinear terms $\lambda_i S_i$ represent the recruitment

of the diseased. Moreover, $\mu(a)$ is the age-dependent natural death rate, $\omega_i(a)$ is the loss rate of immunity, $\sigma_i(a)^{-1}$ the age-dependent incubation period and finally $\gamma_i(a)$ the recovery rate.

In addition, denoting $n(t,a)$ the proportion of the population of age a at time t :

$$n(t,a) := \sum_i s_i(t,a) + e_i(t,a) + \imath_i(t,a) + r_i(t,a),$$

the system is endowed with the boundary conditions

$$\begin{aligned} s_1(\cdot,0) &= \int_0^{+\infty} b(a)n(\cdot,a)da, \\ s_{i+1}(\cdot,0) = e_i(\cdot,0) = \imath_i(\cdot,0) &= r_i(\cdot,0) = 0, \quad i \geqslant 1, \end{aligned} \tag{3.4}$$

reflecting the births generated by the population. On the other hand, we also define the initial conditions at $t = 0$

$$s_i(0,\cdot) = s_i^0(\cdot), \quad e_i(0,\cdot) = e_i^0(\cdot), \quad \imath_i(0,\cdot) = \imath_i^0(\cdot), \quad r_i(0,\cdot) = r_i^0(\cdot), \quad i \geqslant 1, \tag{3.5}$$

for given functions $s_i^0, e_i^0, \imath_i^0, r_i^0 \in L^1_+(0,+\infty)$. For simplicity, we denote $p^0 := (s_i^0, e_i^0, \imath_i^0, r_i^0)_{i\geqslant 1}$. To ensure well-posedness, the coefficients of (3.1) are considered with the following assumptions, which are standard:

- (H1) μ measurable and there exist real numbers $\mu_-, \mu_+ > 0$ such that $0 < \mu_- \leqslant \mu(a) \leqslant \mu_+$ for a.e. $a \in [0,+\infty)$.
- (H2) $b, \omega_i, \gamma_i, \sigma_i \in L^\infty_+[0,+\infty)$ and there exists $k > 0$ such that $0 \leqslant \omega_i(a), \gamma_i(a), \sigma(a) \leqslant k$ for a.e. $a \in [0,+\infty)$ and any $i \geqslant 1$.
- (H3) For all $i, j \geqslant 1$, $\beta_{i,j}(a,\cdot) \in L^\infty_+$ and there exists $\beta_+ > 0$ such that $0 \leqslant \beta_{i,j}(a,a') \leqslant \beta_+$ for all $i, j \geqslant 1$, and a.e. $a, a' \in [0,+\infty)$.

For the analysis, we will consider the state variables of (3.1) as a sequence of elements of the form $(s_i, e_i, \imath_i, r_i), i \geqslant 1$ that can be rewritten more conveniently as $p_i := (p_{i,1}, p_{i,2}, p_{i,3}, p_{i,4}), i \geqslant 1$. In the sequel, we will use both notations. Each element belongs to a four components product of L^1 Lebesgue space $(L^1(0,+\infty))^4$ endowed with the norm

$$\|p_i\| := \sum_{j=1}^4 \int_0^{+\infty} |p_{i,j}(a)|da.$$

Therefore it is natural to consider the solution $p := (p_i)_{i\in\mathbb{N}}$ of the system (3.1) as evolving in the Banach space X befined below.

$$X := \left\{ p = (p_i)_{i\in\mathbb{N}} : \forall i \geqslant 1, p_i \in (L^1(0,+\infty))^4 \text{ and } \|p\|_X < \infty \right\}, \tag{3.6}$$

where $\|p\|_X := \sum_{i\in\mathbb{N}} \|p_i\|$. In addition, a partial ordering will also be required for the definition of the positive solution to the system. Consider then the natural element-wise partial order relation $\geqslant$ on $\mathbb{R}^4$,

$$\forall x^1, x^2 \in \mathbb{R}^4, \quad x^1 \geqslant x^2 \quad \text{if and only if} \quad x^1_j \geqslant x^2_j, \quad 1 \leqslant j \leqslant 4.$$

The cartesian product induces an ordering on $(L^1(0,+\infty))^4$

$$\forall f_1, f_2 \in (L^1(0,+\infty))^4, \quad f_1 \geqslant f_2 \quad \text{if and only if} \quad f_1(a) \geqslant f_2(a), \quad \text{for almost every} \quad a > 0,$$

which itself induces an order relationship on X :

$$\forall f = (f_i)_{i\in\mathbb{N}} \in X, \quad \forall g = (g_i)_{i\in\mathbb{N}} \in X, \quad f \geqslant g \quad \text{if and only if} \quad f_i \geqslant g_i, \quad i \geqslant 1.$$

Finally, we define the positive cone X_+ of X associated to the order relation defined previously

$$X_+ := \left\{ p = (p_i)_{i\in\mathbb{N}} \in X : p \geqslant 0_X \right\}. \tag{3.7}$$

Note that we chose to consider the solution as a sequence of 4 components Cartesian product of Lebesgue integrable elements because it is more meaningful for the SEIRS model, but clearly an equivalent way to consider the solution is to see it as a sequence of simple Lebesgue integrable functions which is the setting of [102], hence the same techniques and results can be easily transposed to the space X. A solution $p : [0, t_{\max}) \to X$ of (3.1) is said to be *classical* if $p \in C^1([0, t_{\max}), X)$ and fulfills (3.1) and the boundary conditions (3.4)-(3.5). Moreover, it is positive if $p(t) \in X_+$ for all $t \in [0, t_{\max})$.

With the preceding definitions, it is possible to demonstrate that the age profile of the total population $n(t, a)$ follows the McKendrick-Von Foerster equation [60] and is independently defined.

Theorem 3.2.1. *(Total population dynamics) Let p be a classical positive solution of* (3.1)*. Then the age density $n : [0, t_{\max}) \to L^1_+(0,+\infty)$ defined in* (3.2) *is the unique classical solution of the McKendrick-Von Foerster equation, i.e., $n \in C^1([0, t_{\max}), L^1_+(0,+\infty))$, $n(t,\cdot) \in W^{1,1}(0,+\infty)$ for $0 \leqslant t < t_{\max}$ and n fulfills:*

$$\begin{aligned} \frac{\partial n(t,a)}{\partial t} + \frac{\partial n(t,a)}{\partial a} &= -\mu(a) n(t,a), \quad t > 0, \quad a > 0, \\ n(t,0) &= \int_0^{+\infty} b(a) n(t,a) da, \quad t > 0, \\ n(0,a) = n^0(a) &= \sum_{i\geqslant 1} \sum_{1\leqslant j\leqslant 4} (p^0)_{i,j}(a), \quad a > 0. \end{aligned} \tag{3.8}$$

Proof of Theorem 3.2.1. Let $0 < T < t_{\max}$ and consider the classical solution p in the interval $[0, T]$. Thanks to the definition of X and the fact that $p \in C^1([0, T], X)$, there exist uniform bounds $k_1, k_2 > 0$ such that

$$\|p(t)\|_X \leqslant k_1, \quad \|\frac{d}{dt} p(t)\|_X \leqslant k_2 \quad \text{for } 0 \leqslant t \leqslant T. \tag{3.9}$$

Let us denote

$$n_l(t,a) := \sum_{1\leqslant i\leqslant l} \sum_{1\leqslant j\leqslant 4} p_{i,j}(t,a).$$

For any $0 \leqslant t \leqslant T$ and for almost every $a > 0$, clearly $|n_l(t,a)| \leqslant k_1$ and $n_l(t,a)$ converges pointwise a.e. to $n(t,a) = \sum_{i\geqslant 1} n_i(t,a)$. Moreover, due to the uniform boundedness in $t \in [0, T]$

of $\|p(t)\|_X$, $n(t,\cdot) \in L^1_+(0,+\infty)$ and it is easy to see that

$$n_i(t,\cdot) \xrightarrow{L^1} n(t,\cdot) \quad \text{and} \quad p_i(t,\cdot) \xrightarrow{(L^1)^4} 0 \quad \text{uniformly in } t \in [0,T].$$

Furthermore the uniform boundedness of $\|\frac{d}{dt}p(t)\|_X$ in (3.9) also implies that $\frac{\partial}{\partial t}n_i(t,\cdot)$ converges uniformly with regard to t and

$$\lim_{i\to+\infty} \frac{\partial}{\partial t} n_i(t,\cdot) \in L^1(0,+\infty).$$

Hence for almost every $a > 0$,

$$\lim_{i\to+\infty} \frac{\partial}{\partial t} n_i(t,a) = \frac{\partial}{\partial t} n(t,a), \quad \text{for a.e.} \quad 0 \leqslant t \leqslant T, \quad a > 0.$$

Thanks to the uniform convergences above and $n_i \in C^1([0,T], L^1_+(0,+\infty))$,

$$n \in C^1([0,T], L^1_+(0,+\infty)).$$

Summing now the equations (3.1), we obtain

$$\frac{\partial}{\partial t} n_i(t,a) = -\frac{\partial}{\partial a} n_i(t,a) - \mu(a) n_i(t,a) - \omega_i(a) p_{i,4}(t,a), \quad \text{for a.e.} \quad 0 \leqslant t \leqslant T, \quad a > 0.$$

Taking $i \to +\infty$, from the convergence results above yields

$$\frac{\partial}{\partial t} n(t,a) = \lim_{i\to+\infty} \frac{\partial}{\partial t} n_i(t,a) = -\lim_{i\to+\infty} \frac{\partial}{\partial a} n_i(t,a) - \mu(a) n(t,a), \quad \text{for a.e.} \quad 0 \leqslant t \leqslant T, \quad a > 0.$$

This also implies that $\lim_{i\to+\infty} \frac{\partial}{\partial a} n_i(t,\cdot) \in L^1(0,+\infty)$, hence $\frac{\partial}{\partial a} n(t,\cdot) = \lim_{i\to+\infty} \frac{\partial}{\partial a} n_i(t,\cdot)$ almost everywhere and $n(t,\cdot) \in W^{1,1}(0,+\infty)$ for every $0 \leqslant t \leqslant T$. Finally,

$$\frac{\partial}{\partial t} n(t,a) = -\frac{\partial}{\partial a} n(t,a) - \mu(a) n(t,a), \quad \text{for a.e.} \quad 0 < t < T, \quad a > 0.$$

We conclude the proof by taking T arbitrarily close to $t_{\max}$. □

It is known that the solution of McKendrick-Von Foerster equation converges asymptotically to an exponential distribution. In fact, the number of births $B(t) := n(t,0)$ at the instant t can be deduced as the solution to the Lotka's renewal equation below.

$$B(t) = \int_0^\omega b(a+t) \frac{\pi(a+t)}{\pi(a)} n^0(a) da + \int_0^t b(a)\pi(a) B(t-a) da,$$

where $\pi(a) = e^{-\int_0^a \mu(s) ds}$ is the survival probability at age a of the model and ω is the maximal attainable age for the individuals in the model. The first term of the equation above is interpreted as the contribution to the birth rate of the initial population at time t and the second term represents the birth generated by newborns after the initial time 0. The asymptotic behavior of this renewal equation is well understood and known in the literature as the Sharpe-Lotka Theorem [115] or the Fundamental Theorem of Demography (e.g. [60, Proposition 1.9]). The result states, for finite $\omega > 0$, that the asymptotic behavior is dominated by an exponential term

$$B(t) = q_0 e^{\lambda_0 t}(1 + O(e^{-\eta t})),$$

where

$$q_0 := \frac{\int_0^\omega e^{-\lambda_0 t} G(t)dt}{\int_0^\omega a e^{-\lambda_0 a} b(a)\pi(a)da}, \quad G(t) := \int_0^\omega b(a+t)\frac{\pi(a+t)}{\pi(a)} n^0(a)da,$$

and η is a positive real and λ_0 is the unique real root of Lotka's characteristic equation

$$\int_0^\omega e^{-\lambda_0} b(a)\pi(a)da = 1.$$

Moreover, notice that we can generalize the result [60, Proposition 1.9] with no particular difficulty to the case $\omega = \infty$, which is assumed in the present model. Due to the Fundamental Theorem, the age profile converges asymptotically to the exponential distribution (Proposition 1.11 [60]) :

$$\lim_{t\to+\infty} e^{-\lambda_0 t} n(t,a) = q_0 e^{-\lambda_0 a} e^{-\int_0^a \mu(s)ds}, \quad \text{uniformly in } a.$$

Introducing the net reproductive rate of the system as

$$\mathcal{R} = \int_0^{+\infty} b(a)\pi(a)da.$$

In order to have a constant stable population, λ_0 must be equal to zero hence $\mathcal{R} = 1$. In the sequel, we take the simplifying assumption below that ensures that the total population has already attained a stable constant distribution, with $N(t)$ constant for $t > 0$.

- (H4) $\mathcal{R} = 1$ and the age profile of the population has already reached the stable age distribution, that is $n(0,a) = ke^{-\int_0^a \mu(s)ds}$, $k \in \mathbb{R}_+$.

Normalizing then the quantities $s_i, e_i, \imath_i, r_i$ in (3.1) by $N(t)$, we may consider only the case where $N \equiv 1$ and the system (3.1) is endowed with a linear force of infection λ_i defined by

$$\lambda_i(t,a) := \sum_j \int_0^{+\infty} \beta_{i,j}(a,a')\imath_j(t,a')da', \tag{3.10}$$

instead of (3.2). For a complete picture on the subject of Kermack-Mackendrick equation, the reader is invited to consult [59, 60].

Remark 3.2.2. *For non-constant total population $N(t)$, it is still possible to consider the well-posedness of* (3.1) *by introducing the alternative normalization*

$$\bar{s}_i(t,a) := \frac{S_i(t,a)}{n(t,a)}, \quad \bar{e}_i(t,a) := \frac{e_i(t,a)}{n(t,a)}, \quad \bar{\imath}_i(t,a) := \frac{\imath_i(t,a)}{n(t,a)}, \quad \bar{r}_i(t,a) := \frac{r_i(t,a)}{n(t,a)}.$$

Similarly to the age-structured SIR model in [60], the resulting infinite system is asymptotically autonomous for sufficiently regular solution, and the well-posedness of the limiting autonomous system may be derived by the same semigroup techniques employed below. The

3.3 Well-posedness results for a perturbed linear abstract Cauchy problem

To verify the existence and uniqueness of solutions of (3.1), the system may be considered as an abstract differential equation of the form (see Section 3.4) :

$$p'(t) = A(\mathbf{I} + H(t))p(t) + (F(t))(p(t)),$$

where A is the generator of a $\mathcal{C}_0$-semigroup and H, F are nonlinear operators. In particular, the multiplicative perturbation $A(\mathbf{I}+H)$ of the linear operator A in the abstract equation takes into account the non-trivial boundary conditions (3.4). We intend then to adapt the perturbed semigroup theory results available in the literature [31, 102] to the proof of well-posedness and positiveness of the present system.

To this end, this section serves as an aside to introduce some definitions and results whose proof can be found in the aforementioned papers. Let $(X, \|\cdot\|)$ be a Banach space with its associated norm and $A : D(A) \to X$ be a linear operator with domain $D(A) \subset X$ generating a strongly continuous $\mathcal{C}_0$-semigroup e^{tA} such that, for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, the following norm inequality is fulfilled

$$\|e^{tA}\| \leqslant Me^{\omega t}, \quad t \geqslant 0.$$

The Favard class of A defined below is required for the analysis.

$$F_A = \left\{ p \in X : \limsup_{t \to 0^+} \frac{1}{t}\|e^{tA}p - p\| < +\infty \right\}.$$

It is easy to see that F_A is a Banach space endowed with the following norm

$$|p|_{F_A} := \|p\| + \limsup_{t \to 0^+} \frac{1}{t}\|e^{tA}p - p\|.$$

The domain $D(A)$ of an operator A generating a $\mathcal{C}_0$-semigroup e^{tA} is the set of all elements $x \in X$ such that the limit below exists :

$$\lim_{t \to +\infty} \frac{1}{t}(e^{tA} - \mathbf{I})x.$$

It is clear from the definition that $D(A) \subset F_A$. Anecdotally, we have $D(A) = F_A$ if X is reflexive (see for instance [38]). On the other hand, let $H : X \to F_A$ and $F : X \to X$ be locally Lipschitz continuous operators, i.e., for all $R > 0$, there exist L_R, $K_R > 0$ such that

$$|H(p) - H(q)|_{F_A} \leqslant L_R\|p - q\|, \quad \|F(p) - F(q)\| \leqslant K_R\|p - q\|, \tag{3.11}$$

for all $p, q \in X$ such that $\|p\|, \|q\| \leqslant R$. Finally, we introduce, for $R > 0$, the projection $\pi_R : X \to X$

$$\pi_R(x) = \begin{cases} x & \text{if } \|x\| \leqslant R, \\ \frac{x}{\|x\|}R & \text{if } \|x\| > R, \end{cases}$$

and the composite functions

$$H_R(x) := H(\pi_R(x)) \text{ and } F_R(x) := F(\pi_R(x)). \tag{3.12}$$

In this setting, the abstract Cauchy problem is proved to be well-posed in [102] which is an adaptation of the results in [31].

Theorem 3.3.1. *(Theorem 2.1 in [102]) Let $A : D(A) \to X$ be a linear operator with $D(A) \subset X$ which generates a $\mathcal{C}_0$-semigroup e^{tA}, $H : X \to F_A$, $F : X \to X$ satisfy* (3.11) *and $p^0 \in X$. Consider the abstract Cauchy problem*

$$\begin{cases} p'(t) = A(\mathbf{I} + H)p(t) + F(p(t)), \\ p(0) = p^0, \end{cases} \tag{3.13}$$

Then

1. *For each $p^0 \in X$, there exists a unique* mild solution *of* (3.13)*, i.e., a continuous function $t \to p(t)$ satisfying the integral equation*

$$p(t) = e^{tA}p^0 + A\int_0^t e^{(t-s)A}H(p(s))ds + \int_0^t e^{(t-s)A}F(p(s))ds \tag{3.14}$$

2. *If $[0, t_{max})$ is the maximal interval of existence of the solution, then $t_{max} = +\infty$ or $\lim_{t\to t_{max}^-} \|p(t)\| = +\infty$,*

3. *If H and F are continuously differentiable and $p^0+(H(0))p^0 \in D(A)$, then $p(t)$ is a classical solution of* (3.13)*, i.e., $p(t)+(H(t))p(t) \in D(A)$ for each $t \in [0, t_{max})$, $p(t)$ is differentiable and satisfies* (3.13) *for each $0 \leqslant t < t_{max}$.*

Remark 3.3.2. *Although not stated explicitly in both Theorem 1 [31] and Theorem 2.1 [102], the classical solution $p(t)$ is proved to be continuously differentiable in [31] and also explicitly stated in Theorem 1 [30] which is an earlier report of the work presented in [31].*

For $p^0 \in X$, we denote in the rest of the paper $p(t, p^0)$ the mild solution to the Cauchy problem (3.13). The next result relates to the continuous dependency of the mild solution on initial data.

Theorem 3.3.3. *(Theorem 2.2 in [102]) Let $p^0 \in X$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence in X converging to p^0. Then for each $t > 0$ such that $p(t, p^0)$ exists, we have*

$$\lim_{n\to\infty} p(t, q_n) = p(t, p^0)$$

and the convergence is uniform for $t \in [0, T]$, where $T > 0$ is such that $p(T, p^0)$ exists.

As the model describes an epidemiological process of positive quantities, we are specifically interested in the existence and uniqueness of positive solutions. In the context of Banach spaces, the natural framework for studying positivity is the Banach lattices, which is a Banach space $(X, \|\cdot\|)$ endowed with an order relation $\leqslant$ such that $(X, \leqslant)$ is a lattice. For a given order relation, we define the *positive cone* of X :

$$X_+ = \{p \in X : p \geqslant 0\},$$

and we denote $p \geqslant q$ if and only if $p - q \in X_+$. In the rest of the chapter, we will only use the natural order between $L^1(\Omega)$ functions, which implies

$$X_+ = \{f \in X : f(x) \geqslant 0, \text{a.e. in } \Omega\}.$$

Definition 3.3.1. *A linear operator T on a Banach lattice X is called* positive *if $Tp \in X_+$ for all $p \in X_+$.*

Finally, we cite the following results on positive solutions which are required for the analysis

Lemma 3.3.4. *(Lemma 3.3 in [102]) Let X be a Banach space, let $\alpha > 0$, $R > 0$, and $p^0 \in X$ and let H_R and F_R be defined as in* (3.12)*. A function $t \to p(t)$ satisfies the integral equation*

$$p(t) = e^{tA}p^0 + A\int_0^t e^{(t-s)A}H_R(p(s))ds + \int_0^t e^{(t-s)A}F_R(p(s))ds, \quad t \geqslant 0, \tag{3.15}$$

if and only if it satisfies the integral equation for $t \geqslant 0$

$$\begin{aligned} p(t) = e^{t(A-\mathbf{I}/\alpha)}p^0 + A\int_0^t e^{(t-s)(A-\mathbf{I}/\alpha)}H_R(p(s))ds \\ +\frac{1}{\alpha}\int_0^t e^{(t-s)(A-\mathbf{I}/\alpha)}(\mathbf{I}+\alpha F_R)(p(s))ds. \end{aligned} \tag{3.16}$$

Theorem 3.3.5. *(Theorem 3.2 in [102]) Let X be a Banach lattice and let A be the generator of a positive $\mathcal{C}_0$-semigroup on X, i.e., $e^{tA}X_+ \subset X_+$ for all $t \geqslant 0$. Suppose that for each $R > 0$ there exists $\alpha > 0$ such that*

$$(\mathbf{I}+\alpha F_R)X_+ \subset X_+$$

and

$$A\int_0^t e^{(t-s)(A-\frac{\mathbf{I}}{\alpha})}H_R(u(s))ds \in X_+ \text{ for all } u \in C([0,T];X_+),$$

where F and H are locally Lipschitz continuous and F_R, H_R are defined as in (3.12)*. Then, if $p^0 \in X_+$, $p(t,p^0) \in X_+$ for all $t \in [0,t_{max})$.*

Remark 3.3.6. *As noticed in [102], Theorem 3.3.5 implies that for existence and uniqueness of positive solution with $p_0 \in X_+$, the operators F and H need only to be defined on X_+.*

3.4 Well-posedness of the age and reinfection structured model

The well-posedness of the abstract Cauchy problem presented in the previous Section 3.3 enables the well-posedness study of the age-structured SEIRS model (3.1), which will be covered in this section. In subsection 3.4.1, we introduce some supplementary notations for the analysis and formulate the model in the abstract framework developed in Section 3.3. Then the main results of our analysis are exposed in subsection 3.4.2. Finally, the proofs of the results can be found in subsections 3.4.3 and 3.4.4.

3.4.1 Adaptation to the abstract Cauchy problem setting

Consider now the same Banach space X and the positive cone X_+ introduced in (3.6) and (3.7), we rewrite then the system (3.1)-(3.10) as an abstract differential equation of the form (3.13)

evolving in X. First, a reformulation of the PDEs gives

$$\begin{aligned}
\frac{\partial s_i}{\partial t} &= -\frac{\partial s_i}{\partial a} + \omega_{i-1}(a) r_{i-1} - \mu(a) s_i - \sum_i \int_0^{+\infty} \beta_{i,j}(a,a') \imath_j(t,a') da' s_i, && i \geqslant 1, \\
\frac{\partial e_i}{\partial t} &= -\frac{\partial e_i}{\partial a} - (\sigma_i(a) + \mu(a)) e_i + \sum_i \int_0^{+\infty} \beta_{i,j}(a,a') \imath_j(t,a') da' s_i, && i \geqslant 1, \\
\frac{\partial \imath_i}{\partial t} &= -\frac{\partial \imath_i}{\partial a} + \sigma_i(a) e_i - (\gamma_i(a) + \mu(a)) \imath_i, && i \geqslant 1, \\
\frac{\partial r_i}{\partial t} &= -\frac{\partial r_i}{\partial a} + \gamma_i \imath_i - (\omega_i(a) + \mu(a)) r_i, && i \geqslant 1.
\end{aligned} \tag{3.17}$$

Let $p := (s_i, e_i, \imath_i, r_i)_{i \in \mathbb{N}}$ be a solution of the system, we see that the equations above can be written under the form

$$\frac{\partial p}{\partial t} = Ap + F(p),$$

where A and F are respectively linear and non-linear operators to be defined.

The operator A assigns a unique element $Ap \in X$ defined below to each element p in the domain $D(A)$.

$$\begin{aligned}
(Ap)_{1,1}(a) &= -\frac{s_1}{da}(a) - \mu(a) s_1(a), \\
(Ap)_{i,1}(a) &= -\frac{s_i}{da}(a) + \omega_{i-1}(a) r_{i-1}(a) - \mu(a) s_i(a), && i > 1, \\
(Ap)_{i,2}(a) &= -\frac{e_i}{da}(a) - (\sigma_i(a) + \mu(a)) e_i(a), && i \geqslant 1, \\
(Ap)_{i,3}(a) &= -\frac{\imath_i}{da}(a) + \sigma_i(a) e_i(a) - (\gamma_i(a) + \mu(a)) \imath_i(a), && i \geqslant 1, \\
(Ap)_{i,4}(a) &= -\frac{r_i}{da}(a) + \gamma_i(a) \imath_i(a) - (\omega_i(a) + \mu(a)) r_i(a), && i \geqslant 1,
\end{aligned}$$

In terms of $(p_i)_{i \in \mathbb{N}}$, where $p_i \in X$, the operator can be rewritten in a more abstract way :

$$\begin{aligned}
(Ap)_{i,1}(a) &= -\frac{p_{i,1}}{da}(a) + \omega_{i-1}(a) p_{i-1,4}(a) - \mu(a) p_{i,1}(a), && i \geqslant 1, \\
(Ap)_{i,2}(a) &= -\frac{p_{i,2}}{da}(a) - (\sigma_i(a) + \mu(a)) p_{i,2}(a), && i \geqslant 1, \\
(Ap)_{i,3}(a) &= -\frac{p_{i,3}}{da}(a) + \sigma_i(a) p_{i,2}(a) - (\gamma_i(a) + \mu(a)) p_{i,3}(a), && i \geqslant 1, \\
(Ap)_{i,4}(a) &= -\frac{p_{i,4}}{da}(a) + \gamma_i(a) p_{i,3}(a) - (\omega_i(a) + \mu(a)) p_{i,4}(a), && i \geqslant 1,
\end{aligned} \tag{3.18}$$

where $\omega_0 p_{0,4} \equiv 0$ is introduced with the aim of simplifying the expression. The domain of the operator A is set as

$$D(A) := \left\{ p \in X \cap (W^{1,1}(0,\infty))^{\mathbb{N}} : \forall i \geqslant 1, p_i(0) = \mathbf{0}_4; \exists N \in \mathbb{N}, \forall j > N, p_j \equiv \mathbf{0}_4 \right\}. \tag{3.19}$$

This domain is obviously dense in X. We will show later that A is closable and replace A with its closure $\bar{A}$ in the analysis of the well-posedness. Moreover, from the equations (3.17), the

non-linear operator $F : X \to X$ is defined by

$$(F(p))_{i,1}(a) = -\sum_{j\geqslant 1}\int_0^{+\infty} \beta_{i,j}(a,a')\imath_j(a')da'S_i(a), \quad i \geqslant 1,$$
$$(F(p))_{i,2}(a) = \sum_{j\geqslant 1}\int_0^{+\infty} \beta_{i,j}(a,a')\imath_j(a')da'S_i(a), \quad i \geqslant 1,$$
$$(F(p))_{i,3} \equiv (F(p))_{i,4} \equiv 0, \quad i \geqslant 1.$$

The nonzero components of $F(p)$ can be alternatively written in terms of $p_{i,j}$

$$(F(p))_{i,1}(a) = -\sum_{j\geqslant 1}\int_0^{+\infty} \beta_{i,j}(a,a')p_{j,3}(a')da'p_{i,1}(a), \quad i \geqslant 1,$$
$$(F(p))_{i,2}(a) = \sum_{j\geqslant 1}\int_0^{+\infty} \beta_{i,j}(a,a')p_{j,3}(a')da'p_{i,1}(a), \quad i \geqslant 1.$$

Finally, it remains to incorporate the boundary condition (3.4) into the abstract Cauchy equation. This is done by adding a multiplicative operator H taking account of the boundary condition (3.4) in the same manner as in [31, 102]. Specifically, we introduce the operator $H : X \to X$ as

$$(H(p))_{1,1}(a) = -\left(\int_0^{+\infty} b(a')[\sum_i s_i(a') + e_i(a') + \imath_i(a') + r_i(a')]da'\right)\pi(a),$$
$$(H(p))_{i,j}(a) \equiv 0 \quad \text{otherwise.}$$

The operator can be equivalently written as,

$$(H(p))_{1,1}(a) = -\left(\int_0^{+\infty} b(a')\sum_{i\geqslant 1}\sum_{1\leqslant j\leqslant 4} p_{i,j}(a')da'\right)\pi(a),$$
$$(H(p))_{i,1}(a) \equiv 0 \quad \text{otherwise.}$$

The addition of the nonlinear operator H in the equation is justified as H solves the stationary equation

$$(AH(p))(a) = 0.$$

Therefore, the evolution equation (3.1) can be equivalently stated as the abstract Cauchy problem

$$\begin{cases} p'(t) = A(\mathbf{I} + H)p(t) + F(p(t)), \\ p(0) = p^0, \end{cases} \tag{3.22}$$

where $\mathbf{I}$ is the identity matrix. In addition, the operator A is proven to be closable below, with the closure denoted $\bar{A}$. Then, from the definition of $D(A)$ in (3.19), if the condition $p + H(p) \in D(\bar{A})$ is verified, we recover the boundary conditions relating to the births (3.4) as $p + H(p) \in D(\bar{A})$ if

and only if the components of p are in $(W^{1,1}(0,+\infty))^4$ and p satisfies the boundary conditions

$$p_{1,1}(0) = \int_0^{+\infty} b(a') \sum_{i\geqslant 1} \sum_{1\leqslant j\leqslant 4} p_{i,j}(a')da',$$
$$p_{i,j}(0) = 0 \quad \text{otherwise.}$$

Furthermore, under the assumptions of Theorem 3.3.1, $p^0+H(p^0) \in D(\bar{A})$ implies $p(t)+H(p(t)) \in D(\bar{A})$ for $t \in [0, t_{\max})$. Thus, the boundary conditions (3.4), equivalent to the constraint $p(t) + H(p(t)) \in D(\bar{A})$, are satisfied for every $t \in [0, t_{\max})$.

3.4.2 Main results

We announce now the main results demonstrated for the age-structured reinfection model, the proof of which is given in the next section. The following Theorem is about the existence and uniqueness of global solution of the infinite system of PDEs (3.1)-(3.10).

Theorem 3.4.1. *(Well-posedness) Let $p^0 := (p_i^0)_{i\geqslant 1} \in X_+$ such that $p^0+H(p^0) \in D(\bar{A})$, i.e., p^0 is such that $p_i^0 \in (W^{1,1}(0,+\infty))^4$ for all $i \geqslant 1$ and fulfills the boundary conditions* (3.4)*. If* (H1)-(H3) *hold, the system* (3.1)-(3.10) *with boundary conditions* (3.4) *and initial conditions* (3.5) *has a unique classical positive and global solution $t \to p(t) := (S_i(t), E_i(t), I_i(t), R_i(t))_{i\in\mathbb{N}} \in X_+$, i.e. $p \in C^1([0,+\infty), X_+)$.*

Corollary 3.4.2. *Under the same assumptions as in Theorem 3.4.1, the classical solution $t \to p(t) := (p_i(t))_{i\geqslant 1}$, where $p_i(t) \in (L^1(0,+\infty))^4$, is such that $p_i(t) \in (W^{1,1}(0,+\infty))^4$ for any $t > 0$.*

Proof of Corollary 3.4.2. This is easy to deduce from the equations (3.1) and the fact that $\frac{\partial p}{\partial t}(t,\cdot) \in X_+$. □

As the system describes a reinfection process with reinfection number counting, it may be interesting to check the behavior of the average number of reinfections in the population at a time t. In this end, we introduce the quantity

$$P(t) := \sum_{i\geqslant 1} i \int_0^{+\infty} \Big[s_{i+1}(t,a) + e_i(t,a) + \imath_i(t,a) + r_i(t,a) \Big] da. \tag{3.24}$$

We also introduce the Banach subspace X_1 of X :

$$X_1 := \left\{ p = (p_i)_{i\in\mathbb{N}} \in X, \sum_{i=1}^{+\infty} i \Big(\int_0^{+\infty} |p_{i+1,1}(a)|da + \sum_{j=2}^{4} \int_0^{+\infty} |p_{i,j}(a)|da \Big) < +\infty \right\}, \tag{3.25}$$

endowed with the norm

$$\|p\|_{X_1} := \int_0^{+\infty} |p_{1,1}(a)|da + \sum_{i=1}^{+\infty} i \Big(\int_0^{+\infty} |p_{i+1,1}(a)|da + \sum_{j=2}^{4} \int_0^{+\infty} |p_{i,j}(a)|da \Big),$$

and define the operator A_1 as the closure of operator A (3.18) restricted to the subset X_1. Then the following result indicates that the mean number of reinfections is defined for the system (3.1) if it is defined initially.

Theorem 3.4.3. *(Average number of reinfections) Let $p^0 \in X_+$ such that $p^0 + H(p^0) \in D(A_1)$, i.e., the initial data fulfil the boundary conditions* (3.4) *with finite average number of reinfections $P(0)$. If (H1)-(H5) hold, there exists M' such that the average number of reinfections $P(t)$ defined in* (3.24) *fulfills $P(t) \leqslant M'$ for any $t \geqslant 0$.*

Remark 3.4.4. *In light of Theorem 3.4.3, one may wonder whether the average number of reinfections $P(t)$ may still be defined with a less restrictive assumption, i.e., in the case where $P(0)$ is infinite initially, can we expect a regularization effect so that $P(t)$ finite ultimately? We present here a counter-example, in the case where the parameters are independent with regard to the age. In this case, after integration, we obtain an SEIRS compartmental system of ODEs:*

$$\begin{aligned}
\dot{S}_i &= \omega R_{i-1} - \beta I S_i - \mu S_i, \quad i \geqslant 1,\\
\dot{E}_i &= \beta I S_i - (\sigma+\mu) E_i, \quad i \geqslant 1,\\
\dot{I}_i &= \sigma E_i - (\gamma+\mu) I_i, \quad i \geqslant 1,\\
\dot{R}_i &= \gamma I_i - (\omega+\mu) R_i, \quad i \geqslant 1,
\end{aligned} \tag{3.26}$$

where we denote $\omega R_0 = \mu$ for simplicity. Notice that the sums $S := \sum_i S_i, E := \sum_i E_i, I := \sum_i I_i, R := \sum_i R_i$ follow the classic SEIRS system with demography,

$$\begin{aligned}
\dot{S} &= \mu + \omega R - \beta I S - \mu S,\\
\dot{E} &= \beta I S - (\sigma+\mu) E,\\
\dot{I} &= \sigma E - (\gamma+\mu) I,\\
\dot{R} &= \gamma I - (\omega+\mu) R.
\end{aligned} \tag{3.27}$$

For $\frac{\beta}{\gamma+\mu} > 1$, this implies the existence of a GAS endemic equilibrium $(S^{EE}, E^{EE}, I^{EE}, R^{EE})$. Taking now

$$c := \frac{6}{\pi^2} = \left(\sum_{i=1}^{+\infty} \frac{1}{i^2} \right)^{-1},$$

and the initial condition $(S_i(0), E_i(0), I_i(0), R_i(0)) = \frac{c}{i^2}(S^{EE}, E^{EE}, I^{EE}, R^{EE}), \qquad i \geqslant 1$, such that

$$\begin{aligned}
P(0) &= \lim_{n\to+\infty} \sum_{i=1}^{n} i(S_{i+1}(0) + E_i(0) + I_i(0) + R_i(0))\\
&= \lim_{n\to+\infty} \left(\sum_{i=1}^{n} (i-1) S_i(0) + \sum_{i=1}^{n} i(E_i(0) + I_i(0) + R_i(0)) \right)\\
&= \lim_{n\to+\infty} c(S^{EE} + E^{EE} + I^{EE} + R^{EE}) \sum_{i=1}^{n} i^{-1} - S^{EE} = +\infty.
\end{aligned}$$

The initial condition implies also that system (3.27) *is at equilibrium. In such case, the system* (3.26) *reduces to an infinite system of linear ODEs with a triangular structure. Summing up the equations and considering $N_i := S_{i+1} + E_i + I_i + R_i$, we have*

$$\dot{N}_i = -\mu N_i + \beta I^{EE}(S_i - S_{i+1}), \quad N_i(t) \quad = N_i(0) e^{-\mu t} + \beta I^{EE} \int_0^t e^{-\mu(t-s)} [S_i(s) - S_{i+1}(s)] ds,$$

$$
\begin{aligned}
P(t) &= \lim_{n\to+\infty} \sum_{i=1}^{n} iN_i(t) \\
&= \lim_{n\to+\infty} \left(e^{-\mu t} \sum_{i=1}^{n} iN_i(0) + \beta I^{EE} [\sum_{i=1}^{n} \int_0^t e^{-\mu(t-s)} iS_i(s)ds - \sum_{i=1}^{n-1} \int_0^t e^{-\mu(t-s)} (i-1)S_i(s)ds] \right) \\
&= \lim_{n\to+\infty} \left(e^{-\mu t} \sum_{i=1}^{n} iN_i(0) + \beta I^{EE} \sum_{i=1}^{n-1} \int_0^t e^{-\mu(t-s)} S_i(s)ds + \beta I^{EE} \int_0^t e^{-\mu(t-s)} nS_n(s)ds \right).
\end{aligned}
$$

Due to $P(0) = \sum_{i=1}^{n} iN_i(0) = +\infty$*, the first sum on the right-hand side diverges to* $+\infty$ *when* $n \to +\infty$*, and because the remaining terms are always positive,* $\lim_{n\to+\infty} \sum_{i=1}^{n} iN_i(t) = +\infty$ *for* $t \geqslant 0$*, which rules out regularization effect.*

3.4.3 Well-posedness — Proof of Theorem 3.4.1

The proof consists in transforming the system (3.1) into the abstract Cauchy form (3.13), which is shown to be well-posed by application of results presented in Chapter 3.3.

For well-posedness of the corresponding abstract Cauchy problem, we first prove that the associated linear homogeneous problem generates a $\mathcal{C}_0$-semigroup of contraction.

Theorem 3.4.5. *The linear operator* A *is closable in* X*, and* $\bar{A}$ *generates a positive, strongly continuous semigroup of contractions.*

Proof. The proof is similar to the proof of Theorem 4.1 in [102]. We begin by showing that A is a densely defined dissipative operator and that the range $\mathcal{R}(\lambda\mathbf{I} - A)$ is dense in X for every $\lambda > 0$. The fact that A is closable and $\bar{A}$ is dissipative follows then from [96, Theorem 4.5]. Moreover, the density of $\mathcal{R}(\lambda\mathbf{I} - A)$ and the fact that A is dissipative imply that $\mathcal{R}(\lambda\mathbf{I} - \bar{A}) = X$ as shown in [102]. Finally, by application of the Lumer-Phillips Theorem ([96, Theorem 4.3]), $\bar{A}$ generates a $\mathcal{C}_0$-semigroup of contractions.

In order to show the dissipativity, we introduce, for $x \in X$, the subdifferential of the norm,

$$
\begin{aligned}
\partial\|x\| &= \{\varphi \in X^* : \langle \varphi, x\rangle = \|x\|, \|\varphi\| = 1\}, \quad x \neq 0, \\
\partial\|0\| &= \{\varphi \in X^* : \|\varphi\| \leqslant 1\}.
\end{aligned}
$$

By definition, A is dissipative if and only if for every $p \in D(A)$ there is $p^* \in \partial\|p\|$ such that $\langle Ap, p^*\rangle \leqslant 0$, where the bracket is the usual duality product. As the case $p = 0$ is trivial, we assume that $p \neq 0$. The dual space of X is

$$
X^* = \{\varphi = (\varphi_i)_{i\in\mathbb{N}} : \varphi_i \in (L^\infty(0,+\infty))^4, \sup_{i\in\mathbb{N}} \|\varphi_i\|_{(L^\infty(0,+\infty))^4} < +\infty\},
$$

and we have the following characterization of $\partial\|p\|$ (see [100, 102]) :

$$
\varphi \in \partial\|p\| \iff \forall i \geqslant 1, 1 \leqslant j \leqslant 4, \begin{cases} \varphi_{i,j}(a) = 1, & \text{if } a \in \Omega_{i,j}^+ = \{s \in [0,+\infty) : p_{i,j}(s) > 0\}, \\ \varphi_{i,j}(a) = -1, & \text{if } a \in \Omega_{i,j}^- = \{s \in [0,+\infty) : p_{i,j}(s) < 0\}, \\ -1 \leqslant \varphi_{i,j}(a) \leqslant 1, & \text{if } a \in \Omega_{i,j}^0 = \{s \in [0,+\infty) : p_{i,j}(s) = 0\}. \end{cases}
$$

Therefore choosing $\varphi \in \partial\|p\|$ such that $\varphi_{i,j}(a) = 0$ for $a \in \Omega^0_{i,j}$, $\forall i \geqslant 1, 1 \leqslant j \leqslant 4$, we obtain,

$$\begin{aligned}
\langle Ap, \varphi\rangle =& \sum_{i=1}^{+\infty}\sum_{j=1}^{4}\left[\int_{\Omega^+_{i,j}}(Ap)_{i,j}(a)da - \int_{\Omega^-_{i,j}}(Ap)_{i,j}(a)da\right] \\
=& \sum_{i=1}^{+\infty}\Bigg[\int_{\Omega^+_{i,1}}(Ap)_{i,1}da + \int_{\Omega^+_{i,2}}(Ap)_{i,2}da + \int_{\Omega^+_{i,3}}(Ap)_{i,3}da + \int_{\Omega^+_{i,4}}(Ap)_{i,4}da - \int_{\Omega^-_{i,1}}(Ap)_{i,1}da \\
& - \int_{\Omega^-_{i,2}}(Ap)_{i,2}da - \int_{\Omega^-_{i,3}}(Ap)_{i,3}da - \int_{\Omega^-_{i,4}}(Ap)_{i,4}da\Bigg] \\
=& \sum_{i=1}^{+\infty}\Bigg[\int_{\Omega^+_{i,1}}(\omega_{i-1}(a)p_{i-1,4}(a) - {p_{i,1}}'(a) - \mu(a)p_{i,1}(a))da - \int_{\Omega^+_{i,2}}({p_{i,2}}'(a) + (\sigma_i(a) \\
& + \mu(a))p_{i,2}(a))da + \int_{\Omega^+_{i,3}}(\sigma_i(a)p_{i,2}(a) - {p_{i,3}}'(a) - (\gamma_i(a) + \mu(a))p_{i,3}(a))da + \int_{\Omega^+_{i,4}}(\gamma_i(a)p_{i,3}(a) \\
& - {p_{i,4}}'(a) - (\omega_i(a) + \mu(a))p_{i,4}(a))da - \int_{\Omega^-_{i,1}}(\omega_{i-1}(a)p_{i-1,4}(a) - {p_{i,1}}'(a) - \mu(a)p_{i,1}(a))da \\
& + \int_{\Omega^-_{i,2}}({p_{i,2}}'(a) + (\sigma_i(a) + \mu(a))p_{i,2}(a))da - \int_{\Omega^-_{i,3}}(\sigma_i(a)p_{i,2}(a) - {p_{i,3}}'(a) - (\gamma_i(a) + \mu(a))p_{i,3}(a))da \\
& - \int_{\Omega^-_{i,4}}(\gamma_i(a)p_{i,3}(a) - {p_{i,4}}'(a) - (\omega_i(a) + \mu(a))p_{i,4}(a))da\Bigg].
\end{aligned}$$

Notice that, as $p \in D(A)$, the sums are in fact finite and rearrangements are possible, then

$$\begin{aligned}
\langle Ap, \varphi\rangle =& \sum_{i=1}^{+\infty}\sum_{j=1}^{4}\left(\int_{\Omega^-_{i,j}} - \int_{\Omega^+_{i,j}}\right)({p_{i,j}}'(a) + \mu(a)p_{i,j}(a))da + \sum_{i=1}^{+\infty}\Bigg[\int_{\Omega^+_{i,1}}\omega_{i-1}(a)p_{i-1,4}(a)da \\
& - \int_{\Omega^+_{i,2}}\sigma_i(a)p_{i,2}(a)da + \int_{\Omega^+_{i,3}}(\sigma_i(a)p_{i,2}(a) - \gamma_i(a)p_{i,3}(a))da + \int_{\Omega^+_{i,4}}(\gamma_i(a)p_{i,3}(a) \\
& - \omega_i(a)p_{i,4}(a))da - \int_{\Omega^-_{i,1}}\omega_{i-1}(a)p_{i-1,4}(a)da + \int_{\Omega^-_{i,2}}\sigma_i(a)p_{i,2}(a)da \\
& - \int_{\Omega^-_{i,3}}(\sigma_i(a)p_{i,2}(a) - \gamma_i(a)p_{i,3}(a))da - \int_{\Omega^-_{i,4}}(\gamma_i(a)p_{i,3}(a) - \omega_i(a)p_{i,4}(a))da\Bigg].
\end{aligned}$$

As remarked in the proof of Theorem 4.1 [102], thanks to $p_{i,j} \in W^{1,1}(0,+\infty)$, $\Omega^+_{i,j}$ is an at most countable union of pairwise disjoint intervals :

$$\Omega^+_{i,j} = \bigcup_{n=1}^{+\infty}(a^{i,j}_{n-1}, a^{i,j}_n),$$

with $p_{i,j}(a^{i,j}_n) = 0$ for $a^{i,j}_n \in \mathbb{R}$ and $\lim_{a\to a^{i,j}_n} p_{i,j}(a) = 0$ if $a^{i,j}_n = +\infty$ due to $p_{i,j} \in \mathcal{BV} \cap L^1(0,+\infty)$ ($\mathcal{BV}$ stands for functions of bounded variation). The same reasoning also applies to $\Omega^-_{i,j}$ and therefore

$$\int_{\Omega^+_{i,j}} p'_{i,j}(a)da = \int_{\Omega^-_{i,j}} p'_{i,j}(a)da = 0.$$

Hence,

$$\begin{aligned}\langle\varphi, Ap\rangle = &-\sum_{i=1}^{+\infty}\sum_{j=1}^{4}\int_0^{+\infty}\mu(a)|p_{i,j}(a)|da+\sum_{i=1}^{+\infty}\Big[\int_{\Omega_{i,1}^+}\omega_{i-1}(a)p_{i-1,4}(a)da-\int_{\Omega_{i,2}^+}\sigma_i(a)p_{i,2}(a)da\\&+\int_{\Omega_{i,3}^+}(\sigma_i(a)p_{i,2}(a)-\gamma_i(a)p_{i,3}(a))da+\int_{\Omega_{i,4}^+}(\gamma_i(a)p_{i,3}(a)-\omega_i(a)p_{i,4}(a))da\\&+\int_{\Omega_{i,2}^-}\sigma_i(a)p_{i,2}(a)da-\int_{\Omega_{i,1}^-}\omega_{i-1}(a)p_{i-1,4}(a)da-\int_{\Omega_{i,3}^-}(\sigma_i(a)p_{i,2}(a)-\gamma_i(a)p_{i,3}(a))da\\&-\int_{\Omega_{i,4}^-}(\gamma_i(a)p_{i,3}(a)-\omega_i(a)p_{i,4}(a))da\Big]\end{aligned}$$

Rearranging again, and as $\omega_0 p_{0,4}\equiv 0$, we have

$$\begin{aligned}\langle\varphi, Ap\rangle = &-\sum_{i=1}^{+\infty}\sum_{j=1}^{4}\int_0^{+\infty}\mu(a)|p_{i,j}(a)|da+\sum_{i=1}^{+\infty}\Big[-\int_{\Omega_{i,2}^+}\sigma_i(a)p_{i,2}(a)da+\int_{\Omega_{i,3}^+}(\sigma_i(a)p_{i,2}(a)\\&-\gamma_i(a)p_{i,3}(a))da+\int_{\Omega_{i,4}^+}(\gamma_i(a)p_{i,3}(a)-\omega_i(a)p_{i,4}(a))da+\int_{\Omega_{i+1,1}^+}\omega_i(a)p_{i,4}(a)da\\&+\int_{\Omega_{i,2}^-}\sigma_i(a)p_{i,2}(a)da-\int_{\Omega_{i,3}^-}(\sigma_i(a)p_{i,2}(a)-\gamma_i(a)p_{i,3}(a))da-\int_{\Omega_{i,4}^-}(\gamma_i(a)p_{i,3}(a)\\&-\omega_i(a)p_{i,4}(a))da-\int_{\Omega_{i+1,1}^-}\omega_i(a)p_{i,4}(a)da\Big]\\=&-\sum_{i=1}^{+\infty}\sum_{j=1}^{4}\int_0^{+\infty}\mu(a)|p_{i,j}(a)|da+\sum_{i=1}^{+\infty}\Big[\Big(\int_{\Omega_{i,3}^+\cap\Omega_{i,2}^-}-\int_{\Omega_{i,2}^+\cap\Omega_{i,3}^-}\Big)\sigma_i(a)p_{i,2}(a)da\\&+\Big(\int_{\Omega_{i,3}^-\cap\Omega_{i,4}^+}-\int_{\Omega_{i,3}^+\cap\Omega_{i,4}^-}\Big)\gamma_i(a)p_{i,3}(a)da+\Big(\int_{\Omega_{i+1,1}^+\cap\Omega_{i,4}^-}-\int_{\Omega_{i,4}^+\cap\Omega_{i+1,1}^-}\Big)\omega_i(a)p_{i,4}(a)da\\&-\Big(\int_{\Omega_{i,3}^-\cap\Omega_{i,2}^+}-\int_{\Omega_{i,2}^-\cap\Omega_{i,3}^+}\Big)\sigma_i(a)p_{i,2}(a)da-\Big(\int_{\Omega_{i,4}^-\cap\Omega_{i,3}^+}-\int_{\Omega_{i,3}^-\cap\Omega_{i,4}^+}\Big)\gamma_i(a)p_{i,3}(a)da\\&-\Big(\int_{\Omega_{i+1,1}^-\cap\Omega_{i,4}^+}-\int_{\Omega_{i,4}^-\cap\Omega_{i+1,1}^+}\Big)\omega_i(a)p_{i,4}(a)da\Big]\leqslant 0.\end{aligned}$$

The last inequality is derived from the definition of $\Omega_{i,j}^{\pm}$. We thus proved that A is dissipative. Moreover, it is obvious that $\overline{D(A)}\subset X$ thus A is closable with $\bar{A}$ also dissipative.

Let us now demonstrate the density of $\mathcal{R}(\lambda\mathbf{I}-A)$. For this, let $\lambda>0$, it is sufficient to show that for every $p\in D(A)$ there exists $q\in D(A)$ such that $\lambda q-Aq=p$. Let $p\in D(A)$ such that $p_i\equiv 0$ for $i>N$ for a $N>0$, we consider $q\in D(A)$ such that $q_i\equiv 0$ for $i>N+1$, $q_{1,1}$ as the solution of the ODE

$$\begin{cases}q_{1,1}'(a)=-(\lambda+\mu(a))q_{1,1}(a)+p_{1,1}(a),\\q_{1,1}(0)=0,\end{cases}$$

that is,

$$q_{1,1}(a) = \int_0^a e^{-\int_s^a (\lambda+\mu(\tau))d\tau} p_{1,1}(s)ds,$$

and we define recursively $q_{i,j}$, $i = 1$, $2 \leqslant j \leqslant 4$ and $1 < i \leqslant N+1$, $1 \leqslant j \leqslant 4$ as the solution of the ODE

$$\begin{cases} q'_{i,1}(a) = -(\lambda + \mu(a))q_{i,1}(a) + \omega_{i-1}(a)q_{i-1,4} + p_{i,j}(a), \\ q'_{i,2}(a) = -(\lambda + \sigma_i(a) + \mu(a))q_{i,2}(a) + p_{i,j}(a), \\ q'_{i,3}(a) = -(\lambda + \gamma_i(a) + \mu(a))q_{i,3}(a) + \sigma_i(a)q_{i,2}(a) + p_{i,j}(a), \\ q'_{i,4}(a) = -(\lambda + \omega_i(a) + \mu(a))q_{i,4}(a) + \gamma_i(a)q_{i,3}(a) + p_{i,j}(a), \\ q_{i,j}(0) = 0. \end{cases} \tag{3.28}$$

Indeed, in this case $q \in D(A)$ and it is a solution of the equation $\lambda q - Aq = p$, therefore $D(A) \subset \mathcal{R}(\lambda \mathbf{I} - A)$ and the range is dense in X, which concludes the proof that $\bar{A}$ generates a $\mathcal{C}_0$-semigroup of contractions.

To demonstrate the positivity of the semigroup $e^{t\bar{A}}$, let us take $q^0 \in D(A) \cap X_+$, we apply as in [102] the method of characteristics solving the linear PDE

$$\begin{cases} q'(t) = Aq, \\ q(0) = q^0, \end{cases}$$

we can check that $q(t) \equiv (q_i(\cdot, t))_{i\in\mathbb{N}} \in X_+$, then by density we have $e^{t\bar{A}}q^0 \in X_+$ for all $q^0 \in X_+$. □

Remark 3.4.6. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(A)$ such that $x_n \to x$ and $Ax_n \to f$ in X. It is easy to check that $x \in D(\bar{A})$ with the domain defined as*

$$D(\bar{A}) := \left\{ p \in X : p_i \in (W^{1,1}(0, +\infty))^4, p_i(0) = 0_4, \forall i \geqslant 1 \right\}.$$

In order to apply Theorem 3.3.1, it is necessary to have $\mathcal{R}(H) \subset F_{\bar{A}}$. As the operator H is a special case of the one defined in p.154 [102] with $\psi \equiv 1$ and $\zeta = 1$, this is easily verified by adapting the proof of Proposition 4.2 in [102], thus

Proposition 3.4.7. *$H(p) \in F_{\bar{A}}$ for all $p \in X$.*

It remains to check that the assumptions of Theorem 3.3.5 are also verified. As the second assumption relates to the operator H, the proof is again similar to the proof of Lemma 4.3 in [102] hence the following result holds.

Lemma 3.4.8. *Let $\alpha > 0$. The operator U_α defined in $W_T = C([0, T], X)$ by*

$$[U_\alpha u](t) := \bar{A} \int_0^t e^{(\bar{A} - \frac{1}{\alpha})(t-s)} H(u(s))ds$$

is positive, i.e., it takes positive functions into positive functions.

Finally, we check the remaining assumption

Lemma 3.4.9. *For any $R > 0$, there exists $\alpha > 0$ such that $(\mathbf{I} + \alpha F_R)X_+ \subset X_+$.*

Proof. For $p \in X_+$, we set $\bar{p} := \pi_R(p)$. By definition, we have $\bar{p}_i \leqslant p_i$, $\forall i \geqslant 1$. It is easy to see that $(\mathbf{I} + \alpha F_R)p \in X_+$ if and only if, for every $i \in \mathbb{N}$

$$p_{i,1} + \alpha(F(\bar{p}))_{i,1} \geqslant 0, \quad p_{i,2} + \alpha(F(\bar{p}))_{i,2} \geqslant 0$$

as we have always $(F(\bar{p}))_{i,2} \geqslant 0$ for $\bar{p} \in X_+$, it is only necessary to verify the first inequality. Thanks to assumption (H_3)

$$\begin{aligned} p_{i,1} - \alpha\bar{p}_{i,1} \sum_{j\geqslant 1} \int_0^{+\infty} \beta_{i,j}(\cdot,a)\bar{p}_{j,3}(a)da &\geqslant p_{i,1} - \alpha\beta_+ p_{i,1} \sum_{j\geqslant 1} \int_0^{+\infty} \bar{p}_{j,3}(a)da \\ &\geqslant (1 - \alpha\beta_+ R)p_{i,1}. \end{aligned}$$

We conclude the proof by taking $\alpha \leqslant \frac{1}{\beta_+ R}$. □

We are now ready for the proof of well-posedness result in Theorem 3.4.1, which is a direct consequence of the following two theorems on well-posedness and global solution of the abstract Cauchy problem (3.13).

Theorem 3.4.10. *If* (H1)-(H3) *hold, the Cauchy problem on* X*,*

$$\begin{cases} p'(t) = \bar{A}(p(t) + H(p(t))) + F(p(t)), \\ p(0) = p^0, \end{cases} \tag{3.29}$$

where $X, \bar{A}, H$ *and* F *have been defined above, has a unique mild solution in* X_+ *for* $p^0 \in X_+$*. Moreover, if* $p^0 + H(p^0) \in D(\bar{A})$*, then the solution is also classical.*

Proof. The fact that H and F are locally Lipschitz and continuously differentiable on X_+ is easy to see, thus the result is an application of Theorem 3.3.1, Theorem 3.4.5 and Lemmas 3.4.8, 3.4.9 (notice that Lemma 3.4.8 can be adapted for operator H_R instead of H as stated in [102]). □

Theorem 3.4.10 gives existence and uniqueness of a local solution for $t \in [0, t_{\max})$, $t_{\max} > 0$. The following result and the point 2 of Theorem 3.3.1 show that the local solution is in fact global.

Theorem 3.4.11. *Let* (H1)-(H3) *hold and* $p(t) = (p_i(\cdot,t))_{i\in\mathbb{N}}$ *be a positive solution of* (3.29) *defined on* $[0, t_{\max})$*. Then there exists* $L > 0$ *such that* $\|p(t)\| \leqslant \|p(0)\|e^{Lt}$ *for each* $t \in [0, t_{\max})$*.*

Proof. Let us begin by assuming that $p^0 \in X_+$ and $p^0 + H(p^0) \in D(\bar{A})$. In this case, Theorem 3.4.10 states that the solution $p(t)$ to the Cauchy problem (3.29) is classical, i.e. $p(\cdot) \in C^1([0, t_{\max}), X)$. For $p \in X$, let us introduce the bounded linear operator L on X :

$$Lp := \sum_{i\in\mathbb{N}} \sum_{1\leqslant j\leqslant 4} \int_0^{+\infty} p_{i,j}(a')da'.$$

Now notice that, for a positive solution $p(t) \in X_+$ of (3.29), we have the equality $\|p(t)\| = Lp(t)$.

Therefore, thanks to the continuously differentiable property of $p(\cdot)$ and the equations (3.17),

$$\begin{aligned}\frac{d}{dt}\|p(t)\| &= \frac{d}{dt}L(p(t)) = L(\frac{d}{dt}p(t)),\\ &= \sum_{i\in\mathbb{N}}\sum_{1\leqslant j\leqslant 4}\int_0^{+\infty}\frac{\partial}{\partial t}p_{i,j}(t,a')da',\\ &= \sum_{i\in\mathbb{N}}\sum_{1\leqslant j\leqslant 4}\int_0^{+\infty}-\frac{\partial}{\partial a}p_{i,j}(t,a')-\mu(a')p_{i,j}(t,a')da',\end{aligned}$$

As $p_{i,j}(t,\cdot)\in W^{1,1}(0,+\infty)$, $\lim_{a\to+\infty}p_{i,j}(t,a)=0$,

$$\begin{aligned}\frac{d}{dt}\|p(t)\| &= \sum_{i\in\mathbb{N}}\sum_{1\leqslant j\leqslant 4}\left[p_{i,j}(t,0)-\int_0^{+\infty}\mu(a')p_{i,j}(t,a')da'\right],\\ &= p_1^1(t,0)-\sum_{i\in\mathbb{N}}\sum_{1\leqslant j\leqslant 4}\int_0^{+\infty}\mu(a')p_{i,j}(t,a')da',\\ &= \int_0^{+\infty}[b(a)-\mu(a)]n(\cdot,a)da\leqslant(\|b\|_{L^\infty}-\mu_-)\|p(t)\|.\end{aligned}$$

Hence, by Grönwall Lemma, $\|p(t)\|\leqslant\|p^0\|e^{(\|b\|_{L^\infty}-\mu_-)t}$ for $p^0\in X_+$ such that $p^0+H(p^0)\in D(\bar{A})$.

For the general case, notice that the operator $\bar{A}(\mathbf{I}+H)$ generates a C_0-semigroup (e.g., see [31, Theorem A]), thus its domain $D(\bar{A}(\mathbf{I}+H))$ is dense in X, but $D(\bar{A}(\mathbf{I}+H))$ is exactly the set of elements p^0 such that $p^0+H(p^0)\in D(\bar{A})$. Therefore, by density, the same estimate holds also for $p^0\in X_+$. □

3.4.4 Mean number of reinfections — Proof of Theorems 3.4.3

Proof of Theorem 3.4.3. We proceed using an argument similar to that used in [102]. One may check that the operator A defined in (3.18) also satisfies Theorem 3.4.5 in the space X_1 (3.25) endowed with the norm $\|\cdot\|_{X_1}$ as the computation of dissipativity can be performed without supplementary complications using as reference the computations done in Theorem 3.4.5 above and Theorem 4.1 in [102]. Therefore, we define A_1 as the closure of A in X_1 and consider the abstract Cauchy problem :

$$\begin{cases}p'(t)=A_1(p(t)+H_1(p(t)))+F_1(p(t)),\\ p(0)=p^0,\end{cases}\tag{3.30}$$

where $F_1:=F|_{X_1\cap E}$ and $H_1:=H|_{X_1}$. Proposition 3.4.7, Lemma 3.4.8, 3.4.9 can be adapted for the space X_1 and Lipschitz continuity and continuous differentiability of H_1 with respect to $|\cdot|_{F(A_1)}$ and F_1 with respect to $\|\cdot\|_{X_1}$ are again easy to see, therefore the Cauchy problem (3.30) is well-posed on $(X_1)_+$. Let then $p^0+H(p^0)\in D(A_1)$ and consider the classical positive solution $p(t)\in X_1$ given by Theorem 3.3.1. Introducing the linear bounded operator L_1 on X_1 :

$$L_1p:=\sum_{i=1}^{+\infty}i\left(\int_0^{+\infty}p_{i+1,1}(a)da+\sum_{j=2}^{4}\int_0^{+\infty}p_{i,j}(a)da\right).$$

Hence $P(t) = L_1(p(t))$, and

$$\begin{aligned}\frac{d}{dt}P(t) =& \frac{d}{dt}L_1(p(t)) = L_1(\frac{d}{dt}p(t)),\\ =& \sum_{i=1}^{+\infty} i\Big(\int_0^{+\infty} \frac{\partial}{\partial t}p_{i+1,1}(t,a)da + \sum_{j=2}^{4}\int_0^{+\infty}\frac{\partial}{\partial t}p_{i,j}(t,a)da\Big),\\ =& -\sum_{i=1}^{+\infty} i\sum_{j=1}^{4}\int_0^{+\infty}(\frac{\partial}{\partial a}p_{i,j}(t,a) + \mu(a)p_{i,j}(t,a))da + \sum_{i=1}^{+\infty}\int_0^{+\infty}\lambda_i(t,a)p_{i,1}(t,a)da\\ &+\sum_{j=1}^{+\infty}\int_0^{+\infty}(\frac{\partial}{\partial a}p_{1,j}(t,a) + \mu(a)p_{1,j}(t,a))da.\end{aligned}$$

As $p_{i,j}(t,\cdot) \in W^{1,1}$, we have $\int_0^{+\infty}\frac{\partial}{\partial a}p_{i,j}(t,a)da = -p_{i,j}(t,0)$ as usual. Notice that $p_{i,j}(t,0) = 0$ for $(i,j) \neq (1,1)$, thus after some further computations,

$$\frac{d}{dt}P(t) = -\sum_{i=1}^{+\infty} i\Big(\int_0^{+\infty}\mu(a)p_{i,j}(t,a)da + \sum_{j=2}^{4}\int_0^{+\infty}\mu(a)p_{i,j}(t,a)da\Big)$$

$$+\sum_{i=1}^{+\infty}\int_0^{+\infty}\lambda_i(t,a)p_{i,1}(t,a)da \leqslant -\mu_- P(t) + \beta_+ N^2.$$

Hence $P(t)$ is bounded for $p^0 + H(p^0) \in D(A_1)$ by applying Grönwall Lemma. Moreover, as the system is normalized, $N = 1$ and the Lemma also implies that

$$\limsup_{t\to+\infty} P(t) \leqslant \frac{\beta_+}{\mu_-},$$

which concludes the proof of Theorem 3.4.3. □

3.5 Underlying age and reinfection structure of a common SEIRS model

We consider in this section the age and reinfection structured model (3.1) in the simplest setting, where the parameters are independent with respect to the age and the number of reinfection i, j. In subsection 3.5.1, the simplified age and reinfection structured model is introduced. Similarly to the SEIRS system counting reinfection (1.10) presented in subsection 1.4.1 of Chapter 1, we show formally how the model (3.1) in this case is linked to the classical SEIRS model. They constitute then a "multi-scale" vision of the same object, with or without its age and reinfection structures. of the age and reinfection structure hidden in the common SEIRS system. Then in subsection 3.5.2 the steady states of the system are exposed along with their asymptotic stability properties. Finally, in subsection 3.5.3 several interesting quantities, such as the mean number of reinfections at given age or the mean age in each compartment at endemic equilibrium of the SEIRS system (3.1) are given, in line with the mean numbers of reinfections presented in subsection 1.4.4 of Chapter 1.

3.5.1 An age-structured SEIRS model counting reinfection

As announced, we consider (3.1) in the simplest setting with $\beta_{i,j}(a,a'), \sigma_i(a), \gamma_i(a), \omega_i(a), \mu(a)$ constant regarding both the age and reinfection variables a and i. We will underline in the present subsection the relationship between the system (3.1), the common SEIRS system (1.2) and the reinfection structured SEIRS model (1.10) studied in Chapter 1 (with the additional assumptions that the disease-related mortality ν is zero and the total number of hosts N is constant). Notice first that the assumption $\mathcal{R} = 1$ in (H4) is here equivalent to

$$b \int_0^{+\infty} e^{-\mu a} da = b\mu^{-1} = 1,$$

hence $b = \mu$. The equations (3.1) subject to the present study can be written as

$$\begin{aligned}
\frac{\partial s_i}{\partial t} + \frac{\partial s_i}{\partial a} &= \omega r_{i-1} - \beta I s_i - \mu s_i, \quad i \geqslant 1, \\
\frac{\partial e_i}{\partial t} + \frac{\partial e_i}{\partial a} &= \beta I s_i - (\sigma_i + \mu) e_i, \qquad i \geqslant 1, \\
\frac{\partial \imath_i}{\partial t} + \frac{\partial \imath_i}{\partial a} &= \sigma e_i - (\gamma + \mu) \imath_i, \qquad i \geqslant 1, \\
\frac{\partial r_i}{\partial t} + \frac{\partial r_i}{\partial a} &= \gamma \imath_i - (\omega + \mu) r_i, \qquad i \geqslant 1,
\end{aligned} \tag{3.31}$$

where $I(t) := \sum_i \int_0^{+\infty} \imath_i(t,a) da$ and $r_0 \equiv 0$. The model is endowed with the following boundary conditions,

$$\begin{aligned}
s_1(\cdot, 0) &= \mu N, \\
s_{i+1}(\cdot, 0) = e_i(\cdot, 0) = \imath_i(\cdot, 0) = r_i(\cdot, 0) &= 0, \quad i \geqslant 1,
\end{aligned} \tag{3.32}$$

and the initial conditions

$$s_i(0,\cdot) = s_i^0(\cdot), \quad e_i(0,\cdot) = e_i^0(\cdot), \quad \imath_i(0,\cdot) = \imath_i^0(\cdot), \quad r_i(0,\cdot) = r_i^0(\cdot), \quad i \geqslant 1,$$

where

$$n(t,a) := s_i(t,a) + e_i(t,a) + \imath_i(t,a) + r_i(t,a), \quad N(t) := \sum_i \int_0^{+\infty} n(t,a) da = 1.$$

It has been proved in Section 3.4 that the system is well-posed and admits a unique global classical solution $p = (s_i, e_i, \imath_i, r_i)_{i \geqslant 1} \in C^1([0, +\infty), X_+)$ for every initial condition $p^0 \in X_+$ fulfilling $p^0 + H(p^0) \in D(\bar{A})$. Moreover, from the assumption (H4), $N(t)$ is constant and equal to one for $t \geqslant 0$, as it represents a normalized quantity. Let us show now that the system (3.31) is, in fact, an "unfolding" of the SEIRS system (1.3) according to both the age and the number of reinfections. For $i \geqslant 1$, we denote

$$\begin{aligned}
S_i(t) &:= \int_0^{+\infty} s_i(t,a) da, \;\; E_i(t) := \int_0^{+\infty} e_i(t,a) da, \\
I_i(t) &:= \int_0^{+\infty} \imath_i(t,a) da, \quad R_i(t) := \int_0^{+\infty} r_i(t,a) da.
\end{aligned} \tag{3.33}$$

Then an integration of (3.31) with respect to the age a gives

$$\begin{aligned}
&\int_0^{+\infty}(\frac{\partial s_1}{\partial t}(t,a')+\frac{\partial s_1}{\partial a}(t,a'))da' = -\beta I(t)S_1(t)-\mu S_1(t),\\
&\int_0^{+\infty}(\frac{\partial s_i}{\partial t}(t,a')+\frac{\partial s_i}{\partial a'}(t,a'))da' = \omega_{i-1}R_{i-1}(t)-\beta I(t)S_i(t)-\mu S_i(t), \qquad i>1,\\
&\int_0^{+\infty}(\frac{\partial e_i}{\partial t}(t,a')+\frac{\partial e_i}{\partial a}(t,a'))da' = \beta I(t)S_i(t)-(\sigma_i+\mu)E_i(t), \qquad i\geqslant 1,\\
&\int_0^{+\infty}(\frac{\partial \imath_i}{\partial t}(t,a')+\frac{\partial \imath_i}{\partial a}(t,a'))da' = \sigma E_i(t)-(\gamma+\mu)I_i(t), \qquad i\geqslant 1,\\
&\int_0^{+\infty}(\frac{\partial r_i}{\partial t}(t,a')+\frac{\partial r_i}{\partial a}(t,a'))da' = \gamma I_i(t)-(\omega+\mu)R_i(t), \qquad i\geqslant 1.
\end{aligned}$$

Thanks to $p\in C^1([0,+\infty),X_+)$ and $p(t,\cdot)\in (W^{1,1}(0,+\infty))^{\mathbb{N}}$ for any $t>0$ (Corollary 3.4.2), therefore, as $W^{1,1}(0,+\infty)\subset \mathcal{BV}\cap L^1(0,+\infty)$, any component of p converges to zero when $a\to\infty$ and the integrals on the left-hand side give

$$\begin{aligned}
&\int_0^{+\infty}(\frac{\partial s_1}{\partial t}(t,a')+\frac{\partial s_1}{\partial a}(t,a'))da' = \dot{S}_1(t)+\lim_{a\to+\infty}s_1(t,a)-s_1(t,0)=\dot{S}_1-\mu,\\
&\int_0^{+\infty}(\frac{\partial s_i}{\partial t}(t,a')+\frac{\partial s_i}{\partial a'}(t,a'))da' = \dot{S}_i(t)+\lim_{a\to+\infty}s_i(t,a)-s_i(t,0)=\dot{S}_i(t), \qquad i>1,\\
&\int_0^{+\infty}(\frac{\partial e_i}{\partial t}(t,a')+\frac{\partial e_i}{\partial a}(t,a'))da' = \dot{E}_i(t)+\lim_{a\to+\infty}e_i(t,a)-e_i(t,0)=\dot{E}_i(t), \qquad i\geqslant 1,\\
&\int_0^{+\infty}(\frac{\partial \imath_i}{\partial t}(t,a')+\frac{\partial \imath_i}{\partial a}(t,a'))da' = \dot{I}_i(t)+\lim_{a\to+\infty}\imath_i(t,a)-\imath_i(t,0)=\dot{I}_i(t), \qquad i\geqslant 1,\\
&\int_0^{+\infty}(\frac{\partial r_i}{\partial t}(t,a')+\frac{\partial r_i}{\partial a}(t,a'))da' = \dot{R}_i(t)+\lim_{a\to+\infty}r_i(t,a)-r_i(t,0)=\dot{R}_i(t), \qquad i\geqslant 1.
\end{aligned}$$

Therefore microscopic quantities defined in (3.33) form a continuously differentiable solution of the ODEs:

$$\begin{aligned}
\dot{S}_i &= \omega R_{i-1}-\beta I S_i-\mu S_i, \quad i\geqslant 1,\\
\dot{E}_i &= \beta I S_i-(\sigma+\mu)E_i, \qquad i\geqslant 1,\\
\dot{I}_i &= \sigma E_i-(\gamma+\mu)I_i, \qquad i\geqslant 1,\\
\dot{R}_i &= \gamma I_i-(\omega+\mu)R_i, \qquad i\geqslant 1,
\end{aligned} \tag{3.34}$$

where $R_0\equiv\frac{\mu}{\omega}$. The solution is moreover endowed with the initial condition

$$\begin{aligned}
S_i(0)&=\int_0^{+\infty}s_i^0(a)da, \quad E_i(0)=\int_0^{+\infty}e_i^0(a)da,\\
I_i(0)&=\int_0^{+\infty}\imath_i^0(a)da, \quad R_i(0)=\int_0^{+\infty}r_i^0(a)da.
\end{aligned} \tag{3.35}$$

We recover then a special case of the infinite ODE system studied in the Section 1.4 of Chapter 1. As studied in the aforementioned Chapter, the C^1 regularity of the solution of (3.34) is sufficient

so that the macroscopic quantities

$$\begin{aligned} S(t) &:= \sum_{i\geqslant 1} S_i(t), \quad E(t) := \sum_{i\geqslant 1} E_i(t), \\ I(t) &:= \sum_{i\geqslant 1} I_i(t), \quad R(t) := \sum_{i\geqslant 1} R_i(t), \end{aligned} \tag{3.36}$$

form indeed a continuously differentiable solution of the SEIRS model:

$$\begin{aligned} \dot{S} &= \mu - \beta IS - \mu S + \omega R, \\ \dot{E} &= \beta IS - (\sigma + \mu)E, \\ \dot{I} &= \sigma E - (\gamma + \mu)I, \\ \dot{R} &= \gamma I - (\omega + \mu)R, \end{aligned} \tag{3.37}$$

endowed with the initial conditions

$$\begin{aligned} S(0) &= \sum_{i\geqslant 1} S_i(0), \quad E(0) = \sum_{i\geqslant 1} E_i(0), \\ I(0) &= \sum_{i\geqslant 1} I_i(0), \quad R(0) = \sum_{i\geqslant 1} R_i(0). \end{aligned} \tag{3.38}$$

We have shown indeed that, integrating the solution of (3.31) according to the age, yields the solution of the reinfection structured SEIRS system (3.34), which is a particular case of the system (1.10) studied in Chapter 1 with $b = \mu$ and $\nu = 0$. Hence, the system (3.31) is related to the previous systems (3.34) and (3.37) studied in Chapter 1 by an additional age structuration. In line with the macroscopic/microscopic scales introduced in Chapter 1 for the reinfection structuration, the system (3.31) can be then described as the *nanoscopic* SEIRS system.

Finally, it is worth pointing out that, if we reverse the order of summation and consider instead the variables

$$\begin{aligned} s(t,a) &:= \sum_{i=1}^{+\infty} s_i(t,a), \quad e(t,a) := \sum_{i=1}^{+\infty} e_i(t,a), \\ \imath(t,a) &:= \sum_{i=1}^{+\infty} \imath_i(t,a), \quad r(t,a) := \sum_{i=1}^{+\infty} r_i(t,a). \end{aligned} \tag{3.39}$$

Then, by the same arguments as in the proof of Theorem 3.2.1, it is easy to deduce that $(s, e, \imath, r)$ constitutes a solution of the evolution equation:

$$\begin{aligned} \frac{\partial s}{\partial t} + \frac{\partial s}{\partial a} &= -\beta IS - \mu S + \omega R, \\ \frac{\partial e}{\partial t} + \frac{\partial e}{\partial a} &= \beta IS - (\sigma + \mu)E, \\ \frac{\partial \imath}{\partial t} + \frac{\partial \imath}{\partial a} &= \sigma E - (\gamma + \mu)I, \\ \frac{\partial r}{\partial t} + \frac{\partial r}{\partial a} &= \gamma I - (\omega + \mu)R, \end{aligned} \tag{3.40}$$

with the boundary conditions

$$\begin{aligned} s(\cdot,0) &= \mu \sum_{c\in\{s,e,\imath,r\}} \int_0^{+\infty} c(\cdot,a)da, \\ e(\cdot,0) = 0, \quad \imath(\cdot,0) &= 0, \quad r(\cdot,0) = 0. \end{aligned} \tag{3.41}$$

Notice then that from the existence and uniqueness of classical solution of (3.31), the (3.40) system is also well-posed and admits a unique global classical positive solution for any initial condition in $(W^{1,1}(0,+\infty))^4$ and fulfilling (3.41). Finally, integrating $s, e, \imath, r$ with regard to the age a on $(0,+\infty)$, we retrieve a solution to the classical SEIRS model (3.37), which shares the same initial condition (3.38) after interchanging summation and integration thanks to Fubini's Theorem.

We summarize the results above in the next Theorem.

Theorem 3.5.1. *Let $p = (s_i, e_i, \imath_i, r_i)_{i\geqslant 1}$ be a classical solution of the nanoscopic system* (3.31)-(3.32)*, then:*

1. *$(S_i, E_i, I_i, R_i)_{i\geqslant 1}$ defined by* (3.33) *is a continuously differentiable solution of the reinfection-structured microscopic system* (3.34)-(3.35)*.*
2. *$(s, e, \imath, r)$ defined by* (3.39) *is a continuously differentiable solution of the age-structured microscopic system* (3.40)-(3.41)*.*
3. *(S, E, I, R) defined by* (3.36) *is a continuously differentiable solution of the macroscopic system* (3.37)-(3.38)*.*

To visualize more easily the result, the relationship between the four SEIRS systems has been represented schematically in Figure 3.1.

3.5.2 Multi-scale picture of the equilibriums and associated stability properties

In the present subsection, we turn our attention to the description of the steady states of (3.31) and (3.40), alongside the steady states of (3.37) and (3.34) and their stability. As a reminder, the asymptotic behavior of solutions of the systems (3.37) and (3.34) can be divided into two situations, see Theorem 1.4.7:

1. If $\mathcal{R}_0 < 1$, both disease-free equilibriums of (3.37) and (3.34) are globally asymptotically stable in their respective biologically feasible sets.
2. If $\mathcal{R}_0 > 1$, both endemic equilibriums of (3.37) and (3.34) are globally asymptotically stable in the interior of their respective biologically feasible sets.

The subsection begins with a description of the disease free equilibrium of (3.31) and (3.40) in Theorems 3.5.2 and 3.5.3. As an illustration, the Figure 3.2 is given in order to expose clearly the relation between the endemic states of the macroscopic, microscopic and nanoscopic systems. Finally, the global asymptotic stability of the disease-free (for $\mathcal{R}_0 < 1$) and endemic equilibrium (for $\mathcal{R}_0 > 1$ and in the presence of the disease) is described in Theorem 3.5.4.

For the description of the steady states, remark first that, as the corresponding macroscopic system (3.37) always admits a disease-free equilibrium and, if $\mathcal{R}_0 > 1$, a unique endemic equilibrium [73, 80]. Then, with this fact and Theorem 3.5.1 in mind, at the disease-free (respectively endemic) steady state of (3.31) and (3.40), (S, E, I, R) must also be at the disease-free (respectively endemic) equilibrium of (3.37).

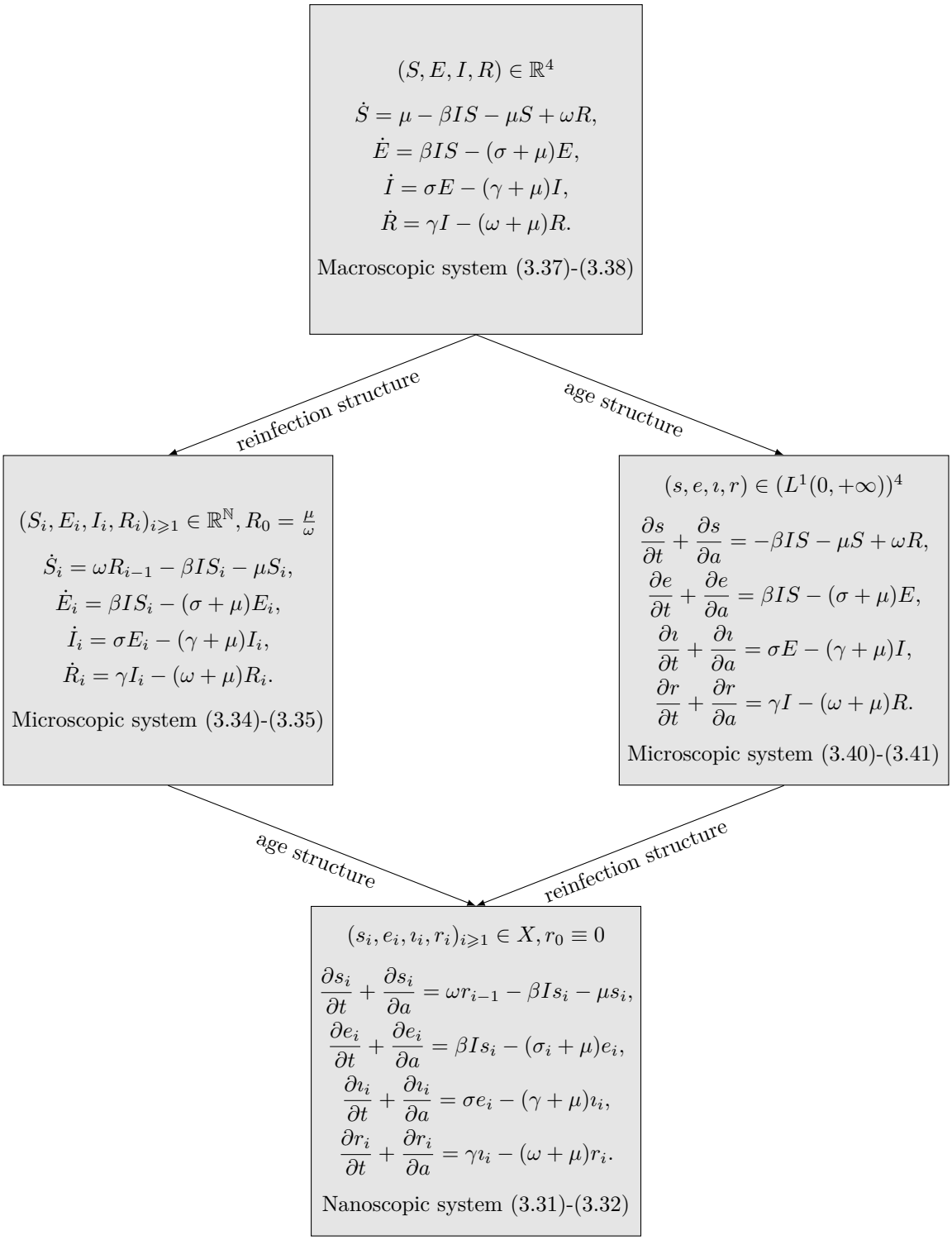


Figure 3.1: Multiscale representation of the SEIRS model

Theorem 3.5.2 (Age profile of the disease-free equilibrium)**.** *The system* (3.40) *admits a unique disease-free distribution at equilibrium* $q^{DFE} = (s^{DFE}, e^{DFE}, \imath^{DFE}, r^{DFE})$*, defined by*

$$s^{DFE}(a) = \mu e^{-\mu a}, \quad a \geqslant 0,$$
$$e^{DFE} \equiv 0, \quad \imath^{DFE} \equiv 0, \quad r^{DFE} \equiv 0.$$

The system (3.31) *admits a unique disease-free distribution at equilibrium* $p^{DFE} = (s_i^{DFE}, e_i^{DFE}, \imath_i^{DFE}, r_i^{DFE})_{i\geqslant 1}$*, defined by*

$$s_1^{DFE}(a) = \mu e^{-\mu a}, \quad a \geqslant 0,$$
$$s_{i+1}^{DFE} \equiv 0, \quad e_i^{DFE} \equiv 0, \quad \imath_i^{DFE} \equiv 0, \quad r_i^{DFE} \equiv 0, \quad i \geqslant 1.$$

Proof. At the disease-free state, (S, E, I, R) defined in (3.36) and $(S_i, E_i, I_i, R_i)_{i\geqslant 1}$ defined in (3.33) are respectively equal to $(1, 0, 0, 0)$ and $(\delta_i^1, 0, 0, 0)_{i\geqslant 1}$ (recall that δ_i^1 is the Kronecker delta). It is then easy to see that the disease-free equilibrium age profile $p^{DFE} := (s_i^{DFE}, e_i^{DFE}, \imath_i^{DFE}, r_i^{DFE})_{i\geqslant 1}$ fulfills the equations :

$$\frac{ds_1^{DFE}}{da}(a) = -\mu s_1^{DFE}(a), \quad s_1^{DFE}(0) = \mu,$$
$$s_{i+1}^{DFE} \equiv 0, \quad e_i^{DFE} \equiv 0, \quad \imath_i^{DFE} \equiv 0, \quad r_i^{DFE} \equiv 0, \quad i \geqslant 1.$$

Similarly, $(s^{DFE}, e^{DFE}, \imath^{DFE}, r^{DFE})$ fulfills

$$\frac{ds^{DFE}}{da}(a) = -\mu s^{DFE}(a), \quad s^{DFE}(0) = \mu,$$
$$e^{DFE} \equiv 0, \quad \imath^{DFE} \equiv 0, \quad r^{DFE} \equiv 0, \quad i \geqslant 1.$$

An integration on s_1^{DFE} and s^{DFE} gives then the result. □

We investigate now the endemic steady states of (3.31) and (3.40). The endemic steady states of the systems (3.37) and (3.34) have already been described in Theorem 1.4.4. Let $x^{EE} := (S_i^{EE}, E_i^{EE}, I_i^{EE}, R_i^{EE})_{i\geqslant 1}$ be the unique endemic equilibrium of (3.34), and $(S^{EE}, E^{EE}, I^{EE}, R^{EE})$ be the corresponding endemic equilibrium of the macroscopic system (3.37), the age profile $p^{EE} := (s_i^{EE}, e_i^{EE}, \imath_i^{EE}, r_i^{EE})_{i\geqslant 1}$ of (3.31) at endemic equilibrium is a solution of the linear system of ODEs:

$$\frac{ds_i^{EE}}{da} = \omega r_{i-1}^{EE} - \beta s_i^{EE} I^{EE} - \mu s_i^{EE}, \quad i \geqslant 1,$$
$$\frac{de_i^{EE}}{da} = \beta s_i^{EE} I^{EE} - (\sigma + \mu) e_i^{EE}, \quad i \geqslant 1,$$
$$\frac{d\imath_i^{EE}}{da} = \sigma e_i^{EE} - (\gamma + \mu) \imath_i^{EE}, \quad i \geqslant 1,$$
$$\frac{dr_i^{EE}}{da} = \gamma \imath_i^{EE} - (\omega + \mu) r_i^{EE}, \quad i \geqslant 1,$$

where $r_0^{EE} \equiv 0$. The associated boundary conditions are

$$s_1^{EE}(0) = \mu,$$
$$s_{i+1}^{EE}(0) = e_i^{EE}(0) = \imath_i^{EE}(0) = r_i^{EE}(0) = 0, \quad i \geqslant 1.$$

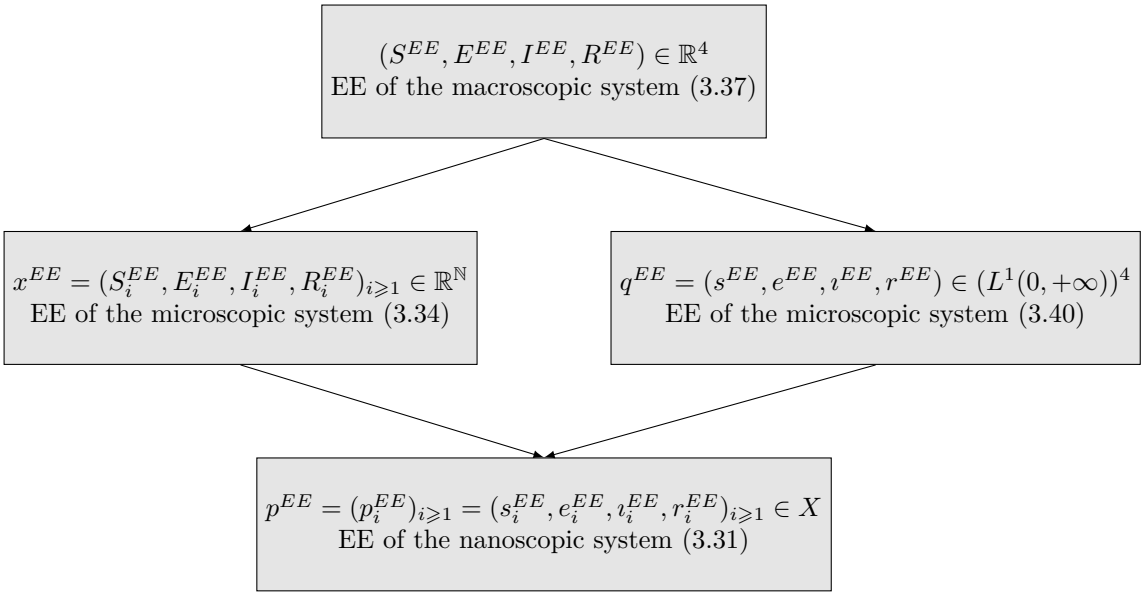


Figure 3.2: Multiscale representation of the endemic equilibrium of the SEIRS model for $\mathcal{R}_0 > 1$

Integrating the differential equations above gives then

$$\begin{aligned}
s_i^{EE}(a) &= \int_0^a \omega r_{i-1}^{EE}(s) e^{-(\beta I^{EE}+\mu)(a-s)} ds, \quad i \geqslant 1,\\
e_i^{EE}(a) &= \int_0^a \beta I^{EE} s_i^{EE}(s) e^{-(\sigma+\mu)(a-s)} ds, \quad i \geqslant 1,\\
\imath_i^{EE}(a) &= \int_0^a \sigma e_i^{EE}(s) e^{-(\gamma+\mu)(a-s)} ds, \quad i \geqslant 1,\\
r_i^{EE}(a) &= \int_0^a \gamma \imath_i^{EE}(s) e^{-(\omega+\mu)(a-s)} ds, \quad i \geqslant 1.
\end{aligned} \tag{3.42}$$

Although the formulas (3.42) is useful in the proof of asymptotic stability of p^{EE}, it is clear that the exact expression of the components of the stable distribution increases rapidly in complexity according to i due to the presence of integrals. Thus, an alternative expression of the endemic equilibrium may be required. Denoting

$$C := \begin{pmatrix} -(\beta I^{EE}+\mu) & 0 & 0 & 0 \\ \beta I^{EE} & -(\sigma+\mu) & 0 & 0 \\ 0 & \sigma & -(\gamma+\mu) & 0 \\ 0 & 0 & \gamma & -(\omega+\mu) \end{pmatrix}, \quad D := \begin{pmatrix} 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.43}$$
$$v := (1,0,0,0)^\intercal$$

the following theorem holds.

Theorem 3.5.3 (Age profile of the endemic equilibrium)**.** *Let $\mathcal{R}_0 > 1$,*

1. *the system* (3.40) *admits a unique endemic distribution at equilibrium $q^{EE} := (s^{EE}, e^{EE}, \imath^{EE},$*

r^{EE}), *which is defined by*

$$q^{EE}(a) = \mu e^{(C+D)a} v,$$

2. *the system* (3.31) *admits a unique endemic distribution at equilibrium* $p^{EE} := (p_i^{EE})_{i\geqslant 1}$, *which is defined by*

$$p_i^{EE}(a) = \mu(u_i^i \otimes \mathbf{I}_4)^\intercal e^{((\mathbf{I}_i \otimes C)+(J_i \otimes D))a}(u_1^i \otimes \mathbf{I}_4)v, \qquad i \geqslant 1,$$

where by convention u_1^i, u_j^i *denote the first and* j*-th vectors of the canonical basis of* $\mathbb{R}^i$, $\mathbf{I}_j$ *is the* $j \times j$ *identity matrix, and* J_j *the* $j \times j$ *matrix with* 1 *on the sub-diagonal and* 0 *elsewhere.*

Proof. Clearly, the endemic steady state of the evolution equations (3.40) $(s^{EE}, e^{EE}, \imath^{EE}, r^{EE})$ is a solution of the equation:

$$\frac{dq^{EE}}{da} = (C + D)q^{EE}, \quad q^{EE}(0) = \mu v.$$

We retrieve then the expression for q^{EE} after integration. On the other hand, $q^{EE} := (s^{EE}, e^{EE}, \imath^{EE}, r^{EE})$ is a solution of

$$\frac{dp_1^{EE}}{da} = Cp_1^{EE}, \quad p_1^{EE}(0) = \mu v, \qquad \frac{dp_{j+1}^{EE}}{da} = Cp_{j+1}^{EE} + Dp_j^{EE}, \quad p_j^{EE}(0) = 0, \quad j \geqslant 1. \tag{3.44}$$

Let $\otimes$ be the Kronecker product, the above equation is equivalent to

$$\frac{d}{da}\begin{pmatrix} p_1^{EE} \\ \vdots \\ p_j^{EE} \end{pmatrix} = ((\mathbf{I}_j \otimes C) + (J_j \otimes D)) \begin{pmatrix} p_1^{EE} \\ \vdots \\ p_j^{EE} \end{pmatrix},$$

so that

$$\begin{pmatrix} p_1^{EE} \\ \vdots \\ p_j^{EE} \end{pmatrix} = e^{((\mathbf{I}_j \otimes C)+(J_j \otimes D))a} \begin{pmatrix} \mu v \\ \vdots \\ 0 \end{pmatrix}.$$

Hence the result. □

Now that both the disease-free and the endemic steady states of (3.31) have been described, let us consider the asymptotic behavior of (3.31) and (3.40). As expected, they correspond to the solutions of systems (3.34) and (3.37).

Theorem 3.5.4 (Global stability result for (3.31))**.** *Let the sets* Y_1 *and* W_1 *be defined as*

$$Y_1 := \{p = (s_i, e_i, \imath_i, r_i)_{i\geqslant 1} \in X_+ : \|p\|_X = 1, p + H(p) \in D(\bar{A}) \text{ and } n(a) = \mu e^{-\mu a}\},$$
$$W_1 := \{p = (s_i, e_i, \imath_i, r_i)_{i\geqslant 1} \in Y_1 : \sum_{i=1}^{+\infty} \int_0^{+\infty} (e_i(a) + \imath_i(a))da > 0\},$$

where $n(a) := \sum_i s_i(a) + e_i(a) + \imath_i(a) + r_i(a)$. *For any initial condition* $p^0 \in Y_1$, *we consider the classical solution* p *of the system* (3.31). *Then, the following convergence results hold*

1. *if* $\mathcal{R}_0 < 1$,

$$\lim_{t\to+\infty} \|p(t) - p^{DFE}\|_X = 0,$$

and the disease-free steady state p^{DFE} *is globally asymptotically stable in* Y_1.

2. *if* $\mathcal{R}_0 > 1$ *and additionally* $p^0 \in W_1 \subset Y_1$,

$$\lim_{t\to+\infty} \|p(t) - p^{EE}\|_X = 0,$$

and the endemic steady state p^{EE} *is globally asymptotically stable in* $W_1 \subset Y_1$.

Theorem 3.5.5 (Global stability result for (3.40))**.** *Consider the product space* $Z := (L^1_+(0,+\infty))^4$ *endowed with the norm* $\|\cdot\|_Z$ *induced by* $\|\cdot\|_{L^1}$. *Let the sets* Y_2 *and* W_2 *be*

$$Y_2 := \{q = (s,e,\imath,r) \in Z : q \in (W^{1,1})^4, \|q\|_Z = 1, \text{ fulfills (3.41) and } (s+e+\imath+r)(a) = \mu e^{-\mu a}\},$$
$$W_2 := \{q = (s,e,\imath,r) \in Y_2 : \int_0^{+\infty} (e(a)+\imath(a))da > 0\}.$$

For any initial condition $q^0 \in Y_2$, *we consider the classical solution* q *of the system* (3.40). *Then, the following convergence results hold*

1. *if* $\mathcal{R}_0 < 1$,

$$\lim_{t\to+\infty} \|q(t) - q^{DFE}\|_Z = 0,$$

and the disease-free steady state q^{DFE} *is globally asymptotically stable in* Y_2.

2. *if* $\mathcal{R}_0 > 1$ *and additionally* $p^0 \in W_1 \subset Y_1$,

$$\lim_{t\to+\infty} \|q(t) - q^{EE}\|_Z = 0,$$

and the endemic steady state q^{EE} *is globally asymptotically stable in* $W_2 \subset Y_2$.

Proof of Theorem 3.5.4. We prove only the convergence $p \to p^{EE}$ for $\mathcal{R}_0 > 1$ and $p^0 \in Y_1$ such that $\sum_{i=1}^{+\infty} \int_0^{+\infty} (e_i^0(a) + \imath_i^0(a))da > 0$, as the proof of the other case is similar. Under the assumption of the Theorem, it is proven that $\lim_{t\to+\infty} I(t) = I^{EE}$ because the macroscopic quantities defined in (3.36) form a solution of the system (3.37). Consider then the solutions of (3.31) along the characteristic curve $s \to (t(s), a(s))$ defined by

$$\frac{dt}{ds} = \frac{da}{ds} = 1, \quad \text{i.e.} \quad t(s) = a(s) + c, \quad c \in \mathbb{R}.$$

Taking the derivative along the characteristic curve, we recover the system of ODEs (recall that

$r_0 \equiv 0$):

$$\begin{aligned}
\frac{ds_i(t(s),a(s))}{ds} &= \omega_{i-1} r_{i-1}(t(s),a(s)) - (\beta I(t(s)) + \mu) s_i(t(s),a(s)), \quad i \geqslant 1,\\
\frac{de_i(t(s),a(s))}{ds} &= \beta I(t(s)) s_i(t(s),a(s)) - (\sigma_i + \mu) e_i(t(s),a(s)), \qquad i \geqslant 1,\\
\frac{d\imath_i(t(s),a(s))}{ds} &= \sigma e_i(t(s),a(s)) - (\gamma + \mu) \imath_i(t(s),a(s)), \qquad i \geqslant 1,\\
\frac{dr_i(t(s),a(s))}{ds} &= \gamma \imath_i(t(s),a(s)) - (\omega + \mu) r_i(t(s),a(s)), \qquad i \geqslant 1.
\end{aligned}$$

With the initial conditions (3.32), an integration of the differential equations leads then to

$$\begin{aligned}
s_i(s+c,s) &= \int_0^s \omega_{i-1} r_{i-1}(\tau + c, \tau) e^{\int_s^\tau (\beta I(\tau+c)+\mu) d\tau} d\tau, \qquad i \geqslant 1,\\
e_i(s+c,s) &= \int_0^s \beta I(\tau + c) s_i(\tau + c, \tau) e^{(\sigma_i+\mu)(s-\tau)} d\tau, \qquad i \geqslant 1,\\
\imath_i(s+c,s) &= \int_0^s \sigma e_i(\tau + c, \tau) e^{(\gamma+\mu)(s-\tau)} d\tau, \qquad i \geqslant 1,\\
r_i(s+c,s) &= \int_0^s \gamma \imath_i(\tau + c, \tau) e^{(\omega+\mu)(s-\tau)} d\tau, \qquad i \geqslant 1.
\end{aligned}$$

Then, for $i = 1$, thanks to the dominated convergence theorem and the formulas (3.42), we compute successively the limits when $t \to +\infty$ for all $a \geqslant 0$:

$$\begin{aligned}
\lim_{t\to+\infty} s_1(t,a) &= \lim_{c\to+\infty} s_1(a+c,a) = \lim_{c\to+\infty} \mu e^{-\int_0^a (\beta I(s+c)+\mu) ds} = \mu e^{-\int_0^a (\beta I^{EE}+\mu) ds} = s_1^{EE}(a),\\
\lim_{t\to+\infty} e_1(t,a) &= \lim_{c\to+\infty} \int_0^a \beta I(s+c) s_1(s+c,s) e^{(\sigma_1+\mu)(a-s)} ds\\
&= \int_0^a \beta I^{EE} s_1^{EE}(s) e^{(\sigma_1+\mu)(a-s)} ds = e_1^{EE}(a),\\
\lim_{t\to+\infty} \imath_1(t,a) &= \lim_{c\to+\infty} \int_0^a \sigma e_1(s+c,s) e^{(\gamma+\mu)(a-s)} ds = \int_0^a \sigma e_1^{EE}(s) e^{(\gamma+\mu)(a-s)} ds = \imath_1^{EE}(a),\\
\lim_{t\to+\infty} r_1(t,a) &= \lim_{c\to+\infty} \int_0^a \gamma \imath_1(s+c,s) e^{(\omega+\mu)(a-s)} ds = \int_0^a \gamma \imath_1^{EE}(s) e^{(\omega+\mu)(a-s)} ds = r_1^{EE}(a).
\end{aligned}$$

It is then easy to see, by performing an induction on $i \geqslant 1$, that for all $i \geqslant 1$,

$$\begin{aligned}
\lim_{t\to+\infty} s_i(t,a) &= s_i^{EE}(a), \quad \lim_{t\to+\infty} e_i(t,a) = e_i^{EE}(a),\\
\lim_{t\to+\infty} \imath_i(t,a) &= \imath_i^{EE}(a), \quad \lim_{t\to+\infty} r_i(t,a) = r_i^{EE}(a),
\end{aligned}$$

that is, each component of the solution $t \to p(t)$ converges pointwise to the corresponding component of p^{EE} as $t \to +\infty$. Moreover, the components are uniformly bounded by the stable age distribution $n(a) = \mu e^{-\mu a}$ thanks to the assumption (H4), thus the dominated convergence theorem implies that every component of p converge asymptotically in the L^1 space. Finally, as $p, p^{EE} \in X_+$ and $\|p^{EE}\|_X, \|p(t)\|_X = 1$ for all $t \geqslant 0$, it is easy to see that the componentwise convergence implies the convergence with regard to the norm $\|\cdot\|_X$.

Hence, for $\mathcal{R}_0 > 1$, the endemic distribution p^{EE} attracts any solution in the bounded

positively invariant set $W_1 \subset Y_1$. Applying again the Theorem 3.3.2 [55] as in the proof of 1.4.7 in Chapter 1, the steady state p^{EE} is moreover stable in Y_1. Similarly, the steady state p^{DFE} is attracting the set Y_1 for $\mathcal{R}_0 < 1$, thus it is stable in this case by the same argument. □

Proof of Theorem 3.5.5. The proof follows the same arguments as in Theorem 3.5.4. □

3.5.3 Mean number of reinfections and average age at the endemic equilibrium

In the previous section, we demonstrated the asymptotic convergence of the age-structured reinfection model (3.31) towards the endemic stable distribution p^{EE} when the threshold condition $\mathcal{R}_0 > 1$ is verified and the system contains disease. Similarly to the study of the reinfection model with infinite ODEs completed in Chapter 1, the additional age structure of (3.31) allows us to obtain more information about the endemic steady state of the disease, such as the mean number of reinfections according to the age of the hosts or the mean age in a given compartment at the endemic stable distribution.

We begin the subsection with a computation of the mean age of the hosts in every compartment $s, e, \imath, r$ of (3.40), which is the content of Theorem 3.5.6. Then, in Theorem 3.5.7, the average age values are again computed, this time in each subcompartments $(s_i, e_i, \imath_i, r_i)$ of (3.31). Finally, in Theorem 3.5.8, we give an expression of the mean number of reinfections in each subcompartments $(s_i, e_i, \imath_i, r_i)$ and according to the age of the hosts.

Theorem 3.5.6 (Mean age of the hosts in the compartments of (3.40))**.** *Let $\mathcal{R}_0 > 1$, the matrices C, D and v be defined as in* (3.43) *and q^{EE} be the endemic steady state of system* (3.40)*. Consider the average age values $\tilde{s}^{EE}, \tilde{e}^{EE}, \tilde{\imath}^{EE}, \tilde{r}^{EE}$ in each compartment $s, e, \imath, r$ of* (3.40) *and denote $\tilde{x}^{EE} := (\tilde{s}^{EE}, \tilde{e}^{EE}, \tilde{\imath}^{EE}, \tilde{r}^{EE}) \in \mathbb{R}^4$. Then $\tilde{x}^{EE}$ is given by*

$$\tilde{x}^{EE} = \operatorname{diag}(S^{EE}, E^{EE}, I^{EE}, R^{EE})^{-1} \int_0^{+\infty} a q^{EE}(a) da,$$

and fulfills

$$\tilde{x}^{EE} = \mu \operatorname{diag}(S^{EE}, E^{EE}, I^{EE}, R^{EE})^{-1} (C + D)^{-2} v,$$

where we recall that $(S^{EE}, E^{EE}, I^{EE}, R^{EE})$ and q^{EE} are respectively the endemic equilibria of the systems (3.37) *and* (3.40)*.*

Proof. One has,

$$\int_0^{+\infty} a q^{EE}(a) da = \mu \left(\int_0^{+\infty} a e^{(C+D)a} da \right) v.$$

On the other hand, the matrix

$$C + D = \begin{pmatrix} -(\beta I^{EE} + \mu) & 0 & 0 & \omega \\ \beta I^{EE} & -(\sigma + \mu) & 0 & 0 \\ 0 & \sigma & -(\gamma + \mu) & 0 \\ 0 & 0 & \gamma & -(\omega + \mu) \end{pmatrix},$$

is Hurwitz (hence invertible) as it is, in fact, the Jacobian matrix of the endemic equilibrium of the SEIRS system (3.37), which is proved to be globally asymptotically stable for $\mathcal{R}_0 > 1$ [73].

Then, an integration by parts yields

$$\int_0^{+\infty} ae^{(C+D)a}da = \left[a(C+D)^{-1}e^{(C+D)a}\right]_0^{+\infty} - (C+D)^{-1}\int_0^{+\infty} e^{(C+D)a}da$$
$$= -(C+D)^{-1}\int_0^{+\infty} e^{(C+D)a}da.$$

Therefore,

$$\int_0^{+\infty} aq^{EE}(a)da = -\mu(C+D)^{-1}\left(\int_0^{+\infty} e^{(C+D)a}\ da\right)v = \mu(C+D)^{-2}v.$$

□

Similarly to the system (3.40), the mean age of the hosts in every compartment $s_i, e_i, \imath_i, r_i$ of (3.31) can be computed with the following result.

Theorem 3.5.7 (Mean age of the hosts in the compartments of (3.31))**.** *Let $\mathcal{R}_0 > 1$, the matrices C, D and v be defined as in* (3.43) *and $p^{EE} = (p_i^{EE})_{i\geqslant 1}$ be the endemic steady state of system* (3.31)*. Consider the average age values $\tilde{s}_i^{EE}, \tilde{e}_i^{EE}, \tilde{\imath}_i^{EE}, \tilde{r}_i^{EE}$ in each compartment $s_i, e_i, \imath_i, r_i$ of* (3.31) *for $i \geqslant 1$ and denote $(\tilde{x}_i^{EE})_{i\geqslant 1} := (\tilde{s}_i^{EE}, \tilde{e}_i^{EE}, \tilde{\imath}_i^{EE}, \tilde{r}_i^{EE})_{i\geqslant 1} \in \mathbb{R}^{\mathbb{N}}$, then $\tilde{x}_i^{EE}$ is given by*

$$\tilde{x}_i^{EE} = \operatorname{diag}(S_i^{EE}, E_i^{EE}, I_i^{EE}, R_i^{EE})^{-1}\int_0^{+\infty} ap_i^{EE}(a)\ da,$$

and fulfills the relation:

$$\tilde{x}_i^{EE} = \mu \operatorname{diag}(S_i^{EE}, E_i^{EE}, I_i^{EE}, R_i^{EE})^{-1}(u_i^i \otimes \mathbf{I}_4)^\intercal((\mathbf{I}_i \otimes C) + (J_i \otimes D))^{-2}(u_1^i \otimes \mathbf{I}_4)v,$$

where the vectors u_i^i, u_1^i are defined in Theorem 3.5.3.

Proof. Thanks to Theorem 3.5.3,

$$\int_0^{+\infty} ap_i^{EE}(a)da = \mu(u_i^i \otimes \mathbf{I}_4)^\intercal\left(\int_0^{+\infty} ae^{((\mathbf{I}_i\otimes C)+(J_i\otimes D))a}da\right)(u_1^i \otimes \mathbf{I}_4)v.$$

Moreover, $(I_i \otimes C) + (J_i \otimes D)$ is Hurwitz as it is block triangular with C as diagonal blocks, and C is Hurwitz. Then, applying again an integration by parts yields

$$\int_0^{+\infty} ae^{((\mathbf{I}_i\otimes C)+(J_i\otimes D))a}da = -((\mathbf{I}_i \otimes C) + (J_i \otimes D))^{-1}\int_0^{+\infty} e^{((\mathbf{I}_i\otimes C)+(J_i\otimes D))a}da$$
$$= ((\mathbf{I}_i \otimes C) + (J_i \otimes D))^{-2}.$$

Finally,

$$\int_0^{+\infty} ap_i^{EE}(a)da = \mu(u_i^i \otimes \mathbf{I}_4)^\intercal((\mathbf{I}_i \otimes C) + (J_i \otimes D))^{-2}(u_1^i \otimes \mathbf{I}_4)v.$$

□

Last, we deliver a tractable expression below for the computation of the mean numbers of reinfections according to the age in each compartment $(s, e, \imath, r)$ of (3.40). For the system (3.37), recall also that the age-independent mean numbers of reinfections in the compartments (S, E, I, R) can be explicitly computed thanks to Theorem 1.4.11.

Theorem 3.5.8 (Mean numbers of reinfections at age a in each compartment of (3.40))**.** *Let $\mathcal{R}_0 > 1$, the matrices C, D and v be defined as in* (3.43) *and $p^{EE} = (p_i^{EE})_{i\geqslant 1}$ be the endemic steady state of system* (3.31) *and define the matrix-valued function $N(s)$ by*

$$N(s) = e^{-s}(C+D)\int_0^s e^{(C+D+\mathbf{I}_4)u}du. \tag{3.45}$$

Consider the mean numbers of reinfections $\bar{s}^{EE}(a), \bar{e}^{EE}(a), \bar{\imath}^{EE}(a), \bar{r}^{EE}(a)$ of hosts at age $a > 0$ in each compartment $s, e, \imath, r$ of (3.40) *and denote the vector $\bar{x}^{EE}(a) := (\bar{s}^{EE}(a), \bar{e}^{EE}(a), \bar{\imath}^{EE}(a), \bar{r}^{EE}(a)) \in \mathbb{R}^4$. Then $\bar{x}^{EE}(a)$ is given by the relation*

$$\bar{x}^{EE}(a) := \operatorname{diag}(s^{EE}(a), e^{EE}(a), \imath^{EE}(a), r^{EE}(a))^{-1}\sum_{i\geqslant 1} ip_i^{EE}(a),$$

and fulfills

$$\bar{x}^{EE}(a) = \mu \operatorname{diag}(s^{EE}(a), e^{EE}(a), \imath^{EE}(a), r^{EE}(a))^{-1}\int_0^a N(a-\tau)De^{C\tau}v d\tau.$$

Remark 3.5.9. *In the case where $C + D + \mathbf{I_4}$ is invertible, we have*

$$N(s) = e^{-s}(C+D)(C+D+\mathbf{I}_4)^{-1}(e^{(C+D+\mathbf{I}_4)s} - \mathbf{I}_4).$$

Proof of Theorem 3.5.8. For $s \in \mathbb{C} \setminus \operatorname{spec}(C)$, where $\operatorname{spec}(C)$ is the spectrum of the matrix C, we apply first a Laplace transform to the differential equation (3.44) to obtain

$$\begin{aligned}\hat{p}_1^{EE}(s) &= (s\mathbf{I}_4 - C)^{-1}\mu v,\\ \hat{p}_{i+1}^{EE}(s) &= (s\mathbf{I}_4 - C)^{-1}D\hat{p}_i^{EE}(s), \quad i \geqslant 1.\end{aligned}$$

We derive recursively

$$\hat{p}_i^{EE}(s) = [(s\mathbf{I}_4 - C)^{-1}D]^{i-1}\hat{p}_1^{EE}(s), \quad i \geqslant 1.$$

The summation gives

$$\begin{aligned}\sum_{i=1}^{+\infty} i\hat{p}_i^{EE} &= \sum_{i=1}^{+\infty} i[(s\mathbf{I}_4 - C)^{-1}D]^{i-1}\hat{p}_1^{EE}(s) = \left(\frac{d}{ds}\sum_{i=1}^{+\infty}((s\mathbf{I}_4 - C)^{-1}D)^i\right)\hat{p}_1^{EE}(s)\\ &= \left(\frac{d}{ds}(\mathbf{I}_4 - (s\mathbf{I}_4 - C)^{-1}D)^{-1}\right)\hat{p}_1^{EE}(s) = \left(\frac{d}{ds}((s\mathbf{I}_4 - C)^{-1}(s\mathbf{I}_4 - C - D))^{-1}\right)\hat{p}_1^{EE}(s)\\ &= \left(\frac{d}{ds}(\mathbf{I}_4 + (s\mathbf{I}_4 - C - D)^{-1}D)\right)\hat{p}_1^{EE}(s)\\ &= -(s\mathbf{I}_4 - C - D)^{-2}D\hat{p}_1^{EE}(s).\end{aligned}$$

Introduce then the system of ODEs

$$\begin{aligned}\dot{\xi}_1 &= (C+D)\xi_1 + Dx_1^{EE}, \quad \xi_1(0) = 0,\\ \dot{\xi}_2 &= (C+D)\xi_2 - \xi_1, \quad \xi_2(0) = 0.\end{aligned} \tag{3.46}$$

Applying again the Laplace transform to ξ_1, ξ_2,

$$\hat{\xi}_1 = (s\mathbf{I}_4 - C - D)^{-1} D\hat{p}_1^{EE}, \quad \hat{\xi}_2 = -(s\mathbf{I}_4 - C - D)^{-1}\hat{\xi}_1.$$

Hence,

$$\hat{\xi}_2 = -(s\mathbf{I}_4 - C - D)^{-2} D\hat{p}_1^{EE},$$

and the Laplace transform of ξ_2 is equal to $\sum_{i=1}^{+\infty} i\hat{p}_i^{EE}$, so that $\xi_2 = \sum_{i=1}^{+\infty} ip_i^{EE}$.

Let us compute now the solution ξ_1, ξ_2 of (3.46). In matrix form, the equation is written as

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} C+D & \mathbf{0}_{4\times 4} \\ C+D & -\mathbf{I}_4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} Dp_1^{EE} \\ \mathbf{0}_{4\times 1} \end{pmatrix}.$$

Define the matrix

$$M := \begin{pmatrix} C+D & \mathbf{0}_{4\times 4} \\ C+D & -\mathbf{I}_4 \end{pmatrix}.$$

The formula in Theorem 3.5.3 with $i = 1$ gives

$$p_1^{EE}(a) = \mu e^{Ca} v,$$

then the method of variation of parameters yields

$$\xi_2(a) = \mu \begin{pmatrix} \mathbf{0}_{4\times 4} & \mathbf{I}_4 \end{pmatrix} \left(\int_0^a e^{M(a-\tau)} \begin{pmatrix} De^{C\tau} v \\ \mathbf{0}_{4\times 1} \end{pmatrix} d\tau \right). \tag{3.47}$$

Moreover, the exponential e^{Ms} may be computed. Indeed, let us introduce the matrix valued function $N(s)$ defined by

$$e^{Ms} = \begin{pmatrix} e^{(C+D)s} & \mathbf{0}_{4\times 4} \\ N(s) & e^{-s}\mathbf{I}_4 \end{pmatrix}. \tag{3.48}$$

The derivation gives

$$\frac{d}{ds} e^{Ms} = \begin{pmatrix} C+D & \mathbf{0}_{4\times 4} \\ C+D & -\mathbf{I}_4 \end{pmatrix} e^{Ms},$$

from which we deduce

$$\dot{N}(s) = (C+D)e^{(C+D)s} - N(s).$$

Therefore, as $N(0) = 0$, we recover the formula (3.45) after integration. Finally, substituting the function $N(s)$ and formula (3.48) into (3.47) gives

$$\xi_2(a) = \mu \int_0^a N(a-\tau) De^{C\tau} v d\tau.$$

□

Chapter 4

Observation and identification of a SIS model measuring primary infections

This chapter is based on the four publications that we co-authored [41, 43, 44, 45], written in collaboration with Denis Efimov and Rosane Ushirobira from Inria Lille.

4.1 Introduction

Since their introduction by Kermack and McKendrik in 1927 [68], compartmental models have been massively employed in mathematical epidemiology in order to study epidemic dynamics. Usually, researchers are interested in the analysis and simulation of epidemic processes, but practical prediction and analysis of epidemics also require reliable estimation of parameters of the models. Nevertheless, it is imperative to consider the state estimation of epidemiological systems for certain matters of significance, such as the surveillance and management of epidemics. Hence, the inverse problem of estimating states (observation) and parameters (identification) remains critical for understanding and supervising the dynamics of epidemics, and it is essential to ensure first and foremost if that the obtained parameter and state estimates are meaningful, in other terms, whether the epidemiological model employed is identifiable and observable [62].

Although identification and observation are well-studied issues in many fields, this is still not the case in mathematical epidemiology, where only a small amount of literature is available. Among them, most are only interested in the parameter identification of epidemic models, even though state estimation is equally significant for understanding and monitoring epidemic processes. Nevertheless, an interested reader may refer, for example, to [27, 92] for recent works about observer design of generic epidemiological model, but also [2, 16] for observer design with a particular set-based approach. The typical difficulty or feature of the estimation/identification problems underlined with epidemiological models is their lack of observability/identifiability in the vicinity of the equilibriums attracting the system behavior.

In the present chapter, we focus on the joint estimation of the state and parameters of a Susceptible-Infected-Susceptible (SIS) epidemic model counting primary infections. The model can be considered as a particular and simplified case of the infinite SEIRS model counting re-infection (1.10) in Chapter 1. Indeed, the SIS system contained in this chapter can be formally obtained by considering the system (1.10) with $\sigma = \omega = \infty$. This simpler case is considered here

as an introduction to identification and observation issues for the more general system (1.10). These issues are treated taking as measurements a same, unknown, portion (denoted α in the sequel) of the infected I and of the primary infected I_1. The usual approach for estimating states and parameters is based on the measurement of prevalence (the number of infected individuals) or incidence (the number of newly infected cases), see [17, 113]. However, in the case of an epidemic with reinfections, other data of interest may be available, such as the number of primary infections. Our main interest here is to see if this additional measurement, a data rarely considered but available, may improve the observation and identification of epidemiological systems with reinfections.

The chapter begins with the analysis of observability and identifiability of a simple SIS model in Section 4.2 with incomplete measurements. In particular, we derive the algebraic observability and identifiability under the condition that both a same portion of infections and primary infectious are measured, and the lack of observability/identifiability when the latter measurement is unavailable. This outcome motivates the subject of the chapter, which is to design an adaptive observer able to deliver a reliable estimation of the states and parameters of the SIS model from the measurements of infected and primary infected hosts.

In Section 4.3, a class of non-linear asymptotic observers for the system is presented and the stability is deduced using both copositive and Lur'e-type Lyapunov functions and linear matrix inequalities (LMI). In Section 4.4, we explore as a first step the adaptive estimation of the SIS model with the simplifying assumption that the whole population of infected and primary infections is measured. Then, the approach is adapted to the observation and identification of the SIS system assuming that only an unknown portion of infected and primary infected is measured.

4.2 An observable and identifiable SIS model

Let us introduce the simple SIS model measuring infections and primary infections, which has the following state-space representation:

$$\dot{S} = \mu - \beta SI - \mu S + \gamma I, \quad \dot{I} = \beta SI - (\mu + \gamma)I, \tag{4.1a}$$

$$\dot{S}_1 = \mu - \beta S_1 I - \mu S_1, \quad \dot{I}_1 = \beta S_1 I - (\mu + \gamma)I_1, \tag{4.1b}$$

$$y = \alpha I, \quad y_1 = \alpha I_1. \tag{4.1c}$$

The four states $S(t)$, $I(t)$, $S_1(t)$, $I_1(t)$, $t \in \mathbb{R}_+$ represent the number of susceptible, infected, never infected susceptible and primary infected individuals, respectively. Moreover, the coefficients μ, β, γ describe the natural mortality rate, the contact rate and the recovery rate. It is assumed that the hosts are initially infection naive and included in the subcompartment S_1 of S, and are transferred to the subcompartment I_1 of I after the first infection. The infection processes are then repeated as the infected returns to the susceptible compartment S thanks to the recovery. However, the reinfection takes place outside the compartments S_1, I_1 that include only disease-naive hosts and first-time infections. In the mathematical epidemiology community, the first stage (S, I) of the system (4.1) is known as the classical SIS model with vital dynamics [58]. However, the system (4.1) includes additional equations (4.1b) modelling the dynamics of primary infections in a cascade connection to (4.1a). This operation may be extended indefinitely to obtain a SIS system modelling infinitely many reinfections, $n \in \mathbb{N}$, similarly to the system studied in Chapter 1.

Last, the outputs of the system y, y_1 describe respectively the measurements of a given portion α of the infected I and of the primary infected I_1. Hence, the model takes the assumption that the proportion α corresponding to both the infected and primary infected individuals detected by

the Public health system (e.g., the symptomatic cases), is identical. Additionally, it is assumed in the remaining of the paper that the basic reproduction number

$$\mathcal{R}_0 := \frac{\beta}{\mu+\gamma}, \tag{4.2}$$

is greater than 1 so that convergence to the (unique, positive) endemic equilibrium occurs for every trajectory such that $I(0) > 0$.

Theorem 4.2.1. *The set* $\mathcal{S} := \{(S, I, S_1, I_1) \in [0,1]^4 \ : \ S_1 \leqslant S \leqslant 1, \ I_1 \leqslant I \leqslant 1, \ S + I = 1\}$ *is positively invariant, as well as its topological interior* $\mathring{\mathcal{S}}$.

Proof. The invariance of $\mathcal{S}$ is deduced from the fact that the derivative at 0 of each state variable is nonnegative; and from the exploitation of the differential inequalities $\dot{S} - \dot{S}_1 = -(\beta I + \mu)(S - S_1) + \gamma I \geqslant -(\beta I + \mu)(S - S_1)$, $\dot{I} - \dot{I}_1 = \beta I(S - S_1) - (\mu + \gamma)(I - I_1) \geqslant -(\mu + \gamma)(I - I_1)$ and $\dot{S} + \dot{I} = \mu(1 - S - I)$. The corresponding result for $\mathring{\mathcal{S}}$ ensues by extending the arguments. □

Theorem 4.2.2. *Assume that* $(S(0), I(0), S_1(0), I_1(0)) \in [0,1]^4$, $S(0) + I(0) = 1$, $I(0) > 0$ *and*

$$\mathcal{R}_0 > 1.$$

System (4.1) *has a unique endemic equilibrium*

$$x^{EE} := (\mathcal{R}_0^{-1}, 1 - \mathcal{R}_0^{-1}, \frac{\mu}{\beta - \gamma}, \frac{\beta - \gamma - \mu}{\beta - \gamma}\frac{\mu}{\gamma + \mu}),$$

which is globally asymptotically stable.

Proof. The proof of global stability of the endemic equilibrium $(\mathcal{R}_0^{-1}, 1 - \mathcal{R}_0^{-1})$ for equations (4.1a) is contained in [69]. The convergence of the states S_1, I_1 of (4.1b) may be deduced by taking advantage of the cascade connection (4.1b) to (4.1a) and employing the same arguments as in the proof of convergence of (1.14) in Theorem 1.4.7 with $i = 1$. □

What is remarkable about the SIS system (4.1) measuring infections and primary infections is that it is both observable and identifiable when the trajectory is not at equilibrium. In fact, the next theorem states that the original SIS system with only the states S, I and output $y = \alpha I$ is neither identifiable nor observable in case where α, β, γ are unknown (the mortality rate μ can be assumed to be available, since it can be easily estimated separately). However, the additional states S_1, I_1 and the output $y_1 = \alpha I_1$ render the system observable and identifiable.

Theorem 4.2.3. *Assume that* β, γ, α *are unknown, and only the measurement* y *is available with* $\dot{y} \neq 0$. *Then system* (4.1a) *with states* (S, I) *is neither observable nor identifiable. On the other hand, if* y_1 *is also available, then system* (4.1) *is both observable and identifiable.*

Proof.

Measuring only y**.** Consider first the use of the measurement y. Because $S + I \equiv 1$, the two formulas in (4.1a) provide indeed the same equation, namely: $\dot{I} = (\beta - (\mu + \gamma) - \beta I)I$. Therefore,

$$\dot{y} = \left(\beta - (\mu + \gamma) - \frac{\beta}{\alpha}y\right) y. \tag{4.3}$$

By differentiation one gets that

$$\frac{d}{dt}\left(\frac{\dot{y}}{y}\right) = -\frac{\beta}{\alpha}\dot{y},$$

and one may express the two quantities $\frac{\beta}{\alpha}$ and $\mu+\gamma$, as

$$\frac{\beta}{\alpha} = -\frac{1}{\dot{y}}\frac{d}{dt}\left(\frac{\dot{y}}{y}\right) = -\frac{1}{\dot{y}}\frac{d^2}{dt^2}(\ln y), \tag{4.4a}$$

$$\beta-\gamma = \frac{\dot{y}}{y}+\mu+\frac{\beta}{\alpha}y = \frac{\dot{y}}{y}+\mu-\frac{y}{\dot{y}}\frac{d^2}{dt^2}(\ln y) = \mu+\frac{d}{dt}(\ln y)-\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right). \tag{4.4b}$$

These two quantities are thus identifiable, but this is not sufficient to obtain each of the three coefficients α,β,γ. On the other hand, it is clear that nothing more may be learned when measuring only y, which fulfills equation (4.3). Therefore, the system (4.1a) is not identifiable over α,β,γ. Also, notice that it is not possible to determine $I=\frac{1}{\alpha}y$, otherwise α would be identifiable, and all other parameters too. Thus the system is not observable.

• **Measuring y and y_1.**

We include now the equations (4.1b) and exploit the knowledge of the supplementary measured output y_1. From the definition of y_1 and the second formula in (4.1b) one deduces, putting $w:=\beta S_1$, that

$$\dot{y}_1 = (\beta S_1)y-(\mu+\gamma)y_1 = wy-(\mu+\gamma)y_1, \tag{4.5}$$

thus:

$$\ddot{y}_1 = \dot{w}y+w\dot{y}-(\mu+\gamma)\dot{y}_1. \tag{4.6}$$

On the other hand, due the first equation in (4.1b) and replacing $\frac{\beta}{\alpha}$ by its value obtained from (4.4a) yields

$$\dot{w} = \beta\dot{S}_1 = \beta(\mu-\beta S_1 I-\mu S_1) = \beta\mu-\frac{\beta}{\alpha}wy-\mu w = \beta\mu+\left(\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu\right)w. \tag{4.7}$$

Inserting in (4.6) the value of $\dot{w}$ extracted from (4.7), we get

$$\ddot{y}_1 = \left(\beta\mu+\left(\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu\right)w\right)y+w\dot{y}-(\mu+\gamma)\dot{y}_1,$$

and gathering the terms in w leads to the following equivalent form:

$$\ddot{y}_1 = \beta\mu y+\left(\dot{y}+y\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu y\right)w-(\mu+\gamma)\dot{y}_1. \tag{4.8}$$

For clarity, let us write (4.5) and (4.8) under matrix form:

$$\begin{pmatrix} -y_1 & y \\ -\dot{y}_1 & \dot{y}+y\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu y \end{pmatrix}\begin{pmatrix}\mu+\gamma\\ w\end{pmatrix} = \begin{pmatrix}\dot{y}_1\\ \ddot{y}_1-\beta\mu y\end{pmatrix}.$$

One deduces by partial inversion of the matrix that

$$\mu+\gamma = \frac{\begin{vmatrix}\dot{y}_1 & y\\ \ddot{y}_1-\beta\mu y & \dot{y}+y\left[\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu\right]\end{vmatrix}}{\begin{vmatrix}-y_1 & y\\ -\dot{y}_1 & \dot{y}+y\left[\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu\right]\end{vmatrix}}. \tag{4.9}$$

On the other hand, one has, by (4.4b), an alternative expression of $\mu+\gamma$, namely

$$\mu+\gamma=\beta-\frac{d}{dt}(\ln y)+\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right). \tag{4.10}$$

Achieving elimination of $\mu+\gamma$ between (4.9) and (4.10) yields

$$\beta-\frac{d}{dt}(\ln y)+\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)=\frac{\begin{vmatrix}\dot{y}_1 & y\\ \ddot{y}_1-\beta\mu y & \dot{y}+y\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu y\end{vmatrix}}{\begin{vmatrix}-y_1 & y\\ -\dot{y}_1 & \dot{y}+y\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu y\end{vmatrix}}.$$

Now, one checks that the previous identity is indeed affine in β. Factorizing all terms in β, it may be written

$$\left(-\left(\dot{y}+y\frac{d}{dt}\left(\ln\left|\frac{d}{dt}(\ln y)\right|\right)-\mu y\right)y_1+y\dot{y}_1-\mu y^2\right)\beta=\Phi(y,\dot{y},\ddot{y},y_1,\dot{y}_1,\ddot{y}_1),$$

where the function Φ depends upon the two outputs y and y_1 and their derivatives up to the second one. In particular, Φ does not contain any occurrence of the unknown coefficients α,β,γ. One deduces that β may be identified, thus also α,γ with the help of (4.4). Therefore, in these conditions, system (4.1) is identifiable.

Once these parameters have been identified, one has $I=\frac{1}{\alpha}y$ and $S=1-I$, while on the other hand, $I_1=\frac{1}{\alpha}y_1$ and $S_1=\frac{1}{\beta}w$, where (see (4.5)) w is given by

$$w=\frac{y_1}{y}\left(\frac{d}{dt}(\ln y_1)+\mu+\gamma\right).$$

The system (4.1) is thus observable. □

4.3 Asymptotic observer

In the last section, algebraic observability and algebraic identifiability were established for a susceptible-infected-susceptible (SIS) model (4.1) counting primary infections, based on the measurement of the number of infected and of the number of primary infections. We take a step further in this section, which is devoted to the observation of the SIS model, assuming that the parameters are known. More precisely, we focus here on providing a class of non-linear observers for the SIS state-space system (4.1), which is transformed in subsection 4.3.1 by a change of coordinates into a Persidskii system form [64, 65, 97], for which a class of observers is proposed. The asymptotic convergence of the latter is established in subsection 4.3.2 under adequate assumptions on the gain coefficients and using techniques inspired from [87, 88]. Finally, some numerical simulations are provided in subsection 4.3.3.

4.3.1 Non-linear observer for the SIS model

The present subsection is organized as follows: paragraph 4.3.1.1 provides first a useful state estimation representation (system (4.11), written in matrix form in (4.12)) obtained by a suitable change of variables. This allows us to propose a class of observers for this system in paragraph 4.3.1.2.

4.3.1.1 Alternative state-space representation of the SIS model

For the observer design, we propose in this subsection a change of variables for (4.1). Setting $s_1 = \ln(S_1)$, $\imath = \ln(I)$, $z_1 = S_1 + I_1$ and $z = S + I$ yields the following equations:

$$\dot{z} = \mu - \mu z, \tag{4.11a}$$
$$\dot{\imath} = \beta z - \beta(e^{\imath} - 1) - (\beta + \mu + \gamma), \tag{4.11b}$$
$$\dot{z}_1 = \mu - (\mu + \gamma)z_1 + \gamma(e^{s_1} - 1) + \gamma, \tag{4.11c}$$
$$\dot{s}_1 = \mu(e^{-s_1} - 1) - \beta(e^{\imath} - 1) - \beta, \tag{4.11d}$$
$$y_1 = \alpha(e^{\imath} - 1) + \alpha,\ y_2 = \alpha z_1 - \alpha(e^{s_1} - 1) - \alpha, \tag{4.11e}$$
$$y_3 = \imath, \quad y_4 = z. \tag{4.11f}$$

Notice that for convenience, we added two additional components $y_3 = \imath = \ln(I)$ and $y_4 = z = S + I = 1$ to the output vector $y(t) := (y_1(t), y_2(t), y_3(t), y_4(t))^\intercal \in \mathbb{R}^4$, this is possible as α is known in the present setting.

Introduce the following notations:

$$A_0 := \begin{pmatrix} -\mu & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & -(\mu+\gamma) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 0 & 0 \\ -\beta & 0 & 0 \\ 0 & \gamma & 0 \\ -\beta & 0 & -\mu \end{pmatrix}, \quad B := \begin{pmatrix} \mu \\ -(\beta+\mu+\gamma) \\ \mu+\gamma \\ -\beta \end{pmatrix},$$

$$C_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D := \begin{pmatrix} \alpha \\ -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\phi(\zeta) := \begin{pmatrix} f_+(\zeta_1) \\ f_+(\zeta_2) \\ f_-(\zeta_2) \end{pmatrix}, \quad \text{where } \zeta := \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \quad f_\epsilon(x) := \epsilon(e^{\epsilon x} - 1), \quad \epsilon = \pm, \quad \text{for all } x \in \mathbb{R}.$$

Denoting $x(t) := \begin{pmatrix} z(t) & \imath(t) & z_1(t) & s_1(t) \end{pmatrix}^\top \in \mathbb{R}^4$, the system (4.11) can be expressed under the following matrix form:

$$\begin{aligned} \dot{x} &= A_0 x + A_1 \phi(Kx) + B, \\ y &= C_0 x + C_1 \phi(Kx) + D. \end{aligned} \tag{4.12}$$

The non-linear system obtained in the form of (4.12) with the nonlinearities ϕ is known as a Persidskii system [88, 97].

4.3.1.2 A class of observer for the SIS model

We introduce now the proposed class of observers for system (4.11), which is given in the formula below, comprising two gain matrices $L^1 := (L^1_{i,j})_{1 \leqslant i,j \leqslant 4} \in \mathbb{R}^{4\times 4}$, $L^2 := (L^2_{i,j})_{1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 3} \in \mathbb{R}^{4\times 3}$.

$$\dot{\hat{x}} = A_0 \hat{x} + A_1 \phi(K\hat{x}) + B + L^1(y - C_0\hat{x} - C_1\phi(K\hat{x}) - D) + L^2 C_2 \phi(K(x - \hat{x})), \tag{4.13}$$

where

$$C_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Notice that we added in (4.13) the error $C_2\phi(K(x-\hat{x}))$, this is possible as $\imath$ is an accessible output and the term $C_2\phi(K(x-\hat{x}))$ is indeed also accessible to the computation from the outputs of (4.11).

With the previous definition, the observation error $e := x - \hat{x}$ behaves according to the differential equation:

$$\dot{e} = (A_0 - L^1C_0)e + (A_1 - L^1C_1)(\phi(Kx) - \phi(K(x-e))) - L^2C_2\phi(Ke). \tag{4.14}$$

Before proceeding to the convergence of the observer (4.13), we rewrite the observer equations in the natural state coordinates $\begin{pmatrix}\hat{S} & \hat{I} & \hat{S}_1 & \hat{I}_1\end{pmatrix}$. For this, let $\Psi : [0,1]\times(0,1]\times[0,1]\times(0,1] \to [0,1]\times\mathbb{R}_-\times[0,1]\times\mathbb{R}_-$ be such that $\Psi(\begin{pmatrix}S & I & S_1 & I_1\end{pmatrix}^\intercal) = x = \begin{pmatrix}z, \imath, z_1, s_1\end{pmatrix}^\intercal$. Then $\Psi^{-1}(x) = \begin{pmatrix}z - e^\imath & e^\imath & e^{s_1} & z_1 - e^{s_1}\end{pmatrix}^\intercal$, and its Jacobian matrix is

$$\mathcal{D}_{(x)}\Psi^{-1} = \begin{pmatrix}1 & -e^\imath & 0 & 0\\ 0 & e^\imath & 0 & 0\\ 0 & 0 & 0 & e^{s_1}\\ 0 & 0 & 1 & -e^{s_1}\end{pmatrix} = \begin{pmatrix}1 & -I & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & S_1\\ 0 & 0 & 1 & -S_1\end{pmatrix}.$$

One then has

$$\begin{pmatrix}\dot{\hat{S}}\\ \dot{\hat{I}}\\ \dot{\hat{S}}_1\\ \dot{\hat{I}}_1\end{pmatrix} = \mathcal{D}_{(\hat{x})}\Psi^{-1}.\dot{\hat{x}} = \begin{pmatrix}1 & -\hat{I} & 0 & 0\\ 0 & \hat{I} & 0 & 0\\ 0 & 0 & 0 & \hat{S}_1\\ 0 & 0 & 1 & -\hat{S}_1\end{pmatrix}\dot{\hat{x}},$$

which after computations yields

$$\begin{pmatrix}\dot{\hat{S}}\\ \dot{\hat{I}}\\ \dot{\hat{S}}_1\\ \dot{\hat{I}}_1\end{pmatrix} = \begin{pmatrix}\mu - \beta\hat{S}\hat{I} - \mu\hat{S} + \gamma\hat{I}\\ \beta\hat{S}\hat{I} - (\mu+\gamma)\hat{I}\\ \mu - \beta\hat{S}_1\hat{I} - \mu\hat{S}_1\\ \beta\hat{S}_1\hat{I} - (\mu+\gamma)\hat{I}_1\end{pmatrix} + \mathcal{D}_{(\hat{x})}\Psi^{-1}(L^1(y - \begin{pmatrix}\alpha\hat{I}\\ \alpha\hat{I}_1\\ \ln\hat{I}\\ \hat{S}+\hat{I}\end{pmatrix} + L^2\begin{pmatrix}\frac{I}{\hat{I}} - 1\\ 0\\ 0\end{pmatrix}). \tag{4.15}$$

Under this form, the nonlinear nature of the observer is evident, due to the presence of the terms $\ln\hat{I}$ and $\frac{I-\hat{I}}{\hat{I}}$.

4.3.2 Observer convergence

Once the candidate observer is introduced, let us analyze now the stability of the origin $e = 0$ of (4.14). We first recall in paragraph 4.3.2.1 the notion of stability that will be considered here, namely of state-independent uniform output stability (SIuOS) [110]. Taking advantage of the positivity of the original system, a natural attempt is to look for copositive Lyapunov functions, see e.g. [85, 97]. This is done in paragraph 4.3.2.2, yielding a first sufficient condition for asymptotic convergence of the observer (Theorem 4.3.2). In preparation for the forthcoming result, some sector estimates for the nonlinearities appearing in the system are provided in paragraph 4.3.2.3. This allows us to give in paragraph 4.3.2.4 a second sufficient condition for asymptotic convergence of the observer (Theorem 4.3.4), based on the search for Lyapunov function equal to a quadratic form plus some Lur'e integral terms.

4.3.2.1 Stability notions

To analyze the stability of the equilibrium $e = 0$ of (4.14), we consider the dynamics of the coupled model (4.12), (4.14) with state variable $(x(t), e(t))$, taking also $e(t)$ as output of the system. Inspired by [110, Definition 3.1], we introduce the following definitions.

Definition 4.3.1. *A system*

$$\dot{x}(t) = f(x(t)), \qquad y(t) = h(x(t)) \tag{4.16}$$

with state $x(t) \in \mathbb{R}^n$ and output $y(t) \in \mathbb{R}^p$ ($f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are continuously differentiable functions) is called state-independent uniformly output stable (SIuOS) *if there exists a $\mathcal{KL}-$function η such that, denoting $y(t, x_0)$ the output at the time t of the system with initial state $x_0 \in \mathbb{R}^n$, one has $|y(t, x_0)| \leqslant \eta(|h(x_0)|, t)$, for any $t \geqslant 0$, any $x_0 \in \mathbb{R}^n$.*

The SIuOS stability is related to the existence of a SIuOS-Lyapunov function. To define the latter, it is required to introduce the definition of classes of function $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$, which are standard in the control theory community [110]:

Definition 4.3.2. *The following definitions hold*

1. *A function $\psi : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ if it is strictly increasing, continuous and satisfies $\psi(0) = 0$.*
2. *In addition, the function $\psi \in \mathcal{K}$ is of class $\mathcal{K}_\infty$ if moreover $\lim_{t\to+\infty} \psi(x) = +\infty$.*
3. *A function $\psi : [0, +\infty)^2 \to [0, +\infty)$ is $\mathcal{KL}$ if it is of class $\mathcal{K}$ on the first argument and decreases to zero on the second argument.*

Definition 4.3.3. *For system (4.16), a smooth function $V_1 : \mathbb{R}^n \to \mathbb{R}_+$ is called a SIuOS-Lyapunov function if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that*

$$\alpha_1(\|y\|) \leqslant V_1(x) \leqslant \alpha_2(\|y\|), \tag{4.17a}$$

$$\nabla V_1(x) f(x) \leqslant -\alpha_3(\|y\|), \tag{4.17b}$$

for all $x \in \mathbb{R}^n$ and $y = h(x)$.

The following result, based on [110, Theorem 3.2], connects the SIuOS property of a system to the existence of a SIuOS-Lyapunov function.

Theorem 4.3.1. *A forward complete system (4.16) is SIuOS iff it admits a SIuOS-Lyapunov function.*

Consequently, if the system is SIuOS, the origin $e = 0$ of the system will be globally attracting all the trajectories, and therefore $\hat{x}$ satisfies the requirement for being an observer of x. The SIuOS property and its conditions are formulated globally for all $x \in \mathbb{R}^n$, and their local counterparts (or ones defined on conic invariant sets) can be easily deduced under suitable restrictions on the amplitudes of x and y.

4.3.2.2 Stability through copositive Lyapunov functions

The result stated below provides a first set of gain matrices ensuring SIuOS for the observer (4.13).

Theorem 4.3.2. *Assume $L^1_{4,2} < 0 < L^1_{2,1}$ and all other coefficients of the matrices L^1, L^2 zero. Then system* (4.12), (4.14) *is SIuOS with respect to the output e.*

Proof. Let a candidate Lyapunov function be

$$V_1(e) = c|e_1| + |e_2| + d|e_3| + |e_4|,$$

for $c, d > 0$ to be chosen afterward. Setting

$$\delta f^i_\epsilon := f_\epsilon(x_i) - f_\epsilon(\hat{x}_i), \qquad \epsilon = \pm, \tag{4.18}$$

we will evaluate the derivative of V_1 along the trajectory, in a point where $e \neq 0$. One has

$$\begin{aligned}
\dot{V}_1(e) =& c\dot{e}_1\mathrm{sgn}(e_1) + \dot{e}_2\mathrm{sgn}(e_2) + d\dot{e}_3\mathrm{sgn}(e_3) + \dot{e}_4\mathrm{sgn}(e_4)\\
=& - c\mu|e_1| + \mathrm{sgn}(e_2)\left(\beta e_1 - (\beta + \alpha L^1_{2,1})\delta f^2_+\right) + d\mathrm{sgn}(e_3)\left(-(\mu+\gamma)e_3 + \gamma\delta f^4_+\right)\\
&+ \mathrm{sgn}(e_4)\left(-\mu\delta f^4_- - \beta\delta f^2_+ - \alpha L^1_{4,2}(e_3 - \delta f^4_+)\right)\\
=& - c\mu|e_1| + \mathrm{sgn}(e_2)\beta e_1 - \mathrm{sgn}(e_4)\alpha L^1_{4,2}e_3 - d(\mu+\gamma)|e_3| - \mathrm{sgn}(e_2)(\beta + \alpha L^1_{2,1})\delta f^2_+\\
&- \mathrm{sgn}(e_4)\beta\delta f^2_+ + d\mathrm{sgn}(e_3)\gamma\delta f^4_+ + \mathrm{sgn}(e_4)(L^1_{4,2}\alpha\delta f^4_+ - \mu\delta f^4_-)\\
\leqslant& - (c\mu - \beta)|e_1| - \left(d(\mu+\gamma) + \alpha L^1_{4,2}\right)|e_3| - (\beta + \alpha L^1_{2,1})\mathrm{sgn}(x_2 - \hat{x}_2)(e^{x_2} - e^{\hat{x}_2})\\
&+ \beta|e^{x_2} - e^{\hat{x}_2}| + L^1_{4,2}\alpha\mathrm{sgn}(x_4 - \hat{x}_4)(e^{x_4} - e^{\hat{x}_4}) + d\gamma|e^{x_4} - e^{\hat{x}_4}|\\
&- \mu\mathrm{sgn}(x_4 - \hat{x}_4)(e^{-\hat{x}_4} - e^{-x_4})\\
=& - (c\mu - \beta)|e_1| - \left(d(\mu+\gamma) + \alpha L^1_{4,2}\right)|e_3| - \alpha L^1_{2,1}|e^{x_2} - e^{\hat{x}_2}| + (\alpha L^1_{4,2} + d\gamma)|e^{x_4} - e^{\hat{x}_4}|\\
&- \mu|e^{-\hat{x}_4} - e^{-x_4}| < 0.
\end{aligned}$$

Choosing c, d positive such that

$$c\mu - \beta > 0, \quad d(\mu+\gamma) + \alpha L^1_{4,2} > 0, \quad \alpha L^1_{4,2} + d\gamma \leqslant 0,$$

V_1 appears indeed as an appropriate Lyapunov function, which is moreover radially unbounded. We thus conclude that (4.12), (4.14) is SIuOS with respect to the output e. □

4.3.2.3 Sector estimates

As preparation for the search of Lyapunov functions with Lur'e terms, we provide first some sector estimates on the nonlinearities. Notice first that the values of f_ε always have the sign of their argument. Furthermore, the following result holds, which provides sector estimates for the nonlinearities present in the dynamics of the observer.

Lemma 4.3.3. *For any trajectory of system* (4.12)-(4.14)*, if there is $\varepsilon' > 0$ such that $\|e(t)\| < \varepsilon'$ for all $t \geqslant 0$, then there exist $0 \leqslant a_i \leqslant b_i$, $i = 1, \ldots, 9$ such that*

$$\begin{gathered}
(b_j e_2 - (\phi_j(K(x+e)) - \phi_j(Kx))) \times (\phi_j(K(x+e)) - \phi_j(Kx) - a_j e_2) \geqslant 0,\\
(b_{j+3}\phi_j(Ke) - (\phi_j(K(x+e)) - \phi_j(Kx))) \times (\phi_j(K(x+e)) - \phi_j(Kx) - a_{j+3}\phi_j(Ke)) \geqslant 0,\\
(b_{j+6}e_2 - \phi_j(Ke)) \times (\phi_j(Ke) - a_{j+6}e_2) \geqslant 0,
\end{gathered}$$

for any $j = 1, 2, 3$.

One may also check with no specific difficulty that the above sector inequalities can be rewritten in matrix format as

$$v_1^\intercal \begin{pmatrix} -E^\intercal \Lambda_1 E & F\Lambda_1 \\ \Lambda_1^\intercal F^\intercal & -\Lambda_1 \end{pmatrix} v_1 \geqslant 0, \quad v_2^\intercal \begin{pmatrix} -\Lambda_2 & \Lambda_2 H \\ \Lambda_2 H & -\Lambda_2 G \end{pmatrix} v_2 \geqslant 0, \tag{4.19a}$$

$$v_3^\intercal \begin{pmatrix} -N^\intercal \Lambda_3 N & M\Lambda_3 \\ \Lambda_3^\intercal M^\intercal & -\Lambda_3 \end{pmatrix} v_3 \geqslant 0, \tag{4.19b}$$

where v_1, v_2, v_3 are defined by

$$v_1(x,e) := \begin{pmatrix} Ke \\ \phi(K(x+e)) - \phi(Kx) \end{pmatrix}, \tag{4.19c}$$

$$v_2(x,e) := \begin{pmatrix} \phi(K(x+e)) - \phi(Kx) \\ \phi(Ke) \end{pmatrix}, \quad v_3(e) := \begin{pmatrix} Ke \\ \phi(Ke) \end{pmatrix}. \tag{4.19d}$$

In addition, $\Lambda_j := \mathrm{diag}\{\lambda_{3j-2}, \lambda_{3j-1}, \lambda_{3j}\}$ for $j = 1,2,3$, $G := \mathrm{diag}\{a_4 b_4, a_5 b_5, a_6 b_6\}$, $H := \frac{1}{2}\mathrm{diag}\{a_4 + b_4, a_5 + b_5, a_6 + b_6\}$, and

$$E = \begin{pmatrix} (a_1 b_1)^{1/2} & 0 \\ 0 & (a_2 b_2)^{1/2} \\ 0 & (a_3 b_3)^{1/2} \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} a_1 + b_1 & 0 & 0 \\ 0 & a_2 + b_2 & a_3 + b_3 \end{pmatrix},$$

$$N = \begin{pmatrix} (a_7 b_7)^{1/2} & 0 \\ 0 & (a_8 b_8)^{1/2} \\ 0 & (a_9 b_9)^{1/2} \end{pmatrix}, \quad M = \frac{1}{2}\begin{pmatrix} a_7 + b_7 & 0 & 0 \\ 0 & a_8 + b_8 & a_9 + b_9 \end{pmatrix}.$$

Let us finally demonstrate Lemma 4.3.3.

Proof of Lemma 4.3.3. Since the basic reproduction number (4.2) is assumed larger than 1, system (4.1) converges towards the endemic equilibrium, which is positive thanks to Theorem 4.2.2. Therefore, there exist $\varepsilon_I, \varepsilon_{S_1} > 0$ and $T \geqslant 0$ such that $I(t) > \varepsilon_I$ and $S_1(t) > \varepsilon_{S_1}$ for any $t \geqslant T$. Assume also that $\|e(t)\| < \varepsilon'$ for some $\varepsilon' > 0$ along the trajectory. Using the shorthand in (4.18), we deduce the following inequalities for $t > T$:

• First, $e^{\ln(\varepsilon_I)-\varepsilon'}|e_2| \leqslant |\delta f_+^2| = |e^{x_2} - e^{\hat{x}_2}| \leqslant |e_2|$, $e^{\ln(\varepsilon_{S_1})-\varepsilon'}|e_4| \leqslant |\delta f_+^4| = |e^{x_4} - e^{\hat{x}_4}| \leqslant |e_4|$ and $|e_4| \leqslant |\delta f_-^4| = |e^{-\hat{x}_4} - e^{-x_4}| \leqslant e^{-\ln(\varepsilon_{S_1})+\varepsilon'}|e_4|$, so that $a_1 = e^{\ln(\varepsilon_I)-\varepsilon'}$, $b_1 = 1$, $a_2 = e^{\ln(\varepsilon_{S_1})-\varepsilon'}$, $b_2 = 1$, $a_3 = 1$, $b_3 = e^{-\ln(\varepsilon_{S_1})+\varepsilon'}$ work.

• Second, $e^{\ln(\varepsilon_I)-\varepsilon'}|f_+(e_2)| \leqslant |\delta f_+^2| = |e^{x_2} - e^{\hat{x}_2}| \leqslant |f_+(e_2)|$, $e^{\ln(\varepsilon_{S_1})-\varepsilon'}|f_+(e_4)| \leqslant |\delta f_+^4| = |e^{x_4} - e^{\hat{x}_4}| \leqslant |f_+(e_4)|$, $|f_-(e_4)| \leqslant |\delta f_-^4| = |e^{-\hat{x}_4} - e^{-x_4}| \leqslant e^{-\ln(\varepsilon_{S_1})+\varepsilon'}|f_-(e_4)|$, and the choices $a_4 = e^{\ln(\varepsilon_I)-\varepsilon'}$, $b_4 = 1$, $a_5 = e^{\ln(\varepsilon_{S_1})-\varepsilon'}$, $b_5 = 1$, $a_6 = 1$, $b_6 = e^{-\ln(\varepsilon_{S_1})+\varepsilon'}$ are convenient.

• Finally $e^{\ln(\varepsilon_I)-\varepsilon'}|e_2| \leqslant |f_+(e_2)| = |e^{x_2} - e^{\hat{x}_2}|e^{-\hat{x}_2} \leqslant |e_2|e^{-\ln(\varepsilon_I)+\varepsilon'}$, $e^{\ln(\varepsilon_{S_1})-\varepsilon'}|e_4| \leqslant |f_+(e_4)| = |e^{x_4} - e^{\hat{x}_4}|e^{-\hat{x}_4} \leqslant |e_4|e^{-\ln(\varepsilon_{S_1})+\varepsilon'}$ and $e^{\ln(\varepsilon_{S_1})-\varepsilon'}|e_4| \leqslant |f_-(e_4)| = |e^{-\hat{x}_4} - e^{-x_4}|e^{\hat{x}_4} \leqslant e^{-\ln(\varepsilon_{S_1})+\varepsilon'}|e_4|$, so that one may take $a_7 = e^{\ln(\varepsilon_I)-\varepsilon'}$, $b_7 = e^{-\ln(\varepsilon_I)+\varepsilon'}$, $a_8 = e^{\ln(\varepsilon_{S_1})-\varepsilon'}$, $b_8 = e^{-\ln(\varepsilon_{S_1})+\varepsilon'}$, $a_9 = e^{\ln(\varepsilon_{S_1})-\varepsilon'}$, $b_9 = e^{-\ln(\varepsilon_{S_1})+\varepsilon'}$.

This completes the demonstration of Lemma 4.3.3. □

4.3.2.4 Stability through quadratic Lyapunov functions with Lur'e terms

We consider now the SIuOS stability of (4.12) (4.14). For a positive linear system, the existence of a copositive Lyapunov function is equivalent to the existence of a quadratic Lyapunov function represented by a diagonal matrix [46]. For a nonlinear system like (4.14), the search for a

quadratic Lyapunov candidate function with Lur'e components seems a priori to be a more powerful method [12, 118], which we will now investigate. Consider the quadratic Lyapunov candidate function with Lur'e components (recall f_ϵ, $\epsilon = \pm$, are defined in paragraph 4.3.1.1)

$$V_2(e) = e^\intercal P e + 2\left(\gamma_1 \int_0^{e_2} f_+(z)dz + \gamma_2 \int_0^{e_4} f_+(z)dz + \gamma_3 \int_0^{e_4} f_-(z)dz\right),$$

parameterized by $P \in \mathbb{R}^{4\times 4}$, $P = P^\intercal$, and the scalars $\gamma_1, \gamma_2, \gamma_3 \geqslant 0$.

We introduce the matrix

$$K_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\Gamma := \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$. Assuming that $P + K_1^\top \Gamma K_1 > 0$ and $P \geqslant 0$, then V_2 is positive definite thanks to Finsler's lemma [19]. Moreover, along the trajectories, the derivative of V_2 can be expressed by

$$\frac{d}{dt}V_2(e) = v^\intercal Q v,$$

where v is defined by

$$v(t) := \begin{pmatrix} e(t) \\ \phi(Kx(t)) - \phi(K(x(t) - e(t))) \\ \phi(Ke(t)) \end{pmatrix},$$

and Q is a 10×10 symmetric matrix defined below.

$$Q := \begin{pmatrix} (A_0 - L^1C_0)^\intercal P + P(A_0 - L^1C_0) & P(A_1 - LC_1) & (A_0 - L^1C_0)^\intercal K_1^\intercal \Gamma - PL^2C_2 \\ (A_1 - L^1C_1)^\intercal P & 0 & (A_1 - L^1C_1)^\intercal K_1^\intercal \Gamma \\ \Gamma K_1(A_0 - L^1C_0) - (L^2C_2)^\intercal P & \Gamma K_1(A_1 - L^1C_1) & -\Gamma K_1 L^2 C_2 - (\Gamma K_1 L^2 C_2)^\intercal \end{pmatrix}.$$

Let us define

$$w = \begin{pmatrix} Ke \\ \phi(K(x+e)) - \phi(Kx) \\ \phi(Ke) \end{pmatrix},$$

then the matrix inequalities (4.19) imply the inequality $w^\intercal R w \geqslant 0$, with $R = R^\intercal \in \mathbb{R}^{8\times 8}$ defined below :

$$R := \begin{pmatrix} -(E^\intercal \Lambda_1 E + N^\intercal \Lambda_3 N) & F\Lambda_1 & M\Lambda_3 \\ \Lambda_1 F^\intercal & -(\Lambda_1 + \Lambda_2) & \Lambda_2 H \\ \Lambda_3 M^\intercal & \Lambda_2 H & -(\Lambda_2 G + \Lambda_3) \end{pmatrix}.$$

The derivative of V_2 along the trajectories then verifies $\dot{V}_2(e) \leqslant v^\intercal Q v + w^\intercal R w$, where the last term is equal to

$$v^\intercal \begin{pmatrix} K & 0_{2\times 3} & 0_{2\times 3} \\ 0_{3\times 4} & I_3 & 0_{3\times 3} \\ 0_{3\times 4} & 0_{3\times 3} & I_3 \end{pmatrix}^\intercal R \begin{pmatrix} K & 0_{2\times 3} & 0_{2\times 3} \\ 0_{3\times 4} & I_3 & 0_{3\times 3} \\ 0_{3\times 4} & 0_{3\times 3} & I_3 \end{pmatrix} v.$$

We have demonstrated, in fact, the following result.

Theorem 4.3.4. Assume the gain matrices L^1, L^2 are such that there exists a symmetric matrix $P \in \mathbb{R}^{4\times 4}$ and diagonal matrices $\Gamma, \Lambda_1, \Lambda_2, \Lambda_3 \in \mathbb{R}^{3\times 3}$ such that

$$P \geqslant 0, \quad P + K_1^\top \Gamma K_1 > 0, \quad \Lambda_1, \Lambda_2, \Lambda_3 \geqslant 0,$$

and for a_i, b_i, $i = 1, ..., 9$ verifying the inequalities (4.19), the matrix inequality

$$Q + \begin{pmatrix} -K^\intercal(E^\intercal\Lambda_1 E + N^\intercal\Lambda_3 N)K & K^\intercal F\Lambda_1 & K^\intercal M\Lambda_3 \\ \Lambda_1 F^\intercal K & -(\Lambda_1+\Lambda_2) & \Lambda_2 H \\ \Lambda_3 M^\intercal K & \Lambda_2 H & -(\Lambda_2 G+\Lambda_3) \end{pmatrix} < 0.$$

is satisfied, then system (4.12), (4.14) is locally SIuOS-stable.

By locally we mean that the initial conditions are chosen such that the constraints $I(t) > \varepsilon_I$, $S_1(t) > \varepsilon_{S_1}$ and $\|e(t)\| < \varepsilon'$ are verified for all $t \geqslant 0$. For the latter restriction on e, due to the substantiated properties of V, it is enough to take $\delta' > 0$ such that $\|e(0)\| < \delta'$ implies $\|e(t)\| < \varepsilon'$ for all $t \geqslant 0$ in such a case.

4.3.3 Numerical simulations

As an application of the previous results, we present some numerical simulations of the system (4.1) and the state observer (4.15). For predefined gain matrices L^1, L^2, and values of $\varepsilon_{S_1}, \varepsilon_I, \varepsilon'$ which depend on the initial conditions, the LMIs of Theorem 4.3.4 are first verified using SDPT3 solvers and YALMIP in Matlab environment. Once the existence of $P, \Gamma, \Lambda_1, \Lambda_2, \Lambda_3$ is guaranteed numerically, the observer is admissible for the chosen value of the gain, and we proceed then to the numerical simulations of state observer dynamics. The simulations presented in the figures 4.1-4.4 correspond to parameter values set to $\gamma^{-1} = 24$ days, $\mu^{-1} = 60$ days, $\alpha = 0.5$ and $\beta = 0.0972$ days^{-1}. The gain matrices are set to $L^1 = \frac{1}{2}C_0^\intercal$ and $L^2 = \frac{1}{2}\begin{pmatrix} 0_{1\times 3} & C_2 \end{pmatrix}^\intercal$. Moreover, to illustrate its efficiency, the trajectories of the observer (in red) are compared to the trajectories (in green) in the case where the gain matrices L^1, L^2 are set to zero.

As seen in the figures, the convergence of the observer $\hat{S}$, $\hat{I}$ to the true states S, I of the system is almost instantaneous. This is expected, as the state I (hence $S = 1 - I$) is directly accessible from the output. Moreover $\hat{S}_1$, $\hat{I}_1$ converge asymptotically to the true state S_1, I_1 as expected.

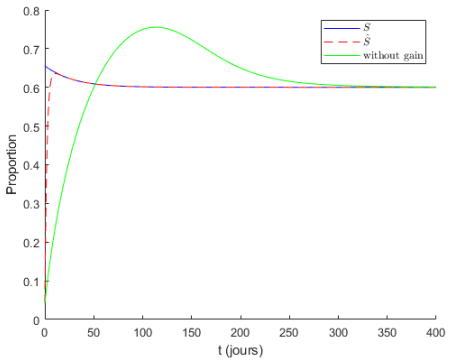


Figure 4.1: Estimation $\hat{S}$ of the susceptible hosts S

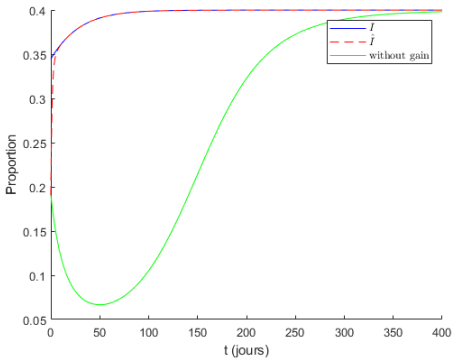


Figure 4.2: Estimation $\hat{I}$ of the infected hosts I

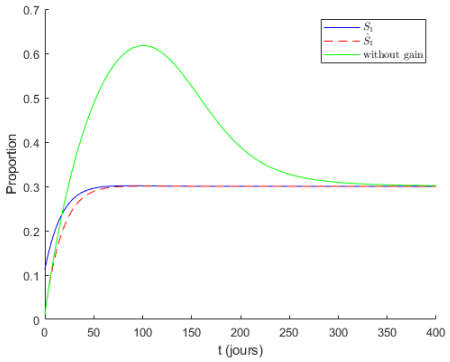


Figure 4.3: Estimation $\hat{S}_1$ of the never infected hosts S_1

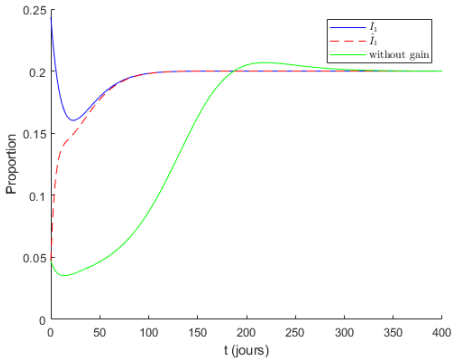


Figure 4.4: Estimation $\hat{I}_1$ of the primary infected hosts I_1

4.4 Adaptive observer

In the present section, we turn our focus to the joint estimation of the state and parameters of the SIS epidemic model counting primary infections (4.1). This was initially done in [44], where we explored the adaptive estimation of the SIS model with β and γ unknown. However, the method exposed requires an estimation of the output derivatives, which may be hardly retrievable. To overcome this issue, we present here an alternative method for designing an adaptive observer independent of output derivatives. In subsection 4.4.1 we transform once again the model into a suitable form for adaptive observer design and introduce some stability notions from the literature, which are used in the analysis. Subsequently, the adaptive observers and the related results are presented in subsections 4.4.2 and 4.4.3 for cases where the entire or a portion of the infected and primary infected are measured, respectively.

4.4.1 Preliminaries

4.4.1.1 Change of coordinates

For the design of the adaptive observer, we introduce the change of variables:

$$z = S + I,\ s_1 = \ln(S_1),\ y = \alpha I,\ y_1 = \alpha I_1.$$

Therefore, (4.1) may be rewritten in the equivalent form:

$$\begin{aligned} \dot{z} &= \mu - \mu z, \\ \dot{s}_1 &= -\mu - \frac{\beta}{\alpha} y + \mu e^{-s_1}, \\ \dot{y} &= \beta y - \frac{\beta}{\alpha} y^2 - y(\mu + \gamma), \\ \dot{y}_1 &= -\mu y_1 - \gamma y_1 + e^{s_1} \beta y, \end{aligned} \tag{4.20}$$

and the states y, y_1 are measured. In the rest of the section, we consider that the mortality μ is a known parameter, which is not a costly assumption as the population's life expectancy is often available. Moreover, as the total normalized population z is constant and equal to 1, the dynamics of the state z can be disregarded.

4.4.1.2 Stability notions

For the stability analysis of non-linear observers, the theory of input-to-state stability (ISS) is a popular framework. In particular, as our objective is to reduce some specific estimation errors of the observer, we will introduce the closely related notions of input-to-output stability (IOS), considering only the components of estimation errors that should be minimized as outputs. We recall in the present subsection some definitions related to IOS stability and its characterization by Lyapunov functions (see [110, 111]).

First, for a Lebesgue measurable function of time $d : \mathbb{K} \to \mathbb{R}$, define the L_∞^m-norm $\|d\|_\infty = \text{ess sup}_{t\in\mathbb{K}}|d(t)|$. We further denote the space of functions d with $\|d\|_\infty < +\infty$ as $L_\infty(\mathbb{K})$. The dimension m is omitted in the notation for simplicity. Consider then the system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{4.21}$$

with state $x(t) \in \mathbb{R}^n$, input $u \in L_\infty(\mathbb{R}_+)$ and output $y(t) \in \mathbb{R}^p$. The functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$,

$h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be continuously differentiable and the system (4.21) is assumed to be forward complete, i.e., for every initial condition $x(0)$ and every input signal u, the corresponding solution is defined for all $t \geqslant 0$. We can now introduce the definition of IOS stability.

Definition 4.4.1. *The system* (4.21) *is called* input-to-output stable *(IOS) if there exist functions $\kappa \in \mathcal{KL}$ and $\lambda \in \mathcal{K}_\infty$ such that*

$$|y(t)| \leqslant \kappa(|x(0)|, t) + \lambda(\|u\|_\infty),$$

for all $t \geqslant 0$, $x(0) \in \mathbb{R}^n$ and $u \in L_\infty(\mathbb{R}_+)$.

Additionally, the following definitions and result provide a characterization of an IOS system by a Lyapunov-like function.

Definition 4.4.2. *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an* IOS-Lyapunov function *of* (4.21) *if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$, such that*

$$\alpha_1(|h(x)|) \leqslant V(x) \leqslant \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n$$

and for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ it holds:

$$V(x) \geqslant \chi(|u|) \implies \nabla V(x) \cdot f(x, u) < -\alpha(V(x), |x|).$$

Definition 4.4.3. *The system* (4.21) *is* uniformly bounded input bounded state stable *(UBIBS) if there exists $\sigma \in \mathcal{K}$ such that the following estimate holds for all $x(0) \in \mathbb{R}^n, u \in L_\infty(\mathbb{R}_+)$ and $t \geqslant 0$:*

$$|x(t)| \leqslant \sigma\left(\max\{|x(0)|, \|u\|_\infty\}\right).$$

Theorem 4.4.1 ([110])**.** *Assume the system* (4.21) *is UBIBS. The system is IOS if and only if it admits an IOS-Lyapunov function.*

4.4.2 Adaptive observation with complete measurements

We investigate in the present subsection the adaptive observation of (4.20) assuming that additionally the parameter α is known, which is equivalent to saying that the total numbers of infections and primary infections are known. Hence, α is assumed to be equal to 1, and the equations of (4.20) are simplified:

$$\begin{aligned} \dot{s}_1 &= -\mu - \beta y + \mu e^{-s_1}, \\ \dot{y} &= \beta y(1 - y) - y(\mu + \gamma), \\ \dot{y}_1 &= -\mu y_1 - \gamma y_1 + e^{s_1}\beta y. \end{aligned} \tag{4.22}$$

As μ is known and y and y_1 are outputs, our goal is to estimate the remaining state s_1 and coefficients β, γ. Our observer design strategy has two steps. First, we design an observer to estimate the unmeasured state s_1 based on auxiliary dummy identification of the uncertain parameters in paragraphs 4.4.2.1. The stability of the observer is analyzed 4.4.2.2. Second, assuming that s_1 is properly reconstructed, the estimates of the parameters are derived in paragraph 4.4.2.3.

4.4.2.1 Observer for s_1

We define the estimates $\hat{s}_1$, $\hat{\beta}$, $\hat{\gamma}$ of s_1, β, γ as follows:

$$\hat{s}_1 = \xi_{\hat{s}_1} + k_{\hat{s}_1} y_1,\ \hat{\beta} = \xi_{\hat{\beta}} + k_{\hat{\beta}} \ln(y),\ \hat{\gamma} = \xi_{\hat{\gamma}} - k_{\hat{\gamma}} \ln(y), \tag{4.23}$$

where $k_{\hat{s}_1}$, $k_{\hat{\beta}}$, $k_{\hat{\gamma}} > 0$ are tuning parameters and $\xi_{\hat{s}_1}$, $\xi_{\hat{\beta}}$, $\xi_{\hat{\gamma}}$ are auxiliary variables whose adaptive laws will be defined below in (4.24). The idea of mixing the states of the observer and the observed system (5) in the parameter estimate is similar to the Immersion & Invariance approach [9]. To clarify the method, we will first introduce the approximations $\hat{y}$, $\hat{y}_1$, which imitate the dynamics of y, y_1:

$$\begin{aligned}\dot{\hat{y}} &= \hat{\beta} y(1-y) - y(\mu + \hat{\gamma}),\\ \dot{\hat{y}}_1 &= e^{\hat{s}_1}\hat{\beta} y - (\hat{\gamma} + \mu) y_1.\end{aligned}$$

If we consider the errors $\rho_1 := y - \hat{y}$ and $\rho_2 := y_1 - \hat{y}_1$, the dynamics are given by

$$\begin{aligned}\dot{\rho}_1 &= \theta_{\hat{\beta}} y - \theta_{\hat{\beta}} y^2 - \theta_{\hat{\gamma}} y,\\ \dot{\rho}_2 &= -\theta_{\hat{\gamma}} y_1 + (e^{s_1} - e^{\hat{s}_1})\hat{\beta} y + e^{s_1}\theta_{\hat{\beta}} y,\end{aligned}$$

where $\theta_{\hat{\beta}} := \beta - \hat{\beta}$, $\theta_{\hat{\gamma}} := \gamma - \hat{\gamma}$ are parameter estimation errors. Thus, from the derivatives of ρ_1 and ρ_2 we obtain terms depending on the errors $\theta_{\hat{\beta}}$, $\theta_{\hat{\gamma}}$, $e^{s_1} - e^{\hat{s}_1}$ which should be put into the adaptive law of $\hat{s}_1$, $\hat{\beta}$, γ as negative feedbacks. This was implemented in [44] under the assumption that the output derivatives $\dot{y}$, $\dot{y}_1$ are estimated using differentiators and put directly into the adaptive law. However, estimating derivatives may be unreliable in the noisy setting with low sampling of measurements, and we propose instead to construct the estimates in the form (4.23) to remove the necessity of estimating $\dot{y}, \dot{y}_1$. Therefore, if we define the dynamics of $\xi_{\hat{s}_1}$, $\xi_{\hat{\beta}}$, and $\xi_{\hat{\gamma}}$ as:

$$\begin{aligned}\dot{\xi}_{\hat{\beta}} &= -k_{\hat{\beta}}(\hat{\beta}(1-y) - \hat{\gamma} - \mu),\\ \dot{\xi}_{\hat{\gamma}} &= k_{\hat{\gamma}}(\hat{\beta}(1-y) - \hat{\gamma} - \mu),\\ \dot{\xi}_{\hat{s}_1} &= -\mu - \hat{\beta} y + \mu e^{-\hat{s}_1} - k_{\hat{s}_1}(e^{\hat{s}_1}\hat{\beta} y - (\hat{\gamma} + \mu) y_1).\end{aligned} \tag{4.24}$$

the dynamics of the errors $\theta_{\hat{\beta}}$, $\theta_{\hat{\gamma}}$ and $\theta_{\hat{s}_1} := s_1 - \hat{s}_1$ become

$$\dot{\theta}_{\hat{\beta}} = -k_{\hat{\beta}}[(1-y)\theta_{\hat{\beta}} - \theta_{\hat{\gamma}}], \tag{4.25a}$$

$$\dot{\theta}_{\hat{\gamma}} = k_{\hat{\gamma}}[(1-y)\theta_{\hat{\beta}} - \theta_{\hat{\gamma}}], \tag{4.25b}$$

$$\dot{\theta}_{\hat{s}_1} = \mu(e^{-s_1} - e^{-\hat{s}_1}) - y\theta_{\hat{\beta}} - k_{\hat{s}_1}[\hat{\beta} y(e^{s_1} - e^{\hat{s}_1}) + e^{s_1} y\theta_{\hat{\beta}} - y_1\theta_{\hat{\gamma}}]. \tag{4.25c}$$

The stability of the errors is checked in the following paragraph.

4.4.2.2 IOS stability through Lyapunov function

Consider first a weighted parameter error norm:

$$V_1 := |\theta_{\hat{\beta}}| + \frac{k_{\hat{\beta}}}{k_{\hat{\gamma}}}|\theta_{\hat{\gamma}}|. \tag{4.26}$$

We compute the derivative of V_1 along the trajectories of the system (3.17)-(4.24),

$$\dot{V}_1 = -k_{\hat{\beta}}[|\theta_{\hat{\beta}}|(1-y) - \mathrm{sgn}(\theta_{\hat{\gamma}})\theta_{\hat{\beta}}(1-y) + |\theta_{\hat{\gamma}}| - \mathrm{sgn}(\theta_{\hat{\beta}})\theta_{\hat{\gamma}}] \leqslant 0.$$

Therefore, the observer has a non-increasing parameters error norm. Consider also,

$$V_2 := |\theta_{\hat{s}_1}|,$$

then

$$\begin{aligned}\dot{V}_2 =& -\mu|e^{-s_1} - e^{-\hat{s}_1}| - y\mathrm{sgn}(\theta_{\hat{s}_1})\theta_{\hat{\beta}} - k_{\hat{s}_1}\mathrm{sgn}(\theta_{\hat{s}_1})[\hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) + e^{s_1}y\theta_{\hat{\beta}} - y_1\theta_{\hat{\gamma}}] \\ \leqslant& -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| + (1 + k_{\hat{s}_1}e^{s_1})|\theta_{\hat{\beta}}|y + k_{\hat{s}_1}|\theta_{\hat{\gamma}}|y_1 \\ \leqslant& -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| + (1 + k_{\hat{s}_1}e^{s_1})V_1y + k_{\hat{s}_1}k_{\hat{\gamma}}k_{\hat{\beta}}^{-1}V_1y_1.\end{aligned} \tag{4.27}$$

Clearly, under the additional assumption that $\hat{\beta}$ has a positive lower bound, V_2 is an IOS-Lyapunov function from the input $V_1 \leqslant V_1(0)$ (measuring the errors of the initial parametric guess) to the output $\theta_{\hat{s}_1}$. Moreover, we notice that the subsystem (4.25a)-(4.25b) is a cooperative system, i.e., the Jacobian has non-negative off-diagonal entries, it is then component-wise order-preserving, and in particular the errors $\theta_{\hat{\beta}}$, $\theta_{\hat{\gamma}}$ remains non-positive if the initial conditions $\theta_{\hat{\beta}}(0)$, $\theta_{\hat{\gamma}}(0)$ are non-positive, *i.e.*, $\hat{\beta}(0) \geqslant \beta$, $\hat{\gamma}(0) \geqslant \gamma$. In this case, $\hat{\beta} \geqslant \beta > 0$, and the assumption above is satisfied.

Finally, the UBIBS property of (4.22)-(4.25) may be easily derived from the global stability property of the system (3.17) in addition to the fact that V_1 is non-increasing, and V_2 also has a negative derivative outside a region containing 0, which size depends on $V_1(0)$.

Therefore, we proved the following result:

Theorem 4.4.2. *For any $k_{\hat{s}_1}, k_{\hat{\beta}}, k_{\hat{\gamma}} > 0$ and $\hat{\beta}(0) > \beta$, $\hat{\gamma}(0) > \gamma$, the system* (4.22) *coupled with the adaptive observer* (4.23), (4.24) *is input-to-output stable with input $V_1(0)$ (measure of the initial parametric estimation gap) and output $\theta_{\hat{s}_1}$.*

4.4.2.3 Estimating β and γ from the measure of s_1

For the previous observer, the simulations in Section 4.4.4 below show excellent accuracy for estimating s_1 at the disease outbreak. This leads us to construct a simple asymptotic estimator of β, γ assuming that s_1 is an additional accessible quantity (for implementation, s_1 will be replaced by its estimated value $\hat{s}_1$ from (4.23)).

$$\check{\beta} = \xi_{\check{\beta}} - k_{\check{\beta}}s_1, \quad \check{\gamma} = \xi_{\check{\gamma}} - k_{\check{\gamma}}\ln(y), \tag{4.28}$$

where $\xi_{\check{\beta}}$, $\xi_{\check{\gamma}}$ are auxiliary variables whose dynamics are defined by

$$\begin{aligned}\dot{\xi}_{\check{\beta}} &= k_{\check{\beta}}(\mu(e^{-s_1} - 1) - \check{\beta}y), \\ \dot{\xi}_{\check{\gamma}} &= k_{\check{\gamma}}(\check{\beta}(1-y) - \check{\gamma} - \mu).\end{aligned} \tag{4.29}$$

Consider now the errors

$$\theta_{\check{\beta}} = \beta - \check{\beta}, \quad \theta_{\check{\gamma}} = \gamma - \check{\gamma},$$

whose dynamics can be written as follows:

$$\begin{aligned}\dot{\theta}_{\check{\beta}} &= -k_{\check{\beta}} y \theta_{\check{\beta}},\\ \dot{\theta}_{\check{\gamma}} &= -k_{\check{\gamma}}(\theta_{\check{\gamma}} - \theta_{\check{\beta}}(1-y)).\end{aligned} \tag{4.30}$$

Taking now the candidate Lyapunov function

$$V_3 := |\theta_{\check{\beta}}| + b|\theta_{\check{\gamma}}|,$$

we obtain the estimate:

$$\dot{V}_3 \leqslant -(k_{\check{\beta}} - b)|\theta_{\check{\beta}}|y - bk_{\check{\gamma}}|\theta_{\check{\gamma}}|,$$

taking $b < k_{\check{\beta}}$, this proves that the estimator (4.28), (4.29) converges exponentially (the output y is separated from zero approaching the endemic equilibrium).

Theorem 4.4.3. *Let $k_{\check{\beta}}, k_{\check{\gamma}} > 0$. The adaptive observer* (4.28), (4.29) *converges exponentially to β, γ.*

In particular, if we take the imperfect estimation $\hat{s}_1$ instead of s_1, the estimators are IOS-stable with input $\theta_{\hat{s}_1}$ and outputs $\theta_{\check{\beta}}$ and $\theta_{\check{\gamma}}$. Then cascade connection of two IOS systems is again IOS (see [109]).

Corollary 4.4.4. *Let $k_{\hat{s}_1}, k_{\hat{\beta}}, k_{\hat{\gamma}}, k_{\check{\beta}}, k_{\check{\gamma}} > 0$. The cascade connection of systems* (4.22)-(4.23)-(4.24)-(4.28)-(4.29) *is input-to-output stable with input $V_1(0)$ and output $\theta_{\hat{s}_1}, \theta_{\check{\beta}}, \theta_{\check{\gamma}}$.*

4.4.3 Adaptive observation with an unknown measured proportion

In this subsection, we examine the case where an unknown proportion α of infected and primary infected is measured. Denoting $\alpha^{-1} = \zeta$, we rewrite again the system (3.17) excluding the dynamics of z.

$$\begin{aligned}\dot{s}_1 &= -\mu - \beta\zeta y + \mu e^{-s_1},\\ \dot{y} &= \beta y - \beta\zeta y^2 - y(\mu + \gamma),\\ \dot{y}_1 &= -\mu y_1 - \gamma y_1 + e^{s_1}\beta y.\end{aligned} \tag{4.31}$$

The additional unknown parameter α governing the outputs y, y_1 renders the estimation problem more delicate and non-linearly parameterized. For the adaptive observer design, we will adapt the same two-step strategy employed for the case with complete measurements in subsection 4.4.2.

4.4.3.1 Adaptive observer for s_1

As previously, the main objective in this subsection is to derive an adaptive observer that, in particular, delivers an accurate estimation of s_1.

We introduce the estimates $\hat{s}_1, \hat{\gamma}, \hat{\zeta}$ in the form:

$$\hat{s}_1 = \xi_{\hat{s}_1} + k_{\hat{s}_1} y_1, \quad \hat{\gamma} = \xi_{\hat{\gamma}} - k_{\hat{\gamma}} \ln y, \quad \hat{\zeta} = \xi_3 - k_{\hat{\zeta}} \ln y. \tag{4.32}$$

Additionally, in light of the inequality upper bounding $\dot{V}_2$ in (4.27), it would be needed for the proof of stability to have $\hat{\beta} > \underline{\beta}$ for a $\underline{\beta} > 0$. However, we could not recover an observer with similar cooperative property governing the error dynamics due to the additional parameter

to estimate. Thus, following [99], we will introduce a projection in the adaptive law instead, to guarantee that the estimate $\hat{\beta}$ remains inside a specific interval $(\underline{\beta}, \overline{\beta})$ with $\underline{\beta}, \overline{\beta} > 0$ determined beforehand. The observer writes:

$$\begin{aligned}
\dot{\xi}_{\hat{\gamma}} &= k_{\hat{\gamma}}(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y), \\
\dot{\xi}_{\hat{\zeta}} &= k_{\hat{\zeta}}(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y), \\
\dot{\xi}_{\hat{s}_1} &= \mu e^{-\hat{s}_1} - \hat{\beta}\hat{\zeta}y - \mu - k_{\hat{s}_1}(e^{\hat{s}_1}\hat{\beta}y - (\hat{\gamma} + \mu)y_1), \\
\dot{\hat{\beta}} &= -\left(1 - \frac{\varepsilon - (\overline{\beta} - \hat{\beta})}{\varepsilon}\mathbb{1}_{h\leqslant 0}\mathbb{1}_{\hat{\beta}\geqslant\overline{\beta}-\varepsilon} - \frac{\varepsilon - (\hat{\beta} - \underline{\beta})}{\varepsilon}\mathbb{1}_{h\geqslant 0}\mathbb{1}_{\hat{\beta}\leqslant\underline{\beta}+\varepsilon}\right)h, \\
h &= f(\hat{\beta}, \hat{\gamma}, \hat{\zeta}, \hat{s}_1, y, y_1),
\end{aligned} \tag{4.33}$$

where $\varepsilon > 0$, $\mathbb{1}_c$ is the condition c indicator and f is a generic C^1 function. For instance, an example of function f is given in paragraph 4.4.4 for simulation.

Due to the projection algorithm, $\hat{\beta}$ stays inside the admissible interval, and our goal is to estimate s_1. The signal f plays then an auxiliary role. Denoting now the errors

$$\theta_{\hat{s}_1} := s_1 - \hat{s}_1, \quad \theta_1 := \beta - \hat{\beta}, \quad \theta_{\hat{\gamma}} := \gamma - \hat{\gamma}, \quad \theta_{\hat{\zeta}} = \zeta - \hat{\zeta},$$

then,

$$\begin{aligned}
\dot{\theta}_{\hat{\gamma}} &= k_{\hat{\gamma}}(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1 - \theta_{\hat{\gamma}}), \\
\dot{\theta}_{\hat{\zeta}} &= k_{\hat{\zeta}}(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1 - \theta_{\hat{\gamma}}), \\
\dot{\theta}_{\hat{s}_1} &= \mu(e^{-s_1} - e^{-\hat{s}_1}) - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1 - k_{\hat{s}_1}(e^{s_1}y\theta_1 + \hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) - y_1\theta_{\hat{\gamma}}).
\end{aligned} \tag{4.34}$$

4.4.3.2 IOS stability through Lyapunov function

We introduce the quantity $\theta_{\hat{\beta}}^{\max} = \overline{\beta} - \underline{\beta}$ and the candidate Lyapunov function

$$V := |\theta_{\hat{s}_1}| + b|\theta_{\hat{\gamma}}| + c|\theta_{\hat{\zeta}}|.$$

Assume that $\beta \in (\underline{\beta}, \overline{\beta})$. The derivative along trajectories is

$$\begin{aligned}
\dot{V} &= -\mu|e^{-s_1} - e^{-\hat{s}_1}| - \hat{\beta}y\mathrm{sgn}(\theta_{\hat{s}_1})\theta_{\hat{\zeta}} - \zeta y\mathrm{sgn}(\theta_{\hat{s}_1})\theta_1 - k_{\hat{s}_1}\mathrm{sgn}(\theta_{\hat{s}_1})(-y_1\theta_{\hat{\gamma}} + \hat{\beta}y(e^{s_1} - e^{\hat{s}_1}) \\
&\quad + e^{s_1}y\theta_1) - b|\theta_{\hat{\gamma}}| + k_{\hat{\gamma}}\mathrm{sgn}(\theta_{\hat{\gamma}})(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1) - c|\theta_{\hat{\zeta}}| + k_{\hat{\zeta}}\mathrm{sgn}(\theta_{\hat{\zeta}})(\theta_1 - \hat{\beta}y\theta_{\hat{\zeta}} - \zeta y\theta_1) \\
&\leqslant -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| + |\theta_{\hat{\zeta}}|\hat{\beta}y + \zeta y|\theta_1| + k_{\hat{s}_1}|\theta_{\hat{\gamma}}|y_1 + k_{\hat{s}_1}e^{s_1}|\theta_1|y - bk_{\hat{\gamma}}|\theta_{\hat{\gamma}}| \\
&\quad + bk_{\hat{\gamma}}(|\theta_1|(1 - \zeta y) + |\theta_{\hat{\zeta}}|\hat{\beta}y) - ck_{\hat{\zeta}}|\theta_{\hat{\zeta}}|\hat{\beta}y + ck_{\hat{\zeta}}(|\theta_1|(1 - \zeta y) + |\theta_{\hat{\gamma}}|y) \\
&\leqslant -\mu|e^{-s_1} - e^{-\hat{s}_1}| - k_{\hat{s}_1}\hat{\beta}y|e^{s_1} - e^{\hat{s}_1}| - ck_{\hat{\zeta}}|\theta_{\hat{\zeta}}|\hat{\beta}y + [\zeta y + k_{\hat{s}_1}e^{s_1}y + (bk_{\hat{\gamma}} + ck_{\hat{\zeta}})(1 - \zeta y)]|\theta_1| \\
&\quad + (k_{\hat{s}_1}y_1 + ck_{\hat{\zeta}}y)|\theta_{\hat{\gamma}}| + (1 + bk_{\hat{\gamma}})\hat{\beta}y|\theta_{\hat{\zeta}}| \\
&\leqslant -\mu|e^{-s_1} - e^{-\hat{s}_1}| - \hat{\beta}yk_{\hat{s}_1}|e^{s_1} - e^{\hat{s}_1}| - (bk_{\hat{\gamma}} - k_{\hat{s}_1}y_1 - ck_{\hat{\zeta}}y)|\theta_{\hat{\gamma}}| - (ck_{\hat{\zeta}} - 1 - bk_{\hat{\gamma}})|\theta_{\hat{\zeta}}|\hat{\beta}y \\
&\quad + [\zeta y + k_{\hat{s}_1}e^{s_1}y + (bk_{\hat{\gamma}} + ck_{\hat{\zeta}})(1 - \zeta y)]\theta_{\hat{\beta}}^{\max}.
\end{aligned}$$

Thus V is an IOS-Lyapunov function with input $\theta_{\hat{\beta}}^{\max}$ and outputs $\theta_{\hat{s}_1}, \theta_{\hat{\gamma}}, \theta_{\hat{\zeta}}$ if

$$bk_{\hat{\gamma}} > k_{\hat{s}_1} y_1 + ck_{\hat{\zeta}} y, \quad ck_{\hat{\zeta}} > 1 + bk_{\hat{\gamma}},$$

These inequalities are equivalent to

$$\frac{ck_{\hat{\zeta}} - 1}{k_{\hat{\gamma}}} > b > \frac{k_{\hat{s}_1} y_1 + ck_{\hat{\zeta}} y}{k_{\hat{\gamma}}}.$$

In order to guarantee that there exists indeed b such that the previous inequality is fulfilled, it is necessary and sufficient to select c sufficiently large to have

$$ck_{\hat{\zeta}}(1 - y) > 1 + k_{\hat{s}_1} y_1.$$

This is possible because, for any trajectory, there exists $\varepsilon_1 > 0$ such that $1 - y = 1 - \alpha I > \varepsilon_1$ since $\alpha < 1$ and the derivative $\dot{I}$ of the SIS system (4.1a) is strictly negative while I is close to 1. Moreover, the IOS stability is deduced from the IOS-Lyapunov property of V and the fact that $\hat{\beta} \in (\underline{\beta}, \overline{\beta})$. Therefore, the following result is proved:

Theorem 4.4.5. *Let f be a C^1 function, $k_{\hat{s}_1}, k_{\hat{\gamma}}, k_{\hat{\zeta}} > 0$ and $\beta, \hat{\beta}(0) \in (\underline{\beta}, \overline{\beta})$. The system (4.31)-(4.33) is input-to-output stable, with input $\theta_{\hat{\beta}}^{\max}$ and output $\theta_{\hat{s}_1}, \theta_{\hat{\gamma}}, \theta_{\hat{\zeta}}$.*

4.4.3.3 Estimating β, γ and α from the measure of s_1

The simulations provided below in Section 4.4.4 show that the observer (4.32)-(4.33) offers again good accuracy for estimating s_1 at the outbreak of the disease. Therefore, we explore once more if a reliable estimation of the unknown parameters can be recovered assuming the state s_1 is measured (again, for implementation, the estimate of s_1 given in (4.32) should be used). We introduce first the notation $\psi = \frac{\beta}{\alpha}$ and rewrite (4.31) as

$$\begin{aligned} \dot{s}_1 &= -\mu - \psi y + \mu e^{-s_1}, \\ \dot{y} &= \beta y - \psi y^2 - y(\mu + \gamma), \\ \dot{y}_1 &= e^{s_1} \beta y - y_1(\mu + \gamma). \end{aligned}$$

The estimators are chosen in the form

$$\begin{aligned} \check{\gamma} = \xi_{\check{\gamma}} - k_{\check{\gamma}} y_1, \quad \check{\psi} = \xi_{\check{\psi}} - k_{\check{\psi}} s_1, \\ \check{\beta} = \xi_{\check{\beta}} + \operatorname{sgn}(e^{s_1} \frac{y}{y_1} - 1) k_{\check{\beta}} (s_1 - \ln y + \ln y_1), \end{aligned} \tag{4.36}$$

and the auxiliary variables $\xi_{\check{\beta}}, \xi_{\check{\gamma}}, \xi_{\check{\psi}}$ are determined by the adaptive law:

$$\begin{aligned} \dot{\xi}_{\check{\gamma}} &= k_{\check{\gamma}} [e^{s_1} \check{\beta} y - y_1(\mu + \check{\gamma})], \\ \dot{\xi}_{\check{\psi}} &= k_{\check{\psi}} (-\mu + \mu e^{-s_1} - \check{\psi} y), \\ \dot{\xi}_{\check{\beta}} &= \operatorname{sgn}(\frac{e^{s_1} y}{y_1} - 1) k_{\check{\beta}} [\mu(1 - e^{-s_1}) + \check{\beta}(1 - \frac{e^{s_1} y}{y_1})]. \end{aligned} \tag{4.37}$$

Let the errors $\theta_{\check{\beta}}, \theta_{\check{\gamma}}, \theta_{\check{\psi}}$ be

$$\theta_{\check{\beta}} := \beta - \check{\beta}, \quad \theta_{\check{\gamma}} = \gamma - \check{\gamma}, \quad \theta_{\check{\psi}} = \psi - \check{\psi}.$$

The errors follow the differential equations :

$$\begin{aligned}\dot{\theta}_{\check{\beta}} &= -k_{\check{\beta}}|1 - e^{s_1}\frac{y}{y_1}|\theta_{\check{\beta}},\\ \dot{\theta}_{\check{\gamma}} &= -k_{\check{\gamma}}(y_1\theta_{\check{\gamma}} - e^{s_1}y\theta_{\check{\beta}}),\\ \dot{\theta}_{\check{\psi}} &= -k_{\check{\psi}}y\theta_{\check{\psi}}.\end{aligned} \tag{4.38}$$

It is then not difficult to deduce that, under the condition that $1-e^{s_1}\frac{y}{y_1} \neq 0$, the estimation errors converge exponentially to zero. Moreover, $e^{s_1}\frac{y}{y_1}$ converges to $\mathcal{R}_0^{-1} < 1$, hence the asymptotic convergence is guaranteed.

Theorem 4.4.6. *Let $k_{\check{\beta}}, k_{\check{\gamma}}, k_{\check{\psi}} > 0$. The adaptive observer* (4.36)-(4.37) *provides estimates of β, γ, ψ with exponential convergence.*

Similarly to the Corollary 4.4.4, the cascade connection is still IOS in the present case.

Corollary 4.4.7. *Let f be a C^1 function, $k_{\hat{s}_1}, k_{\hat{\gamma}}, k_{\hat{\zeta}}, k_{\check{\beta}}, k_{\check{\gamma}}, k_{\check{\psi}} > 0$ and $\beta, \hat{\beta}(0) \in (\underline{\beta}, \overline{\beta})$. The cascade connection of systems* (4.31)-(4.32)-(4.33)-(4.36)-(4.37) *is input-to-output stable with input $\theta_{\hat{\beta}}^{\max}$ and output $\theta_{\hat{s}_1}, \theta_{\check{\beta}}, \theta_{\check{\gamma}}, \theta_{\check{\psi}}$.*

4.4.4 Numerical simulations

We give in the present subsection some numerical simulations of the observers presented in subsection 4.4.2 and 4.4.3.

4.4.4.1 Tuning of the observers

For the parametrization of the observer (4.23)-(4.24), the inequality regarding $\dot{V}_2$ in (4.27) suggests taking $k_{\hat{s}_1}$ and $k_{\hat{\beta}}$ relatively large and $k_{\hat{\gamma}}$ small in order to improve the practical estimation of $\hat{s}_1$ (notice that usually the positive terms $e^{s_1} = S_1$ and $y_1 = \alpha I_1$ in the inequality converge to relatively low values, in fact to $\frac{\mu}{\beta-\gamma}$ and $\alpha\frac{\beta-\gamma-\mu}{\beta-\gamma}\frac{\mu}{\gamma+\mu}$ due to the effects of reinfections). On the other hand, the gains $k_{\check{\beta}}$, $k_{\check{\gamma}}$ of the asymptotic estimator (4.28)-(4.30) will be set sufficiently high to accelerate the convergence.

For the adaptive observer (4.32)-(4.33), $k_{\hat{s}_1}$ is set again sufficiently great in order to have a reliable estimate of s_1. Moreover, the function f in (4.33) is defined as

$$f(\hat{\beta}, \hat{\gamma}, \hat{\zeta}, \hat{s}_1, y, y_1) = k_1(\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y) + k_2(-(\hat{\gamma}+\mu)y_1 + e^{\hat{s}_1}\hat{\beta}y), \tag{4.39}$$

for some positive gains k_1 and k_2. This is motivated by the fact that we expect $\hat{\beta}$ to fulfill

$$\hat{\beta} - \mu - \hat{\gamma} - \hat{\beta}\hat{\zeta}y = 0, \quad e^{\hat{s}_1}\hat{\beta}y - (\hat{\gamma}+\mu)y_1 = 0,$$

because the true coefficients β, γ satisfy these equalities asymptotically. However, as stated in Theorem 4.4.5, the IOS stability is conserved for any C^1 function f. For example, we may choose instead the function

$$f_1 = k_1\hat{\beta}, \tag{4.40}$$

which yields similar results numerically. The advantage of the function defined in (4.39) over (4.40) is to obtain an adaptive estimate $\hat{\beta}$ which does not converge to boundary values $\underline{\beta}, \overline{\beta}$.

4.4.4.2 Case with α known

We consider in this subsection the simulation of the cascade connection of systems (4.22)-(4.23)-(4.24)-(4.28)-(4.29) with the settings: $\mu^{-1} = 70$ years, $\gamma = 12^{-1}$ days^{-1}, $\beta = 0.2$ days^{-1}, $k_{\hat{s}_1} = k_{\hat{\beta}} = k_{\check{\beta}} = 100$ days^{-1}, $k_{\check{\gamma}} = 10$ days^{-1}, $k_{\hat{\gamma}} = 0.01$ days^{-1}. The result indicates that the observer starts converging to an almost exact value of S_1 during the outbreak (Fig. 4.5) before losing its precision slowly. Nevertheless, this initial estimation is sufficient to obtain accurate estimates $\check{\beta}, \check{\gamma}$ of β, γ (Figs. 4.6 and 4.7). In Fig. 4.6, at the end of the estimation interval, when the system has approached the equilibrium, we can see the augmentation of the estimation error and a kind of unstable behavior, which is related to the loss of the previously mentioned identifiability close to the steady state.

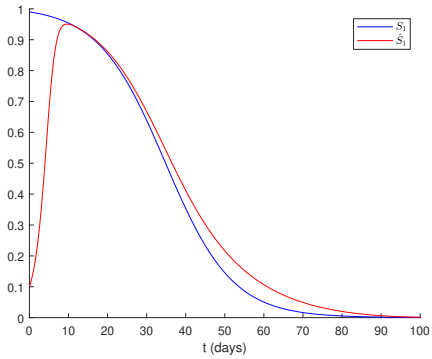


Figure 4.5: Estimation of S_1, α known

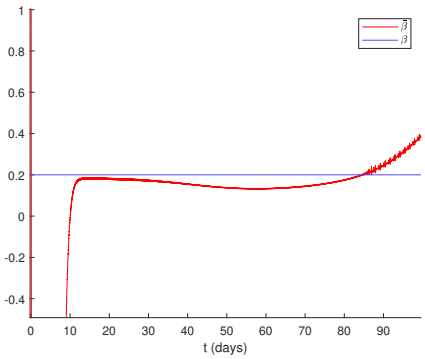


Figure 4.6: Estimation of β, α known

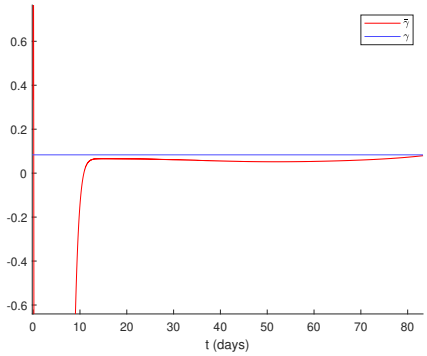


Figure 4.7: Estimation of γ, α known

4.4.4.3 Case with α unknown

The simulation of the cascade connection of systems (4.31)-(4.32)-(4.33)-(4.36)-(4.37) is computed with the same coefficients β, γ, μ as above, additionally we set $\alpha = 0.8$, $k_{\hat{s}_1} = k_{\check{\psi}} = 100$ days^{-1}, $k_{\hat{\gamma}} = k_{\check{\beta}} = k_{\check{\gamma}} = 10$ days^{-1}, $k_{\hat{\zeta}} = k_1 = k_2 = 1$ days^{-1} and $(\underline{\beta}, \overline{\beta}) = (0.01, 2)$ days^{-1}. For better estimation of s_1, fixing $k_{\hat{s}_1}$ and $k_{\hat{\gamma}}$ sufficiently large is necessary. Moreover, $k_{\check{\beta}}$ can take theoretically arbitrarily large values. However, in this case, the estimation would have significant noise. In the present settings, Fig. 4.8 shows that, even for a large confidence interval of β, the

observer for unknown α exhibits the same behavior as above, delivering an accurate estimate of s_1 at the outbreak of the epidemics, however at a slower pace. This estimate is then sufficient for deriving a rather precise estimation of the remaining parameters (figures 4.9-4.11).

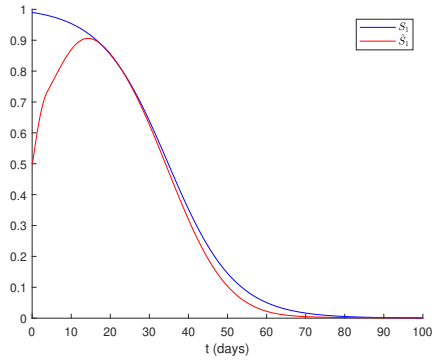


Figure 4.8: Estimation of S_1, α unknown

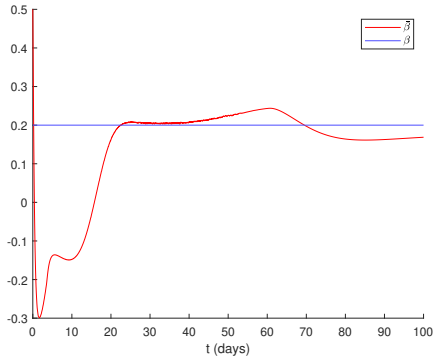


Figure 4.9: Estimation of β, α unknown

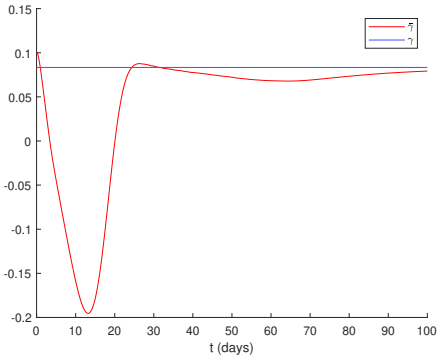


Figure 4.10: Estimation of γ, α unknown

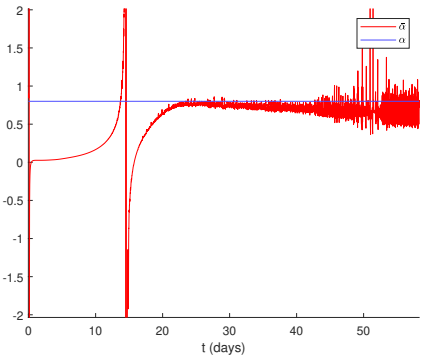


Figure 4.11: Estimation of α, α unknown

Bibliography

[1] L. J. Abu-Raddad and N. M. Ferguson. "Characterizing the symmetric equilibrium of multi-strain host-pathogen systems in the presence of cross immunity." In: *Journal of Mathematical Biology* 50 (5 May 2005).

[2] A. Alanwar, M. U. B. Niazi, and K. H. Johansson. "Data-driven Set-based Estimation of Polynomial Systems with Application to SIR Epidemics". In: *2022 European Control Conference (ECC)*. 2022, pp. 888–893.

[3] R. Anderson and R. May. *Infectious Diseases of Humans: Dynamics and Control*. Infectious Diseases of Humans: Dynamics and Control. OUP Oxford, 1991.

[4] V. Andreasen, J. Lin, and S. A. Levin. "The dynamics of cocirculating influenza strains conferring partial cross-immunity". In: *Journal of Mathematical Biology* 35 (7 Aug. 1997).

[5] D. Angeli. "A Tutorial on Chemical Reaction Network Dynamics". In: *European Journal of Control* 15.3 (2009), pp. 398–406.

[6] D. Angeli and D. Efimov. "Characterizations of Input-to-State Stability for Systems With Multiple Invariant Sets". In: *IEEE Transactions on Automatic Control* 60.12 (2015), pp. 3242–3256.

[7] R. Anguelov, S. M. Garba, and S. Usaini. "Backward bifurcation analysis of epidemiological model with partial immunity". In: *Computers & Mathematics with Applications* 68.9 (2014). BIOMATH 2013, pp. 931–940.

[8] J. Arino, C. C. McCluskey, and P. van den Driessche. "Global Results for an Epidemic Model with Vaccination that Exhibits Backward Bifurcation". In: *SIAM Journal on Applied Mathematics* 64 (1 Jan. 2003).

[9] A. Astolfi and R. Ortega. "Immersion and invariance: a new tool for stabilization and adaptive control of nonlinear systems". In: *IEEE Transactions on Automatic Control* 48.4 (2003), pp. 590–606.

[10] J. M. Ball, J. Carr, and O. Penrose. "The Becker-Döring cluster equations: Basic properties and asymptotic behaviour of solutions". In: *Communications in Mathematical Physics* 104 (4 Dec. 1986).

[11] M. M. Ballyk, C. C. McCluskey, and G. S. K. Wolkowicz. "Global analysis of competition for perfectly substitutable resources with linear response". In: *Journal of Mathematical Biology* 51.4 (2005), pp. 458–490.

[12] E. A. Barbashin. "On construction of Lyapunov functions for nonlinear systems". In: *Proc. 1st IFAC World Congress*. Moscow, 1961, pp. 742–751.

[13] O. Benedictow. *The Black Death, 1346-1353: The Complete History*. Boydell Press, 2004.

[14] D. BERNOULLI. “Essai d’une nouvelle analyse de la mortalite causee par la petite verole, et des avantages de l’inoculation pour la prevenir”. In: *Histoire de l’Acad., Roy. Sci.(Paris) avec Mem* (1760), pp. 1–45.

[15] O. N. Bjørnstad, K. Shea, M. Krzywinski, and N. Altman. “The SEIRS model for infectious disease dynamics”. In: *Nature Methods* 17 (6 June 2020).

[16] P.-A. Bliman and B. D’Avila Barros. “Interval Observers for SIR Epidemic Models Subject to Uncertain Seasonality”. In: *Positive Systems : Theory and Applications (POSTA 2016) Rome, Italy, September 14-16, 2016*. Springer International Publishing, 2017, pp. 31–39.

[17] P.-A. Bliman, D. Efimov, and R. Ushirobira. “A class of nonlinear adaptive observers for SIR epidemic model”. In: *2018 European Control Conference (ECC)*. 2018, pp. 1–6.

[18] O. Boldea, A. Alipoor, S. Pei, J. Shaman, and G. Rozhnova. “Age-specific transmission dynamics of SARS-CoV-2 during the first 2 years of the pandemic”. In: *PNAS Nexus* 3.2 (Jan. 2024).

[19] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*. SIAM, 1994.

[20] R. Breban and S. Blower. “The reinfection threshold does not exist”. In: *Journal of Theoretical Biology* 235.2 (2005), pp. 151–152.

[21] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Ed. by S. V. Gmbh. 2010.

[22] G. Butler and P. Waltman. “Persistence in dynamical systems”. In: *Journal of Differential Equations* 63.2 (1986), pp. 255–263.

[23] Y. Cheng and X. Yang. “On the global stability of SEIRS models in epidemiology”. In: *The Canadian applied mathematics quarterly* 20 (2 2012), pp. 115–133.

[24] H.-D. Chiang, M. Hirsch, and F. Wu. “Stability regions of nonlinear autonomous dynamical systems”. In: *IEEE Transactions on Automatic Control* 33.1 (1988), pp. 16–27.

[25] C. Cohen and J. Pulliam. “COVID-19 infection, reinfection, and the transition to endemicity”. In: *The Lancet* 401.10379 (2023), pp. 798–800.

[26] W. Coppel. *Stability and Asymptotic Behavior of Differential Equations*. Heath mathematical monographs. Heath, 1965.

[27] N. Cunniffe, F. Hamelin, A. Iggidr, A. Rapaport, and G. Sallet. *Identifiability and Observability in Epidemiological Models: A Primer*. SpringerBriefs on PDEs and Data Science. Springer Nature Singapore, 2024.

[28] N. G. Davies et al. “Age-dependent effects in the transmission and control of COVID-19 epidemics”. In: *Nature Medicine* (2020), pp. 1–7.

[29] R. Descartes. *The Geometry of Rene Descartes: With a facsimile of the first edition*. Courier Corporation, 2012.

[30] W. Desch, W. Schappacher, and K. P. Zhang. *Semilinear evolution equations*. Tech. rep. International Centre for Theoretical Physics, 1985.

[31] W. Desch, W. Schappacher, and K. P. Zhang. “Semilinear evolution equations”. In: *Houston Journal of Mathematics* 15.4 (1989), pp. 527–552.

[32] O. Diekmann, H. J. Heesterbeek, and J. A. Metz. “On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations”. In: *Journal of Mathematical Biology* 28 (4 June 1990).

[33] O. Diekmann and J. Heesterbeek. “Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation”. In: *Wiley Series in Mathematical and Computational Biology, Chichester, Wiley* (Jan. 2000).

[34] M. Doumic, K. Fellner, M. Mezache, and H. Rezaei. “A bi-monomeric, nonlinear Becker-Döring-type system to capture oscillatory aggregation kinetics in prion dynamics”. In: *Journal of Theoretical Biology* 480 (Nov. 2019).

[35] P. van den Driessche and J. Watmough. “Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission”. In: *Mathematical Biosciences* 180 (1-2 Nov. 2002).

[36] R. P. Duncan-Jones. “The impact of the Antonine plague”. In: *Journal of Roman Archaeology* 9 (1996), pp. 108–136.

[37] D. Efimov. “Global Lyapunov Analysis of Multistable Nonlinear Systems”. In: *SIAM Journal on Control and Optimization* 50.5 (2012), pp. 3132–3154.

[38] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, 2000.

[39] M. Fan, M. Y. Li, and K. Wang. “Global stability of an SEIS epidemic model with recruitment and a varying total population size”. In: *Mathematical Biosciences* 170.2 (2001), pp. 199–208.

[40] M. Fang and P.-A. Bliman. “Age-structured epidemiological model counting reinfections: Unraveling the hidden age and reinfection structure of reinfection models”. In preparation. 2024.

[41] M. Fang and P.-A. Bliman. “Modelling, Analysis, Observability and Identifiability of Epidemic Dynamics with Reinfections”. In: *2022 European Control Conference (ECC)*. 2022, pp. 278–283.

[42] M. Fang and P.-A. Bliman. “Multistability, persistence and global analysis of epidemiological models with secondary infections”. Submitted for peer-review. 2024.

[43] M. Fang, P.-A. Bliman, D. Efimov, and R. Ushirobira. “A class of nonlinear state observers for an SIS system counting primo-infections”. In: *61st IEEE Conference on Decision and Control*. Cancún, Mexico, Dec. 2022.

[44] M. Fang, P.-A. Bliman, D. Efimov, and R. Ushirobira. “Nonlinear Adaptive Observers for an SIS System Counting Primo-infections”. In: *IFAC-PapersOnLine* 56.2 (2023). 22nd IFAC World Congress, pp. 9727–9732.

[45] M. Fang, P.-A. Bliman, D. Efimov, and R. Ushirobira. “Nonlinear adaptive observers for an SIS system with measurements of primary infections”. Submitted for peer-review. 2024.

[46] L. Farina and S. Rinaldi. *Positive linear systems: theory and applications*. Ed. by J. W. bibinitperiod Sons. Vol. Vol. 50. 2000.

[47] F. Forni and R. Sepulchre. “Differential Dissipativity Theory for Dominance Analysis”. In: *IEEE Transactions on Automatic Control* 64.6 (2019), pp. 2340–2351.

[48] P. Forni and D. Angeli. “Input-to-state stability for cascade systems with multiple invariant sets”. In: *Systems & Control Letters* 98 (2016), pp. 97–110.

[49] H. I. Freedman, S. Ruan, and M. Tang. “Uniform persistence and flows near a closed positively invariant set”. In: *Journal of Dynamics and Differential Equations* 6 (1994), pp. 583–600.

[50] M. C. Gomes, A. Margheri, and C. Rebelo. “Stability and persistence in a compartment model of pulmonary tuberculosis”. In: *Nonlinear Analysis: Theory, Methods & Applications* 48.4 (2002), pp. 617–636.

[51] M. G. M. Gomes, L. J. White, and G. F. Medley. “Infection, reinfection, and vaccination under suboptimal immune protection: epidemiological perspectives”. In: *Journal of Theoretical Biology* 228 (4 June 2004).

[52] J. A. Gondim and L. Machado. “Optimal quarantine strategies for the COVID-19 pandemic in a population with a discrete age structure”. In: *Chaos, Solitons & Fractals* 140 (2020), p. 110166.

[53] D. Greenhalgh. “Hopf bifurcation in epidemic models with a latent period and nonpermanent immunity”. In: *Mathematical and Computer Modelling* 25 (2 Jan. 1997).

[54] D. Greenhalgh, O. Diekmann, and M. C. de Jong. “Subcritical endemic steady states in mathematical models for animal infections with incomplete immunity”. In: *Mathematical Biosciences* 165.1 (2000), pp. 1–25.

[55] J. Hale. “Asymptotic behavior of dissipative system”. In: *Mathematical Surveys and Monographs* 25 (1988).

[56] F. Hamelin, A. Iggidr, A. Rapaport, G. Sallet, and M. Souza. “About the identifiability and observability of the SIR epidemic model with quarantine”. In: *IFAC-PapersOnLine* 56.2 (2023). 22nd IFAC World Congress, pp. 4025–4030.

[57] J. Heesterbeek. “The law of mass-action in epidemiology: A historical perspective”. In: *Ecological Paradigms Lost* (Jan. 2005).

[58] H. W. Hethcote. “Three Basic Epidemiological Models”. In: *Applied Mathematical Ecology*. Ed. by S. A. Levin, T. G. Hallam, and L. J. Gross. Berlin, Heidelberg: Springer Berlin Heidelberg, 1989, pp. 119–144.

[59] M. Iannelli and F. Milner. *The Basic Approach to Age-Structured Population Dynamics: Models, Methods and Numerics*. Lecture Notes on Mathematical Modelling in the Life Sciences. Springer Netherlands, 2017.

[60] H. Inaba. *Age-structured population dynamics in demography and epidemiology*. Springer, 2017.

[61] R. Irving. *Integers, Polynomials, and Rings: A Course in Algebra*. Undergraduate Texts in Mathematics. Springer New York, 2003.

[62] J. A. Jacquez and P. Greif. “Numerical parameter identifiability and estimability: Integrating identifiability, estimability, and optimal sampling design”. In: *Mathematical Biosciences* 77.1-2 (1985), pp. 201–227.

[63] P. Kaklamanos, A. Pugliese, M. Sensi, and S. Sottile. “A Geometric Analysis of the SIRS Model with Secondary Infections”. In: *SIAM Journal on Applied Mathematics* 84.2 (2024), pp. 661–686.

[64] E. Kaszkurewicz and A. Bhaya. *Matrix diagonal stability in systems and computation*. Springer Science & Business Media, 2012.

[65] E. Kaszkurewicz and L. Hsu. “Stability of nonlinear systems: a structural approach”. In: *Automatica* 15.5 (1979), pp. 609–614.

[66] G. Katriel. “Epidemics with partial immunity to reinfection”. In: *Mathematical Biosciences* 228 (2 Dec. 2010).

[67] M. J. Keeling and P. Rohani. *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press, 2008.

[68] W. O. Kermack and A. G. McKendrick. "A contribution to the mathematical theory of epidemics". In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 115 (772 Aug. 1927).

[69] A. Korobeinikov and G. Wake. "Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models". In: *Applied Mathematics Letters* 15.8 (2002), pp. 955–960.

[70] D. Kousoulidis and F. Forni. "An Optimization Approach to Verifying and Synthesizing K-cooperative Systems". In: *IFAC-PapersOnLine* 53.2 (2020). 21st IFAC World Congress, pp. 4635–4642.

[71] P. Laurençot and S. Mischler. "From the Becker-Döring to the Lifshitz-Slyozov-Wagner Equations". In: *Journal of Statistical Physics* 106 (5/6 2002).

[72] M. Li and J. Muldowney. "On Bendixson's Criterion". In: *Journal of Differential Equations* 106.1 (1993), pp. 27–39.

[73] M. Li, J. Muldowney, and P. Driessche. "Global stability of SEIRS models in epidemiology". In: *The Canadian Applied Mathematics Quarterly* 7 (Jan. 1999).

[74] M. Y. Li, J. R. Graef, L. Wang, and J. Karsai. "Global dynamics of a SEIR model with varying total population size". In: *Mathematical Biosciences* 160.2 (1999), pp. 191–213.

[75] M. Y. Li and J. S. Muldowney. "A Geometric Approach to Global-Stability Problems". In: *SIAM Journal on Mathematical Analysis* 27 (4 July 1996).

[76] M. Y. Li and J. S. Muldowney. "Dynamics of Differential Equations on Invariant Manifolds". In: *Journal of Differential Equations* 168.2 (2000), pp. 295–320.

[77] M. Y. Li and J. S. Muldowney. "On R.A. Smith's Autonomous Convergence Theorem". In: *Rocky Mountain Journal of Mathematics* 25.1 (1995), pp. 365–378.

[78] W. Lohmiller and J.-J. E. Slotine. "On Contraction Analysis for Non-linear Systems". In: *Automatica* 34.6 (1998), pp. 683–696.

[79] G. Lu and Z. Lu. "Geometric approach to global asymptotic stability for the SEIRS models in epidemiology". In: *Nonlinear Analysis: Real World Applications* 36 (2017), pp. 20–43.

[80] G. Lu and Z. Lu. "Global asymptotic stability for the SEIRS models with varying total population size". In: *Mathematical Biosciences* 296 (Feb. 2018).

[81] P. Manfredi and J. R. Williams. "Realistic population dynamics in epidemiological models: the impact of population decline on the dynamics of childhood infectious diseases: Measles in Italy as an example". In: *Mathematical Biosciences* 192.2 (2004), pp. 153–175.

[82] A. Margheri and C. Rebelo. "Some examples of persistence in epidemiological models". In: *Journal of mathematical biology* 46.6 (2003), p. 564.

[83] M. Martcheva. *An Introduction to Mathematical Epidemiology*. Texts in Applied Mathematics. Springer US, 2015.

[84] R. H. Martin. "Logarithmic norms and projections applied to linear differential systems". In: *Journal of Mathematical Analysis and Applications* 45.2 (1974), pp. 432–454.

[85] O. Mason and R. Shorten. "On linear copositive Lyapunov functions and the stability of switched positive linear systems". In: *IEEE Transactions on Automatic Control* 52.7 (2007), pp. 1346–1349.

[86] C. C. McCluskey. “A strategy for constructing Lyapunov functions for non-autonomous linear differential equations”. In: *Linear Algebra and its Applications* 409 (2005). Special Issue in honor of Pauline van den Driessche, pp. 100–110.

[87] W. Mei, D. Efimov, and R. Ushirobira. “On input-to-output stability and robust synchronization of generalized Persidskii systems”. In: *IEEE Transactions on Automatic Control* (2021).

[88] W. Mei, D. Efimov, R. Ushirobira, and A. Aleksandrov. “On convergence conditions for generalized Persidskii systems”. In: *International Journal of Robust and Nonlinear Control* (2021).

[89] A. A. Mensah et al. “Disease severity during SARS-COV-2 reinfection: a nationwide study”. In: *Journal of Infection* 84.4 (2022), pp. 542–550.

[90] J. S. Muldowney. “Compound matrices and applications”. In: *Universidad de los Andes, Bogotá, Colombia* (1998).

[91] J. S. Muldowney. “Compound matrices and ordinary differential equations”. In: *The Rocky Mountain Journal of Mathematics* (1990), pp. 857–872.

[92] M. U. B. Niazi and K. H. Johansson. “Observer Design for the State Estimation of Epidemic Processes”. In: *2022 IEEE 61st Conference on Decision and Control (CDC)* (2022), pp. 4325–4332.

[93] W. H. Organization. *Global tuberculosis report 2023*. World Health Organization, 2023.

[94] R. Pagliara, B. Dey, and N. E. Leonard. “Bistability and Resurgent Epidemics in Reinfection Models”. In: *IEEE Control Systems Letters* 2 (2018), pp. 290–295.

[95] L. Pasteur. “De l’extension de la théorie des germes à l’étiologie de quelques maladies communes”. In: *Recueil de Médecine Vétérinaire* 57.1 (1880), pp. 642–654.

[96] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer New York, 1983.

[97] S. K. Persidskii. “Concerning problem of absolute stability”. In: *Automation and Remote Control* (1969), pp. 5–11.

[98] M. E. Peters. “Government finance and imposition of serfdom after the Black Death”. In: *European Review of Economic History* 27.2 (Sept. 2022), pp. 149–173.

[99] J.-B. Pomet and L. Praly. “Adaptive nonlinear regulation: estimation from the Lyapunov equation”. In: *IEEE Transactions on Automatic Control* 37.6 (1992), pp. 729–740.

[100] G. D. Prato. *Applications croissantes et équations d’evolutions dans les espaces de Banach*. Institutiones mathematicae. New York, 1976.

[101] C. C. Pugh. “The Closing Lemma”. In: *American Journal of Mathematics* 89.4 (1967), pp. 956–1009.

[102] A. Pugliese and L. Tonetto. “Well-posedness of an infinite system of partial differential equations modelling parasitic infection in an age-structured host”. In: *Journal of Mathematical Analysis and Applications* 284.1 (2003), pp. 144–164.

[103] G. Rosen. *A History of Public Health*. MD monographs on medical history. MD Publications, 1958.

[104] W. Rosen. *Justinian’s flea: The First Great Plague and the end of the Roman Empire*. Penguin, 2007.

[105] W. Rudin. *Principles of mathematical analysis*. Vol. 3. McGraw-hill New York, 1964.

[106] L. Simonsen, M. J. Clarke, L. B. Schonberger, N. H. Arden, N. J. Cox, and K. Fukuda. "Pandemic versus Epidemic Influenza Mortality: A Pattern of Changing Age Distribution". In: *The Journal of Infectious Diseases* 178.1 (July 1998), pp. 53–60.

[107] H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society. Vol. 41. Mathematical Surveys And Monographs. 1995.

[108] H. L. Smith and H. R. Thieme. *Dynamical Systems and Population Persistence*. Graduate studies in mathematics. American Mathematical Society, 2011.

[109] E. D. Sontag. "Input to State Stability: Basic Concepts and Results". In: *Nonlinear and Optimal Control Theory: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy June 19–29, 2004*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008, pp. 163–220.

[110] E. D. Sontag and Y. Wang. "Lyapunov Characterizations of Input to Output Stability". In: *SIAM Journal on Control and Optimization* 39.1 (2000), pp. 226–249.

[111] E. D. Sontag and Y. Wang. "Notions of input to output stability". In: *Systems & Control Letters* 38.4 (1999), pp. 235–248.

[112] H. R. Thieme. "Uniform persistence and permanence for non-autonomous semiflows in population biology". In: *Mathematical Biosciences* 166.2 (2000), pp. 173–201.

[113] R. Ushirobira, D. Efimov, and P.-A. Bliman. "Estimating the infection rate of a SIR epidemic model via differential elimination". In: *2019 18th European Control Conference (ECC)*. 2019, pp. 1170–1175.

[114] O. Watson, G. Barnsley, J. Turner, A. Hogan, P. Winskill, and A. Ghani. "Global impact of the first year of COVID-19 vaccination: a mathematical modelling study". In: *The Lancet Infectious Diseases* 22 (June 2022).

[115] G. F. Webb. "A semigroup proof of the Sharpe-Lotka theorem". In: *Infinite-Dimensional Systems*. Ed. by F. Kappel and W. Schappacher. Berlin, Heidelberg: Springer Berlin Heidelberg, 1984, pp. 254–268.

[116] World Health Organization. *The top 10 causes of death*. `https://www.who.int/news-room/fact-sheets/detail/the-top-10-causes-of-death`. Accessed: 2024-09-03. 2024.

[117] C. Wu, I. Kanevskiy, and M. Margaliot. "k-contraction: Theory and applications". In: *Automatica* 136 (2022), p. 110048.

[118] V. A. Yakubovich. "Method of matrix unequalities in theory of nonlinear control systems stability. I. Forced oscillations absolute stability". In: *Avtomat. i Telemekh.* 25 (1964), pp. 1017–1029.

[119] L. Zhou, Y. Wang, Y. Xiao, and M. Y. Li. "Global dynamics of a discrete age-structured SIR epidemic model with applications to measles vaccination strategies". In: *Mathematical Biosciences* 308 (2019), pp. 27–37.

[120] S. Zoboli, A. Cecilia, U. Serres, D. Astolfi, and V. Andrieu. "LMI Conditions for $k-$ contraction Analysis: A Step Towards Design". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*. 2023, pp. 1903–1910.

Mathematical modelling, observation and identification of epidemiological models with reinfection

Abstract

The subject of this work is the mathematical modelling of infectious diseases with reinfections and the analysis of the corresponding models. First, we introduce a general class of compartmental models counting the number of reinfections, consisting of an infinite number of ordinary differential equations, and study its well-posedness. The proposed class also allows for the modelling of heterogeneous transmission, whose characteristics depend upon the number of past reinfections. In the homogeneous situation (where the past reinfections do not affect disease transmission), the global behavior obeys a usual compartmental model. Asymptotic results are established, and formulas giving the mean number of reinfections at the endemic equilibrium are provided. Next, we investigate a two-stage reinfection model intended for the modelling of diseases for which the subsequent reinfections behave differently from the primary infection. We describe thoroughly the steady states of the model, which may contain up to three endemic equilibriums, and study the disease persistence. Employing Li and Muldowney theory, we prove asymptotic convergence of every trajectory, in a particular case that may exhibit multiple endemic equilibriums. Using semigroup theory, we then establish the well-posedness for a class of models structured both in age and number of reinfections. The latter are constituted by an infinite number of partial differential equations. This allows the computation of several interesting quantities at endemic equilibrium, such as the average age in each compartment or the mean number of reinfections at each age. Finally, we investigate some issues of Control theory, more precisely whether the use of additional reinfection data may improve parameter and state estimation. To this end, we study identifiability and observability of a SIS model, based on the measure of the number of infected and of primary infected, and propose an asymptotic observer and an adaptive observer respectively for the state, and for the joint state and parameter, estimation.

Keywords: compartmental models in epidemiology; reinfection; age structure; multistability; observation; identification; asymptotic observer; adaptative observer

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Résumé

Le sujet de ce travail est la modélisation mathématique des maladies infectieuses avec réinfections et l'analyse des modèles correspondants. Nous introduisons d'abord une classe générale de modèles compartimentaux comptant les réinfections, constitués d'un nombre infini d'équations différentielles ordinaires, et nous étudions leur caractère bien posé. La classe proposée permet également de modéliser une transmission hétérogène, dont les caractéristiques dépendent du nombre de réinfections antérieures. Dans le cas homogène (où les réinfections passées n'affectent pas la transmission de la maladie), le comportement global obéit à un modèle compartimental usuel. Des résultats asymptotiques sont établis et des formules donnant le nombre moyen de réinfections à l'équilibre endémique sont données. Nous étudions ensuite un modèle de réinfection à deux étages destiné à la modélisation de maladies pour lesquelles les réinfections ultérieures se comportent différemment de l'infection primaire. Nous décrivons en détail les équilibres du modèle, qui peut contenir jusqu'à trois équilibres endémiques, et étudions la persistance de la maladie. Grâce à la théorie de Li et Muldowney, nous prouvons la convergence asymptotique de chaque trajectoire dans un cas particulier qui peut présenter plusieurs équilibres endémiques. En utilisant la théorie des semigroupes, nous établissons ensuite le caractère bien-posé d'une classe de modèles structurés à la fois en âge et en nombre de réinfections. Ces derniers sont constitués d'un nombre infini d'équations aux dérivées partielles. Ceci permet de calculer plusieurs quantités intéressantes à l'équilibre endémique, telles que l'âge moyen dans chaque compartiment ou le nombre moyen de réinfections à chaque âge. Nous examinons enfin des questions de théorie du contrôle, plus précisément si l'utilisation de données supplémentaires sur les réinfections peut améliorer l'estimation des paramètres et de l'état. À cette fin, nous étudions l'identifiabilité et l'observabilité d'un modèle SIS, basées sur la mesure du nombre d'infectés et de primo-infectés, et proposons un observateur asymptotique et un observateur adaptatif respectivement pour l'estimation de l'état, et pour l'estimation conjointe de l'état et des paramètres.

Mots clés : modèles compartimentaux en épidémiologie ; réinfection ; structuration en âge ; multistabilité ; observation ; identification ; observateur asymptotique ; observateur adaptatif

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