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# Topics in string theory and phenomenology: dark sectors, moduli stabilisation, inflation

Anthony Guillen

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Anthony Guillen. Topics in string theory and phenomenology: dark sectors, moduli stabilisation, inflation. High Energy Physics - Phenomenology [hep-ph]. Sorbonne Université, 2024. English. NNT : 2024SORUS330 . tel-04852006

**HAL Id: tel-04852006**

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# Thèse de Doctorat

## DE SORBONNE UNIVERSITE

réalisée au

Laboratoire de Physique Théorique et Hautes Énergies

Sujet de la thèse:

Topics in String Theory and Phenomenology:  
Dark Sectors, Moduli Stabilisation, Inflation

Présentée par:

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Présentée et soutenue publiquement à Paris  
le 16 septembre 2024

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To Ayman

# Résumé

Cette thèse couvre plusieurs sujets à l'interface entre la théorie des cordes et la cosmologie. Elle est organisée en quatre parties. La première partie porte sur l'énergie sombre, plus précisément sur l'analyse, par des techniques de systèmes dynamiques, d'un modèle de quintessence dans la formulation dite de Palatini de la Relativité Générale. Ces techniques sont auparavant passées en revue, au même titre que les bases de la cosmologie moderne. La deuxième partie s'ouvre sur une introduction à la supergravité  $\mathcal{N} = 1$  à quatre dimensions, dans le formalisme superconforme, puis se poursuit par le calcul des amplitudes de diffusion des gravitinos massifs. Ce calcul permet d'exhiber un mécanisme de restauration d'unitarité perturbative, qui a son analogue dans le secteur électrofaible du Modèle Standard. Nous analysons ensuite l'effet sur ce mécanisme des nouveaux termes de Fayet-Iliopoulos. Dans la troisième partie, nous commençons par introduire la théorie des cordes. Cette introduction part des bases de la théorie bosonique, puis présente brièvement la construction de la théorie des supercordes, avant de se concentrer sur la compactification et la stabilisation des modules dans la théorie de type IIB. Le chapitre suivant porte plus spécifiquement sur la stabilisation, par des flux, des modules de structure complexe et du dilaton dans la théorie de type IIB compactifiée sur des orbifolds toroidaux. Dans ce cadre, nous obtenons une relation entre la constante de couplage des cordes et le nombre de flux. Combinée avec la contrainte de tadpole, cette relation empêche la constante de couplage d'être paramétriquement petite. Ce résultat peut être vu comme une conséquence de la finitude du nombre de vides dans ce secteur de la théorie, que nous observons également dans notre analyse. Enfin, la quatrième et dernière partie de cette thèse concerne la Dark Dimension, une dimension supplémentaire prédite à l'échelle du micromètre. Nous commençons par introduire le programme Swampland de la théorie des cordes et les arguments soutenant cette prédiction, avant de nous tourner vers le calcul des spectres des fluctuations primordiales dans un modèle d'inflation à cinq dimensions qui permet d'expliquer l'échelle micrométrique de la Dark Dimension. Ces spectres présentent un changement de comportement à grande échelle, ce qui permet en principe de les distinguer de leurs analogues dans des modèles à quatre dimensions.



# Abstract

This thesis covers several topics, at the interface between string theory and cosmology. It is organized in four parts. The first part is about dark energy. More precisely, it deals with the analysis, using dynamical systems techniques, of a quintessence model in the so-called Palatini formulation of General Relativity. These techniques are reviewed first, along with the basics of modern cosmology. The second part opens with an introduction to  $\mathcal{N} = 1$  supergravity in four dimensions, in the superconformal formalism. It is followed by a calculation of massive gravitino scattering amplitudes within this theory. This calculation reveals a perturbative unitarity restoration mechanism, which has its analogue in the electroweak sector of the Standard Model. We then analyze the effect of the new Fayet-Iliopoulos terms on this mechanism. In the third part, we begin by introducing string theory. This introduction starts from the basics of the bosonic theory, then briefly presents the construction of the superstring, before focusing on compactification and moduli stabilisation in the type IIB theory. The next chapter focuses more specifically on the stabilisation, by fluxes, of the complex structure moduli and dilaton in the type IIB theory compactified on toroidal orbifolds. In this framework, we obtain a relation between the string coupling constant and the flux number. Combined with the tadpole constraint, it prevents the coupling from being parametrically small. This result can be seen as a consequence of the finiteness of the number of flux vacua, which we also observe in our analysis. Finally, the fourth and last part of this thesis concerns the Dark Dimension, an additional dimension predicted at the micrometer scale. We begin by introducing the Swampland program of string theory and the arguments supporting this prediction, before turning to the calculation of the spectra of primordial fluctuation, in a five-dimensional inflation model that helps explain the scale of the Dark Dimension. These spectra change their behaviour at large-scale, which in principle allows them to be distinguished from their analogues in four-dimensional models.



# Remerciements

Qui est-ce qui m'a permis d'en arriver là ? Tout d'abord, je pense que ma famille y est pour quelque chose. Alors, évidemment, il y a ma mère, Sabrina, et mon père, Nicolas, sans qui je ne serais *littéralement* pas là. Mais au delà de m'avoir engendré, je les remercie de m'avoir toujours apporté leur soutien, en particulier pendant mes longues études. Je remercie également mon beau-père, Jean-Pierre, notamment pour m'avoir transmis son sens du détail, ainsi que ma belle-mère Vanessa, puis ma petite soeur Julie et mes demis, Mathieu et Fanette. Je ne serais pas là non plus sans mes grand-mères, Nicole et Odette, ni sans mon arrière-grand-père Robert, qui avait remarquablement prédit que "j'irais loin", alors que j'apprenais tout juste à empiler des cubes ! Je remercie enfin mon oncle Marc, pour avoir pleinement contribué à l'éveil de ma curiosité scientifique en laissant traîner chez sa mère des livres de science-fiction et ses manuels de licence. Et je n'oublie pas de mentionner ma marraine, Lili :D !

Je ne serais pas là non plus sans l'apport des nombreux professeurs que j'ai rencontré sur mon chemin, à commencer par Claire Loubière, au lycée Clément Ader, et François Gaudel, de l'association Science Ouverte. Une mention spéciale va à Dominique Guest qui, en plus d'avoir été une professeure de physique hors du commun au lycée Louis le Grand, m'a convaincu de m'inscrire au concours de l'École polytechnique, quand je croyais que mon agénésie m'empêcherait d'y entrer. Je remercie aussi Jean-Yves Ollitrault, pour m'avoir accordé sa confiance en me prenant en stage, à la dernière minute et en plein confinement, au terme d'un premier M2 catastrophique !

Merci à Ignatios Antoniadis d'avoir bien voulu me prendre sous son aile en tant que directeur de thèse. Malgré la distance qui nous séparait souvent, je dois admettre qu'il était toujours disponible pour discuter et répondre à mes questions. Je lui suis reconnaissant de m'avoir permis de travailler avec un grand degré d'autonomie, et j'espère qu'il n'est pas trop déçu que je ne poursuive pas une carrière académique. Merci aussi aux membres de mon jury, Susha Parameswaran, Jean-Philippe Uzan, Mariana Graña, et en particulier à mes rapporteurs, Dimitrios Tsimpis et David Langlois. Merci enfin à Marios Petropoulos et Grégory Schehr, pour leurs conseils avisés lors de mes comités de suivi de thèse, à Emilian Dudas et Kyriakos Tamvakis, pour leur aide dans ma recherche de postdoc, à Michela Petrini, la directrice du LPTHE, et enfin à Françoise, Carole, Sofiane et Laurent, qui se sont succédés au secrétariat du laboratoire.

Enfin, il y a mes amis. Sans eux, j'aurais peut être quand même fait une thèse, mais elle aurait certainement été beaucoup plus ennuyeuse. Pour commencer, il y a les copains du labo, Andrei, Andriani, Bianca, Diyar, Francesco, Grégoire, Greivin, JordYann, Léo, Mathis,



Pierre, Ramgopal, Simon, Thorsten, Wendy, Wenqi, et en particulier mes camarades de bureau, Jules, Tom et Vincent. Malgré quelques désaccords concernant l'utilisation de la lumière ou de la clim, c'était chouette de passer du temps avec vous, que ce soit au bureau, en manif, à l'escalade, ou à Quy Nhon, et je suis désolé de vous avoir abandonné cette année :'( Je remercie aussi mes copains d'avant, Alexandra, Solène, Amandine, Ayman, Jérôme, Sebastien, Slava, et notamment Léo et Pierre-Louis, avec qui j'ai partagé quelques belles et intenses années de ma vie à la ColoCachan. Je remercie enfin tous mes partenaires d'escalade. A ceux qui ont déjà été nommés et qui se reconnaîtront, je dois ajouter Louis, et surtout Aristote ainsi que tous les membres de l'équipe de France, qui m'ont chaleureusement accueilli cette année. Maintenant que ma thèse se termine, je n'ai plus d'excuse, il va donc falloir que j'aille chercher des médailles ! Pour finir, merci à Doriane, la fille de ma vie depuis déjà deux ans. Merci de supporter mon humour parfois nul depuis tout ce temps et de m'avoir accueilli à Marseille quand il faisait moche à Paris. Maintenant bonne chance pour ta thèse à toi :).

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# Chapter 1

## Introduction

Over the past century, theoretical physics has developed at an amazing rate, and has often been able to make predictions, for instance about the existence of particles before they were observed. Among the examples, we can cite the prediction of the positron by Dirac in 1931, before its discovery by Anderson the year after. Similarly, the  $W^\pm$  and  $Z^0$  bosons, the charm and top quarks, and the Higgs boson were all predicted years or even decades before their discovery, on the basis of theoretical arguments. This century of discoveries is now embodied by the Standard Models of Particle Physics and Cosmology, also known as the  $\Lambda$ CDM model.

However, these models are not complete. Dark Energy and Dark Matter, which are the main ingredients of the  $\Lambda$ CDM model (giving it its name) are not understood within the Standard Model of Particle Physics; nor is inflation, which plays a major role in understanding the structure of our universe. On the other hand, the Standard Model of Particle Physics is rather mysterious in itself. In the framework of Quantum Field Theory, in which it is formulated, its main characteristics (the gauge group, the spectrum of particles, the Higgs potential and the Yukawa couplings) can be thought of as parameters, subject to a few constraints, such as the cancellation of anomalies. But as such, nothing explains the values that are observed experimentally. More so, these values display some puzzling features; for instance: why is the spectrum replicated into three increasingly massive generations? Or why are neutrino masses so small? Theoretical physics has played such a key role in establishing our current understanding, one can be confident it will continue to be central in answering these questions. Especially when experimental investigations are becoming increasingly difficult, with the need to probe more and more subtle phenomena.

From a theoretical point-of-view, a great way to advance our knowledge would be to unify both Particle Physics and Cosmology into a single, consistent framework; so that experimental observations on one side can be used to shed light the other side, and vice versa. Part of the difficulty comes from the fact that Cosmology relies heavily on General Relativity, which cannot be embedded into Quantum Field Theory without running into problems. So far, the main framework which is able to describe aspects of both sides is given by String Theory.

The field that studies the applications of String Theory to Particle Physics and Cosmology is called String Phenomenology. While being able to reproduce some aspects of both sides, at least qualitatively, it has not yet been able to produce a "Standard Model of Everything".

Actually, a crucial aspect that such a model should incorporate, namely Dark Energy, is still poorly understood by itself in this context. In String Theory, the choices that go into defining a model are related to the presence of extra dimensions. The fact that extra-dimensions have not been observed yet can be explained if we assume that the higher-dimensional spacetime is factorised into the four-dimensional one in which we live  $\times$  a compact space, an idea known as compactification. In this framework, what we observe in our four-dimensional universe become tied to geometrical features of the compact space, along with the configuration of additional objects, known as fluxes and D-branes, whose existence is implied by the theory.

A String Theory Model is thus mostly defined by the choice of a compactification manifold, fluxes, and brane configuration, which are all subject to constraints. A first difficulty arises from the vast number of possibilities for these choices, which is poetically referred to as the Landscape. Once the choice is made, one has to understand, as precisely as one can, the behaviour of strings in this background, in order to make the connexion to observables, at an energy scale that is far below the string fundamental scale. At low energies, String Theory can conveniently be described by a supergravity theory, the parameters of which are computed from the higher-dimensional data. Another difficulty comes from the fact that this computation can mostly be done up to approximations, and one has to make sure that these approximations are consistent. In particular, the Supersymmetry of the theory, that is helpful to protect certain quantities against higher-order corrections, has to be broken at some point to make the connection with our observable world.

The goal of String Phenomenology, or more precisely String Model Building, is thus to find models in the Landscape able to reduce to the Standard Model of Particle Physics at low energies, while providing suitable Dark Matter candidates and reproducing the expansion history of the universe, including in particular Dark Energy. Moreover, one has to do so in a way that all approximations are justified, in a regime where the corrections are small.

One of the important requirements for a String Theory Model to be realistic is that its moduli are stabilized. Moduli are parameters controlling deformations of the compact manifold. From our four-dimensional perspective, they would be seen as neutral and gravitationally interacting scalar fields. As such, they could play a big role in Cosmology, especially if their masses are small, since they would then mediate long-range fifth-forces. So they have to be massive, that is, they have to be stabilised in a potential. This requirement is not trivial; in particular, it requires an understanding of the aforementioned corrections to the supergravity description of the theory, because the potential generally has unstable directions at tree-level. Another difficulty in this regard is that the corrections are controlled by the position in field space at which moduli end up, so the consistency of the approximations is hard to impose and must be checked in the end. For example, a universal modulus is the dilaton, whose vacuum expectation value is the string coupling, controlling string-loop corrections.

Looking for a "Standard Model of Everything" into the String Landscape is quite an ambitious program. Instead of this, it might be instructive to understand if there are any common features shared by the low-energy theories descending from String Theory, in terms of gauge group, spectrum, or scalar potential, for example. If so, asking for a Quantum Field Theory to be compatible with String Theory would yield constraints. And if these constraints are strong

enough, they could explain some characteristics of our universe, without having to explicitly build a model. This is the question addressed by the Swampland program. Its scope is in fact broader, since it investigates the constraints imposed on Quantum Field Theory when asking it to be compatible with Quantum Gravity, and not specifically String Theory, even though this latter provides the most fertile testing ground.

For example, in all String Theory examples known so far, whenever the vacuum expectation value of a canonically normalised scalar field takes large values compared to the Planck scale, or in other words when we approach the boundaries of the moduli space, an exponentially light tower of states emerges in the spectrum. Since scalar fields generically control the parameters of the low-energy theory, we can rephrase the previous sentence and say that whenever a parameter of a Quantum Field Theory takes extreme values in Planck units, it must contain a tower of states. In our universe, there is indeed a parameter that has an extremely low value: the cosmological constant. Based on this, one can speculate on the existence of an unobserved tower of states. In String Theory again, all known towers are either towers of string states or Kaluza Klein towers. To fit with the value of the cosmological constant, our tower can only be a Kaluza Klein tower, linked to the existence of a single, large extra dimension, called the Dark Dimension, with a scale of the order of some micrometers. This Dark Dimension has a wide range of phenomenological implications.

This thesis is about four of the aforementioned topics; namely Dark Energy, supersymmetry Breaking, Moduli Stabilisation and the Dark Dimension, and based on [1–6].

In chapter 2, we investigate the effect of the quadratic scalar curvature correction  $\alpha R^2$  and non-minimal scalar curvature coupling  $\xi\phi R$  on the simplest quintessence model with an exponential potential, in the Palatini formulation of gravity [1]. This defines a class of Dark Energy models, in which the late-time acceleration of our universe is driven by a scalar field. Dynamical system techniques are particularly suitable for studying quintessence models, as they allow to understand the general behaviour of the system without solving the equations of motion, and to quickly discard models that cannot describe the homogeneous universe. In the context of cosmological scalar fields, the quadratic correction  $\alpha R^2$  and non-minimal coupling  $\xi\phi R$  are the most simple modifications one can make to minimal models, and they have interesting consequences on inflation. This was the main motivation for us to study their effect on quintessence. We found that the quadratic correction cannot play a role in the late time dynamics. However, there are viable evolutions with the dynamics driven by the non-minimal coupling  $\xi$ . In [2], we also investigated the effect of both terms on Higgs inflation in the Palatini formulation, but we will not expand on these results in this thesis.

Chapter 3 is an introduction to  $\mathcal{N} = 1$  supersymmetry and supergravity, in the superconformal formalism. It contains all the necessary ingredients to understand the construction of the  $\mathcal{N} = 1$  supergravity action in  $D = 4$ , which is used in following chapters.

In chapter 4, we compute the  $2 \rightarrow 2$  gravitino scattering amplitudes at tree-level in spontaneously broken  $\mathcal{N} = 1$  supergravity theories with one chiral and one vector multiplet, in a Minkowski background [3, 4]. In the unitary gauge, the gravitino becomes massive, of mass  $m_{3/2}$ , by absorbing the Goldstino. The scattering amplitudes of its "longitudinal" polarisa-

tions, of spin  $\pm 1/2$ , naively grow with energy as  $\kappa^2 E^4/m_{3/2}^2$ , signaling a potential breakdown of unitarity at a scale  $\Lambda \sim (m_{3/2}/\kappa)^{1/2} \sim M_{\text{SUSY}}$ . We find that this leading term is in fact proportional to the value of the scalar potential; it thus vanishes in the Minkowski background that we consider, restoring perturbative unitarity up to the Planck scale. This situation is analogous to what happens in the electroweak sector of the Standard Model. There, the scattering amplitude of longitudinal massive gauge bosons exhibits a divergence as  $E^2/M_W$ , which is cancelled by the channels involving the Higgs boson exchange. In our case, the role of the Higgs boson is played by the scalar partner of the Goldstino. In this context, we also study the inclusion of the so-called new Fayet-Iliopoulos terms. These are non-linear terms that can be added to the  $\mathcal{N} = 1$  supergravity action, with the property that they reduce to standard Fayet-Iliopoulos terms in the unitary gauge, without requiring a gauged R-symmetry. These terms generically modify the scalar potential without contributing to the  $2 \rightarrow 2$  gravitino scattering amplitudes at tree level. As such, they affect the aforementioned cancellation and lead to the perturbative cutoff  $\Lambda \sim (m_{3/2}/\kappa)^{1/2} \sim M_{\text{SUSY}}$ , except for fine-tuned cases.

Chapter 5 is an introduction to string theory and moduli stabilisation. It starts from the formulation of bosonic string theory, and covers various topics used in the following chapters, up to the description of the KKLT construction of de Sitter vacua in type IIB superstring theory.

Chapter 6 focuses on moduli stabilisation [6], in type IIB string theory compactified with 3-form fluxes on toroidal orientifolds  $T^6/\Gamma$ . In this framework, the fluxes have to satisfy the tadpole constraint, of the form  $1/2N_{\text{flux}} + N_{D3} = 1/4N_{O3}$ , where  $N_{D3}$  is the net number of  $D3$ -branes and  $N_{O3}$  the number of orientifold  $O3$ -planes. If we do not consider the addition of anti- $D3$ -branes, which can introduce instabilities, we have  $N_{D3} \geq 0$ . On the other hand,  $N_{O3} \geq 0$  but orientifold planes are geometric in nature and cannot be introduced at will. So overall, the tadpole constraint sets an upper limit to the value of  $N_{\text{flux}}$ . We study the implications of this limit on the value of the string coupling, when complex structure moduli and the dilaton are all stabilised by the flux superpotential. For orientifolds with zero and one complex structure moduli, we exhibit a relation  $g_{s,\text{min}} \sim 1/N_{\text{flux}}$ . For  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , which has three complex structure moduli at the orbifold point, we exhibit a relation  $g_{s,\text{min}} \sim 16/N_{\text{flux}}^2$ . For vacua satisfying the tadpole constraint, this leads to  $g_s \geq 0.669$ , which is arguably too large to ignore string loop corrections. In addition, we show how the finiteness of flux vacua manifests itself in all cases, and elaborate on the lack of parametric control.

Finally, in chapter 7, we compute the power spectra of primordial fluctuations emerging from a 5-dimensional slow-roll inflation model, which has been proposed as an explanation for the micrometric scale of the Dark Dimension [5]. In this model, all five dimensions expand homogeneously during inflation. After inflation though, matter extends into our four dimensions, while it is localised in the fifth one, which is compact, no longer expands. Remarkably, its size ends up naturally in the micrometer range in the Einstein frame when our observable universe acquires the right size after inflation. In this model, the five-dimensional metric perturbations that are generated during inflation are seen as a tower of perturbations that must be summed over from the four-dimensional perspective. The resulting spectra deviate from scale invariance at a large scale, when the perturbations start to feel the presence of the extra-dimension. This is a smoking gun signature of this scenario. This change of behaviour happens for angles greater than about 10 degrees in our sky.

# Chapter 2

## Late time acceleration in Palatini gravity

We start this chapter with an introduction to homogeneous Cosmology in section 2.1 (cosmological perturbations and inflation will be addressed in chapter 7). We will also introduce the main aspects of dynamical system analysis, closely following [7]. In section 2.2, we review the Palatini formulation in the context of modified gravity, and some of its applications. Finally, in section 2.3, we apply the dynamical system analysis to the study of a quintessence model, with an exponential potential  $V(\phi) = M^4 e^{-\lambda\phi}$ , a quadratic correction  $\alpha R^2$  and a non-minimal coupling  $\xi\phi R$ , in the Palatini formulation [1]. We find that, for reasonable values of  $\alpha$ , the quadratic correction  $\alpha R^2$  plays virtually no role in the late-time dynamics of the universe. However, the non-minimal coupling  $\xi\phi R$  modifies the effective scalar potential in a non-trivial way, creating a new local minimum where the scalar field can act as a cosmological constant. In addition, in the presence of the non-minimal coupling, the initial condition for the quintessence field could naturally be set by inflation, in a way that links the duration of the radiation and matter-domination era with the energy scales of inflation and quintessence.

### 2.1 Cosmology and dynamical systems

Modern Cosmology rests on two pillars: the Cosmological Principle and General Relativity. The Cosmological Principle asserts that the properties of the universe are the same for any observer. It implies that the universe is spatially homogeneous and isotropic at large scales, which in turns implies the existence of coordinates in which the metric takes the FLRW form

$$ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (2.1)$$

Note that in all of this work we use the mostly-plus convention for the signature of the metric. This metric has only one discrete parameter  $k = 0, \pm 1$ , describing the spatial curvature. If  $k = 1$ , the universe is said to be spatially close; if  $k = -1$ , it is open; if  $k = 0$ , it is flat. So far, experimental observations are in fact compatible with a flat universe, so we can set  $k = 0$ . In addition to this parameter, the metric involves the scale factor  $a(t)$ , depending only on the time variable  $t$ . The scale factor is a dynamical quantity, whose evolution is governed by Einstein's field equations (written here in natural units  $\kappa = 8\pi G/c^4 = 1$ )

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}. \quad (2.2)$$



On the right-hand side, the energy-momentum tensor  $T_{\mu\nu}$  encodes everything that is present in the universe. On cosmological scales, homogeneity and isotropy allow us to use a perfect fluid description

$$T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu, \quad (2.3)$$

where  $(\rho, p)$  are respectively the energy density and isotropic pressure of the fluid, and  $u^\mu$  is the four-velocity of an observer comoving with the fluid. In comoving coordinates (2.1), we have  $u^\mu = (-1, 0, 0, 0)$ . In addition, the energy density and pressure are linked together by the equation of state  $p = f(\rho)$ , which depends on the fluid. The most relevant examples are a non-relativistic matter fluid, which has  $p = 0$ , a relativistic radiation fluid, which has  $p = \rho/3$  and a cosmological constant, which has  $p = -\rho$ . In general, we write  $p = w\rho$ . Combining (2.1) and (2.3) in Einstein's equations (2.2) leads to the two equations

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3} \quad \text{and} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} = -p, \quad (2.4)$$

where  $H = \dot{a}/a$  is the Hubble rate and dots stand for time derivatives. These two equations fully determine the time-dependence of the scale factor once the matter-energy content of the universe is specified. Note that the Friedmann equations can be combined into

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p),$$

so in a universe filled by a single fluid with equation of state  $p = w\rho$ , accelerated expansion, defined by  $\ddot{a} > 0$ , is equivalent to  $w < -1/3$ . The Friedmann equations can also be combined into the energy conservation equation, which can equivalently be obtained from  $\nabla_\mu T^{\mu\nu} = 0$

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (2.5)$$

In a flat universe, with  $k = 0$  and constant  $w$ , it can be combined with (2.4) to give

$$\rho \sim a^{-3(1+w)}. \quad (2.6)$$

### 2.1.1 The $\Lambda$ CDM model

At early times, our universe was much denser and hotter than it is today, and the matter it was filled with was relativistic matter, behaving as a radiation fluid. As it expanded, it became cooler and less dense, and non-relativistic matter came to dominate. The ordinary, or baryonic matter, that we are able to observe in the form of galaxies or other objects, is only a part of the matter present in the universe. The remaining part is Dark Matter, it is observed only indirectly by its gravitational influence, and its precise nature is still unknown. In addition to radiation and matter, we need to add a third, even more mysterious component to explain the observation that the universe has recently entered an era of accelerated expansion [8, 9]. This third component is called Dark Energy, and it can simply be modeled by a positive cosmological constant  $\Lambda$  added to the Einstein's field equation (2.2)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}.$$

In fact, the cosmological constant has been considered in modern cosmology before the discovery of the accelerated expansion of the universe [10], in order to reconcile the observations that  $\Omega_{m,0} = 0.3$  and that the universe is spatially flat, imposing  $\Omega_{\text{tot}} = 1$ .

For multiple components in the universe, the first Friedmann equation can be written using dimensionless variables, in a way highlighting that it is actually a constraint

$$3H^2 = \rho_r + \rho_m + \rho_\Lambda \quad \rightarrow \quad \Omega_r + \Omega_m + \Omega_\Lambda = 1 \quad \text{with} \quad \Omega_i = \frac{\rho_i}{3H^2}.$$

The values of  $(\Omega_{m,0}, \Omega_{\Lambda,0})$  at our current time are around  $(0.3, 0.7)$  in the latest Planck measurements [11]. In addition, the conservation equation (2.5) can be written for each component, with eventual terms on the right-hand side taking into account interactions between them. Neglecting these interactions, radiation is negligible today, because its energy density  $\rho_r \sim a^{-4}$  decreases faster than the energy density of matter  $\rho_m \sim a^{-3}$ , from equation (2.6). As its name indicates, a cosmological constant has  $\rho_\Lambda \sim \text{const}$ , and it came to dominate over matter recently in the history of the universe. This model describing a universe filled with Cold (i.e. non-relativistic) Dark Matter with a cosmological constant is called  $\Lambda\text{CDM}$ .

Once the content of the universe is specified, one can solve the Friedmann equations to know everything about the homogeneous, or background evolution of the universe. This can be hard to do analytically, especially once we consider additional components, such as cosmological scalar fields. But it is not actually necessary to solve the equations to understand the overall behavior of the system. Rather, one can use a dynamical system approach.

Let us introduce this approach by applying it to the  $\Lambda\text{CDM}$  model. First, it is convenient to rename  $(\Omega_m, \Omega_r, \Omega_\Lambda) = (x, y, z)$ , so that the first Friedmann equation reads  $x + y + z = 1$ . It can be used to express  $z = 1 - (x + y)$ . In addition, since  $(x, y, z) \geq 0$ , it constrains  $(x, y, z) \leq 1$ . Then, introducing the number of e-folds  $N = \ln a$ , we compute

$$\frac{dx}{dN} = \frac{dx}{Hdt} = \frac{\dot{\rho}_m}{3H^3} - \frac{2\rho_m \dot{H}}{3H^2 H}.$$

Using equation (2.5) for  $\rho_m$  neglecting its interaction with  $\rho_r$ , and the Friedmann equations (2.4) then yields

$$\frac{dx}{dN} = x(3x + 4y - 3), \tag{2.7}$$

and doing the same for  $y$

$$\frac{dy}{dN} = y(3x + 4y - 4). \tag{2.8}$$

This is an autonomous system on  $(x, y)$ . It possesses three fixed points  $\mathbf{P} = (x_*, y_*)$ , where  $dx/dN = dy/dN = 0$ . For example, there is the origin  $O$  with  $x_* = y_* = 0$  and  $z_* = 1$ , so  $\rho_m = \rho_r = 0$ , this point corresponds to a universe dominated by the cosmological constant. The evolution of the system around fixed points is then related to their stability, which can be assessed using linear analysis. For example, around the point  $O$ , the system (2.7)-(2.8) is expanded to first order as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \simeq \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix that is involved in this expression is called the stability matrix. Since both its eigenvalues are real and negative, the fixed point  $O$  is stable and is a future attractor of the system, in the sense that starting from generic initial conditions  $(x_0, y_0)$ , trajectories evolve towards this point in the future. A fixed point with positive (real part of the) eigenvalues is unstable, and is called a past attractor. A fixed point with both positive and negative eigenvalues is a saddle point, having trajectories approaching such points for a long period of time requires some tuning of the initial conditions. For vanishing eigenvalues, linear analysis is not able to assess the stability of the fixed point and other methods must be used. In table 2.1, we summarise the fixed point structure of the  $\Lambda$ CDM system (2.7)-(2.8).

point	$x_*$	$y_*$	eigenvalues	stability	what dominates
$O$	0	0	$\{-4, -3\}$	attractive	cosmo. constant
$R$	0	1	$\{1, 4\}$	repulsive	radiation
$M$	1	0	$\{-1, 3\}$	saddle	matter

Table 2.1: fixed points of the system (2.7)-(2.8) with their properties.

Since  $z = 1 - (x + y)$ , the system is effectively two-dimensional, and we can conveniently plot its phase space in figure 2.1. This gives a clear representation of the fixed point structure of table 2.1 and allows to understand the overall behaviour system in the blink of an eye: the universe was dominated by radiation at early-time, and ends up dominated by Dark Energy.

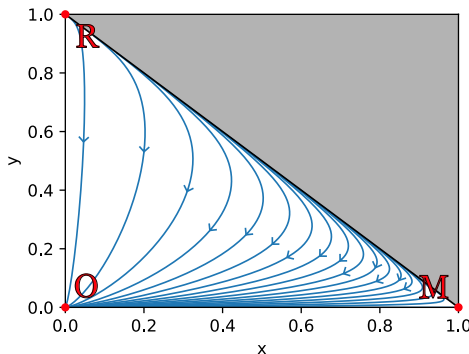


Figure 2.1: Phase space of the system (2.7)-(2.8). The shaded area is excluded by the Friedmann constraint  $x + y + z = 1$ , with  $(x, y, z) \geq 0$ . The fixed point of table 2.1 are shown.

We can observe that, since the point  $M$ , corresponding to matter domination, is a saddle point, generic trajectories do not really approach it. As a consequence, having a long period of matter domination, as in our universe, requires special initial conditions if we naively stick to the system (2.7)-(2.8). But remember that this system it is derived using (2.5) for  $(\rho_m, \rho_r)$  with the assumption that these two fluid components do not interact with each other. In reality, in the early universe's thermal bath, the different particle constituents are converted from the radiation fluid to the matter fluid one after the other, when the temperature drops under their rest mass. Therefore, we cannot really neglect the interactions between the two

fluids. Since our analysis of the  $\Lambda$ CDM system using the dynamic system approach is only illustrative, we will not dwell on a more precise description of this transition.

Another point to note is that the transition between matter domination (with  $\Omega_\Lambda = 0$ ) and Dark Energy domination (with  $\Omega_\Lambda = 1$ ) happens rather quickly on cosmological timescales, as illustrated by figure 2.2. Since the value measured at our current time is  $\Omega_{\Lambda,0} = 0.7$ , this would mean that we live precisely during the transition, which is an intriguing coincidence.

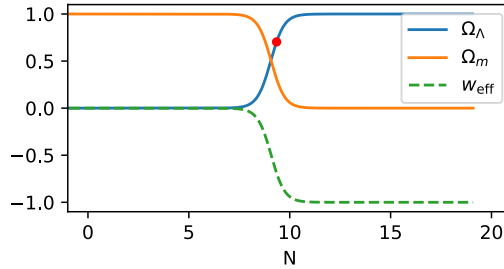


Figure 2.2: Evolution of  $(\Omega_m, \Omega_\Lambda, w_{\text{eff}} = -\Omega_\Lambda)$  for the  $\Lambda$ CDM system (2.7)-(2.8), with initial conditions tuned for illustrative purposes. The red dot represents the current time  $\Omega_{\Lambda,0} = 0.7$ .

There are several problems associated with the cosmological constant. In the first place, its value must be extremely low to be compatible with the observations

$$\Lambda \sim 10^{-47} \text{ GeV}^4 \sim 10^{-122} \text{ in Planck units.} \quad (2.9)$$

There is thus a rather extreme hierarchy between the energy scale associated with the cosmological constant  $E_\Lambda \sim 10^{-12} \text{ GeV}$  and, for instance, the Electro-Weak scale  $E_{\text{EW}} \sim 10^2 \text{ GeV}$ . This hierarchy only becomes more pronounced when considering higher scales such as the Grand Unification or the Planck scales, and it is hard to explain from the point-of-view of Quantum Field Theory without resorting to fine-tuning. This fact has quickly been identified as a major theoretical problem [12], and several solutions have been proposed. Interestingly, the cosmological constant could not be much larger without altering the conditions that led to the development of intelligent life, leading to an anthropic bound on its value [13].

In chapter 7, we will explain how the very smallness of the cosmological constant has been combined with insight from String Theory, to predict the existence of an additional spatial dimension, the Dark Dimension, which is compact and lies at the micrometer scale. This prediction should be taken with a grain of salt, however, because so far there is no consensus on the possibility of obtaining a positive cosmological constant from String Theory, as we will review in chapter 5.

## 2.1.2 Quintessence models

Another puzzle, that we mentioned above figure 2.2, is the cosmic coincidence problem: why is the transition between a matter dominated universe and a Dark Energy dominated universe happening, at a time when we are here to observe it? Maybe it is just a coincidence indeed, but it is still interesting to explore other explanations, that may come with modifications of the  $\Lambda$ CDM model. The simplest extension is to replace the cosmological constant with a canonical scalar field in a potential. This possibility also has been considered before the discovery of late-time accelerated expansion [14–16]. Models for which a scalar field accounts for Dark Energy are now called quintessence models. The simplest action is

$$\mathcal{S}_\phi = \int d^4x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \quad (2.10)$$

Being coupled to gravity, the scalar field acts as a source for Einstein's field equations (2.2), with an energy-momentum tensor given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 - g_{\mu\nu} V(\phi). \quad (2.11)$$

At the homogeneous level, the background value of the scalar field depends only on the cosmic time  $t$ . In the FLRW background (2.1), the energy-momentum tensor (2.11) then takes the perfect fluid form (2.3) with

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V, \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V, \quad w_\phi = \frac{1/2 \dot{\phi}^2 - V}{1/2 \dot{\phi}^2 + V}.$$

We can see that  $w_\phi \in [-1, 1]$ , so the scalar field can mimick all the other components of the universe, depending on its dynamics. In particular, if  $\dot{\phi} \simeq 0$ , the scalar field has  $w_\phi \simeq -1$  and produces the same effect as the cosmological constant. Note that in addition to the two Friedmann equations (2.4), we must consider the equation of motion of the scalar field for the system to be complete

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \quad (2.12)$$

Let us apply again the dynamical system approach to the study of this system. The first step is to define appropriate dynamical variables. The Friedmann constraint is a good guide for this. In the case at hand, in the presence of an additional fluid component with an arbitrary, constant equation of state parameter  $w$ , it reads

$$\frac{\dot{\phi}^2}{6H^2} + \frac{V}{3H^2} + \Omega_m = 1,$$

Which led [17] to define, in the case of a positive potential

$$x = \frac{\dot{\phi}}{\sqrt{6}H} \quad \text{and} \quad y = \frac{\sqrt{V}}{\sqrt{3}H}, \quad (2.13)$$

so that the equation  $\Omega_\phi + \Omega_m = x^2 + y^2 + \Omega_m = 1$  constrains  $(x, y)$  to lie in the unit disk  $x^2 + y^2 \leq 1$ , because  $\Omega_m > 0$ . Defining the third variable

$$\lambda = -\frac{V_{,\phi}}{V}, \quad (2.14)$$

we can use the second Friedmann equation (2.4) and the equation of motion of the scalar field (2.12) to obtain the following system

$$\begin{aligned}\frac{dx}{dN} &= -\frac{3}{2}\left(2x + (w-1)x^3 + x(w+1)(y^2-1) - \sqrt{\frac{2}{3}}\lambda y^2\right), \\ \frac{dy}{dN} &= -\frac{3}{2}y\left((w-1)x^2 + (w+1)(y^2-1) + \sqrt{\frac{2}{3}}\lambda x\right), \\ \frac{d\lambda}{dN} &= -\sqrt{6}\left(\frac{VV_{,\phi\phi}}{V_{,\phi}^2} - 1\right)\lambda^2 x.\end{aligned}\tag{2.15}$$

At first sight, it seems that this system is not autonomous, i.e. it does not only involve the dynamical variables  $(x, y, \lambda)$ , since  $\phi$  appears in the last equation through  $V$  and derivatives. However, from its definition (2.14) we can think of  $\lambda$  as a function of  $\phi$ . If we can invert this relation to obtain  $\phi(\lambda)$ , we can replace  $\phi$  where it appears in (2.15) and close the system. If we cannot invert the relation  $\lambda(\phi)$ , we shall define other variables.

The dynamics of the system is now dependent on the choice of scalar potential. The most simple choice is to consider an exponential potential  $V(\phi) = M^4 e^{-\lambda\phi}$ , corresponding to a constant  $\lambda$ . Not only is this choice simple, but it is also theoretically motivated, since exponential potentials arise naturally in supergravity and string theory, although with a slope  $\lambda$  that is often too large for accelerated expansion to happen, as we will see in following chapters. Note that the system (2.15) is always invariant under  $y \rightarrow -y$ . In the case of a constant  $\lambda$  it is also invariant under  $(x, \lambda) \rightarrow -(x, \lambda)$ , so we can restrict to positive  $y$  and  $\lambda$ . Before we show the fixed point structure of this system, as we did for the  $\Lambda$ CDM model, let us introduce an additional quantity, the effective equation of state parameter

$$w_{\text{eff}} = w(1 - \Omega_\phi) + w_\phi \Omega_\phi = w(1 - x^2 - y^2) + x^2 - y^2,\tag{2.16}$$

whose value at any time determines if the universe's expansion is accelerating ( $w_{\text{eff}} < -1/3$ ) or decelerating ( $w_{\text{eff}} > -1/3$ ). In table 2.2, we summarise the fixed point structure of the quintessence model with an exponential potential  $V(\phi) = V_0 e^{-\lambda\phi}$ , in the presence of an additional fluid component with equation of state parameter  $w$ , and no interactions.

With this analysis, we can immediately determine the qualitative behaviour of the system. The only point that does not depend on  $\lambda$  is  $O$ , corresponding to a universe where the scalar field is completely subdominant. It is always a saddle point. For positive  $\lambda$  the point  $A_-$  is always unstable, and  $A_+$  is also unstable if  $\lambda^2 < 6$ . In this case they are both past attractors, and they correspond to a universe dominated by the kinetic energy of the scalar field and  $w_{\text{eff}} = 1$ , thus undergoing a so-called kination era.

If  $\lambda^2 < 3(1+w)$ , the future attractor is point  $C$ , and it corresponds to a universe dominated by the scalar field. Under the additional condition  $\lambda^2 < 2$ , it undergoes accelerated expansion.

If  $3(1+w) < \lambda^2 < 6$ , point  $C$  becomes a saddle point and the future attractor is point  $B$ . It corresponds to a universe where the scalar field and the other fluid component both share the energy of the universe, but the scalar field mimics the behaviour of the fluid component, so that  $w_{\text{eff}} = w$ . Such a solution is called a scaling solution, and was studied in detail

point	$x_*$	$y_*$	eigenvalues	stability and existence	$\Omega_\phi$	$w_{\text{eff}}$
$O$	0	0	$\{\frac{3}{2}(w \pm 1)\}$	saddle	0	$w$
$A_\pm$	$\pm 1$	0	$\{3(1-w), 3 \mp \sqrt{\frac{3}{2}}\lambda\}$	unstable if $\lambda \geq \pm\sqrt{6}$ saddle if $\lambda < \pm\sqrt{6}$	1	1
$B$	$\sqrt{\frac{3}{2}}\frac{1+w}{\lambda}$	$\sqrt{\frac{3(1-w^2)}{2\lambda^2}}$	$\{\frac{3}{4\lambda}((w-1)\lambda \pm \Delta)\}$	exists if $\lambda^2 \geq 3(1+w)$ stable	$\frac{3(1+w)}{\lambda^2}$	$w$
$C$	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1 - \frac{\lambda^2}{6}}$	$\{\frac{\lambda^2}{2} - 3, \lambda^2 - 3(w+1)\}$	exists if $\lambda^2 < 6$ stable if $\lambda^2 < 3(1+w)$ saddle if $3(1+w) \leq \lambda^2$	1	$\frac{\lambda^2}{3} - 1$

Table 2.2: fixed points of the system (2.15) with their properties, in the case of a constant  $\lambda$ . In this table,  $\Delta^2 = (w-1)((7+9w)\lambda^2 - 24(w+1)^2)$ .

in [18, 19]. It is interesting, because it provides a natural mechanism to hide the effect of the scalar field on cosmological scales, at the background level. If  $\lambda^2 > 6$ , the future attractor is still point  $B$  but point  $C$  stops existing.

Again, the system is effectively two-dimensional, so we can conveniently plot its phase space in figure 2.3, for values of  $\lambda$  that illustrate the three different regimes explained above.

To correctly describe our universe, a quintessence model should have several features. First: the scalar field has to be subdominant during the radiation dominated era, in particular, it should not perturb Big Bang Nucleosynthesis, which predicts the abundance of light elements with a good accuracy. During the matter dominated era, the scalar field should either be subdominant or evolve along a scaling solution so as not to perturb structure formation. Finally, after  $N \sim 10$  e-folds of matter domination, the scalar field must come to dominate and induce the current accelerated expansion of the universe.

In the case of the exponential potential, accelerated expansion only occurs at point  $C$  in the phase space when  $\lambda^2 < 2$ . In this case, point  $C$  is the future attractor and the scalar field dominates with  $\Omega_\phi = 1$ . Therefore, the only realistic trajectory of this model are the ones with  $\lambda^2 < 2$  and initial conditions tuned to obtain a period of matter domination, at the saddle point  $O$ . So this model alone does not solve the coincidence problem. In order to do so, one either needs to find a mechanism to explain the tuning of the initial conditions, or modify the model. In this sense, the point  $B$  is interesting, because it corresponds to a scaling solution where the energy of the universe is shared between the scalar field and the

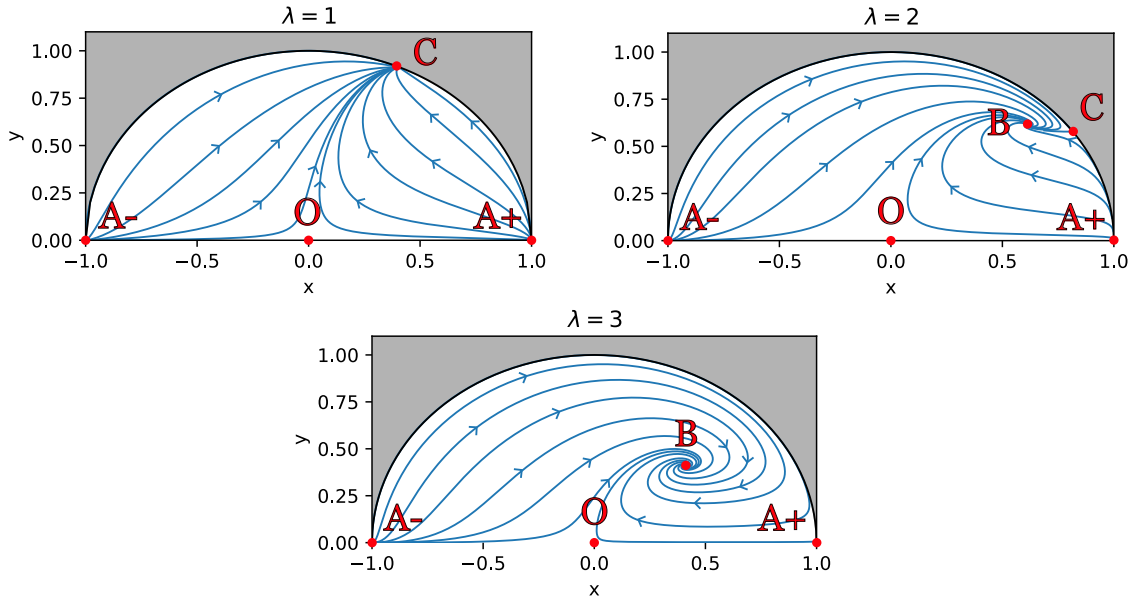


Figure 2.3: Phase space of the system (2.15) in the case of a constant  $\lambda$  and  $w = 0$ , for different values of  $\lambda$ . We restricted to  $y > 0$  by symmetry. The shaded area is excluded by the Friedmann constraint  $x + y + \Omega_m = 1$ , with  $\Omega_m \geq 0$ . The fixed point of table 2.1 are shown.

other component, with  $\Omega_\phi = 3(1 + w)/\lambda^2$ , so it naturally leads to  $\Omega_\phi \simeq 0.7$  with  $\lambda \simeq 2.1$ , but without accelerated expansion. A natural way to solve the coincidence problem would be to have a fixed point, either a saddle or a future attractor, such that  $\Omega_\phi \simeq 0.7$  and  $w_\phi \simeq -1$ .

This simple model can be modified in many ways. Among others, one can consider different scalar potentials, scalar fields with non-canonical kinetic terms, interactions between the scalar field and matter, or modifications in the gravity sector. A large sample of possibilities are reviewed and systematically analysed using dynamical system methods in [7], see also [20]. In the following, we will for instance study a simple quintessence model in the Palatini formulation of gravity. But before that, let us tell a few words about the observational status of evolving Dark Energy.

In order to study the evolution of Dark Energy in a model independent way, one can parameterise its equation of state as a function of the scale factor as

$$w_\phi(a) = w_0 + w_a(1 - a) \quad (2.17)$$

Using this parametisation, the Planck collaboration reported in 2018 [11] that  $w_0 = -0.957 \pm 0.080$  and  $w_a = -0.29^{+0.32}_{-0.26}$ , when combining their observations with Baryon Acoustic Oscillation (BAO) and Type Ia supernovae (SNe) data, which is compatible with a cosmological constant. However, in 2024, the DESI collaboration refined the measurements of BAO and reported in [21] that, although the  $\Lambda$ CDM model is still consistent with observations, allowing Dark Energy to vary according to (2.17) yields  $w_0 = -0.827 \pm 0.063$  and  $w_a = -0.75^{+0.29}_{-0.25}$  when combined with CMB data from the Planck and SNe data from the PantheonPlus collaboration.



## 2.2 Palatini and modified gravity

It is well-known that the action of General Relativity, giving Einstein's field equations (2.2) from the principle of least action, is the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2} \int d^4x \sqrt{-g} R,$$

where  $g$  is the determinant of the metric, and  $R = g^{\mu\nu} R_{\mu\nu}$  the Ricci scalar. In turn, the Ricci tensor  $R_{\mu\nu}$  is obtained from the Riemann tensor  $R_{\mu\nu} = R_{\mu}{}^{\sigma}{}_{\nu\sigma}$ , which is itself obtained from the affine connection

$$R_{\mu\nu\sigma}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\tau}^{\rho} \Gamma_{\nu\sigma}^{\tau} - \Gamma_{\nu\tau}^{\rho} \Gamma_{\mu\sigma}^{\tau}.$$

Then, under the assumptions of metricity and symmetry

$$\nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma_{\mu\nu}^{\sigma} g_{\sigma\rho} - \Gamma_{\mu\rho}^{\sigma} g_{\nu\sigma} = 0 \quad \text{and} \quad \Gamma_{\mu\nu}^{\rho} - \Gamma_{\nu\mu}^{\rho} = 0,$$

the affine connection is completely determined from the metric

$$\Gamma_{\mu\nu}^{\rho}(g) = 1/2 g^{\rho\sigma} (\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}).$$

But it turns out that there are other possibilities. One of them is to let the affine connection be an independent variable, with its own equation of motion. This possibility is called the first-order, or Palatini formulation of General Relativity [22], because the equations of motion of the metric and of the connection are first order in derivatives. On the other hand, the formulation where  $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho}(g)$  is imposed is called the second-order, or metric formulation.

In the pure Einstein-Hilbert case, the equation of motion of the affine connection is algebraic and sets  $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho}(g)$ , so both formulations are in fact equivalent. It stops being the case when considering modifications of the Einstein-Hilbert action [23], for instance in the presence of a coupling between the Ricci scalar and a scalar field, with an action

$$\mathcal{S}_J = \int d^4x \sqrt{-g} \left( \frac{1}{2} \Omega^2(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right). \quad (2.18)$$

In this case, we have  $\Gamma_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho}(g) - K_{\mu\nu}^{\rho}$  where  $K_{\mu\nu}^{\rho}$  is the so-called contortion tensor that we could compute explicitly by solving the equations of motion of the affine connection. But there is in fact a simpler way to deal with a non-minimal coupling, which consists in performing a Weyl rescaling to go in a frame where it is absent, called the Einstein frame. In contrast, the initial frame where the non-minimal coupling is present in (2.18) is called the Jordan frame. At this point, the first and second order formulations start to diverge: in the first-order formulation,  $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$  and only  $g^{\mu\nu}$  is affected by the Weyl rescaling, while in the second-order formulation,  $R = g^{\mu\nu} R_{\mu\nu}(\Gamma(g))$  so  $R_{\mu\nu}$  is also affected. This leads to

$$g_{\mu\nu} \rightarrow \Omega^{-2} g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \Omega^2 g^{\mu\nu}, \quad \sqrt{-g} \rightarrow \Omega^{-4} \sqrt{-g}, \quad R \rightarrow \Omega^2 R - \underline{3/2 \Omega^{-2} (\partial \Omega^2)^2}, \quad (2.19)$$

where the underlined term is present only in the second-order formulation and  $(\partial \cdot)^2$  is a short hand notation for  $g^{\mu\nu} \partial_{\mu} \cdot \partial_{\nu} \cdot$ . Applying this transformation to (2.18) gives

$$\mathcal{S}_E = \int d^4x \sqrt{-g} \left( \frac{1}{2} R - \frac{3}{4} \frac{(\partial \Omega^2)^2}{\Omega^4} - \frac{(\partial \phi)^2}{2 \Omega^2} - \frac{V}{\Omega^4} \right). \quad (2.20)$$

The only place where the affine connection appears is in  $R$ , so if we are working with the first-order formulation, writing and solving its equation of motion from the action (2.20) yields  $\Gamma_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho(g)$ . In conclusion, the underlined term in (2.20), present in the second-order formulation, is the only difference between the two in the Einstein frame. Alternatively, it is possible to write and solve the equation of motion of the affine connection directly from the Jordan frame action (2.18), and the result is of course equivalent to (2.20). This is shown explicitly in [2], where we computed scattering amplitudes in Higgs inflation models in both the metric and Palatini formulations, in order to study the unitarity of these models, as we do in chapter 4 in a different context. Higgs inflation [24] is a well-studied model of inflation in which the Higgs boson of the Standard Model plays the role of the inflaton, and where the non-minimal coupling of the form (2.18) plays a central role.

Apart from non-minimal couplings, another well-studied class of modifications of the Einstein-Hilbert action are  $f(R)$  theories, where the Ricci scalar  $R$  is replaced by a function  $f(R)$ . A simple possibility is to include a quadratic correction

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} \left( R + \frac{\alpha}{2} R^2 \right). \quad (2.21)$$

This  $R^2$  term can be incorporated in an auxiliary scalar field as follows

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-g} \left( (1 + \alpha\chi)R - \frac{\alpha}{2}\chi^2 \right), \quad (2.22)$$

because it is clear that solving the algebraic equation of the motion of  $\chi$  gives  $\chi = R$ , which gives back (2.21) after substitution. The action (2.22) is of the form (2.18), with  $\Omega^2(\chi) = 1 + \alpha\chi$ , and we can rewrite it in the Einstein frame using (2.20). There, we can notice a significant difference between the first and second-order formulations. In the second-order formulation, the underlined term is present, and it is a kinetic term for the scalar  $\chi$ . Therefore, this field is not auxiliary in the Einstein frame, and propagates a degree of freedom. This degree of freedom is of course also present in the initial action (2.21), but there it is carried by an additional polarisation of the graviton and not as clearly visible. The scalar field  $\chi$  can successfully play the role of the inflaton. This is the essence of the Starobinsky model of inflation [25], although it was not initially proposed as such, but as a way to resolve the initial singularity of the universe.

On the other hand, in the first-order formulation, the scalar field  $\chi$  remains auxiliary in the Einstein frame, so it can be integrated out. This process leaves an imprint on the rest of the action, especially on the scalar potential, but it does not bring any additional degree of freedom with respect to the pure Einstein-Hilbert case. For this reason, models of inflation with this  $R^2$  term in the Palatini formulation have been considered in [26, 27], where it was found that it can effectively flatten the potential and reconcile models with observations.

A nice review of  $f(R)$  theories of gravity, including in the first-order formulation of gravity, is conducted in [28]. Note that non-minimal couplings and higher-order corrections to the Einstein-Hilbert action are generally expected to arise from quantum corrections [29].

## 2.3 Application to a quintessence model

In this section, we investigate the effect of the quadratic correction and non-minimal coupling on a simple quintessence model with an exponential potential, in the Palatini formulation of General Relativity [1]. We use the dynamical system techniques that we reviewed in 2.1 to analyze the attractor structure of the model and uncover its possible trajectories. We find that the quadratic correction  $\alpha R^2$  does not play any role in the late-time dynamics of the system, unless the dimensionless parameter  $\alpha$  is given an extreme value. However, the non-minimal coupling  $\xi\phi^2 R$  can drive the evolution of the system from the vicinity of an early-time matter dominated phase towards a late-time attractor corresponding to accelerated expansion. To be precise, it induces a new local minimum of the potential, and we find that, for a realistic evolution of the universe, the only viable trajectories are the ones where the canonical field ends up in this new minimum and acts as a cosmological constant.

### 2.3.1 Description of the model

The action we consider is the following, with a scalar field  $\phi$  in a potential  $V(\phi)$ , a non-minimal coupling  $\xi\phi^2 R$ , and a quadratic correction  $\alpha R^2$ , in the Palatini formulation of gravity

$$\mathcal{S}_J = \int d^4x \sqrt{-g} \left( \frac{1}{2}(1 + \xi\phi^2)R + \frac{\alpha}{4}R^2 - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right). \quad (2.23)$$

Our goal is to understand the influence of these additional terms on the late-time accelerated expansion of the universe. The scalar field is a generic one, that would start evolving from a nonzero vacuum expectation value after inflation and reheating. Replacing the  $R^2$  term by an auxiliary scalar, like in (2.22) and going to the Einstein frame using (2.20), we get

$$\mathcal{S}_E = \int d^4x \sqrt{-g} \left( \frac{1}{2}R - \frac{1}{2} \frac{(\partial\phi)^2}{(1 + \xi\phi^2 + \alpha\chi)} - \frac{V(\phi) + \alpha/4\chi^2}{(1 + \xi\phi^2 + \alpha\chi)^2} \right), \quad (2.24)$$

solving for  $\chi$  yields

$$\chi = \frac{4V(\phi) + (1 + \xi\phi^2)(\partial\phi)^2}{1 + \xi\phi^2 - \alpha(\partial\phi)^2},$$

and substituting in (2.24) we obtain

$$\mathcal{S}_\phi = \int d^4x \sqrt{-g} \left( -\frac{1}{(1 + \xi\phi^2)^2 + 4\alpha V(\phi)} \left( \frac{1}{2}(1 + \xi\phi^2)(\partial\phi)^2 + V(\phi) - \frac{\alpha}{4}(\partial\phi)^4 \right) \right). \quad (2.25)$$

In order to describe our universe, we would have to include the action of the Standard Model  $\mathcal{S}_m$ . Assuming there is no coupling between the scalar field  $\phi$  and the Standard Model in the Jordan frame, it will appear in the Einstein frame as a result of the Weyl transformation (2.19). Such a coupling can potentially be problematic, because the quintessence field is generically very light, with a mass of order  $m_\phi \sim H_0 \sim 10^{-33}$  eV, where  $H_0$  is the present value of the Hubble rate. It would thus easily mediate a fifth force, which is tightly constrained experimentally. In order to avoid this, one can invoke something to suppress the coupling between the scalar field  $\phi$  and the Standard Model [30]. In [31,32], it was also proposed that the mass of the quintessence field may depend on the local matter density, making it heavier and more difficult to detect on earth, where experiments are conducted. Such a

field has been called a chameleon field, because of its ability to adapt to its environment.

At the Cosmological level, matter and radiation are described effectively as perfect fluids, with the energy-momentum tensor (2.3). Therefore, the coupling between the scalar field and the Standard Model would have to be described at the level of the energy-momentum tensor as well. A coupling of the form  $\partial^\mu T_{\mu\nu,(\phi)} = CT_{(m)}\partial_\nu\phi$  and  $\partial^\mu T_{\mu\nu,(m)} = -CT_{(m)}\partial_\nu\phi$  has been considered in [33]. The value of  $C$  is then constrained by observations as  $|C| < 0.1$ , but this bound can be relaxed to  $|C| < 5$  if the quintessence field is assumed to only couple to dark matter [34], or if  $C$  is time-dependent [33]. In the present work, for simplicity, this coupling has not been considered, leaving it for future investigations. The energy density and pressure of the scalar field  $\phi$  resulting from the action (2.25) are

$$\rho_\phi = \frac{1/2(1 + \xi\phi^2)\dot{\phi}^2 + V + 3/4\alpha\dot{\phi}^4}{(1 + \xi\phi^2)^2 + 4\alpha V}, \quad p_\phi = \frac{1/2(1 + \xi\phi^2)\dot{\phi}^2 - V + 1/4\alpha\dot{\phi}^4}{(1 + \xi\phi^2)^2 + 4\alpha V}, \quad (2.26)$$

they enter in the Friedmann equations (2.4), that we recall here

$$H^2 + \frac{k}{a^2} = \frac{\rho_m + \rho_\phi}{3} \quad \text{and} \quad 2\dot{H} + 3H^2 + \frac{k}{a^2} = -(p_m + p_\phi) = -w\rho_m - p_\phi,$$

notice that we consider the presence of an additional perfect fluid component, sometimes called the background fluid, with an arbitrary constant equation of state parameter  $w$ . The equation of motion of the scalar field is

$$\begin{aligned} & \ddot{\phi}(1 + \xi\phi^2 + 3\alpha\dot{\phi}^2) + 3H\dot{\phi}(1 + \xi\phi^2 + \alpha\dot{\phi}^2) + V' + \xi\phi\dot{\phi} \\ & - \frac{\xi\phi(1 + \xi\phi^2) + \alpha V'}{(1 + \xi\phi^2)^2 + 4\alpha V} (2(1 + \xi\phi^2)\dot{\phi}^2 + 4V + 3\alpha\dot{\phi}^4) = 0. \end{aligned} \quad (2.27)$$

As we explained in section 2.2, the system would be much different in the metric formulation, because in that case the quadratic correction  $\alpha R^2$  introduces a second scalar degree of freedom, resulting in a two-fields dynamics. Nevertheless, the non-minimal coupling  $\xi\phi^2 R$  has been considered in this formulation, see for instance [35, 36].

In the rest of this chapter, we study the system (2.26)-(2.27) with an exponential potential  $V(\phi) = M^4 \exp(-\lambda\phi)$ . More precisely study whether the system possesses realistic solutions that realize the sequence *radiation domination*  $\rightarrow$  *matter domination*  $\rightarrow$  *accelerated expansion* with the correct duration and the scalar field staying subdominant during the two first eras. We start by considering the quadratic correction  $\alpha R^2$  alone.

### 2.3.2 Quadratic correction

We start by considering the case where the only non-standard term in (2.25) is the quadratic correction  $\alpha R^2$ . In this case, the action of the scalar field is

$$\mathcal{S}_\phi = \int d^4x \sqrt{-g} \left( -\frac{1}{1+4\alpha V(\phi)} \left( \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{\alpha}{4}(\partial\phi)^4 \right) \right). \quad (2.28)$$

It can be thought of as the action of a scalar field with a non-canonical kinetic term in the effective potential  $U(\phi) = V(\phi)/(1+4\alpha V(\phi))$ . As soon as  $4\alpha V(\phi) \gg 1$ , this new potential exhibits a plateau  $U(\phi) \simeq 1/(4\alpha)$ . This observation is at the origin of the mechanism proposed by [26,27] which makes it possible to flatten inflationary potentials and reconcile them with observations. In the case of an exponential potential  $V(\phi) = M^4 \exp(-\lambda\phi)$ , the effective potential  $U(\phi)$  has this plateau for  $\phi \rightarrow -\infty$ , and an exponential tail for  $\phi \rightarrow +\infty$ .

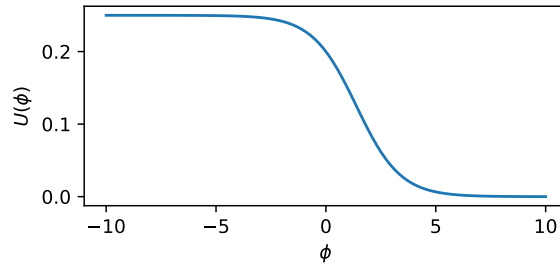


Figure 2.4: Shape of  $U(\phi) = V(\phi)/(1+4\alpha V(\phi))$  with  $V(\phi) = M^4 \exp(-\lambda\phi)$ .

with respect to the standard exponential potential model that we have exposed in 2.1.2, the only new regimes visible on the action (2.28) are when  $4\alpha V \gg 1$ , corresponding to the plateau, and/or  $\alpha/2\dot{\phi}^2 \gg 1$ . So in order to diverge from the previous analysis, the field must enter one of these regimes. If it starts on the plateau, it will roll down and end up in the exponential tail, where accelerated expansion takes place as a future attractor if and only if  $\lambda^2 < 2$ . If it starts with  $\alpha/2\dot{\phi}^2 \gg 1$ , i.e. with a large kinetic energy, it will quickly be slowed down by the Hubble friction term of its equations of motion. From this, we can have the intuition that the quadratic correction does not modify the future attractors, with respect to the analysis of 2.1.2, but only affects past attractors and saddle points. We can also guess that, if we want the  $\alpha R^2$  term to play a role in the late-time dynamics, we need the dimensionless parameter  $\alpha$  to take an extreme value. For instance, if the transition between matter domination and accelerated expansion happens when  $4\alpha V \sim 1$ , estimating that at this time  $V \sim \Lambda \sim 10^{-122}$ , see (2.9), sets  $\alpha \sim 10^{120}$ . For more reasonable values of  $\alpha$ , the  $\alpha R^2$  is not expected to play any role in the late time evolution. Nevertheless, it is interesting, at least academically, to study how dynamical systems methods apply to this case. When  $\xi = 0$ , the equations (2.26)-(2.27) become

$$H^2 = \frac{2\dot{\phi}^2 + 3\alpha\dot{\phi}^4 + 4V}{12(1+4\alpha V)} + \frac{\rho_m}{3} \quad \text{and} \quad \dot{H} = -\frac{\dot{\phi}^2(1+\alpha\dot{\phi}^2)}{2(1+4\alpha V)} - \frac{1}{2}(1+w)\rho_m. \quad (2.29)$$

and

$$\ddot{\phi}(1+3\alpha\dot{\phi}^2) + 3H\dot{\phi}(1+\alpha\dot{\phi}^2) + \frac{V'}{1+4\alpha V}(1-2\alpha\dot{\phi}^2-3\alpha^2\dot{\phi}^4) = 0, \quad (2.30)$$

As in (2.13), the starting point is to choose dynamical variables in order to rewrite (2.29) and (2.30) as an autonomous system. In the case of a positive potential, this is achieved with

$$x = \frac{\dot{\phi}}{2\sqrt{V}}, \quad y = \frac{\dot{\phi}}{2H}, \quad \lambda = -\frac{V'}{V} \quad \text{and} \quad \nu = 4\alpha V. \quad (2.31)$$

As we said earlier, we restrict our analysis to an exponential potential, for which  $\lambda$  is constant. These are not the only variables that can be chosen to analyze the system, but they are convenient in the  $\nu \gg 1$  regime as we will soon see. They obey the following system

$$\frac{dx}{dN} = \frac{\lambda y - 6(1 + \nu)x^2(1 + \nu x^2) + \lambda x^2 y(2 + 3\nu(2 + \nu)x^2)}{2(1 + \nu)x(1 + 3\nu x^2)}, \quad (2.32)$$

and

$$\begin{aligned} \frac{dy}{dN} = & -\frac{y}{2(1 + \nu)x^2(1 + 3\nu x^2)} \\ & \times \left( y((1 + w)y - \lambda) - 3(1 - 3w)\nu^2 x^6 y^2 \right. \\ & - \nu x^4(3 + 3\nu - 3\lambda\nu y + 7y^2 + 9w(1 + \nu - y^2)) \\ & \left. + x^2((1 - w)(3 - 2y^2) + \nu(2\lambda y + 3(1 + y^2) + 3w(y^2 - 1))) \right), \end{aligned} \quad (2.33)$$

and finally

$$\frac{d\nu}{dN} = -2\lambda\nu y. \quad (2.34)$$

For non exponential potentials, there would also be an equation for  $\lambda$  as in (2.15). The system is invariant under  $x \rightarrow -x$ , so we can restrict ourselves to  $x > 0$  and obtain trajectories with  $x < 0$  by reflection. It is also invariant under  $(y, \lambda) \rightarrow -(y, \lambda)$ , so we can restrict ourselves to  $\lambda > 0$ . Furthermore,  $x > 0$  implies  $\dot{\phi} > 0$ . Then, we would have  $y < 0$ , if and only if  $H < 0$ , see (2.31). Therefore, restricting ourselves to expanding cosmologies implies that  $y > 0$ . Since  $dy/dN = 0$  when  $y = 0$ , the system cannot dynamically cross  $y = 0$  and  $y$  remains positive during the entire evolution. Assuming  $\alpha > 0$  also implies  $\nu > 0$ . So in the end, all variables can be taken positive. Note that the scalar field energy density parameter  $\Omega_\phi = \rho_\phi/H^2$  and the equation of state parameter  $w_\phi = p_\phi/\rho_\phi$  can be expressed as follows

$$\Omega_\phi = \frac{y^2(1 + 2x^2 + 3\nu x^4)}{3x^2(1 + \nu)} \quad \text{and} \quad w_\phi = -\frac{1 - 2x^2 - \nu x^4}{1 + 2x^2 + 3\nu x^4}. \quad (2.35)$$

The values taken by the  $(x, y, \nu)$  are constrained by  $\Omega_\phi < 1$ , in order to avoid a negative energy  $\rho_m < 0$ . We will also make use of the effective equation of state parameter, already defined in (2.16)

$$w_{\text{eff}} = w(1 - \Omega_\phi) + w_\phi\Omega_\phi.$$

As before, we are interested in the fixed points of the system, which contain its qualitative dynamics. Let us start by considering equation (2.34). Since the variable  $\nu$  can take any positive value, it is convenient to define variable  $\tilde{\nu} = \nu/(1 + \nu)$ , that take value in the compact interval  $\tilde{\nu} \in [0, 1]$ . In terms of this variable, (2.34) becomes

$$\frac{d\tilde{\nu}}{dN} = 2\lambda\tilde{\nu}(\tilde{\nu} - 1)y, \quad (2.36)$$

so the fixed points satisfying  $d\tilde{\nu}/dN = 0$  have either  $\tilde{\nu} = 0$  or  $\tilde{\nu} = 1$ , corresponding respectively to  $\nu = 0$  and  $\nu \rightarrow +\infty$ , or  $y = 0$ . Note that since  $(y, \nu) > 0$ , equation (2.34) implies that  $\nu$  decreases with  $N$ . So  $\nu \rightarrow +\infty$  (resp.  $\nu = 0$ ) corresponds to the past (resp. the future). In fact, the point  $\nu = 0$  corresponds to the standard exponential model exposed in 2.1.2. Thus, as we guessed earlier, the only modifications to the previous analysis take place at early times, when  $\nu \rightarrow +\infty$ . In this limit, the system (2.32)-(2.33) becomes

$$\frac{dx}{dN} \simeq \frac{x}{2}(\lambda y - 2) \quad \text{and} \quad \frac{dy}{dN} \simeq \frac{y}{2} \left( (1 - 3w)x^2 y^2 - \lambda y + (1 + 3w) \right), \quad (2.37)$$

while the cosmological parameters (2.35) are

$$\Omega_\phi \simeq x^2 y^2 \quad \text{and} \quad w_\phi \simeq \frac{1}{3}. \quad (2.38)$$

Therefore, the equation  $\Omega_\phi \leq 1$  does not constrain the variables  $(x, y)$  to lie in a compact region of the plane. Just like we did for  $\nu$ , we will define new variables to study the behaviour of  $x$  and  $y$  around infinity. But before that, notice that in the case where the of a background fluid of radiation ( $w = 1/3$ ), the system (2.37) can be simplified and solved explicitly

$$x(N) = \frac{x_0(2 + \lambda y_0(e^N - 1))}{2e^N} \quad \text{and} \quad y(N) = \frac{2y_0 e^N}{2 + \lambda y_0(e^N - 1)}. \quad (2.39)$$

In particular, the scalar field energy density (2.38) is constant; namely,  $\Omega_\phi \simeq x^2 y^2 = x_0^2 y_0^2$ . So if the scalar field is subdominant initially, enough to satisfy, for example, the nucleosynthesis constraint, it will remain subdominant throughout the radiation dominated era. The transition to matter domination then proceeds as usual without being affected by its presence. In the case of a background fluid of non-relativistic matter ( $w = 0$ ), the fixed points of the system not at infinity are summarised in table 2.3 with their properties:

point	$x_*$	$y_*$	eigenvalues	stability	$\Omega_\phi$	$w_{\text{eff}}$
$O$	0	0	$\{-1, 1/2\}$	saddle	0	0
$A$	$\lambda/2$	$2/\lambda$	$\{-1, 1\}$	saddle	1	$1/3$
$B$	0	$1/\lambda$	$\{-1/2, -1/2\}$	attractive	0	0

Table 2.3: fixed points of the system (2.37) that are not at infinity, with their properties.

One can see that  $B$  is the only stable point and is thus the future attractor of the system, as long as  $\nu \gg 1$ . It corresponds to a matter dominated universe, where the scalar field is subdominant, in accordance with our expectations. In that respects, we might be concerned that the system could get too close to the scalar dominated saddle point  $A$  in the course of its evolution. But from (2.37) and (2.38) we can show that that  $d\Omega_\phi/dN = \Omega_\phi(\Omega_\phi - 1) < 0$ , so  $\Omega_\phi$  only decreases. Once again, if the scalar field is subdominant initially, it remains subdominant throughout the matter dominated era, or at least as long as  $\nu \gg 1$ .

In order to study the behaviour of  $(x, y)$  at infinity, we can make a change of variable analogous to the one we made in (2.36); namely,  $\tilde{x} = x/(1 + x)$  and  $\tilde{y} = y/(1 + y)$ , so that  $(\tilde{x}, \tilde{y}) \in [0, 1]$ . The system on  $(\tilde{x}, \tilde{y})$  obtained this way has divergences, for example

$$\frac{d\tilde{x}}{dN} = -\frac{\tilde{x}(\tilde{x} - 1)(2 - (2 + \lambda)\tilde{y})}{2(\tilde{y} - 1)}, \quad (2.40)$$

and similarly for  $d\tilde{y}/dN$ , but the formula is long and not more informative. These divergences can be handled knowing that for any dynamical system  $d\mathbf{P}/dN = \mathbf{f}(\mathbf{P})$ , the system  $d\mathbf{P}/dN = \xi(\mathbf{P})\mathbf{f}(\mathbf{P})$  for positive defined  $\xi(\mathbf{P}) > 0$  has the same fixed points with the same properties. For instance, we can remove the pole in (2.40) by multiplying the entire system by  $(\tilde{y} - 1)$ . Applying this result to our case, we obtain two additional fixed points at infinity:

Point	$\tilde{x}_*$	$\tilde{y}_*$	eigenvalues	stability	$\Omega_\phi$	$w_{\text{eff}}$
$P_x$	1	0	$\{0, 0\}$	repulsive	1	1/3
$P_y$	0	1	$\{\lambda, \lambda\}$	repulsive	1	1/3

We cannot conclude about the stability of  $P_x$  using its eigenvalues, but numerical computations show that it is repulsive, see figure 2.5. For both  $P_x$  and  $P_y$ , the energy density  $\Omega_\phi = \tilde{x}^2\tilde{y}^2/((1-\tilde{x})^2(1-\tilde{y})^2)$  is undetermined but can also be found numerically to be  $\Omega_\phi = 1$ . Therefore, from (2.38) we have  $w_{\text{eff}} = 1/3$  at these points. These two points constitute the past attractors of the complete system (2.32)-(2.34).

The previous analysis applies to  $\nu \rightarrow +\infty$ , but in reality  $\nu$  is finite. In particular, in this strict limit, we found that the future attractor of the system is  $B$ , and that it has  $x_* = 0$ , see table 2.3. So as the system evolves towards  $B$ , we have  $\nu x^4 \ll 1$  at some point, even though  $\nu$  is large, and we need to study the system (2.32)-(2.33) in another limit. Namely, we consider the limit  $\nu \gg 1$  and  $x \ll 1$  such that  $\nu x^2 \sim 1$ . In that case, the system becomes

$$\begin{aligned} \frac{dx}{dN} &\simeq \frac{x}{2}(\lambda y - 2) + \frac{1}{\nu x} \left( \frac{\lambda y}{6\nu x^2} - 1 \right), \\ \frac{dy}{dN} &\simeq \frac{y}{2} \left( 1 - \lambda y + \frac{y(\lambda - y)}{\nu x^2} - \frac{4(1 + \lambda y)}{(1 + 3\nu x^2)} \right), \end{aligned} \quad (2.41)$$

where we considered  $w = 0$  for simplicity. The cosmological parameters (2.38) become

$$\Omega_\phi \simeq \frac{y^2}{3\nu x^2} \quad \text{and} \quad w_\phi \simeq -1. \quad (2.42)$$

The system (2.41) has one real fixed point, with an expression that is too complicated to be useful. We can simplify it further by neglecting the first term  $x/2(\lambda y - 2)$  in (2.41), using  $x \ll 1$ . This is not strictly correct, because the system would then be attracted to a fixed point that cancels the second term only. But when the second term vanishes, the first one is not negligible anymore. If we do it anyway, we get a simple approximation of the fixed point

$$x_*^T \simeq \sqrt{\frac{\lambda y_*^T}{6\nu}}, \quad \text{and} \quad y_*^T \simeq \frac{12\lambda}{12 + 3\lambda^2 + \sqrt{3(48 + 72\lambda^2 + 11\lambda^4)}}, \quad (2.43)$$

and it can be shown to be stable. This fixed point, called  $T$ , is the one to which the system is attracted, after it got so close to  $B$  that  $\nu x^2 \sim 1$ . It depends on  $\nu$ , which is decreasing according to (2.34), so it is not really a fixed point, but a point that is tracked by the system. Since the system lags behind this point as it moves, the expressions (2.43) do not describe its behaviour very precisely. In particular, using (2.43) with (2.42) gives  $\Omega_\phi \simeq 2y_*^T/\lambda$  but we will observe numerically that  $\Omega_\phi \simeq 1$  when the system is tracking point  $T$ .



In Figure 2.5, we plot the phase space of the system (2.32)-(2.33) for  $\nu \gg 1$  and  $\lambda = 1$ . In the case  $w = 1/3$ , the trajectories are lines of constant  $\Omega_\phi = x^2 y^2$  which all converge to the fixed line  $y = 2/\lambda$ , as found in (2.39). The origin  $O$  is a saddle point. In the case  $w = 0$ , we can observe the presence of the fixed points listed in table 2.3, with the right properties. The behavior at infinity also corresponds to that discussed earlier. Since  $\nu$  is finite in this plot,  $B$  is not the true future attractor. When  $x$  becomes small, so that  $\nu x^2 \sim 1$ , the system is actually attracted to the point  $T$  located approximately at coordinates (2.43). As  $\nu$  decreases, this point moves to higher values of  $x$ , until the approximation  $\nu \gg 1$  is no longer valid.

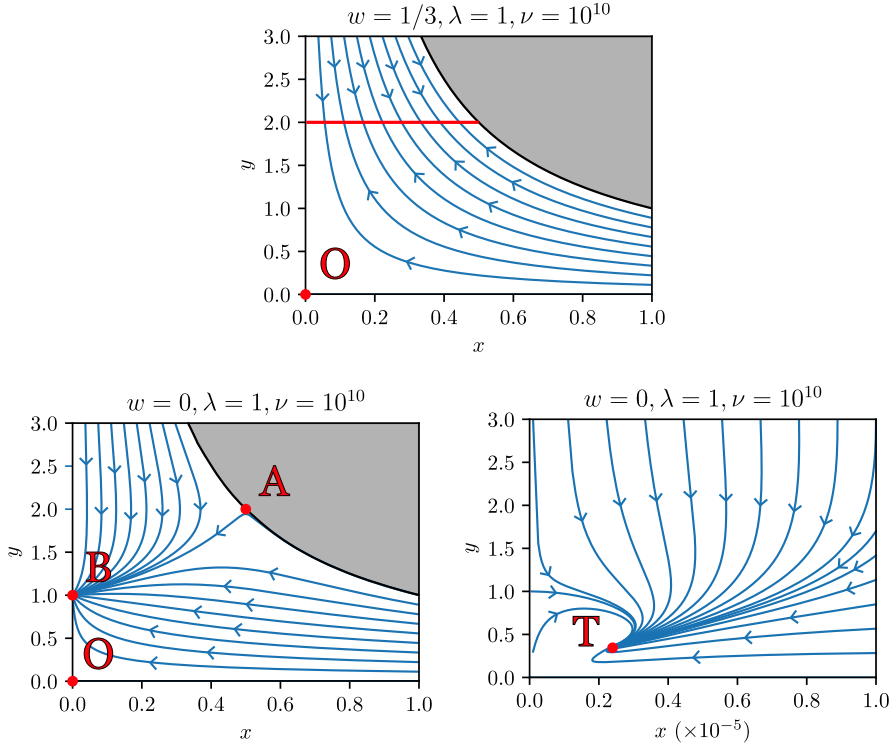


Figure 2.5: Phase space of the system (2.32)-(2.33) with  $\nu = 10^{10}$  and  $\lambda = 1$ . We have  $w = 1/3$  on the top row and  $w = 0$  on the bottom row. The bottom right panel is a zoom for small values of  $x$ . Fixed points are labeled as in table 2.3, and point  $T$  is given in (2.43). The shaded region is the one excluded by the positive energy constraint  $\Omega_\phi < 1$ .

At late times, we have  $\nu \rightarrow 0$ , so the  $\alpha R^2$  correction is negligible and the behaviour of the system is as described in 2.1.2. Let us recall the main features. If  $\lambda^2 < 3(1 + w)$ , the future attractor corresponds to a universe dominated by the scalar field, with  $w_\phi = \lambda^2/3 - 1$ , so there is accelerated expansion if  $\lambda^2 < 2$ . More precisely, [11] gives the experimental constraint  $w_\phi < -0.94$  which implies  $\lambda < 0.42$ . If  $3(1 + w) < \lambda^2$ , the future attractor corresponds to a scaling solution where the scalar field and the background fluid both share the energy of the universe, with  $\Omega_\phi = 3(1 + w)/\lambda^2$ , and the scalar field mimics the background fluid,  $w_{\text{eff}} = w$ .

The evolution of the scalar field parameters (2.35) dictated by the system (2.32)-(2.34), is shown in figure 2.6 for different values of  $\lambda$ . In all cases, we observe evolutions that confirm the previous analysis. We consider  $\nu \gg 1$  initially, so that (2.38) applies and  $w_\phi \simeq 1/3$ . The system is attracted towards point  $B = (0, 1/\lambda)$  of table 2.3 and quickly reaches  $\Omega_\phi \simeq 0$ . Then, as  $x \rightarrow 0$ , we enter the regime where (2.42) applies and  $w_\phi \simeq -1$ . The system tracks the point  $T$ , and in doing so  $\Omega_\phi \simeq 1$ . Finally, as  $\nu$  keeps on decreasing, following (2.34), we end up with  $\nu \simeq 0$  where we recover the standard exponential quintessence behaviour.

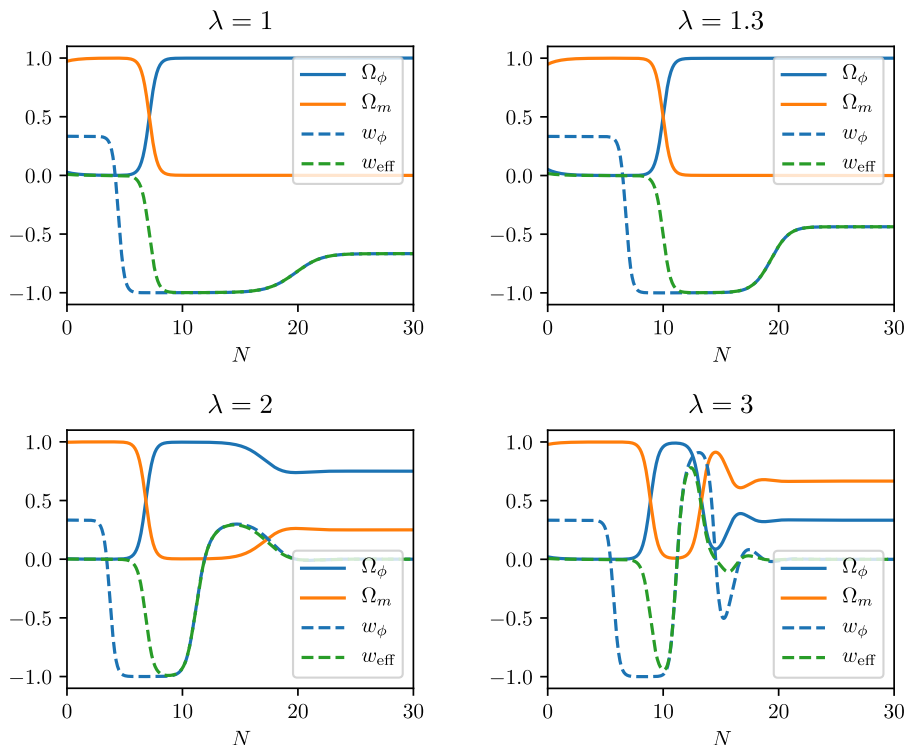


Figure 2.6: Evolution of the scalar field parameters (2.35) dictated by (2.32)-(2.34), for  $w = 0$  and different values of  $\lambda$ . The initial condition  $\nu_i = 10^{10}$  has been chosen to obtain between 8 and 10 e-folds of matter domination and  $(x_i, y_i)$  to have  $\Omega_\phi < 0.2$  initially.

An interesting feature of this evolution is the presence of an accelerated expansion era, with  $w_\phi \simeq -1$ , between matter-scalar equality and the standard quintessence era. It corresponds to when the system is tracking the point  $T$  of figure 2.5. This era only gets shorter as  $\lambda$  increases, effectively disappearing beyond  $\lambda \simeq 3$ . Its presence alleviates the bound  $\lambda < 0.42$  that ensured  $w_\phi < -0.95$  for the standard quintessence in an exponential potential.

So, we have shown that, as long as  $\lambda < 3$ , the system (2.32)-(2.34) can lead to a realistic background evolution of the universe, where the duration of the matter domination era is set by the initial value of  $\nu$ , and  $w_{\text{eff}} \sim -1$  for some time afterwards. Now let us discuss the value of  $\alpha$  that achieves this result. From the definition of the dynamical variables (2.31), at matter-scalar equality

$$\alpha = \frac{\nu_{\text{eq}} x_{\text{eq}}^2}{4y_{\text{eq}}^2} \frac{1}{(\kappa H_{\text{eq}})^2} \simeq \frac{\nu_{\text{eq}} x_{\text{eq}}^2}{4y_{\text{eq}}^2} \times 10^{121},$$

where we reintroduced  $\kappa = 1.1 \times 10^{-19} \text{ GeV}^{-1}$  and used the fact that matter-scalar equality happens now, so  $H_{\text{eq}} \sim H_0 \sim 1.46 \times 10^{-42} \text{ GeV}$ . As we can see, the value of  $H_0$  gives a huge factor  $10^{120}$  in this equation, that implies a huge value for the parameter  $\alpha$  unless  $\nu_{\text{eq}} x_{\text{eq}}^2 / (4y_{\text{eq}}^2) \ll 1$ . But matter-scalar equality happens when the system starts tracking the point  $T$ , and even if the formula (2.42) that gives  $\Omega_\phi$  at this time is approximate, we can use it to obtain

$$\Omega_{\phi, \text{eq}} \simeq \frac{y_{\text{eq}}^2}{3\nu_{\text{eq}} x_{\text{eq}}^2} \sim 0.5 - 0.7 \quad \rightarrow \quad \frac{\nu_{\text{eq}} x_{\text{eq}}^2}{4y_{\text{eq}}^2} \sim 0.1,$$

which is not much smaller than 1. So there is no way to lower the value of  $\alpha \sim 10^{120}$ , with the current Hubble rate being what it is. This value, that we already guessed in the introduction to this section, comes from the attempt to make the  $\alpha R^2$  term play a major role in the dynamic. Reversing this conclusion, it is safe to say that for reasonable values of  $\alpha$ , the  $\alpha R^2$  term never plays a role in the late time evolution, at least in the Palatini formulation.

### 2.3.3 Non-minimal coupling

Let us now consider the case where the only non-standard term in (2.25) is the non-minimal coupling  $\xi\phi^2 R$ . In this case, the action of the scalar field is

$$\mathcal{S}_\phi = \int d^4x \sqrt{-g} \left( -\frac{1}{(1 + \xi\phi^2)^2} \left( \frac{1}{2}(1 + \xi\phi^2)(\partial\phi)^2 + V(\phi) \right) \right),$$

and we could rewrite equations (2.26)-(2.27). In particular, the Hubble constraint becomes

$$H^2 = \frac{\dot{\phi}^2}{6(1 + \xi\phi^2)} + \frac{V}{3(1 + \xi\phi^2)^2} + \frac{\rho_m}{3},$$

so we could think of introducing the following dynamical variables

$$X^2 = \frac{\dot{\phi}^2}{6H^2(1 + \xi\phi^2)}, \quad Y^2 = \frac{V}{3H^2(1 + \xi\phi^2)^2}, \quad Z^2 = \phi^2 \quad \text{and} \quad \lambda = -\frac{V'}{V},$$

before proceeding to the analysis as in the previous sections. In this case, we would have  $\Omega_\phi = X^2 + Y^2$ , constraining  $(X, Y)$  to lie in the unit disk. However, it is more convenient to introduce a canonical scalar field as follows

$$\Phi(\phi) = \int \frac{d\phi}{\sqrt{1 + \xi\phi^2}} = \frac{1}{\sqrt{\xi}} \sinh^{-1}(\sqrt{\xi}\phi) \quad \rightarrow \quad \phi(\Phi) = \frac{1}{\sqrt{\xi}} \sinh(\sqrt{\xi}\Phi). \quad (2.44)$$

In terms of this new scalar, the action (2.25) becomes

$$\mathcal{S}_\Phi = \int d^4x \sqrt{-g} \left( -\frac{1}{1 + 4\alpha U(\Phi)} \left( \frac{1}{2}(\partial\Phi)^2 + U(\Phi) - \frac{\alpha}{4}(\partial\Phi)^4 \right) \right), \quad (2.45)$$

where, in the case of an exponential potential  $V(\phi) = M^4 \exp(-\lambda\phi)$

$$U(\Phi) = \frac{V(\phi(\Phi))}{(1 + \xi\phi(\Phi)^2)^2} = \frac{M^4 \exp(-\lambda/\sqrt{\xi} \sinh(\sqrt{\xi}\Phi))}{\cosh^4(\sqrt{\xi}\Phi)}, \quad (2.46)$$

Thus, when  $\alpha = 0$ , the action (2.45) boils down to a canonical scalar field  $\Phi$  in the potential  $U(\Phi)$ , which contains all the  $\xi$ -dependence. We can then understand most of the dynamics of the system simply by studying the variations of  $U(\Phi)$ . When  $\Phi \rightarrow -\infty$ , it goes to  $+\infty$ , while when  $\Phi \rightarrow +\infty$ , it goes to 0, faster than exponentially in both cases. As we saw in section 2.1.2, for a canonical scalar field in a simple exponential potential  $\exp(-\lambda\phi)$ , accelerated expansion requires it to be not too steep, with  $\lambda^2 < 2$ . Therefore, accelerated expansion should not be expected to occur at the tail of the potential  $U(\Phi)$ , which is steeper than any exponential. If  $4\xi > \lambda^2$ , the potential  $U(\Phi)$  has two local extrema, located at (in terms of non canonical  $\phi$ )

$$\phi_{\min/\max} = -\frac{2}{\lambda} \left( 1 \pm \sqrt{1 - \frac{\lambda^2}{4\xi}} \right),$$

corresponding to

$$U_{\min/\max} = \frac{\lambda^4 M^4 \exp\left(2 \pm 2\sqrt{1 - \lambda^2/(4\xi)}\right)}{64\xi^2 \left(1 \pm \sqrt{1 - \lambda^2/(4\xi)}\right)^2}, \quad (2.47)$$

or, when  $4\xi \gg \lambda^2$

$$U_{\min} \simeq \frac{\exp(4)\lambda^4 M^4}{256\xi^2} \quad \text{and} \quad U_{\max} \simeq M^4. \quad (2.48)$$

In figure 2.7, we give an example of the potential  $U(\Phi)$  for some values of  $\lambda$  and  $\xi$ .

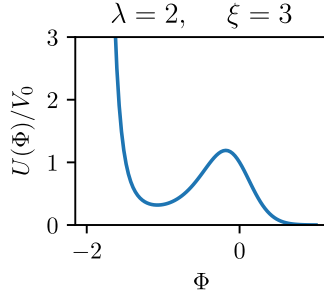


Figure 2.7: Potential  $U(\phi)$  of (2.46), for  $\lambda = 2$  and  $\xi = 3$ .

We will show that the system has two future attractors: one where the field ends up in the local minimum of the potential  $U(\Phi)$ , and another where it rolls down its tail. The choice between the two depends essentially on whether the field starts on one side or the other of the local maximum. The local minimum is the only attractor that corresponds to accelerated expansion. There the field acts as a cosmological constant  $\Lambda_{\text{eff}} \sim U_{\min}$ , in which the parameters  $\lambda, \xi$  play a role, see in (2.48). To reproduce the current value of  $H_0$ , we need

$$\frac{\lambda^2 M^2}{\xi} \sim 10^{-23} \text{ GeV}^2. \quad (2.49)$$

Note that in the region  $\sqrt{\xi}|\Phi| \ll 1$ , the potential  $U(\Phi)$  in (2.46) is approximated by a simple exponential potential  $U(\Phi) \simeq M^4 \exp(-\lambda\Phi)$ . The behavior in this usual exponential discussed in section 2.1.2 can thus be recovered transiently when the field is in this region.

The action (2.45) with  $\alpha = 0$ , is actually the same as in equation (2.10), so it can be studied using the same variables (2.13)

$$x = \frac{\dot{\phi}}{\sqrt{6}H}, \quad y = \frac{1}{H}\sqrt{\frac{V}{3}}, \quad \text{and} \quad \Lambda = -\frac{U'}{U}, \quad (2.50)$$

in terms of which  $\Omega_\phi = x^2 + y^2$  and  $w_\phi = (x^2 - y^2)/(x^2 + y^2)$ , and they follow the system

$$\begin{aligned} \frac{dx}{dN} &= -\frac{3}{2} \left( 2x + (w-1)x^3 + x(w+1)(y^2-1) - \sqrt{\frac{2}{3}}\Lambda y^2 \right), \\ \frac{dy}{dN} &= -\frac{3}{2}y \left( (w-1)x^2 + (w+1)(y^2-1) + \sqrt{\frac{2}{3}}\Lambda x \right), \\ \frac{d\Lambda}{dN} &= -\sqrt{6}(\Gamma-1)\Lambda^2 x, \quad \text{where} \quad \Gamma = \frac{UU''}{U'^2} \end{aligned} \quad (2.51)$$

If  $\Lambda(\Phi)$  is invertible, we can express  $\Gamma$  as a function of  $\Lambda$  to close the system. In our case

$$\Lambda(\Phi) = \lambda \cosh(\sqrt{\xi}\Phi) + 4\sqrt{\xi} \tanh(\sqrt{\xi}\Phi),$$

is not invertible. Therefore, we split the analysis into three regions, the region of large negative  $\sqrt{\xi}\Phi$ , the region of small  $\sqrt{\xi}\Phi$ , and the region of large positive  $\sqrt{\xi}\Phi$ . In the region of small  $\sqrt{\xi}|\Phi|$ , the potential  $U(\Phi)$  in (2.46) is approximated by a simple exponential and the system transiently behaves as discussed in section 2.1.2.

### The region of large negative $\sqrt{\xi}\Phi$ .

In the region  $\sqrt{\xi}|\Phi| \gg 1$  with  $\Phi < 0$

$$\Lambda(\Phi) \simeq \frac{\lambda}{2} \left( -\sqrt{\xi}\Phi \right) - 4\sqrt{\xi} \quad \text{and} \quad \Gamma - 1 \simeq \frac{4\xi + \Lambda\sqrt{\xi}}{\Lambda^2}. \quad (2.52)$$

In particular, this limit corresponds to  $\Lambda \gg -4\sqrt{\xi}$ , so we exit this region when  $\Lambda \sim -4\sqrt{\xi}$ . With this expression of  $\Gamma$ , the system (2.51) has only two fixed points listed in table 2.4.

point	$x_*$	$y_*$	$\Lambda_*$	eigenvalues	stability	$\Omega_\phi$	$w_{\text{eff}}$
$O$	0	0	–	$\left\{0, \frac{3}{2}(w \pm 1)\right\}$	saddle	0	$w$
$C$	0	1	0	$\left\{-3(1+w), -\frac{3}{2} \pm \frac{1}{2}\sqrt{3(3-16\xi)}\right\}$	stable	1	–1

Table 2.4: Fixed points of the system (2.51) in the region  $\sqrt{\xi}|\Phi| \gg 1$  with  $\Phi < 0$ . In the column  $\Lambda_*$ , we put a – when  $\Lambda$  is unconstrained by the fixed point equation.

The only stable point is  $C$ , which is thus the future attractor of the system in the region of large negative  $\sqrt{\xi}\Phi$ . Since  $\Lambda_* = 0$  at this point, the limit  $\Lambda \gg -4\sqrt{\xi}$  that defines the region is satisfied, and point  $C$  is a future attractor of the whole system. It corresponds to the scalar field  $\Phi$  standing still in the local minimum of the potential  $U(\Phi)$ , see figure 2.7, because  $\Lambda_* = 0 \leftrightarrow U' = 0$  from the definition (2.50). At this point, the scalar field dominates and acts as a cosmological constant, so the universe undergoes accelerated expansion.

Since we consider the region  $\Lambda \gg -4\sqrt{\xi}$ , we have  $\Gamma - 1 \ll 1$ , see (2.52), and as a consequence  $\Lambda$  is slowly varying due to equation (2.51). Therefore, we automatically have  $d\Lambda/dN \simeq 0$  and the points where only  $dx/dN = dy/dN = 0$ , with  $\Lambda$  variable, act as would-be fixed points, that are moving but closely tracked by the system. These points are actually the same as in table 2.2, replacing  $\lambda$  by  $\Lambda$ . We recall them in table 2.5 for convenience

point	$x_*$	$y_*$	eigenvalues	stability and existence	$\Omega_\phi$	$w_{\text{eff}}$
$A_{\pm,\Lambda}$	$\pm 1$	0	$\{3(1-w), 3 \mp \sqrt{\frac{3}{2}}\Lambda\}$	unstable if $\Lambda \geq \pm\sqrt{6}$ saddle if $\Lambda < \pm\sqrt{6}$	1	1
$B_\Lambda$	$\sqrt{\frac{3}{2}} \frac{1+w}{\Lambda}$	$\sqrt{\frac{3(1-w^2)}{2\Lambda^2}}$	$\{\frac{3}{4\Lambda}((w-1)\Lambda \pm \Delta)\}$	exists if $\Lambda^2 \geq 3(1+w)$ stable	$\frac{3(1+w)}{\Lambda^2}$	$w$
$C_\Lambda$	$\frac{\Lambda}{\sqrt{6}}$	$\sqrt{1 - \frac{\Lambda^2}{6}}$	$\{\frac{\Lambda^2}{2} - 3, \Lambda^2 - 3(w+1)\}$	exists if $\Lambda^2 < 6$ stable if $\Lambda^2 < 3(1+w)$ saddle if $3(1+w) \leq \Lambda^2$	1	$\frac{\Lambda^2}{3} - 1$

Table 2.5: Would-be fixed points of the system (2.51), solving only for  $dx/dN = dy/dN = 0$ .

Starting from  $\Lambda \gg 1$ , the system is attracted to  $B_\Lambda$ , which is the only future attractor. At this point,  $\Omega_\phi = 3(1+w)/\Lambda^2 \ll 1$ , so it corresponds to a matter dominated universe. At this point,  $\Lambda$  decreases due to equation (2.51), because  $\Gamma - 1 > 0$  in the  $\Lambda \gg -4\sqrt{\xi}$  region. After some time, the stability condition  $\Lambda^2 > 3(1+w)$  breaks and the system is attracted to  $C_\Lambda$ . The point  $C_\Lambda$  then remains stable forever, as  $\Lambda$  continues to decrease, and it converges to point  $C$  of table 2.4, which is the future attractor of the system in the region of large negative  $\sqrt{\xi}\Phi$ . This point corresponds to a scalar dominated universe undergoing accelerated expansion, as the field is trapped in the local minimum of the potential and acts as a cosmological constant.

### The region of large positive $\sqrt{\xi}\Phi$ .

In the region  $\sqrt{\xi}|\Phi| \gg 1$  with  $\Phi > 0$ , only some signs change with respect to (2.52)

$$\Lambda(\Phi) \simeq \frac{\lambda}{2} \exp(\sqrt{\xi}\Phi) + 4\sqrt{\xi} \quad \text{and} \quad \Gamma - 1 \simeq \frac{4\xi - \Lambda\sqrt{\xi}}{\Lambda^2},$$

and the analysis is mostly the same. The only difference is that  $\Gamma - 1 < 0$ , so now  $\Lambda$  increases due to equation (2.51). Therefore, starting from  $\Lambda^2 > 3(1+w)$ , the system is attracted to  $B_\Lambda$ , and remains so forever, because the stability condition  $\Lambda^2 > 3(1+w)$  never breaks. This point corresponds to a matter dominated universe, where the field is rolling down the steep tail of the potential of figure 2.7.

### Initial positions of the scalar field.

From this analysis, we know that the only trajectories leading to a realistic background evolution, with a matter domination followed by accelerated expansion, are with the field starting around the local minimum of the effective potential  $U(\Phi)$ , which is present when  $4\xi > \lambda^2$ . In this case, the period of matter domination corresponds to the point  $C_\Lambda$  in table 2.5. We can use this information to approximate the initial position of the field leading to a given number  $N$  of e-folds of matter domination before accelerated expansion. There are two such positions:  $\Phi_i^L$  (left) and  $\Phi_i^R$  (right), on each side of the minimum.

To compute  $\Phi_i^L$ , we assume that  $\Lambda$  is large initially to approximate  $\Gamma - 1 \simeq \sqrt{\xi}/\Lambda$  in (2.52). With this and the value of  $x_*$  corresponding to  $C_\Lambda$  in table 2.5, we obtain from (2.51):

$$\frac{d\Lambda}{dN} \simeq -3(1+w)\sqrt{\xi} \quad \rightarrow \quad \Lambda \simeq -3(1+w)\sqrt{\xi}N + \Lambda_i.$$

Since the local minimum corresponds to  $\Lambda = 0$ , we reach it after  $N_f$  e-folds if initially

$$\Phi_i^L \simeq -\frac{1}{\sqrt{\xi}} \log \left( 6(1+w) \frac{\sqrt{\xi}}{\lambda} N_f \right). \quad (2.53)$$

For  $\Phi_i^R$ , we can approximate  $\Lambda \simeq -4\sqrt{\xi}$  and write  $\Lambda = -4\sqrt{\xi} + \Lambda'$ , with small  $\Lambda'$ , to get

$$\frac{d\Lambda'}{dN} \simeq \frac{3}{4}(1+w)\Lambda' \quad \rightarrow \quad \Lambda' \simeq \left( \Lambda_i + 4\sqrt{\xi} \right) \exp \left( \frac{3}{4}(w+1)N \right).$$

Once again, we reach the minimum  $\Lambda = 0$  after  $N_f$  e-folds provided that initially

$$\Phi_i^R \simeq -\frac{1}{\sqrt{\xi}} \left( \ln \left( \frac{8\sqrt{\xi}}{\lambda} \right) - \frac{3}{4}(1+w)N_f \right). \quad (2.54)$$

### Some numerical results.

In Figure 2.8, we show the evolution of the system dictated by (2.51), for  $w = 0$ , different values of  $(\lambda, \xi)$ , and initial  $\Phi_i = \Phi_i^R$  given by (2.54), with  $N = 10$ . These numerical results confirm the previous analysis. At first, the system is attracted to the point  $B_\Lambda$  of table 2.5, where  $\Omega_\phi = 3/\Lambda^2$ . If the evolution starts with the field at  $\Phi_i^R$ , as is the case in figure 2.8, we have  $\Lambda \simeq -4\sqrt{\xi}$  and  $\Omega_\phi \simeq 3/(16\xi)$  during this era. If  $\xi$  is not large enough, we can thus have a significant  $\Omega_\phi$ . For instance,  $\xi = 1$  in two of the examples of figure 2.8 leads to  $\Omega_\phi \simeq 0.19$ . If the evolution starts with the field at  $\Phi_i^L$ , we have  $\Lambda \gg 1$  during the matter dominated era and  $\Omega_\phi \gg 1$ , regardless of the value of  $\xi$ . In any case, the system ends up attracted to point  $C$  of Table 2.4 corresponding to accelerated expansion, after the desired number of e-folds.

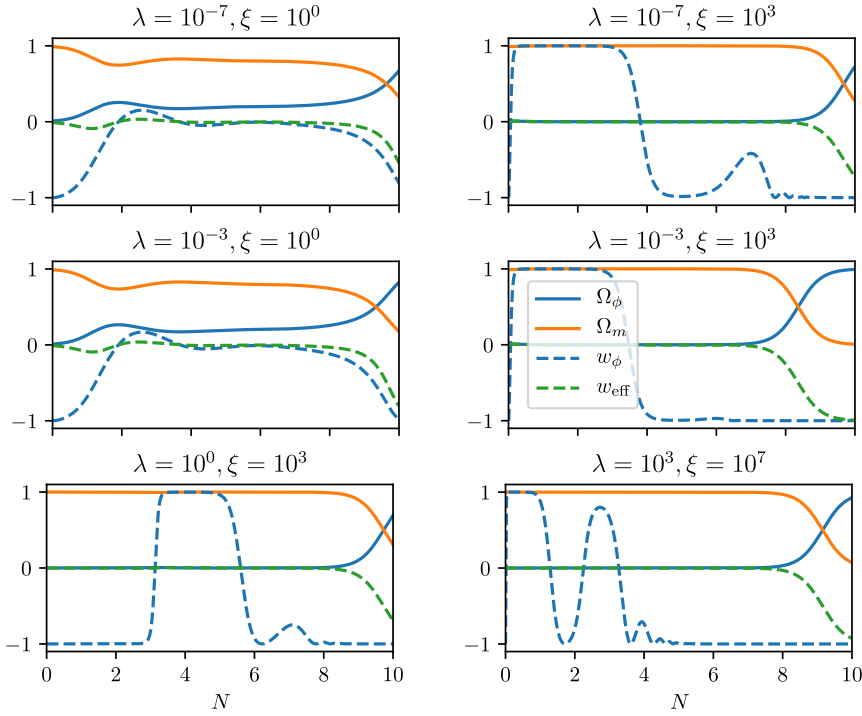


Figure 2.8: Evolution of the system dictated by (2.51), for  $w = 0$  and different values of  $(\lambda, \xi)$ , initial  $\Phi_i = \Phi_i^R$ , given by (2.54) with  $N = 10$ , and initial  $\Omega_{\phi,i} \simeq 0.1$ .

In table 2.6, we give the values of the parameters associated with figure 2.8, which further validate our analysis. In particular, the values  $\Phi_i^{L,R}$ , tuned numerically to have  $N \simeq 10$  e-folds of matter domination, correspond to those computed with (2.53) and (2.54), and do not depend on the initial scalar energy density, as long as this latter is small enough for the field to be subdominant initially. Therefore, for each value of  $\lambda, \xi$ , the initial value of the field is entirely determined by the duration requirement of the matter-dominated era.

Note that in the last line of table 2.6, there is no value for  $\Phi_i^R$ . This is because, at this stage, there is not enough distance between the local minimum and the local maximum of the potential  $U$  (2.47) for the field to roll during enough e-folds of matter domination. So above  $\lambda \sim 10^3$ , the field can only start with initial value  $\Phi_i^L$  to obtain the desired evolution. Note also that the displacement of the scalar field remains subplanckian as long as  $\xi \gtrsim 10$ .

Finally, in table 2.6, we added a column for  $\dot{\Phi}_{\max}^2$ , in order to discuss the importance of the  $\alpha(\partial\Phi)^4$  in the action (2.45). We see that  $\dot{\Phi}^2$  never gets larger than  $10^{-38} \text{ GeV}^4$ , which is around  $10^{-114}$  in Planck units. In consequence, for reasonable values of  $\alpha$ , we can be confident that this term is completely negligible. The quadratic correction  $\alpha R^2$  could also play a role if  $\alpha U \sim 1$ . This does not happen around the local minimum of the potential  $U(\Phi)$  at least for reasonable values of  $\alpha$ , but perhaps in the region of large negative  $\Phi$  where the potential diverges rapidly. However, this region is not involved in late dynamics, and we have not further investigated the implications of the quadratic correction on this model.



$\lambda$	$\xi$	$M$ (GeV)	$\Omega_{\phi,i}^{1/2}$	$\Phi_i^L$	$\Phi_i^R$	$\dot{\Phi}_{\max}^2$ (GeV <sup>4</sup> )
$10^{-7}$	$10^0$	$4 \times 10^{-5}$	$10^{-1}$	-20.4	-11	$10^{-38}$
			$10^{-3}$	-20	-14	$10^{-43}$
	$10^3$	$1 \times 10^{-3}$	$10^{-1}$	-0.75	-0.46	$10^{-39}$
			$10^{-3}$	-0.74	-0.53	$10^{-45}$
$10^7$	$1 \times 10^{-1}$	$10^{-3}$	-0.0089	-0.007	$10^{-47}$	
$10^{-3}$	$10^0$	$1 \times 10^{-8}$	$10^{-1}$	-11.2	-2	$10^{-38}$
			$10^{-3}$	-10.9	-4	$10^{-42}$
	$10^3$	$3 \times 10^{-7}$	$10^{-1}$	-0.46	-0.2	$10^{-40}$
			$10^{-3}$	-0.45	-0.24	$10^{-47}$
$10^7$	$3 \times 10^{-5}$	$10^{-3}$	-0.006	-0.004	$10^{-47}$	
$10^0$	$10^3$	$3 \times 10^{-10}$	$10^{-3}$	-0.23	-0.0004	$10^{-45}$
	$10^7$	$3 \times 10^{-8}$	$10^{-3}$	-0.0038	-0.0017	$10^{-47}$
$10^3$	$10^7$	$3 \times 10^{-11}$	$10^{-3}$	-0.0016	-	$10^{-47}$

Table 2.6: Values of the parameters associated with some numerical evolutions of figure 2.8. The value of  $M$  is tuned to obtain the current value of the Hubble rate at matter-scalar equality. The initial scalar energy density  $\Omega_{\phi,i}$  is varied between. The values  $\Phi_i^{L,R}$  are the two possible initial positions of the field tuned to obtain between 8 and 10 e-folds of matter domination. The value  $\dot{\Phi}_{\max}^2$  is the maximum value that  $\dot{\Phi}^2$  attains during the evolution.

### Initial conditions and inflation.

In equation (2.53) and (2.54), we gave expressions for the initial position of the field giving a desired number of e-folds of matter domination before accelerated expansion. In the presence of a non-minimal coupling, it turns out that this initial position can be linked to inflation. To see this, let us go back to the Jordan frame action (2.23), and assume that, during inflation,  $R \simeq 12H_{\text{inf}}^2$  is a constant, determined by the inflationary dynamics and independent of  $\phi$ . In this case, the non-minimal coupling  $\xi\phi^2 R$  enters as an additional term in the effective potential of  $\phi$

$$\tilde{V}(\phi) = V(\phi) - 6\xi H_{\text{inf}}^2 \phi^2,$$

and, during inflation,  $\phi$  will roll down to the minimum of this potential. As a result, the minimum of  $\tilde{V}$  is a natural initial condition from which the field  $\phi$  can start to roll after inflation, at the beginning of the radiation-dominated era. With an exponential potential  $V(\phi) = M^4 \exp(-\lambda\phi)$ , this minimum is located at

$$\phi_{\min} = \frac{1}{\lambda} W_{-1} \left( -\frac{\lambda^2 \kappa^2 M^4}{12\xi H_{\text{inf}}^2} \right), \quad (2.55)$$

where  $W_{-1}$  is the lower branch of the Lambert function. Now, recall that  $\lambda^2 M^2 / \xi \sim 10^{-23} \text{ GeV}^2$  to reproduce the current value of  $H_0$  at the local minimum of the effective potential  $U(\Phi)$ , see equation (2.49). Thus, we can substitute in (2.55) and get

$$\phi_{\min} = \frac{1}{\lambda} W_{-1} \left( - \left( \frac{M}{H_{\text{inf}}} \right)^2 \times 10^{-60} \right). \quad (2.56)$$

For the values of  $M$  obtained in table 2.6 and reasonable values  $H_{\text{inf}} \sim 10^4 - 10^{13} \text{ GeV}$ , the argument of  $W_{-1}$  is small in (2.56), so we can approximate  $W_{-1}(-x) \simeq \log(x)$ . In order to compare with the initial positions computed in (2.53), we shall convert (2.56) to the canonical field  $\Phi$ , which is given as a function of  $\phi$  in (2.44). Using this definition and the approximation  $\sinh^{-1}(x) \simeq -\log(-2x)$  for large negative  $x$ , we get

$$\Phi_{\min} \simeq -\frac{1}{\sqrt{\xi}} \log \left( \frac{2\sqrt{\xi}}{\lambda} \log \left( \left( \frac{H_{\text{inf}}}{M} \right)^2 \times 10^{60} \right) \right). \quad (2.57)$$

Note that this initial inflation-induced position for  $\Phi$  has the same  $(\lambda, \xi)$ -dependence as one obtained from the dynamical system analysis in (2.53). In the case where the field rolls during  $N_r$  e-folds during the radiation dominated era and  $N_m$  e-folds during the matter dominated era, we can replace  $(1+w)N_f$  by  $4/3N_r + N_m$  in (2.53), leading to

$$\Phi_i^L \simeq -\frac{1}{\sqrt{\xi}} \log \left( \frac{2\sqrt{\xi}}{\lambda} (4N_r + 3N_m) \right).$$

Therefore, if the field starts in the initial inflation-induced position (2.57) at the beginning of the radiation dominated era, we will have the following link between  $N_r, N_m$  and  $H_{\text{inf}}, M$

$$4N_r + 3N_m = 138 + 2 \log \left( \frac{H_{\text{inf}}}{M} \right).$$

If we take  $H_{\text{inf}} \sim 10^4 - 10^{13} \text{ GeV}$  and  $M \sim 10^{-11} - 10^{-1} \text{ GeV}$ , and insist that  $N_m \sim 8 - 10$ , this expression gives  $N_r \simeq 33 - 56$ . If we fix  $H_{\text{inf}}$  to  $10^{13} \text{ GeV}$  instead, we obtain  $N_r \simeq 43 - 56$ . These are reasonable values for the number of e-folds of radiation domination. It is interesting to obtain such a link between the duration of the radiation and matter-domination era on one hand, and the energy scales of inflation and quintessence on the other hand. Whether this link can be obtained in other quintessence models and help solve the coincidence problem remains an open question.



# Chapter 3

## Introduction to $\mathcal{N} = 1, D = 4$ supergravity

This chapter is an introduction to supersymmetry and supergravity in the superconformal formalism, along the lines of [37]. Our goal is to introduce the ingredients leading to the matter-coupled  $\mathcal{N} = 1$  action in  $D = 4$ , which is used in chapters 4 and 6, in a synthetic and self-contained way. The reader who is familiar with or not interested by the details of this construction can safely skip this chapter. In section 3.1, we recall basic facts on internal and spacetime symmetries, mostly to introduce notations. In section 3.2, we introduce global supersymmetry, multiplet calculus and the construction of general supersymmetric actions. Finally, in section 3.3, we turn to supergravity. Starting from the pure  $\mathcal{N} = 1$  supergravity, we introduce the conformal formalism on simple examples before explaining how it is used to build the general action of matter-coupled  $\mathcal{N} = 1$  supergravity.

### 3.1 Internal and spacetime symmetries

Modern Particle Physics is based on the concept of gauge symmetry. A symmetry can be defined in a broad sense as a transformation acting on the constituents of the theory leaving their equations of motion invariant. In our context, these constituents are fields propagating in spacetime. Naively, there are two classes of symmetry: internal and spacetime symmetries.

Internal symmetries are symmetries that do not act on the coordinates. A simple example is to consider a set of free massive scalar fields  $\phi^i(x)$ , with the following action

$$\mathcal{S} = \frac{1}{2} \int d^D x (\eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + m^2 \phi^i \phi^i), \quad (3.1)$$

with a sum over the repeated index  $i$ . This action, and the resulting equations of motion, is manifestly invariant under a transformation  $\phi^i(x) \rightarrow \phi'^i(x) = R^i_j \phi^j(x)$ , with  $R^T R = I$  and  $\det R = 1$ . Thus, the system (3.1) has an internal symmetry under the Lie group  $SO(n)$ , with the field  $\phi$  transforming under the fundamental representation. It is also instructive to consider the Lie algebra of this group, i.e. to consider infinitesimal transformations of the form  $R^i_j = \delta^i_j + \theta r^i_j$ , with a small parameter  $\theta$ . The  $r^i_j$  are called the generators of the Lie algebra, and their commutation relations encode the local structure of the Lie group  $SO(n)$ . More generally, we can consider an algebra with generators  $t_A$  and commutation relations

$$[t_A, t_B] = f_{AB}^C t_C,$$

where the  $f_{AB}^C$  are numerical factors, called the structure constants of the Lie algebra. With this notation, a set of scalar fields  $\phi^i(x)$  in the fundamental representation transforms as

$$\phi^i(x) \rightarrow \phi^i(x) - \theta^A (t_A)^i_j \phi^j(x), \quad (3.2)$$

An internal symmetry is said to be gauged when constant transformation parameters  $\theta$  are promoted to functions of spacetime  $\theta \rightarrow \theta(x)$ . The presence of kinetic terms in (3.1) implies the appearance of derivatives of  $\theta(x)$  when the transformation (3.2) is performed. In order to restore the invariance of the action, we introduce gauge fields, one for each generator, transforming as

$$A_\mu^A \rightarrow A_\mu^A + \partial_\mu \theta^A + \theta^C A_\mu^B f_{BC}^A, \quad (3.3)$$

and replace the spacetime derivatives  $\partial_\mu$  by covariant derivatives  $D_\mu$

$$D_\mu \phi^i = \partial_\mu \phi^i(x) + A_\mu^A (t_A)^i_j \phi^j(x). \quad (3.4)$$

These simple steps are enough to ensure gauge invariance of the action.

On the other hand, spacetime symmetries are symmetries that also involve a motion in spacetime. The simplest example is symmetry under translations  $\phi^i(x) \rightarrow \phi^i(x) = \phi^i(x + a)$ , where  $a^\mu$  is a vector parameter. This symmetry is present in the action (3.1), as can be seen by a simple change of variable. This action is also invariant under Lorentz symmetry, which includes spacetime rotations and boosts. Together with the translations, they form the Poincaré group. As for internal symmetries, we can consider its Lie algebra. In  $D$  dimensions, there are  $D$  generators  $P_\mu$  for translations and  $D(D-1)/2$  generators  $M_{[\mu\nu]}$  for Lorentz transformations, which are antisymmetric. Their commutation relations are

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}. \end{aligned} \quad (3.5)$$

The gauge theory associated with the Poincaré group is actually General Relativity, since local translations can be seen as general coordinate transformations.

## 3.2 Supersymmetry

It is natural to wonder whether internal and spacetime symmetries exhaust all the possible symmetries that can be realized in an interacting relativistic Quantum Field Theory. According to the Coleman-Mandula theorem [38], this is indeed the case up to the addition of the conformal symmetry. But this theorem only considers bosonic symmetries, and there is actually a more general possibility, if we introduce the notion of superalgebra. In a superalgebra, one makes a distinction between two classes of elements: the bosonic elements and the fermionic elements. The structure of the superalgebra is then specified by the commutator of pairs of bosonic elements, the anticommutator of pairs of fermionic elements and the commutator of bosonic-fermionic pairs, or schematically

$$[B, B] = B, \quad [B, F] = F, \quad \{F, F\} = B,$$

Introducing superalgebras allows to extend the Poincaré algebra (3.5) by adding new generators: the supercharges  $Q_\alpha^i$ , that are fermionic. Here, we consider them as four-component

spinors, and the index  $\alpha$  is a spacetime spinor index (just as  $P_\mu$  is a four-component vector, with a spacetime vector index  $\mu$ ). The other index  $i = 1, \dots, \mathcal{N}$  labels distinct supercharges. In the following, we consider only  $\mathcal{N} = 1$  and drop the index  $i$ . With only one supercharge, the resulting super-Poincaré algebra reads, in addition to the commutation relations (3.5)

$$[P_\mu, \mathcal{Q}_\alpha] = 0, \quad [M_{[\mu\nu]}, \mathcal{Q}_\alpha] = -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta \mathcal{Q}_\beta, \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^\beta\} = -\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu, \quad (3.6)$$

where the  $\gamma_\mu$  are the  $\gamma$ -matrices, that satisfy  $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ , and  $\gamma_{\mu\nu} = 1/2[\gamma_\mu, \gamma_\nu]$ . Within the hypotheses of the Haag-Lopuszanski-Sohnius theorem [39], this is the only possible extension of the Poincaré algebra. The supercharges  $\mathcal{Q}_\alpha$  generate new kind of transformations, called supersymmetry (SUSY) transformations. Since these supercharges are fermionic, SUSY transformations turn bosons into fermions, and vice versa. The fields of a supersymmetric theory can be arranged into multiplets, such that fields inside a same multiplet transform into each other. Since  $[P_\mu, \mathcal{Q}_\alpha] = 0$ , fields that belong to the same multiplet have the same mass.

### 3.2.1 Multiplet calculus and supersymmetric actions

The simplest massless multiplets of  $\mathcal{N} = 1$  supersymmetry are the chiral multiplet, which can be described by a complex scalar field  $Z$  and a Majorana spinor field  $\chi$ , and the vector, or gauge multiplet, described by a vector gauge field  $A_\mu$  and a Majorana spinor field  $\lambda$ , the gaugino. In addition, it is convenient to introduce auxiliary fields in these multiplets. These fields have algebraic equations of motion and do not propagate degrees of freedom, but they allow to realise supersymmetry off-shell. Otherwise, the SUSY algebra (3.6) would only close after applying the equations of motion. Auxiliary fields also allow multiplets to contain the same number of bosonic and fermionic degrees of freedom off-shell.

For instance, in a chiral multiplet, we introduce a complex scalar field  $F$ , such that the SUSY transformations of the fields are

$$\delta Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi, \quad \delta P_L\chi = \frac{1}{\sqrt{2}}P_L(\not{\phi}Z + F)\epsilon, \quad \delta F = \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\phi}P_L\chi, \quad (3.7)$$

here,  $\not{\phi} = \gamma^\mu\partial_\mu$  and  $P_L = (1 + \gamma_*)/2$  is the left-handed chiral projector, where  $\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3$ , which is sometimes called  $\gamma_5$ . In addition, the transformation parameter  $\epsilon$  is a Majorana spinor, and the bar over any spinor  $\psi$  denotes the 'Majorana conjugate'  $\bar{\psi} = \psi^T C$ , where  $C$  is the charge conjugation matrix, see [37] for more details. Similarly, in a gauge multiplet, we introduce a real scalar field  $D$ , and the SUSY transformations are

$$\delta A_\mu = -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda, \quad \delta\lambda = \left(\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu} + \frac{1}{2}i\gamma_*D\right)\epsilon, \quad \delta D = \frac{1}{2}i\bar{\epsilon}\gamma_*\gamma^\mu\partial_\mu\lambda, \quad (3.8)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Let us mention the existence of another  $\mathcal{N} = 1$  multiplet: the gravity multiplet, containing a spin-2 graviton and a spin-3/2 field. This spin-3/2 field is related with gauging the supersymmetry transformations, as we will soon see. When this is done, translations are to be gauged as well due to their intimate relation in (3.6). Therefore, the theory of the  $\mathcal{N} = 1$  gravity multiplet is an extension of General Relativity, called supergravity.

More generally, a multiplet can be constructed as follows. First, we choose its lowest component. For the chiral multiplet, it is a complex scalar field  $Z$ ; For the gauge multiplet, it is a gauge vector field  $A_\mu$ . We can also choose a real scalar field  $C$ . We then perform a SUSY transformation of this first component, which necessarily introduces a second component. For instance,  $\delta C \propto \bar{\epsilon}\zeta$ , where  $\zeta$  is a Majorana fermion for  $C$  to be real. We then make a general ansatz for the SUSY transformation of this new component

$$\delta\zeta = (A + \gamma^\mu A_\mu + \gamma^{\mu\nu} A_{\mu\nu})\epsilon, \quad (3.9)$$

where the  $(A, A_\mu, A_{\mu\nu})$  are bosonic fields. The key step is then to implement the anticommutation relation (3.6) to obtain constraints on this ansatz, in terms of fields already present in the multiplet. Indeed, this anticommutation relation implies that

$$[\delta(\epsilon_1), \delta(\epsilon_2)]C = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu C, \quad (3.10)$$

and similarly, replacing  $C$  with any field. The parts of the ansatz (3.9) that remain unconstrained by (3.10) must be added as new components of the multiplet, and the procedure continues until we find do not find any new fields. Starting from a complex scalar field  $Z$  transforming as  $\delta Z = 1/\sqrt{2}\bar{\epsilon}P_L\chi$ , this procedure quickly yields the transformation rules (3.7). Starting from a real scalar field  $C$ , it is a little bit longer. The final multiplet contains

$$(C, \zeta, \mathcal{H}, A_\mu, \lambda, D),$$

with transformation rules

$$\begin{aligned} \delta C &= \frac{1}{2}i\bar{\epsilon}\gamma_*\zeta, & \delta P_L\zeta &= \frac{1}{2}P_L(i\mathcal{H} - \not{A} - i\not{\phi}C)\epsilon, \\ \delta\mathcal{H} &= -i\bar{\epsilon}P_R(\lambda + \not{\phi}\zeta), & \delta A_\mu &= -\frac{1}{2}\bar{\epsilon}(\gamma_\mu\lambda + \partial_\mu\zeta) \\ \delta\lambda &= \frac{1}{2}(\gamma^{\rho\sigma}\partial_\rho A_\sigma + i\gamma_*D)\epsilon, & \delta D &= \frac{1}{2}i\bar{\epsilon}\gamma_*\gamma^\mu\partial_\mu\lambda. \end{aligned} \quad (3.11)$$

The SUSY transformations of  $(A_\mu, \lambda, D)$  shown here are remarkably similar to those in (3.8) for the gauge multiplet. In fact, there is a generalisation of a gauge transformation acting the real multiplet, under which  $C \rightarrow C + \text{Im}Z$ , with  $Z$  being the lowest component of a chiral multiplet parameterising the transformation. The transformation of the other components can be deduced as we just explained. This transformation can be used to go to the Wess-Zumino gauge, where  $(C, \zeta, \mathcal{H}) = 0$ , and recover exactly the field content and transformation rules of a gauge multiplet. The field content and transformation rules (3.7), (3.8) and (3.11) of the multiplets can also be obtained in a somewhat simpler way using the superspace formalism, as described for instance in [40], but this formalism is harder to extend to supergravity.

## General supersymmetric actions

Multiplet calculus, which we have just introduced, is an important ingredient in the computation of general  $\mathcal{N} = 1$  supersymmetric actions. The other ingredient is the observation that the SUSY transformation of  $F$  and  $D$  in (3.7), (3.8) and (3.11) are total derivatives, implying that  $F$ -term and  $D$ -term actions of the form

$$S_F = \int d^4x F \quad \text{and} \quad S_D = \int d^4x D, \quad (3.12)$$

are invariant under supersymmetry. Starting from a set of  $n$  chiral multiplets  $(Z^\alpha, \chi^\alpha, F^\alpha)$ , we can for instance consider a composite chiral multiplet whose lowest component is an arbitrary holomorphic function  $W(Z^\alpha)$  of the scalar components. We can then compute the other components of this multiplet

$$\chi(W) = W_\alpha \chi^\alpha \quad \text{and} \quad F(W) = W_\alpha F^\alpha - \frac{1}{2} W_{\alpha\beta} \bar{\chi}^\alpha P_L \chi^\beta,$$

where  $W_\alpha = \partial W / \partial Z^\alpha$  and  $W_{\alpha\beta} = \partial^2 W / \partial Z^\alpha \partial Z^\beta$ , and conclude that the  $F$ -term action

$$\mathcal{S}_W = \int d^4x \left( W_\alpha F^\alpha - \frac{1}{2} W_{\alpha\beta} \bar{\chi}^\alpha P_L \chi^\beta + \text{h.c.} \right), \quad (3.13)$$

is invariant under supersymmetry. From the same set of  $n$  chiral multiplets, we can also build a composite real multiplet whose lowest component is an arbitrary real function  $1/2K(Z^\alpha, \bar{Z}^{\bar{\alpha}})$ , leading to the  $D$ -term action

$$\begin{aligned} \mathcal{S}_K = \int d^4x & \left( K_{\alpha\bar{\beta}} \left( -\partial^\mu Z^\alpha \partial_\mu \bar{Z}^{\bar{\beta}} - \frac{1}{2} \bar{\chi}^\alpha P_L \not{\partial} \chi^{\bar{\beta}} - \frac{1}{2} \bar{\chi}^{\bar{\beta}} P_R \not{\partial} \chi^\alpha + F^\alpha \bar{F}^{\bar{\beta}} \right) \right. \\ & + \frac{1}{2} \left( K_{\alpha\beta\bar{\gamma}} \left( -\bar{\chi}^\alpha P_L \chi^\beta \bar{F}^{\bar{\gamma}} + \bar{\chi}^\alpha P_L (\not{\partial} Z^\beta) \chi^{\bar{\gamma}} \right) + \text{h.c.} \right) \\ & \left. + \frac{1}{4} K_{\alpha\beta\bar{\gamma}\bar{\delta}} \bar{\chi}^\alpha P_L \chi^\beta \bar{\chi}^{\bar{\gamma}} P_R \chi^{\bar{\delta}} \right), \end{aligned}$$

here again, indices  $(\alpha, \bar{\beta}, \dots)$  denote derivatives with respect to  $Z^\alpha$  and  $\bar{Z}^{\bar{\beta}}$ , for instance  $K_{\alpha\bar{\beta}} = \partial K / \partial Z^\alpha \partial \bar{Z}^{\bar{\beta}}$ . This action contains general kinetic terms for the fields  $(Z^\alpha, \chi^\alpha)$  in the first line, with a target space metric  $g_{\alpha\beta} = K_{\alpha\beta}$ , suggesting to interpret  $K$  as a Kahler potential, see [37] for an introduction to Kahler geometry. In fact, this action can be rewritten in a way that makes the connection to Kahler geometry a little more explicit

$$\begin{aligned} \mathcal{S}_K = \int d^4x & \left( g_{\alpha\bar{\beta}} \left( -\partial^\mu Z^\alpha \partial_\mu \bar{Z}^{\bar{\beta}} - \frac{1}{2} \bar{\chi}^\alpha P_L \not{\nabla} \chi^{\bar{\beta}} - \frac{1}{2} \bar{\chi}^{\bar{\beta}} P_R \not{\nabla} \chi^\alpha \right) \right. \\ & + \left( F^\alpha - \frac{1}{2} \Gamma_{\gamma\beta}^\alpha \bar{\chi}^\gamma P_L \chi^\beta \right) \left( \bar{F}^{\bar{\beta}} - \frac{1}{2} \Gamma_{\bar{\gamma}\bar{\alpha}}^{\bar{\beta}} \bar{\chi}^{\bar{\gamma}} P_R \chi^{\bar{\alpha}} \right) \\ & \left. + \frac{1}{4} R_{\alpha\bar{\gamma}\beta\bar{\delta}} \bar{\chi}^\alpha P_L \chi^\beta \bar{\chi}^{\bar{\gamma}} P_R \chi^{\bar{\delta}} \right), \end{aligned} \quad (3.14)$$

where  $\nabla_\mu$  is a Kahler-covariant derivative,  $\Gamma_{\beta\gamma}^\alpha$  is the affine connection built from  $g_{\alpha\bar{\beta}}$  and  $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$  is the curvature tensor. The formulae involved in this rearrangement are

$$\begin{aligned} P_L \nabla_\mu \chi^\alpha &= P_L (\partial_\mu \chi^\alpha + \Gamma_{\beta\gamma}^\alpha \chi^\gamma \partial_\mu Z^\beta), \quad \Gamma_{\beta\gamma}^\alpha = g^{\alpha\bar{\delta}} \partial_\beta g_{\gamma\bar{\delta}}, \\ R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= \partial_\gamma \partial_{\bar{\delta}} g_{\alpha\bar{\beta}} - g^{\eta\bar{\epsilon}} \partial_\gamma g_{\alpha\bar{\epsilon}} \partial_{\bar{\delta}} g_{\eta\bar{\beta}}. \end{aligned}$$

The kinetic terms in (3.14) are accompanied by interactions between the different fields of the multiplet, which is a general consequence of supersymmetry: different parts of the action (here kinetic terms and interactions) are related to each other. Notice also that no spacetime derivative act on the auxiliary field  $F$ .

It is also possible to build a composite multiplet out of different kinds of multiplets. For instance, we can consider a set of  $N$  abelian gauge multiplets  $(A_\mu^A, \lambda^A, D^A)$  in addition to the set of chiral multiplets  $(Z^\alpha, \chi^\alpha, F^\alpha)$ , and build a chiral multiplet whose lowest component is

$$Z(f) = -\frac{1}{4} f_{AB} (Z^\alpha) \bar{\lambda}^A P_L \lambda^B,$$



where the  $f_{AB}(Z^\alpha)$  are  $N \times N$  holomorphic functions of the  $Z^\alpha$ . This choice is motivated by the fact that the corresponding  $F$ -term action contains general kinetic terms for the  $\lambda^A$

$$\begin{aligned} \mathcal{S}_f = \int d^4x & \left( \frac{1}{4} f_{AB} \left( -\bar{\lambda}^A P_L \not{\partial} \lambda^B - F^{\mu\nu-A} F_{\mu\nu}^{-B} + D^A D^B \right) \right. \\ & \frac{1}{2\sqrt{2}} f_{AB\alpha} \bar{\chi}^\alpha \left( -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^{-A} + iD^A \right) P_L \lambda^B \\ & \left. - \frac{1}{4} f_{AB\alpha} F^\alpha \bar{\lambda}^A P_L \lambda^B + \frac{1}{8} f_{AB\alpha\beta} \bar{\chi}^\alpha P_L \chi^\beta \bar{\lambda}^\alpha P_L \lambda^\beta + \text{h.c.} \right), \end{aligned} \quad (3.15)$$

here  $F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A$  is the abelian field strength of the vector field  $A_\mu^A$  and  $F_{\mu\nu}^{-A} = 1/2(F_{\mu\nu}^A - \tilde{F}_{\mu\nu}^A)$  is an anti self-dual tensor made from it, where  $\tilde{F}^{\mu\nu A} = -1/2 i \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$  and  $\varepsilon^{\mu\nu\rho\sigma}$  is the antisymmetric Levi-Civita tensor, with  $\varepsilon^{0123} = -1$ .

### 3.2.2 Gauged internal symmetry

The action  $\mathcal{S} = \mathcal{S}_W + \mathcal{S}_K + \mathcal{S}_f$  is already a quite general supersymmetric action, but still lacks an essential ingredient of Particle Physics: gauged internal symmetries. In order to gauge global symmetries of chiral multiplets, we must first identify them. Internal symmetries are associated with holomorphic Killing vectors  $k_A^\alpha(Z)$  of the target space, and act infinitesimally on the scalar component as

$$\delta(\theta) Z^\alpha = \theta^A k_A^\alpha(Z). \quad (3.16)$$

From this, we can compute the following action on the other components

$$\begin{aligned} \delta(\theta) P_L \chi^\alpha &= \theta^A \frac{\partial k_A^\alpha(Z)}{\partial Z^\beta} P_L \chi^\beta \\ \delta(\theta) F^\alpha &= \theta^A \frac{\partial k_A^\alpha(Z)}{\partial Z^\beta} F^\beta - \frac{1}{2} \theta^A \frac{\partial^2 k_A^\alpha(Z)}{\partial Z^\beta \partial Z^\gamma} \bar{\chi}^\beta P_L \chi^\gamma \end{aligned} \quad (3.17)$$

Killing vectors generate a Lie algebra, with structure constants involved in the Lie brackets

$$k_A^\beta \partial_\beta k_B^\alpha - k_B^\beta \partial_\beta k_A^\alpha = f_{AB}{}^C k_C^\alpha.$$

In the simplest example of a flat target space metric  $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ , we recover the same transformation as in equation (3.2)

$$\delta(\theta) Z^\alpha = -\theta^A (t_A)^\alpha_\beta Z^\beta,$$

where the  $t_A$  are the matrix generators of the Lie algebra. In that case we can thus easily identify the Killing vectors  $k_A^\alpha = -(t_A)^\alpha_\beta Z^\beta$ . Under the global transformation (3.16)-(3.17), the action  $\mathcal{S}_W$  is invariant provided that  $W(Z)$  is invariant

$$\delta(\theta) W = \theta^A k_A^\alpha W_\alpha = 0.$$

On the other hand, the action  $\mathcal{S}_K$  is invariant, even if  $K(Z, \bar{Z})$  is not. Indeed, Killing vectors are defined by the fact that corresponding transformations preserve the metric, but since the metric is obtained from derivatives of the the Kahler potential  $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$ , the latter is only determined up to

$$K(Z, \bar{Z}) \rightarrow K(Z, \bar{Z}) + f(Z) + \bar{f}(\bar{Z}), \quad (3.18)$$

with  $f(Z)$  an arbitrary holomorphic function. This transformation is called a Kahler transformation. As a result, the Kahler potential does not need to be invariant under a gauge transformation, it can undergo a Kahler transformation

$$\delta(\theta)K = \theta^A(k_A^\alpha \partial_\alpha + k_A^{\bar{\alpha}} \partial_{\bar{\alpha}})K(Z, \bar{Z}) = \theta^A(r_A(Z) + \bar{r}_A(\bar{Z})), \quad (3.19)$$

where the  $r_A(Z)$  are holomorphic. Since  $\mathcal{S}_K$  only depends on  $K$  through at least two derivatives of the form  $K_{\alpha\bar{\beta}\dots}$ , see (3.14), it is clearly invariant.

Using the properties of Kahler geometry, the information contained in Killing vectors can be rewritten into so-called moment maps, which are real functions  $\mathcal{P}_A(Z, \bar{Z})$  such that

$$k_A^\alpha(Z) = -ig^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \mathcal{P}_A(Z, \bar{Z}). \quad (3.20)$$

This relation can be inverted to obtain the moment maps as

$$\mathcal{P}_A(Z, \bar{Z}) = i(k^\alpha \partial_\alpha K(Z, \bar{Z}) - r(Z)). \quad (3.21)$$

In order to gauge internal symmetries, we start in the same way as in (3.3)-(3.4). For each Killing vector, we introduce a gauge multiplet  $(A_\mu^A, \lambda^A, D^A)$  transforming as

$$\delta(\theta)A_\mu^A = \partial_\mu \theta^A + \theta^C A_\mu^B f_{BC}^A, \quad \delta(\theta)\lambda^A = \theta^C \lambda^B f_{BC}^A, \quad \delta(\theta)D^A = \theta^C D^B f_{BC}^A,$$

and replace spacetime derivatives  $\partial_\mu$  by gauge covariant derivatives  $D_\mu$  acting as

$$\begin{aligned} D_\mu Z^\alpha &= \partial_\mu Z^\alpha - A_\mu^A k_A^\alpha(Z), & D_\mu \lambda^A &= \partial_\mu \lambda^A + \lambda^C A_\mu^B f_{BC}^A \\ D_\mu P_L \chi^\alpha &= \partial_\mu P_L \chi^\alpha - A_\mu^A \frac{\partial k_A^\alpha(Z)}{\partial Z^\beta} P_L \chi^\beta. \end{aligned}$$

Replacing spacetime derivatives by gauge covariant derivatives has a major effect on the construction of multiplets, as it replaces the supersymmetry algebra relation (3.10), which was the central element of this construction, by a covariant version

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\frac{1}{2} \bar{\epsilon}_1 \gamma^\mu \epsilon_2 D_\mu \quad (3.22)$$

For instance, the SUSY transformation of chiral multiplets becomes

$$\delta Z^\alpha = \frac{1}{\sqrt{2}} \bar{\epsilon} P_L \chi^\alpha, \quad \delta P_L \chi^\alpha = \frac{1}{\sqrt{2}} P_L (\not{D} Z^\alpha + F^\alpha) \epsilon, \quad \delta F^\alpha = \frac{1}{\sqrt{2}} \bar{\epsilon} \not{D} P_L \chi^\alpha + \bar{\epsilon} P_R \lambda^A k_A^\alpha(Z),$$

not only spacetime derivatives are replaced by covariant derivatives, but gauginos  $\lambda^A$  also enter in the transformation of  $F^\alpha$ , due to the presence of  $A_\mu^A$  in the covariant derivative in the SUSY transformation of  $P_L \chi^\alpha$ . In principle, the same effect affects the composite multiplets of lowest components  $W(Z)$  and  $K(Z, \bar{Z})$ , from which the SUSY invariant actions (3.13) and (3.14) are built. In fact,  $\mathcal{S}_W$  remains unchanged, because of the gauge invariance of  $W(Z)$ , but  $\mathcal{S}_K$  gathers additional terms that depend on the gauginos  $\lambda^A$  and  $D$ -fields  $D^A$

$$\mathcal{S}_K = \int d^4x \left( \dots - D^A \mathcal{P}_A - \sqrt{2} g_{\alpha\bar{\beta}} \bar{\lambda}^A (P_L \chi^\alpha k_A^{\bar{\beta}} + P_R \chi^{\bar{\beta}} k_A^\alpha) + \text{h.c.} \right), \quad (3.23)$$

where the  $\dots$  symbolise the terms already present in (3.14). On the other hand,  $\mathcal{S}_f$  is only updated by replacing field strength with their non-abelian version

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + f_{BC}^A A_\mu^B A_\nu^C,$$

and spacetime derivatives by covariant derivatives. In addition, gauge invariance requires  $f_{AB}(Z)$  to transform in the direct product of adjoint representations

$$\delta(\theta)f_{AB}(Z) = \theta^C k_C^\alpha f_{AB\alpha} = 2\theta^C f_{C(A}{}^D f_{B)D}(Z),$$

here, the parenthesis between indices denotes a symmetrisation,  $M_{(AB)} = 1/2(M_{AB} + M_{BA})$ .

### The Fayet-Iliopoulos terms

Note that the  $r_A(Z)$  are only determined up to an arbitrary imaginary constant in (3.19). Equivalently, the moment maps (3.21) are defined up to an arbitrary real constant. For non-abelian symmetries, these constants are fixed by imposing the moment maps to transform in the adjoint representation, which is actually required by supersymmetry, see [37], but the moment maps of abelian  $U(1)$  factors remain ambiguous.

In parallel, if the gauge group contains abelian factors, the  $D$ -components of the corresponding multiplets  $(A_\mu^A, \lambda^A, D^A)$  transform as total derivatives, see equation (3.8). As a consequence, the term

$$\mathcal{S}_{\text{FI}} = - \int d^4x \xi_{\bar{A}} D^{\bar{A}},$$

is invariant under supersymmetry and can be added to the total action. Comparing with (3.23), we see that the possibility of adding such terms, called Fayet-Iliopoulos terms [41], corresponds exactly to the arbitrary real constant that can be added in the moment maps.

### 3.2.3 Auxiliary fields, scalar potential, SUSY breaking

Following these steps, we obtain a fairly general  $\mathcal{N} = 1$  supersymmetric gauge theory of  $n$  chiral multiplets and  $N$  gauge multiplets, with the action  $\mathcal{S} = \mathcal{S}_K + \mathcal{S}_W + \mathcal{S}_f$ . All that remains is to eliminate the auxiliary fields. Writing and solving their equations of motion, one obtains

$$\bar{F}^{\bar{\beta}} = g^{\alpha\bar{\beta}} \left( -W_\alpha + \frac{1}{2} \Gamma_{\bar{\gamma}\bar{\alpha}}^{\bar{\beta}} \bar{\chi}^{\bar{\gamma}} P_R \chi^{\bar{\alpha}} + \frac{1}{4} f_{AB\alpha} \bar{\lambda}^A P_L \lambda^B \right), \quad (3.24)$$

and

$$\text{Re}(f_{AB}) D^B = \mathcal{P}_A - \frac{1}{2\sqrt{2}} i f_{AB\alpha} \bar{\chi}^\alpha P_L \lambda^B + \frac{1}{2\sqrt{2}} i \bar{f}_{AB\bar{\alpha}} \bar{\chi}^{\bar{\alpha}} P_R \lambda^B. \quad (3.25)$$

They can then be substituted to obtain the physical action. An interesting element of this physical action is the scalar potential, which takes a very particular form

$$V = g^{\alpha\bar{\beta}} W_\alpha \bar{W}_{\bar{\beta}} + \frac{1}{2} (\text{Re} f)^{-1AB} \mathcal{P}_A \mathcal{P}_B. \quad (3.26)$$

Given the potential (3.26), the system will classically evolve towards a configuration that minimise it locally

$$\partial_\alpha V(Z_*, \bar{Z}_*) = \partial_{\bar{\alpha}} V(Z_*, \bar{Z}_*) = 0.$$

Since  $g_{\alpha\bar{\beta}}$  and  $\text{Re}f$  act as metrics in the general kinetic terms of (3.14) and (3.15), they are both positive definite, and as a result  $V(Z_*, \bar{Z}_*) \geq 0$ . It can also be shown from the supersymmetry algebra (3.6) that unbroken supersymmetry requires the vacuum energy to vanish. Therefore, when the system lies in its vacuum state, at the minimum of the potential

$$\begin{aligned} \text{supersymmetry is unbroken} &\leftrightarrow V(Z_*, \bar{Z}_*) = 0 \\ \text{supersymmetry is broken} &\leftrightarrow V(Z_*, \bar{Z}_*) > 0. \end{aligned}$$

Now,  $V(Z_*, \bar{Z}_*) > 0$  is equivalent to either one of the  $W_\alpha(Z_*) \neq 0$  or one of the  $\mathcal{P}_A(Z_*, \bar{Z}_*) \neq 0$ , which are in turn respectively equivalent to either one of the  $F^\alpha \neq 0$  or one of the  $D^A \neq 0$  in a Lorentz invariant vacuum where the fermions vanish anyways, see equations (3.24)-(3.25). So there are essentially two ways to spontaneously break supersymmetry:  $F$ -term and  $D$ -term breaking. Fayet-Iliopoulos terms play a prominent role in  $D$ -term breaking, as they appear as constant shifts in the moment maps.

Breaking supersymmetry is crucial if we want to relate it to the observable world, because it implies that fields of different spin are embedded into multiplets, fields belonging to the same multiplets having the same mass. But this is not what we see in the spectrum of the Standard Model. If supersymmetry is broken, fields in the same multiplets no longer necessarily have the same mass. Supersymmetric partners of the Standard Model can thus be much heavier and effectively inaccessible to current experiments.

When an internal symmetry is spontaneously broken, the theory necessarily contains massless bosons, called Goldstone bosons. There is one for each broken generator. Similarly, when supersymmetry is spontaneously broken, the theory necessarily contains a massless fermion, called the Goldstino. This fermion is a linear combination of the chiral fermions  $\chi^\alpha$  and of the gauginos  $\lambda^A$

$$P_L v = -\frac{1}{\sqrt{2}} P_L \left( W_\alpha \chi^\alpha + \frac{1}{\sqrt{2}} i \mathcal{P}_A \lambda^A \right), \quad (3.27)$$

where  $W_\alpha(Z_*)$  and  $\mathcal{P}_A(Z_*, \bar{Z}_*)$  are evaluated at the minimum of the potential.

### 3.3 Supergravity

The concept of gauged internal symmetry is a very fruitful concept, which is at the heart of the construction of the Standard Model of Particle Physics. After having introduced supersymmetry, one can ask about the possibility of gauging it as well, i.e. consider SUSY transformations with a spacetime dependent parameter  $\epsilon \rightarrow \epsilon(x)$ . Just as with internal symmetries, it necessitates the introduction of a gauge field, which is not only a vector, but also a (Majorana) spinor, due to the fermionic nature of supersymmetry generators. It is thus a vector-spinor of spin 3/2, called the gravitino, sharing a multiplet with the spin-2 graviton. What we call the graviton here is the field  $h_{\mu\nu}(x)$  describing a weak gravitational perturbation around a fixed classical background, e.g. around Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2h_{\mu\nu}(x).$$

The free-field action of the gravitino  $\psi_\mu$  is Rarita-Schwinger

$$\mathcal{S}_{\text{RS}} = -\frac{1}{2} \int d^4x \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho, \quad (3.28)$$

where multi-indices gamma matrices are defined as  $\gamma^{\mu_1 \dots \mu_r} = \gamma^{[\mu_1} \dots \gamma^{\mu_r]}$ , the brackets denoting total antisymmetrisation with a total weight 1, for instance  $\gamma^{\mu\nu} = 1/2(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ . A key property of this free field action is that it is invariant under a fermionic gauge transformation

$$\psi_\mu(x) \rightarrow \psi_\mu(x) + \partial_\mu \epsilon(x). \quad (3.29)$$

On the other hand, the action of gravity is the Einstein-Hilbert action, introduced in 2.2

$$\mathcal{S}_{\text{EH}} = \frac{1}{2} \int d^4x \sqrt{-g} R.$$

Expanding the Einstein-Hilbert action to a certain order, we can obtain the kinetic term and interactions of the graviton and calculate its scattering cross-sections as for any other field.

### Frame field and $\gamma$ -matrices

In order to write spinor actions in curved space, it is necessary to introduce the so-called frame field, defined by

$$g_{\mu\nu}(x) = e_\mu^a(x) \eta_{ab} e_\nu^b(x),$$

where  $\eta_{ab}$  is the Minkowski metric. In what follows, the greek indices  $\mu, \nu, \dots$  are called coordinate or curved indices, while the latin indices  $a, b, \dots$  are called frame or flat indices. At each point, frame field form an orthonormal basis of the local tangent space of the manifold. We can expand any tensor on this basis, for instance  $V^\mu(x) = V^a(x) e_\mu^a(x)$ . From there, indices behave in a straightforward way in computations involving the frame field. For instance,  $e_\mu^a(x)$  is the inverse of  $e_a^\mu(x)$ , such that  $e_\mu^a e_b^\mu = \delta_b^a$ .

The necessity of introducing frame fields when discussing spinors comes from the  $\gamma$ -matrices, which are defined in local frames, as elements of the Clifford algebra  $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ . As such, the flat indices  $\gamma^a$  matrices are constant numeral matrices, while the curved indices  $\gamma^\mu$  depend on spacetime through the frame field

$$\gamma^\mu(x) = e_\mu^a(x) \gamma^a.$$

In addition to the frame field, we can introduce the spin connection  $\omega_\mu^{ab}$ , containing the same information as the affine connection  $\Gamma_{\mu\nu}^\rho$  introduced in section 2.2 and expressed as

$$\omega_\mu^{ab} = \omega_\mu^{ab}(e) + K_{\mu[\nu\rho]} \quad \text{with} \quad \omega_\mu^{ab}(e) = 2e^{\nu[a} \partial_{[\mu} e_{\nu]}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma^c, \quad (3.30)$$

where  $K_{\mu[\nu\rho]}$  is the contortion tensor that we already mentioned in section 2.2. In terms of the frame field and spin connection, the Einstein-Hilbert action reads

$$\mathcal{S}_{\text{EH}} = \frac{1}{2} \int d^4x e e^{a\mu} e^{b\nu} R_{\mu\nu ab}, \quad (3.31)$$

here  $e$  stands for the determinant of  $e_\mu^a$  and  $R_{\mu\nu ab}$  is the Riemann tensor

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac} \omega_\nu^c{}_b - \omega_{\nu ac} \omega_\mu^c{}_b. \quad (3.32)$$

In what follows, we are working in the second order formulation of gravity, where  $\omega_\mu^{ab} = \omega_\mu^{ab}(e)$  is imposed as a constraint, so the spin connection and Riemann tensor must be seen as functions of the frame field.

### 3.3.1 Pure $\mathcal{N} = 1$ supergravity.

In order to obtain the action of pure  $\mathcal{N} = 1, D = 4$  supergravity, a possible strategy is to start from the free-field actions of the gravitino and graviton (3.28) and (3.31), and construct interactions order by order to ensure invariance under local SUSY transformations.

$$\mathcal{S} = \int d^4x e \left( \frac{1}{2} R(\omega(e)) - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \dots \right), \quad (3.33)$$

where  $D_\nu \psi_\rho$  is the covariant derivative of the gravitino in curved space

$$D_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \psi_\rho.$$

The supersymmetry transformation rules can be shown to be

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \quad \text{and} \quad \delta \psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon. \quad (3.34)$$

The transformation of the frame field is in fact the simplest form compatible with the indices, and the one of the gravitino is the curved-space generalisation of (3.29). With these rules, one can show that the action (3.33) is invariant up to linear order in the gravitino. But performing a transformation also generates terms that are cubic in the gravitino. In order to cancel them, one introduces the following four-fermions terms

$$\mathcal{L}_{4f} = -\frac{1}{32} (\bar{\psi}^\mu \gamma^\rho \psi^\nu) (\bar{\psi}_\mu \gamma_\rho \psi_\nu) - \frac{1}{16} (\bar{\psi}^\rho \gamma^\mu \psi^\nu) (\bar{\psi}_\rho \gamma_\nu \psi_\mu) + \frac{1}{8} (\bar{\psi}^\rho \gamma^\mu \psi_\mu) (\bar{\psi}_\rho \gamma^\nu \psi_\nu), \quad (3.35)$$

whose form is inspired by the solution of the equation of motion of the spin connection if we were working in the first-order formulation of gravity, which is not equivalent to the second-order formulation in the presence of fermions. We emphasize that we are actually working in second-order formulation, but the four-fermion terms that are guessed this way are exactly what is needed to complete the action (3.33) and make it invariant under the local transformation (3.34). In the end, the action of pure  $\mathcal{N} = 1$  supergravity is thus

$$\mathcal{S}_{\text{SG}} = \int d^4x e \left( \frac{1}{2} R(\omega(e)) - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \mathcal{L}_{4f} \right), \quad (3.36)$$

### 3.3.2 Conformal construction of General Relativity

The action of pure  $\mathcal{N} = 1, D = 4$  supergravity was not so hard to build. The action of matter-coupled supergravity, i.e. in the presence of chiral and gauge multiplets, is considerably more complicated, it cannot just be guessed from the globally supersymmetric action of section 3.2. A systematic method for constructing this action uses conformal symmetry. Let us first illustrate this method by using it to recover the standard General Relativity in any dimension.

Conformal symmetry can be defined as the transformations of Minkowski spacetime that preserve the angle between any two vectors. These transformations extend the Poincaré group, since translations and rotations, with generators  $P_\mu$  and  $M_{\mu\nu}$ , obviously preserve the angles. In addition to them, the conformal group contains the dilatation transformation,

with generator  $D$ , and special conformal transformations, with generators  $K_\mu$ . With these new generators, the Poincaré algebra (3.5) is extended to

$$\begin{aligned} [P_\mu, M_{\rho\sigma}] &= 2\eta_{\mu[\rho}P_{\sigma]}, & [M_{\mu\nu}, M_{\rho\sigma}] &= 4\eta_{[\mu[\rho}M_{\sigma]\nu]}, & [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu, & [P_\mu, K_\nu] &= 2(\eta_{\mu\nu}D + M_{\mu\nu}), & [K_\mu, M_{\rho\sigma}] &= 2\eta_{\mu[\rho}K_{\sigma]}. \end{aligned} \quad (3.37)$$

Our first step in the conformal construction is to gauge the conformal symmetry. This requires the introduction of a gauge field for each generator. The gauge fields associated with translations  $P_a$  and Lorentz transformations  $M_{ab}$  are in fact the frame field  $e_\mu^a$  and spin connection  $\omega_\mu^{ab}$ , while the gauge fields associated with dilatations and special conformal transformations are new fields denoted  $b_\mu$  and  $f_\mu^a$  respectively. Their transformations follow the same rule as for standard gauged internal symmetries (3.3)

$$\delta(\theta)A_\mu^A = \partial_\mu\theta^A + \theta^C A_\mu^B f_{BC}^A,$$

where the structure constants  $f_{AB}^C$  can be read in the algebra (3.37). With transformation parameters  $(\xi^a, \lambda^{ab}, \lambda_D, \lambda_k^a)$  respectively associated with  $(P_a, M_{ab}, D, K_a)$ , we have

$$\begin{aligned} \delta e_\mu^a &= -\lambda^{ab}e_{\mu b} - \lambda_D e_\mu^a, & \delta\omega_\mu^{ab} &= \partial_\mu\lambda^{ab} + 2\omega_{\mu c}^{[a}\lambda^{b]c} - 4\lambda_K^a e_\mu^b \\ \delta b_\mu &= \partial_\mu\lambda_D + 2\lambda_K^a e_{\mu a}, & \delta f_\mu^a &= \partial_\mu\lambda_K^a - b_\mu\lambda_K^a + \omega_\mu^{ab}\lambda_{Kb} - \lambda^{ab}f_{\mu b} + \lambda_D f_\mu^a. \end{aligned} \quad (3.38)$$

The gauge parameter  $\xi^a$  has been omitted here, because it corresponds to a general coordinate transformation, which are a bit more subtle [37]. In particular, they need to be upgraded to covariant general coordinate transformations. For instance, the naive general coordinate transformation of a scalar field  $\delta(\xi)\phi = \xi^\mu\partial_\mu\phi$  becomes the covariant  $\delta(\xi)\phi = \xi^\mu\mathcal{D}_\mu\phi$ , where  $\mathcal{D}_\mu$  is a covariant derivative containing all the gauge fields under which the scalar is charged.

In order to get back to standard General Relativity in the end, we do not want the additional gauge fields  $b_\mu$  and  $f_\mu^a$  to describe new physical degrees of freedom. For this, we have two tools at our disposal: imposing constraints and gauge fixing. The former is already what we do with the spin connection  $\omega_\mu^{ab}$  when, in the second-order formulation, we impose that  $\omega_\mu^{ab} = \omega_\mu^{ab}(e)$ , with  $\omega_\mu^{ab}(e)$  given in equation (3.30). From a more algebraic point-of-view, in standard General Relativity, this constraint is equivalent to the vanishing of the translation curvature  $R_{\mu\nu}(P^a) = 0$ , where the curvature of a generator, also called field strength in the context of Yang-Mills theories, can be computed from the algebra (3.37)

$$R_{\mu\nu}^A = 2\partial_{[\mu}A_{\nu]}^A + A_\nu^C A_\mu^B f_{BC}^A.$$

In the conformal framework that we are dealing with, we still impose  $R_{\mu\nu}(P^a) = 0$ , along with  $e_b^\nu R_{\mu\nu}(M^{ab}) = 0$ . These two curvatures are given by

$$R_{\mu\nu}(P^a) = 2(\partial_{[\mu} + b_{[\mu]}e_{\nu]}^a) + 2\omega_{[\mu}^{ab}e_{\nu]b} \quad \text{and} \quad R_{\mu\nu}(M^{ab}) = R_{\mu\nu}^{ab} + 8f_{[\mu}^{[a}e_{\nu]}^{b]},$$

where  $R_{\mu\nu}^{ab}$  is the Riemann tensor (3.32). The associated constraints give  $\omega_\mu^{ab}$  and  $f_\mu^a$

$$\omega_\mu^{ab}(e, b) = \omega_\mu^{ab}(e) + 2e_\mu^{[a}e^{b]\nu}b_\nu \quad \text{and} \quad 2(D - 2)f_\mu^a = -R_\mu^a + \frac{1}{2(D - 1)}e_\mu^a R, \quad (3.39)$$

here,  $R_{\mu\nu} = R_{\rho\mu}{}^{ab}e_a^\rho e_{\nu b}$  is the Ricci tensor,  $R = R_\mu{}^\mu$  the Ricci scalar and  $\omega_\mu^{ab}(e)$  is in (3.30). The curvature constraints thus implies that the two fields  $\omega_\mu^{ab}$  and  $f_\mu^a$  are composite, and as

such they do not describe physical degrees of freedom. The remaining unwanted gauge field  $b_\mu$ , can be fixed by choosing a gauge for the special conformal transformation, under which it is shifted  $\delta(\lambda_K)b_\mu = 2\lambda_{K\mu}$ . As a result, we can simply choose a gauge where  $b_\mu = 0$ . In addition, in this gauge, special conformal transformations are no longer independent transformations, bringing us one step closer to the strictly Poincaré invariant General Relativity.

The next step towards this goal is to do the same with the dilatation symmetry. For this, we introduce a scalar field  $\phi$ , called the compensator. First, we need to understand how it transforms under the conformal symmetry. Under dilatations, it is scaled by a constant  $w$ , that we call its Weyl weight

$$\delta(\lambda_D)\phi = \lambda_D w \phi.$$

According to the algebra (3.37), a special conformal transformation decreases the Weyl weight by one unit. But in the presence of only one scalar, there is nothing with the right Weyl weight for the scalar field to transform into. Therefore, the special conformal transformation of the scalar field vanishes

$$\delta(\lambda_K)\phi = 0.$$

In order to obtain the conformal invariant action of the compensating scalar, we can consider the conformal d'Alembertian  $\square^C \phi = \eta^{ab} \mathcal{D}_a \mathcal{D}_b \phi$ . Since the scalar field  $\phi$  only transforms under dilatations, with weight  $w$ , we have  $\mathcal{D}_a \phi = e_a^\mu (\partial_\mu - w b_\mu) \phi$ , and using the transformation rules (3.38) we can show that

$$\square^C \phi = e^{a\mu} (\partial_\mu \mathcal{D}_a \phi - (w+1) b_\mu \mathcal{D}_a \phi + \omega_{\mu ab} \mathcal{D}^b \phi + 2w f_{\mu a} \phi). \quad (3.40)$$

Using again the transformation rules (3.38), we can obtain

$$\delta \square^C \phi = (w+2) \lambda_D \square^C \phi + (2D - 4w - 4) \lambda_K^a \mathcal{D}_a \phi.$$

Consequently, if we choose  $w = 1/2D - 1$ , the conformal d'Alembertian  $\square^C \phi$  is invariant under special conformal transformations. In addition, its Weyl weight is  $w+2$ . Since the Weyl weight of the frame field determinant  $e$  is  $-D$ , the combination  $e \phi \square^C \phi$  has a Weyl weight  $-D + w + (w+2) = 0$ . In other words, it is invariant under local dilatations as well. In addition, it is a Lorentz scalar. This combination is therefore the conformal invariant action we are looking for

$$\mathcal{S} = -\frac{1}{2} \int d^D x e \phi \square^C \phi.$$

Inserting the expressions of the gauge fields (3.39) and in the  $K$ -gauge  $b_\mu = 0$

$$\mathcal{S} = \int d^D x e \left( \frac{1}{2} (\partial\phi)^2 + \frac{D-2}{8(D-1)} R \phi^2 \right). \quad (3.41)$$

As a last step, we can choose a gauge for the dilatations. Since the scalar field transforms as a shift as well  $\delta(\lambda_D)\phi = \lambda_D w \phi$ , we can choose a gauge where  $\phi^2 = 4(D-1)/(D-2)$  is fixed, and end up with the Einstein-Hilbert action

$$\mathcal{S}_{\text{EH}} = \frac{1}{2} \int d^D x e R, \quad (3.42)$$

where only the Poincaré transformations remain as independent gauge transformations. Note that in the process, the kinetic term of the scalar field  $\phi$  has the wrong sign in (3.41). But that is not a problem, because this compensating scalar field is not physical in the final action. The Einstein-Hilbert term, on the other hand, has the right sign.



### 3.3.3 Superconformal construction of supergravity

The advantage of the conformal method is that it can be extended to obtain the action of matter-coupled supergravity. In the case of supergravity, one starts from the superconformal algebra, which contains several new generators in addition to  $(P_a, M_{ab}, D, K_a)$ . First, it contains the usual supercharge  $\mathcal{Q}_\alpha$  that was already present in the super-Poincaré algebra (3.6), but it also contains another supercharge  $\mathcal{S}_\alpha$ , and a new bosonic generator  $T$  that only acts on the supercharges. The commutators involving these new generators are

$$\begin{aligned} [D, \mathcal{Q}] &= \frac{1}{2}\mathcal{Q}, & [D, \mathcal{S}] &= -\frac{1}{2}\mathcal{S}, & [T, \mathcal{Q}] &= -\frac{3}{2}i\gamma_*\mathcal{Q}, & [T, \mathcal{S}] &= \frac{3}{2}i\gamma_*\mathcal{S}, \\ [M_{ab}, \mathcal{Q}] &= -\frac{1}{2}\gamma_{ab}\mathcal{Q}, & [M_{ab}, \mathcal{S}] &= -\frac{1}{2}\gamma_{ab}\mathcal{S}, & [K_a, \mathcal{Q}] &= \gamma_a\mathcal{S}, & [P_a, \mathcal{S}] &= \gamma_a\mathcal{Q}, \end{aligned} \quad (3.43)$$

and the anti-commutators are

$$\begin{aligned} \{\mathcal{Q}_\alpha, \mathcal{Q}^\beta\} &= -\frac{1}{2}(\gamma^a)_\alpha{}^\beta P_a, & \{\mathcal{S}_\alpha, \mathcal{S}^\beta\} &= -\frac{1}{2}(\gamma^a)_\alpha{}^\beta K_a \\ \{\mathcal{Q}_\alpha, \mathcal{S}^\beta\} &= -\frac{1}{2}\delta_\alpha{}^\beta D - \frac{1}{4}(\gamma^{ab})_\alpha{}^\beta M_{ab} + \frac{1}{2}i(\gamma_*)_\alpha{}^\beta T. \end{aligned} \quad (3.44)$$

To each of these generators, we associate a gauge field and a transformation parameter, according to the following table

generator	$P_a$	$M_{ab}$	$D$	$K_a$	$T$	$\mathcal{Q}$	$\mathcal{S}$
gauge field	$e_\mu^a$	$\omega_\mu^{ab}$	$b_\mu$	$f_\mu^a$	$A_\mu$	$\psi_\mu$	$\phi_\mu$
parameter	$\xi^a$	$\lambda^{ab}$	$\lambda_D$	$\lambda_K^a$	$\lambda_T$	$\epsilon$	$\eta$

From the algebra (3.37), (3.43), (3.44) we can deduce the structure constants and use them to compute the transformation rules of the gauge fields and the curvatures, following the two formulae already used

$$\delta(\theta)A_\mu^A = \partial_\mu\theta^A + \theta^C A_\mu^B f_{BC}^A \quad \text{and} \quad R_{\mu\nu}^A = 2\partial_{[\mu}A_{\nu]}^A + A_\nu^C A_\mu^B f_{BC}^A.$$

We will not reproduce all of them here, but only the ones that are useful in the present discussion, and in particular the curvature that are subject to constraints, which are

$$\begin{aligned} R_{\mu\nu}(P^a) &= 2(\partial_{[\mu} + b_{[\mu} e_{\nu]}^a) + 2\omega_{[\mu}{}^{ab} e_{\nu]b} - \frac{1}{2}\bar{\psi}_{[\mu}\gamma^a\psi_{\nu]} \\ R_{\mu\nu}(M^{ab}) &= R_{\mu\nu}{}^{ab} + 8f_{[\mu}{}^{[a} e_{\nu]}^{b]} - \bar{\psi}_{[\mu}\gamma^{ab}\phi_{\nu]}, \\ R_{\mu\nu}(\mathcal{Q}) &= 2\left(\partial_{[\mu} + \frac{1}{2}b_{[\mu} - \frac{3}{2}iA_{[\mu}\gamma_* + \frac{1}{4}\omega_{[\mu}{}^{ab}\gamma_{ab})\psi_{\nu]} - 2\gamma_{[\mu}\phi_{\nu]}. \end{aligned}$$

The first constraint is still  $R_{\mu\nu}(P^a) = 0$ , and yields the composite spin connection  $\omega_\mu{}^{ab}$

$$\omega_\mu{}^{ab}(e, b, \psi) = \omega_\mu{}^{ab}(e, b) + \frac{1}{2}\psi_\mu\gamma^{[a}\psi^{b]} + \frac{1}{4}\bar{\psi}^a\gamma_\mu\psi^b, \quad (3.45)$$

with  $\omega_\mu{}^{ab}(e, b)$  given in (3.39). The second constraint we impose is  $e_\nu^b \hat{R}_{\mu\nu}(M^{ab}) = 0$ , where  $\hat{R}_{\mu\nu}(M^{ab})$  is a modification of  $R_{\mu\nu}(M^{ab})$  making the constraint invariant under  $\mathcal{Q}$ -SUSY, which is not necessary but convenient [37]. The modified curvature reads

$$\hat{R}_{\mu\nu}(M^{ab}) = R_{\mu\nu}(M^{ab}) + \bar{\psi}_{[\mu}\gamma^{[a} R_{\nu]}^{b]}(\mathcal{Q}) + \frac{1}{2}\bar{\psi}_{[\mu}\gamma_{\nu]} R^{ab}(\mathcal{Q}) = \hat{R}_{\mu\nu}{}^{ab} + 8f_{[\mu}{}^{[a} e_{\nu]}^{b]},$$

where  $\hat{R}_{\mu\nu}{}^{ab}$  is a modified Riemann tensor whose expression can easily be deduced from the previous equation. With this notation, the solution of the constraint  $e_b^\nu \hat{R}_{\mu\nu}(M^{ab}) = 0$  is the same as in (3.39), in  $D = 4$

$$f_\mu{}^a = -\frac{1}{4}\hat{R}_\mu{}^a + \frac{1}{24}e_\mu{}^a \hat{R}, \quad (3.46)$$

here,  $\hat{R}_{\mu\nu}$  and  $\hat{R}$  are the modified Ricci tensor and scalar built from  $\hat{R}_{\mu\nu}{}^{ab}$ . The last constraint is  $\gamma^\mu R_{\mu\nu}(\mathcal{Q}) = 0$ , which can be solved to give

$$\phi_\mu = -\frac{1}{2}\gamma^\nu R'_{\mu\nu}(\mathcal{Q}) + \frac{1}{12}\gamma_\mu \gamma^{ab} R'_{ab}(\mathcal{Q}) \quad (3.47)$$

where  $R'_{\mu\nu}(\mathcal{Q}) = R_{\mu\nu}(\mathcal{Q})$  with the  $\phi_\mu$  term omitted, i.e.  $R'_{\mu\nu}(\mathcal{Q}) = R_{\mu\nu}(\mathcal{Q}) + 2\gamma_{[\mu}\phi_{\nu]}$ .

At this stage of the construction, we still have two unwanted gauge fields  $b_\mu$  and  $A_\mu$ . In addition, dilatation, special conformal transformations,  $T$ -symmetry and  $\mathcal{S}$ -supersymmetry must be gauge-fixed to recover the strictly super-Poincaré invariant pure  $\mathcal{N} = 1$  supergravity. The gauge field  $b_\mu$  still transforms as a shift under special conformal transformations  $\delta(\lambda_K)b_\mu = 2\lambda_{K\mu}$ , so we can simply fix the latter by choosing  $b_\mu = 0$ . On the other hand, the gauge field  $A_\mu$  cannot be eliminated and remains as an auxiliary field.

In order to fix dilatations,  $T$ -symmetry and  $\mathcal{S}$ -supersymmetry, we can as in section 3.3.2 introduce a compensating chiral multiplet  $(Z, \chi, F)$ , and the first step is to understand how it transforms under superconformal transformations. Under dilatations, the Weyl weights of the multiplet are  $(w, w + 1/2, w + 1)$ , because of the commutator  $[D, \mathcal{Q}] = 1/2\mathcal{Q}$ , see (3.43). Under  $T$ -symmetry, component  $Z$  is also scaled by a constant  $c$  called its chiral weight

$$\delta(\lambda_T)Z = ic\lambda_T Z.$$

One can show from algebraic considerations that  $c = w$  in any chiral multiplet. The transformations under the other symmetries can also be deduced that way. For instance, the  $\mathcal{S}$ -supersymmetry transformation of  $Z$  has a Weyl weight  $w - 1/2$  because of  $[D, \mathcal{S}] = -1/2\mathcal{S}$ . But since there is no field of this weight in the multiplet, the transformation vanishes  $\delta(\eta)Z = 0$ . Omitting the details, which can be found in [37], we obtain the following transformations

$$\begin{aligned} \delta Z &= (w\lambda_D + iw\lambda_T)Z + \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi \quad (3.48) \\ \delta P_L\chi &= \left(\left(w + \frac{1}{2}\right)\lambda_D + \left(w - \frac{3}{2}\right)i\lambda_T\right)P_L\chi + \frac{1}{\sqrt{2}}P_L(\not{D}Z + F)\epsilon + \sqrt{2}wZP_L\eta \\ \delta F &= ((w + 1)\lambda_D + (w - 3)i\lambda_T)F + \frac{1}{\sqrt{2}}\bar{\epsilon}\not{D}P_L\chi + \sqrt{2}(1 - w)\bar{\eta}P_L\chi. \end{aligned}$$

Since the frame field determinant  $e$  has a Weyl weight  $-4$  and chiral weight  $0$  in  $D = 4$ , the combination  $eF$  has a Weyl weight  $-4 + (w + 1)$  and chiral weight  $(w - 3)$ . Therefore, starting from a chiral multiplet of weight  $w = 3$ , the combination  $eF$  is invariant under dilatations and  $T$ -transformations. But it is not invariant under  $\mathcal{Q}$  and  $\mathcal{S}$ -supersymmetries. To compensate for this, we can consider the action

$$\mathcal{S}_F = \int d^4x e \text{Re}\left(F + \frac{1}{\sqrt{2}}\bar{\psi}_\mu \gamma^\mu P_L\chi + \frac{1}{2}Z\bar{\psi}_\mu \gamma^{\mu\nu} P_R\psi_\nu\right), \quad (3.49)$$

which is indeed superconformally invariant. It is the locally supersymmetric action of a chiral multiplet of weight  $w = 3$ , that replaces the globally supersymmetric version (3.12).

In section 3.3.2, the compensating scalar  $\phi$  had a Weyl weight  $w = 1$  in dimension  $D = 4$ . The compensating multiplet  $(Z, \chi, F)$  that we consider here also has a Weyl weight  $w = 1$ , and in order to use the action (3.49) we should build a multiplet of weight  $w = 3$  from it. First, one can show that  $(\bar{F}, \not{D}P_{R\chi}, \square^C \bar{Z})$  is also a chiral multiplet, with Weyl weight  $w = 2$ . As a result, the chiral multiplet of lowest component  $Z\bar{F}$  has a Weyl weight  $w = 3$  and is a perfect candidate for the construction. Its other components are

$$P_L\chi(Z\bar{F}) = P_L\chi\bar{F} + Z\not{D}P_{R\chi} \quad \text{and} \quad F(Z\bar{F}) = F\bar{F} + Z\square^C\bar{Z} - \bar{\chi}P_L\not{D}\chi, \quad (3.50)$$

where  $\square^C$  is the superconformal covariant d'Alembertian that generalises (3.40)

$$\square^C Z = e^{a\mu} \left( \partial_\mu \mathcal{D}_a Z - 2b_\mu \mathcal{D}_a Z + \omega_{\mu ab} \mathcal{D}^b Z + 2f_{\mu a} Z - iA_\mu \mathcal{D}_a Z - \frac{1}{\sqrt{2}} \bar{\psi}_\mu P_L \mathcal{D}_a \chi + \frac{1}{2} \bar{\phi}_\mu \gamma_a P_L \chi \right).$$

The action (3.49) for the multiplet (3.50) of lowest component  $Z\bar{F}$  reads

$$\mathcal{S} = \int d^4x e \text{Re} \left( F\bar{F} + Z\square^C\bar{Z} - \bar{\chi}P_L\not{D}\chi + \frac{1}{\sqrt{2}} \bar{\psi}_\mu \gamma^\mu (P_L\chi\bar{F} + Z\not{D}P_{R\chi}) + \frac{1}{2} Z\bar{F} \bar{\psi}_\mu \gamma^{\mu\nu} P_R \psi_\nu \right).$$

At this point, we can set  $b_\mu = 0$  to fix the special conformal transformations and insert the expressions of the composite fields  $(\omega_\mu^{ab}, f_\mu^a, \phi_\mu)$  found in (3.45), (3.46), (3.47) and resulting from the curvature constraints. And finally, we can fix dilatations and  $T$ -symmetry by choosing  $Z = \sqrt{3}$ , in order to obtain the Einstein-Hilbert term with the right normalisation, as in (3.42), and  $\mathcal{S}$ -supersymmetry by choosing  $\chi = 0$ , see (3.48). The result is

$$\mathcal{S}_{\text{SG}} = \int d^4x e \left( \frac{1}{2} R(\omega(e, \psi)) - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} (\partial_\nu + \frac{1}{4} \omega_\nu^{ab}(e, \psi) \gamma_{ab}) \psi_\rho + 3A^a A_a - F\bar{F} \right), \quad (3.51)$$

which the action of pure  $\mathcal{N} = 1$  supergravity found in (3.36). The  $\psi$ -dependent terms can be extracted from  $\omega(e, \psi)$  using (3.45) and recombine into the four-fermions term (3.35).

Note that the transformation parameters of gauge-fixed transformations are related to the parameters of the remaining super-Poincaré transformations. For instance, the  $\mathcal{S}$ -gauge condition  $\chi = 0$  is not invariant under a  $\mathcal{Q}$ -SUSY transformation of parameter  $\epsilon$ , unless it is accompanied by a  $\mathcal{S}$ -SUSY transformation of parameter

$$P_L\eta = \frac{1}{2} i P_L \not{A} \epsilon - \frac{1}{2\sqrt{3}} F P_L \epsilon. \quad (3.52)$$

Such a relation is called a decomposition law. Other ones can be found in the same way, by requiring invariance of the gauge-fixing conditions. They play a role in determining the transformation of the different fields under super-Poincaré transformations at the end of the process. For example, the transformation of the gravitino is

$$\delta P_L \psi_\mu = P_L \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{3}{2} i A_\mu + \frac{1}{2} i \gamma_\mu \not{A} + \frac{1}{2\sqrt{3}} \gamma_\mu \bar{F} \right) \epsilon.$$

### 3.3.4 The matter-coupled supergravity

We are now ready to discuss the construction of the matter-coupled  $\mathcal{N} = 1$  supergravity. For this, an ingredient that we have not introduced yet is the superconformally invariant action of a real multiplet  $(C, \zeta, \mathcal{H}, A_\mu, \lambda, D)$ , to replace the globally supersymmetric version (3.12). If the first component  $C$  has a Weyl weight  $w$ , the  $D$ -component has a weight  $w + 2$ , so the combination  $eD$  has a Weyl weight  $-4 + (w + 2)$  and is invariant under dilatations if  $w = 2$ . Since  $D$  is real, it automatically has a vanishing chiral weight. For  $w = 2$ , one can in fact notice that the multiplet of lowest component  $-\bar{\mathcal{H}}$  is chiral with Weyl weight  $w = 3$ . As a consequence, one can apply (3.49) for this multiplet and, after some work, obtain

$$\begin{aligned} \mathcal{S}_D = \int d^4x e \left( D - \frac{1}{2} \bar{\psi} \cdot \gamma i \gamma_* \lambda - \frac{1}{3} CR(\omega) + \frac{1}{6} (C \bar{\psi}_\mu \gamma^{\mu\rho\sigma} - i \bar{\zeta} \gamma^{\rho\sigma} \gamma_*) R'_{\rho\sigma}(\mathcal{Q}) \right. \\ \left. + \frac{1}{4} \varepsilon^{abcd} \bar{\psi}_a \gamma_b \psi_c \left( A_d - \frac{1}{2} \bar{\psi}_d \zeta \right) \right). \end{aligned} \quad (3.53)$$

As in section 3.2, our goal is to obtain a general, action of  $n$  chiral multiplets  $(z^\alpha, \chi^\alpha, F^\alpha)$  and  $N$  gauge multiplets  $(A_\mu^A, \lambda^A, D^A)$ , with gauged supersymmetry and internal symmetry. Recall that in the presence of gauged internal symmetries, the multiplets are constructed according to the prescription explained in 3.2.2, with the covariant version of the SUSY algebra (3.22).

#### General superconformal action

For this construction, we actually start with  $n + 1$  chiral multiplets  $(X^I, \Omega^I, F^I)$ , keeping in mind that one combination of them is a compensating multiplet as introduced in sections 3.3.2 and 3.3.3. In section 3.2.1, a general supersymmetric action was built by applying (3.12) to the composite multiplets of lowest components  $1/2K(Z^\alpha, \bar{Z}^{\bar{\alpha}})$  and  $W(Z^\alpha)$  and  $-1/4f_{AB}(Z^\alpha)\bar{\lambda}^A P_L \lambda^B$ . We will start from the same idea and consider the action

$$\mathcal{S} = \int d^4x \left( [N(X, \bar{X})]_D + [\mathcal{W}(X)]_F + [f_{AB}(X)\bar{\lambda}^A P_L \lambda^B]_F \right), \quad (3.54)$$

where  $[N(X, \bar{X})]_D$  denotes the action obtained by applying (3.53) to the composite real multiplet of components  $N(X, \bar{X})$ , and  $[\mathcal{W}(X)]_F$  and  $[f_{AB}(X)\bar{\lambda}^A P_L \lambda^B]_F$  denote the actions obtained by applying (3.49) to the composite chiral multiplets of respective lowest components. Here,  $N(X, \bar{X})$  is a real function, while  $\mathcal{W}(X)$  and  $f_{AB}(X)$  are holomorphic. Like in the globally supersymmetric case, the way  $N$  appears in the action suggests to interpret it as a Kahler potential, associated to the target space metric  $G_{I\bar{J}} = N_{I\bar{J}}$ , the subscript  $I, \bar{J}$  indicating a derivation with respect to  $X^I, \bar{X}^{\bar{J}}$ .

In order for the action (3.54) to be superconformally invariant,  $[\dots]_D$  and  $[\dots]_F$  must be applied to multiplet of Weyl weight  $w = 2$  and  $w = 3$  respectively. By field redefinition, we can consider that the chiral multiplets  $(X^I, \Omega^I, F^I)$  all have weight  $w = 1$ . On the other hand, it can be shown, starting from a real multiplet and going in the Wess-Zumino gauge, that the weights of a gauge multiplet are  $(0, 3/2, 2)$ . The functions entering in (3.54) must therefore obey the following constraints

$$X^I \mathcal{W}_I = 3\mathcal{W}(X), \quad N(\lambda X, \bar{\lambda} \bar{X}) = (\lambda \bar{\lambda}) N(X, \bar{X}), \quad X^I f_{ABI} = 0. \quad (3.55)$$

## Auxiliary fields and constraints

The complete expansion of the action (3.54) can be found in [37]. From there, one can write and solve the equations of motion of the auxiliary fields  $F^\alpha$  and  $D^A$  and obtain in fact the same expressions as in the globally supersymmetric case (3.24) and (3.25)

$$\bar{F}^{\bar{J}} = N^{I\bar{J}}(-\mathcal{W}_I + \frac{1}{2}N_{I\bar{J}\bar{K}}\bar{\Omega}^{\bar{J}}\Omega^{\bar{K}} + \frac{1}{4}f_{ABI}\bar{\lambda}^A P_L\lambda^B),$$

and

$$\text{Re}(f_{AB})D^B = \mathcal{P}_A - \frac{1}{2\sqrt{2}}if_{ABI}\bar{\Omega}^I\lambda^B + \frac{1}{2\sqrt{2}}if_{ABI}\bar{\Omega}^{\bar{I}}\lambda^B,$$

where the  $\mathcal{P}_A$  are the moment maps, associated to gauged internal symmetries, as explained in section 3.2.2. As in section 3.3.3, the gauge field  $A_\mu$  of the  $T$ -symmetry (not to be confused with the fields  $A_\mu^A$  of gauge multiplets) is also auxiliary and given on-shell by  $A_\mu = \mathcal{A}_\mu + A_\mu^F$ , with

$$\mathcal{A}_\mu = \frac{i}{2N}(N_{\bar{I}}\partial_\mu\bar{X}^{\bar{I}} - N_I\partial_\mu X^I) + \frac{1}{N}A_\mu^A\mathcal{P}_A, \quad (3.56)$$

and

$$A_\mu^F = \frac{i}{4N}\left(\sqrt{2}\bar{\psi}_\mu(N_I\Omega^I - N_{\bar{I}}\bar{\Omega}^{\bar{I}}) + N_{I\bar{J}}\bar{\Omega}^I\gamma_\mu\Omega^{\bar{J}} + \frac{3}{2}\text{Re}(f_{AB})\bar{\lambda}^A\gamma_\mu\gamma_*\lambda^B\right).$$

Once auxiliary fields have been eliminated, the next steps are the same as before. First, we impose curvature constraints to eliminate the gauge fields  $(\omega_\mu^{ab}, f_\mu^a, \phi_\mu)$  from the physical degrees of freedom. These constraints are the same as in 3.3.3 and yield (3.45),(3.46),(3.47).

## Projective parameterisation, gauge fixing and decomposition laws

Next, we must gauge-fix dilatations, special conformal transformations,  $T$ -symmetry and  $\mathcal{S}$ -supersymmetry transformations. Special conformal transformations can as always be fixed by the choice  $b_\mu = 0$ . Then, dilatations can be fixed by choosing  $N = -3$ , in order to obtain the Einstein-Hilbert term with the right normalisation, as in (3.51), and  $\mathcal{S}$ -supersymmetry by choosing  $N_I\Omega^I = 0$ , in order to cancel a mixing with the gravitino. Before fixing the  $T$ -symmetry, it should be noted that the homogeneity property of  $N$  in (3.55) actually indicates that it is the Kahler potential of a projective Kahler manifold, see [37] for more details. This fact suggests to introduce the following parameterisation of the scalars  $X^I$

$$X^I = yZ^I(z), \quad (3.57)$$

where  $y$  is the compensating scalar and the  $z^\alpha$  are the  $n$  physical ones. The functions  $Z^I(z)$  are not uniquely defined in this parameterisation, a simple choice is  $Z^0 = 1$  and  $Z^\alpha = z^\alpha$ . In this parameterisation, the scalar  $y$  carries the Weyl and chiral weights, while the  $z^\alpha$  remains invariant under dilatations and  $T$ -transformations

$$\delta(\lambda_D, \lambda_T)y = (\lambda_D + i\lambda_T)y \quad \text{and} \quad \delta(\lambda_D, \lambda_T)z^\alpha = 0.$$

It can be shown that target space parameterised by the physical scalars  $z^\alpha$  has the following Kahler potential

$$\mathcal{K}(z, \bar{z}) = -3 \ln\left(-1/3Z^I(z)G_{I\bar{J}}\bar{Z}^{\bar{J}}(\bar{z})\right). \quad (3.58)$$

where  $G_{I\bar{J}} = N_{I\bar{J}}$ . In the context of projective Kahler manifolds, the invariance of this Kahler potential under Kahler transformations (3.18) simply comes from the freedom to redefine  $y$  and  $Z^I$  in (3.57)

$$y \rightarrow ye^{f(z)/3} \quad \text{and} \quad Z^I \rightarrow Z^I e^{-f(z)/3} \quad \text{gives} \quad \mathcal{K}(Z, \bar{Z}) \rightarrow \mathcal{K}(Z, \bar{Z}) + f(Z) + \bar{f}(\bar{Z}). \quad (3.59)$$

Note that the homogeneity property of  $N$  in (3.55) forbids its Kahler transformations. On the other hand, inserting the parameterisation (3.57) in the superpotential  $\mathcal{W}$  and using its homogeneity property in (3.55) gives the following relation with the physical superpotential

$$\mathcal{W}(X) = y^3 W(z) \quad (3.60)$$

The decomposition (3.57) between compensating and physical fields can be extended to the fermionic partners as  $\{\Omega^I\} = \{\chi^0, \chi^\alpha\}$ , in such a way that the  $\mathcal{S}$ -gauge condition  $N_I \Omega^I = 0$  translates into  $\chi^0 = 0$  and the  $z^\alpha$  transform under supersymmetry as  $\delta z^\alpha = 1/\sqrt{2}\bar{\epsilon}\chi^\alpha$ . One can compute this decomposition explicitly

$$\chi^0 = -\frac{1}{3}N_I \Omega^I \quad \text{and} \quad \chi^\alpha = \bar{y}\Omega^I G_{I\bar{J}} g^{\alpha\bar{\beta}} \bar{\nabla}_{\bar{\beta}} \bar{Z}^{\bar{J}}, \quad (3.61)$$

where  $\nabla_\alpha Z^I = \partial_\alpha Z^I + 1/3(\partial_\alpha \mathcal{K})Z^I$  is the Kahler-covariant derivative of  $Z^I$ .

With the parameterisation (3.57), we can actually fix the  $T$ -symmetry by the choice  $y = \bar{y}$ . Combined with  $N = -3$  and (3.58), this completely fix the compensating scalar to  $y = e^{\mathcal{K}/6}$ . Let us summarise the different gauge fixing conditions reducing the superconformal algebra to its super-Poincaré subalgebra

$$\text{special conformal transformations:} \quad b_\mu = 0, \quad (3.62)$$

$$\mathcal{S}\text{-supersymmetry:} \quad N_I \Omega^I = 0 \quad \text{or} \quad \chi^0 = 0,$$

$$\text{dilatations and } T\text{-symmetry:} \quad (N = -3 \quad \text{and} \quad y = \bar{y}) \quad \text{or} \quad y = e^{\mathcal{K}/6}.$$

Each of these conditions leads to a decomposition law, as explained around equation (3.52), fixing the parameters of the gauge-fixed symmetries as functions of the remaining ones to ensure gauge invariance of the gauge fixing conditions under super-Poincaré transformations. The decomposition laws of special conformal transformations and dilatations are actually quite simple

$$2\lambda_{K\mu} = -\frac{1}{2}\bar{\epsilon}\phi_\mu + \frac{1}{2}\bar{\eta}\psi_\mu, \quad \lambda_D = 0, \quad (3.63)$$

the one of  $T$ -symmetry is slightly more involved, because the condition  $y = \bar{y}$  is not invariant under gauge transformations,  $\mathcal{Q}$ -supersymmetry and Kahler transformations

$$\lambda_T = \frac{1}{6}i\theta_A(r_A(z) - \bar{r}_A(\bar{z})) + \frac{1}{6\sqrt{2}}i\bar{\epsilon}(\chi^\alpha \partial_\alpha \mathcal{K} - \chi^{\bar{\alpha}} \partial_{\bar{\alpha}} \mathcal{K}) + \frac{i}{6}(f(z) - \bar{f}(\bar{z})), \quad (3.64)$$

where the  $r_A(z)$  have been defined in (3.19) from the gauge transformation of the Kahler potential. The decomposition law of  $\mathcal{S}$ -supersymmetry can be found in [37].

Notice that the decomposition law of the  $T$ -symmetry contains a Kahler transformation with parameter  $f(z)$ . This means that after gauge fixing, the fermions that used to transform under  $T$ -transformations, now transform under Kahler transformations. It is possible to re-define the fermions in such a way that they are invariant under Kahler transformations. In doing so, the final action depends on the Kahler potential  $\mathcal{K}$  and superpotential  $W$  through the Kahler-invariant combination  $\mathcal{G} = \mathcal{K} + \log(W\bar{W})$ , as the one presented in [42].

## Finally, the action

Following all the steps that we just discussed, i.e. starting from the superconformally invariant action (3.54) where the composite multiplets are built using the covariant SUSY algebra (3.22), solving for the auxiliary fields ( $F^I, D^A, A_\mu$ ), applying curvature constraints to eliminate the gauge fields ( $\omega_\mu^{ab}, f_\mu^a, \phi_\mu$ ), introducing the parameterisation (3.57), rewriting everything in terms of the physical scalars  $z^\alpha$ , with the Kahler potential  $\mathcal{K}(z, \bar{z})$  of (3.58) and superpotential  $W(z)$  of (3.60), and in terms of the physical fermions  $\chi^\alpha$  of (3.61), and, finally, applying the gauge fixing conditions (3.62), we can obtain the following action

$$\begin{aligned}
\mathcal{S} = \int d^4x e \Big( & \frac{1}{2} R(\omega(e)) - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \left( \partial_\nu + \frac{1}{4} \omega_\nu^{ab}(e) \gamma_{ab} - \frac{3}{2} i \mathcal{A}_\nu \gamma_* \right) \psi_\rho \\
& - g_{\alpha\bar{\beta}} \left( \hat{\partial}^\mu z^\alpha \hat{\partial}_\mu \bar{z}^{\bar{\beta}} + \frac{1}{2} \bar{\chi}^\alpha \mathcal{D}^{(0)} \chi^{\bar{\beta}} + \frac{1}{2} \bar{\chi}^{\bar{\beta}} \mathcal{D}^{(0)} \chi^\alpha \right) - V \\
& + \text{Re}(f_{AB}) \left( -\frac{1}{4} F_{\mu\nu}^A F^{\mu\nu B} - \frac{1}{2} \bar{\lambda}^A \mathcal{D}^{(0)} \lambda^B \right) \\
& + \frac{i}{4} \left( \text{Im}(f_{AB}) F_{\mu\nu}^A \tilde{F}^{\mu\nu B} - \hat{\partial}_\mu \text{Im}(f_{AB}) \bar{\lambda}^A \gamma_* \gamma^\mu \lambda^B \right) \\
& \frac{1}{8} \text{Re}(f_{AB}) \bar{\psi}_\mu \gamma^{ab} \left( F_{ab}^A + \hat{F}_{ab}^A \right) \gamma^\mu \lambda^B + \left( \frac{1}{\sqrt{2}} g_{\alpha\bar{\beta}} \bar{\psi}_\mu \hat{\partial} \bar{z}^{\bar{\beta}} \gamma^\mu \chi^\alpha + \text{h.c.} \right) \\
& \left( \frac{1}{4\sqrt{2}} f_{AB\alpha} \bar{\lambda}^A \gamma^{ab} \hat{F}_{ab}^B \chi^\alpha + \text{h.c.} \right) + \mathcal{L}_m + \mathcal{L}_{\text{mix}} + \mathcal{L}_{4f},
\end{aligned} \tag{3.65}$$

which is invariant under local super-Poincaré transformations, where the transformation rule of each field is obtained starting from its superconformal transformation, see e.g. (3.48), inserting the values of the auxiliary fields ( $F^I, D^A, A_\mu$ ) and composite gauge fields ( $\omega_\mu^{ab}, f_\mu^a, \phi_\mu$ ), introducing the physical fields (3.57) and (3.61), and finally applying the gauge fixing conditions (3.62) and decomposition laws like (3.63) and (3.64).

Let us make some terms of this action more explicit. First  $\mathcal{A}_\mu$  is the bosonic part of the auxiliary field  $A_\mu$  given in (3.56), sometimes called the Kahler connection. In terms of physical fields

$$\mathcal{A}_\mu = \frac{i}{6} (\partial_\mu z^\alpha \partial_\alpha \mathcal{K} - \partial_\mu \bar{z}^{\bar{\alpha}} \partial_{\bar{\alpha}} \mathcal{K}) - \frac{1}{3} A_\mu \mathcal{P}_A.$$

We also define  $\hat{\partial}_\mu z^\alpha = \partial_\mu z^\alpha - A_\mu^A k_A^\alpha$ , where  $k_A^\alpha$  is the Killing vector associated to the gauge field  $A_\mu^A$ , and the  $\mathcal{D}^{(0)}$  acting on fermions are covariant derivatives from which the fermionic terms have been extracted, to be collected in  $\mathcal{L}_{4f}$  with the four-gravitino term (3.35). The lagrangian  $\mathcal{L}_m$  contains the mass terms of the fermions, and in particular the mass term of the gravitino

$$\mathcal{L}_m \supset \frac{1}{2} m_{3/2} \bar{\psi}_\mu P_R \gamma^{\mu\nu} \psi_\nu + \text{h.c.} \quad \text{with} \quad m_{3/2} = e^{\mathcal{K}/2} W, \tag{3.66}$$

and  $\mathcal{L}_{\text{mix}}$  is a mixing term between the gravitino and the other fermions

$$\mathcal{L}_{\text{mix}} = \bar{\psi} \cdot \gamma \left( \frac{i}{2} P_L \lambda^A \mathcal{P}_A + \frac{1}{\sqrt{2}} \chi^\alpha e^{\mathcal{K}/2} \nabla_\alpha W \right) + \text{h.c.}, \tag{3.67}$$

where  $\nabla_\alpha W = \partial_\alpha W + (\partial_\alpha \mathcal{K}) W$  is the Kahler-covariant derivative of the superpotential. More details on the action (3.65), including the expressions of fermion masses and the four-fermions term  $\mathcal{L}_{4f}$ , can be found in [37]. We discuss the scalar potential  $V$  next.

### 3.3.5 Scalar potential, SUSY breaking, FI term

The scalar potential of  $\mathcal{N} = 1$  supergravity takes the following form

$$V = e^{\mathcal{K}}(g^{\alpha\bar{\beta}}\nabla_{\alpha}W\overline{\nabla_{\bar{\beta}}W} - 3W\overline{W}) + \frac{1}{2}(\text{Re}f)^{-1AB}\mathcal{P}_A\mathcal{P}_B,$$

which is reminiscent of the globally supersymmetric case (3.26), it is the sum of an  $F$ -term and a  $D$ -term. However, one consequent novelty is that the  $F$ -term is not positive definite, it can take negative values. The scalar potential can therefore take any sign, or vanish, and the condition for preserving supersymmetry is no longer  $V(z_*, \bar{z}_*) = 0$  as in global supersymmetry. Supersymmetry is preserved in the vacuum if the SUSY transformation of the fermions vanish, which is equivalent to  $\nabla_{\alpha}W(z_*) = 0$  and  $\mathcal{P}_A(z_*) = 0$ . As a consequence

$$\begin{aligned} \text{supersymmetry is unbroken} &\leftrightarrow V_+(z_*, \bar{z}_*) = 0 \quad \text{and} \quad V(z_*, \bar{z}_*) \leq 0, \\ \text{supersymmetry is broken} &\leftrightarrow V_+(z_*, \bar{z}_*) > 0, \end{aligned} \tag{3.68}$$

where  $V_+$  is the positive part of the potential

$$V_+ = e^{\mathcal{K}}g^{\alpha\bar{\beta}}\nabla_{\alpha}W\overline{\nabla_{\bar{\beta}}W} + \frac{1}{2}(\text{Re}f)^{-1AB}\mathcal{P}_A\mathcal{P}_B.$$

So  $V(z_*, \bar{z}_*) > 0$  implies that supersymmetry is broken, but the converse is not true. When supergravity is spontaneously broken, a supersymmetric version of the Higgs mechanism takes place, where the massless Goldstino is eaten by the gravitino that becomes massive. The Goldstino is a combination of the chiral fermions  $\chi^{\alpha}$  and of the gauginos  $\lambda^A$ , and its expression can actually be found by inspection of the mixing term (3.67), updating the globally supersymmetric (3.27)

$$\mathcal{L}_{\text{mix}} = -\bar{\psi} \cdot P_L v \quad \text{with} \quad P_L v = -\frac{1}{\sqrt{2}}\chi^{\alpha}e^{\mathcal{K}/2}\nabla_{\alpha}W - \frac{i}{2}P_L\lambda^A\mathcal{P}_A.$$

The SUSY transformation of the Goldstino contains a term  $\delta P_L v \supset 1/2V_+P_L\epsilon$ . Since  $V_+ > 0$  when supersymmetry is broken, we can choose a gauge for supersymmetry where the Goldstino vanishes, the so-called

$$\text{unitary gauge: } v = 0. \tag{3.69}$$

### Fayet-Iliopoulos term, R-symmetry, Kahler transformations

As we explained in section 3.2.2, the Kahler potential  $\mathcal{K}(z, \bar{z})$  is not necessarily invariant under a gauge transformation, but can undergo a Kahler transformation  $\delta(\theta)\mathcal{K} = \theta^A(r_A(z) + \bar{r}_A(\bar{z}))$ . For abelian  $U(1)$  factors of the gauge group, the  $r_A(z)$  are only defined up to arbitrary imaginary constants, or equivalently the moment map  $\mathcal{P}_A$  are defined up to real constants.

This arbitrariness corresponds to the freedom of adding a Fayet-Iliopoulos term to the action for any abelian  $U(1)$  factor of the gauge group. Starting from  $\text{Im}(r(z)) = 0$ , adding this constant for a given  $U(1)$  has for consequence that it becomes an R-symmetry, under which the gravitino is charged [43]. It can be seen for instance in its gauge transformation, which contains a term proportional to  $r_A(z) - \bar{r}_A(\bar{z})$  coming from (3.64)

$$\delta(\theta)\psi_{\mu} = \frac{1}{4}(r_A(z) - \bar{r}_A(\bar{z}))\psi_{\mu}\theta^A.$$



In addition,  $r(z)$  is also involved in the gauge transformation of the superpotential

$$\delta(\theta)W(z) = k_A^\alpha(z)\partial W_\alpha(z)\theta^A = -r_A(z)W(z)\theta^A, \quad (3.70)$$

which constrains the form of the superpotential and the charges of the scalars. For instance, with only one scalar and  $r(z) = i\xi_s$ , the superpotential must transform as  $W(z) \rightarrow \exp(-i\xi_s\theta)W(z)$  and the only possibility is  $W(z) = z^b$  with a constraint on the charge of the scalar  $qb = \xi_s$ . A R-symmetry can thus be seen as a symmetry under which  $W$  transforms.

In [44], it has been shown that adding Fayet-Iliopoulos terms in supergravity requires the presence of a global symmetry, which are considered forbidden in quantum gravity, as we will explain in the beginning of chapter 7. In [45], a new way of including Fayet-Iliopoulos terms in supergravity has been introduced, which circumvents this problem. These have been called new Fayet-Iliopoulos terms, and we will introduce them in more details in chapter 4. The connection of these new Fayet-Iliopoulos terms to the effective action of space-filling anti- $D3$ -brane has been explored in [46]. On the other hand, [47] has recently investigated ways of breaking the global symmetry associated with standard Fayet-Iliopoulos terms, while preserving supersymmetry and gauge invariance.

Note that the  $r_A(z)$  are not invariant under Kähler transformations. Let us consider only one scalar field. Under a Kähler transformation  $\mathcal{K}(z, \bar{z}) \rightarrow \mathcal{K}(z, \bar{z}) + J(z) + \bar{J}(\bar{z})$  so we obtain

$$r(z) \rightarrow r(z) + k(z)\partial J(z), \quad (3.71)$$

implying that it is possible to choose  $\partial J(z) = -r(z)/k(z)$  such that  $r(z) \rightarrow 0$  and the  $U(1)$  becomes an ordinary, non R-symmetry. From (3.70), we see that  $-r(z)/k(z) = \partial \log W(z)$ , so choosing  $J(z) = \log W(z) + \text{constant}$  makes  $r(z) \rightarrow 0$ , and the associated Kähler transformation of the superpotential  $W(z) \rightarrow e^{-J}W(z)$  turns it into a constant. In particular it does not transform under any gauge symmetry. In short, R-symmetry is not a Kähler independent concept, since we can always go to a Kähler frame where the superpotential is constant and the gauge symmetry is not an R-symmetry. Note however that the moment map  $\mathcal{P}(z, \bar{z})$  is invariant under Kähler transformations; so if the Fayet-Iliopoulos constant is added in some Kähler frame it is present in any other Kähler frame, even in the one where the  $U(1)$  is not a R-symmetry. For instance, in some frames it can get incorporated into the Kähler potential.

# Chapter 4

## Massive gravitino amplitudes and the new FI terms

In this chapter, we study the  $2 \rightarrow 2$  gravitino scattering amplitudes at tree-level in spontaneously broken  $\mathcal{N} = 1$  supergravity with one chiral and one vector multiplet [3, 4]. The goal of this computation is to exhibit an analogy with a mechanism taking place in the electroweak sector of the Standard Model, that is recalled in section 4.1. When supersymmetry is spontaneously broken, the gravitino becomes massive, of mass  $m_{3/2}$ , by absorbing a Goldstino, and the scattering amplitude naively scales like  $\mathcal{M} \sim \kappa^2 E^4 / m_{3/2}$ , where  $E$  is the energy of the process and  $\kappa$  the reduced Planck mass. In section 4.2, we show that this scaling is reduced to  $\mathcal{M} \sim \kappa^2 E^2$  due to cancellations between different sectors. Amplitudes growing with energy generally signal a loss of unitarity of the perturbative description when  $\mathcal{M} \sim 1$ , so the above cancellation raises the perturbative cutoff of the theory from  $\Lambda \sim (m_{3/2}/\kappa)^{1/2} \sim M_{\text{SUSY}}$  up to the Planck scale, where  $M_{\text{SUSY}}$  is the supersymmetry breaking scale. In section 4.3, we consider the so-called new Fayet-Iliopoulos terms, that have recently been introduced [45] and provide a way to extend the standard Fayet-Iliopoulos terms of global supersymmetry, without the subtleties associated with gauging the R-symmetry. Since these terms generically modify the scalar potential without contributing to the  $2 \rightarrow 2$  gravitino scattering amplitudes at tree-level, they affect the aforementioned cancellation and lead to the perturbative cutoff  $\Lambda \sim (m_{3/2}/\kappa)^{1/2} \sim M_{\text{SUSY}}$ , except for particularly fine-tuned cases.

### 4.1 Unitarity and the Higgs mechanism

The notations and results of this section are based on [48]. Unitarity is an important property that must be satisfied by a consistent theory. If this were not the case, it could mean that some processes occur with a probability greater than one, which is hardly acceptable. Enforcing the unitarity of the S-matrix leads to what is known as the optical theorem, implying in particular that scattering amplitudes cannot be arbitrarily large. At first approximation, we can apply the following bound to the tree-level amplitudes

$$\text{Im}\mathcal{M} \leq |\mathcal{M}|^2 \quad \rightarrow \quad |\mathcal{M}| < 1. \quad (4.1)$$

Beyond this limit, unitarity is not necessarily violated. It may also signal that new states should be included in the theory, or a breakdown of its perturbative description. In any case, it's a threshold where something should be done.

In the Standard Model, it turns out that the Higgs boson plays a crucial role in maintaining perturbative unitarity after electroweak symmetry breaking. Let us briefly review this phenomenon. When electroweak symmetry is broken by the vacuum expectation value  $v$  of the Higgs field, the  $W^\pm$  and  $Z$  bosons become massive and can propagate with a longitudinal polarisation. But longitudinally polarised external vectors scale like  $E/v$  in amplitudes, where  $E$  is the energy of the process. Considering for instance the  $W^+Z \rightarrow W^+Z$  scattering, and ignoring the Higgs boson, there are three graphs contributing: two of them with a  $W$  exchange in the  $s$  and  $u$ -channels, and one coming from the quartic interaction between the gauge bosons. The three-bosons vertex comes with a factor of  $E$ , the  $W$  propagator with a factor of  $1/E^2$ , and the four-bosons vertex has no  $E$  dependence. Therefore, the amplitude with four longitudinal polarisations naively scales like  $E^4/v^4$  and breaks (4.1) when  $E \sim v$ . Computing these three graphs more precisely yields the following amplitude

$$\mathcal{M}_{\text{gauge}}(W_L Z_L \rightarrow W_L Z_L) = \frac{t}{v^2} + \mathcal{O}(1), \quad (4.2)$$

where  $t = (p_1 - p_3)^2 \sim E^2$  is a Mandelstam variable. Instead of  $E^4/v^4$ , this amplitude scales like  $E^2/v^2$ . In fact, its three contributions do separately have terms scaling like  $E^4/v^4$ , but they cancel each other. This can be explained using the Goldstone theorem, since the corresponding Goldstone amplitude scales at most like  $E^2/v^2$  at tree level. A second cancellation occurs when we take into account the Higgs boson, which can be exchanged in the  $t$ -channel, adding a contribution to the amplitude that cancels the first term of (4.2)

$$\mathcal{M}_h(W_L Z_L \rightarrow W_L Z_L) = -\frac{t}{v^2} + \mathcal{O}(1).$$

As a result, perturbative unitarity is maintained after electroweak symmetry breaking. This is reasonable, because in the unbroken phase of the theory, gauge bosons are massless and their amplitudes behave well. So there is no reason to expect a difference in the broken phase, which is described by the same underlying theory.

## 4.2 Massive gravitino scattering

In this section, we compute the massive gravitino scattering amplitudes  $\psi\psi \rightarrow \psi\psi$  in matter-coupled  $\mathcal{N} = 1$  supergravity, with external helicities  $\pm 1/2$ , and compare their behaviour with the case of longitudinal massive gauge boson amplitudes that we just reviewed. We start with the gravitational sector in section 4.2.1, before turning to the matter sector in 4.2.2.

### 4.2.1 Gravitational sector

Our starting point is the pure  $\mathcal{N} = 1$  supergravity action that we presented in (3.36)

$$\mathcal{S}_{\text{SG}} = \int d^4x e \left( \frac{1}{2} R(\omega(e)) - \frac{1}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} \left( \partial_\nu + \frac{1}{4} \omega_\nu{}^{ab}(e) \gamma_{ab} \right) \psi_\rho + \frac{1}{2} m_{3/2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + \mathcal{L}_{4f} \right), \quad (4.3)$$

where

$$\mathcal{L}_{4f} = -\frac{1}{32} (\bar{\psi}^\mu \gamma^\rho \psi^\nu) (\bar{\psi}_\mu \gamma_\rho \psi_\nu) - \frac{1}{16} (\bar{\psi}^\rho \gamma^\mu \psi^\nu) (\bar{\psi}_\rho \gamma_\nu \psi_\mu) + \frac{1}{8} (\bar{\psi}^\rho \gamma^\mu \psi_\mu) (\bar{\psi}_\rho \gamma^\nu \psi_\nu), \quad (4.4)$$

and  $\omega_\mu^{ab}(e)$  is given in (3.30) and  $R((\omega(e))) = e_a^\mu e_b^\nu R_{\mu\nu ab}$ , with  $R_{\mu\nu ab}$  given in (3.32). Recall that the greek letters  $\mu, \nu, \dots$  are used for curved coordinate indices, while latin letters  $a, b, \dots$  are used for local frame indices. The gravitino mass term  $1/2 m_{3/2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu$  has been added by hand in (4.3), and it breaks supersymmetry explicitly. In section 4.2.2, we will consider models where this term is generated by spontaneous supersymmetry breaking in the presence of chiral and gauge multiplets. In all this chapter, we work in a Minkowski background.

Since the gravitino  $\psi_\mu$  is a Majorana fermion, there are four graphs contributing to the  $\psi\psi \rightarrow \psi\psi$  scattering amplitudes at tree level: one for each of the  $s, t, u$  channels, with the exchange of a graviton  $h_{\mu\nu}$ , and one coming from the quartic interaction (4.4). Three ingredients are required to compute the  $s, t, u$  graphs with fixed external helicities  $\pm 1/2$ : the propagator of the graviton, the  $h\psi^2$  vertex, and the external polarisation of the gravitinos.

### Propagator, vertex, polarisations

The graviton is the field  $h_{\mu\nu}$  describing a gravitational fluctuation around the background

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu},$$

where the factor 2 allows the kinetic term of the graviton to be canonically normalised. This kinetic term is found from expanding the Einstein-Hilbert action to second order in  $h_{\mu\nu}$ . In order to get the graviton propagator from there, we must supplement the action with a gauge fixing term. Making the choice  $\mathcal{L}_{g.f.} = -(\partial^\mu h_{\mu\nu} - 1/2 \partial_\nu h)^2$  gives

$$P_{\gamma\delta}^{\alpha\beta}(k) = -\frac{i}{2k^2} (\delta_\gamma^\alpha \delta_\delta^\beta + \delta_\gamma^\beta \delta_\delta^\alpha - \eta^{\alpha\beta} \eta_{\gamma\delta}). \quad (4.5)$$

The  $h\psi^2$  vertex is obtained from expanding the action (4.3) to first order in  $h_{\mu\nu}$ . The frame field expands as

$$e_\mu^a = \delta_\mu^a + h_\mu^a \quad \text{and} \quad e_a^\mu = \delta_a^\mu - h_a^\mu,$$

and after this expansion is made, there is no distinction between flat and curved indices, because we work around a flat background. Before expanding (4.3), it is convenient to write

$$-\frac{1}{2} e \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho = \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_* \gamma_\sigma D_\nu \psi_\rho,$$

where  $\varepsilon^{\mu\nu\rho\sigma}$  is the antisymmetric Levi-Civita tensor and recall that  $\gamma_* = i\gamma_0\gamma_1\gamma_2\gamma_3$ . This is convenient, because there are four factors of the frame field on the left-hand side  $e\gamma^{\mu\nu\rho} = e e_a^\mu e_b^\nu e_c^\rho \gamma^{abc}$ , but only one in the right-hand side  $\gamma_\sigma = e_\sigma^a \gamma_a$ , because the Levi-Civita tensor does not depend on it. The frame field also appears in  $\omega_\mu^{ab}(e)$ , giving at first order

$$\omega_\mu^{ab}(e) = -\partial^{[a} h^b]_\mu.$$

Combining all this information, the  $h\psi^2$  terms of the Lagrangian (4.3) read

$$\begin{aligned} \mathcal{L}_{h\psi^2} = & \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} h_\sigma^a \bar{\psi}_\mu \gamma_* \gamma_a \partial_\nu \psi_\rho - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \partial^a h_\nu^b \bar{\psi}_\mu \gamma_* \gamma_\sigma \gamma_{ab} \psi_\rho \\ & + \frac{1}{2} m_{3/2} h \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu - m_{3/2} h_\rho^\mu \bar{\psi}_\mu \gamma^{\rho\nu} \psi_\nu, \end{aligned} \quad (4.6)$$

which can be simplified using gamma-matrix manipulations. One can show that

$$\varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_a = 4i\gamma^{[\mu\nu\rho}\delta^\sigma]_a} \quad \text{and} \quad \varepsilon^{\mu\nu\rho\sigma}\gamma_*\gamma_\sigma\gamma_{ab} = 6i\gamma^{[\mu\nu}{}_{[b}\delta^{\rho]}_a] + 6i\gamma^{[\mu}\delta^\nu{}_{[b}\delta^{\rho]}_a],$$

leading to

$$\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}h_\sigma^a\bar{\psi}_\mu\gamma_*\gamma_a\partial_\nu\psi_\rho = -\frac{1}{2}h_{\alpha\beta}\bar{\psi}_\mu(\eta^{\alpha\beta}\gamma^{\mu\rho\nu} - \eta^{\alpha\mu}\gamma^{\beta\rho\nu} + \eta^{\alpha\rho}\gamma^{\beta\mu\nu} - \eta^{\alpha\nu}\gamma^{\beta\mu\rho})\partial_\rho\psi_\nu, \quad (4.7)$$

and

$$-\frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}\partial^a h_\nu^b\bar{\psi}_\mu\gamma_*\gamma_\sigma\gamma_{ab}\psi_\rho = -\frac{1}{2}\partial^\rho h_{\alpha\beta}\bar{\psi}_\mu(\delta_\rho^\mu\eta^{\alpha\beta}\gamma^\nu - \delta_\rho^\alpha\eta^{\beta\mu}\gamma^\nu - \delta_\rho^\mu\eta^{\alpha\nu}\gamma^\beta)\psi_\nu. \quad (4.8)$$

In (4.6), (4.7), (4.8), we took care to assign the indices as  $h_{\alpha\beta}\bar{\psi}_\mu \cdots \psi_\nu$ . In order to write the vertex, we consider all the momenta incoming and label them according to  $h(k)\bar{\psi}(p)\psi(q)$ ; we also denote  $t = p - q$ . In principle, it has to be symmetrised with respect to  $(\alpha\beta)$ , but since it is always contracted with the graviton propagator in what follows, which is also  $(\alpha\beta)$ -symmetric, we can keep only one side of the symmetry in the vertex. We also have to take into account that the two gravitinos are Majorana and can be swapped, with a  $-$  sign if there are 1 or 2 matrices  $\gamma$  inserted between them, or with a  $+$  sign if there are 0 or 3, see [37] for details on the properties of spinors and  $\gamma$ -matrices. Finally, there is a factor of  $-i$  associated with each derivative, along with a global factor of  $i$ . At the end of the day, the vertex is

$$\begin{aligned} V_{h\psi^2}^{\mu,\nu,\alpha\beta}(t, k) &= 1/2(\eta^{\alpha\beta}\gamma^\mu k^\nu + \eta^{\alpha\mu}\gamma^\nu k^\beta - \eta^{\alpha\mu}\gamma^\beta k^\nu + \eta^{\alpha\beta}\gamma^{\mu\rho\nu}t_\rho + \eta^{\alpha\mu}\gamma^{\beta\nu\rho}t_\rho \\ &\quad - \eta^{\alpha\beta}\gamma^\nu k^\mu - \eta^{\alpha\nu}\gamma^\mu k^\beta + \eta^{\alpha\nu}\gamma^\beta k^\mu + \gamma^{\alpha\mu\nu}t^\beta - \eta^{\alpha\nu}\gamma^{\beta\mu\rho}t_\rho) \\ &\quad + im_{3/2}(\eta^{\alpha\beta}\gamma^{\mu\nu} - \eta^{\alpha\mu}\gamma^{\beta\nu} + \eta^{\alpha\nu}\gamma^{\beta\mu}). \end{aligned} \quad (4.9)$$

Since in our computation the gravitinos are external and on-shell, we can simplify this vertex using the equations of motion of the free gravitinos. For instance using  $\gamma \cdot \psi = 0$  gives

$$\begin{aligned} V_{h\psi^2}^{\mu,\nu,\alpha\beta}(t, k) &\sim 1/2(-\eta^{\alpha\mu}\gamma^\beta k^\nu + \eta^{\alpha\beta}\eta^{\mu\nu}t - \eta^{\alpha\mu}\gamma^\beta t^\nu + \eta^{\alpha\mu}\eta^{\nu\beta}t \\ &\quad \eta^{\alpha\nu}\gamma^\beta k^\mu - \eta^{\mu\nu}\gamma^\alpha t^\beta + \eta^{\alpha\nu}\gamma^\beta t^\mu - \eta^{\alpha\nu}\eta^{\mu\beta}t) \\ &\quad + im_{3/2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}). \end{aligned} \quad (4.10)$$

The other equations of motions of free gravitinos are  $p \cdot \psi = 0$  and  $(i\not{p} - m_{3/2})\psi_\mu = 0$ , but they do not simplify the vertex much more. Note that our vertex differs from that reported in [49], so we believe there is an error there.

Now that we have the graviton propagator and the  $h\psi^2$  vertex, the last ingredient we need to compute scattering amplitudes are the external polarisations of the gravitinos. These polarizations are classical solutions of the free Rarita-Schwinger equation, which can be reformulated into three equations as we have just mentioned [50]

$$\gamma \cdot \psi^\lambda(p) = 0, \quad p \cdot \psi^\lambda(p) = 0 \quad \text{and} \quad (i\not{p} - m_{3/2})\psi_\mu^\lambda(p) = 0. \quad (4.11)$$

A basis of solutions of this system can be written using the Clebsch-Gordon decomposition

$$\begin{aligned} \psi_\mu^{++} &= \varepsilon_\mu^+ u^+, & \psi_\mu^+ &= \sqrt{2/3}\varepsilon_\mu^0 u^+ + \sqrt{1/3}\varepsilon_\mu^+ u^- \\ \psi_\mu^{--} &= \varepsilon_\mu^- u^-, & \psi_\mu^- &= \sqrt{2/3}\varepsilon_\mu^0 u^- + \sqrt{1/3}\varepsilon_\mu^- u^+, \end{aligned} \quad (4.12)$$

where the  $\varepsilon_{\mu}^{\pm,0}$  are standard transverse and mutually orthogonal polarisation vectors, while the spinors  $u^{\pm}$  are solution of the free Dirac equation. The Dirac equation has two more solutions, usually called  $v^{\pm}$ , which can replace  $u^{\pm}$  in the previous equation. The choice between the two for a given external leg is done in the same way as for spin-1/2 fermions

- $u$  for an ingoing “gravitino”,  $\bar{v}$  for an ingoing “anti-gravitino”
- $\bar{u}$  for an outgoing “gravitino”,  $v$  for an outgoing “anti-gravitino”.

Note that the bar above  $(u, v)$  represents the Dirac conjugate  $\bar{\lambda}^D = \lambda^{\dagger}\gamma^0$ , whereas it represented the Majorana conjugate  $\bar{\lambda}^M = \lambda^T C$  above the gravitino  $\psi_{\mu}$  in the previous equations. Since  $u$  and  $v$  spinors are charge conjugate to one another  $v^{\pm} = (u^{\pm})^c$ , the Dirac conjugate of a  $v$ -spinor is the Majorana conjugate of the corresponding  $u$ -spinor.

Since gravitinos are Majorana, we have to clarify what we mean by “gravitino” and “anti-gravitino”. We simply use “gravitino” when the fermionic current attached to the leg is oriented towards the right of the graph, and “anti-gravitino” when it is oriented towards the left. For Majorana fermions, the direction of this current on each fermionic line is in fact arbitrary, as long as spinor bilinear can be formed consistently. The symmetry properties of the  $h\psi^2$  vertex (4.9) with respect to the exchange of the two gravitinos echo this arbitrariness.

For completeness, let us give the expressions of the  $\varepsilon^{\pm,0}$  and the  $u^{\pm}, v^{\pm}$ . For a momentum parameterised as

$$p^{\mu} = (p^0, |\mathbf{p}| \sin \theta \cos \varphi, |\mathbf{p}| \sin \theta \sin \varphi, |\mathbf{p}| \cos \theta),$$

the  $\varepsilon^{\pm,0}$  are given by, up to a sign that can be adjusted for  $\psi_{\mu}^{\pm}$  to satisfy (4.11)

$$\varepsilon_{\pm}^{\mu} = 1/\sqrt{2}(0, \cos \theta \cos \varphi \mp i \sin \varphi, \cos \theta \sin \varphi \pm i \cos \varphi, -\sin \theta),$$

and

$$\varepsilon_0^{\mu} = 1/m_{3/2}(|\mathbf{p}|, p^0 \mathbf{p}/|\mathbf{p}|). \quad (4.13)$$

To give an expression for  $u^{\pm}$ , we need to choose an explicit representation of the  $\gamma$  matrices. For example, in the Weyl representation

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \quad \text{where} \quad \sigma^{\mu} = (1, \sigma^i), \quad \bar{\sigma}^{\mu} = (-1, \sigma^i),$$

and the  $\sigma^i$  are the Pauli matrices. In the following, when computing the amplitude, we take it as a convention that all gravitinos are incoming. By conservation of energy, some of them will then have  $p^0 > 0$  and others  $p^0 < 0$ , and the expressions of  $u^{\pm}$  differ in these two cases

$$\begin{aligned} u^{\pm}(p) &= \begin{pmatrix} \sqrt{-\sigma \cdot p} \xi^{\pm} \\ i\sqrt{\bar{\sigma} \cdot p} \xi^{\pm} \end{pmatrix} \quad \text{if } p^0 > 0, \\ u^{\pm}(p) &= \begin{pmatrix} \sqrt{\sigma \cdot p} \xi^{\mp} \\ -i\sqrt{-\bar{\sigma} \cdot p} \xi^{\mp} \end{pmatrix} \quad \text{if } p^0 < 0, \end{aligned} \quad (4.14)$$

where

$$\xi^+ = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \quad \text{and} \quad \xi^- = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}.$$

As mentioned above, the  $v^{\pm}$  spinors are obtained as the charge conjugates of the  $u^{\pm}$ .

## Computation of the amplitudes

We now have all the necessary ingredients to calculate the contributions of the  $s, t, u$ -channel to  $\psi\psi \rightarrow \psi\psi$  amplitudes: the propagator (4.5), the vertex (4.10) and the polarisations (4.12). There only question we still need to answer is: what are the relative signs between them? To answer this, let us start by giving our conventions fermionic current orientation in figure 4.1

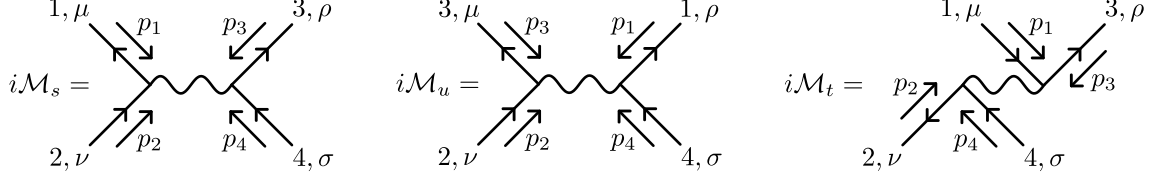


Figure 4.1: Conventions for the orientation of the fermionic lines in the  $s, t, u$  channels.

Without loss of generality, we assume that the  $s$ -channel graph contributes positively. From the  $s$ -channel, the  $u$ -channel is obtained by exchanging the gravitinos 1 and 3, so it comes with a relative  $-$  sign. To obtain the  $t$ -channel from the  $s$ -channel, we reverse the fermionic current around the left vertex before exchanging 1 and 4. Since the reversal of the fermionic current corresponds to the exchange of gravitinos 1 and 2, obtaining the  $t$ -channel from the  $s$ -channel requires two fermions exchanges, therefore it comes with a relative  $+$  sign.

These relative signs ensure that the total amplitude gets a minus sign under the exchange of any pair of gravitinos. This can be verified using the symmetries of the propagator (4.5) and vertex (4.10), but it is easier to see in the contribution of the quartic interaction (4.4). We write this contribution using the same symmetrisation principle as for the vertex (4.10) and with the same relative signs as for the  $s, t, u$  channels, making the four gravitinos equivalent

$$\begin{aligned} \mathcal{M}_{4f} = & -\frac{1}{16}(\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} + 2\eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\mu\rho} - 4\eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\mu\rho} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \sigma)) \\ & \times ((\bar{\psi}_\mu^1\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^4) - (\bar{\psi}_\mu^3\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^1\gamma_\beta\psi_\sigma^4) + (\bar{\psi}_\mu^2\gamma_\alpha\psi_\nu^4)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^1)). \end{aligned} \quad (4.15)$$

Here,  $(\mu \leftrightarrow \nu)$  and  $(\rho \leftrightarrow \sigma)$  stands for a repetition of the first terms with indices exchanged. If we swap for instance the gravitinos 1 and 2 in (4.15), we obtain

$$\begin{aligned} (\bar{\psi}_\mu^1\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^4) & \xrightarrow{1\leftrightarrow 2} (\bar{\psi}_\mu^2\gamma_\alpha\psi_\nu^1)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^4) \sim -(\bar{\psi}_\mu^1\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^4) \\ (\bar{\psi}_\mu^3\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^1\gamma_\beta\psi_\sigma^4) & \xrightarrow{1\leftrightarrow 2} (\bar{\psi}_\mu^3\gamma_\alpha\psi_\nu^1)(\bar{\psi}_\rho^2\gamma_\beta\psi_\sigma^4) \sim (\bar{\psi}_\mu^2\gamma_\alpha\psi_\nu^4)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^1) \\ (\bar{\psi}_\mu^2\gamma_\alpha\psi_\nu^4)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^1) & \xrightarrow{1\leftrightarrow 2} (\bar{\psi}_\mu^1\gamma_\alpha\psi_\nu^4)(\bar{\psi}_\rho^3\gamma_\beta\psi_\sigma^2) \sim (\bar{\psi}_\mu^3\gamma_\alpha\psi_\nu^2)(\bar{\psi}_\rho^1\gamma_\beta\psi_\sigma^4), \end{aligned} \quad (4.16)$$

where the  $\sim$  means up to the symmetry with respect to  $\mu \leftrightarrow \rho$  and  $(\alpha\sigma) \leftrightarrow (\beta\nu)$  and anti-symmetry with respect to  $\mu \leftrightarrow \nu$  that is provided by the first line of (4.15). Note that in the first line of (4.16) we use the Majorana flip  $\bar{\psi}_\mu^2\gamma_\alpha\psi_\nu^1 = \bar{\psi}_\nu^1\gamma_\alpha\psi_\mu^2$ , where the  $-$  sign mentioned before (4.9) is absent because the external polarisations are commuting spinors. Indeed, in the plane wave expansion of a spinor, its anticommutative nature is carried by the coefficients that multiply the polarisations, and not by the polarisations themselves.

Equation (4.16) shows that  $\mathcal{M}_{4f}$  gets a minus sign under the exchange  $1 \leftrightarrow 2$ . One can show that this is true for the exchange of any pair, as well as for the total amplitude including the  $s, t, u$  channels, provided we sum them with the right relative signs

$$\mathcal{M}_{\text{grav}} = \mathcal{M}_s - \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_{4f}.$$

Now that the question of relative signs has been settled, we really have everything we need to compute amplitudes. With four choices for the helicity of each external leg, there are 256 possible helicity assignments. This number is halved to 128 by CPT symmetry. In this work, we are interested in the amplitudes with external helicities  $\pm 1/2$ , which are expected to diverge the most with  $E$  due to  $E/m_{3/2}$  factors in longitudinal polarisation vectors (4.13), where  $2E$  is the center of mass energy. This still gives 8 possibilities after taking into account CPT symmetry. We will report only three of them, since the others behave in a similar way.

The amplitudes with external helicities  $\pm 1/2$  scale at most like  $\kappa^2 E^6/m_{3/2}^4$ , with one factor  $1/E^2$  coming from the propagator (4.5), two factors  $\kappa E$  coming from the vertex (4.10), four factors  $E/m_{3/2}$  coming from the longitudinal polarisations (4.13) and four factors  $E^{1/2}$  coming from the external  $(u, v)$  spinors (4.14). For instance, the subamplitudes contributing to  $\mathcal{M}_{\text{grav}}$  with helicities  $(+, +, -, -)$  have the following leading behaviour

$$\begin{aligned}\mathcal{M}_t^{+,+,-,-} &= \frac{2\kappa^2 E^6}{9m_{3/2}^4} (7 + 4 \cos(\theta) - 3 \cos(2\theta)) + \mathcal{O}\left(\frac{\kappa^2 E^4}{m_{3/2}^2}\right), \\ \mathcal{M}_u^{+,+,-,-} &= \frac{2\kappa^2 E^6}{9m_{3/2}^4} (7 - 4 \cos(\theta) - 3 \cos(2\theta)) + \mathcal{O}\left(\frac{\kappa^2 E^4}{m_{3/2}^2}\right),\end{aligned}$$

and

$$\mathcal{M}_{4f}^{+,+,-,-} = \frac{4\kappa^2 E^6}{9m_{3/2}^4} (-7 + 3 \cos(2\theta)) + \mathcal{O}\left(\frac{\kappa^2 E^4}{m_{3/2}^2}\right),$$

where  $E$  is the energy of the gravitinos 1 and 2 and  $\theta$  is the angle between them in their center of mass frame. It follows that these three contributions cancel in the total amplitude, leaving the next order

$$\mathcal{M}_{\text{grav}}^{+,+,-,-} = \frac{16\kappa^2 E^4}{3m_{3/2}^2} + \mathcal{O}(\kappa^2 E^2). \quad (4.17)$$

This cancellation of the most divergent term also happens for other amplitudes, for instance

$$\mathcal{M}_{\text{grav}}^{+,+,+,-} = \frac{\cancel{\kappa^2 E^5}}{m_{3/2}^3} - \frac{4\kappa^2 E^3}{3m_{3/2}} \sin(2\theta) + \mathcal{O}(\kappa^2 m_{3/2} E), \quad (4.18)$$

and

$$\mathcal{M}_{\text{grav}}^{+,+,+,+} = \frac{\cancel{\kappa^2 E^4}}{m_{3/2}^2} + \frac{16\kappa^2 E^2}{3} \sin^2(\theta) + \mathcal{O}(\kappa^2 m_{3/2}^2). \quad (4.19)$$

These cancellations are reminiscent of the ones we mentioned in the 4.1 section, which lead to scaling  $\mathcal{M}_{\text{gauge}} \sim E^2/v^2$  in (4.2), instead of the naive  $E^4/v^4$ . They can thus also be understood using the Goldstino equivalence theorem, saying that the high energy behaviour of helicity  $\pm 1/2$  gravitino amplitudes is captured by the corresponding Goldstino amplitudes, which scales at most like  $\kappa^2 E^4/m_{3/2}^2$  since their vertex with the graviton scales like  $\kappa E^2/m_{3/2}$ .



The different scalings for the different helicity assignments in (4.17)-(4.19) can also be understood. Indeed, in the next section, we will see that the next-to-leading terms of (4.17)-(4.19) also cancel with contributions from the matter sector, leaving only the  $\mathcal{O}(\dots)$ . After this cancellation, the amplitudes  $(+, +, +, -)$  and  $(+, +, +, +)$  vanish in the massless limit  $m_{3/2} \rightarrow 0$ , because of chirality conservation at the  $h\psi^2$  vertex. The power of  $m_{3/2}$  in the  $\mathcal{O}(\dots)$  of (4.18)-(4.19) reflects the number of potential chirality violations when  $m_{3/2} \rightarrow 0$ .

## 4.2.2 Matter sector

In this section, we consider the addition of a matter sector, with one chiral multiplet  $(z, \chi, F)$  and one gauge multiplet  $(A_\mu, \lambda, D)$ . We consider an arbitrary Kähler potential  $\mathcal{K}(z, \bar{z})$ , superpotential  $W(z)$ , and gauge kinetic function  $f(z)$ . We assume supersymmetry is spontaneously broken by a combination of  $F$  and  $D$ -terms, so the Goldstino is a combination of  $\chi$  and  $\lambda$ . We place ourselves in the unitary gauge, where this Goldstino is eaten by the massive gravitino.

These multiplets open new channels contributing to the  $\psi\psi \rightarrow \psi\psi$  scattering amplitudes, with the exchange of a scalar  $z$  and a gauge vector  $A_\mu$ . To compute these new contributions, the relevant terms in the  $\mathcal{N} = 1$  supergravity Lagrangian are the following

$$e^{-1}\mathcal{L} \supset -\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}(\partial_\nu - \frac{3}{2}i\mathcal{A}_\nu\gamma_*)\psi_\rho - \partial\bar{\partial}\mathcal{K}\hat{\partial}^\mu z\hat{\partial}_\mu\bar{z} - \frac{1}{4}\text{Re}(f)F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(e^{\mathcal{K}/2}W\bar{\psi}_\mu P_R\gamma^{\mu\nu}\psi_\nu + \text{h.c.}). \quad (4.20)$$

Here, partial derivatives without index  $\partial$  and  $\bar{\partial}$  stand for differentiation with respect to the scalar field  $z$  and  $\bar{z}$ , and  $F_{\mu\nu}$  is the abelian field strength of  $A_\mu$ . Recall that  $P_R = 1/2(1 - \gamma_*)$ . The covariant derivative of the scalar is given by  $\hat{\partial}_\mu z = \partial_\mu z - A_\mu k$  where  $k(z)$  is the Killing vector involved in the transformation of the scalar  $z$  under the gauge factor associated to  $A_\mu$ . With only one vector multiplet, the gauge group is  $U(1)$  and  $k(z) = -iqz$ , with  $q$  the charge. Finally,  $\mathcal{A}_\mu$  is the Kähler connection given by

$$\mathcal{A}_\mu = \frac{i}{6}(\partial_\mu z\partial\mathcal{K} - \partial_\mu\bar{z}\bar{\partial}\mathcal{K}) - \frac{1}{3}A_\mu\mathcal{P} = \frac{i}{6}(\hat{\partial}_\mu z\partial\mathcal{K} - \hat{\partial}_\mu\bar{z}\bar{\partial}\mathcal{K} + A_\mu(r - \bar{r})).$$

The moment map  $\mathcal{P}(z, \bar{z})$  appearing there was defined in (3.20). In the case of a single chiral multiplet it satisfies  $\bar{\partial}\mathcal{P}(z, \bar{z}) = ik(z)\partial\bar{\partial}\mathcal{K}$ . In addition,  $r(z)$  was defined in (3.19) from the gauge transformation of the Kähler potential  $(k(z)\partial + \bar{k}(\bar{z})\bar{\partial})\mathcal{K}(z, \bar{z}) = r(z) + \bar{r}(\bar{z})$ . The moment map can then be written as in (3.21)

$$\mathcal{P}(z, \bar{z}) = i(k(z)\partial\mathcal{K}(z, \bar{z}) - r(z)), \quad (4.21)$$

and it is gauge invariant. In order to compute the amplitudes, we assume that the scalar field  $z$  picks a nonvanishing vacuum expectation value  $|z_0| = v$  at the minimum of its potential, and parameterise it as

$$z(x) = (v + \eta(x))e^{i\phi(x)}, \quad \text{where} \quad \partial V(z_0, \bar{z}_0) = \bar{\partial}V(z_0, \bar{z}_0) = 0 \quad \text{and} \quad \langle\eta(x)\rangle = 0.$$

Since  $z$  is charged, this vacuum expectation value breaks the  $U(1)$  symmetry carried by  $A_\mu$ , and in the corresponding unitary gauge, the phase  $\phi(x)$  is eaten by the field  $A_\mu$  that becomes

massive. In this gauge, we simply have  $z(x) = v + \eta(x)$ , with  $\eta(x)$  real. The mass of  $A_\mu$  is obtained from  $\partial\bar{\partial}\mathcal{K}\hat{\partial}^\mu z\hat{\partial}_\mu\bar{z}$ , it reads

$$M_A^2 = 2(qv)^2 \frac{\partial\bar{\partial}\mathcal{K}_0}{\text{Re}(f_0)} = \frac{\partial\bar{\partial}\mathcal{K}_0 W_0 \bar{W}_0}{\nabla W_0 \bar{\nabla} \bar{W}_0} \frac{2\mathcal{P}_0^2}{\text{Re}(f_0)},$$

where  $\nabla$  is the Kähler covariant derivative  $\nabla W = \partial W + (\partial\mathcal{K})W$  and the subscript 0 means evaluated at the minimum  $(z_0, \bar{z}_0)$ . In this expression, we redefined the gauge potential to be canonically normalised, which at lowest order amounts to  $A^\mu \rightarrow A^\mu / (\text{Re}(f_0))^{1/2}$  and gives the factor  $\text{Re}(f_0)$ . In the second equality, we used the definition of  $\mathcal{P}$  in (4.21), along with  $k_0 = -iqz_0 = -iqv$ , and the identity  $-r/k = \partial \log W$  coming from the gauge transformation of the superpotential in (3.70). In the unitary gauge, the propagator of  $A_\mu$  is

$$P_A^{\alpha\beta}(k) = -\frac{i(\eta^{\alpha\beta} + k^\alpha k^\beta / M_A^2)}{k^2 + M_A^2}.$$

In order to obtain the  $\eta\psi^2$  vertex, we expand the Lagrangian (4.20) to first order in  $\eta$ . Like we just did for  $A^\mu$ , we also canonically normalise  $\eta$  by the redefinition  $\eta \rightarrow \eta / (2\partial\bar{\partial}\mathcal{K}_0)^{1/2}$ . We get the following contributions

$$\mathcal{L}_{\eta\psi\psi}^{(1)} = -\frac{1}{8\sqrt{2}} \frac{\partial\mathcal{K}_0 - \bar{\partial}\mathcal{K}_0}{\sqrt{\partial\bar{\partial}\mathcal{K}_0}} (\partial_\rho \eta) \bar{\psi}_\mu \gamma^{\mu\rho\sigma} \gamma_* \psi_\sigma,$$

and

$$\begin{aligned} \mathcal{L}_{\eta\psi\psi}^{(2)} = & + \frac{e^{\mathcal{K}_0/2}}{8\sqrt{2}} \frac{(\partial\mathcal{K}_0 + \bar{\partial}\mathcal{K}_0)(W_0 + \bar{W}_0) + 2(W_0 + \bar{W}_0)}{\sqrt{\partial\bar{\partial}\mathcal{K}_0}} \eta \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \\ & - \frac{e^{\mathcal{K}_0/2}}{8\sqrt{2}} \frac{(\partial\mathcal{K}_0 + \bar{\partial}\mathcal{K}_0)(W_0 - \bar{W}_0) + 2(W_0 - \bar{W}_0)}{\sqrt{\partial\bar{\partial}\mathcal{K}_0}} \eta \bar{\psi}_\mu \gamma^{\mu\nu} \gamma_* \psi_\nu. \end{aligned}$$

We also read the  $A\psi^2$  interaction

$$\mathcal{L}_{A\psi\psi} = -\frac{i}{4} \frac{\mathcal{P}_0}{\sqrt{\text{Re}(f_0)}} A_\rho \bar{\psi}_\mu \gamma^{\mu\rho\sigma} \gamma_* \psi_\sigma.$$

Note that even if the gravitino is not charged under the  $U(1)$ , it can interact with the gauge field  $A^\mu$  through  $\mathcal{P}_0$  when the scalar field  $z$  has a vacuum expectation value. These interactions can be turned into vertices using the same prescriptions as in (4.10), and we can then use the vertices to compute the new contributions to the  $\psi\psi \rightarrow \psi\psi$  scattering amplitudes. Let us for instance give the new contributions to the  $(++--)$  amplitude

$$\mathcal{M}_{\text{scalar}}^{+,+,-,-} = -\frac{8\kappa^2 E^4}{9|m_{3/2}|^2} \frac{\nabla W_0 \bar{\nabla} \bar{W}_0}{W_0 \bar{W}_0 \partial\bar{\partial}\mathcal{K}_0} + \mathcal{O}(\kappa^2 E^2),$$

and

$$\mathcal{M}_{\text{vector}}^{+,+,-,-} = -\frac{8\kappa^2 E^4}{9|m_{3/2}|^2} \left( \frac{2}{M_A^2} + \frac{1}{|m_{3/2}|^2} \right) \frac{\mathcal{P}_0^2}{\text{Re}(f_0)} + \mathcal{O}(\kappa^2 E^2),$$

where the gravitino mass is given by  $m_{3/2} = e^{\mathcal{K}_0/2}W_0$ , see (4.20). Combining these with the pure gravitational contribution that was computed in (4.17), we obtain the total amplitude

$$\mathcal{M}_{\text{total}}^{+,+,-,-} = -\frac{16\kappa^2 E^4}{9|m_{3/2}|^4} \left( e^{\mathcal{K}_0} \left( \frac{\nabla W_0 \bar{\nabla} \bar{W}_0}{\partial \bar{\partial} \mathcal{K}_0} - 3W_0 \bar{W}_0 \right) + \frac{\mathcal{P}_0^2}{2\text{Re}(f_0)} \right) + \mathcal{O}(\kappa^2 E^2). \quad (4.22)$$

Between the parentheses, we observe that the different contributions to the amplitude combined to form the scalar potential  $V(z_0, \bar{z}_0)$ , evaluated at its minimum  $(z_0, \bar{z}_0)$

$$V(z, \bar{z}) = V_F + V_D = e^{\mathcal{K}} \left( \frac{\nabla W \bar{\nabla} \bar{W}}{\partial \bar{\partial} \mathcal{K}} - 3W \bar{W} \right) + \frac{\mathcal{P}^2}{2\text{Re}(f)}. \quad (4.23)$$

As a result, the leading term in  $\mathcal{O}(\kappa^2 E^4/|m_{3/2}|^2)$  cancels when the potential vanishes at the minimum. This is actually required for the consistency of our computation, since we computed the amplitudes in a Minkowski background. This result holds for all helicity assignments. In the case where  $z$  has a vanishing vacuum expectation value  $v = 0$ , the  $U(1)$  is not broken, so the computation involves the complex scalar and a massless gauge vector, but the result is the same.

Therefore, the perturbative unitarity cutoff that was naively sitting at  $\Lambda \sim (m_{3/2}/\kappa)^{1/2} \sim M_{\text{SUSY}}$  is pushed up to the Planck scale  $\Lambda \sim 1/\kappa \sim M_{\text{Pl}}$ , in perfect analogy with the electroweak phenomenon that we discussed in 4.1. This result can certainly be extended with more than one chiral and vector multiplets, resulting in more general gauge groups, but it remains an interesting challenge to extend it beyond a Minkowski backgrounds.

## 4.3 Supergravity breaking and new FI terms

As we explained in 3.3.5, Fayet-Iliopoulos terms are problematic in  $\mathcal{N} = 1$  supergravity, because they are linked with gauged R-symmetry and imply the presence of a global symmetry, which is considered incompatible with quantum gravity. However, in [45], a new way has been found to effectively introduce Fayet-Iliopoulos terms without gauged R-symmetry.

### 4.3.1 Definition of new FI terms

This new Fayet-Iliopoulos term is built from a single gauge multiplet  $V = (A_\mu, \lambda, D)$ . Let us denote by  $y = (y, P_L \chi^0, F^0)$  the chiral compensating multiplet of weights  $(1, 1)$ . Using the superconformal formalism introduced in chapter 3, and in particular (3.53), the new FI term takes the form

$$\mathcal{L}_{\text{FI,I}} = -\xi_n \left[ y \bar{y} \frac{\mathcal{W}^2 \bar{\mathcal{W}}^2}{T(\bar{\mathcal{W}}^2) \bar{T}(\mathcal{W}^2)} (V)_D \right]_D, \quad (4.24)$$

where  $\xi_n$  is a constant of mass dimension two,  $(V)_D$  is the real multiplet of lowest component  $D$ , and  $\mathcal{W}$  is the chiral field strength multiplet of the gauge multiplet  $V$ . More precisely

$$(V)_D = (D, \not{D}\lambda, 0, \mathcal{D}^b \hat{F}_{ab}, -\not{D}\not{D}\lambda, -\square^C D) \quad \text{and} \quad \mathcal{W}^2 = \frac{\bar{\lambda} P_L \lambda}{y^2}$$

As we mentioned before equation (3.55), the Weyl weights of the gauge multiplet  $(A_\mu, \lambda, D)$  are  $(0, 3/2, 2)$ , and the chiral weight of  $P_L\lambda$  is  $3/2$  as well, so the chiral multiplet of lowest component  $\bar{\lambda}P_L\lambda$  has weights  $(3, 3)$ . Its higher components are the following

$$\left(\bar{\lambda}P_L\lambda, \quad \sqrt{2}P_L\left(-\frac{1}{2}\gamma \cdot \hat{F} + iD\right)\lambda, \quad 2\bar{\lambda}P_L\mathcal{D}\lambda + \hat{F}^- \cdot \hat{F}^- - D^2\right),$$

with the covariant field strength  $\hat{F}_{ab}$  and the self-dual/anti self-dual  $\hat{F}_{ab}^\pm$  given by

$$\hat{F}_{ab} = e_a^\mu e_b^\nu (2\partial_{[\mu}A_{\nu]} + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda) \quad \text{and} \quad \hat{F}_{ab}^\pm = \frac{1}{2}(\hat{F}_{ab} \pm \tilde{\hat{F}}_{ab}) \quad \text{where} \quad \tilde{\hat{F}}_{ab} = -\frac{i}{2}\epsilon_{abcd}\hat{F}^{cd}$$

and

$$\mathcal{D}_\mu\lambda = \left(\partial_\mu - \frac{3}{2}b_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \frac{3}{2}i\gamma_*\mathcal{A}_\mu\right)\lambda - \left(\frac{1}{4}\gamma^{ab}\hat{F}_{ab} + \frac{1}{2}i\gamma_*D\right)\psi_\mu,$$

where  $b_\mu$  is the gauge field associated to dilatations, set to  $b_\mu = 0$  in the gauge (3.62), and  $\mathcal{A}_\mu$  is the bosonic part of the auxiliary gauge field associated to the  $T$ -symmetry. Finally, the  $T$  in (4.24) is the chiral projection operator. Acting on an anti-chiral multiplet  $\bar{X} = (\bar{X}, P_R\Omega, \bar{F})$  of weights  $(1, -1)$ , it gives a chiral multiplet of weights  $(2, 2)$  defined as

$$T(\bar{X}) = (\bar{F}, \mathcal{D}P_R\Omega, \square^C\bar{X}),$$

Applied on  $\bar{W}^2$  it yields  $T(\bar{W}^2) = (C_T, P_L\Omega_T, F_T)$ , where in the gauge (3.62) with  $\chi^0 = 0$

$$\begin{aligned} C_T &= \bar{y}^{-2} \left(2\bar{\lambda}P_R\mathcal{D}\lambda + \hat{F}^+ \cdot \hat{F}^+ - D^2\right) - 2\bar{y}^{-3}\bar{F}^0\bar{\lambda}P_R\lambda, \\ P_L\Omega_T &= -\sqrt{2}\mathcal{D} \left(\bar{y}^{-2}P_R\left(\frac{1}{2}\gamma \cdot \hat{F} + iD\right)\lambda\right) \quad \text{and} \quad F_T = \square^C \left(\bar{y}^{-2}\bar{\lambda}P_R\lambda\right), \end{aligned} \quad (4.25)$$

It is important to note that if  $D$  has a vanishing vacuum expectation value, then  $C_T$  vanishes as well in the vacuum. Since  $C_T$  appears in the denominator of (4.24), this makes the new FI term singular and non-local in the limit of unbroken supersymmetry. Note also that the argument of  $[\dots]_D$  in (4.24) has the correct Weyl weight  $w = 2$  to use the formula (3.53).

The action of  $\mathcal{N} = 1$  supergravity with  $n$  chiral multiplets  $(z^\alpha, \chi^\alpha, F^\alpha)$  and a single abelian gauge multiplet associated to the new FI term can be written as in (3.54), where we already introduce the decomposition (3.57)

$$\mathcal{S} = \int d^4x \left(-3[y\bar{y}e^{-\mathcal{K}(z,\bar{z})/3}]_D + [y^3W(z)]_F - \frac{1}{4}[\bar{\lambda}P_L\lambda]_F + \mathcal{L}_{\text{FI,I}}\right). \quad (4.26)$$

Expanding into components, and using the gauge  $y = \bar{y} = e^{\mathcal{K}/6}$  as in (3.62), we get

$$-\frac{1}{4}[\bar{\lambda}P_L\lambda]_F + \mathcal{L}_{\text{FI,I}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2 - \xi_n e^{\mathcal{K}/3}D + \frac{i}{2}\xi_n e^{\mathcal{K}/3}\bar{\psi} \cdot \gamma\lambda + \mathcal{O}(\lambda^2) \quad (4.27)$$

Here,  $\mathcal{O}(\lambda^2)$  denotes terms containing at least two gauginos  $\lambda$ . We can see that couplings between the new FI term and the chiral multiplets are introduced by the choice  $y = e^{\mathcal{K}/6}$ . These couplings are not invariant under Kahler transformations [46, 51]. This can be seen directly in (4.24), if we remember that Kahler transformations act on the compensating scalar

$y$  as  $y \rightarrow ye^{f(z)/3}$ , as we explained in (3.59). In order to avoid this issue, a modification of the new FI term making it Kahler-invariant has been proposed in [52]

$$\mathcal{L}_{\text{FI,II}} = -\xi_n \left[ (y\bar{y}e^{-\kappa/3})^{-3} \frac{(\bar{\lambda}P_L\lambda)(\bar{\lambda}P_R\lambda)}{\bar{T}(\mathcal{W}^2)T(\bar{\mathcal{W}}^2)} (V)_D \right]_D, \quad \text{where} \quad \mathcal{W}^2 = \frac{\bar{\lambda}P_L\lambda}{(y\bar{y}e^{-\kappa/3})^2}. \quad (4.28)$$

In absence of chiral multiplets, the two versions of the new FI term (4.24) and (4.28) are equivalent. The construction of Kahler-invariant new FI terms has been generalised in [53].

As in (4.27), we can expand (4.28) into components and get

$$e^{-1}\mathcal{L}_{\text{FI,II}} = -\xi_n D + \frac{i}{2}\xi_n \bar{\psi} \cdot \gamma \lambda + \mathcal{O}(\lambda^2). \quad (4.29)$$

To first order in  $\lambda$ , we see that the two new FI terms only differ by a factor  $e^{\kappa/3}$ . In what follows, we thus introduce the notation  $\Delta = e^{\kappa/3}$  for the original new FI term (4.24) and  $\Delta = 1$  for the Kähler invariant new FI term (4.28) in order to write both cases at once.

Both new FI terms in (4.27) and (4.29) lead to the equation of motion  $D = \xi_n \Delta$  for  $D$ , so they contribute as follows to the  $D$ -term potential of (4.23)

$$V_{D,\text{FI}} = \frac{(\mathcal{P} + \xi_n \Delta)^2}{2\text{Re}(f)}. \quad (4.30)$$

The new FI terms also contribute to the Goldstino, because  $\lambda$  mixes with the gravitino in (4.27) and (4.29)

$$P_L v = -\frac{1}{\sqrt{2}}e^{\kappa/2}\nabla W \chi - \frac{i}{2}\mathcal{P}P_L\lambda - \frac{i}{2}\xi_n \Delta P_L\lambda,$$

where from now we consider a single chiral multiplet. The supersymmetry variation of this Goldstino contains

$$\delta v_L = \frac{1}{2} \left( e^{\kappa} \frac{\nabla W \bar{\nabla} \bar{W}}{\partial \bar{\partial} \mathcal{K}} + \frac{\mathcal{P}(\mathcal{P} + \xi_n \Delta)}{2\text{Re}(f)} \right) P_L \epsilon + \dots \quad (4.31)$$

In the case  $\xi_n = 0$  where the new FI term is absent, the term between parenthesis is always positive in a SUSY broken phase, so we can make a transformation such that  $v_L \rightarrow 0$ , which defines the unitary gauge (3.69). When  $\xi_n \neq 0$ , this is no longer the case and we need to check that this parenthesis does not vanish when evaluated at the minimum of the potential.

Assuming that we can use the unitary gauge, we can see in (4.27) and (4.29) that the new FI terms do not contain gravitino-gravitino-vector nor gravitino-gravitino-scalar interactions, since all the terms in  $\mathcal{O}(\lambda^2)$  contain at least two gauginos. As a result, they do not contribute to the  $2 \rightarrow 2$  gravitino scattering amplitudes computed in section 4.2.

Even though the addition of the new FI terms does not require gauged R-symmetry, we can consider its combination with the standard FI term discussed in section 3.3.5. With the standard FI term, the  $U(1)$  associated to the multiplet  $(A_\mu, \lambda, D)$  is an R-symmetry, under which the compensating scalar  $y$  transforms, and the original new FI term  $\mathcal{L}_{\text{FI,I}}$  of (4.24) is not gauge invariant for the same reason it is not Kahler invariant. That is, except in the

particular Kähler frame where the  $U(1)$  is not an R-symmetry, and where the superpotential is constant, as discussed after (3.71). In short, if we want to add both the standard and original new FI term  $\mathcal{L}_{\text{FI,I}}$  at the same time, we should start in the Kähler frame where the superpotential is constant. From there, nothing forbids us to perform Kähler transformations, but we shall keep in mind that different Kähler frames are not equivalent.

### 4.3.2 Perturbative cutoff

Let us now study the high energy behaviour of the supergravity theory defined by (4.26). In particular we would like to obtain its cutoff from the perturbative unitarity criterion (4.1). To this purpose, we will first take a look at scaling of the fermionic interaction terms contained in  $\mathcal{L}_{\text{FI,I}}$ , extending an analysis made in [54, 55]. This will show that the cutoff is not lower than the supersymmetry breaking scale. We will then use the results of section 4.2 to show that the cutoff is indeed at  $M_{\text{SUSY}}$ .

#### Dimensional analysis

The original new FI term Lagrangian (4.24) is the  $D$ -term density (3.53) of a real multiplet built from the three chiral multiplets  $y$  and  $W \equiv \bar{\lambda}P_L\lambda$  and  $T \equiv T(\bar{W}^2)$ , their anti-chiral counterparts, and the real multiplet  $(V)_D$ . Its lowest component is  $-\xi_n Df$ , where  $f$  is

$$f(y, \bar{y}, W, \bar{W}, T, \bar{T}) \equiv (y\bar{y})^{-1} \frac{(\bar{\lambda}P_L\lambda)(\bar{\lambda}P_R\lambda)}{C_T C_{\bar{T}}},$$

with  $C_T$  given in (4.25). One can expand the  $D$ -term density in (4.24) following the rules of multiplet calculus explained in chapter 3. Apart from the terms presented in (4.27) this yields a collection of non-renormalisable fermionic interactions of the form

$$\mathcal{L}_{\text{FI,I}} \supset -c\kappa^2 \xi_n D^p (\partial_y^{m_y} \partial_W^{m_W} \partial_T^{m_T} f) \mathcal{O}_F^{(\delta)}, \quad (4.32)$$

where  $c$  is a dimensionless constant and  $p = 0$  or  $1$ . The  $(m_y, m_W, m_T)$  are integers representing the number of derivatives of  $f$  taken with respect to  $(y, W, T)$  to get to the Lagrangian. Such derivatives can also be taken with respect to  $(\bar{y}, \bar{W}, \bar{T})$ , but in the context of this dimensional analysis we can identify  $y \sim \bar{y}$  and similarly for  $W$  and  $T$ . With this in mind, we can simply write  $f \sim y^{-2} W^2 T^{-2}$ . Finally,  $\mathcal{O}_F^{(\delta)}$  is a fermionic operator of mass dimension  $\delta$  fixed by homogeneity. Applying the derivatives on  $f \sim y^{-2} W^2 T^{-2}$  gives

$$\begin{aligned} \mathcal{L}_{\text{FI,I}} &\supset -c\xi_n \kappa^2 D^p y^{-2-m_y} W^{2-m_W} T^{-2-m_T} \mathcal{O}_F^{(\delta)} \\ &\supset -c\xi_n \kappa^{m_y-2m_T} D^{-4-2m_T+p} \mathcal{O}_F^{(10+m_y+2m_T-2p)}. \end{aligned} \quad (4.33)$$

In the second line, we used the gauge condition  $y \sim \kappa^{-1}$  and  $T \sim \kappa^2 D^2$  from (4.25); we also absorbed  $W^{2-m_W}$  into  $\mathcal{O}_F^{(\delta)}$ , because  $W$  is fermionic, and fixed the mass dimension  $\delta$  for  $\mathcal{L}_{\text{FI,I}}$  to have mass dimension 4. Recall that that  $D$  and  $\xi_n$  have mass dimension 2.

At this point, there are still factors of  $\kappa$  in  $\mathcal{O}_F$  coming from the gauge fixing  $y \sim \kappa^{-1}$ . In order to conclude about the cutoff of these higher-dimensional operators, we should extract

this  $\kappa$  dependence. For this, we need to notice that each derivative with respect to  $T$  in (4.32) is contracted with either  $C_T$  or  $P_L \Omega_T$  or  $F_T$  when expanding (4.24). Then from (4.25), we see that they all come with a factor  $\bar{y}^{-2} \sim \kappa^2$ . Similarly, each derivative with respect to  $y$  brings either  $y$  or  $P_L \chi^0$  or  $F^0$ , all contributing with a factor  $\kappa^{-1}$ . There is no factor of  $\kappa$  when taking derivatives with respect to  $W$ . So in the end, we obtain

$$\mathcal{O}_F^{(\delta)} = \kappa^{-m_y+2m_T} \mathcal{O}'_F^{(\delta-m_y+2m_T)},$$

where  $\mathcal{O}'_F$  is an operator containing only fields and derivatives, without factors of  $\kappa$ . Inserting this into (4.33) yields

$$\mathcal{L}_{FI} \supset -c\xi D^{-4-2m_T+p} \mathcal{O}'_F^{(10+4m_T-2p)},$$

which we can put into the form  $\mathcal{O}^{(4+n)}/\Lambda^n$ , with  $\Lambda$  of mass dimension 1. In our case

$$\Lambda^{6+4m_T-2p} = \xi_n^{-1} D^{4+2m_T-p} = D^{3+2m_T-p} \quad \rightarrow \quad \Lambda_{\text{cutoff}}^{\text{FI,I}} \gtrsim \sqrt{D} = M_{\text{SUSY}},$$

where we used  $D \sim \xi_n$  in the second equality, and interpreted the scale  $\Lambda$  as the cutoff of the corresponding operator. It turns out that all the fermionic interactions contained in the new FI term have the same cutoff, which is the supersymmetry breaking scale. When considering all these fermionic interactions together, however, the actual cutoff of the theory could be higher than  $M_{\text{SUSY}}$  because of Fierz identities or potential cancellations in the amplitudes. But we can confirm that it sits indeed at  $M_{\text{SUSY}}$  by using the results of section 4.2. Here, we have performed the analysis for the original  $\mathcal{L}_{\text{FI,I}}$ , but the same can be done with  $\mathcal{L}_{\text{FI,II}}$ .

## Gravitino scattering

As we mentioned after (4.31), in the unitary gauge, the new FI terms do not contribute to the  $2 \rightarrow 2$  gravitino scattering amplitudes computed in section 4.2, but they modify the  $D$ -term scalar potential as (4.30). For this reason, they generically disrupt the cancellation that was taking place in (4.22) and lead to a perturbative cut-off at the supersymmetry breaking scale. More precisely, the condition for cancellation of the leading term in (4.22) is

$$V_F(z_0, \bar{z}_0) + V_D(z_0, \bar{z}_0) = 0, \quad (4.34)$$

where  $V_F$  and  $V_D$  are defined in (4.23), and  $(z_0, \bar{z}_0)$  is an extremum of the scalar potential

$$\partial(V_F(z_0, \bar{z}_0) + V_{D,\text{FI}}(z_0, \bar{z}_0)) = \bar{\partial}(V_F(z_0, \bar{z}_0) + V_{D,\text{FI}}(z_0, \bar{z}_0)) = 0, \quad (4.35)$$

where  $V_{D,\text{FI}}$  is defined in (4.30). In addition, the condition for vanishing cosmological constant at the minimum, as assumed in our amplitude computation, is

$$V_F(z_0, \bar{z}_0) + V_{D,\text{FI}}(z_0, \bar{z}_0) = 0. \quad (4.36)$$

The conditions (4.34) and (4.36) imply  $2\mathcal{P}(z_0, \bar{z}_0) + \xi_n \Delta(z_0, \bar{z}_0) = 0$ , in other words they require that the new FI term does not contribute to the potential at the minimum.

We can easily find concrete examples where (4.34), (4.35), (4.36) are simultaneously satisfied. For instance, let us consider the case of a Kähler potential  $\mathcal{K}(z, \bar{z}) = z\bar{z}$ , superpotential  $W(z) = z^b$ , gauge kinetic function  $f(z) = 1$ , a standard FI term  $\xi_s$ , and either the original

(4.24) or the Kähler invariant (4.28) new FI term with parameter  $\xi_n$ . The presence of the standard FI term implies that the  $U(1)$  associated to the multiplet  $(A_\mu, \lambda, D)$  is an R-symmetry under which the superpotential transforms with a charge  $\xi_s$ , so the charge of the scalar must satisfy  $qb = \xi_s$ . As mentioned after (4.31), the original new FI term should be written in the Kähler frame where the superpotential is constant. In this frame  $\tilde{\mathcal{K}}(z, \bar{z}) = z\bar{z} + b \log(z\bar{z})$ . Taking all of this into account, we can find that for  $0 < b < 0.75$ , there is one value of  $(\xi_s, \xi_n)$  for which all conditions are satisfied. For  $b < 0$ , there are two such values.

For instance, for the original new FI term  $\mathcal{L}_{\text{FI,I}}$

- $b = 0.5, \quad \xi_s = 0.69211, \quad \xi_n = -2.93863, \quad r_0 = 0.96407,$
- $b = -1, \quad \xi_s = 6.43856, \quad \xi_n = -5.04518, \quad r_0 = 0.62367,$
- $b = -1, \quad \xi_s = 2.09581, \quad \xi_n = 4.76335, \quad r_0 = 1.81251,$

where  $r_0 = |z_0|$ .

For the Kähler invariant new FI term  $\mathcal{L}_{\text{FI,II}}$ , we find similarly

- $b = 0.5, \quad \xi_s = 0.50204, \quad \xi_n = -3.92696, \quad r_0 = 1.20644,$
- $b = -1, \quad \xi_s = 5.54153, \quad \xi_n = -7.43073, \quad r_0 = 0.57406,$
- $b = -1, \quad \xi_s = 1.43121, \quad \xi_n = 8.83171, \quad r_0 = 2.02123.$

In figure 4.2, we plot the corresponding potentials for concreteness. In all cases, we checked that the parenthesis in (4.31) does not vanish, allowing us to use the unitary gauge.

The values of the parameters for which (4.34), (4.35), (4.36) are simultaneously satisfied are rather fine tuned. In general, we should expect that the new FI terms lead to a perturbative cutoff at the SUSY breaking scale. Note that even with fine tuning, the new FI terms do contribute to the potential away from the minimum, and they also contain fermionic terms, for instance they contribute to the mass of the physical fermion.



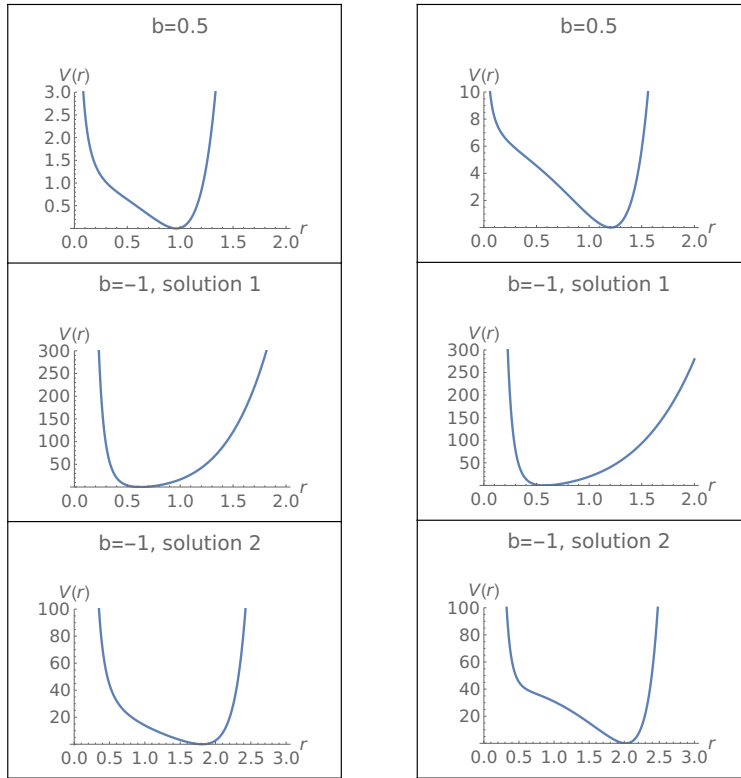


Figure 4.2: Scalar potential for the model with Kähler potential  $\mathcal{K}(z, \bar{z}) = z\bar{z}$ , superpotential  $W(z) = z^b$ , a standard FI term  $\xi_s$ , and either the original (left) or the Kähler invariant (right) new FI terms, with parameters chosen such that (4.34), (4.35), (4.36) are satisfied.

# Chapter 5

## Introduction to superstring theory and moduli stabilisation

This chapter is an introduction to string theory, mostly based on [56, 57], with an emphasis on type IIB compactification and moduli stabilisation, which will be the subject of chapter 6. We start with the basics of bosonic string theory in section 5.1, allowing us to cover some important ideas, as extra dimensions, the massless spectrum of the theory, the role of the dilaton and moduli stabilisation. In 5.2, we turn to superstring theory. We explain how it departs from the bosonic theory and introduce additional notions, such as the presence of spacetime spinors, the GSO projection, the spectrum of type II theories, and the effective action of type IIB theory. Finally, in section 5.3, we introduce the basics of compactification and moduli stabilisation in type IIB string theory. We then discuss Calabi-Yau orientifolds, the use of fluxes to stabilise their complex structure moduli, and conclude with an overview of frameworks that enable complete stabilisation, such as the KKLT and Large Volume Scenario.

### 5.1 Bosonic string theory

Let us start by introducing the simplest version of string theory: the bosonic string theory. Although unrealistic, as we will see, it illustrates several aspects of the more interesting superstring theory. As its name indicates, string theory postulates that the most fundamental constituents of nature are elementary one-dimensional strings, replacing the conventional zero-dimensional point particles. A massive bosonic point particle has no internal degrees of freedom, so its evolution is entirely characterised by its trajectory in spacetime, also called its worldline. This evolution can be determined by applying the variational principle to

$$\mathcal{S} = -m \int ds, \tag{5.1}$$

where  $m$  is the mass of the particle and  $ds$  is the line element  $ds^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$  along the worldline. Here we assume that the particle propagates in a background spacetime of dimension  $D$ , with a metric  $g_{\mu\nu}$ . If we parameterise the worldline with a real parameter  $\tau$ , such that  $x^\mu(\tau)$  describes the successive positions of the particle in spacetime, the action (5.1) takes the form

$$\mathcal{S} = -m \int \sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} d\tau,$$

a dot representing a derivative with respect to  $\tau$ . Applying the variational principle to this action gives the geodesic equation. We can use the same idea to write the following action for a one-dimensional string, called the Nambu-Goto action

$$\mathcal{S}_{\text{NG}} = -T \int d\mu \quad \text{where} \quad d\mu = \sqrt{-\det(g_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu)} d\tau d\sigma, \quad (5.2)$$

where  $T$  is the tension of the string. This action is proportional to the area of the two-dimensional surface in spacetime that the string sweeps as it evolves, called its worldsheet. This worldsheet is parameterised by two coordinates  $(\tau, \sigma)$ , respectively timelike and space-like. The Nambu-Goto action (5.2) has the bad taste of containing a square root, which makes it hard to use. For this reason, it is customary to introduce the Polyakov action

$$\mathcal{S}_{\text{P}} = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu, \quad (5.3)$$

where  $h^{\alpha\beta}$  is an auxiliary worldsheet metric. Writing and solving the equation of motion of the auxiliary metric and putting the result back into (5.3) gives (5.2), so the two actions are indeed classically equivalent. In what follows we consider a flat background  $g_{\mu\nu}(x) = \eta_{\mu\nu}$ .

The Polyakov action (5.3) is invariant under reparameterisations  $\sigma^\alpha \rightarrow f^\alpha(\sigma) = \sigma'^\alpha$ , where  $\sigma^\alpha$  collectively denotes  $(\tau, \sigma)$ . Under this reparameterisation, the worldsheet metric changes as  $h_{\alpha\beta}(\sigma^\alpha) = \partial_\alpha f^\gamma \partial_\beta f^\delta h_{\gamma\delta}(\sigma'^\alpha)$ . In addition, it is invariant under local scale transformations  $h_{\alpha\beta} \rightarrow e^{\phi(\tau, \sigma)} h_{\alpha\beta}$ . With a combination of these two transformations, and in the absence of topological obstructions, it is possible to choose a gauge where the worldsheet metric is flat  $h_{\alpha\beta} = \eta_{\alpha\beta}$  and the Polyakov action becomes

$$\mathcal{S}_{\text{P}} = \frac{T}{2} \int d^2\sigma (\dot{x}^2 - x'^2),$$

the prime representing a derivative with respect to  $\sigma$ . In this case, the equation of motion of the  $x^\mu(\tau, \sigma)$ , which represents the position of the string in spacetime, is a wave equation

$$\left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) x^\mu = 0, \quad (5.4)$$

which can easily be solved after specifying the boundary conditions of the string. But before, let us note that this equation can be simplified by introducing the light-cone coordinates

$$\sigma^\pm = \tau \pm \sigma \quad \rightarrow \quad \partial_+ \partial_- x^\mu = 0 \quad \rightarrow \quad x^\mu(\tau, \sigma) = x_R^\mu(\tau - \sigma) + x_L^\mu(\tau + \sigma),$$

showing that the general solution is a sum of left-moving and right-moving modes.

When choosing the flat gauge  $h_{\alpha\beta} = \eta_{\alpha\beta}$ , it should not be forgotten that the auxiliary metric  $h_{\alpha\beta}$  has an equation of motion implying the vanishing of the energy-momentum tensor

$$T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{-h}} \frac{\delta \mathcal{S}_{\text{P}}}{\delta h^{\alpha\beta}} = \partial_\alpha x^\mu \partial_\beta x_\mu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x_\mu = 0, \quad (5.5)$$

so after choosing the flat gauge,  $T_{\alpha\beta} = 0$  has to be imposed as an additional constraint. In the  $\sigma^\pm$  coordinates, the non trivial components are  $T_{++} = \partial_+ x^\mu \partial_+ x_\mu$  and  $T_{--} = \partial_- x^\mu \partial_- x_\mu$ .

## Boundary conditions, mode expansion, canonical quantisation

Boundary conditions fall into two categories: either the string is closed, or it is open. If it is closed, the boundary condition is periodic  $x^\mu(\tau, \sigma) = x^\mu(\tau, \sigma + \pi)$ , with the coordinate range  $0 \leq \sigma \leq \pi$ ; if it is open, there are still two possibilities, either  $\partial_\sigma x^\mu = 0$  or  $x^\mu = x_*^\mu$  at  $\sigma = 0$  and  $\pi$ , respectively called Neumann and Dirichlet boundary conditions, where  $x_*^\mu$  is a fixed position. Considering a closed string, the general solution of the wave equation (5.4) is

$$\begin{aligned} x_R^\mu(\tau - \sigma) &= \frac{1}{2}x_{\text{cm}}^\mu + \frac{1}{2}l_s^2 p_{\text{cm}}^\mu(\tau - \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)} \\ x_L^\mu(\tau + \sigma) &= \frac{1}{2}x_{\text{cm}}^\mu + \frac{1}{2}l_s^2 p_{\text{cm}}^\mu(\tau + \sigma) + \frac{i}{2}l_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}, \end{aligned} \quad (5.6)$$

where  $x_{\text{cm}}^\mu$  is the center-of-mass position of the string,  $p_{\text{cm}}^\mu$  its total momentum, and the sum represents all its vibration modes. The parameter  $l_s$  introduced here is the string length scale, it is linked to its tension  $T$  and to another parameter called  $\alpha'$  by  $T = 1/(2\pi\alpha')$  and  $l_s^2 = 2\alpha'$ . Note that the reality of  $x_R^\mu$  implies that  $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ , and the same for  $x_L^\mu$ .

We can then apply the canonical quantisation procedure. First, we define the canonical momentum conjugate to  $x^\mu$ , which is  $p^\mu = \delta\mathcal{S}_P/\delta\dot{x}_\mu = T\dot{x}^\mu$ . Next, we impose the canonical commutation relation on  $x^\mu$  and  $p^\mu$ , which implies a commutation relation for the modes

$$[x^\mu(\tau, \sigma), p^\nu(\tau, \sigma')] = i\eta^{\mu\nu} \delta(\sigma - \sigma') \quad \rightarrow \quad [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu} \delta_{m+n,0}.$$

If we redefine the mode operators, we can make this commutation relation more simple

$$a_m^\mu = \frac{1}{\sqrt{m}} \alpha_m^\mu \quad \text{and} \quad a_m^{\mu\dagger} = \frac{1}{\sqrt{m}} \alpha_{-m}^\mu \quad \rightarrow \quad [a_m^\mu, a_n^\nu] = \eta^{\mu\nu} \delta_{m,n}, \quad (5.7)$$

and similarly for the  $\tilde{\alpha}_n^\mu$ . The commutation relation (5.7) looks like the algebra of raising and lowering operators from the quantum harmonic oscillator, so in a similar way we can define the vacuum  $|0\rangle$  as a state that is annihilated by all lowering operators  $a_m^\mu |0\rangle = 0$ , and create states on top of this vacuum by application of raising operators, for instance  $|\phi^\mu\rangle = \alpha_m^{\mu\dagger} |0\rangle$ .

## Energy-momentum tensor, physical states, critical dimension

The energy-momentum tensor can also be expanded in modes, by inserting (5.6) into  $T_{++} = \partial_+ x \cdot \partial_+ x$  and  $T_{--} = \partial_- x \cdot \partial_- x$ . We obtain

$$\begin{aligned} T_{++} &= 2l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau - \sigma)} \quad \text{where} \quad L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} : \alpha_{m-n} \cdot \alpha_n : , \\ T_{--} &= 2l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im(\tau + \sigma)} \quad \text{where} \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} : \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n : , \end{aligned}$$

where the colons  $: :$  stands for the normal ordering prescription, according to which raising operators, with  $n \leq 0$  appear to the left of lowering operators, with  $n \geq 0$ . For example

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n,$$

and  $L_0$  is actually the only operator  $L_m$  for which normal ordering matters. Here  $1/2\alpha_0^2 = \alpha' p^2$  from inspection of (5.6). As we mentioned after equation (5.5), the condition  $T_{++} = T_{--} = 0$  has to be imposed as a physical constraint. For a state  $|\phi\rangle$ , this implies that

$$(L_0 - a) |\phi\rangle = 0 \quad \text{and} \quad L_m |\phi\rangle = 0 \quad \text{for all} \quad m > 0, \quad (5.8)$$

where the  $a$  is a constant that parameterise the arbitrariness of the normal ordering prescription for  $L_0$ . It is not necessary to impose  $L_{-m} |\phi\rangle = 0$ , because  $L_m |\phi\rangle = 0$  already implies  $\langle\phi| L_m^\dagger = \langle\phi| L_{-m} = 0$ . For the closed string, all formulae are duplicated with tilded operators, but we will not always write them explicitly.

The physical state condition  $(L_0 - a) |\phi\rangle = 0$  has some direct consequences. It can be used to determine the mass of a string state

$$\alpha' M^2 = -p_{\text{cm}}^2 = 4(N - a) \quad \text{where} \quad N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \quad (5.9)$$

where  $N$  is the number operator, counting the number of string excitations. So the mass of a state grows linearly with the number of excitations, with a factor  $1/\alpha' = 2/l_s^2$ . In addition

$$(L_0 - a) |\phi\rangle = (\tilde{L}_0 - a) |\phi\rangle = 0 \quad \rightarrow \quad (L_0 - \tilde{L}_0) |\phi\rangle = 0 \quad \rightarrow \quad N = \tilde{N}.$$

This level-matching condition is the only link between the left and right-moving sectors.

An important requirement for the theory to make sense is the absence of negative-norm states. This is not guaranteed a priori, because the commutation relation (5.7) implies that commutators in the timelike direction have a negative sign  $[a_m^0, a_m^{0\dagger}] = -1$ . It can be shown that the absence of negative-norm states puts a constraint on the spacetime dimension  $D$  and the constant  $a$  introduced in (5.8). In bosonic string theory, this is  $a \leq 1$  and  $D \leq 26$ . Therefore, the theory with  $a = 1$  and  $D = 26$  is called critical, and we focus on this case

$$D = 26 \quad \text{and} \quad a = 1 \quad (5.10)$$

Such a large number of dimensions can seem problematic, given that we have only observed four of them so far, but we will see that it is possible to effectively hide extra dimensions, for instance with a method called compactification.

Note that with  $a = 1$ , the mass of the ground state is tachyonic. Indeed, following (5.9) with  $N = 0$ , we have  $M^2 = -4/\alpha' < 0$ . This tachyonic state signals a classical instability, which is a problem of the bosonic string.

### Light-cone gauge and massless spectrum

The choice of flat-gauge  $h_{\alpha\beta} = \eta_{\alpha\beta}$  does not completely fix the reparameterisation and rescaling symmetry of the worldsheet, since there are reparameterisations that leave  $\eta_{\alpha\beta}$  invariant up to a constant factor. Such a transformation is called conformal. An infinitesimal parameter for a conformal transformation thus satisfies

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}.$$

After a conformal transformation, the factor  $\Lambda$  can be reabsorbed by a rescaling. In order to fix this residual symmetry, one can introduce light-cone coordinates for spacetime

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}),$$

and choose a gauge where all the modes in the  $+$  direction vanish  $\alpha_n^+ = 0$ , leading to

$$x^+(\tau, \sigma) = x_{\text{cm}}^+ + ls^2 p_{\text{cm}}^+ \tau. \quad (5.11)$$

The great advantage of this choice of gauge is that the physical condition  $T_{++} = T_{--} = 0$  can be solved explicitly and completely fixes the  $\alpha_n^-$  in terms of the  $\alpha_n^i$ , where  $i = (1, \dots, D-2)$ . Therefore, only the  $\alpha_n^i$  remain as the independent modes used to construct physical states.

For instance, in the case of the closed string, after the tachyonic state at  $N = 0$ , there are  $24^2 = 576$  physical states at the massless level  $N = 1$ , of the form  $|\phi^{ij}\rangle = \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0\rangle$ . These states fall into three representations of  $\text{SO}(24)$ : the symmetric-traceless, the antisymmetric, and the scalar representations. In other words, the massless spectrum of the closed bosonic string contains one spin-two particle, the graviton, one 2-form, and one scalar, the dilaton

$$M^2 = 0 \quad \rightarrow \quad g_{\mu\nu} \text{ (graviton)}, \quad B_{\mu\nu} \text{ (B-field)}, \quad \phi \text{ (dilaton)}.$$

### Background fields, string coupling constant, effective action

In (5.3), we started from the Polyakov action of a string in a background metric  $g_{\mu\nu}$ , but have just learned that the massless spectrum of the closed string also contains the  $B$ -field  $B_{\mu\nu}$  and the dilaton  $\phi$ , on an equal footing with  $g_{\mu\nu}$ . As a result, it is natural to consider adding them to the worldsheet action. One can show that the right way to do so is as follows

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \left( -\sqrt{-h} h^{\alpha\beta} g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu + \varepsilon^{\alpha\beta} B_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu + \alpha' \sqrt{-h} R(h) \phi(x) \right), \quad (5.12)$$

where  $\varepsilon^{\alpha\beta}$  is the Levi-Civita tensor, with  $\varepsilon^{01} = 1$ , and  $R(h)$  is the Ricci scalar of the metric  $h$ .

At this point, we need to pause and think. In order to introduce the notion of string at the beginning, we made an analogy with the point particle. We then realized that the string has several modes of vibration, each corresponding to a distinct type of particle. At the massless level for a closed string, we found a spin-two particle, a 2-form and a scalar. What we are doing now is postulating the existence of fields, of which these particles are quanta. But the precise relation between these fields and strings is rather vague. If we have in mind that an electromagnetic field can be seen as a coherent superposition of photons, the  $(g_{\mu\nu}, B_{\mu\nu}, \phi)$  fields can also be seen as coherent superpositions of strings in the corresponding modes. This would mean that spacetime itself is a coherent superposition of strings.

The position of the dilaton in the action (5.12) is interesting. If we assume that it is a constant  $\phi_0$ , it goes out of the integral and this part of the action becomes proportional to the Euler characteristic  $\chi(\Sigma)$  of the worldsheet  $\Sigma$ , which is a topological invariant

$$\mathcal{S}_\phi = \frac{1}{4\pi} \int_\Sigma d\tau d\sigma \sqrt{-h} R(h) \phi_0 = \phi_0 \chi(\Sigma).$$

If we then consider the partition function of the theory, formulated as a path integral over the worldsheet fields  $h_{\alpha\beta}$  and  $x^\mu$ , the path integral over  $h_{\alpha\beta}$  can be reduced to a sum over all worldsheet topologies, characterised by their Euler characteristic  $\chi$

$$Z \sim \sum_{\text{topologies}} \int [Dx^\mu] \exp(-\mathcal{S}[x^\mu]) = \sum_{\text{topologies}} e^{-\phi_0 \chi} \int [Dx^\mu] \exp(-\mathcal{S}_{g,B}[x^\mu]),$$

and we realise that different topologies are weighed by a factor  $e^{-\phi_0 \chi}$ . In the case of closed strings, the different topologies contributing to the path integral are Riemann surfaces without boundaries, and their Euler characteristic is simply  $\chi = 2 - n_h$ , the number of holes. Defining  $g_s = e^{\phi_0}$ , the different topologies are weighed by  $g_s^{2n_h-2}$ . If  $g_s < 1$ , the sum over worldsheet topologies is thus a perturbative expansion in the string coupling

$$g_s = e^{\phi_0}. \quad (5.13)$$

At energies that are far below the string energy scale  $M_s \sim 1/l_s \sim 1/\sqrt{\alpha'}$ , we can ignore the infinite tower of massive string states and focus on describing its massless sector. To do this, one can derive the following effective action for the fields  $(g_{\mu\nu}, B_{\mu\nu}, \phi)$

$$\mathcal{S} = \frac{1}{2} \int d^{26}x \sqrt{-g} e^{-2\phi} \left( R - \frac{1}{12} H^{\mu\nu\rho} H_{\mu\nu\rho} + 4\partial^\mu \phi \partial_\mu \phi \right), \quad (5.14)$$

where  $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$  is the 3-form field strength of the 2-form  $B_{\mu\nu}$ . This action contains the Einstein-Hilbert term for  $g_{\mu\nu}$ , and kinetic terms for  $(B_{\mu\nu}, \phi)$ . As such, it is formulated in the string frame, where the fields correspond to those of equation (5.12), but as in section 2.2 it is possible to go to the Einstein frame to remove the non-minimal coupling of the dilaton.

It is interesting to see the Einstein-Hilbert term appearing in the effective action (5.14). In fact, this action is only the leading terms of a series in  $\alpha'$ , with corrections coming from the massive string states. So in string theory, the Einstein-Hilbert term is naturally supplemented by higher order terms of the type considered in chapter 2. However, the calculation of these higher-order terms depends on the choices made in compactification.

It is also important to note that the dilaton field has a kinetic term, but no potential. Consequently, nothing determines its vacuum expectation value  $\langle \phi \rangle = \phi_0$ , which plays the role of the string coupling (5.13). More problematic still, this value can change at will at no energy cost, which would undoubtedly have dramatic phenomenological consequences.

Such a scalar field without potential is called a modulus, and in string theory, or in general in theories with compactified extra-dimensions, there are many of them, as we will see. As for the dilaton, the vacuum expectation value of other moduli is linked to physical parameters of the lower-dimensional theory, such as masses or gauge couplings. It is therefore crucial to understand the mechanisms that give a potential to the moduli, in order to stabilise them to a fixed value and avoid uncontrolled variations of the corresponding parameters. We will introduce these mechanisms in section 5.3, in the context of superstring theories.

## 5.2 Superstring theories

There are two problems with bosonic string theory: its spectrum contains a tachyonic state and no fermionic states. Fermions do exist in nature, so they must be incorporated in a realistic theory. In string theory, a natural way to introduce them is to start from the worldsheet and extend the Polyakov action (5.3) as

$$\mathcal{S} = -\frac{1}{\pi} \int d\tau d\sigma (\partial^\alpha x^\mu \partial_\alpha x_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu), \quad (5.15)$$

in the flat gauge  $h_{\alpha\beta} = \eta_{\alpha\beta}$ , and in units where  $T = 1/\pi$ . The new fields  $\psi^\mu$  are worldsheet spinors, and the  $\rho^\alpha$  are two-dimensional Dirac matrices satisfying  $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ . The  $x^\mu(\tau, \sigma)$  represent the position of the string in spacetime, but the  $\psi^\mu(\tau, \sigma)$  are more abstract. The worldsheet spinors  $\psi^\mu$  have two components, that we can denote  $\psi_-^\mu$  and  $\psi_+^\mu$ , and we can choose a representation for the  $\rho^\alpha$  matrices such that their equation of motions are

$$\partial_+ \psi_- = 0 \quad \text{and} \quad \partial_- \psi_+ = 0 \quad \rightarrow \quad \psi_-(\tau - \sigma) \quad \text{and} \quad \psi_+(\tau + \sigma),$$

The worldsheet action (5.15) is invariant under a global supersymmetry

$$\delta x^\mu = \bar{\epsilon} \psi^\mu \quad \text{and} \quad \delta \psi^\mu = \rho^\alpha \partial_\alpha x^\mu \epsilon, \quad (5.16)$$

associated to a supercurrent

$$J^\alpha = -\frac{1}{2} \rho^\beta \rho^\alpha \psi_\mu \partial_\beta x^\mu. \quad (5.17)$$

On the other hand, the energy-momentum tensor (5.5) becomes

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu + \frac{1}{4} \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu - \frac{1}{4} \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - (\text{trace}). \quad (5.18)$$

After equation (5.5), we mentioned that after choosing the flat gauge  $h_{\alpha\beta} = \eta_{\alpha\beta}$ , the vanishing of the energy-momentum tensor, coming from the equation of motion of  $h_{\alpha\beta}$ , has to be imposed as a constraint in the construction of physical states. Here, we started from (5.15) directly in the flat gauge, but it is possible to start from a worldsheet action where the reparameterisation and rescaling invariances are still present. In that action, the supersymmetry (5.16) is local, and associated with a worldsheet gravitino, whose equation of motion implies the vanishing of the supercurrent. After choosing the simple gauge (5.15), the vanishing of the supercurrent also has to be imposed as a constraint in the construction of physical states.

### Boundary conditions, mode expansion, canonical quantisation

With regards to boundary conditions, worldsheet fermions offer a new possibility. Again in the example of a closed string, these conditions can be periodic or antiperiodic

$$\psi_\pm^\mu(\tau, \sigma) = \pm \psi_\pm^\mu(\tau, \sigma + \pi),$$

and similarly for open strings. The periodic choice is called the Ramond (R) boundary condition, while the anti-periodic choice is called the Neveu-Schwarz (NS) boundary condition.



The corresponding mode expansions, are

$$\begin{aligned}\psi_-^\mu(\sigma - \tau) &= \sum_{n \in \mathbb{Z}} d_n^\mu e^{-2in(\tau - \sigma)} \quad \text{or} \quad \psi_-^\mu(\sigma - \tau) = \sum_{r \in \mathbb{Z} + 1/2} b_r^\mu e^{-2ir(\tau - \sigma)}, \\ \psi_+^\mu(\sigma + \tau) &= \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau + \sigma)} \quad \text{or} \quad \psi_+^\mu(\sigma + \tau) = \sum_{r \in \mathbb{Z} + 1/2} \tilde{b}_r^\mu e^{-2ir(\tau + \sigma)}.\end{aligned}$$

Canonical quantisation then imposes the following anticommutation relations

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu} \delta_{m+n,0} \quad \text{and} \quad \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0}.$$

So in superstring theory there are several sectors, and we define a vacuum for each. The Ramond vacuum  $|0\rangle_{\text{R}}$  is annihilated by the lowering operators  $\alpha_m^\mu$  and  $d_m^\mu$  with  $m > 0$ , while the Neveu-Schwarz vacuum  $|0\rangle_{\text{NS}}$  is annihilated by the operators  $\alpha_m^\mu$  and  $b_r^\mu$  with  $(m, r) > 0$ .

The important novelty is that the Ramond vacuum is degenerate, because the operators  $d_0^\mu$  commute with the number operator  $N$  counting the number of string excitations. Moreover, these operators satisfy the Clifford algebra  $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$ . The degenerate Ramond vacuum thus furnishes a representation of this algebra, and behaves as a spacetime spinor.

As for the bosonic closed string, the left-moving and right-moving fermionic modes of the closed superstring are largely independent, and boundary conditions can be imposed separately on the two sectors. The closed superstring therefore comprises four independent sectors: (NS-NS, R-R, NS-R, R-NS).

### Energy-momentum tensor, supercurrent, physical states, critical dimension

As for the bosonic string, we can compute the modes of the energy-momentum tensor (5.18) and of the supercurrent (5.17), in terms of the operators  $(\alpha_m^\mu, b_r^\mu, d_m^\mu)$ . These expansions depend on the sector under consideration

$$\begin{aligned}L_m &= \frac{1}{2} \sum_{n=-\infty}^{+\infty} : \alpha_{m-n} \cdot \alpha_n : + \frac{1}{2} \sum_{r \in \mathbb{Z} + 1/2} \left(r + \frac{m}{2}\right) : b_{-r} \cdot b_{m+r} : \quad (\text{NS sector}) \\ &\quad + \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(n + \frac{m}{2}\right) : d_{-n} \cdot d_{m+n} : \quad (\text{R sector})\end{aligned}$$

and

$$G_r = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n} \quad \text{and} \quad F_m = \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot d_{m+n}.$$

In the case of a closed string we have another collection of the same operators with tildes.

Superstring states are constructed in the same way as bosonic string states: by acting with raising operators  $(\alpha_m^\mu, b_r^\mu, d_m^\mu)$  with  $(m, r) < 0$ , on the Neveu-Schwarz and Ramond vacua. In addition, they are subject to the physical state conditions, as explained after (5.18)

$$\begin{aligned}L_m |\phi\rangle_{\text{NS}} = 0, \quad (L_0 - a_{\text{NS}}) |\phi\rangle_{\text{NS}} = 0, \quad G_r |\phi\rangle_{\text{NS}} = 0 \quad &\text{for } m > 0 \text{ and } r > 0, \\ L_m |\phi\rangle_{\text{R}} = 0, \quad (L_0 - a_{\text{R}}) |\phi\rangle_{\text{R}} = 0, \quad F_n |\phi\rangle_{\text{R}} = 0 \quad &\text{for } m > 0 \text{ and } n \geq 0,\end{aligned}$$

where  $a_{\text{NS}}$  and  $a_{\text{R}}$  are constants that parameterise the arbitrariness of the normal ordering prescription for  $L_0$ . In the light-cone gauge (5.11), which is complemented with  $\psi^+(\tau, \sigma)$  in the superstring theory, only the transverse operators  $(\alpha_m^i, b_r^i, d_m^i)$  are used to construct states, and the physical state conditions are automatically satisfied. In this gauge, the masses of states are

$$\begin{aligned}\alpha' M^2 &= \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} r b_{-r}^i b_r^i - a_{\text{NS}} \quad (\text{NS sector}) \\ \alpha' M^2 &= \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} n d_{-n}^i d_n^i - a_{\text{R}} \quad (\text{R sector})\end{aligned}\tag{5.19}$$

As for the bosonic string, there is a critical dimension  $D$  and values of  $a_{\text{NS}}$  and  $a_{\text{R}}$  for which there are no negative-norm states in the spectrum. These values turn out to be

$$D = 10, \quad a_{\text{NS}} = \frac{1}{2} \quad \text{and} \quad a_{\text{R}} = 0.\tag{5.20}$$

### Modular invariance and GSO projection

So far, superstring theory does not seem much more attractive than bosonic string theory. Admittedly, superstring theory does contain fermionic states in the R sector, which is an improvement. But since  $a_{\text{NS}} = 1/2$ , the NS ground state is still tachyonic, see (5.19). This issue is resolved by the so-called GSO projection, which is actually required by modular invariance.

Modular invariance appears in the closed string sector, when considering the one-loop vacuum amplitude, corresponding to a closed string propagating and closing onto itself, so that its worldsheet has the topology of a 2-torus. When computing this amplitude, one should sum over all possible inequivalent worldsheet geometries with this topology.

A 2-torus can be defined as  $\mathbb{R}^2/\Lambda$ , where  $\Lambda$  is a 2-dimensional lattice. In other words, points are identified as  $x^i \sim x^i + \lambda^i$ , where  $\lambda \in \Lambda$ . In the case of the worldsheet, denoting  $z = \sigma + i\tau$  and  $l$  the length of the string, the 2-torus can be defined by  $z \sim z + l$  and  $z \sim z + \tau l$ , where  $\tau$  is a parameter called the complex structure. This parameter characterises different worldsheet geometries, but there are different values of  $\tau$  leading to the same torus, for instance making  $\tau \rightarrow \tau + 1$  or  $\tau \rightarrow -1/\tau$  does not affect the torus geometry. These two elements can be shown to generate the following set of transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad \text{with} \quad (a, b, c, d) \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1,\tag{5.21}$$

which form the group  $\text{SL}(2, \mathbb{Z})$ , which is thus called the modular group of the 2-torus. Using such a transformation, every  $\tau$  in the upper half complex plane can be brought into

$$\mathcal{F} = \{-1/2 < |\text{Re}(\tau)| \leq 1/2 \quad \text{and} \quad |\tau| \geq 1\},\tag{5.22}$$

called the fundamental domain.

In order to be well-defined, any closed string theory should have its one-loop vacuum amplitude written as the integral over the fundamental domain  $\mathcal{F}$  of a modular invariant

function  $Z(\tau)$ , so that every worldsheet geometry contributes once, and only once.

This requirement is automatically met for the bosonic string, but not for the superstring. In both cases,  $Z(\tau)$  factorises between the left and right-moving sectors, and in superstring theory both sectors get contributions from the NS and R sectors. Omitting the details, these contributions are respectively proportional to

$$\frac{1}{\eta^4} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 \quad \text{and} \quad \frac{1}{\eta^4} \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4,$$

where  $\eta(\tau)$  and  $\vartheta[a, b](\tau)$  are special functions whose definition can be found in [57]. What is important is their transformation properties under the modular transformations (5.21). For example, the contribution from the NS sector transforms as

$$\frac{1}{\eta^4} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 \xrightarrow{\tau \rightarrow \tau+1} -\frac{1}{\eta^4} \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4 \xrightarrow{\tau \rightarrow -1/\tau} -\frac{1}{\eta^4} \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4,$$

under the fundamental transformations  $\tau \rightarrow \tau + 1$  and  $\tau \rightarrow -1/\tau$ . It is already interesting to observe that the contributions of the NS and R sectors are linked by a modular transformation, so they both have to be included in  $Z(\tau)$ , for it to be modular invariant, in addition to two other contributions, one of them coming with an arbitrary sign

$$Z_{\pm} = \frac{1}{2\eta^4} \left( \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^4 - \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}^4 - \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^4 \pm \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^4 \right). \quad (5.23)$$

These two additional contributions may seem rather mysterious at first glance. In fact, they can be interpreted as the consequence of a projection imposed on the physical spectrum, called the GSO projection. In the NS sector, this projection keeps only states with an odd number of  $b_r^i$  oscillators, and in the R sector it keeps states with an even number of  $d_m^i$  oscillators and given chirality, either positive or negative, reflecting the two signs in (5.23).

Since the choice of chirality in the GSO projection can be made independently in the left and right-moving sectors, there are two different closed superstring theories, called the type IIA and type IIB theories. In the type IIA theory, the left and right moving sectors have opposite chirality, while in the type IIB theory, they have the same.

## Type II massless spectra

As a consequence of the GSO projection, the tachyonic NS ground state is projected out of the spectrum, since it contains zero  $b_r^i$  oscillators, which is an even number. The states at the next level are massless. In the type IIA superstring theory, they take the form

$$\tilde{b}_{-1/2}^i |-\rangle_{\text{NS}} \otimes b_{-1/2}^j |+\rangle_{\text{NS}}, \quad |-\rangle_{\text{R}} \otimes |+\rangle_{\text{R}}, \quad \tilde{b}_{-1/2}^i |-\rangle_{\text{NS}} \otimes |+\rangle_{\text{R}}, \quad |-\rangle_{\text{R}} \otimes b_{-1/2}^i |+\rangle_{\text{NS}}.$$

Recall that the Ramond ground state is a spacetime spinor. As a result, states in the NS-R and R-NS sectors are fermionic. On the other hand, states in the R-R sector are bosonic, since they

belong to the tensor product of two spinor representations. In the end, the massless spectrum of type IIA string theory can be decomposed into the following irreducible representations

$$\begin{aligned}
\text{NS-NS sector:} & \quad g_{\mu\nu} \text{ (graviton)}, \quad B_{\mu\nu} \text{ (B-field)}, \quad \phi \text{ (dilaton)}, \\
\text{R-R sector:} & \quad C_\mu \text{ (1-form)}, \quad C_{\mu\nu\rho} \text{ (3-form)}, \\
\text{NS-R sector:} & \quad \psi_\mu^1 \text{ (gravitino)}, \quad \lambda^1 \text{ (dilatino)}, \\
\text{R-NS sector:} & \quad \psi_\mu^2 \text{ (gravitino)}, \quad \lambda^2 \text{ (dilatino)}.
\end{aligned} \tag{5.24}$$

The name given to the states in the NS-R and R-NS sectors suggests the presence of local spacetime supersymmetry, in addition to the worldsheet supersymmetry (5.16). This is indeed the case that the effective action of type II superstring theories is  $\mathcal{N} = 2$  supersymmetric. The 2 comes from the number of gravitini and gives these theories their name.

In type IIB superstring theory the massless spectrum is very similar to (5.24). The only differences concern the chirality of the gravitini/dilatini and the content of the R-R sector

$$\text{R-R sector:} \quad a \text{ (0-form or axion)}, \quad C_{\mu\nu} \text{ (2-form)}, \quad C_{\mu\nu\rho\sigma} \text{ (4-form)}. \tag{5.25}$$

As for the bosonic theory, the string states in these massless spectra can be reinterpreted as the quanta of fields, for which one can derive effective actions, similar to (5.14).

In addition to type II, there exists other string theories. There is the type I theory, which involves both open and closed string and can be seen as a projection of the type IIB theory combined with the addition of an open string sector; and there are the heterotic theories, where the left-moving sector is treated as a bosonic string theory, while the right-moving sector is treated as a superstring. Furthermore, it has been proposed that all these theories are related to a single eleven-dimensional theory, called M-theory. In the following, we will largely focus on type IIB string theory.

### Effective action of type IIB string theory

From now, let us focus on type IIB string theory and give its effective low-energy bosonic action. For this, we first define the field strengths associated to the various form-fields

$$H_3 = dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2, \quad F_5 = dC_4,$$

in other words,  $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$  and similarly for the others. We also define

$$\tilde{F}_3 = F_3 + C_0 H_3 \quad \text{and} \quad \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3, \tag{5.26}$$

with these definitions, we have the action

$$\begin{aligned}
\mathcal{S}_{\text{IIB}} = & \frac{1}{2} \int d^{10}x \sqrt{-g} \left( e^{-2\phi} \left( R + 4\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} |H_3|^2 \right) - \frac{1}{2} |F_1|^2 - \frac{1}{2} |\tilde{F}_3|^2 - \frac{1}{4} |\tilde{F}_5|^2 \right) \\
& - \frac{1}{2} \int C_4 \wedge H_3 \wedge F_3.
\end{aligned} \tag{5.27}$$

Moreover, self-duality of  $\tilde{F}_5$  must be imposed, namely  $\tilde{F}_5 = \star \tilde{F}_5$  where  $\star$  is the Hodge-star operator. It is an equation of motion that cannot be embedded into the action. The action

of the NS-NS sector in (5.27) is actually the same as (5.14). In addition, it contains kinetic terms for the form-fields, and a Chern-Simons term. The fermionic part of the action can be deduced from (5.27) by supersymmetry.

An important feature of the type IIB effective theory is that it is invariant under  $SL(2, \mathbb{R})$  transformations

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \quad \text{and} \quad \mathcal{S} \rightarrow \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \quad (5.28)$$

where  $ad - bc = 1$  and  $\mathcal{S} = C_0 + ie^{-\phi}$  is the axio-dilaton. This invariance is called  $S$ -duality.

## 5.3 Compactification and moduli stabilisation

In (5.10) and (5.20), we saw that string theory requires the presence of extra dimensions, in addition to the four we know. In superstring theory, there are six such dimensions. One of the main ideas for hiding these extra dimensions is compactification. In compactification of type II superstring theories down to four dimensions, one assumes that the ten dimensional spacetime is factorised as  $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6$ . With this assumption, one can make the following metric ansatz

$$ds_{10}^2 = ds_4^2 + ds_6^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \tilde{g}_{mn}(y)dy^m dy^n, \quad (5.29)$$

where from now the greek indices  $\mu, \nu, \dots$  are reserved to the four-dimensional spacetime, while lower case latin indices  $m, n, \dots$  are used for the six-dimensional internal space. Upper case latin indices  $M, N, \dots$  will be used to denote all ten dimensions collectively.

### 5.3.1 The basics of compactification

Let us illustrate important aspects of extra-dimensional theories with simple examples inspired by [58]. First, we consider the five-dimensional theory of a free scalar field compactified on a circle

$$\mathcal{S}_5 = -\frac{1}{2} \int d^5x \partial^M \phi \partial_M \phi. \quad (5.30)$$

If we assume that the fifth dimension, associated to the coordinate  $y$ , is circular, with periodicity  $y \sim y + 2\pi R$ , we can expand the scalar field into Fourier modes

$$\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi R) \quad \rightarrow \quad \phi(x^\mu, y) = \sum_{k \in \mathbb{Z}} \phi_k(x^\mu) e^{iky/R}.$$

Substituting into (5.30) and integrating over  $y$ , we obtain a four-dimensional action

$$\mathcal{S}_4 = -(2\pi R) \int d^4x \left( \partial^\mu \phi_0 \partial_\mu \phi_0 + \sum_{k=1}^{\infty} \left( \partial^\mu \phi_k \partial_\mu \phi_{-k} + m_k^2 \phi_k \phi_{-k} \right) \right) \quad \text{with} \quad m_k^2 = \frac{k^2}{R^2},$$

where the five-dimensional scalar field is represented as an infinite tower of increasingly massive four-dimensional modes, called the Kaluza-Klein (KK) modes. The mass scale of the tower goes as  $1/R$ , so if the compactification radius is small relative to the scale of interest, we can ignore the KK tower and describe the scalar field with its massless zero-mode  $\phi_0(x^\mu)$ .

In string theory compactified on a circle, there is tower in addition to the KK tower. If we take the example of bosonic string theory, we have to take into account in their boundary condition that closed strings can be wound around the circular dimension

$$x^{25}(\tau, \sigma + \pi) = x^{25}(\tau, \sigma) + 2\pi R w \quad \text{with} \quad w \in \mathbb{Z}$$

which modifies the mode expansion (5.6), and ultimately the mass formula (5.9) becomes

$$\alpha' M^2 = \alpha' \left( \frac{k^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} \right) + 4(N - 1), \quad (5.31)$$

where we recognise that the KK tower is accompanied by a tower of winding modes, whose mass scale goes as  $R/\alpha'$ . Therefore, if we want to ignore this tower as well, we need a large compactification radius, more precisely, a large  $R/\alpha'$ . Since  $\alpha'$  is involved and  $\alpha' \sim 1/M_s^2 \ll 1/E^2$  at the energies we are interested in, this condition is not incompatible with the small radius condition to ignore the KK tower.

It is also instructive to consider the compactification of the Einstein-Hilbert action

$$\mathcal{S}_5 = \frac{M_5^3}{2} \int d^5 s \sqrt{-G} R_5(G), \quad (5.32)$$

keeping only the zero-modes, the five dimensional metric can be parameterised as

$$G_{MN}^0(x^\mu) = e^{\sigma/3} \left( \frac{g_{\mu\nu} + e^{-\sigma} A_\mu A_\nu}{e^{-\sigma} A_\mu} \middle| \frac{e^{-\sigma} A_\mu}{e^{-\sigma}} \right). \quad (5.33)$$

In this parameterisation,  $\sigma$  is a scalar field whose expectation value is linked to the radius  $R$  of the compact dimension, as we can see if we consider a constant  $\sigma_0$  and periodicity  $y \sim y+1$

$$2\pi R = \int_0^1 G_{44} dy = e^{-2\sigma_0/3}.$$

Substituting the parameterisation (5.33) in (5.32) leads to the four-dimensional action

$$\mathcal{S}_4 = \frac{M_4^2}{2} \int d^4 x \sqrt{-g} \left( R_4(g) - \frac{1}{6} \partial^\mu \sigma \partial_\mu \sigma - \frac{1}{4e^\sigma} F^{\mu\nu} F_{\mu\nu} \right) \quad \text{where} \quad M_4^2 = (2\pi R) M_5^3. \quad (5.34)$$

Here, we see that the four-dimensional Planck mass  $M_4$  is related to the five-dimensional Planck mass  $M_5$  with a factor corresponding to the volume of the compact dimension. It is an idea that is generally true. Consequently, a large internal volume could explain the weakness of gravity in our four-dimensional perspective. In the limit of infinite volume, gravity is decoupled in four dimensions. This is why we consider compact internal spaces.

The vacuum expectation value of the scalar field  $\sigma$  is involved in both the four-dimensional Planck mass and in the kinetic term of the vector field  $A_\mu$  defined in (5.33). This scalar field has a kinetic term, but no potential: it is a modulus, just like the dilaton in (5.14) and (5.27). Without a potential, its vacuum expectation value can evolve freely, and the strength of our gravitational interactions would vary over time, something we have not observed. We will soon introduce moduli stabilisation mechanisms in type IIB superstring theory.

## Supersymmetry, Calabi-Yau, orientifolds

As we have seen in (5.24), the spectrum of type II string theories contains two gravitini, and they are described at low energy by  $\mathcal{N} = 2$  supergravity theories in 10 dimensions. In  $D = 10$ , a minimal spinor has 16 components, and  $\mathcal{N} = 2$  supergravity has 32 supercharge components. After dimensional reduction, these supercharges must be accommodated in lower-dimensional spinors, which have fewer components. In  $D = 4$ , a minimal spinor has 4 components, so 8 of them are needed for the 32 supercharge components. In conclusion  $\mathcal{N} = 2$  supergravity in  $D = 10$  reduces to at most  $\mathcal{N} = 8$  supergravity in  $D = 4$ .

We say at most, because the 10-dimensional supercharges components are not all transferred to 4-dimensional ones. In fact, the supersymmetries of the 4-dimensional theory correspond to supercharges that are globally defined on  $\mathcal{M}_6$ , which is related to the existence of a covariantly constant spinor  $\eta$ , as is nicely explained in [59]. This condition is itself related to the holonomy group of  $\mathcal{M}_6$ . The holonomy group of a manifold describes how vectors are rotated upon parallel transport around a closed curve. For a Riemannian manifold of dimension  $n$ , the holonomy group is in general  $\text{SO}(n)$ . In the example of a torus, vectors are not rotated when parallel transported around a closed curve, so the holonomy is trivial.

It can be shown that the existence of a covariantly constant spinor on the internal manifold requires that it has  $\text{SU}(3)$  holonomy. If this is the case, each of the 10-dimensional supercharge yields a 4-dimensional supercharge. In other words, a 10-dimensional  $\mathcal{N} = 2$  supergravity reduced on a space of  $\text{SU}(3)$  holonomy yields a 4-dimensional  $\mathcal{N} = 2$  theory. If the holonomy of  $\mathcal{M}_6$  is a subgroup of  $\text{SU}(3)$ , the 4-dimensional theory has more supersymmetry.

Compact manifolds of  $\text{SU}(3)$  holonomy are not trivial to construct. There is a theorem, conjectured by Calabi and proven by Yau, stating that any  $N$ -dimensional complex and Kahler manifold with vanishing first Chern class admits a metric with  $\text{SU}(N)$  holonomy, although no non-trivial metric satisfying this property is known. A manifold satisfying these conditions is called a Calabi-Yau manifold. A  $N$ -dimensional complex manifold has  $2N$  real dimensions.

For  $N = 1$ , there is only one Calabi-Yau manifold, which is the two-torus  $T^2$ , although tori can be considered trivial cases. For  $N = 2$ , there are two Calabi-Yau manifolds, which are  $T^4$  and a more interesting manifold called  $K3$ . For  $N = 3$ , the case of interest for type II compactifications, it is not known whether the number of Calabi-Yau manifolds is finite. The simplest example is defined as a quintic hypersurface in the projective space  $\text{CP}^4$

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0.$$

A vast number of Calabi-Yau manifolds have been constructed this way, or using more advanced tools of algebraic geometry and the Kreuzer-Skarke database of reflexive polyhedra [60]. In chapter 6, we will define and consider toroidal orbifolds, which can be seen as singular limits of Calabi-Yau manifolds.

For phenomenological applications,  $\mathcal{N} = 2$  in four dimensions is too much, because there are no chiral multiplets with this amount of supersymmetry, and the spectrum of the Standard Model is chiral. Fortunately, there is a way to reduce the  $\mathcal{N} = 2$  supersymmetry of type

II theories compactified on Calabi-Yau manifolds, down to  $\mathcal{N} = 1$ , called orientifolding. As we saw in chapter 3, there are chiral multiplets in  $\mathcal{N} = 1$  indeed. At the end of the day, this residual  $\mathcal{N} = 1$  supersymmetry must also be broken if the Standard Model is to be recovered. But maintaining a certain degree of supersymmetry at the level of the compactification protects against certain corrections and greatly simplifies the analysis, as we shall see later.

The orientifold is a quotient of the theory by  $\Omega\mathcal{R}$ , where  $\Omega$  is essentially the worldsheet parity operator, while  $\mathcal{R}$  is a geometric  $\mathbb{Z}_2$ -involution of the compact space  $\mathcal{M}_6$ , for which we have several choices. In type IIB theory, for example, one possibility is to set  $\mathcal{R} = 1$  and just quotient by  $\Omega$ , which in fact defines the type I superstring theory that we mentioned earlier. Part of the spectrum of the theory is projected by the orientifold quotient, in particular one gravitino is removed, leading to  $\mathcal{N} = 1$ .

We can interpret the fixed loci of  $\mathcal{R}$  in  $\mathcal{M}_6$  as the positions of stringy geometrical objects, called orientifold planes, which also fills the four-dimensional  $\mathcal{M}_4$ . With  $\mathcal{R} = 1$ , the orientifold planes fill  $\mathcal{M}_6$  and are called  $O9$ -planes. If we call  $(z_1, z_2, z_3)$  the complex coordinates of  $\mathcal{M}_6$  and  $R_i$  the involution acting as  $z_i \rightarrow -z_i$ , we can also consider

$$\begin{aligned} \mathcal{R} = R_i &\rightarrow O7\text{-planes}, & \mathcal{R} = R_i R_j &\rightarrow O5\text{-planes}, \\ \mathcal{R} = R_i R_j R_k &\rightarrow O3\text{-planes}. \end{aligned} \quad (5.35)$$

In general, it is possible that different kind of  $O_p$ -planes coexist, but in type IIB only  $O3/O7$  and  $O5/O9$  combinations do preserve  $\mathcal{N} = 1$  supersymmetry.

### Moduli of Calabi-Yau manifolds

In the beginning of this section 5.3.1, we showed in a simple case how compactified extra-dimensions lead to the appearance of moduli in the lower-dimensional effective theory. We then explained that an interesting class of compactifications is on orientifolds of Calabi-Yau threefolds, in order to preserve  $\mathcal{N} = 1$  supersymmetry in four dimensions. It is now natural to study the moduli of Calabi-Yau threefolds [61].

It can be shown that Calabi-Yau threefolds, defined as a complex and Kahler manifolds admitting a metric with  $SU(3)$  holonomy, are also Ricci-flat, in the sense that  $R_{mn}(g) = 0$ . Knowing this, moduli can be defined as deformations of this metric that preserve the Ricci-flatness condition, leading to the Lichnerowicz equation

$$R_{mn}(g + \delta g) = 0 \quad \rightarrow \quad \nabla^k \nabla_k \delta g_{mn} + 2R_m^p n^q \delta g_{pq} = 0. \quad (5.36)$$

Using properties of Kahler geometry and introducing complex coordinates, in the same way as throughout chapter 3, the equations for the mixed and pure components  $\delta g_{\alpha\bar{\beta}}$  and  $\delta g_{\alpha\beta}$  decouple. These deformations can then be put in one-to-one correspondence with the following (1, 1) and (2, 1)-forms

$$\delta g_{\alpha\bar{\beta}} dz^\alpha \wedge \bar{d}\bar{z}^{\bar{\beta}} \quad \text{and} \quad \Omega_{\alpha\beta\gamma} g^{\gamma\bar{\delta}} \delta g_{\bar{\delta}\bar{\epsilon}} dz^\alpha \wedge dz^\beta \wedge d\bar{z}^{\bar{\epsilon}}, \quad (5.37)$$

where  $\Omega$  is a nowhere vanishing (3, 0)-form, whose existence and unicity is implied by the Calabi-Yau conditions. The equation (5.36) implies that the  $(p, q)$  forms (5.37) are harmonic,



and thus counted by topological numbers  $h^{p,q}$ , called Hodge numbers.

The  $h^{2,1}$  moduli associated to  $(2,1)$ -forms are called complex structure moduli, because they parameterise deformations of the complex structure, the piece of information making the link between real and complex coordinates on a complex manifold. Explicit examples in the case of toroidal orbifolds will be given in chapter 6. Complex structure moduli are naturally complex scalars.

The  $h^{1,1}$  moduli associated to  $(1,1)$ -forms are called Kahler moduli, because they parameterise deformations of the Kahler form  $J \sim g_{\alpha\bar{\beta}} dz^\alpha \wedge \bar{z}^{\bar{\beta}}$ . This form can be used to compute the volumes of even-dimensional cycles, for example the volume of the entire manifold

$$\int_{\mathcal{M}_6} J \wedge J \wedge J = \text{Vol}(\mathcal{M}_6).$$

Kahler moduli thus have a clear geometrical meaning, they control the volumes of  $2n$ -cycles. As such, they are constrained by the fact that the volumes should not be negative, and they are real scalar fields. But as we saw multiple times already, the metric  $g_{\mu\nu}$  is not alone in the spectrum of string theories (5.24). It is always accompanied by  $p$ -forms, and in particular by the 2-form  $B$ . These forms also give rise to moduli after compactification. Thanks to supersymmetry, these additional moduli are naturally combined with metric moduli to form multiplets. For instance, the R-R 2-form  $C_2$  naturally combines with the Kahler form to give the complexified Kahler form  $\mathcal{J} = C_2 + iJ$ , so that the Kahler moduli become complex scalars by being combined with the  $h^{1,1}$  internal components  $C_{\alpha\bar{\beta}}$ .

Before orientifolding, the effective four-dimensional theory is  $\mathcal{N} = 2$  supersymmetric, and it can be shown that in type IIB, complex structure moduli go into vector multiplets, while Kahler moduli go into hypermultiplets. The dilaton is also combined with the 0-form  $C_0$ , to give the axio-dilaton  $\mathcal{S} = C_0 + ie^{-\phi}$  encountered in (5.28) and goes into a hypermultiplet.

After orientifolding, the supersymmetry of the effective theory is reduced to  $\mathcal{N} = 1$ . The complexified Kahler moduli and the axio-dilaton go into  $h^{1,1} + 1$  chiral multiplets, while complex structure moduli are split into  $h_+^{2,1}$  vector multiplets and  $h_-^{2,1}$  chiral multiplets, where  $h_\pm^{2,1}$  are the numbers of  $(2,1)$ -forms that are even or odd under the geometric involution  $\mathcal{R}$ . In what follows, we always consider  $h_+^{2,1} = 0$ , so all the moduli belong to chiral multiplets.

## Open strings, branes, tadpole conditions

So far, we did not talk about open strings. We only mentioned that they correspond to the boundary conditions  $\partial_\sigma x^\mu = 0$  or  $x^\mu = x_*^\mu$  at  $\sigma = 0$  and  $\pi$ , respectively called Neumann and Dirichlet, where  $x_*^\mu$  is fixed. Dirichlet boundary conditions break Poincaré invariance, but this is not problematic if we interpret the  $x_*^\mu$  as the positions of physical objects, called  $D$ -branes, to which the open string is attached.

Like orientifold planes, there are  $D$ -branes of different dimensions. A  $Dp$ -brane spans  $p$  dimensions of space, and has a  $(p+1)$ -dimensional worldvolume. In the directions transverse to its worldvolume, it is localised. For example, considering 10-dimensional superstring theory compactified on a 6-dimensional space, a  $D3$ -brane can fill  $\mathcal{M}_4$  and be localised in  $\mathcal{M}_6$ .

In this way, Poincaré invariance is preserved on  $\mathcal{M}_4$ . Open strings ending on this  $D3$ -brane have Neumann boundary conditions in  $\mathcal{M}_4$  and Dirichlet boundary conditions in  $\mathcal{M}_6$ .

Open strings and branes play an important phenomenological role in string theory, because they can be used to engineer non-abelian gauge theories. Indeed, the spectrum of open string theories contains a spin-one particle  $A_\mu$  that can be interpreted as the quantum of a gauge field. In the presence of stacks of  $N$  coincident branes, a label should be introduced to keep track of which brane the open strings are attached to, and in the simplest case it can be shown that the corresponding dynamics is that of a  $U(N)$  gauge theory on the worldvolume of the brane stack. Other gauge groups can be obtained with more involved setups.

The existence of branes is also suggested by the massless spectrum of string theories. In the worldline description of a bosonic charged particle, its coupling to the gauge field  $A_\mu$ , seen as a 1-form, can be described by a topological term of the form

$$\mathcal{S} = \int A,$$

the integral being along the worldline. Similarly, the higher forms that are present in the spectra of string theories, see (5.24) and (5.25), are naturally coupled to the higher-dimensional worldvolumes of  $Dp$ -branes. A  $p$ -form couples to a  $p$ -dimensional worldvolume and suggests the existence of  $(p-1)$ -branes. In the type IIB theory, for example, the massless spectrum contains a 2-form and a 4-form, which are respectively coupled to  $D1$  and  $D3$ -branes. Following this logic, the spectrum also contains a 0-form, which is formally coupled to a  $(-1)$ -brane, an object localised in both space and time, called an instanton.

From a  $p$ -form potential  $C_p$ , one can also build a  $(p+1)$ -form field strength  $F_{p+1} = dC_p$ , which can be Hodge-dualised into a  $(D-p-1)$ -form  $\tilde{F}_{D-p-1} = \star F_{p+1}$  corresponding to a  $(D-p-2)$ -form potential  $\tilde{C}_{D-p-2}$ . This new potential can be coupled to the  $(D-p-2)$ -dimensional worldvolume of a  $(D-p-3)$ -brane. By analogy with electromagnetism, this coupling of a  $p$ -form to a  $(D-p-3)$ -brane through Hodge dualisation is called magnetic, while the direct coupling to a  $(p-1)$ -brane discussed above is called electric.

In the type IIB theory, the  $(0, 2, 4)$ -forms of the massless spectrum (5.25) are thus magnetically coupled to  $(D7, D5, D3)$ -branes, respectively. The 4-form  $C_4$  is both electrically and magnetically coupled to  $D3$ -branes, because its 5-form field strength  $F_5$  is self-dual  $\star F_5 = F_5$ .

The fact that branes are charged under  $p$ -form fields has important consequences. Because of this, branes act as sources in the equations of motion and Bianchi identities of these fields. For example, in type IIB, the Bianchi identity of  $\tilde{F}_5$  in (5.26), which is also its equation of motion due to self-duality, is

$$d\tilde{F}_5 = H_3 \wedge F_3 + 2\mu_3 \rho_3, \quad (5.38)$$

where  $\mu_3$  is the  $D3$ -brane charge, which is given in terms of  $\alpha'$  by  $1/\mu_p = g_s(2\pi)^p(\alpha')^{(p+1)/2}$ , and  $\rho_3$  is the  $D3$ -brane density. If the theory is compactified on a 6-dimensional space, the  $D3$ -branes are localised in  $\mathcal{M}_6$ , and  $\rho_3$  is a sum of delta functions. In this case, (5.38) can be integrated over  $\mathcal{M}_6$

$$\frac{1}{2\mu_3} \int_{\mathcal{M}_6} H_3 \wedge F_3 + Q_3 = 0, \quad (5.39)$$

where  $Q_3$  is the total number of  $D3$ -brane on the compact space  $\mathcal{M}_6$ . If we interpret the first term of (5.39) as a contribution to the  $D3$ -brane charge induced by  $(H_3, F_3)$ , we realise that (5.39) is nothing more than the condition that the net  $D3$ -brane charge on  $\mathcal{M}_6$  vanishes.

Something less intuitive is the fact that orientifold planes also carry a  $D3$ -brane charge, which is negative. This charge can be computed by considering the exchange of a closed string between a  $D$ -brane and an  $O$ -plane, using tools that we do not introduce here. The result is that  $\mu_{Op}/\mu_{Dp} = -2^{p-5}$ , so  $Q_3$  in (5.39) can be written as

$$Q_3 = N_{D3} - \frac{1}{4}N_{O3}. \quad (5.40)$$

Later, we will see that in a simple class of solutions, the contribution from  $O3$ -planes is the only negative term in (5.40). Therefore, without them, this equation only has trivial solutions with terms vanishing individually. One might well consider anti- $D3$ -branes, which also contribute negatively, but adding them breaks  $\mathcal{N} = 1$  supersymmetry. In fact,  $O7$ -planes and  $D7$ -branes also contribute negatively in (5.40) as we will see in a moment.

The relation (5.38) is called a tadpole cancellation condition, because it is also linked to the cancellation of tadpole diagrams. Similar relations do exist for all types of brane charges and put fairly strong constraints on string theory model building, as we will see in chapter 6.

## Kahler potential and prepotential

We saw in chapter 3 that a  $\mathcal{N} = 1$  supergravity theory in four dimensions is determined by a Kahler potential  $\mathcal{K}(z, \bar{z})$ , a superpotential  $W(z)$  and a gauge kinetic function  $f_{AB}(z)$ , which are functions of the scalars in chiral multiplets. When type IIB string theory is compactified on a Calabi-Yau orientifold  $\mathcal{M}_6$ , these functions are determined by its geometry.

Let us denote by  $\mathcal{T}^i$  the  $h^{1,1}$  the complexified Kahler moduli of  $\mathcal{M}_6$ , by  $\mathcal{U}^i$  its  $h^{2,1}$  complex structure moduli, and by  $\mathcal{S}$  the axio-dilaton. The Kahler potential obtained from dimensional reduction of the action (5.27) is the following

$$\mathcal{K} = -2 \log \mathcal{V} - \log \left( i \int_{\mathcal{M}_6} \Omega \wedge \bar{\Omega} \right) - \log(-i(\mathcal{S} - \bar{\mathcal{S}})), \quad (5.41)$$

where  $\mathcal{V}$  is the volume  $\mathcal{M}_6$ , controlled by the Kahler moduli,  $\Omega$  is the nowhere vanishing  $(3,0)$ -form already introduced in (5.37), which depends on the complex structure moduli, and bars stand for complex conjugation. Thanks to the  $\mathcal{N} = 2$  supersymmetry present before the orientifold, the Kahler potential of complex structure moduli can be put into a more structured form [62]

$$\mathcal{K}_{c.s.} = -\log \left( -i(X^I \bar{\mathcal{G}}_I - \bar{X}^I \mathcal{G}_I) \right), \quad (5.42)$$

here the  $X^I$  are  $h^{2,1} + 1$  projective coordinates of the complex structure moduli space, which can be set to  $(X^0, X^i) = (1, \mathcal{U}^i)$  after expanding (5.42) to recover (5.41), and  $\mathcal{G}(X)$  is an holomorphic function of the  $X^I$  called the prepotential,  $\mathcal{G}_I$  being its derivative. The Kahler potential of Kahler moduli can also be put in a similar form, with its own prepotential  $\mathcal{F}(Y)$ .

Equivalently, we can introduce a so-called symplectic basis of 3-forms  $(\alpha_I, \beta^I)$  satisfying

$$\int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = \delta_I^J, \quad \text{where } I = 0, \dots, h^{2,1}. \quad (5.43)$$

In such a basis, the 3-form  $\Omega$  expands as

$$\Omega = X^I \alpha_I - \mathcal{G}_I \beta^I,$$

to give back equation (5.42). An explicit example will be given in chapter 6.

### 5.3.2 Moduli stabilisation in type IIB

At this stage, at tree-level, moduli have a Kahler potential, but no superpotential  $W$ , so their scalar potential vanishes identically and they are not stabilised. For convenience, let us recall the form of the scalar potential in four-dimensional  $\mathcal{N} = 1$  supergravity theories containing only chiral multiplets

$$V = e^{\mathcal{K}} (g^{a\bar{b}} \nabla_a W \overline{\nabla_{\bar{b}} W} - 3W\overline{W}), \quad (5.44)$$

where  $g^{a\bar{b}}$  is the inverse of the moduli space metric  $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$  and  $\nabla_a W = \partial_a W + (\partial_a \mathcal{K})W$  is the Kahler-covariant derivative of the superpotential. The index  $a$  and  $\bar{b}$  run over all moduli.

The easiest way to introduce a superpotential that partially stabilises the moduli is to consider 3-form fluxes on  $\mathcal{M}_6$ , as we shall now see.

#### Flux superpotential and no-scale structure

Considering a slightly more general compactification ansatz than (5.29), namely a warped metric ansatz

$$ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} \tilde{g}_{mn}(y) dy^m dy^n, \quad (5.45)$$

there exists solutions of the type IIB equations of motion, resulting from the action (5.27), with non vanishing background fluxes  $(H_3, F_3, \tilde{F}_5)$ . The most simple solution has

$$\tilde{F}_5 = (1 + \star_{10}) d\alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where  $\alpha(y)$  is a function of the internal coordinates, related to the warp factor by  $e^{4A} = \alpha$ . In addition, the complexified 3-form flux, defined as  $G_3 = F_3 + \mathcal{S}H_3$  is imaginary self-dual

$$\star_6 G_3 = iG_3, \quad (5.46)$$

and the sources are  $D3/D7$ -branes and  $O3/O7$ -planes. This is the GKP solution [63]. The condition (5.46) on  $G_3$  implies that the contribution of the 3-form fluxes to the  $D3$ -brane charge is positive

$$N_{\text{flux}} = \int_{\mathcal{M}_6} H_3 \wedge F_3 \geq 0. \quad (5.47)$$

As a result, the tadpole cancellation condition (5.38) has non-trivial solutions only in the presence of  $O3$ -planes if anti- $D3$ -branes are discarded. The imaginary self-duality condition

implies that  $G_3$  is a harmonic  $((2, 1) + (0, 3))$ -form, which is embodied by the GVW superpotential [64], which can also be obtained from a dimensional reduction of (5.27)

$$W = \int_{\mathcal{M}_6} \Omega \wedge G_3, \quad \text{where} \quad G_3 = F_3 + \mathcal{S}H_3. \quad (5.48)$$

This superpotential only depends on the complex structure moduli  $\mathcal{U}^i$  and the dilaton  $\mathcal{S}$ . As we explained in chapter 3, the condition for supersymmetry to be preserved at the minimum of the scalar potential (5.44) is that  $\nabla_a W = 0$ , where  $a$  runs over all the moduli  $(\mathcal{T}^i, \mathcal{U}^i, \mathcal{S})$ . Since  $W$  does not depend on the Kahler moduli  $\mathcal{T}^i$ , we can see that supersymmetry is unbroken iff  $W = 0$ , which is equivalent to  $G_3$  being a  $(2, 1)$ -form. So with this superpotential, we can only have a supersymmetric Minkowski vacuum or broken supersymmetry.

The 3-form fluxes  $(H_3, F_3)$  can be expanded on the symplectic basis  $(\alpha_I, \beta^I)$  defined in (5.43)

$$H_3 = m_I^H \alpha^I + n_I^H \beta^I \quad \text{and} \quad F_3 = m_I^F \alpha^I + n_I^F \beta^I,$$

so that the superpotential (5.48) reads

$$W = n_I X^I + m_I \mathcal{G}_I, \quad \text{where} \quad m_I = m_I^F + \mathcal{S}m_I^H \quad \text{and} \quad n_I = n_I^F + \mathcal{S}n_I^H,$$

and the flux number (5.47) becomes

$$N_{\text{flux}} = m^H \cdot n^F - m^F \cdot n^H, \quad (5.49)$$

where the  $\cdot$  stands for a summation over  $I$ . When the 3-form fluxes are expanded this way, there is an important condition, called the Dirac quantisation condition, which requires the coefficients  $(m_I^H, m_I^F, n_I^H, n_I^F)$  to be integers.

For a given compactification manifold  $\mathcal{M}_6$ , it seems that there are many possible choices of 3-form fluxes satisfying the tadpole constraint. Since the early work of [65–67], this number has been expected to be finite but estimated to be large, on the order of  $10^{500}$  for an average Calabi-Yau. Such a large number of possibilities is an obstacle to the predictivity of the theory.

The fact that (5.48) does not depend on the Kahler moduli  $\mathcal{T}^i$  has an interesting consequence. For simplicity, let us consider the case of a single Kahler modulus  $\mathcal{T}$  controlling the volume of a 4-cycle, so that  $\mathcal{V} \sim \mathcal{T}^{3/2}$ . The Kahler potential (5.41) becomes

$$\mathcal{K}_{\mathcal{T}} = -3 \log(-i(\mathcal{T} - \bar{\mathcal{T}})) \quad \rightarrow \quad g^{\mathcal{T}\bar{\mathcal{T}}} \nabla_{\mathcal{T}} W \bar{\nabla}_{\bar{\mathcal{T}}} \bar{W} = 3W\bar{W},$$

so a cancellation occurs in the scalar potential (5.44), leaving only the first term

$$V_{\text{n.s.}} = e^{\mathcal{K}} g^{i\bar{j}} \nabla_i W \bar{\nabla}_{\bar{j}} \bar{W}, \quad (5.50)$$

where the indices  $i$  and  $\bar{j}$  only run over the complex structure moduli and the axio-dilaton. This property is called the no-scale property. It was introduced in [68] before appearing naturally in string theory. In what remains of the scalar potential, Kahler moduli only appear in the overall factor of  $e^{\mathcal{K}}$  and correspond to runaway and unstable directions.

## Brane moduli, magnetised $D7$ , F-theory

In (5.40), we considered only  $O3$ -planes and  $D3$ -branes, but as we mentioned after (5.35), in type IIB orientifolds it is possible that  $O3$  and  $O7$ -planes are present simultaneously while preserving  $\mathcal{N} = 1$ -supersymmetry. When  $O7$ -planes are present,  $D7$ -branes must also be there to cancel the  $D7$ -brane charge on  $\mathcal{M}_6$ . Since they are located at fixed loci of the orientifold involution,  $O_p$ -planes cannot move. However,  $Dp$ -branes can move in  $\mathcal{M}_6$ , and their positions correspond to additional moduli that must be stabilised.

Let us introduce an additional ingredient in order to stabilise the  $D7$ -brane moduli. Recall that  $Dp$ -branes are associated to the extremities of open strings, whose spectrum contains a spin-one particle. This particle can be interpreted as the quantum of a gauge field  $A_\mu$ , defined on the worldvolume of the brane and associated to a 2-form field strength  $F = dA$ .

This 2-form field strength can be used as  $G_3$  to stabilise the  $D7$ -brane moduli. A  $D7$ -brane wraps a 4-dimensional submanifold  $\mathcal{S}_4$  of  $\mathcal{M}_6$ , and  $F$  can be turned-on in this submanifold. In order to preserve supersymmetry, it must satisfy a condition similar to (5.46)

$$F = - \star_4 F.$$

Such a 2-form  $F$  turned-on on the internal worldvolume of a  $D7$ -brane is called a worldvolume or magnetic flux, and the corresponding  $D7$ -brane is said to be magnetised. It turns out that magnetic fluxes also contribute to the tadpole cancellation condition (5.39)

$$\frac{1}{2} \int_{\mathcal{M}_6} H_3 \wedge F_3 - \frac{1}{2} \int_{\mathcal{S}_4} (\text{Tr}(F \wedge F) - \text{Tr}F \wedge \text{Tr}F) + Q_3 = 0. \quad (5.51)$$

In addition,  $O7$ -planes and  $D7$ -branes appear in the  $D3$ -brane charge (5.40)

$$Q_3 = N_{D3} - \frac{1}{4} N_{O3} - \frac{1}{24} (2\chi(O7) + \chi(D7)), \quad (5.52)$$

where  $\chi(O7/D7)$  are the Euler numbers of the submanifolds of  $\mathcal{M}_6$  wrapped by the  $O7/D7$ . Note that the  $O7$ -planes and  $D7$ -branes are not necessarily coincident, the only requirement being that the  $D7$ -brane charge vanishes globally on  $\mathcal{M}_6$ . Note also that both of them contribute negatively to  $Q_3$ .

Within the framework of F-theory [69], it is possible to simultaneously address the stabilisation of complex structure and  $D7$ -branes moduli. Very schematically, we can think of F-theory as a 12-dimensional theory that reduces to type IIB string theory when compactified on a torus. The axio-dilaton  $\mathcal{S}$  is then interpreted as the complex structure modulus of this torus, and the  $\mathcal{S}$ -duality (5.28) of the theory comes from modular invariance, as in (5.21).

Type IIB string theory compactified on a Calabi-Yau threefold  $\mathcal{M}_6$  is thus described by F-theory compactified on an elliptically fibered Calabi-Yau fourfold  $\mathcal{M}_8$ . The advantage of this description is that the  $O7$  planes and  $D7$  branes are encoded in the geometry of  $\mathcal{M}_8$ , for instance the  $D7$ -branes correspond to singularities of the fibration. Consequently, the  $h^{2,1}$  complex structure moduli of  $\mathcal{M}_6$ , the axio-dilaton and the  $D7$ -brane moduli are all described by the  $h^{3,1}$  complex structure moduli of  $\mathcal{M}_8$ .

In F-theory, the complexified 3-form flux  $G_3$  and 2-form flux  $F$  are all described by a single self-dual 4-form flux  $G_4$ , inducing a superpotential that is very similar to (5.48) for the  $h^{3,1}$  complex structure moduli of the fourfold [70, 71]

$$\star_8 G_4 = G_4 \quad \rightarrow \quad W = \int_{\mathcal{M}_8} \Omega \wedge G_4.$$

Finally, the tadpole cancellation condition (5.51) can be rewritten in a simple way

$$\frac{1}{2} \int_{\mathcal{M}_8} G_4 \wedge G_4 + N_{D3} - \frac{\chi(\mathcal{M}_8)}{24} = 0, \quad (5.53)$$

where  $\chi(\mathcal{M}_8)$  is the Euler characteristic of  $\mathcal{M}_8$  and contains the contribution of  $O7$ -planes and  $D7$ -branes in (5.52).

### The tadpole conjecture

So, we have just seen that it is possible to write a superpotential depending on the complex structure moduli and the dilaton using 3-form fluxes. The question remains whether this superpotential stabilises all the complex structure moduli and the dilaton, while satisfying the tadpole condition. The analysis of several examples in F-theory and type IIB string theory led [72] to conjecture that this indeed is not possible to stabilise a large number of complex structure moduli at a generic point in the moduli space.

More precisely, the conjecture for F-theory is that the stabilisation of the  $h^{3,1}$  complex structure moduli of  $\mathcal{M}_8$  using 4-form fluxes requires a flux contribution to the tadpole constraint (5.53) as

$$N_{\text{flux}} > \alpha h^{3,1} \quad \text{where} \quad \alpha > \frac{1}{3}.$$

On the other hand, the only negative contribution to (5.53) comes from the Euler characteristic  $\chi(\mathcal{M}_8)$ , which is related to the number of moduli by  $\chi(\mathcal{M}_8) = 6(8 + h^{1,1} + h^{3,1} - h^{2,1})$ . So in the limit where  $h^{3,1}$  is large relative to the other numbers

$$\frac{\chi(\mathcal{M}_8)}{24} \sim \frac{h^{3,1}}{4}.$$

As a result, the positive flux contribution required to stabilise all moduli grows more rapidly with  $h^{3,1}$  than the negative contribution, and it becomes impossible to stabilise all moduli while satisfying the tadpole condition when  $h^{3,1}$  is too large. The same conjecture can be formulated for type IIB string theory, where the  $h^{3,1}$  complex structure moduli of  $\mathcal{M}_8$  become the  $h^{2,1}$  complex structure moduli of  $\mathcal{M}_6$  and  $n_7$  brane moduli

$$N_{\text{flux}} > \alpha h^{2,1} + \beta n_7.$$

Since its inception, the tadpole conjecture has been investigated in a variety of contexts. In [73] and [74], investigations have been carried out for F-theory on  $K3 \times K3$  and on a Calabi-Yau fourfold with a weak Fano base, respectively. In [75, 76], further examples have been considered in the large complex structure limit in type IIB string theory. Using tools from asymptotic Hodge theory [77, 78] gave more arguments in favor of the conjecture in asymptotic regimes, including, but not limited to, the large complex structure limit. A counter example has been proposed in [79], but was refuted by [80]. The conjecture has also been studied in the interior of moduli space, using simplifications occurring at symmetric points [81, 82], and in the context of non-geometric compactifications in [83].

## Finiteness of flux vacua

As we mentioned above, the number of flux vacua for a given Calabi-Yau manifold has long been expected to be finite. However, early studies and estimates of the number of vacua did not properly take into account the quantisation of the flux numbers introduced after (5.49). This problem has recently been solved, first in [84] for a single complex structure modulus, then in [85] where finiteness of flux vacua was proven in the general multi-variable case by using advances in the field of tame geometry, see also [86]. This new framework calls for a new estimate of the number of flux vacua, about which conjectures are formulated in [87].

## Kahler moduli stabilisation, KKLT, LVS

To stabilise Kahler moduli, other ingredients are needed on top of the flux superpotential. In fact, the Kahler potential (5.41) and superpotential (5.48) we have considered so far arise at tree-level, from the description of type IIB string theory by its effective action (5.27), with branes and orientifold planes as sources. But this description is only a leading term, which is subject to corrections from several potential sources.

On the one hand, there are perturbative corrections. This includes string-loop corrections, controlled by the string coupling  $g_s = e^\phi = 1/\text{Im}(\mathcal{S})$ , and  $\alpha'$  corrections, coming from the extended nature of the string. As long as  $\mathcal{N} = 1$  supersymmetry is preserved, there is a non-renormalization theorem that prevents the superpotential  $W$  from receiving perturbative corrections, but nothing protects the Kahler potential  $\mathcal{K}$ . For example, there is a well-known correction, resulting from the dimensional reduction of a  $R^4$  term coming at order  $(\alpha')^3$  in the 10-dimensional action (5.27)

$$\mathcal{K}_{\text{pert.}} = -2 \log \left( \mathcal{V} + \frac{\xi}{2} \left( \frac{\mathcal{S} - \bar{\mathcal{S}}}{2i} \right)^{3/2} \right) \quad \text{where} \quad \xi = -\frac{\zeta(3)\chi(\mathcal{M}_6)}{2(2\pi)^3}, \quad (5.54)$$

where  $\zeta$  is the Riemann zeta function and  $\chi(\mathcal{M}_6)$  is the Euler number of  $\mathcal{M}_6$ .

On the other hand, there are non-perturbative corrections. In the type IIB theory, the most widely studied sources of such corrections are  $D3$ -brane instantons and gaugino condensation on stacks of  $D7$ -branes. In both cases, these corrections involve a 4-cycle wrapped by the instanton or by the stack of branes. These corrections contribute to the superpotential with terms of the form

$$W_{\text{n.p.}} = A(\mathcal{U}^i) \exp(ia\mathcal{T}), \quad (5.55)$$

where  $\mathcal{T}$  is the Kahler modulus controlling the volume of the aforementioned 4-cycle,  $a$  is a constant and  $A(\mathcal{U}^i)$  is a holomorphic function of the complex structure moduli.

As we saw in chapter 2 the accelerated expansion of our universe requires the presence of an as yet unknown ingredient, which could be a positive cosmological constant or a scalar field rolling in a flat enough potential. In the end of chapter 3, more precisely in (3.68), we explained that in the framework of supergravity, a positive cosmological constant requires supersymmetry to be broken. But as we mentioned just above  $\mathcal{N} = 1$  supersymmetry protects the superpotential  $W$  from perturbative corrections. So supersymmetry must be broken with



care if these corrections are to be kept under control.

We should also bear in mind that the parameters controlling various corrections are linked to the vacuum expectation value of moduli, such as the string coupling  $g_s = e^\phi$  or the radii  $R$  of the internal space  $\mathcal{M}_6$  that enter into  $\alpha'/R$  corrections as we mentioned after (5.31). When using a given set of correction terms to stabilise a set of moduli, one should subsequently verify that the other corrections that these moduli control are negligible. If there were only one modulus controlling one series of corrections, this would actually be problematic, because to be stabilised, this modulus would have to be in a region where the corrections it controls are important, but then the whole series of corrections would become important. This is called the Dine-Seiberg problem [88]. In type IIB string theory, it can be alleviated thanks to fluxes or by considering multiple sources of corrections, as explained for instance in [89]. Nevertheless, keeping control over the corrections that are not taken into account is one of the main challenges of moduli stabilisation.

The main framework for constructing metastable de Sitter vacua with all moduli stabilised is called KKLT, after the authors of [90]. It is based on the GKP solution presented above, and consists of three steps. In the first step, all the complex structure moduli of the internal Calabi-Yau orientifold, as well as the axio-dilaton, are stabilised by 3-form fluxes. This should be done in such a way that the string coupling  $g_s$  is small to neglect loop corrections. In addition, the vacuum expectation value  $W_0$  of the flux superpotential (5.48) must not vanish.

In a second step, we assume that the complex structure moduli and axio-dilaton are frozen and treat  $W_0$  as a constant. We also assume that non-perturbative corrections such as (5.55) are generated for each Kahler modulus. In the case of a single modulus, the total superpotential thus takes the following form

$$W = W_0 + A \exp(ia\mathcal{T}).$$

From there, we can find the vacuum expectation value of  $\mathcal{T}$  by asking supersymmetry to be preserved  $\nabla_{\mathcal{T}}W = 0$ . To simplify the discussion, let us consider that  $\mathcal{T} = i\sigma$  is imaginary. We then obtain

$$\nabla_{\mathcal{T}}W = 0 \quad \rightarrow \quad W_0 = -A\left(1 + \frac{2a}{3}\sigma\right) \exp(-a\sigma), \quad (5.56)$$

leading to a  $\mathcal{N} = 1$  supersymmetric AdS vacuum with negative potential

$$V_{\min} = -\frac{a^2 A^2}{6\sigma} \exp(-2a\sigma). \quad (5.57)$$

Since  $\mathcal{N} = 1$  supersymmetry is preserved at this stage, the superpotential  $W$  receives no perturbative correction. In order to neglect  $\alpha'/R$  corrections to the Kahler potential (5.41), we need to ensure that the internal manifold is large, which amounts to  $\sigma$  being large. As we see in (5.56), this requires  $W_0$  to be exponentially small after the first step. Another consequence is that the negative potential (5.57) is small.

To obtain a Sitter vacuum, we must now introduce a positive contribution to the potential. In [90], this is done with anti- $D3$  branes, which add terms of the form

$$\delta V = \frac{2A(y_0)T_3}{g_s^4} \frac{1}{\text{Im}(\mathcal{T})^3},$$

where  $A(y_0)$  is the value of the warp factor, defined in (5.45), at the position of the brane. If we want to reproduce the observed value of the cosmological constant, this contribution should be small and fine tuned to almost exactly balance the negative part (5.57). In any case, the anti- $D3$  branes must be located in a region of  $\mathcal{M}_6$  where the warp factor is exponentially small for the de Sitter minimum to exist, as can be seen by analysing the potential. Such regions are called Klebanov-Strassler throats [91], and indeed yield an exponentially small warp factor as a function of two flux integers  $K$  and  $M$

$$e^{A_{\min}} \sim \exp\left(-\frac{2\pi K}{3g_s M}\right).$$

As we can see, the KKLT construction is quite complex and requires a large number of different ingredients. Since it was first proposed, its validity has been thoroughly examined. And while no reason has been found to rule out this scenario, neither has it been realised in a totally explicit and controlled manner.

Stabilising the complex structure moduli and axio-dilaton in such a way that  $g_s$  and  $W_0$  are small is already hard. As we saw above, the tadpole conjecture suggests that it is not possible to stabilise too many moduli using 3-form fluxes that satisfy the tadpole condition. Moreover, the finiteness of flux vacua prevents any parametric control over  $g_s$  and  $W_0$  that result from the stabilisation. By parametric control, we mean tuning the flux integers such that an integer  $n$  or a combination of them does not contribute to  $N_{\text{flux}}$  in (5.49) and can be sent to infinity, and finding a solution where  $g_s \sim 1/n$  or  $W_0 \sim 1/n$ . Solutions with such a property would constitute an infinite tail of vacua, which contradicts finiteness theorems. In chapter 6, we exhibit this finiteness in the simple case of toroidal orbifolds and show how it constrains the minimal value of the string coupling.

Note that there are other frameworks for constructing de Sitter vacua that do not require an exponentially small  $W_0$ , such as the Large Volume Scenario [92] in which the perturbative correction (5.54) plays a central role. More recently, a fully perturbative framework was proposed in [93–96], which is based on logarithmic corrections to the Kahler potential. But they do not evade the tadpole conjecture, and still require that  $g_s$  is small after stabilisation by fluxes. These frameworks are not free from control issues either, see [97, 98].



# Chapter 6

## Minimal string coupling and finiteness in IIB compactification

This chapter focuses on the stabilisation of complex structure moduli in type IIB string theory compactified on toroidal orientifolds [6], a simple setup that allows for very explicit computations. In section 6.1, we introduce toroidal orbifolds. We explain the computation of their complex structure, their flux superpotential, the tadpole constraint, and twisted moduli. In section 6.2, we study the orbifolds with zero and one complex structure moduli and exhibit a relation of the form  $g_{s,\min} \sim 1/N_{\text{flux}}$  between the minimal value of the string coupling  $g_{s,\min}$  and the flux number  $N_{\text{flux}}$  that appears in the tadpole constraint. We also show how the finiteness of flux vacua manifests itself in these cases. Finally, in section 6.3, we tackle the case of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , which has three complex structure moduli at the orbifold point. We exhibit a relation of the form  $g_{s,\min} \sim 16/N_{\text{flux}}^2$ , which leads to  $g_s \geq 0.669$  for the subset of flux vacua satisfying the tadpole constraint. This value of the string coupling is too large for this orientifold to be used without considering string loop corrections, or without introducing other ingredients, as magnetised  $D7$ -branes. We also show how the finiteness of flux vacua and the absence of parametric control manifest themselves in this case.

### 6.1 Constructing toroidal orbifolds

The notations and definitions of this section are mostly based on [99–102]. To construct a toroidal orbifold, we start from a 6-torus, defined as  $T^6 = R^6/\Lambda$ , where  $\Lambda$  is a 6 dimensional lattice. In other words, points are identified as  $x^i \sim x^i + l^i$  where  $l \in \Lambda$ . Once we have specified this lattice, we can choose one of its automorphisms  $\Gamma$  as the orbifold group, or point group, to quotient by. Imposing that the resulting orbifold has  $SU(3)$ -holonomy, in order to preserve  $\mathcal{N} = 1$  supersymmetry as explained in 5.3.1, restricts  $\Gamma$  to be a subgroup of  $SU(3)$ . If we further restrict to abelian orbifold groups, and require that  $\Gamma$  acts crystallographically on the torus lattice, we end up with the following list of groups

$$\begin{aligned} \Gamma &= \mathbb{Z}_N && \text{with } N = 3, 4, 6, 7, 8, 12 \\ \Gamma &= \mathbb{Z}_N \times \mathbb{Z}_M && \text{with } M = kN \text{ and } N = 2, 3, 4, 6 \end{aligned}$$

The action of the group on the torus takes a simple form in complex coordinates  $(z^1, z^2, z^3)$ . Indeed, for each group element  $\theta_N = (n_1, n_2, n_3) \in \Gamma$ , this action is written as

$$\theta_N : (z^1, z^2, z^3) \rightarrow (e^{2i\pi n_1/N} z^1, e^{2i\pi n_2/N} z^2, e^{2i\pi n_3/N} z^3). \quad (6.1)$$

The groups  $\mathbb{Z}_N$  are generated by one element  $\theta_N$  while the groups  $\mathbb{Z}_N \times \mathbb{Z}_M$  are generated by two elements  $\theta_N$  and  $\theta_M$ . There are two inequivalent possibilities for  $(\mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_{12})$ . For instance, for  $\mathbb{Z}_6$  we can choose  $\theta_{6,I} = (1, 1, -2)$  or  $\theta_{6,II} = (1, 2, -3)$ , acting as

$$\begin{aligned} \theta_{6,I} : (z^1, z^2, z^3) &\rightarrow (e^{2i\pi/6} z^1, e^{2i\pi/6} z^2, e^{-4i\pi/6} z^3) \\ \theta_{6,II} : (z^1, z^2, z^3) &\rightarrow (e^{2i\pi/6} z^1, e^{4i\pi/6} z^2, e^{-6i\pi/6} z^3). \end{aligned}$$

In real coordinates  $x^i$ , the action of the orbifold is represented by a  $6 \times 6$  matrix  $Q$  acting as

$$Q : x^i \rightarrow Q^i_j x^j \quad (6.2)$$

This matrix can be taken as the Coxeter element of the torus lattice  $\Lambda$ , but there are other possibilities. The orbifold actions in real and complex coordinates are compatible if the eigenvalues of  $Q$  are equal to those of  $\theta_N$ . This selects one or a few possible lattices for each group.

In table 6.1, we give the list of toroidal orbifolds considered in [99], along with the corresponding torus lattices and group actions  $(\theta_N, \theta_M)$ . In this table, we also give the number of untwisted Kahler and complex structure moduli, respectively  $h^{1,1}$  and  $h^{2,1}$ . These number are obtained by counting the number of  $(1, 1)$ -forms  $dz^i \wedge d\bar{z}^j$  (resp. the number of  $(2, 1)$ -forms  $dz^i \wedge dz^j \wedge d\bar{z}^k$ ) left invariant by the action of the group. We also give the number of twisted moduli  $\tilde{h}^{1,1}$  and  $\tilde{h}^{2,1}$ ; on which we will expand in section 6.1.3.

In what follows, we study the stabilisation, by fluxes, of the complex structure moduli. Looking at table 6.1, we see that  $h^{2,1} \in \{0, 1, 3\}$  for all the orbifolds we consider. We will ignore the twisted complex structure moduli, which are expected to have vanishing vacuum expectation values as long as fluxes are not turned-on on their corresponding twisted 3-cycles. We will show this in section 6.1.3 for the orbifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion.

The first step is to compute the superpotential generated by the 3-form fluxes  $(F_3, H_3)$ , as a function of the flux integers. In the solution that we described in 5.3.2, this superpotential is given by (5.48)

$$W = \int \Omega \wedge G_3, \quad \text{where} \quad G_3 = F_3 + \mathcal{S}H_3. \quad (6.3)$$

where  $\mathcal{S} = C_0 + ie^{-\phi} = C_0 + i/g_s$  is the axio-dilaton and  $\Omega = dz^1 \wedge dz^2 \wedge dz^3$  is the nowhere-vanishing  $(3, 0)$  form of the orbifold. We can expand  $G_3$  on a complex cohomology basis

$$G_3 = A^i \omega_{A_i} + B^i \omega_{B_i}, \quad (6.4)$$

where  $i = 0, \dots, 3$  and

$$\begin{aligned} \omega_{A_0} &= dz^1 \wedge dz^2 \wedge dz^3, & \omega_{A_1} &= d\bar{z}^1 \wedge dz^2 \wedge dz^3, & \omega_{A_2} &= dz^1 \wedge d\bar{z}^2 \wedge dz^3, & \dots \\ \omega_{B_0} &= d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3, & \omega_{B_1} &= dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3, & \omega_{B_2} &= d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3, & \dots \end{aligned}$$

orbifold	torus lattice $\Lambda$	$\theta_N$	$\theta_M$	$h^{1,1}$	$h^{2,1}$	$\tilde{h}^{1,1}$	$\tilde{h}^{2,1}$
$\mathbb{Z}_3$	$SU(3)^3$	(1, 1, -2)		9	0	27	0
$\mathbb{Z}_{4,a}$	$SU(4)^2$	(1, 1, -2)		5	1	20	0
$\mathbb{Z}_{4,b}$	$SU(2) \times SU(4) \times SO(5)$	(1, 1, -2)		5	1	22	2
$\mathbb{Z}_{4,c}$	$SU(2)^2 \times SO(5)^2$	(1, 1, -2)		5	1	26	6
$\mathbb{Z}_{6, Ia}$	$G_2 \times SU(3)^2$ (*)	(1, 1, -2)		5	0	20	1
$\mathbb{Z}_{6, Ib}$	$G_2^2 \times SU(3)$	(1, 1, -2)		5	0	24	5
$\mathbb{Z}_{6, IIa}$	$SU(2) \times SU(6)$	(1, 2, -3)		3	1	22	0
$\mathbb{Z}_{6, IIb}$	$SU(3) \times SO(8)$	(1, 2, -3)		3	1	26	4
$\mathbb{Z}_{6, IIc}$	$SU(2)^2 \times SU(3)^2$ (*)	(1, 2, -3)		3	1	28	6
$\mathbb{Z}_{6, IID}$	$G_2 \times SU(2)^2 \times SU(3)$	(1, 2, -3)		3	1	32	10
$\mathbb{Z}_7$	$SU(7)$	(1, 2, -3)		3	0	21	0
$\mathbb{Z}_{8, Ia}$	$SU(4) \times SU(4)$ (*)	(1, 2, -3)		3	0	21	0
$\mathbb{Z}_{8, Ib}$	$SO(5) \times SO(9)$	(1, 2, -3)		3	0	24	3
$\mathbb{Z}_{8, IIa}$	$SU(2) \times SO(10)$	(1, 3, -4)		3	1	24	2
$\mathbb{Z}_{8, IIb}$	$SO(4) \times SO(9)$	(1, 3, -4)		3	1	28	6
$\mathbb{Z}_{12, Ia}$	$E_6$	(1, 4, -5)		3	0	22	1
$\mathbb{Z}_{12, Ib}$	$SU(3) \times F_4$	(1, 4, -5)		3	0	26	5
$\mathbb{Z}_{12, II}$	$SO(4) \times F_4$	(1, 5, -6)		3	1	28	6
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$SU(2)^6$	(1, 0, -1)	(0, 1, -1)	3	3	48	0
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$SU(2)^2 \times SO(5)^2$	(1, 0, -1)	(0, 1, -1)	3	1	58	0
$\mathbb{Z}_2 \times \mathbb{Z}_{6, I}$	$G_2 \times SU(2)^2 \times SU(3)$	(1, 0, -1)	(0, 1, -1)	3	1	48	2
$\mathbb{Z}_2 \times \mathbb{Z}_{6, II}$	$G_2^2 \times SU(3)$	(1, 0, -1)	(1, 1, -2)	3	0	33	0
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$SU(3)^3$	(1, 0, -1)	(0, 1, -1)	3	0	81	0
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$G_2^2 \times SU(3)$	(1, 0, -1)	(0, 1, -1)	3	0	70	1
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$SO(5)^3$	(1, 0, -1)	(0, 1, -1)	3	0	87	0
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$G_2^3$	(1, 0, -1)	(0, 1, -1)	3	0	81	0

Table 6.1: List of toroidal orbifolds considered in [99], along with the corresponding torus lattices, group actions  $(\theta_N, \theta_M)$  and number of untwisted and twisted Kahler and complex structure moduli, respectively  $(h^{1,1}, h^{2,1})$  and  $(\tilde{h}^{1,1}, \tilde{h}^{2,1})$ . The matrix  $Q$  of (6.2) is the Coxeter element of the torus lattice, except for the three entries marked by a (\*), see [99].

In principle, this basis also contains elements of the form  $dz^1 \wedge d\bar{z}^1 \wedge dz^2$ , but they are projected out in all the orbifolds we consider. In the complex basis, we have  $\Omega = \omega_{A_0}$  and the superpotential (6.3) is simply

$$W = B^0 \int \omega_{A_0} \wedge \omega_{B_0}. \quad (6.5)$$

On the other hand, we can also expand  $G_3$  on a real cohomology basis

$$G_3 = m_i \alpha_i + n_i \beta^i + p^j \gamma_j + q^j \delta^j \quad \text{where} \quad m_i = m_i^F + \mathcal{S}m_i^H \quad \text{etc,} \quad (6.6)$$

where  $i = 0, \dots, 3$  and  $j = 1, \dots, 6$ . Let us introduce the notation  $(i_1 \dots i_k) = dx^{i_1} \wedge \dots \wedge dx^{i_k}$  and use the convention that  $(123456) = -1$  after integration. The real basis can be chosen as

$$\begin{array}{l}
\alpha_0 = (135), \quad \beta^0 = (246) \\
\alpha_1 = (235), \quad \beta^1 = -(146) \\
\alpha_2 = (145), \quad \beta^2 = -(236) \\
\alpha_3 = (136), \quad \beta^3 = -(245)
\end{array}
\left\| \begin{array}{l}
\gamma_1 = (123), \quad \delta^1 = -(456) \\
\gamma_2 = (125), \quad \delta^2 = -(346) \\
\gamma_3 = (134), \quad \delta^3 = -(256)
\end{array} \right.
\left| \begin{array}{l}
\gamma_4 = (345), \quad \delta^4 = -(126) \\
\gamma_5 = (156), \quad \delta^5 = -(234) \\
\gamma_6 = (356), \quad \delta^6 = -(124)
\end{array} \right.$$

The elements of this real basis satisfy  $\alpha_i \wedge \beta^j = \delta_i^j$  and  $\gamma_i \wedge \delta^j = \delta_i^j$  after integration. Because of the Dirac quantisation condition, the coordinates  $(m_i^F, m_i^H, n_i^F, n_i^H, p_j^F, p_j^H, q_j^F, q_j^H)$  of  $(H_3, F_3)$  on the real basis are integers, that we call flux integers. By extension, we also sometimes call the complexified  $m_i = m_i^F + \mathcal{S}m_i^H$  flux integers, even though they are not really integers.

### 6.1.1 Complex structures and flux superpotential

The expression (6.5) of the superpotential on the complex basis is not very informative, as it can be written without knowing anything about the orbifold. It is thus much more interesting to express the superpotential as a function of the flux integers. For this, we have to match the two expressions of  $G$  in (6.4) and (6.6). More precisely, we have to expand (6.4) on the real basis, and identify relations between the  $(A^i, B^i)$  and the flux integers.

But before we can do this, we must determine the relation between the complex coordinates  $z^i$  and the real ones  $x^i$ . In other words, we need to compute the complex structures. This is done by writing the complex coordinates as linear functions of the real ones

$$z^i = A^i_j x^j,$$

and applying the orbifold action (6.1) and (6.2) on both sides

$$\theta_N(z^i) = A^i_j Q(x^j) \quad \rightarrow \quad e^{2i\pi n_i/N} A^i_j x^j = A^i_j Q^j_k x^k. \quad (6.7)$$

Identifying the coefficients of the  $x^i$  on both sides of (6.7) gives a system of  $3 \times 6$  equations and allows to determine the  $3 \times 6$  coefficients  $A^i_j$  of the complex structure. Note that these coefficients are not entirely determined by the system, and are either complex structure moduli or coefficients that can be chosen arbitrarily. For instance, it is clear in equation (6.7) that the coefficients  $A^i_j$  can be rescaled simultaneously without affecting the system.

This procedure gives the complex structures listed in appendix 6.4, a sample of which we give in table 6.2. When the orbifold group acts symmetrically on the  $z^i$ , we expect this to be visible on the complex structure. For instance,  $\mathbb{Z}_3$  acts the same on the three  $z^i$  and we see this in table 6.2. On the other hand,  $\mathbb{Z}_4$  acts the same on  $z^1$  and  $z^2$ , which is manifest for  $\mathbb{Z}_{4,a}$  but not for  $\mathbb{Z}_{4,b}$ . For  $\mathbb{Z}_{4,b}$ , the  $z^1 \leftrightarrow z^2$  symmetry is present in the solution of the system (6.7)

$$\begin{aligned}
z^1 &= A_1^1(x^1 + (e^{3i\pi/4}/\sqrt{2})x^2) + A_3^1(x^3 + ix^4 - x^5) \\
z^2 &= A_1^2(x^1 + (e^{3i\pi/4}/\sqrt{2})x^2) + A_3^2(x^3 + ix^4 - x^5),
\end{aligned}$$

but since we require that  $z^1$  is not proportional to  $z^2$ , this symmetry must be broken by the choice of the remaining  $A^i_j$ , which is otherwise arbitrary. Another example is  $\mathbb{Z}_3 \times \mathbb{Z}_3$ , which acts the same on  $z^1$  and  $z^2$ , but not on  $z^3$ , and this is again clear in table 6.2.

orbifold	coefficients of the complex structure						
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
$\mathbb{Z}_3$	$z_1$	1	$e^{2i\pi/3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$e^{2i\pi/3}$
$\mathbb{Z}_{4,b}$	$z_1$	1	$e^{3i\pi/4}/\sqrt{2}$	0	0	0	0
	$z_2$	0	0	1	$i$	-1	0
	$z_3$	0	0	1	-1	1	$\mathcal{U}$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$z_1$	1	$e^{2i\pi/3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/3}$

Table 6.2: Complex structures of some orbifolds, where  $\mathcal{U}$  is a complex structure modulus.

Knowing the complex structures of the orbifolds, we can obtain the superpotential as a function of the flux integers. Matching the two expressions of  $G$  in (6.4) and (6.6) indeed yields the expression of the  $A^i$  and  $B^i$  as a function of the flux integers, and in particular the expression of  $B^0$  that appears in (6.5). It also gives multiple constraints between the flux integers  $(m_i, n_i, p_j, q_j)$ . To solve these constraints, we must choose a set of basis integers, of which the others will be functions. In the process, we must make sure that the integers remain integers. For instance if one of the constraint is of the form  $m = 2n$ , then we can choose  $n$  as a basis integer, but it would be impractical to choose  $m$ , because we would have to remember that it is a multiple of 2 for  $n$  to be integer.

In addition to the superpotential, an important piece of information is the number  $N_{\text{flux}}$  that enters in the tadpole constraint. We gave its expression in (5.47). Expanding  $(H_3, F_3)$  in the real basis of equation (6.6), we get

$$N_{\text{flux}} = \int H_3 \wedge F_3 = (m^H \cdot n^F - m^F \cdot n^H) + (p^H \cdot q^F - p^F \cdot q^H), \quad (6.8)$$

the  $\cdot$  denotes a sum over  $i = 0, \dots, 3$  in the first parenthesis or  $j = 1, \dots, 6$  in the second.

Let us take the orbifold  $T^6/\mathbb{Z}_3$  as an example. If we choose  $m_0$  and  $m_1$  as basis integers, we obtain the following superpotential

$$W = -3e^{2i\pi/3}(m_0 + e^{i\pi/3}m_1) \quad \text{where} \quad m_i = m_i^F + \mathcal{S}m_i^H,$$

and the other integers are given by  $m_1 = m_2 = m_3$  and  $n_0 = m_0$  and  $n_1 = n_2 = n_3 = m_0 + m_1$  and  $p_j = q_j = 0$ . Replacing in (6.8) then yields  $N_{\text{flux}} = -3(m_0^H m_1^F - m_0^F m_1^H)$ . Note that it is a multiple of 3 only because of the orbifold's geometry.

For orbifolds with  $h^{2,1} = 0$ , we can parameterise the flux superpotential (6.3) as

$$W = K(a + \gamma b) \quad \text{where} \quad a = a^F + \mathcal{S}a^H \quad \text{and} \quad b = b^F + \mathcal{S}b^H, \quad (6.9)$$

where  $a, b$  are flux integers. It also turns out that we can write  $N_{\text{flux}}$  as

$$N_{\text{flux}} = k(a^H b^F - a^F b^H), \quad (6.10)$$



with  $k$  an integer that depends only on the orbifold. In particular,  $N_{\text{flux}}$  is a multiple of  $k$ .

For orbifolds with  $h^{2,1} = 1$ , we can parameterise similarly

$$W = \mathcal{U}A + B, \quad (6.11)$$

where  $A$  and  $B$  are simple combination of basis flux integers, and again  $A = A^F + \mathcal{S}A^H$  etc. With this notation

$$N_{\text{flux}} = k(\text{Re}(A^H \bar{B}^F) - \text{Re}(A^F \bar{B}^H)), \quad (6.12)$$

where  $k$  is again a factor that depends only on the orbifold. For these orbifolds,  $N_{\text{flux}}$  is not a multiple of  $k$ , but of an other integer that we call  $l$ . For instance, for the orbifold  $T^6/\mathbb{Z}_{4,a}$

$$N_{\text{flux}} = 4(m_0^H n_0^F - m_0^F n_0^H + m_1^H n_0^F - m_1^F n_0^H + m_0^H m_2^F - m_0^F m_2^H + 2(m_1^H m_2^F - m_1^F m_2^H)),$$

so  $l = 4$  for this orbifold.

In tables 6.3 and 6.4, we give the superpotential and  $N_{\text{flux}}$  for the orbifolds of table 6.1 with  $h^{2,1} = 0$  and  $h^{2,1} = 1$  respectively, using the parameterisations (6.9) and (6.11).

orbifold	$a$	$b$	$K$	$\gamma$	$k$
$\mathbb{Z}_3$	$m_0$	$m_1$	$-3e^{2i\pi/3}$	$e^{i\pi/3}$	$-3$
$\mathbb{Z}_{6, Ia}$	$m_0$	$m_1$	$2\sqrt{3}e^{5i\pi/6}$	$\sqrt{3}e^{i\pi/6}$	$3$
$\mathbb{Z}_{6, Ib}$	$n_0$	$n_1$	$-\sqrt{3}e^{i\pi/6}$	$e^{5i\pi/6}/\sqrt{3}$	$3$
$\mathbb{Z}_7$	$m_0$	$m_1$	$7/2(7 + i\sqrt{7})$	$(1 + i\sqrt{7})/4$	$7$
$\mathbb{Z}_{8, Ia}$	$m_0$	$m_1$	$16i\sqrt{2}$	$\sqrt{2}e^{i\pi/4}$	$8$
$\mathbb{Z}_{8, Ib}$	$n_0$	$n_1$	$-4i\sqrt{2}$	$e^{3i\pi/4}/\sqrt{2}$	$4$
$\mathbb{Z}_{12, Ia}$	$m_0$	$m_1$	$12e^{i\pi/6}$	$e^{i\pi/3}$	$3$
$\mathbb{Z}_{12, Ib}$	$m_0$	$m_1$	$-3e^{5i\pi/6}$	$e^{i\pi/3}$	$3$
$\mathbb{Z}_2 \times \mathbb{Z}_{6, II}$	$n_0$	$n_1$	$-\sqrt{3}e^{i\pi/6}$	$e^{5i\pi/6}/\sqrt{3}$	$3$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$m_0$	$m_1$	$3e^{2i\pi/3}$	$e^{i\pi/3}$	$3$
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$n_0$	$n_1$	$-1$	$e^{2i\pi/3}$	$3$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$n_0$	$n_1$	$-2$	$e^{3i\pi/4}/\sqrt{2}$	$4$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$n_0$	$n_1$	$-1$	$e^{5i\pi/6}/\sqrt{3}$	$3$

Table 6.3: Superpotential and  $N_{\text{flux}}$  for orbifolds with  $h^{2,1} = 0$ , in the parameterisation (6.9).

orbifold	basis integers	$A$	$B$	$k$	$l$
$\mathbb{Z}_{4,a}$	$m_0, m_1, m_2, n_0$	$4\sqrt{2}e^{3i\pi/4}m_0 - 8m_1$	$4\sqrt{2}e^{3i\pi/4}n_0 - 8m_2$	$1/8$	$4$
$\mathbb{Z}_{4,b}$	$m_1, m_3, n_0, n_3$	$2m_1 - 2\sqrt{2}e^{i\pi/4}n_3$	$-2\sqrt{2}e^{3i\pi/4}m_3 - 4in_0$	$1/2$	$2$
$\mathbb{Z}_{4,c}$	$m_1, n_0, n_1, n_3$	$m_1 - \sqrt{2}e^{i\pi/4}n_3$	$-\sqrt{2}e^{i\pi/4}n_0 + n_1$	$2$	$2$
$\mathbb{Z}_{6,IIa}$	$m_1, m_3, n_2, p_1$	$-6m_1 + 12e^{2i\pi/3}p_1$	$-6i\sqrt{3}m_3 - 18n_2$	$1/18$	$6$
$\mathbb{Z}_{6,IIb}$	$m_0, m_1, m_2, m_3$	$A_{6,IIb}$	$B_{6,IIb}$	$-2/9$	$3$
$\mathbb{Z}_{6,IIc}$	$m_0, m_3, p_2, q_2$	$-6m_0 - 2i\sqrt{3}p_2$	$6m_3 - 2i\sqrt{3}q_2$	$1/6$	$2$
$\mathbb{Z}_{6,IId}$	$m_0, m_1, n_0, n_1$	$-e^{i\pi/3}m_0 - i\sqrt{3}m_1$	$-\sqrt{3}e^{i\pi/6}n_0 + n_1$	$2$	$1$
$\mathbb{Z}_{8,IIa}$	$m_0, m_1, n_0, n_2$	$4im_0 - 4\sqrt{2}m_1$	$-8in_0 + 4(2i + \sqrt{2})n_2$	$-1/8$	$4$
$\mathbb{Z}_{8,IIb}$	$m_1, m_2, n_0, n_2$	$2(2i - \sqrt{2})m_1 - 4im_2$	$4in_0 - 2(2i + \sqrt{2})n_2$	$1/4$	$2$
$\mathbb{Z}_{12,II}$	$m_0, m_3, p_4, q_2$	$-\sqrt{3}m_0 + \sqrt{6}e^{i\pi/4}p_4$	$\sqrt{3}m_3 + \sqrt{6}e^{i\pi/4}q_2$	$2/3$	$2$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$m_2, n_0, n_1, n_2$	$im_2 - \sqrt{2}e^{i\pi/4}n_1$	$\sqrt{2}e^{3i\pi/4}n_0 - in_2$	$-2$	$2$
$\mathbb{Z}_2 \times \mathbb{Z}_{6,I}$	$m_0, m_2, n_0, n_2$	$-e^{i\pi/3}m_0 - i\sqrt{3}m_2$	$-\sqrt{3}e^{i\pi/6}n_0 + n_2$	$2$	$1$

Table 6.4: Superpotential and  $N_{\text{flux}}$  for orbifolds with  $h^{2,1} = 1$ , in the parameterisation (6.11). We introduced the notations,  $A_{6,IIb} = -3e^{i\pi/3}m_0 + 3i\sqrt{3}m_1 + 3e^{2i\pi/3}m_2 - 3i\sqrt{3}m_3$  and  $B_{6,IIb} = 3m_0 + 3i\sqrt{3}m_1 + 3e^{2i\pi/3}m_2$  for the table not to be too wide. Here,  $l$  is a factor of quantization of  $N_{\text{flux}}$  (i.e.  $N_{\text{flux}} \in l\mathbb{N}$ ), that comes solely from the geometry of the orbifold.

## 6.1.2 The tadpole constraint

In addition to being integers, the  $(m, n)$  and  $(p, q)$  have to satisfy the tadpole constraint, translating the fact that the total  $D3$ -brane charge on the compact manifold vanishes. This constraint was given in (5.38)

$$\frac{1}{2}\tilde{N}_{\text{flux}} + N_{D3} = \frac{1}{4}N_{O3}, \quad (6.13)$$

where  $\tilde{N}_{\text{flux}}$  will be defined shortly and  $(N_{D3}, N_{O3})$  are the net numbers of  $D3$ -branes and  $O3$ -planes. As explained after (5.40), we do not consider anti- $D3$ -branes, so  $N_{D3} \geq 0$ . Since  $N_{\text{flux}} \geq 0$  because of the imaginary self-dual condition (5.46), it cannot be arbitrarily large.

To talk about  $O3$  planes, we assumed that we have performed an orientifold quotient. As explained in section 5.3.1, this is a quotient of the theory by a geometric involution combined with a reversal of worldsheet orientation. This introduces the contribution in the right hand side of (6.13), without which  $N_{\text{flux}} = N_{D3} = 0$ . The geometric involution we consider here is simply  $x^i \rightarrow -x^i$ , which corresponds to  $\mathcal{R} = R_i R_j R_k$  in the notations of (5.35).

At this stage, we shall introduce an additional fact about the quantisation of the flux integers. Namely, to avoid subtleties associated with additional 3-cycles that are not present in the covering  $T^6$ , we must take integers to be multiples of  $2|\Gamma|$  when working with  $T^6/\Gamma$ , where  $|\Gamma|$  is the order of  $\Gamma$ , so  $|\mathbb{Z}_N| = N$  and  $|\mathbb{Z}_N \times \mathbb{Z}_M| = NM$ . The factor  $|\Gamma|$  come from the orbifold action, and the factor of 2 comes from the  $\mathbb{Z}_2$  involution of the orientifold.

This can be understood from the fact that in (6.6), we defined the flux integers by expanding  $F_3$  and  $H_3$  on the cohomology basis of the torus. Reciprocally, we have

$$m_i^H = \int_{A^i} H,$$

where  $A^i$  is the 3-cycle that is Poincaré dual to the 3-form  $\alpha_i$  in the real cohomology basis. If we quotient the torus by  $\Gamma$ , this cycle becomes  $\tilde{A}^i$ , which is  $|\Gamma|$  times smaller. More precisely, the 3-cycles generically have  $|\Gamma|$  homologically equivalent images under the orbifold group from the torus point-of-view, which are all identified to only one 3-cycle on the orbifold. The integral over this cycle is also  $|\Gamma|$  times smaller, and defines other flux integers  $\tilde{m}_i^H$ , that we call orientifold flux integers, while the original  $m_i^H$  can be called torus flux integers

$$\tilde{m}_i^H = \int_{\tilde{A}^i} H = \frac{1}{|\Gamma|} \int_{A^i} H \quad \rightarrow \quad m_i^H = |\Gamma| \tilde{m}_i^H.$$

The Dirac quantisation, applies to the  $\tilde{m}_i^H$  and implies that the  $m_i^H$  are multiples of  $|\Gamma|$ . This applies for all the integers  $(m, n)$  and  $(p, q)$ , as well as with the  $\mathbb{Z}_2$  involution of the orientifold.

In the same way, the flux number  $\tilde{N}_{\text{flux}}$  that appears in the tadpole constraint (6.13) has to be computed on the orientifold rather and not on the torus. This  $\tilde{N}_{\text{flux}}$  differs by a factor of  $1/(2|\Gamma|)$  from the untilded  $N_{\text{flux}}$  defined in (6.8)

$$\tilde{N}_{\text{flux}} = \int_{\text{ori.}} H_3 \wedge F_3 = \frac{1}{2|\Gamma|} \int_{T^6} H_3 \wedge F_3 = \frac{1}{2|\Gamma|} N_{\text{flux}}, \quad (6.14)$$

Therefore, if we keep using the torus flux integers, we need to bear in mind that they are multiples of  $2|\Gamma|$ , and that it is  $\tilde{N}_{\text{flux}}$  that appears in the tadpole constraint (6.13). Note that as a result of these considerations,  $\tilde{N}_{\text{flux}}$  is a multiple of  $2|\Gamma|$ , while  $N_{\text{flux}}$  is a multiple of  $4|\Gamma|^2$ .

As we explained in section 5.3.1, the  $O3$ -planes are located at fixed point of  $\mathcal{R}$ , so their number  $N_{O3}$  that appears in (6.13) is obtained by counting the fixed points of this involution. From the point-of-view of the torus, there are  $2^6 = 64$  fixed points, located at  $(\iota_1, \dots, \iota_6)$  in the basis of the torus lattice vectors, where  $\iota_i = 0$  or  $1/2$ . But some points are identified by the orbifold action, which is represented by the matrix  $Q$  in real coordinates (6.2). Taking this into account, we obtain the numbers of  $O3$ -planes reported in table 6.5.

orientifold	$N_{O3}$	orientifold	$N_{O3}$	orientifold	$N_{O3}$
$\mathbb{Z}_3$	22	$\mathbb{Z}_7$	10	$\mathbb{Z}_2 \times \mathbb{Z}_2$	64
$\mathbb{Z}_{4,a}$	22	$\mathbb{Z}_{8, Ia}$	12	$\mathbb{Z}_2 \times \mathbb{Z}_4$	40
$\mathbb{Z}_{4,b}$	28	$\mathbb{Z}_{8, Ib}$	22	$\mathbb{Z}_2 \times \mathbb{Z}_{6, I}$	24
$\mathbb{Z}_{4,c}$	40	$\mathbb{Z}_{8, IIa}$	16	$\mathbb{Z}_2 \times \mathbb{Z}_{6, II}$	22
$\mathbb{Z}_{6, Ia}$	14	$\mathbb{Z}_{8, IIb}$	24	$\mathbb{Z}_3 \times \mathbb{Z}_3$	10
$\mathbb{Z}_{6, Ib}$	22	$\mathbb{Z}_{12, Ia}$	8	$\mathbb{Z}_3 \times \mathbb{Z}_6$	17
$\mathbb{Z}_{6, IIa}$	16	$\mathbb{Z}_{12, Ib}$	14	$\mathbb{Z}_4 \times \mathbb{Z}_4$	28
$\mathbb{Z}_{6, IIb}$	24	$\mathbb{Z}_{12, II}$	16	$\mathbb{Z}_6 \times \mathbb{Z}_6$	17
$\mathbb{Z}_{6, IIc}$	16				
$\mathbb{Z}_{6, IId}$	24				

Table 6.5: numbers of  $O3$ -planes in the orientifolds listed in table 6.1

In a general Calabi-Yau compactifications, the  $O7$ -planes and  $D7$ -branes, wrapped around a 4-dimensional submanifold of the internal space, also contribute to the  $D3$ -brane charge. This contribution is proportional to the Euler number of the wrapped submanifold. In the case

of toroidal orbifold, these submanifolds have vanishing Euler number, so this contribution vanishes. However we could consider the presence of magnetic fluxes on the worldvolume of  $D7$ -branes, which also contribute to the  $D3$ -brane charge. We come back to this in 6.3.4.

### 6.1.3 Words on twisted moduli and discrete torsion

We mentioned earlier the existence of twisted Kahler and complex structure moduli. They are additional degrees of freedom corresponding to strings that are closed up to the action of the orbifold group on the covering space of the orbifold. They have to be taken into account for the theory to be well-defined on the singular geometry of the orbifold, in particular to ensure modular invariance of the one-loop vacuum amplitude, as explained in section 5.2. Geometrically, they correspond to deformation parameters that can be used to resolve the singularities located at fixed loci of the orbifold action. In the effective theory, they correspond to additional moduli counted by the twisted Hodge numbers  $(\tilde{h}^{1,1}, \tilde{h}^{2,1})$ .

In the orbifolds  $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$ , there is an additional possibility, called discrete torsion. This is a discrete phase that can be introduced between different twisted sectors of the one-loop vacuum amplitude, without altering the modular invariance of the theory. Discrete torsion has a non-trivial effect on the geometrical interpretation of the twisted moduli, and it affects the numbers of twisted moduli  $(\tilde{h}^{1,1}, \tilde{h}^{2,1})$ , as we see in table 6.6, taken from [103]

orbifold	$(h^{1,1} + \tilde{h}^{1,1}, h^{2,1} + \tilde{h}^{2,1})$ without vs with discrete torsion			
$\mathbb{Z}_2 \times \mathbb{Z}_2$	(51, 3)	(3, 51)		
$\mathbb{Z}_2 \times \mathbb{Z}_4$	(61, 3)	(21, 9)		
$\mathbb{Z}_2 \times \mathbb{Z}_{6,I}$	(51, 3)	(19, 19)		
$\mathbb{Z}_2 \times \mathbb{Z}_{6,II}$	(36, 0)	(15, 15)		
$\mathbb{Z}_3 \times \mathbb{Z}_3$	(84, 0)	(3, 27)		
$\mathbb{Z}_3 \times \mathbb{Z}_6$	(73, 1)	(13, 13)		
$\mathbb{Z}_4 \times \mathbb{Z}_4$	(90, 0)	(42, 0)	(6, 12)	
$\mathbb{Z}_6 \times \mathbb{Z}_6$	(84, 0)	(51, 3)	(27, 3)	(9, 9)

Table 6.6: Twisted Hodge numbers  $(\tilde{h}^{1,1}, \tilde{h}^{2,1})$  in  $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$  orbifolds with discrete torsion.

In the case of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , discrete torsion exchanges  $h^{1,1}$  with  $h^{2,1}$ . This is a particular case where the orbifolds with and without discrete torsion are related by mirror symmetry. But it does not happen for the other orbifolds. Still in the case of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , discrete torsion has been used in some phenomenological works, to reduce the number of Kahler moduli, from  $3 + 48$  to only the 3 untwisted ones, because the stabilisation of that many Kahler moduli is hard to study. In this case,  $3 + 48$  is the number of complex structure moduli.

But as long as fluxes are not turned-on on the twisted 3-cycles corresponding to the twisted moduli, they are expected to be stabilised at the orbifold point, i.e. with vanishing vevs. Let us consider the example of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with discrete torsion. It has  $h^{2,1} = 3$  untwisted complex structure moduli  $\mathcal{U}^i$ , with  $i = 1, 2, 3$  and  $\tilde{h}^{2,1} = 48$  twisted ones  $\mathcal{D}_{\alpha\beta}^i$ , with  $(\alpha, \beta) = 1, \dots, 4$ . More precisely, in this case, the  $T^6$  is factorised as  $T^2 \times T^2 \times T^2$ , as shown by its complex structure in section 6.4. Each of the three  $\mathcal{U}^i$  is inherited from a  $T^2$ . The group  $\mathbb{Z}_2 \times \mathbb{Z}_2$

contains three elements that each leave one  $T^2$  invariant and have  $4 \times 4$  fixed points on the remaining two  $T^2$ . Each of these fixed points corresponds to one of the  $\mathcal{D}_{\alpha\beta}^i$ .

The Kahler potential of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  can be expanded as follows around the orbifold point

$$\begin{aligned} \mathcal{K} \simeq & -\log\left(-i(\mathcal{U}^1 - \bar{\mathcal{U}}^1)(\mathcal{U}^2 - \bar{\mathcal{U}}^2)(\mathcal{U}^3 - \bar{\mathcal{U}}^3) - \frac{i}{2}(\mathcal{U}^1 - \bar{\mathcal{U}}^1)(\mathcal{D}_{\alpha\beta}^1 - \bar{\mathcal{D}}_{\alpha\beta}^1)^2 \right. \\ & \left. - \frac{i}{2}(\mathcal{U}^2 - \bar{\mathcal{U}}^2)(\mathcal{D}_{\alpha\beta}^2 - \bar{\mathcal{D}}_{\alpha\beta}^2)^2 - \frac{i}{2}(\mathcal{U}^3 - \bar{\mathcal{U}}^3)(\mathcal{D}_{\alpha\beta}^3 - \bar{\mathcal{D}}_{\alpha\beta}^3)^2 + \mathcal{O}(\mathcal{D}^4)\right), \end{aligned} \quad (6.15)$$

Where the twisted moduli appear in pair because they are acted upon by a  $\mathbb{Z}_2$  symmetry of the orbifold group, and they do not mix among themselves, which is reminiscent of the fact that the exceptional divisors of a  $T^4/\mathbb{Z}_2$  do not intersect one another. Using equation (5.42), this Kahler potential can be traced back to the following prepotential

$$\mathcal{G} = \frac{X^1 X^2 X^3}{X^0} + \frac{1}{2} \frac{X^i (X_{\alpha\beta}^i)^2}{X^0} + \mathcal{O}((X_{\alpha\beta}^i)^4) = \mathcal{U}^1 \mathcal{U}^2 \mathcal{U}^3 + \frac{1}{2} \mathcal{U}^i (\mathcal{D}_{\alpha\beta}^i)^2 + \mathcal{O}(\mathcal{D}^4),$$

here we introduced projective coordinates  $(X^0, X^i, X_{\alpha\beta}^i)$  in the first equality, which are set to  $(X^0, X^i, X_{\alpha\beta}^i) = (1, \mathcal{U}^i, \mathcal{D}_{\alpha\beta}^i)$  after expanding (5.42) to recover the Kahler potential (6.15). This prepotential can then be used to compute the flux superpotential by using the formula

$$W = n_I X^I + m^I \mathcal{G}_I,$$

which gives

$$\begin{aligned} W = & n_0 + n_1 \mathcal{U}^1 + n_2 \mathcal{U}^2 + n_3 \mathcal{U}^3 + m_1 \mathcal{U}^2 \mathcal{U}^3 + m_2 \mathcal{U}^1 \mathcal{U}^3 + m_3 \mathcal{U}^1 \mathcal{U}^2 - m_0 \mathcal{U}^1 \mathcal{U}^2 \mathcal{U}^3 \\ & + (n_{i,\alpha\beta} + m_{i,\alpha\beta} \mathcal{U}^i) \mathcal{D}_{\alpha\beta}^i + (m_i - m_0 \mathcal{U}^i) (\mathcal{D}_{\alpha\beta}^i)^2 + \mathcal{O}(\mathcal{D}^4) \end{aligned} \quad (6.16)$$

Using all of this information, we can expand the no-scale scalar potential (5.50) to first order in the  $\mathcal{D}_{\alpha\beta}^i$ . That this first order vanishes is then a sufficient condition for all of the twisted moduli to be stabilised at  $\mathcal{D}_{\alpha\beta}^i = 0$ , and we can show that this is the case when  $m_{i,\alpha\beta} = n_{i,\alpha\beta} = 0$ , i.e. when fluxes are not turn-on on the corresponding twisted 3-cycles.

## 6.2 Relation between $g_{s,\min}$ and $N_{\text{flux}}$

In this section, we study the stabilisation of the untwisted complex structure moduli and axio-dilaton for the orbifolds of table 6.1, except  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  that is treated in 6.3. For this, we use the superpotentials computed and parameterised in section 6.1.1. We will restrict ourselves to the vacua satisfying the supersymmetry condition  $\nabla_i W = \partial_i W + (\partial_i \mathcal{K})W = 0$ , where the index  $i$  runs over the complex structure moduli and the axio-dilaton and  $\nabla$  is the Kahler-covariant derivative. The Kahler potential is in (5.41).

For the vacua with all complex structure moduli and the axio-dilaton stabilised, we exhibit evidence for a relation between the minimal string coupling  $g_s = \exp(\phi)$  and  $N_{\text{flux}}$ . In orbifolds with  $h^{2,1} = 0$  or  $h^{2,1} = 1$ , this relation takes the form

$$g_{s,\min} \sim \frac{1}{N_{\text{flux}}} \quad \text{when} \quad N_{\text{flux}} \gg 1. \quad (6.17)$$

More precise relations will be given in the next subsections. Since  $N_{\text{flux}}$  is bounded due to the tadpole constraint (6.13), this relation implies a lower bound on the string coupling, depending only on the orbifold. We also show how the finiteness of flux vacua manifests itself and give their numbers in some concrete examples. Similar results on toroidal orbifolds have already been presented in [104], and more recently in [105, 106].

### 6.2.1 Orbifolds with $h^{2,1} = 0$

In toroidal orbifolds with no complex structure moduli, the relation (6.17) is simple to obtain. In such orbifolds, we have parameterised the flux superpotential as (6.9)

$$W = K(a + \gamma b) \quad \text{where} \quad a = a^F + S a^H \quad \text{and} \quad b = b^F + S b^H,$$

where the values of  $K$  and  $\gamma$  can be found in table 6.3. We also have  $N_{\text{flux}} = k(a^H b^F - a^F b^H)$ , see (6.10). Solving  $\nabla_S W = 0$  with this superpotential yields

$$\bar{\mathcal{S}} = -\frac{a^F + \gamma b^F}{a^H + \gamma b^H}, \quad (6.18)$$

since  $\mathcal{S} = C_0 + i/g_s$ , we can take the imaginary part of (6.18) and obtain

$$\frac{1}{g_s} = \frac{\text{Im}(\gamma)(a^H b^F - a^F b^H)}{(a^H + b^H \text{Re}(\gamma))^2 + (b^H \text{Im}(\gamma))^2} = \frac{\text{Im}(\gamma) N_{\text{flux}}/k}{(a^H + b^H \text{Re}(\gamma))^2 + (b^H \text{Im}(\gamma))^2}. \quad (6.19)$$

The denominator in the right-hand side of (6.19) cannot be made arbitrarily small when varying the integers  $(a^H, b^H)$ , so we end up with the relation advertised in (6.17)

$$g_{s,\text{min}} = \frac{1}{C N_{\text{flux}}}, \quad (6.20)$$

where the value of  $C$  depends on  $\gamma$  and  $k$  for each orbifold and are given in table 6.7. Note that the sign of  $C$  is the sign of  $\text{Im}(\gamma)/k$ , and we can see from table 6.3 that  $C < 0$  for  $T^6/\mathbb{Z}_3$ . Since  $g_s > 0$ , it implies  $N_{\text{flux}} < 0$  for this orbifolds, but  $N_{\text{flux}}$  is also supposed to be positive. The logical conclusion is that there are no physical vacua with  $\nabla_S W = 0$  for  $T^6/\mathbb{Z}_3$ , so this orbifold is not included in table 6.7.

orbifold	$1/C$	orbifold	$1/C$
$\mathbb{Z}_{6, Ia}$	$2\sqrt{3}$	$\mathbb{Z}_2 \times \mathbb{Z}_{6, II}$	$2\sqrt{3}$
$\mathbb{Z}_{6, Ib}$	$2\sqrt{3}$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$2\sqrt{3}$
$\mathbb{Z}_7$	$2\sqrt{7}$	$\mathbb{Z}_3 \times \mathbb{Z}_6$	$2\sqrt{3}$
$\mathbb{Z}_{8, Ia}$	8	$\mathbb{Z}_4 \times \mathbb{Z}_4$	4
$\mathbb{Z}_{8, Ib}$	4	$\mathbb{Z}_6 \times \mathbb{Z}_6$	$2\sqrt{3}$
$\mathbb{Z}_{12, Ia}$	$2\sqrt{3}$		
$\mathbb{Z}_{12, Ib}$	$2\sqrt{3}$		

Table 6.7: Values of  $C$  in the relation (6.20) for toroidal orbifold with  $h^{2,1} = 0$ .

## Tadpole constraint

Let us consider the tadpole constraint for the orbifolds with  $h^{2,1} = 0$  listed in table 6.7. According to the discussion around equation (6.14), the flux number  $\tilde{N}_{\text{flux}}$  entering in (6.13) is a multiple of  $2|\Gamma|$  because of the quantisation of the orientifold flux integers. But as we saw in equation (6.10), the untilded  $N_{\text{flux}}$  is also a multiple of  $k$  as a consequence of the geometry. As a result,  $\tilde{N}_{\text{flux}}$  is a multiple of  $2k|\Gamma|$ , and this turns out to be very restrictive.

For example, in the orientifold  $T^6/\mathbb{Z}_{6,1a}$ , we have  $k = 3$  and  $|\Gamma| = 6$ , so  $\tilde{N}_{\text{flux}}$  is a multiple of  $2 \times 3 \times 6 = 36$ . In table 6.5, we see that this orientifold has  $N_{O3} = 14$ . The tadpole constraint (6.13) imposes that  $\tilde{N}_{\text{flux}} \leq N_{O3}/2 = 7$ , which is impossible with  $\tilde{N}_{\text{flux}} \in 36\mathbb{N}^*$ . The same conclusion is reached, in fact, for all of the orientifolds with  $h^{2,1} = 0$ .

In other words, the tadpole constraint forbids to turn-on fluxes on these orientifolds, and the classical flux superpotential cannot be used to stabilise the dilaton. Consequently, if we want to use these orientifolds for phenomenological purposes, we need to invoke other mechanisms to stabilise the dilaton or introduce negative  $D3$ -brane charge.

## Number of flux vacua

On the other hand, it seems that every choice of fluxes gives a different solution (6.18). However, this is only the case if we forget about the  $\mathcal{S}$ -duality of the type IIB string theory, introduced in (5.28)

$$\begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \quad \text{and} \quad \mathcal{S} \rightarrow \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \quad (6.21)$$

with  $(a, b, c, d)$  integers such that  $ad - bc = 1$ . As we explained around (5.21), these transformations form the group  $\text{SL}(2, \mathbb{Z})$ , and are all generated by two elementary transformations

$$\begin{aligned} \mathcal{S} &\rightarrow \mathcal{S} + 1 & \text{and} & & (H_3, F_3) &\rightarrow (H_3, F_3 - H_3) \\ \mathcal{S} &\rightarrow -1/\mathcal{S} & \text{and} & & (H_3, F_3) &\rightarrow (F_3, -H_3) \end{aligned}$$

In particular,  $\mathcal{S}$ -duality leave  $N_{\text{flux}}$  invariant

$$N_{\text{flux}} = m^H \cdot n^F - m^F \cdot n^H \quad \rightarrow \quad (ad - bc)N_{\text{flux}} = N_{\text{flux}}. \quad (6.22)$$

As in (5.22), a  $\mathcal{S}$ -duality transformations can be used to bring the dilaton  $\mathcal{S}$  into its fundamental domain

$$\mathcal{F} = \{-1/2 < |\text{Re}(\mathcal{S})| \leq 1/2 \quad \text{and} \quad |\mathcal{S}| \geq 1\}, \quad (6.23)$$

this transformation also affects the integers.

It is interesting to verify if different choices of integers  $(a^H, b^H, a^F, b^F)$  really lead to different vacua, after mapping  $\mathcal{S}$  to its fundamental domain by means of  $\mathcal{S}$ -duality. To illustrate, let us set  $\gamma = 1 + i$ , even if it does not correspond to a particular orbifold in table 6.3. If we restrict ourselves to  $N_{\text{flux}} = 1$  and integers  $|a|, |b| \leq 1$ , we can obtain 20 combinations that

lead to  $\mathcal{S}$  stabilised with  $\text{Im}(\mathcal{S}) \neq 0$ . Among these 20 combinations, some do not give a value of  $\mathcal{S}$  in the fundamental domain, for instance

$$(a^H, b^H, a^F, b^F) = (1, 1, 0, 1) \quad \rightarrow \quad \mathcal{S} = -\frac{3}{5} + \frac{i}{5},$$

but by a  $\mathcal{S}$ -duality transformation with parameters  $(a, b, c, d) = (2, 1, 1, 1)$ , we can map this  $\mathcal{S}$  to its representative in  $\mathcal{F}$ , which turns out to be  $i$ . This  $\mathcal{S}$ -duality transformation brings the flux integers to  $(a^H, b^H, a^F, b^F) = (1, 0, -1, 1)$ .

We find that all of the 20 combinations with  $N_{\text{flux}} = 1$  and integers  $|a|, |b| \leq 1$  have the same representative  $\mathcal{S} \sim i$  in the fundamental domain. In fact, even with  $|a|, |b| > 1$  and as far as we checked, all integer combinations with  $N_{\text{flux}} = 1$  lead to  $\mathcal{S} \sim i$ . So after taking  $\mathcal{S}$ -duality into account, there seems to be only one class of vacua with  $N_{\text{flux}} = 1$ . By class of vacua we mean a set of vacua where the vevs of  $\mathcal{S}$  can be mapped to each other by  $\mathcal{S}$ -duality. The class of vacua with  $N_{\text{flux}} = 1$  is degenerate in the sense that there is an infinity of non  $\mathcal{S}$ -dual combination of integers leading to it.

$N_{\text{flux}}$	1	2	3	4	5	6	7	8	9	10
# of vacua	1	2	2	4	3	6	4	8	7	8

Table 6.8: Number of classes of vacua for the toy orbifold with  $h^{2,1} = 0$  and  $\gamma = 1 + i$

For  $N_{\text{flux}} \geq 1$ , we obtain new classes of vacua, but they still come in finite number for each value of  $N_{\text{flux}}$ . In table 6.8, we give the number of classes of vacua found for the first values of  $N_{\text{flux}}$  for the toy orbifold under consideration, with  $\gamma = 1 + i$ . In figure 6.1, for illustration, we also plot the location of the vacua in the  $\mathcal{S}$ -plane for  $N_{\text{flux}} = 8$ .

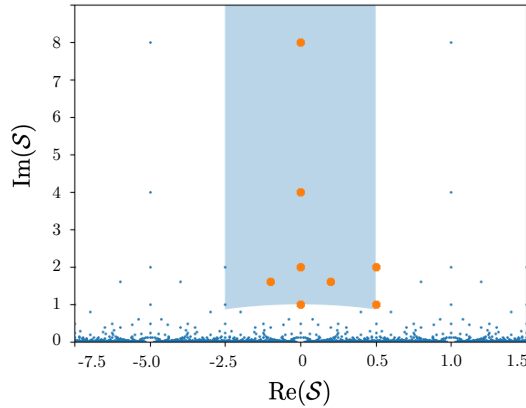


Figure 6.1: Location of the vacua in the  $\mathcal{S}$  plane for  $N_{\text{flux}} = 8$ . Each blue dot corresponds to a different combination of integers. The corresponding values of  $\mathcal{S}$  are all mapped to one of the eight orange dot in the fundamental domain (shaded region) under  $\mathcal{S}$ -duality. Note that all of the eight inequivalent vacua are found within the range  $|a|, |b| \leq 4$



## 6.2.2 Orbifolds with $h^{2,1} = 1$

In toroidal orbifolds with one complex structure modulus, we have parameterised the flux superpotential as (6.11)

$$W = \mathcal{U}A + B \quad \text{where} \quad A = A^F + \mathcal{S}A^H \quad \text{and} \quad B = B^F + \mathcal{S}B^H, \quad (6.24)$$

and the  $(A, B)$  as a function of the flux integers can be found in table 6.4. We also have  $N_{\text{flux}} = k(\text{Re}(A^H \bar{B}^F) - \text{Re}(A^F \bar{B}^H))$ , see (6.12). Solving  $\nabla_{\mathcal{U}}W = 0$  with this  $W$  yields

$$\bar{\mathcal{U}} = -\frac{\mathcal{S}B^H + B^F}{\mathcal{S}A^H + A^F}, \quad (6.25)$$

reinjecting this solution in  $\nabla_{\mathcal{S}}W = 0$ , we get  $\mathcal{S}$  as the solution of a second order equation

$$(A^H \bar{B}^H - \bar{A}^H B^H)\mathcal{S}^2 + (A^F \bar{B}^H - \bar{A}^F B^H + A^H \bar{B}^F - \bar{A}^H B^F)\mathcal{S} + (A^F \bar{B}^F - \bar{A}^F B^F) = 0. \quad (6.26)$$

Due to the fact that  $z_1 \bar{z}_2 - z_2 \bar{z}_1 = 2i\text{Im}(z_1 \bar{z}_2)$ , the coefficients of the equation are all imaginary, and we directly obtain the imaginary part of  $\mathcal{S}$

$$\frac{1}{g_s} = \frac{\sqrt{4\text{Im}(A^H \bar{B}^H)\text{Im}(A^F \bar{B}^F) - (\text{Im}(A^H \bar{B}^F) + \text{Im}(A^F \bar{B}^H))^2}}{2\text{Im}(A^H \bar{B}^H)}, \quad (6.27)$$

if the argument of the square root is positive; otherwise, the imaginary part of  $\mathcal{S}$  vanishes. Contrary to the case of orbifolds with  $h^{2,1} = 0$ , it is rather hard to identify  $N_{\text{flux}}$  in (6.27).

### Number of flux vacua

Let us verify whether the apparently infinite number of flux integer choices leads to an infinite number of classes of vacua, with dualities taken into account. Again, by class of vacua, we mean a set of vacua where the vevs of  $(\mathcal{S}, \mathcal{U})$  can be mapped to each other by dualities. These dualities still include the  $\mathcal{S}$ -duality (6.21), under which  $N_{\text{flux}}$  and  $\mathcal{U}$  are invariant, as can be checked explicitly by transforming its expression in (6.25).

In addition, the superpotential (6.24) is invariant under  $\mathcal{S} \leftrightarrow \mathcal{U}$  and  $A^F \leftrightarrow B^H$ , which translates into an invariance under the following  $\mathcal{U}$ -duality transformation

$$\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad \mathcal{U} \rightarrow \frac{a\mathcal{U} + b}{c\mathcal{U} + d} \quad (6.28)$$

Just as  $N_{\text{flux}}$  and  $\mathcal{U}$  are invariant under  $\mathcal{S}$ -duality, we can check using the previous expressions that  $N_{\text{flux}}$  and  $\mathcal{S}$  are invariant under this  $\mathcal{U}$ -duality.

We can thus bring  $\mathcal{S}$  and  $\mathcal{U}$  into the fundamental domain with duality transformations. As before, we find a finite number of classes of vacua for each value of  $N_{\text{flux}}$ . We proceed in the following way. For each value of  $N_{\text{flux}}$ , we fix the range  $k$  of the integers and build all the combinations leading to the value  $N_{\text{flux}}$  within this range. For each combination of integers, we compute  $(\mathcal{S}, \mathcal{U})$  given by (6.25) and (6.26). We then bring these values to the fundamental domain (6.23) using  $\mathcal{S}$  and  $\mathcal{U}$ -duality transformations (6.21) and (6.28), and

finally count the number of distinct pairs  $(\mathcal{S}, \mathcal{U})$  obtained in this way, i.e. the number of classes of vacua. By increasing the value of the range  $k$ , we observe that this number stops increasing after a given value, which we interpret as a sign that we have found all the classes of vacua for this value of  $N_{\text{flux}}$ . We will show this more explicitly in table 6.12 for  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ .

In table 6.9, we give the number of classes of vacua found in this way for  $T^6/\mathbb{Z}_{4,a}$ . The factor  $1/4$  that appears next to  $N_{\text{flux}}$  in this table is the parameter  $l$  in the table 6.4, coming only from the geometry of the orbifold, and not from the additional quantisation of the flux integers discussed around (6.14).

$N_{\text{flux}}/4$	1	2	3	4	5	6	7	8	9	10
# of vacua	0	1	3	9	20	37	66	104	159	226
$1/g_{s,\text{min}}^2$	–	0.75	2	3.75	6	8.75	12	15.75	20	24.75

Table 6.9: number of classes of vacua and  $1/g_{s,\text{min}}^2$  as a function of  $N_{\text{flux}}/4$  for  $T^6/\mathbb{Z}_{4,a}$

Hence, the boundedness of  $g_s$  announced in (6.17), is a byproduct of the finite number of classes of vacua. In table 6.9, we also gave the values of  $1/g_{s,\text{min}}^2$  as a function of  $N_{\text{flux}}$ . From there, we can guess that

$$g_{s,\text{min}} = \sqrt{\frac{64}{N_{\text{flux}}^2 - 16}}, \quad (6.29)$$

realising the relation (6.17) for the orbifold  $T^6/\mathbb{Z}_{4,a}$ . The result is in fact stronger than announced, because the relation (6.29) is exact and holds for every value of  $N_{\text{flux}}$ , at least as far as we checked. Similar finiteness results and relations can be obtained for other orbifolds. It turns out that we can always parameterise  $g_{s,\text{min}}$  as follows

$$g_{s,\text{min}} = \left( \frac{N_{\text{flux}}^2}{c_2} - \frac{1}{c_0} \right)^{-1/2}. \quad (6.30)$$

The values of the parameters  $c_i$ , for the orbifolds that we considered are given in table 6.10. Note the absence of the orbifolds  $T^6/\mathbb{Z}_{6,IIb}$  and  $T^6/\mathbb{Z}_{8,IIa}$  and  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_4$  from this table. A glance at table 6.4 shows us that they are the ones with  $k < 0$ , where  $k$  is defined in (6.12). These orbifolds suffer from the same issue as the orbifold  $T^6/\mathbb{Z}_3$  explained after (6.20), we found that they do not admit solutions of  $\nabla_i W = 0$  with  $N_{\text{flux}} > 0$ .

## Tadpole constraint

Let us now consider the tadpole constraint. As we already discussed, the  $\tilde{N}_{\text{flux}}$  in (6.13) is a multiple of  $2|\Gamma|$ . For instance, in the orientifold  $T^6/\mathbb{Z}_{4,a}$ , we have  $l = 4$  and  $|\Gamma| = 4$ , so  $\tilde{N}_{\text{flux}}$  is a multiple of  $2 \times 4 \times 4 = 32$ . In table 6.5, we see that this orientifold has  $N_{O3} = 22$ . So the tadpole constraint (6.13) imposes  $\tilde{N}_{\text{flux}} \leq N_{O3}/2 = 11$ , which is impossible with  $\tilde{N}_{\text{flux}} \in 32\mathbb{N}^*$ . The tadpole constraint cannot be satisfied for this orientifold without negative contributions.

The same goes for all the orientifolds of table 6.10, except for  $T^6/\mathbb{Z}_{4,c}$  and  $T^6/\mathbb{Z}_{6,IIId}$ . By the same reasoning, the former has  $\tilde{N}_{\text{flux}}$  multiple of 16 and  $\leq 20$ , while the latter has  $\tilde{N}_{\text{flux}}$  multiple of 12 and  $\leq 12$ . However, we still do not find vacua with  $\tilde{N}_{\text{flux}}$  below this bound, because there are generically no vacua for the first values of  $N_{\text{flux}}$ , as we see in table 6.9.

orbifold	$l$	$c_2$	$c_0$
$\mathbb{Z}_{4,a}$	4	64	4
$\mathbb{Z}_{4,b}$	2	16	1 for $N_{\text{flux}}/l$ even 2 for $N_{\text{flux}}/l$ odd
$\mathbb{Z}_{4,c}$	2	16	4
$\mathbb{Z}_{6,IIa}$	6	48	4 for $N_{\text{flux}}/l$ even 4/3 for $N_{\text{flux}}/l$ odd
$\mathbb{Z}_{6,IIc}$	2	48	4 for $N_{\text{flux}}/l \cong 0 \pmod{6}$ 12 for $N_{\text{flux}}/l \cong 1$ or $5 \pmod{6}$ 3 for $N_{\text{flux}}/l \cong 2$ or $4 \pmod{6}$ 4/3 for $N_{\text{flux}}/l \cong 3 \pmod{6}$
$\mathbb{Z}_{6,II d}$	1	12	1 for $N_{\text{flux}}/l \cong 0 \pmod{3}$ 3 for $N_{\text{flux}}/l \cong 1$ or $2 \pmod{3}$
$\mathbb{Z}_{8,II b}$	2	32	4 for $N_{\text{flux}}/l \cong 0 \pmod{4}$ 8 for $N_{\text{flux}}/l \cong 1$ or $3 \pmod{4}$ 2 for $N_{\text{flux}}/l \cong 2 \pmod{4}$
$\mathbb{Z}_{12,II}$	2	16	4
$\mathbb{Z}_2 \times \mathbb{Z}_{6,I}$	1	12	1 for $N_{\text{flux}}/l \cong 0 \pmod{3}$ 3 for $N_{\text{flux}}/l \cong 1$ or $2 \pmod{3}$

Table 6.10: values of the parameters ( $c_2, c_0$ ) in the relation (6.30) between  $g_{s,\text{min}}$  and  $N_{\text{flux}}$  for the toroidal orbifolds with one complex structure modulus of table 6.4.

### 6.3 $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , with three complex structure moduli

We now treat the case of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , which has three untwisted complex structure moduli. This orbifold also has 3 untwisted Kahler moduli and 48 twisted moduli, which are either Kahler or complex structure, depending on whether discrete torsion is absent or present, respectively, as we explained in section 6.1.3. In the case with discrete torsion, we have also shown that the 48 twisted complex structure moduli  $\mathcal{D}_{\alpha\beta}^i$  are stabilised at the orbifold point  $\mathcal{D}_{\alpha\beta}^i = 0$  when fluxes are not turned-on on the twisted 3-cycles to which they correspond.

We are thus left with the stabilisation of the three untwisted complex structure moduli  $\mathcal{U}^i$ , with  $i = 1, 2, 3$  and the axio-dilaton  $\mathcal{S}$ , with the flux superpotential (6.16)

$$W = n_0 + n_1\mathcal{U}^1 + n_2\mathcal{U}^2 + n_3\mathcal{U}^3 + m_1\mathcal{U}^2\mathcal{U}^3 + m_2\mathcal{U}^1\mathcal{U}^3 + m_3\mathcal{U}^1\mathcal{U}^2 - m_0\mathcal{U}^1\mathcal{U}^2\mathcal{U}^3, \quad (6.31)$$

where the dependence on the dilaton is contained in the  $(m_a, n_a)$  through  $m_a = m_a^F + \mathcal{S}m_a^H$ . In addition,  $N_{\text{flux}}$  is

$$N_{\text{flux}} = m^H \cdot n^F - m^F \cdot n^H, \quad (6.32)$$

the dot representing a summation over  $a = 0, \dots, 3$ . In what follows, we start by presenting an analytic solution of the system  $\nabla_i W = 0$  and  $\nabla_S W = 0$ , before showing evidence that the number of classes of vacua is finite for any given value of  $N_{\text{flux}}$  and realise the relation

$$g_{s,\text{min}} \sim \frac{16}{N_{\text{flux}}^2}. \quad (6.33)$$

Combined with the tadpole constraint on  $N_{\text{flux}}$ , this leads to  $g_s \geq 0.669$  for this orientifold.

### 6.3.1 Solutions of $\nabla_a W = 0$

With the flux superpotential (6.31), the system  $\nabla_{U^i} W = 0$  and  $\nabla_{\mathcal{S}} W = 0$  reads

$$\begin{aligned} \bar{n}_0 + \bar{n}_1 \mathcal{U}^1 + \bar{n}_2 \bar{\mathcal{U}}^2 + \bar{n}_3 \bar{\mathcal{U}}^3 + \bar{m}_1 \bar{\mathcal{U}}^2 \bar{\mathcal{U}}^3 + \bar{m}_2 \mathcal{U}^1 \bar{\mathcal{U}}^3 + \bar{m}_3 \mathcal{U}^1 \bar{\mathcal{U}}^2 - \bar{m}_0 \mathcal{U}^1 \bar{\mathcal{U}}^2 \bar{\mathcal{U}}^3 &= 0, \\ n_0 + n_1 \mathcal{U}^1 + n_2 \bar{\mathcal{U}}^2 + n_3 \bar{\mathcal{U}}^3 + m_1 \bar{\mathcal{U}}^2 \mathcal{U}^3 + m_2 \mathcal{U}^1 \mathcal{U}^3 + m_3 \mathcal{U}^1 \bar{\mathcal{U}}^2 - m_0 \mathcal{U}^1 \bar{\mathcal{U}}^2 \mathcal{U}^3 &= 0, \\ n_0 + n_1 \mathcal{U}^1 + n_2 \mathcal{U}^2 + n_3 \bar{\mathcal{U}}^3 + m_1 \mathcal{U}^2 \bar{\mathcal{U}}^3 + m_2 \mathcal{U}^1 \bar{\mathcal{U}}^3 + m_3 \mathcal{U}^1 \mathcal{U}^2 - m_0 \mathcal{U}^1 \mathcal{U}^2 \bar{\mathcal{U}}^3 &= 0, \\ \bar{n}_0 + \bar{n}_1 \mathcal{U}^1 + \bar{n}_2 \mathcal{U}^2 + \bar{n}_3 \mathcal{U}^3 + \bar{m}_1 \mathcal{U}^2 \mathcal{U}^3 + \bar{m}_2 \mathcal{U}^1 \mathcal{U}^3 + \bar{m}_3 \mathcal{U}^1 \mathcal{U}^2 - \bar{m}_0 \mathcal{U}^1 \mathcal{U}^2 \mathcal{U}^3 &= 0. \end{aligned} \quad (6.34)$$

In what follows, we look for solutions of this system where the complex structure and axio-dilaton are fully stabilised, and satisfy, in the conventions that we are using

$$\text{Im}(\mathcal{U}^i) < 0 \quad \text{and} \quad \text{Im}(\mathcal{S}) > 0. \quad (6.35)$$

The condition on  $\mathcal{S}$  simply comes from its definition as  $\mathcal{S} = C_0 + ie^{-\phi}$ , and the condition on the  $\mathcal{U}^i$  arises from asking that their kinetic term is positive definite. These conditions are also consistent with the Kahler potentials (5.41) and (6.15), namely

$$\mathcal{K} = -2 \log(-i(\mathcal{U}^1 - \bar{\mathcal{U}}^1)(\mathcal{U}^2 - \bar{\mathcal{U}}^2)(\mathcal{U}^3 - \bar{\mathcal{U}}^3)) - \log(-i(\mathcal{S} - \bar{\mathcal{S}})),$$

because they guarantee that the arguments of logarithms are positive.

The high degree of symmetry of the system (6.34) makes it solvable with simple steps. The first step is to notice that all four equations are linear on e.g.  $\mathcal{U}^1$ , so they can be written in a matrix form

$$L_k \times \begin{pmatrix} 1 \\ \mathcal{U}^1 \end{pmatrix} = 0, \quad (6.36)$$

where  $L_k$  is a matrix of size  $1 \times 2$  that depends on  $(\mathcal{U}^2, \mathcal{U}^3)$ . For instance,

$$L_1 = (\bar{n}_0 + \bar{n}_2 \bar{\mathcal{U}}^2 + \bar{n}_3 \bar{\mathcal{U}}^3 + \bar{m}_1 \bar{\mathcal{U}}^2 \bar{\mathcal{U}}^3, \quad \bar{n}_1 + \bar{m}_2 \bar{\mathcal{U}}^3 + \bar{m}_3 \bar{\mathcal{U}}^2 - \bar{m}_0 \bar{\mathcal{U}}^2 \bar{\mathcal{U}}^3).$$

With any pair of  $L_k$ , we can form a  $2 \times 2$  matrix. By virtue of (6.36), this matrix has 0 as an eigenvalue with eigenvector  $(1, \mathcal{U}^1)$ , so its determinant vanishes. One combination of such determinants turns out to be particularly useful

$$\det \begin{pmatrix} \bar{L}_1 \\ \bar{L}_4 \end{pmatrix} - \det \begin{pmatrix} L_2 \\ L_3 \end{pmatrix} = 0,$$

because the  $\mathcal{U}^2$  dependence of this combination completely factorises as  $(\mathcal{U}^2 - \bar{\mathcal{U}}^2)$ , which is nonzero thanks to (6.35), so that we obtain a second order equation on  $\mathcal{U}^3$  only

$$n_1 n_2 - m_3 n_0 + (m_0 n_0 + m_1 n_1 + m_2 n_2 - m_3 n_3) \text{Re} \mathcal{U}^3 + (m_0 n_3 + m_1 m_2) |\mathcal{U}^3|^2 = 0$$

The same equation can be obtained for  $\mathcal{U}^1$  and  $\mathcal{U}^2$  by symmetry of the system (6.34); we thus parameterise them as

$$a_i + b_i \text{Re} \mathcal{U}^i + c_i |\mathcal{U}^i|^2 = 0, \quad \mathcal{U}^i = \rho_i e^{i\theta_i}, \quad (6.37)$$

with, for instance

$$a_1 = n_2 n_3 - m_1 n_0, \quad b_1 = m_0 n_0 - m_1 n_1 + m_2 n_2 + m_3 n_3, \quad c_1 = m_0 n_1 + m_2 m_3, \quad (6.38)$$

and similarly for  $(a_2, b_2, c_2)$  and  $(a_3, b_3, c_3)$ . This equation can be solved and we get

$$x_i = \rho_i \cos \theta_i = -\frac{\text{Im}(\bar{a}_i c_i)}{\text{Im}(\bar{b}_i c_i)}, \quad \rho_i^2 = \frac{\text{Im}(\bar{a}_i b_i)}{\text{Im}(\bar{b}_i c_i)}. \quad (6.39)$$

Note that  $\mathcal{U}^i$  is uniquely determined, because  $\text{Im}\mathcal{U}^i < 0$  uniquely determines  $\theta_i$ . Moreover, for this solution to make sense, we must ensure that  $\rho_i^2 > 0$  and  $\rho_i^2 \geq x_i^2$  a posteriori.

So far, we have obtained the  $\mathcal{U}^i$  as a function of  $\mathcal{S}$ , which contained in the  $(m_a, n_a)$ . From there, we could obtain an equation on  $\mathcal{S}$  by inserting this solution in any of equations of (6.34), but there is a more convenient way of proceeding using the symmetry of the system. The idea is to rewrite the system (6.34), making the dependence in  $\mathcal{S}$  explicit and hiding the dependence in e.g.  $\mathcal{U}^1$ . In other words, we rewrite the superpotential (6.31) as

$$W = q_0 + q_1 \mathcal{S} + q_2 \mathcal{U}^2 + q_3 \mathcal{U}^3 + p_1 \mathcal{U}^2 \mathcal{U}^3 + p_2 \mathcal{S} \mathcal{U}^3 + p_3 \mathcal{S} \mathcal{U}^2 - p_0 \mathcal{S} \mathcal{U}^2 \mathcal{U}^3, \quad (6.40)$$

with

$$\begin{aligned} p_0 &= -m_1^H + m_0^H \mathcal{U}^1, & p_1 &= m_1^F - m_0^F \mathcal{U}^1, & p_2 &= n_3^H + m_2^H \mathcal{U}^1, & p_3 &= n_2^H + m_3^H \mathcal{U}^1, \\ q_0 &= n_0^F + n_1^F \mathcal{U}^1, & q_1 &= n_0^H + n_1^H \mathcal{U}^1, & q_2 &= n_2^F + m_3^F \mathcal{U}^1, & q_3 &= n_3^F + m_2^F \mathcal{U}^1. \end{aligned} \quad (6.41)$$

The system we obtain from this superpotential has exactly the same form as the previous one, with  $\mathcal{U}^1 \leftrightarrow \mathcal{S}$  and  $(m_a, n_a) \leftrightarrow (p_a, q_a)$ . We can therefore solve it in the same way to obtain  $\mathcal{S}$  as a function of  $\mathcal{U}^1$ . More precisely, we can define variables  $(a_s, b_s, c_s)$  as in (6.38)

$$a_s = q_2 q_3 - p_1 q_0, \quad b_s = p_0 q_0 - p_1 q_1 + p_2 q_2 + p_3 q_3, \quad c_s = p_0 q_1 + p_2 p_3,$$

and use these variables to write the same solution as (6.39)

$$x_s = \rho_s \cos \theta_s = -\frac{\text{Im}(\bar{a}_s c_s)}{\text{Im}(\bar{b}_s c_s)}, \quad \rho_s^2 = \frac{\text{Im}(\bar{a}_s b_s)}{\text{Im}(\bar{b}_s c_s)}, \quad \text{where } \mathcal{S} = \rho_s e^{i\theta_s}. \quad (6.42)$$

At this stage, equations (6.38)-(6.39) provide  $\mathcal{U}^1$  as a function of  $\mathcal{S}$  and (6.41)-(6.42) give  $\mathcal{S}$  as a function of  $\mathcal{U}^1$ . Combining the two, we can get an equation on  $\mathcal{S}$ . But for that, we have to make the  $\mathcal{S}$ -dependence of  $\mathcal{U}^1$  more explicit, and vice versa. Upon inspection, we find that the terms appearing in the solutions (6.39) and (6.42) all take the following form

$$\begin{aligned} \text{Im}(\bar{a}_1 b_1) &= y_s (N_{ab}^0 + N_{ab}^1 x_s + N_{ab}^2 \rho_s^2), & \text{Im}(\bar{a}_s b_s) &= y_1 (N_{ab}^0 + 2N_{ac}^0 x_1 + N_{bc}^0 \rho_1^2), \\ \text{Im}(\bar{a}_1 c_1) &= y_s (N_{ac}^0 + N_{ac}^1 x_s + N_{ac}^2 \rho_s^2), & \text{Im}(\bar{a}_s c_s) &= \frac{1}{2} y_1 (N_{ab}^1 + 2N_{ac}^1 x_1 + N_{bc}^1 \rho_1^2), \\ \text{Im}(\bar{b}_1 c_1) &= y_s (N_{bc}^0 + N_{bc}^1 x_s + N_{bc}^2 \rho_s^2), & \text{Im}(\bar{b}_s c_s) &= y_1 (N_{ab}^2 + 2N_{ac}^2 x_1 + N_{bc}^2 \rho_1^2), \end{aligned} \quad (6.43)$$

where  $(x_{s,i}, y_{s,i}, \rho_{s,i})$  are the real, imaginary parts and moduli of  $\mathcal{S}$  and  $\mathcal{U}^i$ , respectively, and the  $N$  are integers expressed as combinations of the integers  $(m_a^H, n_a^H)$  and  $(m_a^F, n_a^F)$

$$N_{ab}^0 = a_1^{FF} b_1^{HF} - a_1^{HF} b_1^{FF}, \quad N_{ab}^1 = 2(a_1^{FF} b_1^{HH} - a_1^{HH} b_1^{FF}), \quad N_{ab}^2 = a_1^{HF} b_1^{HH} - a_1^{HH} b_1^{HF},$$

and similarly for the  $N_{ac}$  and  $N_{bc}$ . Here, we used the following notation

$$a_1^{FF} = n_2^F n_3^F - m_1^F n_0^F, \quad a_1^{HH} = n_2^H n_3^H - m_1^H n_0^H, \quad a_1^{HF} = n_2^H n_3^F + n_2^F n_3^H - m_1^H n_0^F - m_1^F n_0^H,$$

and similarly for the  $b_i$  and  $c_i$ . Combining (6.39), (6.42) and (6.43), we finally obtain

$$x_s = -\frac{A_0 + A_1x_s + A_2\rho_s^2}{B_0 + B_1x_s + B_2\rho_s^2}, \quad \rho_s^2 = \frac{C_0 + C_1x_s + C_2\rho_s^2}{B_0 + B_1x_s + B_2\rho_s^2}, \quad (6.44)$$

here,  $A_i = \mathbf{A} \cdot \mathbf{N}_i$ ,  $B_i = \mathbf{B} \cdot \mathbf{N}_i$ ,  $C_i = \mathbf{C} \cdot \mathbf{N}_i$ , with

$$\mathbf{A} = \frac{1}{2}(N_{ab}^1, -2N_{ac}^1, N_{bc}^1), \quad \mathbf{B} = (N_{ab}^2, -2N_{ac}^2, N_{bc}^2), \quad \mathbf{C} = (N_{ab}^0, -2N_{ac}^0, N_{bc}^0),$$

and

$$\mathbf{N}_i = (N_{bc}^i, N_{ac}^i, N_{ab}^i).$$

The coefficients  $(A_i, B_i, C_i)$  appearing in (6.44) are integers expressed as combinations of the  $(m_a^H, n_a^H)$  and  $(m_a^F, n_a^F)$ . They can be obtained by combining all the previous formulae, but their expressions are extremely long.

The system (6.44) can be solved by using one of the equation to express  $\rho_s^2$  as a function of  $x_s$ , and inserting the result into the other equation. This yields a polynomial equation on  $x_s$  of degree at most three. We can solve this polynomial to obtain  $x_s$ , feed the result back into the system (6.44) to obtain  $\rho_s^2$ , and finally obtain the  $\mathcal{U}^i$  via (6.39). This yields a solution of the system (6.34) if the conditions  $\rho_s^2 > x_s^2$  and  $\rho_i^2 > x_i^2$  are satisfied.

However, we have found that this method of solving the system (6.44) is too naive and misses solutions. To be more accurate, let us rewrite this system as

$$\begin{aligned} B_1x_s^2 + B_2x_s\rho_s^2 + (A_1 + B_0)x_s + A_2\rho_s^2 + A_0 &= 0 \\ B_2\rho_s^4 + B_1x_s\rho_s^2 - C_1x_s + (B_0 - C_2)\rho_s^2 - C_0 &= 0. \end{aligned}$$

Under this form, we can see that the system amounts to finding the real intersections of two conic sections, which is a classical geometry problem. There can be at most four such intersections, except in the case where the two conics are degenerate and share a line. This case corresponds physically to vacua with a flat direction, which we are not interested in. As explained in [6], we implemented an algorithm to solve this problem, following [107], allowing us to find the isolated solutions of the system (6.44), when they exist.

## Comments

Let us make a few comments on the solution we have just obtained. First, the denominators in (6.39) and (6.42) must not vanish for the solution to be well-defined. In cases where a denominator vanishes, the solution breaks down, and special treatment is needed. We found that this happens when a modulus is unstabilised. For instance, if  $b_1 = 0$ , the denominator of (6.39) vanishes, but  $\text{Re}\mathcal{U}^1$  also vanishes from the equation (6.37) and  $\text{Re}\mathcal{U}^1$  is not stabilised.

Second, the system (6.44) generically has multiple solutions, so it seems that some choices of fluxes can lead to multiple vacua. However, after imposing the consistency conditions  $\rho_s^2 > x_s^2$  and  $\rho_i^2 > x_i^2$ , we did not find any choice of fluxes leading to multiple vacua.

## Dualities

In the next section, we will use dualities to see that the infinite number of flux integer choices lead to a finite number of classes of vacua. These dualities include the  $\mathcal{S}$ -duality (6.21)

$$\begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \quad \text{and} \quad \mathcal{S} \rightarrow \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \quad (6.45)$$

under which  $N_{\text{flux}}$  is unmodified, see (6.22). We can also check that the  $\mathcal{U}^i$  given by (6.39) are invariant. This is of course expected, since the equations (6.34) are themselves invariant, but this is a non-trivial consistency check.

Now recall that for orbifolds with one complex structure modulus, in section 6.2.2, the superpotential was invariant under  $\mathcal{S} \leftrightarrow \mathcal{U}$  and  $A^F \leftrightarrow B^H$ , leading to invariance under the  $\mathcal{U}$ -duality transformation (6.28) acting on  $\mathcal{U}$ . In the present case of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , we exhibited an invariance of the superpotential under  $\mathcal{U}^1 \leftrightarrow \mathcal{S}$  and  $(m_a, n_a) \leftrightarrow (p_a, q_a)$ , see (6.40). This leads to invariance under a  $\mathcal{U}^1$ -duality acting on  $\mathcal{U}^1$  as in (6.28) and leaving  $(\mathcal{S}, \mathcal{U}^2, \mathcal{U}^3)$  invariant. Similarly, there are  $\mathcal{U}^2$  and  $\mathcal{U}^3$ -dualities acting only on  $\mathcal{U}^2$  and  $\mathcal{U}^3$  respectively. Using these dualities, we can bring  $\mathcal{S}$  and all of the  $\mathcal{U}^i$  to their fundamental domain independently

$$\mathcal{F} = \{-1/2 < |\text{Re}(\mathcal{A})| \leq 1/2 \quad \text{and} \quad |\mathcal{A}| \geq 1\} \quad \text{with} \quad \mathcal{A} = \mathcal{S} \text{ or } \mathcal{U}^i, \quad (6.46)$$

### 6.3.2 Finite number of vacua

The superpotential (6.31) of  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  depends on 16 flux integers  $(m_a^H, n_a^H)$  and  $(m_a^F, n_a^F)$ . Consequently the number of combinations for given  $N_{\text{flux}}$  grows rapidly with the range  $k$  of the integers. The number of independent integers and of combinations can be reduced if we impose symmetries among the three tori of  $T^6 = T^2 \times T^2 \times T^2$ , as shown in table 6.11.

$ m ,  n  \leq k$	number of flux integer combinations, with		
	three eq. tori	two eq. tori	no symmetry
1	416	21492	975968
2	18276	14325076	8726828016
3	99428	386921556	1099101964400
4	622732	5566156388	...
5	1999388	37625301028	
6	4905948	213491079460	
7	11893404	...	

Table 6.11: number of combinations giving  $N_{\text{flux}} = 4$ , as a function of the range  $k$ .

The numbers in table 6.11 can be further reduced by taking into account symmetries of the system. For instance, (6.34) is invariant if we reverse the sign of all the integers at once. In any case, these numbers show that exhaustive algorithmic explorations are challenging. Nevertheless, we will show evidence that the number of classes of vacua for fixed  $N_{\text{flux}}$  is finite, and that they realise the relation (6.33).

Let us start with the symmetric case where the three tori are equivalent. In this case, it is possible to compute the solution presented in section 6.3.1 for every combination of integers, with given  $N_{\text{flux}}$  and range  $k$ , for values of  $k$  up to  $k \sim 30$ . We then bring the values of both  $\mathcal{S}$  and  $\mathcal{U}^i$  given by this solution in the fundamental domain (6.46) using the  $\mathcal{S}$  and  $\mathcal{U}^i$ -dualities introduced above. Finally, we count the number of distinct pairs  $(\mathcal{S}, \mathcal{U}^i)$  obtained in this way, which we take to be the number of classes of vacua. We report these numbers in table 6.12.

$ m ,  n  \leq k$	$N_{\text{flux}}$	(in the case with three equivalent tori)									
		1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	5	7	1	5	1	0	0
2	2	0	0	1	6	13	28	56	82	101	114
3	3	0	0	1	6	13	30	65	102	168	234
4	4	0	0	1	6	13	30	65	102	171	252
5	5	0	0	1	6	13	30	65	102	171	252
6	6	0	0	1	6	13	30	65	102	171	252

Table 6.12: number of classes of vacua for  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , as a function of  $N_{\text{flux}}$  and the range  $k$  of the integers. In red are the values of  $k$  from which now new vacua are found.

In table 6.12, we observe the behaviour mentioned in section 6.2.2. For fixed  $N_{\text{flux}}$ , if we gradually increase the range  $k$  of the integers, there is a value beyond which no new vacua are found. The table stops at  $k = 6$ , but we checked beyond, up to  $k = 30$  for  $N_{\text{flux}} = 4$ . This observation is the basis of our claim that we have found all the vacua for these  $N_{\text{flux}}$ . Note that all the vacua are often found with small ranges of integers. Note also that the duality transformations used to bring the  $\mathcal{S}$  and  $\mathcal{U}^i$  in the fundamental domain also affect the integers and can modify their range. In our definition,  $k$  is the range of the integers we start with, not the one obtained after the duality transformations.

Hence, the boundedness of  $g_s$  announced in (6.33), is a byproduct of the finiteness of the number of classes vacua. In table 6.13, we give the values of  $1/g_{s,\text{min}}$  as a function of  $(N_{\text{flux}}, k)$ .

$ m ,  n  \leq k$	$N_{\text{flux}}$	(in the case with three equivalent tori)							
		3	4	5	6	7	8	9	10
1	1	none	1.148	1.263	1.520	1.990	1.000	none	none
2	2	0.866	1.495	2.076	2.632	3.056	4.000	2.760	3.331
3	3	0.866	1.495	2.076	2.632	3.175	4.000	5.061	6.249
4	4	0.866	1.495	2.076	2.632	3.175	4.000	5.061	6.249
5	5	0.866	1.495	2.076	2.632	3.175	4.000	5.061	6.249

Table 6.13: values of  $1/g_{s,\text{min}}$  for  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  as a function of  $N_{\text{flux}}$  and the range  $k$  of the integers, for three equivalent tori. In red are the values of  $k$  from which the  $g_{s,\text{min}}$  are found.

For  $N_{\text{flux}} \in 4\mathbb{N}$  and  $N_{\text{flux}} \geq 8$ , the relation (6.33) holds exactly, as we verified up to  $N_{\text{flux}} = 28$

$$g_{s,\text{min}} = \frac{16}{N_{\text{flux}}^2} \quad \text{for } N_{\text{flux}} \in 4\mathbb{N} \quad \text{and} \quad N_{\text{flux}} \geq 8 \quad (6.47)$$

For other values of  $N_{\text{flux}}$ , this formula does not hold exactly, but still gives a good overall fit, except for the very first values of  $N_{\text{flux}}$ , as we can see in figure 6.2.



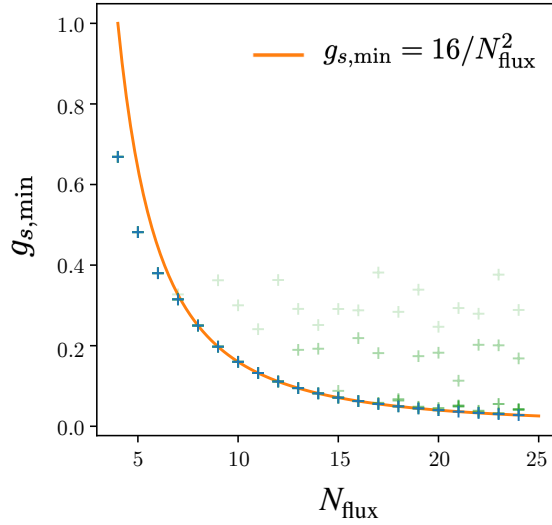


Figure 6.2: the blue points are the  $g_{s,\min}$  as a function of  $N_{\text{flux}}$ , found as explained in the text. The orange curve is the relation  $g_{s,\min} = 16/N_{\text{flux}}^2$ . The green shaded points are the  $g_{s,\min}$  with lower ranges of integers. Increasing the range, the points converge towards the orange curve.

So far, we discussed the case where the three tori are equivalent. Let us now relax this assumption and consider the case where only two tori are equivalent. In this case, we still expect to find a finite number of classes of vacua for fixed  $N_{\text{flux}}$ , but this number should be higher, since less symmetry allows more freedom. We report these numbers in table 6.14.

$ m ,  n  \leq k$	$N_{\text{flux}}$	(in the case with only two equivalent tori)									
		1	2	3	4	5	6	7	8	9	10
1	1	0	0	1	9	66	16	17	1	0	0
2	2	0	0	1	11	94	408	1182	2730	4891	6899
3	3	0	0	1	11	94	412	1249	3051	6881	13388

Table 6.14: number of classes of vacua for  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , as a function of  $N_{\text{flux}}$  and the range  $k$  of the integers. In red are the values of  $k$  from which no new vacua are found.

In principle, the values of  $g_{s,\min}$  can also differ from the more symmetric case. As we can see in table 6.15, this is the case for some values of  $k$ , but as far as we explored the absolute  $g_{s,\min}$  as a function of  $N_{\text{flux}}$  remain the same as in the symmetric case in table 6.13.

$ m ,  n  \leq k$	$N_{\text{flux}}$	(in the case with only two equivalent tori)							
		3	4	5	6	7	8	9	10
1	1	0.866	1.148	1.654	1.925	1.990	1.000	none	none
2	2	0.866	1.495	2.076	2.632	3.056	4.000	5.032	5.396
3	3	0.866	1.495	2.076	2.632	3.175	4.000	5.061	6.249

Table 6.15: values of  $1/g_{s,\min}$  for  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  as a function of  $N_{\text{flux}}$  and the range  $k$  of the integers, for two equivalent tori. In red are the values of  $k$  from which the  $g_{s,\min}$  are found.

From there, it is easy to believe that the relation between  $g_{s,\min}$  and  $N_{\text{flux}}$  remains unchanged even when there is no symmetry among the three tori, even if the number of combinations of integers is way too large to verify this in the same way.

### Tadpole constraint

Let us now consider the tadpole constraint (6.13). As we discussed there,  $\tilde{N}_{\text{flux}}$  is a multiple of  $2|\Gamma| = 8$ , and the orientifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  has  $N_{O_3} = 64$ , see table 6.5. So the tadpole constraint imposes  $\tilde{N}_{\text{flux}} \leq N_{O_3}/2 = 32$  and the  $N_{\text{flux}}$  computed by (6.32) and used throughout this section verifies

$$N_{\text{flux}} \leq 4 \quad \rightarrow \quad g_s \geq \frac{1}{5^{1/4}} \simeq 0.669, \quad (6.48)$$

where  $5^{1/4} \simeq 1.495$  is the exact value found in tables 6.13 and 6.15 for  $N_{\text{flux}} = 4$ . This lower bound on the string coupling does not allow us to neglect string loop corrections. We will see in section 6.3.4 how this conclusion can be mitigated by considering negative contribution to (6.13) induced by magnetised  $D7$ -branes.

### Value of $W_0$

As we saw in the end of chapter 5, the vacuum expectation value  $W_0$  of the flux superpotential resulting from the stabilisation of the complex structure moduli and axio-dilaton is an important parameter in the KKLT and other scenarios. So let us show the values we found with our procedure. In fact, we are not going to show the values of  $W_0$ , but the values of

$$\tilde{W}_0 = e^{\tilde{\mathcal{K}}/2} W_0, \quad (6.49)$$

where  $\tilde{\mathcal{K}}$  is the Kahler potential (5.41) of the complex structure moduli and axio-dilaton

$$\tilde{\mathcal{K}} = -\log(-i(\mathcal{U}^1 - \bar{\mathcal{U}}^1)(\mathcal{U}^2 - \bar{\mathcal{U}}^2)(\mathcal{U}^3 - \bar{\mathcal{U}}^3)) - \log(-i(\mathcal{S} - \bar{\mathcal{S}})).$$

The reason is simply that (6.49) is invariant under the Kahler transformations introduced in chapter 3, see (3.59). It is related to the mass parameter  $m_{3/2}$  of the gravitino, see (3.66). It is also invariant under the  $\mathcal{S}$  and  $\mathcal{U}$ -dualities described above, see (6.45), as can be checked.

In figure 6.3, we show the distributions of  $\tilde{W}_0$  for our vacuum solutions with three equivalent tori and  $N_{\text{flux}} \leq 9$ . In this plot we did not bring the moduli to their fundamental domains. We see that the values of  $\tilde{W}_0$  are organised in the complex plane into circles of constant  $|\tilde{W}_0|$ , corresponding to different orbits of these dualities. The values of  $|\tilde{W}_0|$  are discrete, which is again a consequence of the finiteness of flux vacua. In particular, there is no parametric control over the value of  $\tilde{W}_0$ , and it is not possible to make it exponentially small, as required in the KKLT scenario, without taking additional ingredients into account.

The number of circles approximatively matches the number of vacua found in table 6.12. There is, however, a slight discrepancy, as some distinct classes of vacua give the same value of  $|\tilde{W}_0|$ . In other words, some of the circles correspond to superposed duality orbits.

Finally, note that for three equivalent tori, we obtain solutions with a vanishing  $\tilde{W}_0 = 0$  only for  $N_{\text{flux}}$  multiple of 3. These are the solutions studied in detail in [105].

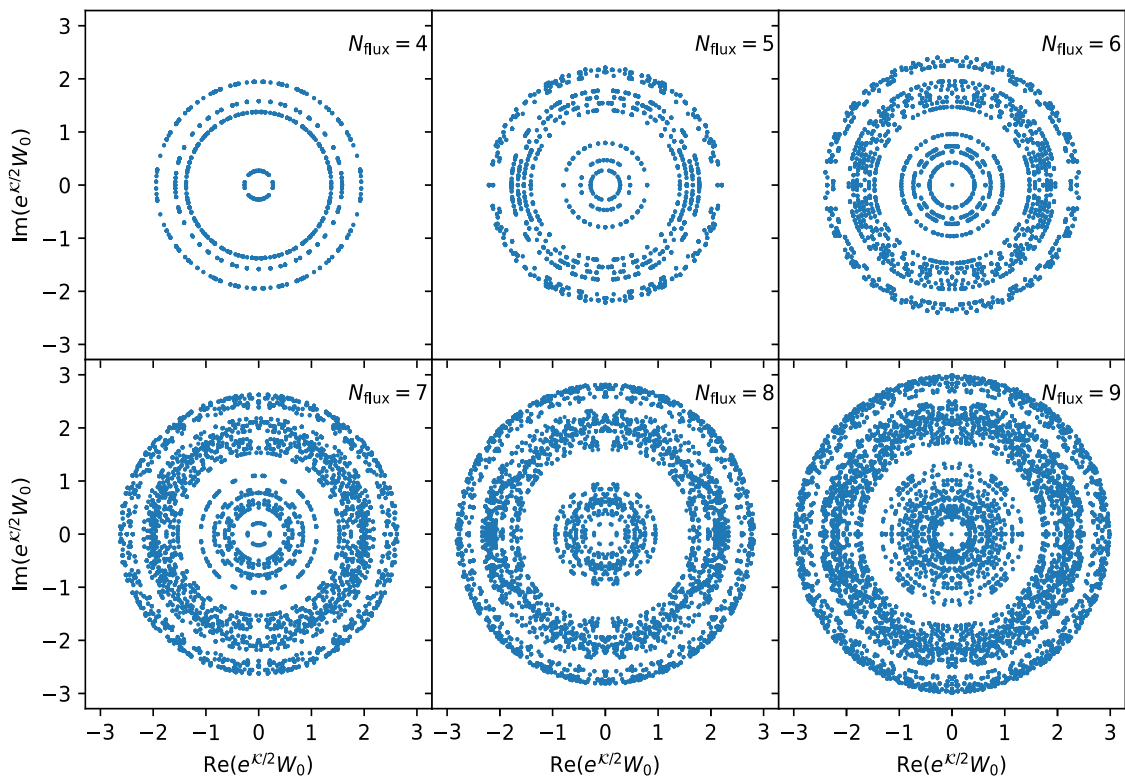


Figure 6.3: Distributions of  $\tilde{W}_0$  in  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  with three equivalent tori, for  $N_{\text{flux}} = 4, \dots, 9$ .

### 6.3.3 Parametric control attempts

The value  $g_s \simeq 0.669$  is the minimal string coupling allowed for the orientifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , if moduli are stabilised only by fluxes at tree level and in absence of negative  $D3$ -brane charge. This is a consequence of the finiteness of the number of flux vacua that we introduced in chapter 5 and explored in section 6.3.2. More generally, finiteness forbids parametric control. By parametric control, we mean tuning the flux integers such that an integer  $n$  or a combination of them does not contribute to  $N_{\text{flux}}$  in (6.32) and can be sent to infinity, and finding a physical solution where  $g_s \sim 1/n$ . If such a solution exists, we should be able to solve the system order by order in  $n$ , and it would provide an infinite family of flux vacua.

In this section, we present some unsuccessful attempts for parametric control and show different ways in which obstructions appear at the level of the SUSY equations  $\nabla_a W = 0$ . Expanded into real and imaginary parts, these equations are equivalent to

$$\begin{aligned}
& n_1^F x_1 + n_2^F x_2 + n_3^F x_3 + m_1^F x_2 x_3 + m_2^F x_1 x_3 + m_3^F x_1 x_2 \\
& + x_s (n_1^H x_1 + n_2^H x_2 + n_3^H x_3 + m_1^H x_2 x_3 + m_2^H x_1 x_3 + m_3^H x_1 x_2) \\
& + n_0^F + n_0^H x_s - m_0^F x_1 x_2 x_3 - m_0^H x_1 x_2 x_3 x_s + m_0^H y_1 y_2 y_3 y_s = 0,
\end{aligned} \tag{6.50}$$

$$\begin{aligned}
& n_1^H x_1 y_s + n_2^H x_2 y_s + n_3^H x_3 y_s + m_1^H x_2 x_3 y_s + m_2^H x_1 x_3 y_s \\
& + m_3^H x_1 x_2 y_s + n_0^H y_s - m_0^F y_1 y_2 y_3 - m_0^H x_s y_1 y_2 y_3 - m_0^H x_1 x_2 x_3 y_s = 0,
\end{aligned} \tag{6.51}$$

$$\begin{aligned}
& m_1^F y_2 y_3 + m_1^H x_s y_2 y_3 - m_2^H x_3 y_1 y_s - m_3^H x_2 y_1 y_s - n_1^H y_1 y_s \\
& - m_0^F x_1 y_2 y_3 - m_0^H x_1 x_s y_2 y_3 + m_0^H x_2 x_3 y_1 y_s = 0,
\end{aligned} \tag{6.52}$$

$$\begin{aligned}
& n_1^F y_1 + m_2^F x_3 y_1 + m_3^F x_2 y_1 + n_1^H x_s y_1 + m_1^H y_2 y_3 y_s + m_2^H x_3 x_s y_1 \\
& + m_3^H x_2 x_s y_1 - m_0^F x_2 x_3 y_1 - m_0^H x_2 x_3 x_s y_1 - m_0^H x_1 y_2 y_3 y_s = 0.
\end{aligned} \tag{6.53}$$

and cyclic permutations of these two last equations.

### With $y_s \rightarrow +\infty$ and one large integer

The simplest possibility we can consider are limits where only one integer  $m$  or  $n$  goes to infinity, with  $y_s \sim m$  or  $n$  and  $x_s, x_i, y_i \sim \mathcal{O}(1)$ . For instance  $n_0^F \rightarrow +\infty$ . In this case (6.50) becomes, at leading order,  $n_0^F + m_0^H y_1 y_2 y_3 y_s = 0$ . But we must also impose  $m_0^H = 0$  for  $N_{\text{flux}}$  to remain finite, so we end up with  $n_0^F = 0$ , which is contradiction with  $n_0^F \rightarrow \infty$ .

Another example is  $m_0^F \rightarrow +\infty$ , with  $n_0^H = 0$ . In this case the system (6.50)-(6.51)-(6.52)-(6.53) at leading order gives  $y_s = (m_0^F x_1 x_2 x_3) / (m_0^H y_1 y_2 y_3)$  and  $x_i^2 + y_i^2 = (m_i^H / m_0^H) x_i$ , along with the constraints  $m_0^H n_1^H + m_2^H m_3^H = 0$ , with cyclic permutations, and  $m_1^H m_2^H m_3^H = 0$ . Hence, one of the  $m_i^H$  needs to be zero, but then for the corresponding  $i$  we have  $x_i^2 + y_i^2 = 0$ , leading to a vanishing untwisted complex structure modulus, which is not physical.

### With $y_s \rightarrow +\infty$ and two large integers

A more elaborate possibility is to consider two large integers, for instance  $n_0^F, m_0^F \rightarrow +\infty$ , with  $m_0^H = n_0^H = 0$  for  $N_{\text{flux}}$  to remain finite. We can further simplify by assuming that the three tori are equivalent ( $1 = 2 = 3 = i$ ). In this case, the system (6.50)-(6.51)-(6.52)-(6.53) becomes, at leading order

$$\begin{aligned}
& n_0^F - m_0^F x_i^3 = 0, & -m_0^F y_i^3 + 3m_i^H x_i^2 y_s + 3n_i^H x_i y_s = 0, \\
& m_0^F x_i y_i^2 + 2m_i^H x_i y_i y_s + n_i^H y_i y_s = 0, & m_0^F x_i^2 y_i - m_i^H y_i^2 y_s = 0.
\end{aligned}$$

The first, third and last equations give directly

$$x_i^3 = \frac{n_0^F}{m_0^F}, \quad y_s^3 = \frac{m_0^F (n_0^F)^2}{(m_i^H)^3 y_i^3} \quad \text{and} \quad y_i^2 = -\frac{(n_0^F)^{1/3} n_i^H}{(m_0^F)^{1/3} m_i^H} - \frac{2(n_0^F)^{2/3}}{(m_0^F)^{2/3}},$$

and the second equation becomes a constraint on the integers that can be written

$$((m_0^F)^{1/3} n_i^H + (n_0^F)^{1/3} m_i^H)^2 - (m_0^F)^{1/3} (n_0^F)^{1/3} m_i^H n_i^H = 0.$$

This equation is of the form  $(x + y)^2 = xy$ , and it has no solutions with real  $x$  and  $y$ .

### With $x_s, y_s \rightarrow +\infty$ and two large integers

Finally, we can consider an even more elaborate possibility where both  $x_s$  and  $y_s$  go to infinity with two large integers, while  $x_i, y_i \sim \mathcal{O}(1)$ . Let us still take  $n_0^F, m_0^F \rightarrow +\infty$  and  $m_0^H = n_0^H = 0$ .

In this case, the system (6.50)-(6.51)-(6.52)-(6.53) becomes, at leading order

$$\begin{aligned}
3n_i^H x_i x_s + 3m_i^H x_i^2 x_s + n_0^F - m_0^F x_i^3 &= 0, \\
3n_i^H x_i y_s + 3m_i^H x_i^3 y_s - m_0^F y_i^3 &= 0, \\
m_i^H x_s y_i^2 - 2m_i^H x_i y_i y_s - n_i^H y_i y_s - m_0^F x_i y_i^2 &= 0, \\
n_i^H x_s y_i + m_i^H y_i^2 y_s + 2m_i^H x_i x_s y_i - m_0^F x_i^2 y_i &= 0.
\end{aligned} \tag{6.54}$$

The first two equations immediatly give

$$x_s = \frac{m_0^F x_i^3 - n_0^F}{3x_i(m_i^H x_i + n_i^H)} \quad \text{and} \quad y_s = \frac{m_0^F y_i^3}{3x_i(m_i^H x_i + n_i^H)},$$

and the third one

$$y_i^2 = -\frac{2m_0^F m_i^H x_i^3 + 3m_0^F n_i^H x_i^2 + n_0^F m_i^H}{2m_0^F m_i^H x_i + m_0^F n_i^H}, \tag{6.55}$$

finally, the last equation becomes a cubic equation on  $x_i$

$$\begin{aligned}
(4m_0^F n_0^F (m_i^H)^3 + 2(m_0^F)^2 (n_i^H)^3) x_i^3 + 6m_0^F n_0^F (m_i^H)^2 n_i^H x_i^2 \\
+ 6m_0^F n_0^F m_i^H (n_i^H)^2 x_i - (n_0^F)^2 (m_i^H)^3 + m_0^F n_0^F (n_i^H)^3 = 0,
\end{aligned} \tag{6.56}$$

which can be used to reexpress the  $y_i^2$  given by (6.55) as

$$y_i^2 = -\frac{3(n_0^F (m_i^H)^2 - m_0^F (n_i^H)^2 x_i)^2}{m_0^F (m_0^F (n_i^H)^3 + 2n_0^F (m_i^H)^3) (2m_i^H x_i + n_i^H)}, \tag{6.57}$$

where the numerator is manifestly positive. Consequently, if we want  $y_i^2$  to be positive, we need the denominator to be negative when  $x_i$  solves (6.56). As illustrated by figure 6.4, numerical investigations for all combinations of integers up to  $|m|, |n| = 10$  suggest that this is never the case. So it seems that there are no physical solutions to the system (6.54).

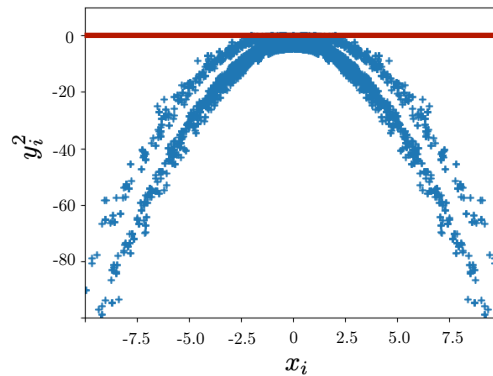


Figure 6.4:  $y_i^2$ , given by (6.57), as a function of  $x_i$  when  $x_i$  is a solution of (6.56)

## Conclusion

We considered other limits in addition to the ones presented here, and they all fail to be consistent for similar reasons. For instance they lead to the vanishing of the imaginary part of a complex structure modulus, a constraint on the integers that cannot be satisfied. There are of course too many possible limits to be exhaustive, and some of them are hard to analyse, but all hints towards the absence of solutions with parametric control over the string coupling.

This conclusion is not surprising, because it is a consequence of the finiteness of flux vacua, which has been proven in [85] as a mathematical theorem. A family of solutions with parametric control would be infinite and contradict this theorem.

### 6.3.4 Magnetised $D7$ -branes

Let us come back to the possibility of evading the conclusion (6.48) that  $g_s \geq 0.669$  in the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  orbifold, with negative  $D3$ -brane charges carried by magnetised  $D7$ -branes [108,109]. Note that negative  $D3$ -brane charges are related to supersymmetry-breaking objects. As we discuss below,  $D7$ -branes are naturally present in most toroidal orientifolds. Magnetised  $D7$ -branes also play a key role in the fully perturbative Kahler moduli stabilisation mechanism using logarithmic loop corrections [93,96].

Including magnetised  $D7$ -branes does not change the relation (6.47) between the minimal string coupling  $g_{s,\min}$  and the flux number  $N_{\text{flux}}$ , since this relation stems directly from complex structure moduli and axio-dilaton stabilisation by quantised background 3-form fluxes. Magnetised  $D7$ -branes contributing negatively to the tadpole constraint only allow larger values of  $N_{\text{flux}}$  to be used, thus reducing the value of  $g_{s,\min}$ .

### Worldvolume fluxes and RR charges

In the  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orbifold, the  $T^6$  is factorised as  $T^2 \times T^2 \times T^2$ , and a  $D7$ -brane wraps two tori and is localised in the third one. Let us label with an index  $a$  the different stacks of branes. Each stack carries magnetic fields  $F_a$  associated with  $U(1)$  gauge groups. Like 3-form fluxes, the internal magnetic fluxes are subject to the Dirac quantisation condition

$$m_a^i \int_{T^2_i} F_a^i = 2\pi n_a^i, \quad (6.58)$$

where the index  $i$  labels the two tori wrapped by the brane stack. The wrapping numbers  $m_a^i$  and flux quanta  $n_a^i$  are coprime integers. Moreover, due to the  $\mathbb{Z}_2$  orientifold quotient, the  $n_a^i$  can take half-integer values. Through the Chern-Simons couplings that are ubiquitous in string theoretic actions, as in (5.27), magnetised  $D7$ -branes induce  $D3$ -brane charge in addition to their  $D7$ -brane charge, see (5.51). Note that magnetised  $D7$ -branes can themselves be seen as magnetised  $D9$ -branes with vanishing wrapping number on the third torus, where they are localised [108]. We can thus assign vanishing wrapping numbers  $m_a^k = 0$  and unit fluxes  $n_a^k = 1$  on the torus where  $D7$ -branes are localised. For example, a stack  $a$  of  $N_a$

$D7$ -branes localised in the first torus  $T_1^2$  has magnetic numbers

$$N_a \rightarrow D7_a \quad \begin{matrix} T_1^2 & T_2^2 & T_3^2 \\ (1, 0) & (n_a^2, m_a^2) & (n_a^3, m_a^3) \end{matrix} \quad (6.59)$$

In terms of these magnetic numbers, the RR charges of a stack  $a$  of  $D7$ -branes read [108]

$$Q_{D3}^a = N_a n_a^1 n_a^2 n_a^3 \quad \text{and} \quad {}^{(i)}Q_{D7}^a = -N_a n_a^i m_a^j m_a^k \quad \text{with} \quad i \neq j \neq k \neq i, \quad (6.60)$$

where  $N_a$  is the number of branes in the stack. The stack  $a$  only has non-vanishing  $D7$ -brane charge on the torus where it is localised. For instance, for the stack (6.59), we have  ${}^{(1)}Q_{D7}^a \neq 0$  and  ${}^{(2)}Q_{D7}^a = {}^{(3)}Q_{D7}^a = 0$ .

From (6.60), we see that magnetised  $D7$ -branes can induce negative  $D3$ -brane charge, with one flux quanta  $n_a^i$  of opposite sign to the other two. For example, a single  $D7_a$  brane as in (6.59) with opposite fluxes  $n_a^2 = -n_a^3 = 1$ , induces a negative unit charge  $Q_{D3}^a = -1$ .

### Tadpole conditions with magnetised $D7$ -branes

The induced charges  $Q_{D3}^a$  are additional contributions to the tadpole condition (6.13)

$$\frac{1}{2} \tilde{N}_{\text{flux}} + \sum_a Q_{D3}^a + N_{D3} = \frac{1}{4} N_{O3}.$$

Thus, magnetised  $D7$ -branes inducing negative  $D3$ -brane charge can relax the conclusion (6.48) that  $g_s \geq 0.669$ , which was a consequence of the tadpole constraint  $N_{\text{flux}} < 4$ . However, consistency of the orientifold construction also requires the cancellation of the RR tadpoles related to  $D7$ -brane charges. This additional tadpole condition links the charges (6.60) to the negative charges of  $O7$ -planes present in the construction. As a result, for a fixed orientifold geometry, the  $D7$  magnetic fluxes  $n_i^a$  and wrapping numbers  $m_i^a$  are not arbitrary.

In the  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$  orientifold constructed with the geometric involution  $x^i \rightarrow -x^i$ , which corresponds to  $\mathcal{R} = R_i R_j R_k$  in the notations of (5.35), there are  $3 \times 4$   $O7$ -planes. Each of them wraps two tori and is localised in the third one at one of the four fixed points of the orbifold action. For instance, four  $O7_1$ -planes are localised in the first torus at  $(\iota_1, \iota_2, 0, 0, 0, 0)$  in the basis of the lattice vectors, where  $\iota_i = 0$  or  $1/2$ , a notation introduced before table 6.5.

The RR tadpole cancellation thus requires the magnetised  $D7_a$  charges (6.60) to satisfy [108]

$$\frac{1}{2} \tilde{N}_{\text{flux}} + \sum_a N_a n_a^1 n_a^2 n_a^3 + N_{D3} = \frac{1}{4} N_{O3} = 16, \quad (6.61)$$

and

$$\begin{aligned} \sum_a {}^{(1)}Q_{D7}^a &= -\sum_a N_a n_a^1 m_a^2 m_a^3 = 4N_{O7_1} = 16, \\ \sum_a {}^{(2)}Q_{D7}^a &= -\sum_a N_a m_a^1 n_a^2 m_a^3 = 4N_{O7_2} = 16, \\ \sum_a {}^{(3)}Q_{D7}^a &= -\sum_a N_a m_a^1 m_a^2 n_a^3 = 4N_{O7_3} = 16. \end{aligned} \quad (6.62)$$

These four tadpole conditions are the same as in the  $\mathcal{T}$ -dual model of [110, 111] with  $D6$ -branes at angles [112, 113].

## Solutions relaxing the constraint on $N_{\text{flux}}$

According to (6.61), stacks  $a$  with an odd number of negative magnetic fluxes  $n_a^i$  contribute negatively to the  $D3$ -brane charge and relax the constraint on  $\tilde{N}_{\text{flux}}$ . In the convention (6.59), the flux number on the torus where the stack is localised is  $n_a^k = 1$ , so to have negative contributions we need one positive and one negative flux numbers on the two other tori. Moreover, as explained under (6.60), a stack  $a$  only has non-vanishing  $D7$ -brane charge on the torus where it is localised. For this  $D7$ -brane charge to be positive and fulfill the tadpole constraint (6.62), we need one positive and one negative wrapping number.

Let us consider a simple configuration with three  $D7$ -branes stacks and magnetic numbers

$$\begin{array}{rcccl}
 & & T_1^2 & T_2^2 & T_3^2 \\
 N_1 \rightarrow D7_1 & (1, 0) & (n_1^2, -1) & (-n_1^3, 1) & \\
 N_2 \rightarrow D7_2 & (-n_2^1, 1) & (1, 0) & (n_2^3, -1) & (6.63) \\
 N_3 \rightarrow D7_3 & (n_3^1, -1) & (-n_3^2, 1) & (1, 0) & 
 \end{array}$$

with  $n_a^i > 0$ . This configuration satisfies the  $D7$ -brane tadpole constraints (6.62) for  $N_a = 16$ . We have chosen to take unit wrapping numbers  $m_a^i = \pm 1$ . Configurations satisfying the  $D7$ -brane tadpole condition (6.62) with larger wrapping numbers  $m_a^i$  require lower values of  $N_a$  and lead to a smaller value of  $Q_{D3}^a$ . The total  $D3$ -brane charge of the configuration (6.63) is

$$\sum_{a=1}^3 Q_{D3}^a = \sum_{a=1}^3 N_a n_a^1 n_a^2 n_a^3 = -16(n_1^2 n_1^3 + n_2^1 n_2^3 + n_3^1 n_3^2). \quad (6.64)$$

For a stack  $a$  to preserve supersymmetry, its flux numbers must satisfy a constraint which, in our context, reads

$$\sum_{i=1}^3 \zeta_a^{(i)} \equiv \sum_i \frac{1}{\pi} \text{Arctan}(2\pi\alpha' F_a^i) = \sum_i \frac{1}{\pi} \text{Arctan}\left(\frac{m_a^i \alpha'}{n_a^i \mathcal{A}_i}\right) = 0. \quad (6.65)$$

The  $\zeta_a^{(i)}$  correspond to oscillator shifts of open string modes caused by the modification of boundary conditions by magnetic fields [114, 115]. The second equality used the explicit magnetic flux quantisation condition (6.58) for wrapped tori  $T_i^2$  of area  $4\pi^2 \mathcal{A}_i$ .

We see that in the configuration (6.63), all  $D7$ -brane stacks break supersymmetry. Supersymmetry breaking configurations generically induce tachyons, originating from open strings ending on the same or different stacks, called doubly charged or mixed states. The masses of both states were explicitly computed in e.g. [96]. Doubly charged tachyons can be eliminated by introducing separations between branes and their orientifold images [96]. This allows to increase the squared mass of these states to positive values. On the other hand, the following



conditions on  $\zeta_a^{(i)}$  allow to cancel the masses of all mixed tachyonic states [96]

$$\begin{aligned}
(A-1) \quad & \zeta_1^{(3)} = \zeta_2^{(1)} = \zeta_3^{(2)}, & \zeta_1^{(2)} = \zeta_2^{(3)} = \zeta_3^{(1)}, \\
& 2) \quad \zeta_1^{(3)} = \zeta_2^{(1)} = -\zeta_3^{(2)}, & \zeta_1^{(2)} = \zeta_2^{(3)} = -\zeta_3^{(1)}, \\
& 3) \quad \zeta_1^{(3)} = -\zeta_2^{(1)} = \zeta_3^{(2)}, & \zeta_1^{(2)} = -\zeta_2^{(3)} = \zeta_3^{(1)}, \\
& 4) \quad \zeta_1^{(3)} = -\zeta_2^{(1)} = -\zeta_3^{(2)}, & \zeta_1^{(2)} = -\zeta_2^{(3)} = -\zeta_3^{(1)}, \\
(B-1) \quad & \zeta_1^{(2)} = \zeta_1^{(3)}, & \zeta_2^{(1)} = \zeta_2^{(3)}, & \zeta_3^{(1)} = \zeta_3^{(2)}, \\
& 2) \quad \zeta_1^{(2)} = -\zeta_1^{(3)}, & \zeta_2^{(1)} = -\zeta_2^{(3)}, & \zeta_3^{(1)} = -\zeta_3^{(2)}.
\end{aligned}$$

In particular, solution  $(B - 2)$  satisfies (6.65) and thus preserves supersymmetry, with all lowest states remaining massless. In the solutions  $(A - i)$ , all the doubly charged  $D7_a - D7_a$  states have identical tachyonic masses  $\alpha' m^2 = -2 |\zeta_1^{(2)} + \zeta_1^{(3)}|$ , while for solution  $(B - 1)$  they can have different masses. The configuration (6.63) satisfies either  $(A - 1)$  or  $(B - 1)$ . In each case, the induced  $D3$ -brane charge (6.64) reads

$$\begin{aligned}
(A-1) \quad & \sum_{a=1}^3 Q_{D3}^a = -48 n_1^2 n_1^3, \\
(B-1) \quad & \sum_{a=1}^3 Q_{D3}^a = -16((n_1^2)^2 + (n_2^1)^2 + (n_3^1)^2).
\end{aligned}$$

We see that in both cases, the induced  $D3$ -brane charge can be made arbitrary large and negative by choosing large values of  $n_a^i$ , relaxing the conclusion (6.48) that  $g_s \geq 0.669$ . For instance, with  $n_a^i = 1$ , the induced  $D3$ -brane charge is  $-48$  in both cases  $(A - 1)$  or  $(B - 1)$ , so the tadpole condition (6.61) imposes  $\tilde{N}_{\text{flux}} \leq 128$ , leading to  $N_{\text{flux}} \leq 16$ . According to the relation (6.47), the minimal value of the string coupling thus becomes

$$g_s \geq g_{s,\text{min}} = \frac{16}{N_{\text{flux}}^2} = \frac{1}{16} = 0.0625.$$

As explained below (6.65), the  $D7$ -brane configuration (6.63) breaks supersymmetry. According to (6.65), supersymmetry is recovered in the limit of large volumes  $\mathcal{A}_i \rightarrow +\infty$ , which corresponds to diluted magnetic fluxes. This supersymmetry breaking configuration induces Fayet-Iliopoulos terms in the four-dimensional effective theory, which depend on the Kahler moduli related to the areas  $\mathcal{A}_i$  [96, 116, 117]. When these Fayet-Iliopoulos terms are associated with logarithmic loop corrections to the Kahler potential, the full potential can develop a metastable de Sitter vacuum [93, 95]. In other words, these ingredients could lead to a toroidal orbifold model with all moduli stabilised explicitly in a de Sitter vacuum.

## 6.4 Complex structures of the orbifolds

In this Appendix, we give the complex structure of all the orbifolds listed in table 6.1. Details about how they are computed are in section 6.1.

orbifold	coefficients of the complex structure						
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
$\mathbb{Z}_3$	$z_1$	1	$e^{2i\pi/3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$e^{2i\pi/3}$
$\mathbb{Z}_{4,a}$	$z_1$	1	$i$	-1	0	0	0
	$z_2$	0	0	0	1	$i$	-1
	$z_3$	1	-1	1	$\mathcal{U}$	$-\mathcal{U}$	$\mathcal{U}$
$\mathbb{Z}_{4,b}$	$z_1$	1	$e^{3i\pi/4}/\sqrt{2}$	0	0	0	0
	$z_2$	0	0	1	$i$	-1	0
	$z_3$	0	0	1	-1	1	$\mathcal{U}$
$\mathbb{Z}_{4,c}$	$z_1$	1	$e^{3i\pi/4}/\sqrt{2}$	0	0	0	0
	$z_2$	0	0	1	$e^{3i\pi/4}/\sqrt{2}$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}$
$\mathbb{Z}_{6, Ia}$	$z_1$	1	$e^{5i\pi/6}/\sqrt{3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	-1	$e^{i\pi/3}$
	$z_3$	0	0	1	$e^{2i\pi/3}$	1	$-e^{i\pi/3}$
$\mathbb{Z}_{6, Ib}$	$z_1$	1	$e^{5i\pi/6}/\sqrt{3}$	0	0	0	0
	$z_2$	0	0	1	$e^{5i\pi/6}/\sqrt{3}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/3}$
$\mathbb{Z}_{6, IIa}$	$z_1$	1	$e^{i\pi/3}$	$e^{2i\pi/3}$	-1	$-e^{i\pi/3}$	0
	$z_2$	1	$e^{2i\pi/3}$	$-e^{i\pi/3}$	1	$e^{2i\pi/3}$	0
	$z_3$	1	-1	1	-1	1	$\mathcal{U}$
$\mathbb{Z}_{6, IIb}$	$z_1$	1	$e^{i\pi/3}$	-1	-1	0	0
	$z_2$	0	0	0	0	1	$e^{2i\pi/3}$
	$z_3$	1	-1	$\mathcal{U}$	$1 - \mathcal{U}$	0	0
$\mathbb{Z}_{6, IIc}$	$z_1$	1	$-e^{2i\pi/3}$	$-e^{i\pi/3}$	$-e^{2i\pi/3}$	0	0
	$z_2$	1	$e^{i\pi/3}$	$-e^{2i\pi/3}$	$-e^{i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}$
$\mathbb{Z}_{6, IIId}$	$z_1$	1	$e^{5i\pi/6}/\sqrt{3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}$

Table 6.16: complex structure of the orbifolds listed in table 6.1 (part. 1 of 3). Here,  $\mathcal{U}$  is a complex structure modulus. For instance, for the orbifold  $T^6/\mathbb{Z}_3$ , the complex coordinates are given by  $z^1 = x^1 + e^{2i\pi/3}x^2$ ,  $z^2 = x^3 + e^{2i\pi/3}x^4$ ,  $z^3 = x^5 + e^{2i\pi/3}x^6$ , etc.

orbifold	coefficients of the complex structure						
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
$\mathbb{Z}_7$	$z_1$	1	$e^{2i\pi/7}$	$e^{4i\pi/7}$	$e^{6i\pi/7}$	$-e^{i\pi/7}$	$-e^{3i\pi/7}$
	$z_2$	1	$e^{4i\pi/7}$	$-e^{i\pi/7}$	$-e^{5i\pi/7}$	$e^{2i\pi/7}$	$e^{6i\pi/7}$
	$z_3$	1	$-e^{i\pi/7}$	$e^{2i\pi/7}$	$-e^{3i\pi/7}$	$e^{4i\pi/7}$	$-e^{5i\pi/7}$
$\mathbb{Z}_{8, Ia}$	$z_1$	1	$i$	-1	$-e^{i\pi/4}$	$e^{3i\pi/4}$	$e^{i\pi/4}$
	$z_2$	1	-1	1	$i$	$-i$	$i$
	$z_3$	1	$i$	-1	$e^{i\pi/4}$	$-e^{3i\pi/4}$	$-e^{i\pi/4}$
$\mathbb{Z}_{8, Ib}$	$z_1$	1	$e^{i\pi/4}$	$i$	$-(1 + \sqrt{2} + i)/2$	0	0
	$z_2$	0	0	0	0	$i$	$e^{3i\pi/4}/\sqrt{2}$
	$z_3$	1	$-e^{i\pi/4}$	$i$	$-(1 - \sqrt{2} + i)/2$	0	0
$\mathbb{Z}_{8, IIa}$	$z_1$	1	$e^{i\pi/4}$	$i$	$-(1 + \sqrt{2} + i)/2$	$-(1 + \sqrt{2} + i)/2$	0
	$z_2$	1	$e^{3i\pi/4}$	$-i$	$-(1 - \sqrt{2} - i)/2$	$-(1 - \sqrt{2} - i)/2$	0
	$z_3$	0	0	0	1	-1	$\mathcal{U}$
$\mathbb{Z}_{8, IIb}$	$z_1$	1	$e^{i\pi/4}$	$i$	$-(1 + \sqrt{2} + i)/2$	0	0
	$z_2$	1	$e^{3i\pi/4}$	$-i$	$-(1 - \sqrt{2} - i)/2$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}$
$\mathbb{Z}_{12, Ia}$	$z_1$	1	$e^{i\pi/6}$	$e^{i\pi/3}$	-1	$-e^{i\pi/6}$	$-\sqrt{2}e^{i\pi/12}$
	$z_2$	1	$e^{2i\pi/3}$	$-e^{i\pi/3}$	1	$e^{2i\pi/3}$	0
	$z_3$	1	$-e^{i\pi/6}$	$e^{i\pi/3}$	-1	$e^{i\pi/6}$	$-\sqrt{2}e^{7i\pi/12}$
$\mathbb{Z}_{12, Ib}$	$z_1$	1	$e^{i\pi/6}$	$e^{11i\pi/12}/\sqrt{2}$	$-e^{i\pi/12}/\sqrt{2}$	0	0
	$z_2$	0	0	0	0	1	$e^{2i\pi/3}$
	$z_3$	1	$-e^{i\pi/6}$	$e^{5i\pi/12}/\sqrt{2}$	$-e^{7i\pi/12}/\sqrt{2}$	0	0
$\mathbb{Z}_{12, II}$	$z_1$	1	$e^{i\pi/6}$	$e^{11i\pi/12}/\sqrt{2}$	$-e^{i\pi/12}/\sqrt{2}$	0	0
	$z_2$	1	$e^{5i\pi/6}$	$-e^{7i\pi/12}/\sqrt{2}$	$e^{5i\pi/12}/\sqrt{2}$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}$

Table 6.17: complex structure of the orbifolds listed in table 6.1 (part. 2 of 3). Here,  $\mathcal{U}$  is a complex structure modulus. For instance, for the orbifold  $T^6/\mathbb{Z}_7$ , the complex coordinates are given by  $z^1 = x^1 + e^{2i\pi/7}x^2 + e^{4i\pi/7}x^3 + e^{6i\pi/7}x^4 - e^{i\pi/7}x^5 - e^{3i\pi/7}x^6$ , etc.

orbifold	coefficients of the complex structure						
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$z_1$	1	$\mathcal{U}^1$	0	0	0	0
	$z_2$	0	0	1	$\mathcal{U}^2$	0	0
	$z_3$	0	0	0	0	1	$\mathcal{U}^3$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$z_1$	1	$\mathcal{U}$	0	0	0	0
	$z_2$	0	0	1	$e^{3i\pi/4}/\sqrt{2}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/4}/\sqrt{2}$
$\mathbb{Z}_2 \times \mathbb{Z}_{6,I}$	$z_1$	1	$\mathcal{U}$	0	0	0	0
	$z_2$	0	0	1	$e^{5i\pi/6}/\sqrt{3}$	0	0
	$z_3$	0	0	0	0	1	$e^{2i\pi/3}$
$\mathbb{Z}_2 \times \mathbb{Z}_{6,II}$	$z_1$	1	$e^{5i\pi/6}/\sqrt{3}$	0	0	0	0
	$z_2$	0	0	1	$-e^{i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$e^{5i\pi/6}/\sqrt{3}$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$z_1$	1	$e^{2i\pi/3}$	0	0	0	0
	$z_2$	0	0	1	$e^{2i\pi/3}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/3}$
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$z_1$	1	$e^{2i\pi/3}$	0	0	0	0
	$z_2$	0	0	1	$e^{5i\pi/6}/\sqrt{3}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/6}/\sqrt{3}$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$z_1$	1	$e^{3i\pi/4}/\sqrt{2}$	0	0	0	0
	$z_2$	0	0	1	$e^{3i\pi/4}/\sqrt{2}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/4}/\sqrt{2}$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$z_1$	1	$e^{5i\pi/6}/\sqrt{3}$	0	0	0	0
	$z_2$	0	0	1	$e^{5i\pi/6}/\sqrt{3}$	0	0
	$z_3$	0	0	0	0	1	$-e^{i\pi/6}/\sqrt{3}$

Table 6.18: complex structure of the orbifolds listed in table 6.1 (part. 3 of 3). Here, the  $\mathcal{U}$  are a complex structure moduli. For instance, for the orbifold  $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ , the complex coordinates are given by  $z^1 = x^1 + \mathcal{U}^1 x^2$ ,  $z^2 = x^3 + \mathcal{U}^2 x^4$ ,  $z^3 = x^5 + \mathcal{U}^3 x^6$ .



# Chapter 7

## Cosmological perturbations from five-dimensional inflation

This chapter opens with an introduction to the swampland program in section 7.1, covering the main ideas leading to the prediction of the dark dimension, which is an extra dimension at the micrometer scale. In section 7.2, we then introduce the traditional four-dimensional inflation and the computation of cosmological perturbations. Finally, in 7.3, we repeat this calculation in a five-dimensional model [5], following the recent proposal proposal that five-dimensional inflation could be a way of explaining the large size of the dark dimension [118]. We find that the power spectra of all primordial fluctuations from five-dimensional inflation exhibit a change of behaviour at large angles leading to more power compared to the four-dimensional case, with a nearly vanishing spectral index. This prediction can in principle be confronted in future CMB observations if the precision at low multipole moments increases.

### 7.1 The Swampland and the Dark Dimension

As we explained in chapter 5, string theory is a self-consistent theory that manages to unify aspects of both particle physics and cosmology. In particular, it is a theory of quantum gravity. It is, however, a supersymmetric higher-dimensional theory, so relating it to our observable world requires a lot of work and involves many choices. In the KKLT framework we introduced in section 5.3.2 for type IIB compactification, we must indeed choose a compactification manifold, an orientifold projection, internal fluxes and the configuration of branes. Moreover, KKLT is not the only possibility, nor is type IIB string theory.

This vast number of possibilities has poetically been called the landscape of string theory, and it represents a challenge for phenomenology. Ultimately, the question is whether the landscape contains a model that can describe both our particle physics and cosmology. Whether we would be able to use this model to precisely compute low energy observables, such as the mass of the electron, or make new predictions, is another question. A problem would arise, however, if the landscape were so vast that any low-energy effective theory could be derived from string theory. If this were the case, string theory would still be useful as a consistent framework for quantum gravity, but it would not be able to provide new predictions for particle physics or cosmology.

Fortunately, the landscape is not that vast, and there are effective theories that have been proven not to come from string theory. Such theories are said to be in the swampland [119, 120]. In fact, we can be more general and extend the notion of swampland to effective theories that cannot be extracted from a theory of quantum gravity, without referring to string theory.

If the condition of not belonging to the swampland is strong enough, it can be used to put constraints on the low-energy effective theories we use to describe our universe. If we are optimistic, these constraints can then be used to make testable predictions. In any case, understanding the boundary between the landscape and the swampland is an excellent way to connect the high-energy theory of quantum gravity to our observable world at low energy.

This section is an introduction to some ideas of the swampland program, mainly based on [121, 122]. Our goal is to introduce the ingredients leading to the dark dimension proposal, according to which there should be one large extra dimension at the micrometer scale.

### 7.1.1 Introduction to the Swampland

Since we do not know everything about quantum gravity, it is not really possible to prove with certainty any statement about the effective theories that can be extracted from it. For this reason, we refer to these statements as swampland conjectures.

One of the most established swampland conjectures, that we already mentioned in chapters 3 and 4, is that a theory of quantum gravity cannot have exact global symmetries [123]. There is in fact a simple argument in favor of this conjecture, which appeals to the semi-classical properties of black holes. Let us consider a Schwarzschild black hole

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + d^2d\Omega^2 \quad \text{where} \quad f(r) = 1 - \frac{2M}{r}. \quad (7.1)$$

According to the no-hair theorem [124], this metric is unique if the black hole is neutral and stationary. If the black hole absorbs matter that is charged under an exact global symmetry of the theory, this charge becomes invisible to an outside observer. So there is an infinity of black holes labeled by their global charge but described by the unique metric (7.1). As a result, the entropy of the black hole (7.1) is infinite, which contradicts the Bekenstein-Hawking formula, telling that its entropy of the black hole is proportional to its area. Another motivation comes from perturbative string theory, where symmetries of the worldsheet are gauged in the target space [125]. To date, there is no counter-example to this conjecture.

Another well-established swampland conjecture is the weak-gravity conjecture [126]. It says that in any theory of quantum gravity with a U(1) gauge symmetry associated to a gauge coupling  $g$ , there must be a state with mass  $m$  and charge  $q$  for which gravity is the weakest force, in the sense that

$$m \leq \sqrt{2}gqM_p. \quad (7.2)$$

This conjecture too can be motivated by the semi-classical properties of black holes. Let us consider a Reissner–Nordström black hole, with charge  $Q$  under the U(1) gauge symmetry

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + d^2d\Omega^2 \quad \text{where} \quad f(r) = 1 - \frac{2M}{r} + \frac{2g^2Q^2}{r^2}.$$

If the following is satisfied

$$M \geq \sqrt{2}gQM_p, \quad (7.3)$$

the black hole has two horizons. Otherwise, it has none, and the central singularity is naked. A black hole saturating the inequality (7.3) is called extremal. In its form (7.2), the weak gravity conjecture allows the decay of extremal black holes by Hawking radiation. Indeed, if there is no state satisfying (7.2) in the theory, an extremal black hole cannot decay, because any quantum it emits would make it super extremal and reveal a naked singularity. Again, there are other arguments in favor of this conjecture, see [121], and so far it is supported by all examples in string theory.

### Distance and Emergent String Conjectures

We have seen in chapter 5 that a characteristic property of string theory is the presence of extra-dimensions. These extra-dimensions are associated to moduli, which are scalar fields whose vacuum expectation value is related to physical parameters of the lower-dimensional effective theory, such as masses and gauge couplings. In particular, asymptotic limits in the moduli space correspond to particular values of these parameters. For instance, there may be a limit corresponding to a vanishing gauge coupling. However, a vanishing gauge coupling corresponds to the restoration of a global symmetry, which is thought to be forbidden. So something should happen when we approach this limit of the moduli space.

According to the Swampland Distance Conjecture [120], what happens when we approach any boundary in moduli space is that a tower of states becomes exponentially light. More precisely, the conjecture says that for two points  $(P, Q)$  in the moduli space  $\mathcal{M}$ , the characteristic mass  $M$  of the tower satisfies

$$M(Q) \sim M(P)e^{-\alpha d(P,Q)}, \quad (7.4)$$

where  $\alpha$  is a positive constant and  $d(P, Q)$  is the geodesic distance between  $P$  and  $Q$ , measured by the moduli space metric  $g_{ij}$ , which appears in the effective action in the kinetic term of the scalars  $g_{ij}(\phi)\partial^\mu\phi^i\partial_\mu\phi^j$ . According to a refined conjecture proposed in [127], this behaviour sets in when  $d(P, Q) \gtrsim 1$  in Planck units and also holds in the presence of a potential.

Let us illustrate this conjecture with the example of bosonic string theory compactified on a circle. According to the dimensional reduction performed in (5.34), the radius  $R$  of the circle is linked to a canonically normalised scalar field  $\sigma \sim \log R$ . The vev of this canonical scalar field has two asymptotic limits  $\sigma \rightarrow \pm\infty$ , associated to  $R \rightarrow +\infty$  and  $R \rightarrow 0$ . Recalling the mass formula (5.31)

$$\alpha' M^2 = \alpha' \left( \frac{k^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} \right) + \text{osc.}, \quad (7.5)$$

we indeed see that the two asymptotic limits correspond to a tower of states becoming light. In the limit  $R \rightarrow +\infty$ , this is the Kaluza-Klein tower, indexed by  $k$ , and in the limit  $R \rightarrow 0$  this is the tower of winding modes, indexed by  $w$ . Note that this latter is intimately related to the extended nature of the string. Moreover, the two limits  $R \rightarrow +\infty$  and  $R \rightarrow 0$  are related by so-called T-duality, under which  $R \leftrightarrow \alpha'/R$  and  $k \leftrightarrow w$ . For this reason, the distance conjecture is closely related to the duality conjecture, predicting the existence of dualities relating



infinite distance limits with each other.

The appearance of towers of states implies that the description of the theory must change as we approach the asymptotic limits of the moduli space. Otherwise, it would have to include the infinite tower of states, which is hardly consistent with gravity [128, 129]. In the above example, the limit  $R \rightarrow +\infty$  corresponds to a decompactification limit, in which the theory can no longer be described as effectively lower-dimensional. In the  $R \rightarrow 0$  limit, the extended nature of the string plays a central role and the theory can no longer be described by a field theory. By T-duality, this limit is related to another decompactification limit.

According to the Emergent String Conjecture [130], any infinite distance limit is either a decompactification limit or a limit where a string becomes weakly coupled. This conjecture can be illustrated by considering the dilaton  $\phi$  in type IIA string theory. Its vev has two asymptotic limits  $\phi \rightarrow \pm\infty$ , associated to  $g_s \rightarrow 0$  and  $g_s \rightarrow +\infty$ , with  $g_s = e^\phi$ , see (5.13). The limit  $g_s \rightarrow 0$  corresponds to the fundamental string becoming weakly coupled. In this case, the tower of state becoming light is the tower of string excitations, with a mass scale

$$M_s \sim M_p g_s^{1/4} \sim M_p \exp\left(-\frac{\sqrt{2}}{4}\Delta\phi\right), \quad (7.6)$$

where the factors  $g_s^{1/4}$  comes from converting (7.5) from string frame to Einstein frame. On the other hand, the limit  $g_s \rightarrow +\infty$  can be seen as a decompactification limit to M-theory, which is 11-dimensional. In this case, the tower of states becoming light can be shown to be a tower of  $D0$ -branes

$$m_{D0} \sim \frac{M_s}{g_s} \sim M_p \exp\left(-\frac{3\sqrt{2}}{4}\Delta\phi\right) \quad (7.7)$$

In type IIB, the limit  $g_s \rightarrow +\infty$  is related by  $\mathcal{S}$ -duality (5.28) to another weakly coupled limit.

Contrary to the no global symmetry and weak gravity conjectures, there are few arguments for the distance and emergent string conjectures apart from the fact that they are satisfied by all examples from string and M-theory. In particular, the coefficient  $\alpha$  in (7.4) is not universal, it depends on the theory and limit considered, as we can see in (7.6) and (7.7).

### AdS/dS Distance Conjectures

In [131], a generalisation of the distance conjecture was proposed, according to which an infinite tower of states also becomes light in the flat-space limit  $\Lambda \rightarrow 0$  of any AdS vacuum

$$m \sim |\Lambda|^\alpha, \quad (7.8)$$

where  $\alpha$  is a constant. A stronger form of the conjecture implies that  $\alpha = 1/2$  if the AdS vacuum is supersymmetric, and  $\alpha \geq 1/2$  if it is not. It is believed that this conjecture also applies to dS vacua, with a positive cosmological constant  $\Lambda$ , even though there is no explicit example of such a vacua in string theory, as we reviewed in chapter 5. In dS space, the Higuchi bound [132] implies that  $\alpha \leq 1/2$ , and it has been argued in [133–135] that  $\alpha > 1/D$ , where  $D$  is the dimension of spacetime. So we have

$$\frac{1}{D} \leq \alpha \leq \frac{1}{2}. \quad (7.9)$$

This conjecture lies at the heart of the dark dimension proposal, which we will now address.

## 7.1.2 The Dark Dimension proposal

The prediction of the dark dimension [135] is based on the observation that our cosmological constant has a very small value

$$\Lambda \sim 10^{-122},$$

Assuming that the dS distance conjecture (7.8) is true, this implies the existence of a light tower of states of characteristic mass  $m \sim |\Lambda|^\alpha$ . According to the Emergent String Conjecture, this is either a tower of string states or a tower of Kaluza-Klein states. In our universe, the tower cannot be a tower of string states, because the associated string scale would be of the same order as the neutrino scale, and field theory still works well beyond that scale.

So the only possibility is the presence of a Kaluza-Klein tower associated to one or more extra-dimensions. According to the experimental bound of [136, 137], the  $1/r^2$  gravitational force law holds down to  $30 \mu\text{m}$ , so the extra-dimensions should be smaller. This pushes the value of the exponent  $\alpha$  in (7.8) to the upper limit  $\alpha = 1/4$  in (7.9). In other words,  $\Lambda^{1/4} \simeq \lambda m$  where  $\lambda$  is a  $\mathcal{O}(1)$  parameter. In [135], it is argued that  $10^{-4} < \lambda < 1$ , so that  $\lambda \sim 10^{-1} - 10^{-3}$ , leading to the following estimate for the scale of the dark dimension

$$m \sim 0.1 - 10 \text{ meV}, \quad l \sim 0.1 - 10 \mu\text{m}. \quad (7.10)$$

All that remains is to determine the number of extra-dimensions associated to the KK tower, which can be done using astrophysical data, because the emission of Kaluza-Klein modes can cause neutron stars to cool too quickly if the tower is too light. This puts upper constraints on the size of large extra-dimensions. According to [138], for the case of a single extra-dimension we have  $l < 44 \mu\text{m}$  and for the case of two we have  $l < 1.6 \cdot 10^{-4} \mu\text{m}$ . The more extra-dimensions there are, the stricter the constraint. Already with two extra-dimensions, the astrophysical constraint is not compatible with the scale (7.10) from the dS distance conjecture. As a result, [135] predicted the existence of a single large dark dimension.

If it exists, this dark dimension would have far-reaching phenomenological consequences. For example, [139–141] explored the idea that the Kaluza-Klein tower of gravitons could be a candidate for dark matter. On the other hand, [142–144] studied the possibility that dark matter is composed of five-dimensional primordial black holes, decaying more slowly than their four-dimensional counterparts. This and other phenomenological applications of the dark dimension proposal are reviewed in [145, 146].

According to [118], the scale of the dark dimension can be explained by inflation. In this article, the authors assume that the dark dimension was also expanding exponentially during inflation, so that the universe was described by a five-dimensional quasi de Sitter solution. After inflation, the size  $R$  of the dark dimension is assumed to be stabilised, and the four and five-dimensional Planck scales,  $M_p$  and  $M_*$  respectively, are related by  $M_p^2 = M_*^3 R$ .

As a result, of going to the Einstein frame, inflation is not perceived as uniform from our four-dimensional point-of-view, and it can be shown that if the dark dimension expands  $N$  e-folds, the three non-compact spatial dimensions expand  $3N/2$  e-folds. If the latter expanded by about 60 e-folds to solve the horizon problem, the dark dimension expanded by about 40 e-folds, which is exactly what is needed to bring it from the fundamental length  $M_*^{-1}$  to the

micron size, if  $M_* \sim 10^9$  GeV.

Inflation is not only useful for solving the horizon problem of the big bang model, it is also known to predict an approximate scale invariant power spectrum of primordial density fluctuations [147, 148] that act as sources for structure formation in our universe and are consistent with observations of the Cosmological Microwave Background [149]. It is therefore important to assess the impact of five-dimensional inflation on the power spectra of primordial fluctuations, to know if this proposition is compatible with observations.

The computation of these power spectra is the subject of the following sections. We start by reviewing this computation in four-dimensional slow-roll inflation in section 7.2, before extending to the five-dimensional case in section 7.3. These sections are based on [5].

## 7.2 Perturbations in four dimensions

In this section, we review the computation of the primordial scalar and tensor perturbations generated by slow-roll inflation, following [150–152]. Let us first recall a few formulae from chapter 2. The background metric during inflation takes the following form

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad (7.11)$$

where the scale factor  $a(t)$  is subject to the Friedmann equations

$$H^2 = \frac{\rho}{3} \quad \text{and} \quad 2\dot{H} + 3H^2 = -p, \quad (7.12)$$

where  $H = \dot{a}/a$  and  $(\rho, p)$  are the energy density and isotropic pressure of the perfect fluid filling the universe. In the case of a canonical scalar field with a potential  $V$ , we have

$$T_\nu^\mu = g^{\mu\rho} \partial_\nu \phi \partial_\rho \phi - \delta_\nu^\mu \left( \frac{1}{2} (\partial\phi)^2 + V \right) \quad \rightarrow \quad \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V, \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V,$$

so (7.12) becomes

$$3H^2 = \frac{1}{2} \dot{\phi}^2 + V \quad \text{and} \quad \dot{H} = -\frac{1}{2} \dot{\phi}^2. \quad (7.13)$$

The starting point of the computation is to perturb the metric around (7.11) and the inflaton  $\phi$  around a time-dependent background  $\bar{\phi}(t)$ , which is a solution of (7.13)

$$ds^2 = -(1 + 2\Phi) dt^2 + 2a(t) B_i dx^i dt + a^2(t) ((1 - 2\Psi) \delta_{ij} + E_{ij}) dx^i dx^j, \quad (7.14)$$

and

$$\phi(t, x) = \bar{\phi}(t) + \delta\phi(t, x).$$

We can further decompose  $B_i$  and  $E_{ij}$  into scalar, vector and tensor modes

$$B_i = \partial_i B + C_i \quad \text{and} \quad E_{ij} = 2\partial_i \partial_j E + 2\partial_{(i} E_{j)} + h_{ij}. \quad (7.15)$$

So in the end, we have five scalar perturbations  $(\delta\phi, \Phi, \Psi, B, E)$ , two vector perturbations  $(C_i, E_i)$ , and one tensor perturbation  $h_{ij}$ . They transform under the diffeomorphisms

$$t \rightarrow t + \xi^t \quad \text{and} \quad x^i \rightarrow x^i + \delta^{ij} \partial_j \xi^x, \quad (7.16)$$

as

$$\delta\phi \rightarrow \delta\phi - \dot{\phi}\xi^t, \quad \Phi \rightarrow \Phi - \xi^t, \quad \Psi \rightarrow \Psi + H\xi^t, \quad B \rightarrow B + a^{-1}\xi^t - a\dot{\xi}^x, \quad E \rightarrow E - \xi^x, \quad (7.17)$$

and similarly for the vectors and tensors perturbations. The perturbation  $\delta\phi$  of the inflaton also induce perturbations of the energy-momentum tensor  $T_\nu^\mu$ , written as

$$\delta T_0^0 = -\delta\rho, \quad \delta T_i^0 = a\partial_i\delta q, \quad T_j^i = \delta_j^i\delta p + \Sigma_j^i,$$

but the anisotropic stress tensor  $\Sigma_j^i$  vanishes with a single field. The perturbations  $(\delta\rho, \delta q, \delta p)$  of the energy-momentum tensor also transform under (7.16). They can be combined with perturbations of the metric into gauge invariant quantities, as the comoving curvature

$$\mathcal{R} = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q. \quad (7.18)$$

This is the variable that we are interested in.

## 7.2.1 Scalar perturbations

Using a transformation with  $\xi^t$ , we can put ourselves in the gauge  $\delta\phi = 0$ , see (7.17). In this gauge,  $\delta q = 0$  as well, and  $\mathcal{R} = \Psi$ , see (7.18). We can also choose either  $B = 0$  or  $E = 0$  since they transform with  $\xi^x$ . Even if  $\delta\phi = 0$ , the energy and pressure density perturbations  $(\delta\rho, \delta p)$  are not vanishing at first order, they are

$$\delta\rho = -\frac{1}{2}g^{00}\dot{\phi}^2 \simeq -\dot{\phi}^2\Phi \quad \text{and} \quad \delta p \simeq -\dot{\phi}^2\Phi,$$

where from now we omit the bar over the time-dependent background of the scalar field  $\phi(t)$ . The other components of the energy momentum tensor remain unperturbed. From there, we can write the Einstein equations with the metric (7.14) in the gauge  $\delta\phi = E = 0$ . In Fourier space, they read

$$\begin{aligned} 3H(-\dot{\mathcal{R}} + H\Phi) - \frac{k^2}{a^2}(\mathcal{R} + aHB) &= \frac{1}{2}\dot{\phi}^2\Phi, & -\dot{\mathcal{R}} + H\Phi &= 0, \\ -\ddot{\mathcal{R}} - 3H\dot{\mathcal{R}} + H\dot{\Phi} + (3H^2 + 2\dot{H})\Phi &= -\frac{1}{2}\dot{\phi}^2\Phi, & (\partial_t + 3H)\frac{B}{a} + \frac{\mathcal{R} + \Phi}{a^2} &= 0. \end{aligned} \quad (7.19)$$

the first one is the 00 component, the second is the 0i component, and the last two come from the ij components. The first two equations can be combined to obtain

$$\Phi = \frac{\dot{\mathcal{R}}}{H} \quad \text{and} \quad B = -\frac{2k^2H\mathcal{R} + a^2\dot{\mathcal{R}}\dot{\phi}^2}{2ak^2H^2},$$

so we see that  $\Phi$  and  $B$  are not dynamical fields, their equations of motion are algebraic. Plugging these expressions into the two remaining equations of (7.19) yields

$$\dot{\mathcal{R}}(2\dot{H} + \dot{\phi}^2) = 0, \quad (7.20)$$

and

$$a^3H\dot{\phi}^2\ddot{\mathcal{R}} + a^3\dot{\phi}(3H^2\dot{\phi} - 2\dot{H}\dot{\phi} + 2H\ddot{\phi})\dot{\mathcal{R}} - 2k^2H(2\dot{a}H - 2aH^2 + a\dot{H})\mathcal{R} = 0. \quad (7.21)$$

We see that (7.20) is reduced to an equation on the background which is satisfied, see (7.13). The other one, (7.21), can be rewritten as

$$\frac{a^3 \dot{\phi}^2}{H^2} \ddot{\mathcal{R}} + \frac{d}{dt} \left( \frac{a^3 \dot{\phi}^2}{H^2} \right) \dot{\mathcal{R}} + \frac{a \dot{\phi}^2}{H^2} k^2 \mathcal{R} = 0, \quad (7.22)$$

which can be obtained from applying the variational principle to the following action

$$\mathcal{S} = \frac{1}{2} \int d^4 x \frac{a^3 \dot{\phi}^2}{H^2} ((\dot{\mathcal{R}})^2 - a^{-2} (\partial_i \mathcal{R})^2). \quad (7.23)$$

Defining  $z = a \dot{\phi} H^{-1}$  and  $v = z \mathcal{R}$ , and conformal time  $a d\tau = dt$ , this action becomes

$$\mathcal{S} = \frac{1}{2} \int d\tau d^3 x \left( (v')^2 + (\partial_i v)^2 + \frac{z''}{z} v^2 \right), \quad (7.24)$$

where a prime denotes a derivation with respect to conformal time.

In Fourier space, the action (7.24) leads to the Mukhanov-Chibisov equation [153]

$$v_k'' + \left( k^2 - \frac{z''}{z} \right) v_k = 0, \quad (7.25)$$

which can be solved numerically or analytically once the background evolution is specified. Let us introduce the slow-roll parameters, along with the conformal Hubble rate  $\mathcal{H} = a'/a$

$$\varepsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \quad \text{and} \quad \delta = -\frac{\ddot{\phi}}{H \dot{\phi}} = 1 - \frac{\phi''}{\mathcal{H} \phi'}. \quad (7.26)$$

The slow-roll limit is defined by  $\varepsilon \ll 1$  and  $\delta \ll 1$ , and in this limit the equation (7.25) can be solved explicitly. We could also introduce the second Hubble slow-roll parameter  $\varepsilon_2 = \dot{\varepsilon}/(\varepsilon H) = -2\delta + 2\varepsilon$  as in [154], but we will use  $\delta$  here. We also define the potential slow-roll parameters

$$\varepsilon_V = \frac{1}{2} \left( \frac{V_{\phi\phi}}{V} \right)^2 \quad \text{and} \quad \eta_V = \frac{V_{\phi\phi}}{V},$$

where the  $\phi$  index denotes a derivative with respect to  $\phi$ . In the slow-roll limit, we can show that  $(\varepsilon_V, \eta_V) \ll 1$  as well and  $(\varepsilon, \delta) \simeq (\varepsilon_V, \eta_V - \varepsilon_V)$ . From the definition of  $\varepsilon$ , we can also get  $\mathcal{H} \simeq -(1 + \varepsilon)/\tau$ . Noticing that  $z = a \phi' \mathcal{H}^{-1}$ , we can then obtain  $z'/z = \mathcal{H}(1 + \varepsilon + \delta)$  and

$$\frac{z''}{z} \simeq \mathcal{H}^2 (2 + 2\varepsilon - 3\delta) \simeq \frac{2 + 6\varepsilon - 3\delta}{\tau^2}, \quad (7.27)$$

at first order in the slow-roll parameters. In the de Sitter limit  $z''/z \simeq 2/\tau^2$ , the solution of equation (7.25) is

$$v_k = c_1 \left( 1 - \frac{i}{k\tau} \right) \frac{e^{-ik\tau}}{\sqrt{2k}} + c_2 \left( 1 + \frac{i}{k\tau} \right) \frac{e^{ik\tau}}{\sqrt{2k}}.$$

Then, Bunch-Davies boundary condition (7.29) fixes  $c_1 = 1, c_2 = 0$ , and we get the following power spectrum for  $\mathcal{R}$

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2} \frac{H^2}{a^2 \dot{\phi}^2} |v_k|^2 = \frac{1}{2\varepsilon} \left( \frac{H}{2\pi} \right)^2,$$

where we used  $a(\tau) = -1/(H\tau)$  and considered the super horizon limit  $k\tau \ll 1$ . If we don't neglect the slow-roll parameters, the general solution of (7.25) is

$$v_k = c_1 \tau^{1/2} J_\nu(k\tau) + c_2 \tau^{1/2} Y_\nu(k\tau) \quad \text{where} \quad \nu = \left(\frac{9}{4} + 6\varepsilon - 3\delta\right)^{1/2} \simeq \frac{3}{2} + 2\varepsilon - \delta \quad (7.28)$$

The  $c_i$  are fixed again by the Bunch-Davies condition

$$\lim_{\tau \rightarrow -\infty} v_k = \frac{\exp(-ik\tau)}{\sqrt{2k}}, \quad (7.29)$$

giving

$$c_1 = \frac{1-i}{2} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{1}{2}i\pi\nu\right) \quad \text{and} \quad c_2 = -\frac{1+i}{2} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{1}{2}i\pi\nu\right), \quad (7.30)$$

and in the super horizon limit  $k\tau \ll 1$ , we have

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{2\varepsilon} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}, \quad (7.31)$$

where we can read the spectral tilt

$$n_{\mathcal{R}} - 1 = 3 - 2\nu = 2\delta - 4\varepsilon = 2\eta_V - 6\varepsilon_V$$

## 7.2.2 Tensor perturbations

To compute the power spectrum of tensor perturbations, we can follow a similar procedure. The second order action for the tensor perturbation  $h_{ij}$  can be obtained from the Einstein-Hilbert action

$$\mathcal{S}_h = \frac{1}{8} \int d^4x a^2 ((h'_{ij})^2 - (\partial_l h_{ij})^2) \quad (7.32)$$

Going in Fourier space, where the two polarisations of  $h_{ij}$  are described by two modes  $h_k^s$ , and defining  $2v_k^s = ah_k^s$ , the equation of motion takes the same form as (7.25)

$$(v_k^s)'' + \left(k^2 - \frac{a''}{a}\right)v_k^s = 0. \quad (7.33)$$

At first order in slow-roll parameters, we have  $a''/a \simeq (2 + 3\varepsilon)/\tau^2$ , and the solution is the same as (7.28) with  $\nu = (9/4 + 3\varepsilon)^{1/2} \simeq 3/2 + \varepsilon$ . The resulting tensor power spectrum is

$$\mathcal{P}_h = \frac{2H^2}{\pi^2} \left(\frac{k}{aH}\right)^{-2\varepsilon}, \quad (7.34)$$

where a factor 2 comes from the fact that there are two polarisations, and a factor of 4 comes from the normalisation in  $2v_k^s = ah_k^s$ . The biggest difference with the scalar spectrum (7.31) is the absence of the factor of  $1/\varepsilon$  that previously came from the factor of  $z$  in  $v = z\mathcal{R}$ , see before (7.24). This makes the tensor-to-scalar ratio  $r \simeq 16\varepsilon$  small in the slow-roll limit.

### 7.3 Perturbations in five dimensions

Let us now reproduce this computation with an additional compact dimension. In this section, we will follow the notations of [118], where five-dimensional inflation was proposed as a mechanism explaining the micrometric size of the dark dimension. In what follows, the five-dimensional quantities will be denoted with a hat. This distinction will be important in the discussion above Figure 7.1. The five-dimensional background metric is

$$ds^2 = -d\hat{t}^2 + \hat{a}^2(\hat{t})\delta_{ij}dx^i dx^j + R^2(\hat{t})dy^2,$$

with periodicity  $y \sim y + 2\pi$ , so that  $R(\hat{t})$  is the radius of the compact circular dimension. In the de Sitter limit, both  $\hat{a}(\hat{t})$  and  $R(\hat{t})$  are expanding exponentially

$$\hat{a}(\hat{t}) = e^{H\hat{t}} \quad \text{and} \quad R(\hat{t}) = R_0 e^{H\hat{t}}.$$

Introducing the conformal time  $\tau = -H^{-1} \exp(-H\hat{t})$ , the metric becomes

$$ds^2 = \hat{a}^2(\tau)(-d\tau^2 + \delta_{ij}dx^i dx^j + R_0^2 dy^2), \quad (7.35)$$

and

$$\hat{a}(\tau) = -\frac{1}{H\tau} \quad \text{and} \quad R(\tau) = -\frac{R_0}{H\tau}.$$

The extra dimension is compactified on an interval  $S^1/\mathbb{Z}_2$  with  $\mathbb{Z}_2$  acting as  $y \rightarrow -y$ , leading at two fixed points, where branes with our observable universe can be localised. In the string theory context, the  $D$ -branes are described by open strings ending on them, with the matching conditions automatically satisfied due to the requirement of tadpole cancellation in the presence of orientifolds. The overall time dependence arises from the bulk cosmological constant and amounts to considering a slice of five-dimensional de Sitter spacetime along the compact direction. This should be contrasted to the more general warped cosmological solutions in the braneworld context studied in the past with less isometries [155–158], or in an anti-de Sitter bulk [159, 160].

During inflation, as in four dimensions, the background is not exactly de Sitter, but quasi de Sitter, sourced by a canonically normalised and minimally coupled scalar field slowly rolling in a flat potential, following the equation of motion

$$\phi'' + 3\mathcal{H}\phi' + \hat{a}^2(\tau)\frac{dV}{d\phi} = 0, \quad (7.36)$$

where, from now on, we stick to conformal time and denote by  $\mathcal{H} = \hat{a}'/\hat{a}$  the conformal Hubble rate. The five-dimensional Friedmann equations generalising (7.13) read

$$6\mathcal{H}^2 = \hat{a}^2\rho \quad \text{and} \quad 3(\mathcal{H}' + \mathcal{H}^2) = -\hat{a}^2 p, \quad (7.37)$$

with

$$\rho = \frac{1}{2\hat{a}^2}(\phi')^2 + V \quad \text{and} \quad p = \frac{1}{2\hat{a}^2}(\phi')^2 - V.$$

These equations imply in particular

$$3(\mathcal{H}' - \mathcal{H}^2) + (\phi')^2 = 0, \quad (7.38)$$

In the following, we still use the slow-roll parameters  $\varepsilon$  and  $\delta$  defined in (7.26). But the definition of  $\varepsilon_V$  and  $\eta_V$  has to be adapted, because the Friedmann equations are not exactly the same in five dimensions. In order to keep  $\varepsilon \simeq \varepsilon_V$  and  $\delta \simeq \eta_V - \varepsilon_V$  in the slow-roll limit, we take

$$\varepsilon_V = \frac{3}{4} \left( \frac{V_\phi}{V} \right)^2 \quad \text{and} \quad \eta_V = \frac{3}{2} \frac{V_{\phi\phi}}{V}.$$

### 7.3.1 The scalar perturbations

The formalism for studying cosmological perturbations in higher-dimensional braneworld theories has been developed in the past, see for example [161]. The framework we consider here is a direct generalization of the four-dimensional case discussed in the previous section. We begin by perturbing the metric around (7.35) as we did in (7.14)

$$ds^2 = \hat{a}^2(\tau) \left( - (1 + 2\Phi)d\tau^2 + 2B_i dx^i d\tau + 2C dy d\tau \right. \\ \left. + ((1 - 2\Psi)\delta_{ij} + E_{ij}) dx^i dx^j + 2F_i dx^i dy + (R_0^2 - 2\Xi) dy^2 \right), \quad (7.39)$$

where as in (7.15) we can further decompose  $(B_i, E_{ij}, F_i)$  into scalar, vector and tensor modes

$$B_i = \partial_i B + C_i, \quad E_{ij} = 2\partial_i \partial_j E + 2\partial_{(i} E_{j)} + h_{ij}, \quad F_i = \partial_i F + G_i.$$

This gives us eight scalar perturbations  $(\delta\phi, \Phi, \Psi, \Xi, B, C, E, F)$ , three vector perturbations  $(C_i, E_i, G_i)$  and one tensor perturbation  $h_{ij}$ . Under a five-dimensional diffeomorphism with parameter  $\xi^\mu = (\xi^t, \partial^i \xi_x + \tilde{\xi}^i, \xi^y)$ , these perturbations transform according to

$$\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - 2\nabla_{(\mu} \xi_{\nu)}$$

giving

$$\begin{aligned} \Phi &\rightarrow \Phi - \mathcal{H}\xi^t - (\xi^t)', & \Psi &\rightarrow \Psi + \mathcal{H}\xi^t, & \Xi &\rightarrow \Xi + R_0^2 \mathcal{H}\xi^t + R_0^2 \partial_y \xi^y, \\ B &\rightarrow B + \xi^t - (\xi^x)', & C &\rightarrow C + \partial_y \xi^t - R_0^2 (\xi^y)', & E &\rightarrow E - \xi^x, \\ F &\rightarrow F - R_0^2 \xi^y - \partial_y \xi^x, & C^i &\rightarrow C^i - (\tilde{\xi}^i)', & E^i &\rightarrow E^i - \tilde{\xi}^i, & G^i &\rightarrow G^i - \partial_y \tilde{\xi}^i. \end{aligned}$$

When computing these transformations, we must take into account  $a(\tau)$  factors that appear when we lower the indices from  $\xi^\mu$  to  $\xi_\mu$ . In addition to the metric, the inflaton perturbation  $\delta\phi$  transforms as before

$$\delta\phi \rightarrow \delta\phi - \phi' \xi^t.$$

In what follows, we will use this freedom to put ourselves in a gauge where  $\delta\phi = E = F = 0$ .

### Einstein equations

Let us now write the five-dimensional Einstein equations generalising (7.19) for the scalar perturbations of the metric (7.39), in the gauge  $\delta\phi = E = F = 0$ . The 00 component is

$$R_0^2 (3\mathcal{H}(3\Psi' + 4\mathcal{H}\Phi) - \Delta(2\Psi - 3\mathcal{H}B)) \\ - \Delta\Xi - 3\partial_y^2 \Psi + 3\mathcal{H}\partial_y C + 3\mathcal{H}\Xi' = R_0^2 (\phi')^2 \Phi, \quad (7.40)$$



the  $0i$  and  $0y$  components are

$$R_0^2(4\Psi' + 6\mathcal{H}\Phi) - \partial_y^2 B + \partial_y C + 2\Xi' = 0, \quad (7.41)$$

$$\Delta(C - \partial_y B) - 6\partial_y \Psi' - 6\mathcal{H}\partial_y \Phi = 0, \quad (7.42)$$

the  $ii$  component gives

$$\begin{aligned} & R_0^2(2\Psi'' + 6\mathcal{H}\Psi' + 3\mathcal{H}\Phi' + 6(\mathcal{H}' + \mathcal{H}^2)\Phi) \\ & + 3\mathcal{H}\partial_y C + 3\mathcal{H}\Xi' + (\partial_y C' + \Xi'') + \partial_y^2(\Phi - 2\Psi) \\ & + 2/3\Delta(R_0^2(B' + 3\mathcal{H}B + \Phi - \Psi) - \Xi) = -R_0^2(\phi')^2\Phi, \end{aligned} \quad (7.43)$$

the  $ij$  and  $iy$  components give

$$R_0^2(B' + 3\mathcal{H}B + \Phi - \Psi) - \Xi = 0, \quad (7.44)$$

$$3\mathcal{H}(\partial_y B + C) + \partial_y(B' + 2\Phi - 4\Psi) + C' = 0, \quad (7.45)$$

and finally, the  $yy$  component is

$$6(\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}(\Delta B + \Phi' + 3\Psi') + \Delta B' + \Delta\Phi - 2\Delta\Psi + 3\Psi'' = -(\phi')^2\Phi. \quad (7.46)$$

In these equations,  $\Delta$  is the three dimensional Laplacian. Note that in (7.40), (7.41), (7.43), (7.44), the terms with  $R_0^2$  factorised are very similar to their four dimensional equivalents, see (7.19), up to numerical factors. Note also that the last line of the left hand side of (7.43), with a  $\Delta$  factorised, vanishes due to (7.44).

Let us now move to Fourier space. Taking into account the compactness of the  $y$  coordinate, we expand all five dimensional fields  $A = (\Phi, \Psi, \Xi, B, C)$  as

$$A(\tau, x, y) = \int d^3k \sum_n A_n(k, \tau) e^{ikx} e^{iny}, \quad (7.47)$$

This amounts to replacing all  $\Delta$  by  $-k^2$  and the  $\partial_y$  by  $in$  in the previous equations, along with adding  $n$  indices to all fields, which we will omit to avoid cluttering the equations.

In the four dimensional case, we started with five scalar perturbations. Two of these were set to zero by a gauge choice. We then had four Einstein equations. Two of them were algebraic constraint eliminating two perturbations, one of them reduced to an equation on the background, and only one dynamical equation remained. In the five-dimensional case, we expect similar results. Starting with eight scalar perturbations, we set three of them to zero by a gauge choice, and we have seven Einstein equations. We will indeed see that three of them are algebraic constraint eliminating two perturbations, another two reduce to equations on the background, and only two remain as genuine dynamical equations.

As a first step, we can use (7.40), (7.41) and (7.42) to express  $(\Phi, B, C)$  in terms of the other perturbations. To do this, we note that (7.41) and (7.42) span a close, linear system on the variables  $\Phi$  and  $C - \partial_y B$ , which gives

$$\Phi = -\frac{(3n^2 + 2R_0^2 k^2)\Psi' + k^2\Xi'}{3\mathcal{H}(n^2 + R_0^2 k^2)} \quad \text{and} \quad C - inB = -\frac{2in(R_0^2\Psi' - \Xi')}{n^2 + R_0^2 k^2}. \quad (7.48)$$

We can then use (7.40) to obtain  $C$  and  $B$  separately, but their expressions are not useful here.

By inserting the expressions for  $(\Phi, B, C)$  into the remaining equations (7.43)-(7.46), we find that two of these equations boil down to equations on the background which are therefore automatically satisfied. To be more precise, we find that

$$\begin{aligned} 2i \times (7.43) + n \times (7.45) &\rightarrow 2iR_0^2(3(\mathcal{H}' - \mathcal{H}^2) + (\phi')^2)\Phi = 0 \\ k^2 \times (7.43) - n^2 \times (7.46) &\rightarrow -(n^2 - R_0^2k^2)(3(\mathcal{H}' - \mathcal{H}^2) + (\phi')^2)\Phi = 0 \end{aligned}$$

so that these two combinations vanish due to the background equation (7.38). In other words, (7.45) and (7.46) are not independent, and we are left with two equations on  $(\Psi, \Xi)$ .

We now introduce a new variable  $\Theta \equiv \Psi - \Xi/R_0^2$ , for which we find that

$$\begin{aligned} R_0^2 \times (7.43) + n^2 \times (7.44) &\rightarrow -R_0^4/\mathcal{H}(3(\mathcal{H}' - \mathcal{H}^2) + (\phi')^2)\Psi' \\ &+ (\text{terms that depend only on } \Theta) = 0. \end{aligned}$$

Using (7.38) again, the first line disappears and we get the following equation on  $\Theta$

$$\Theta'' + 3\mathcal{H}\Theta' + \left(k^2 + \frac{n^2}{R_0^2}\right)\Theta = 0, \quad (7.49)$$

which is the equation of motion of a five-dimensional minimally coupled scalar field of square mass  $(k^2 + n^2/R_0^2)/\hat{a}^2$  in a time-dependent background, see (7.36).

Finally, from (7.43), we can obtain another equation on  $\Psi$ . Let us introduce another variable

$$\Omega = \frac{(3n^2 + 2R_0^2k^2)\Psi + k^2\Xi}{3(n^2 + R_0^2k^2)},$$

note that  $\Omega' = -\mathcal{H}\Phi$ , with  $\Phi$  given by (7.48). We can combine the previous equations to obtain the following on  $\Omega$

$$\Omega'' + \left(3\mathcal{H} + \frac{2(\mathcal{H}')^2 - \mathcal{H}\mathcal{H}''}{\mathcal{H}^3 - \mathcal{H}\mathcal{H}'}\right)\Omega' + \left(k^2 + \frac{n^2}{R_0^2}\right)\Omega = 0.$$

This equation can be put in a more suggestive form, reminiscent of (7.22) in the four-dimensional case

$$\frac{\hat{a}^3(\phi')^2}{\mathcal{H}^2}\Omega'' + \frac{d}{d\tau}\left(\frac{\hat{a}^3(\phi')^2}{\mathcal{H}^2}\right)\Omega' + \frac{\hat{a}^3(\phi')^2}{\mathcal{H}^2}\left(k^2 + \frac{n^2}{R_0^2}\right)\Omega = 0. \quad (7.50)$$

So in the end, both (7.49) and (7.50) can be obtained from applying the variational principle to the action

$$\begin{aligned} \mathcal{S} = \frac{1}{2} \int d\tau d^3k \sum_n &\left\{ \hat{a}^3 \left( (\Theta')^2 - \left( k^2 + \frac{n^2}{R_0^2} \right) \Theta^2 \right) \right. \\ &\left. + \frac{\hat{a}^3(\phi')^2}{\mathcal{H}^2} \left( (\Omega')^2 - \left( k^2 + \frac{n^2}{R_0^2} \right) \Omega^2 \right) \right\}. \end{aligned} \quad (7.51)$$

Note that the only difference between  $\Theta$  and  $\Omega$  is the factor of  $(\phi')^2/\mathcal{H}^2 = 3\varepsilon$  in front of  $\Omega$ , where  $\varepsilon = -\dot{H}/H^2 = 1 - \mathcal{H}'/\mathcal{H}^2$  is the Hubble slow-roll parameter, see (7.26) and (7.38).

## Power spectrum

The action (7.51) has a form similar to (7.23), so let us treat it in a similar way. If we define

$$y = \hat{a}^{3/2}, \quad z = \frac{\hat{a}^{3/2}\phi'}{\mathcal{H}}, \quad \theta = y\Theta, \quad \omega = z\Omega, \quad m_{k,n}^2 = k^2 + \frac{n^2}{R_0^2}, \quad (7.52)$$

we obtain two copies of the Mukhanov-Chibisov equation as in (7.25) and (7.33)

$$\theta''_{k,n} + \left(m_{k,n}^2 - \frac{y''}{y}\right)\theta_{k,n} = 0 \quad \text{and} \quad \omega''_{k,n} + \left(m_{k,n}^2 - \frac{z''}{z}\right)\omega_{k,n} = 0,$$

where we have restored the indices  $k, n$  for greater clarity. At linear order in slow-roll, recall from the discussion between (7.26) and (7.27) that  $\mathcal{H} \simeq -(1+\varepsilon)/\tau$  and  $\hat{a}(\tau) \simeq -1/(H\tau^{1+\varepsilon})$ , so we can obtain directly  $y''/y$ . We can also compute  $z''/z$  as in (7.27)

$$\frac{y''}{y} \simeq \frac{15/4 + 6\varepsilon}{\tau^2} \quad \text{and} \quad \frac{z''}{z} \simeq \frac{15/4 + 10\varepsilon - 4\delta}{\tau^2}.$$

Thus, as in (7.28), we end up with the following solutions for  $\theta_{k,n}$  and  $\omega_{k,n}$ , in terms of the Bessel functions ( $J_\nu, Y_\nu$ )

$$f_{k,n} = c_1^f \tau^{1/2} J_{\nu_f}(m_{k,n}\tau) + c_2^f \tau^{1/2} Y_{\nu_f}(m_{k,n}\tau), \quad \text{where } f = \theta \text{ or } \omega, \quad (7.53)$$

where we have defined

$$\nu_\theta = 2 + \frac{3\varepsilon}{2} \quad \text{and} \quad \nu_\omega = 2 + \frac{5\varepsilon}{2} - \delta. \quad (7.54)$$

Finally, the Bunch-Davies boundary conditions (7.29) give the same expressions for the constants ( $c_1^f, c_2^f$ ) in terms of  $\nu_f$ , as in (7.30).

We have thus obtained the solutions for  $\theta_{k,n}$  and  $\omega_{k,n}$ , from which we can write the following expression for the adiabatic curvature perturbation  $\mathcal{R} = \Psi$

$$\mathcal{R}_{k,n} = \Omega_{k,n} + \frac{k^2}{3m_{k,n}^2} \Theta_{k,n} = \frac{1}{\sqrt{3\varepsilon}\hat{a}^{3/2}} \left( \omega_{k,n} + \sqrt{\frac{\varepsilon}{3}} \frac{k^2}{m_{k,n}^2} \theta_{k,n} \right). \quad (7.55)$$

As in the four dimensional case, after obtaining the modes, we will take the super horizon limit  $k\tau \ll 1$  to compute the power spectrum (7.31). This corresponds to perturbations with wavelengths exiting the Hubble radius, which is constant during inflation, in cosmic time  $\hat{t}$ . These modes then remain constant until they re-enter the horizon, which grows in subsequent eras after the end of inflation.

In our five dimensional case, the analogue limit is  $m_{k,n}\tau \ll 1$ . Again, this corresponds to five dimensional modes exiting the Hubble radius, which is constant during inflation, when all spatial dimensions expand homogeneously. After inflation, only the three spatial dimensions are still expanding, while the fifth dark dimension is assumed to be stabilised [118].

In the super-horizon limit  $m_{k,n}\tau \ll 1$ , (7.53) behaves as

$$f_{k,n} \simeq (1+i) \sqrt{\frac{2\tau}{\pi}} \frac{1}{(m_{k,n}\tau)^{\nu_f}}. \quad (7.56)$$

Let us now consider a mode of given 3D momentum  $k$ . In (7.39), we denoted by  $\hat{a}(\hat{t}) \sim e^{H\hat{t}}$  the scale factor of the 5D universe. Following [118], we also denote by  $a(t) \sim t^\alpha$  the scale factor of the 4D universe in the Einstein frame. Here,  $\hat{t}$  and  $t$  are the 5D and 4D cosmic times.

After inflation, the dark dimension is stabilised and the universe follows its standard four-dimensional evolution, with  $H = 1/(2t)$  and physical momenta  $q = k/a \sim k/t^{1/2}$ , so that modes of physical scales that exited the Hubble radius during inflation eventually re-enter. Note that after inflation, we consider  $H = \dot{a}(t)/a(t)$ , without the hats.

During inflation, the universe is effectively five-dimensional, so the physical wavelengths to compare with the Hubble radius are  $1/\hat{q}$ , with  $\hat{q} = \hat{k}/\hat{a}$  and  $\hat{k} = m_{k,n}$  in our notation. The limit  $m_{k,n}\tau \ll 1$  is equivalent to  $\hat{k}/\hat{a} \ll H$ , corresponding to physical scales that exit the Hubble radius well before the end of inflation. Since  $\hat{k}/\hat{a} \sim \hat{k}/e^{H\hat{t}}$  during inflation while  $H$  is constant, a large band of scales do exit the horizon.

Now, a given 3D momentum  $k$  corresponds to a tower of 4D momenta  $\hat{k}$ , according to  $k^2 = \hat{k}^2 - n^2/R_0^2$ . So as  $n$  increases with  $k$  fixed,  $\hat{k}$  increases. As a consequence, the high modes of the tower fail to satisfy  $\hat{k}/\hat{a} \ll H$  and do not exit the horizon during inflation. However, since 5D inflation last for about 40 e-folds there are a large number of Kaluza-Klein modes that exit the horizon for a sufficiently large region of 3D momenta, which justifies the limit  $m_{k,n}\tau \ll 1$  in (7.56). Corrections from the expansion of the Bessel functions in (7.53) are expected to be irrelevant. This discussion is summarised in figure 7.1.

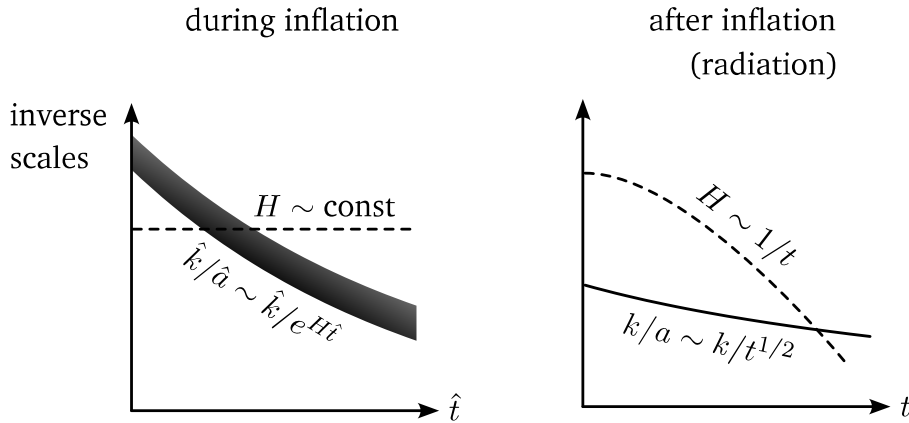


Figure 7.1: Right panel shows evolution of physical scales and Hubble radius *after* inflation. Since  $H \sim 1/t$  while  $q = k/a \sim k/t^{1/2}$ , scales that exited the Hubble radius during inflation eventually re-enter. Left panel shows evolution of physical scales and Hubble radius *during* inflation. Since  $H \sim \text{const}$  while  $\hat{q} = \hat{k}/\hat{a} \sim \hat{k}/e^{H\hat{t}}$ , a large band of scales exit the Hubble radius during inflation. A given 3D momentum  $k$  corresponds to a tower of 4D momenta  $\hat{k}$ , represented by the black band.

We can only observe perturbations in our four-dimensional universe which is localised on a 3-brane in the dark dimension. From a four-dimensional perspective, a perturbation measured at a given spatial momentum  $k$  corresponds to any momentum component along

the extra dimension. So in order to evaluate the four-dimensional power spectrum of  $\mathcal{R}$ , we have to sum over all modes

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2} \sum_n |\mathcal{R}_{k,n}|^2, \quad (7.57)$$

where from (7.55)

$$\frac{k^3}{2\pi^2} |\mathcal{R}_{k,n}|^2 \simeq \frac{2H^3 k^3 \tau^4}{3\pi^3 \varepsilon} \left( \frac{1}{(m_{k,n} \tau)^{2\nu_\omega}} + \frac{\varepsilon}{3} \frac{k^4}{m_{k,n}^4 (m_{k,n} \tau)^{2\nu_\theta}} \right). \quad (7.58)$$

The cross term that comes naively by squaring  $|\mathcal{R}_{k,n}|^2$  is absent, because the perturbations  $\omega_{k,n}$  and  $\theta_{k,n}$  must be considered as Gaussian random variables. Evaluating the sum (7.57) requires computing

$$\sum_n \frac{1}{(R_0 m_{k,n})^{2\alpha}} = \sum_n \frac{1}{((R_0 k)^2 + n^2)^\alpha}, \quad (7.59)$$

where  $\alpha$  is close to 4. If  $\alpha$  is an integer greater than 1, it is rather direct to establish that

$$S_\alpha(x) = \sum_n \frac{1}{(x + n^2)^\alpha} = \frac{(-1)^{\alpha-1} \pi}{\Gamma(\alpha)} \left( \frac{\partial}{\partial x} \right)^{\alpha-1} \left( \frac{\coth(\pi \sqrt{x})}{\sqrt{x}} \right). \quad (7.60)$$

In general

$$S_\alpha(x) = x^{[\alpha]-\alpha} \sum_n \frac{1}{(x + n^2)^{[\alpha]}} \left( \frac{x}{x + n^2} \right)^{\alpha-[\alpha]},$$

where  $[\alpha]$  denotes the integer part of  $\alpha$ . We can then bound  $S_\alpha(x)$  as

$$x^{[\alpha]-\alpha+1} S_{[\alpha]+1}(x) < S_\alpha(x) \leq x^{[\alpha]-\alpha} S_{[\alpha]}(x). \quad (7.61)$$

In our case  $\alpha - [\alpha]$  is proportional to the slow-roll parameters and is small, see (7.54). As a consequence, the upper bound given in (7.61) is a good approximation

$$\sum_n \frac{1}{((R_0 k)^2 + n^2)^\alpha} \simeq (R_0 k)^{2([\alpha]-\alpha)} \sum_n \frac{1}{((R_0 k)^2 + n^2)^{[\alpha]}}.$$

This formula works quite well for all  $x > 0$ . The error is proportional to  $|\alpha - [\alpha]|$ , and it is less than 1% when  $|\alpha - [\alpha]| < 0.06$ . Using this formula in (7.57) yields

$$\mathcal{P}_{\mathcal{R}} \simeq \frac{2R_0 H^3 (R_0 k)^3}{3\pi^3 \varepsilon} \left( \left( \frac{k}{\hat{a}H} \right)^{4-2\nu_\omega} S_2((R_0 k)^2) + \frac{\varepsilon}{3} \left( \frac{k}{\hat{a}H} \right)^{4-2\nu_\theta} (R_0 k)^4 S_4((R_0 k)^2) \right)$$

where  $S_2, S_4$  can be expressed using (7.60), and in the limits  $R_0 k \ll 1$  and  $R_0 k \gg 1$

$$S_{[\alpha]}((R_0 k)^2) \underset{R_0 k \ll 1}{\simeq} \frac{1}{(R_0 k)^{2\alpha}}, \quad S_{[\alpha]}((R_0 k)^2) \underset{R_0 k \gg 1}{\simeq} \frac{\sqrt{\pi} \Gamma(\alpha - 1/2)}{\Gamma(\alpha)} \frac{1}{(R_0 k)^{2\alpha-1}}. \quad (7.62)$$

As a result, the power spectrum in these two limits is

$$\mathcal{P}_{\mathcal{R}} \underset{R_0 k \ll 1}{\simeq} \frac{2H^3}{3\pi^3 \varepsilon k} \left( \left( \frac{k}{\hat{a}H} \right)^{2\delta-5\varepsilon} + \frac{\varepsilon}{3} \left( \frac{k}{\hat{a}H} \right)^{-3\varepsilon} \right), \quad (7.63)$$

and

$$\mathcal{P}_{\mathcal{R}} \underset{R_0 k \gg 1}{\simeq} \frac{R_0 H^3}{3\pi^2 \varepsilon} \left( \left( \frac{k}{\hat{a}H} \right)^{2\delta-5\varepsilon} + \frac{5\varepsilon}{24} \left( \frac{k}{\hat{a}H} \right)^{-3\varepsilon} \right). \quad (7.64)$$

## Short discussion

In the end, we recover the result of [118], up to the normalisation factor  $1/(3\varepsilon)$  which determines the amplitude, and we obtained the spectral tilt in the region  $R_0 k \gg 1$

$$n_{\mathcal{R}} - 1 = 2\delta - 5\varepsilon = 2\eta_V - 7\varepsilon_V. \quad (7.65)$$

In the opposite limit  $R_0 k \ll 1$ , the spectral tilt becomes  $n_{\mathcal{R}} - 1 = 2\eta_V - 7\varepsilon_V - 1$ , and the spectrum is not scale invariant.

The contribution of  $\Omega$  to  $\mathcal{R}$  in (7.55) comes with a factor of  $1/\sqrt{\varepsilon}$  that becomes  $1/\varepsilon$  in the power spectrum. By analogy with the four-dimensional case (7.31), we can say that  $\Omega$  is the contribution of the inflaton to the adiabatic curvature perturbation. Conversely, the contribution of  $\Theta$  to  $\mathcal{R}$  does not come with this factor, and gives the second term in (7.58). In analogy with the tensor perturbations in (7.34), we can say that this is a gravitational contribution to the curvature perturbation coming from the radion which is a second scalar field from the four-dimensional point-of-view. However, in (7.63), (7.64), we can see that this contribution is proportional to  $\varepsilon$  and thus subleading in the power spectrum.

At the end of the five-dimensional inflation, the radion acquires a runaway potential of quintessence type and a stabilization mechanism of the fifth dimension is most likely required. This is discussed in [118], where some possible contributions to the scalar potential are presented. In general, in addition to that from the five dimensional vacuum energy at the minimum of the inflaton potential, there are three additional contributions, corresponding to 3-brane tensions, kinetic gradients of bulk fields and their Casimir energy. It is worth mentioning that the brane tension does not modify the bulk equations of motion, while bulk fields have vanishing expectation values during inflation, which occurs around a flat region of the inflaton potential away from its minimum. Finally, the Casimir energy falls off exponentially for bulk masses larger than the compactification scale and with a power much faster than the five-dimensional cosmological constant when they are light, thus remaining negligible during inflation. Consequently, all these contributions, together with the five-dimensional cosmological constant, can stabilize the fifth dimension without affecting the five-dimensional inflation period discussed in our paper.

### 7.3.2 Isocurvature perturbations

When only one scalar degree of freedom is involved in inflation, perturbations are adiabatic. This implies that after inflation, the relative density of different matter components is constant; only the total density is perturbed. The underlying intuition is that if there is only the inflaton, it decays in the same way everywhere to produce the thermal bath.

When more scalar fields are involved, this is no longer true, as there may be perturbations leaving the total density unperturbed, called isocurvature or entropy perturbations. If such a perturbation occurs between two fields, one of them decaying into a component that, for instance, does not thermalise with the rest of the bath, like dark matter after decoupling, it leaves an imprint in the relative density of different components, such as matter ( $m$ ) and

radiation ( $\gamma$ )

$$\mathcal{S}_m = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4} \frac{\delta\rho_\gamma}{\rho_\gamma} \neq 0.$$

Such fluctuations would have an effect on the CMB. So far, this effect has not been observed. Isocurvature perturbations can be characterised by their relative amplitude  $\beta_{\text{iso}}$  to the curvature perturbations

$$\beta_{\text{iso}} = \frac{\mathcal{P}_S}{\mathcal{P}_R + \mathcal{P}_S} < 0.038, \quad (7.66)$$

where the upper limit is set by Planck observations [149].

In our setup, the radion, coming from the four-dimensional decomposition of the metric acts as an extra scalar from the four-dimensional point-of-view, in addition to the inflaton. So we have to worry about isocurvature perturbations. Such perturbations are hard to compute, as they depend on details about reheating and how the various fields decay after inflation. To give a rough estimate of their magnitude, let us compute the total entropy perturbation [162]

$$\mathcal{S} = \mathcal{H} \left( \frac{\delta p}{p'} - \frac{\delta \rho}{\rho'} \right).$$

Using  $\delta p = \delta \rho = -(\phi')^2 \Phi / \hat{a}^2$  and equations (7.37), (7.38)

$$\mathcal{S} = \left( \frac{-6\mathcal{H}^3 + 4\mathcal{H}'\mathcal{H} + \mathcal{H}''}{4(2\mathcal{H}^3 - \mathcal{H}'')} \right) \Phi. \quad (7.67)$$

Recalling that  $\varepsilon = 1 - \mathcal{H}'/\mathcal{H}^2$ , we also have  $\varepsilon' = 2\mathcal{H}\varepsilon(\varepsilon - \delta)$ , and

$$\mathcal{H}^2 - \mathcal{H}' \simeq \mathcal{H}^2 \varepsilon \quad \text{and} \quad 2\mathcal{H}^3 - \mathcal{H}'' \simeq 4\mathcal{H}^3 \varepsilon - 2\mathcal{H}^3 \varepsilon \delta.$$

Thus, at leading slow-roll order and using the definitions (7.52) we get

$$\mathcal{S} \simeq -\frac{\Phi}{2} \simeq \frac{\Omega'}{2\mathcal{H}} \simeq \frac{\Omega}{2\mathcal{H}} \left( \frac{\omega'}{\omega} - \frac{z'}{z} \right).$$

Using now the expression of  $\omega$  in the super horizon limit (7.56), we can see that  $\omega'/\omega \simeq (1/2 - \nu_\omega)/\tau$ , and  $z'/z \simeq \mathcal{H}(3/2 + \varepsilon - \delta)$ , leading to the following result

$$\mathcal{S} \simeq \frac{3\varepsilon}{4} \Omega \quad \rightarrow \quad \mathcal{P}_S \simeq \frac{9\varepsilon^2}{16} \mathcal{P}_R \quad \rightarrow \quad \beta_{\text{iso}} \simeq \frac{9\varepsilon^2}{16}.$$

Thus, we obtain that  $\beta_{\text{iso}}$  is second order in the slow-roll parameter  $\varepsilon$ , which easily satisfies the constraint (7.66). But again, the total entropy perturbation is at best a rough estimate of the isocurvature perturbations. Further analysis would be required to compute them more precisely, but that is beyond the scope of this chapter.

Note that  $\mathcal{S}$  being proportional to  $\Phi$  in (7.67) is a consequence of our gauge choice  $\delta\phi = 0$ . In “genuine” four-dimensional multifield inflation with two fundamental scalar fields  $\phi_1$  and  $\phi_2$ , it is not possible to set both  $\delta\phi_1 = 0$  and  $\delta\phi_2 = 0$  with a gauge choice. This leads to an additional contribution in (7.67), from the entropy field, defined as  $\delta s$  in [162], which is a linear combination of  $\delta\phi_1$  and  $\delta\phi_2$ . In our case, this contribution is absent, and the total entropy perturbation ends up being suppressed by  $\varepsilon^2$ .

### 7.3.3 Tensor and vector perturbations

For the tensor perturbation  $h_{ij}$  of the metric (7.39), since  $h_i^i = 0$  and  $\partial^i h_{ij} = 0$  by definition, the only non trivial components of the Einstein equation are the  $ij$  ones, which read

$$h''_{ij} + 3\mathcal{H}h'_{ij} + \left(k^2 + \frac{n^2}{R_0^2}\right)h_{ij} = 0. \quad (7.68)$$

Therefore the two independent helicities of  $h_{ij}$  satisfy the same equation as  $\Theta$  in (7.49), and can be studied in the same way. In the end, the power spectrum of tensor perturbations is the same as (7.63), up to a factor of  $2 \times 4 \times 3\varepsilon$ . The factor of 2 comes from the two polarisations, the factor of 4 comes from the normalisation of  $h$ , see (7.32) and discussion after (7.34), and the factor of  $3\varepsilon$  comes from the factor of  $\phi'/\mathcal{H}$  in (7.51) which is absent here. In addition, the tensor spectral tilt is  $4 - 2\nu_\theta$ , see (7.54). In the end we thus have

$$\mathcal{P}_\mathcal{T} \underset{R_0 k \ll 1}{\simeq} \frac{16H^3}{\pi^3 k} \left(\frac{k}{\hat{a}H}\right)^{-3\varepsilon} \quad \text{and} \quad \mathcal{P}_\mathcal{T} \underset{R_0 k \gg 1}{\simeq} \frac{8R_0 H^3}{\pi^2} \left(\frac{k}{\hat{a}H}\right)^{-3\varepsilon}. \quad (7.69)$$

It follows that the tensor-to-scalar ratio  $r$  and the tensor spectral tilt  $n_\mathcal{T}$  are

$$r = 24\varepsilon = 24\varepsilon_V,$$

and

$$n_\mathcal{T} = -3\varepsilon_V \quad \text{for} \quad R_0 k \gg 1, \quad \text{and} \quad n_\mathcal{T} = -3\varepsilon_V - 1 \quad \text{for} \quad R_0 k \ll 1$$

The experimental constraint  $r < 0.06$  reported by [149] then sets a limit  $\varepsilon_V < 0.003$ . On the other hand, using the experimental value of the scalar spectral index  $n_\mathcal{R} - 1 \simeq -0.04$  and the result (7.65), one finds that  $\eta_V$  should be in the range  $[-0.02, -0.01]$ .

### Vector perturbations

In ordinary four-dimensional inflation, vector perturbations are not sourced at linear order and are therefore irrelevant. However, the four-dimensional decomposition of the five-dimensional metric contains vector fields from the four-dimensional point-of-view. In (7.39), there are three vector perturbations  $(C_i, E_i, G_i)$ , and we can choose a gauge where  $E_i = 0$ . Then the only non-trivial components of the Einstein equations are the  $0i$ ,  $ij$  and  $iy$  ones

$$R_0^2 \Delta C_i + \partial_y^2 C_i - \partial_y G'_i = 0, \quad R_0^2 (3\mathcal{H}C_i + C'_i) - \partial_y G_i = 0,$$

and

$$3\mathcal{H}(G'_i - \partial_y C_i) - \Delta G_i - \partial_y C'_i + G''_i = 0.$$

In Fourier space (7.47), we can use the first equation to obtain

$$C_i = -\frac{inG'_i}{n^2 + R_0^2 k^2}.$$

Then, the two other equations turn out to be equivalent to

$$G''_i + 3\mathcal{H}G'_i + \left(k^2 + \frac{n^2}{R_0^2}\right)G_i = 0,$$



which is the same as (7.68) and (7.49).

Thus, the power spectrum of primordial vector perturbations is the same as for tensor perturbations, and twice the contribution of the radion  $\Theta$  to the scalar perturbations, as expected from the fact that all of them emerge from the five-dimensional metric. Usually, in brane-world constructions, the extra-dimension is orbifolded into an interval  $S^1/\mathbb{Z}_2$ , so that the vector zero mode as well as half of the massive modes are projected out by the  $\mathbb{Z}_2$  action. Projecting out the zero-mode boils down to extract  $1/(R_0k)^{2\alpha}$  from the sum in (7.59). This affects only the result in the limit  $R_0k \ll 1$  in (7.62), that becomes

$$S_\alpha((R_0k)^2) - \frac{1}{(R_0k)^{2\alpha}} \underset{R_0k \ll 1}{\simeq} \zeta(2\alpha).$$

Therefore

$$\mathcal{P}_\mathcal{V} \underset{R_0k \ll 1}{\simeq} \frac{4\pi R_0 H^3}{45} (R_0k)^3 \left( \frac{k}{\hat{a}H} \right)^{-3\varepsilon} \quad \text{and} \quad \mathcal{P}_\mathcal{V} \underset{R_0k \gg 1}{\simeq} \frac{4R_0 H^3}{\pi^2} \left( \frac{k}{\hat{a}H} \right)^{-3\varepsilon}. \quad (7.70)$$

And the power spectrum of vector perturbations has a reduced amplitude at large distances, in contrast with all other perturbations which are more important due to an approximate  $1/k$  behaviour. On the other hand, vector perturbations could also lead to  $B$ -mode polarization of the CMB which should be added to  $\mathcal{P}_\mathcal{T}$  found previously in (7.69), but this goes beyond our scope. Note that the  $\mathbb{Z}_2$  action also truncates the sum (7.57) over the modes to  $n > 0$ , effectively adding a factor of  $1/2$  everywhere this sum appears. This factor was not included in the previous power spectra (7.63), (7.64), (7.69), but is taken into account in (7.70).

# Bibliography

- [1] I. Antoniadis, A. Guillen and K. Tamvakis, *Late time acceleration in Palatini gravity*, *JHEP* **11** (2022) 144 [[2207.13732](#)].
- [2] I. Antoniadis, A. Guillen and K. Tamvakis, *Ultraviolet behaviour of Higgs inflation models*, *JHEP* **08** (2021) 018 [[2106.09390](#)].
- [3] I. Antoniadis, A. Guillen and F. Rondeau, *Massive gravitino scattering amplitudes and the unitarity cutoff of the new Fayet-Iliopoulos terms*, *JHEP* **01** (2023) 043 [[2210.00817](#)].
- [4] I. Antoniadis, J. Cunat and A. Guillen, *More on massive gravitino scattering amplitudes and the unitarity cutoff of the new Fayet-Iliopoulos terms*, *JHEP* **04** (2023) 002 [[2302.01247](#)].
- [5] I. Antoniadis, J. Cunat and A. Guillen, *Cosmological perturbations from five-dimensional inflation*, [2311.17680](#).
- [6] I. Antoniadis, A. Guillen and O. Lacombe, *Flux vacua in type IIB compactifications on orbifolds: their finiteness and minimal string coupling*, [2404.18995](#).
- [7] S. Bahamonde, C.G. Böhm, S. Carloni, E.J. Copeland, W. Fang and N. Tamanini, *Dynamical systems applied to cosmology: dark energy and modified gravity*, *Phys. Rept.* **775-777** (2018) 1 [[1712.03107](#)].
- [8] SUPERNOVA COSMOLOGY PROJECT collaboration, *Measurements of  $\Omega$  and  $\Lambda$  from 42 High Redshift Supernovae*, *Astrophys. J.* **517** (1999) 565 [[astro-ph/9812133](#)].
- [9] SUPERNOVA SEARCH TEAM collaboration, *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009 [[astro-ph/9805201](#)].
- [10] G. Efstathiou, W.J. Sutherland and S.J. Maddox, *The cosmological constant and cold dark matter*, *Nature* **348** (1990) 705.
- [11] PLANCK collaboration, *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6 [[1807.06209](#)].
- [12] S. Weinberg, *The Cosmological Constant Problem*, *Rev. Mod. Phys.* **61** (1989) 1.
- [13] S. Weinberg, *Anthropic Bound on the Cosmological Constant*, *Phys. Rev. Lett.* **59** (1987) 2607.

- [14] P.J.E. Peebles and B. Ratra, *Cosmology with a Time Variable Cosmological Constant*, *Astrophys. J. Lett.* **325** (1988) L17.
- [15] B. Ratra and P.J.E. Peebles, *Cosmological Consequences of a Rolling Homogeneous Scalar Field*, *Phys. Rev. D* **37** (1988) 3406.
- [16] C. Wetterich, *The Cosmon model for an asymptotically vanishing time dependent cosmological 'constant'*, *Astron. Astrophys.* **301** (1995) 321 [[hep-th/9408025](#)].
- [17] E.J. Copeland, A.R. Liddle and D. Wands, *Exponential potentials and cosmological scaling solutions*, *Phys. Rev. D* **57** (1998) 4686 [[gr-qc/9711068](#)].
- [18] P.G. Ferreira and M. Joyce, *Structure formation with a selftuning scalar field*, *Phys. Rev. Lett.* **79** (1997) 4740 [[astro-ph/9707286](#)].
- [19] P.G. Ferreira and M. Joyce, *Cosmology with a primordial scaling field*, *Phys. Rev. D* **58** (1998) 023503 [[astro-ph/9711102](#)].
- [20] E.J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, *Int. J. Mod. Phys. D* **15** (2006) 1753 [[hep-th/0603057](#)].
- [21] DESI collaboration, *DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations*, [2404.03002](#).
- [22] A. Palatini, *Deduzione invariante delle equazioni gravitazionali dal principio di Hamilton*, *Rend. Circ. Mat. Palermo* **43** (1919) 203.
- [23] F. Bauer and D.A. Demir, *Inflation with Non-Minimal Coupling: Metric versus Palatini Formulations*, *Phys. Lett. B* **665** (2008) 222 [[0803.2664](#)].
- [24] F.L. Bezrukov and M. Shaposhnikov, *The Standard Model Higgs boson as the inflaton*, *Phys. Lett. B* **659** (2008) 703 [[0710.3755](#)].
- [25] A.A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, *Phys. Lett. B* **91** (1980) 99.
- [26] V.-M. Enckell, K. Enqvist, S. Rasanen and L.-P. Wahlman, *Inflation with  $R^2$  term in the Palatini formalism*, *JCAP* **02** (2019) 022 [[1810.05536](#)].
- [27] I. Antoniadis, A. Karam, A. Lykkas and K. Tamvakis, *Palatini inflation in models with an  $R^2$  term*, *JCAP* **11** (2018) 028 [[1810.10418](#)].
- [28] T.P. Sotiriou and V. Faraoni,  *$f(R)$  Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451 [[0805.1726](#)].
- [29] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, Cambridge, UK (2, 1984), [10.1017/CBO9780511622632](#).
- [30] S.M. Carroll, *Quintessence and the rest of the world*, *Phys. Rev. Lett.* **81** (1998) 3067 [[astro-ph/9806099](#)].

- [31] J. Khoury and A. Weltman, *Chameleon fields: Awaiting surprises for tests of gravity in space*, *Phys. Rev. Lett.* **93** (2004) 171104 [[astro-ph/0309300](#)].
- [32] J. Khoury and A. Weltman, *Chameleon cosmology*, *Phys. Rev. D* **69** (2004) 044026 [[astro-ph/0309411](#)].
- [33] L. Amendola, *Coupled quintessence*, *Phys. Rev. D* **62** (2000) 043511 [[astro-ph/9908023](#)].
- [34] T. Damour, G.W. Gibbons and C. Gundlach, *Dark Matter, Time Varying G, and a Dilaton Field*, *Phys. Rev. Lett.* **64** (1990) 123.
- [35] J.-P. Uzan, *Cosmological scaling solutions of nonminimally coupled scalar fields*, *Phys. Rev. D* **59** (1999) 123510 [[gr-qc/9903004](#)].
- [36] T. Chiba, *Quintessence, the gravitational constant, and gravity*, *Phys. Rev. D* **60** (1999) 083508 [[gr-qc/9903094](#)].
- [37] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge Univ. Press, Cambridge, UK (5, 2012), [10.1017/CBO9781139026833](#).
- [38] S.R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, *Phys. Rev.* **159** (1967) 1251.
- [39] R. Haag, J.T. Lopuszanski and M. Sohnius, *All Possible Generators of Supersymmetries of the s Matrix*, *Nucl. Phys. B* **88** (1975) 257.
- [40] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton, NJ, USA (1992).
- [41] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, *Phys. Lett. B* **51** (1974) 461.
- [42] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Yang-Mills Theories with Local Supersymmetry: Lagrangian, Transformation Laws and SuperHiggs Effect*, *Nucl. Phys. B* **212** (1983) 413.
- [43] D.Z. Freedman, *Supergravity with Axial Gauge Invariance*, *Phys. Rev. D* **15** (1977) 1173.
- [44] Z. Komargodski and N. Seiberg, *Comments on the Fayet-Iliopoulos Term in Field Theory and Supergravity*, *JHEP* **06** (2009) 007 [[0904.1159](#)].
- [45] N. Cribiori, F. Farakos, M. Tournoy and A. van Proeyen, *Fayet-Iliopoulos terms in supergravity without gauged R-symmetry*, *JHEP* **04** (2018) 032 [[1712.08601](#)].
- [46] N. Cribiori, F. Farakos and M. Tournoy, *Supersymmetric Born-Infeld actions and new Fayet-Iliopoulos terms*, *JHEP* **03** (2019) 050 [[1811.08424](#)].
- [47] F. Farakos, A. Kehagias and N. Liatsos, *On Global Symmetries and Fayet-Iliopoulos Terms*, [2404.12799](#).

- [48] M.D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (3, 2014).
- [49] N.E.J. Bjerrum-Bohr and O.T. Engelund, *Gravitino Interactions from Yang-Mills Theory*, *Phys. Rev. D* **81** (2010) 105009 [[1002.2279](#)].
- [50] T. Moroi, *Effects of the gravitino on the inflationary universe*, other thesis, 3, 1995, [[hep-ph/9503210](#)].
- [51] I. Antoniadis, A. Chatrabhuti, H. Isono and R. Knoops, *Fayet–Iliopoulos terms in supergravity and D-term inflation*, *Eur. Phys. J. C* **78** (2018) 366 [[1803.03817](#)].
- [52] I. Antoniadis, A. Chatrabhuti, H. Isono and R. Knoops, *The cosmological constant in Supergravity*, *Eur. Phys. J. C* **78** (2018) 718 [[1805.00852](#)].
- [53] I. Antoniadis and F. Rondeau, *New Kähler invariant Fayet–Iliopoulos terms in supergravity and cosmological applications*, *Eur. Phys. J. C* **80** (2020) 346 [[1912.08117](#)].
- [54] H. Jang and M. Porrati, *Component actions of liberated  $\mathcal{N} = 1$  supergravity and new Fayet–Iliopoulos terms in superconformal tensor calculus*, *JHEP* **11** (2021) 075 [[2108.04469](#)].
- [55] H. Jang and M. Porrati, *Realization of slow-roll inflation and the MSSM in supergravity theories with new Fayet–Iliopoulos terms*, *Phys. Rev. D* **106** (2022) 045024 [[2207.01889](#)].
- [56] K. Becker, M. Becker and J.H. Schwarz, *String theory and M-theory: A modern introduction*, Cambridge University Press (12, 2006), [10.1017/CBO9780511816086](#).
- [57] L.E. Ibanez and A.M. Uranga, *String theory and particle physics: An introduction to string phenomenology*, Cambridge University Press (2, 2012).
- [58] T. Van Riet and G. Zoccarato, *Beginners lectures on flux compactifications and related Swampland topics*, *Phys. Rept.* **1049** (2024) 1 [[2305.01722](#)].
- [59] M.B. Green, J.H. Schwarz and E. Witten, *SUPERSTRING THEORY. VOL. 2: LOOP AMPLITUDES, ANOMALIES AND PHENOMENOLOGY* (7, 1988).
- [60] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four-dimensions*, *Adv. Theor. Math. Phys.* **4** (2000) 1209 [[hep-th/0002240](#)].
- [61] P. Candelas and X. de la Ossa, *Moduli Space of Calabi-Yau Manifolds*, *Nucl. Phys. B* **355** (1991) 455.
- [62] A. Ceresole, R. D’Auria and S. Ferrara, *The Symplectic structure of  $N=2$  supergravity and its central extension*, *Nucl. Phys. B Proc. Suppl.* **46** (1996) 67 [[hep-th/9509160](#)].
- [63] S.B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys. Rev. D* **66** (2002) 106006 [[hep-th/0105097](#)].

- [64] S. Gukov, C. Vafa and E. Witten, *CFT's from Calabi-Yau four folds*, *Nucl. Phys. B* **584** (2000) 69 [[hep-th/9906070](#)].
- [65] M.R. Douglas, *The Statistics of string / M theory vacua*, *JHEP* **05** (2003) 046 [[hep-th/0303194](#)].
- [66] S. Ashok and M.R. Douglas, *Counting flux vacua*, *JHEP* **01** (2004) 060 [[hep-th/0307049](#)].
- [67] F. Denef and M.R. Douglas, *Distributions of flux vacua*, *JHEP* **05** (2004) 072 [[hep-th/0404116](#)].
- [68] E. Cremmer, S. Ferrara, C. Kounnas and D.V. Nanopoulos, *Naturally Vanishing Cosmological Constant in N=1 Supergravity*, *Phys. Lett. B* **133** (1983) 61.
- [69] C. Vafa, *Evidence for F theory*, *Nucl. Phys. B* **469** (1996) 403 [[hep-th/9602022](#)].
- [70] M. Haack and J. Louis, *M theory compactified on Calabi-Yau fourfolds with background flux*, *Phys. Lett. B* **507** (2001) 296 [[hep-th/0103068](#)].
- [71] T.W. Grimm, *The N=1 effective action of F-theory compactifications*, *Nucl. Phys. B* **845** (2011) 48 [[1008.4133](#)].
- [72] I. Bena, J. Blåbäck, M. Graña and S. Lüster, *The tadpole problem*, *JHEP* **11** (2021) 223 [[2010.10519](#)].
- [73] I. Bena, J. Blåbäck, M. Graña and S. Lüster, *Algorithmically Solving the Tadpole Problem*, *Adv. Appl. Clifford Algebras* **32** (2022) 7 [[2103.03250](#)].
- [74] I. Bena, C. Brodie and M. Graña, *D7 moduli stabilization: the tadpole menace*, *JHEP* **01** (2022) 138 [[2112.00013](#)].
- [75] E. Plauschinn, *The tadpole conjecture at large complex-structure*, *JHEP* **02** (2022) 206 [[2109.00029](#)].
- [76] K. Tsagkaris and E. Plauschinn, *Moduli stabilization in type IIB orientifolds at  $h^{2,1} = 50$* , *JHEP* **03** (2023) 049 [[2207.13721](#)].
- [77] T.W. Grimm, E. Plauschinn and D. van de Heisteeg, *Moduli stabilization in asymptotic flux compactifications*, *JHEP* **03** (2022) 117 [[2110.05511](#)].
- [78] M. Graña, T.W. Grimm, D. van de Heisteeg, A. Herraez and E. Plauschinn, *The tadpole conjecture in asymptotic limits*, *JHEP* **08** (2022) 237 [[2204.05331](#)].
- [79] F. Marchesano, D. Prieto and M. Wiesner, *F-theory flux vacua at large complex structure*, *JHEP* **08** (2021) 077 [[2105.09326](#)].
- [80] S. Lüster, *Large complex structure flux vacua of IIB and the Tadpole Conjecture*, [2109.05033](#).
- [81] S. Lüster and M. Wiesner, *The tadpole conjecture in the interior of moduli space*, *JHEP* **12** (2023) 029 [[2211.05128](#)].

- [82] T. Coudarchet, F. Marchesano, D. Prieto and M.A. Urkiola, *Symmetric fluxes and small tadpoles*, *JHEP* **08** (2023) 016 [[2304.04789](#)].
- [83] K. Becker, E. Gonzalo, J. Walcher and T. Wrase, *Fluxes, vacua, and tadpoles meet Landau-Ginzburg and Fermat*, *JHEP* **12** (2022) 083 [[2210.03706](#)].
- [84] T.W. Grimm, *Moduli space holography and the finiteness of flux vacua*, *JHEP* **10** (2021) 153 [[2010.15838](#)].
- [85] B. Bakker, T.W. Grimm, C. Schnell and J. Tsimmerman, *Finiteness for self-dual classes in integral variations of Hodge structure*, [2112.06995](#).
- [86] T.W. Grimm, *Taming the landscape of effective theories*, *JHEP* **11** (2022) 003 [[2112.08383](#)].
- [87] T.W. Grimm and J. Monnee, *Finiteness Theorems and Counting Conjectures for the Flux Landscape*, [2311.09295](#).
- [88] M. Dine and N. Seiberg, *Is the Superstring Weakly Coupled?*, *Phys. Lett. B* **162** (1985) 299.
- [89] M. Cicoli, J.P. Conlon, A. Maharana, S. Parameswaran, F. Quevedo and I. Zavala, *String cosmology: From the early universe to today*, *Phys. Rept.* **1059** (2024) 1 [[2303.04819](#)].
- [90] S. Kachru, R. Kallosh, A.D. Linde and S.P. Trivedi, *De Sitter vacua in string theory*, *Phys. Rev. D* **68** (2003) 046005 [[hep-th/0301240](#)].
- [91] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities*, *JHEP* **08** (2000) 052 [[hep-th/0007191](#)].
- [92] V. Balasubramanian, P. Berglund, J.P. Conlon and F. Quevedo, *Systematics of moduli stabilisation in Calabi-Yau flux compactifications*, *JHEP* **03** (2005) 007 [[hep-th/0502058](#)].
- [93] I. Antoniadis, Y. Chen and G.K. Leontaris, *Perturbative moduli stabilisation in type IIB/F-theory framework*, *Eur. Phys. J. C* **78** (2018) 766 [[1803.08941](#)].
- [94] I. Antoniadis, Y. Chen and G.K. Leontaris, *Logarithmic loop corrections, moduli stabilisation and de Sitter vacua in string theory*, *JHEP* **01** (2020) 149 [[1909.10525](#)].
- [95] I. Antoniadis, O. Lacombe and G.K. Leontaris, *Inflation near a metastable de Sitter vacuum from moduli stabilisation*, *Eur. Phys. J. C* **80** (2020) 1014 [[2007.10362](#)].
- [96] I. Antoniadis, O. Lacombe and G.K. Leontaris, *Hybrid inflation and waterfall field in string theory from D7-branes*, *JHEP* **01** (2022) 011 [[2109.03243](#)].
- [97] D. Junghans, *LVS de Sitter vacua are probably in the swampland*, *Nucl. Phys. B* **990** (2023) 116179 [[2201.03572](#)].

- [98] B. Valeixo Bento, D. Chakraborty, S. Parameswaran and I. Zavala, *De Sitter vacua — when are ‘subleading corrections’ really subleading?*, *JHEP* **11** (2023) 075 [[2306.07332](#)].
- [99] S. Reffert, *Toroidal Orbifolds: Resolutions, Orientifolds and Applications in String Phenomenology*, Ph.D. thesis, Munich U., 2006. [hep-th/0609040](#).
- [100] D. Lust, S. Reffert, W. Schulgin and S. Stieberger, *Moduli stabilization in type IIB orientifolds (I): Orbifold limits*, *Nucl. Phys. B* **766** (2007) 68 [[hep-th/0506090](#)].
- [101] D. Lust, S. Reffert, E. Scheidegger and S. Stieberger, *Resolved Toroidal Orbifolds and their Orientifolds*, *Adv. Theor. Math. Phys.* **12** (2008) 67 [[hep-th/0609014](#)].
- [102] D. Lust, S. Reffert, E. Scheidegger, W. Schulgin and S. Stieberger, *Moduli Stabilization in Type IIB Orientifolds (II)*, *Nucl. Phys. B* **766** (2007) 178 [[hep-th/0609013](#)].
- [103] A. Font, L.E. Ibanez and F. Quevedo,  *$Z(N) \times Z(m)$  Orbifolds and Discrete Torsion*, *Phys. Lett. B* **217** (1989) 272.
- [104] A. Font,  *$Z(N)$  orientifolds with flux*, *JHEP* **11** (2004) 077 [[hep-th/0410206](#)].
- [105] K. Ishiguro, T. Kobayashi and H. Otsuka, *Landscape of Modular Symmetric Flavor Models*, *JHEP* **03** (2021) 161 [[2011.09154](#)].
- [106] K. Ishiguro, T. Kai, T. Kobayashi and H. Otsuka, *Flux Landscape with enhanced symmetry not on  $SL(2, \mathbb{Z})$  elliptic points*, *JHEP* **02** (2024) 099 [[2311.12425](#)].
- [107] J. Richter-Gebert, *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry*, Springer Berlin Heidelberg (2011).
- [108] J.F.G. Cascales and A.M. Uranga, *Chiral 4d string vacua with D branes and NSNS and RR fluxes*, *JHEP* **05** (2003) 011 [[hep-th/0303024](#)].
- [109] F. Marchesano and G. Shiu, *Building MSSM flux vacua*, *JHEP* **11** (2004) 041 [[hep-th/0409132](#)].
- [110] M. Cvetič, G. Shiu and A.M. Uranga, *Three family supersymmetric standard - like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [[hep-th/0107143](#)].
- [111] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional  $N=1$  supersymmetric type 2A orientifolds from intersecting D6 branes*, *Nucl. Phys. B* **615** (2001) 3 [[hep-th/0107166](#)].
- [112] G. Aldazabal, S. Franco, L.E. Ibanez, R. Rabadan and A.M. Uranga, *Intersecting brane worlds*, *JHEP* **02** (2001) 047 [[hep-ph/0011132](#)].
- [113] G. Aldazabal, S. Franco, L.E. Ibanez, R. Rabadan and A.M. Uranga,  *$D = 4$  chiral string compactifications from intersecting branes*, *J. Math. Phys.* **42** (2001) 3103 [[hep-th/0011073](#)].



- [114] A. Abouelsaood, C.G. Callan, Jr., C.R. Nappi and S.A. Yost, *Open strings in background gauge fields*, *Nucl. Phys. B* **280** (1987) 599.
- [115] C. Bachas, *A Way to break supersymmetry*, [hep-th/9503030](#).
- [116] H. Jockers and J. Louis, *The Effective action of D7-branes in  $N = 1$  Calabi-Yau orientifolds*, *Nucl. Phys. B* **705** (2005) 167 [[hep-th/0409098](#)].
- [117] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *Phys. Rept.* **445** (2007) 1 [[hep-th/0610327](#)].
- [118] L.A. Anchordoqui and I. Antoniadis, *Large extra dimensions from higher-dimensional inflation*, *Phys. Rev. D* **109** (2024) 103508 [[2310.20282](#)].
- [119] C. Vafa, *The String landscape and the swampland*, [hep-th/0509212](#).
- [120] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21 [[hep-th/0605264](#)].
- [121] E. Palti, *The Swampland: Introduction and Review*, *Fortsch. Phys.* **67** (2019) 1900037 [[1903.06239](#)].
- [122] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, *Phys. Rept.* **989** (2022) 1 [[2102.01111](#)].
- [123] T. Banks and N. Seiberg, *Symmetries and Strings in Field Theory and Gravity*, *Phys. Rev. D* **83** (2011) 084019 [[1011.5120](#)].
- [124] W. Israel, *Event horizons in static vacuum space-times*, *Phys. Rev.* **164** (1967) 1776.
- [125] T. Banks and L.J. Dixon, *Constraints on String Vacua with Space-Time Supersymmetry*, *Nucl. Phys. B* **307** (1988) 93.
- [126] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [[hep-th/0601001](#)].
- [127] D. Klaewer and E. Palti, *Super-Planckian Spatial Field Variations and Quantum Gravity*, *JHEP* **01** (2017) 088 [[1610.00010](#)].
- [128] G. Dvali, *Black Holes and Large  $N$  Species Solution to the Hierarchy Problem*, *Fortsch. Phys.* **58** (2010) 528 [[0706.2050](#)].
- [129] G. Dvali and M. Redi, *Black Hole Bound on the Number of Species and Quantum Gravity at LHC*, *Phys. Rev. D* **77** (2008) 045027 [[0710.4344](#)].
- [130] S.-J. Lee, W. Lerche and T. Weigand, *Emergent strings from infinite distance limits*, *JHEP* **02** (2022) 190 [[1910.01135](#)].
- [131] D. Lüst, E. Palti and C. Vafa, *AdS and the Swampland*, *Phys. Lett. B* **797** (2019) 134867 [[1906.05225](#)].

- [132] A. Higuchi, *Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time*, *Nucl. Phys. B* **282** (1987) 397.
- [133] T. Rudelius, *Dimensional reduction and (Anti) de Sitter bounds*, *JHEP* **08** (2021) 041 [2101.11617].
- [134] A. Castellano, A. Herráez and L.E. Ibáñez, *IR/UV mixing, towers of species and swampland conjectures*, *JHEP* **08** (2022) 217 [2112.10796].
- [135] M. Montero, C. Vafa and I. Valenzuela, *The dark dimension and the Swampland*, *JHEP* **02** (2023) 022 [2205.12293].
- [136] J.G. Lee, E.G. Adelberger, T.S. Cook, S.M. Fleischer and B.R. Heckel, *New Test of the Gravitational  $1/r^2$  Law at Separations down to 52  $\mu\text{m}$* , *Phys. Rev. Lett.* **124** (2020) 101101 [2002.11761].
- [137] W.-H. Tan et al., *Improvement for Testing the Gravitational Inverse-Square Law at the Submillimeter Range*, *Phys. Rev. Lett.* **124** (2020) 051301.
- [138] S. Hannestad and G.G. Raffelt, *Supernova and neutron star limits on large extra dimensions reexamined*, *Phys. Rev. D* **67** (2003) 125008 [hep-ph/0304029].
- [139] E. Gonzalo, M. Montero, G. Obied and C. Vafa, *Dark dimension gravitons as dark matter*, *JHEP* **11** (2023) 109 [2209.09249].
- [140] J.A.P. Law-Smith, G. Obied, A. Prabhu and C. Vafa, *Astrophysical constraints on decaying dark gravitons*, *JHEP* **06** (2024) 047 [2307.11048].
- [141] G. Obied, C. Dvorkin, E. Gonzalo and C. Vafa, *Dark dimension and decaying dark matter gravitons*, *Phys. Rev. D* **109** (2024) 063540 [2311.05318].
- [142] L.A. Anchordoqui, I. Antoniadis and D. Lust, *Dark dimension, the swampland, and the dark matter fraction composed of primordial black holes*, *Phys. Rev. D* **106** (2022) 086001 [2206.07071].
- [143] L.A. Anchordoqui, I. Antoniadis and D. Lust, *The dark universe: Primordial black hole  $\rightleftharpoons$  dark graviton gas connection*, *Phys. Lett. B* **840** (2023) 137844 [2210.02475].
- [144] L.A. Anchordoqui, I. Antoniadis and D. Lust, *Dark dimension, the swampland, and the dark matter fraction composed of primordial near-extremal black holes*, *Phys. Rev. D* **109** (2024) 095008 [2401.09087].
- [145] L. Anchordoqui, I. Antoniadis and D. Lust, *Landscape, Swampland, and Extra Dimensions*, 5, 2024 [2405.04427].
- [146] C. Vafa, *Swamplandish Unification of the Dark Sector*, 2402.00981.
- [147] E.R. Harrison, *Fluctuations at the threshold of classical cosmology*, *Phys. Rev. D* **1** (1970) 2726.
- [148] Y.B. Zeldovich, *A Hypothesis, unifying the structure and the entropy of the universe*, *Mon. Not. Roy. Astron. Soc.* **160** (1972) 1P.

- [149] PLANCK collaboration, *Planck 2018 results. X. Constraints on inflation*, *Astron. Astrophys.* **641** (2020) A10 [[1807.06211](#)].
- [150] A. Riotto, *Inflation and the theory of cosmological perturbations*, *ICTP Lect. Notes Ser.* **14** (2003) 317 [[hep-ph/0210162](#)].
- [151] D. Baumann, *Inflation*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, pp. 523–686, 2011, DOI [[0907.5424](#)].
- [152] D. Langlois, *Lectures on inflation and cosmological perturbations*, *Lect. Notes Phys.* **800** (2010) 1 [[1001.5259](#)].
- [153] V.F. Mukhanov and G.V. Chibisov, *Quantum Fluctuations and a Nonsingular Universe*, *JETP Lett.* **33** (1981) 532.
- [154] H. Motohashi, A.A. Starobinsky and J. Yokoyama, *Inflation with a constant rate of roll*, *JCAP* **09** (2015) 018 [[1411.5021](#)].
- [155] A. Lukas, B.A. Ovrut and D. Waldram, *Cosmological solutions of Horava-Witten theory*, *Phys. Rev. D* **60** (1999) 086001 [[hep-th/9806022](#)].
- [156] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, *Brane cosmological evolution in a bulk with cosmological constant*, *Phys. Lett. B* **477** (2000) 285 [[hep-th/9910219](#)].
- [157] A. Lukas, B.A. Ovrut and D. Waldram, *Boundary inflation*, *Phys. Rev. D* **61** (2000) 023506 [[hep-th/9902071](#)].
- [158] P. Binetruy, C. Deffayet and D. Langlois, *Nonconventional cosmology from a brane universe*, *Nucl. Phys. B* **565** (2000) 269 [[hep-th/9905012](#)].
- [159] L. Randall and R. Sundrum, *A Large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [[hep-ph/9905221](#)].
- [160] L. Randall and R. Sundrum, *An Alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [[hep-th/9906064](#)].
- [161] C. van de Bruck, M. Dorca, R.H. Brandenberger and A. Lukas, *Cosmological perturbations in brane world theories: Formalism*, *Phys. Rev. D* **62** (2000) 123515 [[hep-th/0005032](#)].
- [162] C. Gordon, D. Wands, B.A. Bassett and R. Maartens, *Adiabatic and entropy perturbations from inflation*, *Phys. Rev. D* **63** (2000) 023506 [[astro-ph/0009131](#)].