Tropical linear series, combinatorial flag arrangements and applications to the study of Weierstrass points

Lucas Gierczak-Galle

INSTITUT POLYTECHNIQUE DE PARIS

NNT : 2024IPPAX083

Thèse de doctorat

ÉCOLE POLYTECHNIQUE

IP PARIS

Tropical linear series, combinatorial flag arrangements and applications to the study of Weierstrass points

Thèse de doctorat de l'Institut polytechnique de Paris
préparée à l'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales

Thèse présentée et soutenue à Palaiseau, le 24 octobre 2024, par

LUCAS GIERCZAK-GALLE

Composition du Jury :

Stéphane Gaubert Directeur de recherche, Inria Saclay	Président
Erwan Brugallé *(absent lors de la soutenance)* Professeur, Nantes Université	Rapporteur
Jan Draisma Professeur, Université de Berne	Rapporteur
Ana-Maria Castravet Professeure, Université Paris-Saclay	Examinatrice
Enrica Mazzon Maîtresse de conférences, Université Paris Cité	Examinatrice
Martin Ulirsch Professeur, Université Goethe de Francfort-sur-le-Main	Examinateur
Omid Amini Professeur chargé de cours, École polytechnique	Directeur de thèse

Remerciements

Écrire des remerciements est rarement une tâche facile. Il s'agit de condenser en quelques paragraphes plusieurs années de vie, de travail et de rencontres avec des personnes formidables. Tentons toutefois de rester simples.

Mes remerciements vont tout d'abord à Omid Amini, mon directeur de thèse. Merci, Omid, de m'avoir encadré pour mon stage de recherche en master, de m'avoir fait confiance pour la thèse et de m'avoir ouvert de magnifiques horizons mathématiques, d'une grande diversité. Merci pour ton encadrement de qualité, ton écoute attentive et ton implication sincère dans le bon déroulement de ma thèse. Tu as joué un rôle important pour me donner confiance en moi. Merci pour tes qualités mathématiques et humaines exceptionnelles.

Many thanks to Erwan Brugallé et Jan Draisma for accepting to review my PhD thesis, to Stéphane Gaubert for being the president of my jury, and to Ana-Maria Castravet, Martin Ulirsch and Enrica Mazzon for being members of my jury.

Je pense ensuite à toutes et tous les collègues du CMLS, avec qui il a toujours été si plaisant de discuter maths, vulgarisation, montagne, trains, escalade, ou politique de l'École polytechnique : Stefano Aloé, Paulo Alves Sampaio, Marine Amier, Diego Berger, Stéphane Bijakowski, Margaret Bilu, Étienne Bonnafoux, Claudio Bravo-Castillo, Kleber Carrapatoso, Anne-Sophie De Suzzoni, Eleonora Di Nezza, Guillaume Dubach, Melvyn El Kamel-Meyrigne, Lucas Ertzbischoff, Lorenzo Fantini, Charles Favre, Siarhei Finski, Béatrice Fixois, Gerard Freixas, Felipe Gambardella, Paul Gauduchon, Quentin Gazda, François Golse, Chen Gong, Pascale Harinck, Cécile Huneau, Hernán Iriarte, Diego Izquierdo, Thibault Juillard, Carole Juppin, Raphaël Krikorian, Seungkyu Lee, Jialun Li, Christophe Margerin, Mirko Mauri, Majdi Medjahed, Pascal Millet, Francesco Morabito, Noema Nicolussi, Frank Pacard, Nicolas Perrin, Matthieu Piquerez, Chuan Qin, David Renard, Gabriel Ribeiro, Claude Sabbah, Tony Salvi, Pierre Schatz, Annalaura Stingo, Arthur

Touati et Lamberto Tresoldi. Le CMLS a été pour moi un endroit très accueillant pour mener ma thèse, et cela tient en grande partie aux personnes qui m'y ont entouré.

Merci en particulier à Marine Amier, Béatrice Fixois et Carole Juppin pour leur aide précieuse et efficace tout au long de la thèse, pour les missions, l'organisation de la soutenance, les TD du Bachelor. Le laboratoire et le département vous doivent beaucoup.

Merci également à David Delavennat, Yannick Fitamant et Ilan Settouraman, de la cellule informatique, toujours disponibles pour résoudre nos problèmes techniques et veiller sur nos bons usages de l'informatique.

Merci à Charles Favre et Nicolas Perrin pour leur grande disponibilité et leur aide précieuse en tant que directeurs de laboratoire. Un grand merci également à Stéphane Bijakowski, Anne-Sophie De Suzzoni et François Golse pour leur grandes qualités dans la gestion du département et l'attribution des enseignements.

Merci à Frédéric Bourgeois, Emmanuel Fullenwarth, Olivier Le Maître, Stéphane Nonnenmacher et Benjamin Schraen pour la qualité de leur suivi pour l'école doctorale. Merci à la FMJH pour son financement m'ayant permis de voyager à Berlin.

Next, I would like to thank my Bachelor students in Integration, Taylor expansion, Ordinary differential equations and Linear algebra II. It was pleasant to have you in class – even correcting your tests was agreeable. Merci à Jérémie Bettinelli et Guillaume Dubach pour les moments passés à préparer des examens, harmoniser des barèmes et gérer des faits de plagiat.

Je souhaiterais remercier, plus largement, toutes les personnes que j'ai côtoyées à l'École polytechnique pendant ma thèse. Je pense en particulier à mes collègues représentant·es élu·es au Conseil d'administration : Octave Berbigier, Antoine Bessemoulin, Silke Biermann, Nina Bili-Rossi, Jérémy Huet, Selim Khouaja, Paul Lesur, Sébastien Maron, Aldjia Mazari, Jean-Luc Moncel, Léna Rodrigues, Loman Sezestre et Benjamin Werner. Merci également aux membres du collectif doctorant d'IP Paris.

I have a friendly thought for the nice people I met during workshops, summer schools and conferences, from the algebraic geometry, tropical geometry and combinatorics communities, in Le Croisic, Tatihou, Orsay, Cambridge, London, Frankfurt, Berlin, Warwick, Lausanne and Bordeaux. Special thoughts to Ulysses Alvarez, Matt Baker, Jefferson Baudin, Mara Belotti, Thomas Blomme, Anna Bot, Tess Bouis, Bilson Castro, Daniel Corey, Lucie Devey, Matthew Dupraz, Gavril Farkas, Alex Fink, Cécile Gachet, Manoel Jarra, Lukas Gustafsson, Junaid Hasan, Dante Luber, Isaac Martin, Leonid Monin, Anne Pichon, Matthieu Piquerez, Constantin Podelski, Thibault Poiret, Zéphyr Salvy, Farbod Shokrieh, Connor Simpson, Pedro Souza, Pim Spelier, and Alejandro Vargas. Many thanks to David Jensen, Diane Maclagan and Sam Payne for discussions about tropical linear series, and to Stéphane Gaubert, Harry Richman and Farbod Shokrieh for our pleasant and fruitful collaborations.

I am deeply grateful to the people who invited me to give talks in conferences or seminars: Tess Bouis, Julien Boulanger, Erwan Brugallé, Vincent Delecroix, Thibault Juillard, Sébastien Labbé, and Martin Ulirsch.

J'ai une pensée toute particulière pour les excellent·es enseignant·es auprès de qui j'ai eu la chance d'apprendre au fil de ma vie. Cela inclut Omid Amini, Bérénice Bonnardot, Raphaël Cerf, Laurent Charles, Pierre Charollois, Gérard Colin, Jean-François Dat, Antoine Ducros, François Loeser, Ilia Itenberg, Jean-François Logeais, Julien Marché, Alexandru Oancea, Odile Pauffard, Alain Pommellet, Lydie Prieur, Béatrice Rativeau, Anne-Sophie Rivat, Julien Rochet et Vincent Thouard. Merci pour votre humanité et vos cours passionnants qui m'ont toujours donné envie d'aller plus loin.

Un grand merci également à la si charmante équipe de *Pour la science*, pour m'avoir accueilli en stage dans la vulgarisation scientifique entre mon master et ma thèse.

Je souhaite ensuite remercier les ami·es de divers horizons, pour les bons moments passés ensemble, hier ou aujourd'hui, mais aussi pour leur aide, leur franchise, et les belles discussions. AA, LA, AB, JB, MB, SB, TB, TB, BC, EC, MNC, PC, LD, MD, PD, RD, AF, CF, LF, RF, AG, CG, EG, FG, LAG, MG, MG, PG, VG, PH, TJ, RK, EL, GDL, ML, PL, QL, SL, SL, CM, DM, EM, LM, YMH, EP, LP, TPH, AR, AR, GR, JPR, OR, RR, AS, BS, ES, LS, NS, AT, BT, VT, AV, LV, MV, QV, TV, PW, LZ, merci pour tout.

Merci également à Sylvie, Laurent et Sophie, Marie, Hervé et Françoise, Michèle et Laurent, Mathilde, Noémie, Philippe et Anne, Éric et Marianne pour votre amitié et les beaux moments passés ensemble.

Un immense merci à ma famille, qui m'a accompagné, soutenu et fait grandir pendant toutes ces belles années : mes parents, mes frères, Erika, Anne, Arnaud, Joséphine, Victoria, Omi, Grand-père, France, les cousins ; mais aussi toutes celles et tous ceux qui ne sont plus là et à qui je pense avec tendresse.

Merci, enfin, à Pauline d'illuminer chaque jour de ma vie.

Préface

Dans cette thèse, nous introduisons d'abord de nouveaux objets combinatoires nommés "matricubes", qui sont une généralisation assez naturelle des matroïdes. De la même façon que les matroïdes fournissent une axiomatisation combinatoire des arrangements d'hyperplans (ou, de façon duale, de vecteurs) dans un espace vectoriel, les matricubes abstraient les arrangements de drapeaux initiaux. Puisque les matroïdes peuvent être définis de manière équivalente par différents systèmes d'axiomes, nous proposons, de même, plusieurs définitions cryptomorphes des matricubes en termes de fonction de rang, de collection de plats, de circuits, et d'indépendants. Cela se fait, en particulier, en étudiant la structure combinatoire de certaines de ces collections. Nous formulons aussi un concept de dualité pour les matricubes, et deux opérations de délétion et de contraction qui étendent celles sur les matroïdes. Nous exhibons des liens précis entre les matricubes et les tableaux de permutation et proposons une description des matricubes comme des collections de matroïdes locaux vérifiant certaines propriétés. Nous explorons certains problèmes de représentabilité pour les matricubes, et posons plusieurs questions ouvertes.

Ce concept de matricube nous aide ensuite à développer une nouvelle théorie purement combinatoire des séries linéaires limites sur les graphes métriques. Cette dernière est également basée sur le formalisme des structures de pentes, qui contraignent les pentes et les vecteurs de pentes de fonctions affines par morceaux à valeurs réelles et à pentes entières sur les graphes métriques. En effet, les matricubes s'avèrent être le bon formalisme pour prescrire les pentes conjointes des fonctions autour des sommets du graphe. Notre définition de série linéaire combinatoire combine ensuite des propriétés combinatoires et algébriques pour donner un concept fort. Certaines séries linéaires combinatoires s'obtiennent naturellement en tropicalisant des séries linéaires sur des courbes algébriques, ce qui soulève des questions de réalisabilité. En outre, nous explorons les propriétés topologiques des séries linéaires combinatoires, définissons une bonne notion de diviseur réduit, et proposons une classification complète des séries linéaires combinatoires de rang 1 : elles sont en correspondance bijective avec les morphismes harmoniques depuis le graphe vers des arbres métriques. Ceci entraîne un théorème de lissification. Nous discutons aussi d'autres applications et de liens avec d'autres concepts de la géométrie algébrique combinatoire.

Enfin, nous étudions les points de Weierstrass tropicaux, qui sont des analogues, sur les courbes tropicales, de points spéciaux sur les courbes algébriques nommés "points de Weierstrass". Ces derniers peuvent être décrits comme les "points d'inflexion" d'une courbe lisse plongée ou, de façon équivalente, comme les points de ramification de fibrés en droites sur une courbe. Puisque le lieu de Weierstrass tropical d'un diviseur sur un

graphe métrique peut être infini, nous étudions ses propriétés topologiques, et associons un poids intrinsèque à chacune de ses composantes connexes. Ceci suppose d'analyser les pentes des fonctions affines par morceaux appartenant à la série linéaire complète du diviseur. Nous démontrons que ces poids ont de bonnes propriétés combinatoires, en ce sens que leur somme dépend uniquement du degré du diviseur, de son rang, et du genre du graphe. De surcroît, dans le cas où le graphe métrique est la tropicalisation d'une courbe algébrique, les tropicalisations des points de Weierstrass géométriques atterrissent dans le lieu de Weierstrass tropical, et nous montrons que nos poids comptent le nombre de points de Weierstrass géométriques qui dégénèrent sur chaque composante connexe du lieu de Weierstrass tropical. Ce résultat utilise les séries linéaires limites combinatoires, qui contiennent la bonne quantité d'information combinatoire provenant du monde géométrique pour permettre ce dénombrement. Nous commentons de multiples exemples de lieux de Weierstrass tropicaux pour lever partiellement le voile sur la diversité qui émane de ce concept.

Preface

In this thesis, we first introduce new combinatorial objects called "matricubes", which are a rather natural generalization of matroids. In the same way that matroids provide a combinatorial axiomatization of hyperplane arrangements (or, dually, vectors) in a vector space, matricubes abstract arrangements of initial flags. Since matroids can be equivalently defined by a variety of axiomatic systems, we provide, likewise, several cryptomorphic definitions of matricubes in terms of rank function, collections of flats, circuits, and independent sets. This is done, in particular, by studying the combinatorial structure of some of these collections. We also formulate a duality concept for matricubes, and two operations of deletion and contraction which extend those on matroids. We provide precise connections between matricubes and permutation arrays, and propose a description of matricubes as collections of local matroids satisfying certain properties. We explore some representability problems for matricubes, and ask several open questions.

The concept of matricube then helps to develop a new, purely combinatorial theory of limit linear series on metric graphs. This is also based on the formalism of slope structures, which constrains the slopes and slope vectors of piecewise affine linear real-valued functions with integral slopes on metric graphs. Indeed, matricubes turn out to be the relevant formalism to prescribe the joint slopes of functions around vertices of the graph. Our definition of combinatorial linear series then combines combinatorial and algebraic properties to yield a strong concept. Some combinatorial linear series are naturally obtained by tropicalizing linear series on algebraic curves, which raises realizability questions. Besides, we explore the topological properties of combinatorial linear series, define a relevant notion of reduced divisor, and provide a full classification for combinatorial linear series of rank one: they are in one-to-one correspondence with harmonic morphisms from the graph to metric trees. This entails a smoothing theorem. We also discuss some other applications and connections to other concepts in combinatorial algebraic geometry.

Finally, we study tropical Weierstrass points, which are analogues, on tropical curves, of special points on algebraic curves called "Weierstrass points". Those can be described as the "flex points" of smooth embedded curves or, equivalently, as the ramification points of line bundles on curves. Since the tropical Weierstrass locus of a divisor on a metric graph can be infinite, we study its topological properties, and associate intrinsic weights to its finitely many connected components. This is done by analyzing the slopes of piecewise linear functions in the complete linear series of the divisor. We prove that these weights have a nice combinatorial behavior, as their sum depends only on the degree of the divisor, its rank, and the genus of the graph. Furthermore, in the case the metric graph is the

tropicalization of an algebraic curve, the tropicalizations of geometric Weierstrass points land inside the tropical Weierstrass locus, and we show that our weights count the number of geometric Weierstrass points degenerating to each connected component of the tropical Weierstrass locus. This uses combinatorial limit linear series, which contain the right amount of combinatorial information from the geometric world to enable this count. We discuss multiple examples of tropical Weierstrass loci to show some of the diversity arising from this concept.

Table of contents

List of figures

1 Introduction

We provide a French, and then an English version of the introduction to the manuscript.

Contents of the chapter

Version française

Cette introduction a pour objectif de résumer le contenu du manuscrit de thèse, en évitant les détails techniques, qui seront développés dans les chapitres suivants. Elle devrait être utile à la fois pour introduire le propos à une personne intéressée à lire le manuscrit en entier, et pour une personne non-spécialiste voulant jeter un œil à ces sujets sans pour autant lire le texte en entier.

Cette thèse a but d'étudier certains aspects de la géométrie tropicale en lien avec la géométrie algébrique.

En quelques mots, de nombreux objets de la géométrie algébrique – comme les courbes, les diviseurs sur les courbes, les espaces de fonctions sur les courbes, ou certains points spéciaux sur les courbes – peuvent être déformés en des objets géométriques d'une nature beaucoup plus simple, ayant des propriétés combinatoires, au moyen d'une procédure nommée "tropicalisation" – voir ci-dessous pour de plus amples détails. Un fil rouge dans cette thèse consiste à étudier la tropicalisation de certains de ces objets de la géométrie algébrique. Premièrement, en développant des théories métriques et combinatoires pour décrire les objets tropicalisés que l'on peut obtenir ; ensuite, en établissant des liens entre les propriétés de ces objets tropicalisés et celles des objets d'origine ; enfin, en tentant de déterminer, parmi les objets métriques ou combinatoires, lesquels peuvent être obtenus à partir d'un objet classique par la procédure de tropicalisation.

L'inspiration pour ce projet de thèse provient de l'étude des séries linéaires du point de vue de la géométrie tropicale, qui a eu un succès considérable dans l'application des méthodes tropicales. La procédure de tropicalisation permet de transformer les sections globales d'un fibré en droites en fonctions affines par morceaux avec des pentes entières sur un objet

polyédral. L'inégalité de spécialisation de Baker en dimension 1 [Bak08, Lemma 2.8], et sa généralisation en dimension supérieure donnée par Cartwright [Car19, Car21], permettent ensuite de comparer la dimension de l'espace des sections globales du fibré en droites avec une notion de rang combinatoire associée à la tropicalisation.

Dans le présent manuscrit, nous nous occuperons principalement d'objets de dimension 1, c'est-à-dire de courbes algébriques, de graphes métriques, et des structures géométriques qui leurs sont associées. Ceci dit, nous discuterons dans la Section 5.4 des perspectives de généralisation de nos résultats en dimension supérieure, en lien avec les travaux de Cartwright mentionnés ci-dessus qui, d'un point de vue personnel, étaient notre point d'entrée dans la géométrie tropicale lors de la préparation de notre mémoire de master.

Dans la section suivante de cette introduction, nous proposerons une approche intuitive de la procédure de tropicalisation, que nous espérons utile pour comprendre comment fonctionne la tropicalisation.

La procédure de tropicalisation et l'esprit de la géométrie tropicale

L'intuition visuelle que la plupart des gens ont de la géométrie tropicale est qu'elle fournit des méthodes pour *déformer* des variétés algébriques jusqu'au point où elles se transforment en des objets *polyédraux*, tels des complexes de cônes ou des complexes polyédraux, parfois avec des données combinatoires supplémentaires.

L'intérêt de ce type de transformation est que les objets polyédraux qu'elle produit sont fondamentalement faciles à étudier, parce qu'ils peuvent être décrits avec une quantité finie d'information combinatoire (les équations définissant chaque polyèdre, par exemple).

En outre, et c'est essentiel, les transformations tropicales n'engendrent pas des objets "quelconques" : elles déforment les variétés algébriques (ou d'autres objets) de telle manière que *certaines informations pertinentes sont conservées* et peuvent être *mesurées* du côté polyédral pour mieux connaître les objets géométriques d'origine.

C'est pourquoi certaines personnes décrivent la géométrie tropicale comme "une ombre combinatoire (ou affine par morceaux) de la géométrie tropicale" [MR18, MS21].

Les premières procédures de tropicalisation effectives furent définies au début des années 1970 par Bergman [Ber71]. Elles s'appliquent principalement à des variétés affines ou projectives sur le corps $\mathbb{C}$ des nombres complexes, et utilisent les fonctions logarithmes pour transformer ces objets géométriques.

À titre d'exemple, considérons l'ensemble $\mathcal{L}$ des points à coordonnées complexes $(z_1, z_2) \in \mathbb{C}^2$ tels que $z_1 + z_2 - 1 = 0$. C'est tout simplement une droite complexe dans le plan complexe. Maintenant, définissons, pour $0 < t < 1$, l'ensemble L_t des points de $\mathbb{R}^2$ définis par le logarithme du module des points de $\mathcal{L}$, coordonnée par coordonnée :

$$L_t := \big\{(x, y) = (\log_t(|z_1|), \log_t(|z_2|)) \;\big|\; (z_1, z_2) \in \mathcal{L}\big\},$$

où $\log_t \colon \mathbb{R}_{>0} \to \mathbb{R}$ est le logarithme réel en base t. L_t est un objet *courbe*, *réel*, de dimension réelle 2 (comme $\mathcal{L}$), et est nommé *amibe*, voir la partie gauche de la Figure 1.1.

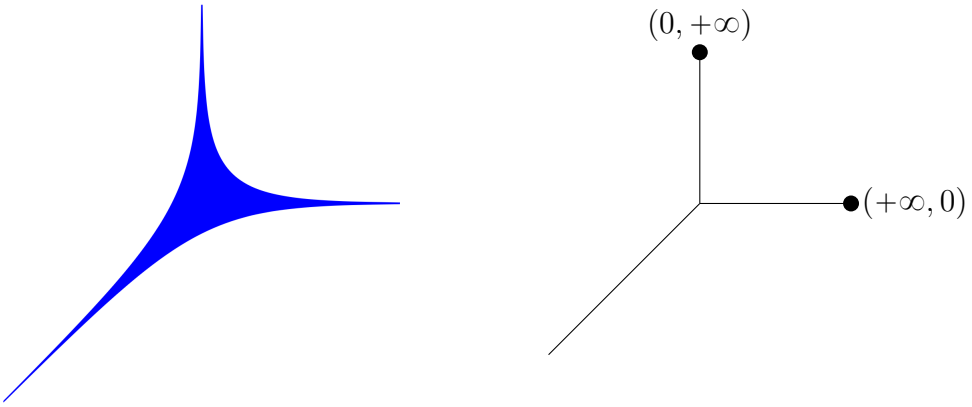


Figure 1.1: Amibe et tropicalisation de la droite complexe $\mathcal{L}$.

Si l'on choisit un autre nombre réel t' tel que $0 < t' < t$, alors l'amibe $L_{t'}$ est simplement obtenue à partir de l'amibe L_t par une homothétie de facteur $\frac{\log(t)}{\log(t')} < 1$: l'amibe se "contracte" de plus en plus lorsque t diminue.

Le phénomène intéressant, au cœur de la géométrie tropicale, se produit lorsque $t \to 0$. Dans ce cas, l'objet courbe L_t *dégénère* en un certain *objet limite* L_0, qui a la propriété d'être entièrement polyédral. L_0 est appelé la *tropicalisation* de la courbe complexe $\mathcal{L}$. Comme $\mathcal{L}$ est une droite complexe, L_0 est en fait un objet absolument fondamental en géométrie tropicale, une brique élémentaire des tropicalisations nommée *droite tropicale dans le plan*. Il est unidimensionnel et constitué de trois demi-droites réelles : une verticale, une horizontale et une diagonale (voir la partie droite de la Figure 1.1). Chacune de ces demi-droites correspond à un comportement conjoint spécifique des variables z_1 et z_2. Par exemple, la demi-droite verticale correspond à la situation où $|z_1|$ est proche de 1 et z_2 est proche de 0.

Notons que, à strictement parler, puisque $\mathcal{L}$ intersecte les axes de coordonnées de $\mathbb{C}^2$, la droite tropicale contient deux points "à l'infini", ce qui peut être décrit dans un cadre rigoureux. Ce détail technique est évité si l'on considère une sous-variété de $(\mathbb{C}^*)^2$.

De surcroît, remarquons que, pour chaque point $(z_1, z_2) \in \mathcal{L}$, chaque coordonnée du point limite $\lim_{t\to 0}(\log_t(|z_1|), \log_t(|z_2|))$ appartient à l'ensemble fini $\{0, +\infty\}$. Cependant, la droite tropicale L_0 est l'objet *unidimensionnel* obtenu comme l'ensemble limite (pour la topologie de Hausdorff) de L_t lorsque $t \to 0$, c'est-à-dire, dans ce cas particulier, l'intersection de toutes les amibes L_t pour $t \neq 0$.

Un théorème d'une importance capitale, le *théorème fondamental de la géométrie tropicale*, donne une description effective de la tropicalisation d'une variété algébrique. Si, pour prendre le cas le plus simple, une variété algébrique $\mathcal{V} \subseteq \mathbb{C}^n$ est définie par l'annulation d'un certain polynôme en n variables $P = \sum_{\mathbf{k}\in\mathbb{N}^n} a_{\mathbf{k}} X_1^{k_1} \cdots X_n^{k_n}$, avec $a_{\mathbf{k}} \in \mathbb{C}$, alors on peut associer à P un *polynôme tropical* $\mathrm{trop}(P)$ obtenu en remplaçant chaque occurrence de la *multiplication* par une *addition*, et chaque occurrence de l'*addition* par un *minimum*

(ou un maximum, selon les conventions). C'est pourquoi le formalisme algébrique utilisant ces opérations tropicales et nommé *algèbre tropicale*, ou *algèbre max-plus*. De cette façon, $\mathrm{trop}(P)$ est une fonction à valeurs réelles, linéaire par morceaux $\mathbb{R}^n \to \mathbb{R}$, définie par

$$\mathrm{trop}(P)(x_1, \ldots, x_n) \coloneqq \min_{\mathbf{k} \in S}(k_1 \cdot x_1 + \cdots + k_n \cdot x_n),$$

où $S \subseteq \mathbb{N}$ est fini. Dans un cas plus général, les coefficients $a_{\mathbf{k}}$ dépendent de t, on est alors en présence d'une *famille de sous-variétés de* $\mathbb{C}^n$; la tropicalisation de P est alors une fonction *affine* par morceaux. Cette situation fait l'objet d'une discussion dédiée plus bas.

Le théorème fondamental de la géométrie tropicale décrit la tropicalisation V de $\mathcal{V}$ comme l'ensemble des points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ où le polynôme tropical $\mathrm{trop}(P)$ s'*annule tropicalement* ; et l'on déclare que cela est le cas en un point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ lorsque le minimum dans $\min_{\mathbf{k} \in S}(k_1 \cdot x_1 + \cdots + k_n \cdot x_n)$ est *atteint au moins deux fois*.

Dans notre exemple ci-dessus, $n = 2$, $P = Z_1 + Z_2 - 1$ et $\mathrm{trop}(P) = \min(x, y, 0)$, dont l'ensemble des zéros tropicaux est en effet formé des trois demi-droites de la Figure 1.1. En effet, la demi-droite horizontale (resp. verticale, resp. diagonale) correspond au cas où les termes x et 0 (resp. y et 0, resp. x et y) de $\mathrm{trop}(P)$ réalisent le minimum simultanément, et l'origine correspond à la réalisation simultanée par les trois termes. Si l'on remplace le polynôme P par le polynôme $P_a \coloneqq Z_1 + Z_2 - a$ avec $a \in \mathbb{C}^*$, on obtient exactement la même tropicalisation. Le cas $a = 0$, toutefois, est à part, puisque la tropicalisation du polynôme $P_0 \coloneqq Z_1 + Z_2$ est la fonction définie par $\mathrm{trop}(P_0)(x, y) = \min(x, y, +\infty) = \min(x, y)$. C'est la raison pour laquelle, dans l'exemple ci-dessus, nous n'avons pas choisi la droite complexe définie par $z_1 + z_2 = 0$, dont la tropicalisation est une version dégénérée de la courbe tropicale ci-dessus, avec seulement une demi-droite diagonale d'équation $x = y$.

Pour une sous-variété quelconque $\mathcal{V} \subseteq (\mathbb{C}^*)^n$, il s'avère que la tropicalisation V sera toujours le support d'un *éventail polyédral*, c'est-à-dire le support d'un bon arrangement de cônes polyédraux, de dimension pure égale à la dimension complexe de $\mathcal{V}$. La Figure 1.2 montre des exemples d'éventails polyédraux de dimension 1 dans $\mathbb{R}^2$.

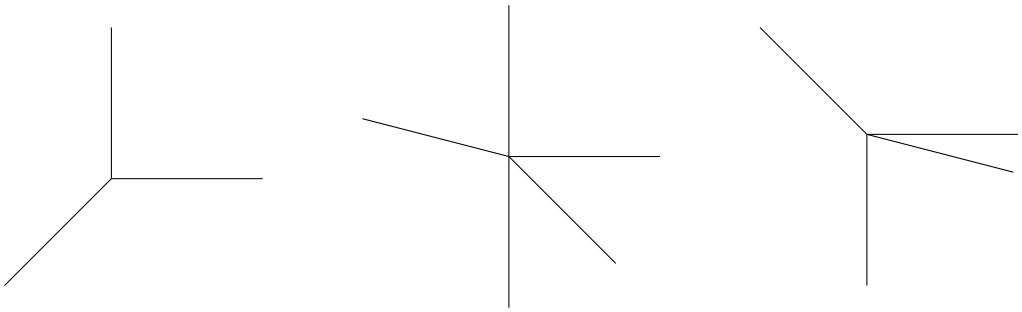


Figure 1.2: Trois éventails polyédraux de dimension 1 dans $\mathbb{R}^2$.

Peut-on généraliser ce type de procédure à des contextes plus généraux, par exemple pour des variétés sur un corps autre que $\mathbb{C}$? La réponse est *oui* : il existe une procédure

de tropicalisation pertinente pour des variétés sur des corps munis d'une *valuation non archimédienne*, ce qui ouvre de nouvelles perspectives.

Une valuation non archimédienne **val** sur un corps $\mathbf{K}$ est une fonction

$$\mathbf{val}\colon \mathbf{K} \to \overline{\mathbb{R}},$$

où $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, vérifiant les propriétés suivantes.

(1) Pour tout $x \in \mathbf{K}$, on a $\mathbf{val}(x) = +\infty$ si, et seulement si, $x = 0$.

(2) Pour tous $x, y \in \mathbf{K}$, $\mathbf{val}(xy) = \mathbf{val}(x) + \mathbf{val}(y)$.

(3) Pour tous $x, y \in \mathbf{K}$, $\mathbf{val}(x + y) \geqslant \min(\mathbf{val}(x), \mathbf{val}(y))$, avec égalité lorsque $\mathbf{val}(x) \neq \mathbf{val}(y)$.

Ces propriétés sont en partie semblables aux propriétés algébriques de la fonction

$$\lim_{t \to 0} \log_t(|\cdot|)\colon \mathbb{C} \to \{0, +\infty\} \subset \overline{\mathbb{R}}$$

utilisée dans le premier type de procédure de tropicalisation, qui porte en fait le nom de *valuation triviale* sur $\mathbb{C}$ et est définie par $\mathbf{val}(z) = 0$ pour tout $z \in \mathbb{C}^*$ et $\mathbf{val}(0) = +\infty$. Un exemple clé de valuation non archimédienne est le *degré en t*, ou *valuation t-adique*, définie sur le corps $\kappa(\!(t)\!)$ des séries formelles de Laurent sur κ, pour κ un corps, ou sur le corps $\kappa\{\{t\}\}$ des séries de Puiseux sur κ, qui a la propriété supplémentaire – et souvent utile – d'être algébriquement clos.

À titre d'illustration, la tropicalisation d'une variété affine $\mathcal{V} \subseteq (\kappa\{\{t\}\}^*)^n$ déterminée par un idéal de $\kappa\{\{t\}\}[Z_1, \ldots, Z_n]$ est définie, par analogie avec la procédure précédente, comme l'adhérence de l'image de $\mathcal{V}$ par la fonction

$$\begin{aligned} \operatorname{trop}\colon \quad & (\kappa\{\{t\}\}^*)^n \to \mathbb{R}^n \\ & (R_1, \ldots, R_n) \mapsto (\mathbf{val}(R_1), \ldots, \mathbf{val}(R_n)), \end{aligned}$$

où **val** désigne la valuation t-adique. De façon similaire, $\operatorname{trop}(\mathcal{V})$ sera toujours un espace polyédral ayant de bonnes propriétés, avec des parties finies mais aussi des parties infinies qui doivent être traitées avec soin. La Figure 1.3 montre un exemple de courbe tropicalisée qui peut être obtenue de cette façon.

La procédure de tropicalisation pour des corps munis d'une valuation non archimédienne est particulièrement utile dans l'étude de la *dégénérescence* d'une famille de variétés algébriques lisses vers une variété limite possiblement non lisse.

Imaginons une famille (X_t) de variétés algébriques lisses sur un corps κ, dépendant algébriquement d'un paramètre t appartement à une certaine base, comme un disque épointé Δ^*. La géométrie algébrique et la théorie des singularités tentent de répondre à des questions telles que les suivantes.

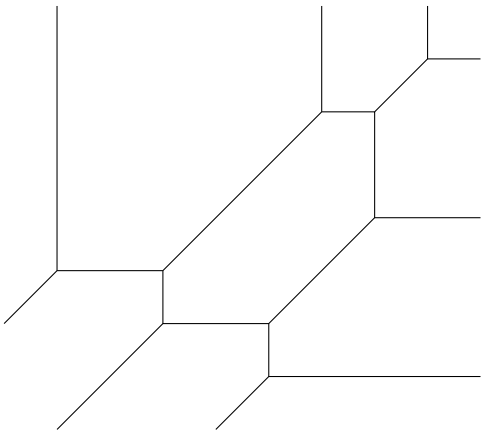


Figure 1.3: La tropicalisation d'une courbe algébrique.

- Quel est le comportement possible de la variété X_t lorsque $t \to 0$?
- Quelles sont les variétés limites X_0 possibles ?
- Qu'en est-il du comportement limite d'une famille de fibrés en droites sur X_t, ou d'autres objets géométriques définis sur X_t ?

Une telle famille qui dégénère, où chaque variété est définie sur un même corps κ, peut être modélisée par une variété sur une base de dimension 1 sur κ, comme $\kappa(\!(t)\!)$ ou $\kappa\{\{t\}\}$ (disons $\kappa(\!(t)\!)$ pour faire simple). Alors, toute extension de cette famille à $t = 0$ sera nommée un *modèle*, et sera décrite par un schéma $\mathcal{X}$ sur l'anneau $R := \kappa[\![t]\!]$, dont le corps des fractions est $\kappa(\!(t)\!)$. Les modèles qui nous intéressent, ayant de bonnes propriétés, sont dits *semi-stables*. Leur existence est assurée en caractéristique nulle, et nécessite typiquement de passer à une extension de corps finie de $\kappa(\!(t)\!)$. La *fibre générique* de $\mathcal{X}$ (la fibre au-dessus du complémentaire de 0 dans Δ^*) se souvient essentiellement de la partie de la famille dégénérescente où les variétés sont lisses, et la *fibre spéciale* (la fibre au-dessus de 0) est la variété limite X_0, qui aura typiquement des points singuliers. Comme R est muni d'une valuation non archimédienne, on peut tropicaliser la famille $\mathcal{X}$ au moyen de la procédure décrite plus haut, en mettant des coordonnées sur la famille.

Ceci est, pour résumer, la raison pour laquelle, pour étudier les dégénérescences de familles de variétés algébriques, on peut utiliser la tropicalisation de cette famille en un objet polyédral. Celui-ci peut, selon la procédure utilisée, être un objet non borné, tel un complexe de cônes (ou, plus généralement, un complexe polyédral, voir la Figure 1.3), ou un objet *borné*, tel un graphe métrique. Un tel graphe apparaît en fait comme un *squelette* d'un complexe polyédral, ces deux objets étant obtenus à partir de la famille de variétés par des méthodes de tropicalisation alternatives.

Une explication plus précise nécessite quelques rudiments de théorie de Berkovich. Dans ce domaine, on définit un espace topologique associé à une variété affine $\mathcal{V} \subseteq (\mathbf{K}^*)^n$ déterminée par un idéal $\mathcal{I}$ de $\mathbf{K}[Z_1, \ldots, Z_n]$. Les points x de cet espace topologique paramétrisent les

semi-valuations $\mathfrak{v}_x$ sur l'anneau $\mathbf{K}[Z_1, \ldots, Z_n]/\mathcal{I}$ qui étendent la valuation **val** de $\mathbf{K}$. Il se trouve que cet espace, appelé l'*analytifié de Berkovich* $\mathcal{V}^{\mathrm{an}}$ de $\mathcal{V}$, a de bonnes propriétés : il est séparé, connexe par arcs, assimilable localement à un espace polyédral, et il se rétracte par déformation sur des sous-espaces polyédraux compacts, ses *squelettes*. De plus, $\mathcal{V}^{\mathrm{an}}$ contient naturellement $\mathcal{V}$, les tropicalisations possibles de $\mathcal{V}$ sont toutes issues de l'analytifié par des projections, et cet analytifié est la limite projective de toutes les tropicalisations de $\mathcal{V}$ [Pay09, FGP14].

À ce stade, élargissons un peu la perspective. Une procédure de tropicalisation pour les variétés sur un corps muni d'une valuation non archimédienne permet de tropicaliser des *familles de variétés*, ce qui aidera à étudier les singularités dans un contexte de dégénérescence. En fait, si certains objets géométriques (comme un fibré en droites, ou un espace de sections d'un tel fibré) sont définis de manière lisse sur une telle famille, il existe aussi des procédures de tropicalisation pour ces objets. Cela sera un fil rouge dans le présent manuscrit, qui portera, entre autres, sur le tropicalisation de *séries linéaires* (des espaces vectoriels de sections d'un fibré en droites) et de *points de Weierstrass* (des points géométriques spéciaux sur les courbes algébriques). Nous renvoyons à la discussion à propos des Chapitres 4 et 3, plus loin dans cette introduction, pour la suite de cette histoire.

Maintenant que la lectrice ou le lecteur a quelques clés pour comprendre la tropicalisation, nous allons décrire, dans la prochaine section, l'organisation du manuscrit.

Organisation du manuscrit

Le manuscrit de thèse est divisé en cinq chapitres.

Le premier chapitre est la présente introduction, dont l'objectif est de résumer le contenu de la thèse et de l'introduire à des non-spécialistes.

Les Chapitres 2, 3 et 4 correspondent approximativement aux trois *preprints* [AG24, AG22, AGR23], respectivement. Plus exactement, ces chapitres consistent en des élargissements plus ou moins importants des articles correspondants : les résultats, figures ou commentaires qui avaient été retirés ou raccourcis dans les papiers pour des raisons de longueur ou de concision ont été gardés dans leur intégralité dans le présent manuscrit.

Les Chapitres 2 et 3 ont été écrits avec Omid Amini. Le Chapitre 4 a été écrit en collaboration avec Omid Amini et Harry Richman, de l'université de Washington à Seattle.

Le Chapitre 2, basé sur [AG24], définit de nouveaux objets combinatoires, nommés *matricubes*, qui se trouvent être une nouvelle généralisation d'objets combinatoires très étudiés : les matroïdes.

Le Chapitre 3, basé sur [AG22], utilise, entre autres, le formalisme des matricubes pour jeter les bases d'une nouvelle théorie purement combinatoire des séries linéaires limites, permettant d'étudier la tropicalisation de séries linéaires algébriques sur des courbes algébriques vers des graphes métriques de façon plus approfondie.

Le Chapitre 4, basé sur [AGR23], prolonge des travaux antérieurs liés aux points de Weierstrass tropicaux, et fournit de nouveaux résultats de localisation pour la tropicalisation de points de Weierstrass sur des courbes algébriques vers des graphes métriques.

Enfin, le Chapitre 5 synthétise les différents types de questions ouvertes qui se sont posées en cours de route, et propose quelques directions de recherche potentiellement intéressantes en partant des travaux de la thèse.

Une section de remerciements est située en page 3, suivie d'une préface dans les deux langues, avant la table des matières.

Une bibliographie unique pour l'intégralité du manuscrit est proposée, combinant les bibliographies de tous les chapitres et située à la fin du manuscrit.

Nous allons maintenant introduire les Chapitres 2, 3 et 4 du manuscrit dans de plus amples détails.

Matroïdes et matricubes — Chapitre 2

Le Chapitre 2 est un chapitre purement combinatoire, qui est basé sur [AG24] et prend ses racines dans le concept fondamental de *matroïde*. Un matroïde est un objet combinatoire qui abstrait et axiomatise les relations de dépendance linéaire entre des vecteurs dans un espace vectoriel, se souvenant uniquement de "qui est linéairement indépendant de qui", sans conserver toute l'information sur ces vecteurs.

Par exemple, soit (e_i) la base canonique dans $\mathbb{R}^3$, et définissons la configuration de vecteurs suivante.

$$\begin{cases} x_1 = e_1, x_2 = e_2, x_3 = e_3 \\ x_4 = 0 \\ x_5 = e_1 + e_2 \end{cases}$$

Cette configuration de vecteurs peut être représentée par la matrice suivante, dont chaque colonne représente un vecteur :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Faisons maintenant une liste de tous les *indépendants*, c'est-à-dire les sous-familles linéairement indépendantes de vecteurs parmi les (x_i), donnés par leurs indices.

$$\mathscr{I} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{3,5\}, \{1,2,3\}, \{1,3,5\}, \{2,3,5\}\}$$

Ici, les indépendants peuvent s'imaginer comme des sous-ensembles de l'ensemble des colonnes de la matrice ci-dessus, c'est-à-dire des sous-ensembles de $\{1,2,3,4,5\}$, en oubliant la valeur exacte des vecteurs.

Mais l'ensemble $\mathscr{I}$ des indépendants d'une configuration de vecteurs ne peut pas être *n'importe quelle* collection de sous-ensembles de $\{1,2,3,4,5\}$, parce qu'il doit satisfaire certaines propriétés combinatoires non triviales. Un *matroïde* (littéralement, "qui ressemble

à une matrice") est précisément une façon d'axiomatiser et d'abstraire les propriétés combinatoires de l'ensemble des indépendants d'une configuration de vecteurs.

Plus formellement, un matroïde peut être défini comme un ensemble E muni d'une collection $\mathscr{I}$ de parties de E vérifiant les trois propriétés suivantes :

(I1) $\varnothing \in \mathscr{I}$ (ou, de façon équivalente en utilisant (I2), $\mathscr{I} \neq \varnothing$).

(I2) (Propriété d'hérédité) Si $I \in \mathscr{I}$ et $I' \subseteq I$, alors $I' \in \mathscr{I}$.

(I3) (Propriété d'échange pour les indépendants) Si $I_1, I_2 \in \mathscr{I}$ et $|I_1| < |I_2|$, alors il existe un élément $e \in I_2 \smallsetminus I_1$ tel que $I_1 \cup \{e\} \in \mathscr{I}$.

Il est aisé de montrer que l'ensemble des indépendants d'une configuration de vecteurs satisfait les propriétés ci-dessus. Ce qui est moins évident est que cette axiomatique définit un objet combinatoire intéressant et mathématiquement riche.

Il se trouve qu'il y a de nombreuses définitions différentes des matroïdes qui donnent le même concept de matroïde. Par exemple, au lieu de définir un matroïde par ses indépendants, on pourrait le définir par ses *bases*, une collection de sous-ensembles vérifiant un autre système d'axiomes spécifique mimant le comportement des vraies bases d'une configuration de vecteurs. Pour assouvir la curiosité de la lectrice ou du lecteur, voici une version possible de l'axiomatique définissant un matroïde par sa collection $\mathscr{B}$ de bases :

(B1) $\mathscr{B} \neq \varnothing$.

(B2) (Propriété d'échange pour les bases) Si $B_1, B_2 \in \mathscr{B}$ et $x \in B_1 \smallsetminus B_2$, alors il existe $y \in B_2 \smallsetminus B_1$ tel que $B_1 \smallsetminus \{x\} \cup \{y\} \in \mathscr{B}$.

Si je vous donne l'ensemble des indépendants d'un matroïde, vous pourriez également en déduire facilement l'ensemble des bases de ce matroïde, tout simplement en gardant seulement les indépendants *maximaux*. Par exemple, la configuration de vecteurs décrite ci-dessus a trois bases :

$$\mathscr{B} = \{\{1,2,3\}, \{1,3,5\}, \{2,3,5\}\}.$$

Comme manière alternative de décrire un matroïde, l'on pourrait aussi retenir, pour chaque sous-famille de vecteurs, la dimension du sous-espace vectoriel engendré par ces vecteurs, ce qui donnerait une *fonction de rang*. Une telle fonction de rang satisfait également certaines propriétés combinatoires précises qui donnent lieu à une axiomatique, et toute fonction prenant ses valeurs dans $\mathbb{N}$ n'apparaît pas comme une fonction de rang. Ces définitions, de même que des dizaines d'autres, donnent exactement la même théorie des matroïdes : elles sont dites *cryptomorphes*. Elles ont été étudiées depuis près d'un siècle, voir par exemple [Whi92, Oxl06].

Pour pouvoir introduire le Chapitre 2, il est utile de noter qu'à chaque configuration de vecteurs dans un espace vectoriel peut être associé un *arrangement d'hyperplans* dans

l'espace vectoriel dual (ici, par un abus de langage, un hyperplan désignera soit un vrai hyperplan, soit l'espace total, lorsque le vecteur dual est le vecteur nul).

Ce faisant, on peut aussi associer un matroïde à chaque arrangement d'hyperplans. Si $\mathrm{F}_1, \dots, \mathrm{F}_m$ sont des hyperplans dans un espace vectoriel H, on peut définir la fonction de rang qui, à chaque $A \subseteq \{1, \dots, m\}$, associe l'entier naturel $\mathbf{r}(A) := \mathrm{codim}_{\mathrm{H}}(\bigcap_{i \in A} \mathrm{F}_i)$. Cette fonction de rang satisfait les propriétés de la fonction de rang d'un matroïde.

Nous pouvons maintenant introduire les matricubes. Ces nouveaux objets combinatoires, définis dans le Chapitre 2, sont apparus naturellement comme la généralisation de la construction précédente : quel type d'objet combinatoire obtient-on si l'on remplace ci-dessus un *arrangement d'hyperplans* par un *arrangement de drapeaux* ? Plus précisément, imagions que pour tout $j \in \{1, \dots, m\}$, $\mathrm{F}_j^\bullet$ soit un drapeau décroissant

$$\mathrm{H} = \mathrm{F}_j^0 \supseteq \mathrm{F}_j^1 \supseteq \cdots \supseteq \mathrm{F}_j^{r-1} \supseteq \mathrm{F}_j^r,$$

où $r \leqslant \dim \mathrm{H}$ et, pour tout i, F_j^{i+1} est de codimension au plus 1 dans F_j^i. Si l'on définit, pour tout $\underline{x} = (x_1, \dots, x_m)$, $x_i \in \{0, \dots, r\}$, le rang de $\underline{x}$ comme

$$\mathbf{r}(\underline{x}) := \mathrm{codim}_{\mathrm{H}}(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m}),$$

alors une telle fonction de rang est une généralisation de la fonction de rang d'un matroïde. Elle satisfait un système d'axiomes similaire à celui d'une fonction de rang de matroïde, adapté au fait que sa source est un hypercube $\square_r^m := \{0, \dots, r\}^m$ au lieu de $\mathcal{P}(\{1, \dots, m\}) \simeq \{0, 1\}^m$, et en utilisant la structure de *poset* (ensemble partiellement ordonné) de $\square_r^m$. Nous appelons ceci un *matricube*, un nom que nous justifions dans la Section 2.2.3.

Ci-dessous figure un exemple de matricube, décrit par sa fonction de rang, et provenant d'une paire de drapeaux à quatre sous-espaces dans un espace vectoriel de dimension 4, pioché dans la Section 2.2.2. L'origine du tableau (les petits indices) est le coin en bas à gauche, selon la convention que nous utiliserons pour tous les tableaux dans le Chapitre 2.

$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Notons que les fonctions de rang des matricubes ne sont pas nécessairement symétriques comme celle-ci. De plus, comme ce matricube provient d'une vraie configuration de vecteurs, il sera dit *représentable*. L'étude de la représentabilité des matroïdes est un domaine de recherche à part entière. Dans le Chapitre 2, nous posons quelques questions de représentabilité sur les *matricubes*, et répondons à certaines d'entre elles (voir la Section 2.9.3). Nous montrons aussi qu'un matricube contient une collection de "matroïdes locaux" qui gouvernent ses propriétés combinatoires (voir Section 2.8).

Le travail présenté dans le Chapitre 2 explore principalement les propriétés combinatoires des matricubes, en lien avec leur structure de *poset* et avec les propriétés de convexité discrète

de leurs fonctions de rang. Nous proposons également quelques définitions différentes, mais cependant équivalentes (plus exactement, cryptomorphes), des matricubes, utilisant la fonction de rang (Section 2.2), la collection des indépendants (Section 2.5), et d'autres collections (les plats et les circuits dans les Sections 2.3 et 2.4) qui généralisent leurs homologues matroïdaux et miment des propriétés concrètes des configurations de vecteurs. Dans ce contexte, les indépendants, plats et circuits sont des points particuliers de l'hypercube $\square_r^m$. De façon surprenante, à ce stade de l'exploration des matricubes, nous ne sommes pas parvenus à proposer une axiomatisation pertinente et intéressante de la collection des *bases* d'un matricube, ce qui soulève des questions ouvertes intéressantes (voir Section 2.9.1).

En outre, en plus d'un système d'axiomes dédié, les indépendants, plats et circuits peuvent être décrits explicitement et facilement pour un matricube donné par sa fonction de rang. Par exemple, les plats d'un matricube sont tous les points de l'hypercube $\square_r^m$ tels qu'en se déplaçant d'une unité dans n'importe quelle direction positive (lorsque c'est possible), la valeur de la fonction de rang augmente. À titre d'exemple, les plats de la fonction de rang définie ci-dessus sont représentés en bleu ci-dessous. Notons que l'ensemble des plats possède une structure de treillis gradué, ce qui est une propriété générale.

$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Une autre propriété combinatoire des matricubes, démontrée dans le Chapitre 2, est leur lien avec un objet combinatoire qui généralise les matrices de permutation en dimension supérieure. Une *matrice de permutation* est une matrice carrée $n \times n$ remplie de 0 et de 1 encodant une permutation σ de n éléments : le coefficient a_{ij} est égal à 1 si, et seulement si, $\sigma(i) = j$. Par exemple, la matrice ci-dessous (représentée avec la convention usuelle pour les matrices, l'origine étant donc en haut à gauche) est celle associée à la permutation $\sigma = (14)(235)$ de cinq éléments.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Ces objets sont intéressants en soi, mais l'on pourrait légitimement se demander si cette notion admet des généralisations intéressantes en dimension supérieure. Il existe, en effet, plusieurs généralisations. On peut par exemple penser aux définitions suivantes (toutes équivalentes) des matrices de permutation, et suggérer en conséquence les généralisations (non équivalentes) correspondantes en dimension supérieure.

(a) Une matrice de permutation est une matrice carrée de 0 et de 1 telle que chaque ligne droite complète (où l'on fait varier une seule coordonnée), c'est-à-dire chaque ligne

et chaque colonne, contient 1 exactement une fois. ⇝ Une matrice de permutation de dimension supérieure est un tableau de 0 et de 1 de dimension supérieure tel que chaque ligne complète (où l'on fait varier une seule coordonnée) contient 1 exactement une fois.

(b) Une matrice de permutation est une matrice carrée de 0 et de 1 telle que chaque couche de codimension 1 (avec exactement une coordonnée fixée), c'est-à-dire chaque ligne et chaque colonne, contient 1 exactement une fois. ⇝ Une matrice de permutation de dimension supérieure est un tableau de 0 et de 1 de dimension supérieure tel que chaque couche de codimension 1 (avec exactement une coordonnée fixée) contient 1 exactement une fois.

(c) Une matrice de permutation est une matrice carrée de 0 et de 1 telle que pour tout choix d'indices (i, j), la sous-matrice formée des éléments d'indices au moins égaux à (i, j) (coordonnée par coordonnée) a la propriété que le nombre de lignes contenant 1 au moins une fois est égal au nombre de colonnes contenant 1 au moins une fois. ⇝ Une matrice de permutation de dimension supérieure est un tableau de 0 et de 1 de dimension supérieure tel que pour tout multi-indice $\underline{x}$, le sous-tableau formé des éléments de multi-indice au moins égal à $\underline{x}$ (coordonnée par coordonnée) a la propriété que le nombre de couches de codimension 1 contenant 1 au moins une fois ne dépend pas de la direction dans laquelle les couches sont découpées.

Bien que ces définitions coïncident en dimension 2, elles donnent lieu, en dimension supérieure, à des notions distinctes ayant chacune leurs applications et liens avec d'autres objets combinatoires. Elles ont été très étudiées, et ces objets ont été dénombrés, dans de nombreux travaux.

La Définition (c) donne naissance à des objets appelés *tableaux de permutation* (*permutation arrays*), étudiés entre autres par Eriksson et Linusson [EL00a]. Il se trouve, comme nous le démontrons dans la Section 2.7, qu'un certain type de matricubes est en fait en correspondance bijective avec les tableaux de permutation. Cela donne une nouvelle perspective sur un objet combinatoire précédemment connu.

Il apparaît que les matricubes sont une notion combinatoire nouvelle et intéressante, digne d'intérêt en soi. Cependant, d'un point de vue chronologique, ils sont apparus, dans le travail de cette thèse, au détour de l'étude de la tropicalisation des séries linéaires, comme nous l'expliquerons dans la prochaine section.

Une théorie combinatoire pour les séries linéaires limites — Chapitre 3

Le Chapitre 3 est basé sur [AG22] et son objectif est de proposer une nouvelle théorie combinatoire pour étudier la tropicalisation des séries linéaires sur les courbes algébriques.

Les séries linéaires sur les variétés algébriques sont des objets géométriques importants et très étudiés, car elles gouvernent la géométrie de ces variétés. Elles sont basées sur la théorie

des diviseurs, que nous présentons brièvement ci-après. Dans ce manuscrit, nous nous concentrerons sur les courbes algébriques (c'est-à-dire les variétés algébriques de dimension 1).

Soit X une courbe algébrique propre et lisse sur un corps κ. Un *diviseur* sur X est une somme formelle finie à coefficients dans $\mathbb{Z}$ de points de X : $\mathcal{D} = \sum_{i\in I} n_i\,(x_i)$, $x_i \in X$. Le coefficient de $\mathcal{D}$ en un point $x \in X$ est noté $\mathcal{D}(x)$. Chaque fonction rationnelle (ou méromorphe) $f \in \kappa(X)$ sur la courbe donne naissance à un diviseur $\operatorname{div}(f)$ d'une façon naturelle, en retenant les zéros et les pôles de f ainsi que leurs ordres :

$$\operatorname{div}(f) \coloneqq \sum_{x\in X} \operatorname{ord}_x(f)\,(x).$$

De cette façon, chaque diviseur $\mathcal{D}$ définit un espace vectoriel de fonctions rationnelles sur X respectant $\mathcal{D}$, défini comme suit :

$$\operatorname{Rat}(\mathcal{D}) \coloneqq \{f \in \kappa(X),\, \mathcal{D} + \operatorname{div}(f) \geqslant 0\},$$

où $\mathcal{D} + \operatorname{div}(f) \geqslant 0$ signifie que ce diviseur n'a que des coefficients positifs (est *effectif*).

En résumé, les diviseurs sont utilisés pour imposer des contraintes sur les ordres des pôles et zéros d'une fonction méromorphe sur une courbe algébrique. Un espace de la forme $\operatorname{Rat}(\mathcal{D})$ est précisément ce qu'on appelle une *série linéaire complète*, et une *série linéaire* de *rang* r sera définie comme un sous-espace vectoriel $\mathrm{H} \subseteq \operatorname{Rat}(\mathcal{D})$ de dimension $r+1$. (Le décalage d'une unité provient du fait que multiplier une fonction méromorphe par un scalaire donne le même diviseur, $\operatorname{div}(cf) = \operatorname{div}(f)$, et par conséquent r est en fait la dimension de l'espace des tels diviseurs, et aussi la dimension du projectivisé de $\operatorname{Rat}(\mathcal{D})$.) Il s'avère que les séries linéaires sur une courbe algébrique X correspondent aux applications de X vers des espaces projectifs, ce qui entraîne que la donnée de toutes les séries linéaires "contient" la géométrie de X ; c'est pourquoi les séries linéaires sont un objet d'étude aussi central en géométrie algébrique.

La situation est moins bien comprise lorsque la courbe algébrique X n'est pas lisse, par exemple s'il s'agit d'une *courbe stable* (une courbe dont les seuls points singuliers sont des *nœuds*, le type de singularité le plus "gentil", avec deux branches s'intersectant transversalement). En fait, comprendre comment les séries linéaires dégénèrent sur des familles de courbes est une question ouverte depuis longtemps.

Pour énoncer le problème d'une façon plus intuitive, introduisons l'*espace de modules* $\mathscr{M}_g$ de toutes les courbes projectives lisses de genre g. C'est un champ de Deligne–Mumford (une généralisation des schémas) qui, comme espace, paramétrise les courbes, en ce sens que chaque point de $\mathscr{M}_g$ représente une courbe projective lisse, d'une manière bijective et cohérente. L'espace $\mathscr{M}_g$ n'est pas compact, mais il se trouve qu'il admet une compactification $\overline{\mathscr{M}_g}$ ayant de bonnes propriétés, dont le bord est exactement constitué des courbes stables. D'une certaine façon, les courbes stables sont les courbes non lisses que l'on peut obtenir en faisant dégénérer des courbes lisses. Introduisons à présent le problème de la dégénérescence pour les séries linéaires.

Question 1.0.1. *Soit X une courbe* stable *de genre g et soit x le point de $\overline{\mathscr{M}}_g$ correspondant. Quelles sont toutes les limites possibles de séries linéaires pour n'importe quelle suite de courbes projectives* lisses *de genre g dont les points correspondants dans $\mathscr{M}_g$ convergent vers x ?*

La géométrie tropicale fournit une approche possible pour répondre à cette question. En effet, une famille de courbes à un paramètre donne naissance à une suite de points de $\mathscr{M}_g$ convergeant vers un point de $\overline{\mathscr{M}}_g$. On peut modéliser cette famille par une courbe $\mathbf{C}$ sur un corps $\mathbf{K}$ muni d'une valuation non archimédienne non triviale, de telle sorte que κ est le corps résiduel de $(\mathbf{K}, \mathbf{val})$ et X est la fibre spéciale du modèle semi-stable de $\mathbf{C}$ sur l'anneau de valuation de $\mathbf{val}$. Ainsi, la procédure de tropicalisation ébauchée plus tôt dans cette introduction permet de tropicaliser $\mathbf{C}$ en un graphe métrique Γ (qu'on peut voir comme une *courbe tropicale*), c'est-dire un graphe avec une longueur associée à chaque arête.

Il s'avère que dans cette configuration, il y a également une manière naturelle de tropicaliser une fonction méromorphe f sur la courbe $\mathbf{C}$ en une fonction $\mathrm{trop}(f)\colon \Gamma \to \mathbb{R}$, qui soit continue et affine par morceaux à pentes entières.

La géométrie tropicale encode des informations combinatoires concernant la famille dégénérescente de courbes $\mathbf{C}$ qui sont complémentaires de la donnée de la courbe stable limite X et de la réduction des fonctions $f \in \mathbf{K}(\mathbf{C})$ sur X. En effet, ces dernières capturent certains aspects algébriques des fonctions limites sur la courbe stable limite, tandis que le graphe métrique Γ associé et les fonctions tropicalisées $\mathrm{trop}(f)$ se souviennent également d'informations plus fines liées à la *façon* (depuis quelle direction, à quelle vitesse) la suite de courbes lisses dégénère vers X.

Par conséquent, on peut raisonnablement espérer que l'étude des fonctions tropicalisées $\mathrm{trop}(f)$ sur la tropicalisation Γ de $\mathbf{C}$ donne des informations intéressantes sur les séries linéaires sur $\mathbf{C}$.

C'est précisément le but du travail présenté dans le Chapitre 3, qui développe une théorie purement combinatoire des séries linéaires sur les graphes métriques, et tisse des liens avec les séries linéaires géométriques, c'est-à-dire les séries linéaires sur les courbes algébriques.

Le point de départ de ce projet fut l'observation que dans le contexte, présenté plus haut, d'une famille dégénérescente de courbes munies d'une série linéaire, il existe une donnée combinatoire finie qui peut être extraite de la famille dégénérescente et qui devrait régir la tropicalisation des fonctions de la série linéaire. Cette donnée est liée aux *ordres d'annulation* des fonctions. Plus précisément, étant donné une fonction f appartenant à une série linéaire sur $\mathbf{C}$, on souhaite tout d'abord mesurer, pour chaque composante irréductible X_v de la courbe stable limite X et pour chaque point singulier x de X_v, l'ordre d'annulation de la réduction de f à X_v en ce point x.

Un résultat classique affirme que le nombre d'ordres d'annulation différents en x que l'on obtient en faisant varier la fonction f dans la série linéaire considérée est exactement la dimension de cette série linéaire. Cet ensemble d'entiers est le premier type d'information combinatoire que l'on peut mesurer pour une dégénérescence de série linéaire. Le second

type d'information est lié à la *corrélation* entre ces ordres d'annulation en chaque point singulier d'une composante irréductible X_v donnée. En bref, on retient l'ensemble des *ordres d'annulation conjoints* que l'on obtient en faisant varier la fonction f dans la série linéaire. Cette donnée s'avère être encodée par des matricubes, les objets présentés dans la section précédente de l'introduction.

Les propriétés de la procédure de tropicalisation pour les fonctions entraînent que ces données combinatoires ont des interprétations intéressantes liées aux fonctions tropicalisées sur le graphe métrique. Chaque sommet v du graphe métrique correspond à une composante irréductible X_v de la courbe stable X, et chaque arête $e = uv$ joignant deux sommets u et v du graphe correspond à un point d'intersection entre les composantes irréductibles X_u et X_v associées à u et v. Alors, les ordres d'annulation de la réduction d'une fonction f sur une composante irréductible X_v en les points singuliers de X situés sur X_v sont égaux aux *pentes* de la fonction tropicalisée $\mathrm{trop}(f)$ sur le graphe métrique, le long des directions sortantes autour du sommet v ; et la donnée conjointe des ordres d'annulation de f en tous les points singuliers de X situés sur une composante irréductible X_v se traduit par un *vecteur de pentes* de $\mathrm{trop}(f)$ autour du sommet v de Γ associé à X_v. Ceci illustre la nature des contraintes combinatoires qui seront utilisées pour définir les séries linéaires combinatoires sur les graphes métriques, que nous présenterons un peu plus tard.

Notons de prime abord que les graphes métriques possèdent une théorie des fonctions méromorphes et des diviseurs, tout comme les courbes algébriques. Une fonction méromorphe $f \in \mathrm{Rat}(\Gamma)$ sur le graphe métrique Γ est une fonction continue, affine par morceaux $\Gamma \to \mathbb{R}$ avec des pentes entières ; un diviseur sur Γ est une somme formelle finie à coefficients dans $\mathbb{Z}$ de points de Γ : $D = \sum_{i \in I} n_i (x_i)$, $x_i \in \Gamma$. De plus, toute fonction méromorphe $f \in \mathrm{Rat}(\Gamma)$ donne naturellement naissance à un diviseur $\mathrm{div}(f)$ sur Γ :

$$\mathrm{div}(f) \coloneqq \sum_{x \in \Gamma} \mathrm{ord}_x(f)\,(x),$$

où $\mathrm{ord}_x(f) \in \mathbb{Z}$ est, au signe près, la somme des pentes sortantes de f sur les arêtes de Γ incidentes à x. Dans le contexte des graphes métriques, un *zéro* (resp. un *pôle*) d'une fonction méromorphe est un changement de pente négatif (resp. positif). Par ailleurs, de même qu'il y a une procédure de tropicalisation pour les fonctions sur les courbes algébriques, il existe une procédure de tropicalisation pour les diviseurs, envoyant tout diviseur $\mathbf{D}$ sur $\mathbf{C}$ sur un diviseur $D = \mathrm{trop}(\mathbf{D})$ sur Γ. Cette procédure commute opportunément avec l'application $f \mapsto \mathrm{div}(f)$. Ainsi, une fonction méromorphe (resp. un diviseur) sur une courbe algébrique $\mathbf{C}$ se tropicalise en une fonction méromorphe (resp. un diviseur) sur le graphe métrique Γ obtenu en tropicalisant $\mathbf{C}$.

Sans entrer dans des détails trop techniques, l'idée générale à propos de cette théorie des diviseurs et des fonctions méromorphes sur les graphes métriques, qui a déjà été très étudiée, est qu'elle se comporte de façon sympathique et similaire à son homologue sur les variétés algébriques. Par exemple, il a pu être démontré, par plusieurs méthodes différentes, qu'il y a un théorème de Riemann–Roch pour les graphes métriques [AC13, GK08, MZ08],

tout comme il y en a un sur les graphes combinatoires [BN07]. Il y a, de plus, de nombreuses applications de cette théorie à la géométrie algébrique, comme des techniques de dégénérescence pour la théorie de Brill–Noether, des résultats sur la dimension de Kodaira d'espaces de modules de courbes [FJP20], des réponses partielles à la *maximal rank conjecture*, de nouvelles bornes en géométrie arithmétique, ou des outils cruciaux pour l'étude des points de Weierstrass ; voir [BJ16] pour une revue exhaustive.

À ce stade, il devient assez naturel de décider comment définir une série linéaire complète sur un graphe métrique Γ. Considérons simplement un diviseur D sur Γ, et définissons $\mathrm{Rat}(D)$ comme l'ensemble des fonctions $f \in \mathrm{Rat}(\Gamma)$ telles que $D + \mathrm{div}(f) \geqslant 0$: un diviseur *contrôle* les zéros et les pôles des fonctions de sa série linéaire complète.

Mais le monde tropical a des différences majeures avec le monde de la géométrie algébrique, l'une d'entre elles étant qu'un espace comme $\mathrm{Rat}(\Gamma)$ ou $\mathrm{Rat}(D)$ n'a rien d'un espace vectoriel. Au lieu de cela, le bon formalisme pour étudier ces espaces est la théorie des *semi-modules tropicaux*, c'est-à-dire des ensembles munis d'une opération interne, $\oplus$, et d'une multiplication externe, $\odot$, qui utilise des scalaires réels. Dans le contexte qui nous intéresse, ces opérations sont définies comme suit : pour tous $f, g \in \mathrm{Rat}(\Gamma)$ et $c \in \mathbb{R}$,

$$f \oplus g \coloneqq \min(f, g) \qquad \text{et} \qquad c \odot f \coloneqq f + c.$$

Ce sont les *opérations tropicales*, qui peuvent être définies de façon semblable sur divers espaces tropicaux. Elles sont bien entendu étroitement liées à l'algèbre max-plus évoquée plus tôt.

Maintenant que nous avons ce formalisme tropical en tête, on pourrait être tenté de définir une série linéaire sur le graphe métrique Γ comme tout sous-semi-module M d'un semi-module tropical de la forme $\mathrm{Rat}(D)$, pour D un diviseur sur Γ, tout comme une série linéaire sur une courbe algébrique $\mathbf{C}$ est un sous-espace vectoriel $\mathbf{H}$ d'une série linéaire complète de la forme $\mathrm{Rat}(\mathbf{D})$. C'est en effet une bonne idée, puisque avec cette définition, *la tropicalisation d'une série linéaire sur une courbe algébrique donne une série linéaire sur le graphe métrique associé*, dans le sens où si $\mathbf{H}$ est une série linéaire sur $\mathbf{C}$, alors

$$\mathrm{trop}(\mathbf{H}) \coloneqq \big\{ \mathrm{trop}(f) \;\big|\; f \in \mathbf{H} \smallsetminus \{0\} \big\}$$

est une série linéaire sur Γ, comme nous le démontrons dans la Section 3.9. En fait, considérer des espaces de la forme $\mathrm{Rat}(D)$ ou des sous-semi-modules de tels espaces s'avère être une façon pertinente de définir les séries linéaires sur les graphes métriques.

Cependant, comme souvent en géométrie tropicale, nous aimerions répondre à des *questions de représentabilité*, c'est-à-dire des questions de la forme : "est-ce que tel objet tropical peut se réaliser comme la tropicalisation d'un objet algébro-géométrique du type attendu ?" Dans le cas présent, quels sont les sous-semi-modules M d'un certain $\mathrm{Rat}(D)$ qui sont des tropicalisations de séries linéaires sur une courbe algébrique ?

Il se trouve que ce n'est pas le cas de tous les sous-semi-modules. C'est ce fait même qui a mené au travail présenté dans le Chapitre 3. L'idée de base est qu'une théorie

des séries linéaires combinatoires plus pertinente devrait comporter quelques contraintes supplémentaires sur ces objets, de telle façon qu'on puisse formuler, au moins dans certains cas, des résultats positifs de représentabilité.

C'est d'ailleurs le moment où la théorie des matricubes présentée dans le Chapitre 2 entre en scène. Nous allons ajouter certaines contraintes combinatoires supplémentaires pour définir les séries linéaires sur les graphes métriques, sous la forme de ce qu'on appellera une *structure de pentes*. Comme exposé plus amplement dans la Section 3.3, une structure de pentes $\mathfrak{S}$ de largeur r sur Γ consiste en :

- pour chaque arête orientée e de Γ, la donnée d'un ensemble de $r+1$ pentes entières admises $s_0^e < s_1^e < \cdots < s_r^e$; et
- pour chaque sommet v de Γ, la donnée d'un matricube (ou *fonction de rang*) ρ_v sur $\square_r^{d_v}$, où d_v est la valence de v.

Pour rendre les choses plus visuelles, voici un exemple concret de structure de pentes, pioché dans la Section 3.6.3. Considérons le graphe Γ en forme d'"haltère" avec des arêtes de longueur arbitraire, comme sur la Figure 1.4. Définissons une structure de pentes de largeur 1 sur Γ. Autorisons les pentes $-1 < 1$ sur l'arête centrale et, pour $i = 1, 2$, autorisons les pentes $0 < 1$ sur chacune des deux arêtes orientées $u_i v_i$, dans la direction donnée par les flèches.

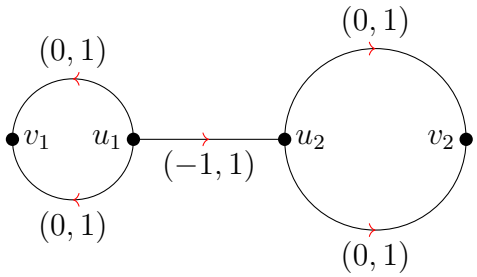


Figure 1.4: Le graphe "haltère" et la structure de pentes $\mathfrak{S}$.

On définit des matricubes appropriés sur les sommets, comme suit. Pour $i = 1, 2$, on munit v_i de la fonction de rang sur $\square_1^2$ définie par le tableau $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, et on munit u_i du matricube sur $\square_1^3$ dont les restrictions aux différentes couches sont définies par les deux tableaux suivants, respectivement : $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Ici, les plats des matricubes (voir la section précédente de l'introduction) figurent en bleu – pour comprendre pourquoi l'élément -1 dans le coin en haut à droite du dernier tableau n'est pas un plat dans ce contexte, on se référera à la Remarque 3.2.16. De plus, la convention utilisée pour représenter les fonctions de rang est celle choisie pour les séries linéaires, en dualité avec les matricubes ; cette dualité est précisée dans la Proposition 3.2.3, et implique que les

fonctions de rang représentées ci-dessus, contrairement aux matricubes, sont des fonctions *décroissantes* pouvant prendre la valeur -1. Concernant les deux sommets u_1 et u_2, la troisième coordonnée dans les fonctions de rang correspond à l'arête centrale de Γ. Cette donnée définit une structure de pentes $\mathfrak{S}$ de largeur 1 sur Γ.

On peut désormais définir l'espace $\mathrm{Rat}(D, \mathfrak{S})$ comme l'espace des fonctions méromorphes $f \in \mathrm{Rat}(D)$ telles que :

- f est compatible avec les *données d'arêtes*, c'est-à-dire que sa pente sur chaque arête e figure dans l'ensemble des $r+1$ entiers prescrits par $\mathfrak{S}$ sur e ; et
- f est compatible avec les *données de sommets*, dans le sens où, autour de chaque sommet v, le vecteur constitué de ses pentes sortantes le long de chaque arête incidente, vu comme un point de $\square_r^{d_v}$, est un *plat* du matricube ρ_v associé à v.

Il peut sembler étrange de demander que les vecteurs de pentes de f soient des *plats* des matricubes attachés à chaque sommet, mais cela a en fait un sens géométrique dans le contexte de la tropicalisation. Pour le voir, prenons X_v une composante irréductible (celle associée à v) de la courbe stable limite dans la famille dégénérescente. Soit H l'espace vectoriel des réductions sur X_v des fonctions appartenant à la série linéaire $\mathbf{H}$ de la courbe $\mathbf{C}$. Soit p un point de X_v. Alors p définit un drapeau complet $\mathrm{F}_p^\bullet$ dans H en considérant les ordres d'annulation, en p, des fonctions de H. Concrètement, définissons $S_p \coloneqq \left\{ \mathrm{ord}_p(f) \;\middle|\; f \in \mathrm{H} \smallsetminus \{0\} \right\}$, qui est un ensemble fini de cardinal $r+1$, comme déjà mentionné, et notons ses éléments $s_0^p < \cdots < s_r^p$. Le drapeau $\mathrm{F}_p^\bullet$ est ensuite défini en posant, pour $j \in \{0, \ldots, r\}$,

$$\mathrm{F}_p^j \coloneqq \left\{ f \in \mathrm{H} \smallsetminus \{0\} \;\middle|\; \mathrm{ord}_p(f) \geqslant s_j^p \right\} \cup \{0\}.$$

En faisant cela pour chaque point *singulier* p de X situé sur X_v, on obtient une collection finie de drapeaux de H, et donc un matricube, ou fonction de rang.

Demander qu'un point du matricube soit un *plat* équivaut en fait à demander que le vecteurs de pentes correspondant autour de v soit *réalisé* par une certaine fonction f de la série linéaire. Par conséquent, si un sous-semi-module M de $\mathrm{Rat}(D)$ est la tropicalisation d'une série linéaire géométrique $\mathbf{H}$, et si on note $\mathfrak{S}$ la structure de pentes définie à partir de la donnée des ordres d'annulation des fonctions des réductions H de $\mathbf{H}$ en tous les points singuliers de la courbe stable limite X, alors, en fait, $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$.

Nous avons maintenant une définition plus restrictive des séries linéaires sur les graphes, mais qui englobe toujours toutes les tropicalisations de séries linéaires géométriques. Nous montrons dans le Chapitre 3 qu'une telle tropicalisation a, en fait, des propriétés supplémentaires, que l'on peut par conséquent, et en toute sécurité, ajouter à la définition des séries linéaires combinatoires $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ de *rang* (ou "dimension") r. Ces propriétés sont de diverse nature ; nous en donnons un aperçu ci-dessous.

- Une condition de rang permet d'assurer que M *contient suffisamment de fonctions* pour engendrer tous les diviseurs effectifs de degré r (voir la Section 3.4.1) ; ceci est

l'équivalent tropical du fait que, comme $\mathbf{H}$ est un espace vectoriel de dimension $r+1$, on peut imposer r contraintes linéaires et toujours avoir des fonctions non nulles à disposition.

- Une condition algébrique permet d'assurer que M est finiment engendré, en d'autres termes qu'il existe un ensemble fini de fonctions de M qui engendrent M en utilisant les opérations tropicales (voir la Définition 3.6.2).

- Une autre condition algébrique permet d'assurer que M a un *rang tropical* au plus égal à r, ce qui signifie que, si $f_0, \ldots, f_{r+1}$ est une collection de $r+2$ fonctions de M, alors elles sont nécessairement tropicalement liées, en ce sens qu'il existe des scalaires $c_0, \ldots, c_{r+1} \in \mathbb{R}$ tels que pour tout $x \in \Gamma$, le minimum dans

$$\min_{0 \leqslant i \leqslant r+1} (f_i(x) + c_i)$$

est atteint au moins deux fois (voir la Définition 3.6.2) ; cette condition est l'équivalent tropical du fait que, comme $\mathbf{H}$ est un espace vectoriel de dimension $r+1$, toute famille de vecteurs de cardinal $r+2$ est liée.

(Notons que le rang défini dans la première condition et le rang tropical défini dans la dernière condition ne coïncident pas nécessairement. À l'inverse, en algèbre linéaire, la dimension d'un espace vectoriel est égale à la fois au cardinal de toute base, et au nombre maximal d'éléments linéairement indépendants. Le fait que ces notions diffèrent dans le monde tropical participe de sa saveur mathématique particulière.)

Dans le Chapitre 3, on définit par conséquent une série linéaire combinatoire comme un semi-module M vérifiant toutes ces conditions, qui sont toutes compatibles avec la procédure de tropicalisation.

Mais comment cela pourrait-il être utile de complexifier de la sorte la définition ? La raison est qu'en ajoutant ces conditions et la donnée combinatoire constituée par la structure de pentes $\mathfrak{S}$, on peut maintenant répondre positivement à des questions de représentabilité, et démontrer d'autres types de résultats.

Voici un échantillon de ce que nous démontrons dans le Chapitre 3, et qui repose, notamment, sur le travail concernant les matricubes présenté dans le Chapitre 2. Certains de ces résultats nécessitent seulement un sous-ensemble des conditions requises pour les séries linéaires combinatoires.

- Si M est une série linéaire sur Γ et v est un sommet de Γ, alors *tout* plat du matricube ρ_v est réalisé par une fonction $f \in M$ (Théorème 3.5.13).

- Par conséquent, si $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ est une série linéaire, alors la donnée de $\mathfrak{S}$ peut être entièrement reconstituée à partir de M seulement (Corollaire 3.5.14).

- Il y a essentiellement un nombre fini de structures de pentes distinctes sur un graphe métrique donné (Théorème 3.4.4) ; cela peut s'interpréter comme un premier pas

pour définir un espace de modules des séries linéaires sur l'espace de modules des graphes métriques de genre fixé.

- Dans le cas $r = 1$, nous proposons une classification complète des séries linéaires sur un graphe métrique fixé : elles sont essentiellement en correspondance bijective avec les morphismes harmoniques finis depuis le graphe vers des arbres métriques (pour de plus amples détails sur ce que cela signifie, voir la Section 3.8). Cela est analogue à la description des séries linéaires algébro-géométriques de rang 1, et résulte d'une description explicite de la variation des *diviseurs réduits* – en bref, des diviseurs qui concentrent le plus grand coefficient possible en un point donné – définies par les séries linéaires, par rapport au point base (voir Section 3.7).

- Par conséquent, *toute* série linéaire combinatoire de rang 1 est réalisable, c'est-à-dire peut se réaliser comme tropicalisation d'une certaine série linéaire géométrique (Théorème 3.8.8).

Comme cela a été discuté plus haut, la définition d'une structure de pentes utilise la notion de matricube développée dans la section précédente. En fait, ce nouvel objet combinatoire est d'abord apparu naturellement dans ce contexte, comme une donnée combinatoire extraite d'une série linéaire géométrique, et nous nous sommes rendu compte seulement dans un second temps que les matricubes étaient des généralisations – relativement naturelles – des matroïdes qui devraient être étudiées en soi, d'un point de vue algébrique et combinatoire (voir Chapitre 2).

Il y a également un lien étroit entre la théorie des séries linéaires combinatoires qui vient d'être présentée et l'étude de la tropicalisation des points de Weierstrass, sujet du Chapitre 4, que nous résumons dans la dernière section de l'introduction.

Localiser la tropicalisation des points de Weierstrass grâce aux séries linéaires combinatoires — Chapitre 4

Dans le même esprit que le Chapitre 3, le Chapitre 4 concerne la tropicalisation d'un objet géométrique. De la même manière, il comporte, d'une part, le développement d'une nouvelle théorie du côté tropical et combinatoire et, d'autre part, l'étude du lien entre les objets issus d'une tropicalisation et leurs homologues purement combinatoires.

Les objets géométriques dont il est question dans ce chapitre sont appelés *points de Weierstrass*. Ce sont des points géométriques spéciaux définis sur des courbes algébriques. Essentiellement, les points de Weierstrass sont une généralisation de ce qui est plus communément nommé les "points d'inflexion", c'est-à-dire les points où le signe de la courbure change, par exemple dans l'étude des courbes réelles. À titre d'illustration, la Figure 1.5 ci-dessous montre les points d'inflexion d'une courbe elliptique réelle plongée dans le plan réel. Néanmoins, on peut en fait définir les points de Weierstrass, de façon plus générale,

sur n'importe quelle courbe algébrique lisse X sur un corps algébriquement clos $\mathbf{K}$. Pour simplifier, nous supposerons que $\mathbf{K}$ est de caractéristique nulle.

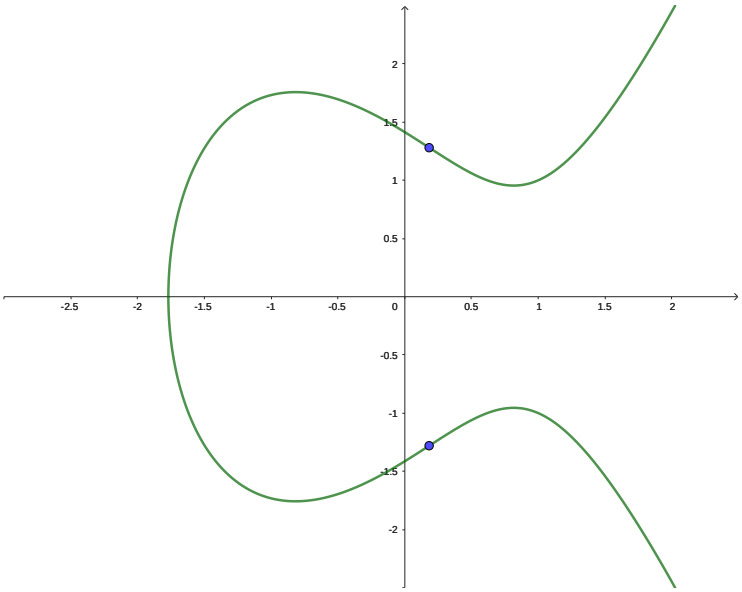


Figure 1.5: La courbe elliptique réelle d'équation $y^2 = x^3 - 2x + 2$ et ses deux points d'inflexion.

Une manière de définir les points de Weierstrass dans le contexte plus général d'une courbe algébrique, sans qu'elle soit forcément plongée dans un espace projectif, utilise de façon cruciale la théorie des diviseurs et des fonctions méromorphes sur les courbes algébriques, exposée dans la section précédente de cette introduction.

Définissons pour cela quelques notions de base que nous n'avions pas encore fait l'effort de définir jusqu'alors. Si $\mathcal{D}$ est un diviseur sur X, le *système linéaire* $|\mathcal{D}|$ de $\mathcal{D}$ est l'ensemble des diviseurs de la forme $\mathcal{D} + \mathrm{div}(f)$ avec f dans la série linéaire $\mathrm{Rat}(\mathcal{D})$ de $\mathcal{D}$. Plus simplement, c'est l'ensemble des diviseurs *effectifs* qui peuvent être obtenus à partir de $\mathcal{D}$ en ajoutant le diviseur d'une fonction méromorphe. Le *degré* de $\mathcal{D}$ est la somme de tous ses coefficients. Le *rang* de $\mathcal{D}$ est la dimension de l'espace vectoriel $\mathrm{Rat}(\mathcal{D})$ (moins un). Il s'avère que le rang de $\mathcal{D}$ admet une définition alternative intéressante : il est égal au plus grand entier r tel que pour tout point $x \in X$, il existe un diviseur (effectif) $\mathcal{E}$ dans $|\mathcal{D}|$ dont le coefficient au point x est au moins r, c'est-à-dire $\mathcal{E} \geqslant r\,(x)$; ou, de manière équivalente, c'est le plus grand entier r tel que pour tout choix de points $x_1, \ldots, x_r$ on X, il existe un diviseur (effectif) $\mathcal{E}$ dans $|\mathcal{D}|$ tel que $\mathcal{E} \geqslant (x_1) + \cdots + (x_r)$. En résumé, et comme noté précédemment, le rang mesure la *taille de la série linéaire* (ou du système linéaire).

L'étude systématique des points de Weierstrass des courbes algébriques remonte aux années 1870, lorsque Weierstrass et Schottky ont compris que les points de Weierstrass étaient liés aux *ordres d'annulation des fonctions méromorphes*. En effet, ils ont d'abord remarqué qu'étant donné un diviseur $\mathcal{D}$ de rang r, l'ensemble $S(y)$ des ordres d'annulation

décalés de toutes les fonctions méromorphes de $\mathrm{Rat}(\mathcal{D})$ en le point y, c'est-à-dire

$$\begin{aligned} S(y) &:= \big\{\mathcal{D}(y) + \mathrm{div}(f)(y) \;\big|\; f \in \mathrm{Rat}(\mathcal{D})\big\} \\ &= \big\{\mathcal{D}(y) + \mathrm{ord}_y(f) \;\big|\; f \in \mathrm{Rat}(\mathcal{D})\big\} \\ &= \big\{\mathcal{E}(y) \;\big|\; \mathcal{E} \in |\mathcal{D}|\big\}, \end{aligned}$$

qui est par définition toujours constitué d'entiers positifs, était en fait égal à l'ensemble $\{0, \ldots, r\}$ pour tout $y \in X$ sauf pour un nombre fini de points ; et, de plus, que ces points en nombre fini étaient exactement les points de Weierstrass de X.

Notons que la condition $S(y) \neq \{0, \ldots, r\}$ est équivalente à la condition $\max S(y) \geqslant r+1$. Par conséquent, en utilisant l'une des définitions du rang proposées ci-dessus, on voit qu'un point de Weierstrass est un point y pour lequel il existe un diviseur dans le système linéaire de $\mathcal{D}$ ayant un coefficient en y strictement plus grand que r, la plus grande valeur attendue. L'existence d'un tel diviseur explique pourquoi les points de Weierstrass sont les points où se produit une certaine forme de "résonance", qui correspond géométriquement à un point d'inflexion. Si $S(y)$ est même strictement supérieur à $r+1$, le point de Weierstrass y a en quelque sorte une *multiplicité* supérieure, ce qui indique qu'un phénomène géométrique d'ordre supérieur a lieu. En additionnant les points de Weierstrass de $\mathcal{D}$ – en nombre fini –, comptés avec leur multiplicité, on obtient un nouveau diviseur $\mathcal{W}(\mathcal{D})$, le *diviseur de Weierstrass* de $\mathcal{D}$.

Ceci est bien connu et a été très étudié depuis plus d'un siècle, mais a priori, on peut définir les points de Weierstrass seulement sur les courbes *lisses*. On *ne sait pas* comment les définir de façon pertinente, par exemple, sur des courbes *stables* en toute généralité. Ce problème est, aujourd'hui, toujours ouvert.

Un premier pas dans cette direction est l'étude de ce qui *arrive aux* points de Weierstrass sur des courbes lisses lorsqu'on fait dégénérer une famille de courbes lisses vers une courbe stable limite, dans le contexte de dégénérescence des familles discuté dans la section précédente de cette introduction. Si l'on se permet des notations moins générales mais plus intuitives, imaginons que l'ont ait une famille à un paramètre $(X_t)_{t\neq 0}$ de courbes *lisses*, munie d'une famille lisse de diviseurs $(\mathcal{D}_t)_{t\neq 0}$, chaque $\mathcal{D}_t$ étant un diviseur sur X_t. Pour tout $t \neq 0$, on dispose des points de Weierstrass de $\mathcal{D}_t$, en nombre fini, et donc d'un diviseur de Weierstrass $\mathcal{W}(\mathcal{D}_t)$, sur X_t. Nous pouvons maintenant reformuler notre question :

Question 1.0.2. *Supposons que la famille peut être complétée en une famille $(X_t)_t$, la fibre spéciale X_0 étant une courbe stable. Quel peut être le comportement asymptotique de $\mathcal{W}(\mathcal{D}_t)$ lorsque $t \to 0$?*

Ou encore, en adoptant un autre point de vue, si X_0 est une courbe stable donnée, peut-on localiser, sur X_0, toutes les limites possibles de points de Weierstrass pour des familles (X_t) qui dégénèrent en X_0 ? Plus largement, le problème visant à construire un espace de modules paramétrisant tous les points de Weierstrass limites possibles d'une courbe stable donnée a été soulevé par Eisenbud et Harris [EH86, EH87a].

Pour une famille (X_t) comme ci-dessus, il y a deux types de points de Weierstrass sur la courbe lisse X_t :

(a) ceux qui, lorsque $t \to 0$, dégénèrent *hors des nœuds* de la courbe stable X_0 ; et

(b) ceux qui, lorsque $t \to 0$, dégénèrent *vers un nœud* de X_0.

Les points de type (b) sont plus difficiles à comprendre, et c'est sur eux que se concentre le Chapitre 4. Des progrès graduels ont été faits sur cette question, sous certaines conditions combinatoires sur la courbe stable X_0. Dans [Ami14], Amini démontre un résultat d'équidistribution pour les points de Weierstrass des puissances d'un fibré en droites, en utilisant une description concrète des points de Weierstrass limites. Nous proposons une étude plus raffinée des points de Weierstrass, avec l'apport de la géométrie tropicale.

Pourquoi la géométrie tropicale est-elle utile pour étudier les points de Weierstrass dégénérant vers un nœud ? C'est parce que la géométrie tropicale fournit des outils pour "dérouler" les singularités et regarder de plus près ce qui se passe "à l'intérieur". En effet, comme exposé précédemment dans cette introduction, la tropicalisation de la famille (X_t) donne naissance à un graphe métrique dont chaque sommet v correspond à une composante irréductible X_v de X_0, et chaque arête $e = uv$ correspond à un point d'intersection entre les composantes irréductibles X_u et X_v. De surcroît, les propriétés de la procédure de tropicalisation impliquent que si un point p_t de X_t, $t \neq 0$, dégénère, lorsque $t \to 0$, vers un nœud $p \in X_u \cap X_v$ de X_0, alors il atterrira sur l'arête $e = uv$ du graphe métrique *à une position qui dépend de la manière dont il dégénère vers* p. C'est pourquoi étudier où les points de Weierstrass dégénèrent sur chaque arête du graphe métrique associé à la famille $(X_t)_t$ donne de l'information sur la géométrie de cette famille. C'est ce chemin que nous suivons dans le Chapitre 4. Nous allons maintenant expliquer comment.

L'approche naturelle, dans le même esprit que la théorie combinatoire des séries linéaires, est de définir ce que devraient être les *points de Weierstrass tropicaux*, et ensuite d'étudier leurs liens avec les *tropicalisations de points de Weierstrass géométriques*.

Pour définir les points de Weierstrass tropicaux d'un diviseur D sur un graphe métrique Γ, on peut raisonner par analogie avec la définition géométrique. Le *degré* de D est la somme de ses coefficients, et son *rang* est le plus grand entier r tel que pour tout choix de points $x_1, \ldots, x_r \in \Gamma$, il existe un diviseur (effectif) E dans $|D|$ tel que $E \geqslant (x_1) + \cdots + (x_r)$. Pour définir les points de Weierstrass de D, on utilise également une analogie avec la géométrie : un point $x \in \Gamma$ est dit de Weierstrass pour D lorsque la valeur maximale de $D(x) + \operatorname{div}(f)(x)$ pour $f \in \operatorname{Rat}(D)$ est au moins égale à $r + 1$.

À titre d'exemple, la figure ci-dessous, issue de la Section 4.3.1, montre le graphe complet à quatre sommets, muni du diviseur K (pour "canonique") de degré 4 et de rang 2, ayant pour coefficient 1 sur chaque sommet. (Le diviseur canonique est défini par $K(x) = d_x - 2$ pour tout $x \in \Gamma$, où d_x est la valence de x. Son degré est toujours $2g - 2$, et son rang $g - 1$, où g est le genre de Γ.) L'ensemble des points de Weierstrass tropicaux est représenté en rouge sur la partie droite de la figure. Notons qu'à chaque point de Weierstrass peut être

attribuée une *multiplicité* de 2, parce que la valeur maximale de $K(x) + \mathrm{div}(f)(x)$ pour $f \in \mathrm{Rat}(K)$ est en fait $4 = r + 2$.

Figure 1.6: Graphe complet à quatre sommets, et le diviseur de Weierstrass du diviseur canonique K.

Cette définition donne déjà des résultats intéressants. Baker a montré [Bak08, Lemma 2.8] que les points de Weierstrass tropicaux sont *compatibles* avec les points de Weierstrass géométriques, dans le sens où si on dispose d'une famille dégénérescente de courbes munies de diviseurs, les limites des points de Weierstrass géométriques sont *parmi* les points de Weierstrass tropicaux.

Mais en géométrie tropicale, les choses ne se passent pas toujours aussi harmonieusement qu'en géométrie algébrique classique, comme on l'a déjà noté pour ce qui est des séries linéaires... Il se trouve que contrairement aux points de Weierstrass géométriques, *les points de Weierstrass tropicaux ne sont pas nécessairement en nombre fini* ! Par exemple, sur le graphe "haltère" ci-dessous, en choisissant le diviseur canonique K qui a pour coefficient 1 sur chacun des deux sommets, l'arête centrale est entièrement constituée de points de Weierstrass. C'est pourquoi on parle du *lieu de Weierstrass* (potentiellement infini) d'un diviseur D, que l'on notera $L_W(D)$.

Figure 1.7: Le graphe "haltère" et le lieu de Weierstrass tropical du diviseur K. Chaque point de Weierstrass isolé, en rouge, a multiplicité 1 mais, à ce stade, on ne sait pas associer une multiplicité à l'arête centrale.

Ceci entraîne que, si une famille dégénérescente de courbes avec des diviseurs a ce graphe métrique et ce diviseur pour tropicalisation, on ne sera pas capable de déterminer exactement où les points de Weierstrass atterrissent si leur limite appartient à l'arête rouge. La seule chose que l'on *pourrait* tenter de déterminer est *le nombre de* points de Weierstrass qui atterrissent sur l'arête rouge, si tant est qu'il soit possible de répondre à cette question. Mais comment trouver une façon naturelle de définir une multiplicité pour l'arête centrale semblable à celle pour les points isolés, dans le but de répondre à cette question d'une manière satisfaisante ?

C'est précisément ce que nous faisons dans la Section 4.3, en deux étapes.

(1) Premièrement, remarquer que le lieu de Weierstrass tropical $L_W(D)$ est un sous-ensemble "sympathique" de Γ : il est *fermé* et a *un nombre fini de composantes connexes*.

(2) Deuxièmement, trouver une façon signifiante d'étendre la multiplicité d'un point de Weierstrass isolé à n'importe quelle composante connexe du lieu de Weierstrass tropical.

Voici comment définir cette multiplicité. Soit D un diviseur de rang r, et soit C une composante connexe du lieu de Weierstrass tropical $L_W(D)$. On définit le *poids de Weierstrass tropical* de C comme

$$\mu_W(C; D) := \deg\left(D|_C\right) + (g(C) - 1)r - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(D) \tag{1.1}$$

où

- $\deg\left(D|_C\right)$ est le degré total de D dans C, défini par $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$;
- $g(C)$ est le genre de C, c'est-à-dire son premier nombre de Betti $\dim H_1(C, \mathbb{R})$;
- $\partial^{\text{out}} C$ est l'ensemble des directions tangentes sortantes à partir de C ; et
- $s_0^\nu(D)$ est la pente minimale d'une fonction $f \in \text{Rat}(D)$ le long de la direction tangente ν.

Observons que cette définition n'utilise que des données combinatoires finies sur Γ et D. Le fait que la pente minimale $s_0^\nu(D)$ est bien définie et facilement calculable découle du travail présenté dans la Section 4.2.

Il s'avère que le poids tropical d'une composante connexe réduite à un singleton coïncide avec la multiplicité usuelle évoquée pour un point isolé. Mais ces poids sont-ils intéressants pour répondre à notre question de dégénérescence ? Oui, parce qu'ils mesurent des phénomènes intéressants, à la fois du point de vue tropical et du point de vue géométrique.

Du côté tropical, les poids se comportent bel et bien comme s'ils *comptaient quelque chose*. En effet,

- pour toute composante connexe C, $\mu_W(C; D)$ est un *entier strictement positif* ; et
- la somme des poids $\mu_W(C; D)$ sur toutes les composantes connexes vaut $d - r + rg$, donc est fonction seulement des caractéristiques combinatoires de Γ et D.

Ces résultats répondent notamment à une question soulevée par Baker, qui a fait le commentaire suivant [Bak08, Remark 4.14] concernant un éventuel analogue tropical d'un fait bien connu en géométrie algébrique pour le diviseur canonique : *"it is not clear if there*

is an analogue for metric graphs of the classical fact that the total weight of all Weierstrass points on a smooth curve of genus $g \geqslant 2$ is $g^3 - g$".

Du côté géométrique, de bonnes nouvelles nous attendent également. Comme nous le montrons dans la Section 4.3, nous disposons du théorème suivant (Théorème 4.1.10).

Theorem 1.0.3. *Supposons que nous ayons une famille de courbes lisses $(X_t)_{t\neq 0}$, munie d'une famille de diviseurs $\mathcal{D}_t$, et dégénérant vers une courbe stable X_0. Soit Γ le graphe métrique associé, par la procédure de tropicalisation, à la famille $(X_t)_t$, et soit D la tropicalisation des $\mathcal{D}_t$. Soit $\mathcal{W}(\mathcal{D}_t)$, pour tout t, le diviseur de Weierstrass de $\mathcal{D}_t$, et soit C une composante connexe du lieu de Weierstrass tropical $L_W(D)$ de D. Supposons de plus que $\mathcal{D}_t$ et D aient le même rang.*

Alors le nombre de points de $\mathcal{W}(\mathcal{D}_t)$ qui convergent vers un point de C lorsque $t \to 0$ est dénombré par le poids $\mu_W(C; D)$, multiplié par $r + 1$.

Ce théorème a pour conséquence que les points de Weierstrass géométriques se tropicalisent par paquets de $r + 1$, ce qui généralise un résultat obtenu par Brugallé et De Medrano [BDM12] sur les multiplicités des points de Weierstrass tropicaux en rang 2.

Dans la Section 4.5, nous démontrons, en fait, une version locale et plus générale de ce résultat qui, entre autres, implique le résultat suivant.

Theorem 1.0.4. *Reprenant les notations du Théorème 1.0.3, si de plus D est de rang au moins 1, alors n'importe quel cycle du graphe métrique Γ intersecte le lieu de Weierstrass tropical $L_W(D)$.*

Ce résultat, qui est *local*, ne semble pas avoir d'analogue pour les courbes algébriques : la géométrie tropicale a ses propres idiosyncrasies et ne s'aligne pas nécessairement sur la géométrie algébrique ! En outre, ceci généralise un résultat *global* obtenu par Baker [Bak08], énonçant que le lieu de Weierstrass tropical $L_W(K)$ du diviseur canonique est non vide si Γ a un genre au moins égal à deux.

Le Théorème 1.0.4 implique, en particulier, que si le lieu de Weierstrass tropical est fini, alors *tout cycle dans Γ contient un point de Weierstrass limite de la famille $(X_t)_t$*, ce qui est une contrainte relativement forte sur le comportement limite des points de Weierstrass.

J'ai promis qu'il y avait un lien étroit entre les points de Weierstrass tropicaux et les séries linéaires combinatoires, mais cela n'est toujours pas visible ! En fait, les séries linéaires combinatoires sont cruciales dans la preuve du Théorème 1.0.3. En effet, pour étudier la tropicalisation des points de Weierstrass de la famille de diviseurs $\mathcal{D}_t$ sur X_t, il est capital de ne pas seulement considérer le diviseur tropicalisé D, mais aussi la structure de pentes définie par la série linéaire géométrique $\mathrm{Rat}(\mathcal{D}_t)$, en se souvenant des ordres d'annulation (conjoints) des fonctions méromorphes réduites en les nœuds de X_0, comme décrit dans la section précédente de cette introduction. En résumé, cette preuve illustre le fait que les séries linéaires combinatoires (incluant la donnée de leur structure de pentes) portent une quantité d'information combinatoire qui se trouve être la bonne dans l'étude de la

tropicalisation de certains objets géométriques, que ce soit les séries linéaires ou les points de Weierstrass.

Bien sûr, certaines complications techniques ont ici été mises sous le tapis. Notamment :

- Une formulation rigoureuse des théorèmes ci-dessus nécessite de considérer non seulement des graphes métriques, mais plus généralement des graphes métriques *augmentés*, c'est-à-dire des graphes métriques munis d'une fonction de genre $\mathfrak{g}$ associant à chaque sommet un entier positif. Le nombre $\mathfrak{g}(v)$ encode le genre de la composante irréductible X_v de X_0.

- Il est de plus nécessaire de définir les points de Weierstrass non seulement d'un diviseur D, mais aussi de tout sous-semi-module $M \subseteq \mathrm{Rat}(D)$ raisonnable ; en particulier, de toute série linéaire combinatoire.

Ces raffinements sont l'objet de la Section 4.4. Notons que les résultats présentés ci-dessus ne valent pas seulement si le corps de base $\mathbf{K}$ est de caractéristique nulle ; nous proposons des adaptations au cas de la caractéristique positive, ce qui laisse entrevoir des applications potentielles en géométrie arithmétique.

Pour une gamme d'exemples commentés de lieux de Weierstrass tropicaux, montrant la diversité combinatoire de ces objets, nous renvoyons la lectrice ou le lecteur à la Section 4.6.

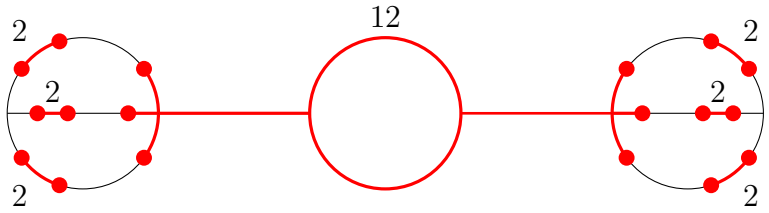


Figure 1.8: Un graphe avec deux ponts (ou isthmes) et le lieu de Weierstrass $L_W(K)$ du diviseur canonique K.

English version

This introduction aims to sum up the content of the thesis manuscript, avoiding technical details which will be developed in the subsequent chapters. It should be useful both for a person interested to read the full manuscript and for a non-specialist wanting to peek an eye into these topics without reading the full text.

The goal of this PhD thesis is to study some aspects of tropical geometry related to algebraic geometry.

In a nutshell, many classical algebro-geometric objects – such as curves, divisors on curves, spaces of functions on curves, or special points on curves – can be deformed into other geometric objects of a much simpler kind, having combinatorial features, through a procedure called "tropicalization" – see below for more details. A central theme in this thesis consists in studying the tropicalization of some of these algebro-geometric objects. Firstly, by developing metric and combinatorial theories to describe the tropicalized objects we can obtain; secondly, by drawing links between the properties of these tropicalized objects and the original objects; and thirdly, by exploring the question of whether a given metric or combinatorial object can be obtained from a classical object through the tropicalization procedure.

The inspiration for this PhD project stems from the study of linear series from the point of view of tropical geometry, which has been a tremendous success in applying tropical methods. The tropicalization procedure allows to transform the global sections of a line bundle into piecewise affine linear functions with integral slopes on a polyhedral object. Baker's specialization inequality in dimension one [Bak08, Lemma 2.8], as well as its generalization to higher dimensions given by Cartwright [Car19, Car21], then permit to compare the dimension of the space of global sections of the line bundle with a notion of combinatorial rank associated to the tropicalization.

In the present manuscript, we will mainly care about one-dimensional objects, that is, algebraic curves, metric graphs, and the associated geometric structures. That being said, we will discuss in Section 5.4 some generalization perspectives for our results to higher dimensions, in connection with the work by Cartwright mentioned above, which, from a personal standpoint, has been my point of entry into tropical geometry during the preparation of my master's thesis.

In the next section of this introduction, we give an intuitive approach on the tropicalization procedure which we hope to be helpful to understand how tropicalization works.

The tropicalization procedure and the spirit of tropical geometry

The visual intuition most people have about tropical geometry is that it provides procedures to *deform* algebraic varieties to the point where they turn into *polyhedral*

objects, like cone complexes or polyhedral complexes, sometimes with some well-chosen additional combinatorial data.

The point of this type of transformation is that the polyhedral objects it spits out are fundamentally easy to study, because they can be described using a finite amount of combinatorial information (the equations defining of each polyhedron, for instance).

Also, and crucially, tropical transformations do not yield "random" objects: they deform algebraic varieties (or other objects) in such a way that *some relevant information is preserved* and can be *measured* on the polyhedral side to learn more about the original geometric objects.

This is why some people describe tropical geometry as "a combinatorial (or piecewise linear) shadow of algebraic geometry" [MR18, MS21].

The first actual tropicalization procedures have been defined in the early 1970's by Bergman [Ber71]. They applied mainly to affine or projective algebraic varieties over the field $\mathbb{C}$ of complex numbers, and used logarithmic functions to transform these geometric objects.

As an example, consider the set $\mathcal{L}$ of complex points $(z_1, z_2) \in \mathbb{C}^2$ such that $z_1 + z_2 - 1 = 0$. This is simply a complex line in the complex plane. Now define, for some $0 < t < 1$, the set L_t of points in $\mathbb{R}^2$ defined by taking the logarithm of the modulus of points of $\mathcal{L}$, coordinate-wise:

$$L_t := \big\{(x, y) = (\log_t(|z_1|), \log_t(|z_2|)) \;\big|\; (z_1, z_2) \in \mathcal{L}\big\},$$

where $\log_t \colon \mathbb{R}_{>0} \to \mathbb{R}$ denotes the base t real logarithm. L_t is a *curved*, *real* object, of real dimension two (like $\mathcal{L}$), and called an *amoeba*, see the left part of Figure 1.9.

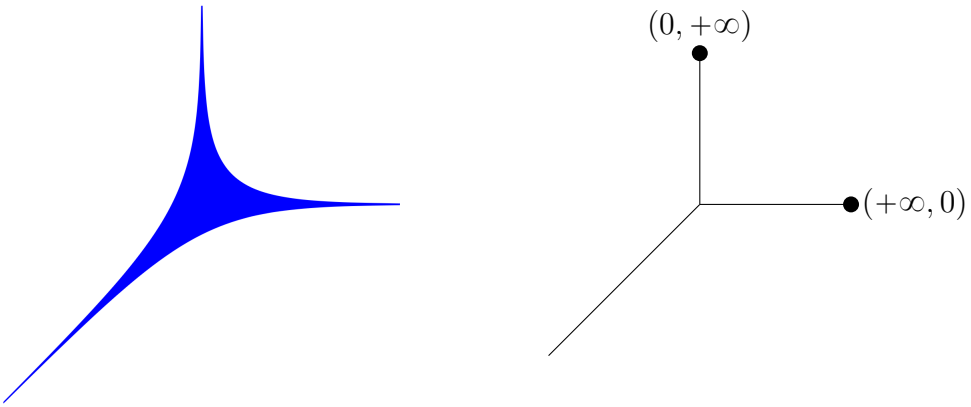


Figure 1.9: Amoeba and tropicalization of the complex line $\mathcal{L}$.

If we choose another real number t' such that $0 < t' < t$, then the amoeba $L_{t'}$ is simply obtained from the amoeba L_t by a homothety of factor $\frac{\log(t)}{\log(t')} < 1$: the amoeba "shrinks" more and more as t becomes smaller.

The interesting phenomenon, at the core of tropical geometry, happens if we let $t \to 0$. In this case, the curved object L_t will *degenerate* to some *limit object* L_0, which has the property of being fully polyhedral. L_0 is called the *tropicalization* of the complex curve $\mathcal{L}$. Since $\mathcal{L}$ is a complex line, L_0 is in fact a very fundamental object in tropical geometry, an elementary building block of tropicalizations called the *tropical line in the plane*. It is one-dimensional and made up of three real half-lines: a vertical one, a horizontal and a diagonal one (see the right part of Figure 1.9). Each of those half-lines corresponds to a specific joint behavior of the variables z_1 and z_2. For instance, the vertical half-line corresponds to the situation where $|z_1|$ is close to 1 and z_2 close to 0.

Note that, strictly speaking, since $\mathcal{L}$ intersects the coordinate axes of $\mathbb{C}^2$, the tropical line contains two points "at infinity", in a framework that can be made rigorous. This technicality is avoided if we consider a subvariety of $(\mathbb{C}^*)^n$.

Moreover, note that, for each particular point $(z_1, z_2) \in \mathcal{L}$, each coordinate of the limit point $\lim_{t\to 0}(\log_t(|z_1|), \log_t(|z_2|))$ belongs to the finite set $\{0, +\infty\}$. Nevertheless, the tropical line L_0 is the *one-dimensional* object obtained as the limit set (in the Hausdorff topology) of L_t as $t \to 0$, i.e., in this particular case, the intersection of all amoebas L_t for $t \neq 0$.

A theorem of paramount importance, the *fundamental theorem of tropical geometry*, gives an effective description of the tropicalization of an algebraic variety. If, to take the simplest situation, an algebraic variety $\mathcal{V} \subseteq \mathbb{C}^n$ is defined by the vanishing of some polynomial in n variables $P = \sum_{\mathbf{k}\in\mathbb{N}^n} a_{\mathbf{k}} X_1^{k_1} \cdots X_n^{k_n}$, with $a_{\mathbf{k}} \in \mathbb{C}$, then we can associate to P a *tropical polynomial* $\mathrm{trop}(P)$ obtained by replacing every occurrence of the *multiplication* by an *addition*, and every occurrence of the *addition* by a *minimum* (or maximum, depending on the convention). This is why the algebraic formalism using these tropical operations is called the *tropical algebra*, or *max-plus algebra*. This way, $\mathrm{trop}(P)$ becomes a real-valued, piecewise linear function $\mathbb{R}^n \to \mathbb{R}$, defined by

$$\mathrm{trop}(P)(x_1, \ldots, x_n) := \min_{\mathbf{k}\in S}(k_1 \cdot x_1 + \cdots + k_n \cdot x_n),$$

where $S \subseteq \mathbb{N}$ is finite. In a more general case, the coefficients $a_{\mathbf{k}}$ depend on t, and we therefore have a *family of subvarieties of* $\mathbb{C}^n$; the tropicalization of P is, in this broader case, a piecewise *affine linear* function. This will be discussed in more detail later on.

The fundamental theorem of tropical geometry describes the tropicalization V of $\mathcal{V}$ as the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ where the tropical polynomial $\mathrm{trop}(P)$ *vanishes tropically*; and this is said to happen at a point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ whenever the minimum in $\min_{\mathbf{k}\in S}(k_1 \cdot x_1 + \cdots + k_n \cdot x_n)$ is *achieved at least twice*.

In our example above, $n = 2$, $P = Z_1 + Z_2 - 1$ and $\mathrm{trop}(P) = \min(x, y, 0)$, whose set of tropical roots is indeed the three half-lines shown in Figure 1.9. Indeed, the horizontal (resp. vertical, resp. diagonal) half-line corresponds to the case where the terms x and 0 (resp. y and 0, resp. x and y) of $\mathrm{trop}(P)$ realize the minimum together, and the origin corresponds to the joint realization by all three terms. If we replace the polynomial P by the polynomial $P_a := Z_1 + Z_2 - a$ with $a \in \mathbb{C}^*$, we get exactly the same tropicalization. The

case $a = 0$, however, stands out, as the tropicalization of the polynomial $P_0 := Z_1 + Z_2$ is the function defined by $\mathrm{trop}(P_0)(x, y) = \min(x, y, +\infty) = \min(x, y)$. This is why in the above example, we didn't chose the complex line defined by $z_1 + z_2 = 0$, whose tropicalization is a degenerate version of the above tropical half-line, with only a diagonal line of equation $x = y$.

For a general subvariety $\mathcal{V} \subseteq (\mathbb{C}^*)^n$, it turns out that the tropicalization V will always be the support of a *polyhedral fan*, i.e., the support of a nice arrangement of polyhedral cones, of pure dimension the complex dimension of $\mathcal{V}$. Figure 1.10 shows examples of one-dimensional polyhedral fans in $\mathbb{R}^2$.

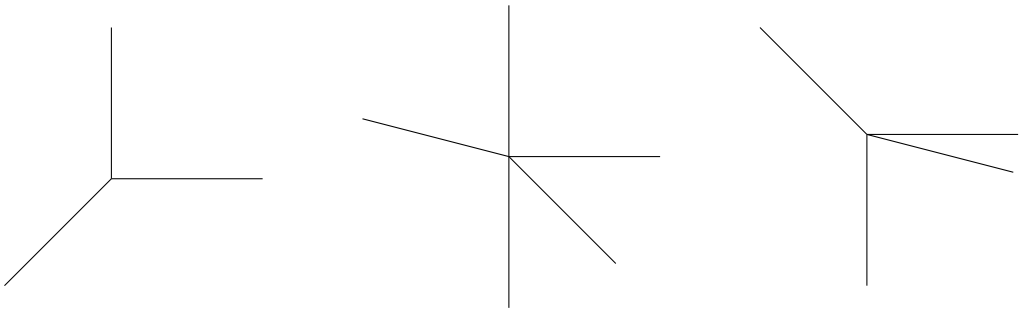


Figure 1.10: Three one-dimensional polyhedral fans in $\mathbb{R}^2$.

Can we generalize this kind of procedure to more general contexts, for example, for a variety over a field other than $\mathbb{C}$? The answer is *yes*: there is a nice tropicalization procedure for varieties over fields endowed with a *non-Archimedean valuation*, which opens new perspectives.

A non-Archimedean valuation $\mathbf{val}$ on a field $\mathbf{K}$ is a function

$$\mathbf{val} \colon \mathbf{K} \to \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, such that the following properties hold.

(1) For every $x \in \mathbf{K}$, we have $\mathbf{val}(x) = +\infty$ if, and only if, $x = 0$.

(2) For every $x, y \in \mathbf{K}$, $\mathbf{val}(xy) = \mathbf{val}(x) + \mathbf{val}(y)$.

(3) For every $x, y \in \mathbf{K}$, $\mathbf{val}(x + y) \geqslant \min(\mathbf{val}(x), \mathbf{val}(y))$, and the equality holds if $\mathbf{val}(x) \neq \mathbf{val}(y)$.

These properties are partially similar to the algebraic properties of the function

$$\lim_{t \to 0} \log_t(|\cdot|) \colon \mathbb{C} \to \{0, +\infty\} \subset \overline{\mathbb{R}}$$

used in the first kind of tropicalization procedure, which is in fact called the *trivial valuation* on $\mathbb{C}$ and is defined by $\mathbf{val}(z) = 0$ for every $z \in \mathbb{C}^*$ and $\mathbf{val}(0) = +\infty$. A key example of non-Archimedean valuation is the *degree in* t, or t*-adic valuation*, defined on the field $\kappa(\!(t)\!)$

of formal Laurent series over κ, for a field κ, or on the field $\kappa\{\{t\}\}$ of Puiseux series over κ, which has the additional – and often useful – property of being algebraically closed.

As an illustration, the tropicalization of an affine variety $\mathcal{V} \subseteq (\kappa\{\{t\}\}^*)^n$ cut out by an ideal of $\kappa\{\{t\}\}[Z_1, \dots, Z_n]$ is defined, by analogy with the above procedure, as the closure of the image of $\mathcal{V}$ by the function

$$\begin{aligned} \text{trop}: \qquad & (\kappa\{\{t\}\}^*)^n \to \mathbb{R}^n \\ & (R_1, \dots, R_n) \mapsto (\mathbf{val}(R_1), \dots, \mathbf{val}(R_n)), \end{aligned}$$

where **val** denotes the t-adic valuation. Again, $\text{trop}(\mathcal{V})$ will be a nice polyhedral space, with some finite parts together with some infinite parts which have to be handled with care. Figure 1.11 shows an example of a tropicalized curve which can be obtained in this way.

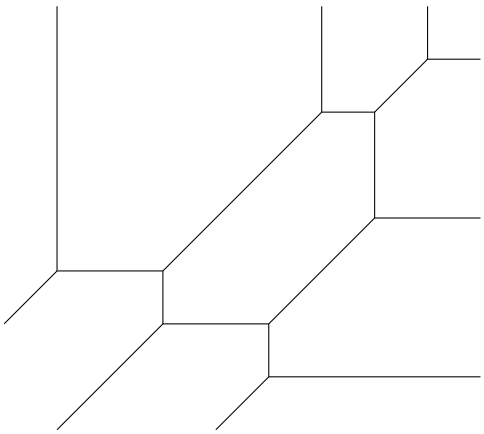


Figure 1.11: The tropicalization of an algebraic curve.

The tropicalization procedure for fields endowed with a non-Archimedean valuation is particularly useful in the study of *degenerations* of a family of smooth algebraic varieties to a (possibly non-smooth) limit variety.

Imagine a family (X_t) of smooth algebraic varieties over a field κ, varying algebraically with a parameter t belonging to some base, like a punctured disk Δ^*. Algebraic geometry and singularity theory strive to answer questions such as the following ones.

- What is the possible behavior of the variety X_t as $t \to 0$?
- What are the possible limit varieties X_0?
- What about the limit behavior of a family of line bundles on X_t, or other geometric objects defined on X_t?

Such a degenerating family, where each variety is defined over a common field κ, can be modeled by a variety over a base of dimension one, like $\kappa(\!(t)\!)$ or $\kappa\{\{t\}\}$ (let's say $\kappa(\!(t)\!)$ for simplicity). Then, every extension of this family to $t = 0$ will be called a *model*, and

will be described by a scheme $\mathcal{X}$ over the ring $R = \kappa[\![t]\!]$, whose field of fractions is $\kappa(\!(t)\!)$. The models which are of interest to us, with nice properties, are called *semistable*. Their existence is assured in characteristic zero, and typically requires passing to a finite field extension of $\kappa(\!(t)\!)$. The *generic fiber* of $\mathcal{X}$ (the fiber over the complement of 0) essentially remembers the part of the degenerating family where the varieties are smooth, and the *special fiber* (the fiber over 0) is the limit variety X_0, which will typically have singular points. Since R is endowed with a non-Archimedean valuation, the family $\mathcal{X}$ can be tropicalized using the procedure described above, by putting coordinates on the family.

This is, in a nutshell, the reason why studying the degeneration of a family of algebraic varieties can make use of the tropicalization of this family into a polyhedral object. This object, depending on the procedure we use, can either be an unbounded object like cone complex (more generally, a polyhedral complex, see Figure 1.11), or a *bounded* object like a metric graph. Such a graph appears in fact as a *skeleton* of a polyhedral complex, both of these objects being obtained from the family of varieties by alternative tropicalization methods.

A more precise explanation of how this works involves some basics of Berkovich theory. In this domain, we define a topological space associated to an affine variety $\mathcal{V} \subseteq (\mathbf{K}^*)^n$ cut out by an ideal $\mathcal{I}$ of $\mathbf{K}[Z_1, \dots, Z_n]$. The points x of this topological space parametrize semi-valuations $\mathfrak{v}_x$ on the ring $\mathbf{K}[Z_1, \dots, Z_n]/\mathcal{I}$ which extend the valuation $\mathbf{val}$ on $\mathbf{K}$. It turns out that this space, called the *Berkovich analytification* $\mathcal{V}^{\mathrm{an}}$ of $\mathcal{V}$, is well-behaved: it is Hausdorff, path-connected, locally akin to a polyhedral space, and it deformation retracts to compact polyhedral subspaces, its *skeleta*. Moreover, $\mathcal{V}^{\mathrm{an}}$ naturally contains $\mathcal{V}$, the possible tropicalizations of $\mathcal{V}$ all come from projections of the analytification, and this analytification is the projective limit of all the tropicalizations of $\mathcal{V}$ [Pay09, FGP14].

At this point, let us broaden the perspective a little bit. A tropicalization procedure for varieties defined over a field with a non-Archimedean valuation allows to tropicalize *families of varieties*, which will help study singularities in a degeneration context. In fact, if some geometric object (e.g. a vector bundle, or a space of sections of such a bundle) is defined smoothly over such a family, there also are tropicalization procedures for these objects. This will be a red thread in the present manuscript, which will revolve, among others, around the tropicalization of *linear series* (vector spaces of sections of a line bundle) and *Weierstrass points* (special geometric points on algebraic curves). We refer to the discussion about Chapters 4 and 3 later in this introduction for the follow-up of this story.

Now that the reader has some keys to understand tropicalization, we describe the organization of the manuscript in the next section.

Organization of the manuscript

The thesis manuscript is divided into five chapters.

The first chapter is the present introduction, which aims to sum up the content of the thesis, also introducing it to non-specialists.

Chapters 2, 3 and 4 correspond approximately to the three preprints [AG22, AGR23, AG24], respectively. More exactly, these chapters consist in more or less extensive expansions on the corresponding articles: the results, figures or comments that have been removed or shortened in the papers for length or conciseness reasons are kept in full length in the present manuscript.

Chapters 2 and 3 have been written with Omid Amini. Chapter 4 has been written in collaboration with Omid Amini and Harry Richman, from the University of Washington, Seattle.

Chapter 2, based on [AG22], defines new combinatorial objects, dubbed *matricubes*, which turn out to be a new generalization of well-known combinatorial objects, called matroids.

Chapter 3 uses, among others, the formalism of matricubes to lay down the foundations for a new, purely combinatorial theory of limit linear series, allowing to study the tropicalization of algebraic linear series from algebraic curves to metric graphs in greater depth.

Chapter 4 expands on previous work related to tropical Weierstrass points and yields new localization results for the tropicalization of Weierstrass points from algebraic curves to metric graphs.

Finally, Chapter 5 sums up the various kinds of open questions which arose along the way and proposes a few possibly interesting research directions starting from there.

An acknowledgments section is located on page 3, followed by a bilingual preface, before the table of contents.

There is a unique bibliography for the whole manuscript, combining the bibliographies of all chapters and located at the end of the manuscript.

We will now introduce Chapters 2, 3 and 4 of the manuscript in greater detail.

Matroids and matricubes — Chapter 2

Chapter 2 is a purely combinatorial chapter which is based on [AG24] and takes its roots in the fundamental concept of *matroid*. A matroid is a combinatorial object which abstracts and axiomatizes the linear dependence relations between vectors in a vector space, remembering only "who is linearly dependent with whom", without keeping the whole information on those vectors.

For example, let (e_i) be the canonical basis in $\mathbb{R}^3$, and define the following configuration of vectors.

$$\begin{cases} x_1 = e_1, x_2 = e_2, x_3 = e_3 \\ x_4 = 0 \\ x_5 = e_1 + e_2 \end{cases}$$

This configuration of vectors can be represented by the following matrix, where each column

represents a vector:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let us now make a list of all *independents*, i.e., linearly independent sub-families of vectors among the (x_i), described by their indices.

$$\mathscr{I} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{3,5\}, \{1,2,3\}, \{1,3,5\}, \{2,3,5\}\}$$

Here, independents can be viewed as subsets of the set of columns of the matrix above, i.e., subsets of $\{1,2,3,4,5\}$, forgetting the exact values of the vectors.

But the set $\mathscr{I}$ of independents of a configuration of vectors cannot be *any* collection of subsets of $\{1,2,3,4,5\}$, because it has to satisfy some non-trivial combinatorial properties. A *matroid* (literally, "resembling a matrix") is precisely a way to axiomatize and abstract the combinatorial properties of the set of independents of a configuration of vectors.

More formally, a matroid can be defined as a set E together with a collection $\mathscr{I}$ of subsets of E verifying the following three properties:

(I1) $\varnothing \in \mathscr{I}$ (or, equivalently using (I2), $\mathscr{I} \neq \varnothing$).

(I2) (Hereditary property) If $I \in \mathscr{I}$ and $I' \subseteq I$, then $I' \in \mathscr{I}$.

(I3) (Independence augmentation property) If $I_1, I_2 \in \mathscr{I}$ and $|I_1| < |I_2|$, then there exists an element $e \in I_2 \smallsetminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

It is easily shown that the set of independents of a configuration of vectors satisfies the above properties. What is less obvious is that this axiomatic system leads to an interesting and mathematically rich combinatorial object.

It turns out that there are numerous different definitions of matroids which lead to the same concept of matroid. For example, instead of defining a matroid by its independents, we could define it by its *bases*, a collection of subsets verifying another specific axiomatic system mimicking the behavior of actual bases in configuration of vectors. To satiate the reader's curiosity, here is one possible version of the axiomatic system defining a matroid using its collection $\mathscr{B}$ of bases:

(B1) $\mathscr{B} \neq \varnothing$.

(B2) (Basis exchange property) If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \smallsetminus B_2$, then there exists $y \in B_2 \smallsetminus B_1$ such that $B_1 \smallsetminus \{x\} \cup \{y\} \in \mathscr{B}$.

If I give you the set of independents of a matroid, you could also easily deduce the set of bases of that matroid, just by choosing only the *maximal* independents. For example, the configuration of vectors described above has three bases:

$$\mathscr{B} = \{\{1,2,3\}, \{1,3,5\}, \{2,3,5\}\}.$$

As an alternative way of describing a matroid, we could also remember, for each subset of the family of vectors, the dimension of the vector subspace spanned by these vectors, obtaining a *rank function*. Such a rank function also satisfies some precise combinatorial properties which give rise to an axiomatic system, and not every function whose values are non-negative integers can arise this way. These definitions, along with dozens of others, yield the exact same theory of matroids: they are called *cryptomorphic*. They have been widely studied for almost a century, see for instance [Whi92, Oxl06].

For the purpose of introducing Chapter 2, it is useful to note that to each configuration of vectors in a vector space can be associated a hyperplane arrangement in the dual vector space (here, by an abuse of language, a hyperplane can be either an actual hyperplane or the whole space, when the dual vector is the zero vector).

This way, we can also associate a matroid to each hyperplane arrangement. If $\mathrm{F}_1, \dots, \mathrm{F}_m$ are hyperplanes in a vector space H, we can define the rank function which, to each $A \subseteq \{1, \dots, m\}$, associates the natural number $\mathbf{r}(A) := \operatorname{codim}_{\mathrm{H}}(\bigcap_{i \in A} \mathrm{F}_i)$. This rank function satisfies the properties of the rank function of a matroid.

We can now introduce matricubes. These new combinatorial objects, defined in Chapter 2, appeared naturally as a generalization of the preceding construction: what kind of combinatorial object do we get if we replace the *hyperplane arrangement* above by a *flag arrangement*? More precisely, imagine that for each $j \in \{1, \dots, m\}$, $\mathrm{F}_j^\bullet$ is a decreasing flag

$$\mathrm{H} = \mathrm{F}_j^0 \supseteq \mathrm{F}_j^1 \supseteq \cdots \supseteq \mathrm{F}_j^{r-1} \supseteq \mathrm{F}_j^r,$$

where $r \leqslant \dim \mathrm{H}$ and, for each i, F_j^{i+1} is of codimension at most 1 in F_j^i. If we define, for every $\underline{x} = (x_1, \dots, x_m)$, $x_i \in \{0, \dots, r\}$, the rank of $\underline{x}$ as

$$\mathbf{r}(\underline{x}) := \operatorname{codim}_{\mathrm{H}}(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m}),$$

then such a rank function is a generalization of a matroid's rank function. It satisfies an axiomatic system similar to that of matroid rank functions, adapted to the fact that its source is the hypercube $\square_r^m := \{0, \dots, r\}^m$ instead of $\mathcal{P}(\{1, \dots, m\}) \simeq \{0, 1\}^m$, and using the poset structure on $\square_r^m$. We dub this a *matricube*, a name that we justify in Section 2.2.3.

Below is an example of a matricube, described by its rank function, and originating from a pair of four-step flags in a four-dimensional vector space, taken from Section 2.2.2. The origin of the array (the smallest indices) is in the bottom left-hand corner, following the convention that we use for all arrays throughout Chapter 2.

$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Note that rank functions of matricubes need not be symmetric like this one. Moreover, since this matricube comes from an actual configuration of vectors, it will be called *representable*.

The study of representability questions about *matroids* is an area of research on its own. In Chapter 2, we ask some representability questions about *matricubes*, and answer some of them (see Section 2.9.3). We also show that a matricube contains a collection of "local matroids" that govern its combinatorial properties (see Section 2.8).

The main part of the work presented in Chapter 2 explores the combinatorial properties of matricubes, related to their poset structure and to the discrete convexity properties of their rank functions. We also propose a few different, however equivalent (more exactly, *cryptomorphic*), definitions of matricubes, using the rank function (Section 2.2), the collection of independents (Section 2.5), and other collections (flats, circuits in Sections 2.3 and 2.4) generalizing their matroidal counterparts and mimicking concrete features of configurations of vectors. In this context, independents, flats and circuits are special points of the hypercube $\square_r^m$. Surprisingly, at this point of the exploration of matricubes, we were not able to provide a relevant and interesting axiomatization of the collection of *bases* of a matricube, which raises interesting open questions (see Section 2.9.1).

Furthermore, in addition to a dedicated axiomatic system, independents, flats and circuits have explicit and concrete descriptions for a matricube given by its rank function. For instance, the flats of a matricube are all the points of the hypercube $\square_r^m$ such that moving by a unit in any positive direction (whenever this is possible) leads to an increase of the value of the rank function. As an example, the flats of the rank function defined above are represented in blue below. Note that the set of flats has the structure of a graded lattice, which is a general property.

$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Another combinatorial property of matricubes, proved in Chapter 2, is their link with a combinatorial object generalizing permutation matrices to higher dimensions. A *permutation matrix* is an $n \times n$ square matrix filled with ones and zeroes and encoding a permutation σ of n elements: the coefficient a_{ij} is equal to 1 if, and only if, $\sigma(i) = j$. For example, the matrix below (represented using the usual notational convention for matrices, the origin thus being the top left-hand corner) is the one associated to the permutation $\sigma = (14)(235)$ of five elements.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

These objects are interesting as such, but one could rightfully ask: does this notion admit interesting generalizations to higher dimensions? There are, indeed, several generalizations. You could for example think of the following (all equivalent) definitions of permutation

matrices, and consequently suggest the corresponding (non-equivalent) higher-dimensional generalizations.

(a) A permutation matrix is a 0-1 square matrix such that each complete straight line (where only one coordinate varies), i.e., each row and column, contains 1 exactly once. $\rightsquigarrow$ A higher-dimensional permutation matrix is a 0-1 square higher-dimensional array such that each complete straight line (where only one coordinate varies) contains 1 exactly once.

(b) A permutation matrix is a 0-1 square matrix such that each one-codimensional layer (with only one coordinate fixed), i.e., each row and column, contains 1 exactly once. $\rightsquigarrow$ A higher-dimensional permutation matrix is a 0-1 square higher-dimensional array such that each one-codimensional layer (with only one coordinate fixed) contains 1 exactly once.

(c) A permutation matrix is a 0-1 square matrix such that for each choice of indices (i, j), the sub-matrix made up of the elements of indices at least (i, j) (coordinate-wise) has the property that the number of rows containing 1 at least once is equal to the number of columns containing 1 at least once. $\rightsquigarrow$ A higher-dimensional permutation matrix is a 0-1 square higher-dimensional array such that for each multi-index $\underline{x}$, the sub-array made up of the elements of multi-index at least $\underline{x}$ (coordinate-wise) has the property that the number of one-codimensional layers containing 1 at least once does not depend on the direction in which the layers are sliced up.

Although these definitions coincide in dimension two, they give rise, in higher dimension, to distinct notions with their own applications and links with other combinatorial objects. They have been well-studied, and such objects have been counted, in numerous works.

Definition (c) leads to objects called *permutation arrays*, studied among others by Eriksson and Linusson [EL00a]. It turns out, as we prove in Section 2.7, that a certain kind of matricubes is in fact in one-to-one correspondence with permutation arrays. This gives a new perspective on a previously known combinatorial object.

It appears that matricubes are a new and interesting combinatorial notion, worthy of interest in their own right. However, chronologically speaking, they appeared in the work of this PhD thesis as a combinatorial by-product of the study of the tropicalization of linear series, as we explain in the next section.

A combinatorial theory for limit linear series — Chapter 3

Chapter 3 is based on [AG22] and its aim is to propose a new, combinatorial theory to study the tropicalization of linear series on algebraic curves.

Linear series on algebraic varieties are important and well-studied geometric objects because they govern the geometry of these varieties. They are based on the theory of

divisors, which we briefly present below. In this manuscript, we will focus on algebraic curves (i.e., one-dimensional varieties).

Let X be a smooth and proper algebraic curve over a field κ. A *divisor* on X is a finite formal sum over $\mathbb{Z}$ of points of X: $\mathcal{D} = \sum_{i \in I} n_i (x_i)$, $x_i \in X$. The coefficient of $\mathcal{D}$ at a point $x \in X$ is denoted $\mathcal{D}(x)$. Each rational (or meromorphic) function $f \in \kappa(X)$ on the curve gives rises to a divisor $\operatorname{div}(f)$ in a natural way, by remembering the zeroes and poles of f together with their orders:

$$\operatorname{div}(f) \coloneqq \sum_{x \in X} \operatorname{ord}_x(f)\,(x).$$

This way, each divisor $\mathcal{D}$ gives rise to a vector space of rational functions on X respecting $\mathcal{D}$, defined as follows:

$$\operatorname{Rat}(\mathcal{D}) \coloneqq \{f \in \kappa(X),\, \mathcal{D} + \operatorname{div}(f) \geqslant 0\},$$

where $\mathcal{D} + \operatorname{div}(f) \geqslant 0$ means that this divisor only has non-negative coefficients (is *effective*).

In a nutshell, divisors are used to put constraints on the orders of the poles and zeroes of a meromorphic function on an algebraic curve. A space of the form $\operatorname{Rat}(\mathcal{D})$ is precisely what is called a *complete linear series*, and a *linear series* of *rank* r will be defined as any vector subspace $\mathrm{H} \subseteq \operatorname{Rat}(\mathcal{D})$ of dimension $r + 1$. (The shift of one follows the observation that multiplying a meromorphic function by scalar yields the same divisor, $\operatorname{div}(cf) = \operatorname{div}(f)$, and therefore r is actually the dimension of the space of such divisors, but also the dimension of the projectivization of $\operatorname{Rat}(\mathcal{D})$.) It turns out that linear series on an algebraic curve X correspond to maps from X to projective spaces, and therefore the data of all linear series "contains" the geometry of X, which is why linear series are such a central object of study in algebraic geometry.

Less is known when the algebraic curve X is not smooth, for example, if it is a *stable curve* (a curve whose only singular points are *nodes*, the "nicest" form of singularity, with two branches intersecting transversely). In fact, understanding how linear series degenerate on families of curves has been widely open for a long time.

To state the problem in a more intuitive way, we introduce the *moduli space* $\mathscr{M}_g$ of all smooth projective curves of genus g. This is a Deligne–Mumford stack (a generalization of schemes) which, as a space, parametrizes curves, in the sense that every point in $\mathscr{M}_g$ represents a smooth projective curve, in a bijective and coherent way. The space $\mathscr{M}_g$ is not compact, but it turns out that it admits a nice compactification $\overline{\mathscr{M}}_g$, whose boundary is exactly made up of stable curves. In a way, stable curves are the non-smooth curves you can get by letting smooth curves degenerate. We now introduce the degeneration problem for linear series.

Question 1.0.5. *Let X be a* stable *curve of genus g and let x be the corresponding point in $\overline{\mathscr{M}}_g$. What are all the possible limits of linear series for any sequence of* smooth *projective curves of genus g when their corresponding points in $\mathscr{M}_g$ converge to x?*

Tropical geometry provides a possible approach to answer this question. Indeed, a one-parameter family of curves gives rise to a sequence of points of $\mathscr{M}_g$ converging to a point in $\overline{\mathscr{M}}_g$. The family can be modeled by a curve $\mathbf{C}$ over a field $\mathbf{K}$ endowed with a non-trivial non-Archimedean valuation $\mathbf{val}$, such that κ is the residue field of $(\mathbf{K}, \mathbf{val})$ and X is the special fiber of the semistable model of $\mathbf{C}$ over the valuation ring of $\mathbf{val}$. Then, the tropicalization procedure outlined sooner in this introduction allows to tropicalize $\mathbf{C}$ into a metric graph Γ (which can be seen as a *tropical curve*), that is, a graph with a length assigned to each edge.

It turns out that in this configuration, there is also a natural way to tropicalize a meromorphic function f on the curve $\mathbf{C}$ into a function $\mathrm{trop}(f)\colon \Gamma \to \mathbb{R}$, which is continuous and piecewise affine linear with integral slopes.

Tropical geometry encodes combinatorial information about the degenerating family of curves $\mathbf{C}$ which is complementary to the data of the limit stable curve X and the reduction of the functions $f \in \mathbf{K}(\mathbf{C})$ to X. Indeed, the latter data captures some algebraic aspects of the limit functions on the limit stable curve, whereas the associated metric graph Γ and the tropicalized functions $\mathrm{trop}(f)$ also remember finer information about *how* (from which direction, at which rate) the sequence of smooth curves degenerates to X.

Therefore, one can rightfully hope that the study of tropicalized functions $\mathrm{trop}(f)$ on the tropicalization Γ of $\mathbf{C}$ will yield interesting information about the linear series on $\mathbf{C}$.

This is precisely the goal of the work presented in Chapter 3, which develops a purely combinatorial theory of linear series on metric graphs, and draws links with geometric linear series, i.e., linear series on algebraic curves.

The starting point of this project was the observation that in the context presented above of a degenerating family of curves bearing a linear series, there is some finite combinatorial data that can be extracted from the degenerating family and that should govern the tropicalizations of the functions in the linear series. This data is related to the *orders of vanishing* of functions. More precisely, given a function f in a linear series on $\mathbf{C}$, we first want to measure, for every irreducible component X_v of the limit stable curve X, and for every singular point x of X_v, the order of vanishing of the reduction of f to X_v at that point x.

A classical result implies that the number of different orders of vanishing at x we get as we let the function f vary in the linear series is exactly the dimension of the linear series we started with. This set of integers is the first kind of combinatorial data that we can measure for a degenerating linear series. The second kind of information is related to the *correlation* between those orders of vanishing at every singular point of a given irreducible component X_v. In short, we record the set of *joint orders of vanishing* we get as we let the function f vary in the linear series. This data turns out to be encoded by matricubes, the objects presented in the previous section of the introduction.

By the properties of the tropicalization procedure for functions, this combinatorial data has nice interpretations for the tropicalized functions on the metric graph. Each vertex v of the metric graph corresponds to an irreducible component X_v of the stable curve

X, and each edge $e = uv$ joining two vertices u and v of the graph corresponds to a point of intersection between the irreducible components X_u and X_v associated to u and v. Then, the orders of vanishing of the reduction of a function f on an irreducible component X_v at the singular points of X located on X_v are equal to the *slopes* of the tropicalized function $\mathrm{trop}(f)$ on the metric graph, along outgoing directions around the vertex v; and the joint data of the orders of vanishing of f at all singular points of X located on an irreducible component X_v translates into a *vector of slopes* of $\mathrm{trop}(f)$ around the vertex v of Γ associated to X_v. This illustrates the nature of the combinatorial constraints that will be used to define combinatorial linear series on metric graphs, which we will present a bit later.

Let us first note that metric graphs have a theory of meromorphic functions and divisors, just like algebraic curves. A meromorphic function $f \in \mathrm{Rat}(\Gamma)$ on the metric graph Γ is a continuous, piecewise affine linear function $\Gamma \to \mathbb{R}$ with integral slopes; a divisor on Γ is a finite formal sum over $\mathbb{Z}$ of points of Γ: $D = \sum_{i \in I} n_i\,(x_i)$, $x_i \in \Gamma$. Furthermore, each meromorphic function $f \in \mathrm{Rat}(\Gamma)$ naturally gives rise to a divisor $\mathrm{div}(f)$ on Γ:

$$\mathrm{div}(f) \coloneqq \sum_{x \in \Gamma} \mathrm{ord}_x(f)\,(x),$$

where $\mathrm{ord}_x(f) \in \mathbb{Z}$ is, up to a sign, the sum of outgoing slopes of f on the edges of Γ starting at x. In the context of metric graphs, a *zero* (resp. a *pole*) of a meromorphic function is a negative (resp. positive) change of slope. Besides, just like there is a tropicalization procedure for functions on algebraic curves, there is a tropicalization procedure for divisors, sending each divisor $\mathbf{D}$ on $\mathbf{C}$ to a divisor $D = \mathrm{trop}(\mathbf{D})$ on Γ. This procedure welcomely commutes with the map $f \mapsto \mathrm{div}(f)$. This way, a meromorphic function (resp. a divisor) on an algebraic curve $\mathbf{C}$ tropicalizes to a meromorphic function (resp. a divisor) on the metric graph Γ obtained by tropicalizing $\mathbf{C}$.

Without going into more technical detail, the general idea about this theory of divisors and meromorphic functions on metric graphs, which has already been much studied, is that if behaves nicely and very similarly to its algebro-geometric counterpart. For example, it has been shown using several distinct approaches that a Riemann–Roch theorem holds on metric graphs [AC13, GK08, MZ08], just like on combinatorial graphs [BN07]. There are, moreover, numerous applications of this theory to algebraic geometry, such as degeneration techniques in Brill–Noether theory, results about the Kodaira dimension of moduli spaces of curves [FJP20], partial answers to the maximal rank conjecture, new bounds in arithmetic geometry, or crucial tools for the study of Weierstrass points; see [BJ16] for a comprehensive survey.

At this point, it should be rather obvious how to define a complete linear series on a metric graph Γ. Just take a divisor D on Γ, and define $\mathrm{Rat}(D)$ as the set of all functions $f \in \mathrm{Rat}(\Gamma)$ such that $D + \mathrm{div}(f) \geqslant 0$: a divisor *controls* the zeroes and poles of the functions in its complete linear series.

But the tropical world has major differences with the algebro-geometric world, one of which being that a space like $\mathrm{Rat}(\Gamma)$ or $\mathrm{Rat}(D)$ is nothing like a vector space. Instead, the right formalism to study these spaces is the theory of *tropical semimodules*, i.e., sets endowed with one internal operation, $\oplus$, and one external multiplication, $\odot$, which uses real scalars. In the present context, these operations are defined as follows: for every $f, g \in \mathrm{Rat}(\Gamma)$ and $c \in \mathbb{R}$,

$$f \oplus g := \min(f, g) \qquad \text{and} \qquad c \odot f := f + c.$$

These are the *tropical operations*, which can be defined in similar ways on various tropical spaces. They are, of course, closely related to the max-plus algebra mentioned earlier.

Now that we have this tropical algebraic formalism in mind, we could be tempted to define a linear series on the metric graph Γ as any particular sub-semimodule M of a tropical semimodule of the form $\mathrm{Rat}(D)$, for D a divisor on Γ, just like a linear series on an algebraic curve $\mathbf{C}$ is any vector subspace $\mathbf{H}$ of a complete linear series of the form $\mathrm{Rat}(\mathbf{D})$. This is indeed a good idea, since with this definition, *the tropicalization of a linear series on an algebraic curve yields a linear series on the associated metric graph*, in the sense that if $\mathbf{H}$ is a linear series on $\mathbf{C}$, then

$$\mathrm{trop}(\mathbf{H}) := \left\{ \mathrm{trop}(f) \;\middle|\; f \in \mathbf{H} \smallsetminus \{0\} \right\}$$

is a linear series on Γ, as we prove in Section 3.9. In fact, considering spaces of the form $\mathrm{Rat}(D)$ or sub-semimodules thereof turns out to be a relevant way of defining linear series on metric graphs.

Nevertheless, like often in tropical geometry, we would like to answer *representability questions*, that is, questions of the form: "can this tropical object be realized as the tropicalization of an algebro-geometric object of the expected type?" In the present case, what are the sub-semimodules M of some $\mathrm{Rat}(D)$ which are tropicalizations of some linear series on an algebraic curve?

It turns out that not all of them are. This very fact has led to the work presented in Chapter 3. The basic idea is that a more relevant theory of combinatorial linear series should add a few more constraints on these objects, so that we can formulate, at least in some cases, positive representability results.

This is, by the way, where the theory of matricubes presented in Chapter 2 comes into place. We will add some finite combinatorial constraints to define linear series on metric graphs, in the form of what we call a *slope structure*. Following Section 3.3, a slope structure $\mathfrak{S}$ of width r on Γ consists of:

- for every oriented edge e of Γ, the data of a set of $r+1$ allowed integral slopes $s_0^e < s_1^e < \cdots < s_r^e$; and
- for every vertex v of Γ, the data of a matricube (or *rank function*) ρ_v on $\square_r^{d_v}$, where d_v is the valence of v.

To make things more visual, here is a concrete example of a slope structure, taken from Section 3.6.3. Consider the barbell graph Γ with edges of arbitrary length, see Figure 1.12. We define a slope structure of width one on Γ. Allow slopes $-1 < 1$ on the middle edge and, for $i = 1, 2$, allow slopes $0 < 1$ on both oriented edges $u_i v_i$, in the direction of the arrows.

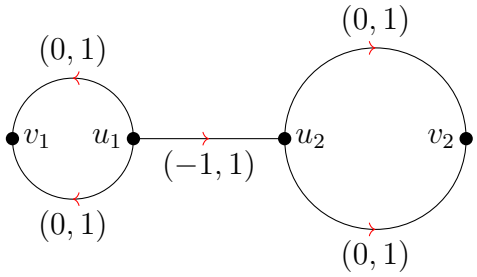


Figure 1.12: The barbell graph and the slope structure $\mathfrak{S}$.

We define suitable matricubes on vertices, as follows. For $i = 1, 2$, we endow v_i with the rank function on $\square_1^2$ defined by the array $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and endow u_i with the matricube on $\square_1^3$ whose restrictions to each layer are defined by the following two arrays, respectively: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Here, flats of the matricubes (see the previous section of the introduction) are depicted in blue – to understand why the element -1 in the top-right hand corner of the last array is not a flat in this context, we refer to Remark 3.2.16. Besides, the convention used to represent rank functions is the one chosen for linear series, which is dual to matricubes ; this duality is specified in Proposition 3.2.3, and implies that the rank functions represented above, contrary to matricubes, are *non-increasing* functions which can take the value -1. For the two vertices u_1 and u_2, the third coordinate in the two rank functions corresponds to the middle edge of Γ. This data defines a slope structure $\mathfrak{S}$ of width one on Γ.

We can now define the space $\mathrm{Rat}(D, \mathfrak{S})$ as the space of all meromorphic functions $f \in \mathrm{Rat}(D)$ such that:

- f is compatible with the *edge data*, i.e., its slope on every edge e is among the set of $r + 1$ integers prescribed by $\mathfrak{S}$ on e; and
- f is compatible with the *vertex data*, i.e., around every vertex v, the vector made up of its outgoing slopes along every incident edge, viewed as a point of $\square_r^{d_v}$, is a *flat* of the matricube ρ_v associated to v.

It can seem strange to require the vectors of slopes of f to be *flats* of the matricubes attached to each vertex, but in fact this has a geometric meaning in the context of tropicalization. To see this, let X_v be an irreducible component (the one associated to v) of the limit stable curve in the degenerating family. Let H be the vector space of reductions

to X_v of the functions belonging to the linear series $\mathbf{H}$ on the main curve $\mathbf{C}$. Let p be a point of X_v. Then p defines a complete flag $\mathrm{F}_p^\bullet$ in H by considering the orders of vanishing, at p, of the functions of H. Concretely, define $S_p \coloneqq \big\{\mathrm{ord}_p(f) \;\big|\; f \in \mathrm{H} \smallsetminus \{0\}\big\}$, which is a finite set of cardinality $r+1$, as previously discussed, and whose elements we denote by $s_0^p < \cdots < s_r^p$. The flag $\mathrm{F}_p^\bullet$ is then defined by setting, for $j \in \{0, \dots, r\}$,

$$\mathrm{F}_p^j \coloneqq \big\{f \in \mathrm{H} \smallsetminus \{0\} \;\big|\; \mathrm{ord}_p(f) \geqslant s_j^p\big\} \cup \{0\}.$$

Doing this for every *singular* point p of X located on X_v yields a finite collection of flags in H, and thus a matricube, or rank function.

Asking that a point of the matricube is a *flat* is in fact equivalent to asking that the corresponding vector of slopes around v is *realized* by some function f in the linear series. Consequently, if a sub-semimodule M of $\mathrm{Rat}(D)$ is the tropicalization of a geometric linear series $\mathbf{H}$, and if we call $\mathfrak{S}$ the slope structure originating from the data of the orders vanishing of functions of the reductions H of $\mathbf{H}$ at all the singular points of the limit stable curve X, then, in fact, $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$.

We now have a more restrictive definition of linear series on graphs which still encompasses all tropicalizations of geometric linear series. We show in Chapter 3 that such a tropicalization has, in fact, supplementary properties, which can therefore safely add to the definition of combinatorial linear series $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ of *rank* (or "dimension") r. These properties are of various kinds; we give an outline below.

- A rank condition ensures that M *contains sufficiently many functions* to generate all effective divisors of degree r (see Section 3.4.1); this is the tropical counterpart of the fact that, since $\mathbf{H}$ is a vector space of dimension $r+1$, we can cut r linear constraints through it and still have non-zero functions.

- An algebraic condition ensures that M is finitely generated, i.e., there exists a finite set of functions of M which generate M using the tropical operations (see Definition 3.6.2).

- Another algebraic condition ensures that M has *tropical rank* at most r, i.e., if $f_0, \dots, f_{r+1}$ is a collection of $r+2$ functions of M, then they are necessarily tropically dependent, in the sense that there exist some scalars $c_0, \dots, c_{r+1} \in \mathbb{R}$ such that for each $x \in \Gamma$, the minimum in
$$\min_{0 \leqslant i \leqslant r+1} (f_i(x) + c_i)$$
is achieved at least twice (see Definition 3.6.2); this condition is the tropical counterpart of the fact that, since $\mathbf{H}$ is a vector space of dimension $r+1$, every family of vectors of cardinality $r+2$ is linearly dependent.

(Note that the rank defined in the first condition and the tropical rank defined in the last condition do not necessarily coincide. On the contrary, in linear algebra, the dimension of a vector space is both equal to the cardinality of every base, and to the maximal number

of linearly independent elements. The fact that these notions differ in the tropical realm is part of its particular mathematical flavor.)

In Chapter 3, we therefore define a combinatorial linear series as a semimodule M satisfying all those conditions, which are all compatible with the tropicalization procedure.

But how is it useful to complexify the definition like this? The answer is that adding those conditions and the combinatorial data of the slope structure $\mathfrak{S}$ now permits to answer representability questions positively, and prove other kinds of results.

Here is a sample of what we show in Chapter 3, and which relies, among others, on the work on matricubes presented in Chapter 2. Some of these results only require a subset of the conditions put on combinatorial linear series.

- If M is a linear series on Γ and v is a vertex of Γ, then *every* flat of the matricube ρ_v is realized by some function $f \in M$ (Theorem 3.5.13).

- As a consequence, if $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ is a linear series, then the data of $\mathfrak{S}$ can be entirely retrieved from M alone (Corollary 3.5.14).

- There are an essentially finite number of distinct slope structures on a given metric graph (Theorem 3.4.4); this can be seen as a first step toward defining a moduli space of linear series over the moduli space of metric graphs of fixed genus.

- In the case $r = 1$, we provide a full classification of linear series on a fixed metric graph: they are essentially in one-to-one correspondence with finite harmonic morphisms from the graph to metric trees (for more details about what this means, see Section 3.8). This is analogous to the description of algebraic geometric linear series of rank one, and follows from an explicit description of the variation of *reduced divisors* – in a nutshell, divisors which concentrate the largest possible coefficient at a given point – defined by linear series, with respect to the base-point (see Section 3.7).

- As a consequence, *every* combinatorial linear series of rank one is realizable, i.e., can be realized as the tropicalization of some geometric linear series (Theorem 3.8.8).

As it was discussed above, the definition of a slope structure uses the notion of matricube developed in the previous section. In fact, this new combinatorial object first appeared naturally in this context, as a combinatorial data extracted from a geometric linear series, and we realized only afterward that matricubes are a – somewhat natural – generalization of matroids that should be studied from an algebraic and combinatorial viewpoint, for its own sake (see Chapter 2).

There is also a strong connection between the theory of combinatorial linear series that was just presented and the study of the tropicalization of Weierstrass points, the subject of Chapter 4, which we sum up in the next and final section of the introduction.

Localizing the tropicalization of Weierstrass points using combinatorial linear series — Chapter 4

In the same spirit as Chapter 3, Chapter 4 is about the study of the tropicalization of a geometric object. Likewise, it involves, on the one hand, the development of a new theory on the tropical and combinatorial side, and, on the other hand, the study of the relationship between tropicalized objects and their purely combinatorial counterparts.

The geometric objects at hand in this chapter are called *Weierstrass points*. Those are very special geometric points defined on algebraic curves. Basically, Weierstrass points are a generalization of what is more often called "flex points" or "inflection points", i.e., points where the sign of the curvature changes, for example in the study of real curves. As an illustration, Figure 1.13 below shows the flex points of a real elliptic curve embedded in the real plane. However, Weierstrass points can in fact be defined more generally on every smooth algebraic curve X over an algebraically closed field $\mathbf{K}$. For simplicity, let us assume that $\mathbf{K}$ has characteristic zero.

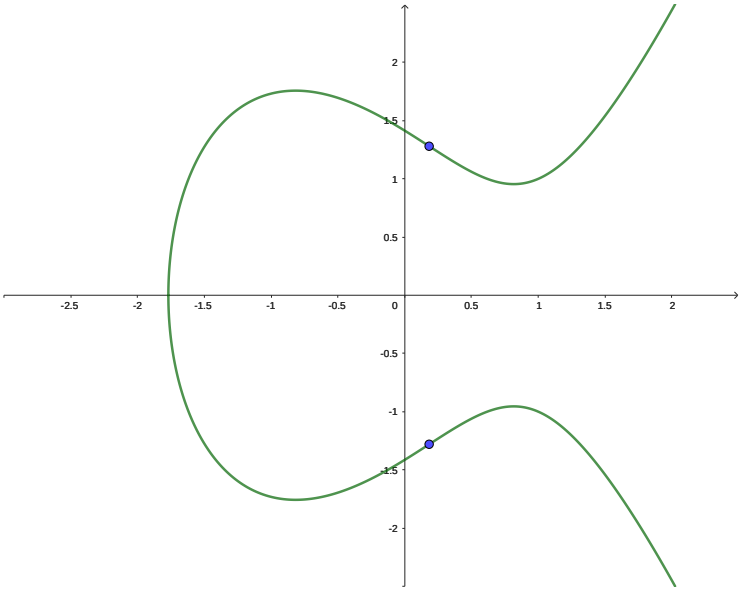


Figure 1.13: The real elliptic curve of equation $y^2 = x^3 - 2x + 2$ and its two inflection points.

One way of defining Weierstrass points in the broader context of an algebraic curve, not necessarily embedded in a projective space, makes crucial use of the theory of divisors and meromorphic functions on algebraic curves, as exposed in the previous section of this introduction.

Let us first define a few basic notions that we have been shying away from defining since now. If $\mathcal{D}$ is a divisor on X, the *linear system* $|\mathcal{D}|$ of $\mathcal{D}$ is the set of all divisors of the form $\mathcal{D} + \mathrm{div}(f)$ with f in the linear series $\mathrm{Rat}(\mathcal{D})$ of $\mathcal{D}$. It is, put more simply, the set of all *effective* divisors that can be obtained from $\mathcal{D}$ by adding the divisor of some meromorphic function. The *degree* of $\mathcal{D}$ is the sum of all its coefficients. The *rank* of $\mathcal{D}$ is the dimension

of the vector space $\mathrm{Rat}(\mathcal{D})$ (minus one). It turns that the rank of $\mathcal{D}$ has an interesting alternate definition: it is equal to the largest integer r such that for every point $x \in X$, there exists an (effective) divisor $\mathcal{E}$ in $|\mathcal{D}|$ having coefficient at least r at the point x, i.e. $\mathcal{E} \geqslant r\,(x)$; or, equivalently, it is the greatest integer r such that for every choice of points $x_1, \ldots, x_r$ on X, there exists an (effective) divisor $\mathcal{E}$ in $|\mathcal{D}|$ such that $\mathcal{E} \geqslant (x_1) + \cdots + (x_r)$. In a nutshell, and as previously noted, the rank measures the *size of the linear series* (or system).

The systematic study of the Weierstrass points of algebraic curves dates back to the 1870's, when Weierstrass and Schottky understood that Weierstrass points had to do with the *orders of vanishing of meromorphic functions*. Indeed, it was first noticed that, given a divisor $\mathcal{D}$ of rank r, the set $S(y)$ of shifted orders of vanishing of all meromorphic functions of $\mathrm{Rat}(\mathcal{D})$ at the point y, namely,

$$\begin{aligned} S(y) &:= \big\{\mathcal{D}(y) + \mathrm{div}(f)(y) \;\big|\; f \in \mathrm{Rat}(\mathcal{D})\big\} \\ &= \big\{\mathcal{D}(y) + \mathrm{ord}_y(f) \;\big|\; f \in \mathrm{Rat}(\mathcal{D})\big\} \\ &= \big\{\mathcal{E}(y) \;\big|\; \mathcal{E} \in |\mathcal{D}|\big\}, \end{aligned}$$

which is by definition always made up of non-negative integers, is in fact equal to the set $\{0, \ldots, r\}$ for every $y \in X$, except a finite number of points; and, moreover, that those finitely many points are exactly the Weierstrass points of X.

Note that the condition $S(y) \neq \{0, \ldots, r\}$ is equivalent to the condition $\max S(y) \geqslant r+1$. Consequently, using one of the definitions of the rank provided above, we see that a Weierstrass point is a point y where some divisor in the linear system of $\mathcal{D}$ has a coefficient at y larger than the expected largest value r. The existence of such a divisor explains why Weierstrass points are the points where some kind of "resonance" happens, which corresponds geometrically to an inflection point. If $S(y)$ is even larger than $r+1$, the Weierstrass point y has some higher *multiplicity*, indicating a geometric phenomenon of higher order. Adding up the finitely many Weierstrass points of $\mathcal{D}$ counted with their multiplicities gives rise to a new divisor $\mathcal{W}(\mathcal{D})$, the *Weierstrass divisor* of $\mathcal{D}$.

This has been well-known and thoroughly studied for more than a century, but Weierstrass points are, a priori, only defined on *smooth* curves. We *do not* know how to define them in a relevant way, for instance, on *stable* curves in full generality. This problem is, at the time, still open.

One first step in this direction is the study of what *happens to* Weierstrass points on smooth curves when we let a family a smooth curves degenerate to a limit stable curve, in the setting of degenerations of families we discussed in the previous section of this introduction. If we grant ourselves less general but more intuitive notations, imagine that we have a one-parameter family $(X_t)_{t\neq 0}$ of *smooth* curves, endowed with a smoothly varying family of divisors $(\mathcal{D}_t)_{t\neq 0}$, each $\mathcal{D}_t$ being a divisor on X_t. For every $t \neq 0$, we have some Weierstrass points of $\mathcal{D}_t$, and therefore a Weierstrass divisor $\mathcal{W}(\mathcal{D}_t)$, on X_t. We can now reformulate our question:

Question 1.0.6. *Assume that the family can be completed into a family $(X_t)_t$, the special fiber X_0 being a stable curve. What can the asymptotic behavior of $\mathcal{W}(\mathcal{D}_t)$ be, as $t \to 0$?*

Or, changing the standpoint, if X_0 is a fixed stable curve, can we locate on X_0 all the possible limits of Weierstrass points for families (X_t) which degenerate to X_0? More broadly, the problem of constructing a moduli space parametrizing all possible limit Weierstrass points of a given stable curve was raised by Eisenbud and Harris [EH86, EH87a].

For a family (X_t) as above, there are two types of Weierstrass points on the smooth curve X_t:

(a) those who, as $t \to 0$, degenerate *away from the nodes* of the stable curve X_0; and

(b) those who, as $t \to 0$, degenerate *to a node* of X_0.

Points of type (b) are harder to understand, and they are the focus of Chapter 4. Some incremental progress has been made on this question, under some combinatorial conditions on the stable curve X_0. In [Ami14], Amini shows an equidistribution result for the Weierstrass points of powers of a line bundle, using a concrete description of limit Weierstrass points. We propose a more refined study of Weierstrass points, using tropical geometry.

Why is tropical geometry useful to study Weierstrass points degenerating to nodes? Because tropical geometry provides tools to "unwind" singularities and look more closely what happens "inside them". Indeed, as exposed previously in this introduction, tropicalizing the family (X_t) yields a metric graph where each vertex v corresponds to an irreducible component X_v of X_0, and each edge $e = uv$ corresponds to a point of intersection between the irreducible components X_u and X_v. Moreover, the properties of the tropicalization procedure imply that if a point p_t on X_t, $t \neq 0$, degenerates to a node $p \in X_u \cap X_v$ of X_0 as $t \to 0$, then it will land on the edge $e = uv$ of the metric graph *at a position which depends on how it degenerates to p*. This is why studying where the Weierstrass points land on each edge of the metric graph associated to the family $(X_t)_t$ gives information about the geometry of this family. This is the path we follow in Chapter 4. We will now explain how.

The natural approach, in the same spirit as the combinatorial theory of linear series, is to define what *tropical Weierstrass points* should be, and then study their connection with the *tropicalizations of geometric Weierstrass points*.

To define the tropical Weierstrass points of a divisor D on a metric graph Γ, we can work by analogy with the geometric definition. The *degree* of D is the sum of its coefficients, and its *rank* is the largest integer r such that for every choice of points $x_1, \dots, x_r \in \Gamma$, there exists an (effective) divisor E in $|D|$ such that $E \geqslant (x_1) + \cdots + (x_r)$. To define the Weierstrass points of D, we also use analogy with geometry: a point $x \in \Gamma$ is said to be Weierstrass for D if the maximal value of $D(x) + \operatorname{div}(f)(x)$ for $f \in \operatorname{Rat}(D)$ is at least $r + 1$.

As an example, the figure below, taken from Section 4.3.1, shows the complete graph over four vertices, endowed with the divisor K (for "canonical") of degree 4 and rank 2

having coefficient 1 on each vertex. (The canonical divisor is defined by $K(x) = d_x - 2$ for every $x \in \Gamma$, where d_x is the valence of x. It has degree $2g-2$ and rank $g-1$, where g is the genus of Γ.) The set of tropical Weierstrass points appears in red on the right part of the figure. Note that each Weierstrass point can be assigned a *multiplicity* of 2, because the maximal value of $K(x) + \mathrm{div}(f)(x)$ for $f \in \mathrm{Rat}(K)$ is in fact $4 = r + 2$.

Figure 1.14: Complete graph on four vertices, and the Weierstrass points of the canonical divisor K.

This definition already gives interesting results. Baker proved [Bak08, Lemma 2.8] that tropical Weierstrass points are *compatible* with geometric Weierstrass points, in the sense that if we have a degenerating family of curves endowed with divisors, the limits of geometric Weierstrass points are *among* the tropical Weierstrass points.

But in tropical geometry, things sometimes do not go as smoothly as in classical algebraic geometry, as we already realized for linear series... It turns out that contrary to the geometric Weierstrass points, *the tropical Weierstrass points are not necessarily finitely many*! For instance, on the barbell graph below, choosing the canonical divisor K having coefficient 1 on both vertices, the whole central edge is Weierstrass. This is why we speak about the (potentially infinite) *Weierstrass locus* of a divisor D, which we denote by $L_W(D)$.

Figure 1.15: The barbell graph and the Weierstrass locus of the divisor K. Each red isolated Weierstrass point has multiplicity 1, but at this point we cannot associate a multiplicity to the central edge.

This entails that, if some degenerating family of curves with divisors has this associated metric graph and divisor as tropicalization, we will not be able to pinpoint exactly where the Weierstrass end up if their limit lies on the red edge. The only thing we *could* try to determine is *how many* Weierstrass end up on the red edge, if there can be an answer to that question. But how could we find a natural way of defining a multiplicity for the central edge like for the isolated points, in order to answer this question in a satisfyingly quantitative manner?

This is precisely what we do in Section 4.3, in two steps.

(1) Firstly, notice that the tropical Weierstrass locus $L_W(D)$ is a "nice" subset of Γ: it is *closed* and has *finitely many connected components*.

(2) Secondly, find a meaningful way to extend the multiplicity of an isolated Weierstrass point to any connected component of the tropical Weierstrass locus.

Here is how we define this generalized multiplicity. Let D be a divisor of rank r, and let C be a connected component of the tropical Weierstrass locus $L_W(D)$. We define the *tropical Weierstrass weight* of C as

$$\mu_W(C; D) := \deg\left(D|_C\right) + (g(C) - 1)r - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(D) \tag{1.2}$$

where

- $\deg\left(D|_C\right)$ is the total degree of D in C, defined by $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$;
- $g(C)$ is the genus of C, i.e., its first Betti number $\dim H_1(C, \mathbb{R})$;
- $\partial^{\text{out}} C$ is the set of outgoing tangent directions from C; and
- $s_0^\nu(D)$ is the minimum slope along the tangent direction ν of any function $f \in \text{Rat}(D)$.

Observe that this definition only uses finite combinatorial data on Γ and D. The fact that the minimum slope $s_0^\nu(D)$ is well-defined and easily computed follows from the work presented in Section 4.2.

It turns out that the tropical weight of a connected component which consists of a single point coincides with the usual multiplicity of an isolated point. But are these weights interesting to answer our degeneration question? Yes, because they measure interesting phenomena, both from a combinatorial and geometric viewpoint.

On the combinatorial side, the weights do behave like if they were *counting something*. Indeed,

- for every connected component C, $\mu_W(C; D)$ is a *positive integer*; and
- the sum of weights $\mu_W(C; D)$ over all connected components equals $d - r + rg$, that is, it is a function only of combinatorial features of Γ and D.

These results notably answer a question raised by Baker, who made the following comment [Bak08, Remark 4.14] regarding a potential tropical analogue of a well-known fact in algebraic geometry about the canonical divisor: *"it is not clear if there is an analogue for metric graphs of the classical fact that the total weight of all Weierstrass points on a smooth curve of genus $g \geqslant 2$ is $g^3 - g$"*.

On the geometric side, good news also lies in store for us. As we show in Section 4.3, the following theorem (Theorem 4.1.10) holds.

Theorem 1.0.7. *Suppose that we have a family of smooth curves $(X_t)_{t\neq 0}$ endowed with a family of divisors $\mathcal{D}_t$, degenerating to a stable curve X_0. Let Γ be the metric graph associated, by the tropicalization procedure, to the family $(X_t)_t$, and let D be the tropicalization of the $\mathcal{D}_t$. Let $\mathcal{W}(\mathcal{D}_t)$ be, for every t, the Weierstrass divisor of $\mathcal{D}_t$, and let C be a connected component of the tropical Weierstrass locus $L_W(D)$ of D. Assume moreover that $\mathcal{D}_t$ and D have the same rank.*

Then, the number of points of $\mathcal{W}(\mathcal{D}_t)$ which converge to a point of C as $t \to 0$ is counted by the weight $\mu_W(C; D)$, multiplied by $r + 1$.

This theorem implies that geometric Weierstrass points tropicalize in packs of size $r + 1$, which generalizes a result obtained by Brugallé and De Medrano [BDM12] on the multiplicities of tropical Weierstrass points in rank two.

In Section 4.5, we prove, in fact, a local and more general version of this result, which, among others, implies the following result.

Theorem 1.0.8. *Using the notations of Theorem 1.0.7, if moreover the rank of D is at least one, then every cycle of the metric graph Γ intersects the tropical Weierstrass locus $L_W(D)$.*

This result, which is *local*, does not seem to have an analogue for algebraic curves: tropical geometry has its own unique features and does not necessarily align on algebraic geometry! Besides, this generalizes a *global* result by Baker [Bak08] stating that the tropical Weierstrass locus $L_W(K)$ of the canonical divisor is nonempty if Γ has genus at least two.

Theorem 1.0.8 implies, in particular, that if the tropical Weierstrass locus is finite, then *every cycle in Γ contains a limit Weierstrass point of the family $(X_t)_t$*, which is a rather strong constraint on the limiting behavior of Weierstrass points.

I have promised that there was a strong connection between tropical Weierstrass points and combinatorial linear series, but this is still not apparent! In fact, combinatorial linear series are key to the proof of Theorem 1.0.7. Indeed, to study the tropicalization of the Weierstrass points of the family of divisors $\mathcal{D}_t$ on X_t, it is crucial to not only consider the tropicalized divisor D, but also the slope structure defined by the geometric linear series $\mathrm{Rat}(\mathcal{D}_t)$, remembering the (joint) orders of vanishing of reduced meromorphic functions at the nodes of X_0, as we described in the preceding section of this introduction. In a nutshell, this proof illustrates the fact that combinatorial linear series (including their slope structure data) carry an amount of combinatorial information which turns out to be relevant in the study of the tropicalization of some geometric objects, be it linear series or Weierstrass points.

Of course, some technical complexity was put under the rug here. Notably:

- An accurate statement of the above theorems requires to consider not simply metric graphs, but more generally *augmented* metric graphs, i.e., metric graphs together with a genus function $\mathfrak{g}$ associating a non-negative integer to each vertex. The number $\mathfrak{g}(v)$ stands for the genus of the irreducible component X_v of X_0.

- It is moreover necessary to define the Weierstrass points not only of a divisor D, but also of any well-behaved sub-semimodule $M \subseteq \mathrm{Rat}(D)$; in particular, to any combinatorial linear series.

These refinements are the subject of Section 4.4. Note that the results presented above do not only hold if the base field $\mathbf{K}$ is of characteristic zero; we provide adaptations to the positive characteristic case for potential applications to arithmetic geometry.

For a variety of commented examples of tropical Weierstrass loci, showing the combinatorial diversity arising among these objects, we refer the reader to Section 4.6.

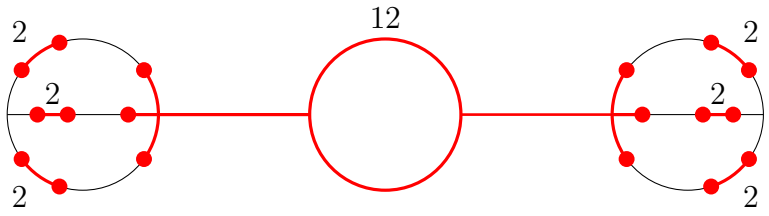


Figure 1.16: A graph with two bridge edges and the Weierstrass locus $L_W(K)$ of the canonical divisor K.

2 Combinatorial flag arrangements

This chapter is slightly adapted from the preprint [AG24].

Contents of the chapter

Abstract

We introduce combinatorial objects named *matricubes* that provide a generalization of the theory of matroids. As matroids provide a combinatorial axiomatization of hyperplane arrangements, matricubes provide a combinatorial axiomatization of arrangements of initial flags in a vector space. We give cryptomorphic axiomatic systems in terms of rank function, flats, circuits, and independent sets, and formulate a duality concept. We also provide precise links between matricubes, permutation arrays and matroids, and raise several open questions.

2.1 Introduction

Consider a vector space H of finite dimension over a ground field κ and a collection $\mathcal{A}$ of m initial flags $\mathrm{F}_1^\bullet, \ldots, \mathrm{F}_m^\bullet$. For $j = 1, \ldots, m$, the flag $\mathrm{F}_j^\bullet$ consists of a chain of $r_j + 1$ vector subspaces, $0 \leqslant r_j \leqslant \dim_\kappa(\mathrm{H})$,

$$\mathrm{H} = \mathrm{F}_j^0 \supseteq \mathrm{F}_j^1 \supseteq \cdots \supseteq \mathrm{F}_j^{r_j - 1} \supseteq \mathrm{F}_j^{r_j} \supsetneq (0)$$

where, for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, r_j\}$, F_j^i is a vector subspace of codimension 0 or 1 in F_j^{i-1}. We call this collection a *flag arrangement*. In the case $r_j = 1$ for all j, and codimension of F_j^1 in H equal to one, we obtain a hyperplane arrangement.

The aim of this chapter is to introduce mathematical structures called *matricubes* that provide a combinatorial axiomatization for the intersection patterns of a finite collection of initial flags in a vector space (as the one above). The case $r_1, \dots, r_m = 1$ recovers the theory of matroids. Like matroids which in the representable case come from matrices, representable matricubes come from cubical matrices (i.e., three-dimensional matrices).

Let us start with a few notations. For n a non-negative integer, we set $[n] \coloneqq \{0, \dots, n\}$. Let m be a positive integer, and $r_1, \dots, r_m$ be non-negative integers. Let $\underline{\varrho} \coloneqq (r_1, \dots, r_m)$. The hypercuboid $\square_{\underline{\varrho}}$ of width $\underline{\varrho}$ is the product $\prod_{j=1}^m [r_j]$. It is endowed with a natural partial order $\preceq$ defined by declaring $\underline{x} \preceq \underline{y}$ for elements $\underline{x} = (x_1, \dots, x_m)$ and $\underline{y} = (y_1, \dots, y_m)$ in $\square_{\underline{\varrho}}$, if $x_j \leqslant y_j$ for all j. The minimum and maximum elements of $\square_{\underline{\varrho}}$ are the points $\underline{0} \coloneqq (0, \dots, 0)$ and $\underline{\varrho}$. We define two operations $\vee$ and $\wedge$ of join and meet by taking the maximum and the minimum coordinate-wise, respectively:

$$\underline{x} \vee \underline{y} \coloneqq (\max(x_1, y_1), \dots, \max(x_m, y_m)), \qquad \underline{x} \wedge \underline{y} \coloneqq (\min(x_1, y_1), \dots, \min(x_m, y_m)),$$

for any pair of elements $\underline{x}, \underline{y} \in \square_{\underline{\varrho}}$. For $i \in \{1, \dots, m\}$ and $t \in [r_i]$, we denote by $t\,\underline{\mathfrak{e}}_i$ the point of $\square_{\underline{\varrho}}$ whose i-th coordinate is t and whose other coordinates are zero.

We first give the definition of matricubes in terms of their rank functions, and then provide cryptomorphic axiomatic systems in terms of their flats, circuits and independent sets.

2.1.1 Definition in terms of rank function

A function $f \colon \square_{\underline{\varrho}} \to \mathbb{Z}$ is called *submodular* if for every two elements $\underline{x}$ and $\underline{y}$, we have

$$f(\underline{x}) + f(\underline{y}) \geqslant f(\underline{x} \vee \underline{y}) + f(\underline{x} \wedge \underline{y}).$$

A *matricube* $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ is defined in terms of a function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{Z}_{\geqslant 0}$ called *the rank function of* $\mathscr{M}$ that satisfies the following conditions:

(R1) $\mathbf{r}(\underline{0}) = 0$, and for every $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant r_i$, we have $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) - \mathbf{r}((t-1)\,\underline{\mathfrak{e}}_i) \leqslant 1$.

(R2) $\mathbf{r}$ is non-decreasing, that is, if $\underline{x} \preceq \underline{y}$, then $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{y})$.

(R3) $\mathbf{r}$ is submodular.

We call the quantity $r = r(\mathscr{M}) \coloneqq \mathbf{r}(\underline{\varrho})$, the maximum value taken by the function $\mathbf{r}$, the *rank* of $\mathscr{M}$. In the case $r_j = 1$ for all j, $\mathscr{M}$ gives a matroid with ground set $E = \{1, \dots, m\}$.

Note that it follows from (R1) and (R2) that $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) \leqslant t$. We say that $\mathscr{M}$ is *simple* if the following stronger version of (R1) holds:

(R1*) $r_i > 0$ and $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$ for all $i = 1, \dots, m$ and $t \in [r_i]$.

For a collection $\mathcal{A}$ of initial flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$ in a vector space of dimension n, as above, the codimensions of the intersection patterns of their elements define a rank function. That is, the function $\mathbf{r}\colon \square_{\underline{\varrho}} \to \mathbb{Z}$ defined, for every $\underline{x} = (x_1, \dots, x_m)$, by

$$\mathbf{r}(\underline{x}) := \mathrm{codim}_\kappa(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m}) = n - \dim_\kappa(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m})$$

is the rank function of a matricube that we denote by $\mathscr{M}_\mathcal{A}$. Note that $\mathscr{M}_\mathcal{A}$ is simple if all the inclusions in each flag are strict. Like for matroids, a matricube appearing in this way will be called *representable* over the field κ. Note that by duality of vector spaces, a representable matricube can be described equivalently by a collection of initial increasing flags in the dual vector space. This point of view allows to associate a matricube to any three-dimensional matrix with entries in a given field. We refer to Section 2.2.3 for more details.

Abstracting the example given above of an arrangement of initial flags in a vector space, we show in Section 2.2.4 that a finite collection of initial flag matroids, all defined on the same ground set, defines a matricube.

In the next three sections, we present alternative axiomatic systems for matricubes, that will be discussed more thoroughly in this chapter.

2.1.2 Flats of matricubes

Let $\mathscr{M}$ be a matricube of rank r with ground set $\square_{\underline{\varrho}}$ and rank function $\mathbf{r}$. A point $\underline{a}$ in $\square_{\underline{\varrho}}$ is called a *flat* of $\mathscr{M}$ if

(∗) for every $i = 1, \dots, m$ such that $\underline{a} + \underline{\mathfrak{e}}_i$ belongs to $\square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a}) + 1$.

Note that in particular, $\underline{\varrho}$ is a flat of $\mathscr{M}$. We denote by $\mathscr{F}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of flats of $\mathscr{M}$. In the case $\mathscr{M}$ is a matroid, $\mathscr{F}(\mathscr{M})$ is the set of flats of that matroid.

As in the case of matroids, a matricube can be defined in terms of its flats. The axiomatic system of flats of a matricube is (F1)-(F2)-(F3), provided below.

Given a poset $(P, \preceq)$ and two elements $x, y \in P$, we say that that y *covers* x, and write $y \succ x$, if $y \succ x$ in P and there is no element $z \in P$ such that $y \succ z \succ x$. Let $\mathscr{F}$ be a subset of $\square_{\underline{\varrho}}$. Endowed with the partial order $\preceq$ of the hypercuboid $\square_{\underline{\varrho}}$, $\mathscr{F}$ is a poset.

We prove in Section 2.3 that $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ is the set of flats of a matricube with underling ground set $\square_{\underline{\varrho}}$ if, and only if, the following properties hold.

(F1) $\underline{\varrho}$ is in $\mathscr{F}$.

(F2) $\mathscr{F}$ is closed under meet.

(F3) If $\underline{a}$ is an element of $\mathscr{F}$ and i is such that $\underline{a} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b} \succeq \underline{a} + \underline{\mathfrak{e}}_i$ in $\mathscr{F}$ such that $\underline{b} \succ \underline{a}$ in $\mathscr{F}$.

In other words, the axiomatic systems (F1)-(F2)-(F3) and (R1)-(R2)-(R3) are equivalent.

2.1.3 Duality, and circuits of matricubes

Again, let $\mathscr{M}$ be a matricube on the ground set $\square_{\underline{\varrho}}$. In Section 2.2.6, we define the dual matricube $\mathscr{M}^*$ on the same ground set $\square_{\underline{\varrho}}$. In terms of rank functions, the rank function $\mathbf{r}^*$ of $\mathscr{M}^*$ is given by

$$\mathbf{r}^*(\underline{x}) := |\underline{x}|_{\ell_1} + \mathbf{r}(\underline{x}^c) - \mathbf{r}(\mathscr{M}) \qquad \forall\, \underline{x} \in \square_{\underline{\varrho}},$$

where $\underline{x}^c := \underline{\varrho} - \underline{x}$ is the *complement* of $\underline{x}$ in $\square_{\underline{\varrho}}$, and $\mathbf{r}$ denotes the rank function of $\mathscr{M}$.

Denote by $\mathscr{F}(\mathscr{M}^*)$ the set of flats of the dual matricube, and consider

$$\check{\mathscr{C}} := \big\{\underline{a}^c \;\big|\; \underline{a} \in \mathscr{F}(\mathscr{M}^*)\big\} \subseteq \square_{\underline{\varrho}}.$$

A point $\underline{c}$ in $\square_{\underline{\varrho}}$ is called a *circuit* of $\mathscr{M}$ if

(∗) $\underline{c}$ is an element of $\check{\mathscr{C}}$ which is not the join of any set of elements of $\check{\mathscr{C}} \smallsetminus \{\underline{c}\}$.

We denote by $\mathscr{C}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of circuits of $\mathscr{M}$. This definition extends that of circuits in matroids. Moreover, as in the case of matroids, a matricube can be defined in terms of its circuits, via the following axiomatic system. We prove in Section 2.4 that a subset $\mathscr{C} \subseteq \square_{\underline{\varrho}}$ is the set of circuits of a matricube with underlying ground set $\square_{\underline{\varrho}}$ if, and only if, the following properties hold.

(C1) $\underline{0}$ is not in $\mathscr{F}$.

(C2) All elements of $\mathscr{C}$ are join-irreducible in $\mathscr{C}$.

(C3) If $\underline{a} \in \check{\mathscr{C}}$ and $i \in \{1, \ldots, m\}$ is such that $\underline{a} - \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b} \preceq \underline{a} - \underline{\mathfrak{e}}_i$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$ such that $\underline{b} \prec \underline{a}$ in $\check{\mathscr{C}} \cup \{\underline{0}\}$.

In other words, the axiomatic systems (C1)-(C2)-(C3) and (R1)-(R2)-(R3) are equivalent.

2.1.4 Independents of matricubes

Let $\mathscr{M}$ be a matricube on the ground set $\square_{\underline{\varrho}}$. We say that a point $\underline{a}$ of $\square_{\underline{\varrho}}$ is an *independent of* $\mathscr{M}$ if

(∗) for every $i = 1, \ldots, m$ such that $\underline{a} - \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} - \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a}) - 1$.

We denote by $\mathscr{I}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of independents of $\mathscr{M}$.

The set of independents of a matricube is nonempty and closed under meet $\wedge$. Moreover, removing unit vectors from an independent reduces the rank in the following sense: for every independent $\underline{a} \in \mathscr{I}(\mathscr{M})$, and every distinct elements $i_1, \ldots, i_k \in \{1, \ldots, m\}$ with $a_{i_j} \neq 0$, $1 \leqslant j \leqslant k$, we have $\mathbf{r}\big(\underline{a} - \underline{\mathfrak{e}}_{i_1} - \cdots - \underline{\mathfrak{e}}_{i_k}\big) = \mathbf{r}(\underline{a}) - k$.

We provide an axiomatic system for independent sets of a matricube. In order to do this, we need to define an operation of *removal* in independents.

Let $\mathcal{J}$ be a subset of $\square_{\underline{\varrho}}$. Let $\underline{a}$ be an element of $\mathcal{J}$ and $i \in \{1, \dots, m\}$ such that $a_i \neq 0$. If there is at least one element $\underline{b} \prec \underline{a}$ in $\mathcal{J}$ which differs from $\underline{a}$ only in the i-th component, we define $\underline{a} \smallsetminus i$ to be such an element in $\mathcal{J}$ with the largest i-th coordinate. In this case, we say that $\underline{a} \smallsetminus i$ is the *removal of* i *in* $\underline{a}$.

Definition 2.1.1. Let $\mathcal{J}$ be a subset of $\square_{\underline{\varrho}}$.

(a) We say that *removals exist in* $\mathcal{J}$ if for every $\underline{a} \in \mathcal{J}$ and $i \in \{1, \dots, m\}$, if $a_i \geqslant 1$, the removal $\underline{a} \smallsetminus i$ exists in $\mathcal{J}$.

If removals exist in $\mathcal{J}$, then $\underline{0} \in \mathcal{J}$ and for every element $\underline{a}$, there exists a sequence of removals that reduces $\underline{a}$ to $\underline{0}$.

(b) We say that $\mathcal{J}$ is *orderable* if removals exist in $\mathcal{J}$ and for every $\underline{a} \in \mathcal{J}$, all the sequences of removals that bring $\underline{a}$ to $\underline{0}$ have the same length.

If $\mathcal{J}$ is orderable, we define the *size* of $\underline{a}$, denoted by $|\underline{a}|$, as the number of removals needed to reduce $\underline{a}$ to $\underline{0}$. ◇

The axiomatic system of independents can be formulated as follows. Let $\mathscr{I}$ be a subset of $\square_{\underline{\varrho}}$ that verifies the following property:

(I1) Removals exist in $\mathscr{I}$ and the following holds. For all $\underline{p} \in \mathscr{I}$ and removals $\underline{p} \smallsetminus i$ and $\underline{p} \smallsetminus j$, with $i, j \in \{1, \dots, m\}$, the meet $\underline{q} := (\underline{p} \smallsetminus i) \wedge (\underline{p} \smallsetminus j)$ belongs to $\mathscr{I}$ and, moreover, the two intervals $[\underline{q}, \underline{p} \smallsetminus i]$ and $[\underline{q}, \underline{p} \smallsetminus j]$ in $\mathscr{I}$ have the same size.

Here, the interval $[\underline{a}, \underline{b}]$ in $\mathscr{I}$ means the set of all $\underline{c} \in \mathscr{I}$ which verify $\underline{a} \preceq \underline{c} \preceq \underline{b}$.

We prove in Section 2.5 that (I1) is equivalent to the orderability of the set $\mathscr{I}$. In particular, if $\mathscr{I}$ verifies (I1), we can define the size of $\underline{a}$ as the number of removals needed to reduce $\underline{a}$ to $\underline{0}$. This enables to formulate the second property of interest. For $\underline{a}, \underline{b} \in \square_{\underline{\varrho}}$, denote by $\Delta(\underline{a}, \underline{b})$ the set of elements in $\{1, \dots, m\}$ such that $a_i < b_i$. The following is understood as a matricube analogue of the augmentation property for independents of matroids.

(I2) $|\cdot|$ is increasing on independents, i.e., for all $\underline{a}, \underline{b} \in \mathscr{I}$ such that $\underline{a} \prec \underline{b}$, we have $|\underline{a}| < |\underline{b}|$. Moreover, let $\underline{a}, \underline{b}$ be two elements of $\mathscr{I}$ such that $|\underline{a}| < |\underline{b}|$ and $\Delta(\underline{a}, \underline{b})$ contains at least two elements. Then, there exists $\underline{c} \in \mathscr{I}$ that verifies:

- $\underline{c} \preceq \underline{a} \vee \underline{b}$,
- $|\underline{c}| > |\underline{a}|$.
- There exists $i \in \Delta(\underline{a}, \underline{b})$ such that $c_i < b_i$.

We prove in Theorem 2.5.5 that (I1)-(I2) are equivalent to (R1)-(R2)-(R3).

2.1.5 Permutation arrays

A combinatorial approach to the study of intersection patterns of a configuration of complete flags was introduced by Eriksson–Linusson in the notion of permutation arrays [EL00a, EL00b]. Our Theorem 2.7.1 proved in Section 2.7 shows that permutation arrays are in a one-to-one correspondence with matricubes of rank r or $r+1$ on the ground set $\square_r^d := \square_{\underline{\varrho}}$ with $\underline{\varrho} = (r, r, \ldots, r)$, that is, with all $r_j = r$.

2.1.6 Matricubes as coherent complexes of matroids

As we explain in Section 2.8, a matricube locally gives rise to a collection of matroids. Local obstructions for the representability of a matricube can then be formulated in terms of matroid representability. In the case of permutation arrays, via our Theorem 2.7.1, this gives obstructions for representability that generalize the examples found in the work of Billey and Vakil [BV08]. We moreover go further by proving Theorem 2.9.1, which shows that the representability of matricubes over infinite fields can be reduced to matroid representability, see Section 2.1.7.

In Section 2.8.3, we provide a matroidal characterization of matricubes by establishing an equivalence between matricubes and coherent complexes of matroids labeled by the elements of a hypercuboid satisfying Properties (CC1) and (CC2) below. Namely, let $(\mathrm{M}_{\underline{a}})_{\underline{a} \in \square_{\underline{\varrho}}}$ be a family of matroids indexed by $\square_{\underline{\varrho}}$, with $\mathrm{M}_{\underline{a}}$ a matroid on the set $I_{\underline{a}}$ consisting of all $j \in \{1, \ldots, m\}$ with $a_j < r_j$. Denote the rank function of $\mathrm{M}_{\underline{a}}$ by $\mathbf{r}_{\underline{a}}$. We say that the collection $(\mathrm{M}_{\underline{a}})$ forms a *coherent complex of matroids* if the following two conditions are met:

(CC1) For all $i \in \{1, \ldots, m\}$ and $0 \leqslant t < r_i$, we have $\rho_{t\,\underline{\mathfrak{e}}_i}(\underline{\mathfrak{e}}_i) \leqslant 1$.

(CC2) The matroids $\mathrm{M}_{\underline{a}}$ satisfy the following relation:

$$\mathrm{M}_{\underline{a}+\underline{\mathfrak{e}}_i} = \begin{cases} \mathrm{M}_{\underline{a}} \diagup i & \text{if } a_i = r_i - 1 \\ \mathrm{M}_{\underline{a}} \diagup i \sqcup \{i\} & \text{else} \end{cases}.$$

Theorem 2.8.6 provides an equivalence between (CC1)-(CC2) and (R1)-(R2)-(R3).

2.1.7 The natural matroid of a matricube and representability

Remembering only the data of the subspaces in a flag arrangement results in a subspace arrangement, whose combinatorics is encoded in an integer polymatroid. In the same way, any matricube on the ground set $\square_{\underline{\varrho}}$, $\underline{\varrho} = (r_1, \ldots, r_m)$, gives rise to an integer polymatroid on the ground set the disjoint union $[r_1] \sqcup \cdots \sqcup [r_m]$.

Bases and exchange properties for integer polymatroids have been studied by Herzog and Hibi [HH02]. Csirmaz [Csi20] gives axiomatic systems for cyclic flats. In recent work, Bonin, Chun and Fife [BCF23] study bases, circuits, and cyclic flats in integer

polymatroids, connecting them to a classical construction going back to McDiarmid [McD73], Lovász [Lov77], and Helgason [Hel06], which shows that the data of an integer polymatroid on a ground set E is equivalent to the data of a matroid on a larger set $\widehat{E}$ obtained from E by replacing each element e of E with $\rho(e)$ distinct copies of that element, called the *natural matroid*. (This has gained recent interest in the work [OSW19], in developing a decomposition theorem for 2-polymatroids, as well as in the works [CHL+22, EL23] related to combinatorial Hodge theory.)

In our setting, starting from a matricube, we can thus associate to it first an integer polymatroid and then use the above construction to replace the integer polymatroid by a matroid on a larger ground set. We will review this construction in Section 2.9.2. As we explain there, this leads to a story different from the theory presented in this manuscript.

This point of view is however useful for treating the question of the representability of matricubes. We show in Theorem 2.9.1 that a matricube is representable over an infinite field (or a field of large enough cardinality) if, and only if, the corresponding natural matroid is representable over the same field.

2.1.8 Further related work

Our original motivation for developing the theory exposed here comes from the problem of describing tropical degenerations of linear series on algebraic curves. In companion work [AG22] (Chapter 3), matricubes are used as the combinatorial structure underlying a combinatorial theory of limit linear series on metric graphs (the geometric situation behind this theory is briefly discussed in Section 2.9.7). While working on the degeneration problem for linear series, we gradually realized how similarly matricubes and matroids behave. Apart from bases, for which we do not provide a definition and an axiomatic system, the other relevant constructions in the theory of matroids have their matricube analogues.

The recent work of Baker and Bowler [BB19] develops a theory of matroids over hyperstructures. The extension of this theory to flag matroids is given by Jarra and Lorscheid [JL24], and a generalization to quiver matroids is the subject of a forthcoming work of Jarra, Lorscheid and Vital. The work by Baker and Lorscheid [BL21, BL20] studies the moduli space of matroids and deduce applications to representability questions for matroids. It seems plausible and interesting to generalize these results to the context of matricubes.

In [BDP18], Bollen, Draisma and Pendavingh show that each representation of an algebraic matroid M over a field of positive characteristic comes naturally with a valuation, that they name the *Lindström valuation* of that representation. To this end, using the Frobenius map of the base field, they associate to any such representation what they call a *matroid flock*, an infinite family of linear matroids of the same rank as M, indexed by $\mathbb{Z}^E$, where E is the ground set of M. It is interesting to note that, although these notions arise in totally different contexts, the axiomatic systems of coherent complexes and matroid flocks are reminiscent of each other. There are however some major differences. Namely, the matroids appearing in a matroid flock all have the same rank, and there is an invariance

property with respect to the direction $(1, \dots, 1)$. Besides, the boundary condition imposed on the coherent complexes does not appear in matroid flocks.

Submodular functions on distributive lattices are a central topic in the study of a large class of combinatorial optimization problems. We refer to the books by Schrijver [Sch03] and Fuji [Fuj05] for a discussion of these aspects.

Murota [Mur98] investigates a theory of convex analysis in the discrete setting that involves functions $f\colon \mathbb{Z}^n \to \mathbb{Z}$. Classical duality theorems about real convex functions are proved in the discrete setting. Discrete convexity in that setting is similar in spirit to the submodularity property studied in the present chapter.

2.1.9 Organization of the text

In Section 2.2, we define matricubes using rank functions and give basic examples, including uniform and representable matricubes. We define operations of deletion and contraction on matricubes, and formulate a duality concept.

In Sections 2.3, 2.4 and 2.5, we explore alternative axiomatic systems for matricubes, relying on flats, circuits and independents, respectively.

In Section 2.6, we prove elementary combinatorial results useful throughout the chapter, which provide a simpler way of checking whether a function on a hypercuboid is a rank function.

In Section 2.7, we show that particular kinds of matricubes are in a natural one-to-one correspondence with permutation arrays.

In Section 2.8, we provide the equivalence of matricubes with coherent complexes of matroids, and provide local obstructions for representability.

Finally, in Section 2.9, we discuss further interesting features of matricubes and raise several open questions.

2.2 Basic properties

Let n be a non-negative integer and $[n] = \{0, 1, \dots, n\}$. For elements $r_1, \dots, r_m \in [n]$, the *hypercuboid* $\square_{\underline{\varrho}}$ of *width* $\underline{\varrho} = (r_1, \dots, r_m)$ is the product $\prod_{j=1}^{m} [r_j]$. When $r_1 = \cdots = r_d = r$, we simply denote the hypercuboid by $\square_r^m$. We denote the elements of $\square_{\underline{\varrho}}$ by vectors $\underline{x} = (x_1, \dots, x_m)$, for $x_1 \in [r_1], \dots, x_m \in [r_m]$. In the hypercuboid, we define, for every $i = 1, \dots, m$ and $t \in [r_i]$, the t-th layer in the direction i as $L_t^i := \left\{ \underline{x} \in \square_{\underline{\varrho}}, x_i = t \right\}$.

We endow $\square_{\underline{\varrho}}$ with the partial order $\preceq$: For a pair of elements $\underline{x}, \underline{y} \in \square_{\underline{\varrho}}$, we have $\underline{x} \preceq \underline{y}$ provided that $x_j \leqslant y_j$ for all $j = 1, \dots, m$. The smallest and largest elements with respect to $\preceq$ are $\underline{0}$ and $\underline{\varrho}$, respectively. Moreover, there is a lattice structure on $\square_{\underline{\varrho}}$, where the two operations of join $\vee$ and meet $\wedge$ correspond to taking the maximum and the minimum coordinate-wise, respectively.

A function $f\colon \boxplus_{\underline{\varrho}} \to \mathbb{Z}$ is called *submodular* if for every pair of elements $\underline{x}$ and $\underline{y}$, we have

$$f(\underline{x}) + f(\underline{y}) \geqslant f(\underline{x} \vee \underline{y}) + f(\underline{x} \wedge \underline{y}).$$

We will be interested in a special kind of submodular function on $\boxplus_{\underline{\varrho}}$. For each integer $i \in \{1, \dots, m\}$, we denote by $\underline{\mathfrak{e}}_i$ the vector whose coordinates are all zero except the i-th coordinate, which is equal to one. For $0 \leqslant t \leqslant r_i$, the vector $t\,\underline{\mathfrak{e}}_i$ lies in $\boxplus_{\underline{\varrho}}$.

Definition 2.2.1 (Matricube)**.** A *matricube* $\mathscr{M}$ with ground set $\boxplus_{\underline{\varrho}}$ is defined in terms of a function $\mathbf{r}\colon \boxplus_{\underline{\varrho}} \to \mathbb{Z}$ called *the rank function of* $\mathscr{M}$ that satisfies the following conditions:

(R1) $\mathbf{r}(\underline{0}) = 0$, and for every $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant r_i$, we have $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) - \mathbf{r}((t-1)\,\underline{\mathfrak{e}}_i) \leqslant 1$.

(R2) $\mathbf{r}$ is non-decreasing with respect to $\preceq$, that is, if $\underline{a} \preceq \underline{b}$, then $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{b})$.

(R3) $\mathbf{r}$ is submodular.

We call $r = r(\mathscr{M}) := \mathbf{r}(\underline{\varrho})$, the maximum value taken by the function $\mathbf{r}$, the *rank* of $\mathscr{M}$. ◇

Note that (R1) implies that $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) \leqslant t$ for all $i = 1, \dots, m$ and $t \in [r_i]$. We say that $\mathscr{M}$ is *simple* if the following alternate form of (R1) holds:

(R1*) $r_i > 0$ and $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$ for all $i = 1, \dots, m$ and $t \in [r_i]$.

Remark 2.2.2. In $\mathscr{M}$ is simple, then the above properties imply that if $\underline{x} \in \boxplus_{\underline{\varrho}}$ has rank j, then $x_i \leqslant j$ for all $i = 1, \dots, m$. In particular, $\underline{0}$ is the only element of rank 0 in $\boxplus_{\underline{\varrho}}$. ◇

To be able to present examples of rank functions easily, we adopt the following convention.

Convention 2.2.3 (Cases $m = 1, 2, 3$)**.** In this article, for $m = 1$, a function on $\boxplus_r$ is described by a tuple with $r+1$ entries $(t_0, \dots, t_r)$, which means that the value of the function on the i-th entry of $\boxplus_r$ is t_i.

In the same way, for $m = 2$, a function on $\boxplus_{(r_1,r_2)}$ will often be described by an array of size $(r_1+1) \times (r_2+1)$, $(t_{ij})_{0 \leqslant i \leqslant r_1, 0 \leqslant j \leqslant r_2}$, which means that the function takes value t_{ij} on $(i,j) \in \boxplus_{(r_1,r_2)}$. We choose the convention that the first direction is horizontal, the second direction is vertical, and the origin is the bottom left-hand corner.

When $m = 3$ and $\underline{\varrho} = (r_1, r_2, r_3)$, a function defined on $\boxplus_{\underline{\varrho}}$ will be specified by $r_3 + 1$ arrays $R_0, \dots, R_{r_3}$ of size $(r_1+1) \times (r_2+1)$, where R_k describes the values of the function on $\boxplus_{(r_1,r_2)} \times \{k\} \subseteq \boxplus_{\underline{\varrho}}$. ◇

Here are two examples of matricubes with $\underline{\varrho} = (4, 3)$.

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 2 & 3 \\ 0 & 1 & 2 & 2 & 3 \end{pmatrix}$$

The one on the left is simple, the one on the right is not.

For future use, we state the following proposition which implies that the set of values of the rank function in a matricube of rank r is the interval $[r]$.

Proposition 2.2.4. *Let* $\mathbf{r}$ *be a rank function on* $\square_{\underline{\varrho}}$. *Let* $i \in \{1, \dots, m\}$. *For an element* $\underline{x} \in \square_{\underline{\varrho}}$ *such that* $\underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, *we have*

$$\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) \leqslant \mathbf{r}(\underline{x}) + 1.$$

Proof. Let $\underline{y} = (x_i + 1)\,\underline{\mathfrak{e}}_i$, and note that $\underline{x} \vee \underline{y} = \underline{x} + \underline{\mathfrak{e}}_i$ and $\underline{x} \wedge \underline{y} = x_i\,\underline{\mathfrak{e}}_i$. Applying the submodularity of $\mathbf{r}$ to the vectors $\underline{x}$ and $\underline{y}$, and using (R1) in Definition 2.2.1, we get $\mathbf{r}(\underline{x}) + 1 \geqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i)$. The first inequality follows from the non-decreasing property of $\mathbf{r}$. □

2.2.1 Uniform matricubes

Notation as in the previous section, let $\underline{\varrho} = (r_1, \dots, r_m)$, and consider the corresponding hypercuboid $\square_{\underline{\varrho}}$. Let $r \in [r_1 + \cdots + r_m]$ be a non-negative integer. We define the uniform matricube $\mathscr{U}_{\underline{\varrho},r}$ of width $\underline{\varrho}$ and rank r as the matricube defined by the *standard rank function* defined as follows

$$\mathbf{r}^{\mathrm{st}}(\underline{a}) := \min(r, a_1 + \cdots + a_m) \qquad \text{for } \underline{a} = (a_1, \dots, a_m) \in \square_{\underline{\varrho}}.$$

Notice that the uniform matricube $\mathscr{U}_{\underline{\varrho},r}$ is simple if, and only if, $r \geqslant \max_i r_i$.

Below are the uniform matricubes $\mathscr{U}_{(4,3),3}$ and $\mathscr{U}_{(4,3),5}$.

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 \\ 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 & 5 & 5 & 5 \\ 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Proposition 2.2.5. *The rank function* $\mathbf{r}$ *of any matricube* $\mathscr{M}$ *of rank* r *on the ground set* $\square_{\underline{\varrho}}$ *is dominated by the rank function* $\mathbf{r}^{\mathrm{st}}$ *of the uniform matricube* $\mathscr{U}_{\underline{\varrho},r}$. *That is, for every* $\underline{x} \in \square_{\underline{\varrho}}$, *we have* $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}^{\mathrm{st}}(\underline{x})$.

Proof. It follows directly from Proposition 2.2.4 that we have $\mathbf{r}(\underline{x}) \leqslant x_1 + \cdots + x_m$. Combined with $\mathbf{r}(\underline{x}) \leqslant r$, we deduce the result. □

2.2.2 Representable matricubes

Let n be a non-negative integer, and let H be a vector space of dimension n over some field κ. An *initial (decreasing) flag* of H of length r consists of a chain of vector subspaces

$$\mathrm{H} = \mathrm{F}^0 \supseteq \mathrm{F}^1 \supseteq \cdots \supseteq \mathrm{F}^{r-1} \supseteq \mathrm{F}^r \supsetneq (0),$$

where for each positive $i \in [r]$, F^i is a vector subspace of codimension 0 or 1 in F^{i-1}. We say that $\mathrm{F}^\bullet$ is *simple* if each F^i has codimension i in H. A *complete flag* is a simple initial flag of length $n - 1$.

Let m be a positive integer, and let $\mathcal{A}$ be a collection of m initial flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$ of H of lengths $r_1, \dots, r_m$, respectively. Define the function $\mathbf{r}\colon \square_{\underline{\varrho}} \to \mathbb{Z}$ by

$$\mathbf{r}(\underline{x}) \coloneqq \operatorname{codim}_\kappa(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m}) \quad \forall\, \underline{x} = (x_1, \dots, x_m) \in \square_{\underline{\varrho}}. \tag{2.1}$$

Proposition 2.2.6. *The hypercuboid $\square_{\underline{\varrho}}$ endowed with the function $\mathbf{r}$ defined in (2.1) is a matricube. This matricube is simple if, and only if, all the initial flags are simple.*

Proof. Let $\underline{a}$ and $\underline{b}$ be two points of $\square_{\underline{\varrho}}$, and let $\underline{x} \coloneqq \underline{a} \wedge \underline{b}$ and $\underline{y} \coloneqq \underline{a} \vee \underline{b}$. We have an injection

$$\big(\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_m^{a_m}\big)\big/\big(\mathrm{F}_1^{y_1} \cap \cdots \cap \mathrm{F}_m^{y_m}\big) \hookrightarrow \big(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_m^{x_m}\big)\big/\big(\mathrm{F}_1^{b_1} \cap \cdots \cap \mathrm{F}_m^{b_m}\big),$$

from which, comparing the dimensions, we get $\mathbf{r}(\underline{b}) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\big(\underline{y}\big) - \mathbf{r}(\underline{a})$. This proves the submodularity of $\mathbf{r}$. Properties (R1) and (R2) in Definition 2.2.1 are trivially verified. This proves the first assertion. The matricube is simple if, and only if, each F_j^i has codimension i in H, that is if, and only if, $\mathrm{F}_j^\bullet$ is simple, for $j = 1, \dots, m$. □

We denote by $\mathscr{M}_{\mathcal{A}}$ the matricube associated to $\mathcal{A}$.

Definition 2.2.7 (Representable matricube)**.** A matricube $\mathscr{M}$ on ground set $\square_{\underline{\varrho}}$ is called *representable over a field κ* if it is the matricube associated to an arrangement of m initial flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$ of lengths $r_1, \dots, r_m$, respectively, in a κ-vector space H. ◇

Example 2.2.8. The matricube $\mathscr{U}_{\underline{\varrho},r}$ is representable over every field of large enough cardinality. Indeed, it is the matricube associated to an arrangement of m initial flags of lengths $r_1, \dots, r_m$ in H of dimension r which are in *general relative position*, that is, whose intersection patterns have the smallest possible dimensions. ◇

Example 2.2.9. We provide the minimal example of a non-representable hypercube rank function. Let $\mathbf{r}$ be the function on $\square_1^3$ defined by the following two arrays of size 2×2 (encoding the restrictions of $\mathbf{r}$ to $\square_1^2 \times \{k\}$ for $k \in \{0, 1\}$):

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

It is easy to check that $\mathbf{r}$ is a hypercube rank function. However, let us suppose by contradiction that $\mathbf{r}$ is associated to a collection of three flags $\mathrm{F}_i^\bullet$, $i \in \{1, 2, 3\}$ of a vector space H of dimension 2. For every i, the flag $\mathrm{F}_i^\bullet$ is determined by a line Δ_i:

$$\mathrm{H} = \mathrm{F}_i^0 \supsetneq \mathrm{F}_i^1 =: \Delta_i \supsetneq \mathrm{F}_i^2 = (0).$$

The restrictions of the function $\mathbf{r}$ to each of the 0-th layers are:

$$\mathbf{r}|_{L_0^1} = \mathbf{r}|_{L_0^2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \mathbf{r}|_{L_0^3} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The first two restrictions imply that $\Delta_2 = \Delta_3$ and $\Delta_1 = \Delta_3$, respectively, whereas the last restriction implies that $\Delta_1 \cap \Delta_3 = (0)$, which cannot hold simultaneously. ◇

Let $\mathscr{M}$ be a matricube associated to an arrangement of m initial (decreasing) flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$ inside H. For every $i \in \{1, \dots, m\}$, duality transforms the initial (decreasing) flag in H

$$\mathrm{H} = \mathrm{F}_i^0 \supseteq \mathrm{F}_i^1 \supseteq \cdots \supseteq \mathrm{F}_i^{r_i - 1} \supseteq \mathrm{F}_i^{r_i} \supsetneq (0)$$

into an *initial (increasing) flag* in H^*

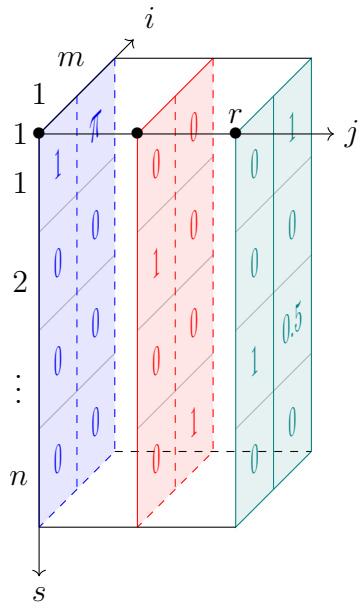
$$(0) = \mathrm{G}_0^i \subseteq \mathrm{G}_1^i \subseteq \cdots \subseteq \mathrm{G}_{r_i - 1}^i \subseteq \mathrm{G}_{r_i}^i \subsetneq \mathrm{H}^*,$$

where G_j^i is the orthogonal to F_i^j for the duality pairing $\langle \cdot, \cdot \rangle \colon \mathrm{H} \times \mathrm{H}^* \to \kappa$, that is, $\mathrm{G}_j^i := \bigcap_{v \in \mathrm{F}_i^j} \ker(v \colon \mathrm{H}^* \to \kappa)$ and $\mathrm{F}_i^j := \bigcap_{\ell \in \mathrm{G}_j^i} \ker(\ell \colon \mathrm{H} \to \kappa)$, so that one filtration can be recovered from the other. Note that in the case of matroids, this duality corresponds to the one between arrangements of hyperplanes in H and arrangements of vectors in the dual H^*.

The rank function $\mathbf{r}$ of $\mathscr{M}$, defined in Equation (2.1) using intersections of the F_i^j, can be alternatively described using the flags G_j^i in the following way:

$$\mathbf{r}(\underline{x}) = \dim_\kappa\left(\mathrm{G}_{x_1}^1 + \cdots + \mathrm{G}_{x_m}^m\right) \qquad \forall\ \underline{x} = (x_1, \dots, x_m) \in \square_{\underline{\varrho}}. \tag{2.2}$$

We will discuss the representability of matricubes further in Section 2.9.3.



$$\begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Figure 2.1: The left figure represents a three-dimensional matrix $A = (A^i_{js})$ of size $m \times r \times n$ with $m = 2$, $r = 3$ and $n = 4$. The blue (resp. red, resp. green) layer contains vertically the coordinates of the vectors v_1^1 and v_1^2 (resp. v_2^1 and v_2^2, resp. v_3^1 and v_3^2). The associated matricube is given on the right.

2.2.3 Matricube induced by a cubical matrix

Using the duality between initial (decreasing) flags in H and initial (increasing) flags in H^*, we explain a procedure that associates a representable matricube to any three-dimensional

matrix with coordinates in a field κ. This construction extends the representation of representable matroids by matrices. As in the case of matroids, this justifies the terminology *matricube*, which encompasses both the idea of *cubical matrix* (like "matroid", coming from "matrix") and *hypercuboid* (a matricube is described by a hypercuboid of numbers, given by the rank function).

Notation as in the previous section, first assume that $r_1 = \cdots = r_m = r$ and let $i \in \{1, \dots, m\}$. We choose vectors $v_1^i, \dots, v_r^i \in \mathrm{H}^*$ such that for every $j \in \{1, \dots, r\}$, $\mathrm{G}_j^i = \langle v_1^i, \dots, v_j^i \rangle$. This shows that a representable matricube $\mathscr{M}$ can be determined by the collection of vectors v_j^i, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, r\}$. Said otherwise, choosing a basis of H^*, $\mathscr{M}$ is determined by a three-dimensional matrix $A = (A_{js}^i)$ where $i \in \{1, \dots, m\}$, $j \in \{1, \dots, r\}$ and $s \in \{1, \dots, n\}$, n being the dimension of H^*. Inversely, using the definition of rank function given in (2.2), this procedure gives a way to associate to every three-dimensional matrix $A = (A_{js}^i)$ of size $m \times r \times n$ with entries in a field κ a matricube $\mathscr{M}_A$ on the hypercube $\square_r^m$.

In the general case, if not all r_i are equal, we set $r := \max_i r_i$ and choose, for every $i \in \{1, \dots, m\}$, a family of vectors $v_1^i, \dots, v_{r_i}^i, 0, \dots, 0$, with v_j^i for $j \leqslant r_i$ as before, completed now with $r - r_i$ copies of the zero vector. This gives a matrix A of size $m \times r \times n$. In the matricube $\mathscr{M}_A$ associated to A, we now delete, for every $i \in \{1, \dots, m\}$, $r - r_i$ times the element i (we refer to Section 2.2.5 below for the definition of the operation of deletion). This gives the matricube associated to the original family of flags.

It follows from the construction above that every representable matricube is associated to some three-dimensional matrix, possibly after a few deletions corresponding to zero vectors.

An example of a matricube associated to a three-dimensional real matrix with $m = 2$, $r_1 = r_2 = 3$ and $n = 4$, is depicted in Figure 2.1. The ground set of the corresponding matricube is $[3] \times [3]$.

2.2.4 Matricube induced by a collection of flag matroids

We show that a finite collection of initial flag matroids on the same ground set E gives rise to a (simple) matricube. We refer to [BGW03, Chapter 1] and [CDMS17] for a nice introduction to flag matroids.

Let E be a finite set and r be a positive integer. An *initial flag* of size s is an increasing chain of subsets $F_0 = \varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_s \subseteq E$ with $|F_j| = j$ for $j = 1, \dots, s$. Note that the data of an initial flag is equivalent to an ordered sequence $\varepsilon_1, \dots, \varepsilon_s$ of distinct elements of E, F_j consisting of the first j elements $\varepsilon_1, \dots, \varepsilon_j$.

A total order $<_o$ on E induces a partial, element-wise order on E^s. Through the bijection between initial flags of size s and ordered sequences of size s in E, $<_o$ induces a partial order on initial flags of size s.

An *initial flag matroid* M of rank s is a collection $\mathscr{F}$ of initial flags of size s as above such that for any total order $<_o$ on E, there exists a unique flag in $\mathscr{F}$ maximal with respect to the induced partial order on initial flags of size s. In this case, the following properties hold:

- the collection consisting of the terms F_j of flags $F_\bullet$ appearing in $\mathscr{F}$ forms the set of bases of a matroid M_j of rank j on the ground set E, for $j = 0, \dots, s$;
- the matroid M_j is a quotient of the matroid M_{j+1};
- any sequence $F_0 = \varnothing \subset F_1 \subset F_2 \subset \cdots \subset F_s$ with F_j a basis of M_j is an element of $\mathscr{F}$.

(These properties are equivalent to $\mathscr{F}$ defining a flag matroid, see [BGW03, Theorem 1.7.1].) Elements of $\mathscr{F}$ are called *bases*, and M_j is called the *j-th constituent* of M.

Let now $\underline{\varrho} = (r_1, \dots, r_m)$ be a vector with positive integer entries. Consider a collection $\mathrm{M}^1, \dots, \mathrm{M}^m$ of initial flag matroids of respective ranks $r_1, \dots, r_m$ on the ground set E. For each $i = 1, \dots, m$, and each $j \in [r_i]$, denote by M^i_j the j-th constituent of M^i. This is a matroid of rank j.

For each $\underline{x} \in \square_{\underline{\varrho}}$, denote by $\mathrm{M}_{\underline{x}}$ the matroid union $\mathrm{M}^1_{x_1} \cup \cdots \cup \mathrm{M}^m_{x_m}$ of $\mathrm{M}^1_{x_1}, \dots, \mathrm{M}^m_{x_m}$. Recall that the independent sets of the matroid union $\mathrm{M}_{\underline{x}}$ are subsets of E of the form $I_1 \cup \cdots \cup I_m$ where each I_i is an independent of the matroid $\mathrm{M}^i_{x_i}$ for $i = 1, \dots, m$.

Consider the function $\mathbf{r} \colon \square_{\underline{\varrho}} \to \mathbb{Z}_{\geqslant 0}$ on the hypercuboid defined by

$$\mathbf{r}(\underline{x}) := r(\mathrm{M}_{\underline{x}}) \qquad \forall \underline{x} \in \square_{\underline{\varrho}},$$

where $r(\mathrm{M}_{\underline{x}})$ is the rank of the matroid $\mathrm{M}_{\underline{x}}$.

Theorem 2.2.10. *Notation as above, $\mathbf{r}$ is the rank function of a simple matricube $\mathscr{M} = \mathscr{M}(\mathrm{M}^1, \dots, \mathrm{M}^m)$ with ground set the hypercuboid $\square_{\underline{\varrho}}$.*

Proof. We first note that $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = r(\mathrm{M}^i_t) = t$ for all $t \in [r_i]$. This shows that (R1*) is verified. The axiom (R2) is obviously verified by the definition of the matroid union. It thus remains to show (R3), i.e., that $\mathbf{r}$ is submodular. By Theorem 2.6.2, it will be enough to show that $\mathbf{r}$ verifies the diamond property, that is, for all $\underline{x} \in \square_{\underline{\varrho}}$ and distinct $1 \leqslant i, j \leqslant m$ with $\underline{x} + \underline{\mathfrak{e}}_i, \underline{x} + \underline{\mathfrak{e}}_j \in \square_{\underline{\varrho}}$, we have

$$\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) + \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_j) \geqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) + \mathbf{r}(\underline{x}). \tag{2.3}$$

Removing an element from each independent set of $\mathrm{M}^i_{x_i+1}$ results in an independent set of $\mathrm{M}^i_{x_i}$. This implies that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) \leqslant \mathbf{r}(\underline{x}) + 1$. We thus have $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) \leqslant \mathbf{r}(\underline{x}) + 2$.

Let $\underline{a} = \underline{x} + \underline{\mathfrak{e}}_i$, $\underline{b} = \underline{x} + \underline{\mathfrak{e}}_j$, and $\underline{c} = \underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j$. Three cases can occur, depending on whether $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$, $\mathbf{r}(\underline{x}) + 1$, or $\mathbf{r}(\underline{x}) + 2$.

• In the first case, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$, inequality (2.3) holds trivially.

• Consider the third case $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) + 2$. In this case, using the inequality $\mathbf{r}\big(\underline{y} + \underline{\mathfrak{e}}_k\big) \leqslant \mathbf{r}\big(\underline{y}\big) + 1$ for all $\underline{y}, \underline{y} + \underline{\mathfrak{e}}_k \in \square_{\underline{\varrho}}$, we infer that $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{b}) = \mathbf{r}(\underline{x}) + 1$, and inequality (2.3) holds again trivially.

• It remains to treat the case $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) + 1$. Let $I = I_1 \cup \cdots \cup I_m$ be a basis of $\mathrm{M}_{\underline{x}}$ with I_k an independent of $\mathrm{M}^k_{x_k}$ for $k = 1, \dots, m$. There exists a basis J of $\mathrm{M}_{\underline{c}}$ which contains I

and an extra element ε of E. Write $J = J_1 \cup \cdots \cup J_m$ with J_k an independent of $\mathrm{M}^k_{c_k}$, for $k = 1, \ldots, m$. Since J is not an independent of $\mathrm{M}_{\underline{x}}$, ε appears in either J_i or J_j. Removing it if necessary from one of the two, we can suppose that ε appears in exactly one of the two sets J_i or J_j, say, without loss of generality, in J_i. Then, J will be an independent set of $\mathrm{M}_{\underline{a}}$, and so $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{x}) + 1$. This shows that inequality (2.3) holds. The theorem follows. □

Note that if, in the definition of an initial flag matroid, we relax strict inclusions, the same construction as above gives rise as well to matricubes which are not necessarily simple.

2.2.5 Operations on matricubes

Let $\mathscr{M}$ be a matricube with underlying ground set $\square_{\underline{\varrho}}$, $\underline{\varrho} = (r_1, \ldots, r_m)$.

Deletion

Let $i \in \{1, \ldots, m\}$. We define the *deletion* of i in $\mathscr{M}$, denoted by $\mathscr{M} \setminus i$, as the matricube with ground set $\square_{\underline{\varrho}'}$, $\underline{\varrho}' = (r_1, \ldots, r_i - 1, \ldots, r_m)$, defined as follows. We view $\square_{\underline{\varrho}'}$ as the subset of $\square_{\underline{\varrho}}$ consisting of all the points $\underline{x}$ with $x_i < r_i$ and define the rank function $\mathbf{r}'$ of $\mathscr{M} \setminus i$ to be the restriction of $\mathbf{r}$ to $\square_{\underline{\varrho}'}$. Obviously, $\mathbf{r}'$ verifies the axiomatic system (R1)-(R2)-(R3) of matricube rank functions. Furthermore, note that if $\mathscr{M}$ is simple, then so is $\mathscr{M} \setminus i$.

As an example, here is a (simple) matricube $\mathscr{M}$ with $\underline{\varrho} = (4, 3)$ (left) and its deletion $\mathscr{M} \setminus 2$ in the vertical direction (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Contraction

Let $i \in \{1, \ldots, m\}$. We define the *contraction* of i in $\mathscr{M}$, denoted by $\mathscr{M} \diagup i$, as the matricube with ground set $\square_{\underline{\varrho}'}$, $\underline{\varrho}' = (r_1, \ldots, r_i - 1, \ldots, r_m)$, defined as follows. We define an embedding of $\square_{\underline{\varrho}'}$ in $\square_{\underline{\varrho}}$ by sending each point $\underline{x}$ to $\underline{x} + \underline{\mathfrak{e}}_i$. We then define the rank function $\mathbf{r}'$ of $\mathscr{M} \diagup i$ by setting $\mathbf{r}'(\underline{x}) := \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{\mathfrak{e}}_i)$. The embedding of $\square_{\underline{\varrho}'}$ in $\square_{\underline{\varrho}}$ respects the two operations of $\wedge$ and $\vee$. It is easy to see that $\mathbf{r}'$ verifies the axiomatic system (R1)-(R2)-(R3) of matricube rank functions. Note that $\mathscr{M} \diagup i$ is not necessarily simple, even if $\mathscr{M}$ is so.

As an example, here is a (simple) matricube $\mathscr{M}$ with $\underline{\varrho} = (4, 3)$ (left) and its (non-simple) contraction $\mathscr{M} \diagup 2$ in the vertical direction (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Minors

A matricube $\mathscr{M}'$ is a *minor* of another matricube $\mathscr{M}$ if it can be obtained by a sequence of deletions and contractions from $\mathscr{M}$.

Both operations of contraction and deletion respect the representability over a given field κ. It follows that if $\mathscr{M}$ is representable over κ, then all of its minors $\mathscr{M}'$ are also representable over κ. We will discuss the connection between representability and minors in Section 2.9.3.

2.2.6 Duality

Let $\mathscr{M}$ be a matricube on the ground set $\square_{\underline{\varrho}}$ with rank function $\mathbf{r}$. The *dual matricube* $\mathscr{M}^*$ is the matricube on $\square_{\underline{\varrho}}$ with rank function $\mathbf{r}^*$ defined by

$$\mathbf{r}^*(\underline{x}) := |\underline{x}|_{\ell_1} + \mathbf{r}(\underline{x}^c) - \mathbf{r}(\mathscr{M}) \qquad \forall\, \underline{x} \in \square_{\underline{\varrho}}.$$

Here,

$$\underline{x}^c := \underline{\varrho} - \underline{x} = (r_1 - x_1, \ldots, r_m - x_m)$$

is the *complement* of $\underline{x}$ and $|\underline{x}|_{\ell_1} := x_1 + \cdots + x_m$ is the ℓ_1-norm of $\underline{x} = (x_1, \ldots, x_m)$. A direct verification shows that $\mathbf{r}^*$ verifies the axiomatic system (R1)-(R2)-(R3) of matricube rank functions. Moreover, $\mathscr{M}^*$ has rank $\mathbf{r}^*(\mathscr{M}^*) = \left|\underline{\varrho}\right|_{\ell_1} - \mathbf{r}(\mathscr{M})$, and we have $(\mathscr{M}^*)^* = \mathscr{M}$. Note however that $\mathscr{M}$ can be simple without $\mathscr{M}^*$ being so, and vice-versa.

Here is a (simple) matricube $\mathscr{M}$ with $\underline{\varrho} = (4, 3)$ (left) and its (non-simple) dual $\mathscr{M}^*$ (right).

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

2.3 Flats

In this section, we define flats of matricubes and provide an axiomatic system for them. This extends the axioms of flats in matroid theory.

2.3.1 Definition and basic properties

Let $\mathscr{M} = (\square_{\underline{\varrho}}, \mathbf{r})$ be a matricube of rank r.

Definition 2.3.1 (Flats of a matricube)**.** A point $\underline{a} \in \square_{\underline{\varrho}}$ is called a *flat* for $\mathbf{r}$ if for every $1 \leqslant i \leqslant m$ such that $\underline{a} + \underline{\mathfrak{e}}_i$ belongs to $\square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a}) + 1$. We denote by $\mathscr{F} = \mathscr{F}(\mathscr{M}) \subseteq \square_{\underline{\varrho}}$ the set of flats of $\mathscr{M}$. ◇

Here are two (simple) matricubes with $\underline{\varrho} = (4, 3)$ for the first, and $\underline{\varrho} = (5, 4)$ for the second. The flats in each case are depicted in blue.

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Proposition 2.3.2 (Stability of flats under meet)**.** *The set $\mathscr{F}(\mathscr{M})$ of flats of a matricube $\mathscr{M}$ is stable under $\wedge$.*

We need the following lemma.

Lemma 2.3.3. *Let $i \in \{1, \dots, m\}$ and let $\underline{x}, \underline{y}, \underline{x} + \underline{\mathfrak{e}}_i, \underline{y} + \underline{\mathfrak{e}}_i$ be elements of $\square_{\underline{\varrho}}$ with $\underline{x} \preceq \underline{y}$ and $x_i = y_i$. If $\mathbf{r}(\underline{y} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{y}) + 1$, then we have $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) + 1$.*

Proof. This follows from the submodularity of $\mathbf{r}$ applied to $\underline{x}+\underline{\mathfrak{e}}_i$ and $\underline{y}$, and Proposition 2.2.4. □

Proof of Proposition 2.3.2. Let $\underline{a}$ and $\underline{b}$ be two flats and let $\underline{c} = \underline{a} \wedge \underline{b}$. Let $i \in \{1, \dots, m\}$ be such that $\underline{c} + \underline{\mathfrak{e}}_i$ belongs to $\square_{\underline{\varrho}}$. We have to show that $\mathbf{r}(\underline{c} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{c}) + 1$. By symmetry, we can suppose that $a_i \leqslant b_i$, that is, $c_i = a_i$. Since $\underline{a}$ is a flat, we have $\mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a}) + 1$. Applying Lemma 2.3.3 to $\underline{x} = \underline{c}$ and $\underline{y} = \underline{a}$, we conclude. □

The above result implies the following.

Theorem 2.3.4. *The set $\mathscr{F}(\mathscr{M})$ of flats of a matricube endowed with the partial order $\preceq$ is a graded lattice. The grading is induced by the rank function.*

Proof. $\mathscr{F}(\mathscr{M})$ has a minimum and a maximum element, and is stable under meet. It follows that it is a lattice, with the operation $\vee$ between two elements $\underline{a}$ and $\underline{b}$ in $\mathscr{F}(\mathscr{M})$ defined as the meet of all the upper bounds $\underline{c}$ for $\underline{a}$ and $\underline{b}$.

Note that for $\underline{a} \prec \underline{b}$ two distinct and comparable flats of $\mathscr{M}$, we have $\mathbf{r}(\underline{a}) < \mathbf{r}(\underline{b})$. The statement that $\mathscr{F}(\mathscr{M})$ is graded is a consequence of Proposition 2.3.6 below. □

Lemma 2.3.5. *Let $\mathscr{M}$ be a matricube on the ground set $\square_{\underline{\varrho}}$. Let $\underline{x}$ be an element of $\square_{\underline{\varrho}}$, and let $\underline{c}$ be the minimum flat with $\underline{c} \succeq \underline{x}$. Then, we have*

(1) $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$.

(2) *Let $i \in \{1, \dots, m\}$ be such that $\underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$. Then,*
- *if $c_i > x_i$, then we have $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x})$.*
- *if $c_i = x_i$, then we have $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) + 1$.*

Proof. To prove (1), it will be enough to show there exists a flat $\underline{b} \succeq \underline{x}$ with $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{x})$. Then, since flats are closed under meet, $\underline{b}$ will coincide with $\underline{c}$ and (1) follows. We proceed by a reverse induction on the ℓ_1-norm of $\underline{x}$. If $\underline{x}$ is a flat, in particular, if $\underline{x} = \underline{\varrho}$, there is nothing to prove. Otherwise, there exists $i \in \{1, \dots, m\}$ with $\underline{y} := \underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$ and $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x})$. By the induction hypothesis, there is a flat $\underline{b} \succeq \underline{y}$ with $\mathbf{r}(\underline{b}) = \mathbf{r}\big(\underline{y}\big) = \mathbf{r}(\underline{x})$, and we conclude.

We prove (2). If $c_i \geqslant x_i + 1$, then $\underline{x} \preceq \underline{x} + \underline{\mathfrak{e}}_i \preceq \underline{c}$, and thus $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) \leqslant \mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x})$. We infer that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x})$.

If $c_i = x_i$, then using $\mathbf{r}(\underline{c} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{c}) + 1$, we apply Lemma 2.3.3 and deduce that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) + 1$, as required. □

We get the following corollary.

Proposition 2.3.6. *Let $\underline{a} \prec \underline{b}$ be two distinct flats of $\mathscr{M}$ with $\mathbf{r}(\underline{b}) \geqslant \mathbf{r}(\underline{a}) + 2$. There exists a flat $\underline{a} \prec \underline{c} \prec \underline{b}$ with $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{a}) + 1$.*

Proof. Since $\underline{a} \prec \underline{b}$, there is an index i such that $a_i < b_i$. Let $\underline{x} := \underline{a} + \underline{\mathfrak{e}}_i \preceq \underline{b}$. Since $\underline{a}$ is a flat, $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{a}) + 1$. Let $\underline{c}$ be the minimum flat with $\underline{c} \succeq \underline{x}$. Obviously, $\underline{a} \prec \underline{c}$. Also $\underline{c} \preceq \underline{b}$, as $\underline{b}$ is a flat and $\underline{c}$ is minimum. By Property (1) in the previous lemma, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) = \mathbf{r}(\underline{a}) + 1$. We thus have strict inequality $\underline{c} \prec \underline{b}$, and the result follows. □

2.3.2 Axiomatic system of flats

For a subset $\mathscr{F}$ of $\square_{\underline{\varrho}}$, consider the following properties.

(F1) $\underline{\varrho}$ is in $\mathscr{F}$.

(F2) $\mathscr{F}$ is closed under meet.

(F3) If $\underline{a}$ is an element of $\mathscr{F}$ and $i \in \{1, \dots, m\}$ is such that $\underline{a} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b}$ in $\mathscr{F}$ such that $\underline{b} \succeq \underline{a} + \underline{\mathfrak{e}}_i$, and $\underline{b} \succ\!\!\succ \underline{a}$ in $\mathscr{F}$.

We recall that $\underline{b} \succ\!\!\succ \underline{a}$ means that $\underline{b}$ *covers* $\underline{a}$, i.e., $\underline{b} \succ \underline{a}$ in $\mathscr{F}$ and there is no element $\underline{c} \in \mathscr{F}$ such that $\underline{b} \succ \underline{c} \succ \underline{a}$.

We also introduce the following non-degeneracy property.

(F*) Each layer L_t^i, $i = 1, \dots, m$, $t \in [r_i]$, contains an element of $\mathscr{F}$.

We prove the following result.

Theorem 2.3.7. *The set of flats $\mathscr{F}$ of a matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ verifies (F1)-(F2)-(F3). Conversely, let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying (F1)-(F2)-(F3). Then, $\mathscr{F}$ is the set of flats of a matricube $\mathscr{M}$ with underling ground set $\square_{\underline{\varrho}}$.*

Moreover, the matricube $\mathscr{M}$ is simple if, and only if, Property (F) holds.*

Remark 2.3.8. The first three conditions (F1)-(F2)-(F3) are the matricube analogs of the three axioms that define the set of flats of a matroid. Property (F*) requires that the matricube does not contain any *loop*: if a layer does not verify this condition, deleting it, we get a smaller hypercuboid with the same collection $\mathscr{F}$ that verifies the same axioms. ◇

Remark 2.3.9. The axioms (F2) and (F*) together imply that $\underline{0}$ is in $\mathscr{F}$. Indeed, (F*) yields a flat in L_0^i for each $i = 1, \dots, m$. The meet of these flats is $\underline{0}$, and by (F2) belongs to $\mathscr{F}$. ◇

Remark 2.3.10. It is easy to see that if $\mathscr{F}$ verifies (F2), then the element $\underline{b}$ in (F3) is unique. ◇

2.3.3 Flats of a matricube verify the axioms

Let $\mathscr{M}$ be a matricube $\square_{\underline{\varrho}}$ and let $\mathscr{F} = \mathscr{F}(\mathscr{M})$ be the set of flats of $\mathscr{M}$. We prove that $\mathscr{F}$ verifies properties (F1)-(F2)-(F3). Moreover, if $\mathscr{M}$ is simple, then we show that (F*) holds.

Proof of the first part of Theorem 2.3.7. Property (F1) follows from the definition of flats of a matricube. We already proved property (F2) in Proposition 2.3.2. It remains to show (F3). If $\mathbf{r}(\underline{\varrho}) = \mathbf{r}(\underline{a}) + 1$, then $\underline{b} = \underline{\varrho}$ satisfies (F3). Otherwise, we have $\mathbf{r}(\underline{\varrho}) \geqslant \mathbf{r}(\underline{a}) + 2$, and by Proposition 2.3.6, we have an element $\underline{b}$ in $\mathscr{F}$ of rank $\mathbf{r}(\underline{a}) + 1$ with $\underline{b} \geq \underline{a} + \underline{\mathfrak{e}}_i$. Again, $\underline{b} \succ \underline{a}$, as required.

Now suppose that $\mathscr{M}$ is simple. Property (F*) is a consequence of Lemma 2.3.5 above applied to $t\,\underline{\mathfrak{e}}_i$. Let $\underline{c}$ be the minimum flat with $\underline{c} \geq t\,\underline{\mathfrak{e}}_i$. If Property (F*) does not hold for the layer L_t^i, then necessary $c_i > t$ and thus $\mathbf{r}((t+1)\,\underline{\mathfrak{e}}_i) = \mathbf{r}(t\,\underline{\mathfrak{e}}_i)$, contradicting the simpleness of $\mathscr{M}$. □

In the rest of the section, we prove the second part of the theorem.

2.3.4 Diamond property

We recall the following definition.

Definition 2.3.11 (Diamond property for lattices)**.** Let $(L, \leq)$ be a lattice with the meet and join operations $\wedge$ and $\vee$, respectively. We say that L has the *diamond property* if for every triple of elements $a, b, c \in L$ such that $b \neq c$ and b and c both cover a, the join $b \vee c$ covers both b and c. ◇

Lemma 2.3.12. *Let L be a lattice that satisfies the diamond property. Then, it admits a grading, i.e., all its maximal chains have the same length.*

Proof. This is well-known. We give a rather informal proof. We apply the diamond property multiple times to show that two maximal chains C and C' in L have the same length. This is done by induction on the elements of C and C'; the "diamonds" drawn in the Hasse diagram by repeated application of the diamond property provide a finite sequence of chains of constant length between C and C', ultimately proving that C and C' have the same length. □

2.3.5 The axiomatic system of flats implies the diamond property

Lemma 2.3.13. *Let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying axioms (F1) and (F2). Then, $\mathscr{F}$ is a lattice. If additionally $\mathscr{F}$ verifies axioms (F3), then it verifies the diamond property.*

Proof. Since $\mathscr{F}$ is closed under meet and it has a maximum element, it is a lattice. The join of two elements $\underline{a}$ and $\underline{b}$ in $\mathscr{F}$ is the meet of all the elements $\underline{c} \in \mathscr{F}$ that verify $\underline{c} \succeq \underline{a}$ and $\underline{c} \succeq \underline{b}$.

Now let $\underline{a}, \underline{b}, \underline{c} \in \mathscr{F}$ be such that $\underline{b} \neq \underline{c}$, and $\underline{b}$ and $\underline{c}$ both cover $\underline{a}$. By assumption, $\underline{b}$ and $\underline{c}$ are not comparable, so there exist $j \neq k \in \{1, \dots, m\}$ such that $b_j > c_j$ and $c_k > b_k$. In fact, $c_j = a_j$, because otherwise, we would have $\underline{a} \prec \underline{b} \wedge \underline{c} \prec \underline{b}$, which is impossible because $\underline{b} \succ \underline{a}$.

Now, applying (F3) yields $\underline{x} \in \mathscr{F}$ such that $\underline{x} \succeq \underline{c} + \underline{\mathfrak{e}}_j$ and $\underline{x} \succ \underline{c}$. We show that $\underline{x} \succ \underline{b}$. Indeed, first, if $\underline{x} \not\succeq \underline{b}$, then we would have $\underline{a} \prec \underline{x} \wedge \underline{b} \prec \underline{b}$, the strictness of the first inequality coming from the inequalities $x_j \geqslant a_j + 1$ (because $\underline{x} \succeq \underline{c} + \underline{\mathfrak{e}}_j$ and $c_j = a_j$) and $b_j > a_j$. This would be in contradiction with $\underline{b} \succ \underline{a}$. Second, $\underline{x} = \underline{b}$ is not possible because $\underline{x} \succeq \underline{c} + \underline{\mathfrak{e}}_j \succeq \underline{c}$ and $\underline{b}$ and $\underline{c}$ are not comparable. Therefore, $\underline{x} \succ \underline{b}$.

Symmetrically, (F3) provides an element $\underline{y} \in \mathscr{F}$ such that $\underline{y} \succeq \underline{b} + \underline{\mathfrak{e}}_k$, $\underline{y} \succ \underline{b}$ and $\underline{y} \succ \underline{c}$.

Let $\underline{u} \coloneqq \underline{x} \wedge \underline{y}$ and notice that $\underline{u}$ verifies the chains of inequalities $\underline{b} \preceq \underline{u} \preceq \underline{y}$ and $\underline{c} \preceq \underline{u} \preceq \underline{x}$. In other words, $\underline{u}$ belongs to the interval $\big[\underline{b}, \underline{y}\big]$, defined as the set of all elements $\underline{z} \in \mathscr{F}$ such that $\underline{b} \preceq \underline{z} \preceq \underline{y}$. This interval is equal to $\big\{\underline{b}, \underline{y}\big\}$ since $\underline{y} \succ \underline{b}$. Likewise, $\underline{u} \in [\underline{c}, \underline{x}] = \{\underline{c}, \underline{x}\}$. Since $\underline{b}$ and $\underline{c}$ are not comparable, $\underline{y} \succ \underline{b}$ and $\underline{x} \succ \underline{c}$, the only possibility for the sets $\big\{\underline{b}, \underline{y}\big\}$ and $\{\underline{c}, \underline{x}\}$ to have the element $\underline{u}$ in common is that $\underline{u} = \underline{x} = \underline{y}$. This shows that $\underline{u}$ covers $\underline{b}$ and $\underline{c}$. Then, $\underline{u}$ is necessarily equal to $\underline{b} \vee \underline{c}$. We conclude that $\mathscr{F}$ verifies the diamond property. □

Applying Lemma 2.3.12, we infer the following.

Proposition 2.3.14. *Let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying axioms (F1)-(F2)-(F3). Then, $\mathscr{F}$ is a graded lattice.*

We denote by $\mathbf{r} \colon \mathscr{F} \to \mathbb{N} \cup \{0\}$ the corresponding grading. The following properties hold.

(a) The function $\mathbf{r}$ is increasing on $\mathscr{F}$, in the following sense: if $\underline{a} \prec \underline{b} \in \mathscr{F}$, then $\mathbf{r}(\underline{a}) < \mathbf{r}(\underline{b})$.

(b) If $\underline{a}, \underline{b} \in \mathscr{F}$ are such that $\underline{b} \succ \underline{a}$, then $\mathbf{r}(\underline{b}) = \mathbf{r}(\underline{a}) + 1$.

2.3.6 Proof of the second part of Theorem 2.3.7

Let $\mathscr{F} \subseteq \square_{\underline{\varrho}}$ be a subset verifying the axioms (F1)-(F2)-(F3). We define a map

$$\varphi \colon \square_{\underline{\varrho}} \to \mathscr{F} \subseteq \square_{\underline{\varrho}}$$

as follows. For each $\underline{x} \in \square_{\underline{\varrho}}$, we define $\varphi(\underline{x})$ to be the minimum flat $\underline{b} \in \mathscr{F}$ such that $\underline{b} \succeq \underline{x}$.

Lemma 2.3.15. *Notation as above, the map φ is well-defined, and φ and $\mathscr{F}$ have the following properties.*

(i) The map $\varphi\colon \square_{\underline{\varrho}} \to \mathscr{F} \subseteq \square_{\underline{\varrho}}$ is non-decreasing.

(ii) Let $\underline{x} \in \square_{\underline{\varrho}}$ and $i \in \{1, \dots, m\}$ such that $\underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$. Then, either $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) = \varphi(\underline{x})$, or $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) \succ \varphi(\underline{x})$.

Proof. The first part is immediate by definition. We prove the second one. If we assume that $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) \neq \varphi(\underline{x})$, then we must have $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) \geq \varphi(\underline{x}) + \underline{\mathfrak{e}}_i$. Indeed, otherwise, $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) > \varphi(\underline{x}) \geq \underline{x} + \underline{\mathfrak{e}}_i$, contradicting the minimality of $\varphi(\underline{x} + \underline{\mathfrak{e}}_i)$.

Now, (F3) yields the existence of $\underline{b} \geq \varphi(\underline{x}) + \underline{\mathfrak{e}}_i$ in $\mathscr{F}$ such that $\underline{b} \succ \varphi(\underline{x})$. Then, using Property (i) above and the definition of φ, we get $\underline{b} \geq \varphi(\varphi(\underline{x}) + \underline{\mathfrak{e}}_i) \geq \varphi(\underline{x} + \underline{\mathfrak{e}}_i)$. The chain of inequalities $\underline{b} \geq \varphi(\underline{x} + \underline{\mathfrak{e}}_i) \geq \varphi(\underline{x})$ and the fact that $\underline{b} \succ \underline{x}$ imply $\underline{b} = \varphi(\underline{x} + \underline{\mathfrak{e}}_i)$. Therefore, we have $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) \succ \varphi(\underline{x})$, as required. □

We can now complete the proof of Theorem 2.3.7.

Proof of the second part of Theorem 2.3.7. Let $\mathscr{F}$ be a subset of $\square_{\underline{\varrho}}$ verifying axioms (F1)-(F2)-(F3). As we have shown already in Proposition 2.3.14, $\mathscr{F}$ is a graded lattice.

We first extend the function $\mathbf{r}\colon \mathscr{F} \to \mathbb{N} \cup \{0\}$ to a function $\mathbf{r}\colon \square_{\underline{\varrho}} \to \mathbb{N} \cup \{0\}$ by setting, for each $\underline{x} \in \square_{\underline{\varrho}}$, $\mathbf{r}(\underline{x}) := \mathbf{r}(\varphi(\underline{x}))$. We claim that $\mathbf{r}$ is the rank function of a matricube. We will use the properties proven in Lemma 2.3.15.

Part (ii) of the lemma implies directly that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) \leqslant \mathbf{r}(\underline{x}) + 1$ for each $\underline{x} \in \square_{\underline{\varrho}}$, proving (R1).

The fact that $\mathbf{r}$ is non-decreasing on $\square_{\underline{\varrho}}$ is a consequence of Property (i) in the lemma and Fact (a) stipulating that $\mathbf{r}$ is increasing on elements of $\mathscr{F}$.

We show that $\mathbf{r}$ is submodular on $\square_{\underline{\varrho}}$. By Theorem 2.6.2 that we will prove in Section 2.6, it is sufficient to prove the diamond property for functions on hypercuboids (see Section 2.6). Let $\underline{x} \in \square_{\underline{\varrho}}$ and $i \neq j \in \{1, \dots, m\}$ be such that $\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j \in \square_{\underline{\varrho}}$. Using Property (ii), we may assume that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) = \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_j) + 1$, and then need to prove that $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) + 1$. The equality $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) = \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_j) + 1$ means that $\varphi(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) \succ \varphi(\underline{x} + \underline{\mathfrak{e}}_j)$. This implies that $\varphi(\underline{x} + \underline{\mathfrak{e}}_j) \in L^i_{x_i}$ (as otherwise, we would get $\varphi(\underline{x} + \underline{\mathfrak{e}}_j)_i \geqslant x_i + 1$, that is, $\varphi(\underline{x} + \underline{\mathfrak{e}}_j) \geqslant \underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j$, and so we would have $\varphi(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) = \varphi(\underline{x} + \underline{\mathfrak{e}}_j)$). Since $\varphi(\underline{x} + \underline{\mathfrak{e}}_j) \geq \varphi(\underline{x})$, this in turn implies that $\varphi(\underline{x}) \in L^i_{x_i}$. However, $\varphi(\underline{x} + \underline{\mathfrak{e}}_i)_i \geqslant x_i + 1$. We infer that $\varphi(\underline{x} + \underline{\mathfrak{e}}_i) \succ \varphi(\underline{x})$, and thus $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) + 1$, as desired.

We have shown that $\mathbf{r}$ is the rank function of a matricube with ground set $\square_{\underline{\varrho}}$. The fact that the set of flats of $\mathbf{r}$ is exactly $\mathscr{F}$ is immediate by the definition of $\mathbf{r}$ and the map φ, and the fact that $\mathbf{r}$ is increasing on $\mathscr{F}$, see (a).

It remains to show that if (F*) holds, then $\mathscr{M}$ is simple. Let $i \in \{1, \dots, m\}$. We show by induction that for every $0 \leqslant t \leqslant r_i$, we have $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$. The base case $t = 0$ holds by definition. We now suppose that $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$ with $0 \leqslant t < r_i$ and show that $\mathbf{r}((t+1)\,\underline{\mathfrak{e}}_i) = t+1$. (F*) implies that $\varphi(t\,\underline{\mathfrak{e}}_i) \in L^i_t$ and $\varphi((t+1)\,\underline{\mathfrak{e}}_i) \in L^i_{t+1}$. In particular, $\varphi((t+1)\,\underline{\mathfrak{e}}_i) \neq \varphi(t\,\underline{\mathfrak{e}}_i)$.

Again, Lemma 2.3.15 implies that $\varphi((t+1)\,\underline{\mathfrak{e}}_i) \succ \varphi(t\,\underline{\mathfrak{e}}_i)$ which, using that $\mathbf{r}$ is increasing on $\mathscr{F}$, implies $\mathbf{r}((t+1)\,\underline{\mathfrak{e}}_i) = \mathbf{r}(\varphi((t+1)\,\underline{\mathfrak{e}}_i)) = \mathbf{r}(\varphi(t\,\underline{\mathfrak{e}}_i)) + 1 = \mathbf{r}(t\,\underline{\mathfrak{e}}_i) + 1 = t+1$, as desired. □

2.4 Circuits

We define circuits in matricubes and provide an intrinsic axiomatic system for them.

2.4.1 Duality and circuits

Let $\mathscr{M}$ be a matricube on the ground set $\square_{\underline{\varrho}}$, and denote by $\mathscr{M}^*$ its dual. Denote by $\mathscr{F}(\mathscr{M}^*)$ the set of flats of the dual matricube, and consider

$$\breve{\mathscr{C}} := \left\{ \underline{a}^c \;\middle|\; \underline{a} \in \mathscr{F}(\mathscr{M}^*) \right\} \subseteq \square_{\underline{\varrho}},$$

where, we recall, $\underline{a}^c = \underline{\varrho} - \underline{a}$. Since $\mathscr{F}(\mathscr{M})$ is closed under meet, $\breve{\mathscr{C}}$ will be closed under the join operation.

Given a subset $\mathcal{A} \subseteq \square_{\underline{\varrho}}$, we say that an element $\underline{a}$ of $\mathcal{A}$ is *join-irreducible in* $\mathcal{A}$ if it is not the join of any set of elements of $\mathcal{A} \smallsetminus \{\underline{a}\}$.

Definition 2.4.1 (Circuits)**.** The collection of circuits of $\mathscr{M}$, denoted by $\mathscr{C}$, is defined as the set of nonzero join-irreducible elements of $\breve{\mathscr{C}}$. ◇

Here is a (simple) matricube $\mathscr{M}$ with $\underline{\varrho} = (5,4)$. On the left, $\mathscr{M}$ is represented by its rank function $\mathbf{r}$, with its circuits in red and the join-reducible elements of $\breve{\mathscr{C}}$ in blue. On the right, the dual $\mathscr{M}^*$ of $\mathscr{M}$, which is not simple, is represented by its rank function $\mathbf{r}^*$, with its flats in teal.

$$\begin{pmatrix} 4 & 4 & 4 & 4 & 5 & 6 \\ 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix} \tag{2.4}$$

Obviously, by definition, $\mathscr{C}$ determines $\breve{\mathscr{C}}$, and therefore, gives the set of flats of the dual matroid $\mathscr{M}^*$. By Theorem 2.3.7, this implies that $\mathscr{C}$ determines $\mathscr{M}$.

2.4.2 Axiomatic system of circuits

For a subset $\mathscr{C}$ of $\square_{\underline{\varrho}}$, denote by $\breve{\mathscr{C}}$ the join-closure of $\mathscr{C}$, obtained by taking the join of any set of elements of $\mathscr{C}$. Consider the following set of properties:

(C1) $\underline{0}$ is *not* in $\mathscr{F}$.

(C2) All elements of $\mathscr{C}$ are join-irreducible in $\mathscr{C}$.

(C3) If $\underline{a} \in \breve{\mathscr{C}}$ and $i \in \{1, \dots, m\}$ is such that $\underline{a} - \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, then there exists an element $\underline{b} \preceq \underline{a} - \underline{\mathfrak{e}}_i$ in $\breve{\mathscr{C}} \cup \{\underline{0}\}$ such that $\underline{b} \prec \underline{a}$ in $\breve{\mathscr{C}} \cup \{\underline{0}\}$.

We also introduce the following simpleness property.

(C*) For every $i \in \{1, \dots, m\}$ and $t \in [r_i]$, $t\,\underline{\mathfrak{e}}_i$ is *not* in $\mathscr{C}$.

We prove the following result.

Theorem 2.4.2. *The set of circuits $\mathscr{C}$ of a matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ verifies (C1)-(C2)-(C3). Conversely, let $\mathscr{C} \subseteq \square_{\underline{\varrho}}$ be a subset verifying (C1)-(C2)-(C3). Then, $\mathscr{C}$ is the set of circuits of a matricube $\mathscr{M}$ with underlying ground set $\square_{\underline{\varrho}}$.*

Moreover, the matricube $\mathscr{M}$ is simple if, and only if, Property (C) holds.*

Proof. ($\Longrightarrow$) Let $\mathscr{C}$ be the set of circuits of a matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$. Properties (C1) and (C2) hold by definition of $\mathscr{C}$ (see Definition 2.4.1). As for Property (C3), it is a translation through duality of Property (F3) which holds for the set of flats of the dual matricube $\mathscr{M}^*$.

Assume moreover that $\mathscr{M}$ is simple. Let $i \in \{1, \dots, m\}$ and $t \in [r_i]$. We have $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$ and therefore, denoting by $\mathbf{r}^*$ the rank function on $\mathscr{M}^*$, we have $\mathbf{r}^*\big(\underline{\varrho} - t\,\underline{\mathfrak{e}}_i\big) = \mathbf{r}^*\big(\underline{\varrho}\big)$. This implies that for every i and t, $\underline{\varrho} - t\,\underline{\mathfrak{e}}_i$ is not a flat in $\mathscr{M}^*$, which means that $t\,\underline{\mathfrak{e}}_i$ is not in $\breve{\mathscr{C}}$. As a consequence, $t\,\underline{\mathfrak{e}}_i \notin \mathscr{C}$.

($\Longleftarrow$) Let $\mathscr{C} \subseteq \square_{\underline{\varrho}}$ be a subset verifying (C1)-(C2)-(C3), and $\breve{\mathscr{C}}$ the join-closure of $\mathscr{C}$. Define

$$\mathscr{F} := \Big\{\underline{a}^c \;\Big|\; \underline{a} \in \breve{\mathscr{C}}\Big\} \cup \big\{\underline{\varrho}\big\}.$$

We claim that $\mathscr{F}$ is the set of flats of a matricube. We need to show that it satisfies (F1)-(F2)-(F3). By construction, (F1) and (F2) hold. Property (F3) is a translation through duality of Property (C3) which holds for $\mathscr{C}$. Therefore, $\mathscr{F}$ is the set of flats of a matricube. We denote the dual of this matricube by $\mathscr{M}$, so that $\mathscr{F} = \mathscr{F}(\mathscr{M}^*)$. It is immediate by construction that $\mathscr{C}$ is its set of circuits of $\mathscr{M}$.

Let $\mathbf{r}$ be the rank function of $\mathscr{M}$ and $\mathbf{r}^*$ that of $\mathscr{M}^*$.

To prove the last assertion, assume that for every $i \in \{1, \dots, m\}$ and $t \in [r_i]$, $t\,\underline{\mathfrak{e}}_i$ is not in $\mathscr{C}$. Then, for every i and t, $\underline{\varrho} - t\,\underline{\mathfrak{e}}_i$ is not a flat of $\mathscr{M}^*$. By a simple induction, this implies that for every i and t, $\mathbf{r}^*\big(\underline{\varrho} - t\,\underline{\mathfrak{e}}_i\big) = \mathbf{r}^*\big(\underline{\varrho}\big)$, and consequently, $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) = t$. We infer that $\mathscr{M}$ is simple. □

2.5 Independents

In this section, we define the independents of a matricube and study their properties. As in the case of flats and circuits, we give the axiomatic system of independents of a matricube.

2.5.1 Definition and basic properties

Let $\mathscr{M}$ be a matricube on the ground set $\boxplus_{\underline{\varrho}}$.

Definition 2.5.1 (Independents of a matricube)**.** We say that a point $\underline{p}$ of $\boxplus_{\underline{\varrho}}$ is called an *independent of* $\mathscr{M}$ if for each $i = 1, \ldots, m$ such that $\underline{p} - \underline{\mathfrak{e}}_i \in \boxplus_{\underline{\varrho}}$, we have $\mathbf{r}\big(\underline{p} - \underline{\mathfrak{e}}_i\big) = \mathbf{r}\big(\underline{p}\big) - 1$. We denote by $\mathscr{I}(\mathscr{M}) \subseteq \boxplus_{\underline{\varrho}}$ the set of independents of $\mathscr{M}$. ◇

Here are two matricubes with $\underline{\varrho} = (4, 3)$ for the first, and $\underline{\varrho} = (5, 4)$ for the second. The independents in each case are depicted in blue.

$$\begin{pmatrix} 3 & 3 & 3 & 4 & 5 \\ 2 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

The following proposition provides a list of properties of independent sets in a matricube.

Proposition 2.5.2. *Let $\mathscr{I}(\mathscr{M})$ be the set of independents of a matricube $\mathscr{M}$. The following properties hold.*

- *$\mathscr{I}(\mathscr{M})$ is non-empty and closed under meet.*
- *For every independent $\underline{p} \in \mathscr{I}(\mathscr{M})$ and every distinct elements $i_1, \ldots, i_k \in \{1, \ldots, m\}$ with $p_{i_j} \neq 0$, $j = 1, \ldots, k$, we have $\mathbf{r}\big(\underline{p} - \underline{\mathfrak{e}}_{i_1} - \cdots - \underline{\mathfrak{e}}_{i_k}\big) = \mathbf{r}\big(\underline{p}\big) - k$.*

Proof. Both statements follow from Lemma 2.3.3, as in the proof of Proposition 2.3.2. □

Note that $\mathscr{I}(\mathscr{M})$ in general does not have a maximum, and lacks the existence of a join.

In order to study more refined properties of independents, we will associate a notion of *size* to each independent element in $\mathscr{M}$ by defining a *removal* operation on elements of $\mathscr{I}(\mathscr{M})$.

2.5.2 Removal and size

Let $\mathcal{J}$ be a subset of $\boxplus_{\underline{\varrho}}$. Let $\underline{a}$ be an element of $\mathcal{J}$ and $i \in \{1, \ldots, m\}$ such that $a_i \neq 0$. If there is at least one element $\underline{b} \prec \underline{a}$ in $\mathcal{J}$ that differs from $\underline{a}$ only in the i-th component, we define $\underline{a} \smallsetminus i$ to be such an element in $\mathcal{J}$ with the largest i-th coordinate. In this case, we say that $\underline{a} \smallsetminus i$ is the *removal of i in $\underline{a}$ in $\mathcal{J}$*.

Definition 2.5.3. Let $\mathcal{J}$ be a subset of $\boxplus_{\underline{\varrho}}$.

(a) We say that *removals exist in* $\mathcal{J}$ if for every $\underline{a} \in \mathcal{J}$ and $i \in \{1, \ldots, m\}$, if $a_i \geqslant 1$, the removal $\underline{a} \smallsetminus i$ exists in $\mathcal{J}$.

(If removals exist in $\mathcal{J}$, then necessarily, we have $\underline{0} \in \mathcal{J}$. Moreover, for every element $\underline{a} \in \mathcal{J}$, there exists a sequence of removals in $\mathcal{J}$, that reduces $\underline{a}$ to $\underline{0}$.)

(b) We say that $\mathcal{J}$ is *orderable* if removals exist in $\mathcal{J}$ and for every $\underline{a} \in \mathcal{J}$, all the sequences of removals in $\mathcal{J}$ that bring $\underline{a}$ to $\underline{0}$ have the same length.

If $\mathcal{J}$ is orderable, we define the *size* of each element $\underline{a} \in \mathcal{J}$ denoted by $|\underline{a}|$ as the number of removals needed to reduce $\underline{a}$ to $\underline{0}$. ◇

In Lemma 2.5.7 below, we formulate a simple orderability criterion.

2.5.3 Axiomatic system of independents

We first make the following definition, which turns out to be useful in the proof of the main theorem of this section.

Definition 2.5.4. Let $\underline{a}, \underline{b} \in \square_{\underline{\varrho}}$ be two elements. We define:

$$\Delta(\underline{a}, \underline{b}) := \big\{k = 1, \dots, m \;\big|\; a_k < b_k\big\} \qquad \text{and} \qquad E(\underline{a}, \underline{b}) := \sum_{k \in \Delta(\underline{a}, \underline{b})} (b_k - a_k). \qquad \diamond$$

For a subset $\mathscr{I}$ of $\square_{\underline{\varrho}}$, consider the following property:

(I1) Removals exist in $\mathscr{I}$ and the following holds. For all $\underline{p} \in \mathscr{I}$ and removals $\underline{p} \smallsetminus i$ and $\underline{p} \smallsetminus j$, with $i, j \in \{1, \dots, m\}$, the meet $\underline{q} := \big(\underline{p} \smallsetminus i\big) \wedge \big(\underline{p} \smallsetminus j\big)$ belongs to $\mathscr{I}$ and, moreover, the two intervals $\big[\underline{q}, \underline{p} \smallsetminus i\big]$ and $\big[\underline{q}, \underline{p} \smallsetminus j\big]$ in $\mathscr{I}$ have the same size.

(The interval $[\underline{a}, \underline{b}]$ in $\mathscr{I}$ is defined as the set of all $\underline{c} \in \mathscr{I}$ such that $\underline{a} \preceq \underline{c} \preceq \underline{b}$.)

It follows from Lemma 2.5.7, proved in Section 2.5.5, that a subset $\mathscr{I} \subseteq \square_{\underline{\varrho}}$ that verifies (I1) is orderable. We can thus define the size $|\underline{a}|$ of each element $\underline{a} \in \mathscr{I}$. This enables us to formulate the second property of interest:

(I2) $|\cdot|$ is increasing on independents, i.e., for all $\underline{a}, \underline{b} \in \mathscr{I}$ such that $\underline{a} \prec \underline{b}$, we have $|\underline{a}| < |\underline{b}|$. Moreover, let $\underline{a}$ and $\underline{b}$ be two elements of $\mathscr{I}$ such that $|\underline{a}| < |\underline{b}|$ and $\Delta(\underline{a}, \underline{b})$ contains at least two elements. Then, there exists $\underline{c} \in \mathscr{I}$ that verifies:

- $\underline{c} \preceq \underline{a} \vee \underline{b}$,
- $|\underline{c}| > |\underline{a}|$.
- There exists $i \in \Delta(\underline{a}, \underline{b})$ such that $c_i < b_i$.

We also introduce the following notion of simpleness.

(I*) For $i = 1, \dots, m$, the points $t\,\underline{\mathfrak{e}}_i$ for $t \in [r_i]$ are all in $\mathscr{I}$.

This is the main result of this section.

Theorem 2.5.5. *The set of independents $\mathscr{I}(\mathscr{M})$ of a matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ verifies (I1)-(I2). Conversely, let $\mathscr{I} \subseteq \square_{\underline{\varrho}}$ be a subset that verifies (I1)-(I2). Then, $\mathscr{I}$ is the set of independents of a matricube $\mathscr{M}$ with underling ground set $\square_{\underline{\varrho}}$.*

Moreover, the matricube $\mathscr{M}$ is simple if, and only if, $\mathscr{I}$ verifies (I).*

Remark 2.5.6. Axiom (I1) is an analog of the hereditary property for independents of matroids. It also implies that $\underline{0} \in \mathscr{I}$, analog of the first axiom of independents in matroids. Axiom (I2) plays the role of the augmentation property for independents. These axioms take into account the more singular nature of independents in the context of matricubes: for example, in matroids, all maximal (for inclusion) independents have the same cardinality, whereas in matricubes, maximal independents (for the partial order $\preceq$) can have different sizes as one of the two examples above shows (see Section 2.9.1 for further discussion). Axiom (I*) requires that there is no "loop". ◇

2.5.4 Independents of a matricube verify the axioms

Let $\mathscr{M}$ be a matricube $\square_{\underline{\varrho}}$. We prove that $\mathscr{I}(\mathscr{M})$ verifies properties (I1)-(I2). Moreover, if $\mathscr{M}$ is simple, then we prove that (I*) holds.

The proof shows that the size function on $\mathscr{I}(\mathscr{M})$ coincides with the rank function.

Proof of the first part of Theorem 2.5.5. We start by proving (I1). Let $\underline{p} \in \mathscr{I}(\mathscr{M})$ and let $i \in \{1, \dots, m\}$ be such that $p_i \neq 0$. We claim that $\underline{a} := \underline{p} - p_i\, \underline{\mathfrak{e}}_i \in \mathscr{I}(\mathscr{M})$. By design, $a_i = 0$. Now, let $j \in \{1, \dots, m\}$ be an integer different from i such that $a_j \neq 0$. We have $p_j = a_j \neq 0$ and $\mathbf{r}\big(\underline{p} - \underline{\mathfrak{e}}_j\big) = \mathbf{r}\big(\underline{p}\big) - 1$. Applying Lemma 2.3.3 with $\underline{x} = \underline{a} - \underline{\mathfrak{e}}_j$ and $\underline{y} = \underline{p} - \underline{\mathfrak{e}}_j$, we get $\mathbf{r}(\underline{a} - \underline{\mathfrak{e}}_j) = \mathbf{r}(\underline{a}) - 1$. This shows that $\underline{a} \in \mathscr{I}(\mathscr{M})$. Therefore, removals exist in $\mathscr{I}(\mathscr{M})$.

We next show that $\mathbf{r}\big(\underline{p} \smallsetminus i\big) = \mathbf{r}\big(\underline{p}\big) - 1$. For the sake of a contradiction, suppose this not being the case, that is, $\mathbf{r}\big(\underline{p} \smallsetminus i\big) \leqslant \mathbf{r}\big(\underline{p}\big) - 2$. Then, there would exist $\underline{p} \smallsetminus i \prec \underline{b} \prec \underline{p}$ such that $\mathbf{r}(\underline{b}) = \mathbf{r}\big(\underline{p}\big) - 1$ and $\mathbf{r}(\underline{b} - \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{b}) - 1$. Note that $b_j = p_j$ for all $j \neq i$. Applying again Lemma 2.3.3 as above, we infer that $\underline{b}$ belongs to $\mathscr{I}(\mathscr{M})$. This would be a contradiction to the definition of the removal.

This implies that a sequence of removals bringing $\underline{p} \in \mathscr{I}(\mathscr{M})$ to $\underline{0}$ has size precisely $\mathbf{r}\big(\underline{p}\big)$.

Now, for distinct $i, j \in \{1, \dots, m\}$, we consider the removals $\underline{p} \smallsetminus i$ and $\underline{p} \smallsetminus j$ in $\mathscr{I}(\mathscr{M})$, as well as $\underline{q} := \big(\underline{p} \smallsetminus i\big) \wedge \big(\underline{p} \smallsetminus j\big)$. By Proposition 2.5.2, $\underline{q} \in \mathscr{I}(\mathscr{M})$. Note that we have $\mathbf{r}\big(\underline{p} \smallsetminus j\big) = \mathbf{r}\big(\underline{p}\big) - 1 = \mathbf{r}\big(\underline{p} \smallsetminus i\big)$.

The element $\underline{q}$ differs from $\underline{p} \smallsetminus i$ only in the j-th component, and therefore can be obtained from it by a sequence of removals of j. It follows that $\big[\underline{q}, \underline{p} \smallsetminus i\big]$ has cardinality $\mathbf{r}\big(\underline{p} \smallsetminus i\big) - \mathbf{r}\big(\underline{q}\big) + 1$. Similarly, $\big[\underline{q}, \underline{p} \smallsetminus j\big]$ has cardinality $\mathbf{r}\big(\underline{p} \smallsetminus j\big) - \mathbf{r}\big(\underline{q}\big) + 1$.

We conclude that the two intervals $\big[\underline{q}, \underline{p} \smallsetminus i\big]$ and $\big[\underline{q}, \underline{p} \smallsetminus j\big]$ in $\mathscr{I}(\mathscr{M})$ have the same cardinality, and (I1) follows. We thus get a well-defined size function $|\cdot|$ on $\mathscr{I}(\mathscr{M})$. As the proof shows, we have $|\underline{a}| = \mathbf{r}(\underline{a})$ for every $\underline{a} \in \mathscr{I}$.

The first half of Property (I2) results from the fact that if $\underline{a} \prec \underline{b}$ are two independents, then $\Delta(\underline{a}, \underline{b}) \neq \varnothing$. Then, taking $k \in \Delta(\underline{a}, \underline{b})$, we get $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{b} - \underline{\mathfrak{e}}_k) = \mathbf{r}(\underline{b}) - 1$.

For the second half of Property (I2), let $\underline{a}, \underline{b}$ be two independents such that $|\underline{a}| < |\underline{b}|$ and $|\Delta(\underline{a}, \underline{b})| \geqslant 2$. We consider two cases depending on whether $|\underline{a}| \leqslant |\underline{b}| - 2$ or $|\underline{a}| = |\underline{b}| - 1$.

First, consider the case $|\underline{a}| \leqslant |\underline{b}| - 2$. Let $i \in \Delta(\underline{a}, \underline{b})$. Since $b_i > a_i \geqslant 0$, we can define $\underline{c} := \underline{b} \smallsetminus i$. Note that $|\underline{c}| = |\underline{b}| - 1 > |\underline{a}|$. Furthermore, by construction, $\underline{c} \preceq \underline{a} \vee \underline{b}$ and $c_i < b_i$. This shows that $\underline{c}$ is suitable.

We now consider the case $|\underline{a}| = |\underline{b}| - 1$. Let $\underline{d} := \underline{a} \vee \underline{b}$. Let $\underline{y} \preceq \underline{d}$ be an element of $\boxplus_{\underline{\varrho}}$ minimal for $\preceq$ under the constraints that $\mathbf{r}(\underline{y}) = \mathbf{r}(\underline{d})$ and for all $k \notin \Delta(\underline{a}, \underline{b})$, $y_k = d_k$ (note that for all those k, we have $d_k = a_k$). For all $k \in \Delta(\underline{a}, \underline{b})$ with $\underline{y} - \underline{\mathfrak{e}}_k \in \boxplus_{\underline{\varrho}}$, we thus have $\mathbf{r}(\underline{y} - \underline{\mathfrak{e}}_k) = \mathbf{r}(\underline{y}) - 1$.

Next, let $\underline{x} \preceq \underline{y}$ be an element of $\boxplus_{\underline{\varrho}}$ minimal for $\preceq$ under the constraint that $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{y}) = \mathbf{r}(\underline{d})$, and for all $k \in \Delta(\underline{a}, \underline{b})$, $x_k = y_k$. Since for all $k \in \Delta(\underline{a}, \underline{b})$ with $x_k = y_k > 0$, we have $\mathbf{r}(\underline{y} - \underline{\mathfrak{e}}_k) = \mathbf{r}(\underline{y}) - 1$, Lemma 2.3.3 implies that $\mathbf{r}(\underline{x} - \underline{\mathfrak{e}}_k) = \mathbf{r}(\underline{x}) - 1$. Moreover, by the choice of $\underline{x}$, we also have, for all $i \notin \Delta(\underline{a}, \underline{b})$, $\mathbf{r}(\underline{x} - \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{x}) - 1$ provided that $\underline{x} - \underline{\mathfrak{e}}_i$ belongs to $\boxplus_{\underline{\varrho}}$. Therefore, combining all this, we conclude that $\underline{x} \in \mathscr{I}(\mathscr{M})$.

If $\underline{y} \prec \underline{d}$, let $\underline{c} := \underline{x} \in \mathscr{I}(\mathscr{M})$. There exists then $k \in \Delta(\underline{a}, \underline{b})$ such that $y_k < d_k = b_k$. As a consequence, $c_k < b_k$. Moreover, $\mathbf{r}(\underline{c}) = \mathbf{r}(\underline{d}) \geqslant \mathbf{r}(\underline{b}) = \mathbf{r}(\underline{a}) + 1$ and therefore $|\underline{c}| > |\underline{a}|$. Since by construction $\underline{c} \prec \underline{d}$, $\underline{c}$ is suitable.

It remains to consider the case $\underline{y} = \underline{d}$. This means that for every $k \in \Delta(\underline{a}, \underline{b})$, $\mathbf{r}(\underline{d} - \underline{\mathfrak{e}}_k) < \mathbf{r}(\underline{d})$. We claim that in this case, the strict inequality $\mathbf{r}(\underline{d}) > \mathbf{r}(\underline{b})$ holds. Indeed, for the sake of a contradiction, suppose $\mathbf{r}(\underline{d}) = \mathbf{r}(\underline{b})$. Since $|\Delta(\underline{a}, \underline{b})| \geqslant 2$, there are two distinct elements $i, j \in \Delta(\underline{a}, \underline{b})$, and for these i, j, we would have $\mathbf{r}(\underline{d} - \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{d} - \underline{\mathfrak{e}}_j) = \mathbf{r}(\underline{d}) - 1$. By submodularity (see Lemma 2.3.3), we would get $\mathbf{r}(\underline{d} - \underline{\mathfrak{e}}_i - \underline{\mathfrak{e}}_j) = \mathbf{r}(\underline{d}) - 2$. Since $i, j \in \Delta(\underline{a}, \underline{b})$, we have $\underline{a} \preceq \underline{d} - \underline{\mathfrak{e}}_i - \underline{\mathfrak{e}}_j$, and therefore, we would have $\mathbf{r}(\underline{a}) \leqslant \mathbf{r}(\underline{d}) - 2 = \mathbf{r}(\underline{b}) - 2$, contradicting the assumption that $|\underline{a}| = |\underline{b}| - 1$. This proves the claim that $\mathbf{r}(\underline{d}) > \mathbf{r}(\underline{b})$.

Let now $\underline{c} := \underline{x} \smallsetminus k$ for an element $k \in \Delta(\underline{a}, \underline{b})$. We have $|\underline{c}| = \mathbf{r}(\underline{c}) = \mathbf{r}(\underline{x}) - 1 = \mathbf{r}(\underline{d}) - 1 \geqslant \mathbf{r}(\underline{b}) = |\underline{b}| > |\underline{a}|$, and therefore $|\underline{c}| > |\underline{a}|$. Besides, $c_k < x_k = d_k = b_k$, and obviously $\underline{c} \preceq \underline{d}$. This shows that $\underline{c}$ is suitable in this last case. This ends the proof of (I2). We have proved that $\mathscr{I}(\mathscr{M})$ verifies (I1) and (I2).

To finish the proof, note that if $\mathscr{M}$ is simple, then (R1*) immediately implies (I*). □

2.5.5 Orderability lemma

Before going to the proof of the second part of Theorem 2.5.5, we show the following criterion for orderability.

Lemma 2.5.7. *Let $\mathcal{J}$ be a subset of $\boxplus_{\underline{\varrho}}$. The following are equivalent:*

(1) $\mathcal{J}$ satisfies (I1).

(2) $\mathcal{J}$ is orderable in the sense of Definition 2.5.3.

Proof. We first prove (1) ⇒ (2). Assume (I1) holds. Proceeding by induction under the partial order $\preceq$, we show that for every $\underline{a} \in \mathcal{J}$, the following property holds:

$$P(\underline{a}): \quad \text{All the sequences of removals that bring } \underline{a} \text{ to } \underline{0} \text{ have the same length.}$$

Obviously, $P(\underline{0})$ holds. Let $\underline{a} \in \mathcal{J}$ be an element such that for every $\underline{b} \in \mathcal{J}$ with $\underline{b} \prec \underline{a}$, $P(\underline{b})$ holds. We prove that $P(\underline{a})$ is true. Let

$$\underline{a} = \underline{b}_0 \succ \underline{b}_1 \succ \cdots \succ \underline{b}_k = \underline{0} \quad \text{and} \quad \underline{a} = \underline{c}_0 \succ \underline{c}_1 \succ \cdots \succ \underline{c}_\ell = \underline{0}$$

be two sequences of removals bringing $\underline{a}$ to $\underline{0}$. We need to prove that $k = \ell$. Let $i, j \in \{1, \dots, m\}$ be such that $\underline{b}_1 = \underline{a} \smallsetminus i$ and $\underline{c}_1 = \underline{a} \smallsetminus j$, and let $\underline{q} := \underline{b}_1 \wedge \underline{c}_1 \in \mathcal{J}$. Let

$$\underline{q} = \underline{x}_0 \succ \underline{x}_1 \succ \cdots \succ \underline{x}_m = \underline{0}$$

be any sequence of removals bringing $\underline{q}$ to $\underline{0}$.

By (I1), the linear intervals $[\underline{q}, \underline{b}_1]$ and $[\underline{q}, \underline{c}_1]$ have the same size, that we denote by $s+1$, with $s \geqslant 0$. Let

$$[\underline{q}, \underline{b}_1] = \left\{\underline{b}_1 = \underline{y}_0 \succ \underline{y}_1 \succ \cdots \succ \underline{y}_s = \underline{q}\right\} \text{ and } [\underline{q}, \underline{c}_1] = \left\{\underline{c}_1 = \underline{z}_0 \succ \underline{z}_1 \succ \cdots \succ \underline{z}_s = \underline{q}\right\}.$$

Then, $\underline{b}_1 = \underline{y}_0 \succ \underline{y}_1 \succ \cdots \succ \underline{y}_s = \underline{q} = \underline{x}_0 \succ \underline{x}_1 \succ \cdots \succ \underline{x}_m = \underline{0}$ is a sequence of removals that brings $\underline{b}_1$ to $\underline{0}$. Property $P(\underline{b}_1)$ therefore implies that the length of this sequence is equal to the length of the sequence $\underline{b}_1 \succ \cdots \succ \underline{b}_k = \underline{0}$, that is, $k - 1 = s + m$. The same argument applied to $\underline{c}_1$ yields $\ell - 1 = s + m$. We conclude that $k = \ell$.

The implication (2) ⇒ (1) follows from the identities

$$|\underline{q}| + |[\underline{q}, \underline{p} \smallsetminus i]| = |\underline{p}| = |\underline{q}| + |[\underline{q}, \underline{p} \smallsetminus j]|,$$

using a sequence of removals of j (resp. i) that brings $p \smallsetminus i$ (resp. $p \smallsetminus j$) to q. □

2.5.6 Proof of the second part of Theorem 2.5.5

We need the following lemma.

Lemma 2.5.8. *Let $\mathcal{J}$ be a subset of $\mathbb{B}_{\underline{\varrho}}$ that verifies (I2). Then, for two elements $\underline{a} \prec \underline{b}$ of $\mathcal{J}$ such that $\Delta(\underline{a}, \underline{b})$ has at least two elements, we have $|\underline{a}| \leqslant |\underline{b}| - 2$.*

Proof. The first part of Property (I2) ensures that $|\underline{a}| < |\underline{b}|$. The second part of Property (I2) now implies that there exists an element $\underline{c} \in \mathcal{J}$ and $i \in \Delta(a, b)$ such that $\underline{c} \preceq \underline{a} \vee \underline{b} = \underline{b}$, $|\underline{c}| > |\underline{a}|$, and $c_i < b_i$. Combining the latter with $\underline{c} \preceq \underline{b}$ yields that $\underline{c} \prec \underline{b}$. Applying (I2) again, we get $|\underline{c}| < |\underline{b}|$. All in all, we get $\underline{a} \prec \underline{c} \prec \underline{b}$, and the inequality $|\underline{a}| \leqslant |\underline{b}| - 2$ follows. □

We now prove the second part of the main theorem.

Proof of the second part of Theorem 2.5.5. Notation as in the statement of the theorem, by Lemma 2.5.7, we have a well-defined size function $|\cdot|$ on $\mathscr{I}$. We define a function $\mathbf{r}$ on $\square_{\underline{\varrho}}$ by setting

$$\mathbf{r}(\underline{x}) \coloneqq \max_{\underline{a}\in\mathscr{I} \text{ with } \underline{a}\preceq\underline{x}} |\underline{a}| \qquad \forall\, \underline{x} \in \square_{\underline{\varrho}},$$

and show that $\mathbf{r}$ is the rank function of a matricube. Note that by (I2), $\mathbf{r}(\underline{a}) = |\underline{a}|$ for $\underline{a} \in \mathscr{I}$.

Obviously, $\mathbf{r}(\underline{0}) = 0$. Moreover, by orderability of $\mathscr{I}$, for $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant r_i$, we have either $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) - \mathbf{r}((t-1)\,\underline{\mathfrak{e}}_i) = 0$ or $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) - \mathbf{r}((t-1)\,\underline{\mathfrak{e}}_i) = 1$. Therefore (R1) holds.

Since $|\cdot|$ is increasing by (I2), $\mathbf{r}$ is non-decreasing on $\square_{\underline{\varrho}}$. That is, Property (R2) holds.

We show $\mathbf{r}$ is submodular. Using Theorem 2.6.2 proved in Section 2.6, it will be enough to show that $\mathbf{r}$ verifies the diamond property for functions on the hypercuboid.

We first observe that, by orderability of $\mathscr{I}$, for every $\underline{x} \in \square_{\underline{\varrho}}$ and $i \in \{1, \dots, m\}$, we have $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}) \leqslant 1$, provided that $\underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$.

Now let $\underline{x} \in \square_{\underline{\varrho}}$ and let $i \neq j$ be elements of $\{1, \dots, m\}$ such that $\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j \in \square_{\underline{\varrho}}$. Let $\underline{y} \coloneqq \underline{x} + \underline{\mathfrak{e}}_i$, $\underline{z} \coloneqq \underline{x} + \underline{\mathfrak{e}}_j$ and $\underline{w} \coloneqq \underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j$. Proving the diamond property for $\underline{x}, \underline{y}, \underline{z}, \underline{w}$ reduces to showing that the situation where $\mathbf{r}(\underline{x}) = \mathbf{r}\big(\underline{y}\big) = \mathbf{r}(\underline{z})$ and $\mathbf{r}(\underline{w}) = \mathbf{r}(\underline{x}) + 1$ never happens. For the sake of a contradiction, assume that we are in the situation where the above equalities hold. This implies in particular that $\underline{y}, \underline{z} \notin \mathscr{I}$. The rest of the argument is a case-by-case analysis. We first treat the case $\underline{w} \in \mathscr{I}$, then, $\underline{w} \notin \mathscr{I}$ but $\underline{x} \in \mathscr{I}$, and then generalize the argument to treat the remaining case $\underline{w}, \underline{x} \notin \mathscr{I}$.

First consider the case where $\underline{w} \in \mathscr{I}$. Let $\underline{a} \preceq \underline{x}$ be an element of $\mathscr{I}$ such that $\mathbf{r}(\underline{a}) = \mathbf{r}(\underline{x})$. Since $\underline{a} \prec \underline{w}$ and $|\Delta(\underline{a}, \underline{w})| \geqslant |\Delta(\underline{x}, \underline{w})| = 2$, applying Lemma 2.5.8, we get the inequality $\mathbf{r}(\underline{x}) \leqslant \mathbf{r}(\underline{w}) - 2$, which is a contradiction. This implies that $\underline{w} \notin \mathscr{I}$.

At this point, we have deduced $\underline{y}, \underline{z}, \underline{w} \notin \mathscr{I}$. Now consider the case where $\underline{x} \in \mathscr{I}$. Let $\underline{b} \prec \underline{w}$ be an element of $\mathscr{I}$ such that $\mathbf{r}(\underline{w}) = \mathbf{r}(\underline{b})$. Notice that $|\underline{b}| = \mathbf{r}(\underline{w}) > \mathbf{r}(\underline{x}) = |\underline{x}|$. Moreover, $b_i = w_i = x_i + 1$ because otherwise we would have $\underline{b} \preceq \underline{z}$ and $\mathbf{r}(\underline{b}) > \mathbf{r}(\underline{z})$, which would be impossible since $\mathbf{r}$ is non-decreasing. Likewise, we have $b_j = w_j = x_j + 1$. Since $\underline{b} \prec \underline{w} = \underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j$, this shows that $\Delta(\underline{x}, \underline{b}) = \{i, j\}$. By (I2), there exists an independent $\underline{c} \in \mathscr{I}$ such that $\underline{c} \preceq \underline{x} \vee \underline{b} = \underline{w}$, $|\underline{c}| > |\underline{x}|$ and $c_k < b_k$ for some $k \in \{i, j\}$. But if $k = i$, then $\underline{c} \preceq \underline{y}$ and therefore $|\underline{c}| \leqslant \mathbf{r}\big(\underline{y}\big) = \mathbf{r}(\underline{x}) = |\underline{x}|$, a contradiction; we conclude similarly if $k = j$. We have shown that $\underline{x} \notin \mathscr{I}$.

We now treat the remaining case. We define a finite procedure by applying repeatedly an analogue of the preceding construction, as follows. Let $\underline{a} \prec \underline{x}$, $\underline{b} \prec \underline{w}$ be elements of $\mathscr{I}$ such that $\mathbf{r}(\underline{x}) = |\underline{a}|$ and $\mathbf{r}(\underline{w}) = |\underline{b}|$. We have $|\underline{b}| > |\underline{a}|$. We claim $b_i = w_i > x_i \geqslant a_i$. Indeed, otherwise, we would have $\underline{b} \preceq \underline{z}$, impossible by the inequality $|\underline{b}| = \mathbf{r}(\underline{w}) > \mathbf{r}(\underline{z})$. Likewise, we have $b_j = w_j > a_j$. Consequently, we have $\Delta(\underline{a}, \underline{b}) \supseteq \{i, j\}$. By (I2), there exists an independent $\underline{c}^1 \in \mathscr{I}$ such that

$$\underline{c}^1 \preceq \underline{a} \vee \underline{b} \preceq \underline{w}, \qquad |\underline{c}^1| > |\underline{a}|, \qquad \text{and} \qquad c^1_{k_1} < b_{k_1} \quad \text{for some } k_1 \in \Delta(\underline{a}, \underline{b}).$$

Next, if $\Delta(\underline{a}, \underline{c}^1)$ contains itself at least two elements, since we have $|\underline{c}^1| > |\underline{a}|$, we can apply (I2), and the same procedure as above, replacing the pair $\underline{a}, \underline{b}$ by the pair $\underline{a}, \underline{c}^1$, yields

an element $\underline{c}^2 \in \mathscr{I}$ such that

$$\underline{c}^2 \preceq \underline{a} \vee \underline{c}^1 \preceq \underline{w}, \qquad |\underline{c}^2| > |\underline{a}|, \qquad \text{and} \qquad c^2_{k_2} < c^1_{k_2} \quad \text{for some } k_2 \in \Delta(\underline{a}, \underline{c}^1).$$

Repeating the procedure while it is possible, we get a sequence $\underline{c}^1, \underline{c}^2, \ldots, \underline{c}^j, \ldots$ of elements of $\mathscr{I}$, satisfying, for every $j \geqslant 1$,

$$\underline{c}^j \preceq \underline{a} \vee \underline{c}^{j-1} \preceq \underline{w}, \qquad |\underline{c}^j| > |\underline{a}|, \qquad \text{and} \qquad c^j_{k_j} < c^{j-1}_{k_j} \quad \text{for some } k_j \in \Delta(\underline{a}, \underline{c}^{j-1}),$$

with $\underline{c}^0 = \underline{b}$. We claim that this sequence is necessarily finite. Indeed, we observe that for every $j \geqslant 1$, we have by construction:

$$\Delta(\underline{a}, \underline{c}^{j-1}) \supseteq \Delta(\underline{a}, \underline{c}^j) \quad \text{and} \quad E(\underline{\mathfrak{e}}, \underline{c}^{j-1}) > E(\underline{\mathfrak{e}}, \underline{c}^j).$$

Since the integers $E(\underline{a}, \underline{c}^j)$ are all non-negative, we infer that the sequence $\underline{c}^\bullet$ ends at some integer $j > 0$. This means that the condition $|\Delta(\underline{a}, \underline{c}^j)| \geqslant 2$ fails, and thus it is impossible to have both i and j included in $\Delta(\underline{a}, \underline{c}^j)$.

Without loss of generality, assume $i \notin \Delta(\underline{a}, \underline{c}^j)$. This implies that $c^j_i \leqslant a_i$, and so, we have $\underline{c}^j \preceq \underline{z}$. We infer that $|\underline{a}| < |\underline{c}^j| \leqslant \mathbf{r}(\underline{z}) = \mathbf{r}(\underline{x}) = |\underline{a}|$, which is a contradiction.

At this point, we have shown the diamond property, and therefore we conclude that $\mathbf{r}$ is submodular, and (R3) follows.

It follows that $\mathbf{r}$ is the rank function of a matricube $\mathscr{M}$. Moreover, by definition of the rank function, $\mathscr{I}$ coincides with the set of independents of $\mathscr{M}$.

Finally, by definition of $\mathbf{r}$, Property (R1*) is seen to be equivalent to (I*), and so $\mathscr{M}$ is simple if, and only if, (I*) holds. □

2.6 Diamond property for functions

The aim of this section is to generalize to the setting of matricubes the well-known result in matroid theory that the submodularity of the rank function of a matroid is equivalent to the diamond property for its graded lattice of flats [Sta11, Proposition 3.3.2]. To this end, we here introduce a weaker version of submodularity.

Definition 2.6.1 (Diamond property for functions on hypercuboids)**.** We say an integer-valued function $\mathbf{r}$ on $\boxplus_{\underline{\varrho}}$ satisfies the *diamond property* if the following holds. For every point $\underline{x} \in \boxplus_{\underline{\varrho}}$ and distinct integers $i \neq j \in \{1, \ldots, m\}$ such that $\underline{x} + \underline{\mathfrak{e}}_i, \underline{x} + \underline{\mathfrak{e}}_j \in \boxplus_{\underline{\varrho}}$, we have

$$\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) - \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_j). \qquad \diamond \quad (2.5)$$

The following theorem shows that the above property is equivalent to submodularity.

Theorem 2.6.2 (Equivalence of submodularity and the diamond property)**.** *Let* $\mathbf{r}$ *be an integer-valued function on* $\boxplus_{\underline{\varrho}}$*. The following properties are equivalent:*

(i) $\mathbf{r}$ *is submodular.*

(ii) $\mathbf{r}$ *verifies the diamond property.*

In preparation for the proof, we provide generalizations of the diamond property, that allow to proceed by induction. We say an integer-valued function $\mathbf{r}$ on $\square_{\underline{\varrho}}$ satisfies the *unidirectional submodularity at distance one* if the following holds. For all $i \in \{1, \dots, m\}$ and for all points $\underline{x} \preceq \underline{y} \in \square_{\underline{\varrho}}$ such that $x_i = y_i$ and $\underline{x} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, the inequality

$$\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\big(\underline{y} + \underline{\mathfrak{e}}_i\big) - \mathbf{r}\big(\underline{y}\big) \tag{2.6}$$

holds. More generally, we have the following generalization of Property (2.6) in *several directions* and at *higher distance*.

Definition 2.6.3 (Multidirectional submodularity at a given distance)**.** For positive integers k and n, we define the *k-directional submodularity at distance up to n*, denoted $(*)_k^n$, as follows.

$(*)_k^n$: Pick any integer $1 \leqslant s \leqslant k$, any integers $1 \leqslant i_1 < \cdots < i_s \leqslant m$ and $0 \leqslant n_{i_1}, \dots, n_{i_s} \leqslant n$.

Then, for every pair of elements $\underline{x} \preceq \underline{y} \in \square_{\underline{\varrho}}$ such that $x_{i_j} = y_{i_j}$ for all $1 \leqslant j \leqslant s$, and $\underline{x} + \sum_{1 \leqslant j \leqslant s} n_j \, \underline{\mathfrak{e}}_{i_j} \in \square_{\underline{\varrho}}$, we have

$$\mathbf{r}\Big(\underline{x} + \sum_{j=1}^{s} n_{i_j} \, \underline{\mathfrak{e}}_{i_j}\Big) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\Big(\underline{y} + \sum_{j=1}^{s} n_{i_j} \, \underline{\mathfrak{e}}_{i_j}\Big) - \mathbf{r}\big(\underline{y}\big). \qquad \diamond$$

Notice that the property stated in (2.6) is exactly $(*)_1^1$ as defined above, and the terminologies are consistent. Moreover, any $(*)_k^n$ with $k, n \geqslant 1$ implies $(*)_1^1$.

Remark 2.6.4 (Alternative description of $(*)_k^n$)**.** Using the notation of Definition 2.6.3, after the change of variables $\underline{a} := \underline{x}$, $\underline{b} := \underline{y} - \sum_{j=1}^{s} n_{i_j} \, \underline{\mathfrak{e}}_{i_j}$, property $(*)_k^n$ can be rewritten as follows.

For all elements $\underline{a}$ and $\underline{b} \in \square_{\underline{\varrho}}$, we have the submodularity inequality

$$\mathbf{r}(\underline{a}) + \mathbf{r}(\underline{b}) \geqslant \mathbf{r}(\underline{a} \vee \underline{b}) + \mathbf{r}(\underline{a} \wedge \underline{b})$$

as long as there exist an integer $1 \leqslant s \leqslant k$ and integers $1 \leqslant i_1 < \cdots < i_s \leqslant m$ such that $\underline{b} + \sum_{1 \leqslant j \leqslant s} (a_{i_j} - b_{i_j}) \, \underline{\mathfrak{e}}_{i_j}$ is an element of $\square_{\underline{\varrho}}$ greater than or equal to $\underline{a}$ and such that, for all $1 \leqslant j \leqslant s$, we have $0 \leqslant a_{i_j} - b_{i_j} \leqslant n$.

This parametrization using $\underline{a}$ and $\underline{b}$ enables to see instantaneously that the submodularity property of $\mathbf{r}$ in the hypercuboid implies all the properties $(*)_k^n$. The other parametrization, using $\underline{x}$ and $\underline{y}$, will be useful to prove Theorem 2.6.2 below, in that it behaves linearly (contrary to formulas involving the symbols $\wedge$ and $\vee$). $\diamond$

Proof of Theorem 2.6.2. Obviously, (i) implies (ii).

We explain how to deduce (2.6), that is $(*)_1^1$, from (ii). Let $\underline{x}$ and $\underline{y}$ be as in Definition 2.6.1. The fact that $x_i = y_i$ implies that $\underline{y}$ can be written as $\underline{y} = \underline{x} + \sum_{j \neq i} n_j \, \underline{\mathfrak{e}}_j$ with $n_j \geqslant 0$, and we can sum inequalities of the form (2.5) to get the inequality (2.6).

We then explain how to deduce (i) from $(*)_1^1$. Proceeding by induction, we show that the property $(*)_1^1$ implies $(*)_k^n$ for all $k, n \geqslant 1$. We first show that $(*)_1^1$ implies $(*)_1^n$ for all $n \geqslant 1$. Let $i \in \{1, \dots, m\}$ and $0 \leqslant n_i \leqslant n$, and let $\underline{x} \preceq \underline{y}$ be elements of $\boxplus_{\underline{\varrho}}$ such that $\underline{x} + n_i\,\underline{\mathfrak{e}}_i \in \boxplus_{\underline{\varrho}}$ and $x_i = y_i$. For all $0 \leqslant t < n_i$, the pair $\big(\underline{x} + t\,\underline{\mathfrak{e}}_i, \underline{y} + t\,\underline{\mathfrak{e}}_i\big)$ satisfies the hypotheses needed to apply $(*)_1^1$ in direction i, so we know that

$$\mathbf{r}(\underline{x} + (t+1)\,\underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x} + t\,\underline{\mathfrak{e}}_i) \geqslant \mathbf{r}\big(\underline{y} + (t+1)\,\underline{\mathfrak{e}}_i\big) - \mathbf{r}\big(\underline{y} + t\,\underline{\mathfrak{e}}_i\big).$$

Summing all these inequalities for $0 \leqslant t < n_i$, and canceling out the terms which appear on both sides, yields

$$\mathbf{r}(\underline{x} + n_i\,\underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\big(\underline{y} + n_i\,\underline{\mathfrak{e}}_i\big) - \mathbf{r}\big(\underline{y}\big),$$

which gives $(*)_1^n$.

We now show that properties $(*)_1^n$ for $n \geqslant 1$ imply properties $(*)_2^n$. Let $i, j \in \{1, \dots, m\}$ and $0 \leqslant n_i, n_j \leqslant n$, and let $\underline{x} \preceq \underline{y}$ be elements of $\boxplus_{\underline{\varrho}}$ such that $\underline{x} + n_i\,\underline{\mathfrak{e}}_i + n_j\,\underline{\mathfrak{e}}_j \in \boxplus_{\underline{\varrho}}$, $x_i = y_i$ and $x_j = y_j$. We apply $(*)_1^n$ to the pair $\big(\underline{x}, \underline{y}\big)$ in direction i and get

$$\mathbf{r}(\underline{x} + n_i\,\underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}) \geqslant \mathbf{r}\big(\underline{y} + n_i\,\underline{\mathfrak{e}}_i\big) - \mathbf{r}\big(\underline{y}\big).$$

The pair $\big(\underline{x} + n_i\,\underline{\mathfrak{e}}_i, \underline{y} + n_i\,\underline{\mathfrak{e}}_i\big)$ satisfies the hypotheses required for applying $(*)_1^n$ again, but this time in direction j. This yields

$$\mathbf{r}(\underline{x} + n_i\,\underline{\mathfrak{e}}_i + n_j\,\underline{\mathfrak{e}}_j) - \mathbf{r}(\underline{x} + n_i\,\underline{\mathfrak{e}}_i) \geqslant \mathbf{r}\big(\underline{y} + n_i\,\underline{\mathfrak{e}}_i + n_j\,\underline{\mathfrak{e}}_j\big) - \mathbf{r}\big(\underline{y} + n_i\,\underline{\mathfrak{e}}_i\big).$$

Summing up these two inequalities shows that $\mathbf{r}$ satisfies $(*)_2^n$. The same procedure inductively proves that $\mathbf{r}$ satisfies all $(*)_k^n$, i.e., $\mathbf{r}$ is submodular. □

Remark 2.6.5 (Discrete partial derivatives and transverse local convexity)**.** For $i \in \{1, \dots, m\}$, we can define the *discrete partial derivative of* $\mathbf{r}$ *in the direction* i as the function $\partial_i \mathbf{r}$ defined by

$$\partial_i \mathbf{r}(\underline{x}) \coloneqq \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x}), \qquad \forall \underline{x} \in \boxplus_{\underline{\varrho}} \text{ such that } x_i < r_i.$$

We notice that property $(*)_1^1$ is equivalent to the fact that for all $i \in \{1, \dots, m\}$ and for all $0 \leqslant t < r_i$, $\partial_i \mathbf{r}|_{L_t^i}$ is non-increasing. This is why $(*)_1^1$ may be alternatively called *transverse local concavity*. Submodularity is thus equivalent to transverse local concavity.

In other contexts, submodularity is sometimes referred to as the discrete analogue of concavity: see, for example, [Sch03, Theorem 44.1]. While this is fully relevant for a function $\mathbf{r}$ defined on the collection $\mathcal{P}(S)$ of all subsets of a given set S, that is, on the hypercube $\boxplus_1^m$, it is not exactly true for supermodular functions on $\boxplus_{\underline{\varrho}}$ for larger values of $r_1, \dots, r_m$. This is because the functions $\partial_i \mathbf{r}$ are non-decreasing only in directions different from i. For example, for the function $\mathbf{r}$ defined on $\boxplus_{(2,2)}$ by $\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$, we have $\mathbf{r}((2,1)) - \mathbf{r}((2,0)) \not\geqslant \mathbf{r}((2,2)) - \mathbf{r}((2,1))$, i.e., $\partial_2 \mathbf{r}$ is not non-increasing in direction 2. ◇

Remark 2.6.6 (Complexity)**.** To make notations easier, we assume in this remark that $\underline{\varrho} = (r, \dots, r)$. Theorem 2.6.2 provides a significant improvement for an algorithm checking whether a given integer-valued function on $\square_r^m$ is a rank function. Indeed, the naive number of operations necessary to check Properties (R1) and (R2) in Definition 2.2.1 is as follows.

- Property (R1) is checked in $\mathcal{O}(mr)$ operations.
- Property (R2) is checked in $\mathcal{O}(m \cdot (r+1)^m)$ operations (check that $\mathbf{r}$ does not increase when adding any unit vector to any point of $\square_r^m$).

Regarding the supermodularity property, whose complexity dominates, a naive check over all pairs of elements $\underline{a}, \underline{b} \in \square_r^m$ needs $\mathcal{O}\left((r+1)^{2m}\right)$ operations a priori, but Theorem 2.6.2 provides a way to reduce to only $\mathcal{O}\left(m^2 \cdot (r+1)^m\right)$ operations, which is a substantial improvement in most cases. ◇

2.7 Permutation arrays

The aim of this section is to study simple matricubes with ground set an actual hypercube ($r_1 = r_2 = \cdots = r_m = r$) of minimum possible rank r or $r+1$. In the representable case, this corresponds to a collection of m complete flags in a vector space of dimension $r+1$. Theorem 2.7.1 establishes a one-to-one correspondence between these matricubes and permutation arrays introduced by Eriksson–Linusson [EL00a, EL00b].

2.7.1 Permutation arrays

First, we recall some terminology from [EL00a]. Our presentation differs slightly from the original setting as our indexing of flags is by codimension while in their work, Eriksson and Linusson use an indexing by dimension. (Concretely, this amounts to having lower blocks in [EL00a, EL00b] replaced here by upper blocks.)

Let $r_1, \dots, r_m$ be m non-negative integers. An *m-dimensional dot array* P is an m-dimensional array of type $[r_1] \times \cdots \times [r_m]$ where some of the entries are dotted.

For a dot array P, and $\underline{x} \in \square_{\underline{\varrho}}$, we denote by $P[\underline{x}]$ the *upper principal subarray* of P, which consists of all $\underline{y}$ with $\underline{y} \geq \underline{x}$. It is naturally a dot array itself.

To be precise, for $P[\underline{x}]$ to become a dot array, we must coordinate-wise subtract the point $(x_1, \dots, x_m)$ to all its elements. In the following, we will use both parametrization conventions freely for the sake of convenience.

For a dot array P and $j \in \{1, \dots, m\}$, the *rank along the j-axis*, denoted by $\operatorname{rank}_j(P)$, is the total number of $0 \leqslant t \leqslant r_j$ such that there is at least one dot in some position whose j-th index is equal to t, i.e., there is at least one dot in the layer L_t^j of P. A dot array P is called *rankable* if we have $\operatorname{rank}_j(P) = \operatorname{rank}_i(P)$ for all $i, j \in \{1, \dots, m\}$. If P is rankable, then we call $\operatorname{rank}_j(P)$ the *rank* of P for any $j \in \{1, \dots, m\}$.

A dot array P is called *totally rankable* if every upper principal subarray of P is rankable.

We recall that in the terminology of [EL00a] and [EL00b], a position $\underline{x}$ is *redundant* if there exist dot positions $\underline{y}_1, \ldots, \underline{y}_m \neq \underline{x}$, for some $m \geqslant 2$, such that each $\underline{y}_i$ has at least one coordinate in common with $\underline{x}$, and such that $\underline{x} = \bigwedge_{i=1}^{m} \underline{y}_i$. The set of redundant positions of P is denoted by $R(P)$. A *redundant dot* is a redundant position that is dotted. The reason for the term "redundant" is that placing or removing a redundant dot does not change the rank of any upper principal subarray of P. (In the language of lattices, a non-redundant position is meet-irreducible in the set of dotted positions.)

If A is a subset of $\boxplus_{\underline{\varrho}}$, then $P \cup A$ (resp. $P \smallsetminus A$) denotes the dot array based on P where, for every $\underline{x} \in A$, we dot (resp. undot) the position $\underline{x}$ in P, if necessary.

A *permutation array* of width r and dimension m is a totally rankable dot array P of shape $\boxplus_r^m = \boxplus_{\underline{\varrho}} = [r]^m$, $\underline{\varrho} = (r, \ldots, r)$, of rank $r+1$, and with no redundant dots.

2.7.2 Equivalence of permutation arrays with simple matricubes of rank r or $r+1$ on $\boxplus_r^m$

Our next theorem establishes an equivalence between permutation arrays and simple matricubes of rank r or $r+1$ on the hypercube $\boxplus_r^m$.

Theorem 2.7.1. *Let P be a permutation array of width r and dimension m. The function $\mathbf{r}_P$ defined by $\mathbf{r}_P(\underline{a}) := r + 1 - \mathrm{rank}(P[\underline{a}])$ for every $\underline{a} \in \boxplus_r^m$ is the rank function of a simple matricube $\mathscr{M}_P$ with ground set $\boxplus_r^m$. This matricube is of rank r or $r+1$ depending on whether the position $\underline{\varrho}$ in $\boxplus_r^m$ is dotted or not. The set of flats of $\mathbf{r}_P$ is precisely the union of the set of dot positions in P with $R(P)$, and $\underline{\varrho}$.*

Conversely, the rank function $\mathbf{r}$ of every simple matricube $\mathscr{M}$ of rank r or $r+1$ on the hypercube $\boxplus_r^m$ defines a dot array $P_{\mathscr{M}}$ on $\boxplus_r^m = [r]^m$ with dots positioned on the set of flats $\underline{a} \neq \underline{\varrho}$ of $\mathscr{M}$, and also a dot positioned on $\underline{\varrho}$ if $\mathbf{r}(\mathscr{M}) = r$. Then, $P := P_{\mathscr{M}} \smallsetminus R(P_{\mathscr{M}})$ is a permutation array.

The proof of this theorem is given in the next section.

2.7.3 Proof of Theorem 2.7.1

We start by proving the first part of the theorem. Let P be a permutation array on $\boxplus_r^m = [r]^m$. We claim that the function

$$\rho_P(\underline{x}) := r + 1 - \mathrm{rank}(P[\underline{x}]), \qquad \forall \underline{x} \in \boxplus_r^m,$$

is the rank function of a simple matricube on the ground set $\boxplus_r^m$. We need to show properties (R1*)-(R2)-(R3).

Since $\underline{x} \preceq \underline{y}$ implies $P[\underline{x}] \supseteq P[\underline{y}]$, we deduce that $\mathbf{r}_P$ is non-decreasing, which shows (R2).

We now prove (R1). Let $i \in \{1, \ldots, m\}$. We have to show that $\mathbf{r}_P(t\,\underline{\mathfrak{e}}_i) = t$ for $t \in [r]$. By definition, $\mathrm{rank}(P[t\,\underline{\mathfrak{e}}_i]) \leqslant r + 1 - t$, which implies $\mathbf{r}_P(t\,\underline{\mathfrak{e}}_i) \geqslant t$. The reverse inequality

is shown by induction on t. The case $t = 0$ is true by the definition of permutation arrays, which requires $\mathrm{rank}(P) = r + 1$. Assuming that $\mathrm{rank}(P[t\,\underline{\mathfrak{e}}_i]) \geqslant r + 1 - t$, we show that $\mathrm{rank}(P[(t+1)\,\underline{\mathfrak{e}}_i]) \geqslant r - t$. This follows from the inequality $\mathrm{rank}(P[\underline{a} + \underline{\mathfrak{e}}_i]) = \mathrm{rank}_i(P[\underline{a} + \underline{\mathfrak{e}}_i]) \geqslant \mathrm{rank}_i(P[\underline{a}]) - 1 = \mathrm{rank}(P[\underline{a}]) - 1$, valid for every $\underline{a} \in \square_r^m$ such that $\underline{a} + \underline{\mathfrak{e}}_i \in \square_r^m$.

It remains to show that $\mathbf{r}_P$ is submodular. Thanks to Theorem 2.6.2, it is sufficient to show that $\mathbf{r}_P$ satisfies the diamond property for functions. We thus take two distinct integers $i \neq j \in \{1, \ldots, m\}$ and an element $\underline{x} \in \square_r^m$ such that $\underline{x} + \underline{\mathfrak{e}}_i, \underline{x} + \underline{\mathfrak{e}}_j \in \square_r^m$. We assume that $\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_i+\underline{\mathfrak{e}}_j)-\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_j) = 1$ and show that $\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_i)-\mathbf{r}(\underline{x}) = 1$. The hypothesis implies that the layer $L^i_{x_i}$ in the dot array $P[\underline{x} + \underline{\mathfrak{e}}_j]$ contains a dotted point. This point will be counted in the difference $\mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i) - \mathbf{r}(\underline{x})$, which proves the result.

The matricube $\mathscr{M}_P$ is of rank r or $r+1$ depending on whether $\mathrm{rank}\big(P\big[\underline{\varrho}\big]\big) = 1$ or 0, that is, whether $\underline{\varrho}$ is dotted or not.

Finally, to see the statement about the flats, consider $\underline{x} \neq \underline{\varrho}$ and assume first that $\underline{x}$ is dotted. Then, for each direction $\underline{\mathfrak{e}}_i$ with $\underline{x}+\underline{\mathfrak{e}}_i \in \square_r^m$, we get $\mathrm{rank}_i(P[\underline{x}])-\mathrm{rank}_i(P[\underline{x}+\underline{\mathfrak{e}}_i]) = 1$. This shows that $\underline{x}$ is a flat.

Next, assume that $\underline{x}$ is not dotted. Since flats are closed under meet, if $\underline{x}$ is a redundant position, then it is a flat. It remains to consider the case where $\underline{x}$ is neither dotted nor a redundant point. This means there is an $i \in \{1, \ldots, m\}$ such that the layer $L^i_{x_i}$ in $P[\underline{x}]$ does not contain any dot. Two cases can happen:

- If $\underline{x} + \underline{\mathfrak{e}}_i \in \square_r^m$, then
$$\mathrm{rank}(P[\underline{x}]) = \mathrm{rank}_i(P[\underline{x}]) = \mathrm{rank}_i(P[\underline{x} + \underline{\mathfrak{e}}_i]) = \mathrm{rank}(P[\underline{x} + \underline{\mathfrak{e}}_i]),$$
and thus $\mathbf{r}(\underline{x}) = \mathbf{r}(\underline{x} + \underline{\mathfrak{e}}_i)$, and $\underline{x}$ is not a flat of $\mathscr{M}_P$.
- Otherwise, $x_i = r$, and so $\mathrm{rank}_i(P[\underline{x}]) = 0$, that is, $\mathbf{r}(\underline{x}) = r + 1$. This implies that $\mathscr{M}$ is of rank $r + 1$, and since $\underline{x} \neq \underline{\varrho}$, then, again $\underline{x}$ is not a flat.

This finishes the proof of the first direction.

We now show the other direction. Suppose that $\mathscr{M}$ is a simple matricube of rank $r + 1$ or r on $\square_r^m$ with rank function $\mathbf{r}$. Let $P_{\mathscr{M}}$ be the corresponding dot array where a dot is positioned on every flat $\underline{a}$ of $\mathscr{M}$ different from $\underline{\varrho}$, and if the rank of $\mathscr{M}$ is r, then a dot is also positioned on $\underline{\varrho}$. Let $P = P_{\mathscr{M}} \smallsetminus R(P_{\mathscr{M}})$. We show that P is a permutation array.

By construction, P has no redundant dots. We thus need to show that P is totally rankable and has rank $r + 1$. We have to prove that for every $\underline{x} \in P_{\mathscr{M}}$ and $i, j \in \{1, \ldots, m\}$, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_j(P_{\mathscr{M}}[\underline{x}])$. This is a direct consequence of Proposition 2.7.2 below, which also shows that the rank of $P_{\mathscr{M}}$ is $r + 1$. We conclude that P is a permutation array. □

Proposition 2.7.2. *Suppose that $\mathscr{M}$ is a simple matricube of rank r or $r+1$ on the ground set $\square_r^m$ and denote by $\mathbf{r}$ its rank function. Let $P_{\mathscr{M}}$ be the corresponding dot array with a dot positioned at each flat $\underline{a} \neq \underline{\varrho}$, and also a dot positioned at $\underline{\varrho}$ in the case $\mathbf{r}(\mathscr{M}) = r$.*

Let $\underline{x}$ be an element of the dot array $P_{\mathscr{M}}$ and $1 \leqslant i \leqslant m$. Then, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$.

Proof. We proceed by reverse induction in the lattice $\boxplus_r^m$, starting from $\underline{\varrho}$. For $\underline{\varrho}$, we have $\mathrm{rank}_i\big(P_{\mathscr{M}}[\underline{\varrho}]\big) = 0$ or 1 depending on whether $\mathbf{r}(\mathscr{M}) = r+1$ or r, respectively, for each $i \in \{1,\dots,m\}$, as required. Assume $\underline{x} \neq \underline{\varrho}$. We suppose the following equalities hold:

$$\mathrm{rank}_i\big(P_{\mathscr{M}}[\underline{y}]\big) = r+1-\mathbf{r}\big(\underline{y}\big) \quad \forall\ \underline{y} > \underline{x} \qquad \text{and} \qquad \forall\ i \in \{1,\dots,m\}.$$

We prove that the equalities hold as well for $\underline{x}$.

Suppose first that $\underline{x}$ is a flat. Two cases can occur.

(I.1) If $\underline{x}$ has rank r, then for each $i \in \{1,\dots,m\}$ with $\underline{x}+\underline{\mathfrak{e}}_i \in \boxplus_r^m$, we have $\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_i) = r+1$. By induction, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+\underline{\mathfrak{e}}_i]) = r+1-\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_i) = 0$. It follows that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+\underline{\mathfrak{e}}_i])+1 = 1$, as required. For the other values of i, we have $\underline{x}+\underline{\mathfrak{e}}_i \notin \boxplus_r^m$, that is, $x_i = r$, and, in this case, we have as well $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1$.

(I.2) If $\mathbf{r}(\underline{x}) < r$, then using the inequality $x_i = \mathbf{r}(x_i\,\underline{\mathfrak{e}}_i) \leqslant \mathbf{r}(\underline{x})$, we get $x_i < r$. This implies that $\underline{x}+\underline{\mathfrak{e}}_i \in \boxplus_r^m$ for all $i \in \{1,\dots,\delta\}$. We get

$$\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}+\underline{\mathfrak{e}}_i])+1 = r+1-\mathbf{r}(\underline{x}+\underline{\mathfrak{e}}_i)+1 = r+1-\mathbf{r}(\underline{x}),$$

as required.

Now suppose that $\underline{x}$ is not a flat of $\mathscr{M}$. Again, two cases can occur.

(II.1) If $x_i < r$ for all $i \in \{1,\dots,m\}$, then let $\underline{a}$ be the minimum flat with $\underline{a} \geq \underline{x}$ and let $i \in \{1,\dots,m\}$. Applying the induction hypothesis to $\underline{y} = \underline{x}+\underline{\mathfrak{e}}_i$, we get $\mathrm{rank}_i\big(P_{\mathscr{M}}[\underline{y}]\big) = r+1-\mathbf{r}\big(\underline{y}\big)$. By Lemma 2.3.5, we deduce that

$$\mathbf{r}\big(\underline{y}\big) = \begin{cases} \mathbf{r}(\underline{x})+1 & \text{if } a_i = x_i, \\ \mathbf{r}(\underline{x}) & \text{if } a_i > x_i. \end{cases}$$

In the first case, when $a_i = x_i$, there is a dot in the layer $L^i_{x_i}$ of $P_{\mathscr{M}}[\underline{x}]$, and thus $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}\big(P_{\mathscr{M}}[\underline{y}]\big)+1$. In the second case, when $a_i > x_i$, there is no dot in the layer $L^i_{x_i}$ of $P_{\mathscr{M}}[\underline{x}]$, and therefore $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = \mathrm{rank}\big(P_{\mathscr{M}}[\underline{y}]\big)$. We infer that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$, as required.

(II.2) We now treat the remaining case where $x_i = r$ for some indices i among $1,\dots,m$. In this case, $\mathbf{r}(\underline{x})$ is either r or $r+1$. We treat each of these possibilities separately.

(1) Firstly, suppose that $\mathbf{r}(\underline{x}) = r$. Consider an index i with $x_i = r$. Then, if $\mathbf{r}(\mathscr{M}) = r$, there is a dot positioned at $\underline{\varrho}$, and thus $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1 = r+1-\mathbf{r}(\underline{x})$. If $\mathbf{r}(\mathscr{M}) = r+1$, the minimum flat $\underline{a} \geq \underline{x}$ has rank r, and lives in the layer L^i_r of $P_{\mathscr{M}}[\underline{x}]$. Again, we get $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 1$.

Now consider an index j with $x_j < r$, so that $\underline{x}+\underline{\mathfrak{e}}_j \in \boxplus_r^m$. Let $\underline{a}$ be the minimum flat dominating $\underline{x}$. A reasoning similar to (II.1), based on the use of Lemma 2.3.5, shows that $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = r+1-\mathbf{r}(\underline{x})$, as required.

(2) Secondly, suppose that $\mathbf{r}(\underline{x}) = r+1$. The unique flat $\underline{a}$ that dominates $\underline{x}$ is $\underline{\varrho}$, which is not dotted. Therefore, $\mathrm{rank}_i(P_{\mathscr{M}}[\underline{x}]) = 0 = r+1-\mathbf{r}(\underline{x})$, as required. □

2.8 Local matroids

In this section, we define local matroids of matricubes, and formulate a local obstruction to their representability. We then turn this into an equivalent characterization of matricubes.

2.8.1 Local matroids of a matricube

In the following, for all $\underline{a} \in \square_{\underline{\varrho}}$, we define $I_{\underline{a}}$ as the set of all $i \in \{1, \dots, m\}$ such that $\underline{a} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$.

To motivate the definition, first consider a representable matricube, given by m initial flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_m^\bullet$ of length $r_1, \dots, r_m$, respectively, in a κ-vector space H. Let $\mathrm{F}^{\underline{a}} := \bigcap_{1 \leqslant i \leqslant m} \mathrm{F}_i^{a_i}$. The arrangement of subspaces $\mathrm{F}^{\underline{a}+\underline{\mathfrak{e}}_i}$ in $\mathrm{F}^{\underline{a}}$ given by $i \in I_{\underline{a}}$ defines a matroid $\mathrm{M}_{\underline{a}}$ on the ground set $I_{\underline{a}}$. The rank one elements of this matroid correspond to those i with $\mathrm{F}^{\underline{a}+\underline{\mathfrak{e}}_i}$ a proper vector subspace of $\mathrm{F}^{\underline{a}}$; all the other elements are loops.

This picture generalizes to any matricube, and defines local matroids associated to elements of the hypercuboid $\square_{\underline{\varrho}}$. Let $\mathbf{r}$ be a rank function on $\square_{\underline{\varrho}}$. Let $\underline{a} \in \square_{\underline{\varrho}}$, and define a function $\rho_{\underline{a}} \colon 2^{I_{\underline{a}}} \to \mathbb{Z}_{\geqslant 0}$ as follows. For every subset $X \subseteq I_{\underline{a}}$, set

$$\rho_{\underline{a}}(X) := \mathbf{r}\Big(\underline{a} + \sum_{i \in X} \underline{\mathfrak{e}}_i\Big) - \mathbf{r}(\underline{a}).$$

Proposition 2.8.1. *The pair $(I_{\underline{a}}, \rho_{\underline{a}})$ defines a matroid $\mathrm{M}_{\underline{a}}$ on the set of elements $I_{\underline{a}}$.*

Proof. By Proposition 2.2.4, $\rho_{\underline{a}}$ takes values in the set $\{0, \dots, |I_{\underline{a}}|\}$, and $\rho_{\underline{a}}(X) \leqslant |X|$. Since $\mathbf{r}$ is non-decreasing, we also have $\rho_{\underline{a}}(Y) \leqslant \rho_{\underline{a}}(X)$ for $Y \subseteq X$. Therefore, it is enough to show that ρ_a is submodular, that is,

$$\forall\, X, Y \subseteq I_{\underline{a}} \quad \rho_{\underline{a}}(X) + \rho_{\underline{a}}(Y) \geqslant \rho_{\underline{a}}(X \cup Y) + \rho_{\underline{a}}(X \cap Y).$$

This follows from the submodularity of $\mathbf{r}$ applied to $\underline{a} + \sum_{i \in X} \underline{\mathfrak{e}}_i$ and $\underline{a} + \sum_{i \in Y} \underline{\mathfrak{e}}_i$ in $\square_{\underline{\varrho}}$. □

Proposition 2.8.2. *A necessary condition for the representability of a matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$ on a field κ is the representability of all the matroids $\mathrm{M}_{\underline{a}}$, $\underline{a} \in \square_{\underline{\varrho}}$, on κ.*

Proof. This follows directly from the above discussions. □

2.8.2 The case of permutation arrays

Let κ be a field. By Theorem 2.7.1, the representability of a simple matricube $\mathscr{M}$ of rank r or $r+1$ with the ground set the hypercube $\square_r^m$ is equivalent to the representability of the corresponding permutation array in the terminology of [EL00b]. Billey and Vakil [BV08] provide several examples of permutation arrays which are non-representable. Proposition 2.8.2 above provides a conceptual explanation of the examples treated in [BV08]. Theorem 2.9.1, combined with Theorem 2.7.1, shows that over an infinite field, the representability for permutation arrays reduce to the representability of matroids.

2.8.3 Coherent complexes of matroids and matricubes

We show that the data of a matricube on the ground set $\square_{\underline{\varrho}}$ is equivalent to the data of a set of matroids indexed by $\square_{\underline{\varrho}}$ satisfying compatibility properties (CC1) and (CC2) listed below.

Notation as in the previous section, for the sake of convenience, if $i \in I_{\underline{a}}$, we write $\rho_{\underline{a}}(i)$ instead of $\rho_{\underline{a}}(\{i\})$. We start with the definitions below.

Definition 2.8.3 (Increasing path)**.** Let $\underline{a}$ and $\underline{b}$ be points of $\square_{\underline{\varrho}}$ with $\underline{a} \preccurlyeq \underline{b}$. We define an *increasing path* from $\underline{a}$ to $\underline{b}$ to be any finite sequence

$$\underline{a} = \underline{c}_0, \underline{c}_1, \ldots, \underline{c}_k = \underline{b}$$

such that for every $0 \leqslant j < k$, we have $\underline{c}_{j+1} = \underline{c}_j + \underline{\mathfrak{e}}_\ell$ for some $\ell \in \{1, \ldots, m\}$. ◇

Note that the integer k is equal to $\sum_{i=1}^{m}(b_i - a_i)$.

Definition 2.8.4 (Coherent complex of matroids)**.** Let $(\mathrm{M}_{\underline{a}})_{\underline{a} \in \square_{\underline{\varrho}}}$ be a set of matroids indexed by $\square_{\underline{\varrho}}$, with $\mathrm{M}_{\underline{a}}$ a matroid on the set $I_{\underline{a}}$ and with rank function $\mathbf{r}_{\underline{a}}$. We say $(\mathrm{M}_{\underline{a}})$ form a *coherent complex of matroids* if the following two properties are satisfied:

(CC1) For all $i \in \{1, \ldots, m\}$ and $0 \leqslant t < r_i$, we have $\rho_{t\,\underline{\mathfrak{e}}_i}(i) \leqslant 1$.

(CC2) The matroids satisfy the following relation.

$$\mathrm{M}_{\underline{a}+\underline{\mathfrak{e}}_i} = \begin{cases} \mathrm{M}_{\underline{a}} \diagup i & \text{if } a_i = r_i - 1 \\ \mathrm{M}_{\underline{a}} \diagup i \sqcup \{i\} & \text{else} \end{cases}.$$

Here, $\mathrm{M} \diagup e$ denotes the contraction of a matroid M by its element e, and i is the element of the matroid set corresponding to the direction i. Moreover, $\mathrm{M}_{\underline{a}} \diagup i \sqcup \{i\}$ denotes an extension of $\mathrm{M}_{\underline{a}} \diagup i$ by a single element denoted i. ◇

In the following, we denote by $\rho_{\underline{a}\diagup i}$ the rank function on $I_{\underline{a}} \smallsetminus \{i\}$ that defines the matroid $\mathrm{M}_{\underline{a}} \diagup i$. It is explicitly given by the following equation, in terms of the rank function $\rho_{\underline{a}}$:

$$\rho_{\underline{a}\diagup i}(X) = \rho_{\underline{a}}(X \cup \{i\}) - \rho_{\underline{a}}(i) \qquad \text{for all } X \subseteq I_{\underline{a}} \smallsetminus \{i\}.$$

Remark 2.8.5. Property (CC2) above implies the following: let $\underline{x} \preccurlyeq \underline{y}$ be two points of $\square_{\underline{\varrho}}$ and $i \in \{1, \ldots, m\}$ such that $x_i = y_i$ and $i \in I_{\underline{x}}$. Then, i being a loop in $\mathrm{M}_{\underline{x}}$ implies that i is a loop in $\mathrm{M}_{\underline{y}}$. Indeed, $\mathrm{M}_{\underline{y}}$ is obtained from $\mathrm{M}_{\underline{x}}$ through a sequence of operations consisting of either the contraction of an element different from i or an extension. These operations do not change the property of i being a loop. ◇

Theorem 2.8.6 (Matroidal characterization of matricubes)**.** *There is a one-to-one correspondence between coherent complexes of matroids indexed by the hypercuboid $\square_{\underline{\varrho}}$ and matricubes with ground set $\square_{\underline{\varrho}}$.*

Proof. ($\Longleftarrow$) If we start with a matricube $\mathscr{M}$ of rank function $\mathbf{r}$ on $\square_{\underline{\varrho}}$, the collection of matroids $\mathrm{M}_{\underline{a}}$ defined above forms a coherent complex of matroids. Indeed, Property (CC1) is trivially satisfied because of Property (R1) in Definition 2.2.1. We check Property (CC2). Let $\underline{a} \in \square_{\underline{\varrho}}$ and $i \in I_{\underline{a}}$. If $a_i = r_i - 1$, then $I_{\underline{a}+\underline{\mathfrak{e}}_i} = I_{\underline{a}} \smallsetminus \{i\}$, which is the ground set of the matroid $\mathrm{M}_{\underline{a}} \diagup i$. If $a_i < r_i - 1$, then $I_{\underline{a}+\underline{\mathfrak{e}}_i} = I_{\underline{a}}$, which is the ground set of the matroid $\mathrm{M}_{\underline{a}} \diagup i \sqcup \{i\}$. We now check the equality of the rank functions on subsets of $I_{\underline{a}} \smallsetminus \{i\}$. Consider $X \subseteq I_{\underline{a}+\underline{\mathfrak{e}}_i}$ not containing the element i. We need to show that

$$\mathbf{r}_{\underline{a}+\underline{\mathfrak{e}}_i}(X) = \mathbf{r}_{\underline{a}}(X \cup \{i\}) - \mathbf{r}_{\underline{a}}(i).$$

The left-hand side is by definition $\mathbf{r}\big(\underline{a} + \underline{\mathfrak{e}}_i + \sum_{j\in X} \underline{\mathfrak{e}}_j\big) - \mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i)$, and the right-hand side is

$$\mathbf{r}\Big(\underline{a} + \underline{\mathfrak{e}}_i + \sum_{j\in X} \underline{\mathfrak{e}}_j\Big) - \mathbf{r}(\underline{a}) - \mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) + \mathbf{r}(\underline{a}).$$

Both sides are therefore equal.

($\Longrightarrow$) The other way around, we consider a coherent complex of matroids $(\mathrm{M}_{\underline{a}})_{\underline{a}}$ and associate a matricube $\mathscr{M}$ on $\square_{\underline{\varrho}}$ by specifying its rank function $\mathbf{r}$. Let $\underline{a} \in \square_{\underline{\varrho}}$. We take any increasing path $\underline{0} = \underline{b}_0, \underline{b}_1, \dots, \underline{b}_k = \underline{a}$ from $\underline{0}$ to $\underline{a}$, and define

$$\mathbf{r}(\underline{a}) := \sum_{j=0}^{k-1} \rho_{\underline{b}_j}\big(\underline{b}_{j+1} - \underline{b}_j\big).$$

We first prove that $\mathbf{r}$ is well-defined, which amounts to showing that $\mathbf{r}(\underline{a})$ does not depend on the choice of the increasing path $\big(\underline{b}_j\big)$. Two different such paths can be linked by a finite sequence of increasing paths such that between two consecutive increasing paths in the sequence, the only change is an inversion between two consecutive elementary moves $\underline{\mathfrak{e}}_i$ and $\underline{\mathfrak{e}}_j$, $i \neq j$. We thus have to check that, for every $\underline{a} \in \square_{\underline{\varrho}}$ and $i, j \in I_{\underline{a}}$ with $i \neq j$, we have

$$\rho_{\underline{a}}(\underline{\mathfrak{e}}_i) + \rho_{\underline{a}+\underline{\mathfrak{e}}_i}(\underline{\mathfrak{e}}_j) = \rho_{\underline{a}}(\underline{\mathfrak{e}}_j) + \rho_{\underline{a}+\underline{\mathfrak{e}}_j}(\underline{\mathfrak{e}}_i).$$

But by (CC2), we have $\rho_{\underline{a}+\underline{\mathfrak{e}}_i}(\underline{\mathfrak{e}}_j) = \rho_{\underline{a}\diagup i}(\underline{\mathfrak{e}}_j) = \rho_{\underline{a}}(\underline{\mathfrak{e}}_j + \underline{\mathfrak{e}}_i) - \rho_{\underline{a}}(\underline{\mathfrak{e}}_i)$ in the left-hand part and $\rho_{\underline{a}+\underline{\mathfrak{e}}_j}(\underline{\mathfrak{e}}_i) = \rho_{\underline{a}\diagup j}(\underline{\mathfrak{e}}_i) = \rho_{\underline{a}}(\underline{\mathfrak{e}}_i + \underline{\mathfrak{e}}_j) - \rho_{\underline{a}}(\underline{\mathfrak{e}}_j)$ in the right-hand part, so the desired equality holds.

We now check (R1)-(R2)-(R3). It is obvious by construction that $\mathbf{r}$ takes integer values, is non-decreasing and that $\mathbf{r}(t\,\underline{\mathfrak{e}}_i) - \mathbf{r}((t-1)\,\underline{\mathfrak{e}}_i) = 0$ or 1 for all $i \in \{1, \dots, m\}$ and $0 \leqslant t \leqslant r_i$. These imply (R1) and (R2).

It remains to show that $\mathbf{r}$ is submodular. By Theorem 2.6.2, it is sufficient to check the diamond property. We show unidirectional submodularity at distance one stated in (2.6). Let thus $i \in \{1, \dots, m\}$ and $\underline{x}, \underline{y} \in \square_{\underline{\varrho}}$ such that $\underline{x} \preceq \underline{y}$, $i \in I_{\underline{x}}$ and $x_i = y_i$. We assume that $\mathbf{r}_{\underline{x}}(\underline{\mathfrak{e}}_i) = 0$ and show that $\mathbf{r}_{\underline{y}}(\underline{\mathfrak{e}}_i) = 0$. But this has been shown to be the case in Remark 2.8.5.

We have defined two maps linking coherent complexes of matroids and matricubes. It is straightforward to check that they are inverse of each other. □

2.9 Further discussions

In this final section, we discuss further related results and questions.

2.9.1 Bases of matricubes and special features of independents

We do not know how to define a good notion of bases for matricubes. We review some natural attempts in this section. For each definition, we show with an example that the data of the set of bases according to that definition does not determine the matricube in a unique way. Below, $\mathscr{M}$ refers to a matricube of rank r on the ground set $\square_{\underline{\varrho}}$ and $\underline{a}$ is an element of $\square_{\underline{\varrho}}$.

First, consider the idea closest to that of matroids.

(a) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is maximal for the partial order $\preceq$.*

This does not carry enough information. The two matricubes below, with $\underline{\varrho} = (2, 2)$, have the same set of bases according to Definition (a), but not the same sets of independents. The elements verifying (a) are highlighted in red, the other independents in blue.

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \tag{2.7}$$

We note that, unlike matroids, a maximal independent of a matricube is not necessarily of maximal rank. This is not visible in Example (2.7), but the following matricube provides such an example. Two maximal independents (in red) have distinct ranks.

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \tag{2.8}$$

Consider the following alternative to (a).

(b) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ of maximal rank r.*

The same examples given in (2.7) show that this does not work neither.

Definitions (a) and (b) are global. Seeking for local counterparts, similar to flats, circuits and independents, treated in the previous sections, leads to the following candidates.

(c) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is locally maximal, in the sense that for every $i \in \{1, \ldots, m\}$ with $\underline{a} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, we have $\underline{a} + \underline{\mathfrak{e}}_i \notin \mathscr{I}$.*

(d) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is locally of maximal rank, in the sense that for every $i \in \{1, \ldots, m\}$ with $\underline{a} + \underline{\mathfrak{e}}_i \in \square_{\underline{\varrho}}$, we have $\mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a})$.*

It is immediate that (d) implies (c). Definitions (c) and (d) are in fact equivalent (the proof is omitted). The two matricubes below, with $\underline{\varrho} = (5, 4)$, have the same set of bases according to Definition (c), but not the same sets of independents. The elements verifying (c) are depicted in red, and the other independents are in blue.

$$\begin{pmatrix} 4 & 4 & 4 & 4 & 5 & 6 \\ 3 & 3 & 4 & 4 & 5 & 6 \\ 2 & 2 & 3 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \qquad \begin{pmatrix} 4 & 4 & 5 & 5 & 5 & 6 \\ 3 & 4 & 5 & 5 & 5 & 6 \\ 2 & 3 & 4 & 4 & 4 & 5 \\ 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{2.9}$$

The removal operation defined in Section 2.5.2 takes an independent and produces smaller independents. Consider the following candidate.

(e) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ which is not the removal $\underline{b} \smallsetminus i$ for some $\underline{b} \in \mathscr{I}$ and $i \in \{1, \dots, m\}$.*

This is genuinely new, but, if we consider the matricubes in Example (2.7), the bases given by (e) and (a) are the same.

In a matroid with ground set E and rank function $\mathbf{r}$, every subset $S \subseteq E$ satisfies $\mathbf{r}(S) + \mathbf{r}^*(S^c) \leqslant |E|$, where $\mathbf{r}^*$ is the dual rank function and S^c is the complement of S, with equality if, and only if, S is a basis. This leads to the following candidate.

(f) *A basis of $\mathscr{M}$ is an independent $\underline{a} \in \mathscr{I}$ such that $\mathbf{r}(\underline{a}) + \mathbf{r}^*(\underline{a}^c) = \left|\underline{\varrho}\right|_{\ell_1}$, where $\underline{a}^c = \underline{\varrho} - \underline{a}$ is the complement of $\underline{a}$ and $\left|\underline{\varrho}\right|_{\ell_1} = \sum_i r_i$.*

The inequality $\mathbf{r}(\underline{x}) + \mathbf{r}^*(\underline{x}^c) \leqslant \left|\underline{\varrho}\right|_{\ell_1}$ does hold for every element $\underline{x} \in \square_{\underline{\varrho}}$. However, some matricubes have no bases at all according to this definition. This is for instance the case for both matricubes in Example (2.9).

The question of finding a good notion of bases in matricubes therefore remains open.

2.9.2 The natural polymatroid and the natural matroid associated to a matricube

We refer to [HH02] for the definition and basic properties of polymatroids. Let P be an integer polymatroid on the ground set E with rank function $\rho\colon 2^E \to \mathbb{Z}_{\geqslant 0}$. Replace each element e of E with $\rho(e)$ elements, and let $\widehat{E}$ be the resulting set. For each subset $S \subseteq E$, let $\widehat{S} \subseteq \widehat{E}$ be the union of all the elements associated to each $e \in S$. Define $\widehat{\rho}\colon 2^{\widehat{E}} \to \mathbb{Z}_{\geqslant 0}$ by the formula

$$\widehat{\rho}(Y) := \min_{S \subseteq E} \Big(\rho(S) + |Y \smallsetminus \widehat{S}| \Big).$$

This defines a matroid on the ground set $\widehat{E}$, called the *natural matroid* of P, which is symmetric with respect to the permutation of the $\rho(e)$ elements associated to each e.

To a given matricube $\mathscr{M}$ on the ground set $\square_{\underline{\varrho}}$, we can associate an integer polymatroid P on the ground set $E \coloneqq [r_1] \sqcup \cdots \sqcup [r_m]$, that we name the *natural polymatroid* of $\mathscr{M}$. The rank function ρ of P is given by associating to any subset $S \subseteq E$ the integer value $\rho(S) \coloneqq \mathbf{r}\big(\bigvee_{\underline{a} \in S} \underline{a}\big)$. The join is taken in $\square_{\underline{\varrho}}$ and $\mathbf{r}$ is the rank function of $\mathscr{M}$.

There is, moreover, a natural way to send elements of $\mathscr{M}$ to subsets of E, by mapping every $\underline{x} = (x_1, \ldots, x_m) \in \square_{\underline{\varrho}}$ to the subset $\psi(x) \coloneqq [0, x_1] \sqcup \cdots \sqcup [0, x_m] \subseteq E$, where each interval $[0, x_i]$ is taken inside $[0, r_i]$. Note that the polymatroid P can be viewed as a polytope in the vector space $\mathbb{R}^E$, but there does not seem to be a natural way of associating vectors in this polytope to elements of the matricube.

Proceeding as above, we can thus associate to a given matricube $\mathscr{M}$ a natural matroid M on the ground set $\widehat{E}$. Note that M has $\sum_{i=1}^{m} \sum_{t=1}^{r_i} \mathbf{r}(t\, \underline{\mathfrak{e}}_i)$ elements.

Using this construction, it seems natural to transfer the notions of flats, independents, bases, and circuits from the matroid M to the matricube, in the spirit of the work [BCF23] on integer polymatroids. This however gives a story complementary to the theory exposed in this manuscript.

In the case of flats, for example, our definition coincides with the definition of flats in the corresponding polymatroid, in the sense that the map ψ described above establishes a bijection between the flats of $\mathscr{M}$ and the flats of P. However, we are not aware of any intrinsic axiomatic system for flats in polymatroids, and the one for flats in matricubes given in the present manuscript does not seem to be directly related to the one for cyclic flats in polymatroids, due to Csirmaz [Csi20]; see as well [BCF23, Section 5].

When it comes to independents of matricubes, our definition differs entirely from that of independents in a polymatroid [BCF23, Section 3]. As we observed in the previous section, maximal independents in matricubes can be of various ranks, whereas maximal independents in polymatroids all have the same rank. The independents in polymatroids are only defined as vectors in the corresponding polytope, not set-theoretically. Besides, as we mentioned previously, we do not have yet a good notion of bases for matricubes.

The same situation holds for circuits: the definition and axiomatic system we give in the present manuscript differ from the ones given for polymatroids in [BCF23, Section 4]. Like independents, circuits in polymatroids can only be defined as vectors. Our definition of circuits does not rely on independents, and yields a different story. Note in particular that in matricubes, we can have comparable circuits for the partial order $\preceq$ (see Example (2.4) in Section 2.4.1), whereas two distinct circuits in a polymatroid are never comparable.

2.9.3 Representability and minors

We do not know whether the local obstructions given by Proposition 2.8.2 are the only obstructions for the representability of a matricube. On the other hand, the representability of a matricube over an infinite field is equivalent to the representability of the corresponding natural matroid, as we show in the present section.

Theorem 2.9.1. *A matricube $\mathscr{M}$ is representable over an infinite field κ if, and only if, the corresponding natural matroid* M *is representable over κ.*

We will deduce this from the observation, firstly, that the representability of a matricube by a flag arrangement over *any* field is equivalent to the representability of the associated natural polymatroid by a subspace arrangement over the same field. And secondly, we show that the representability of an integer polymatroid by a subspace arrangement over an *infinite* field is equivalent to the representability of the natural matroid associated to the polymatroid over that field. As the proof shows, the latter statement, as well as the theorem, remains valid for a *finite* field of large enough cardinality.

Proof. We first prove the equivalence between the representability of a matricube and the representability of the associated natural polymatroid. We take the dual point of view described in Section 2.2.2, using increasing initial flags. Let $\mathscr{M}$ be a representable matricube. The associated natural polymatroid P is obviously representable, remembering only the data of the subspace arrangement coming from the flag arrangement that represents $\mathscr{M}$. In the other direction, we assume that the natural polymatroid P associated to a matricube $\mathscr{M}$ is represented by a subspace arrangement $\left\{\mathrm{G}_j^i,\, i \in \{1,\dots,m\}, j \in [r_i]\right\}$. Let $i \in \{1,\dots,m\}$ and $j \in [r_i - 1]$. By construction of P, we have $\rho([j+1]) = \mathbf{r}((j+1)\,\underline{\mathfrak{e}}_i) \geqslant \mathbf{r}(j\,\underline{\mathfrak{e}}_i) = \rho([j])$, where $[j]$ and $[j+1]$ are included in the interval $[r_i]$ in the ground set of P. This shows that for every i and j, we have $\mathrm{G}_j^i \subseteq \mathrm{G}_{j+1}^i$, and therefore that the subspace arrangement can be arranged into a flag arrangement in a compatible way. It is immediate to see that this flag arrangement is a representation of $\mathscr{M}$.

We now prove the equivalence between the representability of an integer polymatroid (whether or not associated to a matricube) and the representability of its natural matroid, using vector representations. Let first M be a matroid on a ground set $\widehat{E}$, with rank function $\widehat{\rho}$, and consider a partition $\widehat{E} = \bigsqcup_{e\in E} A_e$ of $\widehat{E}$, indexed by a set E. We assume that M is represented by a configuration of vectors $\left\{v_x \in \mathrm{H},\, x \in \widehat{E}\right\}$ in a κ-vector space H. We define, for every $S \subseteq E$, $\rho(S) \coloneqq \widehat{\rho}(\widehat{S}) = \widehat{\rho}(\bigcup_{e\in S} A_e)$. It is easy to see that ρ is submodular on 2^E, and therefore it is the rank function of a polymatroid $\mathrm{P}(\mathrm{M})$ on the ground set E. Moreover, $\mathrm{P}(\mathrm{M})$ is represented by the subspace arrangement $\{\mathrm{G}_e \subseteq \mathrm{H},\, e \in E\}$, where, for every $e \in E$, $\mathrm{G}_e \coloneqq \langle v_x,\, x \in A_e\rangle$ is the vector subspace generated by the v_x, $x \in A_e$. Finally, if M is the natural matroid of some integer polymatroid P, then $\mathrm{P}(\mathrm{M})$ is in fact the polymatroid P, which concludes.

In the other direction, we use the notation of Section 2.9.2. Let P be an integer polymatroid on a ground set E, represented by a subspace arrangement $\{\mathrm{G}_e \subseteq \mathrm{H},\, e \in E\}$. Let M be the natural matroid of P, on the ground set $\widehat{E}$. For every $e \in E$, let $\mathcal{B}_e$ be a generic vector basis of the subspace G_e, i.e., let $\mathcal{B}_e$ be chosen in a Zariski dense open subset (to be specified afterwards) of the variety of bases of G_e. The (disjoint) union $\mathcal{B}$ of the bases $\mathcal{B}_e$ is indexed by $\widehat{E}$ in the natural way, say $\mathcal{B} = \left\{v_x,\, x \in \widehat{E}\right\}$. Let $\bar{\rho}$ be the rank function on $\widehat{E}$ defined by $\bar{\rho}(Y) \coloneqq \dim_\kappa\langle v_x,\, x \in Y\rangle$ for every $Y \subseteq \widehat{E}$. We show the natural matroid M

to be representable by proving that $\bar{\rho} = \widehat{\rho}$. We therefore fix any $Y \subseteq \widehat{E}$. For every $S \subseteq E$, the inequality $\bar{\rho}(Y) \leqslant \rho(S) + |Y \smallsetminus \widehat{S}|$ is immediate. In the rest of the proof, we show the reverse inequality.

Let $S \subseteq E$ be the set of all $e \in E$ such that $\mathrm{G}_e \subseteq \langle v_x,\, x \in Y\rangle$. Let G_0 be the subspace of H defined by $\mathrm{G}_0 \coloneqq \sum_{e\in S} \mathrm{G}_e$, and let $\mathrm{G}_1, \ldots, \mathrm{G}_k$ be the other subspaces G_e for $e \in E \smallsetminus S$. Here, $k = |E| - |S|$. Rearrange the ground set $\widehat{E}$ so that the first k disjoint intervals $[r_1], \ldots, [r_k]$ that make it up are those indexed by $E \smallsetminus S$. For every $j \in \{1, \ldots, k\}$, let Y_j be the family of vectors v_x with $x \in Y \cap [r_j]$ in the ground set $\widehat{E}$ of M. We then admit, for now, that for every $j \in \{1, \ldots, k\}$, the following inequality holds:

$$\dim(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_j) > \dim(\mathrm{G}_0) + |Y_1| + \cdots + |Y_j|. \tag{2.10}$$

We show by induction that if Inequality (2.10) is true for every $j \in \{1, \ldots, k\}$, then, for every $j \in \{0, \ldots, k\}$, the configuration of vectors $Y_1 \sqcup \cdots \sqcup Y_j$ is linearly independent in the quotient space $\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_j\big)/\mathrm{G}_0$. This is obviously true for $j = 0$. If now this is true for some $j < k$, Inequality (2.10) for $j + 1$ reads

$$|Y_{j+1}| < \dim(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j+1}) - (\dim(\mathrm{G}_0) + |Y_1| + \cdots + |Y_j|).$$

The induction hypothesis then implies that the quantity on the right is the dimension of the quotient space

$$\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j+1}\big)/\big(\mathrm{G}_0 + \langle Y_1, \ldots, Y_j\rangle\big).$$

The linear projection map

$$\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_j + \mathrm{G}_{j+1}\big)/\mathrm{G}_0 \twoheadrightarrow \big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_j + \mathrm{G}_{j+1}\big)/\big(\mathrm{G}_0 + \langle Y_1, \ldots, Y_j\rangle\big)$$

now enables to view the family of vectors Y_{j+1} in the quotient space $\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j+1}\big)/\big(\mathrm{G}_0 + \langle Y_1, \ldots, Y_j\rangle\big)$. Since the number of vectors in Y_{j+1} is less than the dimension of this quotient space, and since all the vectors v_x are generic, then Y_{j+1} is linearly independent in this quotient space. This means exactly that $Y_1 \sqcup \cdots \sqcup Y_j \sqcup Y_{j+1}$ is linearly independent in the quotient space $\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_j + \mathrm{G}_{j+1}\big)/\mathrm{G}_0$, which concludes the induction. Specializing the independence property to $j = k$, we get the desired equality:

$$\bar{\rho}(Y) = \rho(S) + |Y \smallsetminus \widehat{S}|.$$

To finish the proof, before turning to Inequality (2.10), notice that the choice of the vectors v_x determines the subset $S \subseteq E$. But since there is a finite number of such S, there is still a non-empty Zariski open set of choices of vectors v_x for which the same set S is associated to all these choices. Now, we have the above equality for a fixed Y. Since there is a finite number of such subsets $Y \subseteq \widehat{E}$, there is still a non-empty Zariski open set of choices of the vectors v_x for which the above equality holds for *every* Y.

We now explain why Inequality (2.10) is true for every j. If by contradiction it was not, let $j_0 \geqslant 1$ be the smallest j such that it does not hold, i.e.,

$$|Y_{j_0}| \geqslant \dim(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j_0}/\mathrm{G}_0 + \langle Y_1, \ldots, Y_{j_0-1}\rangle).$$

The preceding argument shows that the vectors in Y_j for $j < j_0$ are independent in the quotient space $\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j_0-1}\big)/\mathrm{G}_0$. Since the vectors in Y_{j_0} are generic, the above inequality on the size of Y_{j_0} implies that they generate the quotient space $\big(\mathrm{G}_0 + \mathrm{G}_1 + \cdots + \mathrm{G}_{j_0}\big)/\big(\mathrm{G}_0 + \langle Y_1, \ldots, Y_{j_0-1}\rangle\big)$. This leads to the inclusion $\mathrm{G}_{j_0} \subseteq \langle v_x,\, x \in Y\rangle$, contradicting the definition of S. □

We call a matricube $\mathscr{M}$ *regular* if it is representable over every field. For matroids, Seymour's theorem describes regular matroids in terms of sums of graphic, cographic, and an exceptional regular matroid on 10 elements [Sey80]. A theorem by Tutte characterizes regular matroids as those representable over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ with two and three elements, respectively. Another result by Tutte characterizes regular matroids as those that contain as a minor neither the Fano matroid F_7 nor its dual F_7^*. We refer to [Tru92, Chap. 9] for a presentation of these results.

Question 2.9.2. *Provide a characterization of regular matricubes.*

In analogy with Rota's conjecture on the characterization of the representability of matroids over finite fields using a finite set of forbidden minors, we formulate the following question.

Question 2.9.3 (Rota's conjecture for matricubes)**.** *Let κ be a finite field. Does there exist a finite collection of matricubes such that a matricube is representable over κ if, and only if, it does not contain any of the matricubes in the collection as a minor?*

We note that as it was recently shown by Oxley, Semple, and Whittle [OSW16], the analogue of Rota's conjecture for 2-polymatroids fails in general. This does not exclude a positive answer to the above question, as matricubes behave more like matroids than polymatroids.

2.9.4 Stratification of products of flag varieties

Let κ be a field and n be a positive integer. Let H be a κ-vector space of dimension n. For each positive integer $r \leqslant n$, denote by $F(r,n)$ the flag variety parametrizing initial flags of vector subspaces $\mathrm{G}_1 \subset \mathrm{G}_2 \subset \cdots \subset \mathrm{G}_r$ of dimensions $\dim(\mathrm{G}_j) = j$, for $j = 1, \ldots, r$. Given a vector $\underline{\varrho} = (r_1, \ldots, r_m)$ with m positive integers, consider the product variety $F\big(\underline{\varrho}, n\big) := F(r_1, n) \times \cdots \times F(r_m, n)$. We get a natural stratification of $F\big(\underline{\varrho}, n\big)$ by matricubes as follows. Given a simple matricube $\mathscr{M}$ with ground set $\square_{\underline{\varrho}}$, the cell $\mathscr{Z}_{\mathscr{M}}$ parametrizes those collections of m flags $\mathrm{G}^i_\bullet \in F(r_i, n)$, $i = 1, \ldots, m$, whose associated matricube, through the constructions of Section 2.2.2, coincides with $\mathscr{M}$. This stratification is analogous to that induced by matroids for Grassmannians. Theorem 2.9.1 provides a correspondence between strata of given rank r in $F\big(\underline{\varrho}, n\big)$ and strata of the Grassmannian $\mathrm{Gr}(r, N)$, for $N = \frac{1}{2}\sum_{i=1}^m (r_i^2 + r_i)$, see Section 2.9.2. It would be interesting to study the combinatorics of this stratification, and the geometric meaning of this correspondence.

2.9.5 Polymatricubes

A polymatricube is a function $f\colon \square_{\underline{\varrho}} \to \mathbb{R}$ with $f(\underline{0}) = 0$ which is non-decreasing and submodular, that is, it verifies (R2) and (R3). Examples of *integer* polymatricubes are the representable ones which, by definition, are those associated to a collection of *arbitrary* (instead of initial) flags in a vector space. Generalizing the discussion of Section 2.9.2, we can associate a natural polymatroid and a natural matroid to any integer polymatricube. Theorem 2.9.1 extends to this setting: an integer polymatricube is representable over an infinite field if, and only if, the corresponding natural matroid is representable over the same field. In particular, the discussion of Section 2.9.4 can be extended to arbitrary collections of flag varieties.

2.9.6 Tutte polynomial

An important algebraic invariant associated to a matroid is its Tutte polynomial. This is a two-variable polynomial that specializes to the characteristic polynomial of the matroid. The Tutte polynomial of a matroid M on the ground set E is the unique polynomial $T_{\mathrm{M}}(X, Y)$ that verifies the following recursive equation for every $e \in E$:

$$T_{\mathrm{M}}(X,Y) = \begin{cases} X\, T_{\mathrm{M}\setminus e}(X,Y) & \text{if } e \text{ is a coloop} \\ Y\, T_{\mathrm{M}/e}(X,Y) & \text{if } e \text{ is a loop} \\ T_{\mathrm{M}\setminus e}(X,Y) + T_{\mathrm{M}/e}(X,Y) & \text{if } e \text{ is neither a loop nor a coloop,} \end{cases}$$

and is defined for the matroid $\varnothing$ with empty ground set by $T_{\varnothing} \equiv 1$.

We can define the notion of loop and coloop in matricubes. We say $i \in \{1, \dots, m\}$ is a *loop* of $\mathscr{M}$ if $r_i > 0$ and $\mathbf{r}(\underline{\mathfrak{e}}_i) = 0$. We say i is a *coloop* of $\mathscr{M}$ if i is a loop of the dual matricube $\mathscr{M}^*$. This is equivalent to having $\mathbf{r}(\mathscr{M} \setminus i) = \mathbf{r}(\mathscr{M}) - 1$. The recursive equation above, however, does not lead to an invariant of matricubes.

The Tutte polynomial of a matroid M on the ground set E can be defined directly by the following formula:

$$T_{\mathrm{M}}(X,Y) = \sum_{S \subseteq E} (X-1)^{r-\mathbf{r}(S)} (Y-1)^{|S|-\mathbf{r}(S)}.$$

This definition naturally extends to any matricube.

Definition 2.9.4. Let $\mathscr{M}$ be a matricube of rank r on the ground set $\square_{\underline{\varrho}}$. The *Tutte polynomial* of $\mathscr{M}$ is the two-variable polynomial

$$T_{\mathscr{M}}(X,Y) := \sum_{\underline{x} \in \square_{\underline{\varrho}}} (X-1)^{r-\mathbf{r}(\underline{x})} (Y-1)^{|\underline{x}|_{\ell_1} - \mathbf{r}(\underline{x})},$$

where $|\underline{x}|_{\ell_1}$ is the ℓ_1-norm of $\underline{x}$. ◇

Tutte polynomials of matricubes verify the following properties:

- Let $\mathscr{M}^*$ denote the dual of the matricube $\mathscr{M}$. Then, we have $T_{\mathscr{M}^*}(X,Y) = T_{\mathscr{M}}(Y,X)$.
- For two matricubes $\mathscr{M}$ and $\mathscr{M}'$, we have $T_{\mathscr{M}\oplus\mathscr{M}'} = T_{\mathscr{M}} \cdot T_{\mathscr{M}'}$.

Here, the direct sum $\mathscr{M} \oplus \mathscr{M}'$ of matricubes $\mathscr{M}$ and $\mathscr{M}'$ on hypercuboids $\square_{\underline{\varrho}}$ and $\square_{\underline{\varrho}}$, respectively, has ground set $\square_{\underline{\varrho}} \times \square_{\underline{\varrho}}$ and rank function defined by

$$\mathbf{r}_{\mathscr{M}\oplus\mathscr{M}'}(\underline{x} \oplus \underline{x}') = \mathbf{r}_{\mathscr{M}}(\underline{x}) + \mathbf{r}_{\mathscr{M}'}(\underline{x}').$$

Question 2.9.5. *Does there exist a recursive identity which defines $T_{\mathscr{M}}$, in terms of deletions and contractions?*

There is a version of the Tutte polynomial for polymatroids defined by Cameron and Fink [CF22]. This polynomial satisfies a relation involving elementary operations reminiscent of deletion and contraction in polymatroids, and specializes to the Tutte polynomial for matroids. We do not know if there is any relation between the Tutte polynomial of a matricube and the Tutte polynomial of the corresponding natural polymatroid.

2.9.7 Matricubes arising from linear series on curves

As pointed out in Section 2.1.8, matricubes naturally arise in our work on tropical degenerations of linear series on algebraic curves. We provide a brief hint to this by explaining how a finite collection of points and a finite dimensional space of rational functions on an algebraic curve gives rises to a matricube.

Let κ be an algebraically closed field, and let C be a smooth proper curve over κ. Let r be a non-negative integer, and let p be a κ-point on C. Let $\kappa(C)$ be the function field of C, and let $\mathrm{H} \subseteq \kappa(C)$ be a vector subspace of rational functions of dimension $r+1$ over κ. The point p leads to a complete flag $\mathrm{F}_p^\bullet$ of H by looking at the orders of vanishing at p of functions in H, as follows. Define the set $S_p \coloneqq \{\mathrm{ord}_p(f) \mid f \in \mathrm{H} \smallsetminus \{0\}\}$. The set S_p is finite of cardinality $r+1$. Denote by $s_0^p < \cdots < s_r^p$ the elements of S_p, enumerated in increasing order. The flag $\mathrm{F}_p^\bullet$ is defined by setting, for $j = 0, \ldots, r$,

$$\mathrm{F}_p^j \coloneqq \{f \in \mathrm{H} \smallsetminus \{0\} \mid \mathrm{ord}_p(f) \geqslant s_j^p\} \cup \{0\}.$$

It follows that F_p^j has codimension j in H.

Let now m be a natural number, and let $A = \{p_1, \ldots, p_m\}$ be a collection of m distinct κ-points on C. By the construction above, each point p_i leads to a complete flag $\mathrm{F}_i^\bullet$. Denoting $S_i \coloneqq \{\mathrm{ord}_{p_i}(f) \mid f \in \mathrm{H} \smallsetminus \{0\}\}$, and enumerating the elements of S_i in increasing order $s_0^i < \cdots < s_r^i$, the flag $\mathrm{F}_i^\bullet$ is defined by setting

$$\mathrm{F}_i^j \coloneqq \{f \in \mathrm{H} \smallsetminus \{0\} \mid \mathrm{ord}_{p_i}(f) \geqslant s_j^i\} \cup \{0\}.$$

The data of $C, \mathrm{H}, p_1, \ldots, p_m$ defines a matricube on the ground set $\square_r^m = [r]^m$.

We may call *geometric* a matricube with ground set $\square_r^m$ that arises from the above construction for a curve C over an algebraically closed field κ. By construction, geometric matricubes are all representable over the field κ over which the curve C is defined.

Question 2.9.6. *Is it true that all representable matricubes on* $\square_r^m$ *of rank* r *or* $r+1$ *are geometric? What is the smallest possible genus of a curve representing a geometric matricube?*

3 Limit linear series: combinatorial theory

This chapter is adapted from an extended version of the preprint [AG22].

Contents of the chapter

Abstract

We develop a purely combinatorial theory of limit linear series on metric graphs. This will be based on the theory of matricubes, presented in the previous chapter, and the formalism of slope structures. We show that some of these combinatorial objects are naturally obtained by tropicalizing linear series on algebraic curves. We provide a full classification of combinatorial limit linear series of rank one, parametrized by harmonic morphisms from the graph to metric trees. We also discuss some other applications and connections to other concepts in combinatorial algebraic geometry, and raise several open questions.

3.1 Overview

One of the longstanding open questions regarding the asymptotic geometry of curves is the problem of degeneration of linear series on smooth curves of given genus when they

approach the boundary of their corresponding moduli space. That is, to fully describe all the possible limits of linear series of given rank and degree when smooth curves degenerate to singular ones. This question was studied in a series of works by Eisenbud and Harris [EH89, EH86, EH87b, EH87a, EH87c] for which they managed to provide a satisfactory answer in the case the limit curve is of compact type, and used this to make major progress in the study of curves. For curves of pseudo-compact type, these results were generalized by Esteves–Medeiros [EM02] (for curves with two components) and by Osserman [Oss19b, Oss19a, He19]. The case of rank zero linear series in the pluricanonical systems is studied in recent work [BCG+18, BCG+19, MUW20, TT22].

Tropical geometry provides a modern perspective on degeneration methods in algebraic geometry and a new approach to classical questions in algebraic geometry. Developing the mathematics behind the tropical approach usually requires the introduction of new combinatorial structures, and it has become apparent now that from the viewpoint of applications, it is enough in many cases to understand the geometry behind these combinatorics. Two such examples are given in the development of a combinatorial theory of divisors on graphs and metric graphs [BN07, BJ16], and, more recently, in the development of tropical and combinatorial Hodge theories [AHK18, AP20].

In a previous work [AB15], Amini and Baker introduced linear series on hybrid objects called *metrized complexes* and used them to recover and partially generalize the Eisenbud–Harris theory of limit linear series. In a subsequent work [Ami14], the formalism of slope structures on metric graphs was introduced as a way to describe the limiting behavior of Weierstrass points on degenerating families of curves. Slope structures were used in a recent work of Farkas–Jensen–Payne [FJP20] in the study of the geometry of the moduli space of curves.

The aim of this chapter is to take the tropical approach one step further by introducing a purely combinatorial theory of linear series of arbitrary rank and degree on metric graphs. This can be regarded as a combinatorial theory of limit linear series.

More precisely, we aim to draw relevant combinatorial properties of tropicalizations of linear series, regarding:

- the slopes taken by the tropicalizations of functions;
- the vectors of slopes taken by these functions around points;
- the tropical dependence between these functions, and
- the topological properties of tropicalizations,

in order to develop a formalism of linear series on metric graphs. This will be based on two ingredients: hypercube rank functions (which are in close connection to a particular case of the matricubes defined in the companion work [AG24] (Chapter 2)), and slope structures. In the rest of this introduction, we give an overview of the setup and the results.

3.1.1 Degeneration problem for linear series

Recall that a linear series $\mathfrak{g}^r_d$ on a projective curve Y is by definition a vector subspace of dimension $r+1$ of the space of global sections $H^0(Y, L)$ of a line bundle L of degree d on Y.

Let $\mathscr{M}_g$ be the moduli space of smooth projective curves of genus g, and $\overline{\mathscr{M}_g}$ its Deligne–Mumford compactification. Let X be a stable curve of genus g over an algebraically closed field κ and x the corresponding point in $\overline{\mathscr{M}_g}$. The degeneration problem for linear series can be informally stated as follows.

Question 3.1.1. *Describe all the possible limits of linear series over any sequence of smooth projective curves of genus g when their corresponding points in $\mathscr{M}_g$ converge to x.*

3.1.2 Metric graphs and their divisor theory

Metric graphs arise as tropical limits of one-parameter families of smooth projective curves.

We denote by $\mathbb{R}_+$ the set of positive real numbers. Recall that a metric graph Γ is a compact metric space isomorphic to the metric realization of a pair (G, ℓ) consisting of a finite graph $G = (V, E)$ and a length function $\ell\colon E \to \mathbb{R}_+$: this is obtained by associating to each edge e a copy of the interval $\mathcal{I}_e = [0, \ell_e]$, with the two extremities identified with those of e, and then further identifying the ends of different intervals corresponding to a same vertex v. The quotient topology on Γ is metrizable by the path metric. The pair (G, ℓ) is called a *model* of Γ.

In the context related to the degeneration of algebraic curves, a metric graph Γ with model $(G = (V, E), \ell)$ is endowed with a function $\mathfrak{g}\colon V \to \mathbb{Z}_{\geqslant 0}$ associating to each vertex v the genus of some algebraic curve C_v represented by v. Such a triple $(G, \ell, \mathfrak{g})$ will be called an *augmented metric graph*. However, we will mostly handle non-augmented metric graphs in this chapter, and make comments about the relevance of the genus function in the theory developed here.

The set of *rational functions* on Γ is denoted by $\mathrm{Rat}(\Gamma)$, and, by definition, consists of all continuous piecewise affine functions $f : \Gamma \to \mathbb{R}$ with integral slopes. The tropicalization of rational functions on curves gives rise to rational functions on metric graphs.

As in the algebraic setting, rational functions on metric graphs are linked to divisors. A divisor D on a metric graph Γ is a finite formal sum with integer coefficients of points of Γ. For any rational function $f \in \mathrm{Rat}(\Gamma)$, the corresponding divisor of zeros and poles is defined by

$$\mathrm{div}(f) \coloneqq \sum_x \mathrm{ord}_x(f)\,(x), \qquad \text{with } \mathrm{ord}_x(f) \coloneqq -\sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu(f),$$

where $\mathrm{T}_x(\Gamma)$ is the set of outgoing unit tangent vectors to Γ at x, and $\mathrm{slope}_\nu(f)$ is the slope of f along ν at x. A divisor obtained in this way is called *principal*, and two divisors D_1 and D_2 whose difference $D_1 - D_2$ is principal are called *linearly equivalent*.

Each divisor D gives rise to a line bundle on Γ whose space of global sections is denoted by $\mathrm{Rat}(D)$. Concretely, $\mathrm{Rat}(D)$ is the set of $f \in \mathrm{Rat}(\Gamma)$ such that $\mathrm{div}(f) + D$ is effective

(that is, has only non-negative coefficients). Contrary to the algebraic setting, $\mathrm{Rat}(D)$ is not a vector space. Nevertheless, Baker and Norine discovered a way to associate a combinatorial notion of rank to $\mathrm{Rat}(D)$ [BN07, MZ08, GK08]. This is defined as the maximum integer among -1 and integers $r \geqslant 0$ such that for all points $x_1, \ldots, x_r$ in Γ, the divisor $D - (x_1) - \cdots - (x_r)$ is linearly equivalent to an effective divisor. We refer to the survey paper by Baker and Jensen [BJ16] for details, extensions, and several applications.

The results of the present chapter are motivated by the question of describing the tropical limits of linear series, when the metric graph arises as the tropical limit of a one-parameter family of smooth proper curves. In the next two sections, we describe two combinatorial structures that allow to approach this question.

3.1.3 Rank functions on hypercubes

We first describe a combinatorial way of encoding intersection patterns of a flag arrangement.

Let r be a non-negative integer. We set $[r] := \{0, \ldots, r\}$. For a positive integer δ, the hypercube $\square_r^\delta$ of dimension δ and width r is the product $[r]^\delta$.

We define a partial order $\preceq$ on $\square_r^\delta$ where for a pair of elements $\underline{a}, \underline{b}$ in $\square_r^\delta$, we write $\underline{a} \preceq \underline{b}$ whenever for every $j \in \{1, \ldots, \delta\}$, we have $a_j \leqslant b_j$. We define two operations $\vee$ and $\wedge$ by taking the maximum and the minimum coordinate-wise, respectively: for $\underline{a} = (a_1, \ldots, a_\delta)$ and $\underline{b} = (b_1, \ldots, b_\delta)$ in $\square_r^\delta$,

$$\underline{a} \vee \underline{b} := (\max(a_1, b_1), \ldots, \max(a_\delta, b_\delta)), \qquad \underline{a} \wedge \underline{b} := (\min(a_1, b_1), \ldots, \min(a_\delta, b_\delta)).$$

A function $f : \square_r^\delta \to \mathbb{Z}$ is called *supermodular* if for every two elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \leqslant f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

A function $\rho : \square_r^\delta \to \mathbb{Z}$ is called a *(hypercube) rank function* if it is supermodular and, in addition, satisfies the following conditions:

(1) For every $i \in \{1, \ldots, \delta\}$, and each $t \in [r]$, we have $\rho(t\,\underline{\mathfrak{e}}_i) = r - t$.

(2) ρ is non-increasing with respect to the partial order of $\square_r^\delta$, that is, if $\underline{a} \preceq \underline{b}$, then $\rho(\underline{b}) \leqslant \rho(\underline{a})$.

(3) The values of ρ are in the set $\{-1, 0, 1, \ldots, r\}$.

The term "hypercube rank function" defined above is borrowed from [AG24] (Chapter 2) which defines combinatorial objects called "matricubes", consisting of a ground set endowed with a rank function which is *submodular* and verifies a set of properties reminiscent of the ones listed above. In fact, this is an abuse of language because matricubes are more general objects than the rank functions we define in the present chapter. Additionally, it is more convenient for us in this chapter to work with *supermodular* functions, as it allows to considerably simplify the presentation. However, there is a simple way to transform a

supermodular function into a submodular function by an operation we call *conjugation*. This transforms the hypercube rank functions of this section into matricubes on the same ground set. For more details about the link between hypercube rank functions and matricubes, see Section 3.2.1, and more specifically Proposition 3.2.3. Using this, we can transfer to the setting of hypercube rank functions those basic combinatorial properties of matricubes proved in [AG24] (Chapter 2) which are useful in the present chapter; however, for this chapter to be self-contained, we fully reprove them here in the language of hypercube rank functions.

The geometric situation to have in mind in order to interpret hypercube rank functions is a vector space H of dimension $r+1$ over some field κ, and a collection of δ complete flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_\delta^\bullet$. That is, for $j = 1, \dots, \delta$, $F_j^\bullet$ consists of a chain of vector subspaces

$$\mathrm{H} = \mathrm{F}_j^0 \supsetneq \mathrm{F}_j^1 \supsetneq \cdots \supsetneq \mathrm{F}_j^{r-1} \supsetneq \mathrm{F}_j^r \supsetneq (0),$$

with $\operatorname{codim}(\mathrm{F}_j^i) = i$. In this case, the function $\rho\colon \square_r^\delta \to \mathbb{Z}$ defined by

$$\rho(a_1, \dots, a_\delta) \coloneqq \dim_\kappa(\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_\delta^{a_\delta}) - 1$$

is a rank function. Hypercube rank functions appearing in this way are called *representable*. On a smooth projective curve, given a finite dimensional vector space of rational functions, taking the orders of vanishing at a point leads to a complete flag (see Section 3.2.2). Such flags appear naturally in the degeneration of linear series, see Section 3.9.2.

We say that a point $\underline{a}$ of the hypercube is a *jump* of the rank function ρ if we have $\rho_v(\underline{a} + \underline{\mathfrak{e}}_j) < \rho_v(\underline{a})$ whenever $\underline{a} + \underline{\mathfrak{e}}_j$ is in the hypercube. Here, $\underline{\mathfrak{e}}_j$ is the point whose j-th coordinate is equal to one and whose other coordinates are all zero. In the representable case, the jumps correspond to extremal points beyond which the dimension drops. It turns out that the jumps uniquely determine the rank function.

3.1.4 Slope structures on graphs and metric graphs

Slope structures encode the information regarding possible slopes of functions arising from tropicalizations of linear series.

For a combinatorial graph $G = (V, E)$, we denote by $\mathbb{E}$ the set of all the orientations of edges of G, that is, each edge $\{u, v\} \in E$ gives rise to two oriented edges uv and vu in $\mathbb{E}$. We suppose that G is simple, i.e., has no parallel edges. In our setting, this can be assumed without loss of generality, because we may subdivide the parallel edges and the loops of a non-simple graph to make it simple. The subset $\mathbb{E}_v \subseteq \mathbb{E}$ is the set of all the orientations vu of edges $\{v, u\}$ in G. For a vertex v in G, we denote by d_v the valence of v in the graph; note that $d_v = |\mathbb{E}_v|$.

A *slope structure* $\mathfrak{S}$ *of width* r on G, or simply an *r-slope structure*, is the data of

- For each oriented edge $e = uv \in \mathbb{E}$ of G, a collection S^e of $r+1$ integers $s_0^e < s_1^e < \cdots < s_r^e$ subject to the requirement that $s_i^{uv} + s_{r-i}^{vu} = 0$ for each edge $\{u, v\} \in E$.

- For each vertex v of G, a rank function ρ_v on the hypercube $\square_r^{d_v}$.

We denote by $S^v \subseteq \prod_{e\in\mathbb{E}_v} S^e$ the set of all points $s_{\underline{a}} = (s_{a_1}, \dots, s_{a_\delta})$ for $\underline{a} = (a_1, \dots, a_\delta)$ a jump of the rank function ρ_v. Since jumps determine the rank function, using the above notation, we then write $\mathfrak{S} = \{S^v; S^e\}_{v\in V, e\in\mathbb{E}}$.

Let Γ be a metric graph. By an r-slope structure on Γ, we mean an r-slope structure $\mathfrak{S}$ on a simple graph model $G = (V, E)$ of Γ that we naturally extend to any point of Γ by associating to every point $x \in \Gamma \smallsetminus V$ the so-called *standard rank function*, see Section 3.2.2. More precisely, for every point $x \in \Gamma \smallsetminus V$ and each outgoing unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$, we define $S^\nu = S^{uv}$, where uv is the unique oriented edge of G which is parallel to ν, and define $S^x \subseteq S^{uv} \times S^{vu}$ as the set of all pairs (s_i^{uv}, s_j^{vu}) with $i + j \leqslant r$.

Taking into account the slope structure $\mathfrak{S}$, we can define a relevant notion of rational functions. A function f in $\mathrm{Rat}(\Gamma)$ is said to be compatible with $\mathfrak{S}$ if the two conditions (i) and (ii) below are verified. First,

(i) for any point $x \in \Gamma$ and any tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the outgoing slope of f along ν lies in S^ν.

Denote by $\partial_x(f)$ the vector in $\prod_{\nu\in\mathrm{T}_x(\Gamma)} S^\nu$ which consists of outgoing slopes of f along $\nu \in \mathrm{T}_x(\Gamma)$. Then the second condition is:

(ii) for any point $x \in \Gamma$, the vector $\partial_x(f)$ belongs to S^x.

We denote by $\mathrm{Rat}(\Gamma, \mathfrak{S}) \subset \mathrm{Rat}(\Gamma)$, or simply $\mathrm{Rat}(\mathfrak{S})$ if Γ is understood, the set of rational functions on Γ compatible with $\mathfrak{S}$. Endowed with the operations $c \odot f := f + c$ and $f \oplus g := \min(f, g)$ for every $f, g \in \mathrm{Rat}(\mathfrak{S})$ and $c \in \mathbb{R}$, $\mathrm{Rat}(\mathfrak{S})$ has the structure of a tropical semimodule. Moreover, it is naturally endowed with the norm $\|\cdot\|_\infty$.

If D is a divisor on Γ, we denote by $\mathrm{Rat}(D, \mathfrak{S}) := \mathrm{Rat}(D) \cap \mathrm{Rat}(\mathfrak{S})$ the set of all $f \in \mathrm{Rat}(\mathfrak{S})$ such that $D + \mathrm{div}(f)$ is effective. This is a sub-semimodule of $\mathrm{Rat}(\mathfrak{S})$ (see Proposition 3.5.1).

3.1.5 Admissible semimodules and combinatorial limit linear series

A semimodule $M \subseteq \mathrm{Rat}(D)$ is called *admissible of rank r* if it is closed for the topology induced by $\|\cdot\|_\infty$ (equivalently, for that of point-wise convergence), and if there exists an r-slope structure $\mathfrak{S}$ such that $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ and the following holds:

$(**)$ For every effective divisor E of degree r, there exists $f \in M$ such that

(1) For every point $x \in \Gamma$, $\rho_x(\partial_x(f)) \geqslant E(x)$; and in addition,

(2) $D + \mathrm{div}(f) - E \geqslant 0$.

In the degeneration picture for linear series in a one-parameter family of smooth projective curves, viewing x inside the Berkovich analytification of the generic fiber in the family, the first condition reflects the dimension counts underlying the reduction at x of rational functions in the linear series, imposing vanishing conditions along the incident branches. The second one is the analogue of the Baker–Norine rank condition in this setting.

In order to define a notion of linear series, we use moreover the concept of tropical rank introduced by Jensen and Payne in their work on applications of tropical divisor theory to the study of the geometry of generic curves [JP14, JP16] (for the definition, see Section 3.5.1).

A *(combinatorial limit) linear series* of rank r and degree d, more simply called a $\mathfrak{g}^r_d$, is a pair (D, M) consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$ of rank r which is moreover *finitely generated* and has *tropical rank* r.

Here, we mean by "finitely generated" that there exist finitely many elements of M which generate M using the tropical operations of scalar addition and minimum; the tropical rank is defined as the maximal number of tropically independent elements of M. See Sections 3.5.3 and 3.6.1 for more details.

If M is a $\mathfrak{g}^r_d$, the *linear system* $|M|$ is the space of all effective divisors E on Γ of the form $D + \mathrm{div}(f)$ for $f \in M$.

We also define *refined linear series* to be those linear series which in addition verify the following stronger version of $(**)$:

$(\overset{*}{**})$ For any effective divisor E on Γ of degree $s \leqslant r$, there exists a linear series M_E of rank $r - s$ associated to $(D, \mathfrak{S}_E)$ with $\mathfrak{S}_E$ a slope substructure of $\mathfrak{S}$ of rank $r - s$, such that for every function $f \in M_E$, we have

(1) For every point $x \in \Gamma$, $\rho_x(\partial_x(f)) \geqslant E(x)$; and in addition,

(2) $D + \mathrm{div}(f) - E \geqslant 0$.

3.1.6 Basic properties

Here is a list of interesting properties satisfied by slope structures and linear series.

(1) The vector of allowed slopes defined by a (crude) linear series is non-increasing along each edge (Proposition 3.4.3).

That is, as we move from one extremity of an edge to the other, the coordinates of the vector $(s^\nu_0, \dots, s^\nu_r)$ do not increase. This property turns out to be crucial in proving finiteness theorems about slopes structures underlying linear series on metric graphs.

(2) The space of rational functions $\mathrm{Rat}(D, \mathfrak{S})$ is a semimodule over $\mathbb{R}$ (Proposition 3.5.1).

This is a consequence of the supermodularity of rank functions.

(3) If $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ is a closed sub-semimodule, then it is generated by its extremal points (Proposition 3.5.7). Besides, fixing a function $f \in M$ and taking another $g \in M$, testing on a finite number of points is sufficient to determine whether $g = f$ (Lemma 3.5.10).

Extremal points are elements of the semimodule which cannot be obtained as the minimum of other elements of M in a non-trivial way, see Section 3.5.4. The above result yields an effective method to check whether functions of M are equal by checking equality at finitely many points.

3.1.7 Realization property for slope vectors

An important feature of admissible semimodules is the following realization property for jumps, proved in Section 3.5.6.

Theorem 3.1.2 (Realization of slope vectors)**.** *Let $(D, \mathfrak{S})$ be a pair consisting of a divisor D and a slope structure $\mathfrak{S}$ of width r on Γ, and let $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ be an admissible semimodule of rank r. Let v be a point of Γ and let $\underline{a}$ be a jump of ρ_v. Then, there exists $f \in M$ such that $\partial_v(f) = s_{\underline{a}}$.*

One immediate consequence is that $\mathfrak{S}$ can be entirely retrieved from M.

Corollary 3.1.3. *The data of an admissible semimodule $M \subset \mathrm{Rat}(D)$ determines the slope structure $\mathfrak{S}$ uniquely.*

3.1.8 Finiteness of slope structures

Let Γ be a metric graph, let D be a divisor on Γ. Let $G = (V, E)$ be a combinatorial graph underlying Γ and supporting D. Let $M \subset \mathrm{Rat}(D)$ be an admissible semimodule with underlying slope structure $\mathfrak{S}$ defined on some model of Γ (possibly different from G). We prove in Proposition 3.4.3 that the vector of slopes in $\mathfrak{S}$ is non-increasing along each edge. That is, as we move from one extremity of an edge to the other, the coordinates of the vector $(s_0^\nu, \dots, s_r^\nu)$ do not increase. This property turns out to be crucial in proving the following finiteness theorem (see Section 3.4.2 for a more general result).

Theorem 3.1.4 (Finiteness of slopes structures)**.** *For each integer r, there are finitely many subdivisions $H_1, \dots, H_k$ of G, and finitely many slope structures $\mathfrak{S}_1, \dots, \mathfrak{S}_k$ of rank r defined on them, respectively, such that every admissible semimodule $M \subset \mathrm{Rat}(D)$ with underlying slope structure $\mathfrak{S}$ has a combinatorial model H_j among $H_1, \dots, H_k$ such that $\mathfrak{S} = \mathfrak{S}_j$.*

This can be regarded as a first result in the direction of defining the moduli space of $\mathfrak{g}_d^r$'s over the moduli space of tropical curves of given genus.

3.1.9 Classification of $\mathfrak{g}^1_d$'s

In the case $r = 1$, we prove in Section 3.8 that the data of a $\mathfrak{g}^1_d$ on Γ is equivalent to the data of a finite harmonic map to a tree.

Theorem 3.1.5 (Classification of $\mathfrak{g}^1_d$'s on metric graphs)**.** *Let (D, M) be a $\mathfrak{g}^1_d$ on Γ with D a divisor of degree d. Suppose that the constant functions are in M. Then, there exist a tropical modification $\alpha\colon \widetilde{\Gamma} \longrightarrow \Gamma$ of Γ and a finite harmonic morphism $\varphi\colon \widetilde{\Gamma} \to T$ of degree d to a metric tree T such that M is the preimage of the unique $\mathfrak{g}^1_1$ on T, restricted to Γ.*

Using the smoothing theorems proved in [ABBR15a, ABBR15b] for finite harmonic morphisms to trees, we deduce the following smoothing theorem for combinatorial $\mathfrak{g}^1_d$'s (see below for the tropicalization, and Section 3.9.2).

Theorem 3.1.6 (Smoothing theorem for $\mathfrak{g}^1_d$'s)**.** *A $\mathfrak{g}^1_d$ (D, M) on Γ is smoothable, that is, it is the tropicalization of a $\mathfrak{g}^1_d$ from a smooth curve.*

The question of the existence of harmonic morphisms to metric trees of smallest degree, called *geometric gonality*, is thoroughly studied by Draisma and Vargas [DV21a], and by Cool and Draisma [CD18]. The above theorem provides an algebraic characterization of geometric gonality, as the least integer d such that the metric graph admits a $\mathfrak{g}^1_d$, see [DV21b] and the references there for the gonality of metric graphs.

In order to prove the above theorem, we generalize in Section 3.7 the theory of reduced divisors to the setting of linear series, consider the map defined by reduced divisors in Section 3.8, and connect it to the tropical rank to conclude.

3.1.10 Tropicalization

We now discuss the connection between linear series on algebraic curves and their combinatorial counterparts. We assume familiarity with the Berkovich theory of algebraic curves, see Section 3.9 and [BJ16] for more details.

Let $\mathbf{K}$ be an algebraically closed field with a non-trivial non-Archimedean valuation $\mathbf{val}$ and $\mathbf{C}$ be a smooth proper curve over $\mathbf{K}$. We assume that $\mathbf{K}$ is complete with respect to $\mathbf{val}$ and we denote by κ the residue field of $\mathbf{K}$, which is also algebraically closed. Denote by $\mathbf{C}^{\mathrm{an}}$ the Berkovich analytification of $\mathbf{C}$. Let $\Gamma \subset \mathbf{C}^{\mathrm{an}}$ be a metric graph skeleton of $\mathbf{C}^{\mathrm{an}}$. For each point $x \in \Gamma$, let $\mathfrak{v}_x$ be the valuation on the function field $\mathbf{K}(\mathbf{C})$ defined by x. For each nonzero rational function $\mathbf{f} \in \mathbf{K}(\mathbf{C})$, the tropicalization of $\mathbf{f}$, denoted $\mathrm{trop}(\mathbf{f})\colon \Gamma \to \mathbb{R}$, is defined by $\mathrm{trop}(\mathbf{f})(x) = \mathfrak{v}_x(f)$ for all $x \in \Gamma$. This is a piecewise affine function on Γ with integral slopes.

Let $\mathbf{D}$ be a divisor of degree d and rank at least r on $\mathbf{C}$, and $(\mathcal{O}(\mathbf{D}), \mathbf{H})$ be a $\mathfrak{g}^r_d$ on $\mathbf{C}$, so $\mathbf{H}$ is an $(r+1)$-dimensional vector subspace of $H^0(\mathbf{C}, \mathcal{O}(\mathbf{D}))$, the space of global sections of the line bundle associated to $\mathbf{D}$. We can identify $\mathbf{H}$ with a subspace of $\mathbf{K}(\mathbf{C})$ of dimension $r+1$. We define the *tropicalization*

$$M := \mathrm{trop}(\mathbf{H}) = \big\{\mathrm{trop}(\mathbf{f}) \;\big|\; \mathbf{f} \in \mathbf{H} \smallsetminus \{0\}\big\}.$$

The following theorem is proved in Section 3.9.2.

Theorem 3.1.7 (Tropicalization of linear series)**.** *Notation as above, let* $(\mathcal{O}(\mathbf{D}), \mathbf{H})$*,* $\mathbf{H} \subseteq H^0(\mathbf{C}, \mathcal{O}(\mathbf{D})) \subset \mathbf{K}(\mathbf{C})$*, be a* g^r_d *on* $\mathbf{C}$*. Let* Γ *be a skeleton of* $\mathbf{C}^{\mathrm{an}}$*. The slopes of rational functions* F *in* M *along edges in* Γ *yield a well-defined slope structure* $\mathfrak{S}$ *on* Γ*. Let* D *be the tropicalization of* $\mathbf{D}$ *to* Γ*. Then,* $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ *is a refined* $\mathfrak{g}^r_d$ *on* Γ*.*

In particular, the tropicalization $\mathrm{trop}(\mathbf{H})$ is finitely generated (see Proposition 3.9.4).

A pluricanonical linear series of rank r and order n is a vector subspace $\mathbf{H} \subseteq H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)$ of rank r, i.e., of dimension $r+1$, with $\omega_{\mathbf{C}}$ the canonical sheaf of $\mathbf{C}$.

Using Temkin metrization [Tem16], one can define tropicalizations of subspaces of global sections of pluricanonical sheaves. We explain how they fit into our theory of combinatorial limit linear series.

Let Γ be the metric graph skeleton of the Berkovich analytification $\mathbf{C}^{\mathrm{an}}$, and K the canonical divisor of Γ. The following theorem is proved in Section 3.10.3.

Theorem 3.1.8 (Tropicalization of pluricanonical linear series)**.** *Let* $\mathbf{H} \subseteq H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)$ *be a pluricanonical linear series of rank* r *and order* n*. Let*

$$M := \mathrm{trop}(\mathbf{H}) = \big\{\mathrm{trop}(\alpha) \;\big|\; \alpha \in \mathbf{H} \smallsetminus \{0\}\big\}.$$

Then $M \subseteq \mathrm{Rat}(nK, \mathfrak{S})$*, for the pluricanonical slope structure* $\mathfrak{S}$ *defined by* $\mathbf{H}$*. Moreover,* M *is a refined* $\mathfrak{g}^r_d$ *on* Γ*, for* $d = n(2g-2)$*.*

The finiteness theorem implies that there are only finitely many combinatorial types for pluricanonical slope structures of order n on augmented metric graphs Γ of a given combinatorial type. It is an interesting open question to classify all the pluricanonical slope structures on a given graph G.

3.1.11 Discussion of applications

The formalism of this chapter has applications to the geometry of curves. In particular, the equidistribution theorem proved in [Ami14] is a consequence of the formalism of slope structures and the behavior of reduced divisors in a given combinatorial linear series. The results of [FJP20] also use slope structures and the notion of tropical independence, the underlying concepts of the materials presented in this chapter. In a joint work with Amini and Richman [AGR23] (Chapter 4), we apply the formalism of this chapter to associate a Weierstrass weight to each connected component of the naive Weierstrass locus of a given divisor on a metric graph. This solves a problem posed by Matt Baker from his original work on the specialization of linear series from curves to graphs [Bak08]. Using these ideas, we explain the discrepancy between the naive counting of Weierstrass points on metric graphs in the work of Richman [Ric24] and the correct count of multiplicities.

The hypercube rank functions considered here are linked to matricubes and a theory of combinatorial flag arrangements developed in a companion work [AG24] (Chapter 2), which provide a generalization of the theory of matroids. For more information about the connection, we refer to Section 3.2.1, and more specifically to Proposition 3.2.3.

3.1.12 Further notation

For a subset $S \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, and an element $a \in \mathbb{R}^d$, we define $S + a$ as the set of all elements $b + a$ for $b \in S$. For the rest of the article, Λ will denote a divisible subgroup of $\mathbb{R}$ – a group is called *divisible* when multiplication by every positive integer n is surjective. Examples of such Λ are $\mathbb{Q}$ or $\mathbb{R}$ itself.

A Λ-*metric graph*, or Λ-*rational metric graph*, is a metric graph Γ whose edge lengths are Λ-rational, that is, lie in Λ. A point $x \in \Gamma$ is said to be Λ-*rational* if its distances to the endpoints of its incident edges are in Λ. In the case Γ is Λ-rational, a divisor whose support is made up of Λ-rational points is said to be a Λ-*rational* divisor. We also denote by $\mathrm{Rat}_\Lambda(\Gamma)$ the set of functions of $\mathrm{Rat}(\Gamma)$ which only change slope at Λ-rational points of Γ and which take a value in Λ on some (equivalently, on every) vertex v of Γ.

We denote by $\mathbb{T}$ the semifield of tropical numbers, which is the set $\mathbb{R}$ endowed with the internal operations of tropical addition $\oplus := \min$ and tropical multiplication $\odot := +$. A (divisible) subgroup Λ endowed with the operations $\oplus$ and $\odot$ defines a sub-semifield of $\mathbb{T}$.

3.2 Rank functions on hypercubes

3.2.1 Definition

Let r be a non-negative integer and $[r] = \{0, 1, \dots, r\}$. For a positive integer δ, the *hypercube* $\square_r^\delta$ of *dimension* δ and *width* r is the product $[r]^\delta$. We denote the elements of $\square_r^\delta$ by vectors $\underline{a} = (a_1, \dots, a_\delta)$, for $0 \leqslant a_1, \dots, a_\delta \leqslant r$.

The hypercube $\square_r^\delta$ is endowed with a natural partial order $\preceq$ defined by declaring $\underline{a} \preceq \underline{b}$ for elements $\underline{a} = (a_1, \dots, a_\delta)$ and $\underline{b} = (b_1, \dots, b_\delta)$ in $\square_r^\delta$, if $a_j \leqslant b_j$ for all $j \in \{1, \dots, \delta\}$. The smallest and largest elements of $\square_r^\delta$ with respect to this partial order are $\underline{0} := (0, \dots, 0)$ and $\underline{r} = (r, \dots, r)$, respectively. Moreover, there is a lattice structure on $\square_r^\delta$, where the two operations of join $\vee$ and meet $\wedge$ are defined by

$$\underline{a} \vee \underline{b} = (\max(a_1, b_1), \dots, \max(a_\delta, b_\delta)), \quad \underline{a} \wedge \underline{b} = (\min(a_1, b_1), \dots, \min(a_\delta, b_\delta)) \quad \forall \underline{a}, \underline{b} \in \square_r^\delta.$$

A function $f \colon \square_r^\delta \to \mathbb{Z}$ is called *supermodular* if for every pair of elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \leqslant f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

If the inequalities above are all reversed, then we say that f is *submodular*. We define the *conjugate* of f, denoted by $\bar{f}$, as the integer-valued function on the hypercube $\square_r^\delta$ given by

$$\bar{f}(\underline{a}) := r - f(\underline{a}) \qquad \forall\, \underline{a} \in \square_r^\delta.$$

Note that $\bar{\bar{f}} = f$. Moreover, f is supermodular, resp. submodular, if, and only if, $\bar{f}$ is submodular, resp. supermodular.

In this chapter, we will be working with a special kind of supermodular function on $\square_r^\delta$. For each integer $i \in \{1, \dots, \delta\}$, denote by $\underline{\mathfrak{e}}_i$ the vector whose coordinates are all zero except the i-th coordinate, which is equal to one. For $t \in [r]$, the vector $t\,\underline{\mathfrak{e}}_i$ lies in $\square_r^\delta$.

Definition 3.2.1 (Hypercube rank function)**.** A function $\rho\colon \square_r^\delta \to \mathbb{Z}$ is called a *rank function* if it satisfies the following conditions:

(HR1) for every $i \in \{1, \dots, \delta\}$, and each $t \in [r]$, we have $\rho(t\,\underline{\mathfrak{e}}_i) = r - t$.

(HR2) ρ is non-increasing with respect to the partial order of $\square_r^\delta$, that is, if $\underline{a} \preceq \underline{b}$, then $\rho(\underline{b}) \leqslant \rho(\underline{a})$.

(HR3) ρ is supermodular.

(HR4) The values of ρ are in the set $\{-1, 0, 1, \dots, r\}$.

The integer $\rho(\underline{a})$, for $\underline{a} \in \square_r^\delta$, is called the *rank* of $\underline{a}$. A hypercube $\square_r^\delta$ endowed with a rank function will be called a *ranked hypercube*. ◇

Remark 3.2.2. The above properties imply that if $\underline{a} \in \square_r^\delta$ has rank j, then $a_i \leqslant r - j$ for all $i \in \{1, \dots, \delta\}$. In particular, $\underline{0}$ is the only element of rank r in $\square_r^\delta$. ◇

The following proposition relates hypercube rank functions to matricubes, defined in [AG24] (Chapter 2).

Proposition 3.2.3. *Let $\rho\colon \square_r^\delta \to \mathbb{Z}$ be an integer-valued function, and $\mathbf{r} = \bar{\rho}$ be its conjugate. The following are equivalent:*

- *ρ is a hypercube rank function in the sense of Definition 3.2.1.*
- *$\mathbf{r}$ is the rank function of a simple matricube of rank r or $r+1$, in the sense of [AG24] (Chapter 2).*

Proof. Properties (HR1)-(HR2)-(HR3) are equivalent to the axioms (R1*)-(R2)-(R3), respectively, in the definition of matricubes in [AG24] (Chapter 2). Property (HR4) for ρ is equivalent to requiring $\mathscr{M}$ being of rank r or $r+1$, via [AG24, Proposition 2.4] (Proposition 2.2.4 in the present manuscript). □

Remark 3.2.4. It turns out that working with hypercube rank functions, instead of their conjugate matricube rank functions, considerably simplifies the mathematical expressions appearing in the treatment of combinatorial linear series. For this reason, we prefer them. We do not use any non-trivial result from [AG24] (Chapter 2). ◇

Although this is not used in the following, we mention that, combining the above result with [AG24, Theorem 7.1] (Theorem 2.7.1 in the present manuscript), we get the following corollary.

Corollary 3.2.5. *The data of a hypercube rank function ρ on $\square_r^\delta$ is equivalent to the data of a permutation array in the terminology of [EL00a, EL00b].*

In order to give examples in low dimension, we choose the following notational convention.

Convention 3.2.6 (Cases $\delta = 1, 2, 3$)**.** In this article, for $\delta = 1$, a function on $\square_r^1 = [r]$ is described by a tuple with $r+1$ entries $(t_0, \dots, t_r)$, which means that the value of the function on the i-th entry of $\square_r^1$ is t_i.

In the same way, for $\delta = 2$, a function on $\square_r^2$ will often be described by an array of size $(r+1) \times (r+1)$, $(t_{ij})_{0 \leqslant i,j \leqslant r}$, which means that the function takes value t_{ij} on $(i, j) \in \square_r^2$. We choose the convention that the first direction is horizontal, the second direction is vertical, and the origin is the bottom left-hand corner.

When $\delta = 3$, a function defined on $\square_r^3$ will be specified by $r+1$ arrays $R_0, \dots, R_r$ of size $(r+1) \times (r+1)$, where R_k describes the values of the function on $\square_r^2 \times \{k\} \subseteq \square_r^3$. ◇

Here are two examples of two-dimensional rank functions, with $r = 3$ and $r = 4$ respectively.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 2 & 2 & 2 & 1 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

3.2.2 Rank functions induced by complete flags

Let r be a non-negative integer, and let H be a vector space of dimension $r+1$ over some field κ. A complete flag of H consists of a chain of vector subspaces

$$\mathrm{H} = \mathrm{F}^0 \supsetneq \mathrm{F}^1 \supsetneq \cdots \supsetneq \mathrm{F}^{r-1} \supsetneq \mathrm{F}^r \supsetneq \mathrm{F}^{r+1} = (0),$$

where, for each $i \in [r+1]$, F^i is a vector subspace of codimension i in H.

Let δ be a positive integer, and let $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_\delta^\bullet$ be a collection of δ complete flags of H. Define the function $\rho\colon \square_r^\delta \to \mathbb{Z}$ by

$$\rho(a_1, \dots, a_\delta) := \dim_\kappa(\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_\delta^{a_\delta}) - 1. \tag{3.1}$$

Proposition 3.2.7. *The function ρ defined in (3.1) is a rank function on $\square_r^\delta$.*

Proof. This is [AG24, Proposition 2.6] (Proposition 2.2.6 in the present manuscript). Let $\underline{a}$ and $\underline{b}$ be two points of $\square_r^\delta$, and let $\underline{x} := \underline{a} \wedge \underline{b}$ and $\underline{y} := \underline{a} \vee \underline{b}$. We have an injection

$$\big(\mathrm{F}_1^{a_1} \cap \cdots \cap \mathrm{F}_\delta^{a_\delta}\big)/\big(\mathrm{F}_1^{y_1} \cap \cdots \cap \mathrm{F}_\delta^{y_\delta}\big) \hookrightarrow \big(\mathrm{F}_1^{x_1} \cap \cdots \cap \mathrm{F}_\delta^{x_\delta}\big)/\big(\mathrm{F}_1^{b_1} \cap \cdots \cap \mathrm{F}_\delta^{b_\delta}\big),$$

from which, comparing the dimensions, we get $\rho(\underline{a}) - \rho\big(\underline{y}\big) \leqslant \rho(\underline{x}) - \rho(\underline{b})$. This proves the supermodularity of ρ. Properties (HR1)-(HR2)-(HR4) in Definition 3.2.1 are trivially verified. □

A hypercube rank function ρ on $\square_r^\delta$ is called *representable over a field κ* if it comes from a collection of δ complete flags $\mathrm{F}_1^\bullet, \dots, \mathrm{F}_\delta^\bullet$ as above, in a κ-vector space H of dimension $r+1$.

This is equivalent to the representability of the corresponding matricube. Examples of non-representable rank functions can therefore be obtained from examples of non-representable matricubes of rank r or $r+1$ over $\square_r^\delta$.

Standard rank functions

The simplest kind of rank function is the following.

Definition 3.2.8 (Standard rank function)**.** The *standard rank function* of dimension δ and width r is the rank function ρ^{st} on $\square_r^\delta$ given by

$$\rho^{\mathrm{st}}(\underline{a}) := \max\{-1, r - a_1 - \cdots - a_\delta\} \qquad \forall \underline{a} \in \square_r^\delta. \qquad \diamond$$

For instance, the standard rank function of dimension 2 and width 4 is given by

$$\begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & -1 & -1 \\ 3 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Via Proposition 3.2.3, a rank function ρ on $\square_r^\delta$ is standard if, and only if, its conjugate $\mathbf{r} = \bar{\rho}$ is the rank function of the uniform matricube $\mathscr{U}_{\underline{r},r+1}$, see [AG24] (Chapter 2) for the definition.

Proposition 3.2.9. *Every rank function ρ on the hypercube $\square_r^\delta$ dominates the standard rank function ρ^{st}. That is, for every $\underline{a} \in \square_r^\delta$, we have $\rho(\underline{a}) \geqslant \rho^{\mathrm{st}}(\underline{a})$.*

Proof. It will be enough to show that for every $\underline{a} = (a_1, \ldots, a_\delta) \in \square_r^\delta$, we have $\rho(\underline{a}) \geqslant r - a_1 - \cdots - a_\delta$. This can be proved by induction on δ, using the supermodularity of ρ. □

Remark 3.2.10. The standard rank function is induced by complete flags, over an infinite field, which are in *general relative position*, that is, whose intersection patterns have the smallest possible dimensions. ◇

Geometric rank functions

Let C be a smooth proper curve over an algebraically closed field κ. Let $\kappa(C)$ be the function field of C, and let $\mathrm{H} \subset \kappa(C)$ be a vector subspace of rational functions of dimension $r+1$ over κ. Each κ-point p of C gives a complete flag $\mathrm{F}_p^\bullet$ of H by considering the orders of vanishing at p of functions in H. Define $S_p := \{\mathrm{ord}_p(f) \mid f \in \mathrm{H} \smallsetminus \{0\}\}$. This is a finite set of cardinality $r+1$. Denote by $s_0^p < \cdots < s_r^p$ its elements, enumerated in increasing order. The flag $\mathrm{F}_p^\bullet$ is defined by setting, for $j \in [r]$,

$$\mathrm{F}_p^j := \{f \in \mathrm{H} \smallsetminus \{0\} \mid \mathrm{ord}_p(f) \geqslant s_j^p\} \cup \{0\}.$$

Each F_p^j has codimension j in H.

Let now δ be a positive integer, and let $A = \{p_1, \ldots, p_\delta\}$ be a collection of δ distinct κ-points on C. By the construction above, each point p_i leads to a complete flag $\mathrm{F}_i^\bullet$. Letting

$S_i := \{\mathrm{ord}_{p_i}(f) \mid f \in \mathrm{H} \setminus \{0\}\}$, and enumerating the elements of S_i in increasing order $s_0^i < \cdots < s_r^i$, the flag $\mathrm{F}_i^\bullet$ is defined by setting

$$\mathrm{F}_i^j := \{f \in \mathrm{H} \setminus \{0\} \mid \mathrm{ord}_{p_i}(f) \geqslant s_j^i\} \cup \{0\}.$$

This leads to a rank function ρ on the hypercube $\square_r^\delta$ using (3.1) in Section 3.2.2.

A rank function ρ on $\square_r^\delta$ that arises from the above construction for a curve C over an algebraically closed field κ is called *geometric*.

3.2.3 Jumps of a rank function

For $\underline{a} \in \square_r^\delta$, let $I_{\underline{a}}$ be the set of all $i \in \{1, \ldots, \delta\}$ such that $\underline{a} + \underline{\mathfrak{e}}_i \in \square_r^\delta$.

Proposition 3.2.11. *Let ρ be a rank function on $\square_r^\delta$. For an element $\underline{a} \in \square_r^\delta$ and $i \in I_{\underline{a}}$, we have $\rho(\underline{a}) - 1 \leqslant \rho(\underline{a} + \underline{\mathfrak{e}}_i) \leqslant \rho(\underline{a})$.*

Proof. The first inequality results from the supermodularity property applied to the vectors $\underline{a}$ and $(a_i + 1)\,\underline{\mathfrak{e}}_i$, using Property (HR1) in Definition 3.2.1; the second inequality comes from the non-increasing property of ρ. □

The proposition leads to the following definition.

Definition 3.2.12 (Jumps of a rank function)**.** Let ρ be a rank function on $\square_r^\delta$. A point $\underline{a}$ of $\square_r^\delta$ is called a *jump* for ρ if

(1) $\rho(\underline{a}) \geqslant 0$, and

(2) for every $i \in \{1, \ldots, \delta\}$ such that $\underline{a} + \underline{\mathfrak{e}}_i$ belongs to $\square_r^\delta$, we have $\rho(\underline{a} + \underline{\mathfrak{e}}_i) = \rho(\underline{a}) - 1$.

We denote by J_ρ the set of jumps of ρ. ◇

Here are three rank functions of dimension two, with $r = 3$ for the first one, and $r = 4$ for the second and third ones. The jumps of each rank function are depicted in blue.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 2 & 2 & 2 & 1 & 0 \\ 3 & 2 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 2 & 1 & 0 & -1 & -1 \\ 3 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

The set of jumps of a rank function has the following properties.

Proposition 3.2.13. *The set of jumps J_ρ of a rank function ρ on $\square_r^\delta$ is stable under $\wedge$.*

Proof. Let $\underline{a}, \underline{b} \in J_\rho$ and let $\underline{c} = \underline{a} \wedge \underline{b}$. The non-increasing property of ρ ensures that $\rho(\underline{c}) \geqslant 0$. Let $i \in \{1, \ldots, \delta\}$ be such that $\underline{c} + \underline{\mathfrak{e}}_i$ belongs to $\square_r$. We have to show that $\mathbf{r}(\underline{c} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{c}) + 1$. By symmetry, we can suppose that $a_i \leqslant b_i$, that is, $c_i = a_i$. Since $\underline{a}$ is a jump, we have $\mathbf{r}(\underline{a} + \underline{\mathfrak{e}}_i) = \mathbf{r}(\underline{a}) + 1$. We conclude by applying the supermodularity property to the points $\underline{a} + \underline{\mathfrak{e}}_i$ and $\underline{c}$. □

Proposition 3.2.14. *The set of jumps J_ρ is a graded poset. The grading is given by the conjugate $\bar{\rho}$. In particular, if $\underline{a} \prec \underline{b}$ are two distinct jumps of ρ, then we have $\rho(\underline{a}) > \rho(\underline{b})$.*

Proof. It is sufficient to show that if $\underline{a} \prec \underline{b}$ are two distinct jumps of ρ with $\rho(\underline{b}) \leqslant \rho(\underline{a}) - 2$, then there exists a jump $\underline{a} \prec \underline{c} \prec \underline{b}$ such that $\rho(\underline{c}) = \rho(\underline{a}) - 1$. Let i be an index such that $a_i < b_i$, and let $\underline{x} := \underline{a} + \underline{\mathfrak{e}}_i$. Since $\underline{a}$ is a jump, we have $\rho(\underline{x}) = \rho(\underline{a}) - 1$. Let $\underline{c}$ be the meet of all the points $\underline{c}'$ of $\square_r^\delta$ verifying $\underline{c}' \succeq \underline{x}$ and such that either $\underline{c}'$ is a jump or $\underline{c}' = \underline{r}$. Since $\underline{x} \preceq \underline{b}$ and $\underline{b}$ is a jump, $\underline{c}$ is also a jump by virtue of Proposition 3.2.13, and moreover $\underline{a} \prec \underline{c} \preceq \underline{b}$. It is easy to see, by induction on the ℓ_1-norm of $\underline{x}$, that $\rho(\underline{c}) = \rho(\underline{x}) = \rho(\underline{a}) - 1$, which implies that $\underline{c} \prec \underline{b}$. This shows that $\underline{c}$ is as desired and that the poset J_ρ is graded. By construction, the grading is given by $\bar{\rho}$. □

The following fact will be useful in the sequel.

Fact 3.2.15. Let ρ be a rank function on $\square_r^\delta$. If $\underline{r}$ is a jump for ρ, then it is the only element of J_ρ having some coordinate equal to r. ◇

Proof. For the sake of a contradiction, let $\underline{x} \neq \underline{r}$ be an element of J_ρ with some coordinate equal to r, say the first one. By (1) in Definition 3.2.12, we have $\rho(\underline{r}) \geqslant 0$. The inequality $\underline{x} \succeq r\,\underline{\mathfrak{e}}_1$ implies that $\rho(\underline{x}) \leqslant 0$, and Proposition 3.2.14 implies $\rho(\underline{r}) < \rho(\underline{x}) \leqslant 0$, a contradiction. □

Remark 3.2.16. Let $\mathscr{M}$ be the matricube on the ground set $\square_r^\delta$ defined by the conjugate $\mathbf{r} = \bar{\rho}$. The set of jumps J_ρ coincides with the set of flats $\mathscr{F}(\mathscr{M})$ of $\mathscr{M}$ if $\mathbf{r}(\mathscr{M}) = r$ and with $\mathscr{F}(\mathscr{M}) \smallsetminus \{\underline{r}\}$ if $\mathbf{r}(\mathscr{M}) = r + 1$. Propositions 3.2.13 and 3.2.14 can be deduced from the analogue properties of flats of matricubes. For the sake of completeness, we provided the short proofs of these results. Note that contrary to the set of flats of matricubes, the set of jumps is not necessary a lattice (this happens only in the case $\mathbf{r}(\mathscr{M}) = r + 1$). ◇

3.2.4 Partition Lemma

In this section, we prove a result about hypercube rank functions which turns out to be useful in the sequel.

Let ρ be a rank function on $\square_r^\delta$. The point $\underline{0}$ is the only point of $\square_r^\delta$ whose image by ρ is r (Remark 3.2.2). Besides, the set of jumps J_ρ of ρ contains the point $\underline{0}$ (because $\rho(\underline{\mathfrak{e}}_i) = r - 1$ for all i). Every jump of ρ of rank $r - 1$ has only coordinates equal to zero or one (Remark 3.2.2), among which at least one is equal to one. For each $\underline{a} \in J_\rho$ such that $\rho(\underline{a}) = r - 1$, denote by $P_{\underline{a}}$ the subset of $\{1, \dots, \delta\}$ consisting of all the indices i with $a_i = 1$, that is, $P_{\underline{a}}$ is the *support* of $\underline{a}$. Denote by $\mathcal{P}_\rho$ the collection of all sets $P_{\underline{a}}$ for $\underline{a} \in J_\rho \smallsetminus \{\underline{0}\}$ verifying $\rho(\underline{a}) = r - 1$. We have the following proposition.

Lemma 3.2.17 (Partition Lemma)**.** *Notation as above, $\mathcal{P}_\rho$ provides a partition of $\{1, \dots, \delta\}$.*

Proof. We first prove that the elements of $\mathcal{P}_\rho$ are pairwise disjoint. Let $\underline{a}$ and $\underline{b}$ be two distinct elements of $J_\rho \smallsetminus \{\underline{0}\}$ with $\rho(\underline{a}) = \rho(\underline{b}) = r-1$. Since $\underline{a} \not\leqslant \underline{b}$ and $\underline{b} \not\leqslant \underline{a}$, we have $\rho(\underline{a} \vee \underline{b}) \leqslant r-2$. Using the supermodularity property for $\underline{a}$ and $\underline{b}$, we get $\rho(\underline{a} \wedge \underline{b}) \geqslant r$ and therefore $\rho(\underline{a} \wedge \underline{b}) = r$. This forces $\underline{a} \wedge \underline{b} = \underline{0}$, from which we can conclude that $P_{\underline{a}} \cap P_{\underline{b}} = \varnothing$.

It remains to prove that the sets $P_{\underline{a}}$ cover $\{1, \dots, \delta\}$. For an $i \in \{1, \dots, \delta\}$, we need to show the existence of $\underline{a} \in J_\rho$ with $\rho(\underline{a}) = r-1$ and $a_i = 1$. We define $\underline{a}$ as the meet of all the jumps $\underline{a}' \in J_\rho$ such that $\underline{a}' \geq \mathfrak{e}_i$. As in the proof of Proposition 3.2.14, $\underline{a}$ is well-defined, belongs to J_ρ and has rank $\mathbf{r}(\underline{a}) = \mathbf{r}(\mathfrak{e}_i) = r-1$. The non-increasing property of ρ implies that $a_i \leqslant 1$, as desired. □

Remark 3.2.18. In the case $r = 1$, in the construction above, the condition $\rho(\underline{a}) = r-1$ is automatic. This will be crucially used in Section 3.8. ◇

3.3 Slope structures

In the following sections, we define combinatorial linear series on metric graphs with the help of an auxiliary data called a *slope structure*. A slope structure is the data of a family of hypercube rank functions of given width r, parametrized by the points of the metric graph, of varying dimension given by the valences of points, and verifying a finiteness condition.

3.3.1 Slope structures on graphs

Let first $G = (V, E)$ be a simple graph. We denote by $\mathbb{E}$ the set of all the orientations of edges of G, so that for an edge $\{u, v\}$ in E, we have two orientations $uv, vu \in \mathbb{E}$. For an oriented edge $e = uv \in \mathbb{E}$, we call u the tail and v the head of e. We denote by $\overline{e} = vu$ the oriented edge in $\mathbb{E}$ with reverse orientation. For a vertex $v \in V$, we denote by $\mathbb{E}_v \subseteq \mathbb{E}$ the set of oriented edges in $\mathbb{E}$ which have tail v, that is, all $vu \in \mathbb{E}$ for edges $\{v, u\} \in E$.

A *slope structure* $\mathfrak{S} = \{S^v; S^e\}_{v \in V, e \in \mathbb{E}}$ *of width* r on G, or simply an *r-slope structure*, is the data of

(SLS1) For every oriented edge $e = uv \in \mathbb{E}$ of G, a collection S^e of $r+1$ integers

$$s_0^e < s_1^e < \cdots < s_r^e$$

subject to the requirement that $s_i^{uv} + s_{r-i}^{vu} = 0$ for every edge $\{u, v\} \in E$.

(SLS2) For every vertex v of G, a rank function ρ_v on the hypercube $\square_r^{d_v}$.

If J_{ρ_v} denotes the set of jumps of ρ_v (see Definition 3.2.12), we denote by $S^v \subseteq \prod_{e \in \mathbb{E}_v} S^e$ the set of all points $s_{\underline{a}}$ for $\underline{a} \in J_{\rho_v}$.

Here, for a point $\underline{a} = (a_e)_{e \in \mathbb{E}_v}$ of the hypercube, the element $s_{\underline{a}} \in \prod_{e \in \mathbb{E}_v} S^e$ denotes the point in the product which has coordinate at $e \in \mathbb{E}_v$ equal to $s_{a_e}^e$. In other words, S^v fits

into the following natural commutative diagram:

$$\begin{array}{ccc} J_{\rho_v} & \hookrightarrow & \square_r^{d_v} \\ \downarrow{\scriptstyle \underline{a}\mapsto s_{\underline{a}}} & & \downarrow{\scriptstyle \underline{a}\mapsto s_{\underline{a}}} \\ S^v & \hookrightarrow & \Pi_{e\in\mathbb{E}_v} S^e \end{array}$$

We will sometimes need to separate the data relative to edges and the data relative to vertices. In this case, we will denote by $\mathfrak{S}^{\mathrm{e}}$ the data of a set of prescribed slopes on each edge, and by $\mathfrak{S}^{\mathrm{v}}$ the data of a rank function for each vertex.

3.3.2 Slope structures on metric graphs

Let now Γ be a metric graph. By an *r-slope structure* on Γ we mean an r-slope structure $\mathfrak{S}$ on a simple graph model $G = (V, E)$ of Γ, extended to each point of Γ as follows.

For every point x and each outgoing unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$, there exists a unique oriented edge uv of G which is parallel to ν. Define $S^\nu = S^{uv}$. Also, for every point $x \in \Gamma \smallsetminus V$ in the interior of an edge $\{u, v\}$, define ρ_x to be the standard rank function on $\square_r^2$. In particular, $S^x \subseteq S^{uv} \times S^{vu}$ can be identified with the set of all pairs (s_i^{uv}, s_j^{vu}) with $i + j \leqslant r$. We call the collection $\left\{S^x; S^\nu \;\middle|\; x \in \Gamma,\, \nu \in \mathrm{T}_x(\Gamma)\right\}$ a *slope structure of width r*, or simply an *r-slope structure* on Γ. We denote it by $\mathfrak{S}_\Gamma$, or simply $\mathfrak{S}$, if there is no risk of confusion. We extend the notation $\mathfrak{S}^{\mathrm{e}}$ and $\mathfrak{S}^{\mathrm{v}}$ in the natural way. Note that a slope structure on a metric graph can arise from choices of slope structures on different graph models of Γ.

Example 3.3.1. We give an example of a 2-slope structure on a metric graph. Consider the metric graph Γ depicted below, with edges of arbitrary positive lengths.

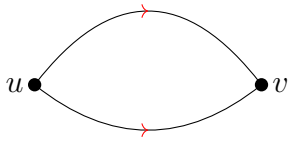


Figure 3.1: An example of a 2-slope structure on the circle.

Let $s_0 < s_1 < s_2$ and $s'_0 < s'_1 < s'_2$ be two sets of distinct integers. We define $\mathfrak{S}^{\mathrm{e}}$ by allowing slopes $s_0 < s_1 < s_2$ on the top edge and $s'_0 < s'_1 < s'_2$ on the bottom edge. We define $\mathfrak{S}^{\mathrm{v}}$ by choosing the rank functions at u and v to be given by the array

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

(jumps depicted in blue). The rank functions at all the other points of Γ are standard. This fully describes a 2-slope structure $\mathfrak{S}$ on Γ. ◇

3.3.3 Rational functions compatible with a slope structure

We define a notion of rational function on metric graphs compatible with a slope structure.

Let Γ be a metric graph and let $\mathfrak{S} = \left\{S^x; S^\nu \;\middle|\; x \in \Gamma,\, \nu \in \mathrm{T}_x(\Gamma)\right\} = (\mathfrak{S}^{\mathrm{e}}, \mathfrak{S}^{\mathrm{v}})$ be a slope structure of width r on Γ. Recall that we denote by $\mathrm{Rat}(\Gamma)$ the set of continuous piecewise affine functions $f\colon \Gamma \to \mathbb{R}$ with integral slopes. For each point $x \in \Gamma$, and each $\nu \in \mathrm{T}_x(\Gamma)$, we denote by $\mathrm{slope}_\nu(f)$ the slope of f at x along ν.

A function f in $\mathrm{Rat}(\Gamma)$ is said to be *compatible* with $\mathfrak{S}$ if the two conditions (i) and (ii) below are verified:

(i) for each point $x \in \Gamma$ and each $\nu \in \mathrm{T}_x(\Gamma)$, the outgoing slope of f along ν lies in S^ν.

Denote by

$$\partial_x(f) \in \prod_{\nu \in \mathrm{T}_x(\Gamma)} S^\nu, \qquad \partial_x(f)(\nu) = \mathrm{slope}_\nu(f) \quad \forall\, \nu \in \mathrm{T}_x(\Gamma)$$

the vector in $\prod_{\nu \in \mathrm{T}_x(\Gamma)} S^\nu$ with ν-coordinate consisting of the outgoing slope of f along $\nu \in \mathrm{T}_x(\Gamma)$. Then, the second condition is:

(ii) for every point $x \in \Gamma$, the vector $\partial_x(f)$ belongs to S^x.

We denote by $\mathrm{Rat}(\Gamma, \mathfrak{S})$, or simply $\mathrm{Rat}(\mathfrak{S})$ if there is no risk of confusion, the space of rational functions on Γ compatible with $\mathfrak{S}$. We also denote by $\mathrm{Rat}(\Gamma, \mathfrak{S}^{\mathrm{e}})$ or $\mathrm{Rat}(\mathfrak{S}^{\mathrm{e}})$ the space of rational functions satisfying (i).

If Γ is Λ-rational, we define the spaces $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S})$, $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S}^{\mathrm{e}})$ and $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S}^{\mathrm{v}})$ accordingly, adding the constraint that $f(x)$ is in Λ for all Λ-rational points of Γ.

3.3.4 Slope substructures

Let Γ be a metric graph and let $\mathfrak{S} = \left\{S^x; S^\nu \;\middle|\; x \in \Gamma,\, \nu \in \mathrm{T}_x(\Gamma)\right\}$ be an r-slope structure on Γ.

A *slope substructure* of $\mathfrak{S}$ of width $s \leqslant r$ is a slope structure

$$\mathfrak{S}' = \left\{S'^x; S'^\nu \;\middle|\; x \in \Gamma,\, \nu \in \mathrm{T}_x(\Gamma)\right\}$$

of width s on Γ such that for every $x \in \Gamma$, we have:

(1) for every $\nu \in \mathrm{T}_x(\Gamma)$, the set of prescribed slopes S'^ν is a subset of S^ν;

(2) the set of prescribed vectors of slopes S'^x is a subset of S^x.

Note that if $\mathfrak{S}'$ is a slope substructure of $\mathfrak{S}$, then the inclusion $\mathrm{Rat}(\mathfrak{S}') \subseteq \mathrm{Rat}(\mathfrak{S})$ holds. We will see in Corollary 3.5.15 that in the case of interest to us, the converse will be also true.

3.3.5 Divisors on a metric graph and their rank

A divisor D on a metric graph Γ is a finite formal sum over $\mathbb{Z}$ of points of Γ, that is, $D = \sum_{i\in I} n_i\,(x_i)$ with $n_i \in \mathbb{Z}$ and distinct points $x_i \in \Gamma$, for a finite set I. The coefficient of a point x of Γ in D is denoted by $D(x)$. A divisor D is called *effective*, written $D \geqslant 0$, if $D(x) \geqslant 0$ for all $x \in \Gamma$. For any rational function $f \in \mathrm{Rat}(\Gamma)$, the corresponding divisor is denoted by

$$\mathrm{div}(f) := \sum_{x\in\Gamma} \mathrm{ord}_x(f)\,(x), \qquad \text{where } \mathrm{ord}_x(f) := -\sum_{\nu\in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu(f).$$

A divisor obtained in this way is called *principal*. In the case Γ is Λ-rational, a divisor whose support is made up of Λ-rational points is said to be a Λ-*rational* divisor. Notice that the space $\mathrm{Rat}_\Lambda(\Gamma)$ defined in Section 3.1.12 can be redefined as the set of functions of $\mathrm{Rat}(\Gamma)$ such that $\mathrm{div}(f)$ is Λ-rational.

We have the following elementary fact.

Proposition 3.3.2. *For $f, g \in \mathrm{Rat}(\Gamma)$, $\mathrm{div}(f) = \mathrm{div}(g)$ if, and only if, $f - g$ is constant on Γ.*

Two divisors D_1 and D_2 are called *linearly equivalent* if their difference $D_1 - D_2$ is principal. The Baker–Norine rank $r(D)$ of a divisor D is defined as the maximum integer among -1 and the integers $r \geqslant 0$ such that for all points $x_1, \dots, x_r$ in Γ, the divisor $D - (x_1) - \cdots - (x_r)$ is linearly equivalent to an effective divisor.

3.3.6 Linear equivalence of slope structures

We define a notion of linear equivalence for slope structures on a metric graph as follows.

Let $\mathfrak{S}_1 = \left\{S_1^x; S_1^\nu \;\middle|\; x \in \Gamma,\ \nu \in \mathrm{T}_x(\Gamma)\right\}$ and $\mathfrak{S}_2 = \left\{S_2^x; S_2^\nu \;\middle|\; x \in \Gamma,\ \nu \in \mathrm{T}_x(\Gamma)\right\}$ be two slope structures on a metric graph Γ. We say $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are *linearly equivalent*, and write $\mathfrak{S}_1 \simeq \mathfrak{S}_2$, if there exists a rational function f on Γ such that for every point x of Γ and every $\nu \in \mathrm{T}_x(\Gamma)$, we have $S_1^\nu = S_2^\nu - \mathrm{slope}_\nu(f)$, and $S_1^x = S_2^x - \partial_x(f)$. In this case, we write $\mathfrak{S}_1 = \mathfrak{S}_2 + \mathrm{div}(f)$. Note that if $\mathfrak{S}$ is a slope structure, then $\mathfrak{S} + \mathrm{div}(f)$ is a slope structure for every rational function $f \in \mathrm{Rat}(\Gamma)$.

3.3.7 Divisors endowed with a slope structure on Γ

A *divisor endowed with an r-slope structure of degree d* is a pair $(D, \mathfrak{S})$ consisting of a divisor D of degree d and a slope structure $\mathfrak{S}$ of width r. We extend the definition of linear equivalence between slope structures to all pairs $(D, \mathfrak{S})$ with D a divisor of degree d and $\mathfrak{S}$ an r-slope structure on Γ by declaring that $(D_1, \mathfrak{S}_1) \simeq (D_2, \mathfrak{S}_2)$ if there exists a rational function f on Γ such that $D_1 = D_2 + \mathrm{div}(f)$ and $\mathfrak{S}_1 = \mathfrak{S}_2 + \mathrm{div}(f)$.

Definition 3.3.3. A *divisor class endowed with an r-slope structure of degree d* on Γ is the linear equivalence class of a pair $(D, \mathfrak{S})$ where D is a divisor of degree d and $\mathfrak{S}$ is an r-slope structure on Γ. ◇

We now define the space of rational functions relative to a divisor and a slope structure.

Definition 3.3.4 (Space of rational functions and linear system associated to a divisor endowed with a slope structure)**.** Let $(D, \mathfrak{S})$ be a divisor endowed with a slope structure on Γ. We denote by $\mathrm{Rat}(D, \mathfrak{S})$ the space of all $f \in \mathrm{Rat}(\mathfrak{S})$ with the property that $D + \mathrm{div}(f) \geqslant 0$, and define the linear system $|(D, \mathfrak{S})|$ associated to $(D, \mathfrak{S})$ as the space of all effective divisors E on Γ of the form $D + \mathrm{div}(f)$ for some $f \in \mathrm{Rat}(D, \mathfrak{S})$. ◇

Remark 3.3.5. Recall that we define $\mathrm{Rat}(D)$ as the set of all functions $f \in \mathrm{Rat}(\Gamma)$ such that $D + \mathrm{div}(f) \geqslant 0$. In a similar way, we can define the space $\mathrm{Rat}(D, \mathfrak{S}^{\mathrm{e}})$, see Section 3.3.3. ◇

Remark 3.3.6. Note that $|(D, \mathfrak{S})|$ is independent of the choice of the pair $(D, \mathfrak{S})$ in its linear equivalence class. Also note that if $D(x) > 0$ for some $x \in \Gamma$ in the interior of an edge of a model G on which $\mathfrak{S}$ is defined, then, we have

$$|(D, \mathfrak{S})| = |(D - (x), \mathfrak{S})| + (x).$$ ◇

Definition 3.3.7. A divisor endowed with a slope structure $(D, \mathfrak{S})$ is called *effective* if $\mathrm{Rat}(D, \mathfrak{S})$ contains the null function. ◇

This is equivalent to asking that D is effective and that we have $0 \in S^{\nu}$ and $\underline{0} \in S^{x}$ for every point x and every $\nu \in \mathrm{T}_x(\Gamma)$.

For future use, we make the following remark.

Remark 3.3.8. Since $\mathrm{Rat}(D, \mathfrak{S}) \subset \mathcal{C}^0(\Gamma, \mathbb{R})$, it is naturally endowed with the norm $\|\cdot\|_\infty$. The corresponding topology shall be used later on to study linear series (see Definition 3.6.2). We note that the slopes of all functions in $\mathrm{Rat}(D, \mathfrak{S})$ are trivially bounded in magnitude by $\max_{1 \leqslant i \leqslant r} |s_i^e|$. ◇

3.4 Crude linear series

We define *crude linear series* which are the simplest notion of combinatorial linear series. The requirement in the definition is reminiscent of the rank condition on divisors on metric graphs. Moreover, it takes into account the data of the hypercube rank function on points.

3.4.1 Definition

A *crude linear series of degree d and rank r*, or *crude* $\mathfrak{g}^r_d$, is the equivalence class of a divisor D of degree d endowed with a slope structure $\mathfrak{S}$ of width r on Γ subject to the following property:

$(*)$ For every effective divisor E on Γ of degree r, there exists a rational function $f \in \mathrm{Rat}(D, \mathfrak{S})$ such that

(CL1) For every point $x \in \Gamma$, $\rho_x(\partial_x(f)) \geqslant E(x)$, and in addition,

(CL2) $D + \operatorname{div}(f) - E \geqslant 0$.

We call a crude linear series *effective* if the underlying divisor endowed with the slope structure is so. ◇

We make a set of comments and provide examples in order to clarify the definition.

First, note that Property (CL2) in Definition 3.4.1 implies that the Baker–Norine rank $r(D)$ is greater than or equal to r.

Second, for given E and f, (CL1) does not necessarily imply (CL2), for example at points x such that $D(x) < 0$ or such that ρ_x is not standard. On the other hand, (CL1) implies (CL2) generically for all E and f. More precisely, let $G = (V, E)$ be a graph model of Γ such that $\mathfrak{S}$ comes from an r-slope structure on G, and D has support on V. Then, for every point $x \in \Gamma \smallsetminus V$ lying on an edge $\{u, v\}$, the rank function ρ_x is standard. The first condition above is thus equivalent to $i + j \leqslant r - E(x)$ for s_i^{uv} and s_j^{vu} the two slopes of f at x. In particular, since the slopes are all integral, it is easy to see that condition (CL2), written $\operatorname{div}_x(f) = -s_i^{uv} - s_j^{vu} \geqslant E(x)$, is automatically implied by condition (CL1) for x. This means that (CL1) implies (CL2) outside the (finite) set of vertices of G. Note that (CL1) can be *strictly* stronger than (CL2) in the interior of edges e (as long as the possible slopes on e do not form an integral interval, i.e., in the case there are gaps in S^e.

The relevance of (CL1) will be justified in Section 3.9, which treats the geometric situation in which the slope structure comes from tropicalization.

We finally note that the definition generalizes to Λ-rational divisors on Λ-metric graphs. In this case, we require f to be in $\operatorname{Rat}_\Lambda(\Gamma)$.

Example 3.4.1. Consider the metric graph Γ below, with two edges of equal length this time.

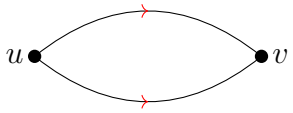


Figure 3.2: An example of a crude linear series on the circle.

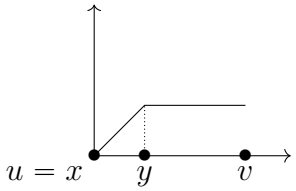
We allow the slopes $0 < 1 < 2$ on both edges in the direction of the arrows. We define the rank functions at u and v by the array

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

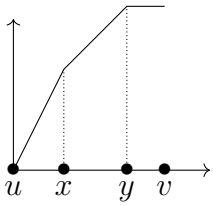
(jumps depicted in blue), and the rank functions at all the other points of Γ are chosen to be standard. This fully describes $\mathfrak{S}$. Consider the effective divisor $D = 4\,(u)$. The pair $(D, \mathfrak{S})$ is an effective crude linear series of degree 4 and rank 2. To see this, we need to check Property $(*)$ in Definition 3.4.1 for $E = (x) + (y)$, for points x, y in Γ. This can be

done by a case analysis depending on whether x or y coincide with a vertex, or they are in the interior of the same edge, or in the interior of two different edges of Γ.

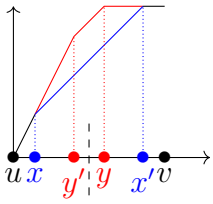
- If $x = y = u$, we can take f to be the zero (or any constant) function on Γ.
- If $x = y = v$, we can take for f any function having constant slope 2 on both edges in the direction of the arrows.
- If $x = u$, $y = v$, we can take for f any function having constant slope 1 on both edges in the direction of the arrows.
- If $x = u$ and $y \notin \{u, v\}$, we can take f with slopes 1 and 0 on both edges as follows:

  The remaining case $x \notin \{u, v\}$, $y = v$ is similar.
- If $x, y \notin \{u, v\}$ and x, y are on the same edge, then we can take f to behave as the following function (with slopes 2, 1 and 0) on both edges:

  This works even if $x = y$, in which case f does not take the slope 1.
- If $x, y \notin \{u, v\}$ and x, y are on different edges, we define x' (resp. y') to be the point of the edge containing x (resp. y) symmetrical about the middle of this edge. Then, we can take f to be the function

  with values on one edge represented in blue, values on the other edge represented in red, and values common to both edges represented in black.

On the same graph, $D = 2\,(u) + 2\,(v)$ with slopes $-1 < 0 < 1$ on both edges, and the same slope structure as above, provides another crude linear series of degree 4 and rank 2. ◇

Example 3.4.2. Here is another simple example that will be used later (see Example 3.7.6). We consider the metric graph Γ depicted in Figure 3.3 with arbitrary positive edge lengths.

Figure 3.3: The path graph Γ with three vertices, and edges of arbitrary positive lengths.

We allow slopes $-2 < 0 < 2$ in the direction of the arrows and take $D = 2\,(u)+4\,(v)+2\,(w)$. We choose the rank function ρ_v on the vertex v to be the same as in Example 3.4.1. The rank functions on u and w are automatically standard, since these vertices are of valence 1. Then, $(D, \mathfrak{S})$ is a crude linear series of degree 8 and rank 2.

Without changing D, we can also consider a slope substructure $\mathfrak{S}'$ of $\mathfrak{S}$ of rank one by allowing slopes $0 < 2$ on the edge uv and slopes $-2 < 0$ on the edge vw. We adapt ρ'_v at v as follows (jumps depicted in blue): $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The pair $(D, \mathfrak{S}')$ is an effective crude linear series of degree 8 and rank 1.

We divide the coefficients of D to obtain the divisor $D' = 2\,(v) + (u) + (w)$. We then consider the slope substructure $\mathfrak{S}''$ with allowed slopes $0 < 1$ on uv and $-1 < 0$ on vw. This makes $(D', \mathfrak{S}'')$ an effective crude linear series of degree 4 and rank 1. ◇

3.4.2 Non-increasing property of slope vectors and a finiteness theorem

In the rest of this section, we prove two important results about crude linear series.

Let $(D, \mathfrak{S})$ be a crude linear series on Γ of rank r. Let $G = (V, E)$ be a model of Γ such that $\mathfrak{S}$ is defined on G and D is supported on V. Let $e = uv$ be an oriented edge of G. For each point x in the interior of e, let $\nu \in \mathrm{T}_x(\Gamma)$ be the unit tangent vector consistent with the orientation of e. Let $s_0^\nu(x) < s_1^\nu(x) < \cdots < s_r^\nu(x)$ be the corresponding slopes in S^e. By an abuse of notation, for each point y on e, we still denote by ν the tangent vector in $\mathrm{T}_y(\Gamma)$ parallel to e, and thus to $\nu \in \mathrm{T}_x(\Gamma)$, and denote by $s_0^\nu(y) < s_1^\nu(y) < \cdots < s_r^\nu(y)$ the corresponding slopes.

Proposition 3.4.3 (Non-increasing property of slope vectors)**.** *Notation as above, the collection of vectors $(s_0^\nu, \ldots, s_r^\nu)$, as a vector-valued function on the segment corresponding to e, forms a coordinate-wise non-increasing collection of vectors.*

In other words, for every $\varepsilon > 0$ small enough, denoting by $y = x + \varepsilon\nu$ the point at distance ε from x in the direction of ν, we have

$$s_j^\nu(y) \leqslant s_j^\nu(x) \qquad \text{for every } j = 0, 1, \ldots, r.$$

Proof. Since the vector of slopes is piecewise constant, we can suppose that x is in the interior of e. Changing the model by adding x as a new vertex of Γ, if necessary, we can

suppose that x is a vertex of G. Denoting by e' the edge emanating from x in the direction ν and by e'' the other edge incident to x, oriented toward x (that is, with an orientation compatible with that of e'). To prove the proposition, it is sufficient to show that for every $j \in \{1, \dots, r\}$, $s_j^{e'}(x) \leqslant s_j^{e''}(x)$. We will use the divisorial rank property in the definition of crude linear series. In the following, $\overline{e''}$ denotes the edge e'' with the reverse orientation.

Let $p_1, \dots, p_j$ be j distinct points on e' close enough to x, in this order away from x, such that the slope structure is constant between x and p_j. Likewise, let $q_1, \dots, q_{r-j}$ be $r-j$ distinct points on e'' close enough to x, in this order away from x, such that the slope structure is constant between x and q_{r-j}. Consider the effective divisor of degree r

$$E \coloneqq \sum_{i=1}^{j} (p_i) + \sum_{i=1}^{r-j} (q_i).$$

By property $(*)$ in Definition 3.4.1, there exists $f \in \mathrm{Rat}(D, \mathfrak{S})$ such that $D + \mathrm{div}(f) - E$ is an effective divisor. The vector of outgoing slopes of f around x, $\partial_x(f)$, corresponds to some jump $\underline{a} \in J_{\rho_x}$. Since by construction D has no support between q_{r-j} and p_j, the inequality $D + \mathrm{div}(f) - E \geqslant 0$ implies that f has a positive order of vanishing on all the points p_i and q_i. This in turn implies that $a_{e'} \geqslant j$ and $a_{\overline{e''}} \geqslant r - j$. The fact that $\mathrm{div}(f)(x) \geqslant 0$ implies that $s^{e'}_{a_{e'}}(x) + s^{\overline{e''}}_{a_{\overline{e''}}}(x) \leqslant 0$. Finally,

$$s_j^{e'}(x) \leqslant s^{e'}_{a_{e'}}(x) \leqslant -s^{\overline{e''}}_{a_{\overline{e''}}}(x) \leqslant -s^{\overline{e''}}_{r-j}(x) = s_j^{e''}(x).$$

□

Theorem 3.4.4 (Theorem 3.1.4 on the finiteness of slopes structures in crude linear series)**.** *Let Γ be a metric graph and let D be a divisor on Γ. Let $G = (V, E)$ be a combinatorial model of Γ that supports D on its vertices. For each integer r, there exist finitely many subdivisions $H_1, \dots, H_k$ of G, and an r-slope structure $\mathfrak{S}_j$ defined on H_j for $j = 1, \dots, k$, such that every crude linear series $(D, \mathfrak{S})$ of rank r has a model H_j of Γ among $H_1, \dots, H_k$ on which it is defined, and moreover, the equality $\mathfrak{S} = \mathfrak{S}_j$ holds.*

Proof. Using [GK08, Lemma 1.8], we infer that the slopes appearing in the slope structure $\mathfrak{S}$ are all bounded. Applying Proposition 3.4.3, this implies that the number of graph models over which the slope structure is defined is finite, and there are only finitely many possibilities for rank functions on the vertices of each of these graph models. The result follows. □

3.5 Admissible semimodules

In this section, we introduce admissible semimodules of rational functions. The idea is to replace the full space of rational functions $\mathrm{Rat}(D, \mathfrak{S})$ in a crude linear series (which is in general not closed, in the topological sense) by a closed semimodule, still enjoying the properties of crude linear series.

3.5.1 Semimodule structure on spaces of rational functions

Let $\mathbb{T} = (\mathbb{R}, \oplus, \odot)$ be the semifield of tropical numbers. The space $\mathbb{R}^\Gamma$ of real-valued functions on Γ is naturally a $\mathbb{T}$-semimodule: for $f, g \in \mathbb{R}^\Gamma$ and $c \in \mathbb{R}$, we have operations of tropical addition and tropical multiplication by scalars

$$f \oplus g := \min(f, g) \qquad \text{and} \qquad c \odot f := f + c.$$

The relevance of this discussion is in the following basic result, which shows that the space of rational functions associated to a divisor endowed with a slope structure is a $\mathbb{T}$-semimodule.

Proposition 3.5.1. *The space* $\mathrm{Rat}(\Gamma)$ *is a* $\mathbb{T}$*-semimodule. Both subsets* $\mathrm{Rat}(D)$ *and* $(\mathrm{Rat}(D, \mathfrak{S})$*, for a divisor* D *and a divisor endowed with a slope structure), respectively, are* $\mathbb{T}$*-semimodules.*

Proof. For the first two statements see [HMY12, Lemma 4]. In order to prove the last statement, we need to show that $\min(f, g)$ belongs to $\mathrm{Rat}(D, \mathfrak{S})$ if f and g do. Let v be a vertex of Γ and let d_v be the valence of v. Denote by $\underline{a}$, $\underline{b}$ and $\underline{c}$ the elements of $\square_r^{d_v}$ such that $s_{\underline{a}} = \partial_v(f)$, $s_{\underline{b}} = \partial_v(g)$ and $s_{\underline{c}} = \partial_v(\min(f, g))$. If $f(v) = g(v)$, then we have $\partial_v(\min(f, g)) = \min(\partial_v(f), \partial_v(g))$, which implies $\underline{c} = \underline{a} \wedge \underline{b}$. Otherwise, we have $\underline{c} = \underline{a}$ or $\underline{b}$ depending on whether $f(v) < g(v)$ or $f(v) > g(v)$, respectively. Using that J_{ρ_v} is stable under $\wedge$ (see Proposition 3.2.13), we conclude that, in either case, $\underline{c}$ belongs to J_{ρ_v}, and the proposition follows. □

We introduce some terminology that we use later.

Definition 3.5.2. Let S be a subset of $\mathbb{R}^\Gamma$ and $v \in \Gamma$. By S_v, we mean the space of all functions f of S such that $f(v) = 0$. This will be used, in particular, when S is a sub-semimodule M of some linear series $\mathrm{Rat}(D, \mathfrak{S})$.

For a subset S of $\mathbb{R}^\Gamma$ and $f \in \mathbb{R}^\Gamma$, we also define $S(-f) := S - f = \{h - f \mid h \in S\}$. ◇

The latter definition mimics the linear equivalence relation between divisors or slope structures (Section 3.3.7). Notice that if M is a sub-semimodule of $\mathbb{R}^\Gamma$, then this is also true of $M(-f)$.

Definition 3.5.3. A semimodule $M \subseteq \mathbb{R}^\Gamma$ is called *effective* if it contains the null function. ◇

Remark 3.5.4. This definition extends Definition 3.3.7 where $M = \mathrm{Rat}(D, \mathfrak{S})$. Note that if $f \in M$, then $M(-f)$ is effective. ◇

3.5.2 Restriction and extension of scalars

The above notions generalize easily to the case of Λ-rational divisors on Λ-metric graphs.

If Γ and D are Λ-rational and if M is a sub-$\mathbb{T}$-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$, we define $M_\Lambda \coloneqq M \cap \mathrm{Rat}_\Lambda(\Gamma)$, the sub-$\Lambda$-semimodule of $\mathrm{Rat}_\Lambda(D, \mathfrak{S})$ made up of elements $f \in M$ which are Λ-rational. We say that M_Λ is obtained from M by *restriction of scalars*.

Assume that Γ and D are Λ-rational. Let Λ' be another sub-semi-field of $\mathbb{R}$ such that $\Lambda \subseteq \Lambda'$, and M be a sub-Λ-semimodule of $\mathrm{Rat}_\Lambda(D, \mathfrak{S})$. We define $M^{\Lambda'}$ to be the sub-Λ'-semimodule of $\mathrm{Rat}_{\Lambda'}(D, \mathfrak{S})$ generated by M. We say that $M^{\Lambda'}$ is obtained from M by *extension of scalars*.

3.5.3 Finite generation and closedness

A semimodule M over a commutative semi-ring R is *finitely generated* if there exist $f_1, \dots, f_n \in M$ such that for all $g \in M$, there exist $c_1, \dots, c_n \in R$ such that $g = \bigoplus_{1 \leqslant i \leqslant n} c_i \odot f_i$.

Using cut sets and extremal points, Haase, Musiker and Yu showed in [HMY12] that for every divisor D, $\mathrm{Rat}(D)$ is a finitely generated $\mathbb{T}$-semimodule. This is not necessarily the case for spaces of the form $\mathrm{Rat}(D, \mathfrak{S})$ for crude linear series $(D, \mathfrak{S})$ (see Example 3.7.6).

A semimodule $M \subseteq \mathrm{Rat}(\Gamma)$ is called *closed* if it is closed for the norm $\| \cdot \|_\infty$.

We have the following basic result on the connection between finite generation and closedness properties.

Proposition 3.5.5. *Let $M \subseteq \mathrm{Rat}(\Gamma)$ be a finitely generated semimodule. Then, M is closed.*

Proof. Let $(f_n)_n$ be a sequence of functions of M converging to a function f in $\mathrm{Rat}(\Gamma)$ for the $\| \cdot \|_\infty$ topology. Assume that M is generated by elements $h_1, \dots, h_r$, and write, for all n, $f_n = \min_{1 \leqslant i \leqslant r}(h_i + c_i^n)$ with $c_i^n \in \mathbb{R}$. We can suppose that all h_i are zero at some point v. It follows that the sequences $(c_i^n)_n$ are bounded. By extraction, we can assume that for every i, $(c_i^n)_n$ converges to some $c_i \in \mathbb{R}$, implying that $f_n \xrightarrow{\|\cdot\|_\infty} \min_{1 \leqslant i \leqslant r}(h_i + c_i)$, and thus $f \in M$. □

We next give an alternative characterization of closedness for semimodules $M \subseteq \mathrm{Rat}(D)$. First, notice that each effective divisor D of degree d is written in the form $D = (x_1) + \cdots + (x_d)$ for points $x_j \in \Gamma$, and can be viewed as a point $[x_1, \dots, x_d]$ in the d-th symmetric product $\mathrm{Sym}^d(\Gamma)$ of the metric graph, which is a compact metric space. Furthermore, we can view $\mathrm{Sym}^d(\Gamma)$ as the subspace of $\mathrm{Div}^d(\Gamma)$ consisting of effective divisors of degree d. This defines a map

$$\varphi \colon \mathrm{Rat}(D) \longrightarrow \mathrm{Sym}^d(\Gamma) \hookrightarrow \mathrm{Div}^d(\Gamma)$$

by $\varphi(f) \coloneqq D + \mathrm{div}(f)$. We set $|M| = \varphi(M)$, and view it in $\mathrm{Sym}^d(\Gamma)$. We endow $\mathrm{Rat}(D)$ with the $\| \cdot \|_\infty$ topology.

Proposition 3.5.6. *The map φ is continuous. Furthermore, a semimodule $M \subseteq \mathrm{Rat}(D)$ is closed if, and only if, $|M| \subseteq \mathrm{Sym}^d(\Gamma)$ is closed.*

Proof. We omit the proof of the first assertion, and prove only the second.

$\Longrightarrow$ If M is closed then so is M_v, and $|M| = \varphi(M_v)$. Let k be a universal bound for the slopes in $\mathfrak{S}$ (see [GK08, Lemma 1.8]), and let B be the space of continuous functions $\Gamma \to \mathbb{R}$ whose slopes are bounded by k. By the Arzelá–Ascoli theorem, B is compact, and since $M_v \subseteq B$ is closed, we infer that M_v is compact. Since φ is continuous, $|M|$ is compact and thus closed in $\mathrm{Sym}^d(\Gamma)$.

$\Longleftarrow$ Let $(f_n)_n$ be a sequence of functions of M converging to some $f \in \mathrm{Rat}(D)$. Then by continuity $\varphi(f_n) \longrightarrow \varphi(f)$. By closedness, $\varphi(f) \in |M|$, so there exists some $g \in M$ such that $\varphi(f) = \varphi(g)$. Since $\mathrm{div}(f) = \mathrm{div}(g)$, f and g differ by some constant, and thus $f \in M$. □

3.5.4 Closedness and extremal generators

In this section, we provide a result linking the closedness property and extremal generators. We recall that if M is a subset of a semimodule, then $x \in M$ is called *extremal* if an equality of the form $x = y \oplus z$ with $y, z \in M$ implies $y = x$ or $z = x$. If M is a finitely generated sub-semimodule of $\mathbb{R}^\Gamma$, then it is generated by its extremals, of which there is a finite number up to tropical scaling, see [HMY12, Proposition 8]. This fact is related to the following result (see Proposition 3.5.5 for the link between finite generation and closedness).

Proposition 3.5.7. *Let D be a divisor on Γ and M be a sub-semimodule of $\mathrm{Rat}(D)$. Assume that M is closed in $\mathrm{Rat}(D)$. Then M is generated by its extremals.*

Remark 3.5.8. A consequence of the above proposition is that a closed sub-semimodule of $\mathrm{Rat}(D)$ is finitely generated if, and only if, it contains a finite number of extremals. ⋄

The proof relies on the following lemmas, which have their own significance.

Lemma 3.5.9. *Using the notations of Proposition 3.5.7 and under the same hypotheses, let $f \in M$ and $v \in \Gamma$. Then there exists a function $g \in M$ which is extremal in M and such that $g \geqslant f$ and $g(v) = f(v)$.*

Proof. Without loss of generality, we assume that $f(v) = 0$. Let P be the (non-empty) set of all functions $h \in M$ such that $h(v) = f(v)$ and $h \geqslant f$. P is closed in M_v, which is compact. Let $(h_s)_{s \in S}$ be a chain in P (i.e., a totally ordered subset). Since P is bounded, the function $h := \sup_s h_s$ is well-defined. By an argument similar to that used in the proof of Proposition 3.7.3, and using the fact that $\{h_s,\ s \in S\}$ is totally ordered, h can be written as the limit of some sequence $(h_n)_n$ of functions of P. Since P is closed, we get $h \in P$.

We have shown that every chain in P has an upper bound, so by Zorn's lemma, P admits a maximal element g. Since g is maximal, it is extremal, which concludes. □

Lemma 3.5.10. *Using the notations of Proposition 3.5.7, let $f \in M$. Then there exist an integer n and points $x_1, \ldots, x_n \in \Gamma$ such that for all $g \in M$, we have $g = f$ if, and only if, $g(x_i) = f(x_i)$ for all i.*

Proof. We choose a model G of Γ such that the supports of D and $\mathrm{div}(f)$ are included in the set of vertices. Then we consider the set of all vertices of G, and add an extra point strictly between every pair of adjacent vertices: this gives a set of points $x_1, \dots, x_n$. Let now $g \in M$ be such that for all i, $g(x_i) = f(x_i)$, so f and g coincide at every vertex and at some point in the interior of every edge. Let now $e = x_i x_j$ be an edge, containing the marked point x_k in its interior. Since the interior of e contains no point of the support of $\mathrm{div}(f)$, we know that f is linear on e. Since the interior of e contains no point of the support of D, the slopes of g along e are non-increasing. Combined with the fact that $g(x_i) = f(x_i)$ and $g(x_j) = f(x_j)$, we get that $g \geqslant f$ on e. Since $g(x_k) = f(x_k)$, we have in fact $g = f$ on e, and this is true on every edge of Γ, so $g = f$. □

Proof of Proposition 3.5.7. Let f be an element of M. For every $x \in \Gamma$, Lemma 3.5.9 provides an extremal $g^x \in M$ such that $g^x \geqslant f$ and $g^x(x) = f(x)$. We apply this to every point x_i given by Lemma 3.5.10, which yields extremal functions $g^{x_1}, \dots, g^{x_n} \in M$ such that for all i, $g^{x_i} \geqslant f$ and $g^{x_i}(x_i) = f(x_i)$. Define $g := \min_i g^{x_i}$. For all i, we have $g(x_i) = f(x_i)$, and thus $g = f$, which shows that f is generated by extremals. □

We end this section with an open question about the topological properties of linear series.

Question 3.5.11. *Let M be a closed sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$. Is there any connection between M being finitely generated and M being of finite tropical rank? In other words (in light of Remark 3.5.8), is there any connection between M having a finite number of extremals and M being of finite tropical rank?*

3.5.5 Admissible semimodules

Let d be an integer and D a divisor of degree d on Γ.

A semimodule $M \subseteq \mathrm{Rat}(D)$ is called *admissible of rank r* if it is closed for the topology induced by $\|\cdot\|_\infty$, and if there exists an r-slope structure $\mathfrak{S}$ such that $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ and such that the following holds:

($**$) For every effective divisor E on Γ of degree r, there exists $f \in M$ such that

(AS1) For every point $x \in \Gamma$, $\rho_x(\partial_x(f)) \geqslant E(x)$; and in addition,

(AS2) $D + \mathrm{div}(f) - E \geqslant 0$.

When Γ and D are Λ-rational, we say a semimodule $M \subseteq \mathrm{Rat}_\Lambda(D, \mathfrak{S})$ is admissible if the extension $M^{\mathbb{R}} \subseteq \mathrm{Rat}(D, \mathfrak{S})$ is admissible. ◇

Note that if M is effective as a semimodule, then so is $\mathrm{Rat}(D, \mathfrak{S})$ and therefore $(D, \mathfrak{S})$ is by definition effective as a crude linear series.

Although it is not clear from the definition, we will show in the next section that admissibility is a property of the pair (D, M), that is, both the rank r and the slope structure $\mathfrak{S}$ (and thus the crude linear series $(D, \mathfrak{S})$) can be extracted from M.

Example 3.7.6 shows that, in general, the semimodule $\mathrm{Rat}(D, \mathfrak{S}) \subseteq \mathrm{Rat}(D)$ for a crude linear series $(D, \mathfrak{S})$ may not be closed in $\mathrm{Rat}(D)$, and therefore may not be admissible. Furthermore, a crude linear series $(D, \mathfrak{S})$ of rank r might not necessarily admit an admissible semimodule $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ of the same rank, although we do not have an example thereof.

Remark 3.5.12. For further use, we note that if $(f_n)_n$ is a sequence of functions in $\mathrm{Rat}(D)$ converging uniformly to a function f, then $f \in \mathrm{Rat}(D)$ and, for every $x \in \Gamma$, the inequality $\partial_x(f) \leqslant \partial_x(f_n)$ holds coordinate-wise for all large n. This inequality can be strict in general, and the corresponding point of the hypercube may not be a jump of the rank function at x. This explains why a semimodule of the form $\mathrm{Rat}(D, \mathfrak{S})$ is not necessarily closed. ◇

3.5.6 Realization of jumps in admissible semimodules

Let D be a divisor of degree d. In this section, we prove the following theorem.

Theorem 3.5.13 (Realization of jumps in admissible semimodules)**.** *Let $(D, \mathfrak{S})$ be a crude linear series of degree d and rank r, and let $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ be an admissible semimodule of rank r. Let v be a point of Γ and let $\underline{a} \in J_{\rho_v}$ be a jump. Then, there exists $f \in M$ such that $\partial_v(f) = s_{\underline{a}}$.*

For example, taking $E = r(v)$ for $v \in \Gamma$, we deduce from property $(**)$ the existence of a function $f \in M$ such that $\rho_v(\partial_v(f)) \geqslant r$. This implies that f takes all minimum slopes around v, that is, $\partial_v(f) = s_{\underline{0}}$. The proof for other values of $\underline{a}$ is more involved.

This theorem immediately implies the following important results.

Corollary 3.5.14 (Corollary 3.1.3)**.** *Keeping the notation of Theorem 3.5.13, the slope structure $\mathfrak{S}$ can be entirely retrieved from M.*

In light of this corollary, we will often drop $\mathfrak{S}$ and only say that M is an admissible semimodule of $\mathrm{Rat}(D)$.

Corollary 3.5.15. *Let $(D, \mathfrak{S})$ and $(D', \mathfrak{S}')$ be two crude linear series on Γ of rank r and r', respectively, and $M' \subseteq \mathrm{Rat}(D', \mathfrak{S}')$ and $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ be admissible. Then $\mathfrak{S}'$ is a slope substructure of $\mathfrak{S}$ provided that $M' \subseteq M$ holds.*

Remark 3.5.16. Note that the divisor D cannot be uniquely determined by M, because if $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$, then $M \subseteq \mathrm{Rat}(D', \mathfrak{S})$ for every divisor $D' \geqslant D$. However, there is a unique base-point free choice for D, that is, a unique minimal choice $D_{\min}$ for D. Simply define the coefficient $D_{\min}(x)$ of $D_{\min}$ as the minimum possible integer such that all the functions $f \in M$ verify $\mathrm{ord}_x(f) + D_{\min}(x) \geqslant 0$. This is well-defined, and every other choice D with $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ verifies $D \geqslant D_{\min}$. ◇

The rest of the section is devoted to the proof of the realization theorem.

3.5.7 Proof of Theorem 3.5.13

Consider a model $G = (V, E)$ such that $\mathfrak{S}$ is defined on G and let v be a vertex of V. Let $\delta = d_v$. We show that for each jump $\underline{a} \in J_{\rho_v}$, there exists $f \in M$ such that $\partial_v(f) = s_{\underline{a}}$. The idea is to use the divisorial rank condition $(**)$ satisfied by the admissible semimodule M in order to find a sequence of functions with the desired slope at a given tangent direction at v. We then find f as the limit of a subsequence, using the closedness of M.

We start with a lemma.

Lemma 3.5.17. *Let $\underline{q} \in \square_r^\delta$ be any point in the hypercube at v such that $\rho_v(\underline{q}) \geqslant 0$. Then, there exists a unique jump $\underline{a} \in J_{\rho_v}$ of rank $\rho_v(\underline{q})$ with $\underline{a} \succeq \underline{q}$.*

Proof. We reason by contradiction to show existence. If no such jump exists, we can construct an increasing path (i.e., a finite sequence of points $\underline{x}^i$ of $\square_r^\delta$ such that $\underline{x}^i \prec \underline{x}^{i+1}$ for every i), starting at $\underline{q}$ and made up of points of constant rank. This process necessarily ends up at the point $\underline{r} = (r, \dots, r)$. We know that $\rho_v(\underline{r}) = \rho_v(\underline{q}) = 0$. Therefore, $\underline{r}$ is a jump, $\underline{r} \succeq \underline{q}$ and they have the same rank, a contradiction.

We now show uniqueness. Suppose there are two different jumps $\underline{a}$ and $\underline{b}$ such that $\underline{a} \geqslant \underline{q}$, $\underline{b} \geqslant \underline{q}$, $\rho_v(\underline{a}) = \rho_v(\underline{b}) = \rho_v(\underline{q})$. Since $\underline{a} \neq \underline{b}$, then $\underline{a} \wedge \underline{b}$ is different from at least one among $\underline{a}$ and $\underline{b}$, say $\underline{a}$. Proposition 3.2.14 yields that $\rho_v(\underline{a} \wedge \underline{b}) > \rho_v(\underline{a})$, which is impossible because $\underline{q} \leqslant \underline{a} \wedge \underline{b} \leqslant \underline{a}$ implies $\rho_v(\underline{a}) = \rho_v(\underline{q}) \geqslant \rho_v(\underline{a} \wedge \underline{b}) \geqslant \rho_v(\underline{a})$. □

We now come to the proof of the theorem. We start by defining an increasing path in $\square_r^\delta$ that starts at $\underline{0}$, stays below $\underline{a}$ and moves only along the first direction in the hypercube $\square_r^\delta$ at the beginning, then only along the second direction, and so on, until direction δ.

Let, for all $i \in \{1, \dots, \delta\}$, e_i be the edge incident to v corresponding to the direction i. For convenience, we also define, for all $i \in \{1, \dots, \delta\}$, $e'_i := e_{\delta - i + 1}$, just reversing the order of the edges around v.

Given a point $\underline{y} \in \square_\delta^r$ with $\underline{y} \preceq \underline{a}$ and a direction $\underline{\mathfrak{e}}_i$ in the hypercube, we say that $\underline{y}$ is *a fall in the direction $\underline{\mathfrak{e}}_i$ with respect to $\underline{a}$* if $y_i = 0$ or $\rho_v(\underline{y}) < \rho_v(\underline{y} - \underline{\mathfrak{e}}_i)$. We say that $\underline{y}$ is *a largest fall in the direction $\underline{\mathfrak{e}}_i$ with respect to $\underline{a}$* if $\underline{y}$ is a fall in the direction $\underline{\mathfrak{e}}_i$ with respect to $\underline{a}$ and if for all non-negative integers n such that $\underline{y} + n\,\underline{\mathfrak{e}}_i \preceq \underline{a}$, we have $\rho_v(\underline{y} + n\,\underline{\mathfrak{e}}_i) = \rho_v(\underline{y})$. Saying in words, moving in the direction of $\underline{\mathfrak{e}}_i$ from a largest fall, remaining bounded by $\underline{a}$, the rank does not change.

We construct our increasing path starting from $\underline{0}$, going each time to a largest fall in the given direction relative to $\underline{a}$ and remembering only the falls in that direction. More precisely, suppose that we have already built the path along directions $1, \dots, i-1$ with $1 \leqslant i \leqslant \delta$, consisting of all the falls in the direction of $\underline{\mathfrak{e}}_1$, then the falls in the direction of $\underline{\mathfrak{e}}_2$, …, and the falls in the direction of $\underline{\mathfrak{e}}_{i-1}$. Therefore, the path currently ends at the point $\sum_{k=1}^{i-1} t_k\,\underline{\mathfrak{e}}_k$, $t_k \in [r]$. We will now let the path continue only in the direction i, by adding multiples of $\underline{\mathfrak{e}}_i$, adding the falls in the direction of $\underline{\mathfrak{e}}_i$ to the path, until we reach the point $\underline{y} = \sum_{k=1}^{i-1} t_k\,\underline{\mathfrak{e}}_k + t_i\,\underline{\mathfrak{e}}_i$ with $\underline{y} \preceq \underline{a}$, which is a largest fall in the direction $\underline{\mathfrak{e}}_i$ with respect to $\underline{a}$. This way, we have built an increasing path starting at $\underline{0}$, composed only of

falls in some direction, staying below $\underline{a}$ and moving successively in directions $\underline{\mathfrak{e}}_1, \dots, \underline{\mathfrak{e}}_\delta$. The turning point, where we move to a different direction, is a largest fall with respect to $\underline{a}$ in that direction. The ending point of the path

$$\underline{z} = \sum_{k=1}^{\delta} t_k \, \underline{\mathfrak{e}}_k$$

is a largest fall in the direction $\underline{\mathfrak{e}}_\delta$ with respect to $\underline{a}$. In particular, note that $\underline{z} \preceq \underline{a}$.

For every $i \in \{1, \dots, \delta\}$, let ℓ_i be the rank drop of the path in the direction i:

$$\ell_i := \rho_v\left(\sum_{k=1}^{i-1} t_k \, \underline{\mathfrak{e}}_k\right) - \rho_v\left(\sum_{k=1}^{i} t_k \, \underline{\mathfrak{e}}_k\right).$$

Then, $\sum_{i=1}^{\delta} \ell_i = r - \rho_v(\underline{z})$.

Lemma 3.5.18. *Let $\underline{z}$ be the endpoint of the increasing path constructed above. We have $\rho_v(\underline{z}) = \rho_v(\underline{a})$. Therefore, $\rho_v(\underline{a}) + \sum_{i=1}^{\delta} \ell_i = r$.*

Proof. Let $\underline{w} \in \square_r^\delta$ be such that $\underline{z} \preceq \underline{w} \preceq \underline{a}$ and let $i \in \{1, \dots, \delta\}$ be such that $\underline{w} + \underline{\mathfrak{e}}_i \preceq \underline{a}$. By construction, we know that

$$\rho_v\left(\sum_{k=1}^{i} t_k \, \underline{\mathfrak{e}}_k + \underline{\mathfrak{e}}_i\right) = \rho_v\left(\sum_{k=1}^{i} t_k \, \underline{\mathfrak{e}}_k\right).$$

This equality, together with supermodularity, implies that $\rho_v(\underline{w} + \underline{\mathfrak{e}}_i) = \rho_v(\underline{w})$. Applying this fact recursively yields $\rho_v(\underline{z}) = \rho_v(\underline{a})$. The last equality follows from $\rho_v(\underline{z}) + \sum_{i=1}^{\delta} \ell_i = r$. □

For every $i \in \{1, \dots, \delta\}$, let $p_1^i, \dots, p_{\ell_i}^i$ be distinct points on the edge $e_i' = e_{n-i+1}$, ordered increasingly with respect to their distance from v. Let

$$E := \rho_v(\underline{a})\,(v) + \sum_{i=1}^{\delta} \sum_{j=1}^{\ell_i} (p_j^i).$$

By the preceding lemma, this is an effective divisor of degree r. Since M is admissible, there exists $f \in M$ such that $f(v) = 0$, and

$$\rho_v(\partial_v(f)) \geqslant \rho_v(\underline{a}) \quad \text{and} \quad \operatorname{div}(f) + D - E \geqslant 0. \tag{3.2}$$

Let $\underline{b} := \partial_v(f) \in \square_r^\delta$. Note that $\underline{b}$ is a jump of ρ_v. The following lemma will imply that $\underline{b} = \underline{a}$.

Lemma 3.5.19. *We have $\underline{b} \succeq \underline{z}$.*

Proof. The first of the two properties in (3.2) tells us that $\rho_v(\underline{b}) \geqslant \rho_v(\underline{a}) = \rho_v(\underline{z})$.

We will now show that the second property in (3.2), when applied to a sequence of effective divisors of degree r that starts at E, implies that $\underline{b} \succeq \underline{z}$. The idea is to make all

the points in the support of E converge to v one after the other, apply the admissibility of M, and define a decreasing sequence of jumps.

Let $\left(p_{1,n}^1\right)_{n\geqslant 0}$ be a sequence of points of e_1' starting at p_1^1 and which converges to v. We replace p_1^1 in E by $p_{1,n}^1$ to obtain an effective divisor $E_{1,n}^1$ of degree r. Formally,

$$E_{1,n}^1 := \rho_v(\underline{a})\,(v) + \left(p_{1,n}^1\right) + \sum_{j=2}^{\ell_1}\left(p_j^1\right) + \sum_{i=2}^{\delta}\sum_{j=1}^{\ell_i}\left(p_j^i\right).$$

Applying the admissibility of M, we find a function $f_{1,n}^1 \in M$ with $f_{1,n}^1(v) = 0$ such that $\operatorname{div}(f_{1,n}^1) + D - E_{1,n}^1 \geqslant 0$. Replacing $f_{1,n}^1$ by $\min\left(f, f_{1,n}^1\right) \in M$ ensures that for all n, $\partial_v\left(f_{1,n}^1\right) \preceq \partial_v(f)$. Moreover, since M is closed, we can extract a subsequence so that the sequence $\left(f_{1,n}^1\right)_n$ converges to an element $f_1^1 \in M$. Let $E_1^1 = E - (p_1^1) + (v)$.

By Remark 3.7.9, for all n,

$$\partial_v\left(f_1^1\right) \preceq \partial_v\left(f_{1,n}^1\right) \preceq \partial_v(f).$$

Besides, the fact that $\left(\partial_v(f_1^1)\right)_1 < \left(\partial_v(f_{1,n})\right)_1$ implies that $\partial_v(f_1^1) \neq \partial_v(f)$.

What precedes yields a jump $\underline{b}_1^1 := \partial_v(f_1^1) \preceq \partial_v(f) = \underline{b}$, different from $\underline{b}$ because the first coordinates verify $\left(\underline{b}_1^1\right)_1 < b_1$. Therefore, $\underline{b}_1^1 \prec \underline{b}$.

We repeat the same process as above, starting from E_1^1. We take a sequence $\left(p_{2,n}^1\right)_n$ of points of e_1' starting at p_2^1, and define

$$E_{2,n}^1 := \left(\rho_v(\underline{a}) + 1\right)(v) + \left(p_{2,n}^1\right) + \sum_{j=3}^{\ell_1}\left(p_j^1\right) + \sum_{i=2}^{\delta}\sum_{j=1}^{\ell_i}\left(p_j^i\right),$$

and find an element $f_{2,n}^1 \in M$ with $f_{2,n}^1(v) = 0$ such that $\operatorname{div}(f_{1,n}^1) + D - E_{2,n}^1 \geqslant 0$. After taking the minimum $\min\left(f_1^1, f_{2,n}^1\right) \in M$, and passing to a subsequence, we obtain a limit $f_2^1 \in M$. This yields a jump $\underline{b}_2^1 \preceq \underline{b}_1^1$, different from $\underline{b}_1^1$ because $\left(\underline{b}_2^1\right)_1 < \left(\underline{b}_1^1\right)_1$.

We repeat the same process, exhausting first all the points p_j^1 on e_1' in the support of E, then all the points p_j^2 on e_2', and so on, until finally all the points p_j^δ on the last edge. The above reasoning yields a decreasing path of jumps

$$\underline{b} \succ \underline{b}_1^1 \succ \underline{b}_2^1 \succ \cdots \succ \underline{b}_{\ell_1}^1 \succ \cdots \succ \underline{b}_1^\delta \succ \cdots \succ \underline{b}_{\ell_\delta}^\delta.$$

Using Proposition 3.2.14, the sequence of ranks $\left(\rho_v(\underline{b}_j^i)\right)$ is increasing. As a consequence,

$$\rho_v(\underline{b}_{\ell_\delta}^\delta) \geqslant \rho_v(\underline{b}) + \sum_{i=1}^{\delta}\ell_i \geqslant r.$$

We thus infer that $\underline{b}_{\ell_\delta}^\delta = \underline{0}$ and all the rank differences between consecutive jumps in the sequence are exactly one.

Reversing the order in the sequence of jumps constructed above, we get an increasing path of jumps starting at $\underline{0}$ and ending at $\underline{b}$, whereas we defined beforehand an increasing

path starting at $\underline{0}$ and ending at $\underline{z}$. The two increasing paths have the same length, equal to $r - \rho_v(\underline{a})$. To show that $\underline{b} \succeq \underline{z}$, we will prove that the path leading to $\underline{b}$ remains greater than or equal to the path leading to $\underline{z}$ at each step, that is, the k-th element of the former dominates the k-th element of the latter.

We proceed by induction. The claim is true at the beginning because the starting point of both paths is $\underline{0}$. Suppose that the inequality is true at some step j_0 with $0 \leqslant j_0 < r - \rho_v(\underline{a})$. We denote by $\underline{z}_{j_0}$ the current fall and by $\underline{b}_{j_0}$ the current jump. The inequality reads $\underline{b}_{j_0} \succeq \underline{z}_{j_0}$. We suppose that the next fall $\underline{z}_{j_0+1}$ differs from $\underline{z}_{j_0}$ (only) in the direction i_0. Let

$$\underline{c}_{j_0} \coloneqq \underline{z}_{j_0} + \Big(\big(\underline{b}_{j_0}\big)_{i_0} - \big(\underline{z}_{j_0}\big)_{i_0} \Big)\, \underline{\mathfrak{e}}_{i_0}$$

be the point obtained by starting at $\underline{z}_{j_0}$ and moving in the direction i_0 as much as possible without overtaking $\underline{b}_{j_0}$ along this axis.

Since both paths are parametrized by the same integers ℓ_i, we know that the next jump $\underline{b}_{j_0+1}$ will also differ from $\underline{b}_{j_0}$ (at least) in the direction i_0. The latter statement implies that $\underline{b}_{j_0+1} \succeq \underline{b}_{j_0} + \underline{\mathfrak{e}}_{i_0}$. Since $\underline{b}_{j_0}$ is a jump, by supermodularity and using $\underline{b}_{j_0} \succeq \underline{c}_{j_0}$, we get that

$$\rho_v\big(\underline{c}_{j_0} + \underline{\mathfrak{e}}_{i_0}\big) = \rho_v\big(\underline{c}_{j_0}\big) - 1$$

so $\underline{c}_{j_0} + \underline{\mathfrak{e}}_{i_0} \succeq \underline{z}_{j_0}$ is a fall in the direction i_0 which coincides with $\underline{z}_{j_0}$ in all directions but i_0. Therefore,

$$\underline{z}_{j_0+1} \preceq \underline{c}_{j_0} + \underline{\mathfrak{e}}_{i_0} \preceq \underline{b}_{j_0} + \underline{\mathfrak{e}}_{i_0} \preceq \underline{b}_{j_0+1}.$$

We have proved the claim for $j_0 + 1$. We infer that $\underline{b} \succeq \underline{z}$, as desired. □

Proof of Theorem 3.5.13. The preceding lemma shows that $\underline{b} \succeq \underline{z}$, and thus $\rho_v(\underline{b}) \leqslant \rho_v(\underline{z}) = \rho_v(\underline{a})$. Combined with the inequality $\rho_v(\underline{b}) \geqslant \rho_v(\underline{a})$, we deduce that $\rho_v(\underline{b}) = \rho_v(\underline{a})$. On the other hand, $\underline{b} \succeq \underline{z}$ is a jump of ρ_v. Consequently, $\underline{a}$ and $\underline{b}$ are two jumps of ρ_v which dominate $\underline{z}$ and have the same rank as $\underline{z}$. The uniqueness in Lemma 3.5.17 implies that $\underline{b} = \underline{a}$, which finishes the proof. □

3.6 Combinatorial limit linear series

In this section, we define combinatorial limit linear series on metric graphs.

3.6.1 Tropical rank

In linear algebra, the dimension d of a vector space can be characterized as either the size of a minimal generating set or as the least integer such that every collection of $d + 1$ elements is linearly dependent. For spaces of tropical functions, however, there is *a priori* no direct link between the two above notions. The former one corresponds to the finite generation property in semimodules, discussed in Section 3.5.5. We will need a second notion of finiteness based on tropical independence from [JP14, JP16], see also [AGG09] where a finite version of this was developed.

Definition 3.6.1 (Tropical rank)**.** Let $M \subseteq \mathrm{Rat}(\Gamma)$ be a semimodule. We call *tropical rank* of M the least integer $r \in \mathbb{Z}_{\geqslant 0}$ such that for all elements $f_0, \dots, f_{r+1} \in M$, there exist $c_0, \dots, c_{r+1} \in \mathbb{R}$ such that for each $x \in \Gamma$, the minimum in

$$\min_{0 \leqslant i \leqslant r+1} (f_i(x) + c_i)$$

is achieved at least twice, that is, for at least two indices $i \in [r+1]$. If such r does not exist, we say that M is of infinite tropical rank.

For a Λ-semimodule, the tropical rank is defined the same way by imposing $c_i \in \Lambda$. ◇

3.6.2 Limit linear series: Definition

Definition 3.6.2. A *combinatorial limit linear series* of rank r and degree d on a metric graph Γ, more simply called a *linear series* or a $\mathfrak{g}^r_d$, is a pair (D, M) consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$ of rank r which is moreover finitely generated and has tropical rank r. The linear series is called *effective* if M is so.

When Γ and D are Λ-rational, we define a Λ-linear series, or simply Λ-$\mathfrak{g}^r_d$, as a finitely generated semimodule $M \subseteq \mathrm{Rat}_\Lambda(D)$ of tropical rank r such that the semimodule $M^{\mathbb{R}} \subseteq \mathrm{Rat}(D)$, obtained by extension of scalars, is admissible of rank r. ◇

Note that $M^{\mathbb{R}}$ is finitely generated, and by Proposition 3.5.5, it is automatically closed.

Remark 3.6.3. For a linear series (D, M), by Corollary 3.5.14, there is a unique slope structure $\mathfrak{S}$ with $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$, that is, $\mathfrak{S}$ is entirely determined by M. This explains why we do not include the data of $\mathfrak{S}$ in the linear series.

Moreover, in light of Remark 3.5.16, there is a minimal divisor associated to M. However, for clarity and since specific divisors on metric graphs appear naturally, we keep track of D in the definition. ◇

The relevance of this will be explained by Theorem 3.9.1, which states that a semimodule $M \subseteq \mathrm{Rat}(D)$ that comes from the tropicalization of a linear series is finitely generated, is of tropical rank r, and also verifies property $(**)$ in Section 3.5.5. In fact, such an M verifies the stronger property $(\overset{*}{**})$ below. For the notion of slope substructure, we refer to Section 3.3.4.

Definition 3.6.4 (Refined linear series)**.** A *refined linear series*, or *refined* $\mathfrak{g}^r_d$, is a pair (D, M) which is a $\mathfrak{g}^r_d$ and which in addition verifies the following stronger version of $(**)$:

$(\overset{*}{**})$ For every effective divisor E of degree $s \leqslant r$ on Γ, there exists a semimodule $M_E \subseteq M$ such that (D, M_E) is a $\mathfrak{g}^{r-s}_d$ and for every element $f \in M_E$, we have

(1) $\rho_x(\partial_x(f)) \geqslant E(x)$ for each point $x \in \Gamma$, and in addition,

(2) $D + \mathrm{div}(f) - E \geqslant 0$. ◇

Each linear series (D, M) with underlying slope structure $\mathfrak{S}$ gives rise to a linear system $|(D, M)| \subseteq |(D, \mathfrak{S})|$, defined as follows.

Definition 3.6.5 (Linear system associated to a $\mathfrak{g}^r_d$)**.** For a linear series (D, M), we define the linear system $|(D, M)|$ as the set of all effective divisors E on Γ of the form $D + \operatorname{div}(f)$ for $f \in M$. ◇

Remark 3.6.6. For a finitely generated admissible semimodule $M \subseteq \operatorname{Rat}(D)$ of rank r, being a linear series, that is, having tropical rank r, is equivalent to having tropical rank *at most* r. In fact, the tropical rank of an admissible semimodule of given rank r is always at least r. To see this, we observe using Theorem 3.5.13 that at every point x of Γ, in every direction, each of the $r + 1$ slopes in the slope structure underlying M is taken by an element f_i of M. The functions f_i therefore take pairwise distinct constant slopes on a small segment I based at x. This implies that for all $c_0, \dots, c_r \in \mathbb{R}$, the minimum in $\min_{0 \leqslant i \leqslant r}(f_i(y) + c_i)$ is achieved at least twice only for a finite number of points $y \in I$, which shows that the $r + 1$ functions $f_0, \dots, f_r$ are tropically independent. Therefore, the tropical rank of M is at least r. ◇

3.6.3 Examples

We give two simple examples of linear series, and refer to Sections 3.10.1 and 3.10.1 for more examples.

Example 3.6.7 (A $\mathfrak{g}^1_2$ on the barbell graph)**.** Consider the barbell graph Γ with edges of arbitrary length, see Figure 3.4. This metric graph has genus two and the canonical divisor K has rank one. We define a linear series (K, M), $M \subseteq \operatorname{Rat}(K)$, of rank one on Γ. Allow slopes $-1 < 1$ on the middle edge and, for $i = 1, 2$, allow slopes $0 < 1$ on both oriented edges $u_i v_i$, in the direction of the arrows.

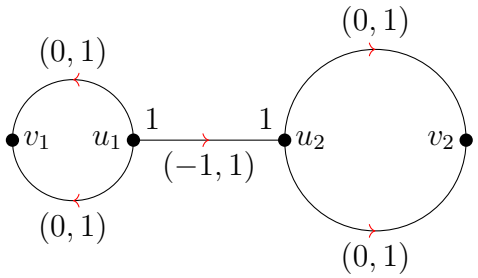


Figure 3.4: The barbell graph, the canonical divisor and the slope structure $\mathfrak{S}$.

We define suitable rank functions on vertices, as follows. For $i = 1, 2$, we endow v_i with the rank function on $\square^2_1 = [1]^2$ defined by the array $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and endow u_i with the rank function on $\square^3_1 = [1]^3$ whose restrictions to $[1]^2 \times \{0\}$ and $[1]^2 \times \{1\}$ are defined by the

following two matrices, respectively: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Here, jumps are depicted in blue. For the two vertices u_1 and u_2, the third coordinate in each of the two rank functions corresponds to the middle edge of Γ. We further endow each other point of Γ with the standard rank function. Set $M := \mathrm{Rat}(K, \mathfrak{S})$. The pair (K, M) is a linear series of degree two and rank one on Γ.

Note that M is not effective, and thus the constant functions do not belong to M. Moreover, (K, M) is the unique $\mathfrak{g}^1_2$ on Γ with the underlying divisor K. This shows that the canonical divisor K on Γ is not realizable, that is, K is not the tropicalization of a divisor $\mathcal{K} \in |\omega_C|$ in the canonical linear system $|\omega_C|$ on a smooth proper curve C over an algebraically closed field with a non-trivial non-Archimedean valuation (otherwise, since $\mathcal{K}$ is effective, the constant functions would belong to the semimodule M). Here, ω_C is the canonical sheaf of C. ◇

Example 3.6.8. Consider the following metric graph Γ with two edges of equal length. Let

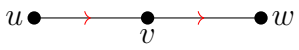


Figure 3.5: The metric graph Γ with edges of equal length.

$D = (u) + (w)$ and define a slope structure $\mathfrak{S}$ by allowing slopes $0 < 1$ on the edge uv and slopes $-1 < 0$ on the edge vw, in the direction of the arrows. Let M be the sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$ made up of all functions which are symmetric with respect to v. We define ρ_v by the array

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here, the jumps of ρ_v are depicted in blue. We endow every other point of Γ with the standard rank function. The pair (D, M) is a $\mathfrak{g}^1_2$. ◇

In the next two sections, we provide a classification of $\mathfrak{g}^1_d$'s.

3.7 Reduced divisors

We establish an extension of the machinery of reduced divisors to linear series. The results of this section are valid without extra effort for the linear system $|(D, M)|$ associated to a pair (D, M) consisting of a divisor D of degree d and an admissible semimodule M of rank r, so we present the results in this generality. Moreover, replacing (D, M) with a linearly equivalent admissible pair, we can assume for the full section that M is effective.

3.7.1 Reduced divisors in the chip-firing context

We briefly recall the definition of reduced divisors in the "chip-firing" context. In terms of the chip-firing game, the x-reduced divisor is obtained from D by firing chips the closest

possible to x.

More formally, a *cut* X in a metric graph Γ is a compact subset of Γ with finitely many connected components. The (finite) set of boundary points of X, denoted by ∂X, is the set of all points of X which are in the closure of the complement of X in Γ. For a point $x \in \partial X$, we denote by $\mathrm{out}_X(x)$ the set of all outgoing branches (given by unit tangent vectors) at x from X, and by $\mathrm{outval}_X(x)$ the number of such branches.

Definition 3.7.1 (Reduced divisors in the classical context)**.** Let x be a point of Γ. A divisor D on Γ is *x-reduced* if the following two conditions are met:

(1) For each $y \in \Gamma \smallsetminus \{x\}$, we have $D(y) \geqslant 0$, and

(2) For each cut $X \subset \Gamma \smallsetminus \{x\}$, there exists a point $y \in \partial X$ with $D(y) < \mathrm{outval}_X(y)$. ◇

For each divisor D and each point x in Γ, there exists a unique x-reduced divisor linearly equivalent to D, which we denote by D_x. For graphs, this was proved by Baker and Norine in [BN07]. The extension to metric graphs was given in [MZ08], see also [Ami13].

3.7.2 Reduced divisors in linear systems

We need the following definition.

Definition 3.7.2. Let $M \subseteq \mathrm{Rat}(D)$ be an admissible semimodule of rank r, and let v be a point of Γ. We define the rational function f_v^M, denoted f_v when the context is clear, by

$$f_v^M(x) := \inf_{g \in M} [g(x) - g(v)] = \inf_{f \in M_v} f(x)$$

for every point x of Γ. ◇

(Recall that M_v is the set of all functions of M which vanish at v. In particular, $f_v(v) = 0$.)

The closedness of M implies that the function f_v^M belongs to M, as we show next.

Proposition 3.7.3. *The function f_v^M is well-defined and belongs to M_v.*

Note that, by this proposition, since the infimum is reached, we can write $f_v = \min_{f \in M_v} f$.

Proof. We first show that f_v is well-defined. This does not require that M is closed in $(\mathrm{Rat}(D), \|\cdot\|_\infty)$ and boils down to proving that the set

$$\{f(x),\ f \in M_v\}$$

is bounded (from below) for any given $x \in \Gamma$. Let k be a bound for the slopes of all functions of $M \subseteq \mathrm{Rat}(D, \mathfrak{S})$ (see Remark 3.3.8).

It yields a universal bound for values $f(x)$ with $x \in \Gamma$ and $f \in M_v$:

$$|f(x)| \leqslant k \,\mathrm{Diam}(\Gamma),$$

where $\mathrm{Diam}(\Gamma)$ is the diameter of Γ as a compact metric space. This shows that f_v is well-defined.

We then show that f_v can be written as the uniform limit of a (decreasing) sequence of functions in M_v, essentially following the proof of Dini's theorem. Since the slopes of functions in M_v are universally bounded, this set is uniformly Lipschitz, so f_v is itself Lipschitz and therefore continuous.

Now let $n \geqslant 1$. For every $x \in \Gamma$, there exists $f_n^x \in M_v$ such that

$$0 \leqslant f_n^x(x) - f_v(x) \leqslant \frac{1}{n}.$$

By continuity of f_n^x and f_v, there is an open set $U_n^x \ni x$ such that for all $y \in U_n^x$,

$$0 \leqslant f_n^x(y) - f_v(y) \leqslant \frac{3}{n}.$$

Since this is true for all x and Γ is compact, we can cover Γ using a finite number of open sets $U_n^{x_i}$, $1 \leqslant i \leqslant s(n)$, with corresponding functions $f_n^{x_i}$ satisfying inequalities as above. We define

$$f_n \coloneqq \min_{1 \leqslant i \leqslant s(n)} f_n^{x_i},$$

which is an element of M_v satisfying, for all $x \in \Gamma$, $0 \leqslant f_n(x) - f_v(x) \leqslant \frac{3}{n}$. This implies that (f_n) converges uniformly to f_v. Note that replacing f_n by

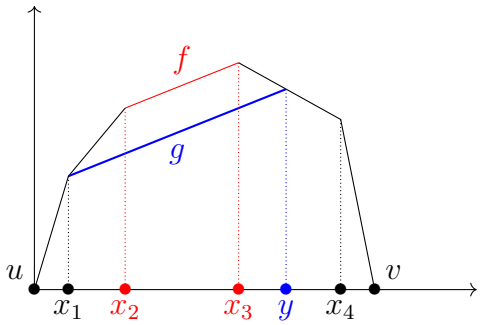
$$g_n \coloneqq \min_{1 \leqslant i \leqslant n} f_i,$$

we can make this sequence of functions a decreasing sequence, but this it not necessary in the current proof.

To conclude, we use the closedness of M in $\mathrm{Rat}(D)$, which yields $f_v \in M$. □

Before giving an example, we make two remarks about the functions of the form f_v^M.

Remark 3.7.4. Let $M \subseteq \mathrm{Rat}(D)$ be an admissible semimodule of rank r, and let $\mathfrak{S}$ be the associated slope structure. We assume to be in the particular case where the equality $M = \mathrm{Rat}(D, \mathfrak{S})$ holds. Up to subdividing some edges, we suppose that the support of D is included in V. Then in the computation of f_v, we can ignore functions $f \in \mathrm{Rat}(D, \mathfrak{S})$ which have more than 3 changes of slope on some edge.



Indeed, assume that on an edge $e = uv$, the function f (in red and black) changes slope at points x_1, x_2, x_3 and x_4, placed on e in this order. Then the function g (in blue and black), which is defined by "pushing downwards" the central segment of the graph of f until reaching a point where the slope of f changes, is and element of $M_v = \mathrm{Rat}(D, \mathfrak{S})_v$ which is $\leqslant f$ and which has at most 3 changes of slope on e (at most 2 if by chance $y = x_4$ in the configuration above). ◇

Remark 3.7.5. A very special case of Theorem 3.5.13 implies that f_v^M takes minimum slopes around v, that is, for all $\nu \in \mathrm{T}_v(\Gamma)$, $\mathrm{slope}_\nu(f_v^M) = s_0^\nu$, see the discussion right after that theorem. ◇

Example 3.7.6. The following example shows that the closedness condition on M is necessary for Proposition 3.7.3 to hold, that is, the statement might fail for crude linear series and their submodules, in general. Consider the metric graph Γ and crude linear series of degree four and rank one defined at the end of Example 3.4.2. We allow slopes $0 < 1$

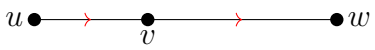
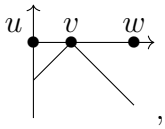


Figure 3.6: The metric graph Γ with edges of arbitrary lengths.

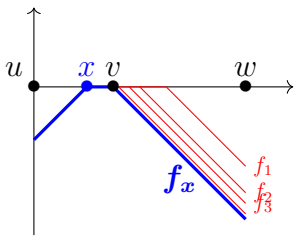
on the edge uv and slopes $-1 < 0$ on the edge vw, in the direction of the arrows. We set $D = (u) + (w)$ and we adapt ρ_v as follows:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here, the jumps of ρ_v are depicted in blue. This makes $(D, \mathfrak{S})$ a crude linear series of degree two and rank one. We take $M = \mathrm{Rat}(D, \mathfrak{S})$. The function f_v



takes slopes 1 and -1 on the segments $[u, v]$ and $[v, w]$, respectively, and belongs to $\mathrm{Rat}(D, \mathfrak{S})$. But if we choose any $x \neq v$, then $f_x \notin \mathrm{Rat}(D, \mathfrak{S})$.



Indeed, consider a sequence of functions $(f_i)_i$ which coincide with f_x on the segment $[u, v]$ and whose graphs on the segment $[v, w]$ are represented in thin red in the above figure,

converging uniformly to f_x as $i \to +\infty$. All the functions f_i belong to $\mathrm{Rat}(D, \mathfrak{S})_x$; but f_x does not itself belong to $\mathrm{Rat}(D, \mathfrak{S})$ since it does not take symmetrical slopes around v, a mandatory condition given the expression of ρ_v. In particular, this shows that $\mathrm{Rat}(D, \mathfrak{S})$ is not closed in $\mathrm{Rat}(D)$. ◇

Remark 3.7.7 (Bounds on slopes)**.** In the proof of Proposition 3.7.3, we have used a universal bound for the slopes of functions of $\mathrm{Rat}(D, \mathfrak{S})$ using the finiteness of the set of slopes prescribed by $\mathfrak{S}^{\mathrm{e}}$. In fact, there is a universal bound for these slopes, even without a slope structure: we can bound uniformly the slopes of all functions of $\mathrm{Rat}(D)$, for D any effective divisor.

The first such bound was provided by [GK08, Lemma 1.8], which states that all the slopes of functions in $\mathrm{Rat}(D)$ are bounded by

$$(\deg(D) + N)^e,$$

where N is the maximal valence of a point in Γ, and e is the number of edges of Γ.

However, we believe that there is a small gap in the (purely combinatorial) proof given in [GK08]. The issue is that e depends on the graph model $G = (V, E)$ that was chosen for Γ such that all points in the support of D or $\mathrm{div}(f)$ are in V. For this reason, the bound above is not universal in f. But we can make it universal (though weaker) without fundamentally changing the reasoning: we choose, once and for all, a model G of Γ adapted to D and, for each f, we subdivide the edges of G so that all *zeroes* of f (points where $\mathrm{div}(f) \geqslant 0$) are vertices, leaving alone the possible *poles* of f in the interior of edges (they do not change the inductive reasoning in the proof of [GK08, Lemma 1.8]). Then, we follow the proof in that paper, noting that the number of edges we had to add is smaller than the number of distinct poles of f, which is smaller than the number of poles counted with multiplicities, which is itself smaller than $\deg(D)$. This leads to the universal (highly non-sharp) bound, depending only on $\deg(D)$ and the combinatorics of Γ:

$$(\deg(D) + N)^{e+\deg(D)}.$$

Anyway, a much stronger bound was given a few years later by [HMY12, Lemma 7], which states that the slopes are simply bounded by $\deg(D)$. The proof is however less elementary than the one above. ◇

Remark 3.7.8 (Topological discussion)**.** Let k be a universal bound for the slopes in $\mathfrak{S}$. Let us denote by C the space $(\mathcal{C}^0(\Gamma, \mathbb{R})_v, \|\cdot\|_\infty)$ and by B the subspace formed by the functions of C whose slopes are bounded by k. Since B is closed in C, bounded and equicontinuous, it is compact by the Arzelà–Ascoli theorem. In particular, a space of the form

$$A \coloneqq (\mathrm{Rat}(D, \mathfrak{S})_v, \|\cdot\|_\infty) \subseteq B$$

is compact as long as it is closed. Furthermore, if M is a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$ which is closed in $\mathrm{Rat}(D)$ (in particular, if M is admissible, or if (D, M) is a $\mathfrak{g}^r_d$), then M_v

is again compact. But Example 3.7.6 shows that when the finiteness condition is dropped, A is not necessarily compact, that is, it is not necessarily closed in B. It is possible however to show that the space

$$A^{\mathrm{e}} \coloneqq (\mathrm{Rat}(D, \mathfrak{S}^{\mathrm{e}})_v, \|\cdot\|_\infty) \subseteq B$$

is always closed in B, and thus compact, even if A is not finitely generated.

To prove this, let us consider a sequence (f_n) of functions in A^{e} converging uniformly to $f \in B$ and show that $f \in A^{\mathrm{e}}$. The crucial point here is that, by the very nature of the slope structure $\mathfrak{S}$, the slopes of the functions f_n on each edge live in a fixed finite set, and that the f_n change slope a uniformly bounded number of times on each edge.

Indeed, without loss of generality, we assume that all points of the support of D are vertices, so that each f_n is a concave affine linear function on each edge. As a consequence, on a given edge $e = uw$ of Γ, f_n changes slope at most r times, because its slope can only decrease as we travel along e in any direction. We can thus encode the graph of f_n on e by telling its value on the extremity u and giving r numbers

$$0 \leqslant x^e_{1,n} \leqslant x^e_{2,n} \leqslant \cdots \leqslant x^e_{r,n} \leqslant \ell(e),$$

where $\ell(e)$ is the length of e. In short, f_n has slope s^e_r between u and $x^e_{1,n}$, has slope s^e_{r-1} between $x^e_{1,n}$ and $x^e_{2,n}$, and so on (here, we also denote by $x^e_{i,n}$ the point on e at distance $x^e_{i,n}$ from the extremity u). The fact that the inequalities are not strict translates the possibility that f_n does not necessarily realize all possible slopes on each edge (and even globally on Γ).

For every edge e, we define an arbitrary path in Γ from the starting vertex v to one of the extremities of e, also chosen arbitrarily. For every n, we see by immediate induction (and thanks to the fact that $f_n(v) = 0$) that we do not need to specify the values of f_n on one extremity of each edge: the data of the $x^e_{i,n}$ is sufficient to reconstruct f_n entirely.

Then, since the edges are compact, we can extract subsequences at most $r\,|E|$ times and assume that on every edge e, for all i, the sequence $\left(x^e_{i,n}\right)_n$ converges to a certain x^e_i. The inequalities

$$x^e_1 \leqslant \cdots \leqslant x^e_r$$

still hold for every e, so the data of these points defines a new function $g \in B$ which is affine linear and respects $\mathfrak{S}^{\mathrm{e}}$ by construction. It is easy to see that (f_n) converges uniformly to g, and therefore $g = f$. We have proved that f is affine linear and respects $\mathfrak{S}^{\mathrm{e}}$.

To conclude, we now prove that $D + \mathrm{div}(f) \geqslant 0$. Let v be a vertex of Γ such that $\mathrm{div}(f)(v) < 0$ (we know that $\mathrm{div}(f)$ is effective outside V since D is supported on V). We have to prove that $D(v) + \mathrm{div}(f)(v) \geqslant 0$. We denote by $e_1, \ldots, e_s$ the edges starting at v and take the convention that v is their preferred extremity. For $1 \leqslant k \leqslant s$, we define two integers $0 \leqslant \alpha_k \leqslant \beta_k \leqslant r$ by the following properties:

- For all $1 \leqslant i \leqslant \alpha_k$, the sequence $\left(x^{e_k}_{i,n}\right)$ converges to zero (the points converge to v), and $x^{e_k}_{i,n} = 0$ for an infinite number of n.

- For all $\alpha_k < i \leqslant \beta_k$, the sequence $\left(x_{i,n}^{e_k}\right)$ converges to zero, but $x_{i,n}^{e_k} = 0$ only for a finite number of n.
- For all $\beta_k < i \leqslant r$, the sequence $\left(x_{i,n}^{e_k}\right)$ does not converge to zero.

We can now extract subsequences so that for all $n \geqslant 0$ and $1 \leqslant i \leqslant \alpha_k$, $x_{i,n}^{e_k} = 0$ and assume that n is large enough such that for all $\alpha_k < i \leqslant \beta_k$, $x_{i,n}^{e_k} > 0$, which implies the same inequality for $\beta_k < i \leqslant r$. Then, only the "genuinely converging" sequences contribute to changing the value of the limit divisor at v:

$$\operatorname{div}(f)(v) = \operatorname{div}(f_n)(v) + \sum_{1 \leqslant k \leqslant s} \left(s_{\beta_k}^{e_k} - s_{\alpha_k}^{e_k}\right).$$

But the extra term is non-negative, so from $(D + \operatorname{div}(f_n))(v) \geqslant 0$ we deduce

$$(D + \operatorname{div}(f))(v) \geqslant 0.$$

Since this is true for all vertex v where the inequality was not automatically true, we have

$$D + \operatorname{div}(f) \geqslant 0,$$

which finishes the proof that $f \in A^{\mathrm{e}}$.

To sum up, we have the inclusions

$$A \subset A^{\mathrm{e}} \subset B \subset C,$$

with B compact and A^{e} closed in B, so A^{e} is compact. The fact that A is not necessarily compact comes from the fact that it is not necessarily closed in A^{e}. It is cut out by the non-necessarily closed constraints at the level of vertices given by $\mathfrak{S}^{\mathrm{v}}$.

Note that the argument here also shows that $\operatorname{Rat}(D)_v$ is closed in B, and thus compact. ◇

Remark 3.7.9. It can be shown that if $(f_n)_n$ is a sequence of functions in $\operatorname{Rat}(D)$ converging uniformly to a function f and such that for all n, then $f \in \operatorname{Rat}(D)$ and, for every $x \in \Gamma$, the inequality $\partial_x(f) \leqslant \partial_x(f_n)$ holds for all large n. ◇

3.7.3 Definition of the reduced divisor

We now define reduced divisors for linear series.

Definition 3.7.10 (Reduced divisor)**.** Let (D, M) be a pair consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \operatorname{Rat}(D)$. The effective divisor defined by

$$D_v^M := D + \operatorname{div}(f_v^M),$$

more simply denoted by D_v when M is contextually clear, is called the *v-reduced divisor linearly equivalent to D in the linear system* $|(D, M)|$. Denoting by $\mathfrak{S}$ the r-slope structure underlying M, we denote the slope structure $\mathfrak{S} + \operatorname{div}(f_v)$ by $\mathfrak{S}_v$. The effective pair $(D_v, \mathfrak{S}_v)$ is then equivalent to the pair $(D, \mathfrak{S})$. Finally, f_v gives rise to a modification of M, which is denoted by $M(-f_v)$ according to Definition 3.5.2, and which is effective. ◇

Remark 3.7.11. We will see in Proposition 3.7.16 that D_v is the *unique* effective divisor in the linear system $|(D, M)|$ that satisfies an interesting property involving unsaturated cuts (see Definition 3.7.13). ◇

3.7.4 Coefficient at the base-point

We have the following useful result.

Proposition 3.7.12. *Let (D, M) be pair consisting of a divisor D and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$. Let $\mathfrak{S}$ be the underlying slope structure. For every point $x \in \Gamma$, we have*

$$D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu.$$

In addition, this quantity is greater than or equal to r.

Proof. This is a direct consequence of Remark 3.7.5. □

3.7.5 Unsaturated cuts

In this section, we provide another characterization of reduced divisors in terms of unsaturated cuts in a metric graph with respect to admissible pairs (D, M).

Definition 3.7.13 (Unsaturated cut with respect to an admissible pair)**.** Let (D, M) be a pair consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$ of rank r. Let $\mathfrak{S}$ be the underlying slope structure. Let v be a point of Γ. Consider a cut X in Γ and assume that the point v does not belong to X.

We say that X is *unsaturated with respect to v and M* if, for a sufficiently small $\varepsilon > 0$, there exists a function $f \in M$ which satisfies the following properties:

- for every point $x \in \partial X$, f is linear of positive slope $s^\nu > 0$ on a small segment I_ν on each adjacent outgoing branch $\nu \in \mathrm{out}_X(x)$;
- f is identically equal to $-\varepsilon$ on X; and
- f is zero everywhere else.

In this case, we say that f *fires* the unsaturated cut X *at level ε*. Otherwise, if no such f exists, X is called *saturated*. ◇

Note that if $f \in M$ fires the cut X at level ε for some $\varepsilon > 0$, then for every $\varepsilon' \in (0, \varepsilon)$, the element $f' := \min(f + \varepsilon - \varepsilon', 0)$ of M fires X at level ε'. Therefore, the saturation property for cuts is well-defined.

Remark 3.7.14. If X is an unsaturated cut with respect to v and M, then for every point $x \in \partial X$, there exists an element $\underline{s}(x) = \prod_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu \in S^x$ such that $s^\nu \geqslant 0$ for all $\nu \in \mathrm{T}_x(\Gamma)$, with equality $s^\nu = 0$ if and only if $\nu \in \mathrm{T}_x(\Gamma) \smallsetminus \mathrm{out}_X(x)$, and, in addition,

$$D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu \geqslant 0. \tag{3.3}$$

Such vectors of slopes $\underline{s}(x)$ are provided by the vector of slopes $\partial_x(f)$ of any function $f \in M$ which fires the unsaturated cut X. The data of these vectors of slopes, together with a small number $\varepsilon > 0$, determine the function f entirely. ◇

Remark 3.7.15. Inequality (3.3) in the previous remark implies that each point x of the boundary ∂X of the cut X is in the support of the divisor D.

In particular, since the support of D is finite, for given v and M, there are finitely many unsaturated cuts with respect to v and M. ◇

The following result gives an alternative characterization of reduced divisors.

Proposition 3.7.16 (Characterization of reduced divisors by unsaturated cuts)**.** *Let (D, M) be a pair consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$. Then, D is v-reduced if, and only if, there is no unsaturated cut with respect to v and M.*

Proof. We prove the equivalence of the negations.

First, we assume that there is an unsaturated cut X with respect to v and M, and then show that D is not reduced. By definition, this means that $v \notin X$ and that there exists $\varepsilon > 0$ and a function $f \in M$ which fires X at level ε. Obviously, $f \leqslant 0$ and f takes negative values on a non-empty set of points. Moreover, if ε is small enough, then $f(v) = 0$. This shows that D is not v-reduced.

To prove the other direction, we assume that D is not v-reduced, and prove that there exists an unsaturated cut with respect to v and M. Since D is not v-reduced, there exists a non-constant function $f \in M \smallsetminus \{0\}$ such that $f(v) = 0$ and $f(x) \leqslant 0$ for every $x \in \Gamma$. Let X be the set of points of Γ where f takes its minimum value. This is a compact set which does not contain v, and since f is piecewise linear, X has finitely many connected components. X is therefore a cut, and the properties of f imply that it is unsaturated with respect to v and M. This establishes the proposition. □

3.7.6 Behavior of reduced divisors with respect to the base point

We give an explicit description of reduced divisors under an infinitesimal change of the base point. This will be used to prove the continuity of the reduced divisor map, Theorem 3.7.24.

Let (D, M) be a pair consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$ of rank r with underlying slope structure $\mathfrak{S}$.

Let (D, M) be a pair consisting of a divisor D of degree d and an admissible semimodule $M \subseteq \mathrm{Rat}(D)$ of rank r with underlying slope structure $\mathfrak{S}$. Let v be a point of Γ, $\nu \in \mathrm{T}_v(\Gamma)$,

and e the oriented edge of Γ parallel to ν, all fixed for the remainder of this section. We give an explicit description of D_u for u in a small segment $I^e \subset e$ with an endpoint equal to v.

Replacing $(D, \mathfrak{S})$ with the linearly equivalent $(D_v, \mathfrak{S}_v)$ and M with $M(-f_v)$, we can assume that D is v-reduced. Note that M remains effective. In particular, the zero function belongs to M. Also, adapting the combinatorial model accordingly, we suppose that the slope structure $\mathfrak{S}$ is constant on every edge. Now, there exists a sufficiently small segment I^e on e adjacent to v which does not contain any point of the support of D apart from v. For a point $u \neq v$ on this segment, we have $D(u) = 0$ and $D_u(u) > 0$, and thus $D \neq D_u$. We infer that D is not u-reduced. It follows from Proposition 3.7.16 that there exist unsaturated cuts with respect to u and M. Since D is v-reduced, we infer that every such cut Y contains v. In addition, ν must belong to $\mathrm{out}_Y(v)$, and therefore $v \in \partial Y$, since otherwise, the boundary of that cut would contain a point of e between v and u (see Remark 3.7.15). This would be impossible by the assumption made on the support of D. We have proved

Claim 3.7.17. *For every unsaturated cut Y with respect to u and M, we have $v \in \partial Y$.*

Since D is v-reduced, we have $s_0^\nu = 0$ for every $\nu \in \mathrm{T}_v(\Gamma)$ (see Remark 3.7.5). It follows that $s_r^\nu > 0$, and, by Proposition 3.7.12 and the definition of slope structures, the coefficient of D_u at u is precisely equal to $s_r^\nu + s_0^\nu = s_r^\nu$, given that $-s_r^\nu$ and $s_0^\nu = 0$ are the smallest possible slopes at u in the direction of v and away from v, respectively. We now claim

Claim 3.7.18. *There exists an unsaturated cut Y with respect to u and M for which we can choose a function $f \in M$ firing Y at some level $\varepsilon > 0$ with the additional property that its slope along the tangent vector ν is the maximal possible slope s_r^ν.*

Proof. Let $f = f_u$ be the element of M with $D + \mathrm{div}(f) = D_u$ and $f(u) = 0$, as in Definition 3.7.2. Since $-s_r^\nu$ is the minimum slope at u along e in the direction of v, Remark 3.7.5 implies that f takes slope $-s_r^\nu$ away from u on a sufficiently small segment included in $[v, u]$ containing u. The fact that D_u is effective then implies that f has constant slope along $[v, u]$. Consequently, f has slope s_r^ν in the direction of u on the whole segment $[u, v]$, and in particular along the tangent vector ν.

Now, let η be the distance between v and u on I^e, and set $\varepsilon := s_r^\nu \eta$. Then, we have $f(v) = -\varepsilon$. Also, note that, by the hypothesis made on D, $\mathfrak{S}$ and M, we have $f_v \equiv 0$ on Γ. Therefore, following Definition 3.7.2, we get $f \geqslant f(v) + f_v = -\varepsilon$ on Γ. Consequently, $-\varepsilon = f(v)$ is the minimum of f on Γ.

Consider the set Y of all points of Γ where f takes this minimum value. Y is a compact set with finitely many connected components. Moreover, Y does not contain u and, for a sufficiently small $\varepsilon' \in (0, \varepsilon]$, the function $f' := \min(f + \varepsilon - \varepsilon', 0)$ fires Y at level ε'. (Note that, contrary to f, the function f' does not necessarily have a constant slope on the whole segment $[v, u]$.) As a sanity check and as expected (see Claim 3.7.17), $v \in \partial Y$. Since in addition the slope of f (and thus the slope of f') along ν is equal to s_r^ν, the unsaturated cut Y has the desired properties. □

Now consider the family $\mathcal{X}$ of all the unsaturated cuts Y with respect to u and M such that Y verifies the properties of Claim 3.7.18. Let $X \coloneqq \bigcup_{Y\in\mathcal{X}} Y$.

Since there are finitely many unsaturated cuts with respect to u and M (see Remark 3.7.15), the family $\mathcal{X}$ is finite. Therefore, X is still compact and has finitely many connected components. i.e., it is itself a cut.

Fact 3.7.19. We notice that if we choose another point $u' \neq v$ on the segment $[v, u]$, a cut Y is unsaturated with respect to u' if, and only if, it is unsaturated with respect to u. Verifying the stronger property of Claim 3.7.18 also remains unchanged. This shows that $\mathcal{X}$ (and thus X) does not change when we choose u to be even closer to v, a fact that will be used in the discussions below and in the proof of Theorem 3.7.22. ◇

We now claim that

Claim 3.7.20. *Notation as above, the cut X itself belongs to $\mathcal{X}$.*

Proof. To prove that X is unsaturated with respect to u and M, we first note that $u \notin X$.

Then, denote by $Y_1, \dots, Y_N$ the elements of $\mathcal{X}$. By assumption, for every $i \in \{1, \dots, N\}$, there exists a function $f_i \in M$ such that for $\varepsilon_i > 0$ small enough, f_i fires the unsaturated cut Y_i at level ε_i, and such that the slope of f_i along ν is equal to s_r^ν. For a given i and for a fixed $\varepsilon_i > 0$, there are finitely many such functions $f_i \in M$, one for each choice of joint outgoing slopes (see Remark 3.7.14). By choosing the minimum of these possible choices, we ensure that f_i takes the minimal possible set of joint slopes on edges leaving Y_i, under the constraint, if $v \in \partial Y_i$, of having slope s_r^ν on ν. Furthermore, we replace all the numbers ε_i with $\varepsilon \coloneqq \min_i \varepsilon_i$.

For each $i \in \{1, \dots, N\}$, each $x \in \partial Y_i$, and each $\nu' \in \mathrm{out}_{Y_i}(x)$, we call $I_{\nu'}$ the small segment parallel to ν' on which f_i takes a positive slope away from Y_i. We then reduce $\varepsilon > 0$ again so that the outgoing segments $I_{\nu'}$ are essentially disjoint and therefore the functions f_i fire in a "decoupled" way. More precisely, up to reducing ε further, we can assume that for every $i, j \in \{1, \dots, N\}$, every $x \in \partial Y_i$, $y \in \partial Y_j$ and every $\nu' \in \mathrm{out}_{Y_i}(x)$, $\nu'' \in \mathrm{out}_{Y_j}(y)$, if $\mathring{I}_{\nu'} \cap \mathring{I}_{\nu''} \neq \varnothing$, then $x = y$ (and thus one of $I_{\nu'}, I_{\nu''}$ contains the other).

For a sufficiently small ε chosen this way, the function $f^\varepsilon \coloneqq \min_i f_i$ belongs to M and fires X at level ε. Indeed,

- on $X = \bigcup_i Y_i$, f^ε takes value $-\varepsilon$, and away from X and the union of $I_{\nu'}$, for $x \in \partial Y_i$, $i = 1, \dots, N$, and $\nu' \in \mathrm{out}_{Y_i}(x)$, it is identically zero.
- on a segment of the form $I_{\nu'}$ based at some $x \in \partial X$, the slope of f^ε away from X is the minimum of the slopes of the functions f_i for i such that $x \in \partial Y_i$.

This shows that X is unsaturated. Moreover, the slope of every f_i along ν is s_r^ν, which concludes. Note that f inherits the slope minimality properties of the functions f_i, which will be useful in the proof of Theorem 3.7.22. □

Now, as in the proof of Claim 3.7.18, let η be the distance between v and u on I^e, and set $\varepsilon_0 := s_r^\nu \eta$. We look at which number between ε_0 and the number ε appearing in the proof of Claim 3.7.20 is smaller. If $\varepsilon_0 < \varepsilon$, we freely reduce ε to ε_0 in the proof of Claim 3.7.20. If, on the contrary, $\varepsilon < \varepsilon_0$, we move u closer to v so that both numbers are equal; we can do this properly and without self-reference because, as u moves closer to v, the family $\mathcal{X}$ remains constant, by Fact 3.7.19. At this point, X is an unsaturated cut with respect to u and M, and there exists a function $f^u := f^\varepsilon \in M$, constructed in the proof of Claim 3.7.20, with fires X at level exactly $\varepsilon = \varepsilon_0$, with constant slope s_r^ν on the whole segment $[v, u]$. Moreover, by construction, among the functions with the previous properties, f^u has the smallest possible set of joint slopes on outgoing tangent vectors.

Remark 3.7.21. Given the way it was constructed in the proof of Claim 3.7.20, and thanks to Fact 3.7.19, the function f^u is entirely determined by ε_0, and its non-zero slopes are independent of u. More precisely, if $u' \in (v, u]$, denoting by η' the distance between v and u', and letting $\varepsilon_0' := s_r^\nu \eta'$, then the function $f^{u'}$ is given by $f^{u'} = \min(f^u + \varepsilon_0 - \varepsilon_0', 0)$. This shows that f^u depends continuously on u. ◇

We can now formulate the main theorem of this section.

Theorem 3.7.22. *The u-reduced divisor with respect to M is $D_u = D + \operatorname{div}(f^u)$.*

Proof. Let f_u be the rational function in M which defines the u-reduced divisor D_u, and which takes value zero at u (see Definition 3.7.2). We will prove that in fact $f_u = f^u$. Note that f_u verifies the following properties.

- f_u is linear on the segment $[v, u]$ with slope s_r^ν.

This follows from Remark 3.7.5 and the fact that D_u is effective (see the first paragraph of the proof of Claim 3.7.18).

- f_u takes it minimum value at v, and its maximum value at u.

The fact that f_u takes its maximum value at u comes from the definition of the u-reduced divisor map, given that $0 \in M$. The fact that f_u takes its minimum value at v was shown in the proof of Claim 3.7.18.

- Let X_0 be the set of all points where f_u takes its minimum value. Then, $X_0 = X$.

Obviously, X_0 is an unsaturated cut with respect to u and M which in addition verifies the property of Claim 3.7.18 (it is the set Y in the proof of that statement). Therefore $X_0 \in \mathcal{X}$. This shows by Claim 3.7.20 that $X_0 \subseteq X$. The reverse inclusion comes from the definition of reduced divisors, which implies $f_u \leqslant f^u$, globally on Γ. This shows that $X_0 = X$.

- Up to reducing η further (i.e., up to moving u even closer to v), f_u and f^u coincide on all segments of length η around X.

Note that $f_u = f^u$ on X by the previous claim. Then, the slope minimality property of f^u (see the proof of Claim 3.7.20) ensures, on the one hand, that the slopes taken by f^u on tangent vectors leaving X do not exceed those taken by f_u. On the other hand, the inequality $f_u \leqslant f^u$ holds on Γ, as previously noted. All in all, we can move u closer to v (and thus reduce η), without changing X (see Fact 3.7.19), and keeping the properties of f_u and f^u established so far in the current proof, in such a way that f_u and f^u coincide on all segments of length η leaving X.

- $f_u = f^u$ everywhere.

To finish the proof of the theorem, let X' be the closure of the complement of $X \cup \bigcup_{x\in\partial X,\nu'\in \mathrm{out}_X(x)} I_{\nu'}$. We need to show that $f_u = f^u$ on X'. In other words, we need to show that $f_u \equiv 0$ on X'. Suppose this is not the case, and consider the minimum locus Y of f_u on X'. Note that Y lies in the interior of X', i.e., $Y \cap \partial X' = \varnothing$, and $v \notin Y$. This shows that Y is an unsaturated cut with respect to v and M, which contradicts Proposition 3.7.16, given that D is v-reduced. This finally establishes the theorem. □

Remark 3.7.23. Note that in the proof of Theorem 3.7.22, we did not refer to the admissibility of M, but this does not contradict Example 3.7.6. Indeed, Proposition 3.7.3 states that, assuming admissibility, D_v exists for all v, whereas Theorem 3.7.22 only implies that if D_v *does exist* for some v, then D_u exists for all u in a neighborhood of v, and behaves as stated in the theorem. In other words, the set of u for which D_u exists is an open subset of Γ. ◇

3.7.7 Continuity of the map to $|(D, M)|$ defined by reduced divisors

Theorem 3.7.22 has the following direct consequence, which will be crucial in the next section.

Let D be a divisor of degree d and $M \subset \mathrm{Rat}(D)$ an admissible semimodule of rank r on a metric graph Γ with underlying slope structure $\mathfrak{S}$. Reduced divisors with respect to points of Γ define a map from Γ to the linear system $|(D, M)|$. More precisely, define the map

$$\mathrm{Red}\colon \Gamma \to |(D, M)| \subseteq |(D, \mathfrak{S})| \subseteq \mathrm{Sym}^d(\Gamma)$$

by sending a point v of Γ to D_v, the v-reduced divisor linearly equivalent to D with respect to M.

Theorem 3.7.24. *Let $M \subseteq \mathrm{Rat}(D)$ be an admissible semimodule on Γ, with D a divisor of degree d. The map $\mathrm{Red}\colon \Gamma \to |(D, M)| \subseteq \mathrm{Sym}^d(\Gamma)$ is continuous and non-contracting.*

Proof. By Proposition 3.7.12, the coefficient of D_v at v is precisely $D(v) - \sum_{\nu\in \mathrm{T}_v(\Gamma)} s_0^\nu > 0$ for every point v of Γ. This obviously shows that Red cannot be constant on any segment of positive length, proving that it is non-contracting. Furthermore, it follows from Theorem 3.7.22 and Remark 3.7.21 that the function $f_u = f^u$ depends continuously on the point $u \in \Gamma$. Since $D_u = D + \mathrm{div}(f^u)$, the continuity of Red follows. □

Remark 3.7.25. The above result is a generalization of [Ami13, Theorem 3]. Note that the map Red is also integer affine linear with respect to the natural integer affine structures on Γ and $\mathrm{Sym}^d(\Gamma)$ as described in [Ami13, Section 2]. Thus, the image of Red is a metric graph. In the case (D, M) is a $\mathfrak{g}^1_d$, we will see in Proposition 3.8.16 that the image is a metric tree. ◇

3.8 Classification of $\mathfrak{g}^1_d$'s

In this section, we consider the case $r = 1$, and prove, roughly speaking, that the data of a $\mathfrak{g}^1_d$ on Γ is equivalent to the data of a finite harmonic map to a metric tree (see Theorem 3.8.6 below for a precise statement). Then, we formulate a smoothing theorem for combinatorial $\mathfrak{g}^1_d$'s. In this regard, $\mathfrak{g}^1_d$'s on metric graphs are well-behaved, and our theorem can be regarded as a generalization of the Eisenbud–Harris smoothing result for their limit $\mathfrak{g}^1_d$'s [EH86, Proposition 3.1].

Let (D, M) be a $\mathfrak{g}^1_d$ on Γ with underlying slope structure $\mathfrak{S}$ of width one. By Remarks 3.3.6 and 3.5.4, we can assume that (D, M) is effective. This implies that for every point x and every outgoing tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ at x, one of the two integers s^ν_0 or s^ν_1 is equal to zero.

Assume that $(D, \mathfrak{S})$ is defined on a model $G = (V, E)$ of Γ.

Definition 3.8.1 (Orientation associated to a $\mathfrak{g}^1_d$)**.** Notation as above, we define an orientation of G in such a way that the edge $\{u, v\}$ gets orientation uv if $s^{uv}_0 = 0 < s^{uv}_1$. ◇

Let ρ be a rank function on $\square^\delta_1$. The point $\underline{0}$ is the only point of $\square^\delta_1$ of rank one (Remark 3.2.2). Besides, the set J_ρ of jumps of ρ contains the point $\underline{0}$ (because $\rho(\underline{\mathfrak{e}}_i) = r - 1 = 0$ for all i) and every other point $\underline{a} \neq \underline{0}$ has at least one coordinate equal to one. For each $\underline{a} \in J_\rho$, denote by $P_{\underline{a}}$ the subset of $\{1, \dots, \delta\}$ consisting of all the indices i with $a_i = 1$ (the support of $\underline{a}$). Denote by $\mathcal{P}_\rho$ the collection of all sets $P_{\underline{a}}$ for $\underline{a} \in J_\rho \smallsetminus \{\underline{0}\}$. By Lemma 3.2.17 and Remark 3.2.18, $\mathcal{P}_\rho$ provides a partition of $\{1, \dots, \delta\}$.

This construction provides a direct proof of the representability of rank functions of rank one, which also follows easily from the fact that such a rank function is geometric in the sense of Section 3.2 (see the results in [ABBR15a] and [ABBR15b]).

Proposition 3.8.2. *Every rank function on $\square^\delta_1$ is representable, and the field can be chosen to be of characteristic zero.*

Proof. Chose κ to be any infinite field of characteristic zero. Let $P_\rho = \{P_1, \dots, P_s\}$ be the partition of $\{1, \dots, \delta\}$ as previously described, each P_i being a subset of $\{1, \dots, \delta\}$. In the plane κ^2, let $L_1, \dots, L_s$ be distinct lines. Now let, for each $i \in \{1, \dots, \delta\}$, $\mathrm{F}^1_i := L_{\tau(i)}$, where $\tau(i) \in \{1, \dots, \delta\}$ is the integer such that $i \in P_{\tau(i)}$. This, in turn, defines δ complete flags in κ^2, completing the lines with κ^2 and (0). Then it is easy to check that the collection of flags $\mathrm{F}^\bullet_1, \dots, \mathrm{F}^\bullet_\delta$ is a representation of the rank function ρ. □

For every point p of Γ (resp. G) of valence d_p, consider the rank function ρ_p on $\square_1^{d_p}$, and define $\mathcal{P}_p$ as the partition of $\mathrm{T}_p(\Gamma)$ (resp. $\mathbb{E}_p$) given by the previous proposition. Note that S^p consists of the point $(s_0^\nu)_{\nu\in\mathrm{T}_p(\Gamma)}$ (resp. $(s_0^e)_{e\in\mathbb{E}_p}$), and vectors $(s_{a_\nu}^\nu)$ (resp. $(s_{a_e}^e)$) for $\underline{a}\in\square_1^{d_p}$ with $P_{\underline{a}}\in\mathcal{P}_p$.

Proposition 3.8.3. *Let v be a vertex of G, and denote by $I_v\subset\mathbb{E}_p$ the set of all the edges vw with $s_0^{vw}<0$. Then we have $I_v\in\mathcal{P}_v$ or $I_v=\varnothing$.*

Proof. We know by hypothesis that $0\in M$. Let $\underline{a}$ be the element of $\square_1^{d_v}$ such that $\partial_v(0)=s_{\underline{a}}$. Then, if vw is an edge incident to v, since exactly one of the possible slopes along vw is zero, we have $s_0^{vw}<0\Leftrightarrow a_{vw}=1$, which concludes. □

A consequence of this proposition is the following corollary.

Corollary 3.8.4. *Let p be a point of Γ and let $\mathrm{T}_p(\Gamma)=\{\nu_1,\dots,\nu_{d_p}\}$. For each $P\in\mathcal{P}_p$, all the coordinates $s_j^{\nu_i}$ with $i\in P$ and $j\in\{0,1\}$ have the same sign. In addition, there exists at most one $P\in\mathcal{P}_p$ such that all $s_0^{\nu_i}$ for $i\in P$ are negative (and, if it exists, this is I_p).*

Proposition - Definition 3.8.5. *Let T be a metric tree. Then, there exists a unique effective $\mathfrak{g}^1_1$ on T up to linear equivalence.*

Proof. Let (D,N) be an effective $\mathfrak{g}^1_1$ on T with underlying slope structure $\mathfrak{S}$. Then, we can assume that $D=(v)$ for some vertex v of the tree. For every point y in the tree, let $f_{v\to y}$ to be the unique function taking value zero at v with $\mathrm{div}(f_{v\to y})+(v)=(y)$. Since N has rank one and $N\subseteq\mathrm{Rat}(D)$, $f_{v\to y}$ must belong to N. This shows that $\mathrm{Rat}(D)=N$. It is easy to see that N is finitely generated and has tropical rank one and from this, the proposition follows.

Note that, orienting the edges of T away from v, $\mathfrak{S}^{\mathrm{e}}$ is fully determined to allow slopes $0<1$ on each oriented edge, and that $\mathfrak{S}^{\mathrm{v}}$ is standard at each vertex of T. We moreover have $N=\mathrm{Rat}(D,\mathfrak{S})=\mathrm{Rat}(D)=\{f_{v\to y}+c\mid y\in T,\,c\in\mathbb{R}\}$. Finally, $\mathrm{Rat}(D)$ is finitely generated (for every metric graph by [HMY12]), and it is easy to see that $\mathrm{Rat}(D)$ has tropical rank one. □

The following is the main theorem of this section. We recall that a map between metric graphs is called a *morphism* if it is integer affine with respect to the natural integer affine structures of the metric graphs, and it is called *finite* if the preimage of every point is finite.

Theorem 3.8.6. *Let (D,M) be an effective $\mathfrak{g}^1_d$ on a metric graph Γ with underlying slope structure $\mathfrak{S}$. We suppose that D is x_0-reduced for some point $x_0\in\Gamma$. Then, we have the following results.*

- *The image of the map $\mathrm{Red}\colon\Gamma\to|M|\subseteq\mathrm{Sym}^d(\Gamma)$ is a metric tree T, and Red is a finite morphism. Moreover, we have $M=\mathrm{Red}^*(N)$, where N is the semimodule on T defined in Proposition–Definition 3.8.5 using the point $\mathrm{Red}(x_0)\in T$.*

- *There exist a tropical modification* $\alpha\colon \widetilde{\Gamma} \longrightarrow \Gamma$ *of* Γ *and a finite harmonic morphism* $\varphi\colon \widetilde{\Gamma} \to T$ *of degree* d *such that* $\varphi_{|\Gamma} = \mathrm{Red}$.

Conversely, let $\psi\colon \widetilde{\Gamma} \to T$ *be a finite harmonic morphism from a tropical modification* $\widetilde{\Gamma}$ *of* Γ *to a metric tree* T*, and denote by* $\tau\colon \widetilde{\Gamma} \to \Gamma$ *the contraction map. We put the semimodule* N *on* T *defined in Proposition–Definition 3.8.5 using the point* $y_0 = \psi(x_0)$*. Then,* $(\tau(\psi^{-1}(y_0)), \psi^*(N))$ *is an effective* $\mathfrak{g}^1_d$ *on* Γ *for* d *equal to the degree of* ψ.

Remark 3.8.7. The theorem shows that the combinatorial $\mathfrak{g}^1_d$'s are the precise analogues of algebraic geometric linear series of rank one. In the context of linear series on metrized complexes introduced in [AB15], a smoothing theorem in rank one was previously obtained by Luo and Manjunath in [LM18]. However, in that context, there are obstructions to smoothing, and the results are different. ◇

Using [ABBR15b, Theorem 3.11] and [ABBR15a, Theorem 7.7], we deduce the following smoothing theorem. See Section 3.9 for more details about Berkovich curves and tropicalization of linear series.

Theorem 3.8.8 (Smoothing theorem 3.1.6 for $\mathfrak{g}^1_d$'s)**.** *Every effective* $\mathfrak{g}^1_d$ (D, M)*, with* $M \subseteq \mathrm{Rat}(D)$*, on* Γ *is smoothable. That is, there exists a smooth proper curve* X *over an algebraically closed field with a non-trivial non-Archimedean valuation such that* Γ *is a skeleton of* X^{an}*, a divisor* E *on* X*, and a vector subspace* $\mathrm{H} \subseteq H^0(X, \mathcal{O}(E))$ *of rank one on* X *such that* (D, M) *is the tropicalization of the* $\mathfrak{g}^1_d$ (E, H) *from* X *to* Γ.

Remark 3.8.9. This crucially uses the fact that rank functions of rank one are geometric in the terminology of Section 3.2 (see [ABBR15a] and [ABBR15b]). This is applied to every ρ_v with $v \in V$ for $G = (V, E)$ a model a Γ on which $(D, \mathfrak{S})$ is defined. ◇

The rest of this section is devoted to the proof of Theorem 3.8.6.

3.8.1 Proof of Theorem 3.8.6

To this end, we will define an equivalence relation and a partial preorder on the points of Γ, show that the equivalence classes correspond to the map Red, and prove that the quotient $\Gamma/\sim$ is a metric tree.

Let (D, M) be an effective $\mathfrak{g}^1_d$ with underlying slope structure $\mathfrak{S}$ on Γ. We define an equivalence relation $\sim_M$ on the set of points of Γ as follows. For two points $x, y \in \Gamma$, we write $x \sim_M y$ (or simply $x \sim y$) if, for all $f \in M$, we have $f(x) = f(y)$. We also define a partial order by writing $x \leqslant_M y$ (or simply $x \leqslant y$) if, for all $f \in M$, we have $f(x) \leqslant f(y)$. Note that, if x and x' belong to the same edge e, then x and x' are comparable for $\leqslant_M$: in this case, the comparability is given by the orientation of e. We also write $x <_M y$ (or simply $x < y$) the corresponding strict partial preorder, when $x \leqslant y$ and there exists some $f \in M$ such that $f(x) < f(y)$. The statement $x <_M y$ is equivalent to $x \leqslant_M y$ and $x \not\sim_M y$.

It is immediate that $\leqslant_M$ is indeed reflexive and transitive, and $<_M$ irreflexive, transitive and asymmetric. Also, $\leqslant_M$, resp. $<_M$, induces a well-defined partial order, resp. a well-defined strict partial order, on the quotient $\Gamma/\sim$.

We first show that $\sim$ is a well-behaved equivalence relation.

Proposition 3.8.10. *For all $x \in \Gamma$, the class of x under $\sim$ is finite.*

Proof. Let x and x' belong to a common edge e. Suppose without loss of generality that $x \leqslant x'$. Then, by Theorem 3.5.13, there exists $f \in M$ that takes all minimal slopes around x' (for example $f = f_{x'}$), in particular in the direction of x in which the minimal slope is negative. Thus $f(x) < f(x')$ and $x \not\sim x'$. This shows that we can only have $x \sim x'$ if x and x' do not belong to the same edge, from which the result follows. □

This shows, in particular, that Red is a finite morphism.

Remark 3.8.11. A consequence of Proposition 3.8.10 is that, at each point, each jump is realized, which is a particular case of Theorem 3.5.13. ◇

We now show that the equivalence relation corresponds to the map Red.

Proposition 3.8.12. *Let $x, y \in \Gamma$. Then $x \sim y \iff \mathrm{Red}(x) = \mathrm{Red}(y)$.*

Proof. $\implies$ If $x \sim y$, then the sets M_x and M_y coincide, which shows, according to Definition 3.7.2, that $f_x = f_y$ and, therefore, $\mathrm{Red}(x) = \mathrm{Red}(y)$.

$\impliedby$ If $\mathrm{Red}(x) = \mathrm{Red}(y)$, by Proposition 3.3.2, there exists a constant c such that $f_y = f_x + c$. Suppose that $c > 0$. Then $f_y(x) = f_x(x) + c = c > 0$, which is impossible because for all $z \in \Gamma$, $f_z \leqslant 0$. So we have, in fact, $f_x = f_y$ by symmetry. Now, let g be any function of M such that $g(x) = 0$. We need to show that $g(y) = 0$. First, we note that

$$g(y) \geqslant f_x(y) = f_y(y) = 0.$$

Second, let $h := g - g(y)$. h belongs to M and verifies $h(y) = 0$, so we know that $h \geqslant f_y = f_x$, and thus, $-g(y) = h(x) \geqslant f_x(x) = 0$, which concludes. □

The map Red is equal to the projection map $\Gamma \to \Gamma/\sim$. We denote $\Gamma/\sim$ by T, and the goal is now to show that T is a metric tree. This amounts to showing that T has no cycle (T is already a metric graph). The proof uses the condition that the tropical rank is one. In the following, we will transfer the orientation of the edges of Γ to the edges of T.

Lemma 3.8.13. *Let $\overline{x} \in T$. Then $\overline{x}$ has in-degree at most one in T.*

Proof. Suppose, on the contrary, that $\overline{x}$ has in-degree at least two, that is, using the orientation of G defined by $\mathfrak{S}$, that two distinct edges $\overline{e}$ and $\overline{e}'$ incident to $\overline{x}$ are oriented toward $\overline{x}$. There are two cases.

(i) $\overline{e}$ and $\overline{e}'$ originate from two edges of G, e and e' respectively, which are incident to the same preimage x of $\overline{x}$.

We show that this cannot happen by proving that, on the contrary, e and e' will be glued together by Red in the following way: there exist $\lambda, \varepsilon > 0$ such that

- ε and $\lambda \cdot \varepsilon$ are smaller than $\ell(e), \ell(e')$, and
- for every choice of $\eta \in (0, \varepsilon)$, for every $y \in e$ at distance η from x, for $y' \in e'$ at distance $\lambda \cdot \eta$ from x, we have $y \sim y'$.

We know that f_x takes all minimum slopes around x, and, in particular, takes negative slopes on e and e' away from x. These two slopes can be different, but since they are both negative, the image by f_x identifies points of e and e' close to x as described above for some fixed $\varepsilon > 0$, with a dilation factor $\lambda = \frac{s_0^e}{s_0^{e'}}$. The null function 0 also identifies e and e' with the same dilation factor (in fact, with any dilation factor).

Let now g be any function of M. Without loss of generality, we assume that $g(x) = 0$. Since M is assumed to have tropical rank one, there exist $c, d \in \mathbb{R}$ such that for all $z \in \Gamma$, the minimum in

$$\min(0, f_x(z) + c, g(z) + d)$$

is attained at least twice. We will show that the image by g identifies e and e' in the same way, getting a contradiction.

We will consider three cases, depending on whether c is negative, positive, or zero.

Let us first assume that c is negative. This implies that, for y on e or e', $f_x(y) + c < 0$ and thus $g(y) + d = f_x(y) + c$. So $g = f_x$ on e and e' (evaluate at x) and thus identifies e and e' close to x with dilation factor λ.

Now, let us assume that c is positive. Then, for y on e or e' close to x, $f_x(y) + c$ is still positive and thus $g(y) + d = 0$. In fact, d must be zero, and so g is null close to x on e and e'. Therefore, g identifies e and e' with dilation factor λ.

The last case is when $c = 0$. For y on e different from x, we have $f_x(y) = g(y) + d$ and thus, by continuity, g identifies e and e' with dilation factor λ.

We have thus shown that in all cases, g identifies the two edges oriented toward x. Since this is true for all $g \in M$, we infer tha $\overline{x}$ cannot have in-degree at least two.

(ii) $\overline{e}$ and $\overline{e}'$ originate from two edges of G, e and e' respectively, which are incident respectively to two different preimages x and x' of $\overline{x}$. Since, by definition, $f_x = f_{x'}$ and $g(x) = g(x')$ for all $g \in M$, we can use exactly the same argument, *mutadis mutandis*. □

Remark 3.8.14. We have the following fact: the edges incident to some $x \in \Gamma$ that are glued by Red are exactly those belonging to the same set in the partition $\mathcal{P}_x$.

$\Longleftarrow$ Take $g \in M$. Then $\partial_x(g)$ is a jump in S^x, so, using Lemma 3.2.17 and Proposition 3.8.3, we are in one of the three following cases:

(i) g is constant around x.

(ii) g takes negative slopes away from x on all incoming edges, and zero slope on all outgoing edges.

(iii) g takes negative slopes away from x on all incoming edges, positive slopes on some set $S \in \mathcal{P}_x$ of outgoing edges, and slope zero on all other (outgoing) edges.

In each one of these cases, g identifies the edges of each set of $\mathcal{P}_x$. Since this is true for all g, then Red glues together the edges of each set of $\mathcal{P}_x$.

$\Longrightarrow$ If e and e' are two edges that do not belong to the same set of $\mathcal{P}_x$, then we can assume without loss of generality that e is an outgoing edge (see Corollary 3.8.4). Moreover, there is a jump $\underline{a} \in \mathbb{R}_1^{d_x}$ such that $e \in P_{\underline{a}}$ and $e' \notin P_{\underline{a}}$. The jump $\underline{a}$ is realized by some function $f \in M$ (Remark 3.8.11). There are two cases:

(a) e' is also an outgoing edge. Then f has a positive slope on e (away from x) and slope zero on e'.

(b) e' is an incoming edge. Then f has a positive slope on e and negative slope on e'.

In both cases, f does not identify e and e', so Red does not glue these edges.

Finally, every edge incident to x in Γ is sent to an actual edge incident to $\overline{x}$ in T because Red is affine linear and non-contracting (Theorem 3.7.24). $\diamond$

Remark 3.8.15. Remark 3.8.14 has an interesting consequence. At a point $x \in \Gamma$, we know that the set of all incoming edges is a set of the partition $\mathcal{P}_x$ (Proposition 3.8.3). Then, applying the first part of Remark 3.8.14 gives that all incoming edges at x are glued together by the map Red, which yields case (i) in Lemma 3.8.13 automatically. However, case (ii) really requires the argument involving the tropical rank, as developed in the proof. $\diamond$

We now come to the desired result.

Proposition 3.8.16. *T has no cycles, and therefore is a metric tree.*

Proof. First, we claim that T can have no oriented cycle. Indeed, Proposition 3.8.10 states that if x and y are the vertices of an edge oriented from x to y, then $x < y$ (using the strict partial order induced on T). In an oriented cycle, we would get a strict inequality of the form $x < x$, which is absurd.

To conclude, we show that T cannot have any cycle. Indeed, a cycle endowed with an orientation of its edges, if it is not an oriented cycle, gives an in-degree two to at least one of its vertices, which is impossible by Lemma 3.8.13. $\square$

We thus conclude that T is an acyclic metric graph, that is, a metric tree. Note that the metric on T is the one induced from Γ by the gluing. It is the only metric such that the slopes of the functions $T \to \mathbb{R}$ factored from functions $\Gamma \to \mathbb{R}$ in $\mathrm{Rat}(D, \mathfrak{S})$ are in the set $\{-1, 0, 1\}$. Roughly speaking, this amounts to giving an edge $\overline{e}$ of T a length equal to the product of the length of e times the non-zero allowed slope on e, for e any edge of Γ sent to

$\overline{e}$ by Red. Equivalently, the metric is such that the relative slope of Red on an edge is the non-zero slope on this edge in the slope structure $\mathfrak{S}$.

We now show the following claim of Theorem 3.8.6: $M = \mathrm{Red}^*(N)$, where N is associated to $\mathrm{Red}(x_0) \in T$.

Proof of Theorem 3.8.6: $M = \mathrm{Red}^*(N)$. We show the assertion in Theorem 3.8.6 that $M = \mathrm{Red}^*(N)$, where N is associated to $\mathrm{Red}(x_0) \in T$.

(The inclusion $\subseteq$) Let $f \in M$. By definition of Red, f is constant on the equivalence classes for $\sim_M$, so it can be written in the form $f = g \circ \mathrm{Red}$, with g a function $T \to \mathbb{R}$. It is straightforward that g is continuous and affine linear with integral slopes (see Proposition 3.8.10 which implies that Red is a local homeomorphism on its image). By definition of the metric on T, g has slopes zero and one compatible with $\mathfrak{S}^{\mathrm{e}}$, with $\mathfrak{S}$ the slope structure underlying N. To show that g is compatible with D and $\mathfrak{S}^{\mathrm{v}}$, we simply look at the possible sets of slopes of f around a point $y \in \Gamma$ in the cases (i) and (ii) explored in the proof of Proposition–Definition 3.8.5.

(The inclusion $\supseteq$) The other way around, let $g \in N$ and $f := g \circ \mathrm{Red}$. We have to show that $f \in M$. Since g belongs to N, it is of the form $g = f_y = f_{\mathrm{Red}(x_0)\to y} \in \mathrm{Rat}((\mathrm{Red}(x_0)), \mathfrak{S})$ with $y \in T$. Let $w \in \mathrm{Red}^{-1}(y)$. All the functions $f_{x_0 \to w}$ for such a w are equal thanks to Proposition 3.8.12. The fact that f belongs to M then comes from the equality $f = f_w$ which is implied by Theorem 3.5.13. □

To finish the first part of the theorem, we need to show that the morphism Red from Γ to T can be resolved to a finite harmonic morphism of degree d by a tropical modification of Γ. Given a point $\bar{x} \in T$ and $\nu \in \mathrm{T}_{\bar{x}}(T)$, we denote by $T_{\bar{x},\nu}$ the metric subtree of T consisting of the point $\bar{x}$ and all the points z of T which have the property that the unique path from $\bar{x}$ to z in T has tangent vector at $\bar{x}$ equal to ν. For the language of harmonic morphisms and degrees of maps between metric graphs, we refer the reader to [ABBR15a, Section 2.1].

Proof of Theorem 3.8.6: resolution to a finite harmonic morphism. Let $\bar{x}$ be a point of T, and consider a point x of Γ with $\mathrm{Red}(x) = \bar{x}$. We denote by $\mathrm{Red}_* \colon \mathrm{T}_x(\Gamma) \to \mathrm{T}_{\bar{x}}(T)$ the induced map on tangent vectors. For each unit tangent vector $\nu \in \mathrm{T}_{\bar{x}}(T)$, consider the degree of Red at x above ν, denoted by $\deg_\nu \mathrm{Red}(x)$ and defined as the sum

$$\deg_\nu \mathrm{Red}(x) := \sum_{\substack{\mu \in \mathrm{T}_x(\Gamma) \\ \mathrm{Red}_*(\mu) = \nu}} \mathrm{slope}_\mu(\mathrm{Red})(x).$$

Define the degree of Red at x, denoted by $\deg \mathrm{Red}(x)$, to be the maximum quantity $\deg_\nu \mathrm{Red}(x)$ for all $\nu \in \mathrm{T}_{\bar{x}}(T)$.

We define the tropical modification $\widetilde{\Gamma}_0$ as follows. For each point $\bar{x} \in T$ and each point $x \in \Gamma$ as above, and each $\nu \in \mathrm{T}_{\bar{x}}(T)$ with $\deg_\nu \mathrm{Red}(x) < \deg \mathrm{Red}(x)$, we take $\deg \mathrm{Red}(x) - \deg_\nu \mathrm{Red}(x)$ copies of the subtree $T_{\bar{x},\nu}$ of T, glue them to the point x by identifying the point $\bar{x}$ of $T_{\bar{x},\nu}$ in each copy with the point x. We then naturally extend Red to each copy of $T_{\bar{x},\nu}$ by the identity map on $T_{\bar{x},\nu}$. Since $\deg \mathrm{Red}(x) - \deg_\nu \mathrm{Red}(x)$

is non-zero only for finitely many pairs $(\bar{x}, \nu)$, we obtain a tropical modification $\widetilde{\Gamma}_0$ of Γ and a map $\phi!_0 \colon \widetilde{\Gamma}_0 \to T$ which coincides with Red on the metric subgraph $\Gamma \hookrightarrow \widetilde{\Gamma}_0$. By construction, the map $\phi!_0$ is harmonic of degree $\deg \mathrm{Red}(x)$ at all points x of Γ. It is also harmonic of degree one at all other points of $\widetilde{\Gamma}_0 \smallsetminus \Gamma$. Note that $\phi!_0$ is of degree d_0 at most d. In fact, in the gluing process above, in the construction of $\widetilde{\Gamma}_0$, when we add $\deg \mathrm{Red}(x) - \deg_\nu \mathrm{Red}(x)$ at x, the degree remains bounded by d. Indeed, for each point $\bar{y}$ in the subtree $T_{\bar{x},\nu}$, and each $y \in \Gamma$ with $\mathrm{Red}(y) = \bar{y}$, the coefficient of x in D_y^M is at least $\deg \mathrm{Red}(x) - \deg_\nu \mathrm{Red}(x)$, using the explicit description of the variation of reduced divisors with respect to the base point provided in Section 3.7.6. This implies that the fiber $\mathrm{Red}^{-1}(\bar{y})$, counted by multiplicity, is of size bounded by $d - \deg \mathrm{Red}(x) + \deg_\nu \mathrm{Red}(x)$, so that after modification, the degree remains bounded by d. Treating the points one by one in the gluing process, and proceeding by induction, we obtain $d_0 \leqslant d$.

To conclude, let $\widetilde{\Gamma}$ be the tropical modification of $\widetilde{\Gamma}_0$ obtained by plugging $d - d_0$ copies of T at some point y of $\widetilde{\Gamma}_0$, by identification of the point $\phi_0(y)$ in each copy with y. We extend $\phi!_0$ by the identity map on these copies to get the harmonic map $\phi \colon \widetilde{\Gamma} \to T$ of degree d, as required. □

Now we show the second part of Theorem 3.8.6: if $\psi \colon \widetilde{\Gamma} \to T$ is any finite harmonic morphism to a metric tree T on which we put the semimodule N using the point $y_0 = \psi(x_0)$, then $(\tau(\psi^{-1}(y_0)), \psi^*(N))$ is an effective $\mathfrak{g}^1_d$ on Γ.

Proof of Theorem 3.8.6: the converse. It will be enough to show that $(\psi^{-1}(y_0), \psi^*(N))$ is an effective $\mathfrak{g}^1_d$ on $\widetilde{\Gamma}$.

We first have to define the pullback of the slope structure $\mathfrak{S}'$ of N by ψ to $\widetilde{\Gamma}$. Let D be the divisor $\psi^{-1}(y_0)$. Its degree is d, the degree of ψ. At some point $x \in \Gamma$, in the direction $\nu \in \mathrm{T}_x(\Gamma)$, the non-zero possible slope is defined to be the relative slope of ψ in the direction ν: this defines $\mathfrak{S}^{\mathrm{e}}$. We now define $\mathfrak{S}^{\mathrm{v}}$ around x by saying that the jumps of ρ_x are exactly the vector $\underline{0}$, and the vectors having ones for all edges belonging to a certain complete set of edges identified by ψ, and zero on all other edges, which entirely defines ρ_x. We have defined the pair $(D, \mathfrak{S})$.

We now show that $\psi^*(N)$ is a semimodule included in $\mathrm{Rat}(D, \mathfrak{S})$. Firstly, it is stable by the two tropical operations since N is. Secondly, we show that $\psi^*(N) \subseteq \mathrm{Rat}(D, \mathfrak{S})$. If f is a function of $\psi^*(N)$, we can write it $f = g \circ \psi$ with $g \in N$. It is automatic by the construction of $\mathfrak{S}$ on $\widetilde{\Gamma}$ that f is compatible with $\mathfrak{S}^{\mathrm{e}}$ and $\mathfrak{S}^{\mathrm{v}}$. The fact that $D + \mathrm{div}(f) \geqslant 0$ comes from the harmonicity of ψ.

We now check property $(**)$ of Section 3.5.5. Let $x \in \Gamma$ and $E = (x)$. Let $y = \psi(x)$. Then, the function $f_{y_0 \to y} \circ \psi$ has the required properties (1) and (2). This function belongs to $\psi^*(N)$. Therefore, $(D, \psi^*(N))$ verifies property $(**)$ of Section 3.5.5. Finally, the finite generation of $\psi^*(N)$ and the fact that $\psi^*(N)$ has tropical rank one follow from the same properties for N. The same is true for effectivity. We have proved that $(D, \psi^*(N))$ is an effective $\mathfrak{g}^1_d$. □

This finishes the classification of $\mathfrak{g}^1_d$'s on metric graphs.

3.9 Limit linear series on the skeleton of a Berkovich curve

In this section, we show that the tropicalization of linear series on $\mathbf{C}$ gives combinatorial limit linear series.

Let $\mathbf{K}$ be an algebraically closed field with a non-trivial non-Archimedean valuation $\mathbf{val}$ and let $\mathbf{C}$ be a smooth proper curve over $\mathbf{K}$. We assume that $\mathbf{K}$ is complete with respect to $\mathbf{val}$. Let R, $\mathfrak{m}$, and $\kappa = \mathrm{R}/\mathfrak{m}$ be the valuation ring, the maximal ideal of R, and the residue field, respectively. Denote by Λ the value group of $\mathbf{val}$. We also denote by $|\cdot|$ the corresponding norm on $\mathbf{K}$, so that $\mathbf{val}(\cdot) = -\log|\cdot|$. Let $\mathbf{K}(\mathbf{C})$ be the function field of $\mathbf{C}$.

A semistable vertex set for $\mathbf{C}^{\mathrm{an}}$ is a finite set of type 2 points V in $\mathbf{C}^{\mathrm{an}}$ such that the complement $\mathbf{C}^{\mathrm{an}} \smallsetminus V$ is a disjoint union of finitely many open annuli and infinitely many open disks. A semistable vertex set V gives rise to a skeleton Γ for $\mathbf{C}^{\mathrm{an}}$, defined as the union in $\mathbf{C}^{\mathrm{an}}$ of V and the skeleta of the open annuli in $\mathbf{C}^{\mathrm{an}} \smallsetminus V$. The canonical metric on the skeleta of the open annuli gives the skeleton a metric graph structure, naturally embedded in $\mathbf{C}^{\mathrm{an}}$. The underlying graph $G = (V, E)$ has vertex set V and edge set E in bijection with the set of open annuli in $\mathbf{C}^{\mathrm{an}} \smallsetminus V$. There is an edge between a pair of vertices v and u in V for each open annulus whose closure contains the points v and u. Moreover, the edge length function $\ell\colon E \to (0, +\infty)$ associates to each edge of G the modulus of the corresponding annulus.

Let Γ be a metric graph skeleton of $\mathbf{C}^{\mathrm{an}}$ with underlying graph $G = (V, E)$ and denote by $\tau\colon \mathbf{C}^{\mathrm{an}} \to \Gamma$ the canonical retraction map. We call τ the tropicalization map. We get a tropicalization map $\tau_*\colon \mathrm{Div}(\mathbf{C}) \to \mathrm{Div}_\Lambda(\Gamma)$ that sends a divisor $\mathbf{D} = \sum_{x \in \mathbf{C}(\mathbf{K})} a_x(x)$ on $\mathbf{C}$ to the divisor $\tau_*(\mathbf{D}) = \sum_{x \in \mathbf{C}(\mathbf{K})} a_x(\tau(x))$.

We denote by $\mathfrak{v}_x\colon \mathbf{K}(\mathbf{C}) \to \mathbb{R} \cup \{+\infty\}$ the valuation of a point $x \in \mathbf{C}^{\mathrm{an}} \smallsetminus X(\mathbf{K})$ with $\mathfrak{v}_x(f) = +\infty$ only if $f = 0$. The residue field of this valuation is denoted by $\kappa(x)$. We also denote by $|\cdot|_x = \exp(-\mathfrak{v}_x)$ the corresponding norm.

For each non-zero $\mathbf{f} \in \mathbf{K}(\mathbf{C})$, we define the tropicalization of $\mathbf{f}$, denoted $\mathrm{trop}(\mathbf{f})\colon \Gamma \to \mathbb{R}$, as the map that sends each $x \in \Gamma \subseteq \mathbf{C}^{\mathrm{an}} \smallsetminus \mathbf{C}(\mathbf{K})$ to $\mathfrak{v}_x(\mathbf{f})$. This induces a tropicalization map $\mathrm{trop}\colon \mathbf{K}(\mathbf{C}) \smallsetminus \{0\} \to \mathrm{Rat}_\Lambda(\Gamma)$.

Let $\mathbf{D}$ be divisor of degree d and rank at least r on $\mathbf{C}$, and let $(\mathcal{O}(\mathbf{D}), \mathbf{H})$ be a $\mathfrak{g}^r_d$ on $\mathbf{C}$. We identify $\mathbf{H}$ with a subspace of $\mathbf{K}(C)$ of dimension $r + 1$.

Let Γ be a skeleton of $\mathbf{C}^{\mathrm{an}}$. We define

$$M := \mathrm{trop}(\mathbf{H}) = \big\{\mathrm{trop}(\mathbf{f}) \;\big|\; \mathbf{f} \in \mathbf{H} \smallsetminus \{0\}\big\}.$$

Theorem 3.9.1. *The pair (D, M) is a refined Λ-rational $\mathfrak{g}^r_d$ on Γ.*

We will call the linear equivalence class of the pair (D, M) *the combinatorial limit linear series on Γ induced by $(\mathcal{O}(\mathbf{D}), \mathbf{H})$*. It is easy to see that (D, M) is effective provided that $\mathbf{H}$ contains constants.

3.9.1 Reduction

For each point x of type 2 in $\mathbf{C}^{\mathrm{an}}$, the extension $\kappa(x)/\kappa$ is of transcendence degree one. Let C_x be the smooth projective curve over κ with function field $\kappa(x)$. For the point x, the valuation $\mathfrak{v}_x$ has the same value group as $\mathbf{val}$. For each nonzero $\mathbf{f} \in \mathbf{K}(\mathbf{C})$, choosing $a \in \mathbf{K}$ with $\mathbf{val}(a) = \mathfrak{v}_x(\mathbf{f})$, we get that $a^{-1}\mathbf{f}$ has valuation $\mathfrak{v}_x(a^{-1}\mathbf{f}) = 0$, and therefore gives an element in the residue field $\kappa(x)$ that we denote by $\tilde{f}_x$. We call this the *reduction of* $\mathbf{f}$ *at* x, which is nonzero and defined only up to multiplication by a non-zero scalar in κ. For a vector subspace $\mathbf{H} \subset \mathbf{K}(X)$ of dimension $r+1$, denote by $\widetilde{H}_x \subset \kappa(x)$ the κ-vector subspace spanned by the reductions $\tilde{f}_x$ of elements $\mathbf{f} \in \mathbf{H}$ [AB15, Section 4.4]. By [AB15, Lemma 4.3], $\widetilde{H}_x$ has dimension $r+1$ over κ.

3.9.2 Slope structure and crude linear series coming from tropicalization

For a point x in Γ of type 2 in $\mathbf{C}^{\mathrm{an}}$, each unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ gives a point $p_x^\nu \in \mathrm{C}_x(\kappa)$. By the slope formula [BPR16], for any non-zero $\mathbf{f} \in \mathbf{K}(\mathbf{C})$, we have $\mathrm{slope}_\nu(\mathrm{trop}(\mathbf{f})) = \mathrm{ord}_{p_x^\nu}\left(\tilde{f}_x\right)$. Moreover, as a consequence of the slope formula [AB15], we get

$$\tau_*(\mathrm{div}(\mathbf{f})) = \mathrm{div}(\mathrm{trop}(\mathbf{f})).$$

(Note that there is a sign difference between our definition of the divisor of a rational function and that of [AB15].)

If $\mathbf{H} \subset \mathbf{K}(\mathbf{C})$ is a $\mathbf{K}$-vector subspace of dimension $r+1$, for any unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$, we get a collection of integers $\mathrm{slope}_\nu(\mathrm{trop}(\mathbf{f})) = \mathrm{ord}_{p_x^\nu}\left(\tilde{f}_x\right)$, $\mathbf{f} \in \mathbf{H}$. Since $\widetilde{H}_x$ has dimension $r+1$, this collection has size $r+1$. This means that the collection of slopes $\mathrm{slope}_\nu(h)$, for $h \in M = \mathrm{trop}(\mathbf{H})$, has size $r+1$. For each unit tangent vector ν, we order the slopes $\mathrm{slope}_\nu(h)$, for $h \in M$, in the form $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$. Define $S^\nu := \{s_i^\nu\}_{i\in[r]}$. In addition, the collection of points $p_x^\nu \in C_x(\kappa)$ for $\nu \in \mathrm{T}_x(\Gamma)$ defines a geometric rank function ρ_x associated to the corresponding filtrations on $\widetilde{H}_x$ as in Section 3.2.2. We define S^x as the set of jumps of ρ_x. We have the following theorem, which can be regarded as a refinement of [AB15, Theorem 5.9].

Proposition 3.9.2 (Slope structures induced by tropicalization of rational functions)**.** *Notation as above, let* $\mathbf{H} \subset \mathbf{K}(\mathbf{C})$ *be a* $\mathbf{K}$*-vector space of dimension* $r+1$*. Let* Γ *be a skeleton of* $\mathbf{C}^{\mathrm{an}}$*. There exists a semistable vertex set* V *for* $\mathbf{C}$ *such that* $\Sigma(\mathbf{C}, V) = \Gamma$*, and such that the slopes of the tropicalizations of rational functions* $\mathbf{f}$ *in* $\mathbf{H}$ *along edges in* Γ *define a slope structure of width* r *on* Γ*.*

Proof. We already defined S^x and S^ν for type 2 points of $\Gamma \subseteq \mathbf{C}^{\mathrm{an}}$ and $\nu \in \mathrm{T}_x(\Gamma)$. We show that the definitions can be extended to all points of Γ, and that the collection $\mathfrak{S} = \{S^x; S^\nu\}_{x\in\Gamma,\, \nu\in\mathrm{T}_x(\Gamma)}$ is induced by a simple graph model of Γ (or, equivalently, by a semistable vertex set of $\mathbf{C}^{\mathrm{an}}$). To show this, let x be a point of type 2, and let $\nu \in \mathrm{T}_x(\Gamma)$

be a tangent direction at x in Γ. Let $\mathbf{f}_0, \dots, \mathbf{f}_r$ be a basis of $\mathbf{H}$ such that the reductions $f_0 = \tilde{\mathbf{f}}_0, \dots, f_r = \tilde{\mathbf{f}}_r$ to $\kappa(x)$ yield the orders of vanishings $s_0^\nu, \dots, s_r^\nu$ at the point p_x^ν, respectively. By the slope formula, the slope of $\mathrm{trop}(\mathbf{f}_j)$ along ν at x coincides with s_j^ν. There thus exists a half segment $I_x^\nu = [x, y^\nu]$ on the edge supporting the point x and the tangent direction ν such that the slope of the function $\mathrm{trop}(\mathbf{f}_j)$ along ν at every point of I_x^ν is s_j^ν. Since these are $r+1$ integers, they are all the possible slopes along ν in I_x^ν of tropicalizations of functions in $\mathbf{H}$. We extend the definition of $S^\bullet$ to every point in the interior of the segment I_x^ν by taking these slopes and by declaring the rank function to be standard. Applying now the compactness of Γ, we deduce a finite covering of Γ by segments I_x^ν, from which we deduce the statement in the proposition. □

The fact that $\tau(\mathrm{div}(\mathbf{f})) = \mathrm{div}(\mathrm{trop}(\mathbf{f}))$ then shows that $(D, \mathfrak{S})$ has rank r, as defined in Definition 3.4.1. This is analogous to the proof of the specialization theorem for metrized complexes in [AB15], we thus omit the details. This shows that

Proposition 3.9.3. *The pair $(D, \mathfrak{S})$ defined by tropicalization is a crude linear series on Γ of degree d and rank r.*

Note that for two linearly equivalent divisors $\mathbf{D} \sim \mathbf{D}'$ on $\mathbf{C}$, and $\mathbf{H}$ a subspace of the space of global sections of the corresponding line bundles $\mathcal{O}(\mathbf{D}) = \mathcal{O}(\mathbf{D}')$ of dimension $r+1$, the two pairs $(D, \mathfrak{S})$ and $(D', \mathfrak{S}')$ are linearly equivalent.

3.9.3 $M \subseteq \mathrm{Rat}_\Lambda(D)$ is a semimodule

We show that $M \subseteq \mathrm{Rat}_\Lambda(D)$ is a semimodule. Changing the vertex set if necessary, we can assume without loss of generality that the support of D is included in the set of vertices V. Let f and g be two elements in M, and let $\lambda \in \Lambda$. By definition, we can write $f = \mathrm{trop}(\mathbf{f})$ and $g = \mathrm{trop}(\mathbf{g})$ with $\mathbf{f}, \mathbf{g} \in \mathbf{H} \smallsetminus \{0\}$. Let $h := \min(f + \lambda, g)$ for $\lambda \in \Lambda$. We show the existence of $\mathbf{h} \in \mathbf{H}$ such that $\mathrm{trop}(\mathbf{h}) = h$. This will show that M is a Λ-semimodule. Since $\lambda \in \Lambda = \mathbf{val}(\mathbf{K} \smallsetminus \{0\})$, we can write $\lambda = \mathbf{val}(\alpha)$ with $\alpha \in \mathbf{K}$. We take $\mathbf{h} := \alpha\mathbf{f} + \mathbf{g}$. The goal is to show that replacing, if necessary, α with $\alpha\beta$ for $\beta \in \mathbf{K}^\times$ of valuation zero, we have $h = \mathrm{trop}(\mathbf{h})$.

Let $x \in \Gamma$ be such that $f(x) + \lambda \neq g(x)$. Then, by the enhanced non-Archimedean triangular inequality (that is, if $\mathbf{val}(a) \neq \mathbf{val}(b)$, then $\mathbf{val}(a+b) = \min(\mathbf{val}(a), \mathbf{val}(b))$), we get automatically that $\mathfrak{v}_x(\mathbf{h}) = \min(f(x) + \lambda, g(x)) = h(x)$.

Let now $\Gamma_0 := \{x \in \Gamma,\ f(x) + \lambda = g(x)\}$. Since f and g are piecewise affine linear, Γ_0 can be written as a union of finitely many segments of Γ so that both f and g are affine on each of these segments. Let now I be any of them, whose extremities we denote x and y, on an edge e of Γ. A segment of Γ is a segment of an edge of G; it can be reduced to just a point. Let ν be the tangent direction in $\mathrm{T}_x(\Gamma)$ pointing toward y; if $x = y$, we choose an arbitrary $\nu \in \mathrm{T}_x(\Gamma)$. Consider the point p_x^ν of $C_x(\kappa)$ corresponding to ν. By the slope formula and the fact that $f + \lambda = g$ on all I, we have

$$\mathrm{ord}_{p_x^\nu}\left(\widetilde{\mathbf{f}}\right) = \mathrm{slope}_\nu(f) = \mathrm{slope}_\nu(g) = \mathrm{ord}_{p_x^\nu}(\widetilde{\mathbf{g}}).$$

Up to multiplying α by some element β of $\mathbf{K}^\times$ of valuation zero, we can ensure that

$$\mathrm{ord}_{p_x^\nu}\Big(\widetilde{\alpha\mathbf{f}+\mathbf{g}}\Big) = \mathrm{ord}_{p_x^\nu}\Big(\widetilde{\mathbf{f}}\Big) = \mathrm{ord}_{p_x^\nu}(\widetilde{\mathbf{g}}) = \mathrm{ord}_{p_x^\nu}\Big(\widetilde{\alpha}\widetilde{\mathbf{f}}+\widetilde{\mathbf{g}}\Big).$$

Note that only one value for the reduction $\widetilde{\alpha}$ in κ is forbidden.

Using again the slope formula for $\widetilde{\alpha\mathbf{f}+\mathbf{g}}$ yields $\mathrm{ord}_{p_x^\nu}\Big(\widetilde{\alpha}\widetilde{\mathbf{f}}+\widetilde{\mathbf{g}}\Big) = \mathrm{slope}_\nu(h)$, so that

$$\mathrm{slope}_\nu(h) = \mathrm{slope}_\nu(f) = \mathrm{slope}_\nu(g).$$

We can do the same at the other extremity y of I, and ensure that locally, starting at either extremity of I, h has the same slope as f and g. Since f and g are linear on I, D has no support in the interior of I, and h coincides with $f+\lambda$ and g on the extremities of I, we must have $h = f+\lambda = g$ on the full interval I.

Since this can be done for each of the finitely many segments composing Γ_0, forbidding at most one value for the reduction $\tilde{\alpha} \in \kappa$ each time, and since κ is algebraically closed (and thus infinite), there is some $\alpha \in \mathbf{K}$ such that $\mathbf{val}(\alpha) = \lambda$ and

$$\mathrm{trop}(\alpha\mathbf{f}+\mathbf{g}) = \min(f+\lambda, g) = h$$

on all Γ. We have shown that $h \in M$, so M is a semimodule of $\mathrm{Rat}_\Lambda(D, \mathfrak{S})$.

3.9.4 Finite generation property

Proposition 3.9.4. *The semimodule $M = \mathrm{trop}(\mathbf{H})$ is finitely generated.*

We suppose that the slope structure $\mathfrak{S}$ is defined on the model $G = (V, E)$ of Γ associated to the semistable vertex V.

Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbf{K}[T])$ and let $\mathbb{A}^{1,\mathrm{an}}$ be its Berkovich analytification. Let $A(\rho)$ be the closed annulus in $\mathbb{A}^{1,\mathrm{an}}$ of center 0 with outer radius one and inner radius $\rho \in (0,1)$,

$$A(\rho) = \big\{x \in \mathbb{A}^{1,\mathrm{an}} \;\big|\; \rho \leqslant |T|_x \leqslant 1\big\}.$$

Let $R(\rho)$ be the ring of analytic functions on $A(\rho)$. An analytic function f on $A(\rho)$ admits a formal power series expansion

$$f = \sum_{n\in\mathbb{Z}} a_n T^n$$

with $\lim_{n\to\pm\infty} |a_n| s^n = 0$ for all $s \in [\rho, 1]$. The skeleton of $A(\rho)$ is a closed interval, which can be identified with $I \coloneqq [0, -\log\rho]$: each point q in this interval corresponds to the norm $|\cdot|_{\zeta_q}(f) = \sup_{n\in\mathbb{Z}} |a_n| \exp(-qn) = \max_{n\in\mathbb{Z}} |a_n| \exp(-qn)$ on any analytic function f as above. The tropicalization of an analytic function f is the function $\mathrm{trop}(f)$ on the interval I given by

$$\mathrm{trop}(f)(q) = \min\{\mathbf{val}(a_n) + nq \;\big|\; n \in \mathbb{Z}\} \qquad \forall q \in I.$$

Each edge e in E is the skeleton of one of the annuli in the complement $\mathbf{C}^{\mathrm{an}} \smallsetminus V$, that we denote by $A(e)$. We have $A(e) \simeq A(\rho_e)$ with $\rho_e = \exp(-\ell_e)$, where ℓ_e is the length of e in Γ. Using this identification, e is identified in Γ with the interval $[0, \ell_e]$.

Proof of Proposition 3.9.4. By restriction, each element $\mathbf{f} \in \mathbf{H}$ gives rise to an analytic function on the annulus $A(e)$ that we denote by $\mathbf{f}_e$. The tropicalization of $\mathbf{f}$ restricted to e coincides with the tropicalization of $\mathbf{f}_e$, and has slopes among $s_0^e < \cdots < s_r^e$. It follows that, taking the analytic development $\mathbf{f}_e = \sum_{n\in\mathbb{Z}} a_n^e(\mathbf{f})T^n$ in $A(e)$, with $a_n^e(\mathbf{f}) \in \mathbf{K}$, the tropicalization of $\mathbf{f}_e$ is entirely defined by the truncation

$$\underline{\mathbf{f}}_e := \sum_{n=s_0^e}^{s_e^e} a_n^e(\mathbf{f})T^n.$$

Consider the $\mathbf{K}$-linear map

$$\begin{aligned} \varphi\colon \mathbf{H} &\longrightarrow \prod_e \mathbf{K}^{[s_0^e, s_e^r]} \\ \mathbf{f} &\longmapsto \Big((a_n^e(\mathbf{f}))_{n=s_0^e,\dots,s_e^r} \Big)_{e\in E}. \end{aligned}$$

We infer that the tropicalization $\mathrm{trop}(\mathbf{f})$, for $\mathbf{f}$ non-zero in $\mathbf{H}$, is entirely determined by tropicalization

$$\mathrm{trop}(\varphi(\mathbf{f})) := \Big((\mathbf{val}(a_n^e(\mathbf{f})))_{n=s_0^e,\dots,s_e^r} \Big)_{e\in E} \in \prod_e \mathbb{T}^{[s_0^e, s_e^r]}$$

via the expression

$$\mathrm{trop}(\mathbf{f})|_e(q) = \min\big\{ \mathbf{val}(a_n^e(\mathbf{f})) + nq \;\big|\; n = s_0^e, \dots, s_r^e \big\} \qquad \forall q \in [0, \ell_e].$$

Let $\mathbf{H}' = \varphi(\mathbf{H})$, and $M' = \mathrm{trop}(\mathbf{H}') := \big\{ \mathrm{trop}(\varphi(\mathbf{f})) \;\big|\; \mathbf{f} \in \mathbf{H} \smallsetminus \{0\} \big\} \subset \prod_e \mathbb{T}^{[s_0^e, s_e^r]}$. From the above discussion we deduce the existence of a surjective morphism of semimodules $\varphi^{\mathrm{trop}} : M' \to M$. We conclude by observing that M', being the tropicalization of a linear subspace of $\mathbf{K}^n$ for $n = \sum_e (s_r^e - s_0^e + 1)$, is finitely generated, see [Spe08, Theorem 3.1] and [Gau92, Chap. III, Theorem 1.2.2]. A generating set for M' gives a generating set for M via the surjection φ^{trop}. The proposition follows. □

3.9.5 Proof of Theorem 3.9.1

We have proved nearly all the properties to show that (D, M) is a linear series.

We first need to show that M has tropical rank r. Keeping in mind Remark 3.6.6, we have to prove that the tropical rank is at most r. Let $f_0, \dots, f_{r+1} \in M$. For each i, we write $f_i = \mathrm{trop}(\mathbf{f}_i)$ with $\mathbf{f}_i \in \mathbf{H} \smallsetminus \{0\}$. Since $\dim(\mathbf{H}) = r + 1$, there exist some $\lambda_i \in \mathbf{K}$, $i \in [r + 1]$, such that $\sum_{i=0}^{r+1} \lambda_i \mathbf{f}_i = 0$. This shows that for all $x \in \Gamma$, the minimum in

$$\min_{0\leqslant i\leqslant r+1} (f_i(x) + \mathbf{val}(\lambda_i))$$

is achieved at least twice. Therefore the tropical rank of M is at most r.

Finally, we need to show that property $\binom{*}{**}$ in Definition 3.6.4 is verified. Let E be an effective Λ-rational divisor on Γ of degree $s \leqslant r$. Choose $\mathbf{E}$ to be any lift of E to the curve

$\mathbf{C}$. Let $\mathbf{H_E} \subset \mathbf{H}$ be a subspace of $\mathbf{H}$ of rank $r-s$ such that for each $\mathbf{f} \in \mathbf{H_E}$, we have $\mathrm{div}(\mathbf{f}) + \mathbf{D} - \mathbf{E} \geqslant 0$. This space exists since $\mathbf{E}$ imposes at most s linear constraints on $\mathbf{H}$. Let $\mathfrak{S}_E$ be the slope structure of width $r-s$ induced by the tropicalization of $\mathbf{H_E}$, which is a slope substructure of $\mathfrak{S}$. Let $M_E \coloneqq \mathrm{trop}(\mathbf{H_E})$. Then, (D, M_E) is a linear series of rank $r-s$, $M_E \subseteq M$, and the properties (1) and (2) of $\binom{*}{**}$ are verified.

At this point, the proof of Theorem 3.9.1 is complete. □

3.10 Examples and discussions

We provide a set of examples and some complementary results.

3.10.1 The dipole graph

We consider the dipole graph with four edges (of unit length to simplify the notation). The genus is $g = 3$ and the rank of the canonical divisor K is $r = 2$. Denote by u and v the two vertices and by e_1, e_2, e_3 and e_4 the four edges of Γ (see Figure 3.7).

A crude linear series of degree 4 and rank 2

For $i = 1, 2, 3, 4$, let $t_i \in \left(0, \frac{1}{6}\right]$. For each choice of the t_i's, we will construct a slope structure $\mathfrak{S}$ of width two.

Let the t_i's be fixed. For each i, we endow the edge e_i with the slope sets $0 < 1 < 2$ on the interval $\left[0, \frac{1}{2} - t_i\right]$ in the direction of the arrows, $-1 < 0 < 1$ on the interval $\left[\frac{1}{2} - t_i, \frac{1}{2} + t_i\right]$, and $-2 < -1 < 0$ on the interval $\left[\frac{1}{2} + t_i, 1\right]$.

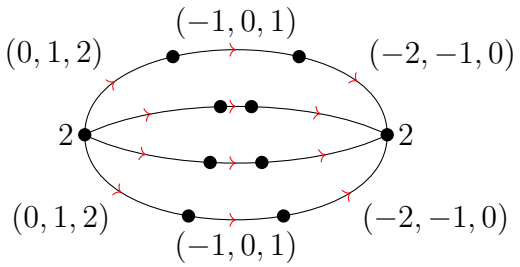


Figure 3.7: A slope structure of width two on the dipole graph.

We now define suitable rank functions. Endow the eight points $\frac{1}{2} - t_i, \frac{1}{2} + t_i$ on the edges e_i for $i = 1, 2, 3, 4$, respectively, with the rank function on $[2]^2$ defined by the array $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ (jumps in blue) and all the other points of Γ, including u and v, with the standard rank function. This defines a slope structure $\mathfrak{S}$ of width two on Γ, and $(K, \mathfrak{S})$ is a crude linear series of rank two on Γ.

An elementary $\mathfrak{g}^2_4$ on the dipole graph

We will present an example of a combinatorial limit linear series of degree 4 and rank 2 on the dipole graph with four edges of unit length, the same graph as in Section 3.10.1.

Like in the aforementioned section, we consider the canonical divisor K on Γ, which has degree 4 and rank 2. We keep the notations of Section 3.10.1. The linear series will be based on a degenerate version of the slope structure defined in that section, essentially corresponding to the limit $t_i = 0$ for every $i = 1, 2, 3, 4$.

More precisely, as shown in Figure 3.8, we endow, for each i, the edge e_i with the slope sets $0 < 1 < 2$ on the interval $\left[0, \frac{1}{2}\right]$, and $-2 < -1 < 0$ on the interval $\left[\frac{1}{2}, 1\right]$, in the direction of the arrows. We moreover endow every point of Γ, including u, v and the middle point m_i of each edge e_i, with the standard rank function. This defines a slope structure $\mathfrak{S}$ of width two on Γ.

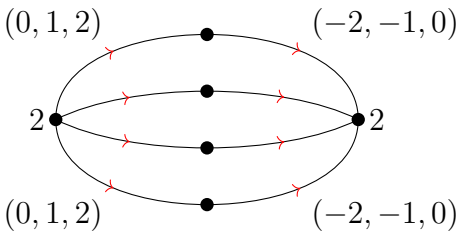


Figure 3.8: A slope structure of width two on the dipole graph.

Denote by $\mathrm{Rat}^{\mathrm{Sym}}(K)$ (resp., $\mathrm{Rat}^{\mathrm{Sym}}(K, \mathfrak{S})$) the set of functions $f \in \mathrm{Rat}(K)$ (resp., $f \in \mathrm{Rat}(K, \mathfrak{S})$) whose restriction to each e_i is symmetric with respect to the middle point m_i. It is not difficult to see that the following holds:

$$\mathrm{Rat}^{\mathrm{Sym}}(K, \mathfrak{S}) = \mathrm{Rat}^{\mathrm{Sym}}(K) \subsetneq \mathrm{Rat}(K, \mathfrak{S}) \subsetneq \mathrm{Rat}(K),$$

where both inclusions are moreover closed. The linear series will be defined as

$$M := \mathrm{Rat}^{\mathrm{Sym}}(K, \mathfrak{S}) = \mathrm{Rat}^{\mathrm{Sym}}(K).$$

Then, M is a sub-semimodule of $\mathrm{Rat}(K, \mathfrak{S})$ of rank two, which here also implies the same for $\mathrm{Rat}(K, \mathfrak{S})$ and $\mathrm{Rat}(K)$. The symmetries of the functions $f \in M$ then imply that M can be viewed as a *complete* linear series: it is isomorphic to the linear series $\mathrm{Rat}(2\,(u))$ on the tree T obtained as the left half of Γ, see Figure 3.9 below.

Now that we have essentially reduced to the case of a complete linear series, we use [HMY12, Theorem 6] and [HMY12, Corollary 9], which show that M is generated by its finitely many extremal points, and provide a characterization of the extremal points which enables to enumerate them all. Using this, M is seen to be generated by the functions defined as follows. For every $i = 1, 2, 3, 4$, let f_i be the function on Γ whose graph on the edge e_i is

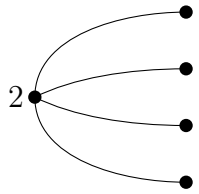
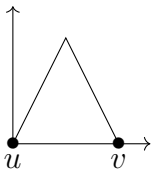
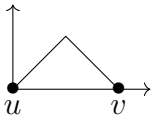


Figure 3.9: The tree T on which the complete linear series $\mathrm{Rat}(2\,(u))$ is defined.



with slopes 2 and -2, and which is identically zero on all other edges. For every choice of indices $1 \leqslant i < j \leqslant 4$, let $g_{i,j}$ be the function on Γ whose graph on the edges e_i and e_j is



with slopes 1 and -1, and which is identically zero on both other edges. Then M is generated by the functions f_i and $g_{i,j}$, which amounts to 10 generators.

Finally, a case-by-case analysis on the generators shows that M has tropical rank two, concluding the proof that M is a $\mathfrak{g}^2_4$.

A more involved $\mathfrak{g}^2_4$ on the dipole graph

Keeping the same dipole graph as in the preceding two sections, we now present another example of a combinatorial limit linear series of degree 4 and rank 2. Unlike in Section 3.10.1, we here specialize the slope structure defined in Section 3.10.1 to the choice of parameters $t_i = \frac{1}{6}$, $i = 1, 2, 3, 4$.

Unlike the linear series in Section 3.10.1, symmetrical functions will not be sufficient to get a $\mathfrak{g}^2_4$ compatible with $\mathfrak{S}$, but we will constrain the functions in another way. Let

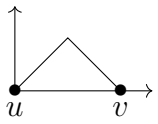
$$M \subsetneq \mathrm{Rat}(K, \mathfrak{S})$$

be the subset of functions $f \in \mathrm{Rat}(K, \mathfrak{S})$ which have slope 2 along *at most one tangent* vector based at u or v. Equivalently, a function $f \in \mathrm{Rat}(K, \mathfrak{S})$ belongs to M if, and only if, on every edge e_i, if f has slope 2 close to u on e_i, then its slope close to v on e_i is less than 2 (in fact, it consequently has to be 1).

It is easily verified that M is a sub-semimodule of $\mathrm{Rat}(K, \mathfrak{S})$. We will show that it is a linear series of degree 4 and rank 2. Firstly, M can be shown to have rank 2; the key observation here is that functions of $\mathrm{Rat}(K, \mathfrak{S})$ which take slope 2 on both endpoints of some edge are not needed.

To prove that M is finitely generated, we then divide the functions $f \in M$ into two categories, and find a finite number of generators for each category, following the kind of strategy deployed in Section 3.10.1.

Namely, we consider, on the one hand, the functions which are non-constant on exactly two distinct edges. Those functions cannot use the slope 2. Like in Section 3.10.1, it is easy to see that they are consequently generated by the functions $g_{i,j}$, for indices $1 \leqslant i < j \leqslant 4$, whose graph on the edges e_i and e_j is



with slopes 1 and -1, and which is identically zero on the other edges.

On the other hand, we look at the sub-semimodule $M' \subseteq M$ of functions which are non-constant on at most one edge. For example, denote by M'_1 the sub-semimodule of M' consisting of the functions which are constant on all edges e_i, $i = 2, 3, 4$. Since all functions in $\mathrm{Rat}(K)$ have the same value on u and v, and thanks to the slope constraint defining M, it follows that M'_1 is isomorphic to the *complete* linear series $\mathrm{Rat}(3\,(u)) = \mathrm{Rat}(3\,(u), \mathfrak{S})$ on the cycle obtained by identifying the endpoints of e_1 and deleting the other edges, see Figure 3.10.

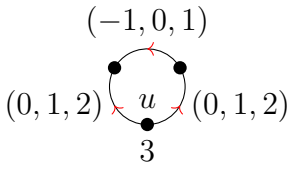
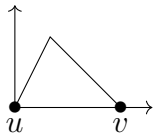


Figure 3.10: A cycle of length one with slope structure inherited from $\mathfrak{S}$.

As in Section 3.10.1, reducing to a complete linear series provides a set of generators for M'_1, obtained as the extremal points. It follows that M'_1 is generated by the function h_1 whose graph on the edge e_1 is



with slopes 2 on $\left[0, \frac{1}{3}\right]$ and -1 on $\left[\frac{1}{3}, 1\right]$, and which is identically zero on both other edges, together with the function h'_1 obtained from h_1 by symmetry with respect to the midpoint of each edge. Likewise, for $i = 2, 3, 4$, we define functions h_i and h'_i. Then the set of all these functions generates M'.

All in all, it follows that

$$M = \left\langle g_{i,j},\ h_k \;\middle|\; 1 \leqslant i < j \leqslant 4,\ 1 \leqslant k \leqslant 4 \right\rangle,$$

which amounts to 14 generators.

Finally, an elementary check on these generators shows that M has tropical rank two and therefore is a $\mathfrak{g}^2_4$.

3.10.2 Realizability and genera

In the context of tropicalization of linear series, the metric graph Γ can be enriched with a supplementary numerical data consisting of a genus function $\mathfrak{g}\colon V \to \mathbb{Z}_{\geqslant 0}$ associating to each vertex v the genus of the curve C_v; the pair $(\Gamma, \mathfrak{g})$ is called an *augmented metric graph*. The study of linear series on metric graphs is not dependent on the data of a genus function. However, considering genus functions becomes important in making connections to geometry when asking realizability questions. Indeed, for a slope structure to be realizable, that is, to be the tropicalization of a linear series on a curve defined over a non-Archimedean field, it is necessary that the divisor defined on Γ by

$$\mu_W^{\mathrm{clls}}(x) \coloneqq (r+1)\,K(x) + \frac{r(r+1)}{2}\,(d_x + \mathfrak{g}(x) - 2) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} s_j^\nu \tag{3.4}$$

be effective (here, d_x is the valence of x and $\mathfrak{g}(x)$ denotes the genus of the curve C_x corresponding to x). See Section [AGR23, Section 5], and more specifically Proposition 5.12 in *loc. cit.*. The realizability of a linear series on an augmented graph will depend on the genus function $\mathfrak{g}$. On the one hand, the general form of Equation (3.4) puts constraints on $\mathfrak{g}$ for the linear series to be realizable; and, on the other hand, the property of each rank function ρ_v being geometric will depend on the genus $\mathfrak{g}(v)$ of C_v.

To give an example, consider the metric graph Γ below with arbitrary edge lengths, and the divisor K with coefficient one at the trivalent vertices. Consider a family of slope structures on Γ, as follows.

First, for each bridge edge oriented outwards (towards the adjacent circle), allow slopes $-1 < 1 < 3$. Divide each circle into three equal parts, in a way compatible with the position of the attachment points. On each circle, on the two edges adjacent to the attachment points, allow slopes $0 < 1 < 2$ away from the attachment point, and on the remaining edge, allow slopes $-1 < 0 < 1$. We endow the vertices of Γ with the following rank functions. The central vertex gets the standard rank function. The three attachment points are endowed with the rank function on $[2]^3$ whose restrictions to $[2]^2 \times \{0\}$, $[2]^2 \times \{1\}$ and $[2]^2 \times \{2\}$ are given respectively by the three matrices

$$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \text{ and } \quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Jumps are depicted in blue and the third coordinate corresponds to the edge connecting the attachment point to the central vertex. Finally, the six other vertices of Γ are endowed with the rank function on $[2]^2$ defined by the first of the three arrays above. The standard

rank function is imposed on all other points of Γ. This defines a slope structure of width two $\mathfrak{S}$ on Γ, and $M \coloneqq \mathrm{Rat}(K, \mathfrak{S})$ is an admissible semimodule.

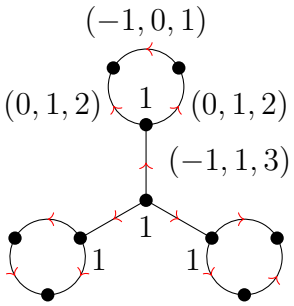


Figure 3.11: A slope structure of width two on the three-cycle graph.

Now, consider the example where, instead of allowing the slopes $-1 < 1 < 3$ on the three central edges, we allow the slopes $-1 < 1 < 3$ on (possibly trivial) intervals incident to the central vertex on these edges, and the slopes $-1 < 1 < 2$ on the rest of the edges, with the same choice of rank functions. Choosing the lengths of these intervals independently yields a three-parameter family of slope structures on Γ.

Among these possibilities, with genus function 0, only the one where all three intervals carrying the slopes $-1 < 1 < 3$ are trivial can be realizable.

3.10.3 Limit linear series defined by pluricanonical sheaves

In this section, we discuss the tropicalization of subspaces of global sections of pluricanonical sheaves, and explain how it fits into the theory presented in the previous sections.

Let $\mathbf{C}$ be a smooth proper curve over an algebraically closed complete non-trivially valued non-Archimedean field $\mathbf{K}$. We assume that the residue field κ of $\mathbf{K}$ has characteristic zero. We denote by $\omega_{\mathbf{C}}$ the canonical sheaf of $\mathbf{C}$, and by $\omega_{\mathbf{C}}^{\otimes n}$ its n-th power, for $n \in \mathbb{N}$.

Definition 3.10.1 (Pluricanonical linear series)**.** By a *pluricanonical linear series* of rank r and order n, we mean a vector subspace $\mathbf{H} \subseteq H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)$ of rank r, i.e., of dimension $r+1$. ◇

We follow the notation of the previous section and denote by Γ a skeleton of $\mathbf{C}^{\mathrm{an}}$ with combinatorial model $G = (V, E)$. The vertex set is included in the set of points of type 2 and, for each $v \in V$, the corresponding curve over the residue field κ is denoted by C_v.

Temkin metrization and tropicalization

Let $\|\cdot\|$ be the Kähler norm on the sheaf of differentials $\omega_{\mathbf{C}}$ defined by Temkin in [Tem16]. It induces a norm on each $\omega_{\mathbf{C}}^{\otimes n}$, $n \in \mathbb{N}$, that we continue to denote by $\|\cdot\|$.

Definition 3.10.2 (Tropicalization of pluricanonical forms)**.** For each non-zero section α of $\omega_{\mathbf{C}}^{\otimes n}$, the tropicalization of α is the function

$$\mathrm{trop}(\alpha)\colon \Gamma \longrightarrow \mathbb{R}, \qquad x \longmapsto -\log \|\alpha\|_x. \qquad \diamond$$

Let $K = K_\Gamma$ be the canonical divisor of the augmented metric graph Γ, given explicitly by

$$K = \sum_{x\in\Gamma} (2\mathfrak{g}(x) - 2 + d_x)\,(x).$$

The tropicalization $F = \mathrm{trop}(\alpha)$ of a pluricanonical differential form of order n is a piecewise affine function with integral slopes. It verifies the slope formula

$$\mathrm{div}(F) + n\,K = \tau_*(\mathrm{div}(\alpha))$$

with $\mathrm{div}(\alpha)$ the divisor of zeroes of α, defined by $\mathrm{div}(\alpha) := \sum_{x\in\mathbf{C}} \mathrm{ord}_x(\alpha)$, $\tau\colon \mathbf{C}^{\mathrm{an}} \longrightarrow \Gamma$ the retraction map to Γ, and τ_* the induced map on the level of divisors. (The first claim is a consequence of [Tem16, Theorem 8.2.4]. For the second, see [BT20], as well as [Ami14, KRZ16, BN16]. The scaled reduction $\widetilde{\alpha}$ of α at a point $x \in \mathbf{C}^{\mathrm{an}}$ of type 2 is a pluricanonical meromorphic form on C_x, which is well-defined up to scaling by a scalar in $\kappa^\times$. And, for every tangent direction ν at x, the order of vanishing of $\widetilde{\alpha}$ at the point p_x^ν of C_x is given by $-n - \mathrm{slope}_\nu F(x)$.

Let $\mathbf{H} \subseteq H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)$ be a pluricanonical linear series of rank r and order n. Let $M := \mathrm{trop}(\mathbf{H}) = \big\{\mathrm{trop}(\alpha) \;\big|\; \alpha \in \mathbf{H} \smallsetminus \{0\}\big\}$, and denote by $\widetilde{H}_x$ the vector space of pluricanonical meromorphic forms of the same order on C_x generated by the reductions $\widetilde{\alpha}$ of $\alpha \in \mathbf{H}$ at x. As in the case of the reduction of functions, the dimension of $\widetilde{H}_x \subset \kappa(C_x)$ is $r+1$, and the orders of vanishing of reductions $\widetilde{\alpha} \in \widetilde{H}_x$ of elements $\alpha \in \mathbf{H}$ at p_x^ν define a sequence of integers $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$. We define $S^\nu := \{s_i^\nu\}$, and denote by ρ_x the geometric rank function associated to the collection of points $p_x^\nu \in C_x(\kappa)$ for $\nu \in \mathrm{T}_x(\Gamma)$. Moreover, we define S^x as the set of jumps of ρ_x. We have the following theorem.

Theorem 3.10.3 (Specialization of pluricanonical linear series)**.** *Let $\mathbf{H} \subseteq H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)$ be a pluricanonical linear series of rank r and order n. Let $M := \mathrm{trop}(\mathbf{H}) = \big\{\mathrm{trop}(\alpha) \;\big|\; \alpha \in \mathbf{H} \smallsetminus \{0\}\big\}$. Then, $M \subseteq \mathrm{Rat}(nK, \mathfrak{S})$. Moreover, (nK, M) is a refined $\mathfrak{g}^r_d$ on Γ, for $d = n(2g-2)$, with g the genus of $\mathbf{C}$.*

Proof. The proof is similar to that of Theorem 3.9.1 given in the previous section. □

Definition 3.10.4 (Pluricanonical limit linear series)**.** Notation as above, the semimodule

$$M_n^{\mathrm{can}} := \mathrm{trop}\big(H^0\big(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n}\big)\big) \subseteq \mathrm{Rat}(nK, \mathfrak{S})$$

is called the *tropical semimodule of pluricanonical differential forms* on Γ induced by $\mathbf{C}$. The slope structure $\mathfrak{S}$ is called the *pluricanonical slope structure of order n on* Γ induced by $\mathbf{C}$. For every pluricanonical linear series $\mathbf{H} \subseteq H^0(\mathbf{C}, \omega_{\mathbf{C}}^{\otimes n})$ of rank r and order n, the sub-semimodule M of M_n^{can} defined by the tropicalization of $\mathbf{H}$ gives rise to the pair (nK, M), called the *limit pluricanonical linear series* $\mathfrak{g}^r_{n(2g-2)}$ of rank r and order n on Γ induced by $\mathbf{H}$. $\diamond$

Combinatorial pluricanonical types

For each pluricanonical slope structure on Γ of order n, its combinatorial type is the pair $(G, \mathfrak{S})$, where G is the minimal graph model of Γ over which $\mathfrak{S}$ is defined.

Theorem 3.10.5 (Finiteness of pluricanonical slope structures)**.** *There are only finitely many combinatorial types for pluricanonical slope structures of order n on augmented metric graphs Γ of the same combinatorial type.*

Proof. This follows directly from Theorem 3.4.4. □

Let $G = (V, E)$ be a given graph that we assume to be augmented with a genus function. A slope structure $\mathfrak{S}$ on G is called *pluricanonical* if there exists a length function $\ell\colon E \to \mathbb{R}$ on the edges of G such that $\mathfrak{S}$ defines a pluricanonical slope structure on Γ.

Problem 3.10.6. *Provide a classification of all pluricanonical slope structures on a given augmented graph G.*

4 Tropical Weierstrass points and Weierstrass weights

This chapter is adapted from the preprint [AGR23].

Contents of the chapter

Abstract

In this chapter, we study tropical Weierstrass points. These are the analogues for tropical curves of ramification points of line bundles on algebraic curves.

For a divisor on a tropical curve, we associate intrinsic weights to the connected components of the locus of tropical Weierstrass points. This is obtained by analyzing the slopes of rational functions in the complete linear series of the divisor. We prove that for a divisor D of degree d and rank r on a genus g tropical curve, the sum of weights is equal to $d-r+rg$. We establish analogous statements for tropical linear series.

In the case D comes from the tropicalization of a divisor, these weights control the number of Weierstrass points which are tropicalized to each component. Our results provide answers to open questions originating from the work of Baker on specialization of divisors from curves to graphs.

We conclude with multiple examples which illustrate interesting features appearing in the study of tropical Weierstrass points, and raise several open questions.

4.1 Overview

Weierstrass points have a rich history in the development of algebraic geometry as they provide an important tool for the study of smooth algebraic curves and their moduli spaces. It is natural to ask how their theory can be extended to stable curves, which correspond to boundary points in the Deligne–Mumford compactification $\overline{\mathscr{M}}_g$ of the moduli space of genus g smooth curves. One strategy is to take the *limit Weierstrass points* induced by a one-parameter family $(X_t)_{t\neq 0}$ of smooth curves degenerating to a stable curve X_0; there will be $g^3 - g$ limit Weierstrass points on X_0 when counted with appropriate weights. However, the limit points generally depend on the chosen family, and a stable curve X_0 has many possible smoothings corresponding to paths in $\overline{\mathscr{M}}_g$ that end at the point representing X_0.

Tropical geometry provides a new perspective on degeneration methods in algebraic geometry by enriching it with polyhedral geometry. Given the successes of tropical methods in the past two decades in the study of algebraic curves and their moduli spaces, it is natural to ask whether tropical geometry can be used to gain insight about the limiting behavior of Weierstrass points on degenerating families of curves. In the tropical perspective, the data of a stable curve X_0 is replaced by the data of its dual graph. The collection of all stable curves having the same dual graph forms a stratum of $\overline{\mathscr{M}}_g$. This gives a correspondence between the strata of $\overline{\mathscr{M}}_g$ and the set of stable graphs of genus g [Cap15].

The prototype of what we can expect to address using tropical techniques is the following natural question.

Question 4.1.1. *Given a stratum of $\overline{\mathscr{M}}_g$, and a log-tangent direction of approaching that stratum, what can be said about the limit Weierstrass points of a smooth family $(X_t)_{t\neq 0}$ degenerating to a stable curve in that stratum along the chosen direction?*

The arithmetic geometric version of the above question is the following.

Question 4.1.2. *Given a smooth proper curve over the field $\mathbb{Q}_p$ of p-adic numbers with stable reduction lying in a given stratum of $\overline{\mathscr{M}}_g$ (over the algebraic closure of the residue field $\mathbb{F}_p$), what can be said about the specialization of the Weierstrass points?*

Previously, there has been much work making incremental progress on the first question [EH87a, EM02, ES07, Dia85, Ami14, Gen21] and on the second question [Ogg78, LN64, Atk67, AP03, Bak08]; see Section 4.1.4 for a more thorough discussion.

Our aim in this chapter is to provide an answer to the above questions from the point of view of tropical geometry. This is done by introducing new tools which allow us to solve problems related to the tropical geometry of curves, whose origin goes back to the beginning of the use of tropical methods in the study of algebraic curves.

Our answer to Question 4.1.1 can be summarized as follows: we can specify how many Weierstrass points degenerate to each component and to each node of a stable curve X_0 lying in the given stratum. This is done without specifying their precise position within each irreducible component, giving instead a more precise location of those degenerating to

a node by specifying their position on the dual metric graph of the family (X_t). Our result also applies to limits of ramification points of arbitrary line bundles, in addition to the case of the canonical bundle.

Similarly, we answer Question 4.1.2 by specifying where Weierstrass points specialize when reducing modulo p.

Moreover, these results lead to an effective way of locating the limit Weierstrass points.

We next give an overview of our results.

4.1.1 Tropical perspective

The central concept studied in this chapter is that of tropical Weierstrass points. The definition is based on divisor theory on metric graphs, and we refer to the survey paper [BJ16] and the references there for more details.

Let Γ be a metric graph, and let D be a divisor of degree d and rank r on Γ.

Definition 4.1.3 (Weierstrass points)**.** A point x in Γ is called a *Weierstrass point*, or *ramification point*, for D if there exists an effective divisor E in the linear system of D whose coefficient at x is at least $r+1$. The *(tropical) Weierstrass locus* of D, denoted by $L_W(D)$, is the set of all such points in Γ. ◇

The set $L_W(D)$ is a closed subset of Γ which can be infinite, in contrast with the classical setting of algebraic curves. In this regard, Baker comments in [Bak08, Remark 4.14], regarding the canonical divisor, that "it is not clear if there is an analogue for metric graphs of the classical fact that the total weight of all Weierstrass points on a smooth curve of genus g is $g^3 - g$." More generally, we can ask the following question.

Question 4.1.4. *Is it possible to associate intrinsic tropical weights to the connected components of $L_W(D)$? What is the total sum of weights associated to these components?*

The following question is a special case.

Question 4.1.5. *Assume the locus of Weierstrass points of D is finite. What is the total weight of these points?*

Our aim in this chapter is to provide answers to the above questions. In order to streamline the presentation which follows, we first discuss our results in the case of non-augmented metric graphs. From the geometric perspective, this corresponds to the situation of a *totally degenerate* stable curve, that is, a stable curve whose irreducible components are all projective lines. This is the same as requiring that the arithmetic genus of the stable curve is equal to the genus of the dual graph. We have an analogue of these statements for augmented metric graphs (respectively, arbitrary stable curves), see the discussion which follows below.

In order to solve Question 4.1.4, we make the following definition.

Definition 4.1.6 (Intrinsic Weierstrass weight of a connected component)**.** Let D be a divisor of rank r, and let C be a connected component of the Weierstrass locus $L_W(D)$. We define the *tropical Weierstrass weight* of C as

$$\mu_W(C; D) := \deg\left(D|_C\right) + (g(C) - 1)r - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(D) \tag{4.1}$$

where

- $\deg\left(D|_C\right)$ is the total degree of D in C, defined by $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$;
- $g(C)$ is the genus of C, i.e., its first Betti number $\dim H_1(C, \mathbb{R})$;
- $\partial^{\text{out}} C$ is the set of outgoing unit tangent directions from C; and
- $s_0^\nu(D)$ is the minimum slope along tangent direction ν of any rational function f on Γ with $\operatorname{div}(f) + D \geqslant 0$.

We abbreviate $\mu_W(C; D)$ simply as $\mu_W(C)$ if D is understood from the context. ◇

Although it is not obvious from the definition, we will show in Theorem 4.3.6 that the tropical Weierstrass weight of any component is positive. Note as well that a connected component of $L_W(D)$ is always a metric subgraph of Γ, see Proposition 4.3.1.

We say that D is *Weierstrass finite* or simply *W-finite* if the tropical Weierstrass locus $L_W(D)$ has finite cardinality. In this case, connected components of $L_W(D)$ are isolated points in Γ, and we define the *tropical Weierstrass divisor* $W(D)$ as the effective divisor

$$W(D) := \sum_{x \in L_W(D)} \mu_W(x)\,(x)$$

where $\mu_W(x) := \mu_W(\{x\})$. The support $|W(D)|$ of the tropical Weierstrass divisor is exactly the tropical Weierstrass locus $L_W(D)$. The tropical Weierstrass weight of x can be identified as $\mu_W(x) = D_x(x) - r$, with D_x denoting the unique x-reduced divisor in the linear system of D, see Remark 4.3.3.

This gives the following geometric meaning to the Weierstrass weights, in the spirit of the classical definition on algebraic curves. The coefficient of the reduced divisor at a point $x \in \Gamma$ corresponds precisely to the maximum order of vanishing at x of any global section of the line bundle $\mathcal{O}(D)$ defined by the divisor. The Weierstrass weight of the point x is thus obtained by comparing this quantity to r, which would be the expected minimum value, over points $y \in \Gamma$, of the largest order of vanishing of global sections at y. (Note, however, that r is not always equal to the actual minimum largest order of vanishing, as examples in Section 4.6.5 show.) That being said, the definition differs from the algebraic setting, where we need to take into account *all* the orders of vanishing of global sections of the line bundle at a given point (and then compare them with the standard sequence, the one obtained for a point in general position on the curve).

The following theorem answers Questions 4.1.4 and 4.1.5, and is proved in Section 4.3.3.

Theorem 4.1.7 (Total weight of the Weierstrass locus)**.** *Let Γ be a metric graph of genus g, and let D be an effective divisor of degree d and rank r on Γ. Then, the total sum of weights of the connected components of $L_W(D)$ is equal to $d - r + rg$. In particular, if D is W-finite, then we have* $\deg(W(D)) = d - r + rg$.

The proof of this theorem will imply in particular the following result, proved in Section 4.3.4.

Theorem 4.1.8. *If the rank r of D is at least one, then every cycle in Γ intersects the tropical Weierstrass locus $L_W(D)$. In particular, if Γ has genus at least two, then every cycle intersects the Weierstrass locus of the canonical divisor K.*

In [Bak08], Baker proves that the tropical Weierstrass locus of the canonical divisor is nonempty if Γ has genus at least two. This earlier tropical result is obtained as a consequence of the analogous algebraic statement, using the specialization lemma. In contrast, our theorem above states that tropical Weierstrass points obey a stronger "local" existence condition, which has seemingly no algebraic analogue. In the case that the canonical divisor of Γ is W-finite, our result implies that for an arbitrary family $(X_t)_{t\neq 0}$ of smooth curves tropicalizing to Γ, every cycle in Γ contains a limit Weierstrass point of the family.

To prove Theorem 4.1.7, we will show that in fact (4.1) defines a consistent notion of Weierstrass weight when applied to any connected, closed subset of Γ whose boundary points are not in the interior of $L_W(D)$; see Theorem 4.3.9. To do so, we retrieve information about the slopes of rational functions in the linear series $\mathrm{Rat}(D)$ along tangent directions in Γ. We have the following description, proved in Section 4.2.

Theorem 4.1.9. *Let D be a divisor of rank r on Γ. We take a model for Γ whose vertex set contains the support of D. Let $x \in \Gamma$ be a point and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction.*

(a) If the open interval $(x, x+\varepsilon\nu)$ is disjoint from $L_W(D)$ for some $\varepsilon > 0$, then the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ consists of $r+1$ consecutive integers $\{s_0^\nu, s_0^\nu + 1, \ldots, s_0^\nu + r\}$.

(b) If the open interval $(x, x+\varepsilon\nu)$ is contained in $L_W(D)$, then the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ is a set of consecutive integers of size at least $r+2$.

4.1.2 Comparison results and extensions

We further justify our definition of weights by making a precise link to tropicalizations of Weierstrass points on algebraic curves.

Suppose that Γ and D come from geometry; that is, let X be a smooth proper curve of genus g over an algebraically closed non-Archimedean field $\mathbf{K}$ of characteristic zero with a non-trivial valuation and a residue field of arbitrary characteristic. Let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be a line bundle of degree d on X. Assume that Γ is a skeleton of the Berkovich analytification X^{an} of X. Denote by τ the tropicalization map from X to Γ, and suppose that $D = \tau_*(\mathcal{D})$ is the tropicalization of $\mathcal{D}$ on Γ where $\tau_* \colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$ the induced map on divisors.

Denote by $\mathcal{W}(\mathcal{D})$ the Weierstrass divisor of $\mathcal{D}$ on X, and by $\tau_*(\mathcal{W}(\mathcal{D}))$ its tropicalization on Γ. The following result, proved in Section 4.5.3, uses the notion of $L_W(D)$-measurable set, for which the connected components of $L_W(D)$ form the atoms, and the natural counting measure $\hat{\mu}_W$ on such sets, induced by Weierstrass weights (see Section 4.3.3 for more details).

Theorem 4.1.10 (Algebraic versus tropical Weierstrass weights)**.** *Assume that D and $\mathcal{D}$ have the same rank, and let $A \subseteq \Gamma$ be a closed, connected subset which is $L_W(D)$-measurable. Then, the total weight of Weierstrass points of $\mathcal{W}(\mathcal{D})$ tropicalizing to points in A is precisely $(r+1)\,\hat{\mu}_W(A; D)$; that is,*

$$\deg\left(\mathcal{W}(\mathcal{D})|_{\tau^{-1}(A)}\right) = (r+1)\left(\deg\left(D|_A\right) + r(g(A)-1) - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(D)\right).$$

In particular, if D is W-finite, then we have the equality

$$\tau_*(\mathcal{W}(\mathcal{D})) = (r+1)\, W(\tau_*(\mathcal{D})).$$

This statement, which involves the metric of Γ in a crucial way, gives an essentially complete description of the behavior of Weierstrass points in the tropical limit. In particular, if the limit divisor is W-finite, then for every family $(X_t)_{t\neq 0}$ of smooth proper curves approaching a stable curve with dual metric graph Γ, the limit Weierstrass points are precisely described by the tropical Weierstrass divisor. This rigidity type theorem on the limiting behavior of Weierstrass points allows us to give a precise count of the number of Weierstrass points going to the nodes or to the smooth parts of a limit stable curve X_0 on the given stratum of $\overline{\mathscr{M}}_g$ along the given log-tangent direction from which the family $(X_t)_{t\neq 0}$ approaches X_0. Moreover, as a special case, the theorem also applies in the context of arithmetic geometry in which the curve X is defined over a finite extension of $\mathbb{Q}_p$. As we will show in Section 4.5.3, this theorem holds as well over a field $\mathbf{K}$ of positive characteristic provided that the gap sequence of $\mathcal{L}$, defined as the sequence of orders of vanishing of the global sections of $\mathcal{L}$ at a general point of X, is the standard sequence $0, 1, \ldots, r$. (In this case, $\mathcal{L}$ is called classical [Lak81, Nee84].)

We provide natural extensions and refinements of the above results to the setting of augmented metric graphs, which, from the degeneration perspective, corresponds to the situation where the limit stable curve has irreducible components of possibly positive genus. Since a given vertex of positive genus hides information about the geometry of the corresponding component, it turns out that there will be an ambiguity when talking about the Weierstrass locus of a divisor D. In fact, the right setup in this context is a divisor D endowed with the data of a closed sub-semimodule M of $\mathrm{Rat}(D)$, which plays the role of a (not necessarily complete) linear series on the augmented metric graph.

In this regard, first, we use the weights defined in Definition 4.1.6 with a relevant notion of divisorial rank associated to the sub-semimodule which we further modify by including the data of the genus function. We get Theorem 4.4.12, which provides a global count of weights in this setting.

To the question of *whether it is still possible to associate a natural Weierstrass locus to a divisor in the augmented setting*, we provide an answer by introducing two special classes of semimodules, the *generic semimodule* associated to any divisor (see Section 4.4.2), and the *canonical semimodule* associated to the canonical divisor on an augmented metric graph (see Section 4.4.3). Both of them require some level of genericity, which we properly justify in Section 4.4.4 using the framework of metrized complexes.

The case of the canonical divisor on an augmented metric graph is particularly interesting as it reveals new facets of divisor theory in the augmented setting. We associate a canonical linear series to any augmented metric graph, show that it has the appropriate rank, and study its Weierstrass locus. To justify the definition and prove these results, we use the setting of metrized complexes and their divisor theory from [AB15]. Using that framework, we show that the canonical linear series on an augmented metric graph is the tropical part of the canonical linear series on any metrized complex with that underlying augmented metric graph, provided that the markings associated to edges on the curves of the metrized complex are in general position. It is interesting to note that this is the assumption made in the works by Esteves and coauthors [EM02, ES07], and our results here complement these works by developing the tropical part of the story in greater generality.

As we show in Theorem 4.5.5, the statement of Theorem 4.1.10 remains valid in these settings (when including the genera of points of A on the right-hand side of the stated equality). The following theorem is a direct application of our results on Weierstrass weights for an augmented metric graph. We use the setting of tropicalization preceding Theorem 4.1.10.

Theorem 4.1.11. *Suppose $\mathcal{D}$ is a divisor on an algebraic curve X over an algebraically closed non-Archimedean field $\mathbf{K}$ of characteristic zero with a non-trivial valuation and a residue field of arbitrary characteristic. Let $(\Gamma, \mathfrak{g})$ be an (augmented) skeleton of X^{an}. Let H be a vector space of global sections of $\mathcal{O}(\mathcal{D})$ of rank r and denote by $\mathcal{W}(\mathrm{H})$ the Weierstrass divisor of H. Let M be the tropicalization of H. Then, for any connected, closed subset $A \subseteq \Gamma$ which is $L_w(M, \mathfrak{g})$-measurable, we have the bound*

$$\deg\left(\mathcal{W}(\mathrm{H})|_{\tau^{-1}(A)}\right) \geqslant \left(r^2 + r\right)\left(g(A) + \sum_{x \in A} \mathfrak{g}(x)\right).$$

The proof of this theorem will be given in Section 4.5.3. As in the case of Theorem 4.1.10, the statement holds as well over a field $\mathbf{K}$ of positive characteristic provided the gap sequence of H is the standard sequence.

In the case $L_w(M, \mathfrak{g})$ is finite, this inequality holds for any closed subset $A \subseteq \Gamma$. In particular, we have the following application to stable curves: suppose X_0 is a stable curve with dual augmented graph $(G, \mathfrak{g})$, and suppose (X_t) is a family degenerating to X_0 with tropicalization $(\Gamma, \mathfrak{g})$. If the locus of canonical Weierstrass points of $(\Gamma, \mathfrak{g})$ is finite, then for every connected subgraph A of G, the number of limit Weierstrass points lying on components and nodes of X_0 which correspond to vertices and edges of A, respectively, is at least $(g^2 - g)(g(A) + \sum_{v \in A} \mathfrak{g}(v))$.

Semimodules inside $\mathrm{Rat}(D)$ that come from the tropicalization of linear series verify an extra set of properties. These are thoroughly studied in recent works [AG22] (Chapter 3) and [JP22] that develop a combinatorial theory of (limit) linear series. In particular, such a semimodule M of rank r satisfies the following:

($\star$) For each point x in Γ and any unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the set of slopes taken by functions in M has size $r+1$.

(We refer to Section 4.8.3 for more details.)

In Section 4.5, we associate a refined notion of Weierstrass divisor to any divisor D and any closed sub-semimodule $M \subseteq \mathrm{Rat}(D)$ that verifies the above property. The definition takes into account the higher orders of vanishing of the combinatorial limit linear series, and is closer to the spirit of the algebraic definition of Weierstrass weights on curves.

Using this together with the results proved in Section 4.8, discussed below, we provide a proof of Theorem 4.1.10 and its extensions to the augmented and incomplete settings. Finally, using combinatorial Weierstrass divisors, we formulate obstructions to the realizability of combinatorial limit linear series.

4.1.3 Tropicalization of Weierstrass divisors

The proof of our comparison results, Theorem 4.1.10 and its extension Theorem 4.5.5, makes use of the results proved by Amini in Section 4.8. An earlier version of these results was written around 2014.

Let X be a smooth proper curve defined over $\mathbf{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle. Let $\mathrm{H} \subseteq H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, \mathrm{H})$ the corresponding Weierstrass divisor. We assume that the gap sequence of H is the sequence $0, 1, \dots, r$, that is, for a general point $x \in X(\mathbf{K})$, the orders of vanishing of sections of $\mathcal{L}$ in H are $0, 1, \dots, r$. Let τ be the tropicalization map from X to Γ. We describe the tropicalization $W = \tau_*(\mathcal{W})$. The divisor $\mathcal{W}$ is equal to $(r+1)\mathcal{D} + \mathrm{div}(\mathrm{Wr}_{\mathcal{F}})$ for a section $\mathrm{Wr}_{\mathcal{F}}$ of the sheaf $\omega_X^{\otimes r(r+1)/2}$ called the Wronskian. The sheaf $\omega_X^{\otimes r(r+1)/2}$ admits a natural norm; using this norm, we can tropicalize the section $\mathrm{Wr}_{\mathcal{F}}$, and define a rational function $F = \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}}) \colon \Gamma \to \mathbb{R}$. Denote by K the canonical divisor of $(\Gamma, \mathfrak{g})$. Using the slope formula for sections of powers of the canonical sheaf, Lemma 4.8.1, it is shown in Theorem 4.8.2 that for any $x \in \Gamma$, we have

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu F.$$

It follows from the results proved in Section 4.8.7, that for a point $x \in \Gamma$ and $\nu \in \mathrm{T}_x(\Gamma)$, if

- either, the residue field κ is of characteristic zero,
- or, the sequence $s_0^\nu, \dots, s_r^\nu$ forms an interval, that is, $s_j^\nu = s_0^\nu + j$,

then the slope $\operatorname{slope}_\nu F$ is given by the sum $s_0^\nu + \cdots + s_r^\nu$, see Proposition 4.8.4 and Theorem 4.8.2. This result is needed to prove in Section 4.5 our comparison results between tropical and algebraic Weierstrass loci.

Note that over a field $\mathbf{K}$ of equicharacteristic zero, the first item in the above condition is verified, and we get all the coefficients $W(x)$,

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{i=0}^{r} s_i^\nu,$$

see Theorem 4.8.5.

4.1.4 Previous work

The study of Weierstrass points from a tropical perspective was initiated by Baker [Bak08, Section 4]. Baker defines Weierstrass points for graphs and metric graphs, and uses his Specialization Lemma [Bak08, Lemma 2.8] to prove an essential compatibility with Weierstrass points on stable curves—namely, that the tropicalization of the algebraic Weierstrass locus is a subset of the tropical Weierstrass locus. To be more precise, for a divisor $\mathcal{D}$ on a non-Archimedean curve with Weierstrass divisor $\mathcal{W}(\mathcal{D})$, if $\tau_*(\mathcal{D})$ has the same rank as $\mathcal{D}$ and $\mathcal{D}$ has classical gap sequence, then we have an inclusion

$$|\tau_*(\mathcal{W}(\mathcal{D}))| \subseteq L_W(\tau_*(\mathcal{D})),$$

which may be strict in general. (This is stated for the canonical divisor in *loc. cit.*, but the proof works in greater generality.) This statement has strong implications for the behavior of Weierstrass points on a family of degenerating Riemann surfaces, and for p-adic reduction of curves over $\mathbb{Q}_p$, discussed earlier in the introduction. Indeed, Baker motivates his study of Weierstrass points on graphs with several results from the arithmetic geometry of modular curves, in particular, as a way to decide whether certain cusps are Weierstrass points, c.f. [Ogg78, LN64, Atk67, AP03].

The question of how to determine the tropicalization of Weierstrass points on a non-Archimedean curve was settled in [Ami14]; these results appear in Section 4.8 and are used to prove our comparison results. The question of determining tropical Weierstrass loci and their weights, and the way to properly count them in the tropical setting remained however open. The work [Ric24] by Richman studies Weierstrass points on tropical curves. Although the tropical Weierstrass locus may be infinite in general, [Ric24] shows that for a generic divisor class (i.e., lying in a nonempty open subset of Pic^d), this locus is finite, and moreover computes its cardinality. It is worth mentioning that important divisor classes such as the canonical divisor are non-generic, so they are not covered by the methods of [Ric24]. The way tropical Weierstrass points distribute when the degree of divisor classes tend to infinity is studied in [Ami14, Ric24]. For an extended discussion of how divisor theory on graphs is connected to the degeneration of smooth curves to nodal curves, with various applications, see the survey by Baker–Jensen [BJ16], in particular Section 12.

For an extensive and informative survey describing the history and applications of Weierstrass points, starting with Weierstrass and Hürwitz [Wei67, Hur92] in the 1800s, see Del Centina [DC08]. The study of Weierstrass points on stable curves was initiated by Eisenbud and Harris [EH87a], who proved results on nodal curves of *compact type*, i.e., curves whose dual graph is a tree. This work served as an application of their newly-developed theory of limit linear series [EH86]. They moreover raised the question of constructing a moduli space parametrizing all possible limit Weierstrass divisors of a given stable curve, a problem which has been widely open since then.

Moving beyond stable curves of compact type, Lax [Lax87] studied Weierstrass points on stable curves consisting of one rational component with nodes; in this case, the dual graph is a single vertex with self-loops. (The term *tree-like* is used in the literature to describe curves whose dual graph consists of a tree after removing self-loops.) A further breakthrough came with Esteves–Medeiros [EM02] who worked with stable curves with two components, i.e., curves whose dual graph is a dipole graph. (We refer to Section 4.6.8 for a discussion of our results applied to dipole graphs and the connection to [EM02].) Esteves–Salyehan [ES07] studied further cases of nodal curves, including when the dual graph is a complete graph. Cumino–Esteves–Gatto [CEG08] studied limits of *special* Weierstrass points on certain stable curves, i.e., Weierstrass points with weight at least two. The problem of describing limits of Weierstrass points away from the nodes in a given one-parameter family in characteristic zero is addressed in [Est98].

Although not directly related to the results of this manuscript, we mention that other works treat the case of irreducible Gorenstein curves, and associate Weierstrass weights to their singular points, see e.g. [LW90, dCS94, GL95, BG95] and the references there. It might be possible to use tropical geometry to describe these weights.

Weierstrass points have appeared in other interesting work on moduli spaces of curves. Arbarello [Arb74] studied subvarieties of the moduli space of curves cut out by Weierstrass points; further results were found in Lax [Lax75] and Diaz [Dia85]. Eisenbud–Harris [EH87b] showed that the moduli space of curves has positive Kodaira dimension, using loci of Weierstrass points as part of their argument. Cukierman(–Fong) [Cuk89, CF91] found the coefficients for the Weierstrass locus in the universal curve $\mathscr{C}_g$ of genus g, in a standard basis for the Picard group of $\mathscr{C}_g$. We discuss the behavior of tropical Weierstrass loci over the moduli space of tropical curves in Section 4.7.1.

4.1.5 Organization of the text

The chapter is organized as follows. We first treat the case of non-augmented metric graphs, and then provide refinements. This choice has the advantage of making the presentation less technical, and we hope this will add to readability.

We define slope sets and prove Theorem 4.1.9 in Section 4.2.

In Section 4.3, we study Weierstrass weights and the Weierstrass measure they define on a metric graph. We state and prove Theorem 4.3.9, which provides a description of

the Weierstrass measure using the slopes, from which we deduce Theorem 4.1.7 and other interesting consequences. This section contains the proof of positivity of Weierstrass weights as well, and a discussion of the case of combinatorial graphs. The case of the canonical Weierstrass locus on non-augmented metric graphs is treated in Section 4.3.7.

Section 4.4 provides several refinements and generalizations of the previous sections. The setting is extended in two ways. First, complete linear series $\mathrm{Rat}(D)$ are replaced with incomplete linear series, by taking closed sub-semimodules of $\mathrm{Rat}(D)$. Second, metric graphs are replaced with augmented metric graphs. We provide justification for our definitions in the augmented setting and provide the corresponding generalizations of Theorem 4.3.9 on the Weierstrass measure and of Theorem 4.1.7.

In Section 4.5, we explain how to associate Weierstrass divisors to combinatorial limit linear series. This is particularly interesting in the case where the locus of Weierstrass points associated to the underlying divisor becomes infinite after forgetting the slopes. We show the compatibility of the definitions appearing in this section with the previous ones.

Using the above materials, we establish a precise link between the tropical Weierstrass divisors with tropicalizations of Weierstrass divisors on smooth curves. This includes the proof of Theorem 4.1.10, and its generalizations.

In the last two Sections 4.6 and 4.7, we provide several examples with the aim of clarifying the concepts introduced in previous sections, and discuss other interesting results related to them. We also raise several open questions.

Section 4.8 proves the results we need on the tropicalization of Weierstrass divisors.

4.1.6 Basic notations

A *(combinatorial) graph* $G = (V, E)$ is defined by a set of vertices V and a set of edges E between certain vertices. In the current chapter, graphs will always be taken to be finite and connected. Moreover, they will allow loops and multiple edges.

A *metric graph* is a compact, connected metric space Γ verifying the following properties:

(i) For every point $x \in \Gamma$, there exist a positive integer n_x and a real number $r_x > 0$ such that the r_x-neighborhood of x is isometric to the star of radius r_x with n_x branches.

(ii) The metric on Γ is given by the path metric, i.e., for points x and y in Γ, the distance between x and y is the infimum (in fact minimum) length of any path from x to y.

The integer n_x above is called the *valence* of x and is denoted by $\mathrm{val}(x)$.

Given a graph $G = (V, E)$ and a length function $\ell\colon E \to (0, +\infty)$ assigning to every edge of G a positive length, we can build from this data a metric graph Γ by gluing a closed interval of length $\ell(e)$ between the two endpoints of the edge e, for every $e \in E$, and endowing Γ with the path metric. The space Γ is then called the *geometric realization* of the pair (G, ℓ).

A *model* of a metric graph Γ is a pair (G, ℓ) consisting of a graph $G = (V, E)$ and a length function $\ell\colon E \to (0, +\infty)$ such that Γ is isometric to the geometric realization of (G, ℓ). By an abuse of notation, we also call G a model of Γ.

For a metric graph Γ and a point $x \in \Gamma$, the tangent space $\mathrm{T}_x(\Gamma)$ is defined as the set of all unit outgoing tangent vectors to Γ at x. This is a finite set of cardinality $\mathrm{val}(x)$. If $G = (V, E)$ is a loopless model for Γ such that $x \in V$, then $\mathrm{T}_x(\Gamma)$ is in one-to-one correspondence with the edges of G incident to x. Through this natural bijection, a tangent direction ν is said to be *supported by* the corresponding edge $e \in E$.

Each edge e supports two tangent directions, which belong to either endpoint of e, respectively. If ν is one of those tangent directions, the opposite direction is denoted by $\overline{\nu}$. For $\nu \in \mathrm{T}(\Gamma)$, we denote by x_ν the point x with $\nu \in \mathrm{T}_x(\Gamma)$.

In this chapter, all the semimodules will be assumed to be nonempty.

4.2 Slope sets

In this section, we prove Theorem 4.1.9. We first recall some terminology for divisors and functions on metric graphs.

Given a metric graph Γ, let $\mathrm{Div}(\Gamma)$ denote the *group of divisors* of Γ, which is the free abelian group generated by points $x \in \Gamma$. Let $\mathrm{Rat}(\Gamma)$ denote the set of real-valued piecewise affine linear functions on Γ whose slopes are all integers. Given a function $f \in \mathrm{Rat}(\Gamma)$, let $\mathrm{div}(f)$ denote the *principal divisor* of f, defined as

$$\mathrm{div}(f) := \sum_{x \in \Gamma} a_x(x) \qquad \text{where} \qquad a_x = -\sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu f(x).$$

Let D be a divisor of rank r on Γ. Let $\mathrm{Rat}(D)$ denote the set of *rational functions in the complete linear series of* D defined as

$$\mathrm{Rat}(D) := \{f \in \mathrm{Rat}(\Gamma) : D + \mathrm{div}(f) \geqslant 0\}.$$

Given a point $x \in D$, there is a unique representative f_x of the linear series of D defined by

$$f_x := \min_{\substack{f \in \mathrm{Rat}(D) \\ f(x) = 0}} f.$$

The corresponding divisor $D + \mathrm{div}(f_x)$, denoted by D_x, is the (unique) *x-reduced divisor* linearly equivalent to D. This statement is a consequence of the maximum principle, see e.g. [BS13, Lemma 4.11].

Definition 4.2.1 (Slope sets and minimum slopes)**.** Let D be an effective divisor on Γ. Given a point $x \in \Gamma$ and a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, let $\mathfrak{S}^\nu(D)$ denote the *slope set*

$$\mathfrak{S}^\nu(D) := \{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}.$$

Let $s_0^\nu(D)$ denote the *minimum slope* in the slope set $\mathfrak{S}^\nu(D)$, i.e.,

$$s_0^\nu(D) := \min\{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}.$$

When the divisor D is clear from context, we will simply use s_0^ν to denote $s_0^\nu(D)$. $\diamond$

Lemma 4.2.2. *Suppose D is a divisor of rank r. Then, for every $x \in \Gamma$ and every $\nu \in \mathrm{T}_x(\Gamma)$, there are at least $r+1$ integers in the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}$.*

Proof. Let $x_1, \ldots, x_r$ be a set of distinct points in the branch incident to x in the direction of ν sufficiently close to x. There exists a function $f \in \mathrm{Rat}(D)$ such that

$$D + \mathrm{div}(f) \geqslant (x_1) + \cdots + (x_r).$$

The function f changes slope at the points $x_1, \ldots, x_r$. Each of the slopes taken at x_j in the direction of ν can be obtained as the slope of a function in $\mathrm{Rat}(D)$ at x along ν. $\square$

The minimum slope $s_0^\nu(D)$ is related to the reduced divisor D_x at x.

Lemma 4.2.3. *Let D be an effective divisor on Γ, and x a point of Γ. Let D_x be the x-reduced divisor linearly equivalent to D.*

(a) Let f_x be the above defined rational function satisfying $\mathrm{div}(f_x) + D = D_x$, then, for any outgoing tangent vector $\nu \in \mathrm{T}_x(\Gamma)$,

$$s_0^\nu(D) = \mathrm{slope}_\nu f_x(x).$$

(b) The coefficient of D_x at x satisfies

$$D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D).$$

Proof. The first result is obtained by observing that $f_x = \min h$ for $h \in \mathrm{Rat}(D)$ verifying $h(x) = 0$. The second result is a direct consequence of (a) and the definition of the principal divisor $\mathrm{div}(f_x)$. $\square$

We now turn to the proof of Theorem 4.1.9. Let D be a divisor of rank r on Γ. Recall (Definition 4.1.3) that the Weierstrass locus of D, denoted by $L_{\scriptscriptstyle W}(D)$, is the subset of Γ formed by the points x such that there exists an effective divisor $E \sim D$ with $E(x) \geqslant r+1$. Equivalently, $L_{\scriptscriptstyle W}(D)$ is defined in terms of reduced divisors as

$$L_{\scriptscriptstyle W}(D) = \{x \in \Gamma : D_x(x) > r\},$$

where D_x denotes the x-reduced divisor linearly equivalent to D.

Proof of Theorem 4.1.9. We first assume that the open interval $(x, x+\varepsilon\nu)$ is disjoint from $L_W(D)$. Along the branch incident to x in the direction of ν, there is a small segment on which s_0^ν is the slope of a function in $\mathrm{Rat}(D)$, and it is the smallest slope taken by a function of $\mathrm{Rat}(D)$ on this segment. If a slope of $s_0^\nu + r + 1$ or larger is achieved at x, then, again on a small segment, it will be achieved at any point of that segment. This means that on the interior of this segment, the two minimum outgoing slopes at every point are s_0^ν and $-s_0^\nu - s'$ with $s' \geqslant r+1$. Therefore, by Lemma 4.2.3, we infer that on the interior of this segment, the reduced divisor at each point has coefficient at least $r+1$. This contradicts the assumption that a neighborhood $(x, x+\varepsilon\nu)$ is disjoint from $L_W(D)$, and shows that the highest possible slope is $s_0^\nu + r$. Combining this with Lemma 4.2.2, the slopes achieved at x along ν must be precisely $s_0^\nu, s_0^\nu + 1, \ldots, s_0^\nu + r$. This proves (a).

We now assume that $(x, x+\varepsilon\nu) \subset L_W(D)$ and that ε is small enough so that the set of slopes of functions of $\mathrm{Rat}(D)$ along ν is constant on this interval. By Lemma 4.2.3, this means that on the interior of a small segment starting at x, the two minimum outgoing slopes at every point are s_0^ν and $-s_0^\nu - s'$ with $s' \geqslant r+1$. Therefore, close to x, a slope of at least $s_0^\nu + r + 1$ is achieved by a function in $\mathrm{Rat}(D)$. To prove (b), it is thus sufficient to show that the set of slopes $\mathrm{slope}_\nu f(x)$ of functions $f \in \mathrm{Rat}(D)$ is always made up of consecutive integers. Take $s_1 < s_2 < s_3$ to be three integers, and suppose that for $i \in \{1,3\}$ there exists a function $f_i \in \mathrm{Rat}(D)$ such that $\mathrm{slope}_\nu f_i(x) = s_i$. Using f_1, f_3 and tropical operations, it is easy to construct a function f taking slopes s_3 and then s_1 away from x, changing slope at a point we denote by y (see Figure 4.1). We can then "chop up" the graph of f to construct a function h equal to f everywhere except on a small interval around y where it takes slope s_2. Since $(x, x+\varepsilon\nu)$ is disjoint from the support of D, h still belongs to $\mathrm{Rat}(D)$. The assumption made on ε at the beginning ensures that in fact there exists a function $f_2 \in \mathrm{Rat}(D)$ taking slope s_2 at x along ν, which concludes the argument. □

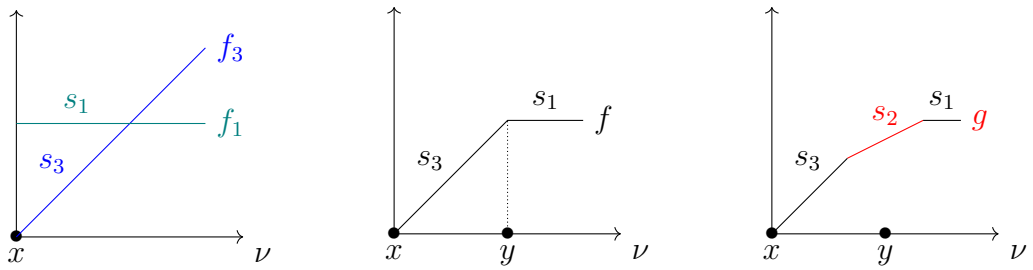


Figure 4.1: Construction of the functions f and g using functions f_1 and f_3 taking slopes $s_1 < s_3$.

Remark 4.2.4. In particular, note that along a given unit tangent vector ν attached to a point x, the slopes $\mathrm{slope}_\nu f(x)$ for $x \in \mathrm{Rat}(D)$ always form a set of consecutive integers. Moreover, if t is a positive integer such that for every $x \in e$, for e an edge of some model of Γ, the x-reduced divisor D_x satisfies $D_x \geqslant t\,(x)$, then for any $x \in \mathring{e}$, the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in \mathrm{Rat}(D)\}$ contains at least $t+1$ consecutive integers. This claim is

analogous to [Ami13, Theorem 14] and is proved using Theorem 3 of the same paper, which gives a concrete description of the variations of the reduced divisor D_x with respect to x. See also [AG22, Section 7.6] (Section 3.7.6 in the present manuscript). ◇

For future use, we note the following generalization of part (b) of Lemma 4.2.3.

Proposition 4.2.5. *Suppose D is a divisor of rank r. Then, for any closed, connected subset $A \subseteq \Gamma$, we have*

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\text{out}} A} s_0^\nu(D) \geqslant r.$$

Proof. Let E be an effective divisor of degree r, with support contained in A. Since D has rank r, there exists a function $f \in \mathrm{Rat}(D)$ such that $D + \mathrm{div}(f) \geqslant E$. Evaluating the respective degrees restricted to A yields

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\text{out}} A} \mathrm{slope}_\nu f(x_\nu) \quad \geqslant \quad \deg\left(E|_A\right) = r,$$

where, we recall, x_ν is the point x of Γ with $\nu \in \mathrm{T}_x(\Gamma)$. By definition of the minimum slope $s_0^\nu(D)$, we have $s_0^\nu(D) \leqslant \mathrm{slope}_\nu f(x_\nu)$ for each $\nu \in \partial^{\text{out}} A$, so the result follows. □

4.3 Weierstrass weights

Using the structure of slope sets in $\mathrm{Rat}(D)$, we prove Theorem 4.1.7, which will follow from the more general Theorem 4.3.9.

4.3.1 Definition of weights and basic properties of the Weierstrass locus

We start by establishing basic properties of Weierstrass loci. (Definition 4.1.3) The Weierstrass locus $L_W(D)$ is defined as the set of points x in Γ such that there exists an effective divisor E in the linear system of D whose coefficient at x is at least $r+1$. This is equivalent to requiring that $D_x(x) \geqslant r+1$. Let us now recall Definition 4.1.6 from the introduction. Given a connected component C of the Weierstrass locus $L_W(D)$, the tropical Weierstrass weight of C is defined as

$$\mu_W(C) = \mu_W(C; D) := \deg\left(D|_C\right) + (g(C) - 1)\, r - \sum_{\nu \in \partial^{\text{out}} C} s_0^\nu(D)$$

where $\deg\left(D|_C\right) = \sum_{x \in C} D(x)$ is the degree of D in C, $g(C) = \dim H_1(C, \mathbb{R})$ is the genus of C, $\partial^{\text{out}} C$ is the set of outgoing unit tangent directions from C, and $s_0^\nu(D)$ is the minimum slope at x along a tangent direction ν, as defined in Definition 4.2.1. The following proposition shows that $L_W(D)$ is topologically nice.

Proposition 4.3.1. *The Weierstrass locus $L_W(D)$ is closed and has finitely many components. Each connected component is a metric graph.*

Proof. By the continuity of variation of reduced divisors proved in [Ami13, Theorem 3], the function $x \mapsto D_x(x)$ is upper semicontinuous, which implies that the subset $L_W(D)$ is closed. Then, by Theorem 4.1.9, the number of connected components of $L_W(D)$ is finite. The last statement follows as any connected component of a closed subset in a metric graph is itself a metric graph. □

Remark 4.3.2. We have the following geometric construction of $L_W(D)$, which gives another proof for Proposition 4.3.1. Let $\mathrm{Pic}^d(\Gamma)$ denote the space of divisor classes of degree d on Γ, and let $\mathrm{Eff}^d(\Gamma)$ denote the space of effective divisor classes of degree d. Let $\varphi : \Gamma \to \mathrm{Pic}^{d-r-1}(\Gamma)$ be the map defined by $\varphi(x) = [D - (r+1)(x)]$.

The condition that $D_x(x) > r$ is equivalent to $D_x \geqslant (r+1)(x)$. This is in turn equivalent to the condition that the divisor class $[D-(r+1)(x)]$ has an effective representative. Using this observation and the above terminology, $L_W(D) = \varphi^{-1}\big(\varphi(\Gamma) \cap \mathrm{Eff}^{d-r-1}(\Gamma)\big)$. In other words, $L_W(D)$ is described by the following pullback diagram.

$$\begin{array}{ccc} L_W(D) & \longrightarrow & \mathrm{Eff}^{d-r-1}(\Gamma) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Gamma & \xrightarrow{\varphi} & \mathrm{Pic}^{d-r-1}(\Gamma) \end{array}$$

Both $\mathrm{Eff}^{d-r-1}(\Gamma)$ and $\varphi(\Gamma)$ are polyhedral subsets of $\mathrm{Pic}^{d-r-1}(\Gamma)$ with finitely many facets. Thus their intersection has finitely many components, and each component is a union of finitely many closed intervals. ◇

Remark 4.3.3. As before, let D_x denote the x-reduced divisor linearly equivalent to D. Since the Weierstrass locus $L_W(D)$ is defined as $\{x \in \Gamma : D_x(x) - r > 0\}$, the expression $D_x(x) - r$ is a natural "naive" candidate for defining a tropical Weierstrass weight. In fact, this ends up being the correct definition when x is an isolated component of $L_W(D)$. When x is not an isolated component, our more technical definition of weight is required.

If the singleton $\{x\}$ is a connected component of $L_W(D)$, then we verify that the weight of x is simply given by $D_x(x) - r$. Since the genus of the component $\{x\}$ is zero, Definition 4.1.6 states that

$$\mu_W(x) = D(x) - r - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D),$$

and Lemma 4.2.3 states that $D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu(D)$. This verifies the claim.

Note that this applies for every connected component of $L_W(D)$ if D is W-finite. ◇

We now give two examples of metric graphs and their Weierstrass loci. The first Weierstrass locus is finite whereas the second one is infinite.

Example 4.3.4. Suppose Γ is the complete graph on four vertices with unit edge lengths; see Figure 4.2. This graph has genus three, and the rank of the canonical divisor K is

$r = g-1 = 2$. It is easy so see that the slopes on each edge are $\{-1, 0, 1\}$. This can be proved by direct verification, or by using the reflexivity of each edge of Γ, see Definition 4.3.21. The Weierstrass locus $L_W(K)$ is finite and consists of the four vertex points. At a vertex v, the reduced divisor at v is $K_v = 4\,(v)$. Thus, $\mu_W(v) = K_v(v) - r = 4 - 2 = 2$.

Figure 4.2: Complete graph on four vertices, and its Weierstrass locus $L_W(K)$.

We will treat the example of the complete graph on five or more vertices in Section 4.6.5. ◇

Example 4.3.5. Suppose Γ is the "barbell graph" consisting of two cycles joined by a bridge edge; see Figure 4.3. (The edge lengths may be arbitrary.) This graph has genus two, and the canonical divisor K has rank $r = g - 1 = 1$.

The Weierstrass locus $L_W(K)$ consists of the middle edge and the outer midpoint on each cycle. The latter have weight one. If we divide each cycle into two equal parts according to its two distinguished points, then the slopes on each half-circle are $\{0, 1\}$ starting on the middle edge. This implies that the weight of the middle edge is also one.

Figure 4.3: The barbell graph and its Weierstrass locus $L_W(K)$.

We will show in greater generality in Section 4.6.4 that if e is a bridge edge of Γ such that each component of $\Gamma \smallsetminus \mathring{e}$ has positive genus, then e is contained in the canonical Weierstrass locus. ◇

4.3.2 Positivity of Weierstrass weights

We now prove the following theorem.

Theorem 4.3.6. *Let D be a divisor on Γ with non-negative rank r, and let C be a connected component of the Weierstrass locus $L_W(D)$. Then, the weight $\mu_W(C)$ given in Definition 4.1.6 is positive.*

Proof. We use the notations introduced previously. Let x be a point in the connected component C, and let D_x be the x-reduced divisor equivalent to D. By definition of the

Weierstrass locus $L_W(D)$, we have $D_x(x) - r > 0$. Let f_x be a rational function such that $D_x = D + \operatorname{div}(f_x)$. We have

$$D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \operatorname{slope}_\nu f_x(x).$$

Let A be any connected subgraph of Γ, and recall that $\deg\left(D_x|_A\right)$ denotes the sum $\sum_{y \in A} D_x(y)$. For a tangent vector $\nu \in \vec{\partial}^{\text{out}} A$, let, as before, x_ν denote the associated boundary point. We have

$$\deg\left(D_x|_A\right) = \deg\left(D|_A\right) - \sum_{\nu \in \vec{\partial}^{\text{out}} A} \operatorname{slope}_\nu f_x(x_\nu)$$

by applying Stokes theorem to the derivative of f_x on the region A.

Because $x \in C$ and D_x is effective, we have $\deg\left(D_x|_C\right) \geqslant D_x(x) > r$. For each tangent direction $\nu \in \vec{\partial}^{\text{out}} C$, the minimum slope $s_0^\nu(D)$ satisfies $s_0^\nu(D) \leqslant \operatorname{slope}_\nu f_x(x_\nu)$ by definition (Definition 4.2.1). Therefore,

$$\begin{aligned} \mu_W(C) &= \deg\left(D|_C\right) + (g(C) - 1)\, r - \sum_{\nu \in \vec{\partial}^{\text{out}} C} s_0^\nu(D) \geqslant \deg\left(D|_C\right) - r - \sum_{\nu \in \vec{\partial}^{\text{out}} C} s_0^\nu(D) \\ &\geqslant \deg\left(D|_C\right) - r - \sum_{\nu \in \vec{\partial}^{\text{out}} C} \operatorname{slope}_\nu f_x(x_\nu) \quad = \quad \deg\left(D_x|_C\right) - r > 0 \end{aligned}$$

as claimed. □

The proof of Theorem 4.3.6 shows the stronger bound $\mu_W(C) > g(C)r$. This is addressed later, in greater generality, in Corollary 4.3.13.

4.3.3 Weierstrass measure

We prove Theorem 4.3.9 below, which will imply Theorem 4.1.7.

Definition 4.3.7. Fix a divisor D on a metric graph Γ, with Weierstrass locus $L_W(D)$. A subset $A \subseteq \Gamma$ is $L_W(D)$-*measurable* if A is a Borel set and, for every component C of the Weierstrass locus $L_W(D)$, we have either

$$C \subseteq A \qquad \text{or} \qquad C \subseteq \Gamma \smallsetminus A.$$

Let $\mathcal{A} = \mathcal{A}(D)$ denote the σ-algebra of $L_W(D)$-measurable subsets of Γ. ◇

In other words, given a Weierstrass locus $L_W(D) \subseteq \Gamma$, we can construct the quotient map $\pi : \Gamma \to \Gamma_0$ in which each component $C_i \subseteq L_W(D)$ is contracted to a single point. Then, the $L_W(D)$-measurable sets of Γ are the preimages of Borel sets of Γ_0. If the divisor D is W-finite, then all Borel sets in Γ are $L_W(D)$-measurable.

Definition 4.3.8 (Weierstrass measure)**.** Notations as above, let D be an effective divisor of rank r on Γ, and let $\mathcal{A}$ denote the σ-algebra of $L_W(D)$-measurable subsets of Γ. We define the *Weierstrass measure* $\hat{\mu}_W$ as the "weighted counting measure" on Γ whose atoms are the connected components in the Weierstrass locus $L_W(D)$. More precisely, $\hat{\mu}_W$ is the measure on $(\Gamma, \mathcal{A})$ defined by

$$\hat{\mu}_W(A) := \sum_{C \subseteq A} \mu_W(C),$$

where the sum is taken over components of $L_W(D)$ contained in A, and $\mu_W(C)$ is given by (4.1). ◇

We have the following description of the Weierstrass measure.

Theorem 4.3.9. *Notations as above, for any closed connected $A \in \mathcal{A}$, we have*

$$\hat{\mu}_W(A) = \deg\left(D|_A\right) + (g(A) - 1)r - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(D). \tag{4.2}$$

Proof. Let $\mathfrak{A} = \{C_1, \ldots, C_n\}$ denote the set of components of $L_W(D)$ contained in A. Let $G = (V, E)$ be a model for Γ whose vertex set V contains the support of D, and let $V \cap (A \smallsetminus L_W(D)) = \{v_1, \ldots, v_m\}$ denote the set of non-Weierstrass vertices in A. For each such vertex v_i, let $C_{n+i} = \{v_i\}$ denote the corresponding singleton, and let $\widetilde{\mathfrak{A}}$ denote the union

$$\widetilde{\mathfrak{A}} = \mathfrak{A} \cup \{\{v_1\}, \ldots, \{v_m\}\} = \{C_1, \ldots, C_n, C_{n+1}, \ldots, C_{\tilde{n}}\} \qquad \text{where} \qquad \tilde{n} = n + m.$$

Finally, let $|\widetilde{\mathfrak{A}}| = \bigcup_{i=1}^{\tilde{n}} C_i$ be the underlying subset of Γ. Note that $|\widetilde{\mathfrak{A}}| \subseteq A$, and $A \smallsetminus |\widetilde{\mathfrak{A}}|$ consists of a union of finitely many open intervals; let k denote their number.

Let $V' := V \smallsetminus L_W(D)$, as in Figure 4.4. For each $v \in V'$, we have $D_v(v) = r$, so

$$\mu_W(\{v\}) = D(v) - r - \sum_{\nu \in \mathrm{T}_v(\Gamma)} s_0^\nu(D) = D_v(v) - r = 0.$$

Thus, the "components" $C_{n+i} = \{v_i\}$ inside $\widetilde{\mathfrak{A}} \smallsetminus \mathfrak{A}$ do not contribute to the total weight, so it suffices to show that $\sum_{C_i \in \widetilde{\mathfrak{A}}} \mu_W(C_i)$ satisfies (4.2).

From Definition 4.1.6, we have

$$\sum_{i=1}^{\tilde{n}} \mu_W(C_i) = \sum_{i=1}^{\tilde{n}} \deg\left(D|_{C_i}\right) + r \sum_{i=1}^{\tilde{n}} (g(C_i) - 1) - \sum_{i=1}^{\tilde{n}} \left(\sum_{\nu \in \partial^{\mathrm{out}} C_i} s_0^\nu(D) \right). \tag{4.3}$$

We treat separately the three terms appearing on the right-hand side of (4.3). The first term $\sum_i \deg\left(D|_{C_i}\right)$ is equal to $\deg\left(D|_A\right)$, since the vertex set V was chosen to contain the support of D.

For the second term, we apply the identity

$$\deg\left(K|_B\right) = 2g(B) - 2 + \mathrm{outval}(B) \qquad \text{for} \quad B \subseteq \Gamma \quad \text{closed and connected}$$

(see Lemma 4.3.16) twice to obtain

$$\begin{aligned} r\sum_{i=1}^{\tilde{n}}(g(C_i)-1) &= \frac{r}{2}\sum_{i=1}^{\tilde{n}}\Big(\deg\left(K|_{C_i}\right)-\mathrm{outval}(C_i)\Big)\\ &= \frac{r}{2}\big(\deg\left(K|_A\right)-\mathrm{outval}(A)-2k\big)\\ &= r(g(A)-1)-rk, \end{aligned}$$

where k, we recall, denotes the numbers of edges of $\Gamma\smallsetminus|\widetilde{\mathfrak{A}}|$ whose endpoints are both in $|\widetilde{\mathfrak{A}}|$.

For the third term, the collection of all tangent directions $\bigcup_{C_i\in\tilde{\mathfrak{A}}}\{\nu\in\partial^{\mathrm{out}}C_i\}$ can be partitioned into "paired" directions, if following ν leads to another component in $\widetilde{\mathfrak{A}}$, and "unpaired" directions, if following ν leads out of A. For any paired tangent direction $\nu\in\partial^{\mathrm{out}}C_i$, there is a matching opposite direction $\overline{\nu}\in\partial^{\mathrm{out}}C_j$ (see Section 4.1.6) and their minimum slopes satisfy $s_0^{\nu}(D)+s_0^{\overline{\nu}}(D)=-r$. For any unpaired tangent direction $\nu\in\partial^{\mathrm{out}}C_i$, the minimum slope $s_0^{\nu}(D)$ is equal to $s_0^{\nu'}(D)$ for some parallel tangent direction $\nu'\in\partial^{\mathrm{out}}A$. Moreover, this gives a bijection between $\partial^{\mathrm{out}}A$ and the unpaired tangent directions. Using this, we have

$$\begin{aligned} \sum_{i=1}^{\tilde{n}}\left(\sum_{\nu\in\partial^{\mathrm{out}}C_i}s_0^{\nu}(D)\right) &= \sum_{\text{unpaired }\nu}s_0^{\nu}(D)+\sum_{\text{paired }\nu}s_0^{\nu}(D)\\ &= \sum_{\nu\in\partial^{\mathrm{out}}A}s_0^{\nu}(D)+\sum_{\ell=1}^{k}\big(s_0^{\nu_\ell}(D)+s_0^{\overline{\nu}_\ell}(D)\big)\\ &= \sum_{\nu\in\partial^{\mathrm{out}}A}s_0^{\nu}(D)-rk. \end{aligned}$$

Combining the above identities shows that $\hat{\mu}_W(A)$ satisfies (4.2). □

Remark 4.3.10. For a closed subset $A\in\mathcal{A}$ with a finite number of connected components, the weight $\hat{\mu}_W(A)$ can be expressed equivalently as

$$\hat{\mu}_W(A)=\deg\left(D|_A\right)+(g(A)-c(A))r-\sum_{\nu\in\partial^{\mathrm{out}}A}s_0^{\nu}(D)$$

where $c(A)=h_0(A)$ denotes the number of connected components of A. Note that $g(A)=h_1(A)$, so that in terms of the Euler characteristic χ, the middle term is $-r\cdot\chi(A)$. ⋄

The following result can be obtained by the same method. Let U be a connected open subset of Γ which is $L_W(D)$-measurable.

Theorem 4.3.11. *Notations as above, the Weierstrass weight $\hat{\mu}_W(U)$ can be recovered from the slopes around the incoming branches as the sum*

$$\hat{\mu}_W(U)=\deg\left(D|_U\right)+(g(U)-1)r+\sum_{\nu\in\partial^{\mathrm{in}}U}s_{\max}^{\nu}(D)$$

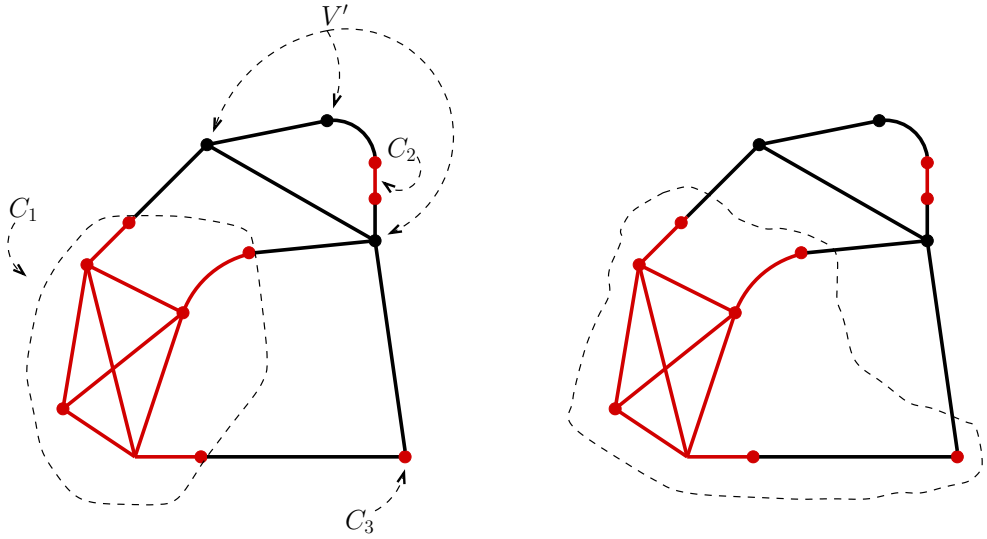


Figure 4.4: The part in red in the left figure is the (hypothetical) locus of Weierstrass points, and consists of three connected components. Red thickened points are on the boundary of the Weierstrass locus. Black vertices are those belonging to V', that is, outside the Weierstrass locus. They are three in number. The right figure is an example of a set A appearing in $\mathcal{A}$. There is no vertex in A outside the Weierstrass locus, so $m = 0$. There are two connected components of the Weierstrass locus in A, so $n = 2$. The subset $A \smallsetminus |\widetilde{\mathfrak{A}}|$ consists of four intervals. This means $k = 4$.

where $\partial^{\text{in}}U$ *denotes the set of incoming unit tangent vectors from the boundary of* U*, and* $s^{\nu}_{\max}(D)$ *the maximum slope along the incoming tangent vector* ν *of any rational function in* $\mathrm{Rat}(D)$.

Note that since U is open and $L_W(D)$ is closed, every $\nu \in \partial^{\text{in}}U$ is tangent to an open interval on Γ which is outside $L_W(D)$ and thus $s^{\nu}_{\max}(D) = s^{\nu}_0(D) + r$ (see Theorem 4.1.9). Thus we have

$$\hat{\mu}_W(U) = \deg\left(D|_U\right) + (g(U) - 1 + \mathrm{inval}(U))\, r + \sum_{\nu \in \partial^{\text{in}}U} s^{\nu}_0(D).$$

Proof of Theorem 4.1.7. We apply Theorem 4.3.9 with $A = \Gamma$. The statement about W-finite divisors follows from the first statement and Remark 4.3.3. □

4.3.4 Consequences

We now provide some direct consequences of the above results, starting with the following remark.

Remark 4.3.12. Theorem 4.1.7 and [Ric24, Theorem A] together imply that a generic divisor D of degree $d \geqslant g$ has a finite Weierstrass locus made up of $g\,(d - g + 1)$ points, all

of weight one. Indeed, the cardinality of $L_W(D)$ given by [Ric24, Theorem A] is $g(d-g+1)$, whereas the total weight given by Theorem 4.1.7 is $d-r+rg$. But $r=d-g$ generically and in this case we have $g(d-g+1)=g(r+1)=d-r+rg$. ◇

Corollary 4.3.13. *Suppose D is a divisor of rank r. For any closed, connected, $L_W(D)$-measurable subset $A \subseteq \Gamma$, we have*

$$\hat{\mu}_W(A) \geqslant g(A)\, r.$$

Proof. This follows from Theorem 4.3.9 and Lemma 4.2.5. □

Corollary 4.3.14 (Theorem 4.1.8)**.** *Suppose that the rank r of D is at least one. Then, the complement of the Weierstrass locus $L_W(D)$ is a disjoint union of (open) metric trees. In other words, every cycle in Γ intersects the tropical Weierstrass locus.*

Proof. For the sake of a contradiction, suppose that A is a cycle in Γ disjoint from the Weierstrass locus $L_W(D)$. Then, A is $L_W(D)$-measureable, and by definition (Definition 4.3.8), $\hat{\mu}_W(A)=0$. However, Corollary 4.3.13 states that $\hat{\mu}_W(A) \geqslant g(A)\, r > 0$, which gives a contradiction. □

4.3.5 Special cases of weights

Here, we point out some special cases of the weight formula.

(i) If a divisor D has rank $r=0$, then $\hat{\mu}_W(\Gamma)=d$. Suppose D is effective in its linear equivalence class. For any tangent direction ν outside the Weierstrass locus, the slope set $\mathfrak{S}^\nu(D)$ contains a single slope, and this slope must be zero since D is effective. Thus, a component C of the Weierstrass locus has weight $\mu_W(C)=\deg\left(D|_C\right)$.

(ii) If the genus $g=0$, then for any divisor $\hat{\mu}_W(\Gamma)=d-r=0$. (In general $0 \leqslant d-r \leqslant g$.) In particular, this implies that the Weierstrass locus $L_W(D)$ is empty.

(iii) If the genus $g=1$, then for a divisor of degree d, the total Weierstrass weight is $\hat{\mu}_W(\Gamma)=d$. Every component C of the Weierstrass locus has weight $\mu_W(C)=1$.

(iv) If the rank satisfies $r=d-g$, then $\hat{\mu}_W(\Gamma)=d-r+rg=g(r+1)$. In particular, this holds for a generic divisor class with degree $d \geqslant g$, and for every divisor with degree $d \geqslant 2g-1$.

(v) If $D=K$ is the canonical divisor, then $d=2g-2$ and $r=g-1$, so $\hat{\mu}_W(\Gamma)=g^2-1$. See Section 4.3.7 below for more discussion of this case.

4.3.6 Combinatorial graphs

In this section we assume Γ is a combinatorial graph. By this we mean Γ admits a model $(G = (V, E), \ell)$ which has unit edge lengths. We assume the divisor D is supported on the vertex set V.

Theorem 4.3.15. *Suppose $e = uv$ is an edge in G whose interior $\mathring{e}$ is $L_W(D)$-measurable. Let f_{uv} be a rational function that satisfies $\operatorname{div}(f_{uv}) = D_u - D_v$. Let ν be the unit tangent vector at v along e, towards u. Then, the Weierstrass weight of the interior of e is*

$$\hat{\mu}_W(\mathring{e}) = r - \operatorname{slope}_\nu(f_{uv}).$$

Proof. Let $U := \mathring{e}$. Since $L_W(D)$ is closed, we can take the open interval U a little bit smaller so that its extremities are distinct from u and v and U still contains the same components of $L_W(D)$. Theorem 4.3.11 states that the sum of Weierstrass weights on $U = \mathring{e}$ is equal to

$$\hat{\mu}_W(U) = \deg\left(D|_U\right) + (g(U) - 1)\, r + \sum_{\nu \in \partial^{\text{in}} U} s^\nu_{\max}(D).$$

Since D is supported on the vertex set, we have $\deg\left(D|_U\right) = 0$, and we also have $g(U) = 0$. Thus, the expression simplifies to

$$\hat{\mu}_W(U) = -r + \left(s^{v,\nu}_{\max}(D) + s^{u,\overline{\nu}}_{\max}(D)\right)$$

where ν and $\overline{\nu}$ are tangent directions towards u and v, respectively. If f_u and f_v satisfy

$$\operatorname{div}(f_u) = D_u - D \qquad \text{and} \qquad \operatorname{div}(f_v) = D_v - D,$$

then we have

$$\operatorname{slope}_{\overline{\nu}} f_u(u) = s_0^{u,\overline{\nu}}(D) = s^{u,\overline{\nu}}_{\max}(D) - r \qquad \text{and} \qquad \operatorname{slope}_\nu f_v(v) = s_0^{v,\nu}(D) = s^{v,\nu}_{\max}(D) - r,$$

and the relation $f_{uv} = f_u - f_v$ implies

$$\begin{aligned} \operatorname{slope}_\nu f_{uv}(u) &= \operatorname{slope}_\nu(f_u - f_v)(u) = -\left(s^{u,\overline{\nu}}_{\max}(D) - r\right) - \left(s^{v,\nu}_{\max}(D) - r\right) \\ &= 2r - \left(s^{v,\nu}_{\max}(D) + s^{u,\overline{\nu}}_{\max}(D)\right). \end{aligned}$$

Note that the slope of f_{uv} is constant along the interior of e, since the reduced divisors D_u and D_v are supported on vertices. □

4.3.7 Canonical Weierstrass locus

In this section we discuss the case of the canonical divisor on a metric graph.

Weierstrass weight

The weight formula (4.1) for $\mu_W(C; D)$ may be specialized to the case of the canonical divisor $D = K$. We need the following lemma.

Lemma 4.3.16. *Let $K = \sum_{x\in\Gamma}(\mathrm{val}(x) - 2)\,(x)$ denote the canonical divisor of Γ, and let $A \subseteq \Gamma$ be a closed connected subset. Then*

$$\deg\left(K|_A\right) = 2g(A) - 2 + \mathrm{outval}(A).$$

Proof. The proof can be obtained by direct calculation using an adapted graph model. The details are omitted. □

By direct summation, this result generalizes to closed subsets with finitely many connected components.

Theorem 4.3.17. *Suppose Γ is a metric graph of genus g, and let K be its canonical divisor. The weight of any component C of the Weierstrass locus $L_W(K)$ is*

$$\mu_W(C; K) = (g+1)(g(C) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}C}(s_0^\nu(K) - 1).$$

More generally, for any closed, connected subset $A \subseteq \Gamma$ that is $L_W(K)$-measurable,

$$\hat{\mu}_W(A; K) = (g+1)(g(A) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}A}(s_0^\nu(K) - 1).$$

Proof. Let $\partial^{\mathrm{out}}C$ denote the set of outgoing tangent directions from C in Γ, and let $\mathrm{outval}(C)$ denote its cardinality. From (4.1) we have

$$\mu_W(C; K) = \deg\left(K|_C\right) + r\,(g(C) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}C} s_0^\nu(K).$$

The canonical divisor K has rank $r = g - 1$. By Lemma 4.3.16, on a closed connected set $B \subseteq \Gamma$, the degree $\deg\left(K|_B\right)$ satisfies $\deg\left(K|_B\right) = 2g(B) - 2 + \mathrm{outval}(B)$. Therefore,

$$\begin{aligned}
\mu_W(C; K) &= \deg\left(K|_C\right) + (g-1)(g(C) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}C} s_0^\nu(K) \\
&= 2(g(C) - 1) + \mathrm{outval}(C) + (g-1)(g(C) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}C} s_0^\nu(K) \\
&= (g+1)(g(C) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}C}(s_0^\nu(K) - 1),
\end{aligned}$$

which concludes. □

If we repeat the same computation for the pluricanonical divisor nK, where $n \geqslant 2$, we find that

$$\hat{\mu}_W(A; nK) = (2n-1)g(g(A) - 1) - \sum_{\nu\in\partial^{\mathrm{out}}A}(s_0^\nu(nK) - n).$$

This next corollary to Theorem 4.3.17 is also a direct consequence of Theorem 4.1.7.

Corollary 4.3.18. *Suppose Γ is a genus g metric graph.*

(a) The sum of Weierstrass weights over all components of $L_W(K)$ is equal to g^2-1.

(b) For any integer $n \geqslant 2$, the sum of Weierstrass weights over all components of $L_W(nK)$ is equal to $(2n-1)g(g-1)$.

The next result is a special case of Corollary 4.3.13.

Corollary 4.3.19. *Suppose Γ is a metric graph of genus g. For any closed, connected, $L_W(K)$-measurable subset $A \subseteq \Gamma$, we have*

$$\hat{\mu}_W(A) \geqslant g(A)\,(g-1).$$

We end this section by providing a geometric interpretation of the tropical canonical Weierstrass locus. For the general description for any divisor D, see Remark 4.3.2.

The tropical canonical Weierstrass locus $L_W(K)$ can be described as an intersection as follows. Suppose $f : \Gamma \to \mathrm{Pic}^{g-2}(\Gamma)$ sends x to the divisor class $[K-g\,(x)]$, and let $h : \mathrm{Eff}^{g-2}(\Gamma) \to \mathrm{Pic}^{g-2}(\Gamma)$ be the inclusion of effective divisor classes in the space of all divisor classes of fixed degree $g-2$. The points in $L_W(K)$ are those such that $[K-g\,(x)] \geqslant 0$, or equivalently $[K-g\,(x)] \in \mathrm{Eff}^{g-2}(\Gamma)$. This description is summarized by the following pullback diagram.

$$\begin{array}{ccc} L_W(K) & \longrightarrow & \mathrm{Eff}^{g-2}(\Gamma) \\ \downarrow & \lrcorner & \downarrow h \\ \Gamma & \xrightarrow{f} & \mathrm{Pic}^{g-2}(\Gamma) \end{array}$$

The bottom horizontal map f sends x to the divisor class $[K-g\,(x)]$. The right vertical map h is the inclusion of effective divisor classes in the space of all divisor classes of fixed degree $g-2$. The points in $L_W(K)$ are those such that $[K-g\,(x)] \geqslant 0$, or equivalently $[K-g\,(x)] \in \mathrm{Eff}^{g-2}(\Gamma)$.

This description, which makes $L_W(K)$ sit inside the polyhedral complex $\mathrm{Eff}^{g-2}(\Gamma)$, brings forward the following open question.

Question 4.3.20. *It is possible to express the Weierstrass weights using this geometric description in a meaningful way?*

Edge symmetry

We now discuss properties of some specific Weierstrass points under some symmetry condition; see as well Section 4.6.5.

Definition 4.3.21. An edge e of a metric graph Γ is *reflexive* if there is an automorphism $\sigma : \Gamma \to \Gamma$ such that $\sigma(e) = \overline{e}$, i.e., σ reverses the direction of e. ◇

We show that the midpoint of a reflexive edge is either a Weierstrass point of K, or a Weierstrass point of nK for all $n \geqslant 2$.

Theorem 4.3.22. *Suppose Γ is a metric graph of genus $g \geqslant 2$, and let K denote the canonical divisor of Γ. Suppose e is a reflexive edge in Γ.*

(a) If g is even, then the midpoint of e is in the Weierstrass locus $L_W(K)$.

(b) If g is odd, then the midpoint of e is in the Weierstrass locus $L_W(nK)$ for any integer $n \geqslant 2$.

Proof. Let x denote the midpoint of the reflexive edge e. The tangent space $\mathrm{T}_x(\Gamma)$ contains two directions $\{\nu_1, \nu_2\}$, and the reflexive assumption implies that the minimum slopes are equal in both directions, i.e., $s_0^{\nu_1}(K) = s_0^{\nu_2}(K)$. If x is outside the Weierstrass locus, then the singleton $\{x\}$ is $L_W(K)$-measurable and we may apply the weight formula from Theorem 4.3.17,

$$\hat{\mu}_W(x; K) = (g+1)(-1) - 2(s_0^{\nu_1}(K) - 1) \equiv g + 1 \mod 2.$$

Hence if g is even, then $\hat{\mu}_W(x)$ is nonzero, which contradicts our assumption that x is outside the Weierstrass locus. This proves part (a).

Now consider $D = nK$ for $n \geqslant 2$. By a similar argument, if x is outside the Weierstrass locus $L_W(nK)$, then its Weierstrass weight is

$$\hat{\mu}_W(x; nK) = (2n-1)g(-1) - 2(s_0^{\nu_1}(nK) - n) \equiv g \mod 2.$$

If g is odd, then the weight $\hat{\mu}_W(x)$ is nonzero, which again gives a contradiction. This proves part (b). □

4.4 Generalizations

In this section, we generalize the setting of the previous sections to the case of augmented metric graphs, that is, in the presence of genera associated to the vertices.

Since the genus of a given vertex hides information about the geometry of the component, it turns out that there will be an ambiguity when talking about the Weierstrass locus of a divisor D. In fact, the right setup in this context is a divisor D endowed with the data of a closed sub-semimodule M of $\mathrm{Rat}(D)$, which plays the role of a (not necessarily complete) linear series on the augmented metric graph. In what follows, we will explain how the preceding definitions and results extend from divisors to semimodules in the more general setting of augmented metric graphs. We then introduce two special classes of semimodules, the *generic* semimodule associated to any divisor, and the *canonical* semimodule associated to the canonical divisor. We properly justify both of them using the framework of metrized complexes.

In the following, we assume all semimodules are nonempty unless specified otherwise.

4.4.1 Weierstrass loci of semimodules and augmented metric graphs

Semimodules

Let Γ be a metric graph, and D a divisor of degree d on Γ. The set of functions $\mathrm{Rat}(D)$ naturally has the structure of a semimodule on the tropical semifield; we refer to [HMY12, AG22] (Chapter 3) for a discussion on this semimodule structure. Let M be a subsemimodule of $\mathrm{Rat}(D)$. We endow $\mathrm{Rat}(D)$ with the topology induced by $\|\cdot\|_\infty$, and say $M \subseteq \mathrm{Rat}(D)$ is *closed* if it is closed with respect to this topology. The following is a direct extension to semimodules of the rank of divisors on graphs introduced by Baker and Norine [BN07].

Definition 4.4.1 (Divisorial rank)**.** The *divisorial rank* or simply *rank* of $M \subseteq \mathrm{Rat}(D)$ (also called the *rank of D with respect to M*) is the greatest integer r such that for any effective divisor E on Γ of degree r, there exists a function $f \in M$ verifying $D + \mathrm{div}(f) \geqslant E$. It is denoted by $r(M, D)$. ◇

In fact, as the following statement shows, the divisorial rank will only depend on the semimodule M, if we additionally assume that M is closed. Therefore, we will work only with closed semimodules in the following, and will denote their rank simply by $r(M)$. Note that any (nonempty) semimodule has rank $r(M) \geqslant 0$. Also note that by definition, we have the immediate inequality $r(M) \leqslant r(D)$.

Proposition 4.4.2. *The divisorial rank $r(M, D)$ of a closed semimodule $M \subseteq \mathrm{Rat}(D)$ depends only on M.*

Proof. First note that there is a unique minimal divisor D_0 such that $M \subseteq \mathrm{Rat}(D_0)$, which is obtained by taking the (point-wise) minimum of all such divisors.

Then, we denote $r(M, D)$ by r and $r(M, D_0)$ by r_0. It is clear from the inequality $D_0 \leqslant D$ that the inequality $r_0 \leqslant r$ holds. We thus prove that $r_0 \geqslant r$. We choose a model $G = (V, E)$ such that the vertex set contains the support of D.

First, we suppose that E is an effective divisor of degree r on Γ whose support is disjoint from the support of D. By definition of r, there exists $f \in M$ such that $D + \mathrm{div}(f) \geqslant E$. Since $M \subseteq \mathrm{Rat}(D_0)$ and D coincides with D_0 outside V, it follows that $D_0 + \mathrm{div}(f) \geqslant E$.

Now, let E be an effective divisor of degree r on Γ whose support may intersect that of D. Let $(E_n)_n$ be a sequence of divisors of degree r converging to E, such that for each n, the support of E_n is disjoint from V. By what precedes, for each n, there exists a function $f_n \in M$ such that $D_0 + \mathrm{div}(f_n) \geqslant E_n$. Without loss of generality, assume that $f_n(x_0) = 0$ for some $x_0 \in \Gamma$. Thanks to the boundedness of the slopes of functions in $\mathrm{Rat}(D_0)$ (see [GK08, Lemma 1.8]), we can assume that $(f_n)_n$ converges uniformly to a function f, which satisfies $D_0 + \mathrm{div}(f) \geqslant E$ at the limit. The limit function f is in M by assumption that M is closed, which concludes the argument. □

Remark 4.4.3. In essence, the above proof shows that the complement of the support of D is a "rank-determining set" for the semimodule M in the sense of [Luo11]. ◇

The notion of minimum slopes naturally extends to closed semimodules.

Definition 4.4.4 (Slope sets and minimum slopes)**.** Let $M \subseteq \mathrm{Rat}(D)$ be a closed subsemimodule. Given a point $x \in \Gamma$ and a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, let $\mathfrak{S}^\nu(M)$ denote the *slope set*

$$\mathfrak{S}^\nu(M) := \{\mathrm{slope}_\nu f(x) : f \in M\}.$$

Let $s_0^\nu(M)$ denote the *minimum slope* along ν of functions in M. More generally, let $s_j^\nu(M)$ denote the $(j+1)$-smallest slope along ν of functions in M, i.e.,

$$s_0^\nu(M) = \min\{\mathfrak{S}^\nu(M)\}, \qquad s_j^\nu(M) = \min\{s \in \mathfrak{S}^\nu(M),\, s > s_{j-1}^\nu\}.$$

When the semimodule M is clear from context, we will simply use s_j^ν to denote $s_j^\nu(M)$. ⋄

The following result is obtained similarly to Proposition 4.2.5; we omit the details.

Proposition 4.4.5. *Suppose $M \subseteq \mathrm{Rat}(D)$ is a closed semimodule of divisorial rank r. Then for any closed, connected subset $A \subseteq \Gamma$, we have*

$$\deg\left(D|_A\right) - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(M) \geqslant r.$$

Reduced divisors

For closed $M \subseteq \mathrm{Rat}(D)$, there is a well-defined and well-behaved notion of x-reduced divisor, denoted D_x^M, linearly equivalent to D with respect to M for every $x \in \Gamma$. Simply, we define $f_x \colon \Gamma \to \mathbb{R}$ by setting

$$f_x(p) := \inf_{\substack{f \in M \\ f(x)=0}} f(p) \quad \forall p \in \Gamma.$$

Using the boundedness of slopes [GK08, Lemma 1.8], the infimum in the definition above turns out to be a minimum, and f_x is the uniform limit of a sequence of elements in M. Therefore, $f_x \in M$. We set $D_x^M := D + \mathrm{div}(f_x)$. It follows from the definition that $\mathrm{slope}_\nu f_x(x) = s_0^\nu$ for all $\nu \in \mathrm{T}_x(\Gamma)$, and $D_x^M(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu$. Therefore, the analogue of Lemma 4.2.3 holds.

Augmented metric graphs

An *augmented metric graph* is a metric graph Γ endowed with a model $(G = (V, E), \ell)$ and a genus function $\mathfrak{g} : V \to \mathbb{Z}_{\geqslant 0}$. The *genus* of $(\Gamma, \mathfrak{g})$, denoted by $g(\Gamma, \mathfrak{g})$ or simply g, is defined by

$$g(\Gamma, \mathfrak{g}) := g(\Gamma) + \sum_{v \in V} \mathfrak{g}(v).$$

This terminology follows [ABBR15a]; "vertex-weighted graph" is used in other places. Augmented metric graphs arise from the semistable reduction of smooth proper curves

over a valued field, when remembering the genera $\mathfrak{g}(v) = g(X_v)$ of the components X_v, for $v \in V$.

Note that any metric graph is naturally an augmented metric graph, by declaring the genus function to be the zero function. This means that what we will discuss below applies equally to the setting of non-augmented metric graphs.

Weierstrass locus

We now extend the notion of tropical Weierstrass locus to semimodules in the general setting of augmented metric graphs. Let $(\Gamma, \mathfrak{g})$ be an augmented metric graph. Let D be a divisor on Γ and M be a closed sub-semimodule of $\mathrm{Rat}(D)$ of divisorial rank $r \leqslant r(D)$.

Definition 4.4.6 (Tropical Weierstrass locus of a closed semimodule)**.** The *tropical Weierstrass locus* of M, denoted by $L_W(M, D, \mathfrak{g})$ (or $L_W(M, \mathfrak{g})$ if D is clear from the context), is the set of all points $x \in \Gamma$ which verify $D_x^M(x) + (\mathfrak{g}(x) - 1)\, r > 0$.

In the case the genus function $\mathfrak{g}$ is zero, we lighten the notations and simply write $L_W(M, D)$, instead of $L_W(M, D, 0)$. We abbreviate $L_W(M, D)$ as $L_W(M)$ if D is clear from context. ◇

The set $L_W(M, \mathfrak{g})$ is a closed subset of Γ that can in general be infinite. Note that for every $x \in \Gamma$, we have $D_x^M(x) \geqslant r$ and therefore $D_x^M(x) + (\mathfrak{g}(x) - 1)\, r \geqslant \mathfrak{g}(x)\, r \geqslant 0$. In particular, if $\mathfrak{g}(x) > 0$ and $r > 0$, then x belongs to the tropical Weierstrass locus.

We now associate an intrinsic weight to each connected component of the Weierstrass locus. The definition is analogous to Definition 4.1.6; here it is adapted to semimodules and depends on the genus function.

Let D be a divisor of degree d on Γ, and let $M \subseteq \mathrm{Rat}(D)$ be a closed sub-semimodule of divisorial rank r. We use the notations of Definition 4.1.6 for $\deg\left(D|_C\right)$, $g(C)$, and $\partial^{\mathrm{out}} C$; $s_0^\nu(M)$ is introduced in Definition 4.4.4.

Definition 4.4.7 (Intrinsic Weierstrass weight of a connected component)**.** Let C be a connected component of the tropical Weierstrass locus $L_W(M, \mathfrak{g})$. The *Weierstrass weight* of C, denoted by $\mu_W(C; M, D, \mathfrak{g})$, is defined by

$$\mu_W(C; M, D, \mathfrak{g}) := \deg\left(D|_C\right) + \left(g(C) + \sum_{x \in C} \mathfrak{g}(x) - 1\right) r - \sum_{\nu \in \partial^{\mathrm{out}} C} s_0^\nu(M). \tag{4.4}$$

It is also denoted simply by $\mu_W(C; M, \mathfrak{g})$ or $\mu_W(C; \mathfrak{g})$ if M and D are understood from the context.

In the case the genus function is zero, we use $\mu_W(C; M, D)$, $\mu_W(C; M)$ or $\mu_W(C)$ for $\mu_W(C; M, D, 0)$. ◇

This quantity is well-defined because any connected component of $L_W(M, \mathfrak{g})$ is a metric graph, a result that adapts directly from Proposition 4.3.1. As in the case of divisors (Proposition 4.3.1), $L_W(M, \mathfrak{g})$ has a finite number of connected components. And since

Theorem 4.3.6 extends directly, we get $\mu_W(C; M, \mathfrak{g}) > 0$. We denote by $g(C, \mathfrak{g})$ the sum $g(C) + \sum_{x \in C} \mathfrak{g}(x)$, that is, the genus of C in the augmented metric graph $(\Gamma, \mathfrak{g})$.

Definition 4.4.8 (Tropical Weierstrass divisor)**.** We say that $(M, D, \mathfrak{g})$ is *Weierstrass finite* or simply *W-finite* if the tropical Weierstrass locus $L_W(M, D, \mathfrak{g})$ is finite. In this case, we define the *tropical Weierstrass divisor* $W(M, D, \mathfrak{g})$ as the effective divisor

$$W(M, D, \mathfrak{g}) \coloneqq \sum_{x \in L_W(M, \mathfrak{g})} \mu_W(x; M, D, \mathfrak{g})\,(x).$$

The tropical weight of x verifies $\mu_W(x; M, D, \mathfrak{g}) = D_x^M(x) + (\mathfrak{g}(x) - 1)\, r$. We abbreviate $W(M, D, \mathfrak{g})$ as $W(M, \mathfrak{g})$ if D is clear from the context. Note that the support $|(W(M, \mathfrak{g}))|$ of the tropical Weierstrass divisor is exactly the tropical Weierstrass locus $L_W(M, \mathfrak{g})$.

In the case the genus function is zero, we simply use $W(M, D)$ or $W(M)$ for $W(M, D, 0)$. ◇

Remark 4.4.9. If we set $M = \mathrm{Rat}(D)$, and if the genus function is $\mathfrak{g} = 0$, then we recover the definitions given in Section 4.3 for a complete linear series on a non-augmented metric graph. Namely,

(i) For every $x \in \Gamma$, we have $D_x^{\mathrm{Rat}(D)} = D_x$.

(ii) We have $L_W(\mathrm{Rat}(D), 0) = L_W(D)$.

(iii) For every connected component C of $L_W(\mathrm{Rat}(D), 0)$, we have

$$\mu_W(C; \mathrm{Rat}(D), 0) = \mu_W(C; D).$$

(iv) D is W-finite if, and only if, $\mathrm{Rat}(D)$ is so. In this case, $W(\mathrm{Rat}(D), 0) = W(D)$. ◇

The following proposition, a direct consequence of the definitions, states how the Weierstrass locus and Weierstrass weights on an augmented graph are related to the non-augmented definition.

Proposition 4.4.10. *If $M \subseteq \mathrm{Rat}(D)$ is a closed semimodule of rank r, then the following equalities hold.*

(a) $L_W(M, \mathfrak{g}) = L_W(M) \cup |\mathfrak{g}|$.

(b) For every connected component C of $L_W(M, \mathfrak{g})$, we have

$$\mu_W(C; M, D, \mathfrak{g}) = \mu_W(C; M, D) + r \sum_{x \in C} \mathfrak{g}(x).$$

Total sum of Weierstrass weights

The following theorem is an analogue of Theorem 4.1.9 for closed sub-semimodules of $\mathrm{Rat}(D)$, and is proved using a natural analogue of Lemma 4.2.3, given in Section 4.4.1. The only difference is that in the case of semimodules, sets of slopes are no longer necessarily made up of consecutive integers.

Theorem 4.4.11. *Let D be a divisor on Γ and M be a closed sub-semimodule of $\mathrm{Rat}(D)$ of divisorial rank r. We take a model for $(\Gamma, \mathfrak{g})$ such that the support of D is made up of vertices. Let $x \in \Gamma$ be a point and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction.*

(a) If the open interval $(x, x+\varepsilon\nu)$ is disjoint from the Weierstrass locus $L_W(M, \mathfrak{g})$, for $\varepsilon > 0$, then the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in M\}$ consists of $r+1$ consecutive integers $\{s_0^\nu, s_0^\nu + 1, \dots, s_0^\nu + r\}$.

(b) If the open interval $(x, x+\varepsilon\nu)$ is contained in the Weierstrass locus $L_W(M, \mathfrak{g})$, then the set of slopes $\{\mathrm{slope}_\nu f(x) : f \in M\}$ consists of integers $\{s_0^\nu < s_1^\nu < \cdots < s_t^\nu\}$ with $t \geqslant r$ and $s_t^\nu - s_0^\nu \geqslant r+1$.

Part (a) implies in particular that for any edge e outside the Weierstrass locus of M, the number of slopes of functions on e is $r+1$ and these slopes are consecutive.

As a corollary, following the same computation as in the case of a divisor, we get an analogue of Theorem 4.3.9.

Theorem 4.4.12 (Sum of Weierstrass weights for an incomplete series on an augmented metric graph)**.** *Suppose $(\Gamma, \mathfrak{g})$ is a genus $g = g(\Gamma, \mathfrak{g})$ augmented metric graph, D is a degree d divisor, and $M \subseteq \mathrm{Rat}(D)$ is a closed semimodule of divisorial rank $r \geqslant 0$.*

Then, the total sum of weights associated to connected components of $L_W(M, \mathfrak{g})$ is equal to $d - r + rg$. In particular, if M is W-finite, then we have $\deg(W(M, \mathfrak{g})) = d - r + rg$.

More generally, let $\mathcal{A}$ denote the σ-algebra of $L_W(M, \mathfrak{g})$-measurable subsets of Γ and $\hat{\mu}_W$ the counting measure on $(\Gamma, \mathcal{A})$ associated to the weights $\mu_W(C; M, \mathfrak{g})$ given as above. Then, for any closed, connected $A \in \mathcal{A}$, we have

$$\hat{\mu}_W(A; M, \mathfrak{g}) = \deg\left(D|_A\right) + (g(A, \mathfrak{g}) - 1)r - \sum_{\nu \in \partial^{\mathrm{out}} A} s_0^\nu(M), \tag{4.5}$$

where $g(A, \mathfrak{g})$ denotes $g(A) + \sum_{x \in A} \mathfrak{g}(x)$.

Theorem 4.4.12 implies the following analogue of Theorem 4.1.8.

Theorem 4.4.13. *If the divisorial rank r of M is at least one, then every closed connected subset A of Γ with $g(A, \mathfrak{g}) \geqslant 1$ contains a point of $L_W(M, \mathfrak{g})$.*

Proof. Theorem 4.4.12 and Proposition 4.4.5 imply that for any closed, connected, $L_W(M, \mathfrak{g})$-measurable subset $A \subseteq \Gamma$, we have

$$\hat{\mu}_W(A; M, \mathfrak{g}) \geqslant g(A, \mathfrak{g})\, r,$$

an analogue of Corollary 4.3.13 for closed semimodules. Then, the argument used in the proof of Theorem 4.1.8 yields the result. □

Coherence under inclusion of semimodules

We have the following coherence property for the Weierstrass loci and weights associated to semimodules.

Proposition 4.4.14. *Let $M \subseteq M' \subseteq \mathrm{Rat}(D)$ be two closed semimodules of rank r. Then, $L_W(M, \mathfrak{g}) \subseteq L_W(M', \mathfrak{g})$ and any $L_W(M', \mathfrak{g})$-measurable subset A of Γ is $L_W(M, \mathfrak{g})$-measurable. Moreover, the equality $\hat{\mu}_W(A; M, \mathfrak{g}) = \hat{\mu}_W(A; M', \mathfrak{g})$ holds.*

Proof. Note that the inclusion $M \subseteq M'$ implies that we have $D_y^M(y) \leqslant D_y^{M'}(y)$ for every $y \in \Gamma$. This, in turn, implies that $L_W(M, \mathfrak{g}) \subseteq L_W(M', \mathfrak{g})$. The claim that A is $L_W(M, \mathfrak{g})$-measurable follows then, since A is assumed to be $L_W(M', \mathfrak{g})$-measurable.

To see that $\hat{\mu}_W(A; M, \mathfrak{g}) = \hat{\mu}_W(A; M', \mathfrak{g})$, it suffices to show that $s_0^\nu(M) = s_0^\nu(M')$ for each $\nu \in \partial^{\mathrm{out}} A$. Suppose ν is such a tangent direction pointing out of A. By part (a) of Theorem 4.4.11, there are exactly $r+1$ consecutive slopes of functions $F \in M'$ along ν. The same statement holds for M. Since $M \subseteq M'$, we infer that these slopes are the same. In particular, $s_0^\nu(M) = s_0^\nu(M')$, as desired. □

In the following two sections, we specialize the above constructions to two special families of closed semimodules M: the generic semimodule associated to any divisor D, and the canonical semimodule.

4.4.2 The generic semimodule associated to a divisor

Let $(\Gamma, \mathfrak{g})$ be an augmented metric graph. Denote by $|\mathfrak{g}|$ the support of $\mathfrak{g}$. For any divisor D on Γ, we define a closed semimodule $\mathrm{Rat}^{\mathrm{gen}}(D, \mathfrak{g}) \subseteq \mathrm{Rat}(D)$.

Definition 4.4.15. The *generic linear series* or *generic semimodule* $\mathrm{Rat}^{\mathrm{gen}}(D, \mathfrak{g})$ consists of all rational functions f on Γ such that for every $x \in \Gamma$, we have the inequality

$$D(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x). \qquad \diamond$$

Equivalently, we have the equality $\mathrm{Rat}^{\mathrm{gen}}(D, \mathfrak{g}) = \mathrm{Rat}(D_0)$ for the divisor D_0 defined by $D_0(x) := D(x) - \mathfrak{g}(x)$, for every $x \in \Gamma$. (The claimed containment $\mathrm{Rat}^{\mathrm{gen}}(D, \mathfrak{g}) \subseteq \mathrm{Rat}(D)$ is clear.)

It follows that $\mathrm{Rat}^{\mathrm{gen}}(D, \mathfrak{g})$ is closed in the $\|\cdot\|_\infty$ topology of $\mathrm{Rat}(D)$.

Remark 4.4.16. The superscript "gen" stands for "generic" because, from the viewpoint of the degeneration of smooth projective curves, augmented metric graphs can be obtained from intermediate geometric objects called metrized complexes of curves. If this is the case, the above definition gives, precisely, the tropical part of the linear series of a divisor on the metrized complex in the case where the restriction of the divisor on every curve component of the metrized complex is generic. See Section 4.4.4 for more details. $\diamond$

The following statement computes the divisorial rank of the generic semimodule associated to a divisor.

Proposition 4.4.17. *Denote by r the divisorial rank of the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$, and let $r(D)$ and $r(D_0)$ denote the respective ranks of the two divisors D and D_0 in Γ without the genus function. We have the following (in)equalities.*

(a) $r \leqslant r(D)$;

(b) $r = r(D_0)$.

Proof. (a) The inequality follows from the containment $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) \subseteq \mathrm{Rat}(D)$.
(b) This follows from Proposition 4.4.2 applied to $M := \mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) = \mathrm{Rat}(D_0)$. □

Now that we have a closed sub-semimodule $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ of $\mathrm{Rat}(D)$ with a well-known divisorial rank, we can apply the machinery developed above.

Definition 4.4.18 (Generic tropical Weierstrass weights and locus of a divisor)**.** Notations as above, let D be a divisor on an augmented metric graph $(\Gamma,\mathfrak{g})$. The tropical Weierstrass locus, the Weierstrass weights, and the Weierstrass divisor (if it exists) are defined by plugging the semimodule $M := \mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g})$ into Definitions 4.4.6, 4.4.7 and 4.4.8.

To lighten the notations while stressing the choice of the generic semimodule and the dependence on D and $\mathfrak{g}$, we write:

(i) $L_W^{\mathrm{gen}}(D,\mathfrak{g})$ for $L_W\big(\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\big)$;

(ii) $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g})$ for $\mu_W\big(C;\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\big)$; and

(iii) $W^{\mathrm{gen}}(D,\mathfrak{g})$ for $W\big(\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}),\mathfrak{g}\big)$.

When D is clear from context, we simply use $\mu_W^{\mathrm{gen}}(C;\mathfrak{g})$ for $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g})$. ◇

Note that when $\mathfrak{g}$ is the zero function, we have the equality $\mathrm{Rat}^{\mathrm{gen}}(D,\mathfrak{g}) = \mathrm{Rat}(D)$, and so the above definition recovers the one given in the previous sections for the Weierstrass divisor associated to a divisor.

Proposition 4.4.10 and a straightforward computation gives the following description of the generic Weierstrass locus.

Proposition 4.4.19. *The following equalities hold:*

(a) $L_W^{\mathrm{gen}}(D,\mathfrak{g}) = L_W(D_0) \cup |\mathfrak{g}|$;

(b) $\mu_W^{\mathrm{gen}}(C;D,\mathfrak{g}) = \mu_W(C;D_0) + (r+1)\sum_{x\in C}\mathfrak{g}(x)$.

In the remainder of this section, we discuss the generic semimodule associated to the canonical divisor. We first recall the definition of the canonical divisor in the augmented setting.

Definition 4.4.20 (Canonical divisor on an augmented metric graph)**.** Given an augmented metric graph $(\Gamma, \mathfrak{g})$, the *canonical divisor* K on $(\Gamma, \mathfrak{g})$ is defined by

$$K(x) \coloneqq \operatorname{val}(x) - 2 + 2\,\mathfrak{g}(x) \tag{4.6}$$

for each $x \in \Gamma$. ◇

Remark 4.4.21. In the context of augmented metric graphs, Lemma 4.3.16 becomes the following statement: for every closed connected subset $A \subseteq \Gamma$,

$$\deg\left(K|_A\right) = 2g(A) - 2 + 2\sum_{x\in A} \mathfrak{g}(x) + \operatorname{outval}(A). \qquad \diamond$$

The following statement gives the rank of the semimodule $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$, which is not $g-1$ as one might expect.

Proposition 4.4.22 (Rank of the generic semimodule $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$)**.** *If the genus function* $\mathfrak{g}$ *is nontrivial, the semimodule* $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$ *has rank* $g-2$.

Proof. The rank of $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$ coincides with the rank of $K_0 \coloneqq K - \sum \mathfrak{g}(x)\,(x)$ within the non-augmented metric graph Γ. Since the genus function is nontrivial, we have $\deg(K_0) = 2\,g(\Gamma) - 2 + \sum_x \mathfrak{g}(x) > 2\,g(\Gamma) - 2$ with $g(\Gamma)$ the genus of the non-augmented metric graph, and so, by Riemann–Roch on Γ, we have $r(K_0) = \deg(K_0) - g(\Gamma) = g(\Gamma, \mathfrak{g}) - 2$. □

In the next section, we define the canonical linear series for an augmented metric graph, and show it has the correct rank $g-1$.

Example 4.4.23. We compute the Weierstrass locus of the generic semimodule $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$ on a cycle with one point of positive genus equal to two.

Let $(\Gamma, \mathfrak{g})$ be the augmented metric graph where Γ is the cycle of length one, parametrized by the interval $[0, 1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = 2$. The genus of this augmented metric graph is $g = 3$.

We consider the canonical divisor K and the associated generic semimodule $\operatorname{Rat}^{\text{gen}}(K, \mathfrak{g})$, as defined in the present section (see Definition 4.4.15). The rank is $r = g - 2 = 1$ according to Proposition 4.4.22, and the total weight of the Weierstrass locus is 6. The Weierstrass locus consists of the vertex v and the point of coordinate $\frac{1}{2}$. It is easy to compute that the weights are $\mu_W^{\text{gen}}(v; K, \mathfrak{g}) = 5$ and $\mu_W^{\text{gen}}\left(\frac{1}{2}; K, \mathfrak{g}\right) = 1$. Figure 4.5 shows the augmented metric graph and its Weierstrass locus. A generalization for any value of $\mathfrak{g}(v)$ is presented in Section 4.6.6. ◇

4.4.3 The canonical linear series on an augmented metric graph

Consider the augmented metric graph $(\Gamma, \mathfrak{g})$ and its canonical divisor K, defined by $K(x) = \operatorname{val}(x) - 2 + 2\,\mathfrak{g}(x)$ for each $x \in \Gamma$. In this section, we define the linear series $\operatorname{KRat}(\mathfrak{g})$ associated to K, that we call the *canonical linear series* or *canonical semimodule*.

Figure 4.5: An augmented cycle graph with one point of genus two, the canonical divisor and its Weierstrass locus $L_W^{\text{gen}}(D, \mathfrak{g})$.

Definition 4.4.24. We define the *canonical semimodule* $\mathrm{KRat}(\mathfrak{g})$ as the set of all functions $f \in \mathrm{Rat}(\Gamma)$ which verify the following conditions:

(1) For every $x \in \Gamma$, we have $K(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x) - 1$.

(2) If x has a tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ such that $\mathrm{slope}_\nu f(x) \leqslant 0$, then $K(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x)$. ◇

The following set of conditions is equivalent to that of Definition 4.4.24.

(1) (local-minimum condition) If $x \in \Gamma$ is an *isolated local minimum* of f, i.e., $\mathrm{slope}_\nu f(x) \geqslant 1$ for every $\nu \in \mathrm{T}_x(\Gamma)$, then we impose $K(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x) - 1$.

(2) (generic condition) For all other points $x \in \Gamma$, we impose the stricter condition $K(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x)$.

Note that according to the above definition, if a point x has $\mathfrak{g}(x) = 0$, then x cannot be an isolated local minimum of $f \in \mathrm{KRat}(\mathfrak{g})$. Indeed, an isolated local minimum of f satisfies $\mathrm{div}(f)(x) \leqslant -\mathrm{val}(x)$, and so $K(x) + \mathrm{div}(f)(x) \leqslant -2$ assuming $\mathfrak{g}(x) = 0$, which would violate both conditions. This means that, for any $x \in \Gamma$ and $f \in \mathrm{KRat}(\mathfrak{g})$, we have $K(x) + \mathrm{div}(f)(x) \geqslant 0$, which implies that $\mathrm{KRat}(\mathfrak{g})$ is a subset of $\mathrm{Rat}(K)$. (It is easy to see that it is in fact a semimodule.) This shows, moreover, that the above definition is equivalent to Definition 4.4.15 outside of the support of $\mathfrak{g}$. Also note that we have the inclusion of semimodules $\mathrm{Rat}^{\text{gen}}(K, \mathfrak{g}) \subseteq \mathrm{KRat}(\mathfrak{g})$.

Remark 4.4.25. The definition of the canonical semimodule differs from the generic semimodule $\mathrm{Rat}^{\text{gen}}(K, \mathfrak{g})$ given by Definition 4.4.15. This is because the earlier definition, suitable for every divisor D on Γ, assumed D has "generic support" in the vertices with "hidden genus." The canonical divisor, however, is not generic. Its specific properties suggest a distinct definition for the complete linear series of K. The relevance of the above modification compared to Definition 4.4.15 will be further clarified in Section 4.4.4. ◇

We have the following theorem which justifies the name given to the linear series $\mathrm{KRat}(\mathfrak{g})$. Recall that $g = g(\Gamma, \mathfrak{g})$.

Theorem 4.4.26. *The divisorial rank of the semimodule* $\mathrm{KRat}(\mathfrak{g})$ *is* $g - 1$.

Proof. The proof of this theorem will be given in Section 4.4.4. □

We have a closed sub-semimodule $\mathrm{KRat}(\mathfrak{g})$ of $\mathrm{Rat}(K)$ of divisorial rank $r = g - 1$, and we can apply the machinery developed for semimodules on augmented metric graphs.

Definition 4.4.27 (Canonical tropical Weierstrass weights and locus)**.** Notations as above, the canonical tropical Weierstrass locus, the Weierstrass weights, and the Weierstrass divisor on an augmented metric graph are defined by plugging the semimodule $M := \mathrm{KRat}(\mathfrak{g})$ into Definitions 4.4.6, 4.4.7 and 4.4.8.

To lighten the notations while stressing the choice of the canonical semimodule and the dependence on $\mathfrak{g}$, we write:

(i) $L_W(K, \mathfrak{g})$ for $L_W(\mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$;

(ii) $\mu_W(C; K, \mathfrak{g})$ for $\mu_W(C; \mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$; and

(iii) $W(K, \mathfrak{g})$ for $W(\mathrm{KRat}(\mathfrak{g}), \mathfrak{g})$. ◇

Example 4.4.28. In this example, we compute the canonical Weierstrass locus on an augmented cycle with a point of genus two. For the case of the generic Weierstrass locus associated to the same divisor K, see Example 4.4.23.

Let $(\Gamma, \mathfrak{g})$ be the augmented metric graph where Γ is the cycle of length one, parametrized by the interval $[0, 1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = 2$. The genus of this augmented metric graph is $g = 3$.

We consider the canonical divisor K and the associated canonical semimodule $\mathrm{KRat}(\mathfrak{g})$, as defined in the present section (see Definition 4.4.24). The rank is $r = g - 1 = 2$ according to Theorem 4.4.26, and the total weight of the Weierstrass locus is $g^2 - 1 = 8$. The Weierstrass locus consists of the vertex v and the points of coordinates $\frac{1}{3}$ and $\frac{2}{3}$. The Weierstrass weights are $\mu_W(v; K, \mathfrak{g}) = 6$ and $\mu_W\left(\frac{1}{3}; K, \mathfrak{g}\right) = \mu_W\left(\frac{2}{3}; K, \mathfrak{g}\right) = 1$. Figure 4.6 shows the locus of Weierstrass points. A generalization for any value of $\mathfrak{g}(v)$ is presented in Section 4.6.6. ◇

Figure 4.6: An augmented cycle graph, the canonical divisor and its Weierstrass locus $L_W(K, \mathfrak{g})$.

In the rest of the chapter, when handling the canonical divisor K on an augmented metric graph, the semimodule $\mathrm{KRat}(\mathfrak{g})$ will be preferred over $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$, unless explicitly specified otherwise.

4.4.4 Justification of the definition of Weierstrass loci for augmented metric graphs, in the generic and canonical case

In this section, we provide a justification for the definitions we gave in Sections 4.4.2 and 4.4.3. This will be through divisor theory on metrized complexes, that we recall first. A purely metric graph justification, using metric graphs with shrinking parts, is sketched in Remark 4.4.34.

Divisor theory on a metrized complex of curves

We fix κ an algebraically closed field. A metrized complex of curves is, roughly speaking, the (metric realization of the) data of an augmented metric graph $(\Gamma, \mathfrak{g})$ endowed with a model $G = (V, E)$ and, for every $v \in V$, of a smooth, proper, connected, marked κ-curve $\mathcal{C}_v$ of genus $\mathfrak{g}(v)$ with marked points A_v in bijection with the branches of Γ incident to v. That is, a metrized complex of curves is a hybrid refinement of an augmented metric graph. For a full definition, see [AB15, Definition 2.17].

Let $\mathfrak{C}$ be a metrized complex of curves. A *divisor* $\mathfrak{D}$ on $\mathfrak{C}$ is a formal sum with integer coefficients of a finite number of points in $\mathfrak{C}$. We denote its hybrid rank on $\mathfrak{C}$ by $r_{\mathfrak{C}}(\mathfrak{D})$. By the forgetful projection map from $\mathfrak{C}$ to Γ, this gives rise to a divisor D on Γ of the same degree. Moreover, by restriction to each curve $\mathcal{C}_v$, for $v \in V$, we get a divisor $\mathcal{D}_v$ on $\mathcal{C}_v$. A rational function $\mathfrak{f}$ on $\mathfrak{C}$ consists of a rational function f on Γ and, for every $v \in V$, a nonzero rational function f_v on $\mathcal{C}_v$. The space of such functions $\mathfrak{f} = (f, f_v : v \in V)$ is denoted by $\mathrm{Rat}(\mathfrak{C})$.

Let now $\mathfrak{C}$ be a metrized complex of curves, with underlying metric graph Γ. Let $\mathfrak{D}$ be a divisor on $\mathfrak{C}$. We follow [AB15] and consider the linear series $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$ defined as the subset of $\mathrm{Rat}(\mathfrak{C})$ consisting of all rational functions $\mathfrak{f} = (f \in \mathrm{Rat}(\Gamma); f_v \in \kappa(\mathcal{C}_v), v \in V)$ on $\mathfrak{C}$ that verify $\mathfrak{D} + \mathrm{div}(\mathfrak{f}) \geqslant 0$. This means that $D + \mathrm{div}(f)$ is effective on Γ, and for each $v \in V$, the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{slope}_\nu f(v)\,(x_v^\nu) + \mathrm{div}(f_v)$ is effective on $\mathcal{C}_v$. Here, x_v^ν is the marked point on $\mathcal{C}_v$ that corresponds to ν.

Definition 4.4.29. We define $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$ to be the subset of $\mathrm{Rat}(D)$ consisting of the tropical parts of all functions $\mathfrak{f} \in \mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$. ◇

We omit the proof of the following result.

Proposition 4.4.30. $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$ *is a closed sub-semimodule of* $\mathrm{Rat}(D)$.

We have the following comparison result, whose proof is direct from the definition of rank of divisors.

Proposition 4.4.31. *Let* $r(D, \mathfrak{C})$ *be the divisorial rank of the semimodule* $\mathrm{Rat}^{\mathrm{trop}}(D, \mathfrak{C})$. *Then we have the inequality*

$$r_{\mathfrak{C}}(\mathfrak{D}) \leqslant r(D, \mathfrak{C}).$$

The inequality in the above proposition can be strict in general. However, in some situations, e.g., for generic divisors on $\mathfrak{C}$ and for the canonical divisor, when the marked curves (C_v, A_v), for all $v \in V$, are generic in their moduli, we have the equality, as we explain now.

The case of augmented metric graphs with generic divisors

The second condition in the definition of $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$ in the previous section justifies Definition 4.4.15. Indeed, take a rational function f on Γ such that for every $x \in \Gamma$, we have $D(x) + \mathrm{div}(f)(x) \geqslant \mathfrak{g}(x)$. Assume that the augmented metric graph $(\Gamma, \mathfrak{g})$ comes from a metrized curve complex $\mathfrak{C}$. Let $v \in \Gamma$ be a point underlying a curve $\mathcal{C}_v$. On the curve $\mathcal{C}_v$, the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu f(v)\,(x_v^\nu)$ has degree $\geqslant \mathfrak{g}(v)$ by assumption. Therefore, by the Riemann–Roch theorem, its rank is non-negative, which is precisely the second condition in the definition of $\mathrm{Rat}(\mathfrak{D}, \mathfrak{C})$. Now, in the other direction, if $\mathcal{D}_v$ is generic in the Picard group of $\mathcal{C}_v$ of relevant degree, then the divisor $\mathcal{D}_v - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{slope}_\nu f(v)\,(x_v^\nu)$ on $\mathcal{C}_v$ appearing in the second condition has non-negative rank only if it has degree at least $\mathfrak{g}(v)$. This means that Definition 4.4.15 is equivalent to the definition given for metrized complexes with a generic choice of divisors on components.

The case of canonical divisor in augmented metric graphs

We now justify Definition 4.4.24 using the terminology of Section 4.4.4, and also prove Theorem 4.4.26.

Let $G = (V, E)$ be a model of Γ whose vertex set contains all the points of valence different from two, and the support of $\mathfrak{g}$. Let $\mathfrak{C}$ be a metrized complex with underlying augmented metric graph $(\Gamma, \mathfrak{g})$. Denote by $\mathfrak{K}$ a canonical divisor for $\mathfrak{C}$ given by the collection of divisors $\mathcal{K}_{\mathcal{C}_v} + A_v = \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu)$ on $\mathcal{C}_v$, where $\mathcal{K}_{\mathcal{C}_v}$ denotes a canonical divisor on $\mathcal{C}_v$, i.e., $\mathcal{O}(\mathcal{K}_{\mathcal{C}_v}) = \omega_{\mathcal{C}_v}$. The following claim justifies our definition of the canonical semimodule. We denote by $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ the tropical part of $\mathrm{Rat}(\mathfrak{K})$.

Proposition 4.4.32. *Notations as above, we have* $\mathrm{KRat}(\mathfrak{g}) \subseteq \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. *Moreover, if the markings A_v on the curves $\mathcal{C}_v$ are in general position, for all $v \in V$, then we have the equality* $\mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$.

Remark 4.4.33. This general position condition is the same as the one imposed in the work by Esteves and coauthors [EM02, ES07] in the special case of stable curves with two irreducible components, and stable curves in which any pair of components intersect. We will treat examples of augmented dipole graphs in Section 4.6.

The results in this special case can be viewed as complementing from the tropical perspective the work by Esteves and Medeiros [EM02], by analyzing the proportion and precise locus of points specializing to the nodes on considered in their paper. ◇

Proof of Proposition 4.4.32. We first prove the inclusion $\mathrm{KRat}(\mathfrak{g}) \subseteq \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. Consider an element $f \in \mathrm{KRat}(\mathfrak{g})$. We claim the existence of rational functions f_v on $\mathcal{C}_v$, for each

$v \in V$, such that the collection $\{f, f_v, v \in V\}$ forms a rational function in $\mathrm{Rat}(\mathfrak{K})$. This proves the claim. Let $v \in V$, and consider the divisor $\mathcal{D}$ on $\mathcal{C}_v$ defined by f as follows:

$$\mathcal{D}_v := \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu) - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{slope}_\nu f(v)\,(x_v^\nu).$$

Note that the degree of $\mathcal{D}_v$ is precisely $K(v) + \mathrm{div}(f)(v)$. If the genus of v is zero, then by the condition $K(v) + \mathrm{div}(f) \geqslant 0$, the degree of $\mathcal{D}_v$ is non-negative and so there exists a rational function f_v on $\mathcal{C}_v$ such that $\mathcal{D}_v + \mathrm{div}(f_v) \geqslant 0$. If $\mathfrak{g}(v) \geqslant 1$ and v is not an isolated local minimum, then by the definition of $\mathrm{KRat}(\mathfrak{g})$, we have $\deg(\mathcal{D}_v) \geqslant \mathfrak{g}(v)$. By Riemann–Roch, this implies the existence of a function f_v such that $\mathcal{D}_v + \mathrm{div}(f_v) \geqslant 0$. Let $v \in \Gamma$ be a vertex of Γ such that $\mathfrak{g}(v) > 0$ and which is an isolated local minimum of f. In this case, by the definition of $\mathrm{KRat}(\mathfrak{g})$, we have $\deg(\mathcal{D}_v) \geqslant \mathfrak{g}(v) - 1$. The divisor $\mathcal{D}_v$ can be rewritten as $\mathcal{K}_{\mathcal{C}_v} - E$, where

$$E := \sum_{\nu \in \mathrm{T}_v(\Gamma)} (\mathrm{slope}_\nu f(v) - 1)\,(x_v^\nu)$$

is effective because v is an isolated local minimum of f. The Riemann–Roch theorem on $\mathcal{C}_v$, combined with the inequality $r(E) \geqslant 0$, thus yields

$$r(\mathcal{D}_v) = r(\mathcal{K}_{\mathcal{C}_v} - E) = r(E) + \deg(\mathcal{D}_v) - \mathfrak{g}(v) + 1 \geqslant 0.$$

That is, there exists a function f_v such that $\mathcal{D}_v + \mathrm{div}(f_v) \geqslant 0$. The rational function $\mathfrak{f} = (f, f_v : v \in V)$ on $\mathfrak{C}$ verifies $\mathfrak{K} + \mathrm{div}(\mathfrak{f}) \geqslant 0$, as desired.

We now prove the inclusion $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} \subseteq \mathrm{KRat}(\mathfrak{g})$ provided that the markings A_v on the curves $\mathcal{C}_v$, for all $v \in V$, are generic. First, we observe that $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} \subseteq \mathrm{Rat}(K)$. Combining this with the results we proved in Section 4.2, it follows that the slopes taken by functions in $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ are bounded. Let f be an element of $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. We claim that under the general position assumption, we have $f \in \mathrm{KRat}(\mathfrak{g})$. Let v be a vertex of Γ. Resuming the notations introduced above, we write $\mathcal{D}_v$ for the divisor on $\mathcal{C}_v$ induced by f, and write it in the form $\mathcal{D}_v = \mathcal{K}_{\mathcal{C}_v} - E$.

First consider the case where v is an isolated local minimum of f. In this case, E is an effective divisor. We need to show that $\deg(E) \leqslant \mathfrak{g}(v) - 1$. Indeed, otherwise, if $\deg(E) \geqslant \mathfrak{g}(v)$, then if the points x_v^ν, for $\nu \in \mathrm{T}_v(\Gamma)$, are in general position on $\mathcal{C}_v$, we will get $r(\mathcal{D}_v) \leqslant r(\mathcal{K}_{\mathcal{C}_v}) - \mathfrak{g}(v) = -1$, which contradicts the assumption that $f \in \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$.

Consider the other case, where v is not an isolated minimum. In this case, the divisor E is not effective. We write $E = E_+ - E_-$ where E_+ and E_- are the positive and negative parts of E, respectively. Note that E_+ and E_- are effective and they have disjoint support. Since E is not effective, E_- is non-zero, and so by Riemann–Roch, we have

$$r(\mathcal{K}_{\mathcal{C}_v} + E_-) = 2\mathfrak{g}(v) - 2 + \deg(E_-) - \mathfrak{g}(v) = \mathfrak{g}(v) - 2 + \deg(E_-).$$

Now, we write

$$\mathcal{D}_v = \mathcal{K}_{\mathcal{C}_v} - E = \mathcal{K}_{\mathcal{C}_v} + E_- - E_+$$

and observe, by the general position assumption on the points of A_v, that

$$r(\mathcal{D}_v) = \max\{-1, r(\mathcal{K}_{\mathcal{C}_v} + E_-) - \deg(E_+)\}.$$

Combining the two observations, we get

$$\begin{aligned} r(\mathcal{D}_v) = \max\{-1, \mathfrak{g}(v) - 2 + \deg(E_-) - \deg(E_+)\} &= \max\{-1, \deg(\mathcal{K}_{\mathcal{C}_v} - E) - \mathfrak{g}(v)\} \\ &= \max\{-1, \deg(\mathcal{D}_v) - \mathfrak{g}(v)\}. \end{aligned}$$

If $\deg(\mathcal{D}_v) < \mathfrak{g}(v)$, we get $r(\mathcal{D}_v) < 0$, which would be a contradiction to the assumption that $f \in \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$. We conclude that $\deg(\mathcal{D}_v) \geqslant \mathfrak{g}(v)$, which leads to the inclusion $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} \subseteq \mathrm{KRat}(\mathfrak{g})$. □

We now show that $\mathrm{KRat}(\mathfrak{g})$ has the expected rank $g - 1$.

Proof of Theorem 4.4.26. We keep the notations as above. We denote by r the divisorial rank of $\mathrm{KRat}(\mathfrak{g})$.

It will be enough to show that if the markings A_v on the curves $\mathcal{C}_v$, for all $v \in V$, are in general position, then we have $r_{\mathfrak{C}}(\mathfrak{K}) = r$. By Riemann–Roch for metrized complexes proved in [AB15], we then obtain the equality $r = g - 1$, as desired.

The inequality $r \geqslant r_{\mathfrak{C}}(\mathfrak{K})$ follows from the case of equality $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} = \mathrm{KRat}(\mathfrak{g})$ proved in the previous proposition, and by the definition of the rank in the metrized complex.

It remains to show the inequality $r_{\mathfrak{C}}(\mathfrak{K}) \geqslant r$. Let $\mathcal{E}$ be an effective divisor of degree r on $\mathfrak{C}$, and let E be the corresponding divisor on Γ. There exists a function $f \in \mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ such that $E + \mathrm{div}(f) \geqslant 0$. If E has support outside the vertices of Γ, that is, $\mathcal{E}$ is entirely supported at the interior of edges of Γ, then using the arguments we used in the first part of Proposition 4.4.32, we deduce the existence of rational functions f_v on $\mathcal{C}_v$, for all $v \in V$, such that the rational function $\mathfrak{f} = (f, f_v : v \in V)$ on $\mathfrak{C}$ gives $\mathfrak{K} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geqslant 0$, as desired.

Otherwise, if $\mathcal{E}$ has support in some of the curves $\mathcal{C}_v$, for $v \in V$, we write E as a limit of effective divisors E_n, for $n \geqslant 0$, of the same degree with support outside the vertices of Γ, and find elements f_n in $\mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}} = \mathrm{KRat}(\mathfrak{g})$ which verify $K - E_n + \mathrm{div}(f_n) \geqslant 0$. Going to a subsequence, and using the boundedness of the slopes in $\mathrm{KRat}(\mathfrak{g})$, we can suppose that all the f_n have the same slopes along tangent directions at v, for each vertex $v \in V$. Moreover, changing the function $f \in \mathrm{KRat}(\mathfrak{g}) = \mathrm{Rat}(\mathfrak{K})^{\mathrm{trop}}$ under the constraint that $E + \mathrm{div}(f) \geqslant 0$ if necessary, we can assume furthermore that f_n converges to f as n tends to infinity.

Denote by s_v^ν the slope of the f_n along the tangential direction $\nu \in \mathrm{T}_v(\Gamma)$. Let

$$\mathcal{D}_v := \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (x_v^\nu) - \sum_{\nu \in \mathrm{T}_v(\Gamma)} \mathrm{slope}_\nu f(v)\, (x_v^\nu)$$

that we rewrite in the form

$$\mathcal{D}_v = \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu \in \mathrm{T}_v(\Gamma)} (1 - s_v^\nu)\, (x_v^\nu) + \sum_{\nu} m_\nu\, (x_v^\nu)$$

with m_ν denoting the (weighted) number of points in the support of E_n tending to v through the tangential direction ν. Note that we have $\sum_{\nu\in \mathrm{T}_v(\Gamma)} m_\nu = E(v)$. Let

$$\mathcal{D}'_v := \mathcal{K}_{\mathcal{C}_v} + \sum_{\nu\in \mathrm{T}_v(\Gamma)} (1 - s_v^\nu)\,(x_v^\nu).$$

Two cases can happen. Either, some of the slopes s_v^ν, for $\nu \in \mathrm{T}_v(\Gamma)$, are not positive, that is, v is not an isolated local minimum of f_n. In this case, the divisor $\mathcal{D}'_v$ has degree at least $\mathfrak{g}(v)$, which implies that it has non-negative rank. Or, all the slopes s_v^ν, for $\nu \in \mathrm{T}_v(\Gamma)$, are positive, that is, v is an isolated local minimum of f_n for all n. In this case, the divisor $\mathcal{D}'_v$ has degree at least $\mathfrak{g}(v) - 1$, and is the difference of $\mathcal{K}_{\mathcal{C}_v}$ and an effective divisor on $\mathcal{C}_v$. So again, it has non-negative rank.

In either case, we conclude that the divisor $\mathcal{D}'_v = \mathcal{D}_v - \sum_{\nu\in \mathrm{T}_v(\Gamma)} m_\nu\,(x_v^\nu)$ has non-negative rank. Since the points x_v^ν are assumed to be in general position on $\mathcal{C}_v$, it follows that the divisor $\mathcal{D}_v$ has rank at least $E(v) = \sum_{\nu\in \mathrm{T}_v(\Gamma)} m_\nu$. This shows the existence of a rational function f_v on $\mathcal{C}_v$ such that $\mathcal{D}_v - \mathcal{E}_v + \mathrm{div}(f_v) \geqslant 0$, with $\mathcal{E}_v$ being the part of $\mathcal{E}$ supported in $\mathcal{C}_v$. We conclude with the existence of a rational function $\mathfrak{f} = (f, f_v : v \in V)$ which verifies $\mathfrak{K} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \geqslant 0$. This implies the inequality $r_{\mathfrak{C}}(\mathfrak{K}) \geqslant r$, and finishes the proof of our theorem. □

Remark 4.4.34. Definition 4.4.24 can be also justified using only the formalism of metric graphs and their limits. We briefly discuss this.

Suppose that the augmented metric graph $(\Gamma_0, \mathfrak{g})$ comes from a "limit family" of non-augmented metric graphs in the following sense. Let Γ be a (non-augmented) metric graph and $\Sigma \subseteq \Gamma$ a closed subset, where Σ has connected components $\Sigma_1, \ldots, \Sigma_n$. For each $\varepsilon > 0$, consider the graph Γ_ε defined by shrinking every edge in Σ by the factor ε. As $\varepsilon \to 0$, the family Γ_ε converges to a metric graph Γ_0 (in the sense of Gromov–Hausdorff convergence). The limit metric graph Γ_0 is naturally equipped with a genus function $\mathfrak{g}$ where $\mathfrak{g}(v_i) = g(\Sigma_i)$ for each $v_i \in \Gamma_0$ that is the limit of a component Σ_i, and $\mathfrak{g}(x) = 0$ for all other $x \in \Gamma_0$. In this situation, we say that the augmented metric graph $(\Gamma_0, \mathfrak{g})$ is the limit of the shrinking family of the pair $\Sigma \subseteq \Gamma$.

Now consider the corresponding family of canonical series $\mathrm{Rat}(K_\varepsilon)$ on Γ_ε. For each $\varepsilon > 0$, the linear series $\mathrm{Rat}(K_\varepsilon)$ has rank $g-1$ on Γ_ε. The limit as $\varepsilon \to 0$ produces a semimodule of rational functions on Γ_0. We claim that this limit semimodule always contains $\mathrm{KRat}(\Gamma_0, \mathfrak{g})$, as described in Definition 4.4.24, and that if Σ is "generic" in an appropriate sense, then this limit is equal to $\mathrm{KRat}(\Gamma_0, \mathfrak{g})$.

We omit a proof of these claims here. The details can be verified using the theory of higher rank tropical curves and their algebro-geometric properties developed in [AN22, AN24]. ◇

4.5 Tropical vs. algebraic Weierstrass loci

In the first sections of this chapter, we associated a Weierstrass locus to a fixed divisor D on a metric graph, and then generalized this to a closed sub-semimodule of M the space

$\mathrm{Rat}(D)$ on an augmented metric graph. However, those semimodules which come from tropicalization verify an extra set of properties, in particular, the following important one (see Section 4.8.3):

($\star$) for each point x in Γ and any unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the set of slopes $\mathfrak{S}^\nu(M)$ taken by functions in M has size $r+1$.

In this section, we associate to any pair (M, D) consisting of a divisor D and a closed sub-semimodule $M \subseteq \mathrm{Rat}(D)$ that verifies ($\star$) a refined notion of Weierstrass divisor. It is inspired from the formula given in Theorem 4.8.5, with the slopes being directly retrieved from M using property ($\star$). We then provide a comparison of this definition with that of Section 4.4. Using this link, we prove the main result of this section, Theorem 4.5.5, which relates tropical Weierstrass loci studied in the previous sections to tropicalization of Weierstrass divisors. We deduce then Theorem 4.1.10 as a special case of this result.

In the following, by an abuse of terminology, we refer to any pair (M, D) as above as a combinatorial limit linear series (clls). The terminology is borrowed from [AG22] (Chapter 3), however, the precise definition of combinatorial linear series requires more properties for the semimodule M. In our setting, we only need property ($\star$). The results can be thus applied more generally, in particular to the setting of tropical linear series developed in [JP22].

4.5.1 Weierstrass divisor of a combinatorial limit linear series

Let D be a divisor on an augmented metric graph $(\Gamma, \mathfrak{g})$ and $M \subseteq \mathrm{Rat}(D)$ a closed sub-semimodule that verifies ($\star$). Since $M \subseteq \mathrm{Rat}(D)$ is closed, we can apply the machinery of Section 4.4. This point of view on Weierstrass loci however results in a loss in information provided by the slopes of M, unless the Weierstrass locus is finite. The following definition relies on the knowledge of the slopes along edges of G prescribed by M.

Definition 4.5.1. Suppose D is a divisor of degree d and M is a closed sub-semimodule in $\mathrm{Rat}(D)$ such that M verifies ($\star$). The *clls Weierstrass divisor* of (M, D) is the divisor $W^{\mathrm{clls}}(M, D, \mathfrak{g})$ defined as

$$W^{\mathrm{clls}}(M, D, \mathfrak{g}) := \sum_{x \in \Gamma} \mu_W^{\mathrm{clls}}(x)\,(x)$$

where *the clls Weierstrass weight* $\mu_W^{\mathrm{clls}}(x)$ of x is defined by

$$\mu_W^{\mathrm{clls}}(x) := (r+1)\,D(x) + \frac{r(r+1)}{2}\,(\mathrm{val}(x) + 2\mathfrak{g}(x) - 2) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} s_j^\nu(M). \tag{4.7}$$

We write $W^{\mathrm{clls}}(M, D, \mathfrak{g})$ simply as $W^{\mathrm{clls}}(M, \mathfrak{g})$, the *clls Weierstrass divisor* of M, if D is understood from the context. If the genus function is trivial, $\mathfrak{g} = 0$, then we abbreviate $W^{\mathrm{clls}}(M, \mathfrak{g})$ to $W^{\mathrm{clls}}(M)$. ◇

Note that $W^{\text{clls}}(M, \mathfrak{g})$ has finite support. Indeed, since elements in M are piecewise affine linear, we can find a model $G = (V, E)$ of Γ such that $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$ is constant in the interior of any edge of $G = (V, E)$ for parallel unit tangent vectors ν based at points of the edge and pointing in the same direction. It follows that if $x \notin V$ and x is outside the supports of D and $\mathfrak{g}$, then $\mu_W^{\text{clls}}(x) = 0$. Also note that the central term in the expression of $\mu_W^{\text{clls}}(x)$ above is equal to $\frac{1}{2}r(r+1)K(x)$, where K is the canonical divisor on $(\Gamma, \mathfrak{g})$.

Example 4.5.2. Consider the non-augmented barbell graph Γ with edges of arbitrary length, see Figure 4.7. This metric graph has genus two and the canonical divisor has rank one. We define a sub-semimodule $M \subseteq \mathrm{Rat}(K)$ of rank one on Γ by prescribing the slopes $-1 < 1$ on the middle edge and, for $i = 1, 2$, slopes $0 < 1$ on both oriented edges $u_i v_i$. Then, we define M as the set of all functions in $\mathrm{Rat}(K)$ that, along any unit tangent vector at a given point of Γ, take one of the two prescribed slopes. It is easy to see that M is closed and verifies $(\star)$.

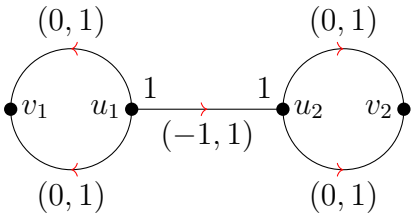


Figure 4.7: The barbell graph, the canonical divisor and the slope structure $\mathfrak{S}$.

The clls Weierstrass divisor is

$$W^{\text{clls}}(M) = (u_1) + (u_2) + 2\,(v_1) + 2\,(v_2)$$

(see Figure 4.8, right). For comparison, the tropical Weierstrass locus $L_W(M, \mathfrak{g})$ of the semimodule M with trivial genus function $\mathfrak{g} = 0$ (as defined in Section 4.4), is shown on the same figure (left). Here, $L_W(M)$ turns out to be identical to the tropical Weierstrass locus of the complete linear series $L_W(K)$ (see Example 4.3.5). ◇

Figure 4.8: The tropical Weierstrass locus $L_W(M)$ (left) and the clls Weierstrass divisor $W^{\text{clls}}(M)$ (right) on the barbell graph.

4.5.2 Comparison with the tropical Weierstrass locus

The following proposition shows that the notion of clls Weierstrass divisor can be viewed as a refinement of the tropical Weierstrass locus defined in Section 4.4.

Proposition 4.5.3 (Comparison of the tropical and clls Weierstrass loci)**.** *Suppose $M \subseteq \mathrm{Rat}(D)$ is a combinatorial limit linear series of rank r with clls Weierstrass divisor $W^{\mathrm{clls}}(M,\mathfrak{g})$. Let $L_W(M,\mathfrak{g})$ denote its Weierstrass locus, defined as in Section 4.4.1. If $A \subseteq \Gamma$ is closed, connected, and $L_W(M,\mathfrak{g})$-measurable, then we have the equality*

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g}).$$

In particular, if M is W-finite as a semimodule, then the following equality holds:

$$W^{\mathrm{clls}}(M,\mathfrak{g}) = (r+1)\,W(M,\mathfrak{g}).$$

Proof. We have

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right)$$

where K denotes the canonical divisor on $(\Gamma,\mathfrak{g})$ (see Definition 4.4.20) and $s_j^\nu = s_j^\nu(M)$. The terms $(r+1)\,D(x)$ add up to the term $(r+1)\deg\left(D|_A\right)$. Remark 4.4.21 yields that the terms $K(x)$ add up to $2g(A) - 2 + 2\sum_{x\in A}\mathfrak{g}(x) + \mathrm{outval}(A)$, where $\mathrm{outval}(A) := \left|\partial^{\mathrm{out}} A\right|$ is the number of outgoing branches from A.

The terms in the third part can be rearranged as a sum over directed edges of A, using some compatible model. Each edge has two in-going tangent directions, and the slope sums cancel out for this pair $(\nu,\overline{\nu})$ of opposing in-going directions since $s_j^\nu + s_{r-j}^{\overline{\nu}} = 0$. The only terms that do not cancel are the tangent directions which point out of A, i.e.,

$$\sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right) = \sum_{\nu\in\partial^{\mathrm{out}} A}\left(\sum_{j=0}^{r} s_j^\nu\right).$$

Combining these terms, we have

$$\begin{aligned}\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) &= (r+1)\deg\left(D|_A\right) + \frac{r(r+1)}{2}(2g(A,\mathfrak{g}) - 2 + \mathrm{outval}(A)) \\ &\quad - \sum_{\nu\in\partial^{\mathrm{out}} A}\sum_{j=0}^{r} s_j^\nu \\ &= (r+1)\deg\left(D|_A\right) + r(r+1)(g(A,\mathfrak{g}) - 1) - \sum_{\nu\in\partial^{\mathrm{out}} A}\sum_{j=0}^{r}(s_j^\nu - j).\end{aligned}$$

Finally, we use the fact that $s_j^\nu = j + s_0^\nu$ for every j and for tangent directions ν outside the Weierstrass locus $L_W(M,\mathfrak{g})$, by Theorem 4.4.11. Thus,

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\left(\deg\left(D|_A\right) + (g(A,\mathfrak{g}) - 1)\,r - \sum_{\nu\in\partial^{\mathrm{out}} A} s_0^\nu\right),$$

which, using the same technique as in the proof of Theorem 4.3.9, gives the first statement. The second statement follows from the first by the expression of the Weierstrass weight of a connected component of the tropical Weierstrass locus which is reduced to a point. □

We have the following extension of the above proposition, using the notion of tangential ramifications introduced later in Section 4.5.5. In particular, the statement holds even if A is not $L_W(M, \mathfrak{g})$-measurable.

Proposition 4.5.4. *Notations as in Proposition 4.5.3, for any closed, connected $A \subseteq \Gamma$, the following equality holds*

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\left(\deg\left(D|_A\right) + (g(A,\mathfrak{g})-1)\,r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(M)\right) - \sum_{\nu\in\partial^{\mathrm{out}}A}\sum_{j=0}^{r}\alpha_j^\nu(M),$$

where $\alpha_j^\nu(M) := s_j^\nu(M) - j - s_0^\nu(M)$ are the tangential ramifications along ν.

4.5.3 Tropicalization of Weierstrass loci

The goal of this section is to prove Theorem 4.5.5, using the machinery developed for semimodules on augmented metric graphs (see Section 4.4.1). This provides a precise link between tropical Weierstrass loci and the tropicalization of Weierstrass divisors on algebraic curves. Using this result, we will deduce Theorem 4.1.11.

Let X be a smooth proper curve of genus g over an algebraically closed non-Archimedean field $\mathbf{K}$ of arbitrary characteristic with a non-trivial valuation. Let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be a line bundle of positive degree d on X. Let H be a vector subspace of global sections of $\mathcal{L}$ of rank r (i.e., $\dim \mathrm{H} = r+1$), that we naturally view in the function field of X. When $\mathbf{K}$ has positive characteristic, we suppose that $\mathcal{L}$ is classical [Lak81, Nee84], that is, the gap sequence of H is the standard sequence $0 < 1 < \cdots < r$. We denote by $\mathcal{W} = \mathcal{W}(\mathrm{H})$ the corresponding Weierstrass divisor on X. Recall that $\mathcal{W}$ is the zero divisor of a global section, called the Wronskian, of the line bundle $\omega_X^{\otimes r(r+1)/2} \otimes \mathcal{L}^{\otimes(r+1)}$, see [Lak81] and Section 4.8.4. In particular, we have

$$\deg(\mathcal{W}) = \frac{r(r+1)}{2}\,(2g-2) + (r+1)\,d = (r+1)\,(d-r+rg).$$

Let $(\Gamma, \mathfrak{g})$ be a skeleton of X^{an}, and let $\tau\colon X^{\mathrm{an}} \to \Gamma$ denote the specialization map. Let $W := \tau_*(\mathcal{W})$ be the specialization of $\mathcal{W}$ to Γ. Note that $(\Gamma, \mathfrak{g})$ is an augmented metric graph. We let $D := \tau_*(\mathcal{D})$ be the specialization of $\mathcal{D}$ to Γ, and let $M \subseteq \mathrm{Rat}(D)$ be the subsemimodule consisting of the tropicalizations of non-zero rational functions in H. It follows from the slope formula that the divisorial rank of M is equal to the rank of H, see [AG22, Theorem 9.1] (Theorem 3.9.1 in the present manuscript) and [JP22, Proposition 4.1].

The following theorem compares the algebraic Weierstrass divisor of H on the curve X with the tropical Weierstrass divisor of M on the augmented metric graph $(\Gamma, \mathfrak{g})$.

Theorem 4.5.5 (Algebraic versus tropical weights: general case)**.** *Notations as above, let A be a closed, connected, $L_W(M,\mathfrak{g})$-measurable subset of Γ. Then, the total weight of Weierstrass points of $\mathcal{W}$ which tropicalize to A is given by*

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g})$$

where

$$\hat{\mu}_W(A;M,\mathfrak{g}) = \deg\left(D|_A\right) + \left(g(A) + \sum_{x\in A}\mathfrak{g}(x) - 1\right)r - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(M).$$

In particular, if M is W-finite, we have the following equality of divisors on Γ:

$$\tau_*(\mathcal{W}) = (r+1)\,W(M,\mathfrak{g}).$$

Before proceeding to the proof, a remark is in order.

Remark 4.5.6. By Proposition 4.4.14, Theorem 4.5.5 holds in a slightly more general setting. Let M' be any closed sub-semimodule of $\mathrm{Rat}(D)$ of divisorial rank r containing M. Then, we have

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = (r+1)\,\hat{\mu}_W(A;M',\mathfrak{g})$$

for every $L_W(M',\mathfrak{g})$-measurable subset A of Γ. ◇

Proof of Theorem 4.5.5. In the case the residue field of $\mathbf{K}$ has characteristic zero, we use Theorem 4.8.5 which provides a description of the divisor $W = \tau_*(\mathcal{W})$ in terms of slope structures. As explained in Section 4.8.3, the slopes at any point x and any unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$ of elements of the tropicalization M of H form a set of $r+1$ integers $s_0^\nu, s_1^\nu, \dots, s_r^\nu$. The definition of the Weierstrass divisor associated to a tropical linear series is chosen to ensure the equality $W = W^{\mathrm{clls}}(M,\mathfrak{g})$, which implies

$$\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) = \deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right).$$

Proposition 4.5.3 states that if A is $L_W(M,\mathfrak{g})$-measurable, then

$$\deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g}),$$

from which the result follows.

In the general case, when the characteristic of $\mathbf{K}$ is arbitrary and the gap sequence of H is standard, we use the description of the reduction of the Weierstrass divisor to the skeleton given in Theorem 4.8.2. Using the notations of Section 4.8, letting $W = \tau_*(\mathcal{W})$, we have

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu\in\mathrm{T}_x(\Gamma)}\mathrm{slope}_\nu F,$$

with $F = \mathrm{trop}(\mathrm{Wr}_{\mathcal{F}})$. Furthermore, $\mathrm{slope}_\nu F = \frac{r(r+1)}{2} + \mathrm{ord}_{p_x^\nu}\widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$.

Since the slopes along the unit tangent vectors $\nu \in \mathrm{T}_x(\Gamma)$ which are outgoing from A form a consecutive sequence of integers, by Proposition 4.8.4 we infer that the quantity $\mathrm{ord}_{p_x^\nu} \widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$ is equal to $s_0^\nu + \cdots + s_r^\nu$. Using Theorem 4.8.2, we get $\mathrm{slope}_\nu F = s_0^\nu + \cdots + s_r^\nu$.

Moreover, since F belongs to $\mathrm{Rat}(\Gamma)$, the total sum of the slopes of F for the edges which appear in the interior of A vanishes. We infer that

$$\begin{aligned}\deg\left(\mathcal{W}|_{\tau^{-1}(A)}\right) &= (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\sum_{\nu\in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu F \\ &= (r+1)\sum_{x\in A} D(x) + \frac{r(r+1)}{2}\sum_{x\in A} K(x) - \sum_{x\in A}\left(\sum_{\nu\in \mathrm{T}_x(\Gamma)}\sum_{j=0}^{r} s_j^\nu\right) \\ &= \deg\left(W^{\mathrm{clls}}(M,\mathfrak{g})|_A\right) = (r+1)\,\hat{\mu}_W(A;M,\mathfrak{g}),\end{aligned}$$

as required. □

4.5.4 Proofs of Theorems 4.1.10 and 4.1.11

We deduce Theorem 4.1.10 from Theorem 4.5.5.

Proof of Theorem 4.1.10. Since D and $\mathcal{D}$ have the same rank r, we can plug $\mathrm{H} := \mathrm{Rat}(\mathcal{D})$ and $M' := \mathrm{Rat}(D)$ into Remark 4.5.6, following Theorem 4.5.5, to get

$$\deg\left(\mathcal{W}(\mathcal{D})|_{\tau^{-1}(A)}\right) = (r+1)\left(\deg\left(D|_A\right) + r(g(A,\mathfrak{g})-1) - \sum_{\nu\in\partial^{\mathrm{out}}A} s_0^\nu(D)\right). \tag{4.8}$$

In the context of Theorem 4.1.10, $\mathfrak{g} = 0$. The result follows. □

Using this result, we can prove Theorem 4.1.11.

Proof of Theorem 4.1.11. This follows from the combination of Theorem 4.4.12, Proposition 4.4.5 and Theorem 4.5.5. □

4.5.5 Tangential ramification sequence and effectivity

Unlike the tropical Weierstrass divisors defined earlier in this chapter, the Weierstrass divisor defined in Definition 4.5.1 is not automatically effective. We can rewrite the Weierstrass

weight as

$$\begin{aligned}\mu_W^{\text{clls}}(x) &= (r+1)\Big(D(x) + \frac{r}{2}\operatorname{val}(x) + (\mathfrak{g}(x)-1)\,r\Big) \\ &\qquad - \sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} \big(s_0^\nu + j + \big(s_j^\nu - s_0^\nu - j\big)\big) \\ &= (r+1)\big(D_x^M(x) + (\mathfrak{g}(x)-1)\,r\big) - \sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} \big(s_j^\nu - s_0^\nu - j\big) \\ &= r(r+1)\mathfrak{g}(x) + \underbrace{(r+1)\big(D_x^M(x) - r\big)}_{\geqslant 0} - \underbrace{\sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} \big(s_j^\nu - s_0^\nu - j\big)}_{\geqslant 0}.\end{aligned}$$

Definition 4.5.7 (Tangential ramification sequence)**.** We call the sequence

$$\{\alpha_j^\nu(M) \coloneqq s_j^\nu(M) - s_0^\nu(M) - j : j = 0, 1, \ldots, r\}$$

the *ramification sequence* of M at x along the tangential direction ν. This sequence is non-decreasing. ◇

This motivates the following definition.

Definition 4.5.8 ($\mathfrak{g}$-effective linear series)**.** Let $\mathfrak{g}$ be a genus function on Γ. The combinatorial limit linear series M is called $\mathfrak{g}$-*effective* if $W^{\text{clls}}(M, \mathfrak{g})$ is effective. That is, for all $x \in \Gamma$,

$$r(r+1)\mathfrak{g}(x) + (r+1)\big(D_x^M(x) - r\big) \geqslant \sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} \alpha_j^\nu(M). \tag{4.9}$$

◇

We say that $M \subseteq \operatorname{Rat}(\Gamma, \mathfrak{g})$ is *realizable* if there exists a smooth proper curve X of genus g over $\mathbf{K}$, a line bundle $\mathcal{L} = \mathcal{O}(\mathcal{D})$ of degree d and a subspace $\mathrm{H} \subseteq H^0(X, \mathcal{L})$ of rank r such that $(\Gamma, \mathfrak{g})$ is a skeleton of X^{an}, and $M = \{\operatorname{trop}(f) : f \in \mathrm{H} \smallsetminus \{0\}\}$. If this happens over $\mathbf{K}$ of equicharacteristic zero, we say M is realizable in equicharacteristic zero.

Proposition 4.5.9. *If M is realizable in equicharacteristic zero, then the following hold:*

(i) $W^{\text{clls}}(M, \mathfrak{g})$ is effective, i.e., M is $\mathfrak{g}$-effective.

(ii) the divisor of degree zero

$$W^{\text{clls}}(M, \mathfrak{g}) - (r+1)\,D - \frac{r(r+1)}{2}\,K = \sum_{x\in\Gamma} \left(\sum_{\nu\in \mathrm{T}_x(\Gamma)} \sum_{j=0}^{r} s_j^\nu \right)(x)$$

is principal.

Proof. Both statements follow from Theorem 4.8.5. □

Example 4.5.10. Consider the non-augmented metric graph Γ below and its canonical divisor K. We consider the following combinatorial limit linear series $M \subseteq \mathrm{Rat}(K)$. For each bridge edge oriented outwards (towards the adjacent circle), allow slopes $-1 < 1 < 3$. Divide each circle into three equal parts, in a way compatible with the position of the attachment points. On the two edges adjacent to the attachment points, allow slopes $0 < 1 < 2$ away from the attachment points, and on the remaining edges, allow slopes $-1 < 0 < 1$ (see Figure 4.9).

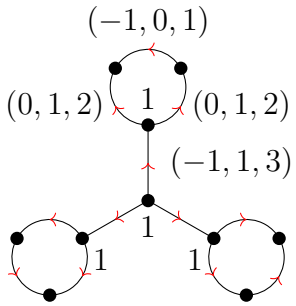


Figure 4.9: Three-cycle graph with a specified slope structure on $\mathrm{Rat}(K)$, defining a combinatorial limit linear series $M \subseteq \mathrm{Rat}(K)$.

We can define a suitable closed sub-semimodule $M \subseteq \mathrm{Rat}(K)$ of rank two of functions compatible with this choice of slopes. The tropical Weierstrass locus $L_W(M)$ of the semimodule M, in the sense of Section 4.4.1 (with $\mathfrak{g} = 0$), contains the bridge edges and the points of coordinates $\frac{1}{3}$ and $\frac{2}{3}$ on the circles (see Figure 4.10, left). In particular, M is not W-finite. The clls Weierstrass divisor $W^{\mathrm{clls}}(M)$ is also shown in the figure (right). In particular, M is not $\mathfrak{g}$-effective. ◇

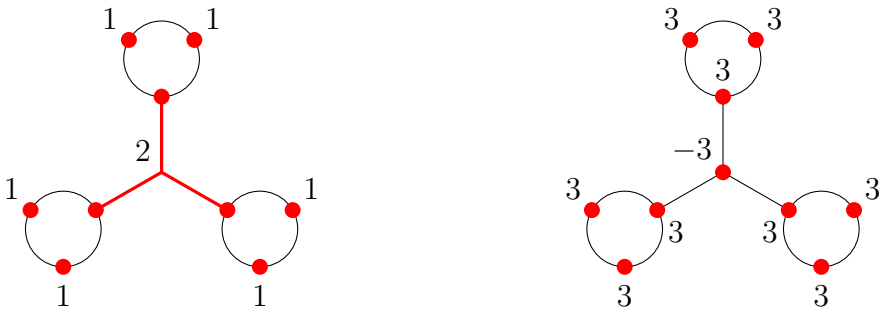


Figure 4.10: The tropical Weierstrass locus $L_W(M)$ (left) and the clls Weierstrass divisor $W^{\mathrm{clls}}(M)$ (right).

Remark 4.5.11. Note that instead of allowing the slopes $-1 < 1 < 3$ on the three central edges, we could allow the slopes $-1 < 1 < 3$ on (possibly trivial) intervals incident to the central vertex on these edges, and the slopes $-1 < 1 < 2$ on the rest of the edges, with the

same choice of rank functions. This gives three degrees of freedom to choose combinatorial limit linear series on Γ and leads to different tropical Weierstrass loci and clls Weierstrass divisors. The clls Weierstrass coefficient on the central vertex is non-negative only when the three edges are entirely endowed with the slopes $-1 < 1 < 2$. This implies that the corresponding combinatorial limit linear series in the case $\mathfrak{g} = 0$ is the only (possibly) realizable one (see Proposition 4.5.9). ◇

4.6 Examples

We here discuss several examples in order to illustrate the results of the previous sections.

4.6.1 Dipole graph

Suppose Γ is a dipole graph of genus $g \geqslant 2$ (also known as a "banana" graph), consisting of two vertices joined by $g+1$ edges, possibly of different lengths. The canonical divisor K has coefficient $g-1$ on each vertex. The Weierstrass locus $L_W(K)$ consists of the interval $[\ell/g,\, \ell - \ell/g]$ on every edge, with ℓ the length of that edge (see Figure 4.11 for $g = 3$). Each component $C \subseteq L_W(K)$ has two outgoing directions, and in each outgoing direction, the minimum slope is $s_0^\nu = -(g-1)$. By Theorem 4.3.17, the Weierstrass weight of each component is

$$\mu_W(C) = (g+1)(g(C)-1) - \sum_{\nu \in \partial^{\mathrm{out}} C} (s_0^\nu - 1) = (g+1)(-1) - (-g-g) = g-1.$$

The total Weierstrass weight of $L_W(K)$ is $g^2 - 1$, as expected (Corollary 4.3.18 (a)).

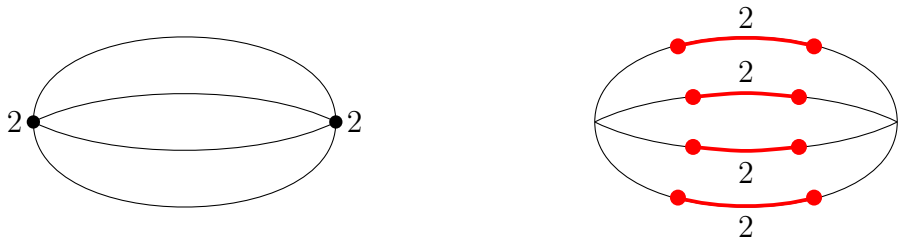


Figure 4.11: Dipole graph of genus $g = 3$ and its Weierstrass locus $L_W(K)$.

4.6.2 Tent graph

Consider the tent graph G, consisting of three vertices and five edges, as shown in Figures 4.12, 4.13 and 4.14. We first consider the case $D = K$, a divisor of rank $r = g - 1 = 2$. We have, for each of the Weierstrass points located at the endpoints of the bottom edge in Figure 4.12, $\mu_W(v) = (3+1)(-1) - (-2-2-2) = 2$. The other four Weierstrass points

are located on either of the four other edges respectively, one third of the distance from the top vertex to the other endpoints. Their weight is 1.

$$\mu_W(x) = (3+1)(-1) - (-2-3) = 1.$$

Figure 4.12: Tent graph and its Weierstrass locus $L_W(K)$.

Now consider the case $D = K + (v)$ for v the vertex of degree four, a divisor also of rank $r = 2$. $L_W(D)$ has a unique component, see Figure 4.13, and by Theorem 4.3.17, its weight is $\mu_W(C) = \deg\left(D|_C\right) + (g(C)-1)\,r - \sum_{\nu\in\partial^{\text{out}}C} s_0^\nu = 5 + (2-1)\cdot 2 - (-1-1) = 9$.

Figure 4.13: A divisor on the tent graph and its Weierstrass locus.

Finally, consider $D = K + (u)$ for u one of the vertices of degree three, a divisor still of rank $r = 2$. See Figure 4.14. The two singleton components of $L_W(D)$ each have weight one. Suppose C is the non-singleton component of $L_W(D)$, whose boundary points on both left-hand edges are located one third of the distance from the top vertex. Theorem 4.3.17 gives $\mu_W(C) = 7$.

$$\mu_W(C) = \deg\left(D|_C\right) + (g(C)-1)\,r - \sum_{\nu\in\partial^{\text{out}}C} s_0^\nu = 3 + (0-1)\cdot 2 - (-2-2-1-1) = 7.$$

Figure 4.14: A divisor on the tent graph and its Weierstrass locus.

4.6.3 Cube graph

The cube graph is shown in Figure 4.15, with all edges of length one. It has genus 5 and the canonical divisor K has rank 4. The Weierstrass locus $L_W(K)$ consists of the closed segment $[2/5, 3/5]$ on each edge, and excludes the vertices. Each component C of $L_W(K)$ has out-valence 2, with minimum slopes in $\mathrm{Rat}(K)$ in each outgoing direction equal to -3. Theorem 4.3.17 gives $\mu_W(C) = 2$.

$$\begin{aligned}\mu_W(C) &= (g+1)(g(C)-1) - \sum_{\nu \in \partial^{\mathrm{out}} C} (s_0^\nu - 1) \\ &= 6 \cdot (-1) - (-4-4) = 2.\end{aligned}$$

There are 12 components, so the total weight is 24.

Figure 4.15: Cube graph with its Weierstrass locus $L_W(K)$.

4.6.4 Bridge edges

We expand on the barbell graph (Example 4.3.5).

Theorem 4.6.1 (Weierstrass loci and bridge edges)**.** *Let Γ be a metric graph which has a bridge edge e such that each component of $\Gamma \smallsetminus \mathring{e}$ has positive genus. Then, the edge e is contained in the canonical Weierstrass locus $L_W(K)$.*

Proof. To show this, let u_1 and u_2 denote the endpoints of e, and Γ_1 and Γ_2 be the components of $\Gamma \smallsetminus \mathring{e}$ containing u_1 and u_2, respectively. If g, g_1 and g_2 are the genera of Γ, Γ_1 and Γ_2 respectively, then $g = g_1 + g_2$. Let $r = g-1$ be the rank of the canonical divisor on Γ. We want to show that we can move $r+1 = g_1+g_2$ chips to every point $x \in e$. For $i = 1, 2$, denoting by K_i the canonical divisor of Γ_i, we have $r_{\Gamma_i}\left(K|_{\Gamma_i} - (u_i)\right) = r_{\Gamma_i}(K_i) = g_i - 1 \geqslant 0$, which implies that, using only functions on Γ that are constant outside Γ_i, we can move g_i chips to u_i. It is easy to see that we can move chips along e to put $g_1 + g_2$ chips at x. □

Figure 4.16 shows an example where the Weierstrass locus strictly contains the bridge edges. Γ has two bridge edges and is of genus 5. All the edges of Γ are taken of unit length. The boundary points of the Weierstrass locus on the left and right circle are the points of coordinates $\frac{1}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$ on each of the six corresponding edges, 1 being the outermost point. The sum of all Weierstrass weights is $12 + 6 \cdot 2 = 24 = g^2 - 1$, as expected.

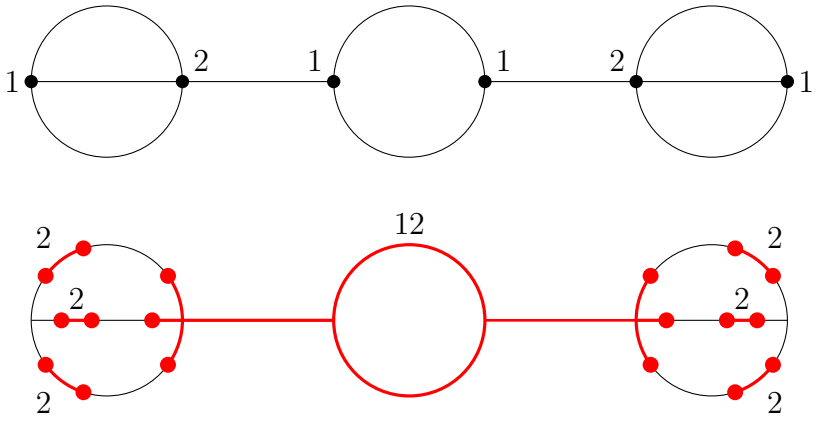


Figure 4.16: A graph with two bridge edges and its Weierstrass locus $L_W(K)$.

4.6.5 Cases where the whole graph is Weierstrass

We provide two infinite families of examples for which the Weierstrass locus is the whole graph, and discuss related questions.

Example 4.3.4 treated the complete graph on four vertices with unit edge lengths, with the Weierstrass locus consisting of the four vertices. Now consider the case Γ is the complete graph on $n \geqslant 5$ vertices with unit edge lengths. This graph has genus $g = \frac{n^2-3n+2}{2}$, and the canonical divisor K has rank $\frac{n^2-3n}{2}$. The Weierstrass locus of K is the whole graph as $K_x(x) \geqslant g$. Indeed, $K(v) = n - 3$ on each vertex v, and the reduced divisor at v is $K_v = (n^2 - 3n)(v)$ (move all chips to v in a single firing). For x in the interior of an edge, the reduced divisor K_x leaves $(n-1)$ chips away from x, i.e., $K_x(x) = n^2 - 4n + 1$. Note that $n^2 - 4n + 1 \geqslant \frac{n^2-3n+2}{2}$ for $n \geqslant 5$. This provides a first infinite family with the whole metric graph Weierstrass.

We now give a second such family. In this family, the choice of the length function is free and there are infinitely many possible choices of divisors with this property on the same metric graph. See also [Ric24, Example 4.6]. Let Γ be the metric graph generalizing the barbell graph (Example 4.3.5) to any number of cycles. More precisely, take $g \geqslant 2$ cycles of arbitrary length and join them all to a central vertex v with a bridge edge of positive length, as in Figure 4.17. Consider the divisor $D = d(v)$, with $d \geqslant 3$. By Clifford's theorem, the rank of D satisfies the bound $r \leqslant d - 2$. Since a divisor of positive degree on a cycle has rank one less than the degree, and since chips can move freely on bridge edges, it is easy to show $D_x(x) \geqslant d - 1 \geqslant r + 1$ for every $x \in \Gamma$. Therefore, the Weierstrass locus is the whole graph.

Note that in the second family the quantity $\min_{x\in\Gamma}(D_x(x) - r)$ can be arbitrarily large. The existence of these two families of examples, with very different combinatorial properties (for example, the first is made up of graphs with high connectivity, whereas the graphs of the second have many bridge edges), suggests the following.

Question 4.6.2. *Provide a classification of all graphs G that admit a length function and*

Figure 4.17: The generalized barbell graph, the divisor D and its Weierstrass locus $L_W(D)$.

a divisor with Weierstrass locus the whole metric graph. Among them, what are the ones for which this property holds for every choice of edge lengths?

4.6.6 Augmented cycle with one point of positive genus

We compute Weierstrass loci for the canonical divisor with respect to the canonical and generic linear systems on an augmented cycle on which one point has positive genus, generalizing Examples 4.4.28 and 4.4.23. The canonical case recovers a result of Diaz [Dia85, Theorem A2.1]: the generic non-separating node on a uninodal stable curve is a limit of exactly $g(g-1)$ Weierstrass points on nearby smooth curves.

Let a be a positive integer, and consider the augmented metric graph $(\Gamma, \mathfrak{g})$ where Γ is the cycle of length one, parametrized by the interval $[0, 1]$, the single vertex v coincides with the endpoints $v = 0 = 1$, and $\mathfrak{g}(v) = a$. The genus of this augmented metric graph is $g = a + 1$.

The case of the canonical linear system

We expand on Example 4.4.28 for which $a = 2$ was fixed. Consider the canonical divisor K and the associated canonical semimodule $\mathrm{KRat}(\mathfrak{g})$, as defined in Section 4.4.3. The rank is $r = g - 1 = a$ according to Theorem 4.4.26, and the total weight of the Weierstrass locus is $g^2 - 1 = a^2 + 2a$. The Weierstrass locus consists of the vertex v and all the points of the form $\frac{k}{a+1}$ for $k = 1, \dots, a$. The Weierstrass weights are $\mu_W(v; K, \mathfrak{g}) = a^2 + a$ and $\mu_W\left(\frac{k}{a+1}; K, \mathfrak{g}\right) = 1$. Figure 4.18 shows the canonical divisor and its (canonical) Weierstrass locus depicted in the middle.

The case of the generic linear system

In the second case, we generalize Example 4.4.23 and consider the same divisor K as above, but take the generic semimodule $\mathrm{Rat}^{\mathrm{gen}}(K, \mathfrak{g})$ as defined in Section 4.4.2. In this case, the rank is $r = g - 2 = a - 1$ (see Proposition 4.4.22) and the total weight of the Weierstrass locus is $a^2 + a$. The Weierstrass points are v, and all the points $\frac{k}{a}$ with $k = 1, \dots, a-1$. The weights are $\mu_W(v; K, \mathfrak{g}) = a^2 + 1$ and $\mu_W\left(\frac{k}{a}; K, \mathfrak{g}\right) = 1$. Figure 4.18 shows the canonical divisor on the left, and its generic Weierstrass locus depicted on the right.

We note that the Weierstrass loci are different even though they are both finite. The total weights are also different, as the underlying semimodules have different ranks.

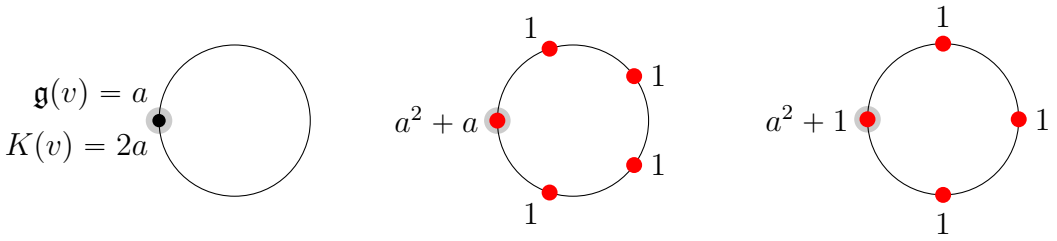


Figure 4.18: An augmented cycle graph, with its canonical Weierstrass locus $L_W(K, \mathfrak{g})$ in the middle, and its Weierstrass locus $L_W^{\text{gen}}(K, \mathfrak{g})$ on the right. The drawing is made for $a = 4$.

4.6.7 Augmented cycle with two points of positive genus

Now consider an augmented cycle with exactly two points of positive genus. We describe the Weierstrass locus $L_W(K, \mathfrak{g})$.

Suppose the augmented metric graph $(\Gamma, \mathfrak{g})$ consists of two vertices u and v connected by two edges of length α and β, and the vertices have genus $\mathfrak{g}(u) = g_1$ and $\mathfrak{g}(v) = g_2$. The genus of $(\Gamma, \mathfrak{g})$ is $g = g_1 + g_2 + 1$, and the rank of the canonical linear system $\mathrm{KRat}(\mathfrak{g})$ is $g_1 + g_2$. We parametrize Γ by the interval $[0, \alpha + \beta]$ with 0 and $\alpha + \beta$ identified, $u = 0$ and $v = \alpha$ (see Figure 4.19).

The canonical system is W-finite, and $W(K, \mathfrak{g}) = W_{\text{aug}} + W_{\text{met}}$ where

- $W_{\text{aug}} = g_1 g(u) + g_2 g(v) = g_1(g_1 + g_2 + 1)(u) + g_2(g_1 + g_2 + 1)(v)$, see Figure 4.20; and
- $W_{\text{met}} = \sum_{i=1}^{g_1}(x_i) + \sum_{j=1}^{g_2}(y_j)$ where

$$x_i := \alpha + \frac{i}{g_1 + g_2 + 1}\beta - \frac{g_1 + 1 - i}{g_1 + g_2 + 1}\alpha, \quad \text{and} \quad y_j := \frac{j}{g_1 + g_2 + 1}\alpha - \frac{g_2 + 1 - j}{g_1 + g_2 + 1}\beta$$

modulo $\alpha + \beta$, for every $1 \leqslant i \leqslant g_1$, and $1 \leqslant j \leqslant g_2$, see Figure 4.20. It turns out that these $g_1 + g_2$ points are all distinct.

If we additionally assume that the edge lengths α and β are generic, then all x_i's and y_j's are also distinct from u and v. The total weight is $g^2 - 1$. If $g_2 = 0$, we recover the example in Section 4.6.6.

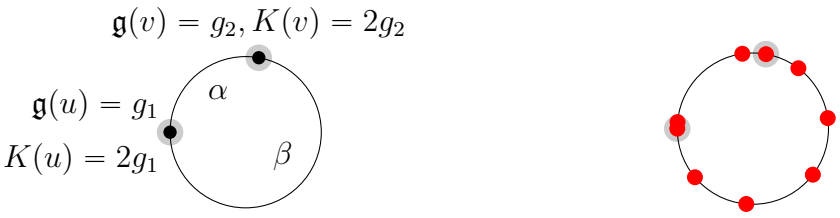


Figure 4.19: An augmented cycle with two points of positive genus and its Weierstrass locus $L_W(K, \mathfrak{g})$ (here, $g_1 = 4$, $g_2 = 3$). Weights are given in Figure 4.20.

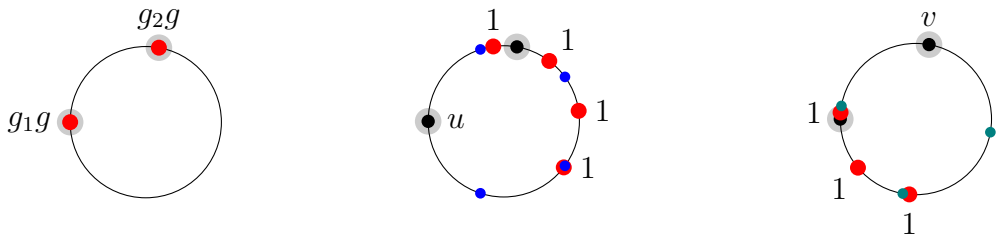


Figure 4.20: The three different types of Weierstrass points with $g_1 = 4$, $g_2 = 3$. The points in blue are the $(g_1 + 1)$-torsion points with respect to u, and the points in teal are the $(g_2 + 1)$-torsion points with respect to v.

4.6.8 Augmented dipole graph

We now consider an augmented dipole graph made up of two vertices u and v joined by $n = h + 1$ edges of arbitrary lengths, where h is the genus of the corresponding metric graph. We assume $\mathfrak{g}$ has support in $\{u, v\}$, and denote by a and b the genus of u and v, respectively, with $a \leqslant b$. This metric graph is the one that appears in the work by Esteves and Medeiros [EM02]. As we explained previously, the canonical linear series reflects the genericity of the points of intersection on each of the two components.

The canonical divisor has coefficients $K(u) = h - 1 + 2a$ and $K(v) = h - 1 + 2b$. The total genus is $g = g(\Gamma, \mathfrak{g}) = h + a + b$, and the rank r of the canonical linear series is equal to $g - 1 = h + a + b - 1$ according to Theorem 4.4.26. We compute $L_W(K, \mathfrak{g})$.

In the case $h = 0$, if a and b are both positive, then $L_W(K, \mathfrak{g}) = \Gamma$. Otherwise, if $a = 0$, and b is at least two, then $L_W(K, \mathfrak{g}) = \{v\}$, and the Weierstrass weight is $b^2 - 1$. If $b = 1$ or 0, then the Weierstrass locus is empty. The case $h = 1$ was treated separately in Section 4.6.7.

We now suppose $h \geqslant 2$. The determination of the Weierstrass locus turns out to be complicated in general, and its shape depends on the values of a, b, h and the edge lengths. We illustrate the computation in two concrete cases.

First particular case

Suppose $a = b = 1$, all the edges have unit length, and the genus of the metric graph is $h \geqslant 2$. We have $r = h + 1$ and $g = h + 2$.

Then the Weierstrass locus is made up of both vertices u and v, along with the segment $\left[\frac{2}{h+2}, \frac{h}{h+2}\right]$ on each edge (see Figure 4.21). The vertices u and v and have weight $2h + 2$ and each segment in the interior of an edge has weight $h - 1$. The total weight is $g^2 - 1$.

Second particular case

Suppose $a = 3$, $b = 5$, $h = 2$, and all the edges have unit length. We have $r = 9$ and $g = 10$. The Weierstrass locus is made up of the vertex v (weight 50), the union of the three segments $[0, 1/10]$ lying on each edge (weight 34), the point of coordinate $6/10$ on

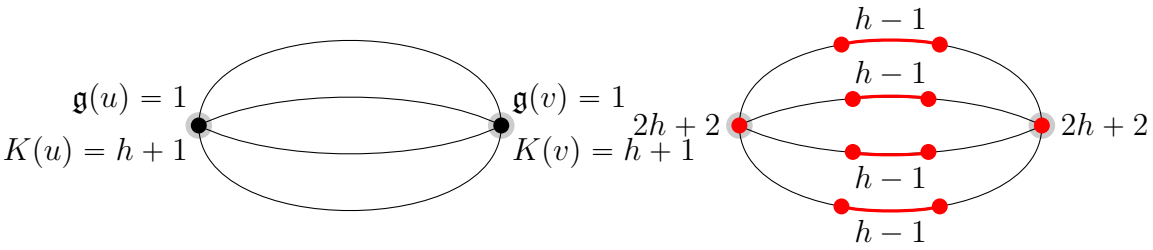


Figure 4.21: Augmented dipole graph with combinatorial genus $h = 3$, genera $a = b = 1$, all edges of unit length, and its Weierstrass locus $L_W(K, \mathfrak{g})$.

each edge (weight 1), and the segments [3/10, 4/10] and [8/10, 9/10] on each edge (each of weight 2). See Figure 4.22. The total weight is $50 + 34 + 3 \cdot (2 + 1 + 2) = 99 = g^2 - 1$.

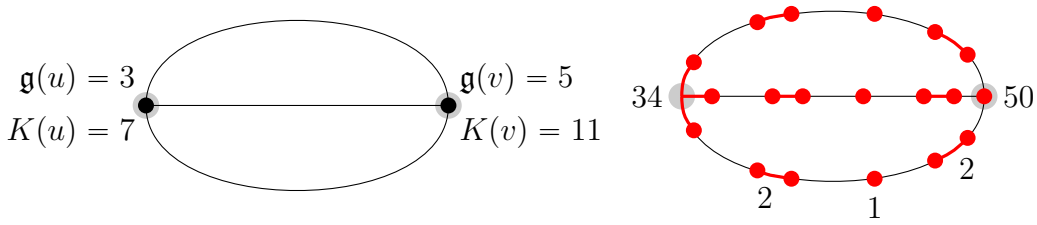


Figure 4.22: Augmented dipole graph with $h = 2$, $a = 3$ and $b = 5$, edges all of unit length, and its Weierstrass locus $L_W(K, \mathfrak{g})$.

4.6.9 Weierstrass divisor of a combinatorial limit linear series

We go back to the non-augmented dipole graph with four edges (of unit length to simplify the notations), a particular case of the class of examples presented in Section 4.6.1. The genus is $g = 3$ and the rank of the canonical divisor K is $r = 2$. Denote by u and v the two vertices and by e_1, e_2, e_3 and e_4 the four edges of Γ (see Figure 4.23, left). For $i = 1, 2, 3, 4$, let $t_i \in \left[0, \frac{1}{6}\right]$. For each choice of the t_i's, we will construct a semimodule M of rank two that verifies condition $(\star)$ from Section 4.5, and compute its clls Weierstrass divisor $W^{\text{clls}}(M)$ (here, $\mathfrak{g} = 0$).

Assume t_i's are fixed. We prescribe the set of slopes taken by functions in M as in Figure 4.23. For each i, we endow the edge e_i with the slope sets $0 < 1 < 2$ on the interval $\left[0, \frac{1}{2} - t_i\right]$, slopes $-1 < 0 < 1$ on the interval $\left[\frac{1}{2} - t_i, \frac{1}{2} + t_i\right]$, and slopes $-2 < -1 < 0$ on the interval $\left[\frac{1}{2} + t_i, 1\right]$.

We define M as the sub-semimodule of $\mathrm{Rat}(K)$ consisting of all the functions that take one of the prescribed slopes above at any point of Γ along any unite tangent vector. We thus get a 4-parameter family of pairs (M, D) of rank two, $M \subseteq \mathrm{Rat}(K)$, that verify $(\star)$.

The clls Weierstrass divisor is given by (4.7), and yields $W^{\text{clls}}(M) = 3\sum_{i=1}^{4}((x_i) + (y_i))$, where x_i and y_i denote the points of coordinates $1/2 - t_i$ and $1/2 + t_i$ on the edge e_i,

respectively. Figure 4.23 gives a visual rendering of $W^{\text{clls}}(M)$ for the choice $(t_1, t_2, t_3, t_4) = \left(\frac{1}{6}, 0, \frac{1}{12}, \frac{1}{8}\right)$.

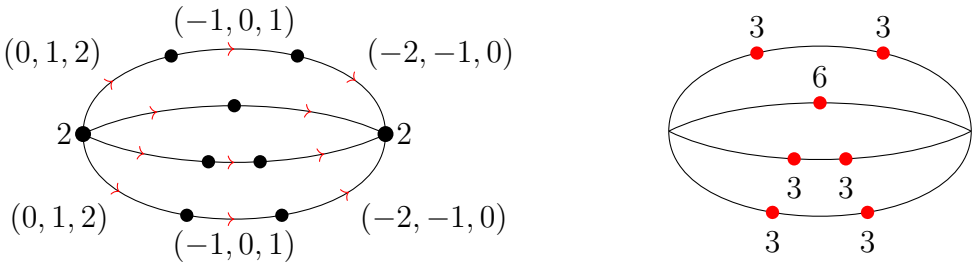


Figure 4.23: Dipole graph and its clls Weierstrass divisor $W^{\text{clls}}(M)$.

4.6.10 Combinatorial graphs without Weierstrass points

There exist combinatorial graphs which do not have any Weierstrass point. Using [HKN13], this is equivalent to saying that in the metric graph obtained by assigning uniform edge lengths equal to one to all edges of G, the connected components of the Weierstrass locus $L_W(K)$ of the canonical divisor live in the interior of the edges of G. Such graphs are interesting from the point of view of arithmetic geometry, see [Bak08, Section 4] and [Ogg78, LN64, Atk67, AP03].

The dipole graph is an example of such a graph, see Figure 4.11. So is the cube graph, see Figure 4.15. Figure 4.24 shows another example. We refer to Section 4.7.5 for further discussion.

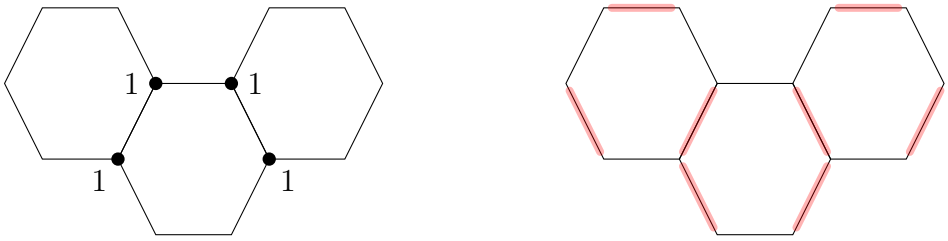


Figure 4.24: The canonical divisor of a combinatorial graph and the distribution of the Weierstrass weights on the edges of the corresponding metric graph with unit lengths. A black edge has total weight zero, and the interior of a light-red edge has total weight one. This indicates that the Weierstrass locus is concentrated in the interior of certain edges and does not contain any vertex.

4.7 Further discussions

We discuss other interesting questions and results related to the content of the chapter.

4.7.1 Total locus of Weierstrass points

Let $G = (V, E, \mathfrak{g})$ be a stable augmented graph of genus g, that is, a combinatorial graph of genus h endowed with a genus function $\mathfrak{g}\colon V \to \mathbb{N} \cup \{0\}$ such that any vertex of genus zero has valence at least three. Its total genus is $g = h + \sum_{v \in V} \mathfrak{g}(v)$. We view G as the dual graph of a stable curve X of total genus g with components X_v, for $v \in V$. Any one-parameter family of curves $\mathfrak{X}_t$ with fiber $\mathfrak{X}_0 = X$ and smooth fibers away from 0 gives rise to an edge length function $\ell\colon E \to (0, +\infty)$. Reparametrization of the family leads to another length function which is a homothety of ℓ. Every length function ℓ arises in this way from a one-parameter family of curves $\mathfrak{X}_t$, see e.g. [ABBR15a, Theorem 3.24].

Given a fixed edge $e \in E$, consider the set of all the edge length functions ℓ which give $\ell(e) = 1$. Any family of curves with a stable curve X as fiber at zero whose dual graph is G gives rise, after a possible reparametrization, to such a length function. We will refer to such a family as being (G, e)-*admissible*.

Denote by Γ_ℓ the augmented metric graph associated to the pair (G, ℓ) augmented with the genus function $\mathfrak{g}$. The metric graphs Γ_ℓ all share an interval of length one corresponding to the edge e. We denote by $L_W^{\mathrm{can}}(\Gamma_\ell)$ the Weierstrass locus of the canonical divisor in $(\Gamma_\ell, \mathfrak{g})$, using the semimodule $\mathrm{KRat}(\mathfrak{g})$ of functions on Γ_ℓ as in Section 4.4.3.

We define the *total Weierstrass locus of the canonical divisor*, denoted $L_W^{\mathrm{tot}}(e)$, as the portion of the edge e covered by Weierstrass points of all the augmented metric graphs Γ_ℓ, for those verifying $\ell(e) = 1$, that is,

$$L_W^{\mathrm{tot}}(e) := \bigcup_{\ell \text{ with } \ell(e)=1} L_W^{\mathrm{can}}(\Gamma_\ell) \cap e.$$

Question 4.7.1.

(i) What is the shape of $L_W^{\mathrm{tot}}(e)$, that is, how many components can it have on the edge e?

(ii) What is the size of $L_W^{\mathrm{tot}}(e)$? That is, what proportion of e is covered by Weierstrass points of metric graphs of combinatorial type $(G, \mathfrak{g})$?

(iii) How is $L_W^{\mathrm{tot}}(e)$ placed on e? That is, characterize the boundary of $L_W^{\mathrm{tot}}(e)$.

(iv) Characterize all the points in $L_W^{\mathrm{tot}}(e)$ which can arise as a limit of Weierstrass points on nearby smooth curves. More precisely, characterize those points p for which there exists a (G, e)-admissible family of curves $\mathfrak{X}_t$ and a Weierstrass point p_t on $\mathfrak{X}_t$ such that p is the tropical limit of p_t.

(v) What is the quantity $\sup_\ell |L_W^{\mathrm{can}}(\Gamma_\ell) \cap e|$, where $|L_W^{\mathrm{can}}(\Gamma_\ell) \cap e|$ refers to the Lebesgue measure of $L_W^{\mathrm{can}}(\Gamma_\ell) \cap e$ and the supremum is taken over all length functions ℓ such that $\ell(e) = 1$?

Inspired by Baker [Bak08, Lemma 4.7], we can prove Theorem 4.7.5 below which shows that the total Weierstrass locus $L_W^{\mathrm{tot}}(e)$ on the edge e is not always connected. This provides

a partial answer to Question (i) above. We do not know of any example with a number of connected components larger than two.

We can define a refined version of $L_W^{\mathrm{tot}}(e)$ by requiring the stable curve in the admissible family to be a fixed stable curve X, as follows. We define $L_W^{\mathrm{tot}}(e, X)$ as the locus of all the points in e that are limits of Weierstrass points in a one-parameter family of smooth curves converging to X.

Question 4.7.2. *What is the quantity* $\sup_X \left|L_W^{\mathrm{tot}}(e, X)\right|$*, where* $\left|L_W^{\mathrm{tot}}(e, X)\right|$ *refers to the Lebesgue measure of* $L_W^{\mathrm{tot}}(e, X)$*?*

Here, the supremum is taken over all stable curves X with the same stable dual graph G.

The discussion above is related to the work of Diaz [Dia85] and Gendron [Gen21]. Translated into the above language, Diaz and Gendron show in *loc. cit.* that the set $L_W^{\mathrm{tot}}(e)$ is nonempty. In fact, they prove that for any X with dual graph G, the set $L_W^{\mathrm{tot}}(e, X)$ is nonempty provided that e is not a bridge edge in G. If e is a bridge, then Gendron has a characterization of the situations where $L_W^{\mathrm{tot}}(e, X)$ is nonempty. The statement on non-bridge edges can be proved by using tropical arguments, by reducing to the example of the augmented cycle 4.6.6.

In a similar vein, we cite the following theorem of Eisenbud and Harris.

Theorem 4.7.3 (Eisenbud–Harris [EH87a])**.** *Suppose X is a smooth curve of genus g, and E is an elliptic curve with identity $e_0 \in E$. Let $X' = X \cup_x E$ denote the nodal curve obtained by joining $e_0 \in E$ to $x \in X$ by a node. If x is not a Weierstrass points of X, then the limit Weierstrass points of X' contained in E are exactly the torsion points of order g on E.*

Remark 4.7.4. Let G be a simple graph of genus g. Assume that G is 2-connected, that is, it does not have bridge edges. Then, we believe the following should be true. Given an edge e, there should exist a choice of edge lengths for which the Weierstrass locus contains a connected component in the interior of e. ◇

The above questions and the results we proved in this chapter provide a tropical refinement of the problem raised by Eisenbud and Harris on the determination of the limit Weierstrass loci on stable curves.

An example with a disconnected locus $L_W^{\mathrm{tot}}(e)$

We first prove the following result.

Theorem 4.7.5. *Let $G = (V, E)$ be a graph containing an edge $e = uv$ such that deleting e along with a small open neighborhood of its endpoints creates a tree. Assume e is parametrized by the interval $[0, 1]$, and suppose that its endpoints have valence $\mathrm{val}(u) = a+2$ and $\mathrm{val}(v) = b + 2$. Then, $L_W^{\mathrm{tot}}(e)$ is disjoint from the interval $\left[\dfrac{b}{a+b+1}, \dfrac{b+1}{a+b+1}\right]$ in e.*

Note that a graph satisfying the conditions in Theorem 4.7.5 has genus $g = a + b + 1$.

Proof. Let Γ_ℓ be metric graph of model G with $\ell(e) = 1$. Consider a point x in the interval $\left[\dfrac{b}{a+b+1}, \dfrac{b+1}{a+b+1}\right]$ and let $D \coloneqq K - g\,(x)$. In order to prove that x is not a Weierstrass point in Γ_ℓ, we will prove that the rank of D is negative. We proceed as follows.

Let $T \coloneqq G - u - v$ be the tree obtained by removing u, v, and all the incident edges to them from G. Let y be a point in the interior of e in Γ_ℓ, that will be determined later as a function of x. The set $V' = V \cup \{y\}$ is the vertex set of another model of Γ_ℓ. We enumerate the vertices of the tree T as $v_0, \ldots, v_n$ such that each vertex v_j for $j \in \{0, 1, \ldots, n\}$ is connected to exactly one vertex among $v_0, \ldots, v_{j-1}$. Consider the total order $\mathcal{o}$ on V' given by the enumeration $v_0, \ldots, v_n, u, v, y$. The corresponding divisor $D_\mathcal{o}$ is explicitly given as $D_\mathcal{o} = a\,(u) + b\,(v) + (y) - (v_0)$. Denote by $\overline{\mathcal{o}}$ the total order on V' opposite to $\mathcal{o}$, and $D_{\overline{\mathcal{o}}}$ the corresponding divisor. The divisors $D_\mathcal{o}$ and $D_{\overline{\mathcal{o}}}$ have degree $g - 1 = a + b$, they are of negative rank, and moreover, $D_\mathcal{o} + D_{\overline{\mathcal{o}}} = K$, see [BN07, BJ16].

We now write

$$D = K - g(x) = D_\mathcal{o} + D_{\overline{\mathcal{o}}} - g\,(x) = D_{\overline{\mathcal{o}}} - E - (v_0)$$

where $E = g\,(x) - D_\mathcal{o} - (v_0) = (a + b + 1)\,(x) - a\,(u) - b\,(v) - (y)$. The claim $r(D) = -1$ now follows by observing that for x in the above interval, there exists y in e such that the divisor E is principal, that is, $E = \operatorname{div}(f)$ for a function $f \in \operatorname{Rat}(\Gamma_\ell)$. Explicitly, using the parametrization of e by the interval $[0, 1]$ for a given x, we take $y = (a + b + 1)\,x - b$. We have $y \in [0, 1]$ because of the assumption that $x \in \left[\dfrac{b}{a+b+1}, \dfrac{b+1}{a+b+1}\right]$. The desired function f on Γ_ℓ is constant outside e, has slopes $\operatorname{slope}_e f(u) = a$ and $\operatorname{slope}_e f(v) = b$, and has orders of vanishing at x and y given by $a + b + 1$ and -1, respectively. □

Now consider a graph G verifying conditions of Theorem 4.7.5. Note that this implies there is a single edge between u and v. Assume that the leaves in the tree T are connected to both u and v. In this case, if $\operatorname{val}(u) > 2$ (which is equivalent, according to the previous assumption, to the fact that T has at least two leaves, i.e., T is not made up of a single vertex), then $v \in L_W^{\text{tot}}(e)$, and similarly, if $\operatorname{val}(v) > 2$, then $u \in L_W^{\text{tot}}(e)$.

To prove this, using symmetry and keeping the notations of Theorem 4.7.5, we assume $b > 0$. Take the union of T and v, as in Figure 4.25. Set the length of edges between u and T equal to b, that of uv equal to one, and the others arbitrary. Let f be the function defined to be affine linear on edges and which takes value b at u and zero at other vertices. Then, $b > 0$ implies that $E = K + \operatorname{div}(f)$ is effective and has coefficient at least $a + b + 1$ at u. So u belongs to the total Weierstrass locus, as required.

In the case a and b are both positive, this implies that u and v are both in the total Weierstrass locus $L_W^{\text{tot}}(e)$. Therefore, $L_W^{\text{tot}}(e)$ will be disconnected.

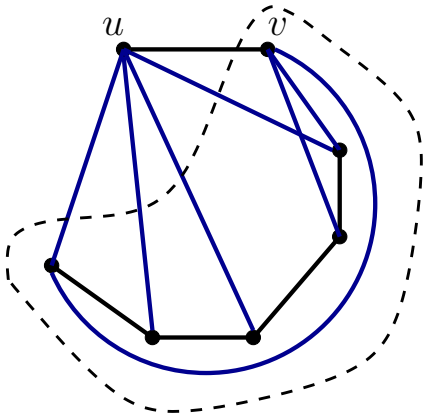


Figure 4.25: The cut in G used to prove that u is in the total Weierstrass locus $L_W^{\text{tot}}(e)$ if $\text{val}(v) > 2$.

4.7.2 Variation of Weierstrass loci over the moduli space of metric graphs

Let $G = (V, E)$ be a stable graph of genus g. Consider the cone $\eta_G := \mathbb{R}_+^E$ of positive metrics on G, and let $\bar{\eta}_G$ be its closure. The (coarse) moduli space of metric graphs of genus g, denoted by $\mathscr{M}_g^{gr}$, is obtained by gluing of the cones $\bar{\eta}_G$, for every stable graph G of genus g. More precisely, it is the direct limit of the diagram of inclusions $\bar{\eta}_H \hookrightarrow \bar{\eta}_G$ for pairs H and G of stable graphs of genus g such that H is obtained by contraction of some edges in G; see [ACP15] for more details. We endow $\mathscr{M}_g^{gr}$ with the topology induced by those on η_G as the corresponding quotient topology on the limit. For each stable graph G of genus g, we get a canonical map $\eta_G \to \mathscr{M}_G^{gr}$. The universal metric graph $\mathscr{G}_g$ is defined over these charts. That is, over the cone η_G, we have the universal metric graph $\mathscr{G}_G$.

Let $\mathcal{D} = (D_t)_{t \in \eta_G}$ be a continuous family of effective divisors of degree d and rank r. At each point $t \in \eta_G$, we consider the Weierstrass locus $L_W(D_t)$ which lives in the metric graph $\mathscr{G}_{G,t}$. We denote by $L_W(\mathcal{D})$ the Weierstrass locus of the family defined as the union of all $L_W(D_t)$, for $t \in \eta_G$. We have the following theorem.

Theorem 4.7.6. *The Weierstrass locus $L_W(\mathcal{D})$ is a closed subset of $\mathscr{G}_G$.*

Sketch of the proof. We need to show that any point x_{t_0} in a fiber $\mathscr{G}_{G,t_0}$ which is a limit of Weierstrass points x_t in $\mathscr{G}_{G,t}$, as t tends to t_0, is Weierstrass. This amounts to showing the existence of a function f in $\text{Rat}(D_{t_0})$ such that $D_{t_0} - (r+1)(t_0) + \text{div}(f) \geqslant 0$. By assumption, there exists $f_t \in \text{Rat}(D_{t_0})$ such that $D_t - (r+1)(t) + \text{div}(f_t) \geqslant 0$, and such that moreover $f_t(x_{t_0}) = 0$. A compactness argument then shows the existence of a subsequence of f_t's converging to a function f on $\mathscr{G}_{G,t_0}$. This limit function is in $\text{Rat}(D_{t_0})$, from which the theorem follows. □

More generally, we can define the Weierstrass locus over the full moduli space $\mathscr{M}_g^{gr}$. Let $\mathcal{D} = (D_t)$, for $t \in \mathscr{M}_g^{gr}$, be a continuous family of effective divisors of degree d and rank r over the moduli space of metric graphs of genus g.

Theorem 4.7.7. *The Weierstrass locus $L_W(\mathcal{D})$ is closed.*

Proof. The proof is similar to that of Theorem 4.7.6. □

4.7.3 Effective determination of minimum slopes

We discuss a concrete way of determining the Weierstrass locus and weights in a given metric graph.

Let D be an effective divisor on Γ. There is an algorithmic way for determining all the minimum slopes of functions in $\mathrm{Rat}(D)$ along unit tangent vectors in Γ. This is based on chip-firing on metric graphs. More precisely, [Luo11] gives a generalization of Dhar's burning algorithm for metric graphs, which allows us to test whether a divisor is x-reduced for any point $x \in \Gamma$ and eventually to compute reduced divisors. See Definition 2.10, Algorithm 2.13 and Theorem 2.15 in [Luo11].

We can extract the minimum slopes from this procedure. Let x be a point of Γ and $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction at x. At step i of the algorithm, following the notations of [Luo11, Definition 2.10], we count the number n_i of indices $1 \leqslant j \leqslant J$ such that $Q_j^{(1)}$ contains a segment of Γ starting at x and supporting the direction ν. The number n_i is either zero or one and represents the number of chips that go through this segment toward the point x at step i. We denote by n the sum of the n_i's. It is the total number of chips that are brought to x by Dhar's algorithm via the branch supporting ν. This means that $s_0^\nu = -n$, which shows that the minimum slope on ν can be computed using Dhar's algorithm.

4.7.4 Tropical Weierstrass points in positive characteristic

The treatment of Weierstrass points for curves over positive characteristic fields suggests the following possible modification of the theory of tropical Weierstrass points in the isolated cases where the whole graph is Weierstrass. We replace the rank r with the integer

$$b = b(\Gamma, D) \coloneqq \min_{x \in \Gamma} D_x(x),$$

and define the Weierstrass locus as the subset of points $x \in \Gamma$ verifying $D_x(x) \geqslant b + 1$. The weight of a connected component C of this modified Weierstrass locus is modified by setting

$$\mu_W(C; D) \coloneqq \deg\left(D|_C\right) + (g(C) - 1)b - \sum_{\nu \in \partial^{\mathrm{out}} C} s_0^\nu(D).$$

This leads to a consistent theory on the tropical side, with the weights of components of the Weierstrass locus adding up to $d - b + bg$ (instead of $d - r + rg$). This is reminiscent of the setting of curves in the situation where the standard sequence of vanishing orders differs from the sequence $0, 1, \ldots, r$, cf. [Lak81]. However, at this point, we are not aware of any geometric meaning to this tropical count.

4.7.5 Weierstrass points of random combinatorial graphs

There exist combinatorial graphs without any Weierstrass points among their vertices (see Section 4.6.10). This seems, however, to be a rare phenomenon, as a computer verification of examples indicates. Examples of random graphs were created and visualized using Python, Matplotlib [Hun07], and NetworkX [HSS08].

Question 4.7.8. *What is the proportion of combinatorial graphs which do not have any Weierstrass point among their vertices? That is, what is the probability that a combinatorial graph on n vertices has no Weierstrass point?*

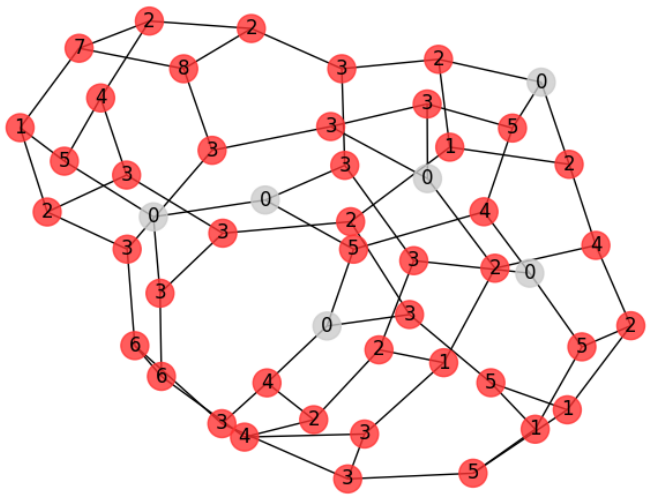


Figure 4.26: Random trivalent graph and its Weierstrass locus $L_W(K)$. The graph has genus 26, and the vertex labels indicate the coefficients $K_v(v) - 25$.

Randomness is understood within a class of graphs, for example regular graphs of given degree, or Erdős–Rényi random graphs. This is related to the following question of Baker.

Question 4.7.9 (Baker [Bak08])**.** *Provide a classification of combinatorial graphs without Weierstrass points among their vertices.*

4.8 Tropicalization of Weierstrass points

In this section, adapted from the Appendix of [AGR23], we describe the tropicalization of the Weierstrass divisor of a line bundle on a smooth curve over a non-Archimedean field. The notations and the presentation of the context have been kept, as well as the statement of the main results. For the proofs, we refer to the preprint cited above.

Let $\mathbf{K}$ be an algebraically closed complete non-Archimedean field with a non-trivial valuation denoted by $\mathbf{val}$. Let R, $\mathfrak{m}$, and $\kappa = \mathrm{R}/\mathfrak{m}$ be the valuation ring, the maximal ideal

of R, and the residue field, respectively. We also denote by $|\cdot|$ the corresponding norm on $\mathbf{K}$, so that $\mathbf{val}(\cdot) = -\log|\cdot|$.

Let X be a smooth proper curve defined over $\mathbf{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle, with $\mathcal{O} = \mathcal{O}_X$, the structure sheaf of X. Denote by ω_X the canonical line bundle on X.

Let $\mathrm{H} \subseteq H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, \mathrm{H})$ the corresponding Weierstrass divisor. We assume that the gap sequence of H is the standard sequence $0, 1, \ldots, r$, that is, for a general point $x \in X(\mathbf{K})$, the orders of vanishing of sections of $\mathcal{L}$ in H are $0, 1, \ldots, r$. In particular, if $\mathbf{K}$ is of characteristic zero, this is automatic.

4.8.1 Tropicalization

We denote by X^{an} the Berkovich analytification of X. We assume familiarity with the theory of Berkovich analytic curves, and refer to [AB15, Section 4] and [BPR16, Section 5] that contain what we need.

A semistable vertex set for X^{an} is a finite set of type 2 points V in X^{an} such that the complement $X^{\mathrm{an}} \smallsetminus V$ is a disjoint union of finitely many open annuli and infinitely many open disks. A semistable model for X is an integral proper relative curve $\mathfrak{X}$ over R with generic fiber $\mathfrak{X}_\eta = X$ and special fiber $\mathfrak{X}_0$ that is reduced and has nodal singularities. Any irreducible component of the special fiber $\mathfrak{X}_0$ of a semistable model $\mathfrak{X}$ gives a valuation on $\mathbf{K}(X)$ and defines a point of type 2 in X^{an}. The set V of points in X^{an} associated to the irreducible components of $\mathfrak{X}_0$ is a semistable vertex set for X^{an}. This process provides, in fact, a bijection between semistable vertex sets of X^{an} and semistable models of $\mathfrak{X}$ (see [BPR16, Thm. 5.38]). Moreover, each point of type 2 appears in a semistable vertex set.

A semistable vertex set V gives rise to a skeleton Γ for X^{an}, defined as the union in X^{an} of V and the skeletons of the open annuli in $X^{\mathrm{an}} \smallsetminus V$. The canonical metric on the skeletons of the open annuli gives the skeleton a metric graph structure, naturally embedded in X^{an}.

The underlying graph $G = (V, E)$ has vertex set V and edge set E in bijection with the set of open annuli in $X^{\mathrm{an}} \smallsetminus V$. There is an edge between a pair of vertices v and u in V for each open annulus whose closure contains the points v and u. Moreover, the edge length function $\ell\colon E \to (0, +\infty)$ associates to each edge of G the modulus of the corresponding annulus. Using the correspondence between semistable models and semistable vertex sets, the graph G is identified with the dual graph of $\mathfrak{X}_0$, the special fiber of $\mathfrak{X}$, with vertices in bijection with the irreducible components of $\mathfrak{X}_0$, and edges in bijection with the nodes of $\mathfrak{X}_0$. There is an edge $e = uv$ in G for each node that lies on the irreducible components associated to u and v. The length of an edge corresponds to the singularity degree in $\mathfrak{X}$ of the corresponding node.

For each point x of type 2 in X^{an}, the extension $\kappa(x)/\kappa$ is of transcendence degree one. We denote by C_x the corresponding smooth proper curve over κ. In a semistable model $\mathfrak{X}$ in which x is in the vertex set, C_x is the normalization of the irreducible component in $\mathfrak{X}_0$

associated to x, and $\kappa(x)$ is the function field of this component.

We denote by $\mathbb{B}_+$ the standard open ball in the Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}$. The complement of Γ in X^{an} is a disjoint union of open balls B_ν in bijection with $\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \smallsetminus \mathrm{T}_x(\Gamma)$ for all points x of type 2 in Γ, each isomorphic to $\mathbb{B}_+$. For a given ball B_ν in $X^{\mathrm{an}} \smallsetminus \Gamma$, the corresponding point x is the unique point in Γ that lies in the closure of B_ν. Denote by p_x^ν the point of $\mathrm{C}_x(\kappa)$ corresponding to $\nu \in \mathrm{T}_x(X^{\mathrm{an}}) \smallsetminus \mathrm{T}_x(\Gamma)$.

Let Γ be a metric graph skeleton of X^{an} with underlying graph $G = (V, E)$ and denote by $\tau\colon X^{\mathrm{an}} \to \Gamma$ the canonical retraction map. We call τ the tropicalization map. In the notations of the previous paragraph, the tropicalization map sends all the points in B_ν to the point x. The restriction of τ to $X(\mathbf{K}) \subseteq X^{\mathrm{an}}$ is compatible with the specialization map from the generic fiber $\mathfrak{X}_\eta$ to $\mathfrak{X}_0$, that is, a point specialized to a node is sent by τ to a point in the corresponding edge, and a point specialized to a smooth point of $\mathfrak{X}_0$ is sent by τ to the vertex of G corresponding to this component. We get a tropicalization map $\tau_*\colon \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$ that sends a divisor $\mathcal{D} = \sum_{x \in X(\mathbf{K})} a_x(x)$ on X to the divisor $\tau_*(\mathcal{D}) = \sum_{x \in X(\mathbf{K})} a_x(\tau(x))$.

We denote by $\mathfrak{v}_x\colon \mathbf{K}(X) \to \mathbb{R} \cup \{+\infty\}$ the valuation of a point $x \in X^{\mathrm{an}} \smallsetminus X(\mathbf{K})$ with $\mathfrak{v}_x(f) = +\infty$ only if $f = 0$. The residue field of this valuation is denoted by $\kappa(x)$. We also denote by $|\cdot|_x = \exp(-\mathfrak{v}_x)$ the corresponding norm.

For each non-zero $f \in \mathbf{K}(X)$, we define the tropicalization of f, denoted $\mathrm{trop}(f)\colon \Gamma \to \mathbb{R}$, as the map that sends each $x \in \Gamma \subseteq X^{\mathrm{an}} \smallsetminus X(\mathbf{K})$ to $\mathfrak{v}_x(f)$. This induces a tropicalization map $\mathrm{trop}\colon \mathbf{K}(X) \smallsetminus \{0\} \to \mathrm{Rat}(\Gamma)$.

For a vector subspace $\mathrm{H} \subset \mathbf{K}(X)$, we call $M = \mathrm{trop}(\mathrm{H} \smallsetminus \{0\})$ the tropicalization of H, and denote it, by a slight abuse of notation, by $\mathrm{trop}(\mathrm{H})$.

We define the genus function $\mathfrak{g}$ on X^{an} to be the genus of C_x for a point of type 2, extended by zero everywhere else. The restriction of $\mathfrak{g}$ to Γ gives an augmented metric graph of genus g equal to that of X. We denote by K the canonical divisor of the augmented metric graph $(\Gamma, \mathfrak{g})$, with $K(x) = 2\mathfrak{g}(x) - 2 + \mathrm{val}(x)$ for all $x \in \Gamma$.

4.8.2 Reduction

For a point of type 2, the valuation $\mathfrak{v}_x$ has the same value group as **val**. For each nonzero $f \in \mathbf{K}(X)$, choosing $a \in \mathbf{K}$ with $\mathbf{val}(a) = \mathfrak{v}_x(f)$, we get that $a^{-1}f$ has valuation $\mathfrak{v}_x(a^{-1}f) = 0$, and therefore gives an element in the residue field $\kappa(x)$ that we denote by $\tilde{f}_x$. We call this the reduction of f at x, which is nonzero and defined only up to multiplication by a non-zero scalar in κ. For a vector subspace $\mathrm{H} \subset \mathbf{K}(X)$ of dimension $r+1$, denote by $\widetilde{H}_x \subset \kappa(x)$ the κ-vector subspace spanned by the reductions $\tilde{f}_x$ of elements $f \in \mathrm{H}$ [AB15, Section 4.4]. By [AB15, Lemma 4.3], $\widetilde{H}_x$ has dimension $r+1$ over κ.

4.8.3 Slopes

For point x in Γ of type 2 in X^{an}, each unit tangent direction $\nu \in \mathrm{T}_x(\Gamma)$ gives a point $p_x^\nu \in \mathrm{C}_x(\kappa)$. By the slope formula [BPR16], for any non-zero $f \in \mathbf{K}(X)$, we have

$\mathrm{slope}_\nu(\mathrm{trop}(f)) = \mathrm{ord}_{p_x^\nu}\left(\tilde{f}_x\right)$. Moreover,

$$\tau_*(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f)).$$

If $\mathrm{H} \subset \mathbf{K}(X)$ is a $\mathbf{K}$-vector subspace of dimension $r+1$, for any point $x \in \Gamma$ and unit tangent vector $\nu \in \mathrm{T}_x(\Gamma)$, we get a collection of integers $\mathrm{slope}_\nu(\mathrm{trop}(f)) = \mathrm{ord}_{p_x^\nu}(\tilde{f}_x)$, $f \in \mathrm{H}$. Since $\widetilde{H}_x$ has dimension $r+1$, this collection has size $r+1$. This means that the collection of slopes $\mathrm{slope}_\nu(h)$, for $h \in M = \mathrm{trop}(\mathrm{H})$, has size $r+1$. In particular, Property $(\star)$ in Section 4.5 is satisfied by $M = \mathrm{trop}(\mathrm{H})$.

For each unit tangent vector ν, we order the slopes $\mathrm{slope}_\nu(h)$, for $h \in M$, in the form $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$. Since elements of $\mathrm{trop}(\mathrm{H})$ are piecewise affine linear, adding more points of Γ to the semistable vertex set, we can suppose that the set of slopes

$$s_0^\nu < s_1^\nu < \cdots < s_r^\nu$$

is constant in the interior of any edge of $G = (V, E)$ for parallel tangent directions ν at the point of the edge that point in the same direction.

4.8.4 Weierstrass divisor and Wronskian

Let X be a smooth proper curve defined over $\mathbf{K}$. Let $\mathcal{D}$ be a divisor of degree d on X and let $\mathcal{L} = \mathcal{O}(\mathcal{D})$ be the corresponding line bundle, with $\mathcal{O} = \mathcal{O}_X$, the structure sheaf of X. Denote by ω_X the canonical line bundle on X.

Let $\mathrm{H} \subseteq H^0(X, \mathcal{L})$ be a space of sections of dimension $r+1$ and denote by $\mathcal{W} = \mathcal{W}(\mathcal{D}, \mathrm{H})$ the corresponding Weierstrass divisor. The Weierstrass divisor $\mathcal{W}$ is the divisor of a global section of the line bundle $\omega_X^{\otimes \frac{r(r+1)}{2}} \otimes \mathcal{L}^{\otimes(r+1)}$ called the *Wronskian*. It is described as follows.

In local coordinates, for any point $p \in X(\mathbf{K})$, the local ring $\mathcal{O}_p$ is a discrete valuation ring. We choose a uniformizer that we denote by $\mathfrak{t}_p$. We have $\mathcal{L}_p \simeq \mathcal{O}_p$ as an $\mathcal{O}_p$-module. Taking the generator $g_p = \mathfrak{t}_p^{\mathcal{D}(p)}$ of $\mathcal{L}_p$, each global section f of $\mathcal{L}$ can be written in the form $f = f_p g_p$ with $f_p \in \mathcal{O}_p$.

We define the Hasse derivative of order j, for $j \in \mathbb{Z}_{\geqslant 0}$, on $\mathbf{K}[\mathfrak{t}_p]$ by

$$\mathrm{D}^{(j)}\mathfrak{t}^m = \binom{m}{j}\mathfrak{t}^{m-j} \qquad \text{for } m > 0,$$

and extend it by linearity to all $\mathbf{K}[\mathfrak{t}]$, and then to all $\mathbf{K}(\mathfrak{t})$. Since the extension $\mathbf{K}(X)/\mathbf{K}(\mathfrak{t})$ is separable, $\mathrm{D}^{(j)}$ is extended to $\mathbf{K}(X)$. Note that if $\mathbf{K}$ has characteristic zero, we can recursively define for any $j \geqslant 0$, the j-th derivative $f_p^{(j)}$ by

$$f_p^{(j)} = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}_p} f_p^{(j-1)}$$

with $f_p^{(0)} = f_p$. In this case, we have $j!\,\mathrm{D}^{(j)} f_p = f_p^{(j)}$.

Let $\mathcal{F} := \{f_0, \dots, f_r\}$ be a basis for $\mathrm{H} \subseteq H^0(X, \mathcal{L})$, and for each i, write $f_i = f_{i,p} g_p$. Viewing $\mathrm{Wr}_{\mathcal{F}}$ as a meromorphic section of $\omega_X^{\otimes \frac{r(r+1)}{2}}$, the stalk of the Wronskian $\mathrm{Wr}_{\mathcal{F}}$ at p is given by

$$\mathrm{Wr}_{\mathcal{F},p} = \det\Big(\mathrm{D}^{(j)} f_{i,p}\Big)_{0 \leqslant i,j \leqslant r} (\mathrm{d}\mathfrak{t}_p)^{\frac{r(r+1)}{2}} \in \omega_p^{\otimes \frac{r(r+1)}{2}}.$$

We have

$$\mathcal{W} = (r+1)\mathcal{D} + \mathrm{div}(\mathrm{Wr}_{\mathcal{F}}).$$

We note, without going into details, that the Wronskian $\mathrm{Wr}_{\mathcal{F}}$ can also be defined without local coordinates, in terms of a filtration of the jet bundle and the diagonal embedding of X.

Wronskian without local coordinates

Consider the diagonal embedding $\Delta\colon X \hookrightarrow X \times X$ and let $I = I_\Delta$ be the ideal of the diagonal in $X \times X$. Denote by π_1 and π_2 the projections of $X \times X$ to the first and second factors, respectively. The *jet bundle* $\mathcal{J}_r := \pi_{1*}(\pi_2^*(\mathcal{L})/I^{r+1})$ defined on X inherits a filtration induced by powers of I, and quotients are identified with $\mathcal{L} \otimes \omega_X^{\otimes i}$, for $i = 0, 1, \dots, r$.

Let $\mathcal{F} := \{f_0, \dots, f_r\}$ be a basis for $\mathrm{H} \subseteq H^0(X, \mathcal{L})$. Each section f_i of $\mathcal{L}$ defines a section $\pi_2^*(f_i)$ of $\mathcal{J}_r$. The determinant of the sections $\pi_2^*(f_i)$ with respect to the filtration by powers of I is well-defined, that is, does not depend on the choice of the basis $\mathcal{F}$, and gives a global section of $\omega_X^{\otimes r(r+1)/2} \otimes \mathcal{L}^{\otimes (r+1)}$. This is $\mathrm{Wr}_{\mathcal{F}}$. Since we assumed that the gap sequence is $0, 1, \dots, r$, the Wronskian $\mathrm{Wr}_{\mathcal{F}}$ is non-vanishing [Lak81].

The divisor $\mathcal{D}$ identifies $\omega_X^{\otimes \frac{r(r+1)}{2}} \otimes \mathcal{L}^{\otimes (r+1)}$ with the twist $\omega_X^{\otimes \frac{r(r+1)}{2}}((r+1)\mathcal{D})$ of the pluricanonical sheaf $\omega_X^{\otimes \frac{r(r+1)}{2}}$. Under this identification, sections become meromorphic pluricanonical forms with poles having order bounded at a point x of X by $(r+1)\mathcal{D}(x)$.

4.8.5 Slope formula for meromorphic differentials

We denote by $\|\cdot\|$ the Kähler norm introduced by Temkin in [Tem16] on the sheaf of differentials ω_X that at any point $x \in X^{\mathrm{an}}$ associates to any section α of ω_X the real number $\|\alpha\|_x$. For each positive integer n, the Kähler norm $\|\cdot\|$ induces a metric on $\omega_X^{\otimes n}$ which, by an abuse of notation, we still denote by $\|\cdot\|$. Given a meromorphic section α of $\omega_X^{\otimes n}$, the tropicalization of α denoted by $\mathrm{trop}(\alpha)$ is the map

$$\mathrm{trop}(\alpha)\colon \Gamma \to \mathbb{R}, \qquad x \mapsto -\log \|\alpha\|_x.$$

The tropicalization of any meromorphic n-form on X is a rational function on Γ, that is, $\mathrm{trop}(\alpha) \in \mathrm{Rat}(\Gamma)$. Moreover, the following slope formula holds.

Lemma 4.8.1 (Slope formula for meromorphic differentials)**.** *For any meromorphic section α of $\omega_X^{\otimes n}$, we have*

$$\tau_*(\mathrm{div}(\alpha)) = nK + \mathrm{div}(\mathrm{trop}(\alpha)).$$

Moreover, for any point $x \in \Gamma$ of type 2 and for any $\nu \in \mathrm{T}_x(\Gamma)$, we have $\mathrm{slope}_\nu(\mathrm{trop}(\alpha)) = \mathrm{ord}_{p_x^\nu}(\tilde{\alpha}_x) + n$.

Here, $\tilde{\alpha}_x$ is the reduction of α at x, and is a meromorphic form on C_x, see [TT22, Section 2].

The proof of Lemma 4.8.1 can be found in [AGR23, Appendix A].

4.8.6 Tropicalization of the Wronskian

Notations as in Section 4.8.4, let $F := \mathrm{trop}(\mathrm{Wr}_\mathcal{F})$ be the tropicalization of the Wronskian $\mathrm{Wr}_\mathcal{F}$, which is a meromorphic $(r(r+1)/2)$-form. Let $W = \tau_*(\mathcal{W})$ be the tropicalization of $\mathcal{W}$ to Γ. Let $D = \tau_*(\mathcal{D})$. The following result is a direct consequence of Lemma 4.8.1, with $\alpha = \mathrm{Wr}_\mathcal{F}$.

Theorem 4.8.2. *Notations as above, we have*

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu F,$$

where $F = \mathrm{trop}(\mathrm{Wr}_\mathcal{F})$. *Furthermore,* $\mathrm{slope}_\nu F = \frac{r(r+1)}{2} + \mathrm{ord}_{p_x^\nu} \widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$.

Here, $\widetilde{\mathrm{Wr}_{\mathcal{F}_x}}$ denotes the reduction of $\mathrm{Wr}_\mathcal{F}$ at x.

4.8.7 The Wronskian of analytic functions on annuli

Let $\mathbb{A}^1 = \mathrm{Spec}(\mathbf{K}[T])$ and $\mathbb{A}^{1,\mathrm{an}}$ be its Berkovich analytification. Let $A(\rho)$ be the closed annulus in $\mathbb{A}^{1,\mathrm{an}}$ of center 0 with outer radius one and inner radius $\rho \in (0,1)$,

$$A(\rho) = \big\{ x \in \mathbb{A}^{1,\mathrm{an}} \,\big|\, \rho \leqslant |T|_x \leqslant 1 \big\}.$$

Let $R(\rho)$ be the ring of analytic functions on $A(\rho)$. An analytic function f on $A(\rho)$ admits a formal power series expansion

$$f = \sum_{n \in \mathbb{Z}} a_n T^n$$

with $\lim_{n \to \pm\infty} |a_n| s^n = 0$ for all $s \in [\rho, 1]$. The skeleton of $A(\rho)$ is a closed interval, which can be identified with $I := [0, -\log \rho]$: each point q in this interval corresponds to the norm $|\cdot|_{\zeta_q}(f) = \sup_{n \in \mathbb{Z}} |a_n| \exp(-qn) = \max_{n \in \mathbb{Z}} |a_n| \exp(-qn)$ on any analytic function f as above. The tropicalization of an analytic function f is the function $\mathrm{trop}(f)$ on the interval I given by

$$\mathrm{trop}(f)(q) = \min\{ \mathbf{val}(a_n) + nq \,\big|\, n \in \mathbb{Z} \} \qquad \forall q \in I.$$

Let $\zeta = \zeta_0 \in A(\rho)$ be the boundary point corresponding to the extremity 0 of I, that is, $|f|_\zeta = \max_{n \in \mathbb{Z}} |a_n|$ on any analytic function f as above. The reduction at ζ of an analytic function f with $|f|_\zeta = 1$ is a Laurent polynomial

$$\tilde{f}_\zeta = \sum_{\substack{n \in \mathbb{Z} \\ |a_n| = 1}} \tilde{a}_n t^n$$

where t is the reduction of T at ζ and $\tilde{a}_n \in \kappa$ is a_n modulo $\mathfrak{m}$. The slope of $\mathrm{trop}(f)$ at 0 along the unit tangent direction $\nu \in \mathrm{T}_0(I)$ is the minimum exponent that appears in $\tilde{f}_\zeta$.

Let $f_0, \dots, f_r$ be $r+1$ $\mathbf{K}$-linearly independent analytic functions on $A(\rho)$ with

$$f_i = \sum_{n \in \mathbb{Z}} a_{i,n} T^n, \quad a_{i,n} \in \mathbf{K}$$

the analytic expansion of f_i. Suppose that $|f_i|_\zeta = 1$ for all $i = 0, 1, \dots, r$ and that $\mathrm{trop}(f_0), \dots, \mathrm{trop}(f_r)$ have slopes $s_0 < \cdots < s_r$ at 0 along the unit tangent direction $\nu \in \mathrm{T}_0(I)$. This means the reduction $\tilde{f}_{i,\zeta}$ of f_i at ζ has initial term t^{s_i}.

Consider the analytic function on $A(\rho)$ defined by

$$h := \det\left(\mathrm{D}^{(j)} f_i\right)_{i,j=0}^{r},$$

where $\mathrm{D}^{(j)} f_i$ denotes the Hasse derivative of f_i. The following proposition describes the slope of $\mathrm{trop}(h)$ at the point 0 along the direction ν.

Proposition 4.8.3. *Notations as above, assume*

- *either, the residue field κ is of characteristic zero,*
- *or, the sequence $s_0, \dots, s_r$ forms an interval, that is, $s_j = s_0 + j$ for all $j = 0, \dots, r$.*

Then, we have

$$\mathrm{slope}_\nu(\mathrm{trop}(h)) = s_0 + \cdots + s_r - \frac{r(r+1)}{2}.$$

The proof of Proposition 4.8.3 can be found in [AGR23, Appendix A].

4.8.8 Order of vanishing of the reduction of the Wronskian

A consequence of Proposition 4.8.3 is the following description of the slopes appearing in Theorem 4.8.2.

Proposition 4.8.4. *Let x be a point of type 2 in Γ and $\nu \in \mathrm{T}_x(\Gamma)$. Denote by $s_0^\nu, \dots, s_r^\nu$ the sequence of slopes associated by tropicalization of H to ν. Assume*

- *either, the residue field κ is of characteristic zero,*
- *or, the sequence $s_0^\nu, \dots, s_r^\nu$ forms an interval, that is, $s_j^\nu = s_0^\nu + j$.*

Then, we have

$$\mathrm{ord}_{p_x^\nu} \widetilde{\mathrm{Wr}_{\mathcal{F}_x}} = s_0^\nu + \cdots + s_r^\nu - \frac{r(r+1)}{2}.$$

The proof of Proposition 4.8.4 can be found in [AGR23, Appendix A].

4.8.9 Reduction of Weierstrass points in equal characteristic zero

Assume the residue field κ has characteristic zero. As in Section 4.8.3, denote by s_i^ν, $i = 0, \dots, r$, the slopes of functions of the form $\mathrm{trop}(f) \in \mathrm{trop}(\mathrm{H})$, for $f \in \mathrm{H}$.

Theorem 4.8.5. *Let $W = \tau_*(\mathcal{W})$. We have*

$$W(x) = (r+1)D(x) + \frac{r(r+1)}{2}K(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} \sum_{i=0}^{r} s_i^\nu.$$

Proof. This follows by combining Proposition 4.8.4 with Theorem 4.8.2. □

5 Perspectives

Numerous open questions and research perspectives emerged from the work presented in the current manuscript. Hereafter, we present some of them.

Contents of the chapter

5.1 Matricubes, representability questions and Schubert calculus

In Chapter 2, we provide cryptomorphic definitions of matricubes, which are a generalization of matroids, in terms of rank function, flats, circuits, and independents. To propose these definitions, we tried to stick as much as possible to their matroidal counterparts. However, some of these axiomatic systems seem to necessarily involve some dedicated combinatorial aspects related to the structure of matricubes. This is for example the case regarding independents, where our definition requires to define the "size" of independents to formulate the independent augmentation property (I2). It would be interesting to try to enhance these definitions and make them "purer", i.e., simpler and more parallel to the corresponding axiomatic systems for matroids.

As explained in Section 2.9.1, we were not able to give a relevant axiomatic system for the *bases* of matricubes. All the natural generalizations of bases of matroids we could think about (locally or globally maximal independents, independents locally or globally of maximal rank, etc.) yield a poor notion in matricubes, not cryptomorphic to the other definitions. A deeper understanding of matricubes would surely involve finding a relevant axiomatic system for bases, or a reasonable explanation why the formalism of matricubes does not allow to define bases.

In Section 2.9.2, we associate to every matricube an integer polymatroid and a matroid in a natural way. This enables, among others, to characterize the representability of the

matricube by that of the associated matroid (see Section 2.9.3). However, the question of characterizing matricubes representable over every field (Question 2.9.2) remains open. We also do not know if the representability of all local matroids of a matricube is sufficient for the matricube to be representable – Proposition 2.8.2 tells us that this condition is necessary. Furthermore, it would be interesting to understand the connections between the definitions of circuits and independents in matricubes and those given for polymatroids in [BCF23].

We explain in Section 2.9.7 (and, more thoroughly, in Chapter 3) how a linear series on a smooth proper curve with marked points naturally gives rise to a matricube, in this case called *geometric*. Question 2.9.6 asks whether every representable matricube is necessarily geometric, and whether one can bound the genus of a curve representing a geometric matricube.

We also define, in Section 2.9.6, a Tutte polynomial for matricubes, defined by a global formula involving the rank function in the exponents of the polynomial. Although this definition is a quite natural generalization of the Tutte polynomial for matroids, it does not appear to verify a recursive equation involving deletions and contractions, like its matroidal counterpart. A better understanding of this object could probably be achieved by exploring its connections with the Tutte polynomial for polymatroids defined in [CF22].

Finally, it is reasonable to think that matricubes could be a useful tool to study the geometry of Grassmannians. As exposed briefly in Section 2.9.4, there is a stratification of products of initial Grassmannians by matricubes, where each stratum is defined by asking that the intersection pattern of the tuple of initial flags be encoded by a given matricube. The equivalence between the realizability of a matricube and that of the associated natural matroid entails a one-to-one-correspondence between this stratification and the stratification of a unique Grassmannian. One could ask whether this correspondence has a geometric meaning. Furthermore, we hope that the formalism of matricubes could be used to formulate positivity results for the intersection theory of flag varieties, in order to extend the results about the Littlewood–Richardson rules proved by Coskun [Cos09] for two-step flag varieties.

5.2 Tropical linear series, Brill–Noether theory and smoothing results

In Chapter 3, we define a new, combinatorial theory of limit linear series, and start exploring the algebraic, combinatorial and topological properties of the spaces of functions defined in this framework. But these tropical spaces, which are infinite-dimensional contrary to the (well-studied) tropical subspaces of $\mathbb{T}^d$, probably conceal some well-guarded mysteries which have to be uncovered.

For instance, Question 3.5.11 asks, for a closed sub-semimodule M of $\mathrm{Rat}(D, \mathfrak{S})$, whether there is a connection between M being finitely generated and M being of finite tropical rank, i.e., between M having a finite number of extremals and M being of finite tropical rank.

Also, not much is known, for these spaces, about the connections between the cardinality of generating sets and several notions of rank (divisorial rank, tropical rank, Kapranov rank, and Barnikov rank) defined by various authors, mostly in the finite-dimensional context [DSS03]. It seems, moreover, that some supplementary combinatorial tools would be needed to check more easily whether a given admissible semimodule is finitely generated, or compute its tropical rank.

A firmer grasp on the properties of tropical semimodules of functions should be useful in tropical Brill–Noether theory. There is a well-known result by Baker stating that in the tropical realm, like in the algebro-geometric world, the property of the Brill–Noether number $\rho(d,r,g) := g-(r+1)(g-d+r)$ being *non-negative* implies the existence of divisors of degree d and rank at least r on a given metric graph Γ [Bak08, Theorem 3.20]. The surprising fact is that the only known proof as of today is based on the geometric analogue of the theorem and the use of Baker's specialization lemma.

We hope that better understanding combinatorial linear series and developing relevant tools in the tropical linear algebra of semimodules could help to give a direct combinatorial proof of the tropical Brill–Noether theorem. Moreover, if the Brill–Noether number is *negative*, we do not know in general whether the tropical counterpart of the converse of the algebro-geometric Brill–Noether theorem holds. Cartwright, Jensen and Payne showed that in the particular case where Γ is a chain of g loops of generic lengths, then $\rho(d,r,g) < 0$ implies that there are no divisors of degree d and rank at least r on Γ [CJP15]; but this question is still open for general metric graphs and may be solved using combinatorial linear series.

In Section 3.8, we classify rank one combinatorial limit linear series on a metric graph, by providing a bijection with finite harmonic morphisms from the metric graph to metric trees (Theorem 3.8.6). Another interesting goal is to do the same for *higher rank* tropical linear series. This means, for instance, finding correspondences with morphisms from the metric graph to some models of tropical projective spaces. To obtain such a classification of combinatorial limit linear series, one could try to make a precise connection between the content of Chapter 3 and a parallel work by Jensen and Payne [JP22] in which they define *strong tropical linear series*, a notion that relies on the data of a valuated matroid.

Another natural open question arises from the study of concrete examples of (crude) linear series, in particular the examples in Sections 3.10.2 (rank two crude linear series on a tripod graph) and 3.10.1 (rank two linear series on a dipole graph). In part of these examples, it turns out that with some fixed given choice of rank functions, there exist whole *families* of crude linear series depending on a choice of locations for the vertices of a combinatorial model of the slope structure. More simply said, we can sometimes move the points where the allowed set of slopes changes, continuously along some edges, which provides a *space* of crude linear series. It would be interesting to study these spaces for their own sake – understand their structure, compute their dimension.

5.3 Further exploration of tropical Weierstrass points

In Chapter 4, we explore the concept of tropical Weierstrass points on metric graphs and the topological properties of tropical Weierstrass loci. We also associate intrinsic weights to each connected component of these loci, and show that, in addition to having nice combinatorial properties, these weights count the number of algebraic Weierstrass points which are being tropicalized to each connected component of the tropical locus, if the graph and divisor come from geometry.

This raises the question of the combinatorial diversity achieved by tropical Weierstrass loci. In Section 4.6 and in other examples disseminated in Chapter 4, we give an overview of this diversity: the tropical Weierstrass locus can be finite or infinite, connected or not (and there can be many connected components), supported by the set of vertices or disjoint from the vertices of a combinatorial model, and can even *cover the whole metric graph*.

Section 4.6.5 provides a countably infinite family of such examples where the whole graph is Weierstrass, using the canonical divisor on complete graphs endowed with a uniform length function; it also presents an uncountably infinite family of examples, using cycles of arbitrary lengths joined to a central vertex by bridge edges of arbitrary lengths, with a divisor concentrated on the central vertex. This second family is particularly striking, since it shows that the quantity

$$\min_{x\in\Gamma}(D_x(x)-r),$$

which compares the minimum, over all points $x \in \Gamma$, of the largest coefficient at x in the linear system of D, with the expected number r, is *unbounded*.

These two families are however very different from a combinatorial point of view: the first one has high connectivity whereas the second one has many bridge edges. This suggests to try and provide a classification of all combinatorial graphs G that admit a length function and a divisor with Weierstrass locus the whole metric graph. Among them, it would also be interesting to determine for which ones this property holds for every choice of edge lengths. See Question 4.6.2.

The observation that the quantity $b := \min_{x\in\Gamma}(D_x(x)-r)$ can be positive suggests the following variant of the theory of tropical Weierstrass points: replace r by b in the definition of Weierstrass points. The total weight of the tropical Weierstrass locus on a metric graph is still a nice function of the basic combinatorial parameters, but a natural direction of research would be to find a possible *geometric meaning* to this.

Another open question related to the tropical Weierstrass loci associated with combinatorial graphs is raised in Section 4.6.10. It suggests to establish a classification of *combinatorial graphs* which have *no Weierstrass points*, i.e., such that endowing them with a uniform length function yields a metric graph whose tropical Weierstrass locus is disjoint from the vertices.

In Section 4.7.1, finally, we define the *total Weierstrass locus* of a divisor on a metric graph as the union of the Weierstrass loci for every choice of length functions, with a relevant normalization. In a nutshell, this locus registers all the possible tropical Weierstrass

points as we change the graph's metric. Theorems 4.7.6 and 4.7.7 imply that this subset is closed, as could be expected. It would be interesting to answer Question 4.7.1, which asks how this total locus behaves: what number of connected components can it have on each edge? What can its Lebesgue measure be? How does its boundary behave? Which of its points come from geometry? And finally, as we vary the edge lengths, what is the maximal proportion of a given edge which is covered by tropical Weierstrass points? So far, we only provide a partial answer to the question related to connectedness, by providing in Theorem 4.7.5 an example where the restriction of the total Weierstrass locus to a given edge is *disconnected*.

5.4 Riemann–Roch theorem in higher dimension and tropical Hodge theory

The combinatorial limit linear series presented in Chapter 3 are defined on algebraic curves and metric graphs (therefore, one-dimensional tropical varieties), and it is natural to look for higher-dimensional generalizations. In [Car21], Cartwright studies a generalization of dual graphs to higher-dimensional algebraic varieties, by associating polyhedral complexes enriched with additional combinatorial data. This allows him to formulate a conjectural version of the Riemann–Roch theorem for surfaces [Car21, Conjecture 3.6].

A promising direction of research could be to try and prove an inequality of this form, using the language of combinatorial Chow rings to expand Cartwright's formalism with hybrid data. Our current ongoing work on this topic is based upon recent developments, including the combinatorial Hodge theory developed by Adiprasito–Huh–Katz [AHK18], and the tropical Hodge theory developed by Amini and Piquerez [AP20, AP23]. It would be interesting as well to extend the formalism of slope structures and combinatorial limit linear series to higher dimension.

Bibliography

[AB15] Omid Amini and Matthew Baker. Linear series on metrized complexes of algebraic curves. *Mathematische Annalen*, 362(1-2):55–106, 2015.

[ABBR15a] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. *Research in the Mathematical Sciences*, 2(1):1–67, 2015.

[ABBR15b] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff. Lifting harmonic morphisms II: tropical curves and metrized complexes. *Algebra & Number Theory*, 9(2):267–315, 2015.

[AC13] Omid Amini and Lucia Caporaso. Riemann–Roch theory for weighted graphs and tropical curves. *Advances in Mathematics*, 240:1–23, 2013.

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):765–809, 2015.

[AG22] Omid Amini and Lucas Gierczak. Limit linear series: combinatorial theory. *arXiv preprint arXiv:2209.15613*, 2022.

[AG24] Omid Amini and Lucas Gierczak. Combinatorial flag arrangements. *arXiv preprint arXiv:2404.01971*, 2024.

[AGG09] Marianne Akian, Stéphane Gaubert, and Alexander Guterman. Linear independence over tropical semirings and beyond. *Contemporary Mathematics*, 495(1):1–38, 2009.

[AGR23] Omid Amini, Lucas Gierczak, and Harry Richman. Tropical Weierstrass points and Weierstrass weights. *arXiv preprint arXiv:2303.07729*, 2023.

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. Hodge theory for combinatorial geometries. *Annals of Mathematics*, 188(2):381–452, 2018.

[Ami13] Omid Amini. Reduced divisors and embeddings of tropical curves. *Transactions of the American Mathematical Society*, 365(9):4851–4880, 2013.

[Ami14] Omid Amini. Equidistribution of Weierstrass points on curves over non-Archimedean fields. *arXiv preprint arXiv:1412.0926*, 2014.

[AN22] Omid Amini and Noema Nicolussi. Moduli of hybrid curves II: Tropical and hybrid Laplacians. *arXiv preprint arXiv:2203.12785*, 2022.

[AN24] Omid Amini and Noema Nicolussi. Moduli of hybrid curves III: Intrinsic geometry of hybrid curves and metric degeneration. Preprint, 2024.

[AP03] Scott Ahlgren and Matthew Papanikolas. Higher Weierstrass points on $X_0(p)$. *Trans. Amer. Math. Soc.*, 355(4):1521–1535, 2003.

[AP20] Omid Amini and Matthieu Piquerez. Hodge theory for tropical varieties. *arXiv preprint arXiv:2007.07826*, 2020.

[AP23] Omid Amini and Matthieu Piquerez. Hodge theory for tropical fans. *arXiv preprint arXiv:2310.15367*, 2023.

[Arb74] Enrico Arbarello. Weierstrass points and moduli of curves. *Compos. Math.*, 29(3):325–342, 1974.

[Atk67] A. O. L. Atkin. Weierstrass points at cusps of $\Gamma_0(n)$. *Ann. of Math.*, pages 42–45, 1967.

[Bak08] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra & Number Theory*, 2(6):613–653, 2008.

[BB19] Matthew Baker and Nathan Bowler. Matroids over partial hyperstructures. *Advances in Mathematics*, 343:821–863, 2019.

[BCF23] Joseph E. Bonin, Carolyn Chun, and Tara Fife. The natural matroid of an integer polymatroid. *SIAM Journal on Discrete Mathematics*, 37(3):1751–1770, 2023.

[BCG+18] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller. Compactification of strata of Abelian differentials. *Duke Mathematical Journal*, 167(12):2347–2416, 2018.

[BCG+19] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller. Strata of k-differentials. *Algebraic Geometry*, 6(2):196–233, 2019.

[BDM12] Erwan A. Brugallé and Lucía M. López De Medrano. Inflection points of real and tropical plane curves. *Journal of Singularities*, 4:74–103, 2012.

[BDP18] Guus P. Bollen, Jan Draisma, and Rudi Pendavingh. Algebraic matroids and Frobenius flocks. *Advances in mathematics*, 323:688–719, 2018.

[Ber71] George M. Bergman. The logarithmic limit-set of an algebraic variety. *Transactions of the American Mathematical Society*, pages 459–469, 1971.

[BG95] Edoardo Ballico and Letterio Gatto. On the monodromy of Weierstrass points on Gorenstein curves. *J. Algebra*, 175(2):633–643, 1995.

[BGW03] Alexandre Borovik, Izrail Moiseevich Gelfand, and Neil White. *Coxeter matroids*. Springer, 2003.

[BJ16] Matthew Baker and David Jensen. Degeneration of linear series from the tropical point of view and applications. In *Nonarchimedean and Tropical Geometry*, pages 365–433. Springer, 2016.

[BL20] Matthew Baker and Oliver Lorscheid. Foundations of matroids I: Matroids without large uniform minors. *arXiv preprint arXiv:2008.00014*, 2020.

[BL21] Matthew Baker and Oliver Lorscheid. The moduli space of matroids. *Advances in Mathematics*, 390:107883, 2021.

[BN07] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Advances in Mathematics*, 215(2):766–788, 2007.

[BN16] Matthew Baker and Johannes Nicaise. Weight functions on Berkovich curves. *Algebra & Number Theory*, 10(10):2053–2079, 2016.

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom*, 3(1):63–105, 2016.

[BS13] Matthew Baker and Farbod Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. *J. Combin. Theory Ser. A*, 120(1):164–182, 2013.

[BT20] Uri Brezner and Michael Temkin. Lifting problem for minimally wild covers of Berkovich curves. *Journal of Algebraic Geometry*, 29(1):123–166, 2020.

[BV08] Sara Billey and Ravi Vakil. Intersections of Schubert varieties and other permutation array schemes. In *Algorithms in algebraic geometry*, pages 21–54. Springer, 2008.

[Cap15] Lucia Caporaso. Algebraic and tropical curves: comparing their moduli spaces. In *Handbook of Moduli. Volume I*, pages 119–160. Somerville, MA: International Press, 2015.

[Car19] Dustin Cartwright. Tropical complexes. *Manuscripta Mathematica*, pages 1–35, 2019.

[Car21] Dustin Cartwright. A specialization inequality for tropical complexes. *Compositio Mathematica*, 157(5):1051–1078, 2021.

[CD18] Filip Cools and Jan Draisma. On metric graphs with prescribed gonality. *Journal of Combinatorial Theory, Series A*, 156:1–21, 2018.

[CDMS17] Amanda Cameron, Rodica Dinu, Mateusz Michałek, and Tim Seynnaeve. Flag matroids: algebra and geometry. In *International Conference on Interactions with Lattice Polytopes*, pages 73–114. Springer, 2017.

[CEG08] Caterina Cumino, Eduardo Esteves, and Letterio Gatto. Limits of special Weierstrass points. *Int. Math. Res. Pap. IMRP*, 2, 2008.

[CF91] Fernando Cukierman and Lung-Ying Fong. On higher Weierstrass points. *Duke Math. J.*, 62(1):179–203, 1991.

[CF22] Amanda Cameron and Alex Fink. The Tutte polynomial via lattice point counting. *Journal of Combinatorial Theory, Series A*, 188:105584, 2022.

[CHL+22] Colin Crowley, June Huh, Matt Larson, Connor Simpson, and Botong Wang. The Bergman fan of a polymatroid. *arXiv preprint arXiv:2207.08764*, 2022.

[CJP15] Dustin Cartwright, David Jensen, and Sam Payne. Lifting divisors on a generic chain of loops. *Canadian Mathematical Bulletin*, 58(2):250–262, 2015.

[Cos09] Izzet Coskun. A Littlewood–Richardson rule for two-step flag varieties. *Inventiones mathematicae*, 176(2):325–395, 2009.

[Csi20] Laszlo Csirmaz. Cyclic flats of a polymatroid. *Annals of Combinatorics*, 24(4):637–648, 2020.

[Cuk89] Fernando Cukierman. Families of Weierstrass points. *Duke Math. J.*, 58(2):317–346, 1989.

[DC08] Andrea Del Centina. Weierstrass points and their impact in the study of algebraic curves: a historical account from the "Lückensatz" to the 1970s. *Annali dell'Universita di Ferrara*, 54(1):37–59, 2008.

[dCS94] Cicero Fernandes de Carvalho and Karl-Otto Stöhr. Higher order differentials and Weierstrass points on Gorenstein curves. *Manuscripta Math.*, 85:361–380, 1994.

[Dia85] Steven Diaz. Exceptional Weierstrass points and the divisor on moduli space that they define. *Mem. Amer. Math. Soc.*, 56(327), 1985.

[DSS03] Mike Develin, Francisco Santos, and Bernd Sturmfels. On the rank of a tropical matrix. *arXiv preprint math/0312114*, 2003.

[DV21a] Jan Draisma and Alejandro Vargas. Catalan-many tropical morphisms to trees; Part I: Constructions. *Journal of symbolic computation*, 104:580–629, 2021.

[DV21b] Jan Draisma and Alejandro Vargas. On the gonality of metric graphs. *Notices of the American Mathematical Society*, 68(5):687–695, 2021.

[EH86] David Eisenbud and Joe Harris. Limit linear series: basic theory. *Inventiones Mathematicae*, 85(2):337–371, 1986.

[EH87a] David Eisenbud and Joe Harris. Existence, decomposition, and limits of certain Weierstrass points. *Inventiones Mathematicae*, 87(3):495–515, 1987.

[EH87b] David Eisenbud and Joe Harris. The Kodaira dimension of the moduli space of curves of genus $\geqslant 23$. *Inventiones Mathematicae*, 90(2):359–387, 1987.

[EH87c] David Eisenbud and Joe Harris. The monodromy of Weierstrass points. *Inventiones Mathematicae*, 90(2):333–341, 1987.

[EH89] David Eisenbud and Joe Harris. Progress in the theory of complex algebraic curves. *Bulletin of the American Mathematical Society*, 21(2):205–232, 1989.

[EL00a] Kimmo Eriksson and Svante Linusson. A combinatorial theory of higher-dimensional permutation arrays. *Advances in Applied Mathematics*, 25(2):194–211, 2000.

[EL00b] Kimmo Eriksson and Svante Linusson. A decomposition of $\mathrm{Fl}(n)^d$ indexed by permutation arrays. *Advances in Applied Mathematics*, 25(2):212–227, 2000.

[EL23] Christopher Eur and Matt Larson. Intersection theory of polymatroids. *arXiv preprint arXiv:2301.00831*, 2023.

[EM02] Eduardo Esteves and Nivaldo Medeiros. Limit canonical systems on curves with two components. *Inventiones Mathematicae*, 149(2):267–338, 2002.

[ES07] Eduardo Esteves and Parham Salehyan. Limit Weierstrass points on nodal reducible curves. *Trans. Amer. Math. Soc.*, 359(10):5035–5056, 2007.

[Est98] Eduardo Esteves. Linear systems and ramification points on reducible nodal curves. *Mat. Contemp.*, 14:21–35, 1998.

[FGP14] Tyler Foster, Philipp Gross, and Sam Payne. Limits of tropicalizations. *Israel Journal of Mathematics*, 201:835–846, 2014.

[FJP20] Gavril Farkas, David Jensen, and Sam Payne. The Kodaira dimensions of $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$. *arXiv preprint arXiv:2005.00622*, 2020.

[Fuj05] Satoru Fujishige. *Submodular functions and optimization*, volume 58 of *Annals of Discrete Mathematics*. Elsevier, 2005.

[Gau92] Stéphane Gaubert. *Théorie des systèmes linéaires dans les dioïdes*. PhD thesis, Thèse, École des mines de Paris, 1992.

[Gen21] Quentin Gendron. Sur les nœuds de Weierstraß. *Annales Henri Lebesgue*, 4:571–589, 2021.

[GK08] Andreas Gathmann and Michael Kerber. A Riemann–Roch theorem in tropical geometry. *Mathematische Zeitschrift*, 259(1):217–230, 2008.

[GL95] Arnaldo García and Robert F. Lax. On canonical ideals, intersection numbers, and Weierstrass points on Gorenstein curves. *J. Algebra*, 178(3):807–832, 1995.

[He19] Xiang He. Smoothing of limit linear series on curves and metrized complexes of pseudocompact type. *Canadian Journal of Mathematics*, 71(3):629–658, 2019.

[Hel06] Thorkell Helgason. Aspects of the theory of hypermatroids. In *Hypergraph Seminar: Ohio State University 1972*, pages 191–213. Springer, 2006.

[HH02] Jürgen Herzog and Takayuki Hibi. Discrete polymatroids. *Journal of Algebraic Combinatorics*, 16:239–268, 2002.

[HKN13] Jan Hladkỳ, Daniel Král, and Serguei Norine. Rank of divisors on tropical curves. *J. Combin. Theory Ser. A*, 120(7):1521–1538, 2013.

[HMY12] Christian Haase, Gregg Musiker, and Josephine Yu. Linear systems on tropical curves. *Mathematische Zeitschrift*, 270(3):1111–1140, 2012.

[HSS08] Aric A. Hagberg, Daniel A. Schult, and Pieter J. Swart. Exploring network structure, dynamics, and function using NetworkX. In Gaël Varoquaux, Travis Vaught, and Jarrod Millman, editors, *Proceedings of the 7th Python in Science Conference*, pages 11–15, Pasadena, CA USA, 2008.

[Hun07] J. D. Hunter. Matplotlib: A 2D graphics environment. *Computing in Science & Engineering*, 9(3):90–95, 2007.

[Hur92] Adolf Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.*, 41(3):403–442, 1892.

[JL24] Manoel Jarra and Oliver Lorscheid. Flag matroids with coefficients. *Advances in Mathematics*, 436:109396, 2024.

[JP14] David Jensen and Sam Payne. Tropical independence I: shapes of divisors and a proof of the Gieseker–Petri theorem. *Algebra & Number Theory*, 8(9):2043–2066, 2014.

[JP16] David Jensen and Sam Payne. Tropical independence II: The maximal rank conjecture for quadrics. *Algebra & Number Theory*, 10(8):1601–1640, 2016.

[JP22] David Jensen and Sam Payne. Tropical linear series and tropical independence. *preprint*, 2022.

[KRZ16] Eric Katz, Joseph Rabinoff, and David Zureick-Brown. Uniform bounds for the number of rational points on curves of small Mordell–Weil rank. *Duke Mathematical Journal*, 165(16):3189–3240, 2016.

[Lak81] Dan Laksov. Weierstrass points on curves. *Astérisque*, 87(88):221–247, 1981.

[Lax75] Robert F. Lax. Weierstrass points of the universal curve. *Math. Ann.*, 216:35–42, 1975.

[Lax87] Robert F. Lax. Weierstrass points on rational nodal curves. *Glasgow Math. J.*, 29(1):131–140, 1987.

[LM18] Ye Luo and Madhusudan Manjunath. Smoothing of limit linear series of rank one on saturated metrized complexes of algebraic curves. *Canadian Journal of Mathematics*, 70(3):628–682, 2018.

[LN64] Joseph Lehner and Morris Newman. Weierstrass points of $\Gamma_0(n)$. *Ann. of Math.*, 85:360–368, 1964.

[Lov77] László Lovász. Flats in matroids and geometric graphs. In *Combinatorial Surveys (Proc. 6th British Combinatorial Conference)*, pages 45–86, 1977.

[Luo11] Ye Luo. Rank-determining sets of metric graphs. *J. Combin. Theory Ser. A*, 118(6):1775–1793, 2011.

[LW90] Robert F. Lax and Carl Widland. Weierstrass points on Gorenstein curves. *Pacific J. Math.*, 142(1):197–208, 1990.

[McD73] Colin McDiarmid. Independence structures and submodular functions. *Bulletin of the London Mathematical Society*, 5(1):18–20, 1973.

[MR18] Diane Maclagan and Felipe Rincón. Tropical ideals. *Compositio Mathematica*, 154(3):640–670, 2018.

[MS21] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161. American Mathematical Society, 2021.

[Mur98] Kazuo Murota. Discrete convex analysis. *Mathematical Programming*, 83:313–371, 1998.

[MUW20] Martin Möller, Martin Ulirsch, and Annette Werner. Realizability of tropical canonical divisors. *Journal of the European Mathematical Society*, 23(1):185–217, 2020.

[MZ08] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and Abelian varieties*, volume 465, pages 203–230. Contemporary Mathematics, 2008.

[Nee84] Amnon Neeman. Weierstrass points in characteristic p. *Invent. Math.*, 75(2):359–376, 1984.

[Ogg78] Andrew P. Ogg. On the Weierstrass points of $X_0(n)$. *Illinois J. Math.*, 22(1):31–35, 1978.

[Oss19a] Brian Osserman. Limit linear series and the Amini–Baker construction. *Mathematische Zeitschrift*, 293(1-2):339–369, 2019.

[Oss19b] Brian Osserman. Limit linear series for curves not of compact type. *Journal für die reine und angewandte Mathematik*, 2019(753):57–88, 2019.

[OSW16] James G. Oxley, Charles Semple, and Geoff Whittle. A wheels-and-whirls theorem for 3-connected 2-polymatroids. *SIAM Journal on Discrete Mathematics*, 30(1):493–524, 2016.

[OSW19] James G. Oxley, Charles Semple, and Geoff Whittle. A splitter theorem for 3-connected 2-polymatroids. *Electronic Journal of Combinatorics*, 26(Paper 2.37):95pp, 2019.

[Oxl06] James G. Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[Pay09] Sam Payne. Analytification is the limit of all tropicalizations. *Mathematical Research Letters*, 16(3):543–556, 2009.

[Ric24] David Harry Richman. The distribution of Weierstrass points on a tropical curve. *Selecta Mathematica*, 30(2):1–48, 2024.

[Sch03] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003.

[Sey80] Paul Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory, Series B*, 28(3):305–359, 1980.

[Spe08] David E. Speyer. Tropical linear spaces. *SIAM Journal on Discrete Mathematics*, 22(4):1527–1558, 2008.

[Sta11] Richard P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge studies in advanced mathematics, 2011.

[Tem16] Michael Temkin. Metrization of differential pluriforms on Berkovich analytic spaces. In *Nonarchimedean and tropical geometry*, pages 195–285. Springer, 2016.

[Tru92] Klaus Truemper. *Matroid decomposition*, volume 6. Academic Press Boston, 1992.

[TT22] Michael Temkin and Ilya Tyomkin. Reduction and lifting problem for differential forms on Berkovich curves. *Advances in Mathematics*, 397:108208, 2022.

[Wei67] Karl Weierstrass. *Mathematische Werke, 7 vols (1894–1927) Berlin, reprint by G.* Olms, Hildesheim, 1967.

[Whi92] Hassler Whitney. On the abstract properties of linear dependence. *Hassler Whitney Collected Papers*, pages 147–171, 1992.

ECOLE
DOCTORALE
DE MATHEMATIQUES
HADAMARD

Titre : Séries linéaires tropicales, arrangements combinatoires de drapeaux et applications à l'étude des points de Weierstrass

Mots clés : Géométrie tropicale, Séries linéaires, Matroïdes, Points de Weierstrass, Espaces de modules

Résumé : Nous introduisons d'abord un nouvel objet combinatoire nommé "matricube", une généralisation naturelle des matroïdes. De même que les matroïdes fournissent une axiomatisation combinatoire des arrangements d'hyperplans dans un espace vectoriel, les matricubes représentent des arrangements de drapeaux. Comme pour les matroïdes, nous donnons des définitions cryptomorphes des matricubes en termes de fonction de rang, plats, circuits et indépendants. Nous donnons des liens précis entre les matricubes et les tableaux de permutation, et proposons une description des matricubes selon des matroïdes locaux.
Nous utilisons ensuite les matricubes pour développer une théorie purement combinatoire des séries linéaires sur les graphes métriques. Ceci se fonde également sur le formalisme des structures de pentes, donnant des contraintes sur les pentes des fonctions méromorphes tropicales. Nous montrons que les séries linéaires combinatoires apparaissent naturellement par tropicalisation de séries linéaires sur des courbes algébriques. Nous explorons leurs propriétés topologiques et développons des outils pour les étudier. Nous proposons une classification complète des séries linéaires combinatoires de rang un, montrant qu'elles sont en bijection avec les morphismes harmoniques du graphe vers des arbres métriques. Ceci donne un théorème de lissification.
Enfin, nous étudions les points de Weierstrass tropicaux, qui sont des analogues, sur les courbes tropicales, de points de ramification de fibrés en droites sur les courbes algébriques. Le lieu de Weierstrass tropical d'un diviseur sur un graphe métrique peut être infini. Cependant, nous associons un poids intrinsèque à chacune de ses composantes connexes. Nous montrons que le poids total du lieu de Weierstrass tropical ne dépend que du degré du diviseur, de son rang, et du genre de la courbe tropicale. De plus, dans le cas d'un graphe métrique obtenu comme tropicalisation d'une courbe algébrique, nous montrons, en utilisant les séries linéaires combinatoires, que ces poids comptent le nombre de points de Weierstrass algébriques tropicalisés sur chaque composante connexe du lieu tropical.
Dans chacune de ces contributions, nous présentons de nombreux exemples et posons des questions ouvertes menant vers d'autres perspectives de recherche.

Title : Tropical linear series, combinatorial flag arrangements and applications to the study of Weierstrass points

Keywords : Tropical geometry, Linear series, Matroids, Weierstrass points, Moduli spaces

Abstract : We first introduce new combinatorial objects called "matricubes", a natural generalization of matroids. In the same way that matroids provide a combinatorial axiomatization of hyperplane arrangements in a vector space, matricubes abstract arrangements of flags. As for matroids, we provide cryptomorphic definitions of matricubes in terms of rank function, flats, circuits, and independent sets. We provide precise connections between matricubes and permutation arrays, and propose a description of matricubes in terms of local matroids.
We then use matricubes to develop a purely combinatorial theory of limit linear series on metric graphs. This is based as well on the formalism of slope structures, which constrains the slopes of tropical meromorphic functions. We show that combinatorial linear series arise naturally by tropicalizing linear series on algebraic curves. We explore their topological properties and develop tools to study them. We provide a full classification of combinatorial linear series of rank one, showing that they are in one-to-one correspondence with harmonic morphisms from the graph to metric trees. This entails a smoothing theorem.
Finally, we study tropical Weierstrass points, which are analogues, on tropical curves, of ramification points of line bundles on algebraic curves. The tropical Weierstrass locus of a divisor on a metric graph can be infinite. Nevertheless, we associate intrinsic weights to each of its connected components. We prove that the total weight of the tropical Weierstrass locus depends only on the degree of the divisor, its rank, and the genus of the tropical curve. Furthermore, in the case the metric graph is the tropicalization of an algebraic curve, we show, using combinatorial linear series, that our weights count the number of algebraic Weierstrass points which are tropicalized to each connected component of the tropical locus.
In each of these contributions, we discuss multiple examples and ask open questions leading to other perspectives of research.

Institut polytechnique de Paris
91120 Palaiseau, France