



Capability approach inspired tools for aiding policy design

Nicolas Fayard

► To cite this version:

Nicolas Fayard. Capability approach inspired tools for aiding policy design. Operations Research [math.OC]. Université Paris sciences et lettres, 2024. English. ⟨NNT : 2024UPSLD043⟩. ⟨tel-04903814⟩

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THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à Université Paris-Dauphine - PSL

Capability approach inspired tools for aiding policy design.

Soutenue par

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Le 9 décembre 2024

École doctorale n°543

SDOSE

Spécialité

Informatique

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Acknowledgements

I would like to express my heartfelt gratitude to my supervisor, Alexis Tsoukiàs, for his invaluable advice and thoughtful guidance throughout this research. His insightful advice greatly enhanced the quality of my work. In addition to academic support, I truly cherished the moments spent at conferences, which enriched my journey beyond the academic setting. To defend, two things needed Alexis's approval: that the manuscript was ready and that his Ph.D. student knew how to make *vrai café*.

I would also like to extend my sincere thanks to my collaborators, Chabane Mazri, David Ríos Insua and Giovanna Fancello, for their contributions and insightful discussions, which played a crucial role in the successful completion of this work. A special thanks to David for graciously hosting me in Madrid for several months, and for helping me refine my work into a human-readable paper. I am equally grateful to Giovanna without her, no case study.

I would also like to thank the members of my jury, Nadia Brauner, Léa Deleris, Patrice Perny, Marc Pirlot, and Remzi Sanver, for their time and careful evaluation of my thesis. A special thanks to the two reviewers, Patrice Perny and Marc Pirlot, and my appreciation to Patrice Perny and Daniel Vanderpooten for their feedback during the pre-defense. Their advice led to the restructuring of the second part of the manuscript and the addition of Chapter 5. My thanks also go to Marc Pirlot for the memorable discussion about *les coins* in Hammamet.

My sincere appreciation goes to all the members of LAMSADE. A special thanks to Meltem Öztürk Escoffier, who inspired me to pursue research and with whom I did my first research internship several years ago. The friendships and memories I have made—whether in bureau C.605, the CROUS, the *salle de convivialité*, the terrace, or at il cottage—are truly irreplaceable. I'm even considering writing a

series inspired by all of you! So many unique personalities, each one contributing to unforgettable moments.

Thanks to my parent who, even though they didn't quite understand what I was doing (something about happiness, but it's in computer science), always supported me. I am also deeply grateful to Dr. Claire and Dr. Audrey for paving the way, and for their moral support.

Last but not least, thanks to my friends and loved ones. They have followed every step, every doubt, and every joy, always surprisingly enthusiastic about my work and endlessly supportive. I feel extremely fortunate to be surrounded by such amazing people. *Je vous aime.*

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Notations

Symbols	Meaning
Indexes	
$h \in \{1, \dots, h^*\}$	Beings dimensions
$i \in \{1, \dots, i^*\}$	Individuals
$j \in \{1, \dots, j^*\}$	Doing
$k \in \{1, \dots, k^*\}$	Resources
$l \in \{1, \dots, l^*\}$	Scenario
$m \in \{1, \dots, m^*\}$	Act
Variables, vectors and matrices	
$a_{j,k} \in A$	Consumption of a resource k when doing j
$\vec{b} = [b_1, b_2, \dots, b_{h^*}]$	Beings
$\vec{b}^* = [b_1^*, b_2^*, \dots, b_{h^*}^*]$	Ideal beings
$\vec{c} = [c_{k'+1}, c_{k'+2}, \dots, c_{k^*}]$	Commons
$\vec{p} = [p_1, p_2, \dots, p_{l^*}]$	Probabilities
$\vec{r} = [r_1, r_2, \dots, r_{k'}]$	Private resources
$\vec{x} = [x_1, x_2, \dots, x_{j^*}]$	Doings
$w_{j,h} \in W$	Transformation of doing j into a being dimension h
A	Consumption matrix
M	A large value
W	Transformation matrix
ϵ	A small value

Symbols	Meaning
Sets	
$\mathcal{C} = \vec{b}^{*c}$	Capability space
$F = \{f_1, f_2 \dots, f_{m^*}\}$	Set of acts
$S = \{s_1, s_2, \dots, s_{l^*}\}$	Set of scenarios
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots \subseteq \mathbf{Q}$	Capability sets
$\mathbf{A}^c = (\mathbf{A} - \mathbb{R}_{+,*}^{h^*}) \cap \mathbb{R}_{+,*}^{h^*}$	The closure of the Capability set $\mathbf{A}$
$\mathbf{Q} \subset \mathbb{R}_{+,*}^{h^*}$	Set of all beings
V	Set of value functions
$\mathbb{R}_+$	Positive reals
$\mathbb{R}_{+,*}$	Strictly positive reals
$\mathbb{R}_{+,*}^{h^*} = \prod_1^{h^*} \mathbb{R}_{+,*}$	Strictly h^* dimensional space
Functions	
$f_m : S \rightarrow \mathbf{Q}$	The consequence(s) of f_m in the case of s_i
$p : S \rightarrow [0; 1]$	The probability of a scenario to be true.
$PF : \mathbf{Q} \rightarrow \mathbf{Q}$	The Pareto frontier of $\mathbf{A}$
$\mathcal{P}(\mathbf{Q})$	The power set $\mathbf{A}$
$E(\mathbf{Q})$	Expected capability set
$v : \mathbf{Q} \rightarrow \mathbb{R}^+$	The value of a being.
$\phi : [0; v(\vec{b}^*)] \rightarrow \mathbb{R}_{+,*}$	Freedom to achieve beings valued as y
$\Phi : \mathbf{Q} \times V \times \phi \rightarrow \mathbb{R}^+$	Freedom value.

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Introduction

A typical activity of decision aiding consists in advising stakeholders involved in public policy making processes. Aiding to decide is a multi-stakeholder decision process (there are at least two of them: the client and the analyst) constructing cognitive artefacts to enable the client to improve their handling of a decision process for which they asked an advice to the analyst (see also [MeinardTsoukiasEJOR2018]). To do so, we need three requirements [tsoukias2007concept]:

- meaningfulness (for the analyst) of any information manipulation [Roberts1994Chapter1L, roberts1981measurement, roberts1985applications];
- usefulness (for the client) of the recommendations constructed by the process;
- legitimacy of such recommendations with respect to the decision process for which the advice has been requested [Meinard2022].

Let's focus upon this last dimension characterising decision aiding. An essential aspect of legitimacy is the pretension of the recommendation constructed to be "rational".

Such rationality needs to be "arguable" (stakeholders should be able to discuss it and contrast it) and "convincing" (when no more argumentation can hold). In other terms such rationality is a common ground among the stakeholders of a public decision process upon which to construct any decisions.

A usual way to establish rationality is to impose it "normatively". As soon as we convince the stakeholders that respecting a certain "norm" is rational, the decision process boils down to check how to remain coherent to the norm. A typical example in many public decision processes is the use of Cost-Benefit Analysis (CBA: see [DasguptaPearce72]). We are not aiming to discuss this tool here, but to mention the fact that CBA is de-facto the standard of rationality when public investments are discussed [HM2020, EU-CBA2015, Worldbank2010]. The result is that the

acceptability of many decisions depends upon going through such an analysis (the reader should note that the same type of reasoning applies for many environmental impact analysis tools which are established as norms of rationality¹).

We think that “rationally speaking” designing a policy which is expected to have an impact upon the citizens’ welfare implies being able to:

- observe the present situation and distribution of welfare;
- anticipate the impact of doing nothing;
- anticipate the impact of implementing a policy.

In other terms if a policy is expected to have any impact upon the citizens, it makes sense to try to measure it: as it stands presently and as it could stand under different possible scenarios. However, this apparently simple idea needs to be better specified. The concept to use here is the one of “welfare” (see [sen1991welfare]), but we need to understand which are the operational difficulties we may have in order to establish and measure the welfare of either a single citizen or of a group of them.

A first difficulty is what exactly welfare should represent. There exist several different dimensions to consider: health, education, shelter, work, culture, mobility, leisure... Should these dimensions be considered separately or should we merge them in a single figure?

A second difficulty arises from the fact that each citizen may have a different perception of what is important for their well-being (welfare) and how each of the possible welfare dimensions contributes to their overall welfare.

A third difficulty is related to the fact that even citizens sharing the same values (appreciation of the different welfare dimensions) may still have different cultural and social backgrounds and/or live in very different environments ending in having totally different realisation options.

Last, but not least, while welfare certainly depends upon the private endowments of each citizen, it also depends upon the access of each citizen to “commons” whose consumption does not depend from the will of that precise private citizen.

Summarising, we need to be able to “measure” welfare and the impact policies can have upon “welfare”, but such measurement needs to consider:

- the multidimensional nature of welfare;

¹<https://ec.europa.eu/environment/eia/eia-legalcontext.htm>

- the subjective dimension of welfare;
- the different impact a policy or an event can have upon different groups of citizens;
- the impact of the commons upon welfare.

The above constraints represent a challenge for decision analysts: if our tools are aimed to help (among others) policy makers to design policies and to improve how policies are designed we need to provide appropriate methods taking into consideration the above discussion. For this purpose, this thesis explores the Capability Approach (CA) ([sen1979equality, sen1993capability, sen1985commodities, sen1999development, sen2009idea]) as a framework allowing to develop appropriate decision aiding for public policy design. Our proposal is that CA could offer a common ground to different stakeholders assessing the impact of a policy to the citizens' welfare, offering a certain number of advantages with respect to other approaches aiming at measuring welfare

The CA aims to represent what an individual is effectively capable of doing and being. However, this theory suffers from a lack of operational development, both in terms of modeling and the availability of mathematical tools to ensure its practical usefulness.

A first concern of this thesis (Chapter 1) is to discuss the fundamental concepts behind different approaches to welfare measurement in case we consider using this as a rationality norm for public policy design and making. It consists of a brief survey of what welfare economics consider reasonable concepts and definitions of welfare. The survey do not cover all definitions of welfare and the controversies presented are not exhaustive, but it shows the interest of using the capability approach as a base for a rational tool of welfare measurement.

The basic idea of Chapter 2 is to design a model that allows to design and choose public policies. To do so, we provide a critic on CA from a decision aiding perspective and nevertheless, argue CA could be useful for decision aiding. Then, we present a simple model that allows an explicit representation of the multidimensional nature of welfare using notions established by the capability approach, and thus allowing alternative designs of policies. This is a linear model providing Pareto frontier as a representation of individual welfare, which we refer to as *capability sets*.

Chapter 3 is dedicated to a case study on the welfare of Parisian elders, serving as a proof of concept for the framework's practical implementation and usefulness.

However, in this specific case, we lack sufficient data to fully inform certain aspects (notably about the preferences and values of studied citizens), so this serves as an experimental validation rather than a complete operational validation.

Using capabilities to inform public policy can be challenging since capability sets are complex, consisting of sets of Pareto optimal multidimensional solutions. To effectively use our framework, we need mathematical tools to facilitate decision-making.

Chapter 4 focuses on assessing the capability sets generated using our model. Capabilities involve both the choices individuals have and how they value those choices. We propose a framework to assess capability sets based on individuals' preferences over the diversity of accessible alternatives that can lead to different levels of well-being. We present three methods for modeling this diversity according to different types of preferences and examine the major theoretical properties underlying these approaches.

Finally, public policies often involve uncertainties regarding their consequences. We propose different approaches based on the information available to policymakers.

In cases of complete uncertainty (where the likelihood of scenarios is unknown), as discussed in Chapter 5, we draw on classical decision-making based on order statistics and address capability order statistics while ensuring that key properties are maintained.

In cases of risk (where subjective probabilities are known), discussed in Chapter 6, we focus on defining expected capability sets that align with desired properties, mirroring the principles of the classical expected utility approach.

Part I

Is the Capability approach a
useful tool for decision aiding in
public policy making?

Chapter 1

Welfare measurement

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This chapter delves into the topic of welfare measurement, drawing from welfare economics and social choice theory.

These disciplines aim to develop techniques for evaluating potential public policies based on their impact on the well-being of individuals. Later, the Rawlsian theory of justice, which can be interpreted as a resource-based approach to well-being and has significantly influenced Sen's theory, will be presented. Criticisms of these two approaches will be discussed, leading to the introduction and exploration of the capability theory. Finally, the empirical application of this framework will be examined.

Sen's Capability Approach is a theoretical framework that focuses on how individuals' well-being can be identified, and how equality and justice can be considered. It emphasizes the freedom of individuals to choose a life that aligns with their values and aspirations.

1.1 Welfare economics and Rawls's theory

Welfare economics, as a branch of economics, aims to evaluate "social welfare" defined as the representation of the "goodness" of the social state [sen1991welfare]. While this is not an exhaustive definition of welfare economics, it provides a necessary background for understanding its influences on the capability approach. Welfare economics has evolved through various historical developments, which we will present in chronological order.

1.1.1 The old welfare

Welfare economics originates from utilitarianism [hicks1939foundations], an ethical theory focusing on the consequences of actions, known as consequentialism [sinnot2003consequentialism]. Utilitarians measure welfare through utility, a concept introduced by Jeremy Bentham [bentham1789introduction], who defined it as the balance between pleasure and pain. Bentham's principle of utility suggests actions are good if they promote "the greatest happiness of the greatest number" [bentham1789introduction].

John Stuart Mill expanded on Bentham's ideas by distinguishing between higher and lower pleasures, arguing that intellectual and moral pleasures are superior to

physical ones [mill1863utilitarianism].

Despite its appeal, utilitarianism has critical limitations [Robbins1932]. It focuses on total utility, ignoring distribution. For example, it may prefer a scenario where person 1's utility is 99 and person 2's is 1 over a scenario where both have a utility of 49, even though the latter is more equitable. Additionally, utilitarianism relies on two fundamental assumptions:

- Utility is *cardinal* (it can be represented by a quantity upon an interval scale).
- Utility allows *interpersonal comparison* (we can compare utility of individuals, and add them).

These assumptions imply that differences in value between any two states of the world can be measured, and that such measurements are commensurable among individuals, enabling comparisons of value differences between different individuals.

Cost-Benefit Analysis (CBA) is a widely-used decision-making technique with deep roots in Welfare economics [pearce1983origins]. It is commonly applied in the evaluation of public sector projects, such as those related to economics, transport, health, and environment [HM2020, EU-CBA2015, Worldbank2010, johansson1991introduction, boadway1974welfare, adler1999rethinking, frank2000cost, DasguptaPearce72]. At its core, CBA is based on the principle that decisions should be assessed in terms of their consequences, and a project should be chosen if its benefits outweigh its costs.

The foundation of CBA lies in the assumption that citizens are viewed as “consumers” and their marginal utility is derived from their consumption of goods. It is further assumed that stakeholders live in a perfect market economy and aim to maximize their utility. As such, a “marginal utility income rate” can be established, indicating that utility can be derived from income ([boadway1974welfare]; [johansson1991introduction], Chapter 9; and [bouyssou2000evaluation], Chapter 5). However, perfect markets do not exist for all goods and services, such as public goods like education or health, or common resources like forests or bridges. In such cases, shadow prices need to be determined, along with the evaluation of externalities, such as noise or aesthetic value, and the valuation of a human life.

In CBA, a project is considered as a “change in the net supplies of commodities” [dreze1987theory], and it is chosen if its benefits exceed its costs over the specified

time period, resulting in an increase in social welfare (*i.e.* the welfare of the society) during that period. Additionally, since the utility of income is assumed to be higher today than in the future, a discount rate is applied to future costs and benefits. It is important to note that CBA can only be applied to projects that marginally change the current state, *i.e.* small projects that only infinitesimally perturb the market prices, as the prices of goods are considered fixed.

CBA is a typical example of how utilitarianism is practically used in decision-making.¹ It views a project's utility as the sum of utilities of each individual citizen/consumer in the affected territory, assuming that all citizens have the same utility (*i.e.* are equally affected). In order to compute this utility, the consequences of the project are assumed to be measurable at a real or proxy market, reflecting the preferences of citizens/consumers. The “prices” of these consequences, which represent the gains and losses of utilities of each citizen/consumer, are summed to determine overall utilities.

However, these assumptions about interpersonally comparable utility have been criticized by scholars such as Robbins [**Robbins1932**, **robbins1938interpersonal**], who argue that utility cannot be objectively compared as it is derived from subjective mental states. They claim that measuring the utility of others is impossible, which challenges the basis of summing individual utilities as a criterion for judging actions. This led to the emergence of “new welfare economics” in the 1930s, which rejects the summation of individual utilities as the basis for evaluating actions while retaining the concept of utility.

1.1.2 New welfare economics

The New Welfarists argue that social states should be evaluated based on their utilities, without making comparisons between the utilities of different individuals. This argument aligns with the Pareto principle, as introduced by [**Pareto06**].

The Pareto principle states that a social state is considered Pareto optimal if and only if it is impossible to increase the utility of one individual without decreasing the utility of another. However, relying only on the Pareto principle as a criterion

¹Harberger [**harberger1971three**] argues that cost-benefit analysis (CBA) can ensure Pareto improvements if winners compensate losers, but this is not always reliable [**boadway1974welfare**].

for evaluating social states has been criticized [sen2018collective], as it imposes an overly restrictive informational basis. In reality, finding social improvements based on Pareto comparison is highly unrealistic when designing public policies, as these policies often result in winners and losers.

Robbins criticized interpersonal comparison of utilities for their lack of objectivity [weymark2016social]. He argued that utilities are normative and cannot be objectively determined. In response, the Bergson-Samuelson social welfare function was introduced [kaushik2011functionings], emphasizing ethical considerations in selecting the best solution from Pareto efficient alternatives.

This approach requires additional ethical criteria to evaluate different Pareto efficient policies. Scientists analyze the outcomes of various value judgments derived from external sources. Individual utility functions, considering factors like consumption, work, and environment, are used to rank alternatives. A social welfare ordering then combines these individual rankings with value judgments.

The new welfare foundations rely on ordinal and non-interpersonal utilities, forming the basis of social choice theory, which derives societal preferences from individual preferences.

The foundations and methods of social choice theory can be traced back to Condorcet [marquis1785essai] (mathematician) and Borda [borda1784memoire] (engineer). These early contributions aimed at constructing a framework that would enable rational and democratic decision-making within a group of citizens. They both developed voting procedures for aggregating individual rankings of alternatives to arrive at a collective decision. They both obtained frameworks containing weaknesses that we do not discuss here (see instead: [black1958theory, farquharson1969theory, sen2018collective]). Yet, social choice theory really started being developed thanks to Arrow's works.

Kenneth Arrow's *General Possibility Theorem* (or *Arrow's Impossibility Theorem*) made significant contributions to both fields [arrow1951social]. Arrow's theorem demonstrates that no social welfare function (SWF) can satisfy all necessary conditions (unrestricted domain, Pareto principle, independence of irrelevant alternatives, and non-dictatorship) simultaneously when there are three or more alternatives and two or more individuals [maskin2014arrow].

This result may be seen as surprising, but [vincke1982arrow] shows that in fact

it is not. The reason behind this lies in the fact that any Social Welfare Function (SWF) serves as a procedure for aggregating preferences, and as such, the resulting aggregated preferences are inherently less informative in terms of their content. In other words, if we aggregate weak orders, using an SWF, the resulting outcome cannot retain the same level of information content as a weak order. This observation is consistent with the fact that Condorcet's procedure, which satisfies Arrow's conditions, does not guarantee the existence of a social complete order.

Sen [sen1977social] further interprets Arrow's impossibility theorem as a proof of the informational limitations of Arrowian SWFs. Merely considering preference orderings and imposing strict non-interpersonal comparisons is inadequate for making decisions about social states. Thus, some welfare economists have proposed SWFs that allow for interpersonal comparisons, for instance see [adler2012well] and [adler2019measuring].

A criticism made in [sen2018collective] is that Welfarists only consider preferences or utilities of individuals, regardless of the nature of the social state being considered. This phenomenon, referred to as "neutrality", arises as a consequence of the axioms (unrestricted Domain, Pareto principle and Independence of irrelevant alternatives). It implies that if an individual is decisive on the determination of a preference between two social states, then this individual should be decisive in every situation. In other words, the social order depends on individual preference only, and not on the *nature* of the alternative. Neutrality is characterized by a lack of information and can potentially give rise to moral difficulties.

To illustrate this problem, [sen2018collective] presents the example of a cake division problem involving two individuals preferring the largest share possible. It could be argued, state (a) that if the cake is shared in the proportion of 99% to person 1 and 1% to person 2, person 2's preference should prevail on person 1's preference due to ethical considerations. However, in light of neutrality, if person 2's preference prevails in state (a), then person 2's preference would prevail over person 1's preference regardless of the social state. Consequently, state (b), where person 1 has 1% and person 2 has 99%, would be socially preferred to a state where person 1 and person 2 each have 50%. This outcome, is clearly not aligned with ethical principles, as the aggregation of individual preferences should take into account the characteristics of the current social state. In order to consider the current social

state, it becomes imperative to consider additional information beyond utility, as argued in [maskin2014arrow].

1.1.3 A Theory of Justice

In *A Theory of Justice*, John Rawls defines a fair society based on distributive justice [rawls2009theory]. Justice, for Rawls, is achieved through fairness and impartiality. He proposed that political, economic, and social institutions should maximize the *primary goods* of the least well-off [rawls1982social]. These primary goods include basic liberties, freedom of movement and occupation, powers and responsibilities in institutions, income and wealth, and the social bases of self-respect.

Rawls' "maxmin" strategy aims to maximize the welfare of the least advantaged is justify by the "original position", a hypothetical scenario where individuals design fair institutions without knowing their future social positions. From this, two principles of justice arise:

-*First Principle*: Equal basic liberties for all.

-*Second Principle (difference principle)*: Inequalities are permissible if they benefit the least advantaged and ensure equal opportunities.

Rawls focus on "ex ante" opportunities distinguishes his theory from welfarism, which centers on "ex post" outcomes.

1.1.4 Criticism of Welfarism and Rawls's theory

The Capability Approach emerges from a new interpretation of egalitarianism presented in Sen's *Tanner Lectures: Equality of what* [sen1979equality]. This lecture presents a critique of Welfarism and Rawl's theory, offering a new perspective on the measurement of welfare.

a. Critique of Welfarism Utility is generally defined as happiness or desire fulfillment, and can be seen as a good representation of the welfare of an individual. Indeed, a lack of happiness or unfulfilled desires indicates a low level of welfare. However, very strong criticisms can be raised against this approach, which can be summarized in four key points.

1. *Subjective Differences*: Utilities are influenced by individual expectations and desires [sen1985commodities, sen1999development]. For example, a per-

son living in poor conditions might have low expectations and feel happy with little, while a wealthy person might have high expectations and feel unhappy. This skews utility comparisons. Moreover, without interpersonal comparisons, individual preference orderings fail to convey differences in welfare accurately, as two individuals with the same preferences might derive different levels of welfare from the same bundle of goods due to personal or environmental differences.

2. *Distribution Effects:* Classical utilitarianism, which focuses on the sum of utilities, often ignores distributional fairness. For instance, more resources might be allocated to someone deriving more utility from them, disadvantaging individuals with disabilities, resulting in a double disadvantage: one due to their condition and another due to resource allocation policies [sen1979equality]. Cost-benefit analysis (CBA), on the other hand, assumes that all individuals derive utility from income at the same rate, which is often unrealistic and overlooks the varying needs and circumstances of different individuals.
3. *Societal Motivations:* Utility-based approaches may overlook important non-utility values like personal liberty. For example, the Pareto principle can conflict with personal freedoms, as shown in scenarios where respecting individual preferences can lead to outcomes that contradict the freedom of choice on personal matters principle [sen1970impossibility].
4. *Individual Motivations:* Individuals act on motivations beyond self-interest. For instance, someone might go on a hunger strike for justice, which utility-based approaches fail to differentiate from involuntary starvation, highlighting the inadequacy of viewing individuals as mere utility-maximizers [rawls2009theory].

b. Critique to Rawls. Sen criticizes Rawls for focusing only on primary goods, arguing that they are merely means to achieving ends, and it is the ends that should be analyzed [sen1979equality]. This focus can lead to unfair situations as individuals with different needs require different allocations to achieve the same well-being.

Sen also critiques Rawls's contractarian approach, labeling it as "transcendental institutionalism" [sen2009idea, thomas2013sen]. This approach seeks perfect

principles of justice and limits justice to institutions, ignoring social interactions and behaviors. Sen argues that justice should encompass both institutions and the broader society. Thus, Sen presents two main criticisms of the transcendental institutionalism inherent in Rawls' theory.

1. *Feasibility*: Sen challenges the feasibility of Rawls' original position leading to two unique principles. In [roemer1998theories], it is shown that the *maxmin* strategy (Difference Principle) would only be chosen by individuals who are infinitely risk-averse, which is a strong assumption. Sen [sen2009idea] argues that multiple impartial arguments can exist, invalidating Rawls' theory. For example, in the “three children and a flute” scenario. Three children, Anne, Bob, and Carla, argue over a flute. Anne claims the flute because she is the only one who knows how to play it, appealing to the principle of utility. Bob argues that he should have the flute because he is the poorest and has no other toys, appealing to the principle of fairness. Carla insists that the flute is hers because she made it, appealing to the principle of entitlement. Different impartial arguments can justify giving a flute to any of the three children, highlighting the complexity of achieving a single impartial solution.
2. *Redundancy*: Sen also criticizes the contractarian tradition for redundancy, suggesting that if the perfect social arrangement is unattainable, the transcendental institutionalism approach is useless for evaluating policies. Knowing an ideal does not necessarily help in choosing the best practical alternative, as proximity in theoretical ideals does not always align with practical preferences. For example, knowing that someone's favorite cuisine is sushi does not help in predicting their choice of dish at an Italian restaurant.

1.2 Capability Approach

1.2.1 Equality of what?

In [sen1979equality] and later in [sen2009idea], Sen argues that the concept of justice is intrinsically linked to the notion of equality, raising the question, “equality of what?” Utilitarians focus on the equality of marginal utility, aiming to maximize total utility by equalizing individual marginal utilities, given their concave nature.

Rawls emphasizes equality of primary goods or opportunities, permitting inequality only if it benefits everyone and respects basic liberties, freedom of movement, and fair chances of occupation.

Other perspectives focus on equal revenue, capital, or redistribution of created value. Sen proposes that equality should center on “capabilities”: the substantive opportunities and freedoms individuals possess.

1.2.2 What is the capability approach ?

The capability approach has garnered significant attention in recent decades, leading to extensive research by various scholars. As a result, multiple conceptions of capabilities and differing terminologies have emerged, with even Sen’s own conceptual and terminological formulations evolving over time. In this Chapter, our primary focus will be on Sen’s presentation of the capabilities in [sen1985commodities].

According to [sen1985commodities, sen1997choice], a distinction should be made between owning a good, using it, and valuing its use. While welfarists mainly focuses on income and/or commodities, Sen argues that, in addition to goods and income, it is crucial to know what individuals can actually do with them. First, a commodity has to be distinguished from its characteristics. For example, owning a laptop may afford access to knowledge, social interaction, work, and entertainment, among other things. However, the characteristics of a commodity do not necessarily indicate how an individual will effectively use it or what they are capable of doing with it. It is often observed that individuals only use a fraction of the potential characteristics of a commodity, and some may be unable to use all of its characteristics or may use them less efficiently.

These disparities in utilization can be attributed to varying *conversion factors* that an individual possess. According to Robeyns [robeyns2005capability] these factors can be divided into three categories:

- *Personal conversion factors*: physical and mental conditions, gender, reading skills, etc. For instance, reading skills influence laptop usage.
- *Social conversion factors*: public policies, laws, social norms, discrimination practices, power relations, etc. For example, government restrictions on internet access can limit laptop usage.

- *Environmental conversion factors*: climate, geographical location, pollution, etc. Living in an area with poor internet connectivity limits laptop usage.

Assessing individuals welfare requires examining their *functionings*, which are what individuals are actually able to do with their commodities. For instance, watching a video or working on a laptop are distinct functionings.

Citizens' *capabilities* represents all the possible combinations of functionings they can potentially achieve. It is important to distinguish capabilities from *achieved functioning*, which refers to the actual set of “doings” an individual has chosen. Capabilities represent the effective freedom to choose from various combinations of functionings they value.

Functionings are typically perceived as the “beings” and “doings” of an individual. For a given combination of functionings, individuals have “beings” states, describing their conditions (e.g., mobile, happy). The realization and valuation of a functioning are distinct; two individuals may watch the same video but have different appreciations due to personal history, values, desires, and other factors.

Individuals are viewed as *agents* capable of acting and changing their values and goals. They may choose actions that do not maximize their happiness, as reflected in their *valuation vectors*. For instance, an individual may fast for political reasons at the cost of their health and happiness.

One of the first efforts of Sen has been to develop his own framework to represent capabilities [sen1985commodities]:

- X_i : Individual's entitlements as described in [sen1982poverty], is the set of bundle of commodities that can be chosen by the individual through legal channels. It depends on the individual's initial ownership and the set of bundles that can be obtained from trade or production.
- x_i : The vector of commodities possessed by individual i i.e. *5000\$, a car, some food...* This vector is chosen from the set of entitlements.
- $c(x_i)$: An exogenous function converting a commodity vector into a vector of characteristics; $c(x_i)$ represents the characteristics of the consumption of x_i i.e. *spending power, possibility of displacement, nutritional aspects ...*

$f_i(\cdot)$: Person i 's “personal utilization function”. It is a function that converts certain characteristics into functionings using the *conversion factors*.

F_i : The set of all “utilization functions” f_i that i can choose from.

b_i : The “beings” a person achieves using the commodities x_i and a utilization function $f_i \in F_i$:

$$b_i = f_i(c(x_i))$$

$v_i(b_i)$: The valuation function. $v_i(b_i)$ is the valuation of the beings b_i .

For individual i , given a vector of commodity x_i , the set of *possible* functionings is:

$$P_i(x_i) = \{b_i | b_i = f_i(c(x_i)), \text{ for some } f_i \in F_i\}$$

Then the *capabilities* is given by:

$$Q_i = \{b_i | b_i = f_i(c(x_i)), \text{ for some } f_i \in F_i \text{ and some } x_i \in X_i\}$$

The mathematical representation of capabilities serves as a formalization to elucidate the relationship between various elements of the capability approach, such as commodities, characteristics, individual conversion factors, functionings, and capabilities. It provides a conceptual framework to understand the intricate interplay among these elements, rather than a practical formula for representing an individual's capabilities.

One of the strengths of the capability approach is its ability to account for human diversity. It considers variations in conversion factors from commodity characteristics to functionings. These differences may arise from individual-level factors as well as structural factors in society [robeyns2003capability]. For instance, one individual may benefit significantly from public transportation due to physical mobility, while another with a disability may not, leading to a lower conversion factor of transportation into access to opportunities. A structural example is the disparity between men and women in converting work into income, revealing gender-based inequalities.

The capability approach also recognizes multidimensional diversity in human life. Inequality or deprivation can manifest in various dimensions beyond income, such

bases, beyond solely relying on wealth to evaluate poverty, can be seen as a consequence of a shift of paradigm raised by the CA.

3. *A Tool to make interpersonal comparison of well-beings*: A procedural approach, typically in the form of formulas or algorithms, that enables the comparison of well-being between individuals.

These levels are interconnected, the paradigms deriving from the critiques of Rawls' theory, and formulas and algorithms serves to study empirically poverty and inequalities. As the first level has already been discussed in this chapter, in this section we will see some applications of the capability approach in the real world, mainly using level 2. Level 3 being further investigated in Chapter 4.

Table 1.1 provides an overview of the literature on the operationalization of the CA, highlighting three major areas of research: inequalities and poverty identification, health, and urban planning. Various methods have been used to measure capabilities;

Sen's early empirical applications of the CA in [sen1985commodities] demonstrate that GDP is an inadequate tool to examine inequalities. He compared countries like Brazil and Mexico with China and Sri Lanka, showing that despite having a lower GDP per capita, the latter countries had higher life expectancy and lower infant mortality. Sen also explored gender inequalities in India, revealing that women experienced worse health outcomes compared to men.

The Human Development Index (HDI) is a notable example derived from the CA, measuring human development through life expectancy, education, and adjusted GDP per capita [UNDP2019]. Another example is the Multidimensional Poverty Index (MPI), which uses household surveys to assess poverty through health, education, and living standards [alkire2011counting].

Non-statistical techniques have been employed to measure urban capabilities. For instance, [blecic2015walkability] assigned scores to city points based on "walkability" factors in Alghero, Italy. Similar methods were used in Lisbon, Portugal [blevcic2015evaluating].

In health economics, several measures have been developed, such as the ICECAP-A for adults, ICECAP-O for older people, and CES for care-related well-being [coast2008valuing, flynn2011assessing, al2011estimation]. These measures

typically involve creating questionnaires and calculating scores based on population values.

The fuzzy set approach, which extends the scaling method, has been used to measure capabilities [**martinetti1994new**, **martinetti2006capability**]. For instance, an individual's nutritional status might be scored on a scale from 0 (severely deprived) to 1 (well-nourished). This approach has been applied in studies on South African and Italian contexts [**qizilbash2002note**, **qizilbash2005capability**, **martinetti2000mult**].

Statistical methods like Factor Analysis and Principal Component Analysis (PCA) have also been employed. Factor Analysis reduces data by identifying common factors among various functionings, as shown in a study of Belgian unemployed citizens [**schokkaert1990sen**]. PCA computes linear combinations of original functionings to capture the variance, as demonstrated in studies on the physical quality of life index [**ram1982composite**, **klasen2000measuring**].

Clustering methods have been used to measure multidimensional poverty. For example, a study in Burkina Faso grouped households into socio-economic classes and assigned capability sets to each cluster to design policies for improving well-being [**zeumo2014new**]. Similar methods have been proposed to define citizen clusters to improve urban capabilities [**Fancellolurning**, **FancelloetalSEPS2020**].

This chapter analyzes the history of welfare economics and social choice theory, which aim to evaluate the impact of public policies on well-being. The Rawlsian theory of justice is introduced as a resource-based approach that has influenced Sen's capability theory. Despite their contributions, these traditional approaches have limitations, leading to the development of the capability theory. The capability theory focuses on individual freedoms and opportunities and addresses the criticisms of earlier approaches.

The next chapter examines the capability approach in detail, exploring its critiques and challenges. It looks at how these criticisms can be addressed in *decision aiding processes*, showing practical applications of the capability approach.

Chapter 2

Capability approach inspired tools for aiding policy design

Contents

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Building on the theoretical foundations established in Chapter 1, this chapter explores a specific tool inspired by the capability approach to facilitate the design of public policies. Although the capability approach has faced criticism, this chapter will analyze these critiques and demonstrate that many can be addressed effectively when viewed through the lens of decision aiding. Furthermore, we will discuss the advantages of using the capability approach in decision aiding and emphasize the importance of adopting a systemic perspective. To illustrate these ideas, we will introduce a mathematical programming framework inspired by the capability approach and provide an example.

2.1 Capability approach, a critical perspective

The Capability approach has been subject to criticism on various theoretical and empirical fronts. However, our focus in this chapter is specifically on its application as a tool for decision aiding in public policies design. In the following section, we outline the key criticisms raised against the Capability approach in this context, with the aim of identifying the attributes that a framework must possess in order to be considered suitable for policy design purposes.

2.1.1 Critics on capability approach from a decision aiding perspective

Interpersonal comparison [robeyns2000unworkable](#) observes that the capability approach, as formulated by Sen, does not fully address the challenge of interpersonal comparison. While certain capabilities like access to safe water, shelter, and nutrition are universally recognized as essential for well-being, more complex functionings such as access to culture, urban quality, and self-respect involve subjective judgments. Comparing these “subjective functionings” across individuals is challenging because they are influenced by personal, social, and environmental factors, as well as individual values.

Thus, to make meaningful comparisons of capabilities, *we need to understand the subjective value individuals place on certain achievements and opportunities, enhancing the information basis for interpersonal comparison.*

Most applications of the capability approach have focused on poverty and inequality, treating all functionings as critically important [sen1993capability]. However, incorporating individuals' values is often limited to weighting different functionings in aggregation methods, particularly in indexing techniques [alkire2014multidimensional, UNDP2020]. This can be misleading, as users and decision-makers might not be aware of these assumptions. *Therefore, it is crucial to explicitly integrate citizens' subjectivity and values into the evaluation of functionings, rather than relying solely on implicit weighting assumptions.*

Functioning and not capabilities. The empirical literature on the capability approach has mainly focused on the measurement and identification of, present individual *functionings* rather than *capabilities* e.g. [sen1985commodities, cerioli1990fuzzy, qizilbash2005capability, alkire2014multidimensional]. This bias towards functionings is not surprising given that much of the empirical work on the CA has studied poverty measurement and identification. In this context, functionings may be considered more relevant than capabilities, as being undernourished is rarely a choice [robeyns2006capability]. However, in contexts like predicting the welfare impact of public policies in developed countries, focusing only on functionings can be limiting. In such cases, *capabilities may be more useful, as they require fewer assumptions about individuals' future choices.*

Individualistic. One prominent critique of the Capability Approach in the literature is its perceived individualistic nature [deneulin2010capability, stewart2002amartya, stewart2005groups, kaushik2011functionings], wherein individuals are treated as atomized entities. The measurement of real individual freedom in the CA framework is complex, as it necessitates considering their utilization of Commons, social structures, and interactions with other individuals. The concept of Commons, in particular, introduces the notion of subtractability, where an individual's actual capability is dependent upon the utilization of Commons by others. However, there is no intrinsic impossibility in the capability approach to take into account those social environments. Most of the work referring to the CA has not paid a lot of attention to groups, but some did [kynch1983indian]. Besides, social aspects are theoretically acknowledged in the form of social conversion factor of resources into functioning. More discussion on the individualistic argument can be found in

[robeyns2017wellbeing].

From an empirical perspective, citizens' welfare is contingent upon complex bundle of goods and services, many of which interact with each other and are part of more complex systems such as cities, communities, and territories. Public policies have the potential to impact the structure of these systems and consequently affect citizens' welfare in ways that may not always be fully comprehended. In this regard, *the CA, in its current form, lacks a systemic vision*, as it has primarily been employed to measure and identify the causes of poverty and inequality using simplistic deterministic representations of these phenomena. The absence of a systemic perspective also poses challenges in identifying the relevant Commons impacting welfare. We will turn back to the benefit of a systemic vision of welfare (Section 2.1.2) and propose a systemic representation of the CA (Section 2.2).

Under-Theorization. Some scholars criticize the capability approach for its perceived under-theorization, especially its lack of a definitive list of functionings, unlike Nussbaum's approach [nussbaum2003capabilities]. This absence is seen as a weakness, making the theory unclear and difficult to apply. However, from a decision-aiding perspective [Tsoukias07aor], this lack of a fixed list is actually a strength. *It allows the framework to be adapted to the specific needs of each problem, enabling stakeholders to customize the list of functionings to better address the unique aspects of each decision-making context. This flexibility makes the capability approach versatile and especially useful for decision-makers.*

No framework to compare. sen2009idea argues that public policies should focus on equalising the individuals' capabilities, but he did not provide a framework to compare them. The challenge of comparing capabilities arises from the multidimensional nature of the capability approach, which encompasses diverse dimensions of human well-being. For instance, comparing the capability of an individual who has the opportunity to be in good health but lacks wealth with another individual who has the opportunity to be wealthy but faces health challenges presents difficulties. Additionally, Sen cautions against equalizing capabilities at any cost, citing an example where equalizing life longevity between men and women could lead to reducing women's access to hospitals, which is not desirable. These challenges, including the complexity of comparing capabilities and the lack of clarity on what

aspects of capabilities should be equalized, pose operational issues in the practical implementation of the capability approach.

Nevertheless, *the absence of a framework for comparing capabilities does not necessarily imply that no comparisons can be made. The available informational base can still serve as a foundation for argumentation among different stakeholders. However, if the informational base is deemed insufficient, techniques need to be developed to discern between different sets of capabilities, in order to facilitate meaningful comparisons (Chapter 4).*

Access to data. Data availability is a major challenge in implementing the capability approach. Unlike income-based methods, which focus on net income, the capability approach requires diverse data. First, information on the means individuals have to achieve their functionings, like income, commodities, and public goods access, must be collected. Additionally, data on personal, social, and environmental conversion factors are needed, as well as the subjective values of individuals. These data often mix quantitative (e.g., income, age) and qualitative (e.g., values, social groups) aspects, making collection challenging. Most data used in current applications, as noted in [robeyns2006capability], are secondhand and not specifically designed for this approach. robeyns2017wellbeing highlights that the required informational richness extends beyond data collection to include mathematical modeling. Modeling social constraints and individual values is difficult, requiring a balance between simplification and preserving the approach’s richness. We believe data quality issues must be explicitly addressed: *while collecting data for the capability approach is costlier and more complex than other welfare measurements, it is worthwhile.* New methodologies for data collection need to be developed.

The model presented in Section 2.2 is a proposition to answer (at least partially) those operational issues. To meet the interpersonal comparison and individualistic critics, we suggest a mathematical program model assessing something close to “capabilities” (rather than achieved functioning), integrating explicitly the evaluation of functioning and being systemic, it is introduced in Section 2.2.1. We argue that the under-theorization concern may not be a significant issue from a decision-aiding perspective, as we provide a method for selecting functionings, as outlined in Section 2.2.2. Additionally, while we discuss the challenges of data collection necessary

to support our model in Section 2.2.3, this manuscript does not fully resolve this issue, which remains an ongoing challenge. Finally, in Chapter 4, we propose a framework to compare our “representation” of capability set.

2.1.2 Why the Capability approach can be useful for decision aiding?

Despite criticisms of its operational applicability, the Capability Approach remains a valuable framework for decision aiding. We can identify four main steps in which a framework can be useful for decision aiding purposes, especially in the context of a public decision within a policy analytics framework [tsoukias2013policy, de2016evidence, daniell2016policy].

1. Help the client to have a better *understanding* of the problem;
2. Help to *imagine and design* potential solutions;
3. *Explore* consequences of different solutions;
4. Provide some *arguments* in favour or against any selected solution.

The reader should note that our aim is to use the CA as a common ground helping to rationalise how public policies are designed, assessed and implemented. Our perspective differs from the mainstream proposal of the CA, as some scholars, such as **nussbaum2011perfectionist** only consider governments as the actors of policy design and improvement, and argue that the CA should provide recommendations to governments exclusively. Yet, as highlighted in [stewart2005groups] and [robeyns2017wellbeing], policy improvement often arises from political pressure exerted by different groups. Clients, such as non-governmental organizations (NGOs), trade unions, or economic actors, may seek assistance for various purposes and objectives, but they all require a common ground for discussion, negotiation, and agreement, if feasible. In the following sections, we will briefly elucidate why the CA can serve as such a rationalizing ground.

We consider three main characteristics that make the capability approach useful for policy design purposes.

Multidimensionality. A single figure may be easy to *understand* and communicate, but it fails to capture the complex reality of welfare. Such simplification can obscure arbitrary aggregation procedures and mask significant differences between welfare dimensions. Multidimensionality addresses this limitation by offering a comprehensive view that reveals the full spectrum of welfare aspects.

A multidimensional representation of welfare allows for a broader range of potential solutions in *design*, including options that may initially seem inconceivable but are potentially feasible. It enables us to use “what-if” questions, such as why performance might be lacking in a specific dimension and what actions can be taken to improve it.

Explicitly considering the multidimensional nature of welfare provides the necessary foundation for effective *argumentation*, as it aligns better with societal norms that view welfare as inherently multidimensional.

Diversity. Recognizing the diversity of well-being levels is crucial for *understanding* the issue at hand. Measuring well-being at the individual level, rather than relying on indices that represent the “general well-being” of citizens, is essential. Measuring the well-being of a population as a homogeneous group could lead to ignoring minority groups and missing the specific resources or conversion factors limiting a particular group of citizens.

We generally *design* policies targeting specific parts of the population, so recognizing diversity within the population is important for any framework aiming to aid policymakers.

Diversity also strengthens *argumentation*, as it allows policymakers to justify their policies by demonstrating how they are designed to improve the well-being of specific groups of citizens, as public policies often benefit only certain segments of the population. It also enables policymakers to better justify their choices to different audiences.

Subjectivity. It is important to realize that welfare is both perceived and assessed subjectively. Policies are generally expected to address a problem situation, but the extent to which the present distribution of welfare is a problem, and for whom, is a matter of subjective appreciation by each citizen. This idea also applies to the impacts of any policy. Explicitly considering this subjective dimension allows

policymakers to *understand* the broader picture and anticipate the different reactions of groups of citizens who share common perceptions and values.

Analyzing the subjective values driving citizens' behavior also helps identify different policy targets and expand the inclusiveness of policy *design*.

This approach aligns more closely with societal norms, as most individuals view welfare as a subjective concept. Taking into account citizens' values also ensures long-term legitimacy, as the subjective basis makes the argumentation process more acceptable.

In addition to these three CA's characteristics, we propose that a fourth aspect, namely the *systemic* modeling, should be further developed in order to improve the efficiency of the CA in a policy analytics context.

Systemic. By adopting a systemic approach to welfare representation, we can account for the multiple interactions between access to private resources and access to the commons, and individual and collective behaviors.

Exploring deep “what-if” questions is only possible when the systemic nature of welfare is explicitly considered. It is through representation of the interactions between resources, actions, and values that different *designs* become visible.

To achieve rational policy design, it is crucial to *explore* and measure the consequences of different policies. Policies should be simulated, projected onto the citizens, and their impacts studied, including possible drawbacks and unforeseen outcomes. A systemic representation of welfare, as proposed in our model, enables such exercises by offering quantitative analysis, simulation, and visualization of impacts. This provides stakeholders with a common ground for discussion and negotiation, making a formal model a necessary condition for an effective participative policy design process.

Table 2.1 provides a summary of how the advantages of the CA can facilitate the different steps of a policy cycle, showcasing its applicability in policy design.

2.2 Model

In this section, we propose a Mathematical Programming (MP) framework with the objective of providing a rational approach to measure welfare and support public

	Understand	Design	Explore	Argue
Multidimensional	✓	✓		✓
Diversity	✓	✓		✓
Subjective	✓	✓		✓
Systemic	✓	✓	✓	

Table 2.1: How the CA is useful to construct a policy analytic framework.

policy. Our model is designed to capture a concept close to the citizens' capability sets, which represents their potential to achieve different sets of actions constrained by their access to resources (private or Commons) and conversion factors. A public policy is viewed as a modification of citizens' access to resources and conversion factors, which in turn affect their set of actions, that is evaluated (through their value functions and their conversion factors) in a multi-dimensional welfare space.

To solve this MP problem for a given citizen, we compute the Pareto frontier, which represents the set of actions that are compatible with the citizen's endowments and are not dominated on any of the welfare dimensions that matter for that citizen. This Pareto frontier, which closely resembles a capability set, is referred to as the "capability set" or "welfare representation" in our framework. The reader should consider that the term "capability set" will be defined hereafter with that sense.

Our approach consists of several distinct steps:

First Step: Establish a generic model The initial step involves creating a generic model to compute the capability set of a given citizen, which serves as a measure of their welfare in the present state.

Second Step: Simulate policy impact In the second step, we use the same generic model to simulate the impact of a public policy on a given citizen, thereby altering the current distribution of welfare. A public policy is assumed to modify certain parameters of the mathematical program. By computing the new welfare representation, we simulate the updated capability set of citizens in light of the new policy.

Decision-Making Process If selecting the most optimal public policy based on simulated capability sets is not straightforward, additional steps are required in the

decision-making process. This involves finding the best solution, which will be the subject of Chapter 4.

Addressing Uncertainty We must also address the inherent uncertainty arising from public policy consequences. This uncertainty may stem from issues related to data quality, modeling assumptions, hypothesis validity, or unexpected events. This topic will be covered in Chapters 5 and 6.

Future Considerations Finally, although it is beyond the scope of this manuscript, an intriguing step would involve clustering the population based on various characteristics, primarily using the similarity of their capability sets or welfare distributions.

Throughout the remainder of this chapter, our focus will predominantly be on presenting the first and second step in detail.

2.2.1 Mathematical programming model

The diagram in Figure 2.1 provides an overview of our model, illustrating the interactions between various components. Let us discuss the different elements in the simple generic case where we have a linear system.

Variables. Our decision variable vector $\vec{x} = [x_1 \ x_2 \ \cdots \ x_{j^*}]$ represents the *Doings*, which are the actions that citizens can undertake in order to “achieve”, “obtain” or “reach” one or more goals. More precisely, a *Doings* is considered as a combination of actions that an individual can perform within a set time period and the *Doings set* encompasses all possible combinations of actions that a citizen can choose to perform within a given interval of time.

Constraints over private resources. A *Doings* consumes private resources, and the consumption of a private resource k by a citizen i to achieve their *Doings* must not exceed the quantity of private resources they possess, denoted as $\vec{r}^i = [r_1^i \ r_2^i \ \cdots \ r_{k'}^i]$, where k' represents the number of different private resources owned or accessible by the citizen. It is worth noting that private resources are defined broadly to include intangible items such as time or knowledge. The amount

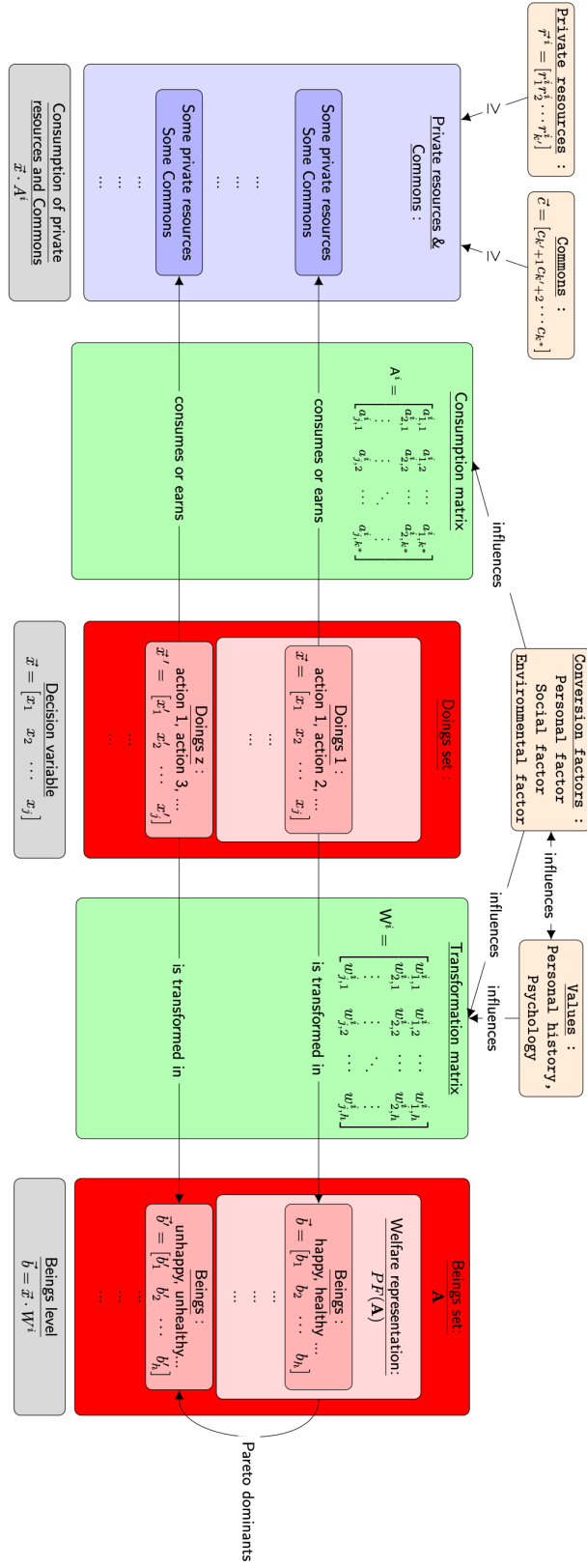


Figure 2.1: A diagram of the welfare representation

of private resources consumed or earned is determined by the combination of Doings and the conversion matrix A^i . The elements $A^i = (a_{j,k}^i; j \in \{1, \dots, j^*\}, k \in \{1, \dots, k^*\})$ are determined through *personal conversion factors*, *social conversion factors* and *environmental conversion factors*.

Assuming the problem is linear, the constraints over private good consumption can be expressed as:

$$\sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq r_k^i \quad \text{for all } k \in \{1, \dots, k'\}.$$

Constraints over Commons. We consider a vector of *Commons*, denoted as $\vec{c} = [c_{k'+1} \ c_{k'+2} \ \dots \ c_{k^*}]$ as defined by [Ostrom1990]. As for private resources, the “quantity” of a Common used or consumed by a citizen i is influenced by elements $a_{j,k}^i \in A^i$ from the conversion matrix. We distinguish between two types of Commons: those that are utilized and those that are consumed. In the case of utilized Commons, the quantity of Common is not reduced when a citizen uses it, but it may reduce the utility derived for other citizens. For example, a road can be considered as a utilized Common, where more people on the road do not consume it but may create traffic jams, decreasing the overall utility of the road. For such Commons, we can use a binary representation, where $c_k \in \{0, 1\}$, indicating whether the Common is available or not. All actions j either use this Common k , ($a_{j,k}^i = \epsilon$ with ϵ being a very small value), or do not use it ($a_{j,k}^i = 0$). If the Common becomes unavailable ($c_k = 0$), for example due to damage or inaccessibility, all actions that use k ($\exists a_{j,k}^i = \epsilon$) are then infeasible and set to 0. A generic way to model the impact of Common is considering similarly to the private resources those type of constraints:

$$\sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq c_k \quad \text{for some } k \in \{k' + 1, \dots, k^*\}.$$

In the case of consumed Commons, resources available for a particular citizen is what is left by the other citizens. For instance, if a citizen consumes a quantity of water from a groundwater, this can not be consumed by someone else. The portion of a Common k left by others is denoted a Δ_k . Consumable Commons use continuous representations; $c_k, \Delta_k \in \mathbb{R}^+$. In the general case, those Commons can be represented as:

$$\sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq c_k - \Delta_k \quad \text{for other } k \in \{k' + 1, \dots, k^*\}.$$

Objective Function. The *Beings vector*, denoted as $\vec{b} = [b_1 \ b_2 \ \cdots \ b_{h^*}]$, represents the mapping of our variables (*Doings*) into the welfare space of h^* dimensions.

The *Beings set* encompasses all possible Beings an individual can potentially experience. Essentially, it is the image of the Doings set. The welfare representation, or *capability set*, denoted as $\mathbf{A}, \mathbf{B}$, etc., for citizens $i = 1, i = 2$, and so on, consists of the Beings not Pareto-dominated within the Beings set representing the Pareto frontier. These Beings represent the “optimal” Doings an individual can achieve regarding welfare.

We assume that if all welfare dimensions impacting citizens’ choices are considered, individuals would naturally choose a Pareto-efficient solution. For instance, consider Sen’s example of a fasting person for political reasons. Focusing solely on two functionings, such as *being able to obtain an adequate amount of food* and *being able to maintain good health*, fails to capture the full scope of citizens’ welfare and decision-making. Fasting is not Pareto-efficient because by eating, the citizen could potentially improve both dimensions. However, by incorporating additional welfare dimensions, such as *expressing political opinion*, the choice becomes Pareto-efficient. From a decision-aiding perspective and considering operational constraints, it is practically impossible to account for all welfare dimensions impacting citizens’ choices. Thus, computing the Pareto frontier serves as an approximation.

The transformation matrix, denoted W^i (where $w_{j,h}^i \in W^i$), determines how the Doings vector $\vec{x}$ is converted into the Beings vector $\vec{b}$. These transformations are influenced by conversion factors, including personal, social, and environmental aspects, as well as individual values.

In the linear case, with $PF(\cdot)$ designating the Pareto frontier, the objective functions are represented as:

$$PF\left(\sum_{j=1}^{j'} x_j \cdot w_{j,1}^i, \cdots, \sum_{j=1}^{j'} x_j \cdot w_{j,h^*}^i\right).$$

Referring back to the diagram in Figure 2.1, we can summarize the relationships as follows: Personal factors influence the consumption matrix, which dictates

how the Doings performed by a citizen translate into the consumption of various resources, both private and common. Since resources are limited, citizens cannot engage in Doings that consume more resources than they have access to. The set of all feasible Doings a person can choose from is referred to as the Doings set.

On the other hand, values, shaped by an individual's psychology and personal history, interact with conversion factors to form the personal transformation matrix. The purpose of the citizen's actions is to achieve certain Beings, which are the outcomes of transforming Doings using the transformation matrix. The Beings set represents all possible Beings as a mapping of the Doings into the welfare dimension space.

Since citizens are considered rational, their capability set is the Pareto frontier of the Beings set. From a mathematical programming perspective, private resources and commons are utilized to achieve specific levels of Beings through the realization of particular Doings.

The relationship can be formalized in the following mathematical problem:

$$\begin{aligned}
 & PF \left(\sum_{j=1}^{j^*} x_j \cdot w_{j,1}^i, \dots, \sum_{j=1}^{j^*} x_j \cdot w_{j,h^*}^i \right) \\
 & s.t. \\
 & \sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq r_k^i, \quad \forall k \in \{1, \dots, k'\}, \\
 & \sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq c_k, \quad \text{for some } k \in \{k' + 1, \dots, k^*\}, \\
 & \sum_{j=1}^{j'} x_j \cdot A_{j,k}^i \leq c_k - \Delta_k, \quad \text{for other } k \in \{k' + 1, \dots, k^*\}.
 \end{aligned}$$

Of course, the consumption of resources is not always linear, and public policies or scenarios (denoted s_l) can influence both how people consume resources and their access to these resources, including commons and private resources. Additionally, the transformation of Doings into Beings is not necessarily linear and can also be affected by certain policies or scenarios.

Thus, a more general way to model our problem is presented below:

$$\begin{aligned}
 & PF \left(\omega_1(\vec{x}, W^i, s_l), \dots, \omega_h(\vec{x}, W^i, s_l) \right) \\
 & s.t. \\
 & \psi_k(\vec{x}, A^i, s_l) \leq r_{k,l}^i, \quad \text{for all } k \in \{1, \dots, k'\}, \\
 & \psi_k(\vec{x}, A^i, s_l) \leq c_{k,l}, \quad \text{for some } k \in \{k', \dots, k^*\}, \\
 & \psi_k(\vec{x}, A^i, s_l) \leq c_{k,l} - \Delta_{k,l}, \quad \text{for other } k \in \{k', \dots, k^*\}.
 \end{aligned}$$

In this model, $\psi_k(\vec{x}, A^i, s_l)$ and $\omega_h(\vec{x}, W^i, s_l)$ are not necessarily linear functions. The terms $r_{k,l}^i$, $c_{k,l}$, and $\Delta_{k,l}$ represent the amounts of private resources, common

resources, and societal consumption, respectively, in the scenario s_t .

2.2.2 Construction of the model

A public policy has direct consequences on citizens' welfare by impacting their private resources and their access to Commons. Private resources are affected through mechanisms such as taxes, flood risk management, and help with energy renovation, while Commons are influenced through urbanism, water, forests, schools, hospitals, laws, and rights, among others. In a symmetrical way the collective threats threaten both private resources and Commons.

The first step in constructing the model is to identify all the private resources ($\vec{r}^i$) and Commons (vectors $\vec{c}$, $\vec{\Delta}$) that will be directly affected by the public policy or an external event. The modification of resources directly affects citizens by enabling them to perform more or fewer actions that consume or earn those resources. The analyst must identify all the relevant actions that are directly impacted by the policy or event, i.e., actions that consume or earn resources that are modified by the policy. For example, if a city council seeks to study the impact of transforming a brownfield site into a park, citizens will be able to choose to engage in various activities in this park, such as running, walking, playing, meeting friends, reading a book, sitting on a bench, or breathing fresh air.

The choice of welfare dimensions should consider both the preferences of the client and the welfare dimensions that are associated with the directly impacted actions. Continuing with the above example, we can identify two main welfare dimensions: leisure (involving running, walking, playing, meeting friends, reading a book, and sitting on a bench) and health (involving running, walking, playing, and breathing fresh air). The first step of the model construction is illustrated in Figure 2.2 with continuous arrows.

The second step of the model is to identify the indirect effects of the public policy. This involves considering a set of activities that have a significant effect on the selected welfare dimensions. Building on the previous example, if the welfare dimensions are leisure and health, other activities such as similar activities in another park, sports activities in local sports facilities, and other leisure activities in places of entertainment need to be considered. These new activities will consume or earn new private and Common resources, which need to be integrated into the model. In our

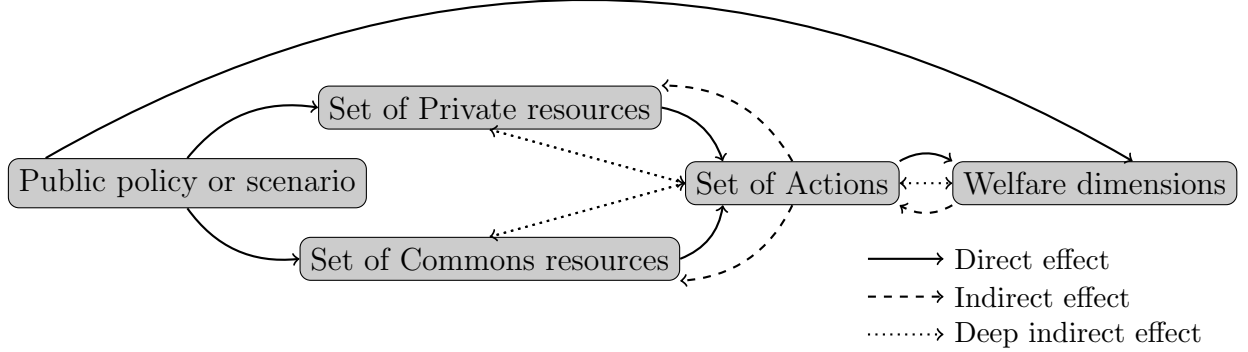


Figure 2.2: A diagram of the construction of the model for welfare representation.

example, this would include the other park, sports facilities, places of entertainment, money (to be used in places of entertainment), and so on. The second step of the model construction is represented by dashed arrows in Figure 2.2.

It is possible to continue adding all the activities that use these new resources until all possible activities and resources that affect the well-being of a citizen are included (as shown by dotted arrows in Figure 2.2). However, due to limitations in time, cognitive capacity, and financial resources, the analyst must decide when to stop the procedure and consider that there are sufficient resources and actions to approximate the problem being modeled.

2.2.3 Data collection

We can broadly distinguish two types of data: objective (or minimally subjective) data with $\vec{r}^i$, $\vec{c}$, $\vec{\Delta}^i$, and A^i and subjective data with W^i . The subjectivity arises primarily through individual values as shown in Figure 2.1. The objective elements can be determined through common sense, observation, and expert opinions.

Subjective data involve the transformation parameters for each citizen, which capture personal values. Learning these parameters is essential for understanding individual perspectives and preferences. This can be accomplished through surveys that ask citizens to rate the impact of specific actions on welfare dimensions or through multi-attribute value theory (MAVT) [keeney1993decisions, winterfeldt, jansen2011multi, dyer2016multiattribute].

Ultimately, data collection methods must evolve from practical experiences, uti-

lizing guidelines developed by practitioners to ensure effective acquisition and application of the necessary data. Successful data acquisition can only be achieved through practice, informed by real-world applications.

2.3 Example

This illustrative example is conceived to showcase the utilization of our framework in the context of welfare measurement. More precisely, the example aims to demonstrate how the influence of the Commons on citizens' welfare can be modeled. In the context of public policies aimed at protecting Commons, the ability to identify critical Commons and measure their impact on citizens' welfare is a crucial step in the design and rationalization of public policy decisions. Measuring the impact of damage to a Common can provide insights into the consequences of policies that involve “not maintaining”, “removing”, or “shutting down” a Common.

In this example, we consider the welfare of two individuals, Alice and Bob, who are in current states denoted as $\mathbf{A}$ and $\mathbf{B}$, respectively. They both reside in the same city, represented by a graph $G(V, E)$ shown in Figure 2.3.

- The set of vertices, denoted as V , represents points of opportunities in the city, such as home, park, city center, and workplace, denoted by the vertices 1, 2, 3, and 4, respectively. Both citizens live in the same neighborhood (vertex 1) and work at the same place (vertex 4). On each vertex, citizens can perform actions, which are transformed into well-being through the conversion matrix W^i (refer to Table 2.3 and 2.4). These transformation parameters are personalized for each citizen and depend on their conversion factors and values. An action consumes private resources and Commons A^i , with the consumption differing only based on the conversion factors.
- The set of edges, denoted as E , consists of roads e_{vwr} and public transport (PT) e_{vwp} that connect two vertices vw . Both roads and PT are considered as Commons in this example. For simplicity, we assume that the transformation of utilizing a road (or PT) and the consumption of private resources are the same regardless of the specific road (or PT) used by the citizen (refer to Table 2.3 and 2.4). The utilization of these edges is considered as actions in the framework.

Citizen	Money	Time in hours	Car in hours
Alice	0	24	24
Bob	0	24	0

Table 2.2: Vectors of private resources

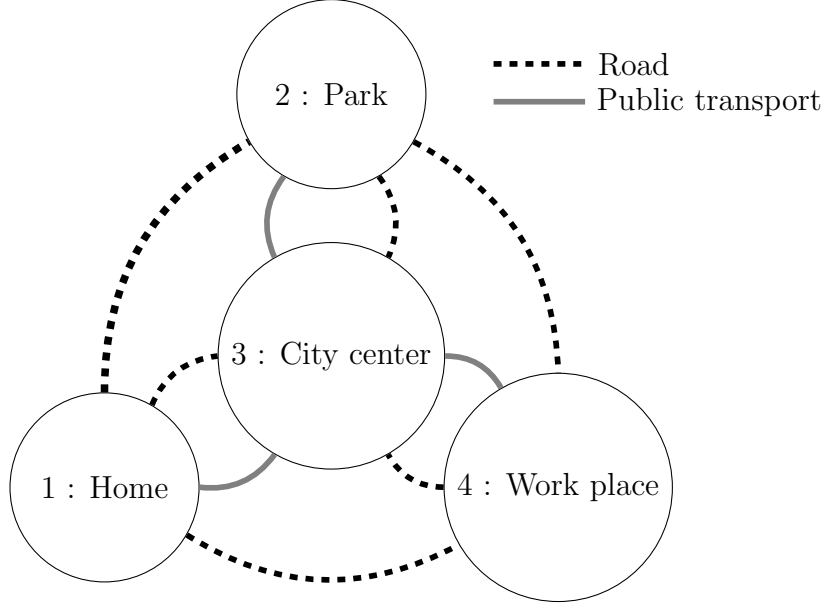


Figure 2.3: Graph of the city of Alice and Bob.

Var	Activity	Transformation Matrix W^a		Conversion matrix A^a on private resources			Conversion matrix A^a on Commons		
		Health	Pleasure	Money	Time	Car	Road ij	PT ij	Park
x_{ijr}	Take the road ij	0	-1	2	0.5	0.5	ϵ	0	0
x_{ijc}	Take the pubic transport ij	0	-2	2	0.5	0.5	0	ϵ	0
y_{11}	Sleep	10	10	0	9	0	0	0	0
y_{12}	Family Time	0	2	0	1	0	0	0	0
y_{21}	Walk	3	2	0	2	0	0	0	ϵ
y_{22}	Run	6	-2	0	1	0	0	0	ϵ
y_{31}	Museum	1	6	10	3	0	0	0	0
y_{32}	After work	-1	1	5	1	0	0	0	0
y_{33}	Doctor	5	-1	30	1	0	0	0	0
y_{41}	Work	1	1	-100	8	0	0	0	0

Table 2.3: Table of the conversion and transformation matrix of Alice

Var	Activity	Transformation Matrix W^b		Conversion matrix A^b on private resources			Conversion matrix A^b on Commons		
		Health	Pleasure	Money	Time	Car	Road ij	PT ij	Park
x_{ijr}	Take the road ij	0	0	2	0.5	0.5	ϵ	0	0
x_{ijc}	Take the pubic transport ij	0	-1	2	0.5	0.5	0	ϵ	0
y_{11}	Sleep	10	10	0	9	0	0	0	0
y_{12}	Family Time	0	1	0	1	0	0	0	0
y_{21}	Walk	1	2	0	2	0	0	0	ϵ
y_{22}	Run	3	4	0	1	0	0	0	ϵ
y_{31}	Museum	0	2	10	3	0	0	0	0
y_{32}	After work	0	2	5	1	0	0	0	0
y_{33}	Doctor	2	-3	30	1	0	0	0	0
y_{41}	Work	1	1	-100	8	0	0	0	0

Table 2.4: Table of the conversion and transformation matrix of Bob

Each citizen possesses a set of private resources denoted as $A^i = \{A_1^i = \text{Money}, A_2^i = \text{Time}, A_3^i = \text{Car}\}$, as described in table 2.2. The objective of a citizen is to maximize their well-being, represented by two dimensions: Health ($h = 1$) and Pleasure ($h = 2$), through their respective transformation functions. The solutions obtained from this maximization process yield the Pareto frontier $PF(\mathbf{A})$ and $PF(\mathbf{B})$ displayed in Figure 2.4. To obtain these solutions, we solved two mathematical programs, with the MP corresponding to $PF(\mathbf{A})$ displayed in Table A.1 in Appendix A.

The impact of damaging a Common on citizens' welfare is now simulated, wherein the corresponding parameter is assigned a value of 0 to denote the resource's unavailability. Thereafter, a new linear program is formulated and solved to obtain the updated welfare representation of Alice after the damage to the park, as presented in Table A.2 in Appendix A. Figure 2.5a shows the welfare representations of Alice $PF(\mathbf{A}|\text{Road14})$ and Bob $PF(\mathbf{B}|\text{Road14})$ after the closure of the road connecting their home (1) and workplace (4). The green (resp. blue) space represents the space that used to be dominated by at least one beings in $PF(\mathbf{A})$ (resp. $PF(\mathbf{B})$), but by none in $PF(\mathbf{A}|\text{Road14})$ (resp. $PF(\mathbf{B}|\text{Road14})$). This serves to graphically show how much a citizen loses in freedom of choice (the assessment of freedom of choice will be discussed in greater detail in Chapter 4). Figure 2.5b depicts the welfare representations of Alice $PF(\mathbf{A}|\text{Park})$ and Bob $PF(\mathbf{B}|\text{Park})$ after the park's damage.

These two figures allow a simple analysis of the negative impact of a Common being not available. For instance, the closure to the Road linking 1 to 4 has a

negligible negative impact on citizens' Welfare representation. This is due to the fact that Bob, who does not possess a car, remains unaffected, while the potential Beings of Alice experience only a marginal decrease in their Welfare representation. In contrast, the damage inflicted upon the park is deemed to have a significant impact, particularly due to its implications for health-related outcomes, as Beings characterized by high levels of Health can no longer be achieved.

Let us now examine the negative impact of the above damage to the Commons from a utilitarian perspective, using a simplified cost-benefit analysis (CBA). Building upon the findings of [metz2008myth], which indicate that 80% of the benefits derived from a road stem from time savings (other benefits will be ignored), we consider the usage patterns of Alice and Bob. In our example, Alice utilizes road $r_{1,4}$ approximately 8 times per week, while Bob does not use it at all. This translates to an average daily usage of 1.5 times (equivalent to 45 minutes/day) for the population under consideration. In the event of the closure of the road between 1 and 4, the fastest alternative path increases the travel time by 30 minutes compared to the direct road. Consequently, this results in an average wasted time of 45 minutes per day (1.5 times 30 minutes). For the purpose of our analysis, we assume a time valuation of 0.11 euro/minute, in line with the estimation for time savings/losses used in the UK [DPUL], which results in a total loss of 3.77 euros per day (time wasted multiplied by the valuation of time loss).

Turning our attention to the impact of the park's damage, we seek to establish a proxy market for its valuation. The primary benefits of the park are associated with health, owing to its facilitation of physical activities and air quality improvement through carbon capture. As a proxy for physical activities, we adopt the price of a gym pass, while the amount of carbon capture by the park is multiplied by the carbon emissions price of the European Union Emissions Trading System (EU ETS)¹ to estimate the value of air quality improvement. This results in an estimated benefit of approximately 3.75 euros per day.

In our cost-benefit analysis, we used the values of time for the road, and health and entertainment/pleasure proxy for the park. It is noteworthy that these dimensions are encompassed in our capability-based approach, where health and pleasure

¹https://ec.europa.eu/clima/policies/ets_en

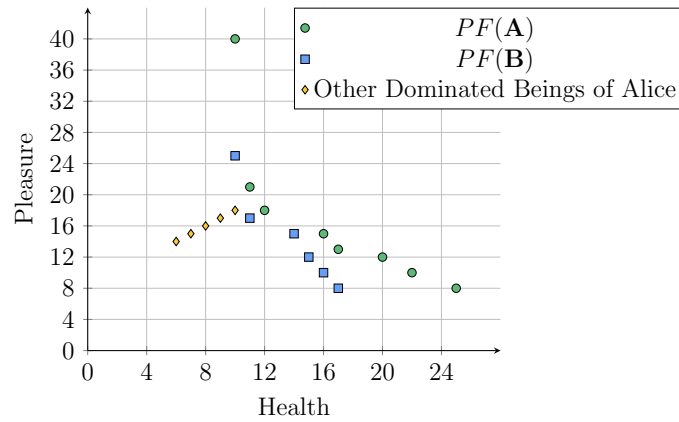


Figure 2.4: Citizens Welfare representation in the initial state.

are considered as welfare dimensions, and time is regarded as a constraint. We view time as a constraint because our focus is not solely on the amount of free time available in a day, but rather on how individuals can utilize it.

Through the illustration of our simplified example, we have demonstrated that our capability-based approach may offer better insights to decision makers compared to a utilitarian approach. In particular, while a CBA may suggest that the loss incurred from the closure of the road and the damage to the park are equivalent, our capability-based approach enables us to identify the following distinctions:

- The closure of the road will not impact Bob, but will have negative consequences for Alice.
- The damage to the park is projected to have a greater negative impact on citizens' welfare.
- The damage to the park will predominantly affect citizens' ability to achieve high level of health.

It is worth noting that the CBA represents an archetype of the utilitarian welfare measurement approach, and the limitations of such an approach, such as the compensation among impacts and the consideration of all citizens as indistinguishable consumers, apply to other utilitarian approaches as well. Thus, our capability-based approach offers a more nuanced and comprehensive perspective on welfare assessment.

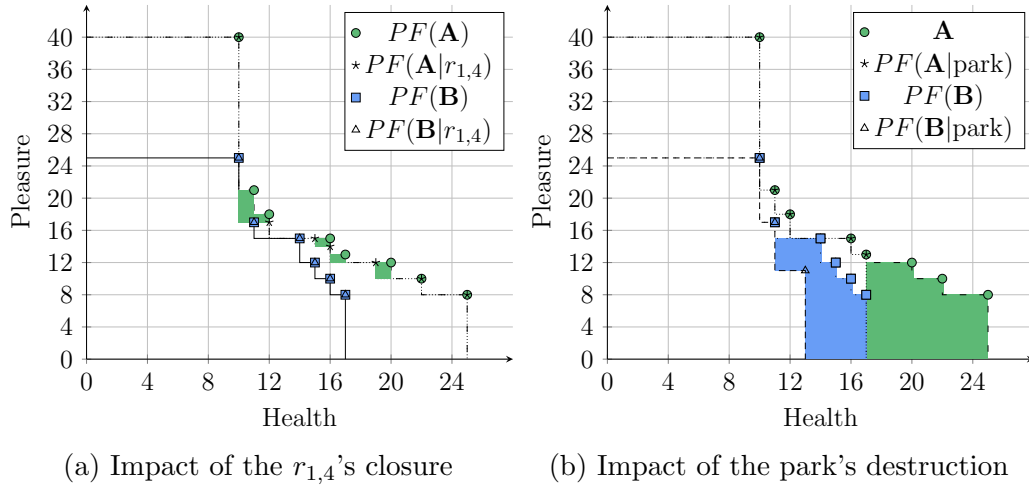


Figure 2.5: Damages

Chapter 3

Case study: Exploring mental health capabilities of older adults in the Paris urban context

The “Neighborhood Environments and Mobility: Effects on Social Health Inequalities” team (NEMESIS) at the *Institut Pierre Louis d’Épidémiologie et de Santé Publique* (IPLESP) investigates whether a range of environmental exposures incurred in urban contexts (whether at home, in visited locations, outdoors, or during travel), have an impact on selected dimensions of health. The primary exposures of interest are related to urban forms (such as built environments, green and blue spaces, and services) as well as air pollutants and noise. While their work significantly enhances statistical analyses related to health, it currently needs to improve the frameworks for designing public health policies. We are collaborating with NEMESIS to fortify their policy-making impact, focusing on effectively developing a method for aiding decision in health policy design integrating the epidemiological study of environment and transportation determinants of wellbeing with policy making methods for aiding decision.

This chapter aims to use NEMESIS data from HANC and MINDMAP study to examine how the access to different urban opportunities and limitations entail a risk or an opportunity for the development of mental wellbeing of elders in the Paris metropolitan area, with a focus on their mental capabilities within their urban environment. The ultimate goal is to help to design public policies that enhance the mental wellbeing of these older adults. This serves as a case study to demonstrate that the model presented in Chapter 2 can be implemented effectively and provide useful insights. In detail, we analyze the set of actions available to individuals (what they can *do* and how it influences their *being*) considering both their personal resources and the shared urban infrastructure they can access. In addition to accessible resources, we identify key barriers that limit the improvements in mental health. This will offer insights to guide policymakers in building policies enhancing population’s mental health.

Mental well-being is a fundamental component of overall health, particularly in older populations, where mental health disorders such as depression and anxiety are becoming increasingly prevalent. According to the OECD and EU [oecd2018], more than one in six individuals in Europe experienced mental health issues, with depression and anxiety being the most widespread concerns. These conditions are even more pronounced among elders, as mental health disorders in this demographic group often contribute to physical disabilities, diminished quality of life, and a

higher risk of mortality. The prevalence of mental health disorders increases significantly with age, making individuals aged 65 and over particularly vulnerable [eurostat2020]. As Europe’s population continues to age, with projections estimating an increase from 90.5 million elderly people in 2019 to 129.8 million by 2050, promoting healthy aging has become a crucial public health priority [eurostat2020].

Urban environments, where a large proportion of the elderly population resides, have been identified as both potential risk and protective factors for mental well-being. Exposure to urban stressors, such as air pollution, noise, and overcrowding degrade mental health. However, positive environmental features such as green spaces, pedestrian-friendly areas, and water bodies have shown restorative effects, promoting mental recovery and reducing symptoms of depression [buttazzoni2021, fancello2023micro]. Recent studies have emphasized the need for a dynamic and people-centered approach to urban mental health, one that considers the positive effects of the exposure through daily mobility paths to a variation of mixture of urban landscapes factors [chaix2017, fancello2023micro]. We believe that a truly people-centered approach should also account for individuals’ private resources and their subjective insights into how they perceive common resources.

Section 3.1 presents the data used in this study, and Section 3.2 focuses on the modeling assumptions and formulation of the model. In Section 3.3, we discuss the results and suggest potential policy interventions based on the findings. Note that due to the number of assumptions made, the results do not serve as definitive real-life conclusions but rather as a demonstration of how the model can be practically useful. Finally, Section 3.4 explores how this study could be improved and expanded in future research, so that future results can have more direct real-life applicability.

3.1 Dataset

The study followed 216 participants aged 60 years and older, from July 2019 to July 2021. They were selected from the Paris metropolitan area as part of the *HANC* and *MINDMAP* studies [fernandes2021linking], and from the RECORD Cohort [chaix2012cohort]. Some socio-economic data are presented in Table 3.1. These participants were monitored for seven consecutive days using a combination of GPS receivers, accelerometers, and surveys to assess their mental well-being in relation

to their mobility and exposure to different urban environments. While the dataset was not specifically designed for a capability study, it offers valuable insights into personal mobility patterns in Paris and their associated mental health impacts.

First, the preprocessed GPS data allow access to the visited places by participants at any given time during the observation week. These places included parks, bakeries, supermarkets, cinemas, and more, as well as how participants moved through the city using various modes of transport, such as trams, walking, buses, metro, cars, and more.

Two key mental wellbeing metrics were measured both before and after the GPS observation period:

- **STAI (State-Trait Anxiety Inventory):** This psychological inventory uses a 4-point scale and consists of 40 self-reported questions to measure the presence and severity of current anxiety symptoms, as well as a general tendency toward anxiety. Scores range from 20 (indicating the lowest level of anxiety) to 80 (indicating the highest level of anxiety) [balsamo2013state].
- **CES-D (Center for Epidemiologic Studies Depression Scale):** A screening tool for depression designed to measure depressive symptoms in the general population. It comprises 20 items, each rated on a 4-point scale, assessing the frequency of depressive symptoms experienced in the past week. Scores range from 0 (indicating the absence of depressive symptoms) to 60 (indicating severe depressive symptoms) [radloff1977ces].

Additionally, a questionnaire was administered to participants just before the observation week. This questionnaire included various socio-economic data, as well as questions related to their neighborhood and subjective assessments of their ability to perform certain actions (e.g., being able to drive, afford public transportation, or feel safe in the metro). Some of the relevant questions used later in this study are provided in Appendix B.

In summary, the key data used in this study are:

1. The GPS location data of participants, collected over the course of one week, along with detailed information about their associated activities.
2. The STAI and CES-D scores, recorded both before and immediately after the observation period. These scores, in combination with the GPS data and

	gender	marital status	income home	level of study
female	76	—	—	—
male	136	—	—	—
couple	—	143	—	—
divorced/widowed/unmarried	—	69	—	—
<2000	—	—	31	—
2000-4000	—	—	128	—
>4000	—	—	53	—
Lower education	—	—	—	63
Secondary education	—	—	—	70
Hihger education	—	—	—	79

Table 3.1: Socio-economic data

additional assumptions, will be used in estimating the transformation matrix for our analysis.

3. A set of survey responses, which include socio-economic information, neighborhood characteristics, and subjective perceptions about participants' capacities and their environment. These responses are crucial for constructing the constraints of the model.

Some temporal data were missing for certain participants. To ensure the reliability of the analysis, we excluded participants with less than 95% of the observation week recorded, reducing the number of participants from 216 to 212.

3.2 Assumptions and modeling

To test the model presented in Chapter 2, we were constrained to use a data set that was not originally collected for our specific purpose, necessitating several assumptions that may not fully align with real-world conditions. Furthermore, due to the data-driven nature of our approach, we will not strictly adhere to the guidelines provided in Section 2.2.2 on the construction of the model.

3.2.1 Selecting the action

The first assumption is that we focus exclusively on free time activities, excluding work-related activities from the set of considered actions. This decision is based on

the fact that work activities vary significantly between individuals in terms of type, intensity, duration, and location. Additionally, we assume that time spent at home does not constitute a specific action but rather represents an absence of action, as our data does not provide details about what individuals actually do inside their homes. Consequently, we restrict our focus to observable activities outside of the home, concentrating only on leisure time.

Actions can be categorized into two main types:

- *End actions*, denoted by a_j (or a_j^i to indicate it is specific to citizen i)
- *Transport actions*, denoted by t_j (or t_j^i to indicate it is specific to citizen i)

This categorization is mainly for practical reasons, as the two types of actions are subject to different constraints.

From the available data, we have identified 43 unique end actions. These can be regrouped into 9 broader categories, as detailed in Table 3.2. Regrouping is necessary to ensure sufficient data is available to assess the impacts of these actions on citizens. The rationale behind regrouping is that if an action is performed by only one or few individuals, its impact, when analyzed using linear regression (as discussed in Section 3.2.2), may appear disproportionately high and unrealistic. On the other hand, we aim to retain a diverse set of actions to capture the variety of activities people can engage in. Ideally, with a much larger sample size, regrouping might not be necessary. We also regrouped the transportation actions, as shown in Table 3.3. We made a distinction between surface and underground public transport because the Parisian metro is poorly accessible, often requiring the use of stairs.

An issue with the data is that most transport modes are intermodal, typically combining walking with another form of transportation. From a modeling perspective, it is simpler to consider that a travel using the underground includes the walking time within the underground journey itself. Therefore, we decided that each journey should be categorized as a single, unified action. To achieve this, we applied the following rule: a “transport set” is defined as all transportation actions occurring between two distinct end actions. We then sum the total time spent on these actions and assign a single action based on a predefined hierarchy: underground public transport, surface public transport, private transport, cycling, and walking.

Activities	Regrouped activities
Home Alternative Accommodation, Hotel Secondary Residence Workplace Other Walks	Not used
Supermarket Grocery Store	General Shopping
Bakery Fruit and Vegetable Vendor Fishmonger Butcher/Delicatessen Cheesemonger Frozen Food Store Organic Store Market Other specialized Food Store Sale of Takeaway Food Preparations	Specialized Food Shopping
Traditional Restaurant Institutional Catering Fast Food Café, Bar	Restaurant
Tobacco Shop Bank Other Errands and Services Post Office Hairdresser	Other Shops and Services
Doctor Dentist Other Health Professional Medical Testing Pharmacy	Medical
Green Spaces	Green Spaces
Sports Facilities	Sports Facilities
Cultural Activity Community, Political, Union, Religious Activity Cinema Other Performances	Cultural activities
Visiting People Accompanying People	Visiting People

Table 3.2: Regrouped end actions

For instance, if an individual walks, then takes a bus, followed by the metro, and walks again, the entire sequence is categorized as underground public transport.

It is important to note that the unified action of taking the underground includes the assumption that individuals typically walk to access it, which has been shown

Action	Regrouped Action
Walking	Pedestrian
Personal Bicycle Bike-sharing Service (Vélib')	Cycling
Moped - Driver Motorcycle - Driver Car-sharing Service (Autolib') - Driver Car - Driver Motorcycle - Passenger Moped - Passenger Car - Passenger Car-sharing Service (Autolib') - Passenger Taxi	Private Transport
Urban Bus Coach Bus Tram	Surface Public Transport
Subway RER (Regional Express Network) TER/SNCF Suburban SNCF Suburban TER (Transport Express Régional)	Underground Public Transport

Table 3.3: Regrouped transportation actions

to improve mental well-being, as discussed in [chaix2019combining].

Table 3.4 gives the label of all our actions. In our model, transportation is conducted with the intent of carrying out a specific end action, but will also have an impact on our welfare dimension. The notation $t_{m,j}$ is employed to signify that the citizen travels using mode m to perform action j . For instance, $t_{4,9}$ refers to using surface public transport to go to a restaurant.

3.2.2 Transformation matrix

We have access to the CES-D and STAI scores measured just before the observation week (denoted as $S_{\text{CES-D, pre}}^i$ and $S_{\text{STAI, pre}}^i$) and immediately after the observation week (denoted as $S_{\text{CES-D, post}}^i$ and $S_{\text{STAI, post}}^i$). Our key assumption is that the difference between these two sets of scores is related to variation in actions performed during the observation week compared to the previous week.

Label	Activities
a_1	General Shopping
a_2	Specialized Food Shopping
a_3	Other Shops and Services
a_4	Medical
a_5	Green Spaces
a_6	Sports Facilities
a_7	Cultural Activity
a_8	Visiting People
a_9	Restaurant
t_1	Pedestrian
t_2	Cycling
t_3	Private Transport
t_4	Surface Public Transport
t_5	Underground Public Transport

Table 3.4: Actions and their associated labels

While this assumption seems intuitive, suggesting that changes in anxiety and depression levels are linked to differences in activities, it is quite strong and carries significant limitations for several reasons.

Firstly, anxiety and depression scores can be influenced by external factors unrelated to the individual's actions, such as weather conditions in Paris or French politics. Secondly, there is a reciprocal relationship between mental health and actions: anxiety and depression can shape the types of activities individuals choose to engage in. For instance, while a person engaging in fewer activities may experience increased depression, it is equally plausible that someone already suffering from depression may engage in fewer activities due to their mental state.

A key challenge with our data is that we only have information on the actions $x_{j,post}^i$, which were performed during the week corresponding to the second set of STAI and CES-D scores. However, we lack data on the actions $x_{j,pre}^i$ that occurred during the week preceding the first set of scores.

Learning the transformation matrix using only the post scores, without considering the difference from the pre scores, would introduce significant bias. This is because post scores alone do not account for pre-existing conditions or baseline factors that could influence the outcomes. To apply our framework effectively, it is

essential to focus on the difference between pre and post scores. Consequently, since we are analyzing the change in scores, we must also consider the change in actions occurred in the previous week.

To do so, we need to have access to pre actions $x_{j,pre}^i$. Our second assumption concerns the method by which we generate these unknown actions. We assume that every action has a positive impact, or at least not a negative one, on the welfare dimensions. The previous alternatives are generated as follows.

$$x_{j,h,pre}^i \sim \mathcal{N}(\mu, \sigma^2), \quad \text{with } x_{j,h,pre}^i \in [\mu - 1.96 \cdot \sigma, \mu + 1.96 \cdot \sigma]$$

where:

$$\mu = x_{j,post}^i \times \left(1 + \frac{S_{h,pre}^i - S_{h,post}^i}{60} \right), \quad h \in \{\text{STAI, CES-D}\}.$$

Note that we generate two sets of actions for the pre-observation week: one determined by the STAI scores and one by the CES-D scores. While it does not align with realistic scenarios, it is mathematically grounded.

Here, μ represents the average number of actions $x_{j,h,pre}^i$, adjusted based on the difference between the scores before and after the observed week. The term $\frac{S_{h,pre}^i - S_{h,post}^i}{60}$ normalizes the change in scores between -1 and 1 . If the score has decreased ($\frac{S_{h,pre}^i - S_{h,post}^i}{60} > 0$), indicating an improvement in well-being, the assumption is that the individual is likely to have engaged in more actions, on average, compared to the previous week.

To capture variability, the variance σ^2 is defined as:

$$\sigma^2 = \frac{S_{h,post}^i}{\alpha},$$

where α is an arbitrarily chosen constant, set to 2, as it appears reasonable based on the patterns observed in Figure 3.1.

The rationale for this choice of variance can be explained with examples:

- If an individual has performed an action only once during the observed week, it is likely they performed it around the same amount the week before—perhaps just once, or slightly more or less. Hence, the variance should be small, reflecting this consistency.

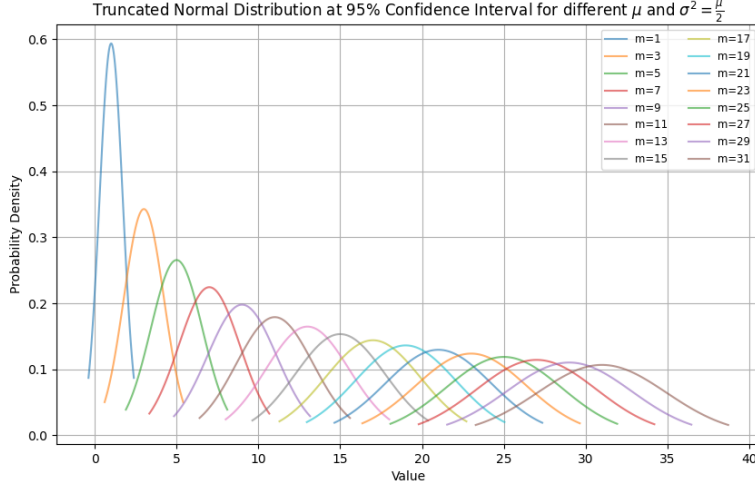


Figure 3.1: Truncated Normal distribution at 95% confidence interval for different values of μ and $\sigma^2 = \frac{\mu}{2}$

- Conversely, if an individual has performed an action 25 times during the observed week, it is reasonable to assume more variability in their behavior for the previous week. For example, the number of times the action could have been performed might range from 20 to 30 times, resulting in a larger variance.

We make a relatively strong assumption that the impact of an action on the two dimensions is the same for all citizens. Yet, it is necessary because we need to determine the transformation matrix W , but we lack information on how each citizen perceives their actions. We only have the list of actions they performed and the changes in their depression and anxiety scores.

We assume that if citizens have similar transformation matrices, we can learn them by performing a linear regression, $(X_{h,\text{post}} - X_{h,\text{pre}}) \cdot W_h = S_{h,\text{post}} - S_{h,\text{pre}}$ for $h \in \{\text{STAI, CES-D}\}$. Using a linear regression is not ideal, as it yields poor results when comparing the observed scores after the observation week with those calculated using the W matrix associated with the actions. Alternative methods, such as rule-based learning or other forms of learning algorithms, might have produced better results. However, this manuscript does not focus on preference elicitation, so we only use linear regression and leave other approaches for future work.

3.2.3 Constraints

The variables represent the number of times an individual performs an action during the week, rather than the duration of the action. We believe that the frequency of performing an action is more relevant for assessing its impact on welfare dimensions.

Additionally, it is easier to model since we need to link the travel required to perform the action with the number of times the action is performed. If we only consider time, the linear programming model might skew towards either performing an action numerous times for very short durations or performing it once for an excessively long duration, both of which do not realistically capture typical human behavior. To address this, we link transportation to the corresponding end actions, ensuring that, for all actions, the citizen must first access them by some means of transport, as follows:

$$\sum_{m=1}^5 t_{m,j} = a_j, \forall j; \quad t_{m,j} \in \mathbb{N}, \forall m, \forall j; \quad a_j \in \mathbb{N}, \forall j$$

However, even when considering the frequency of actions, it is essential to account for the duration of each action, as the time required can vary considerably across different activities. For example, the median time spent on general shopping is 16 minutes, while dining at a restaurant typically takes around 59 minutes. To represent the duration of actions, we use the median time $\tilde{t}(\cdot)$ spent on each activity, denoted as $\tilde{t}(a_j)$ and $\tilde{t}(t_{m,j})$, calculated based on data from all actions performed by all citizens (see Table 3.5). The median was chosen as it provides a robust measure of central tendency, less susceptible to the influence of outliers and skewed data distributions than the average.

To define a realistic upper limit for total leisure time, we consider that $7 \times 24 \times 60$ minutes per week would be unreasonable, as it implies that individuals could be constantly engaged in activities without rest or personal downtime.

Instead, we calculated the total leisure time for each citizen by summing all the minutes spent outside of home and work activities. To set a reasonable upper limit, we used the third quartile of these values, ensuring the model does not unrealistically allocate every minute of the week to active pursuits. As a result, we obtained a limit of 1,482 minutes, or approximately 24.7 hours per week:

$$\sum_{j=1}^9 a_j \tilde{t}(a_j) + \sum_{j=1}^9 \sum_{m=1}^5 t_{m,j} \tilde{t}(t_{m,j}) \leq 1,482$$

In addition, we also need to impose limits on each individual action to ensure they remain reasonable. For example, it is unlikely that a person would visit a restaurant 20 times in a single week. To address this, let $M(a_j)$ (for end actions) and $M(t_m)$ (for transport actions) denote the maximum number of times any action a_j or transport t_m is performed by any individual during the observed week. See Table 3.5 for the maximum of time an end action is performed by all individuals. These constraints can be expressed as:

$$a_j \leq M(a_j) \quad \forall j \in \{1, \dots, 9\}, \quad \sum_{j=1}^9 t_{m,j} \leq M(t_m) \quad \forall m \in \{1, \dots, 5\}.$$

These constraints ensure that the proposed solutions align with typical human behavior patterns, thereby improving the realism and applicability of the model.

Activity	$\tilde{t}(a_j)$	Total observed actions	$M(a_j)$
General Shopping	16	589	13
Specialized Food Shopping	5	430	9
Other Shops and Services	10	651	22
Cultural activities	95	352	10
Visiting People	40	470	15
Medical	22	250	9
Green Spaces	14	201	9
Restaurant	59	196	9
Sports Facilities	75	79	7

Table 3.5: Median of the time an action takes (in minutes) and maximum number a action is performed by an citizen

Next, we need to consider the constraints related to walking distances to various services. The information is available in the questionnaire. Based on the responses to these questions, we establish the constraints on walking distances, as shown in Table 3.6.

We also incorporate constraints derived from subjective evaluations. See Table

Actions	Question	Constraint (If response no)
$t_{1,1}$: Pedestrian to General shopping	Q1.f	$t_{1,1} \leq 0$
$t_{1,2}$: Pedestrian to Specialized food shopping	Q1.e and Q1.g	$t_{1,2} \leq 0$
$t_{1,4}$: Pedestrian to Medical	Q1.a and Q1.c	$t_{1,4} \leq 0$
$t_{1,5}$: Pedestrian to Green space	Q1.i	$t_{1,5} \leq 0$
$t_{1,6}$: Pedestrian to Sports	Q1.j	$t_{1,6} \leq 0$
$t_{1,7}$: Pedestrian to Culture	Q1.k	$t_{1,7} \leq 0$
$t_{1,9}$: Pedestrian to Café, bar	Q1.l	$t_{1,9} \leq 0$

Table 3.6: Constraints on walking distances linked to actions and answer from questions in Appendix B

3.7, where if one of the requirements is not met, the corresponding traveling action is deemed unavailable ($t_j \leq 0$).

In making this assumption, we simplify real life decision by assuming that individuals who find public transport too expensive or unsafe will not use it. In reality, they might still use it sometimes or even regularly, despite feeling unsafe. However, this assumption helps us focus on clear behavior patterns and keeps the model simpler. By doing this, we highlight the main barriers to using public transport, allowing for a more straightforward analysis of accessibility and potential policy interventions.

Finally, we apply additional constraints based on the possibility of visiting relatives. If a citizen indicates that they have no relatives in the neighborhood (answer “no” to questions **Q2.a**, **Q2.c**, and **Q2.e** in Appendix B), the constraint $t_{1,8} \leq 0$ is applied (unable to walk to visit relatives). Furthermore, if they indicate that they have no relatives in the region (answer “no” to all questions **Q2.a**, **Q2.b**, **Q2.c**, **Q2.d**, **Q2.e** and **Q2.f** in Appendix B), then the constraint $a_8 \leq 0$ is enforced (unable to visit relatives).

Mode of Transport	Requirements	Associated questions
t_1 : Pedestrian	• Able to walk one hundred meters	• Q3.i
t_2 : Cycling	• Able to walk one hundred meters • Knows how to cycle • Bike available in the household or valid subscription for Vélib	• Q3.i • Q7.a • Q8 or Q9
t_3 : Private Transport	• Not too expensive • Not scary • Household owns a motorized vehicle • Has a license and can drive	• Q10.a • Q10.e • Q5 • Q4.a
t_4 : Surface Public Transport	• Able to walk one hundred meters • Public transportation allows for safe travel • Using public transportation is not too expensive • There are enough public transport lines	• Q3.i • Q12.b • Q11.b • Q11.c
t_5 : Underground Public Transport	• Able to walk one hundred meters • Public transportation allows for safe travel • Using public transportation is not too expensive • There are enough public transport lines • Climbing several flights of stairs	• Q3.i • Q12.b • Q11.b • Q11.c • Q3.d

Table 3.7: Constraints on different modes of transport using answer from questions in Appendix B.

3.2.4 The Model

Finally, we obtain the following model:

$$\begin{aligned}
& \max \left(\sum_{j=1}^9 a_j w_{j,STAI} + \sum_{j=1}^9 \sum_{m=1}^5 t_{m,j} w_{m,STAI} \right) \\
& \max \left(\sum_{j=1}^9 a_j w_{j,CES-D} + \sum_{j=1}^9 \sum_{m=1}^5 t_{m,j} w_{m,CES-D} \right) \\
& \sum_{j=1}^9 a_j \tilde{t}(a_j) + \sum_{j=1}^9 \sum_{m=1}^5 t_{m,j} \tilde{t}(t_{m,j}) \leq 1,482 \\
& a_j \leq M(a_j) \quad \forall j \in \{1, \dots, 9\} \\
& \sum_{j=1}^9 t_{m,j} \leq M(t_m) \quad \forall m \in \{1, \dots, 5\} \\
& \sum_{m=1}^5 t_{m,j} = a_j \quad \forall j \in \{1, \dots, 9\} \\
& t_{1,j} \leq 0 \quad j \in \{1, 2, 4, 5, 6, 7, 8, 9\}, \text{ if any} \\
& \sum_{j=1}^9 t_{m,j} \leq 0 \quad m \in \{1, \dots, 5\}, \text{ if any} \\
& a_8 \leq 0 \quad \text{possibly.}
\end{aligned}$$

Although all citizens share the same transformation matrix, the model is individualized because certain constraints are linked to personal questionnaire responses. These personal constraints include three categories: eight possible constraints related to walking-accessible activities, five constraints related to different means of transport, and one concerning close relatives. This results in approximately $14^2 = 196$ possible sets of constraints to represent a citizen¹.

The model primarily focuses on transportation actions, as the data were collected to assess the impact of mobility on mental health.

¹This is an approximation, as some combinations of constraints are not possible, such as being unable to walk while being able to take the underground.

3.3 Results and interpretation

Since we needed to simulate the actions performed prior to the observation week, the model results vary. We propose discussing two different simulations. While a Monte Carlo simulation would provide more robust results and is a promising direction for future research, we will settle with these two simulations as a demonstration of the model's usefulness.

The transformation matrices for the two simulations are presented in Figure 3.2. As shown, the effect of performing an action on the two dimensions can vary significantly. For example, in the second simulation, spending time in a green space has no positive impact on the STAI, but a strong effect on the CES-D. Additionally, there is considerable variation across the simulations. For instance, the impact on the STAI in the second simulation is much greater than the effect of green spaces than in the first simulation.

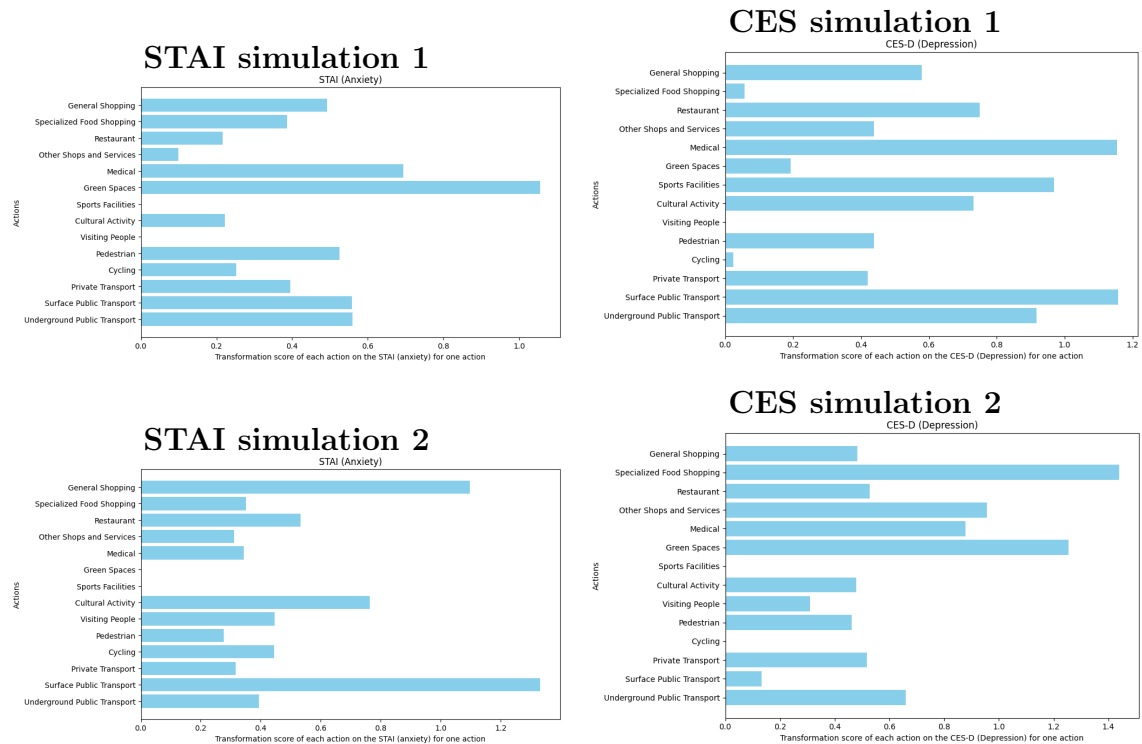


Figure 3.2: Comparison of STAI and CES for two different simulations.

Once the transformation matrix is obtained, we can apply our model to generate the capability sets shown in Figure 3.3. The profiles represent groups of individuals

with the same capability set. The number of profiles varies across simulations, as the number of rational behaviors depends on the variation in the transformation matrix. For example, if five actions are deemed beneficial across both dimensions, while others are not, individuals will likely focus most of their free time on those five actions (if feasible), resulting in less diverse set of actions. In contrast, if all actions are moderately good in both dimensions, or very good in just one, individuals will have a broader range of efficient set of actions.

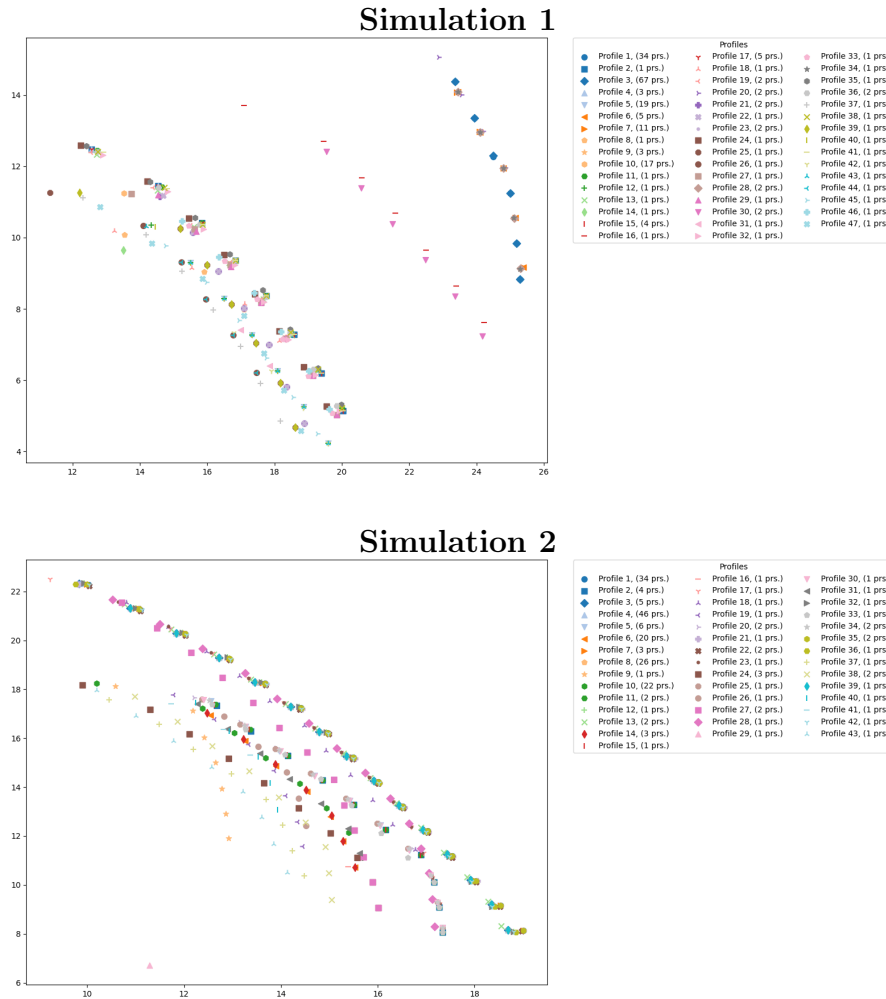


Figure 3.3: Capability sets.

The diversity of the Pareto frontiers varies across the different simulations. Simulation 1 appears to exhibit more diverse profiles, with three distinct clusters emerg-

ing. In contrast, in Simulation 2, the profiles seem to be more similar, with their Pareto frontiers displaying more comparable shapes.

At this stage, the obtained capability sets are not particularly useful, and a more systematic analysis is required. To address this, we propose examining all possible combinations of transportation options given the importance of transportation actions in the modelization. We will analyze their influence by grouping profiles, represented by capital letters, based on their access to different modes of transportation, as outlined in Table 3.8.

Profile	Underground	Surface Transport	Bike	Car
A	✓	✓	✓	✓
B	✓	✓	✓	×
C	✓	✓	×	✓
D	✓	✓	×	×
E	✓	×	✓	✓
F	✓	×	✓	×
G	✓	×	×	✓
H	✓	×	×	×
I	×	✓	✓	✓
J	×	✓	✓	×
K	×	✓	×	✓
L	×	✓	×	×
M	×	×	✓	✓
N	×	×	✓	×
O	×	×	×	✓
P	×	×	×	×

Table 3.8: Combinations of transport modes associated with regrouped profiles.

Figures 3.4 and 3.5 illustrate the capability sets of these regrouped profiles in dark gray, compared to the other capability sets shown in light gray. A key observation is that citizens with access to the underground consistently exhibit better Pareto frontiers than those without access.

Note that profiles *E*, *F*, *G*, and *H* are empty because, in the model, access to the underground requires meeting all the conditions for using surface public transport, in addition to the ability to climb several flights of stairs. Profiles *I* and *L* are also empty, though this is likely due to sample size limitations; with a larger sample,

Simulation 1

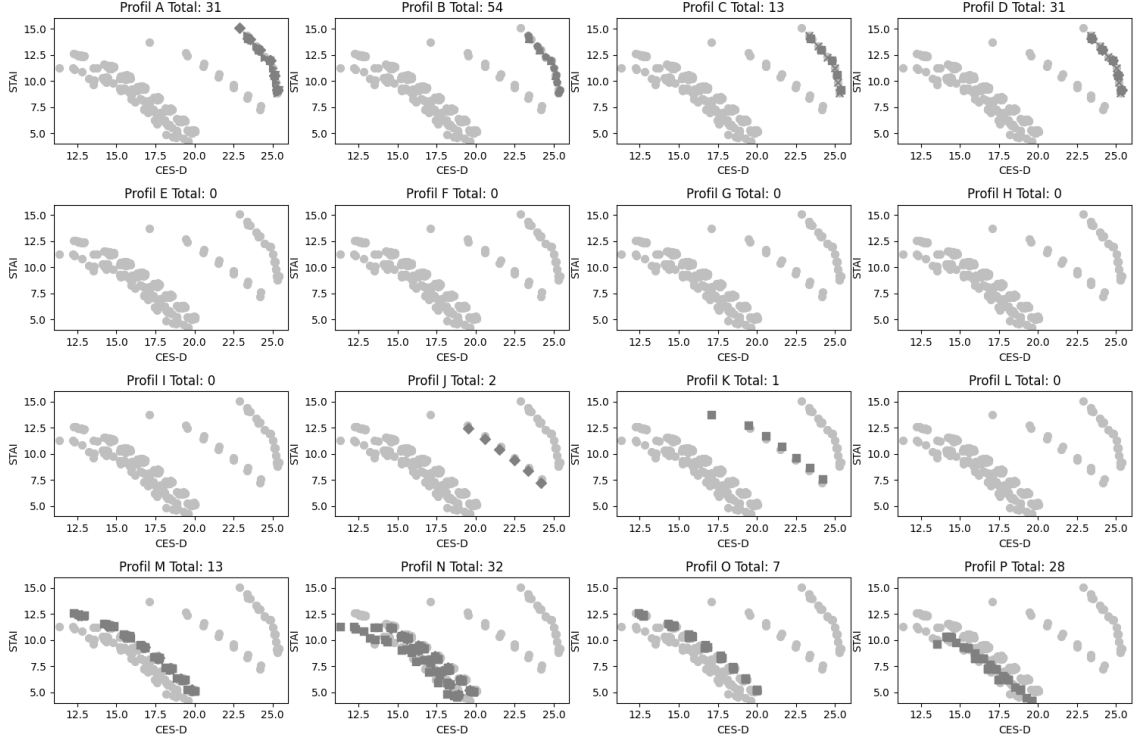


Figure 3.4: Capability sets and mode of transportation in Simulation 1.

these profiles would probably not remain empty.

Based on this analysis, we can further reduce the number of profiles by forming clusters. The idea is to create clusters based on both their constraints (here we focus on transportation-related constraints) and their capability sets.

To achieve this, we can explore clustering using an “explanatory graph,” where the nodes represent clusters and the edges represent both an additional constraint to the cluster (such as the inability to use the underground) and quasi-dominance between the capability sets of the clusters. The transitivity principle applies to both constraints and quasi-dominance.

A possible clustering and explanatory graph is displayed in Figures 3.6 and 3.7 for Simulation 1, and in Figures 3.8 and 3.9 for Simulation 2. We use the term “quasi-dominance” because it does not necessarily imply dominance in the strict sense, as we will define it in Chapter 4. For instance, in the second simulation, some beings in certain profiles in cluster *D* dominate beings in profiles in cluster *A* in the lower right part of the Figure 3.9. However, it should become clear to the reader

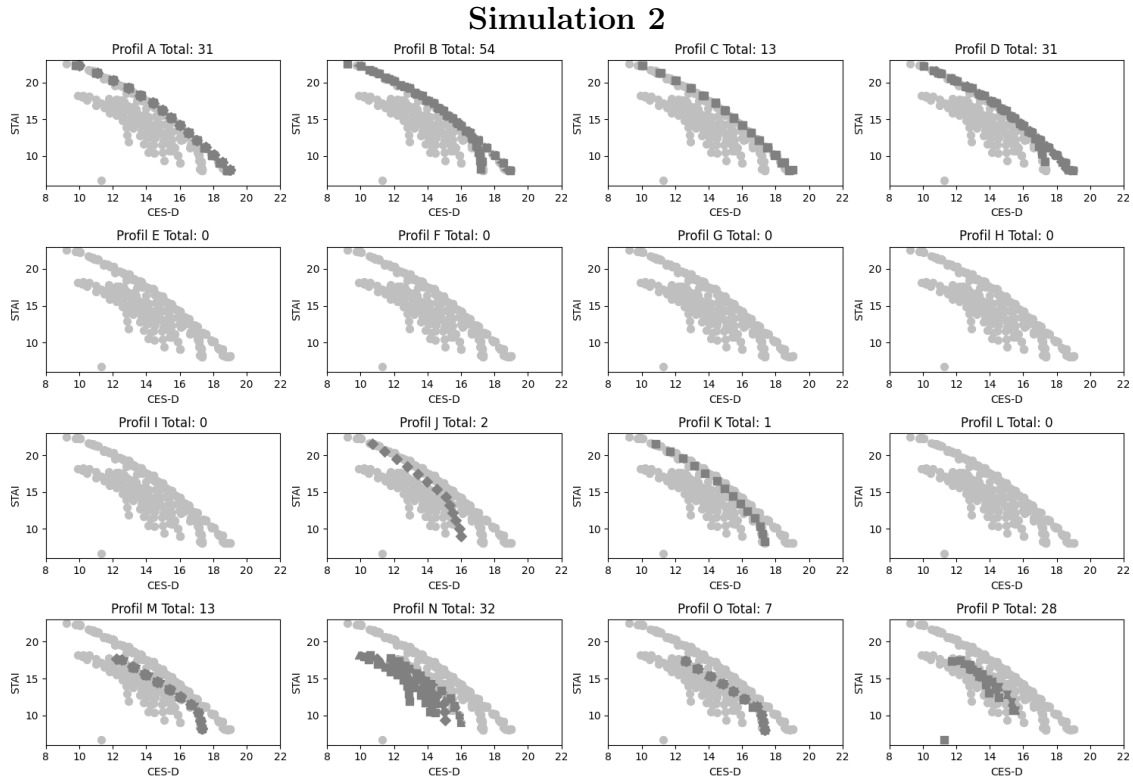


Figure 3.5: Capability sets and modes of transportation in Simulation 2.

that even though the dominance described in Figures 3.6 and 3.8 is presented as quasi-dominance, it reflects the relationships of the clusters shown in Figures 3.7 and 3.9.

Figure 3.6 can be interpreted as follows: individuals in **Cluster A** face no significant mobility constraints, they have the best capability set. Individuals in **Cluster B** are unable to use the underground transport system, resulting in a capability set that is dominated by Cluster A (see Figure 3.7). Individuals in **Cluster C** are further constrained, as they are unable to use both the underground and surface public transport. Those constraints on public transport access lead to the most restricted capability set (domination by both cluster A and B).

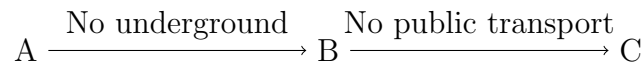


Figure 3.6: Explanatory graph of simulation 1

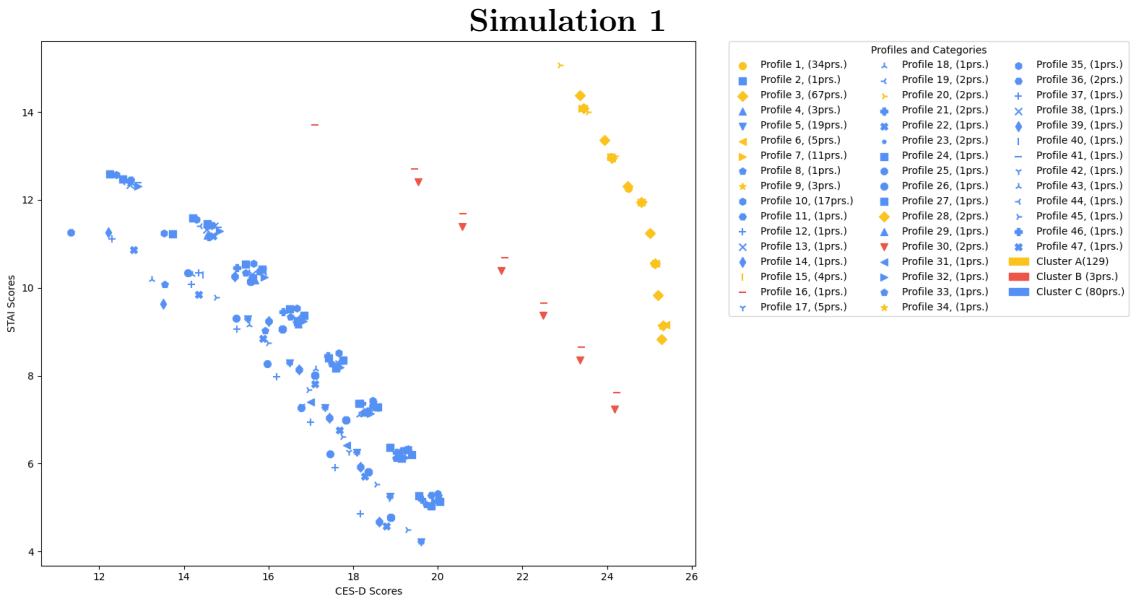


Figure 3.7: Clustered capability sets in Simulation 1.

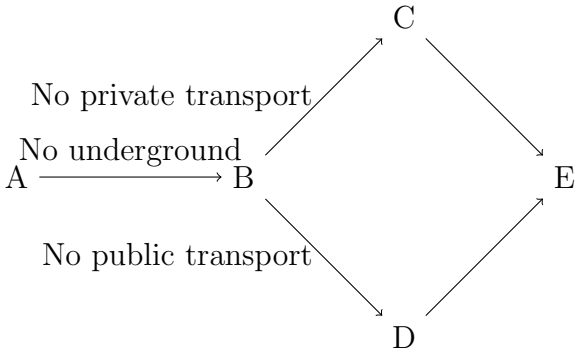


Figure 3.8: Explanatory graph of simulation 2

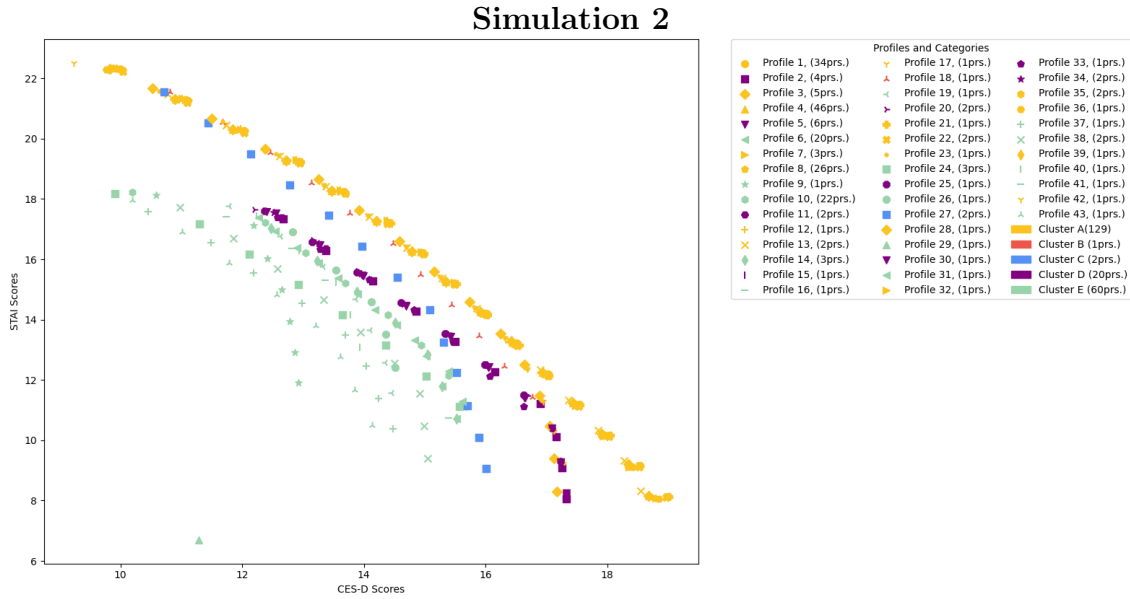


Figure 3.9: Clustered capability sets in Simulation 2.

These figures are particularly useful because they clearly show that certain clusters are better off than others and explain why. For instance, it is evident that access to the underground significantly improves capability sets in both simulations. This kind of analysis highlights the specific constraints that prevent some clusters from having access to better capability sets, making it easier to identify which policy interventions could have the most impact.

By visualizing the relationships between clusters, constraints, and quasi-dominance, policymakers can better understand how removing certain constraints (such as improving access to transportation) would enable entire clusters of individuals to achieve improved capability sets.

Moreover, these figures help policymakers prioritize interventions. Clusters with the worst capabilities or those with the greatest potential for improvement can be targeted first, making policy implementation more efficient. By identifying specific constraints within each cluster, policies can be tailored to address the unique needs of different groups, ensuring that interventions are both effective and fair.

This approach also allows for the identification of policies that could benefit multiple clusters simultaneously. For example, in Simulation 2, improving access to private transport would benefit individuals in cluster *E*, but enhancing access to the underground would benefit a larger group, including clusters *B*, *C*, *D*, and *E*.

This analysis suggests several potential public policy interventions, as illustrated in Figures 3.10 and 3.11. One possible policy could be offering free underground transport for elderly individuals. In Simulation 1, this policy would enable 18 individuals to transition from cluster *C* to cluster *A*, significantly improving their mobility capabilities. Similarly, in Simulation 2, 16 individuals would shift from cluster *E* to cluster *A*, and 2 individuals from cluster *D* to cluster *A*, illustrating the broad impact of improved transportation access on enhancing the capabilities and well-being of the elderly population.

Additionally, measures aimed at enhancing the sense of security for elderly individuals could also improve capability sets. In simulation 1, this could shift 36 individuals from cluster *C* to cluster *A* and 1 person from cluster *C* to *B*. In simulation 2, 26 individuals could move from cluster *E* to cluster *A*, 1 person from cluster *E* to *B*, and 10 individuals from cluster *D* to *A*.

If, for any reason, the public policies mentioned above are not feasible, simulation 1 suggests that implementing policies to improve access to surface public transport would be the next best option. In simulation 2, it is evident that any policy aimed at increasing mobility for those currently without any modes of transport would significantly enhance the capabilities of the most disadvantaged group (profile E).

3.4 Conclusion and future work

Due to the assumptions made in constructing the framework, particularly the generation of random actions to simulate pre-observation data, the results and discussions in the previous section fail to accurately describe real-life scenarios. This limitation highlights the need for more precise and realistic input data.

The ideal solution would be to have access to real-life data from the week preceding the observation period. However, due to practical constraints, this is not feasible. Nevertheless, for future studies, efforts should be made to collect more extensive pre-observation data when possible.

Another potential solution could be to employ Monte Carlo simulations to model a broader range of possible scenarios and outcomes. By running repeated random sampling processes, Monte Carlo methods could provide a more robust framework for policy recommendations and mitigate some of the limitations caused by data

assumptions.

Additionally, we can have access to mental assessments throughout the day in the recieved data, they are based on a random subset of questions from the STAI and CES-D questionnaires. If we could make reliable assumptions about how these subsets relate to the overall scores, we could improve our model by working with more comprehensive mental health data. This would force us to assess mental well-being on a daily scale rather than on a weekly scale, providing a more accurate depiction of participants' capabilities.

The development of such a model would greatly benefit from interdisciplinary collaboration. Input from sociologists, geographers, urban planners, economists, and policymakers would provide valuable insights. Certain behavioral constraints are difficult to model, and consultation with experts from other fields would ensure that the model reflects real-world conditions more accurately and coherently.

Finally, the creation of explanation graphs and the clustering process was done manually. This approach is both time-consuming and potentially suboptimal for constructing clusters and graphs. An interesting area for future research would be the development of algorithms to better identify and visualize the constraints that impact individuals' capabilities.

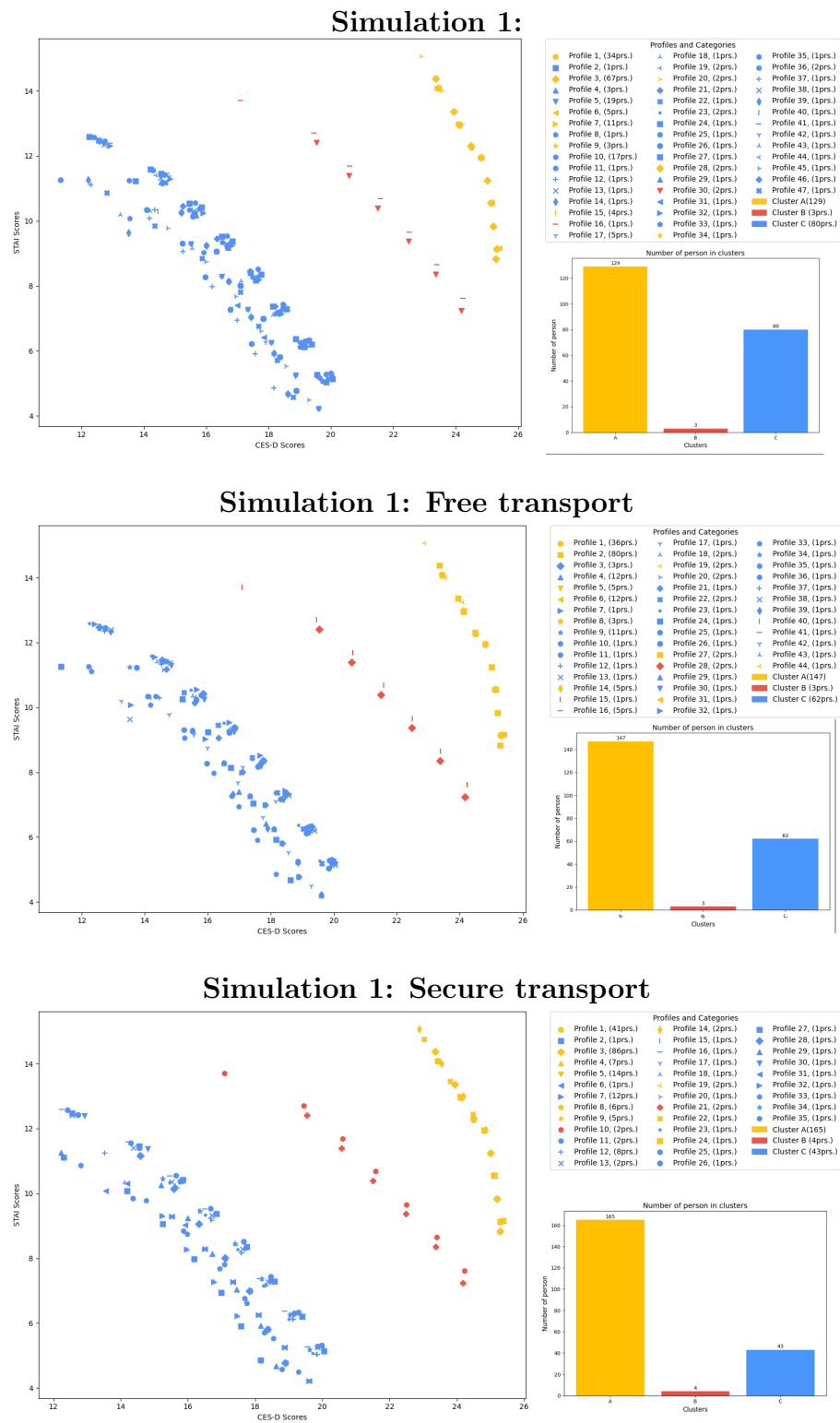
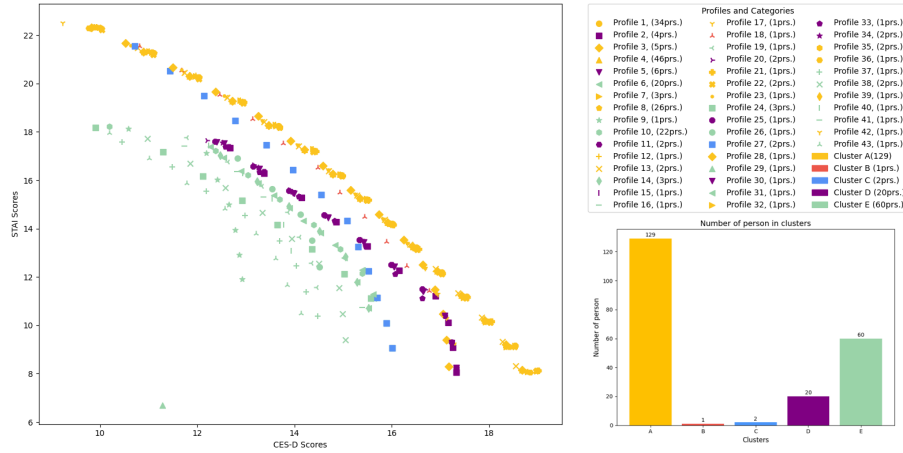
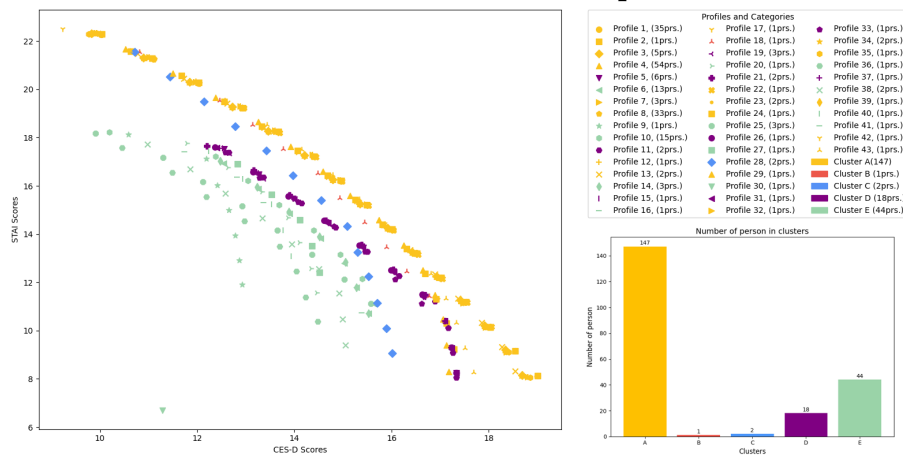


Figure 3.10: Simulation 1: Initial capability sets and public policy implied capability sets

Simulation 2:



Simulation 2: Free transport



Simulation 2: Secure transport

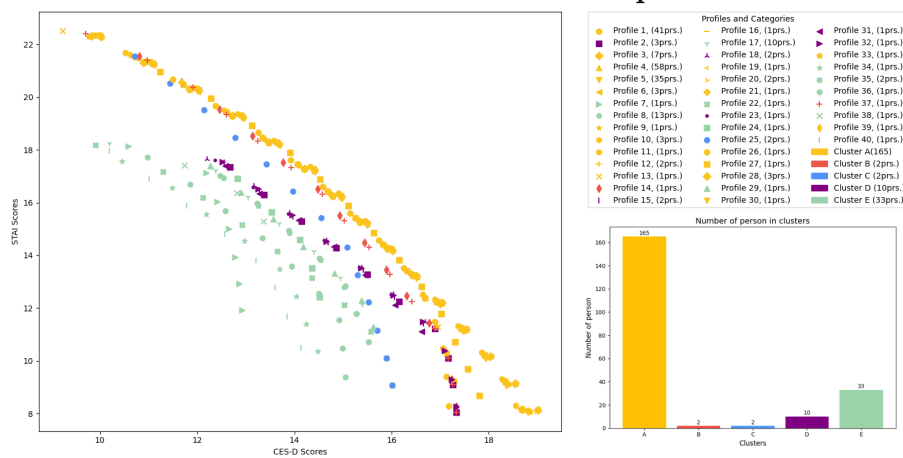


Figure 3.11: Simulation 2: Initial capability sets and public policy implied capability sets

Part II

Aggregation and Capability sets

Introduction

In the first part of the manuscript, we proposed a model to assist decision-makers in designing public policies. We demonstrated the usefulness of this model through a case study. However, the case study also revealed that without additional tools, its utility would be limited. What if two capability sets do not Pareto-dominate each other, as it is the case between Clusters C and D in the second scenario? What if, as in the case study, we have multiple scenarios or simulations? How do we make the trade-off about which cluster to help, the largest (in terms of people), the most deprived, or the one we can improve the most? As you can see, these questions are the same as those found in classical decision-aiding. The main difference from traditional approaches is that we model current situations and the consequences of public policies using Pareto-efficient frontiers, rather than simple (possibly multi-objective) solutions. However, the same challenges arise, and some of them are linked to the question of aggregation. How do we aggregate multi-objective solutions? How do we aggregate different outcomes across multiple scenarios? And how do we aggregate outcomes for different individuals?

As illustrated in Figure 3.12, a public policy problem using our welfare representation involves several attributes [colorni2024decision].

A typical public policy problem would encompass capability sets (sets of multidimensional solutions), multiple citizens, and various scenarios. To simplify the problem, we need to navigate along the edges of the “cube” to reach H , where the problem becomes more manageable. While several well-known algorithms and methods exist to navigate between B, C, D and H , the introduction of capability sets requires us to create methods to navigate from the upper corners of the cube (A, E, F, G) .

The order of moving from any given point to H is crucial. In this part, we will

present methods to navigate the edges of the cube. We will begin in Chapter 4 with a method to aggregate on the welfare representation, typically for moving from G to H . Next, we will address aggregation on scenarios, typically for moving from F to G . We propose different approaches based on the information available to policymakers: in cases of uncertainty (no information on the likelihood of scenarios), discussed in Chapter 5, and in cases of risk (subjective probabilities are known), covered in Chapter 6.

Our research addresses two critical challenges that arise in policy processes, namely comparing the outcomes of different policies and managing uncertainty in policy outcomes.

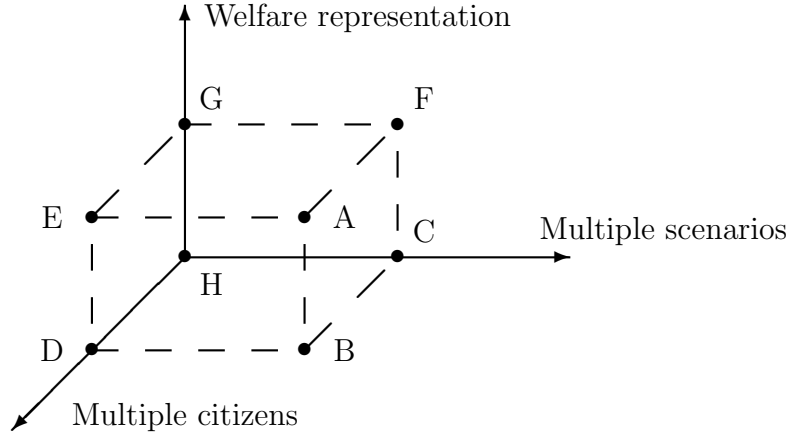


Figure 3.12: Multiple attributes in a public policy design

It is noteworthy that capability sets are essentially sets of Pareto-efficient solutions in a multidimensional space. This concept bears similarities to other notions such as object sets, consequence sets, solution sets, choice sets, or capability sets (as elaborated in [barbera2004ranking]). Multiple formulations can be adapted to our setup, allowing for the flexible application of our methodology to a wide range of problem domains. Firstly, the problem formulated in Chapter 2. The capability set represent a set of beings of citizens (i.e., the set of achievable outcomes for each citizens) resulting from an act (i.e., a public policy) and a scenario (e.g., following

damage to a Common, as discussed in Section 2.3). The objective is to determine the best public policy for an individual (or a ranking).

In an uncertain environment, capability sets could refer to the potential results associated with risk treatments before we are able to undertake a full probabilistic assessment [**bossert2000choice**, **cohen1980rational**].

In the domain of research and development (R&D) management, capability sets can refer to the potential outcomes that can be achieved through projects. R&D decisions are often made over multiple periods, where choices made today impact the options available in the future. The question then arises as to which development option should be chosen today, knowing that it will influence future development choices.

Furthermore, in many cases, the decision-maker may not have precise knowledge of their future preferences due to uncertainties related to factors such as the economy, societal values, environmental conditions, future laws, etc. The sets of possible future choices would also be uncertain due to the inherent uncertainties associated with R&D projects, such as performance, development time, and costs. Representing these choices as capability sets would enable flexibility and resilience [**evans1991strategic**, **koopmans1962flexibility**, **kreps1979representation**], allowing for better adaptation to unforeseen events.

Lastly, in a multi-objective setting [**ehrgott2005multicriteria**], capability sets would refer to the multiple objective levels achievable under various scenarios, when we have not yet undertaken a process to aggregate results.

Chapter 4

Assessing capability sets

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4.1 Introduction

This chapter suggests evaluating a capability set using individuals' preferences over a variety of accessible beings and their preference over their diversity. We propose three ways of modeling this diversity according to different types of preferences. Once again, one way to represent the concept of Capability set is by considering *capability sets* as collections of *beings* within a multidimensional space, and, if all significant welfare dimensions are taken into account, the capability set can be perceived as a Pareto frontier, where, with no loss of generality, we may assume that all such dimensions are to be maximized.

In our discussion, it is crucial to emphasize how the notion of capabilities pertains to freedom. As an example from [sen1985commodities], if a citizen has access to a capability set containing multiple solutions, among which being b is considered to be the best, according to her preferences, a second capability set comprising only option b would be deemed less desirable than the first one, even if the citizen would end up choosing option b in both cases. This is because the second set provides less freedom, in the sense of providing fewer choices.

Assuming the existence of a value function associating a number to beings, the question addressed by this chapter is how to assess different capability sets, taking explicitly into account both the diversity of their options and their values. Previous work in this area includes studies by [foster2011freedom, barbera2004ranking, gaertner2012evaluating, gaertner2006capability, gaertner2008new].

Section 4.2 establishes the foundational notation. Section 4.3 delves into the necessity of developing a function to assess capabilities, focusing on the role of the freedom of choice. In Section 4.4, we explore desired properties when assessing capability sets. Section 4.5 delineates three distinct methodologies for quantifying freedom of choice functions: the classical *max* approach, which interprets freedom as an indirect utility or instrumental value; the *min* approach, considering freedom as having intrinsic value; and an *integral* approach, which amalgamates the full diversity of a capability set. We also provide some computational insight, and show how diverse methods align with different pre-established properties, shedding light on their respective strengths and limitations. Section 4.6 is dedicated to the proofs.

4.2 Notation

Let us begin by presenting a formal definition of our problem. We assume that all possible beings $\vec{b}$ belong to a set $\mathbf{Q} \subset \mathbb{R}_{+*}^{h^*}$. The set $\mathbf{Q}$ may contain a finite or infinite number of beings or vectors $\vec{b}$, and we suppose that all beings $\vec{b} = [b_1, \dots, b_{h^*}]$ are h^* -dimensional vectors in $\mathbb{R}_{+,*}^{h^*}$ describing the performance of $\vec{b}$ on h^* welfare dimensions such as life expectancy, level of nourishment, mobility, etc. With no loss of generality, we assume that all dimensions are to be maximized.

In our framework, the capability sets, denoted as $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and so forth, comprise a collection of beings $\vec{b}$, are subsets of a more extensive set $\mathbf{Q}$, and are compact (closed and bounded) sets. For any given set $\mathbf{A} \subseteq \mathbf{Q}$, we define its closure, denoted $\mathbf{A}^c$, through the expression $\mathbf{A}^c = (\mathbf{A} - \mathbb{R}_+^{h^*}) \cap \mathbb{R}_+^{h^*}$. This closure represents the positive space, that is effectively dominated by $\mathbf{A}$.

For any pair of beings $(\vec{b}, \vec{b}') \in \mathbf{Q}^2$, we employ the notation $\vec{b} \succeq \vec{b}'$ (respectively, $\vec{b} \succ \vec{b}'$) to mean that $b_h \geq b'_h$ for all $h \in \{1, \dots, h^*\}$ (respectively, and for at least one h , $b_h > b'_h$). This notation captures the concept of weak and strict dominance, respectively, in the multidimensional space.

Similarly, for any pair of sets $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, the notation $\mathbf{A} \succeq \mathbf{B}$ (respectively, $\mathbf{A} \succ \mathbf{B}$) is used to denote that for every being $\vec{b} \in \mathbf{B}$, there exists a being $\vec{a} \in \mathbf{A}$ such that $\vec{a} \succeq \vec{b}$ (respectively, and additionally, there exists at least one $\vec{a} \in \mathbf{A}$ for which no $\vec{b} \in \mathbf{B}$ exists such that $\vec{b} \succeq \vec{a}$). The relation $\mathbf{A} \succeq \mathbf{B}$ indicates that $\mathbf{A}$ weakly dominates $\mathbf{B}$; in other words, $\mathbf{A}$ is “above” $\mathbf{B}$, or equivalently, $\mathbf{B}$ is a subset of the set of all beings dominated by $\mathbf{A}$, formally represented as $\mathbf{B} \subseteq \mathbf{A}^c$. The notation $\mathbf{A} \succ \mathbf{B}$ implies that $\mathbf{B}$ is strictly dominated by $\mathbf{A}$, that is $\mathbf{B}^c \subset \mathbf{A}^c$.

Example 4.1 *Figure 4.1 depicts three capability sets in the case $h^* = 2$. Two of the capability sets, denoted $\mathbf{A}$ and $\mathbf{B}$, are finite, while the third one, denoted $\mathbf{C}$, contains an infinite number of solutions. Areas in the plot represent the spaces dominated by $\mathbf{A}$ and $\mathbf{B}$, specifically $\mathbf{A}^c$ and $\mathbf{B}^c$.*

In this example, $\mathbf{A}$ should be preferred over $\mathbf{B}$ because $\mathbf{B}^c$ is entirely contained within $\mathbf{A}^c$. On the other hand, when comparing $\mathbf{C}$ to $\mathbf{A}$ (and $\mathbf{B}$) using the above method, no comparison can be made in principle. This is because neither $\mathbf{C}$ is a subset of $\mathbf{A}^c$, nor is $\mathbf{A}$ a subset of $\mathbf{C}^c$. $\triangle$

Let $\vec{b}^*$ represent the ideal point of $\mathbf{Q}$. This point is characterized by the property

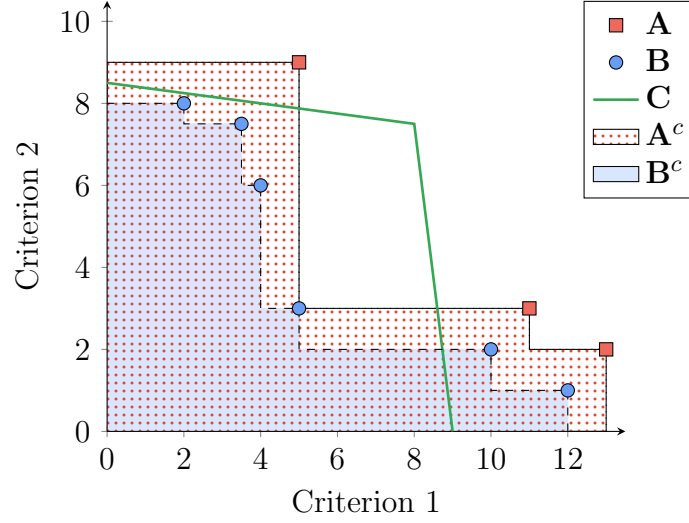


Figure 4.1: Three Capability sets

that for every $\vec{b} \in \mathbf{Q}$ and for each dimension $h \in \{1, \dots, h^*\}$, the condition $b_h \leq b_h^*$ holds. Additionally, for each dimension h , there exists at least one $\vec{b} \in \mathbf{Q}$ such that $b_h = b_h^*$.

The capability space $\mathcal{C} \subseteq \mathbb{R}_{+*}^{h^*}$ is defined as the strictly positive space dominated by $\vec{b}^*$, or $\mathcal{C} = \vec{b}^{*c}$. Essentially, $\mathcal{C}$ is a continuous space encompassing all beings within $\mathbf{Q}$.

We define a set of valuation functions V such that each $v : \mathcal{C} \rightarrow \mathbb{R}^+$ maps the capability space $\mathcal{C}$ to non-negative real numbers. These functions $v(\cdot)$ are increasing such that:

$$\forall v \in V, \forall (\vec{c}, \vec{c}') \in \mathcal{C}^2, \text{ if } \vec{c} \succ \vec{c}', \text{ then } v(\vec{c}) > v(\vec{c}').$$

The requirement for v being defined on $\mathcal{C}$ is strong because it requires the ability to compare all the options in $\mathcal{C}$, which may not always be possible (i.e., $\mathbf{Q} \subseteq \mathcal{C}$ but not necessarily $\mathcal{C} \subseteq \mathbf{Q}$). Nonetheless, the monotonicity of v is a reasonable demand, given the objective of maximizing welfare across all dimensions.

4.3 Freedom of choice

There are two crucial aspects to consider when comparing capability sets: their overall level of welfare and their distribution across different dimensions.

One intuitive approach could be to utilize stochastic dominance and Lorenz dom-

inance concepts. First Degree Stochastic Dominance (FSD) [levy2016stochastic] would correspond to the simple rule where a set $\mathbf{A}$ dominates $\mathbf{B}$ if it is “above” it ($\mathbf{A} \succ \mathbf{B}$). The Second Degree Stochastic Dominance (SSD) would deal both with the mean and “dispersion” of the beings on the capability space. Similarly, the general Lorenz curve dominance [shorrocks1983ranking] would capture both solutions’ mean and distribution of solutions to order them. While these concepts cannot be directly applied to our problem as they deal with probabilities or cumulative distribution functions, their similarities are worth noting.

One important aspect that is often overlooked in the measurement of capability sets *is the value assigned to the freedom of choosing beings valued at a certain level of well-being*. In the example given by [sen1985commodities], it may be reasonable to argue that a person who assigns no value to choice would be indifferent between a capability set where only $\vec{b}$ is achievable and a capability set where $\vec{b}$ is achievable along with other option $\vec{b}'$ such that $v(\vec{b}') < v(\vec{b})$. However, it can also be argued that a person who values choice would prefer the second capability set over the first. This consideration highlights the importance of incorporating individual preferences over freedom and values over beings within the measurement framework.

The assessment of capability sets is a complex task that requires careful consideration. One way to do so is to count the number of options in the capability set, as proposed by [pattanaik1990ranking] in an axiomatic characterization. However, it should be noted that the value of choice cannot be reduced to the cardinality of the capability set due to several important reasons:

- First, the existence of capability sets with an *infinite number of beings*. We need to be able to discern between different infinite sets in order to effectively evaluate their relative goodness. For instance, in Figure 4.2, capability sets $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ are both infinite, yet it is evident that $\mathbf{A}$ dominates $\mathbf{B}$ in terms of the ordering of capabilities. See [pattanaik2000ranking] for a quantity-based axiomatic work on infinite capability sets.
- The second reason is linked to the notion of *diversity* [pattanaik2000diversity] of options (or their range, [klemisch1993freedom]). Let $\mathbf{B} \subseteq \mathbf{Q}$ represents ten possible ways of walking to work, while $\mathbf{A} \subseteq \mathbf{Q}$ represents one way of walking and one way of cycling to work. In this case, although $\mathbf{B}$ offers a higher number of possible beings, it can be argued that the diversity of beings

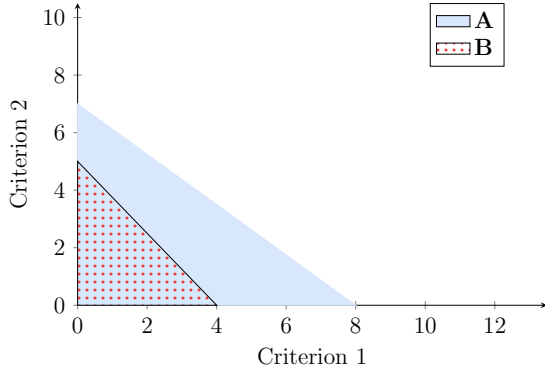


Figure 4.2: Comparison of Capability Sets **A** and **B** with infinite beings (**A** is superior to **B**).

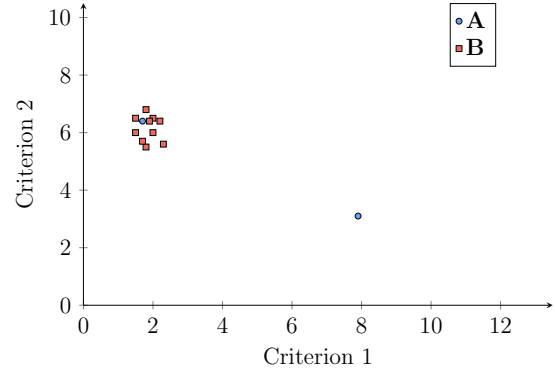


Figure 4.3: Comparison of Capability Sets **A** and **B** (**B** has more potential beings, but **A** is more diverse).

in **A** is greater in terms of freedom of choice, as Figure 4.3 illustrates.

- Thirdly, we need to take into account the *value* of the offered choices according to individual preferences [sen1990welfare, sen1991welfare]. An individual may possess a capability set **B** that contains numerous options, including diverse but low-valued ones, and may perceive as having fewer capabilities compared to a capability set **A** that offers fewer options but of higher value. For instance in Figure 4.4, **B** has more options and is “more diverse” than **A**, but every being of **B** is dominated by a being in **A**.

Following these three remarks, we assume that inclusion of dominated beings in the capability space does not enhance individual freedom. For instance, adding the option to “go to work using only one leg” when “going to work on foot” is already available does not improve an individual’s freedom. As depicted in Figure 4.4, the addition of set **B** to set **A** does not improve individual freedom, as set **B** is dominated by set **A**. We can say that **B** is a set of insignificant beings with respect to **A** [JONES198247]. The underlying idea is that if we consider all relevant welfare dimensions, a rational individual would have no interest in choosing an option from set **B** regardless of their preferences.

Another example of an option that is irrelevant with respect to **A** is adding an option being already in set **A**. For instance, if $\mathbf{A} = \{\vec{a}\}$ represents buying a bike from a store, and $\mathbf{B} = \{\vec{b}\}$ represents buying the same model of bike, we have $\mathbf{A} \cup \mathbf{B} = \{\vec{a}, \vec{b}\}$ with $\vec{a} = \vec{b}$. It would be reasonable to assert that the individual is

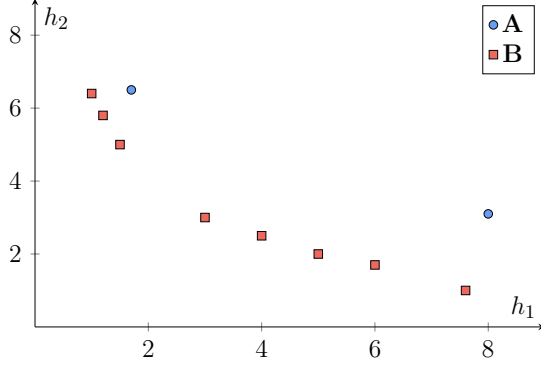


Figure 4.4: Comparison of Capability Sets **A** and **B** (**B** offers more options and is more diverse compared to **A**, but every being of **B** is dominated by a being in **A**).

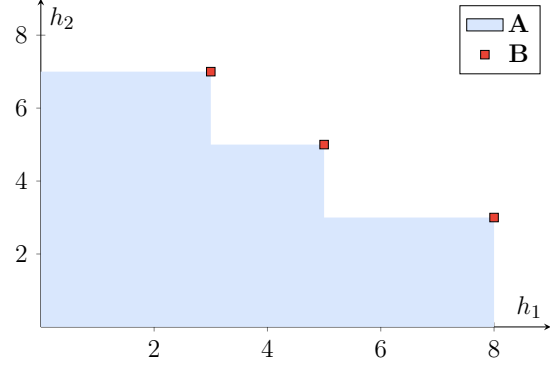


Figure 4.5: Two equivalent sets **A**, **B** according to Axiom 4.1

indifferent between sets **A** and $\mathbf{A} \cup \mathbf{B}$. This axiom leads to the implication that a capability set can be reduced to its Pareto frontier denoted $PF(\mathbf{A})$, such that

$$PF(\mathbf{A}) = \{\vec{b} \in \mathbf{A} \mid \nexists \vec{b}' \in \mathbf{A} \text{ such that } \vec{b}' \succ \vec{b}\}$$

as suggested in Chapter 2.2. Consequently, we consider sets **A** and **B** to be equivalent if $PF(\mathbf{A}) = PF(\mathbf{B})$, or $\mathbf{A}^c = \mathbf{B}^c$, as illustrated in Figure 4.5.

Subsequently, we introduce the function Φ , which evaluates capability sets within the framework of $(\mathcal{P}(\mathbf{Q}) \setminus \emptyset, V)$. Here, $\mathcal{P}(\mathbf{Q}) \setminus \emptyset$ represents all non-empty subsets derived from the power set of **Q**, and V denotes the set of all possible value functions. For every subset **A** of **Q** and for any $v \in V$, the function Φ applied to **A** with respect to v , denoted as $\Phi_v(\mathbf{A})$, yields a value in $\mathbb{R}_+$. Based on this, we propose the following:

Axiom 4.1 (Indifference of irrelevant alternatives) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, such that $\mathbf{A} \succeq \mathbf{B}$ then $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B})$.*

[foster2011freedom] argues that a measure of freedom depends on the quantity and quality of available alternatives. However, we contend that the notion of diversity, rather than quantity, is a more appropriate indicator of freedom. Specifically, we propose that freedom would emerge from a *conjoint measure of diversity*

and valuation of alternatives. Hence, the concept of a Pareto dominant set, which focuses on the set of solutions that are not dominated by any other solution in the space, is perceived more suitable in this context.

For instance, consider options from sets **A** and **B**, where **A** may have an infinite number of solutions while **B** has only three, as depicted in Figure 4.5. Despite this difference, these solutions can be considered equivalent under certain circumstances (i.e., Axiom 4.1). This is because, if all relevant dimensions are accounted, a rational individual endowed with the capability set **A** would invariably opt for a solution from **B**. Therefore, a measure of freedom based on diversity, rather than quantity or quality alone, is more appropriate for capturing the essence of freedom in the capability approach.

4.3.1 Gaertner and Xu method

To the authors' knowledge, the only works that have tried to measure freedom as jointly the diversity and the valuation of alternatives are [gaertner2012evaluating, gaertner2006capability, gaertner2008new, gaertner2011reference]. It is worth noting that these methods have been primarily proposed for assessing human development, whereas the current study addresses capabilities of all types. The fundamental principles of their model are the following (with adaptations to facilitate the comprehension in the context of our problem):

First, a reference beings vector $\vec{b}^0$ in the capability space $\mathcal{C}$ is defined, which represents the direction of societal development. Typically, this vector indicates a boundary of deprivation below which an individual is considered poor, or it can be the existing *status quo*.

Next, a distance function $d(x, y)$ between two points x and y is defined.¹ For instance, the Euclidean distance can be used, given by

$$d(x, y) = \sqrt{\sum_{h=1}^{h^*} (x_h - y_h)^2},$$

where h^* denotes the number of dimensions in the capability space.

Then, we adapted the Gaertner-Xu method to assess **A** using;

¹The distance function need to respect some properties (see [gaertner2006capability])

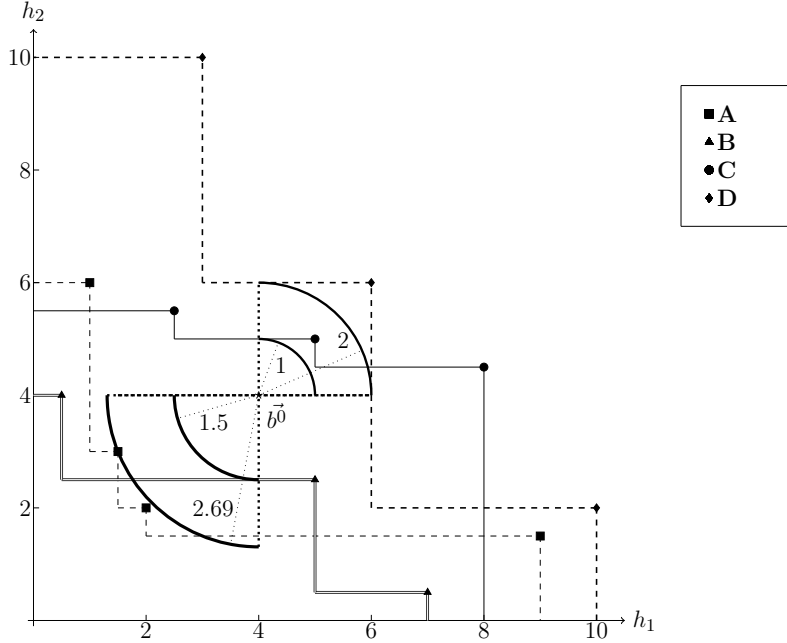


Figure 4.6: An example of the determination of scores for some capability sets using Gaertner-Xu like method.

$$r(\mathbf{A}, d) = \begin{cases} -\min_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : d(\vec{b}, \vec{b}^0) \leq t\} \cap \mathbf{A}^c \neq \emptyset\} & \text{if } \vec{b}^0 \notin \mathbf{A}^c, \\ \max_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : \vec{b} \geq \vec{b}^0, d(\vec{b}, \vec{b}^0) \leq t\} \subseteq \mathbf{A}^c\} & \text{if } \vec{b}^0 \in \mathbf{A}^c. \end{cases}$$

Example 4.2 Figure 4.6 displays an example of the determination of the score of four capability sets using the Gaertner-Xu like method with $\vec{b}^0 = (4, 4)$. We would obtain the order $r(\mathbf{A}, d) < r(\mathbf{B}, d) < r(\mathbf{C}, d) < r(\mathbf{D}, d)$ with $r(\mathbf{A}, d) = -2.69$, $r(\mathbf{B}, d) = -1.5$, $r(\mathbf{C}, d) = 1$, $r(\mathbf{D}, d) = 2$. $\triangle$

Several generalizations of the method have been proposed in [gaertner2008new, gaertner2011reference, gaertner2012evaluating]. However, for the purpose of understanding the main idea, the simple unique point ($\vec{b}^0$) formulation suffices. It is important to highlight that the Gaertner-Xu method can be perceived as a pessimistic approach when $\vec{b}^0$ is in the capability set, and an optimistic approach when it is located “above” it. Specifically, if $\vec{b}^0$ is “above” the capability set, its score is determined by the closest point weakly dominated by $\mathbf{A}^c$ from $\vec{b}^0$, while other points weakly dominated by both $\mathbf{A}^c$ and $\vec{b}^0$ would have a lower score. Conversely, if $\vec{b}^0$ is contained within $\mathbf{A}^c$, the score is equal to the closest point on the frontier of weakly

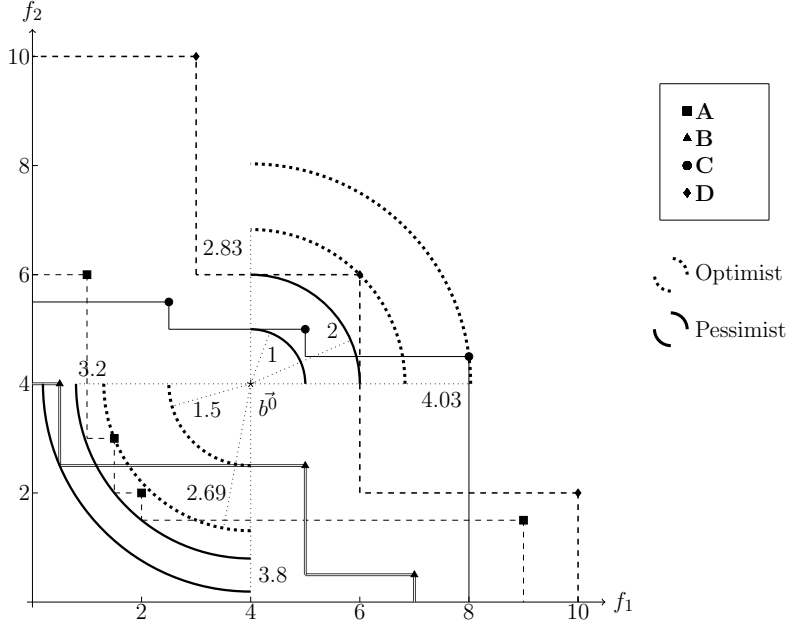


Figure 4.7: An example of the determination of scores for some capability sets using an optimistic and pessimistic Gaertner-Xu like method.

dominated solutions by $\mathbf{A}^c$, but it is possible for solutions that dominate $\vec{b}^0$ to exist at a greater distance.

It is worth mentioning that other rules can be formulated, although they may not necessarily adhere to all the axioms proposed by Gaertner and Xu. For instance, an optimistic version of the procedure would be:

$$r(\mathbf{A}, d) = \begin{cases} -\min_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : d(\vec{b}, \vec{b}^0) \leq t\} \cap \mathbf{A}^c \neq \emptyset\} & \text{if } \vec{b}^0 \notin \mathbf{A}^c, \\ \max_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : \vec{b} \geq \vec{b}^0, d(\vec{b}, \vec{b}^0) \leq t\} \cap \mathbf{A}^c \neq \emptyset\} & \text{if } \vec{b}^0 \in \mathbf{A}^c. \end{cases}$$

whereas a pessimistic approach would be:

$$r(\mathbf{A}, d) = \begin{cases} -\min_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : \vec{b} \leq \vec{b}^0, d(\vec{b}, \vec{b}^0) = t\} \subseteq \mathbf{A}^c\} & \text{if } \vec{b}^0 \notin \mathbf{A}^c, \\ \max_t \{t \in \mathbb{R}_+ : \{\vec{b} \in \mathbb{R}_+^{h^*} : \vec{b} \geq \vec{b}^0, d(\vec{b}, \vec{b}^0) \leq t\} \subseteq \mathbf{A}^c\} & \text{if } \vec{b}^0 \in \mathbf{A}^c. \end{cases}$$

Example 4.3 Using the same capability sets and $\vec{b}^0$ as example 4.2, we obtain different orders and scores, as in Figure 4.7 illustrates, the order in the case of the optimistic procedure is : $r(\mathbf{C}, d) \succ r(\mathbf{D}, d) \succ r(\mathbf{B}, d) \succ r(\mathbf{A}, d)$ (with $r(\mathbf{A}, d) = -2.69$, $r(\mathbf{B}, d) = -1.5$, $r(\mathbf{C}, d) = 4.03$, $r(\mathbf{D}, d) = 2.83$) and in the pessimistic procedure is : $r(\mathbf{D}, d) \succ r(\mathbf{C}, d) \succ r(\mathbf{A}, d) \succ r(\mathbf{B}, d)$ (with $r(\mathbf{A}, d) = -3.2$, $r(\mathbf{B}, d) = -3.8$, $r(\mathbf{C}, d) = 1$, $r(\mathbf{D}, d) = 2$).

Importantly, note that in all cases, the entire distribution of the capabilities is not taken into account. Indeed, we only look at the extreme solutions; those closest or furthest from $\vec{b}^0$ and, moreover, we do not conjointly measure the diversity and the valuation of alternatives.

4.4 Desirable properties

To address the limitations described in the previous section, we suggest the function $\Phi_v(\mathbf{A})$ to be an individual subjective function representing the freedom of choice in $\mathbf{A} \subseteq \mathbf{Q}$ according to the individual's preferences $v \in V$ over beings, and their preference over diversity.

We will provide details on how to define $\Phi_v(\mathbf{A})$ in Section 4.5, one consistent with the optimistic Gaertner-Xu approach, another one with the pessimistic Gaertner-Xu's method, and a final one that takes into account the full diversity of capability sets. This last measure of freedom takes into account the individual's preferences over diversity and valuation of alternatives, allowing for a more nuanced and comprehensive evaluation of capability sets.

In this section, we shall discuss several interesting properties for an assessment procedure of capability set to fulfill, beside Axiom 4.1.

A first interesting property would be its consistency with the classical utility approach in one-dimensional spaces. In this case, it is always possible to identify a best option $b' \in \mathbf{A}$ such that for all $b \in \mathbf{A}$ it stands that $b \leq b'$. This means that if only one dimension is considered important for the individual, the set $\mathbf{A} \setminus b'$ is irrelevant with respect to b' and all individuals should prefer the capability set that includes the best option, regardless of their preference over diversity.

Property 4.1 (1-dim consistency) *For $h^* = 1$, for all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ such that $\Phi_v(\mathbf{A}) > \Phi_v(\mathbf{B})$, there exists an $a \in \mathbf{A}$ such that for all $b \in \mathbf{B}$, $a > b$.*

A second notable property to evaluate capability sets is related to enhancing individual freedom by including non-dominated beings. Consider a scenario where an individual's capability set allows them to “go to work on foot” but not “go to work using a bike”. Introducing the option of “going to work using a bike” should enhance their freedom, even if they continue to prefer walking. This is because,

despite a new being $\vec{b}'$ possibly being subjectively inferior to some existing beings in the set (i.e., $\exists \vec{b} \in \mathbf{A}$ such that $v(\vec{b}) > v(\vec{b}')$), as long as $\vec{b}'$ is not objectively (or Pareto) dominated by any existing being in $\mathbf{A}$ (i.e., $\nexists \vec{b} \in \mathbf{A}$ such that $\vec{b} \succ \vec{b}'$), its addition should still contribute to the expansion of the individual's choices and, consequently, their freedom.

Property 4.2 (Strict monotone consistency:) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, having $\mathbf{A} \not\preceq \mathbf{B}$ is equivalent to $\Phi_v(\mathbf{A} \cup \mathbf{B}) > \Phi_v(\mathbf{A})$.*

Example 4.4 *This is illustrated in Figure 4.8, where adding the set $\mathbf{B}$ to the set $\mathbf{A}$ improves individual freedom because $\mathbf{B}$ is not dominated by $\mathbf{A}$. Note that Axiom 4.1 implies that adding $\mathbf{C}$ to $\mathbf{A}$ should not improve individual freedom since $\mathbf{A} \succ \mathbf{C}$. $\triangle$*

Note that from Axiom 4.1 and Property 4.2, we obtain strict monotonicity with respect to the set domination property, that is if $\mathbf{B} \prec \mathbf{A}$, then $\Phi_v(\mathbf{B}) < \Phi_v(\mathbf{A})$.

Thirdly, a softer version of Property 4.2, meaning that adding more beings to a capability set (irrelevant or not), cannot lead to less freedom.

Property 4.3 (Soft monotone consistency) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) \geq \Phi_v(\mathbf{A})$.*

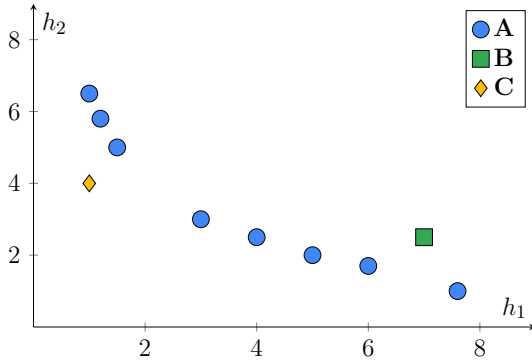


Figure 4.8: Illustration of Example 4.4

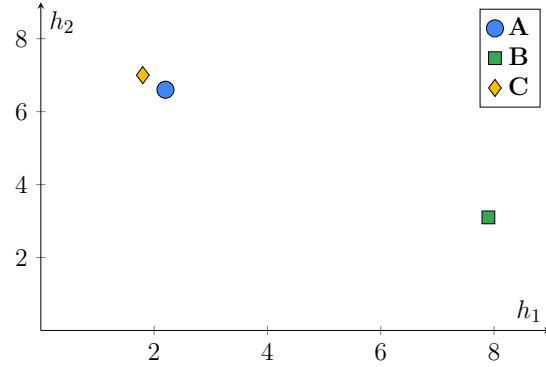


Figure 4.9: Illustration of Example 4.5.

By Axiom 4.1 and Property 4.3, we establish monotonicity for the set domination property, that is if $\mathbf{B} \preceq \mathbf{A}$, then $\Phi_v(\mathbf{B}) \leq \Phi_v(\mathbf{A})$.

Fourth, having more freedom in set $\mathbf{A} \subseteq \mathbf{Q}$ compared to set $\mathbf{B} \subseteq \mathbf{Q}$, implies that the addition of a set $\mathbf{C} \subseteq \mathbf{Q}$ to both $\mathbf{A}$ and $\mathbf{B}$ would lead to a preference for $\mathbf{A} \cup \mathbf{C}$ over $\mathbf{B} \cup \mathbf{C}$.

Property 4.4 (Independence of relevant alternatives) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, it follows that $\Phi_v(\mathbf{A}) \geq \Phi_v(\mathbf{B})$ implies $\Phi_v(\mathbf{A} \cup \mathbf{C}) \geq \Phi_v(\mathbf{B} \cup \mathbf{C})$.*

Example 4.5 *In Figure 4.9, consider a scenario where set $\mathbf{A}$ represents one way of walking to work, and set $\mathbf{B}$ represents one way of cycling to work, with $\mathbf{A}$ being preferred over $\mathbf{B}$. Now, if set $\mathbf{C}$ represents another way of walking to work, we may find that $\mathbf{B}$ union $\mathbf{C}$ are preferred over $\mathbf{A}$ union $\mathbf{C}$, as $\mathbf{C}$ provides a more diverse option in relation to $\mathbf{B}$ compared to $\mathbf{A}$. $\triangle$*

Indeed, this property may not always be desirable in certain contexts, for instance if adding $\mathbf{C}$ to $\mathbf{B}$ improves significantly the diversity whereas adding $\mathbf{C}$ to $\mathbf{A}$ only slightly improves it.

Fifth, as proposed in [10.2307/1910406], if adding a set $\mathbf{B}$ to a set $\mathbf{A}$ does not result in an improvement in freedom, then adding $\mathbf{B}$ to $\mathbf{A}$ union some other beings $\mathbf{C}$ should not lead to an improvement in freedom.

Property 4.5 (Indifference independence) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$ such that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B})$, then $\Phi_v(\mathbf{A} \cup \mathbf{C}) = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})$*

Sixth, the impact of adding a set $\mathbf{B}$ of beings on freedom should be greater or equal when the individual is more deprived. As an example, if $\mathbf{A}$ is walking to work and $\mathbf{C}$ is driving to work, the marginal freedom of $\mathbf{B}$ going to work by bike will be greater or equal added to $\mathbf{A}$ than to $\mathbf{A} \cup \mathbf{C}$.

Property 4.6 (Decreasing marginal freedom) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) \geq \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$.*

Seventh, adding a set $\mathbf{B}$ of new beings to a capability set $\mathbf{A}$ should have a greater marginal impact on freedom compared to adding $\mathbf{B}$ to $\mathbf{A}$ union $\mathbf{C}$, if there exist some beings that are dominated by $\mathbf{B}$ and $\mathbf{C}$ but not in $\mathbf{A}$. This property is based on the fact that the increase in the dominated capability space by adding $\mathbf{B}$ to $\mathbf{A}$ is greater than adding $\mathbf{B}$ to $\mathbf{A}$ union $\mathbf{C}$.

Property 4.7 (Strictly decreasing marginal freedom) *For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, such that $(\mathbf{B}^c \cap \mathbf{C}^c) \not\subseteq \mathbf{A}^c$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) > \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$.*

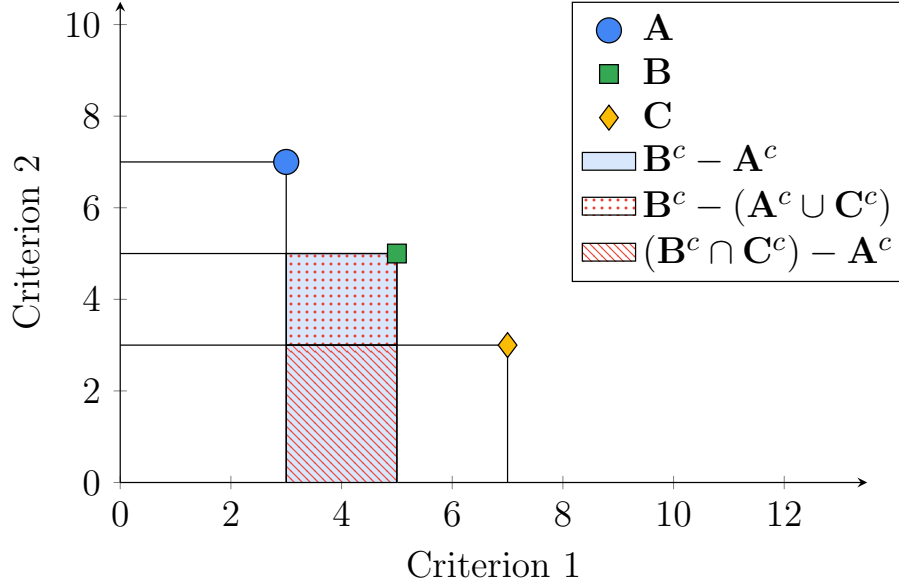


Figure 4.10: Illustration of Example 4.6.

Example 4.6 In Figure 4.10, the marginal dominated space by adding **B** to only **A** is larger than the one obtained by adding **B** to $\mathbf{A} \cup \mathbf{C}$, and thus, it should be valued as increasing the capabilities to a greater extent. $\triangle$

4.5 How to define freedom of choice functions

Functions Φ of freedom of choice can be defined based on individuals' behavior with respect to the freedom to achieve valued beings, according to preferences over diversity. In this section, we discuss three different approaches.

4.5.1 Max approach

If individuals' view freedom as having instrumental value only, they may be solely concerned about the possibility of achieving the being with the highest value in **A**. This represents the case where the individual is indifferent between the highest-valued being and, the highest-valued being with other beings as long as they have the same or lower value. In other words, the individual has no interest in the diversity of beings in the set, and only cares about the best solution. This behavior can be

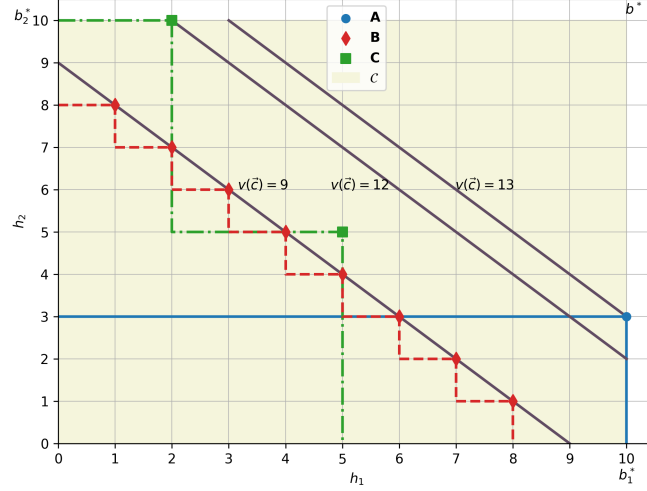


Figure 4.11: Example 4.7: max approach

formalized as follows:

$$\Phi_v(\mathbf{A}) = \max_{\vec{b} \in \mathbf{A}^c} v(\vec{b})$$

Example 4.7 Consider the following sets:

$$\mathbf{A} = \{(10, 3)\},$$

$$\mathbf{B} = \{(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)\},$$

$$\mathbf{C} = \{(2, 10), (5, 5)\},$$

along with a valuation function $v(\vec{c}) = c_1 + c_2$ for all $\vec{c} \in \mathcal{C}$ where $\mathcal{C} = ((10, 10) - \mathbb{R}_+^2) \cap \mathbb{R}_+^2$ as displayed in Figure 4.11. To simplify the example, $v(\vec{c})$ is a linear function, which we represent using indifference curves. Using the max approach, we obtain $\Phi_v(\mathbf{A}) = 13$, $\Phi_v(\mathbf{B}) = 9$, and $\Phi_v(\mathbf{C}) = 12$. $\triangle$

Note that the max approach is equivalent to the “optimistic” Gaertner-Xu method if we choose $\vec{b}^0 = \vec{0}$, $d(\vec{0}, \vec{b}) = v(\vec{b})$.

From a computational perspective, if $PF(\mathbf{A})$ is given and finite, the complexity of computing $\Phi_v(\mathbf{A})$ is $O(|PF(\mathbf{A})|)$. This process essentially involves performing a maximization over the solutions of the Pareto frontier. Since $v \in V$ is continuous and increasing on $\mathcal{C}$, the expression $\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b})$ is equivalent to $\max_{\vec{b} \in PF(\mathbf{A})} v(\vec{b})$.

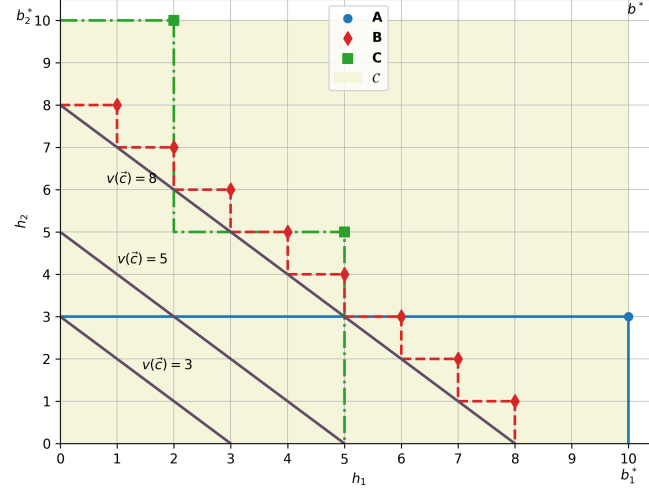


Figure 4.12: Example 4.7: min approach

4.5.2 Min approach

An individual who regards diversity as primarily important in terms of freedom to achieve a certain level of well-being will have a different approach. These individuals will believe that they are only free to achieve a well-being level y if they can achieve all the beings $\vec{c} \in \mathcal{C}$ such that $v(\vec{c}) = y$, or achieve objectively better beings $\vec{c}'$ in the sense that $\vec{c}' \succeq \vec{c}$ and $v(\vec{c}) = y$. In other words, individuals believe that they are not free to achieve a well-being level y if they cannot achieve all the beings (or objectively better beings) that lead to y . This behavior can be formalized as follows:

$$\Phi_v(\mathbf{A}) = \inf_{\forall(\vec{c} \in \mathcal{C} \setminus \mathbf{A}^c)} v(\vec{c})$$

Example 4.7 (Cont.) Continuing with the example, as represented in Figure 4.12, we obtain $\Phi_v(\mathbf{A}) = 3$, $\Phi_v(\mathbf{B}) = 8$, and $\Phi_v(\mathbf{C}) = 5$ $\triangle$

It is worth noting that the min approach is equivalent to the “pessimistic” Gaertner-Xu method, if we choose. $\vec{b}^0 = \vec{0}$, $d(\vec{0}, \vec{b}) = v(\vec{b})$.

From a computational standpoint, the task of calculating the output of this method poses significant challenges. One approach, albeit impractical for $|PF(\mathbf{A})|$

larger than a few, involves employing a mathematical programming model as described below:

$$\text{minimize } v(x_1, \dots, x_{h^*}) \quad (4.1)$$

$$\text{subject to } x_h > b_{j,h} \cdot d_{j,h} \quad \forall h \in \{1, \dots, h^*\}, \quad \forall b_j \in PF(\mathbf{A}) \quad (4.2)$$

$$\sum_{h=1}^{h^*} d_{j,h} \geq 1 \quad \forall b_j \in PF(\mathbf{A}) \quad (4.3)$$

$$x_h \in \mathbb{R} \quad \forall h \in \{1, \dots, h^*\} \quad (4.4)$$

$$d_{j,h} \in \{0, 1\} \quad \forall h \in \{1, \dots, h^*\}, \quad \forall j \in \{1, \dots, |PF(\mathbf{A})|\} \quad (4.5)$$

In our proposed approach, we introduce $|PF(\mathbf{A})| \times h^*$ decision variables $d_{j,h}$, as specified in (4.5). These variables are designed to ensure that for each being b_j in $PF(\mathbf{A})$, the solution vector $\vec{x} = [x_1, \dots, x_{h^*}]$ (referenced in (4.4)) is not dominated (as per (4.2)) in at least one dimension (according to (4.3)), for every solution in $PF(\mathbf{A})$ (as outlined in (4.2) and (4.3)). The goal of this formulation is to minimize $v(x_1, \dots, x_{h^*})$, as detailed in (4.1).

This formulation, however, introduces computational challenges. The generation of a large number of decision variables typically renders the problem NP-hard. New methods should be developed to implement this approach.

When $h^* = 2$, a more efficient method is presented in Algorithm 1. It uses the fact that any vector $\vec{c}$ within $\mathcal{C}$ satisfying $v(\vec{c}) = \inf_{\vec{c} \in \mathcal{C} \setminus \mathbf{A}^c} v(\vec{c})$ will correspond to a local nadir point formed by two adjacent solutions in $PF(\mathbf{A}) \cup \{(b_1^* + \epsilon, 0), (0, b_2^* + \epsilon)\}$, the local nadir point between two vectors $(\vec{b}, \vec{b}') \in \mathbf{Q}^2$ being $(\min(b_1, b'_1), \min(b_2, b'_2))$.

Algorithm 1: Algorithm for Calculating $\Phi(\mathbf{A})$ via Ordered Nadir Points Comparison when $h^* = 2$

Input: $PF(\mathbf{A})$

Output: Calculate $\Phi_v(\mathbf{A})$ using the min approach

```

1  $L \leftarrow \text{Sort } PF(\mathbf{A}) \cup \{(b_1^* + \epsilon, 0), (0, b_2^* + \epsilon)\}$  in lexical order prioritizing  $h_1$ 
2  $\text{Output} \leftarrow v(\vec{b}^*)$ 
3 for  $i \leftarrow 1$  to  $|L| - 1$  do
4   if  $v(L[i+1]_1, L[i]_2) < \text{Output}$  then
5      $\text{Output} \leftarrow v(L[i+1]_1, L[i]_2)$ 
6   end
7 end
```

4.5.3 Integral approach

In the previous approaches, individuals' behaviors did not take into consideration all efficient solutions in the capability set $\mathbf{A}$. The max approach reduces $\Phi_v(\mathbf{A})$ to the best beings according to the value function v , while the min approach only considers the smallest not “fully freely” achievable level of being, ignoring the rest of the beings. To address this limitation, we propose an integral approach that allows for all $\vec{b}$ in the Pareto frontier $PF(\mathbf{A})$, to have a strictly positive impact on $\Phi_v(\mathbf{A})$. We define the integral method as follows:

$$\Phi_v(\mathbf{A}) = \int \cdots \int_{\mathbf{A}^c} \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*}$$

with $\phi : [0; v(b^*)] \rightarrow \mathbb{R}_{+,*}$. Several functions ϕ can be chosen to represent different behaviors towards freedom. Let us give three examples that represent different behaviors.

First, the individual considers the simple case where $\phi(v(\vec{b})) = v(\vec{b})$. By applying this function to Example 4.7, we obtain the integration on the $h^* + 1$ volume defined under the valuation of the region of the capability set as in Figure 4.13. We obtain $\Phi_v(\mathbf{A}) = 195$, $\Phi_v(\mathbf{B}) = 204$ and $\Phi_v(\mathbf{C}) = 207$. To compute $\Phi_v(\mathbf{A})$, we have

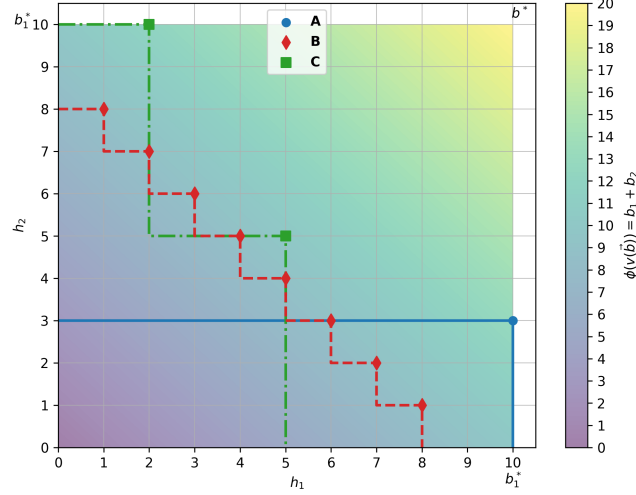


Figure 4.13: Example 4.7: Φ_v using the integral approach with $\phi(v(\vec{b})) = v(\vec{b})$.

to compute double integral over $(b_1, b_2) \in \mathbf{A}^c$, being $b_1 \in [0, 10]$ and $b_2 \in [0, 3]$:

$$\begin{aligned}
 \Phi_v(\mathbf{A}) &= \iint_{\mathbf{A}^c} \phi(v(b_1, b_2)) db_1 db_2 = \int_0^{10} \int_0^3 (b_1 + b_2) db_2 db_1 \\
 &= \int_0^{10} \left[b_1 b_2 + \frac{1}{2} b_2^2 \right]_0^3 db_1 = \int_0^{10} \left(3b_1 + \frac{9}{2} \right) db_1 \\
 &= \left[\frac{3}{2} b_1^2 + \frac{9}{2} b_1 \right]_0^{10} = \frac{3}{2} (10)^2 + \frac{9}{2} (10) = 150 + 45 = 195.
 \end{aligned}$$

Secondly, the individual chooses a function $\phi(v(\vec{b})) = v(\vec{b})^2$ that gives more weight to higher values, that is, beings with higher well-being levels, will have a higher weighting in proportion to their value. Continuing with Example 4.7, a new freedom valued capability set is displayed in Figure 4.14, and $\Phi_v(\mathbf{A}) = 1,540$, $\Phi_v(\mathbf{B}) = 1,302$ and $\Phi_v(\mathbf{C}) = 1,475.8$. To compute $\Phi_v(\mathbf{A})$, we do as follows:

$$\int_0^{10} \int_0^3 (b_1 + b_2)^2 db_2 db_1 = \int_0^{10} \left[\frac{1}{3} (b_1 + b_2)^3 \right]_0^3 db_1 = \int_0^{10} \left(\frac{1}{3} (b_1 + 3)^3 - \frac{1}{3} b_1^3 \right) db_1$$

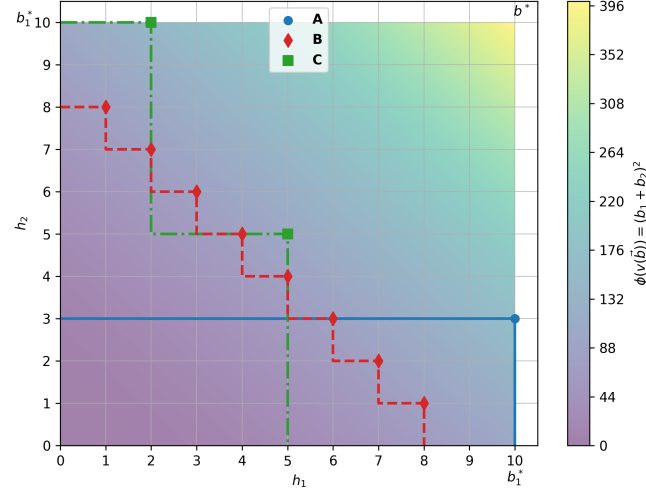


Figure 4.14: Example 4.7: Φ_v using the integral approach with $\phi(v(\vec{b})) = v(\vec{b})^2$.

$$\begin{aligned}
 &= \int_0^{10} \frac{1}{3} (b_1^3 + 9b_1^2 + 27b_1 + 27) db_1 - \int_0^{10} \frac{1}{3} b_1^3 db_1 \\
 &= \frac{1}{3} \left[\frac{1}{4} b_1^4 + 3b_1^3 + \frac{27}{2} b_1^2 + 27b_1 \right]_0^{10} - \frac{1}{3} \left[\frac{1}{4} b_1^4 \right]_0^{10} = 1540.
 \end{aligned}$$

Thirdly, the individual's perspective prioritizes more balanced capability sets by assigning greater weight to lower-valued beings. An illustrative approach to achieve this balance is by applying the function $\phi(v(\vec{b})) = \sqrt{v(\vec{b})}$. Applying this method to our example, we compute the values $\Phi_v(\mathbf{A})$, $\Phi_v(\mathbf{B})$, and $\Phi_v(\mathbf{C})$ as depicted in Figure 4.15. The resulting calculations yield $\Phi_v(\mathbf{A}) \approx 70.01$, $\Phi_v(\mathbf{B}) \approx 83.95$, and $\Phi_v(\mathbf{C}) \approx 83.55$. The computation of $\Phi_v(\mathbf{A})$ is obtained as the following:

$$\begin{aligned}
 \Phi_v(\mathbf{A}) &= \iint_{\mathbf{A}^c} \phi(v(b_1, b_2)) db_1 db_2 = \int_0^{10} \int_0^3 \sqrt{b_1 + b_2} db_2 db_1 \\
 &= \int_0^{10} \left[\frac{2}{3} (b_1 + b_2)^{\frac{3}{2}} \right]_0^3 db_1 = \int_0^{10} \left(\frac{2}{3} (b_1 + 3)^{\frac{3}{2}} - \frac{2}{3} b_1^{\frac{3}{2}} \right) db_1 \\
 &= \left[\frac{4 \left((b_1 + 3)^{\frac{5}{2}} - b_1^{\frac{5}{2}} \right)}{15} \right]_0^{10} = \frac{4 \left(13^{\frac{5}{2}} - 10^{\frac{5}{2}} - 3^{\frac{5}{2}} \right)}{15} \approx 70.01.
 \end{aligned}$$

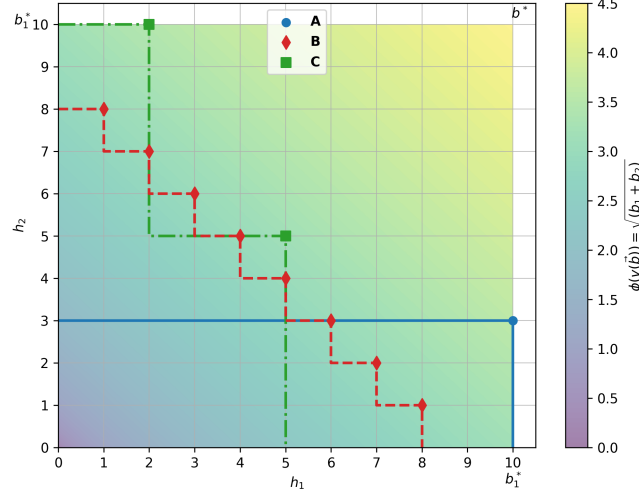


Figure 4.15: Example 4.7: Φ_v using the integral approach with $\phi(v(\vec{b})) = \sqrt{v(\vec{b})}$.

We note that altering the function ϕ allows us to place different emphasis on the value of freedom. For instance, choosing $\phi(v(\vec{b})) = v(\vec{b})^2$ highlights the instrumental value of freedom, while adopting $\phi(v(\vec{b})) = \sqrt{v(\vec{b})}$ shifts the focus towards the intrinsic value of freedom.

From a computational standpoint, calculating $\Phi_v(\mathbf{A})$ presents a complex challenge that is beyond the scope of this chapter. Nevertheless, we introduce Algorithm 2 as a theoretical method to compute $\Phi_v(\mathbf{A})$ when $PF(\mathbf{A})$ is known and finite, despite practical limitations in its application.

Considering $\mathbf{A}^c = PF(\mathbf{A})^c$, we can express $\Phi_v(\mathbf{A})$ as

$$\Phi_v(\mathbf{A}) = \Phi_v \left(\bigcup_{\vec{b} \in PF(\mathbf{A})} \vec{b}^c \right)$$

Applying the Principle of Inclusion-Exclusion further refines this expression to

$$\Phi_v \left(\bigcup_{\vec{b} \in PF(\mathbf{A})} \vec{b}^c \right) = \sum_{Z \subseteq \mathcal{P}(PF(\mathbf{A})) \setminus \{\emptyset\}} (-1)^{|Z|+1} \cdot \Phi_v \left(\bigcap_{\vec{b} \in Z} \vec{b}^c \right)$$

This equation underpins the generation of all subsets of the power set of the Pareto frontier of $\mathbf{A}$ (line 2 of Algorithm 2).

We note that for any set Z within the power set $\mathcal{P}(PF(\mathbf{A}))$, the expression $\bigcap_{\vec{b} \in Z} \vec{b}^c$ is equivalent to $(\min_{\vec{b} \in Z} b_h \mid \forall h \in \{1, \dots, h^*\})^c$. This equivalence forms the basis for deriving lines 3-5 of Algorithm 2. Furthermore, line 6 is developed based on the Principle of Inclusion-Exclusion. A crucial observation is that all vectors $\vec{b} \in \mathbf{Q}$ reside in $\mathbb{R}_{+,*}^{h^*}$, and, for any function $v \in V$, the function $\phi(v())$ maintains strict positivity over $\mathcal{C}$. Consequently, $\Phi_v(\bigcap_{\vec{b} \in Z} \vec{b}^c)$ always has a measurable volume.

To compute the integral indicated in line 6, we can utilize numerical methods, facilitated by the SciPy 1.0 Python package [virtanen2020scipy]. While this approach is computationally intensive, it offers a viable solution for performing the required calculations in the algorithm. It is important to note that the complexity of the integration methods in SciPy 1.0, especially the ‘dblquad’ function, is variable and does not adhere to a standard fixed complexity notation. Consequently, it is advisable to be aware that complex functions, precise requirements, and larger ranges are computationally demanding.

Furthermore, the practical feasibility of the algorithm is limited due to the exponential complexity involved in generating the power set $\mathcal{P}(PF(\mathbf{A}))$, which scales as $O(2^{|PF(\mathbf{A})|})$. This exponential growth in complexity renders the algorithm impractical for large Pareto sets.

Algorithm 2: Finding $\Phi(\mathbf{A})$ for the integral approach using the Inclusion-Exclusion Principle

Input: $PF(\mathbf{A})$

Output: Value of $\Phi(\mathbf{A})$ calculated using the integral approach

```

1 Output  $\leftarrow 0$ 
2 forall  $Z \in \mathcal{P}(PF(\mathbf{A})) \setminus \{\emptyset\}$  do
3   for  $h \leftarrow 1$  to  $h^*$  do
4      $a_h \leftarrow \min_{\vec{b} \in Z} b_h$ 
5   end
6   Output  $\leftarrow$ 
      Output  $+ (-1)^{|Z|+1} \cdot \left( \int_0^{a_1} \dots \int_0^{a_{h^*}} \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*} \right)$ 
7 end
```

To effectively implement this technique in scenarios where $|PF(\mathbf{A})|$ is more than

a few, the development of feasible approximation methods is crucial. However, in the simple case where $h^* = 2$, Algorithm 3 provides an exact method with complexity $O(|PF(\mathbf{A})|)$ integration method calls. By ordering the beings on the Pareto frontier and updating the integration limits during each iteration, the algorithm partitions $\mathbf{A}^c$ in $|PF(\mathbf{A})|$ subsets, meaning that it creates $|PF(\mathbf{A})|$ non empty subsets, such that each element dominated by the capability set is included in one and only one of these subsets.

Algorithm 3: Algorithm for Calculating $\Phi(\mathbf{A})$ via Ordered Integration

when $h^* = 2$

Input: $PF(\mathbf{A})$

Output: Value of $\Phi(\mathbf{A})$ using the integral approach

```

1  $L \leftarrow$  Sort  $PF(\mathbf{A})$  in lexical order prioritizing  $h_1$ 
2 Output  $\leftarrow 0$ 
3  $a \leftarrow 0$ 
4 for  $\vec{b} \in L$  do
5   | Output  $\leftarrow$  Output +  $\int_0^{b_1} \int_a^{b_2} \phi(v(b_1, b_2)) db_1 db_2$ 
6   |  $a \leftarrow b_2$ 
7 end
```

4.5.4 Desirable properties respected by the max, min and integral approaches

Table 4.1 provides a summary of how the max, min, and integral approaches align with the specified properties. Naturally, all approaches adhere to the principle of indifference to irrelevant alternatives (A.1). In the one-dimensional case, the absence of possible diversity with respect to A.1 makes their alignment with 1-dimensional consistency (P.4.1) somewhat predictable. Moreover, all approaches maintain soft monotonicity (P.4.3), which is essential. Any freedom of choice function where, given any $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ and $v \in V$ such that $\Phi_v(\mathbf{A} \cup \mathbf{B}) < \Phi_v(\mathbf{A})$ would be deemed entirely counterintuitive.

Properties	max	min	integral
A.4.1 Indifference of irrelevant alternatives	✓	✓	✓
P.4.1 1-dim consistency	✓	✓	✓
P.4.2 Strict monotone consistency	×	×	✓
P.4.3 Soft monotone consistency	✓	✓	✓
P.4.4 Independence of relevant alternatives	✓	×	×
P.4.5 Indifference independence	✓	×	✓
P.4.6 Decreasing marginal value options	✓	×	✓
P.4.7 Strict decreasing marginal value options	×	×	✓

Table 4.1: Properties respected by max, min and integral approaches.

Interestingly, the max approach is the only one to respect the Independence of Relevant Alternatives (P.4.4), this is due to its indifference towards diversity. Conversely, the integral approach is the only one adherent to strict monotone consistency (P.4.2), as it values the addition of every new relevant being, thereby favoring diversity in all its forms (in accordance with A.1)

The min approach's non-adherence to indifference independence (P.4.5) and decreasing marginal freedom (P.4.6) comes from its pessimistic perspective. It misses to see improvement of freedom of choice in choosing beings valued as y if some other beings valued y are still not available. This tendency is illustrated in the counter-examples provided in the Section 4.6.

Lastly, the integral approach stands alone in adhering to strict decreasing marginal freedom (P.4.7). This mirrors the classical “law” of diminishing marginal utility (see [Gossen1983]).

Proofs concerning Table 4.1 are shown the Section 4.6.

4.6 Proofs

We provide here the proofs of Table 4.1.

Axiom 4.1: Indifference of irrelevant alternatives For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, such that $\mathbf{A} \succeq \mathbf{B}$ then $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B})$.

Proof: The validity of the claim becomes evident when considering that all three

approaches are assessed using the closure $\mathbf{A}^c$, rather than $\mathbf{A}$ itself. Given that $\mathbf{B} \preceq \mathbf{A}$, it follows that the closure of the union of $\mathbf{A}$ and $\mathbf{B}$ is equivalent to the closure of $\mathbf{A}$ alone, i.e., $(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c$. ■

Property 4.1: 1-dim consistency For $h^* = 1$, for all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ such that $\Phi_v(\mathbf{A}) > \Phi_v(\mathbf{B})$, there exists an $a \in \mathbf{A}$ such that for all $b \in \mathbf{B}$, $a > b$.

Proof for the max approach: With $h^* = 1$, the closure of $\mathbf{A}$, can be represented as $\mathbf{A}^c = [0, \max_{a \in \mathbf{A}} a]$. Given that $v(\cdot)$ is increasing, $\max_{a \in \mathbf{A}} a > \max_{b \in \mathbf{B}} b \Leftrightarrow \Phi_v(\mathbf{A}) > \Phi_v(\mathbf{B})$. ■

Proof for the min approach: For $h^* = 1$, consider any set $\mathbf{A} \subseteq \mathbf{Q}$. We have:

$$\Phi_v(\mathbf{A}) = \inf_{c \in \mathcal{C} \setminus \mathbf{A}^c} v(c) \Leftrightarrow \max_{a \in \mathbf{A}^c} v(a)$$

This equivalence demonstrates the property under the min approach. ■

Proof for the integral approach: Since $\phi(v(\cdot))$ is positive, the integral $\int_{\mathbf{A}^c} \phi(v(b)) db$ over any subset of $\mathcal{C}$ is positive. For $h^* = 1$, $\Phi_v(\mathbf{A}) > \Phi_v(\mathbf{B})$ implies $\int_{\mathbf{A}^c} \phi(v(b)) db > \int_{\mathbf{B}^c} \phi(v(b)) db$, and $\mathbf{A}^c = [0, \max_{a \in \mathbf{A}} a]$ and $\mathbf{B}^c = [0, \max_{b \in \mathbf{B}} b]$. Therefore, $\max_{a \in \mathbf{A}} a > \max_{b \in \mathbf{B}} b$ holds, confirming the property for the integral approach. ■

Property 4.2: Strict monotone consistency: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ such that $\mathbf{A} \not\preceq \mathbf{B}$, is equivalent to $\Phi_v(\mathbf{A} \cup \mathbf{B}) > \Phi_v(\mathbf{A})$.

Counter example for the max and min approaches: Consider the function $v(\vec{b}) = b_1 + b_2$. Let $\mathbf{A} = \{(5, 5)\}$ and $\mathbf{B} = \{(1, 6)\}$. We observe that $\mathbf{A} \not\preceq \mathbf{B}$, but in the case of the max approach, we find that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B}) = 10$, and in the case of min approach, we find that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B}) = 5$. ■

Proof for integral approach: For any $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$ and $v \in V$ the value of $\Phi_v(\mathbf{A} \cup \mathbf{B})$ is given by the sum of two integrals

$$\int_{\mathbf{A}^c} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*} + \int_{\mathbf{B}^c \setminus (\mathbf{A}^c \cap \mathbf{B}^c)} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*}$$

Since $\phi(v(\vec{b}))$ is strictly positive on $\mathcal{C}$, it follows that

$$\int \cdots \int_{\mathbf{B}^c \setminus (\mathbf{A}^c \cap \mathbf{B}^c)} \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*} > 0$$

if and only if the set $\mathbf{B}^c \setminus (\mathbf{A}^c \cap \mathbf{B}^c)$ is non-empty. We know that $\mathbf{A} \not\preceq \mathbf{B}$ is equivalent to $\mathbf{B} \not\preceq \mathbf{A}^c$. This implies $\mathbf{B} \not\preceq (\mathbf{A}^c \cap \mathbf{B}^c)$, ensuring that $\mathbf{B}^c \setminus (\mathbf{A}^c \cap \mathbf{B}^c)$ is indeed non-empty.

Therefore, $\Phi_v(\mathbf{A} \cup \mathbf{B}) > \Phi_v(\mathbf{A})$, proving that $\mathbf{A} \not\preceq \mathbf{B}$ if and only if $\Phi_v(\mathbf{A}) < \Phi_v(\mathbf{A} \cup \mathbf{B})$. ■

Property 4.3: Soft monotone consistency: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}) \subseteq \mathbf{Q}^2$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) \geq \Phi_v(\mathbf{A})$.

Proof for the max approach: We simply observe that:

$$\Phi_v(\mathbf{A} \cup \mathbf{B}) = \max \left[\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}) \right] \geq \max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}) = \Phi_v(\mathbf{A})$$

■

Proof for the min approach: Note that $\mathcal{C} \setminus (\mathbf{A}^c \cup \mathbf{B}^c) \subseteq \mathcal{C} \setminus \mathbf{A}^c$. Therefore:

$$\Phi_v(\mathbf{A}) = \inf_{\vec{c} \in (\mathcal{C} \setminus \mathbf{A}^c)} v(\vec{c}) \leq \inf_{\vec{c} \in (\mathcal{C} \setminus (\mathbf{A}^c \cup \mathbf{B}^c))} v(\vec{c})$$

This implies $\Phi_v(\mathbf{A} \cup \mathbf{B}) \geq \Phi_v(\mathbf{A})$, upholding the property under the min approach. ■

Proof for the integral approach: Given that the integral approach respects strict monotone consistency (Property 4.2), it naturally conforms to its softer version. ■

Property 4.4: Independence of relevant alternatives: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, it follows that $\Phi_v(\mathbf{A}) \geq \Phi_v(\mathbf{B})$ implies $\Phi_v(\mathbf{A} \cup \mathbf{C}) \geq \Phi_v(\mathbf{B} \cup \mathbf{C})$.

Proof for the max approach: Assuming

$$\Phi_v(\mathbf{A}) = \max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}) \geq \max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}) = \Phi_v(\mathbf{B})$$

it follows that

$$\Phi_v(\mathbf{A} \cup \mathbf{C}) = \max \left[\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) \right] \geq \max \left[\max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) \right] = \Phi_v(\mathbf{B} \cup \mathbf{C})$$

■

Counterexample for the min approach: Consider a function $v(\vec{b}) = b_1 + b_2$. Let $\mathbf{A} = \{(1, 3), (2, 2), (3, 1)\}$, $\mathbf{B} = \{(2, 4)\}$, and $\mathbf{C} = \{(4, 2)\}$. Under the min approach, we find that $\Phi_v(\mathbf{A}) = 3$ and $\Phi_v(\mathbf{B}) = 2$. However, when considering the unions, $\Phi_v(\mathbf{A} \cup \mathbf{C}) = 3$ and $\Phi_v(\mathbf{B} \cup \mathbf{C}) = 4$.

■

Counterexample for integral approach: Let $v(\vec{b}) = b_1 + b_2$ for all $\vec{b} \in \mathcal{C}$, and define $\phi(v(\vec{b})) = v(\vec{b})$. Suppose $\mathbf{A} = \{(1, 5)\}$, $\mathbf{B} = \{(4, 1)\}$, and $\mathbf{C} = \{(2, 5)\}$. Clearly, $\Phi_v(\mathbf{A}) > \Phi_v(\mathbf{B})$ as $\Phi_v(\mathbf{A}) = 15$ and $\Phi_v(\mathbf{B}) = 10$. However, $\Phi_v(\mathbf{A} \cup \mathbf{C}) = 35$ while $\Phi_v(\mathbf{B} \cup \mathbf{C}) = 42$.

■

Property 4.5: Indifference independence: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$ such that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B})$, then $\Phi_v(\mathbf{A} \cup \mathbf{C}) = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})$.

Proof for the max approach: If

$$\Phi(\mathbf{A}) = \max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}) = \max \left[\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}) \right] = \Phi_v(\mathbf{A} \cup \mathbf{B})$$

it naturally follows that:

$$\max \left[\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) \right] = \max \left[\max \left[\max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}), \max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}) \right], \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) \right]$$

Demonstrating that $\Phi(\mathbf{A} \cup \mathbf{C}) = \Phi(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})$

■

Counterexample for min approach: Let $v(\vec{b}) = b_1 + b_2$ and $\mathbf{A} = \{(2, 1)\}$, $\mathbf{B} = \{(1, 2)\}$ and $\mathbf{C} = \{(3, 1)\}$. We observe that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{B}) = 1$ but $\Phi_v(\mathbf{A} \cup \mathbf{C}) = 1 < 2 = \Phi_v(\mathbf{B} \cup \mathbf{C})$

Proof for the integer approach: Invoking Axiom 1 and Property 2, we deduce that $\Phi_v(\mathbf{A}) = \Phi_v(\mathbf{A} \cup \mathbf{B})$ implies that $\mathbf{B} \subseteq \mathbf{A}^c$, and it follows that $\Phi_v(\mathbf{A} \cup \mathbf{C}) = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})$.

■

Property 4.6: Decreasing marginal value options: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) \geq \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$.

Proof for the max approach: Consider three cases based on the relationship between the maximum values of $v(\vec{b})$ in sets $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

1. If $\max_{\vec{b} \in \mathbf{A}^c} (v(\vec{b})) \leq \max_{\vec{b} \in \mathbf{B}^c} (v(\vec{b}))$, then $\Phi_v(\mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{B}) = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{B})$.

2. If $\max_{\vec{b} \in \mathbf{A}^c} (v(\vec{b})) > \max_{\vec{b} \in \mathbf{B}^c} (v(\vec{b}))$ and $\max_{\vec{b} \in \mathbf{A}^c} (v(\vec{b})) \geq \max_{\vec{b} \in \mathbf{C}^c} (v(\vec{b}))$, then $\Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{B}) = 0 \leq \Phi_v(\mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{B})$.

3. If $\max_{\vec{b} \in \mathbf{A}^c} (v(\vec{b})) > \max_{\vec{b} \in \mathbf{B}^c} (v(\vec{b}))$ and $\max_{\vec{b} \in \mathbf{A}^c} (v(\vec{b})) < \max_{\vec{b} \in \mathbf{C}^c} (v(\vec{b}))$, then $\Phi_v(\mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{B}) = \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) - \max_{\vec{b} \in \mathbf{B}^c} v(\vec{b}) \geq \max_{\vec{b} \in \mathbf{C}^c} v(\vec{b}) - \max_{\vec{b} \in \mathbf{A}^c} v(\vec{b}) = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{B})$. ■

Counterexample for the min approach: This is the same as for Property 4.5. ■

Proof for the integral approach: We know that using the principle of Inclusion-Exclusion

$$\Phi_v(\mathbf{A} \cup \mathbf{B}) = \Phi_v(\mathbf{A}) + \Phi_v(\mathbf{B}) - \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c)$$

or

$$\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) = \Phi_v(\mathbf{B}) - \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c),$$

Similarly, as

$$\begin{aligned} \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) &= \Phi_v(\mathbf{A}) + \Phi_v(\mathbf{B}) + \Phi_v(\mathbf{C}) \\ &\quad - \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c) - \Phi_v(\mathbf{A}^c \cap \mathbf{C}^c) - \Phi_v(\mathbf{B}^c \cap \mathbf{C}^c) \\ &\quad + \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) \end{aligned}$$

We obtain

$$\begin{aligned}
\Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C}) &= \Phi_v(\mathbf{A}) + \Phi_v(\mathbf{B}) + \Phi_v(\mathbf{C}) - \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c) \\
&\quad - \Phi_v(\mathbf{A}^c \cap \mathbf{C}^c) - \Phi_v(\mathbf{B}^c \cap \mathbf{C}^c) \\
&\quad + \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) \\
&\quad - (\Phi_v(\mathbf{A}) + \Phi_v(\mathbf{C}) - \Phi_v(\mathbf{A}^c \cap \mathbf{C}^c)) \\
&= \Phi_v(\mathbf{B}) - \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c) - \Phi_v(\mathbf{B}^c \cap \mathbf{C}^c) \\
&\quad + \Phi_v(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c).
\end{aligned}$$

Then, we can deduce that $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) \geq \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$ holds if and only if

$$\Phi_v(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) \leq \Phi_v(\mathbf{B}^c \cap \mathbf{C}^c),$$

which is indeed true. To demonstrate this, observe that

$$\int_{\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*} \leq \int_{\mathbf{A}^c \cap \mathbf{B}^c} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*},$$

given that $\phi(v(\vec{b}))$ is positive and $(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) \subseteq (\mathbf{B}^c \cap \mathbf{C}^c)$. Thus, the property is respected under the integral approach. $\blacksquare$

Property 4.7: Strict decreasing marginal value options: For all $v \in V$ and $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \subseteq \mathbf{Q}^3$, such that $(\mathbf{B}^c \cap \mathbf{C}^c) \not\subseteq \mathbf{A}^c$, it follows that $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) > \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$.

Counterexample for the max approach: Let us have $v(\vec{b}) = b_1 + b_2$, $\mathbf{A} = \{(1, 3)\}$, $\mathbf{B} = \{(2, 1)\}$ and $\mathbf{C} = \{(2, 4)\}$. Then, $(\mathbf{B}^c \cap \mathbf{C}^c) \setminus \mathbf{A}^c = (2, 1)^c \setminus (1, 1)^c \neq \{\emptyset\}$ and $\Phi_v(\mathbf{A} \cup \mathbf{B}) - \Phi_v(\mathbf{A}) = 4 - 4 = 0 = 6 - 6 = \Phi_v(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}) - \Phi_v(\mathbf{A} \cup \mathbf{C})$. $\blacksquare$

Counterexample for the min approach: Same as for Property 4.5. $\blacksquare$

Proof for the integral approach: Referencing proof of Property 4.6, we observe that the property is satisfied if

$$\Phi_v(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) < \Phi_v(\mathbf{B}^c \cap \mathbf{C}^c),$$

which indeed holds, given that the set $(\mathbf{B}^c \cap \mathbf{C}^c) \setminus \mathbf{A}^c$ is non-empty, it means that $(\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c) \subset (\mathbf{B}^c \cap \mathbf{C}^c)$. In terms of the integral, this can be expressed as

$$\int_{\mathbf{A}^c \cap \mathbf{B}^c \cap \mathbf{C}^c} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*} < \int_{\mathbf{B}^c \cap \mathbf{C}^c} \cdots \int \phi(v(b_1, \dots, b_{h^*})) db_1 \dots db_{h^*},$$

because the function $\phi(v(\cdot))$ is positive, and the additional region contributed by $(\mathbf{B}^c \cap \mathbf{C}^c) \setminus \mathbf{A}^c$ ensures that the right-hand side integral is strictly greater. ■

4.7 Conclusion

This chapter has delved into various methodologies for assessing capabilities within our multiobjective decision-making framework. We introduced three distinct approaches for defining freedom of choice functions: the classic max approach, which views freedom as an indirect utility or instrumental value; the min approach, treating freedom as an intrinsic value; and the integral approach that appreciates the full diversity of the capability set. Each approach is built upon a specific function of freedom choice, and we have provided some solutions for their computation. Our analysis revealed how these diverse methods align with different properties, shedding light on their respective strengths and limitations. Looking forward, several avenues for further research have emerged. Firstly, our study presupposes the existence of a value function, which may not always be applicable. Future work should investigate scenarios lacking a clearly defined value function. Secondly, the computational feasibility of the min and integral approaches poses a challenge when dealing with capability sets with large number of solution in their Pareto frontiers. Developing solutions for cases with infinite Pareto frontiers remains a crucial area for exploration. Lastly, there is a need to develop techniques for learning the function $\phi(\cdot)$ in the integral approach context. Such advancements would enable subjective assessments of capabilities.

Chapter 5

Capability sets under uncertainty

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5.1 Introduction

A basic form of a decision aiding approach involves evaluating the current welfare of a territory by examining the distribution of private assets and the accessibility to common resources. This method includes identifying possible scenarios involving damage to both private properties and commons. If the likelihood of such scenarios is unknown, the optimal choice is selected based on the welfare associated with each scenario and a chosen decision rule.

From a decision support perspective, this necessitates measuring welfare—as capability sets—induced as consequences and impacts of policies. This approach differs from more classical methods such as multi-attribute utility theory [**fishburn1977multiattribute**, **keeney1993decisions**], or portfolio approaches [**durbach2012modeling**, **liesio2012scenario**, **stewart2013integrating**, **vilkumaa2018scenario**], which aim to identify a single, optimal (albeit multidimensional) solution for policymakers.

Considering a classical framework where various events occur under different states of the world, and in the absence of their likelihoods, one approach is to use a decision rule to aggregate the utilities in each state. Several traditional uncertainty rules exist, such as the Minimax and Maximax, which fundamentally rely on order statistics; the Hurwicz, which is a mixture of the pessimistic (Minimax) and optimistic (Maximax) approaches; the Laplace, which assumes the expected utilities with all states being equiprobable; and the Min Max Regret, which focuses on minimizing the maximum regret experienced. The goal of this chapter is to adapt these methods to our capability set framework, focusing on order statistics decision rules as they are the most fundamental. Hurwicz and Laplace will be briefly discussed in Appendix C, as they require aggregation methods presented in Chapter 6.

Indeed, the issue for our policy maker aiming to maximize welfare for the citizens of her territory is that there is no single utility function to consider for each citizen. Instead, there is a Pareto frontier representing the welfare that any policy is supposed to achieve. Therefore, the traditional decision rule based on order statistic must be adapted. The open technical question is how to construct such a “capability order statistic”? To our knowledge, this remains an open question despite its significance for policy design purposes.

We begin by precisely formulating the problem in Section 5.2, where we introduce essential terminology to ensure a comprehensive understanding of the key concepts.

We offer a novel perspective on conventional order statistics within the traditional framework. In Section 5.3, we introduce capability order statistics and discuss several interesting propositions that are analogous to conventional order statistics, but applicable to capability sets. The proofs for these propositions are provided in Section 5.4.

5.2 Problem Formulation.

A Policy Maker (PM) is faced with the task of selecting an act in an environment of uncertainty. To represent this uncertainty, a set S of states of the world is employed. S is assumed to be finite, with $S = \{s_1, s_2, \dots, s_{l^*}\}$.

The PM has to select an act from a finite set $F = \{f_1, f_2, \dots, f_{m^*}\}$ which maps states from S into consequences. To evaluate these consequences, we employ a function U , resulting in a capability set $U(f_m(s_l)) \subseteq \mathbb{R}_+^{h^*}$ for each $m \in \{1, 2, \dots, m^*\}$ and $l \in \{1, 2, \dots, l^*\}$. The *beings* within these sets, denoted as $\vec{b} \in \{U(f_m(s_l)) \mid \forall (f_m, s_l) \in F \times S\}$, are h^* -dimensional vectors that encapsulate various welfare dimensions such as health, security, pleasure, and more. As previously, capability sets are generally not singletons and are guaranteed to be non-empty ($U(f_m(s_l)) \neq \emptyset$), potentially encompassing an infinite number of beings, and must form a compact (closed and bounded) set. For the sake of simplicity in notation, we represent sure acts $U(f_m(S)), U(f_{m'}(S)), U(f_{m''}(S)), \dots$ with $|S| = 1$ as $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ respectively. This notation will be used to discuss capability sets in general terms. Similarly, $U(f_m(s_l)), U(f_{m'}(s_l)), U(f_{m''}(s_l))$ will be denoted as $\mathbf{A}_l, \mathbf{B}_l, \mathbf{C}_l$.

Once again it is important to note that the choice of beings $\vec{b}$ within $\mathbf{A}_l$, made by the citizen after the PM selects f_m and nature determines s_l , is not the concern of the PM. The PM intentionally avoids making assumptions about which being will actually be chosen for two primary reasons: first, the PM may not have access to the individual's preferences across the various dimensions of welfare; second, the objective of the PM is to enhance freedom without dictating a specific lifestyle. Indeed, the *sole* assumption the PM makes about the citizen is that all welfare dimensions relevant to the citizen's choice are included in the h^* welfare dimensions, and that assuming increasing monotonic preferences over welfare, the citizen is rational. Thus, we will choose a $\vec{b}$ in $\mathbf{A}_l$ that is also in $PF(\mathbf{A}_l)$. When a capability set $\mathbf{A}_l$ comprises

a finite number n_l^* of solutions, we represent its beings by $\vec{b}_{l,1}, \dots, \vec{b}_{l,n_l^*}$ (where $b_{l,n,h}$ specifies the value of the n^{th} being for the l^{th} state in the h^{th} dimension).

5.2.1 Order statistics

Following the classical framework established by [savage1972foundations], the methodology for decision support within the set of actions F is outlined as follows: the combination of a chosen action and a state results in a consequence, denoted $f_m(s_l) \in Y$. There exists an utility function $u : Y \mapsto \mathbb{R}$, such that $u(f_m(s_l)) \geq u(f_{m'}(s_{l'}))$ if and only if the outcome of $f_m(s_l)$ is considered at least as preferable as that of $f_{m'}(s_{l'})$.

Within this framework, the notation $u(f_m(s_l))$ is analogously represented as $\max_{x \in \mathbb{R}} u(f_m(s_l)) - \mathbb{R}_+$ (or $\max_{x \in \mathbb{R}} \{u(f_m(s_l)) - x : x \in \mathbb{R}_+\}$), a fact that, although seemingly straightforward, plays an important role in the following part of the chapter. This formulation draws attention to the principle that optimizing weakly dominated elements yields an identical consequence to direct solution consideration. The significance of this principle will increase when considering higher-dimensional spaces and sets of solutions. Moreover, remark that any given pair $(x, x') \in \mathbb{R}^2$ satisfying $x \leq x'$, it can equivalently be asserted that x is encompassed within the solution dominated by x' , expressed as $x \in (x')^c$.

Example 5.1 Let us have only two acts and three states of the world as in Table 5.1. The consequence of act f_1 are uncertain whereas f_2 is certain. The space dominated by each act and scenario is displayed in Figure 5.1. $\triangle$

	s_1	s_2	s_3
$u(f_1(s_l))$	8	3	4
$u(f_2(s_l))$	5.5	5.5	5.5

Table 5.1: Uncertain decision problem

Consider the set $u(f_m(S)) = \{u(f_m(s_1)), \dots, u(f_m(s_{l^*}))\}$. We define a bijective function $\sigma : S \mapsto \{1, 2, \dots, l^*\}$ such that for any two elements $(s_l, s_{l'}) \in S^2$, if $u(f_m(s_l)) \leq u(f_m(s_{l'}))$, then $\sigma(s_l) \leq \sigma(s_{l'})$. This function σ orders the elements of $u(f_m(S))$ based on their values.

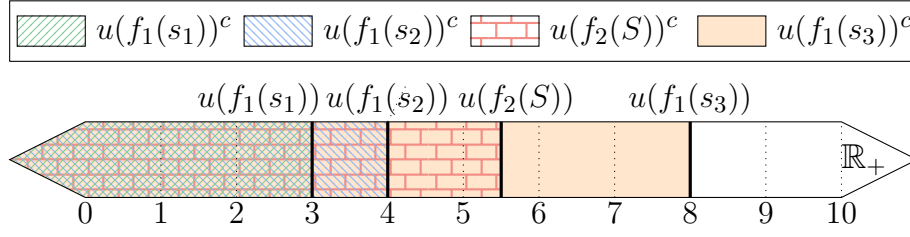


Figure 5.1: Representation of the dominated space in one dimension of solution of example 5.1

For instance, the utility for elements in S can be by:

$$\begin{aligned} u(f_m(s_1)) &= 2, & u(f_m(s_2)) &= 1, & u(f_m(s_3)) &= 2, \\ u(f_m(s_4)) &= 3, & u(f_m(s_5)) &= 5, & u(f_m(s_6)) &= 8, \end{aligned}$$

Denoting $u_{f_m}(k) = u(f_m(\sigma^{-1}(k)))$ as the k^{th} order statistic, we can apply the bijective function σ to obtain the ordered sequence:

$$u_{f_m}(1) = 1, u_{f_m}(2) = 2, u_{f_m}(3) = 2, u_{f_m}(4) = 3, u_{f_m}(5) = 5, u_{f_m}(6) = 8.$$

Here, $u_{f_m}(k)$ represents the k^{th} order statistic in $u(f_m(S))$, and $\sigma^{-1}(k)$ provides the original element in S that corresponds to this order.

An alternative yet equivalent formulation of order statistics is that for $u_{f_m}(k)$, (a) there exist at least $|S| - k + 1$ states of the world, for which $u(f_m(s_l)) \geq u_{f_m}(k)$, and (b) there are at most $k - 1$ states of the world, such that there is $u(f_m(s_l)) \not\geq u_{f_m}(k)$.

From an algorithmic perspective, obtaining the order statistic is straightforward. The process entails sorting the set of utilities $u(f_m(S))$ using any efficient sorting algorithm. Then, the k^{th} order statistic is assigned the k^{th} value in the resulting ordered sequence. The main challenge in determining the order statistic for capability sets arises from the complexity in establishing a clear hierarchy among them, as capability sets are often incomparable, as discussed in Chapter 4.

In decision-making contexts characterized by uncertainty, order statistic is employed as follows: the order statistic k signifies the k^{th} least favorable outcome of a decision across all possible states, where $k = 1$ corresponds to the least desirable outcome, and $k = |S|$ represents the most favorable outcome.

Building on the notion of least favorable, we delve into the concept of dominance.

In terms of dominance, we can formally define the k^{th} order statistic, $u_{f_m}(k)$, as the maximum value denoted x , that is dominated within at least $|S| + 1 - k$ scenarios. Formally as:

$$u_{f_m}(k) = \max_x \left\{ x : x \in \bigcup_{\substack{\forall X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(\bigcap_{s_l \in X} u(f_m(s_l)) - \mathbb{R}_+ \right) \right\} \quad (5.1)$$

Here, $\mathcal{P}(S)$ denotes the power set of S , which includes all possible subsets of S , ranging from the empty set to S itself. The validity of Eq. (5.1) follows from the proposition discussed in Section 5.3.

Within the framework of order statistics, it is pertinent to explore two particular cases that represent extremes in the decision-making process under uncertainty. These approaches are commonly employed to model opposite ends of the decision-making spectrum.

The first, commonly referred to as the “Minimax” approach, focuses on the minimum order statistic. This approach embodies a cautious attitude towards uncertainty, aiming to maximize the less favorable possible outcomes. This can be formulated as:

$$\max_{f_m \in F} \min_{s_l \in S} (u(f_m(s_l))) \text{ or } \max_{f_m \in F} u_{f_m}(1)$$

Or, employing our domination representation (Eq. 5.1), it follows that since there exists a unique $X \in \mathcal{P}(S)$ for which $|X| = |S|$, namely S itself, we deduce that:

$$\max_{f_m \in F} \max_x \left\{ x : x \in \bigcap_{s_l \in S} u(f_m(s_l))^c \right\} \quad (5.2)$$

Conversely, the “maximax” approach is based on the maximum order statistic, or the $|S|^{th}$ order statistic, implying the pursuit of the best possible outcome among all conceivable scenarios. This method represents an optimistic or aggressive perspective, where the goal is to capture the most advantageous opportunity, despite potential risks. It can be formulated as:

$$\max_{f_m \in F} \max_{s_l \in S} (u(f_m(s_l))) \text{ or } \max_{f_m \in F} u_{f_m}(|S|)$$

Adopting our domination representation framework (Eq.5.1), it is evident that for the set $X \in \mathcal{P}(S)$, where $|X| = 1$, each X comprises a singular scenario s_l , so we deduce the following maximization problem:

$$\max_{f_m \in F} \max_x \left\{ x : x \in \bigcup_{s_l \in S} (u(f_m(s_l))^c) \right\} \quad (5.3)$$

It is important to note that any rational PM should conceptualize the outcomes of any given act within the spectrum bounded by the minimum and maximum approaches.

Example 5.1 (Cont) *The order statistic of f_1 and f_2 are $u_{f_1}(1) = 3, u_{f_1}(2) = 4, u_{f_1}(3) = 8$ and $u_{f_2}(1) = u_{f_2}(2) = u_{f_2}(3) = 5.5$.*

When employing the MaxiMin criterion, we select the worst-case scenario for each alternative, which corresponds to the order statistic $u_{f_m}(1)$. This results in values of 3 for f_1 and 5.5 for f_2 . Accordingly, f_2 is chosen due to its higher value.

Alternatively, employing the MaxiMax criteria involves considering the best-case scenario for each alternative, which corresponds to the order statistic $u_{f_m}(3)$. This leads to values of 8 for f_1 and 5.5 for f_2 . In this case, f_1 is selected owing to its higher value.

Lastly, we can use other order statistic based criterion, for instance if $|S|$ is odd, the median is the $\frac{|S|+1}{2}^{th}$ order statistic. When using the Max Median criteria. This yields values of 4 for f_1 and 5.5 for f_2 . Hence, f_2 is chosen due to its higher value. $\triangle$

Methods that rely on order statistic have some limitations. Firstly, these methods do not allow for compensation between states. These approaches also only use the utilities of the consequences $u : Y \mapsto \mathbb{R}$ as ordinal scale. However, if $u : Y \mapsto \mathbb{R}$ contains a higher degree of information (for instance, ratio or interval scale), more meaningful analysis could be conducted [roberts1985measurement]. Furthermore, the MaxiMin criteria is often seen as a bias towards the status quo, as it may inherently favor decisions that maintain current conditions or involve less deviation from known states. The MaxiMax approach, on the other hand, can be seen as a particularly risky approach.

Each criterion consists of two components: the first part, which simplifies the act along the scenario dimension, and the second part, which involves selecting the

optimal act based on the condensed information. The following section aims to develop an equivalent for the first component, employing similar framework but adapted for aggregating capability sets.

5.3 Capability order statistic

In this section, we aim to adapt the order statistic decision rules discussed in Section 5.2.1 to the context of capability sets.

Let us define a method to find the k^{th} capability set of an act denoted $\mathbf{A}(k)$, using an adaptation of Eq. 5.1:

$$\mathbf{A}(k) = \max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in \bigcup_{\substack{X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(\bigcap_{\forall s_l \in X} \mathbf{A}_l^c \right) \right\} \quad (5.4)$$

or equivalently

$$\mathbf{A}(k) = \max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in \bigcup_{\substack{X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(\bigcap_{s_l \in X} \left(\bigcup_{\vec{b} \in \mathbf{A}_l} \vec{b}^c \right) \right) \right\} \quad (5.5)$$

The resolution of Equation (5.4) results in a Pareto frontier. The formulation closely mirrors that of Eq. 5.1, with the stipulation $|X| = |S| + 1 - k$ ensuring the consideration of subsets of S that contain exactly $|S| + 1 - k$ elements. The expression $\mathbf{A}_l^c$ denotes our focus on the domain dominated by $\mathbf{A}_l$, capturing all potential beings that are no better than those within $U(f_m(s_l))$.

In the case where the sets in $f_m(S)$ are composed of a finite set of beings such that $U(f_m(s_l))$ for any given s_l contains a definite number n_l^* of beings, represented as $\{\vec{b}_{l,1}, \dots, \vec{b}_{l,n_l^*}\}$, then the application of a MI-MOLP is appropriate. For ease of comprehension, the program is presented in a step-by-step manner:

Firstly, as derived from Equation (5.5), the condition $\vec{x} \in \vec{b}^c$ can be reformulated as: $x_h \leq b_h \forall h$, where $x_h \in \mathbb{R} \forall h$.

Secondly, the expression $\vec{x} \in \bigcup_{\vec{b} \in U(f_m(s_l))} \vec{b}^c$ is detailed as:
 $x_h \leq b_{l,n,h} + M \cdot \delta_{l,n} \forall n, \forall h$, with $\sum_{n=1}^{n_l^*} \delta_{l,n} \leq n_l^* - 1$, where $x_h \in \mathbb{R} \forall h$, $\delta_{l,n} \in$

$\{0, 1\} \forall n$, and M represents a sufficiently large value.

Consequently, the condition $\vec{x} \in \bigcap_{\forall s_l \in S} \left(\bigcup_{\vec{b} \in \mathbf{A}_l} \vec{b}^c \right)$ is encapsulated by:

$x_h \leq b_{l,n,h} + M \cdot \delta_{l,n} \forall n, \forall h, \forall l$, and $\sum_{n=1}^{n_l^*} \delta_{l,n} \leq n_l^* - 1 \forall l$, with $x_h \in \mathbb{R} \forall h$, $\delta_{l,n} \in \{0, 1\} \forall n, \forall l$, and M a sufficiently large number.

And finally, Equation 5.5 can be interpreted as the following MI-MOLP:

$$PF(x_1, x_2, \dots, x_{h^*})$$

s.t.

$$x_h \leq b_{l,n,h} + M \cdot \delta_{l,n} \quad \forall l, \forall h, \forall n \quad (5.6a)$$

$$\sum_{n=1}^{n_l^*} \delta_{l,n} \leq n_l^* - d_l \quad \forall l \quad (5.6b)$$

$$|S| + 1 - k \leq \sum_{l=1}^{l^*} d_l \quad (5.6c)$$

$$x_h \in \mathbb{R} \quad \forall h \quad (5.6d)$$

$$\delta_{l,n} \in \{0, 1\} \quad \forall l, \forall n \quad (5.6e)$$

$$d_l \in \{0, 1\} \quad \forall l \quad (5.6f)$$

With M a sufficiently large number

With $PF()$ designating the research of the Pareto frontier.

The complexity of the problem increases notably as k approaches $\frac{|S|}{2}$ and as the size of S increases, primarily due to a *combinatorial explosion in the number of subsets* X that need to be evaluated, since the number of sets $X \in \mathcal{P}(S)$ of size $|X| = |S| - k + 1$ is equal to $\binom{|S|}{|S|-k+1}$. Moreover, the complexity of solving such a problem escalates with the expansion in the *number of dimensions* h^* [ehrgott2005multicriteria]. Finally, the *size of the capability sets* also affects the computational complexity. Indeed, we introduce $\sum_{l=1}^{l^*} n_l^*$ decision variables $\delta_{l,n}$, where n_l^* represents the number of solutions in each capability set $\mathbf{A}_l \in f_m(S)$. Thus, larger capability sets involve more decision variables and constraints, increasing computational demands. If solving such MI-MOLP becomes computationally too hard, we may need to resort to heuristics as such in [branke2016mcda, chugh2019survey, deb2002fast].

Example 5.2 Consider a decision-making framework involving two actions and five states of the world, with the associated capability sets depicted in Table 5.2 and illustrated in Figure 5.2. Within the capability set $\mathbf{A}_1$, a citizen has the option to select any beings from the set $\{(1, 14), (3, 8), (10, 6)\}$. In Figure 5.2, the lines indicate Pareto domination among the various capability sets, implying that all beings below the line is dominated by the capability set. For simplicity in this example, the action f_2 is represented as a “sure” act leading to the capability set $\mathbf{B}$ regardless of the true state of world.

$\mathbf{A}_1$	$\{(1, 14)(3, 8)(10, 6)\}$
$\mathbf{A}_2$	$\{(2, 13)(3, 11)(6, 8)(7, 1)\}$
$\mathbf{A}_3$	$\{(5, 10)(8.5, 5)\}$
$\mathbf{A}_4$	$\{(5.5, 9)(9, 4)(11, 2)\}$
$\mathbf{A}_5$	$\{(4, 12)(7, 7)(8, 3)\}$
$\mathbf{B}$	$\{(4.5, 9.5)(6.5, 6.5)(9.5, 1.5)\}$

Table 5.2: Example of Capability sets

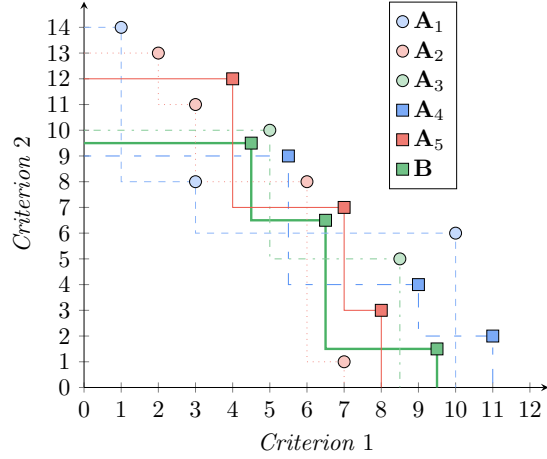


Figure 5.2: Graphical representation of Table 5.2

Figure 5.3 illustrates the first and last order statistic of f_1 and f_2 . Using the maximin approach, f_2 is preferred over f_1 , whereas the maximax criterion reveals a preference for f_1 over f_2 .

When considering the median, as shown in Figure 5.5 (representing the third order statistic), the choice between f_1 and f_2 becomes less clear due to the lack of a definitive advantage for either option.

Similar ambiguity is observed for the second and fourth orders, depicted in Figure 5.4. However, as the order decreases, f_2 tends to be preferred over f_1 , and the opposite trend is noted as the order increases. $\triangle$

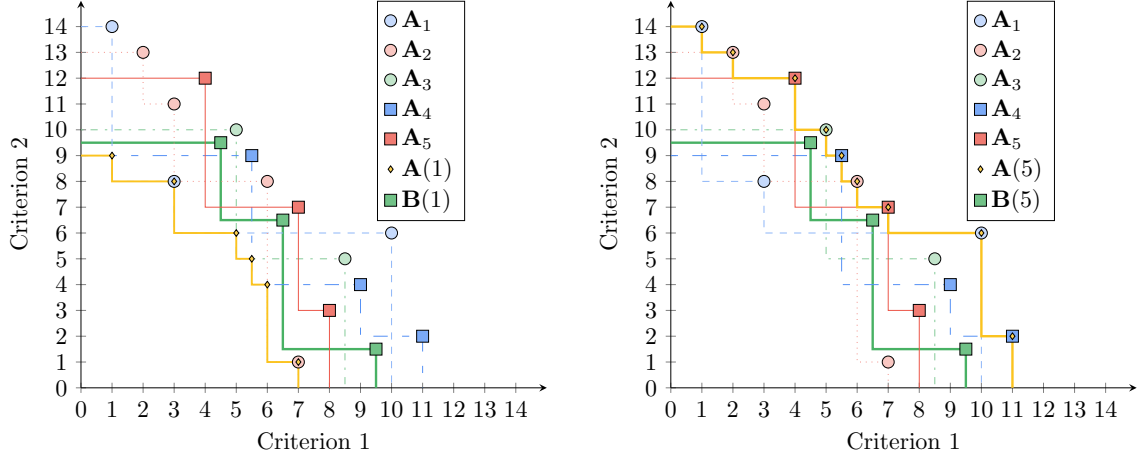


Figure 5.3: Example for the first and last capability order statistic

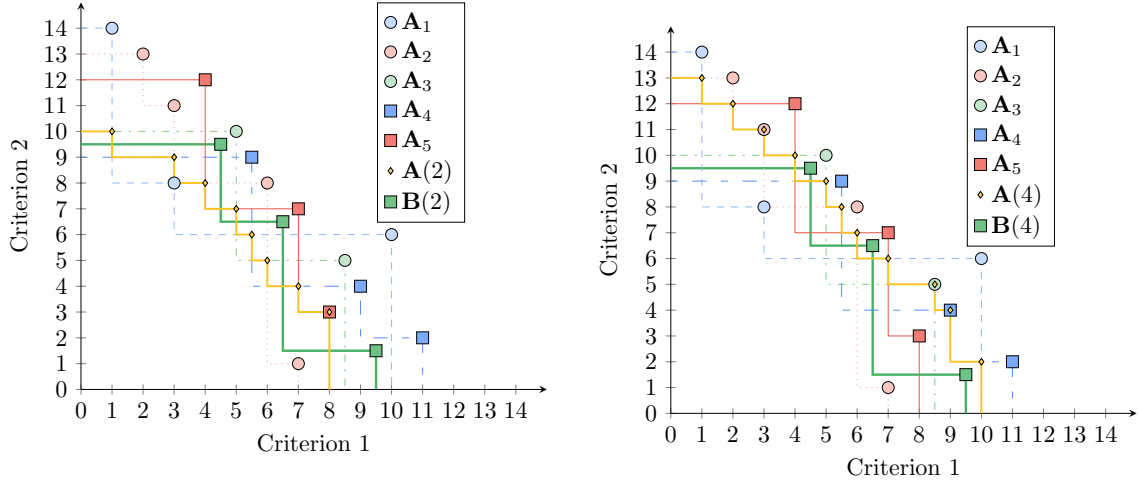


Figure 5.4: Example of the second and fourth capability order statistic

Let us now consider some interesting propositions respected by the capability order statistics, as detailed in Equation 5.5, with proofs provided in the appendix.

A key proposition endorsed by our framework is that it reflects the traditional formulation of order statistics.

Proposition 5.1 (Dominance Consistency) *Given any $\vec{x} \in \mathbf{A}(k)$:*

- (a) *For at least $|S| - k + 1$ states of the world, referred to as $s_l \in S$, there is a vector $\vec{b} \in \mathbf{A}_l$ satisfying $\vec{b} \geq \vec{x}$.*

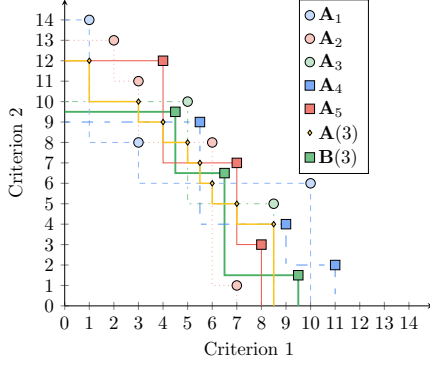


Figure 5.5: Example for the third capability order statistic

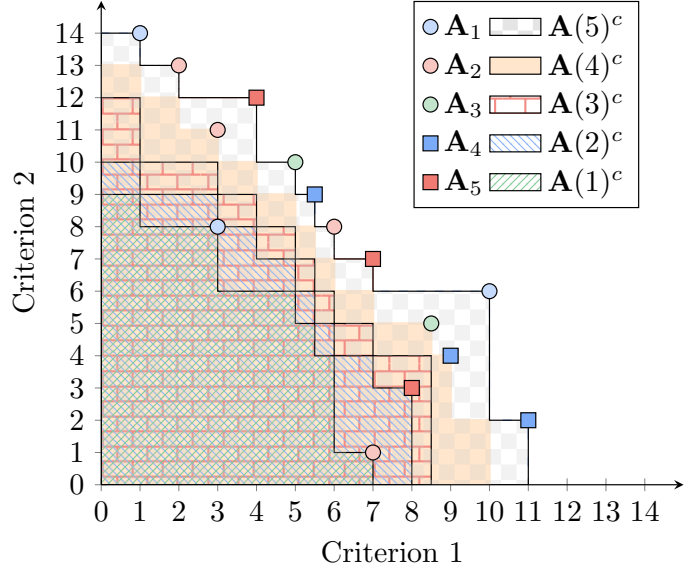


Figure 5.6: Summarising the capability order statistic of $\mathbf{A}$.

- (b) *There are at most $k - 1$ states of the world $s_l \in S$, where no vector $\vec{b} \in \mathbf{A}_l$ meets the condition $\vec{b} \geq \vec{x}$.*

A second proposition, resonating with the classical principles of order statistics, asserts that for any pair (k, k') within $\{1, \dots, |S|\}^2$, where $k' < k$, it follows that $u_{f_m}(k') \leq u_{f_m}(k)$.

Proposition 5.2 (Monotonicity) *For all pairs $(k, k') \in \{1, \dots, |S|\}^2$ with $k' < k$, it holds that $\mathbf{A}(k') \subseteq \mathbf{A}(k)^c$.*

A property within classical order statistic is that for every $u_{f_m}(k)$, there exists a corresponding $u(f_m(s_l)) \in u(f_m(S))$ such that $u_{f_m}(k) = u(f_m(s_l))$. Building upon this, we introduce an adaptation that incorporates the notion of an order among capability sets. Specifically, if the capability sets $\mathbf{A}$ can be systematically ordered (*i.e.* there is a relation of dominance between all of them), then the order statistic should correspond to an existing capability set within $\mathbf{A}$.

Proposition 5.3 (Bijection Consistency) *For any two states $(s_l, s_{l'}) \in S^2$, if one of the two conditions always holds:*

$$\text{either } \mathbf{A}_l \subseteq \mathbf{A}_{l'}^c \text{ or } \mathbf{A}_{l'} \subseteq \mathbf{A}_l^c$$

then we can define a bijective function $\sigma : S \rightarrow \{1, 2, \dots, |S|\}$ such that

$$\sigma(s_l) \leq \sigma(s_{l'}) \text{ whenever } \mathbf{A}_l \subseteq \mathbf{A}_{l'},$$

then, for each $k \in \{1, \dots, |S|\}$, the equality $\mathbf{A}(k) = \mathbf{A}(\sigma^{-1}(k))$ holds.

Finally, if capability sets are unidimensional and contain only one beings, then the capability order statistic is equivalent to the classical order statistic.

Proposition 5.4 (Consistency with classical order statistic) *If $h^* = 1$ and $|\mathbf{A}_l| = 1$ for all $s_l \in S$, then the capability order statistic $\mathbf{A}(k)$ is equivalent to the classical order statistic $u_{f_m}(k)$.*

5.4 Proofs

Proof of Proposition 5.1: The first part of this property is derived from the definition of $U_{f_m}(k)$, given as:

$$\mathbf{A}(k) = \max_x \left\{ x : x \in \bigcup_{\substack{\forall X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(\bigcap_{s_l \in X} \left(\bigcup_{\vec{b} \in U(f_m(s_l))} \vec{b}^c \right) \right) \right\}.$$

There must exist at least one subset $X \subseteq S$ where $\vec{x}$ is included in the intersection $\bigcap_{s_l \in X} \left(\bigcup_{\vec{b} \in \mathbf{A}_l} \vec{b}^c \right)$. It is guaranteed that for every $s_l \in X$, at least one being $\vec{b}$ exists within $\mathbf{A}_l$ such that $\vec{b} \geq \vec{x}$. Given that $|X| = |S| - k + 1$, this proves the first part.

For the second part, we define K as a set containing k states of the world $s_l \in S$, where no vector $\vec{b}$ in $\mathbf{A}_l$ satisfies $\vec{b} \geq \vec{x}$. None of the states s_l in K can be part of X ; inclusion would imply that there exist a $\vec{b} \in \mathbf{A}_l$ such that $\vec{x} \in \vec{b}^c$ for some $s_l \in K$, contradicting the definition of K . Hence, X must be a subset of $S \setminus K$. However, $|S \setminus K| = |S| - k$ is smaller than $|S| - k + 1$ (the size of X), necessitating the inclusion of at least one state $s_l \in K$ in X to meet the size requirement $|X| = |S| + 1 - k$. Consequently, the cardinality of K must be at most $k - 1$. ■

Proof of Proposition 5.2: Assume towards a contradiction that $\mathbf{A}(k') \not\subseteq \mathbf{A}(k)^c$. This implies the existence of some vector $\vec{x}' \in \mathbf{A}(k')$ for which no corresponding vector $\vec{x} \in \mathbf{A}(k)$ satisfies $\vec{x} \geq \vec{x}'$.

However, every vector $\vec{x}' \in \mathbf{A}(k')$ is within $\bigcap_{s_l \in X'} \left(\bigcup_{\vec{b} \in \mathbf{A}} \vec{b}^c \right)$, where $X' \subseteq S$ and $|X'| = |S| - k'$.

Given $k > k'$, we can construct a set $X \subseteq S$ with $|X| = |S| - k < |S| - k'$, such that $X \subseteq X'$. This construction ensures that for every $\vec{x}' \in \mathbf{A}(k')$, there exists a $\vec{x} \in \mathbf{A}(k)^c$ satisfying $\vec{x} = \vec{x}'$, which contradicts the initial assumption. ■

Proof of Proposition 5.3: We begin with the ordered sequence of capability sets under transformation σ :

$$\mathbf{A}(\sigma^{-1}(1))^c \subseteq \mathbf{A}(\sigma^{-1}(2))^c \subseteq \dots \subseteq \mathbf{A}(\sigma^{-1}(|S|))^c,$$

This implies that for any two states $s_l, s_{l'}$ in S with $\sigma(s_l) \leq \sigma(s_{l'})$, the intersection of their dominated spaces aligns with the lower ordered one:

$$\begin{aligned} \forall (s_l, s_{l'}) \in S^2, \text{ with } \sigma(s_l) \leq \sigma(s_{l'}) : \\ \mathbf{A}_l^c \cap \mathbf{A}_{l'}^c = \mathbf{A}_l^c \end{aligned}$$

For any non-empty subset $X \subseteq S$, we have:

$$\forall X \in \mathcal{P}(S) \setminus \{\emptyset\} : \bigcap_{s_l \in X} (U(f_m(s_l)) - \mathbb{R}_+^{h*}) = U(f_m(\sigma^{-1}(\arg \min_{s_l \in X} \sigma(s_l)))) - \mathbb{R}_+^{h*}$$

Considering the maximization over vector $\vec{x}$, and recognizing that each capability $U(f_m(s_l))$ for $s_l \in S$ forms a Pareto frontier, we deduce:

$$\max_{\vec{x}} \vec{x} \in \left\{ \bigcap_{s_l \in X} (U(f_m(s_l)) - \mathbb{R}_+^{h*}) \right\} = U(f_m(\sigma^{-1}(\arg \min_{s_l \in X} \sigma(s_l))))$$

Hence, for each k :

$$u_{f_m}(k) = \max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in \bigcup_{\substack{X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(U(f_m(\sigma^{-1}(\arg \min_{s_l \in X} \sigma(s_l)))) \right) \right\}$$

We observe that within any pair of states $(s_l, s_{l'}) \in S^2$ with $\sigma(s_l) \leq \sigma(s_{l'})$, the

union of their dominated spaces is effectively determined by the greater ordered state:

$$\begin{aligned} & \forall (s_l, s_{l'}) \in S^2, \text{ given } \sigma(s_l) \leq \sigma(s_{l'}) : \\ & \max_{\vec{x}} \vec{x} \in \{U(f_m(s_l)) \cup U(f_m(s_{l'}))\} = \max_{\vec{x}} \vec{x} \in U(f_m(s_{l'})) \end{aligned}$$

For any subset $X \subseteq S$ with $|X| = |S| + 1 - k$, there exists a particular subset $X' = \{\sigma^{-1}(|S|), \sigma^{-1}(|S| - 1), \dots, \sigma^{-1}(k)\}$ also of size $|S| + 1 - k$, such that:

$$\arg \min_{s_l \in X} \sigma(s_l) \leq \arg \min_{s_l \in X'} \sigma(s_l) = \sigma^{-1}(k)$$

Given the ordered sequence, the union of dominated spaces for any subset $X \in \mathcal{P}(S)$ with $|X| = |S| + 1 - k$ yields $U(f_m(\sigma^{-1}(k)))$ as the highest order set within the union:

$$\begin{aligned} u_{f_m}(k) &= \max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in \bigcup_{\substack{X \in \mathcal{P}(S) \\ |X|=|S|+1-k}} \left(U(f_m(\sigma^{-1}(\arg \min_{s_l \in X} \sigma(s_l)))) \right) \right\} \\ &= \max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in U(f_m(\sigma^{-1}(k))) - \mathbb{R}_+^{h^*} \right\} \end{aligned}$$

Given the Pareto optimality of each $U(f_m(s_l))$ for $s_l \in S$ and the maximization of $\vec{x}$,

$$\max_{\vec{x}} \left\{ \vec{x} : \vec{x} \in U(f_m(\sigma^{-1}(k))) - \mathbb{R}_+^{h^*} \right\} = U(f_m(\sigma^{-1}(k)))$$

this leads to the conclusion that $U_{f_m}(k) = U(f_m(\sigma^{-1}(k)))$. ■

Proof of Proposition 5.4: Proposition 5.1 establishes a direct equivalence to the classical order statistic problem. This equivalence is further supported by Proposition 5.3, which affirm the bijective consistency between capability sets and classical outcomes in a single-dimensional and single-outcome context. ■

5.5 Conclusion

We have introduced a new method for defining order statistics in the context of capability sets, resulting in promising propositions. The Hurwicz rule, which involves a weighted expected capability set between the first and last order statistics, and the Laplace rule, which calculates the expected capability set assuming uniform probability across all states, are both compatible with our approach with the expected capability set approach developed in Chapter 6. These methods are detailed in Appendix C.

However, the Savage minimax regret rule is not directly adaptable from our work. This limitation arises from the unique characteristics of the Savage criterion, where an action f_m is not merely aggregated based on its outcomes across various states but also requires comparative evaluation against other potential actions. This distinction opens a promising line of research, suggesting further exploration into decision rules under uncertainty and their application to capability sets.

In this chapter, we have operated without information about the likelihood of each state being the true state. In the following Chapter 6, we introduce insights about the likelihood of each state of the world being the true one.

Chapter 6

Capability sets under risk

6.1 Introduction

The framework is the same as in Chapter 5, except we now have subjective probabilities $p(s_l)$ for each scenario s_l . We will use the same notation.

Under classical assumptions [**savage1972foundations**], decision support within F is carried out as follows. For a given act and state of the world, there exists a unique consequence $f_m(s_l)$. There is also a utility function u such that $u(f_m(s_l)) \in \mathbb{R}$ for all $f_m \in F$ and $s_l \in S$, where $f_{m'}$ is at least as preferred as f_m if and only if the expected utility of $f_{m'}$ is not less than that of f_m :

$$E(f_{m'}) = \sum_{l=1}^{l^*} p(s_l) \cdot u(f_{m'}(s_l)) \geq \sum_{l=1}^{l^*} p(s_l) \cdot u(f_m(s_l)) = E(f_m).$$

Consequently, the policy maker is recommended to select the act that maximizes expected utility:

$$\max_{f_m \in F} (E(u(f_m))).$$

The major difference in our framework is that we use capability sets $U(f_m(s_l)) \subseteq \mathbb{R}_+^{h^*}$, instead of utilities $u(f_m(s_l)) \in \mathbb{R}$. Section 6.2.1 introduces a natural approach toward mixing capability sets, termed *average capability sets*. In Section 6.2.2, we propose a more sophisticated approach called the *expected capability set*. Section 6.3 delves into the major propositions of the expected capability set. Section 6.4 further explores the distinctions between both approaches, expanding our understanding of the possibilities and considerations involved in decision-making within the capability approach context.

6.2 Average and Expected capability sets

In this section, we propose two ways to mix capability sets. One approach, referred to as the *average capability set* seems quite natural, yet does not respect some desirable property. The other one, known as the *expected capability set*, is more sophisticated and generally more relevant to our problem at hand.

6.2.1 Average capability set

A natural approach to addressing this problem, involves considering every combination of beings from various capability sets. These combinations are then aggregated based on the probability of their corresponding states of the world. This method of aggregation is referred to as the *average capability set*, denoted by $\bar{\mathbf{A}}$ for an act f_m :

$$\bar{\mathbf{A}} = \left\{ \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}, \text{ for all } X_i \text{ such that } X_i = (\vec{b}_1, \dots, \vec{b}_{l^*}) \right. \\ \left. \text{with } \vec{b}_l \in \mathbf{A}_l \text{ for every } l \in \{1, \dots, l^*\} \right\}.$$

It is a natural extension of the expected utility concept and is consistent with it: if the capability sets include just one beings and are only assessed using one dimension, then the average capability sets are equivalent to expected utilities (see proof in Appendix D). If our goal is to identify and retain only the efficient Pareto set of the set $\bar{\mathbf{A}}$, denoted as $PF(\bar{\mathbf{A}})$, we calculate it by solving Problem 6.1

$$\begin{aligned} PF(\bar{\mathbf{A}}) = PF \quad & \left(\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \dots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l) \right) \\ \text{s.t.} \quad & \vec{b}_l \in \mathbf{A}_l \quad \forall l \in \{1, \dots, l^*\}. \end{aligned} \tag{6.1}$$

Generally, reducing to the Pareto frontier is not mandatory. However, it serves as a step in elucidating the concept of the expected capability set and facilitates the comparison of both approaches. The decision to use or not the Pareto frontier will be discussed in Section 6.4.

Example 6.1 Consider a case where $U(f_m(s_1)) = \mathbf{A}_1 = \{(2, 7), (3, 4)\}$ and $U(f_m(s_2)) = \mathbf{A}_2 = \{(4, 3), (7, 2)\}$. Should s_1 be the actual state, a citizen would choose between solutions $(2, 7)$ and $(3, 4)$, whereas if s_2 was the actual one, such citizen would choose between $(4, 3)$ and $(7, 2)$. With $p(s_1) = p(s_2) = 0.5$, the average capability set $\bar{\mathbf{A}}$ is $\{(3, 5), (3.5, 3.5), (4.5, 4.5), (5, 3)\}$ and its Pareto frontier $PF(\bar{\mathbf{A}})$ is $\{(3, 5), (4.5, 4.5), (5, 3)\}$. Refer to Table 6.1 for the steps used to construct this set and Figure 6.1 for a visual depiction.

X_i	$\vec{b} \in \overline{\mathbf{A}}$
$\{(2, 7), (4, 3)\}$	$(\mathbf{3}, \mathbf{5})$
$\{(2, 7), (7, 2)\}$	$(\mathbf{4.5}, \mathbf{4.5})$
$\{(3, 4), (4, 3)\}$	$(\mathbf{3.5}, \mathbf{3.5})$
$\{(3, 4), (7, 2)\}$	$(\mathbf{5}, \mathbf{3})$

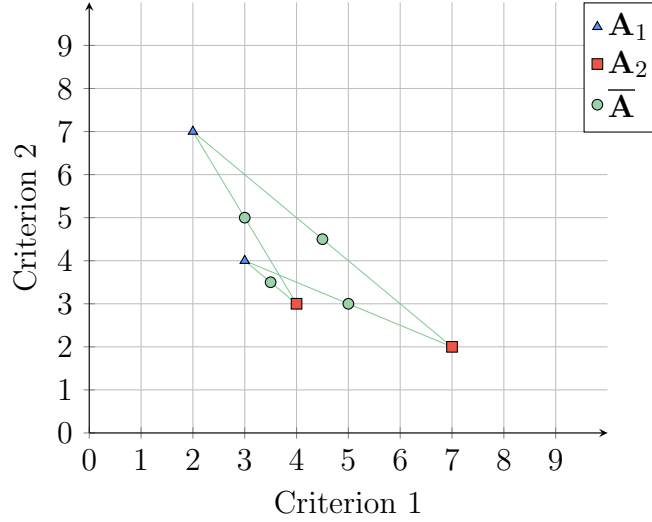
Table 6.1: Construction of solutions of $\overline{\mathbf{A}}$ 

Figure 6.1: Example 6.1: Average capability set

It is important to note that this mixed solution includes the possibility of achieving $(3.5, 3.5)$, which is dominated by $(4.5, 4.5)$. Additionally, in this particular case, the average capability set is not dominated by any beings from $\mathbf{A}_1$ or $\mathbf{A}_2$. In fact, no beings within the average capability set is dominated by beings from either $\mathbf{A}_1$ or $\mathbf{A}_2$.

△

Although the average capability set provides a straightforward method for aggregating sets of capabilities, it will prove inadequate in our problem. In the following subsection, we demonstrate the need to impose stronger properties on such a procedure, which we will refer to as the *expected capability set*. Nevertheless, as Section 6.4 explores, the average capability set can be useful for a problem that slightly differs from ours.

6.2.2 Expected capability set

Let us introduce two properties that naturally capture any citizens' preferences in relation to capability sets that will inform our discussion. The first one, called *greater choice* property, reflects the fundamental property within the capability approach that citizens prefer to have access to more and better beings.

Property 6.1 (Greater choice) *For all capability sets $\mathbf{A}$ and $\mathbf{B}$, it holds that $\mathbf{A} \cup \mathbf{B}$ is at least as preferred as $\mathbf{A}$.*

By iteratively applying this property, we find that the union of the capability sets resulting from an act f_m across all the states, denoted $\cup_{l=1}^{l^*} \mathbf{A}_l$, should be at least as preferred as the capability set corresponding to each individual state. Consequently, we should also expect that this union will be at least as preferred as the expected capability set, denoted $E(\mathbf{A})$.

In other words, similar to the classical framework, the last order statistic is at least as good as the expected outcomes.

The second one, designated the *fewer choice* property, captures another aspect of citizens' preferences: for any capability set, citizens do not prefer having less choice.

Property 6.2 (Fewer choice) *For all capability sets $\mathbf{A}$ and $\mathbf{B} \subseteq \mathbb{R}_+^{h^*}$, it holds that $\mathbf{A}$ is at least as preferred as $\mathbf{A}^c \cap \mathbf{B}^c$.*

Based on this property, the capability set associated with each individual state should not be less preferred to the intersection of the solutions dominated by the capability sets resulting from an act f_m across all states, that is $\cap_{l=1}^{l^*} \mathbf{A}_l^c$. Accordingly, the expected capability sets $E(\mathbf{A})$ should be not less preferred than such set.

In other words, similar to the classical framework, the first order statistic is at least as good as the expected outcomes.

A simple example illustrates these properties.

Example 6.1 (Cont) *Figure 6.2 depicts the capability sets $\mathbf{A}_1$ and $\mathbf{A}_2$, as well as the space dominated by the solutions in the union of the capability sets under both states, denoted $(\mathbf{A}_1 - \mathbb{R}_+^2) \cup (\mathbf{A}_2 - \mathbb{R}_+^2)$, in yellow dots, and the space that can be*

dominated in any state, denoted by $(\mathbf{A}_1 - \mathbb{R}_+^2) \cap (\mathbf{A}_2 - \mathbb{R}_+^2)$, in gray. The average capability set is not contained within the yellow dotted area, so it does not fulfill Property 6.1.

Following Property 6.1, the expected capability set $E(\mathbf{A})$ should be contained within the yellow dotted area, which is not the case for the average capability set. The space above the yellow dotted area represents the beings that we are sure not to be able to dominate, regardless of the actual state of the world. On the other hand, following Property 6.2, the gray area should be dominated by $E(\mathbf{A})$, since in any state, we should find a solution dominating such area and, therefore, are sure to be able to dominate it. $\triangle$

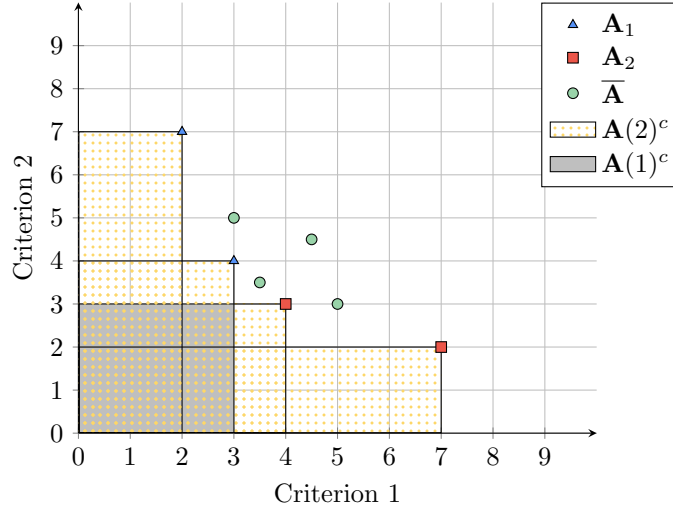


Figure 6.2: Example 6.1 used to illustrate Property 1 and 2.

Our objective is thus to establish a mixing procedure that ensures the condition

$$\bigcap_{l=1}^{l^*} \mathbf{A}_l^c \subseteq E(\mathbf{A})^c \subseteq \bigcup_{l=1}^{l^*} \mathbf{A}_l^c.$$

or

$$\mathbf{A}(1) \subseteq E(\mathbf{A})^c \subseteq \mathbf{A}(l^*)^c$$

Note that this condition is not a definition of the expected capability set $E(\mathbf{A})$, but rather a crucial requirement that such set must fulfill to be considered effective in our context. To operationalize it, we propose a model ensuring that for any $\vec{b} \in E(\mathbf{A})$,

there exists $\vec{b}' \in \bigcup_{l=1}^{l^*} \mathbf{A}_l$ such that $\vec{b} \leq \vec{b}'$ (i.e., $b_h \leq b'_h$ for all h). Additionally, for any $\vec{b}''$ in $\bigcap_{l=1}^{l^*} \mathbf{A}_l^c$, there exists $\vec{b} \in E(\mathbf{A})$ such that $\vec{b}'' \leq \vec{b}$.

To do this, we shall provide several mathematical programming formulations to define and compute $E(\mathbf{A})$. The first formulation, referred to as Problem 6.2, is designed to define the expected capability set. Within this framework, beings $\vec{b}_l$ are not contained in $\mathbf{A}_l$ as in the averaging Problem 6.1, but are just weakly Pareto dominated by $\mathbf{A}_l$ and are combined, using the probabilities associated with their specific states as in Problem 6.1. Additionally, a necessary condition for aggregation is imposed: to preserve weak Pareto dominance across beings in different states. This process ensures that the resulting expected capability set fulfills the desired two properties. Specifically, the problem is formulated as

$$E(\mathbf{A}) = PF \left(\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \dots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l) \right) \quad (6.2a)$$

$$\vec{b}_l \in \mathbf{A}_l^c \quad \forall l \in \{1, \dots, l^*\}$$

$$\vec{b}_l \geq \vec{b}_{l'} \text{ or } \vec{b}_l \leq \vec{b}_{l'} \quad \forall l, l' \in \{1, \dots, l^*\}, l \neq l'. \quad (6.2b)$$

Condition (6.2a) directs our focus towards mixing all beings *dominated* by $\mathbf{A}_l$ for all $s_l \in S$, rather than just the beings of $\mathbf{A}_l$. On the other hand, condition (6.2b) stipulates the existence of a total order among all $\vec{b}_l$ (whereby $\vec{b}_l \geq \vec{b}_{l'}$ if and only if $b_{l,h} \geq b_{l',h}$ for all $h \in \{1, \dots, h^*\}$). This implies that any two beings $\vec{b}_l \in \mathbf{A}_l, \vec{b}_{l'} \in \mathbf{A}_{l'}$ which are incomparable (i.e., neither $\vec{b}_l \geq \vec{b}_{l'}$ nor $\vec{b}_l \leq \vec{b}_{l'}$) are inappropriate to lead to an expected solution of the type $\vec{b} = p(s_l) \cdot \vec{b}_l + p(s_{l'}) \cdot \vec{b}_{l'}$. Note that in the classical settings [savage1972foundations] with one-dimensional utilities ($h^* = 1$) and a single beings in the capability sets, all beings are comparable as $\geq$ is a total order on $\mathbb{R}$.

While Problem 6.2 has been employed to elucidate the concept of expected capability set, we still need to provide computational schemes to build them. When dealing with a collection of capability sets $\mathbf{A}_l$ (with the different $\mathbf{A}_l$ possibly obtained through a MI-MOLP as in Chapter 2 and 3), we turn to a second, more operational formulation to derive $E(\mathbf{A})$. It is based on a MI-MOLP, referred to as Problem 6.3, where M represents a large number.

$$E(\mathbf{A}) = PF \left(\sum_{l=1}^{l^*} b_{l,1} \cdot p(s_l), \dots, \sum_{l=1}^{l^*} b_{l,h^*} \cdot p(s_l) \right)$$

s.t.

$$b_{l,h} \leq z_{l,h} \quad \forall l \in \{1, \dots, l^*\} \quad (6.3a)$$

$$\forall h \in \{1, \dots, h^*\}$$

$$\vec{z}_l \in \mathbf{A}_l \quad \forall l \in \{1, \dots, l^*\} \quad (6.3b)$$

$$d_{l,l'} \in \{0, 1\} \quad l \neq l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.3c)$$

$$b_{l,h} \leq b_{l',h} + d_{l,l'} \cdot M \quad l \neq l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.3d)$$

$$d_{l,l'} + d_{l',l} \leq 1 \quad l < l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.3e)$$

$$b_{l,h} \in \mathbb{R} \quad \forall l \in \{1, \dots, l^*\} \quad (6.3f)$$

$$\forall h \in \{1, \dots, h^*\}$$

In this formulation, inequality (6.3a) guarantees that we aggregate vectors $\vec{b}_l$ that are dominated by a solution $\vec{z}_l$ in $\mathbf{A}_l$, as expressed by (6.3b) which compiles the constraints to find the capability set of $\mathbf{A}_l$, as in Chapter 2. To establish a total order among the solutions $\vec{b}_l = (b_{l,1}, \dots, b_{l,h^*})$, we introduce binary decision variables $d_{l,l'}$ using Equations (6.3c). These variables ensure that if $d_{l,l'} = 0$, then $b_{l,h} \leq b_{l',h}$ for all h , as in inequality (6.3d). We ensure that this order exists between all solutions $\vec{b}_l$ by using inequality (6.3e).

Two key factors influence the computational complexity of Problem 6.3. First, the *number l^* of states of the world* plays a significant role. Indeed, $\sum_{l=1}^{l^*-1} l \times 2$ decision variables $d_{l,l'}$ are introduced to establish a total order between the weakly Pareto dominated solutions in each state. Thus, the bigger the number of states, the more decision variables and constraints need to be considered. Second, the *number h^* of welfare dimensions* affects the complexity of finding the expected Pareto frontier, a crucial aspect in our framework. When the dimensionality of the criteria is large, computational demands become significant, scaling typically exponentially with the number of criteria [ehrgott2005multicriteria].

If the capability sets are finite $\mathbf{A}_l = \{z_{l,1}, \dots, z_{l,n_l^*}\}$, the expected capability set $E(\mathbf{A})$ can be found through the following MI-MOLP (Problem 6.4). Where, again,

M is a sufficiently large number

$$E(\mathbf{A}) = PF\left(\sum_{\forall l=1}^{l^*} b_{l,1} \cdot p(s_l), \quad \dots, \sum_{\forall l=1}^{l^*} b_{l,h^*} \cdot p(s_l)\right)$$

s.t.

$$\sum_{n=1}^{n_l^*} \delta_{l,n} \leq n_l^* - 1 \quad \forall l \in \{1, \dots, l^*\} \quad (6.4a)$$

$$b_{l,h} \leq z_{l,n,h} + \delta_{l,n} \cdot M \quad \forall l \in \{1, \dots, l^*\}, \quad (6.4b)$$

$$\forall n \in \{1, \dots, n_l^*\}$$

$$\forall h \in \{1, \dots, h^*\}$$

$$d_{l,l'} \in \{0, 1\} \quad l \neq l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.4c)$$

$$b_{l,h} \leq b_{l',h} + d_{l,l'} \cdot M \quad l \neq l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.4d)$$

$$d_{l,l'} + d_{l',l} \leq 1 \quad l < l' \quad l, l' \in \{1, \dots, l^*\} \quad (6.4e)$$

$$\delta_{l,n} \in \{0, 1\} \quad \forall l \in \{1, \dots, l^*\} \quad (6.4f)$$

$$\forall n \in \{1, \dots, n_l^*\}$$

$$b_{l,h} \in \mathbb{R} \quad \forall l \in \{1, \dots, l^*\} \quad (6.4g)$$

$$\forall h \in \{1, \dots, h^*\}$$

Constraints (6.4c), (6.4d) and (6.4e) establish an order among the aggregated beings as with constraints (6.3c), (6.3d), and (6.3e). To determine dominated beings, we no longer use constraints (6.3a) and (6.3b) but rather introduce new constraints (6.4a), (6.4b) and (6.4f). We introduce decision variables $\delta_{l,n}$ using constraint (6.4f). Constraint (6.3b) is now replaced by (6.4b). If $\delta_{l,n}$ equals 0, then the beings $\vec{b}_l = (b_{l,1}, \dots, b_{l,h^*})$ is considered weakly dominated by the n -th beings in the capability set $\mathbf{A}_l$. Finally, constraint (6.4a) ensures that at least one of the decision variables $\delta_{l,n}$ equals 0, indicating that $\vec{b}_l$ is dominated by at least one solution in $\mathbf{A}_l$.

In addition to the number of states and the number of dimensions, a third factor influences the computational complexity of Problem 6.4. The *size of the capability sets* affects computational complexity. We introduce $\sum_{l=1}^{l^*} n_l^*$ decision variables $\delta_{l,n}$, where n_l^* represents the number of solutions in each capability set $\mathbf{A}_l$. Thus, larger capability sets involve more decision variables and constraints, increasing

computational demands.

Our proposal provides insights into decision aiding under uncertainty in the context of the capability approach. However, it is important to consider carefully the above computational limitations when applying the proposed framework. Addressing these limitations presents a compelling line of research that could significantly improve the tractability of our model. As a last resort, we would need to draw on multiobjective heuristics as in [branke2016mcda, chugh2019survey, deb2002fast].

Example 6.2 Consider an act f_m with two states:

$$U(f_m(s_1)) = \mathbf{A}_1 = \{(3, 10), (4, 5), (7, 3), (8, 1)\} \text{ and}$$

$U(f_m(s_2)) = \mathbf{A}_2 = \{(2, 5), (5, 4), (10, 2)\}$. Figure 6.3 displays $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_1^c$ and $\mathbf{A}_2^c$ as all the points on and under the blue dotted (resp. red dashed) lines. It also shows $E(f_m)$ according to the formulation given in Problem 6.4, when $p(s_1) = 0.8$ (and $p(s_2) = 0.2$). We obtain $E(f_m) = \{(2.8, 9), (3, 8.8), (4, 4.8), (4.2, 4), (5, 3.2), (6.6, 3), (7, 2.8), (7.6, 2), (8.4, 1.2)\}$. As an example, $(2.8, 9)$ is obtained by aggregating $\vec{b}_1 = (3, 10)$ and $\vec{b}_2 = (2, 5)$. We have $\vec{b}_1 \leq \vec{z}_{1,1} = (3, 10)$ and $\vec{b}_2 \leq \vec{z}_{2,1} = (2, 5)$; since $\vec{b}_1 \geq \vec{b}_2$, we have $d_1, d_2 = (1, 0)$. The other solutions are as shown in Table 6.2. $\triangle$

$\vec{b} \in E(\mathbf{A})$	$\vec{z}_1$	$\vec{b}_1$	$\vec{z}_2$	$\vec{b}_2$	(d_1, d_2)
(2.8, 9)	(3, 10)	(3, 10)	(2, 5)	(2, 5)	(1, 0)
(3, 8.8)	(3, 10)	(3, 10)	(5, 4)	(3, 4)	(1, 0)
(4, 4.8)	(4, 5)	(4, 5)	(5, 4)	(4, 4)	(1, 0)
(4.2, 4)	(4, 5)	(4, 4)	(5, 4)	(5, 4)	(0, 1)
(5, 3.2)	(7, 3)	(5, 3)	(5, 4)	(5, 4)	(0, 1)
(6.6, 3)	(7, 3)	(7, 3)	(5, 4)	(5, 3)	(1, 0)
(7, 2.8)	(7, 3)	(7, 3)	(10, 2)	(7, 2)	(1, 0)
(7.6, 2)	(7, 3)	(7, 2)	(10, 2)	(10, 2)	(0, 1)
(8.4, 1.2)	(8, 1)	(8, 1)	(10, 2)	(10, 2)	(0, 1)

Table 6.2: Example 6.2: Calculation of $\mathbf{E}(\mathbf{A})$

Example 6.1 (Cont) Figure 6.4 depicts the expected capability set $E(f_m)$ when $p(s_1) = p(s_2) = 0.5$. Notably, $E(\mathbf{A})$ is situated between the first capability order

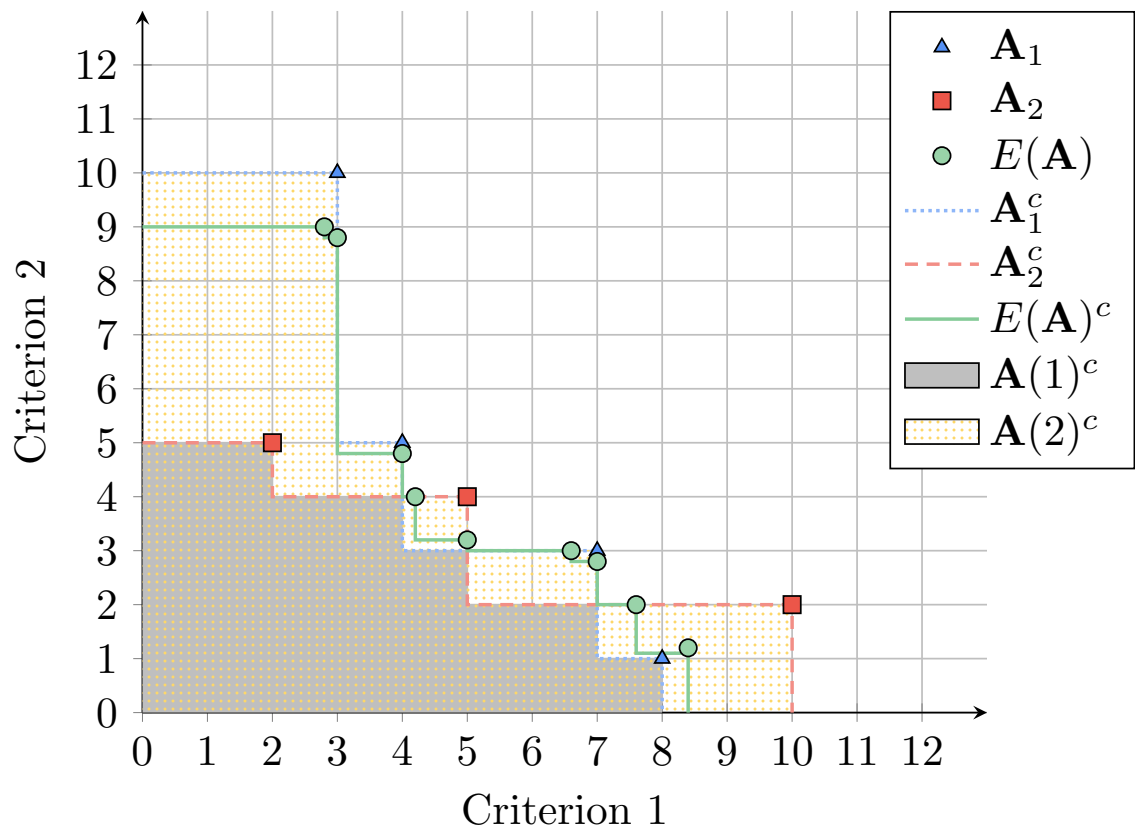


Figure 6.3: Example 6.2: The expected capability set $E(A)$ with $p(s_1) = 0.8$ and $p(s_2) = 0.2$

statistic, i.e., $\mathbf{A}_1^c \cap \mathbf{A}_2^c \subseteq E(\mathbf{A})^c$, and the last capability order statistic, i.e., $E(\mathbf{A}) \subseteq \mathbf{A}_1^c \cup \mathbf{A}_2^c$. $\triangle$

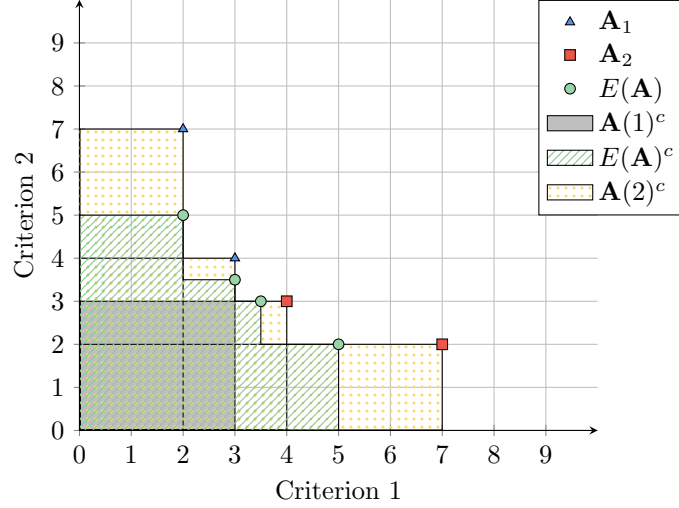


Figure 6.4: Example 6.1: The expected capability set $\mathbf{E}(\mathbf{A})$

6.3 Propositions of expected capability sets

Inspired by Savage [savage1972foundations] axioms, we consider the major propositions satisfied by the proposed expected capability concept. For comparison, propositions for average capability sets are provided in the Appendix D.

The first property expresses consistency with the classical maximum expected utility principle under the original conditions in Savage's setup (univariate utilities and single consequences).

Proposition 6.1 (Consistency with expected utility) *If the capability sets include just one beings and are only assessed using one dimension, then expected capability sets are equivalent to expected utilities.*

Proof Suppose capability sets include only one beings ($|\mathbf{A}_l| = 1$ for all s_l) and there is just one assessment dimension ($h^* = 1$). In such a case, we can express $\mathbf{A}_l$ as a single real number, b_l . Given the total order among all b_l , and considering that there is a unique solution for each state, it follows that $b = \max \sum_{l=1}^{l^*} p(s_l) \cdot (b_l - \mathbb{R}_+)$ if and only if $b = \sum_{l=1}^{l^*} p(s_l) \cdot b_l$. This implies that b is the expected utility of f_m ,

denoted $\sum_{l=1}^{l^*} p(s_l) \cdot \mathbf{A}_l$. ■

The second property, derived from Axiom 6.1, states that the expected capability set should be weakly Pareto dominated by the union of capability sets across all states. This property is deemed desirable because if PMs are certain that they cannot dominate a solution $\vec{b}$ (i.e., for all states, there is no $\vec{b}' \in \mathbf{A}_l$ such that $\vec{b} \leq \vec{b}'$), they should not expect to dominate $\vec{b}$. By adhering to this property, PMs maintain a realistic understanding of the feasible choices and avoid including beings that citizens are certain to not dominate regardless of the state of nature. To illustrate this, consider the case of a one-dimensional expected capability set, where each capability set $\mathbf{A}_l$ is one-dimensional and contains only one beings (i.e., $\mathbf{A}_l = \{b_l\}$ with $b_l \in \mathbb{R}$). Under this setting, the expected capability set is dominated by the best solution among all capability sets from different states. In other words, the expected capability set is dominated by the union of all capability sets. Mathematically, this is expressed as $E(\mathbf{A}) \leq \max_{\forall s_l} U(f_m(s_l))$ or, equivalently $E(\mathbf{A}) \in \bigcup_{l=1}^{l^*} (\mathbf{A}_l - \mathbb{R}_+)$.

Proposition 6.2 (Sure domination of the expected capability) *The expected capability set is dominated by the union of capability sets over all possible states, that is,*

$$E(\mathbf{A}) \subseteq \bigcup_{l=1}^{l^*} \mathbf{A}_l^c$$

Proof Consider any solution $\vec{b}$ in $E(\mathbf{A})$ obtained by aggregating $\sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$. By definition, there is an order among all the beings $\vec{b}_l$ used in this aggregation. Therefore, $\vec{b}$ is at least weakly Pareto dominated by the best beings used in the aggregation. This highest beings is necessarily weakly Pareto dominated by a solution in $\mathbf{A}_l$. ■

The next property derives from Axiom 6.2 and specifies that the expected capability set should dominate the solutions dominated in the capability set in every state. This seems desirable since if $\vec{b} \in \bigcap_{l=1}^{l^*} \mathbf{A}_l^c$, we can find a beings that dominates $\vec{b}$ under every state. To illustrate this, consider the previous one dimension and one beings capability setting. In this case, the expected capability set dominates the worst solution among all capability sets from different states. In other words, the

expected capability set dominates the intersection of the space dominated by capability sets in every state. Mathematically, this is expressed as $\min_{\forall s_l} U(f_m(s_l)) \leq E(\mathbf{A})$ or equivalently $\bigcap_{l=1}^{l^*} (\mathbf{A}_l - \mathbb{R}_+) \in E(\mathbf{A}) - \mathbb{R}_+$.

Proposition 6.3 (Sure domination by the expected capability) *Expected capability sets dominate the intersection of all beings dominated by all states' capability sets, that is*

$$\bigcap_{l=1}^{l^*} \mathbf{A}_l^c \subseteq E(\mathbf{A})^c$$

Proof Consider any beings $\vec{b}$ that is dominated by every capability set, i.e. $\vec{b} \in \bigcap_{l=1}^{l^*} \mathbf{A}_l^c$. For each $l \in \{1, \dots, l^*\}$, there exists $\vec{b}_l \in (\mathbf{A}_l - \mathbb{R}_+^{h^*})$ such that $\vec{b}_l = \vec{b}$. Therefore, the solution $\sum_{i=1}^{l^*} p(s_l) \cdot \vec{b}_l \in E(\mathbf{A})$ is equal to $\vec{b}$. ■

The fourth one shows that the expected procedures are preserved by positive affine transformations. This relates to the positive affine uniqueness property of utility functions [french].

Proposition 6.4 (Linearity) (a) *Preservation of addition:*

$$E(\mathbf{A} + \vec{c}) = E(\mathbf{A}) + \vec{c} \text{ with } \vec{c} \in \mathbb{R}^{h^*}$$

(b) *Preservation of positive multiplication*

$$E(\mathbf{A} \cdot \vec{c}) = E(\mathbf{A}) \cdot \vec{c} \text{ with } \vec{c} \in \mathbb{R}_+^{h^*}$$

Proof To prove (a), consider any $\vec{b}$ in $E(\mathbf{A} + \vec{c})$. This means $\vec{b} = \sum_{l=1}^{l^*} (p(s_l) \cdot \vec{b}_l)$, with an existing order between all $\vec{b}_l$, and $\vec{b}_l \in \mathbf{A}_l + \vec{c} - \mathbb{R}_+^{h^*}$. Similarly, for any $\vec{b}'$ in $E(\mathbf{A}) + \vec{c}$, we have $\vec{b}' \in \sum_{l=1}^{l^*} (p(s_l) \cdot \vec{b}'_l) + \vec{c}$, with an order between all $\vec{b}'_l$, and $\vec{b}'_l \in \mathbf{A}_l - \mathbb{R}_+^{h^*}$. Now, for each $\vec{b}$ in $E(\mathbf{A} + \vec{c})$, we can find a $\vec{b}'$ in $E(\mathbf{A}) + \vec{c}$ such that $\vec{b}' = \vec{b}$, by setting $\vec{b}'_l = \vec{b}_l - \vec{c}$. Conversely, for each $\vec{b}'$ in $E(\mathbf{A}) + \vec{c}$, we can find a $\vec{b} \in E(\mathbf{A} + \vec{c})$ such that $\vec{b} = \vec{b}'$, by setting $\vec{b}_l = \vec{b}'_l + \vec{c}$.

The proof for (b) follows a similar approach. ■

The fifth property stipulates that extending capability sets on states cannot lead to a reduction in the expected capability set.

Proposition 6.5 (Monotonicity over capability domination) *If for all $s_l \in S$ we have $\mathbf{A}_l \subseteq \mathbf{B}_l^c$, then $E(\mathbf{A}) \subseteq E(\mathbf{B})^c$.*

Proof For every beings $\vec{b}$ in $E(\mathbf{A})$, where $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$, there exists a $\vec{b}' = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}'_l$ in $E(\mathbf{B})^c$ such that $\vec{b}' \geq \vec{b}$ by setting $\vec{b}_l = \vec{b}'_l$ for all $l \in \{1, \dots, l^*\}$. ■

Our sixth property shows that increasing the probability of one state of the world dominating another one should lead to the expected capability set of the first one to dominate the expected capability set of the second one.

Proposition 6.6 (Monotonicity over probabilities) *If we have $\mathbf{A}_z \subseteq \mathbf{A}_{z'}^c$ with $(s_z, s_{z'}) \in S^2$ and*

$$p'(s_l) = \begin{cases} p(s_l) + c & , \text{ for } l = z' \\ p(s_l) - c & , \text{ for } l = z \\ p(s_l) & , \text{ otherwise} \end{cases}$$

with $c \in (0; p(s_z)]$. Then, $E(\mathbf{A}, p) \subseteq E(\mathbf{A}, p')^c$.

Proof For every beings $\vec{b}$ in $E(\mathbf{A}, p)$, defined as $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$, there always exists a corresponding beings $\vec{b}'$ in $E(\mathbf{A}, p')$, defined as $\vec{b}' = \sum_{l=1}^{l^*} p'(s_l) \cdot \vec{b}'_l$, such that $\vec{b}' \geq \vec{b}$:

- If $\vec{b}_z \leq \vec{b}_{z'}$, let $\vec{b}_l = \vec{b}'_l$ for all $l \in \{1, \dots, l^*\}$. It then follows that $\sum_{l=1}^{l^*} p'(s_l) \cdot \vec{b}'_l \geq \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$.
- If $\vec{b}_z > \vec{b}_{z'}$, let $\vec{b}'_l$ be defined as:

$$\vec{b}'_l = \begin{cases} \vec{b}_z & \text{for } l = z' \\ \vec{b}_{z'} & \text{for } l = z \\ \vec{b}_l & \text{Otherwise} \end{cases}$$

Then, we can deduce that $\sum_{l=1}^{l^*} p'(s_l) \cdot \vec{b}'_l \geq \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$. ■

6.4 Expected vs Average Capability sets

Average capability sets respect the same properties as expected capability sets except for the no domination of impossible beings (Proposition 6.2), and the monotonicity over probabilities (Proposition 6.6).

An interesting relation between both concepts is that for all solutions in the expected capability set, it is possible to find a solution in the average capability set that is at least as preferred as it, i.e. the expected capability set is dominated by the average capability set.

Proposition 6.7 *We have*

$$E(\mathbf{A}) \subseteq \overline{\mathbf{A}}^c$$

Proof. For every beings $\vec{b}$ in $E(\mathbf{A})$ such that $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}_l$, we have that for each $\vec{b}_l$ there exists a corresponding beings $\vec{b}'_l$ in $\mathbf{A}_l$ such that $\vec{b}'_l \geq \vec{b}_l$. Consequently, the beings $\vec{b}' = \sum_{l=1}^{l^*} p(s_l) \cdot \vec{b}'_l$ is in $\overline{\mathbf{A}}$ and satisfies $\vec{b}' \geq \vec{b}$. ■

To illustrate the distinction between both sets, consider the example of mixing two capability sets as Figure 6.5 shows. In both frameworks, the mixed capability set tends to align more closely with $\mathbf{A}_1$ as the probability $p(s_1)$ increases, and similarly with $\mathbf{A}_2$ as its probability $p(s_2)$ increases. Furthermore, observe that the expected capability sets are always “below” the union of $\mathbf{A}_1$ and $\mathbf{A}_2$, as well as “below” the average capability set $PF(\overline{\mathbf{A}})$.

Expected and average capability sets therefore fulfill different properties. Thus, a careful scrutiny is necessary to determine which set to employ under given circumstances. The expected capability approach is best suited when calculating the anticipated expected beings of a citizen. In this context, Proposition 6.2 becomes highly relevant. Indeed, it is crucial that no PM should expect to dominate beings that are not weakly Pareto dominated in any state of the world. Moreover, when increasing the probabilities of a state of the world leading to a capability set that is preferred over another one generated in a different state (Proposition 6.6), it is crucial that the resulting expected capability set is improved for the citizen.

Conversely, the average capability approach can be used in situations requiring social aggregation, such as moving from E to G or A to F in Figure 3.12. In this

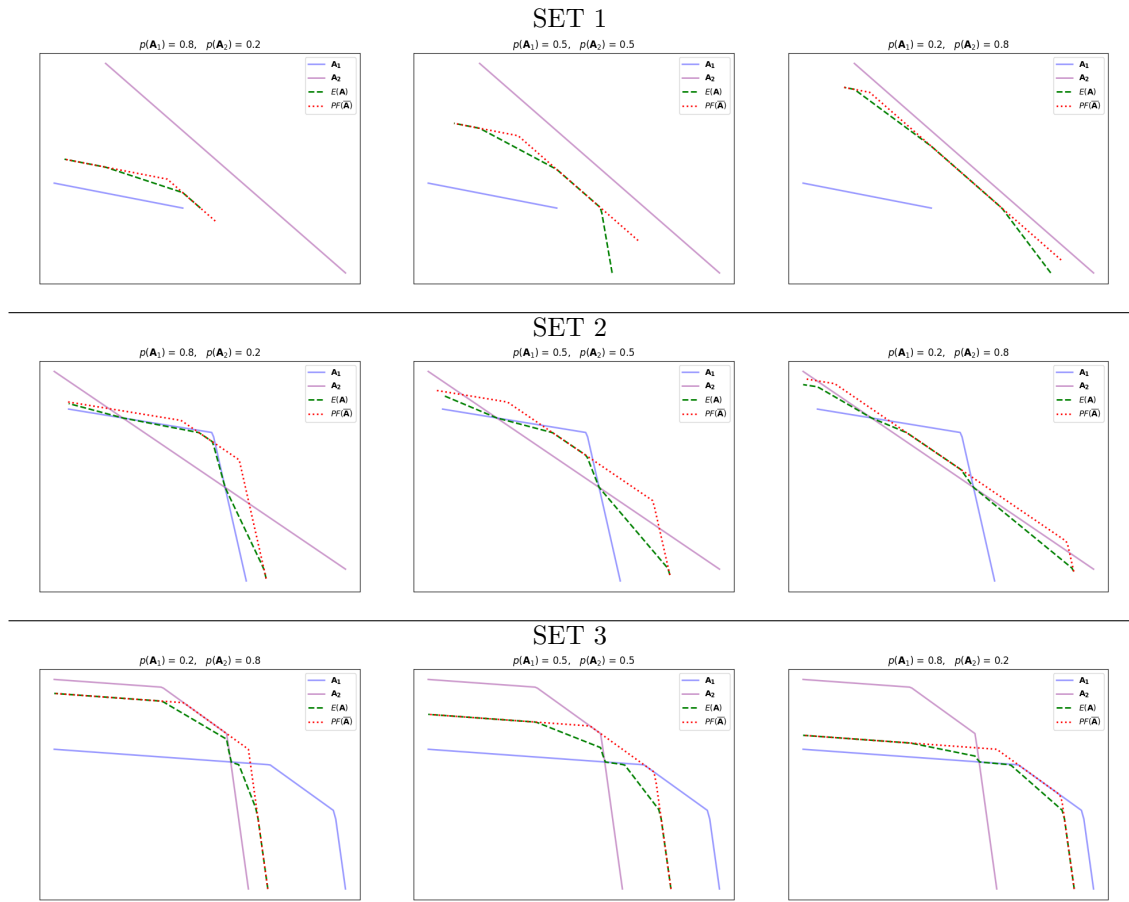


Figure 6.5: Comparison of the expected and average capability sets for three sets and three probability distributions.

approach, the average capability set of a group reflects a composite of all possible beings that any citizen within the group could choose. Consequently, it is not surprising that average capability sets do not fulfill Proposition 6.2. This is because the “social beings” in an average capability set represent a combination of “individual beings.”

Therefore, we recommend compiling $\overline{\mathbf{A}}$ rather than $PF(\overline{\mathbf{A}})$. The set $\overline{\mathbf{A}}$ embodies actual social aggregation, which may not be efficient since individuals maximize their beings independently of the aggregated beings at the social level. In contrast, $PF(\overline{\mathbf{A}})$ represents social aggregation where all individuals choose according to an efficient solution at the social level.

The non-fulfillment of Proposition 6.6 is expected because increasing the probability of a better capability set makes the average capability set “closer” to it, but does not necessarily make it dominate the previous average capability set. Further exploration of social aggregation is needed, but it will not be part of this manuscript.

We illustrate such distinctions with two final examples.

Example 6.3 *Consider a local farmers’ market where two farmers set up identical stalls. Each stall offers two baskets for sale. One basket contains 2 apples and 4 carrots; the other holds 4 apples and 2 carrots. Two shoppers, Alice and Bob, visit the market with enough money to purchase only one basket each. The act f_m represents the random assignment of Alice and Bob to one of these stalls (we obtain $\mathbf{A}_{\text{alice}} = \mathbf{A}_{\text{bob}} = \{(\text{apples} : 2, \text{carrots} : 4), (\text{apples} : 4, \text{carrots} : 2)\}$). Let us determine the average capability set in purchasing apples and carrots in the community ($\overline{\mathbf{A}} = \{(\text{apples} : 2, \text{carrots} : 4), (\text{apples} : 3, \text{carrots} : 3), (\text{apples} : 4, \text{carrots} : 2)\}$). Then,*

- *If both Alice and Bob choose the first basket, the average basket of Alice and Bob contains 2 apples and 4 carrots per person.*
- *If Alice and Bob each select different baskets, the average basket of Alice and Bob contains 3 apples and 3 carrots per person.*
- *If both Alice and Bob opt for the second basket, the average basket of Alice and Bob contains 4 apples and 2 carrots per person.*

Now, consider the particular case of a shopper that is randomly directed to one of the stalls and is allowed to select one of the baskets (we obtain $\mathbf{A}_1 = \mathbf{A}_2 = \{(\text{apples} : 2, \text{carrots} : 4), (\text{apples} : 4, \text{carrots} : 2)\}$). In this context, the expected capability set for the shopper would be $E(\mathbf{A}) = \{(\text{apples} : 2, \text{carrots} : 4), (\text{apples} : 4, \text{carrots} : 2)\}$:

- 2 apples and 4 carrots if they select the first basket
- 4 apples and 2 carrots if they select the second basket.

△

Example 6.4 Consider a region where the local government has two types of land grants available: Land Grant 1 offers plots in a rain-abundant valley, ideal for rice cultivation. On the other hand, Land Grant 2 provides plots on sun-soaked hill terraces, perfect for vineyards. The government, in an attempt to diversify agriculture, randomly assigns Land Grant 1 to farmer Alice and Land Grant 2 to farmer Bob.

In this scenario, the term “average capability set” refers to the expected agricultural products both farmers can yield. Although Alice has the exclusive capability to cultivate rice due to her land’s conditions, and Bob has the unique advantage to grow grapes ($\mathbf{A}_{\text{Alice}} = \{(\text{rice} : [0; M], \text{grapes} : 0)\}$, $\mathbf{A}_{\text{Bob}} = \{(\text{rice} : 0, \text{grapes} : [0; M])\}$ with M a large number), each farmer in this community, on average, possesses the ability to produce both rice and grapes. Therefore, the average capability set aligns with any reasonable amount of rice and grapes per farmer $\bar{\mathbf{A}} = \{(\text{rice} : [0; \frac{M}{2}], \text{grapes} : [0; \frac{M}{2}])\}$.

In contrast, Alice’s expected yield is contingent on the specific land grant she receives $\mathbf{A}_{\text{valley}} = \{(\text{rice} : [0; M], \text{wine} : 0)\}$, $\mathbf{A}_{\text{terrace}} = \{(\text{rice} : 0, \text{wine} : [0; M])\}$. If she is assigned Land Grant 1, she can cultivate an ample amount of rice but no grapes. However, if she were to receive Land Grant 2, she could produce grapes but not rice. Hence, Alice’s expected yield encompasses up to a significant amount of either rice or grapes, but not both, $E(\mathbf{A}) = \{(\text{rice} : \frac{M}{2}, \text{wine} : 0), (\text{rice} : 0, \text{wine} : \frac{M}{2})\}$. △

These examples highlight the critical distinction between average and expected capability sets. While average capability sets reflect the combined possible outcomes in a population, expected capability sets provide a probabilistic forecast of the individual’s potential outcomes. The choice between both methods would be context-dependent, whether it requires a societal or an individual perspective.

6.5 Conclusion

We have provided two concepts that can be used to summarise capability sets in a multiobjective decision-making setup under risk within the capability approach:

expected and average capability sets. Both have similar properties but are crucially different concerning sure domination of the expected capability (Proposition 6.2) and monotonicity over probabilities (Proposition 6.6). The choice of which concept to apply depends on the decision-maker's ultimate goal. Specifically, if the PM is dealing with capability sets linked to states of the world, each having definite probabilities, and only one of these states of the world will materialize, then using expected capability sets is recommended. Conversely, if the situation involves the combination of multiple capability sets, and all beings can be achieved by different individuals, average capability sets seem a more suitable choice.

Conclusion

The framework proposed by **landry1983model** [landry1983model] is structured into four distinct yet interconnected stages for model validation in operations research: conceptual validation, logical validation, experimental validation, and operational validation. Although these stages are interrelated, the achievements of this thesis in developing the framework will be presented by addressing each stage independently.

Conceptual Validation assesses the alignment of the model's foundational assumptions and theories with its intended applications. This stage ensures that the model is grounded in a theoretical framework that is pertinent to the problem it aims to solve. As detailed in Chapter 1, the Capability Approach offers a robust foundation for welfare assessment, providing a multidimensional framework that accounts for diversity and subjectivity. We propose in Chapter 2 enhancing the CA with a systemic approach to better align it with the goal of designing public policies.

Logical Validation scrutinizes the internal consistency of the model. It verifies that the model's formal structure accurately reflects the conceptual framework, ensuring that the model is free from logical errors, thereby guaranteeing its internal coherence and the precision of its construction. Chapter 2 presents the formal representation of the CA and demonstrates its alignment with the principles of the approach.

Experimental Validation involves testing the model through simulations or other experimental methodologies to assess the effectiveness of its solutions. This stage is essential for confirming the model's robustness and its ability to produce reliable outcomes under various conditions. Chapter 3 demonstrates that the framework can effectively inform public policy design. Additionally, the use of mixed-integer linear programming is shown to be feasible within reasonable time, at least for two

dimensions and 14 actions.

Operational Validation assesses the model’s practical applicability and performance in real-world settings. This stage verifies that the model’s outputs are practical and beneficial for decision-making processes, highlighting the model’s usefulness and relevance in actual operational environments. However, this thesis does not provide operational validation, which remains a significant limitation. The next priority is to refine, with the help of the NEMESIS team, the case study on the mental well-being of elderly Parisians to achieve this stage.

In the second part of this manuscript, we demonstrated that capability sets, viewed as Pareto frontiers, can be aggregated similarly to what is done in classical decision-aiding methods, effectively showing the potential of working with capability sets. However, two key aspects remain unexplored in this work.

First, we did not provide a full characterization of the proposed aggregation framework. While we showed that it respects certain desirable properties, a formal characterization would strengthen its theoretical foundation. Capability sets, being sets of solutions in a multidimensional space, present a mathematical challenge, but this represents a promising line of future research.

Secondly, we did not conduct computational simulations to assess the practical feasibility of the proposed frameworks of the second part. Although the algorithms were implemented for two dimensions without computational issues, its scalability to more dimensions remains uncertain. Additionally, as discussed in Chapters 5 and 6, the number of scenarios to aggregate may have a significant impact on computation time. Investigating these practical limitations will also be a valuable avenue for future research.

We conclude by discussing two future works that fall into two broad categories: the implementation of our framework to generate capability sets, and the effective utilization of the generated capabilities.

First, to generate the transformation matrix, we must elicit preferences for actions across different dimensions of welfare. One potential approach is to treat each action as a criterion and apply a multi-attribute utility framework. However, for the model to be practical, the number of actions must be substantial, and the time required to evaluate their impact on various dimensions must remain minimal. Long

questionnaires are not feasible, as citizens are unlikely to engage with them.

Thus, exploring innovative methods to efficiently learn these impacts is crucial. This could involve adaptive or automated approaches to gathering preference data, reducing the burden on participants while ensuring accuracy.

Another approach, closer to what we have done, could involve developing techniques to infer each citizen's transformation matrix by observing their actions in relation to performance measures across different dimensions of welfare. However, the challenge lies in the fact that we cannot infer much from a single individual's data. On the other hand, aggregating data from the entire population (as in Chapter 3) risks overlooking the subjective differences in how individuals transform actions.

Additionally, some individuals may refrain from certain actions due to their personal values, while others may be unable to perform these actions due to external constraints. We would need an algorithm capable of learning these distinctions, taking into account both values and abilities while preserving individual differences.

Secondly, future work should focus on developing explanatory graphs and clustering techniques to support policy design. A promising direction for research lies in creating algorithms that can better identify and visualize the constraints shaping individuals' capabilities. This is particularly interesting because it involves identifying combinations of constraints that lead to distinct clusters of individuals.

To achieve this, we need a reliable method for measuring the similarity or dissimilarity between capability sets. But what exactly constitutes this dissimilarity? One of the most promising approaches may be to measure the "space" that is dominated by only one of the two compared sets. However, how should this space be measured? And should other factors, such as the shape of the Pareto frontier, also be considered?

In any case, the introduction of the Capability Approach into decision-aiding opens up numerous avenues for research, with only a few discussed here. We could have explored comparing capability sets without individual utility functions, the social aggregation of capability sets, how to handle sensitivity analysis, and so on, essentially all the challenges that classical decision-aiding faces should be adapted to deal with capability sets.

Appendix A

Linear problem example

$$\begin{array}{ll}
 \max & (\\
 \max & (- \sum_{ijr \in E} x_{ijr} \quad - \sum_{ijc \in E} 2x_{ijc} \quad 10y_{11} \quad +3y_{21} \quad +6y_{22} \quad +y_{31} \quad -y_{32} \quad 5y_{33} \quad +y_{41}) \\
 & +10y_{11} \quad +2y_{12} \quad +2y_{21} \quad -2y_{22} \quad +6y_{31} \quad +y_{32} \quad -2y_{33} \quad +y_{41}) \\
 \\
 \text{S.T.} & 2 \sum_{ijr \in E} x_{ijr} \quad +2 \sum_{ijc \in E} x_{ijc} \quad +10y_{31} \quad +5y_{32} \quad +30y_{33} \quad -100y_{41} \leq 0 \\
 & \frac{1}{2} \sum_{ijr \in E} x_{ijr} \quad +\frac{1}{2} \sum_{ijc \in E} 2x_{ijc} \quad +9y_{11} \quad +y_{12} \quad +2y_{21} \quad +2y_{22} \quad +3y_{31} \quad +y_{32} \quad +y_{33} \quad +8y_{41} \leq 24 \\
 & \frac{1}{2} \sum_{ijr \in E} x_{ijr} \quad +\frac{1}{2} \sum_{ijc \in E} 2x_{ijc} \leq 24 \\
 & \epsilon x_{ijr} \leq 1 \quad \forall ijr \in E \\
 & \epsilon x_{ijc} \leq 1 \quad \forall ijc \in E \\
 & y_{11} \quad +y_{12} \leq M \left(\sum_{i1r \in E} + \sum_{i1c \in E} \right) \\
 & y_{21} \quad +y_{22} \leq M \left(\sum_{i2r \in E} x_{i2r} + \sum_{i2c \in E} x_{i2c} \right) \\
 & y_{31} \quad +y_{32} \quad +y_{33} \leq M \left(\sum_{i3r \in E} x_{i3r} + \sum_{i3c \in E} x_{i3c} \right) \\
 & +y_{41} \leq M \left(\sum_{i4r \in E} x_{i4r} + \sum_{i4c \in E} x_{i4c} \right) \\
 & = 0 \\
 & = 0 \\
 & = 0 \\
 & = x \quad 0 \\
 & x_{ijr}, \quad x_{ijc} \in \mathbb{N}, \forall ijr, ijc \in E \\
 & y_{11}, \quad y_{21}, \quad y_{22}, \quad y_{31}, \quad y_{33}, \quad y_{41} \in \{0, 1\} \\
 & y_{12}, \quad y_{32} \in \mathbb{N}
 \end{array}$$

Table A.1: LP for **A**

[illegible]

Table A.2: LP for $\mathbf{A}|_{\text{park}}$

Appendix B

Questionnaire

Q1: À partir de votre domicile, peut-on accéder en marchant aux services suivants:

From your home, can you access the following services by walking:

- | | | |
|---|--|---|
| a. médecin généraliste (<i>GP</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| b. médecins spécialistes (<i>Specialists</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| c. pharmacie (<i>Pharmacy</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| d. lab. d'analyses (<i>Lab</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| e. boulangerie (<i>Bakery</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| f. supérette/supermarché (<i>Store</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| g. fruits et légumes (<i>Produce</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| h. autres magasins (<i>Other stores</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| i. parc (<i>Park</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| j. équipements sportifs (<i>Sports</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| k. équipements culturels (<i>Culture</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |
| l. lieux de sociabilité (<i>Social</i>) | ☐ oui (<i>yes</i>) | ☐ non (<i>no</i>) |

Q2: Les questions suivantes portent sur votre quartier.

The following questions are about your neighborhood.

a. Un ou plusieurs de vos enfants habitent-ils dans votre quartier ?

Do one or more of your children live in your neighborhood?

Réponse: ☐ oui ☐ non

b. Un ou plusieurs de vos enfants habitent-ils à l'extérieur de votre quartier, mais à moins de 30 minutes de transport pour s'y rendre ?

Do one or more of your children live outside your neighborhood but within 30 minutes of transportation to get there?

Réponse: ☐ oui ☐ non

c. Avez-vous de la famille qui habite dans votre quartier ?

Do you have family who live in your neighborhood?

Réponse: ☐ oui ☐ non

d. Avez-vous de la famille qui habite à l'extérieur de votre quartier, mais à moins de 30 minutes de transport pour s'y rendre ?

Do you have family who live outside your neighborhood but within 30 minutes of transportation to get there?

Réponse: ☐ oui ☐ non

e. Avez-vous des amis qui habitent dans votre quartier ?

Do you have friends who live in your neighborhood?

Réponse: ☐ oui ☐ non

f. Avez-vous des amis qui habitent à l'extérieur de votre quartier, mais à moins de 30 minutes de transport pour s'y rendre ?

Do you have friends who live outside your neighborhood but within 30 minutes of transportation to get there?

Réponse: ☐ oui ☐ non

Q3: Cette question porte sur les activités que vous pourriez avoir à faire au cours d'une journée normale. Votre état de santé actuel vous limite-t-il dans ces activités ? Si oui, dans quelle mesure ?

(This question is about the activities you might have to do during a normal day. Does your current state of health limit you in these activities? If so, to what extent?)

Réponses possibles: **Oui, il me limite beaucoup, Oui, il me limite un peu, Non, il ne me limite pas du tout**

(Possible answers: Yes, it limits me a lot, Yes, it limits me a little, No, it doesn't limit me at all)

a. Activités exigeant un effort physique important: courir, soulever des objets lourds, pratiquer un sport de façon intensive

Activities requiring significant physical effort: running, lifting heavy objects, intensive sports

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

b. Activités modérées: déplacer une table, passer l'aspirateur, jouer à la pétanque

Moderate activities: moving a table, vacuuming, playing pétanque

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

c. Soulever ou transporter des sacs de courses

Lifting or carrying grocery bags

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

d. Monter plusieurs étages à pied

Climbing several flights of stairs

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

e. Monter un seul étage à pied

Climbing one flight of stairs

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

f. Vous pencher, vous mettre à genoux ou vous accroupir

Bending, kneeling or squatting

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

g. Marcher plus d'un kilomètre

Walking more than one kilometer

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

h. Marcher cinq cents mètres

Walking five hundred meters

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

i. Marcher cent mètres

Walking one hundred meters

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

j. Prendre un bain, une douche ou vous habiller seul(e)

Taking a bath, shower, or dressing oneself

Réponse: ☐ Oui, il me limite beaucoup ☐ Oui, il me limite un peu ☐ Non, il ne me limite pas du tout

Q4. Dans quelle situation êtes-vous vis-à-vis de la conduite auto / moto ?

What is your situation with regard to driving cars / motorcycles?

- a. J'ai le permis et je sais conduire (*I have a license and I can drive*)
- b. J'ai le permis mais je ne sais pas / plus conduire (*I have a license but I cannot / no longer drive*)
- c. Je n'ai pas le permis de conduire (*I don't have a driving license*)

Q5. Est-ce que vous, ou un membre de votre ménage, possédez un véhicule motorisé (voiture, moto, etc.) ? (*Do you, or a member of your household, own a motorized vehicle (car, motorcycle, etc.)?*) ☐ oui (*yes*) ☐ non (*no*)

Q6. Vous arrive-t-il de faire appel pour vous déplacer en voiture ou en moto :

(Do you ever request a ride for transportation by car or motorcycle?)

a. à une personne de votre ménage ?

(From a member of your household?)

Réponse: ☐ oui ☐ non

b. à un proche, un ami, etc. en dehors de votre ménage ?

(From a relative, a friend, etc. outside of your household?)

Réponse: ☐ oui ☐ non

c. à un chauffeur rémunéré (taxi, Uber, service d'aide à la personne, etc.) ?

(From a paid driver (taxi, Uber, care service, etc.)?)

Réponse: ☐ oui ☐ non

Q7. Dans quelle situation êtes-vous vis-à-vis de la pratique du vélo ?

(What is your situation with regard to cycling?)

a. Je sais faire du vélo

(I know how to cycle)

Réponse: ☐ oui ☐ non

b. Je ne sais plus faire du vélo

(I no longer know how to cycle)

Réponse: ☐ oui ☐ non

c. Je n'ai jamais su faire du vélo

(I have never known how to cycle)

Réponse: ☐ oui ☐ non

Q8. Est-ce que vous avez un vélo à disposition dans votre ménage ? ☐ oui ☐ non

Do you have a bike available in your household? ☐ yes ☐ no

Q9. Avez-vous personnellement un abonnement valable pour le Vélib ? ☐ oui ☐ non

Do you personally have a valid subscription for Vélib? ☐ yes ☐ no

Q10. Etes-vous d'accord avec les affirmations suivantes :

(Do you agree with the following statements:)

Réponses possibles: **Tout à fait d'accord, Plutôt d'accord, Plutôt pas d'accord, Pas du tout d'accord**

(Possible answers: Strongly Agree, Agree, Disagree, Strongly Disagree)

a. Utiliser une voiture est trop cher pour moi au quotidien.

(Using a car is too expensive for me on a daily basis.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

b. A l'endroit où j'habite, il est facile de vivre sans voiture.

(Where I live, it's easy to live without a car.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

c. Avec mon rythme de vie, il est (ou serait) utile d'avoir une voiture.

(With my lifestyle, it is (or would be) useful to have a car.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

d. Il est agréable de se déplacer en voiture.

(It's pleasant to travel by car.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

e. J'ai peur quand je suis en voiture.

(I'm afraid when I'm in a car.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

Q11. Etes-vous d'accord avec les affirmations suivantes :

(Do you agree with the following statements:)

Réponses possibles: **Tout à fait d'accord, Plutôt d'accord, Plutôt pas d'accord, Pas du tout d'accord**

(Possible answers: Strongly Agree, Agree, Disagree, Strongly Disagree)

a. J'utilise les transports en commun le moins souvent possible.

(I use public transportation as infrequently as possible.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

b. Utiliser les transports en commun revient trop cher.

(Using public transportation is too expensive.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

c. Autour de l'endroit où j'habite, il y a suffisamment de lignes de transports en commun pour réaliser mes activités.

(Around where I live, there are enough public transport lines to carry out my activities.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

d. Les transports en commun sont un moyen pratique de se déplacer.

(Public transportation is a convenient way to get around.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

Q12. Etes-vous d'accord avec les affirmations suivantes :

(Do you agree with the following statements:)

Réponses possibles: **Tout à fait d'accord, Plutôt d'accord, Plutôt pas d'accord, Pas du tout d'accord**

(Possible answers: Strongly Agree, Agree, Disagree, Strongly Disagree)

a. Les transports en commun ne permettent pas de voyager confortablement.

(Public transportation does not allow for comfortable travel.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

b. Les transports en commun ne permettent pas de voyager en sécurité (risque d'agression).

(Public transportation does not allow for safe travel (risk of assault).)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

c. Dans les transports en commun, il est agréable de côtoyer les autres passagers.

(It is pleasant to interact with other passengers on public transportation.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

d. Il est souvent difficile de trouver son chemin dans les transports en commun.

(It is often difficult to find one's way in public transportation.)

Réponse: ☐ Tout à fait d'accord ☐ Plutôt d'accord ☐ Plutôt pas d'accord ☐
Pas du tout d'accord

Appendix C

Hurwicz and Laplace for Capability sets

C.1 Classical methods

This appendix begins by reviewing the Hurwicz and Laplace methods in their classical context. We then propose adaptations of these methods for application to capability sets, providing examples and discussing key properties. Finally, we prove the adherence to the proposed approach.

C.2 Classical settings

Hurwicz Criterion

The Hurwicz criterion is not only based on order statistics, but rather represents an aggregation of the first and last orders statistic. PMs base their choices on a compromise between caution (MaxiMin) and optimism (MaxiMax). Given a number $\alpha \in [0, 1]$ called the “pessimism coefficient”, the best choice is determined by the following expression:

$$\max_{f_m \in F} \left[\alpha \cdot \min_{s_l \in S} (u(f_m(s_l))) + (1 - \alpha) \cdot \max_{s_l \in S} (u(f_m(s_l))) \right] \quad (\text{C.1})$$

Example C.1 (Cont) *Continuing with Example 5.1, act f_1 is preferred if $\alpha \leq 0.5$ and f_2 is preferred if $\alpha \geq 0.5$.*

The Hurwicz criterion, while offering a pragmatic approach to decision-making under uncertainty, is subject to two main criticisms. The first concern is that it represents a compromise between two potentially “bad” solutions; specifically, it combines the most pessimistic (min) and the most optimistic (max) approaches, which may not always yield a desirable outcome.

Furthermore, a significant challenge lies in the practical determination of the pessimism coefficient α . The choice of α is inherently subjective and can profoundly impact the decision outcome.

Laplace Criterion

Unlike the previous criteria, the Laplace criterion is not predicated upon order statistic but is founded on the principle of “Insufficient Reason”. Under this principle, each possible state of the world is assigned an equal probability, denoted as $p(s_l) = \frac{1}{|S|}$. The decision-maker then opts for the action that yields the most favorable expected outcomes, mathematically expressed as:

$$\max_{f_m \in F} \left[\sum_{l=1}^{l^*} \frac{1}{|S|} u(f_m(s_l)) \right] \quad (\text{C.2})$$

Example C.2 *Applying the Laplace criterion to Example 5.1 leads the DM to favor f_2 (with $E(f_2) = 5.5$) over f_1 (with $E(f_1) = 5$).*

Although the Laplace criterion offers a simple approach to decision-making under uncertainty by assigning equal probability to each state of the world, this assumption raises several critical concerns. Firstly, the presumption that all states carry an equal probability is often questioned. In reality, the likelihood of each state can vary significantly and is not always equally distributed, making this assumption potentially unrealistic.

Moreover, this criterion is particularly sensitive to the choice, generally arbitrary, of the number of states of Nature considered. Nature can always be subdivided into more detailed states, as exemplified by weather conditions, where one state can be split into “It will rain tomorrow” and “It will not rain tomorrow”, significantly impacting the criterion’s outcome.

Lastly, even assuming all states are equally probable, it remains to be seen whether

the expected value is a sensible choice criterion. This approach might not account for the variability or distribution of possible outcomes, limiting its relevance in certain complex decision-making scenarios.

C.3 Adaptation for Capability sets

C.3.1 Hurwicz for Capability sets

The Hurwicz criterion represents an aggregation between the first and last order statistics. The critical question pertains to the method of aggregation. We have opted for the expected capability aggregation method developed in Chapter 6. The Hurwicz criterion can be written as:

$$\text{Hurwicz}(\mathbf{A}, \alpha) = E(\mathbf{A}(1), \mathbf{A}(l^*)) \text{ with } p(\mathbf{A}(1)) = \alpha \text{ and } p(\mathbf{A}(l^*)) = 1 - \alpha$$

Example C.3 *Building upon Example 5.2, Figure C.1 depicts the application of the Hurwicz criterion across a spectrum of α values. Particularly, Figure C.2 provides a clear visualization of the preference relationship between f_1 and f_2 under different levels of pessimism represented by α . For higher pessimism level, specifically at $\alpha = 0.25$, act f_2 is preferred over f_1 . Conversely, as the level of pessimism decrease to $\alpha = \frac{11}{12}$, the preference shifts, favoring f_1 over f_2 . This exemplifies the sensitivity of the Hurwicz criterion to the pessimism parameter α . $\triangle$*

The use of the expected capability sets within this decision-making framework is justified by its alignment with foundational principles that enable the Hurwicz criterion for capability sets to parallel the classical Hurwicz criteria.

Proposition C.1 (Hurwicz capability bound) *The Hurwicz criterion for capability sets establishes a bounded spectrum between the first and last order statistics:*

$$\forall \alpha \in [0, 1], \quad \mathbf{A}(1) \subseteq \text{Hurwicz}(\mathbf{A}, \alpha)^c \subseteq \mathbf{A}(l^*)^c,$$

indicating that the Hurwicz criterion outcome for any given α value is framed within the extremes defined by the first and last order statistics of the capability sets.

An interesting proposition is the behavior of the Hurwicz criterion in response to adjustments in the pessimism coefficient. Intuitively, as the pessimism coefficient

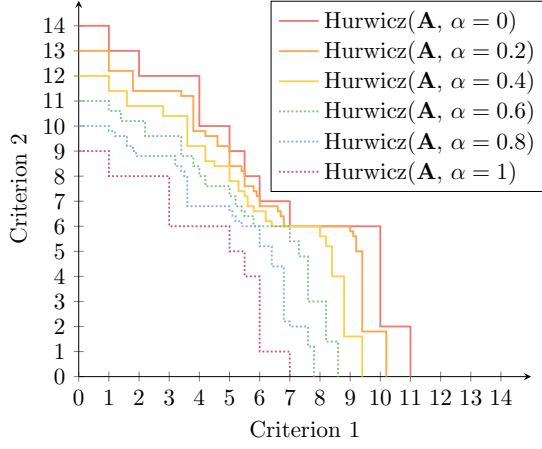


Figure C.1: Hurwicz representation of $\mathbf{A}$ for several α

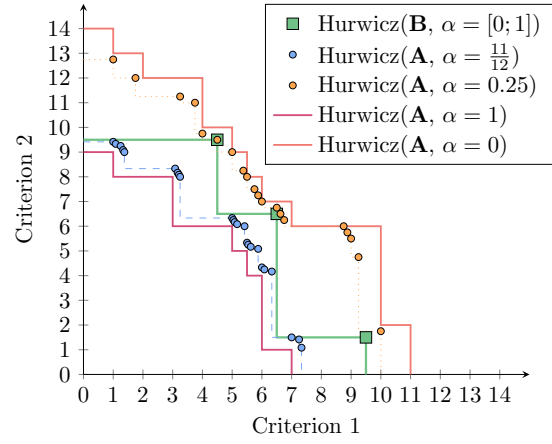


Figure C.2: Domination relation between $\mathbf{A}$ and $\mathbf{B}$ for different values of α

decreases, the assessment under the Hurwicz criterion is expected to become more optimistic.

Proposition C.2 (Hurwicz optimism monotonicity)

If $\alpha \leq \alpha'$, then $\text{Hurwicz}(\mathbf{A}, \alpha') \subseteq \text{Hurwicz}(\mathbf{A}, \alpha)^c$

Of course, we want to be consistent with the classical Hurwicz criterion.

Proposition C.3 (Consistency with Hurwicz criterion)

If $h^* = 1$ and $|\mathbf{A}_l| = 1$ for all $s_l \in S$, then the Hurwicz criterion for a capability set is equivalent to the classical Hurwicz criterion:

$$\text{Hurwicz}(\mathbf{A}, \alpha) = \alpha \cdot \min_{s_l \in S} (\mathbf{A}_l) + (1 - \alpha) \cdot \max_{s_l \in S} (\mathbf{A}_l).$$

C.3.2 Laplace criterion

The Laplace criterion can be expressed as a direct application of the Expected capability set with $p(s_l) = \frac{1}{|S|}$ for all $s_l \in S$.

Example C.4 Building upon Example 5.2, the outcomes obtained from applying the Laplace criterion are illustrated in Figure C.3. In this scenario, it is not evident which of the two act is preferable, as $\text{Laplace}(\mathbf{A}) \not\subseteq \text{Laplace}(\mathbf{B})$ and $\text{Laplace}(\mathbf{B}) \not\subseteq \text{Laplace}(\mathbf{A})$.

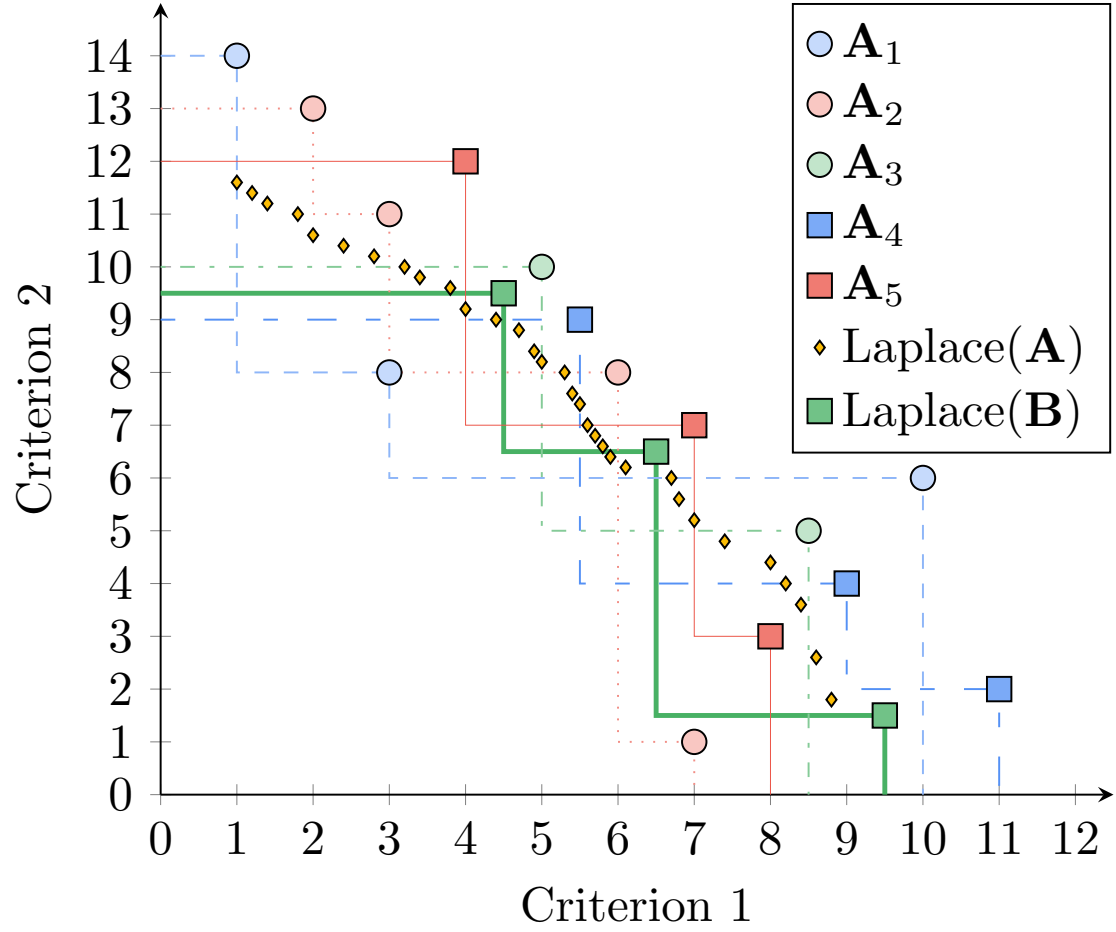


Figure C.3: Laplace criterion

Proposition C.4 (Laplace Capability bound) *The Laplace criterion for capability sets establishes a bounded spectrum between the first and last order statistics: $\mathbf{A}(1) \subseteq \text{Laplace}(f_m)^c \subseteq \mathbf{A}(l^*)^c$.*

Proposition C.5 (Monotonicity over Capability domination) *If for every state $s_l \in S$, it holds that $\mathbf{A}_l \subseteq \mathbf{B}_l^c$, then the Laplace criterion applied to $\mathbf{A}$ is dominated by that applied to $\mathbf{B}$, denoted as $\text{Laplace}(\mathbf{A}) \subseteq \text{Laplace}(\mathbf{B})^c$.*

Proposition C.6 (Consistency with Laplace Criterion) *If $h^* = 1$ and $|\mathbf{A}| = 1$ for all $s_l \in S$, then the Hurwicz criterion for a capability set is equivalent to the classical Hurwicz criterion: $\text{Laplace}(f_m) = \sum_{l=1}^{l^*} \frac{1}{|S|} \mathbf{A}_l$.*

C.4 Proofs

Proof C.1: The concept of expected capability set adheres to the principle of sure domination of the expected capability (Proposition 6.2). Specifically, the expected capability set is a subset of the space dominated by the union of capability sets across all conceivable states, which can be formally expressed as:

$$E(\mathbf{A}) \subseteq \bigcup_{l=1}^{l^*} (\mathbf{A}_l)^c.$$

For any given $\alpha \in [0, 1]$, it follows that:

$$\text{Hurwicz}(\mathbf{A}, \alpha) = E(\mathbf{A}(1), \mathbf{A}(l^*)) \subseteq (\mathbf{A}(1)^c \cup (\mathbf{A}(l^*))^c) \subseteq \mathbf{A}(l^*)^c.$$

Additionally, this framework aligns with the sure domination by expected capability principle (Proposition 6.3), whereby the expected capability set dominates the intersection of all outcomes dominated by the capability sets across all states, articulated as:

$$\bigcap_{l=1}^{l^*} (\mathbf{A}_l)^c \subseteq E(\mathbf{A})^c.$$

Consequently, for any $\alpha \in [0, 1]$:

$$\mathbf{A}(1)^c \subseteq (\mathbf{A}(1)^c \cap \mathbf{A}(l^*)^c) \subseteq E(\mathbf{A}(1), \mathbf{A}(l^*))^c = \text{Hurwicz}(\mathbf{A}, \alpha)^c.$$

Then the Hurwicz criterion both dominates the first order statistics and is itself dominated by the last order statistics, fulfilling the condition:

$$\forall \alpha \in [0, 1], \quad \mathbf{A}(1) \subseteq \text{Hurwicz}(\mathbf{A}, \alpha)^c \subseteq \mathbf{A}(l^*)^c.$$

■

Proof C.2: The principle of expected capability sets adheres the monotonicity over probabilities (Proposition 6.6). If we have $\mathbf{A}_z \subseteq \mathbf{A}_{z'}^c$ with $(s_z, s_{z'}) \in S^2$ and

$$p'(s_l) = \begin{cases} p(s_l) + c & , \text{ for } l = z' \\ p(s_l) - c & , \text{ for } l = z \\ p(s_l) & , \text{ otherwise} \end{cases}$$

with $c \in (0; p(s_z)]$. Then, $E(\mathbf{A}, p) \subseteq E(\mathbf{A}, p')^c$.

Given that $\mathbf{A}(1) \subseteq \mathbf{A}(l^*)^c$, it follows that decreasing pessimism (increasing optimism) by moving from α to α' ($\alpha \leq \alpha'$) results in $\text{Hurwicz}(\mathbf{A}, \alpha') \subseteq \text{Hurwicz}(\mathbf{A}, \alpha)$, reflecting an improved assessment under the Hurwicz criterion. ■

Proof C.3: The formulation of expected capability aligns with that of classical expected utilities when capability sets are limited to a single entity and assessed within a unidimensional framework (Proposition 6.1). Given that the order statistic for capability sets aligns with classical order statistic under these conditions (as established in Proposition 5.4), and considering that the Hurwicz criterion represents an aggregation of the first and last order statistics, this proposition holds true. ■

Proof C.4: The proof is similar to the one of Property C.1. ■

Proof C.5: This relationship is a direct consequence of the expected capability sets' adherence to monotonicity over capability domination, as discussed in Proposition 6.5. ■

Proof C.6: The formulation of expected capabilities aligns with that of classical expected utilities (Proposition 6.1) when capability sets are limited to a single beings and assessed within a one dimension. ■

Appendix D

Proof of the average capability set

The appendix provides proofs of the propositions satisfied by average capability sets.

Proposition 6.1bis: Consistency with expected utility *If the capability sets include just one beings and are only assessed using one dimension, then expected capability sets are equivalent to expected utilities.*

Proof Suppose we are dealing with only one beings ($|\mathbf{A}| = 1$) and one assessment dimension ($h^* = 1$). In such a case, we can express $u(f_m(s_l))$ as a single real number, b_l . Then $\bar{\mathbf{A}} = \left\{ \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}, \text{ for all } X_i \text{ such that } X_i = (b_1, \dots, b_{l^*}) | b_l \in \mathbf{A}_l \text{ for every } l \in \{1, \dots, l^*\} \right\}$ is equivalent to $\sum_{l=1}^{l^*} p(s_l) \cdot b_l$ which is equivalent to $\sum_{l=1}^{l^*} p(s_l) \cdot u(f_m(s_l))$. ■

Proposition 6.3bis: Sure domination by the average capability *The average capability sets include the intersection of all beings dominated by states' capability, that is*

$$\bigcap_{l=1}^{l^*} \mathbf{A}_l^c \subseteq \bar{\mathbf{A}}^c$$

Proof Consider any beings $\vec{b}$ that is dominated by every capability set, i.e. $\vec{b} \in \bigcap_{l=1}^{l^*} \mathbf{A}_l^c$. For each $l \in \{1, \dots, l^*\}$, there exists $\vec{b}_l' \in \mathbf{A}_l$ such that $\vec{b}_l' \geq \vec{b}$. Therefore, the solution $\sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}$ with $X_i = (b_1', \dots, b_{l^*}')^T$ is in $\bar{\mathbf{A}}$ and is at least weakly dominating $\vec{b}$. ■

Proposition 6.4bis: Linearity(a) *Preservation of addition:*

$$\forall \vec{c} \in \mathbb{R}^{h^*}, \overline{\mathbf{A} + \vec{c}} = \overline{\mathbf{A}} + \vec{c}$$

(b) *Preservation of positive multiplication*

$$\forall \vec{c} \in \mathbb{R}_+^{h^*}, \overline{\mathbf{A} \cdot \vec{c}} = \overline{\mathbf{A}} \cdot \vec{c}$$

Proof To prove (a), we simply remark that

$$\overline{\mathbf{A}} + \vec{c} = \left\{ \sum_{l=1}^{l^*} (p(s_l) \cdot X_{i,l}) + \vec{c}, \text{ for all } X_i \text{ such that } X_i = (\vec{b}_1, \dots, \vec{b}_{l^*}) \right. \\ \left. \mid \vec{b}_l \in \mathbf{A}_l \text{ for every } l \in \{1, \dots, l^*\} \right\}$$

and

$$\overline{\mathbf{A} + \vec{c}} = \left\{ \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}, \text{ for all } X_i \text{ such that } X_i = (\vec{b}_1, \dots, \vec{b}_{l^*}) \right. \\ \left. \mid \vec{b}_l \in \mathbf{A}_l + \vec{c} \text{ for every } l \in \{1, \dots, l^*\} \right\}$$

The proof for (b) follows a similar approach. ■**Proposition 6.5bis: Monotonicity over capability domination** *If for all $s_l \in S$ we have $\mathbf{A}_l \subseteq \mathbf{B}_l^c$, then $\overline{\mathbf{A}} \subseteq \overline{\mathbf{B}}^c$.***Proof** For every beings $\vec{b}$ in $\overline{\mathbf{A}}$, where $\vec{b} = \sum_{l=1}^{l^*} p(s_l) \cdot X_{i,l}$ and $X_i = (\vec{b}_1, \dots, \vec{b}_{l^*})$, there is a set of beings $X'_i = (\vec{b}'_1, \dots, \vec{b}'_{l^*})$ such that $b_l \leq b'_l$ for all l (since for all $s_l \in S$ we have $\mathbf{A}_l \subseteq \mathbf{B}_l^c$). Thus $\vec{b}$ is an element of $\overline{\mathbf{B}}^c$ with $\vec{b}' = \sum_{l=1}^{l^*} p(s_l) \cdot X'_{i,l}$ and $\vec{b}'$ is in $\overline{\mathbf{A}'}$. ■

We provide now counterexamples for the propositions not fulfilled by average capability sets.

Proposition 6.2bis: Possible domination of the average capability *The average capability set is not always dominated by the set of all possible states, that is, we can have $\overline{\mathbf{A}} \not\subseteq \bigcup_{l=1}^{l^*} \mathbf{A}_l^c$.***Proof** See Examples 6.3, 6.4, or Set 1 and 3 of Figure 6.5. ■

Proposition 6.6bis: Non-monotonicity over probabilities *If we have $\mathbf{A}_z \subseteq \mathbf{A}_{z'}^c$ with $(s_z, s_{z'}) \in S^2$ and*

$$p'(s_l) = \begin{cases} p(s_l) + c & \text{for } l = z' \\ p(s_l) - c & \text{for } l = z \\ p(s_l) & \text{Otherwise} \end{cases}$$

with $c \in (0; p(s_z)]$. Then, we do not necessarily have $\overline{\mathbf{A}} \subseteq \overline{\mathbf{A}'}^c$.

Proof Consider a decision problem with two states of the world, s_1 and s_2 , and their associated capability set $\mathbf{A}_1 = \{(0, 1)\}$ and $\mathbf{A}_2 = \{(0, 1), (1, 0)\}$. We have $\mathbf{A}_1 \subseteq \mathbf{A}_2^c$

Initially, the policy makers base their decision on $\overline{\mathbf{A}}$ with equal subjective probabilities ($p(s_1) = p(s_2) = 0.5$), resulting in $\overline{\mathbf{A}} = \{(0, 1), (0.5, 0.5)\}$.

Now, with increased importance on the “best” set (e.i. $\mathbf{A}_2$), the new choice is $\overline{\mathbf{A}'} = \{(0, 1), (0.75, 0.25)\}$ when using subjective probabilities $p'(s_1) = 0.25$ and $p'(s_2) = 0.75$, resulting in $\overline{\mathbf{A}} \not\subseteq \overline{\mathbf{A}'}$. ■

RÉSUMÉ

Cette thèse explore l'application de l'Approche par les Capabilités (CA) dans les cadres d'aide à la décision, avec un accent particulier sur la conception des politiques publiques. La CA est présentée comme une alternative aux mesures traditionnelles de bien-être, offrant un cadre multidimensionnel qui prend en compte la diversité et la subjectivité. Nous proposons une amélioration de l'approche en intégrant des facteurs systémiques à travers la programmation linéaire, obtenant ainsi des ensembles de capabilités sous forme de frontières de Pareto. Une étude de faisabilité a été développée en appliquant la CA pour évaluer les capabilités de santé mentale des personnes âgées en Ile de France.

Plutôt que de comparer et d'agréger des solutions, y compris multidimensionnelles, de nouvelles méthodologies doivent être développées pour comparer et agréger des ensembles de solutions Pareto-efficaces. La deuxième partie de ce manuscrit est donc consacrée à l'adaptation des méthodes classiques d'agrégation à la CA, en se concentrant spécifiquement sur l'agrégation des ensembles de capabilités en présence d'une fonction d'utilité et sur la gestion de l'incertitude, avec ou sans probabilités subjectives.

MOTS CLÉS

Politiques publiques ; Approche par les capabilités ; Analyse multicritères ; Aide à la décision ; Théorie du risque et de l'incertitude

ABSTRACT

This research explores the application of the Capability Approach (CA) within decision-aiding frameworks, focusing on public policy design. The CA is presented as an alternative to traditional welfare measures, offering a multidimensional framework that accounts for diversity and subjectivity. We propose an improved approach by incorporating systemic factors through mixed-integer linear programming, obtaining capability sets as Pareto frontiers. A proof of concept was developed by applying the CA to assess mental health capabilities of older adults in the urban context of Paris.

Rather than comparing and aggregating, possibly multidimensional, solutions, new methodologies are required to compare and aggregate sets of Pareto-efficient solutions. The second part of this manuscript is dedicated to adapting classical aggregation methods to the CA, specifically focusing on aggregating capability sets in the presence of a utility function and dealing with uncertainty, both with and without subjective probabilities.

KEYWORDS

Policy analytics; Capability Approach; Multi-criteria analysis; Decision aiding; Risk and Uncertainty Theory