Complexity and bifurcation of the identity

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▶ **To cite this version:**

Mathieu Helfter. Complexity and bifurcation of the identity. Dynamical Systems [math.DS]. Sorbonne Université, 2024. English. NNT : 2024SORUS362 . tel-04906262

HAL Id: tel-04906262

https://theses.hal.science/tel-04906262v1

Submitted on 22 Jan 2025

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THÈSE DE DOCTORAT DE MATHÉMATIQUES

Opérée au sein de :

Sorbonne Univeristé, IMJ-PRG, CNRS, ERC EMERGENCE

École Doctorale de Sciences Mathématiques de Paris Centre, ED 386

Complexité et bifurcation de l'identité

Complexity and bifurcation of the identity

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Remerciements

J'ai bien cru, avant de me lancer dans la grande aventure de la thèse, qu'il s'agirait d'un travail en solitaire. Je suis bien heureux aujourd'hui de vous dire que je me suis trompé. Ces années ont été des années de rencontres et d'échanges. J'en sors grandi, mathématiquement et comme individu.

Il va de soi de commencer par remercier mon directeur de thèse, Pierre Berger, avec la plus grande attention possible. Je suis infiniment reconnaissant pour ton accompagnement exceptionnel tout au long de cette thèse. Pédagogue et patient, tu m'as offert un soutien sans faille. Je me sens privilégié d'avoir pu bénéficier de longs échanges de qualité et de la vision mathématique d'un chercheur de ton envergure. Tu as su me guider dans un domaine aussi vaste que les systèmes dynamiques.

S'ensuit un grand merci aux rapporteurs Jérôme Buzzi et Bassam Fayad, qui ont relu le manuscrit avec enthousiasme et assiduité. Les discussions et corrections que vous m'avez proposées sont des contributions précieuses qui ont permis d'améliorer ce manuscrit. Mes remerciements vont également aux autres membres du jury de cette thèse qui me font l'honneur de leur présence : Aihua Fan, François Ledrappier et Maria Saprykina, des mathématiciens aux contributions variées que j'admire profondément.

Je tiens également à exprimer une reconnaissance particulière à Nicolas Gourmelon qui, par notre collaboration avec Pierre, fait partie des artisans de cette thèse. Je suis marqué par ta créativité mathématique et tes qualités humaines. Je tiens également à remercier les référés anonymes qui ont participé à l'amélioration des travaux présentés. J'aimerais ensuite remercier Sébastien Biebler pour ses conseils avisés, Dimitri Turaev pour ses dicussions passionnantes, Romain Dujardin pour m'avoir fait rencontrer Pierre, Martin Leguil pour m'avoir permis d'exposer mes travaux devant Aihua Fan, David Burguet pour les dicussions et l'intérêt qu'il a porté à mon travail.

Je voudrais remercier toutes les personnes qui m'ont invité à des conférences, tant pour y exposer mes travaux que pour découvrir ceux des autres. Merci aussi à tous ceux que je croise et recroise en séminaires et qui rendent ces moments encore plus chers, notamment : Pierre-Antoine Guihéneuf, Yutaka Ishii, Patrice Le Calvez, Raphaël Krikorian, Pablo Carrasco, Sylvain Crovisier, Anna Florio, Zhiyuan Zhang.

Il y a aussi parmi les doctorants, de l'équipe AA, et d'ailleurs, qui ont marqué mon parcours. D'abord celle de Thomas que je suis particulièrement heureux de compter aujourd'hui comme un ami ; et sans qui je n'aurais pas fait la belle découverte de Roman, Bilal et Anna. Ensuite, Nelson et Lucas, avec qui j'ai partagé un bureau, et dont la bonne humeur et l'intérêt pour les mathématiques sont contagieux. Un grand merci à Odylo Costa mon "petit frère" de thèse avec qui les discussions sont synonymes de partage et de soutien. J'aimerais aussi remercier Yann mon second "petit frère" que je m'empresse de connaitre plus. Je remercie aussi Anna, Pietro, Joaqín, João, Thiago, Francesco, Eva, Nastaran, Alexandre, Sergi, Jérôme, Lucas et Pierre qui ont participé à rendre ces années encore plus agréables.

J’en profite également pour remercier l’ensemble du personnel du laboratoire. Je tiens particulièrement à dire un grand merci à Gaëlle Callouard pour son aide décisive tout au long de ces années de thèse ; ainsi qu’à Corentin Lacombe qui a permis à cette soutenance d’avoir lieu.

J’ai la chance précieuse d’avoir des amis exceptionnels qui m’ont soutenu et encouragé tout au long de ces années : Paul, Firiel, Antoine, Martin, Théo, Anna, Lila, Eliott, Paola, Carol-Anne, Mouna, Pauline, Kaspar.

Il m’est impossible de conclure sans remercier ma famille. Tout d’abord, ma mère, Claire, mon père, Marc, et mon frère Nicolas, dont la tendresse et le soutien infatigables m’accompagnent à chaque instant. Une pensée égale pour le reste de ma famille : grands-parents, oncles, tantes, cousins et cousines, avec qui chaque retrouvaille est un bonheur incommensurable. Enfin, je tiens à remercier Clara, qui partage ma vie et mes émotions, quelles qu’elles soient.

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Chapitre 1

Introduction (version française)

1.1 Organisation du manuscrit de thèse

Cette thèse est articulée autour de trois chapitres. Le premier expose le travail issu de l'article *"Scales"* [Hel24], en français "Echelles". Ce chapitre s'inscrit dans la théorie de la dimension avec des applications en probabilité, en systèmes dynamiques et en analyse fonctionnelle. Les échelles sont des familles d'invariants de plusieurs natures : Hausdorff, paquet, recouvrement, quantisation ... et de différents taux de croissances, contenant ainsi la dimension ou l'ordre et permettant en particulier l'étude d'espaces de dimension infinie. Dans ce chapitre, les théorèmes établis de comparaison entre les différentes versions de dimensions sont étendus au cadre plus général des échelles. Ces résultats de comparaisons sont ensuite appliqués pour décrire la géométrie des espaces fonctionnels, de la mesure de Wiener et de la décomposition ergodique d'un système dynamique conservatif.

Le second chapitre présente une collaboration avec P. Berger et N. Gourmelon qui a mené à l'article intitulé *"Every Diffeomorphism is a total renormalization of an arbitrarily close to identity map"* [BGH24], en français : "Tout difféomorphisme est la renormalisation totale d'une application arbitrairement proche de l'identité". Nous améliorons un résultat fondateur de D. Turaev [Tur15b]. Précisément, nous démontrons que pour tout $1 \leq r \leq \infty$, tout difféomorphisme à support compact dans la composante connexe de l'identité, sur une variété de la forme $\mathbb{R}/\mathbb{Z} \times M$ peut être obtenu comme renormalisation totale d'une application C^r-proche de l'identité. Autrement dit, il existe une application g arbitrairement proche de l'identité telle que l'application de premier retour de g dans un certain domaine soit conjuguée à f, et de plus, l'orbite du domaine recouvre toute la variété. Notre preuve utilise de nouveaux outils qui bénéficient de propriétés de la théorie des groupes et des algèbres de Lie.

Enfin, le dernier chapitre s'inscrit dans la théorie de la bifurcation. Dans cette partie sont discutées les premières étapes établissant le lien entre la typicité au sens de Kolmogorov et la généricité localisable en dimension 3. En particulier, une preuve alternative et géométrique d'un résultat de Gochenko-Meiss-Ovsyannikov [GMO06] est proposée. Leur résultat décrit l'apparition de points périodiques totalement paraboliques par perturbation lisse de difféomorphismes présentant des tangences homoclines. Cette nouvelle preuve permet d'obtenir une version paramétrique de ce résultat.

Mots clés : *Systèmes dynamiques, théorie de la mesure, dimensions, espaces de dimension infinie, systèmes dynamiques différentiables, renormalisation, tangence homocline, bifurcation.*

1.2 Motivations

Plongeons d'abord dans le cadre général de l'étude des systèmes dynamiques et de leur propension au chaos. Nous considérons ici des dynamiques discrètes qui décrivent l'évolution d'un système en temps discret. Dans ce contexte, un système dynamique est donné par un espace de phases X doté d'une certaine structure : topologique, différentielle, mesurable. Les états possibles du système étudié correspondent à des points de cet espace X. Dans le cadre déterministe, l'évolution d'un point est donnée par un endomorphisme $f : X \to X$ compatible avec la structure dont X est muni. Par exemple : continu, différentiable, mesurable. L'information sur l'évolution temporelle d'un état initial $x \in X$ est donnée par son *orbite* :

$$\mathrm{Orb}(x) := \{x, f(x), f \circ f(x), f \circ f \circ f(x), \dots\}.$$

L'orbite est simplement le chemin que le système suit en temps discret dans l'espace des états, étant donné une condition initiale $x \in X$.

Des objets naturels dans l'étude des systèmes dynamiques sont les *attracteurs*. Ce sont des ensembles d'états invariants par la dynamique et qui capturent dans le futur les trajectoires voisines. Les attracteurs peuvent prendre diverses formes, comme des points fixes, des orbites périodiques, ou encore d'autres ensembles présentant des géométries plus complexes et affichant des comportements étranges, voire chaotiques. L'étude de leurs géométries a contribué à la popularisation de la géométrie fractale et au développement de la théorie de la dimension. Plus généralement, la théorie de la dimension cherche à quantifier la "taille" de certains objets mathématiques et fournit un moyen de mesurer la complexité, la structure et les propriétés géométriques des espaces métriques et de leurs mesures. Différentes versions des dimensions existent, dévoilant ainsi divers aspects de la géométrie des objets étudiés.

Berger et Bochi, dans [BB21], ont montré un lien entre la dimension des ensembles invariants hyperboliques basiques d'un difféomorphisme conservatif lisse d'une surface et la taille de l'espace des mesures ergodiques invariantes par ce difféomorphisme. La quantité introduite pour décrire la taille de cet espace par Berger est l'*émergence* [Ber16]. Pour des systèmes complexes, cet espace est de dimension infinie, correspondant à une haute émergence, et peut avoir une géométrie a priori difficile à appréhender. Par ailleurs, nombreux sont les objets naturels de dimensions infinies tels que certains processus aléatoires ou les espaces fonctionnels, qui ont conduit à de nombreuses études de leurs propriétés géométriques (voir par exemple [Kol30], [KT93], [Ber17], [Ber20], [DL05], [Klo15]). Toutes ces considérations ont motivé l'introduction de la notion d'*échelles* dans [Hel24]. Les échelles sont des invariants qui vérifient les mêmes inégalités entre leurs différentes versions (Hausdorff, paquet, recouvrement, quantisation, locales) que dans le cadre de la dimension et permettent en particulier d'étudier la géométrie des espaces de dimension infinie mentionnés ci-dessus.

Une autre approche importante des dynamiques pour étudier la complexité d'un système est la théorie de la *bifurcation*. Elle mène à la naissance de nouveaux attracteurs, à leur destruction ou à des changements dans leur comportement dynamique et leur géométrie. Pour simplifier, une bifurcation correspond à une discontinuité dans le comportement du système lorsque l'un de ses paramètres varie. Cela conduit à la naissance de nouveaux types de comportement qui n'apparaissaient pas auparavant dans le système. Les exemples classiques incluent les applications de Hénon

ou le système de Lorenz. Ces applications exhibent un riche éventail de comportements, y compris la formation d'attracteurs étranges et une dépendance sensible aux conditions initiales.

Un type de bifurcation parmi les plus célèbres est le déploiement des *tangences homoclines*. Étant donnée une régularité $r \geq 1$, possiblement $r = \infty$ ou $r = \omega$, et une variété lisse M, la stabilité d'un difféomorphisme $f \in \mathrm{Diff}^r(M)$ près d'un point fixe est liée aux espaces invariants de sa différentielle $D_P f$. Dans le cas où cette différentielle est diagonalisable, rappelons que le point fixe P est *hyperbolique* si toutes les valeurs propres de la différentielle ont un module différent de 1. De plus, c'est un *puits* si tous les modules des valeurs propres sont inférieurs à 1, une *source* s'ils sont supérieurs à 1, et une *selle* sinon. Lorsque l'on considère un point selle, l'ensemble des points dont l'orbite converge dans le futur vers la selle est non vide et forme un ensemble appelé *variété stable*. Le théorème de la variété stable affirme que c'est une variété lisse de dimension égale au nombre de ses valeurs propres - comptées avec multiplicité - ayant un module strictement inférieur à 1. Cette variété est tangente à l'espace tangent stable de la différentielle. Inversement, la variété instable d'un point selle correspond aux préorbites qui convergent vers le point fixe. Dans le cas des difféomorphismes, c'est également une variété lisse bien définie de dimension égale au nombre de valeurs propres ayant un module strictement supérieur à 1 et est tangente à l'espace instable de la selle. Bien que le point fixe corresponde à une intersection transverse de ces variétés, elles peuvent éventuellement se croiser encore une infinité de fois. Une configuration particulièrement intéressante est celle des *tangences homoclines*. Les tangences homoclines sont des intersections non transverses entre les variétés stable et instable. Elles sont cruciales dans les systèmes dynamiques car elles marquent souvent la transition d'un régime simple vers un régime complexe. Tout particulièrement, Palis a formulé la conjecture suivante :

Conjecture 1.2.1 (Palis [Pal00](Conjecture II))**.** *En toute dimension, les difféomorphismes ayant une tangence homocline ou un cycle heterodimensionel sont C^r-denses dans le complémentaire de la fermeture des difféomorphismes hyperboliques pour tout $r \geq 1$.*

Même à proximité de dynamiques très bien comprises, il peut y avoir des bifurcations menant à la complexité ou au chaos. Un travail majeur, qui est l'une des principales motivations de cette thèse, est le résultat suivant de D. Turaev :

Théorème 1.2.2 (Turaev [Tur15b])**.** *Pour tout $1 \leq r \leq \infty$, dans l'espace des difféomorphismes préservant l'orientation de classe C^r de la boule unité $\mathbb{B}^n$ de dimension n dans $\mathbb{R}^n$ pour $n \geq 2$,il existe un ensemble* résiduel *$\mathcal{S}_r$ tel que pour toute application $F \in \mathcal{S}_r$, pour tout $\delta > 0$ et pour toute boule D de dimension n, il existe une application $g : \mathbb{R}^n \to \mathbb{R}^n$, qui coincide avec l'identité en dehors de D, telle que $\|g - \mathrm{id}\|_{C^r} < \delta$ et que F soit une* renormalisation *de g, c'est-à-dire qu'il existe un changement de coordonnées local tel qu'une itération de g coïncide avec F.*

Dans sa preuve, Turaev associe les "lèvres d'Iljashenko" à des renormalisations d'application de premier retour près de tangences homoclines permettant d'obtenir toute fonction de type Hénon. Cela permit à Turaev d'exhiber des phénomènes dynamiques chaotiques à proximité de l'identité en régularité C^∞. Dans [BGH24], nous améliorons son théorème en obtenant après renormalisation n'importe quel difféomorphisme au lieu d'un sous-ensemble résiduel. Outre la boule unité n-dimensionnelle nous considérons aussi les variétés de la forme $M \times \mathbb{T}$, où $\mathbb{T}$ est le cercle et M une

variété à bords de dimension au moins 1. Pour ces variétés nous obtenons une *renormalisation totale* : l'orbite par g du domaine où l'itération de g est conjuguée à F recouvre toute la variété. Pour obtenir ces différentes améliorations, nous proposons une méthode différente de celle de Turaev [Tur15b] qui utilise des outils de théorie des groupes et d'algèbre de Lie. Cette dernière amélioration conduit à de nouvelles applications telles que l'existence de difféomorphismes de Bernoulli arbitrairement proches de l'identité préservant une forme volume lisse.

Maintenant que nous savons que des perturbations de l'identité permettent d'atteindre toute dynamique par renormalisation, la question inverse et complémentaire reste ouverte : Quelles sont les dynamiques lisses qui se renormalisent en l'identité après une perturbation arbitrairement petite ? Encore une fois, l'étude des tangences homoclines joue un rôle clé. Mentionnons deux autres travaux qui motivent le dernier chapitre de cette thèse. Tout d'abord, le travail de Gonchenko-Meiss-Ovsyannikov dans [GST08]. Dans cet article sont étudiées les bifurcations de difféomorphismes tridimensionnels ayant une tangence homocline pour un point selle périodique ayant une valeur propre complexe non réelle. En considérant une déformation non-dégénérée à trois paramètres du difféomorphisme initial, ils ont montré qu'il existe une infinité de régions ouvertes de paramètres s'accumulant autour de 0, pour lesquelles la forme normale de l'application de premier retour dans un voisinage d'un point de tangence homocline est un difféomorphisme de type Hénon tridimensionnel. En conséquence, si le déterminant du jacobien du point selle est égal à 1, ils obtiennent par petite perturbation un point neutre périodique, c'est-à-dire ayant des valeurs propres égales à 1. Par ailleurs,Gochenko-Shilnikov-Turaev [GST08] ont réalisé un travail similaire pour les difféomorphismes tridimensionnels ayant des cycles hétérocliniques équidimensionnels non transverses. Si le volume est dilaté sur une orbite périodique du cycle et contracté sur une autre, alors l'application de premier retour est un difféomorphisme de type Hénon tridimensionnel. De plus, les points fixes de cette application peuvent avoir n'importe quelle combinaison de valeurs propres lorsque les paramètres du système varient.

Dans le dernier chapitre, une preuve alternative et géométrique d'un théorème de Gonchenko-Meiss-Ovsyannikov [GST08] est fournie. Leur théorème montre en particulier qu'il existe un point fixe totalement parabolique, i.e. dont les valeurs propres sont toutes sur le cercle unité, par petite perturbation lisse d'un difféomorphisme tri-dimensionnel ayant une selle avec une tangence homocline. Dans ce dernier chapitre, la construction géométrique est étendue aux familles prouvant ainsi une version paramétrique de ce résultat. Cette preuve est en partie inspirée de la construction en dimension 2 réalisée par Berger dans [Ber17].

1.3 Échelles

La théorie de la dimension a gagné en popularité grâce à l'article fondateur de Mandelbrot, "*How Long Is the Coast of Britain ?*" [Man67], qui a mis en lumière la question générale de la taille d'objets mathématiques naturels. La catégorie d'objets considérée comprend les espaces métriques possiblement munis d'une mesure. La théorie de la dimension englobe non seulement l'étude des espaces lisses tels que les variétés, mais aussi celle d'espaces à géométries plus complexes comme les fractales. Ainsi, la dimension peut être n'importe quel nombre réel positif. Il existe plusieurs notions de dimension : par exemple, la dimension de Hausdorff [Hau18], de paquet [Tri82] ou de recouvre-

ment (ou boîte). De plus, lorsque l'espace est muni d'une mesure, s'ajoutent la dimension locale et celle de quantisation. Ces différentes versions de la dimension sont des invariants métriques qui ne sont généralement pas égaux. Ainsi, ils révèlent différents aspects de l'espace étudié. Les travaux de Hausdorff [Hau18], Frostman [Fro35], Tricot [Tri82], Fan [Fan94, FLR02], Tamashiro [Tam95], Pötzelberger [Pöt99], Graf-Luschgy [GL07] ont permis de décrire la relation entre ces différentes notions et ont donné certaines des conditions sous lesquelles elles coïncident.

Évidemment, ces invariants ne fournissent pas beaucoup d'informations sur les espaces de dimension infinie. Cependant, de tels espaces font l'objet de nombreuses études. Plus particulièrement, Kolmogorov-Tikhomirov dans [KT93] ont donné des équivalents asymptotiques du nombre de recouvrement des espaces fonctionnels. Dereich-Lifshits dans [DL05] ont donné le comportement asymptotique de la masse des petites boules pour la mesure de Wiener et l'ont comparé avec le problème de quantisation. Voir aussi [DFMS03, CM44, Chu47, BR92, KL93]. De plus, Berger et Bochi [BB21] ont donné l'ordre de recouvrement et de quantisation de la décomposition ergodique de certains systèmes dynamiques conservatifs lisses. Voir également [BR92, Klo15, BB21]. Ces considérations conduisent à la question naturelle suivante :

Question. *Y a-t-il des analogues en dimension infinie des différentes versions de dimension qui révèlent les mêmes différentes propriétés géométriques des objets étudiés et vérifiant les mêmes inégalités qui les comparent ?*

Pour répondre à cette question, nous introduisons la notion d'*échelle*. L'idée est de considérer un *échelonnage*, c'est une famille à un paramètre de fonctions de "jauge" vérifiant certaines hypothèses, qui prescrit l'étalonnage que l'on doit effectuer pour que la taille de l'espace soit correctement étudiée. Par exemple, les familles associées à la *dimension* ou à l'*ordre* données dans Exemple 1.3.2 Page 12 sont des échelonnages. Etant donné un échelonnage, différentes versions d'échelles sont définies. En particulier, la dimension de Hausdorff, la dimension de paquet ou la dimension de recouvrement sont des exemples d'échelles.

Nous généraliserons les théorèmes de comparaison entre les différentes versions de dimensions aux différentes versions d'échelles dans les Théorèmes A, B et C. La définition de l'échelonnage est choisie de sorte que les preuves des résultats des Théorèmes A et B soient des généralisations presque directes du cas établi de la dimension.

La principale difficulté sera alors de prouver le Théorème C qui permet de comparer les échelles de quantisation avec les échelles locales et de recouvrement. Même dans le cadre de la dimension finie, de nouvelles inégalités entre la dimension de quantisation d'une mesure et la dimension de recouvrement des ensembles de mesure positive sont énoncées (inégalités (f) et (h)) dans le Théorème C).

1.3.1 De la dimension à l'échelle

Commençons par rappeler quelques définitions classiques de la théorie de la dimension et voyons comment elles sont naturellement étendues pour définir des invariants finis pour des espaces de dimension infinie. Les dimensions de Hausdorff, de paquet et de recouvrement d'un espace métrique totalement borné (X, d) sont définies en examinant des familles de sous-ensembles de X. Pour les dimensions de recouvrement, étant donnée une erreur $\epsilon > 0$, rappelons que le nombre de recouvrement $\mathcal{N}_\epsilon(X)$ est le cardinal minimal d'un recouvrement de X par des boules de rayon ϵ.

Les dimensions inférieure et supérieure de recouvrement de (X, d) sont données par :

$$\underline{\dim}_B X := \sup\{\alpha > 0 : \phi_\alpha(\epsilon) \cdot \mathcal{N}_\epsilon(X) \to +\infty\} \quad \text{et} \quad \overline{\dim}_B X := \inf\{\alpha > 0 : \phi_\alpha(\epsilon) \cdot \mathcal{N}_\epsilon(X) \to 0\} \ ,$$

où $(\phi_\alpha)_{\alpha>0}$ est la famille de fonctions sur $(0, 1)$ donnée pour $\alpha > 0$ par $\phi_\alpha : \epsilon \mapsto \epsilon^\alpha$.

En général, les dimensions supérieure et inférieure de recouvrement ne coïncident pas (voir par exemple [FF97, Fal04]). Cependant, lorsque X est une variété lisse munie d'une métrique euclidienne, ces définitions coïncident avec la définition usuelle de la dimension. Les propriétés de base des dimensions de recouvrement se révèlent lorsque l'on regarde des sous-espaces d'un espace ambiant munis de la métrique induite. Notamment, ces dimensions ne sont pas invariantes par union dénombrable d'espaces metriques de même dimensions et sont invariantes par fermeture topologique. La version la plus populaire de la dimension qui jouit de la propriété d'être stable par union dénombrable est la *dimension de Hausdorff*. Rappelons sa définition. Étant donnée une erreur $\epsilon > 0$, considérons :

$$\mathcal{H}^\alpha_\epsilon(X) := \inf_{C \in \mathcal{C}_H(\epsilon)} \sum_{B(x,\delta) \in C} \phi_\alpha(\delta) \ ,$$

où $\mathcal{C}_H(\epsilon)$ est l'ensemble des recouvrements dénombrables de X par des boules de rayon au plus ϵ. La *dimension de Hausdorff* de (X, d) est alors donnée par :

$$\dim_H X = \sup\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} \ .$$

Enfin, une autre dimension intéressante est la *dimension de paquet*. Elle est elle aussi stable par unions dénombrables. Sa construction est analogue à celle de la dimension de Hausdorff et a été introduite par Tricot dans sa thèse [Tri82]. Elle est en fait liée à la dimension supérieure de recouvrement par la caractérisation suivante que nous utiliserons ici comme définition :

$$\dim_P X := \inf \sup_{n\geq 1} \overline{\dim}_B E_n \ ,$$

où l'infimum est pris sur les recouvrements dénombrables $(E_n)_{n\geq 1}$ par des sous-ensembles de X.

Ces quatre versions de la dimension sont des invariants bi-Lipschitz. Elles quantifient différents aspects de la géométrie de l'espace métrique étudié, car elles ne coïncident pas a priori. Cependant, les inégalités suivantes sont toujours vérifiées par le théorème classique suivant :

Théorème 1.3.1. *Soit (X, d) un espace métrique. Les inégalités suivantes sont vérifiées :*

$$\dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X \quad \textit{et} \quad \dim_H X \leq \dim_P X \leq \overline{\dim}_B X \ .$$

Les preuves détaillées de ces résultats sont par exemple dans les livres de Falconer [FF97] et [Fal04]. Cependant, nous démontrons à toutes ces inégalités dans le premier chapitre. Ces inégalités sont optimales : il existe des espaces où toutes ces inégalités sont des égalités et d'autres où les inégalités sont toutes strictes.

Introduisons alors la notion d'*échelle*. Une observation simple est que, dans toutes les définitions ci-dessus des différentes versions de la dimension, la famille paramétrée $(\phi_\alpha)_{\alpha>0} = (\epsilon \mapsto \epsilon^\alpha)_{\alpha>0}$ joue un rôle particulier. Il s'agit d'une famille de fonctions de jauge avec un comportement polynomial. Rappelons que, classiquement, une fonction de jauge est une fonction qui remplace la simple

mesure des rayons des boules utilisée dans la définition de la mesure de Hausdorff afin de donner une précision plus fine sur la dimension de l'espace et potentiellement obtenir une mesure finie non triviale.

L'idée de la construction des échelles prend une direction transverse. Au lieu de chercher un affinement de la notion de dimension, nous prendrons des fonctions dont le comportement est potentiellement loin d'être polynomial car nous voulons décrire des espaces de dimension infinie. Nous veillerons tout de même à ce que les échelles englobent les différentes définitions de dimension.

Précisons. Si un espace (X, d) est de dimension infinie, alors son nombre de recouvrement $\mathcal{N}_\epsilon(X)$ croît plus rapidement que tout polynôme en ϵ^{-1} lorsque ϵ décroît vers 0. Ainsi, pour espérer définir des invariants finis pour les espaces de dimension infinie, nous devons autoriser l'utilisation d'autres fonctions de jauge qui décroissent plus rapidement que tout polynôme lorsque le rayon des boules impliquées décroît vers 0.

En conséquence, nous proposons ici de remplacer dans toutes les définitions des dimensions ci-dessus, la famille $(\phi_\alpha)_{\alpha>0} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)\alpha > 0$ par une autre famille de fonctions. Parmi ces familles de fonctions nous voulons inclure les exemples suivants :

Exemple 1.3.2.
1. *La famille* $\mathsf{dim} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$ *qui est utilisée dans les définitions de dimensions,*
2. *la famille* $\mathsf{ord} = (\epsilon \in (0,1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$ *appelée* ordre. *Elle correspond à la croissance du nombre de recouvrement d'espaces de fonctions de régularité finie étudiés par Kolmogorov-Tikhomirov [KT93], ou à celle de l'espace des mesures ergodiques par Berger-Bochi [BB21],*
3. *la famille* $(\epsilon \in (0,1) \mapsto \exp(-(\log \epsilon^{-1})^\alpha))_{\alpha>0}$ *qui correspond à la croissance du nombre de recouvrement de fonctions holomorphes estimée par Kolmogorov-Tikhomirov [KT93].*

Cependant, pour étendre correctement les définitions et les théorèmes de comparaison entre les échelles, c'est-à-dire la généralisation des dimensions de recouvrement, de Hausdorff et de paquet, la famille $(\phi_\alpha)\alpha > 0$ doit satisfaire certaines hypothèses, ce qui conduit à introduire la notion d'*échelonnage* :

Définition 1.3.3 (Echelonnage)**.** *Une famille* $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$ *de fonctions positives et croissantes sur* $(0,1)$ *est un* échelonnage *lorsque pour tout* $\alpha > \beta > 0$ *et tout* $\lambda > 1$ *suffisamment proche de* 1*, les estimations suivantes sont vérifiées :*

$$\mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \textit{et} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right), \ \textit{lorsque } \epsilon \to 0; . \tag{$*$}$$

Remarque 1.3.4. *Il existe des échelles qui permettent d'étudier les espaces de dimension* 0*, par exemple :*

$$(\epsilon \in (0,1) \mapsto \log(\epsilon^{-1})^{-\alpha})_{\alpha>0} \ .$$

Nous montrons que les familles mentionnées dans l'exemple ci-dessus sont effectivement des échelonnages. Regardons maintenant comment les échelonnages permettent de définir les échelles. Etant donné un espace métrique (X, d), et nous remplaçons la famille spécifique dim dans la définition des dimensions de recouvrement par un échelonnage quelconque $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$.

Nous obtenons alors la définition suivante :

Définition 1.3.5 (Echelles de recouvrement). *Les* échelles de boîte inférieure et supérieure *d'un espace métrique* (X, d) *sont définies par :*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

et

$$\overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} .$$

Similairement, les notions de dimensions de Hausdorff et de paquet sont généralisées à celles d'*échelle de Hausdorff* notée $\mathsf{scl}_H X$ et d'*échelle de paquet* notée $\mathsf{scl}_P X$ en remplaçant la famille dim de l' Exemple 1.3.2 par l'échelonnage correspondant. Ces notions dépendent bien évidemment du choix de l'échelonnage. Les constructions sont entièrement détaillées dans le premier chapitre. Passons maintenant à l'énoncé des principaux résultats sur la comparaison des échelles des espaces métriques.

1.3.2 Resultats de comparaison d'échelles

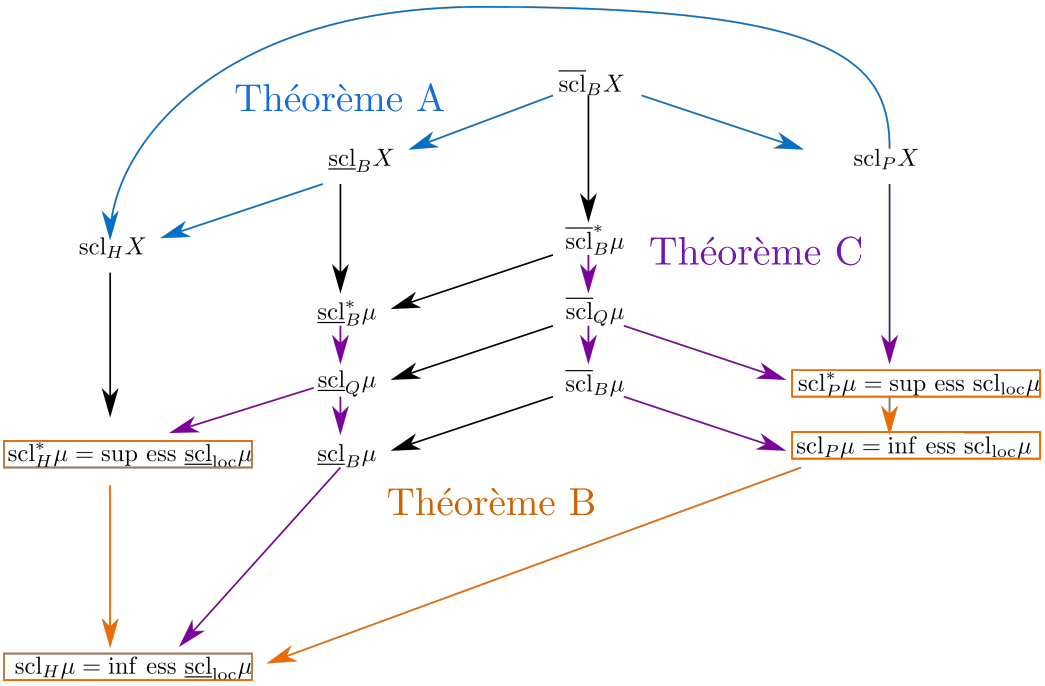

FIGURE 1.1 – Diagramme présentant les résultats des théorèmes sur les échelles d'espaces métriques. Chaque flèche représente une inégalité, l'échelle au point de départ de la flèche est plus grande que celle à son arrivée : " $\to$ " = " $\geq$ ".

Tout d'abord, nous présentons la généralisation suivante des inégalités bien établies comparant les dimensions des espaces métriques :

Théorème A ([Hel24]). *Soit* (X, d) *un espace métrique et* scl *un échelonnage. Les inégalités suivantes sont vérifiées :*

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \quad \textit{et} \quad \mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X .$$

L'essentiel de la preuve consiste à montrer que les échelles de Hausdorff et les échelles de paquet sont des quantités bien définies. Il est alors ensuite aisé d'adapter les preuves rédigées par Falconer dans [FF97] et [Fal04] pour démontrer le Théorème A.

Il est possible de montrer que pour chacune des inégalités dans la Figure 1.3.2 il existe un espace probabilisé tel que l'inégalité est stricte. Par ailleurs il existe des espaces où toutes les inégalités sont des égalités. Ce résultat est bien connu dans le cadre de la dimension, voir par exemple [FF97, Fal04].

Lorsque l'espace métrique (X, d) est équipé d'une mesure μ, nous introduisons les *échelles locales* qui étendent les notions de dimensions locales d'une mesure :

Définition 1.3.6 (Echelles locales)**.** *Soit μ une mesure borélienne sur un espace métrique (X, d) et* scl *un échelonnage. Les* échelles inférieure et supérieure *de μ sont les fonctions qui à un point $x \in X$ associent :*

$$\underline{\mathsf{scl}}_{loc}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} 0\right\}$$

et

$$\overline{\mathsf{scl}}_{loc}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} +\infty\right\} .$$

Comme dans la théorie de la dimension, nous devons comparer les échelles locales avec les échelles des sous-ensembles de X ayant une masse positive. Cela conduit à considérer les quantités suivantes :

Définition 1.3.7 (Echelles de Hausdorff, de paquet et de recouvrement d'une mesure)**.** *Soit* scl *un échelonnage et μ une mesure borélienne non nulle sur un espace métrique (X, d). Pour tout $\mathsf{scl}_\bullet \in \left\{\mathsf{scl}_H, \mathsf{scl}_P, \underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\right\}$, nous définissons les échelles inférieure et supérieure de la mesure μ par :*

$$\mathsf{scl}_\bullet\mu = \inf_{E\in\mathcal{B}}\left\{\mathsf{scl}_\bullet E : \mu(E) > 0\right\} \quad \textit{et} \quad \mathsf{scl}^*_\bullet\mu = \inf_{E\in\mathcal{B}}\left\{\mathsf{scl}_\bullet E : \mu(X\backslash E) = 0\right\} ,$$

où $\mathcal{B}$ est l'ensemble des sous-ensembles boréliens de X.

Dans le cadre de la dimension, Frostman [Fro35], Tricot [Tri82], Fan [Fan94] et [FLR02] avec Lau et Rao, et Tamashiro [Tam95] ont mis en évidence la relation entre les dimensions de Hausdorff et de paquet des mesures et leurs dimensions locales que nous généralisons comme suit :

Théorème B ([Hel24])**.** *Soit μ une mesure borélienne sur un espace métrique (X, d), alors pour tout échelonnage* scl*, les échelles de Hausdorff et de paquet de μ sont caractérisées par :*

$$\mathsf{scl}_H\mu = \operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu, \quad \mathsf{scl}^*_H\mu = \operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu,$$

$$\mathsf{scl}_P\mu = \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu, \quad \mathsf{scl}^*_P\mu = \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu ,$$

où $\operatorname{ess\,sup}$ *et* $\operatorname{ess\,inf}$ *désignent le suprémum essentiel et l'infimum essentiel d'une fonction.*

La preuve de ce résultat est inspirée de celles des théorèmes de Fan-Lau-Rao [Fan94, FLR02].

Introduisons un dernier type d'échelle, l'*échelle de quantisation*. Elle généralise la dimension de quantisation qui a suscité de nombreux travaux [GL07, Pöt99, DFMS03, DL05, Ber17, BB21, Ber20].

Définition 1.3.8 (Echelles de quantisation). *Soit (X, d) un espace métrique et μ une mesure borelienne sur X. Le* nombre de quantisation $\mathcal{Q}_\mu$ *de μ est la fonction qui à une erreur $\epsilon > 0$ associe le cardinal minimal d'un ensemble de points qui est en moyenne ϵ-proche de tout point de X :*

$$\mathcal{Q}_\mu(\epsilon) = \inf\left\{N \geq 0 : \exists\, \{c_i\}_{i=1,\dots,N} \subset X, \int_X d(x, \{c_i\}_{1\leq i\leq N})\, d\mu(x) \leq \epsilon\right\}\ .$$

Alors les échelles de quantisation inférieure et supérieure *de μ pour un échelonnage* scl *donné sont définies par :*

$$\underline{\mathsf{scl}}_Q\mu = \sup\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

et

$$\overline{\mathsf{scl}}_Q\mu = \inf\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

Les différentes échelles de mesure sont comparées dans le résultat suivant :

Théorème C ([Hel24]). *Soit (X, d) un espace métrique. Soit μ une mesure borélienne sur X. Pour tout échelonnage* scl*, les inégalités suivantes sur les échelles de μ sont vérifiées :*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(a)} \underline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(b)} \underline{\mathsf{scl}}_Q\mu \quad ; \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(c)} \overline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(d)} \overline{\mathsf{scl}}_Q\mu$$

et

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(e)} \underline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(f)} \underline{\mathsf{scl}}^*_B\mu \quad ; \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(g)} \overline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(h)} \overline{\mathsf{scl}}^*_B\mu\ .$$

Même dans le cas spécifique de la dimension, ces inégalités (b) et (d) sont nouvelles. Aussi, les inégalités (e) et (g) ont été démontrées par Pötzelberger dans [Pöt99] pour la dimension mais seulement dans le cas où $X = [0, 1]^d$ avec d un entier.

1.3.3 Application des théorèmes de comparaisons d'échelles

Pour conclure cette présentation du premier chapitre, regardons quelques applications des théorèmes de comparaisons des échelles pour l'ordre. Rappelons que l'ordre est le second échelonnage ord donné dans l'Exemple 1.3.2.

Espaces fonctionnels Une première application du Théorème A couplé aux estimations par Kolmogorov et Tikhomirov[KT93][Thm XV] du nombre de recouvrement des espaces de fonctions de régularité finie, ainsi qu'une nouvelle estimée, permet d'obtenir des informations géométriques sur ces espaces.

Soit d un entier positif. Pour tout entier $k \geq 0$ et pour tout $a \in [0, 1]$, introduisons :

$$\mathcal{F}^{d,k,a} := \left\{f \in C^k([0, 1]^d, [-1, 1]) : \|f\|_{C^k} \leq 1,\ D^k f \text{ est } a\text{-Hölder de constante } 1\right\}.$$

Cet espace est muni de la norme C^0. En plongeant un groupe dont l'ordre de Hausdorff est borné inférieurement dans $\mathcal{F}^{k,d,a}$, via une application dilatante ; et comme application directe du Théorème A et du théorème de Kolmogorov-Tikhomirov [KT93][Thm XV] sur l'estimation du nombre de recouvrement de $\mathcal{F}^{d,k,a}$, on obtient :

Théorème D ([Hel24]). *Soit d un entier positif. Pour tout entier k et pour tout $a \in [0, 1]$, on a :*

$$\mathsf{ord}_H\mathcal{F}^{d,k,a} = \mathsf{ord}_P\mathcal{F}^{d,k,a} = \underline{\mathsf{ord}}_B\mathcal{F}^{d,k,a} = \overline{\mathsf{ord}}_B\mathcal{F}^{d,k,a} = \frac{d}{k+a}.$$

Mesure de Wiener Le second exemple concerne les ordres de la mesure de Wiener W qui décrit un mouvement brownien standard unidimensionnel sur $[0,1]$. À partir de l'estimation de la masse des boules aléatoires pour les normes L^p pour $p \in [1,\infty]$ par Dereich-Lifshits [DL05][3.2, 5.1, 6.1, 6.3] et par application directe du Théorème C, on obtient :

Corollaire A ([Hel24]). *La mesure de Wiener sur $C^0([0,1],\mathbb{R})$ munie de la norme L^p, pour $p \in [1,\infty]$, vérifie pour presque tout $\omega \in C^0([0,1])$:*

$$\begin{aligned} 2 = \underline{\mathsf{ord}}_{loc} W(\omega) = \mathsf{ord}_H W = \mathsf{ord}^*_H W = \overline{\mathsf{ord}}_{loc} W(\omega) = \mathsf{ord}_P W \\ = \mathsf{ord}^*_P W = \underline{\mathsf{ord}}_B W = \underline{\mathsf{ord}}_Q W = \overline{\mathsf{ord}}_B W = \overline{\mathsf{ord}}_Q W \ . \end{aligned}$$

Par exemple, soit A un sous-ensemble de $C^0([0,1])$ ayant une masse de Wiener positive. Alors son nombre de recouvrement croît asymptotiquement plus vite que $\exp(\epsilon^{-2+\delta})$ lorsque ϵ tend vers 0 pour tout $\delta > 0$ fixé.

Lien entre émergence locale et globale Le cadre des échelles permet également de répondre à un problème posé par Berger dans [Ber20] sur l'émergence de systèmes dynamiques conservatifs. La réponse à cette question est une application directe du Théorème C. Rappelons d'abord quelques définitions. Soit (X,d) un espace métrique compact, $f : X \to X$ une application mesurable et μ une mesure f-invariante sur X. En vertu du théorème de Birkhoff, pour μ-presque tout $x \in X$, la limite des mesures empiriques $e(x)$ est bien définie :

$$e^f(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \ .$$

L'espace $\mathcal{M}$ des mesures de probabilité sur X est muni de la distance de Wasserstein W_1 définie par :

$$W_1(\nu_1,\nu_2) = \sup_{\phi \in \mathrm{Lip}^{\,1}(X)} \int \phi d(\nu_1 - \nu_2) \ ,$$

pour deux mesures $\nu_1, \nu_2 \in \mathcal{M}$ et où $\mathrm{Lip}^{\,1}(X)$ désigne l'ensemble des fonctions 1-Lipschitziennes sur X à valeurs réelles. Elle induit la topologie faible-étoile pour laquelle $\mathcal{M}$ est compact. Une manière de quantifier la complexité de la décomposition ergodique est d'examiner le comportement asymptotique de l'émergence $\mathcal{E}_\mu$ de la mesure μ :

$$\mathcal{E}_\mu(\epsilon) := \min\{N \in \mathbb{N} \ : \ \exists \nu_1,\dots,\nu_N \in \mathcal{M}(X), \ \int_X W_1(e^f(x), \{\nu_i\}_{1\le i\le N}) d\mu(x) \le \epsilon\} \ ,$$

lorsque $\epsilon \to 0$. Par application directe de l'inégalité (g) du Théorème C, on obtient :

Théorème E ([Hel24]). *Pour μ-presque tout $x \in X$, on a :*

$$\limsup_{\epsilon\to 0} \frac{\log(-\log \mu(\mathcal{B}_\epsilon(x)))}{-\log\epsilon} \le \limsup_{\epsilon\to 0} \frac{\log\log \mathcal{E}_\mu(\epsilon)}{-\log\epsilon} \ ,$$

où $\mathcal{B}_\epsilon(x) := \{y \in X : W_1(e(y),e(x)) < \epsilon\}$.

Observons alors que $\mu(\mathcal{B}_\epsilon(x))$ est la masse de la boule $B_{W^1}(e^f(x),\epsilon)$ pour la decomposition ergodique $e^f_*\mu$. Le théorème ci-dessus met alors en lien le comportement asymptotique de la masse des petites boules pour la décomposition ergodique avec l'ordre d'émergence de la mesure.

1.4 Tout difféomorphisme est la renormalisation totale d'une application proche de l'identité

L'article intitulé "Every diffeomorphism is a total renormalization of a close to identity map" est un travail commun avec Pierre Berger et Nicolaz Gourmelon. Le travail de cet article occupe le second chapitre de cette thèse.

Soit $1 \leq r \leq \infty$ et Vune variété de la forme $\mathbb{R}/\mathbb{Z} \times M$ où M est une variété à bords de dimension au moins 1. Nous montrons que tout difféomorphisme sur V qui est isotope à l'identité par un chemin à support compact dans $V \backslash \partial V$, est une renormalisation totale d'un difféomorphisme C^r-proche de l'identité. En d'autres termes, pour un tel difféomorphisme f, il existe une application g arbitrairement proche de l'identité telle que l'application de premier retour de g dans un certain domaine soit conjuguée à f et de plus l'orbite de ce domaine est égale à $\mathbb{R}/\mathbb{Z} \times M$. Cela nous permet de localiser près de l'identité l'existence de propriétés dynamiques comme le fait d'être Bernoulli pour une forme volume lisse. Rappelons d'abord la définition de renormalisation. Soit $\mathbb{B}^n$ la boule fermée unitaire de $\mathbb{R}^n$.

Définition 1.4.1 (Renormalisation primitive)**.** *Une* renormalisation G *d'un difféomorphisme* $g \in \mathrm{Diff}(\mathbb{B}^n)$ *est une conjuguaison d'une itération de* g*. En d'autres termes, il existe* $N \geq 2$*, un plongement* $\psi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ *tel que* $g^i(\psi(\mathbb{B}^n)) \cap \psi(\mathbb{B}^n) = \emptyset$ *pour tout* $0 < i < N$ *et* $g^N(\psi(\mathbb{B}^n)) = \psi(\mathbb{B}^n)$*, et la renormalisation est donnée par :*

$$G = \psi^{-1} \circ g^N \circ \psi.$$

Un problème de longue date en systèmes dynamiques est le suivant :

Problème 1.4.2. *Quelles dynamiques peuvent être atteintes par renormalisation d'applications proches de l'identité ?*

Ce problème a été étudié pour la première fois par Ruelle et Takens dans [RT71]. Motivés par l'étude de la *turbulence*, ils ont montré que pour tout entier $n \geq 2$, toute dynamique sur le tore de dimension $n \geq 2$ est la renormalisation d'une application C^n-proche de l'identité. Il en suit l'existence arbitrairement proche de l'identité de difféomorphismes ayant un "attracteur étrange". Ils ont alors émis l'hypothèse que ce phénomène apparait naturellement en dynamique des fluides et pourrait être utilisé comme définition mathématique de la notion de turbulence. Voir aussi [Lor63].

Le principal inconvénient mathématique de ce résultat est que la régularité est limitée par la dimension du tore. Cependant, en considérant des flots, ce problème a été résolu par Newhouse, Ruelle et Takens dans [NRT78] : étant donné un champ vectoriel X dont le flot est une rotation sur le tore $\mathbb{T}^n$, $n \geq 3$, et une application $F_0 \in \mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ homotope à l'identité, ils ont perturbé X de sorte que l'application de premier retour d'une section transverse soit F_0. Cependant, en faisant cela, ils rajoutent une dimension. Et comme en effet $\mathrm{Diff}^\infty(\mathbb{T}^n)$ est "bien plus grand" que $\mathrm{Diff}^\infty(\mathbb{T}^{n-1})$, le Problème 1.4.2 reste non résolu.

Une percée majeure a ensuite été réalisée par Turaev [Tur15b], qui a prouvé qu'un *ensemble résiduel* de plongements de $\mathbb{B}^n$ dans $\mathbb{R}^n$ préservant l'orientation peut être obtenu après renormalisation d'une application arbitrairement proche de l'identité, en toute régularité C^r pour tout $0 \leq r \leq \infty$.

Dans [BGH24], nous améliorons le théorème de Turaev pour obtenir ***n'importe quelle*** application préservant l'orientation et C^r de $\mathbb{B}^n$ (au lieu des applications parmi un sous-ensemble résiduel) :

Théorème F ([BGH24]). *Pour tout $1 \leq r \leq \infty$, tout $n \geq 2$ et toute application G préservant l'orientation de $\mathrm{Diff}^r(\mathbb{B}^n)$, dans n'importe quel voisinage $\mathcal{N}$ de l'identité, il existe $g \in \mathcal{N}$ tel qu'une renormalisation de g soit* égal *à G. De plus, la conjuguaison de cette renormalisation peut être choisie affine.*

Il reste alors naturellement à savoir si g peut être pris conservatif ou symplectique lorsque G l'est lui-même. Dans cette direction, mentionnons le travail de Gonchenko-Shilnikov-Turaev [GST07] qui ont montré que, pour tout $0 \leq r \leq \infty$, un sous-ensemble dense de difféomorphismes C^r-conservatifs sur le disque $\mathbb{B}^2$ peut être obtenu après renormalisation "quasi-périodique" (i.e. le domaine de renormalisation ne revient pas exactement sur lui-même après itération) d'une application conservative arbitrairement proche de l'identité. Par ailleurs, Fayad et Saprykina [FS22] ont montré que toute application conservative de la boule n-dimensionnelle peut être obtenue par renormalisation d'une perturbation conservative C^n-proche de l'identité. Dans leur construction, contrairement à celle de Gonchenko-Shilnikov-Turaev [GST07], Fayad et Saprykina ont pu obtenir un domaine de renormalisation exactement périodique.

Si tous ces théorèmes indiquent la richesse des comportements dynamiques possibles près de l'identité, l'objection suivante peut être faite. Dans le cadre de cette définition de la renormalisation, l'orbite de $\bigcup_{k=0}^{N-1} \psi(\mathbb{B}^n)$ du domaine de renormalisation peut être extrêmement petite et donc expérimentalement non observable. Cette objection n'a plus lieu d'être lorsque l'orbite du domaine de renormalisation recouvre toute la variété. Cela nous amène à renforcer la notion de renormalisation par ce qui suit :

Définition 1.4.3. *Soit $r \in \{1, \dots, \infty\} \cup \{\omega\}$ et soit V une variété (possiblement à bords). Une application $g \in \mathrm{Diff}^r(V)$ est* renormalisable *s'il existe une sous-variété stricte (possiblement à coins) $\Delta \subsetneq V$ telle que :*

- *il existe un difféomorphisme local bijectif $H : \Delta \to V$, appelé le* redimensionnement *du* domaine de renormalisation Δ,
- *le temps de premier retour $\tau : \Delta \to \mathbb{N}^*$ dans Δ par g est borné et la* renormalisation *définie par :*

$$G = H \circ g^\tau \circ H^{-1}$$

appartient à $\mathrm{Diff}^r(V)$.

L'application g est totalement renormalisable *si l'orbite en avant de Δ recouvre V. C'est-à-dire que V est égal à $\bigcup_{n \geq 0} g^n(\Delta)$. L'application G est alors une* renormalisation totale *de g.*

Précisons les espaces considérés. Soit V une variété compacte (éventuellement à bords) et $1 \leq r \leq \infty$. On rappelle que le *support* $\mathrm{supp}\, f$ de $f \in \mathrm{Diff}^r(V)$ est la fermeture de l'ensemble des points $x \in V$ tels que $f(x) \neq x$. Notons $\mathrm{Diff}_0^r(V)$ la composante connexe de l'identité dans $\mathrm{Diff}^r(V)$. On considère alors $\mathrm{Diff}_c^r(V)$ le sous-ensemble de $\mathrm{Diff}_0^r(V)$ formé par les applications isotopes à l'identité par des isotopies $(f_t)_{t \in [0,1]}$ dont le support $\bigcup_{t \in [0,1]} \mathrm{supp}\, f_t$ est un sous-ensemble compact de $V \setminus \partial V$. Observons alors que lorsque V est sans bord, on a $\mathrm{Diff}_0^r(V) = \mathrm{Diff}_c^r(V)$.

Une question naturelle est la suivante :

Question 1.4.4. *Pour quelles variétés V, toute application $F \in \mathrm{Diff}_c^r(V)$ est-elle une* renormalisation totale *d'une application proche de l'identité ?*

Jusqu'à présent, aucun exemple d'une telle variété V n'était connu. Dans ce travail avec Berger et Gourmelon nous avons donné une classe complète d'exemples.

Théorème G ([BGH24]). *Soit $0 \leq r \leq \infty$ et soit M une variété compacte de dimension au moins 1. Posons $V := \mathbb{T} \times M$. Soit $\mathcal{N} \subset \mathrm{Diff}^r(V)$ un voisinage de l'identité. Alors tout $G \in \mathrm{Diff}^r_c(V)$ est une renormalisation totale d'une application $g \in \mathcal{N}$.*

Si Théorème F implique que les phénomènes dynamiques *locaux* peuvent être trouvés à proximité de l'identité, Théorème G nous permet d'y obtenir des phénomènes dynamiques *globaux*. Cette nouvelle amélioration apportée par ce dernier résultat vient du fait que le domaine de renormalisation est plus grand que dans toutes les extensions précédentes des théorèmes de Ruelle-Takens : son orbite coïncide avec l'ensemble de la variété où vit la dynamique. Cela a permis de nouvelles applications telles que la preuve de l'existence de difféomorphismes préservant des mesures SRB lisses à proximité de l'identité ou des applications *universelles* dont l'union des domaines de renormalisation est arbitrairement grande dans la variété.

Le cadre de la preuve des théorèmes principaux repose sur un nouvel objet appelé *plugin*. Un plugin est une application renormalisable ayant une forme particulière qui possède une renormalisation canonique appelée *output*. Nous étudions le groupe des difféomorphismes qui sont des outputs de plugins arbitrairement proches de l'identité. Les propriétés topologiques de l'ensemble des plugins et l'étude de l'algèbre de Lie des champs de vecteurs ayant des flots dans ce groupe sont les ingrédients clés de la preuve de nos résultats. En fait, nous démontrons un résultat sur les plugins qui implique directement Théorème G.

Nous donnons également des théorèmes analogues pour les familles à paramètres de tous nos résultats. Cela mène à l'étude suivante.

1.5 Points totalement paraboliques près des tangences homoclines

Le dernier chapitre contient une étude des perturbations des tangences homoclines en dimension 3 pour les familles à paramètres.

Kolmogorov a introduit la notion de *typicité* lors des ICM en 1954 pour décrire les phénomènes qui se produisent pour presque tous les paramètres de familles génériques de dynamiques différentiables pour les espaces de dimension finie. Pour être plus précis :

Définition 1.5.1 (Arnold-Kolmogorov typicality). *Pour tout $k, r \geq 0$, une propriété $(\mathcal{P})$ sur une variété M est dite (k, r)-Kolmogorov* typique *s'il existe un ensemble Baire générique de familles C^r $(f_a)_{a\in\mathbb{R}^k}$ d'applications C^r telles que $(\mathcal{P})$ soit vérifiée par f_a pour presque tous les paramètres $a \in \mathbb{R}^k$.*

Ainsi, la typicité au sens de Kolmogorov contient une dimension probabiliste. Elle est donc a priori indépendante de la généricité de Baire. Par exemple, il existe un sous-ensemble Baire générique G du cercle ayant une mesure de Lebesgue nulle. Ainsi, le fait d'avoir tous les points elliptiques périodiques avec des valeurs propres dans G est une propriété Baire générique mais pas typique au sens de Kolmogorov pour les difféomorphismes de surface conservatifs. Un mécanisme permettant

de montrer la typicité de Kolmogorov de certains phénomènes dynamiques Baire génériques a été introduit par Berger dans [Ber16, Ber17], appelé *para-mélangeurs* qui généralise les *mélangeurs* de Bonatti-Díaz [BD96] aux familles à paramètres. En particulier, cela a permis de montrer dans [Ber16] que le célèbre phénomène de Newhouse est C^r-typique en dimension au moins 3 pour tout $1 \leq r < \infty$. Rappelons que le *phénomène de Newhouse* [New74, New79] correspond à l'existence d'ensembles localement génériques de dynamiques présentant un nombre infini de puits qui s'accumulent sur un fer à cheval de Smale. De plus, les para-mélangeurs ont permis à Berger dans [Ber21] de montrer la typicité au sens de Kolmogorov de la croissance sur-exponentielle du nombre de points périodiques en dimension au moins 2, pour les applications C^r avec $2 \leq r \leq \infty$. Voir aussi [BR21]. Berger dans [Ber20] a considéré la classe suivante de phénomènes génériques au sens de Baire et a conjecturé que cette classe devrait être typique au sens de Kolmogorov :

Définition 1.5.2 (Génériquement localisable)**.** *Une propriété $(\mathcal{P})$ sur une variété M est dite C^r-localisable si pour tout ensemble ouvert $U \subset M$ et pour tout $f \in \mathrm{Diff}^r(M)$ tel que $f^N_{|U} = \mathrm{id}_U$ pour un entier $N \geq 1$, il existe une perturbation arbitrairement petite dans $\mathrm{Diff}^r(M)$ de f qui vérifie $(\mathcal{P})$. De plus, cette propriété est ouvertement localisable s'il existe un ensemble ouvert de perturbations qui réalise la propriété. Enfin, la propriété est* génériquement localisable *si elle est impliquée par une conjonction dénombrable de propriétés ouvertement localisables, c'est-à-dire qu'il existe un ensemble dénombrable $\{(\mathcal{P}_i) : i \in \mathbb{N}\}$ de propriétés ouvertement localisables tel que :*

$$\bigwedge_{i\in\mathbb{N}} (\mathcal{P}_i) \Rightarrow (\mathcal{P}).$$

La définition ci-dessus introduite par Berger dans [Ber20] a été inspirée par le résultat suivant de Turaev :

Théorème 1.5.3 (Turaev [Tur15b])**.** *Pour toute surface M et $r \geq 2$, il existe un ouvert non vide $\mathcal{N}_r \subset \mathrm{Diff}^r(M)$ tel que pour un ensemble dense $\mathcal{D} \subset \mathcal{N}_r$, pour tout $f \in \mathcal{D}$, il existe un ouvert $U \subset M$ et un entier $N \geq 1$ tels que $f^N_{|U} = \mathrm{id}_U$.*

Énonçons maintenant la conjecture reliant les propriétés génériquement localisables à la typicité de Kolmogorov :

Conjecture 1.5.4 (1.5 (Berger [Ber20]))**.** *Soit M une variété lisse de dimension suffisamment grande et $2 \leq r \leq \infty$. Alors toute propriété génériquement localisable est (k, r)-Kolmogorov typique dans $\mathrm{Diff}^r(M)$, pour tout $k \geq 0$.*

Rappelons que nous montrons dans [BGH24] que toute famille lisse de difféomorphismes sur la boule unité de dimension $n \geq 2$ est une renormalisation d'une famille proche de l'identité de difféomorphismes en toute régularité $2 \leq r \leq \infty$. De plus, Turaev a annoncé que si un difféomorphisme présente un point périodique avec une différentielle égale à l'identité, alors une perturbation arbitrairement petite permet de créer un *îlot périodique*, c'est-à-dire un ouvert où une itération de la fonction coïncide avec l'identité. Ainsi, pour prouver Conjecture 1.5.4, il suffit de trouver des mécanismes pour créer des points périodiques pour certains ensembles ouverts de familles de paramètres.

Conjecture 1.5.5 (Berger)**.** *Soit M une variété de dimension au moins 3 et $1 \leq r < \infty$. Alors il existe un ensemble localement dense $\mathcal{D}$ de C^r-familles à paramètres de difféomorphismes de M, tel que pour*

chaque famille $(f_a)_a \in \mathcal{D}$, chaque f_a a un point périodique neutre, c'est-à-dire qu'il existe $P_a \in M$ tel que $\tilde{f}_a^p(P_a) = P_a$ et $D_{P_a}\tilde{f}_a^p = \mathrm{id}$ pour un certain entier p.

La première étape – le Théorème H énoncée ci-dessous – vers une preuve de la Conjecture 1.5.5 pour en dimension 3 est la version paramétrique du résultat suivant :

Théorème 1.5.6 (Gochenko-Meiss-Ovsyannikov)**.** *Soit f un difféomorphisme lisse d'une variété M de dimension 3. Supposons qu'il existe une selle périodique Ω ayant une valeur propre complexe et un déterminant égal à 1. On suppose de plus que Ω présente une tangence homocline en un point H. Alors il existe une perturbation C^∞-petite $\tilde{f}$ de f et un point périodique Q proche de H ayant pour valeurs propres $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$.*

Dans le dernier chapitre de cette thèse nous apportons la version paramétrique du précédent théorème :

Théorème H. *Soit $(f_a)_{a\in\mathbb{A}}$ une famille C^∞ de difféomorphismes lisses d'une variété M de dimension 3 et paramétrée par une variété connexe et compacte $\mathbb{A}$. Supposons qu'il existe des familles lisses de points $(H_a)_{a\in\mathbb{A}}$ et $(\Omega_a)_{a\in\mathbb{A}}$ et un angle non nul $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ tels que pour tout $a \in \mathbb{A}$ on ait :*

1. *Ω_a est une selle de f_a avec une valeur propre complexe d'argument θ et un jacobien égal à 1,*
2. *Ω_a est la continuation hyperbolique d'un certain Ω_{a_0} pour un paramètre $a_0 \in \mathbb{A}$,*
3. *H_a est un point de tangence homocline de Ω_a pour f_a.*

Alors il existe une C^∞-perturbation arbitrairement petite $(\tilde{f}_a)_{a\in\mathbb{A}}$ de $(f_a)_{a\in\mathbb{A}}$ et une famille lisse de points $(Q_a)_{a\in\mathbb{A}}$ tel que Q_a est périodique pour $\tilde{f}_a$ et a pour valeurs propres $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$.

Plus exactement, Gochenko-Meiss-Ovsyannikov ont donné la forme normale (une application Hénon tridimensionnelle) pour la renormalisation de l'application de premier retour à proximité de cette tangence homocline. A partir de ce résultat, le Théorème 1.5.6 peut être déduit. Nous proposons une preuve alternative de ce résultat basée sur une construction géométrique. Cette construction géométrique s'étend facilement aux familles à paramètres afin de montrer Théorème H sans calculs supplémentaires.

Chapter 2

Introduction (English version)

2.1 Organization of the thesis manuscript

This thesis is organized into three chapters. The first chapter presents the work from the article *"Scales"* [Hel24]. This chapter is embedded in dimension theory with applications in probability, dynamical systems, and functional analysis. Scales are families of invariants of various types: Hausdorff, packing, covering, quantization, etc., and with different growth rates encompassing dimension or order, thus allowing the study of infinite-dimensional spaces. In this chapter, well-established comparison theorems between different versions of dimensions are extended to the more general framework of scales. These comparison results are then applied to describe the geometry of functional spaces, the Wiener measure, and the ergodic decomposition of a measurable dynamical system.

The second chapter presents the article titled *"Every Diffeomorphism is a Total Renormalization of an Arbitrarily Close to Identity Map"* [BGH24] which is a joint work with P. Berger and N. Gourmelon. We improve a seminal result of D. Turaev [Tur15b] using a new proof based on tools from group theory and Lie algebras. More precisely, we show that for any $1 \leq r \leq \infty$, every diffeomorphism, in the connected component of the identity and isotopic to identity by a compactly supported path, of a manifold of the form $\mathbb{R}/\mathbb{Z} \times M$ where $\dim M \geq 1$, can be obtained as a total renormalization of a C^r-close to identity map. In other words, there exists a map g arbitrarily close to the identity such that the first return map of g in a certain domain is conjugate to f, and moreover, the orbit of the domain covers the entire manifold.

Finally, the last chapter contains a study of the perturbation of smooth diffeomorphisms presenting homoclinic tangencies. The initial steps of establishing the link between typicality in the sense of Kolmogorov and localizable genericity in dimension 3 are discussed. An alternative and geometric proof of a result of Gonchenko-Meiss-Ovsyannikov in [GST08] is brought. This states in particular that there exist totally parabolic periodic points, i.e. with eigenvalues on the unit circle, by arbitrarily small perturbation of a 3-dimensional diffeomorphism with a saddle with determinant 1 and displaying a homoclinic tangency. In this last chapter, the geometric construction is brought to a new parametric counterpart of this result. The construction takes some ideas from the 2-dimensional case shown by Berger in [Ber17].

Keywords: *Dynamical systems, measure theory, dimensions, infinite-dimensional spaces, differentiable dynamical systems, renormalization, homoclinic tangency, bifurcation.*

2.2 Motivations

Let us delve into the study of dynamics and their propensity for chaotic behavior. We consider discrete dynamics that describe the evolution of a system over discrete time steps. A dynamical system is given by a state space X endowed with some structure: topological, differential, measurable. Possible states of the studied systems correspond to points in X. For discrete deterministic dynamics, the evolution of a point is given by an endomorphism $f : X \to X$ that must be compatible with the structure on X, e.g., continuous, differentiable, measurable. The information of the time evolution of an initial state $x \in X$ is given by its *orbit*:

$$\mathrm{Orb}(x) := \{x, f(x), f \circ f(x), f \circ f \circ f(x), \dots\}.$$

The orbit is simply the path that the system follows in the phase space given the initial condition $x \in X$.

Natural objects of study for dynamical systems are *attractors*. They are sets of states that are invariant under the dynamics and capture nearby trajectories. Attractors can take various forms, including fixed points, periodic orbits, and more complex geometries allowing for strange or chaotic behaviors. The study of their geometries has contributed to the popularization of fractal geometry and the development of dimension theory. Dimension theory seeks to quantify the "size" of mathematical objects and provides a way to measure the complexity, structure, and geometric properties of metric spaces and their measures. Different versions of dimensions exist to encompass various aspects of the involved spaces. For instance Berger and Bochi in [BB21] the dimension of basic hyperbolic sets for a smooth conservative diffeomorphism of surface is related to the size of the space of ergodic measures on this set. The latter space is infinite dimensional and can display some special geometric properties. Moreover, many natural infinite dimensional objects of dynamical systems such as random processes or functional spaces gave birth to many studies of their geometric properties. For instance, one way to observe the complexity of a dynamical system is to see how big its set of ergodic measures is, leading to the notion of *emergence* [Ber16]. In the sense of emergence, complex dynamical systems correspond to dynamics where the space of ergodic measures is infinite dimensional. Following that path, the notion of *scales* is introduced in [Hel24] to study the geometry of infinite dimensional spaces by generalizing different classical notions of dimension. Another much-celebrated view point to study the complexity of a dynamical system is the theory of *bifurcation*. It leads to the birth of new attractors, the destruction of existing attractors, or changes in their dynamical behavior and geometry. Roughly speaking, bifurcations correspond to discontinuity in the behavior of the dynamical system as a parameter of the system varies. This leads to the birth of new types of behavior that did not appear in the system previously. Examples include the Hénon map or the Lorenz system. These maps exhibit a rich array of behaviors, including the formation of strange attractors and sensitive dependence on initial conditions. One rich type of bifurcation is the unfolding of *homoclinic tangencies*. Given a regularity $r \geq 1$ possibly $r = \infty$ or $r = \omega$, and a smooth

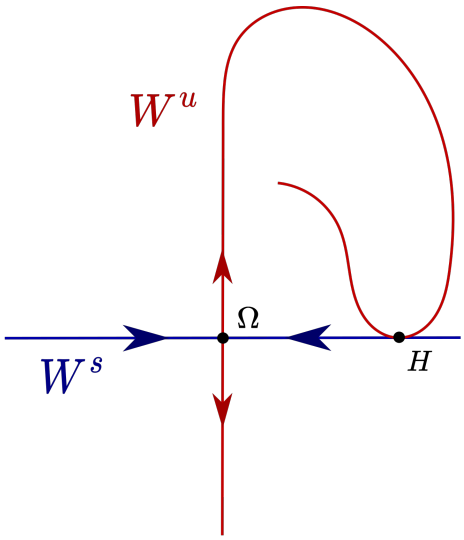

Figure 2.1 – Illustration of a Homoclinic Tangency

manifold M, the stability of a diffeomorphism $f \in \mathrm{Diff}^r(M)$ nearby a fixed point is related to the eigenspaces of its differential $D_P f$. Recall that, when $D_P f$ is diagonalizable, the fixed point P is *hyperbolic* if all the eigenvalues of the derivative have no eigenvalues of the complex unit circle. It is moreover a *sink* if all the moduli of the eigenvalues are smaller than 1, a *source* if they are greater than 1, and a *saddle* otherwise.

When considering a saddle point, the stable manifold theorem states that the set of points whose orbits converge to the saddle form a smooth embedded manifold with dimension equal to the number of eigenvalues with modulus smaller than 1. It is called the *stable manifold* of the saddle. This manifold is tangent to the stable tangent space of the differential at the saddle. Conversely, the *unstable manifold* of a saddle fixed point consists of points that converge to the saddle by backward iterations of the dynamic. In the case of diffeomorphisms, it is also a well-defined smooth manifold with dimension equal to the number of eigenvalues greater than 1 and it is tangent to the unstable space of the saddle. While these manifolds intersect transversely at the saddle, they could possibly intersect again infinitely many times. A specific configuration is the so called *homoclinic tangency*. Homoclinic tangencies are non-transverse intersections between the stable and unstable manifolds. They are crucial in dynamical systems because they often mark the onset of transitions from simple to complex dynamics. Let us state the following famous conjecture of Palis:

Conjecture 2.2.1 (Palis [Pal00](Conjecture II)). *In any dimension, diffeomorphisms displaying a homoclinic tangency or heterodimensional cycles are C^r-dense in the complement of the closure of hyperbolic dynamics for every $r > 1$.*

Consequently, even near simple maps with well-understood dynamics, there could be bifurcation leading to complexity and chaos. A major work that is one of the main motivations of this thesis is the following result from Turaev:

Theorem 2.2.2 (Turaev [Tur15b]). *For $1 \leq r < \infty$, in the space of C^r-smooth orientation-preserving diffeomorphisms of the unit n-dimensional ball $\mathbb{B}^n$ into $\mathbb{R}^n$ for $n \geq 2$, there exists a* residual *set $\mathcal{S}_r$ such that for every map $F \in \mathcal{S}_r$, for every $\delta > 0$ and for every n-dimensional ball D there exists a map*

$g : \mathbb{R}^n \to \mathbb{R}^n$, *equal to identity outside* D, *such that* $\|g - \mathrm{id}\|_{C^r} < \delta$ *and* F *is a* renormalization *of* g, *i.e. there exists a local change of coordinates such that an iteration of* g *coincides with* F.

His proof combines "Iljashenko lips" together with renormalizations of first return maps nearby homoclinic tangencies that allow us to obtain any Hénon-like map. It allowed Turaev to capture chaotic dynamical phenomena nearby the identity. In [BGH24], we improve the latter theorem by capturing any diffeomorphism instead of a dense subset. Moreover, we replace the n-dimensional ball by a manifold of the form $M \times \mathbb{T}$ where $\mathbb{T}$ is the circle and M a manifold possibly with boundaries. We propose a new approach for the proof that allows moreover the orbit of the renormalization domain, i.e. the orbit by g of the domain where the iteration of g is conjugated to F, to cover the whole manifold. This leads to new applications such as the existence of arbitrarily close to identity Bernoulli maps preserving a smooth volume form.

As we now know that perturbations of the identity allow us to reach any dynamic by renormalization, the reverse and complementary question remains open: Which dynamics renormalize into the identity after arbitrarily small perturbations ?

Again, the study of homoclinic tangencies seems to play a key role there. Let us mention the work of Gonchenko-Meiss-Ovsyannikov in [GST08]. They studied bifurcations of a three-dimensional diffeomorphism with a saddle-focus displaying a homoclinic tangency and having non-real eigenvalues and a determinant equal to 1. They have shown that in a three-parameter family deforming the initial map, there are infinitely many open regions of parameters accumulating to 0, for which the normal form of the first return map to a neighborhood of a homoclinic point is a three-dimensional Hénon-like map. As a consequence of their result, we can obtain by small perturbation a periodic neutral point. Also for three-dimensional diffeomorphisms with non-transversal equi-dimensional heteroclinic cycles, a similar work was done by Gochenko-Shilnikov-Turaev [GST08]. If volume is expanded at one periodic orbit of the cycle and contracted at another orbit, then the first-return map is a three-dimensional Hénon map, whose fixed points may have any combination of eigenvalues. In the last chapter, we provide an alternative and geometric proof of the Theorem of Gonchenko-Meiss-Ovsyannikov in [GST08] that extends to the parametric counterpart.

Let us now present the main results of the three chapters.

2.3 Scales

The first article is titled "Scales." Berger posed a question about the complexity of the ergodic decomposition of conservative dynamical systems. Specifically, for the ergodic decomposition of a conservative dynamical system, what is the link between the growth rate of the quantization of the ergodic decomposition and the asymptotic behavior of the mass of small balls ? The answer to this question falls within a broader framework that includes various applications describing geometric aspects of infinite-dimensional spaces.

A first application lies in the study of the geometry of the Wiener measure and complements an estimation by Dereich and Lifshits (2005). Another application refines a theorem by Kolmogorov and Tikhomirov (1958) on the study of functional spaces.

What are "scales"? By prescribing an arbitrarily growth given by a family of gauge functions,

called a *scaling*, we define different versions of scales such as Hausdorff, packing, box, local, and quantization scales that generalize their corresponding dimensions. These different versions of scales do not generally coincide. General theorems comparing them are proved, which extend established results from dimension theory. Thus, scales unify known comparison theorems for dimension with new comparison theorems between different versions of scales for infinite-dimensional objects. For instance, *orders* are different versions of scales providing geometric aspects of spaces of measures of compact metric spaces, classes of differentiable maps, or Brownian motion.

2.3.1 From Dimension to Scaling

The Hausdorff, packing, and box dimensions of a totally bounded metric space (X, d) are defined by looking at families of subsets of X. For the box dimension, given $\epsilon > 0$, consider the *ϵ-covering number* $\mathcal{N}_\epsilon(X)$ of (X, d), which is the minimal cardinality of a covering by ϵ-balls of X. Then the *lower and upper box dimensions* are usually defined by:

$$\underline{\mathsf{dim}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon \cdot \epsilon^\alpha(X) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{dim}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon \cdot \epsilon^\alpha(X) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}.$$

For the *Hausdorff dimension*, given $\epsilon > 0$ and $\alpha > 0$, denote:

$$\mathcal{H}^\alpha_\epsilon(X) := \inf_{(E_n)_{n\in\mathbb{N}}} \sum_{n\in\mathbb{N}} \mathrm{diam}(E_n)^\alpha,$$

where the infimum is taken over countable coverings $(E_n)_{n\in\mathbb{N}}$ of X such that $\mathrm{diam}(E_n) < \epsilon$ for every $n \in \mathbb{N}$. Then the *Hausdorff dimension* is given by:

$$\mathsf{dim}_H X = \sup\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}.$$

Hausdorff and box dimensions enjoy different properties. While the Hausdorff dimension is countably stable (the dimension of a countable union is the supremum of the dimensions of the elements of the union), the box dimensions are stable under topological closure. One way to define *packing dimension* is to modify the upper box dimension to make it countably stable:

$$\mathsf{dim}_P X = \inf \sup_{n\geq 1} \overline{\mathsf{dim}}_B E_n,$$

where the infimum is taken over countable coverings $(E_n)_{n\geq 1}$ of X. These four versions of dimension are bi-Lipschitz invariants; they quantify different aspects of the geometry of the studied metric space.

Now we can ask the following questions: *What can we say when the studied spaces have infinite dimensions? What kind of invariants can we introduce to generalize the latter versions of dimensions that allow us to study some kind of homogeneity and size of the spaces?* I replaced the specific family $(\phi_\alpha)_{\alpha>0} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$ with families of gauge functions, called scalings, defining comparable versions of *scales*, which encompass the following examples of growth:

Example 2.3.1. *1. The family* $\mathsf{dim} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$*, which is used in the definitions of dimensions.*

2. *The family* $\mathsf{ord} = (\epsilon \in (0,1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$*, which is used in the definition of* order *and fits for instance with spaces of finitely regular functions studied by Kolmogorov-Tikhomirov [KT93] or with the one of the space of ergodic measures in chaotic dynamics by Berger-Bochi [Ber20].*
3. *The family* $(\epsilon \in (0,1) \mapsto \exp(-(\log \epsilon^{-1})^{\alpha}))_{\alpha>0}$*, which fits with the growth of the covering number of holomorphic functions estimated by Kolmogorov-Tikhomirov [KT93].*

To keep a comparison between the generalizations of box, Hausdorff, and packing dimensions, the family $(\phi_\alpha)_{\alpha>0}$ must satisfy some mild assumptions, which leads to the notion of scaling:

Definition 2.3.2 (Scaling)**.** *A family* $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha\geq 0}$ *of positive non-decreasing functions on* $(0,1)$ *is a* scaling *when for any* $\alpha > \beta > 0$ *and any* $\lambda > 1$ *close enough to 1, it holds:*

$$\mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \textit{and} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right) \ \textit{as}\ \epsilon \to 0.$$

2.3.2 Scales of metric spaces

Scalings allow us to define *scales* that generalize packing dimension, Hausdorff dimension, box dimensions, quantization dimensions and local dimensions. For each scaling, the different kinds of scales do not a priori coincide on any space. Examples of metric spaces and measures where all those definitions coincide are presented. In those examples, those equalities coincide with some intuitive idea of homogeneity we might assume in those spaces. Now for a metric space (X,d), replacing the specific family dim in the definition of box dimensions by any scaling $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$ gives the following:

Definition 2.3.3 (Box scales)**.** Lower and upper box scales *of a metric space* (X,d) *are defined by:*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon(X)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon(X)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}\ .$$

Moreover, Hausdorff and packing dimensions are generalized to the *Hausdorff scale* denoted $\mathsf{scl}_H X$ and *packing scale* denoted $\mathsf{scl}_P X$ by replacing the family dim by the corresponding scaling scl.

In the specific case of dimension, inequalities between the different versions of dimensions were shown, for instance, by Tricot [Tri82] or Falconer [FF97, Fal04]. The following contains their results and extends to the description of infinite dimensional spaces:

Theorem A ([Hel24])**.** *Let* (X,d) *be a metric space and* scl *a scaling, the following inequalities hold:*

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \quad \textit{and} \quad \mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X\ .$$

An important consequence of the latter theorem is the following that refines an estimation of Kolmogorov and Tikhomirov [KT93][Thm XV].

Let d be a positive integer. For any integer $k \geq 0$ and for any $a \in [0,1]$, let us denote:

$$\mathcal{F}^{d,k,a} := \left\{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1,\ \ D^k f \text{ is } a\text{-Hölder with constant } 1 \right\}.$$

This space is endowed with the C^0 norm. By embedding a group whose Hausdorff order is bounded from below into $\mathcal{F}^{k,d,a}$, via an expanding map; and as a direct application of Theorem A, we obtain the following geometric interpretation of Kolmogorov-Tikhomirov [KT93][Thm XV] estimates of the covering number of $\mathcal{F}^{d,k,a}$:

Theorem B ([Hel24]). *Let d be a positive integer. For any integer k and for any $a \in [0,1]$, it holds:*

$$\mathsf{ord}_H \mathcal{F}^{d,k,a} = \mathsf{ord}_P \mathcal{F}^{d,k,a} = \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,a} = \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,a} = \frac{d}{k+a} .$$

This result gives hope to finding some - in a sense that should be precised - nicely distributed measure on the space $\mathcal{F}^{d,k,a}$.

2.3.3 Scales of measures

When we consider a Borel measure μ on the metric space (X,d), local scales are introduced and extend the notion of local dimensions:

Definition 2.3.4 (Local scales). *Let μ be a Borel measure on a metric space (X,d) and* scl *a scaling. The* lower and upper scales *of μ are the functions that map a point $x \in X$ to:*

$$\underline{\mathsf{scl}}_{loc}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} 0\right\}$$

and

$$\overline{\mathsf{scl}}_{loc}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} .$$

As in dimension theory, local scales should be compared with the scales of subsets of X with positive mass. In the case of dimension, Frostman [Fro35], Tricot [Tri82], Fan [Fan94] , Fan-Lau-Rao [FLR02] and Tamashiro [Tam95] exhibited the relationship between the Hausdorff and packing dimensions of sets of positive and total mass with the essential extrema of local dimensions of a measure. Those results are generalized to the framework of scales. A last kind of scales that are considered are the quantization scales. They generalize the quantization dimensions which dragged much research interest, for instance in [GL07, Pöt99, DFMS03, DL05].

Definition 2.3.5 (Quantization scales). *Let (X,d) be a metric space and μ a Borel measure on X. The* quantization number $\mathcal{Q}_\mu$ *of μ is the function that maps $\epsilon > 0$ to the minimal cardinality of a set of points that is on average ϵ-close to any point in X:*

$$\mathcal{Q}_\mu(\epsilon) = \inf\left\{N \geq 0 : \exists \{c_i\}_{i=1,\dots,N} \subset X, \int_X d(x,\{c_i\}_{1\leq i\leq N}) d\mu(x) \leq \epsilon\right\} .$$

Then lower and upper quantization scales *of μ for a given scaling* scl *are defined by:*

$$\underline{\mathsf{scl}}_Q\mu = \sup\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_Q\mu = \inf\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} .$$

I proved the following inequalities between scales of measures:

Theorem C ([Hel24]). *Let (X, d) be a metric space. Let μ be a non-zero measure on X. For any scaling* scl*, the following inequalities on the scales of μ hold:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu \leq \underline{\mathsf{scl}}_B\mu \leq \underline{\mathsf{scl}}_Q\mu \quad ; \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu \leq \overline{\mathsf{scl}}_B\mu \leq \overline{\mathsf{scl}}_Q\mu$$

and

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \leq \underline{\mathsf{scl}}_Q\mu \leq \underline{\mathsf{scl}}^*_B\mu \quad ; \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \leq \overline{\mathsf{scl}}_Q\mu \leq \overline{\mathsf{scl}}^*_B\mu \ ,$$

where:

$$\mathsf{scl}_B\mu := \inf\{\mathsf{scl}_B E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_B\mu := \inf\{\mathsf{scl}_B E : \mu(X\backslash E) = 0\}$$

for $\mathsf{scl}_B \in \{\underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\}$.

To end this first part, let us present two other applications that motivated the introduction of scales of measures.

Wiener measure The first example concerns the orders of the Wiener measure W that describes uni-dimensional standard Brownian motion on $[0, 1]$. From an estimation of the mass random ball for L^p-norms by Dereich-Lifshits [DL05][3.2, 5.1, 6.1, 6.3], we directly deduce that $\underline{\mathsf{ord}}_{\mathrm{loc}}(\omega) = 2 = \overline{\mathsf{ord}}_Q W$. From this, I deduced together with Theorem C that the following other invariants also coincide:

Theorem D ([Hel24]). *For the Wiener measure on $C^0([0, 1], \mathbb{R})$ endowed with the L^p-norm, for $p \in [1, \infty]$, verifies for W almost every $\omega \in C^0([0, 1])$:*

$$\begin{aligned} 2 = \underline{\mathsf{ord}}_{loc}W(\omega) = \mathsf{ord}_H W = \mathsf{ord}^*_H W = \overline{\mathsf{ord}}_{loc}W(\omega) = \mathsf{ord}_P W \\ = \mathsf{ord}^*_P W = \underline{\mathsf{ord}}_B W = \underline{\mathsf{ord}}_Q W = \overline{\mathsf{ord}}_B W = \overline{\mathsf{ord}}_Q W \ . \end{aligned}$$

For instance, any subset A of $C^0([0, 1])$ with positive Wiener mass verifies that its covering number grows asymptotically faster than $\exp(\epsilon^{-2+\delta})$ when ϵ decreases to 0 for any $\delta > 0$ fixed.

Local and global emergence The framework of scales moreover allows us to answer to a problem set by Berger in [Ber20] on the emergence of measurable dynamical systems. As a direct application of Theorem C, I answered to this question by the following: Let (X, d) be a compact metric space, $f : X \to X$ a measurable map and μ a Borel f-invariant measure on X. Recall by Birkhoff theorem that for μ-almost every $x \in X$, the limit of empirical measures $e(x)$ is well defined. Let us denote $\mathcal{M}$ the space of measure on X that we endow with the Wassertein distance W_1. One way to quantify the complexity of ergodic decomposition is to look at the asymptotic behavior of the emergence of the measure μ:

$$\mathcal{E}_\mu(\epsilon) := \min\{N \in \mathbb{N} \ : \ \exists \nu_1, \dots, \nu_N \in \mathcal{M}(X), \ \int_X W_1(e(x), \{\nu_i\}_{1\leq i\leq N}) d\mu(x) \leq \epsilon\} \ ,$$

when $\epsilon \to 0$. Note that the emergence is the quantization number of the ergodic decomposition of μ.

A last important application of Theorem C is the following:

Theorem E ([Hel24]). *For μ-almost every $x \in X$, it holds:*

$$\limsup_{\epsilon \to 0} \frac{\log(-\log \mu(\mathcal{B}_\epsilon(x)))}{-\log \epsilon} \leq \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(\epsilon)}{-\log \epsilon} \ ,$$

where $\mathcal{B}_\epsilon(x) := \{y \in X : W_1(e(y), e(x)) < \epsilon\}$.

Observe that $\mu(\mathcal{B}_\epsilon(x))$ is the mass of the ball $B_{W^1}(e^f(x), \epsilon)$ for the ergodic decomposition $e^f_* \mu$. Thus the latter theorem compares the asymptotic behaviour of the small balls for the ergodic decomposition with the emergence order of the measure.

2.4 Every diffeomorphism is a total renormalization of a close to identity map

The article titled "Every diffeomorphism is a total renormalization of a close to identity map" is a common work with Pierre Berger and Nicolaz Gourmelon. For any $1 \leq r \leq \infty$, we show that every diffeomorphism of a manifold of the form $\mathbb{R}/\mathbb{Z} \times M$ is a total renormalization of a C^r-close to identity map. In other words, for every diffeomorphism f of $\mathbb{R}/\mathbb{Z} \times M$, there exists a map g arbitrarily close to identity such that the first return map of g to a domain is conjugate to f and moreover the orbit of this domain is equal to $\mathbb{R}/\mathbb{Z} \times M$. This enables us to localize nearby the identity the existence of many properties in dynamical systems, such as being Bernoulli for a smooth volume form. Let $\mathbb{B}^n$ be the unit closed ball of $\mathbb{R}^n$.

Definition 2.4.1. *A* renormalization *G of a diffeomorphism $g \in \mathrm{Diff}(\mathbb{B}^n)$ is a rescaling of an iteration of g. In other words, there exists $N \geq 2$ and an embedding $\psi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ such that:*

$$G = \psi^{-1} \circ g^N \circ \psi.$$

A long standing open problem of dynamical systems theory is:

Problem 2.4.2 (1971). *Which dynamics can be reached by renormalization of close to identity maps?*

This problem was first studied by Ruelle and Takens in [RT71]. Motivated by the study of turbulence, they proved that for any integer $n \geq 2$, any dynamic on the n-dimensional torus is the renormalization of a C^n-close to identity map. This enabled them to construct perturbations of the identity map of the torus with a strange attractor. Based on this, they conjectured that this appears as well in fluid dynamics and could be used as a mathematical definition of the notion of turbulence. The main mathematical issue with this result is that the regularity is limited by the dimension of the torus. However, when considering flows, this problem was solved by Newhouse, Ruelle and Takens in [NRT78]: given any vector field X equal to a rotation on the torus $\mathbb{T}^n$, $n \geq 3$ and any map $F_0 \in \mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ homotopic to the identity, they perturbed X to $\widetilde{X}$ so that its first return map to a global transverse section is F_0. Yet they added one dimension so the mathematical Problem 2.4.2 remains unsolved.

A breakthrough was then performed in the seminal work of Turaev who proved that a C^r-*residual subset* of C^r-orientation preserving embeddings of $\mathbb{B}^n$ could be obtained after renormalization of an arbitrarily close to identity map, for every $0 \leq r \leq \infty$.

A first main result is a solution to Problem 2.4.2, where we improve Turaev's theorem to obtain, via a self-contained and new proof, *any* C^r-orientation preserving map of $\mathbb{B}^n$ (instead of maps among a dense or residual subset):

Theorem F ([BGH24]). *For any $1 \leq r \leq \infty$, any integer $n \geq 2$ and any orientation preserving $G \in \mathrm{Diff}^r(\mathbb{B}^n)$, in any neighborhood $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{B}^n)$ of the identity, there exists $g \in \mathcal{N}$ such that a renormalization of g is **equal** to G. Moreover, the rescaling map of this renormalization can be chosen affine.*

A natural open problem is whether g can be obtained conservative or symplectic when G is conservative or symplectic. In this direction let us mention the work of Gonchenko-Shilnikov-Turaev [GST07] who proved that, for every $0 \leq r \leq \infty$, a C^r- dense subset of volume preserving embeddings of $\mathbb{B}^2$ could be obtained after renormalization of an arbitrarily close to identity volume preserving map. Also, Fayad and Saprykina in [FS22] showed that any conservative map of the n-dimensional ball can be realized by renormalized iteration of a conservative C^n-perturbation of the identity.

If all these theorems tell us about the richness of dynamical behaviors nearby the identity, in the setting of Definition 2.4.1, the orbit of $\bigcup_{k=0}^{N-1} \psi(\mathbb{B}^n)$ of the renormalization domain might be extremely small when the renormalizable map is close to identity. This leads to physically non observable dynamical properties. To lift this objection we brought a different version of the notion of renormalization, allowing us to introduce the notion of total renormalization. We then obtain a renormalization domain intersecting every orbit of the map.

Definition 2.4.3. *Let $r \in \{1, \dots, \infty\} \cup \{\omega\}$ and let V be a manifold (with boundary). A map $g \in \mathrm{Diff}^r(V)$ is* renormalizable *if there exists a strict submanifold with corners $\Delta \subsetneq V$ such that:*

- *there exists a bijective, local diffeomorphism $H : \Delta \to V$, called the* rescaling map *of the* renormalization domain Δ,
- *the first return time $\tau : \Delta \to \mathbb{N}^*$ into Δ by g is bounded and the* renormalization $G = H \circ g^\tau \circ H^{-1}$ *belongs to* $\mathrm{Diff}^r(V)$.

The map g is totally renormalizable *if the forward orbit of Δ covers V, i.e. $\bigcup_{n \geq 0} g^n(\Delta) = V$. The map G is then a* total renormalization *of g.*

Let V be a compact manifold (possibly with corners) and $1 \leq r \leq \infty$. We recall that the *support* $\mathrm{supp}\, f$ of $f \in \mathrm{Diff}^r(V)$ is the closure of the set of points such that $f(x) \neq x$.

Definition 2.4.4. *Let $\mathrm{Diff}^r_0(V)$ be the component of the identity in $\mathrm{Diff}^r(V)$. Let $\mathrm{Diff}^r_c(V)$ be the subset of $\mathrm{Diff}^r_0(V)$ formed by maps isotopic to id through isotopies $(f_t)_{t \in [0,1]}$ whose support $\bigcup_{t \in [0,1]} \mathrm{supp}\, f_t$ is a compact subset of $V \setminus \partial V$.*

Observe that when V is boundaryless, it holds $\mathrm{Diff}^r_0(V) = \mathrm{Diff}^r_c(V)$. A natural question is:

Question 2.4.5. *For which manifold V, any map $F \in \mathrm{Diff}^r_c(V)$ is a* total *renormalization of a close to identity map?*

So far, no example of such a manifold V was known. In this work we gave a full class of examples:

Theorem G ([BGH24]). *Let $0 \leq r \leq \infty$, let M be a compact manifold of dimension ≥ 1 and put $V := \mathbb{T} \times M$. Let $\mathcal{N} \subset \mathrm{Diff}^r(V)$ be a neighborhood of the identity. Then any $G \in \mathrm{Diff}^r_c(V)$ is a total renormalization of some $g \in \mathcal{N}$.*

If Theorem F implies that every *local* dynamical phenomenon can be found nearby the identity, Theorem G implies that every *global* dynamical phenomenon can be found nearby the identity. A new improvement brought by the latter result is that the renormalization domain is larger than in all of the previous extensions of Ruelle-Takens theorems: its orbit coincides with the whole domain of the dynamic. This enabled new applications such as the proof of existence of maps preserving smooth SRB nearby the identity or universal maps whose renormalization domains decrease as slowly as we want.

The framework of the proof of the main theorems relies on a new object called *plugin*. A plugin is a renormalizable map of special form that has a canonical renormalization called output. We study the group of maps that are outputs of plugins arbitrarily close to identity. Topological properties of the set of such maps and the study of the Lie algebra of vector fields having flows in this group are the key ingredients of the proof of our main theorems. We also provide parametric counterparts of our results which lead to the following study.

2.5 Totally parabolic points nearby homoclinic tangencies

Kolmogorov introduced the notion of *typicality* at the ICM in 1954. to encompass phenomena that occur for almost every parameter of generic parameter families of differentiable dynamics for finite dimensional spaces. To be precise:

Definition 2.5.1 (Arnold-Kolmogorov typicality). *For every $k, r \geq 0$, a property $(\mathcal{P})$ on a manifold M is said to be (k, r)-Kolmogorov* typical *if there exists a Baire generic set of C^r-families $(f_a)_{a \in \mathbb{R}^k}$ of C^r-maps so that $(\mathcal{P})$ is satisfied by f_a for Lebesgue almost every parameter $a \in \mathbb{R}^k$.*

Thus, typicality in the sense of Kolmogorov encompasses some probabilistic aspect. It is a priori independent from typicality in the sense of Baire genericity. For instance there are Baire generic subsets G of the circle with null Lebesgue measure. Thus having all periodic elliptic points with eigenvalues in G is Baire generic but not Kolmogorov typical among conservative surface diffeomorphisms. On the other side, any Kolmogorov typical property is Baire generic. This leads to wondering how big is the class of Kolmogorov typical properties. Yet, a mechanism to show Kolmogorov typicality of some Baire generic dynamical phenomena was introduced by Berger in [Ber16, Ber17], called *para-blenders* that generalizes Bonatti-Díaz blenders to parameter families. In particular it enabled to show in [Ber16] that the so-called Newhouse phenomenon is C^r-typical in dimension at least 3 for $1 \leq r < +\infty$. Recall that *Newhouse phenomenon* [New74, New79] corresponds to the existence of locally generic sets of dynamics displaying infinitely many sinks that accumulate onto Smale's horseshoes. Moreover, parablenders allowed Berger in [Ber21] to show the Kolmogorov-typicality of the fast growth of the number of periodic points in dimension at least 2 for C^r-self mappings with $2 \leq r \leq \infty$. See also [BR21]. Berger in [Ber20] considered the following class of Baire generic phenomena and conjectured that it should be Kolmogorov typical:

Definition 2.5.2 (Generically localizable). *A property $(\mathcal{P})$ on a manifold M is said to be C^r- localizable if for every open set $U \subset M$ and for any $f \in \mathrm{Diff}^r(M)$ such that $f^N_{|U} = \mathrm{id}_U$ for some integer $N \geq 1$, then there exists an arbitrarily small perturbation in $\mathrm{Diff}^r(M)$ of f that verifies $(\mathcal{P})$. It is moreover openly localizable if there exists an open set of perturbations that realizes the property. Finally, the property is* generically localizable *if it is implied by a countable conjunction of openly localizable $\{(\mathcal{P}_i) : i \in \mathbb{N}\}$:*

$$\bigwedge_{i\in\mathbb{N}} (\mathcal{P}_i) \Rightarrow (\mathcal{P}) \ .$$

The above definition introduced by Berger in [Ber20] was inspired by the following result of Turaev:

Theorem 2.5.3 ((Turaev [Tur15b])). *For every surface M and $r \geq 2$, there is a non-empty open set $\mathcal{N}_r \subset \mathrm{Diff}^r(M)$ such that for a dense set $\mathcal{D} \subset \mathcal{N}_r$, for every $f \in \mathcal{D}$, there exists $U \subset M$ and an integer $N \geq 1$ such that $f^N_{|U} = \mathrm{id}_U$.*

Let us state the conjecture linking generically localizable properties to Kolmogorov typicality:

Conjecture 2.5.4 (Berger [Ber20](Conjecture 1.5)). *If M is a smooth manifold with sufficiently large dimension and $2 \leq r \leq \infty$. Then any generically localizable property is (k, r) -Kolmogorov typical in $\mathrm{Diff}^r(M)$, for any $k \geq 0$.*

Recall that it is shown in [BGH24] that every smooth family of diffeomorphisms on the n-dimensional ball is a renormalization of a close to identity family of diffeomorphisms for any regularity $2 \leq r \leq \infty$for every $n \geq 2$. This improved and generalized the seminal work of Turaev in [Tur15b].

Also, Turaev has announced that if a diffeomorphism displays a periodic point with Jacobian equal to the identity, then an arbitrarily small perturbation allows us to create a *periodic spot*, that is an open set on which the first return map is equal to the identity. Thus, to prove Conjecture 2.5.4 we need to find mechanisms to create periodic points for some open set of parameter families.

Conjecture 2.5.5 (Berger). *Let M be a manifold of dimension at least 3 and $1 \leq r < \infty$. Then there exists a locally dense set $\mathcal{D}$ of C^r-parameter families of diffeomorphisms of M, such that for every family $(f_a)_a \in \mathcal{D}$, each f_a displays a neutral periodic point, i.e. there exists $P_a \in M$ such that $f^p_a(P_a) = P_a$ and $D_{P_a} f^p_a = \mathrm{id}$ for some integer p.*

Here I present a first step –Theorem H –towards a proof of Conjecture 2.5.5 for the three dimensional case. The next step will be completed soon.

The non-parametric version of Theorem H can be deduced from [GMO06].

Theorem 2.5.6 (Gochenko-Meiss-Ovsyannikov [GMO06]). *Let f be a smooth diffeomorphism of 3-manifold M. Assume that there exists a hyperbolic saddle periodic point Ω with one complex eigenvalue and Jacobian determinant equal to 1 that displays a homoclinic tangency at some point H. Then there exists a small C^∞-perturbation $\tilde{f}$ of f and a periodic point Q close to H with multipliers $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$.*

Here is the parametric version of this result:

Theorem H. *Let $(f_a)_{a\in\mathbb{A}}$ be a C^∞-family of smooth diffeomorphisms of a 3-manifold M parameterized by a compact connected smooth manifold $\mathbb{A}$. Assume that there exists an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}\backslash\{0\}$ and smooth families of points $(H_a)_{a\in\mathbb{A}}$ and $(\Omega_a)_{a\in\mathbb{A}}$ such that for every $a \in \mathbb{A}$ it holds:*

1. *Ω_a is a saddle periodic point of f_a with one complex eigenvalue with argument θ and determinant equal to 1,*
2. *Ω_a is the hyperbolic continuation of Ω_{a_0} for some $a_0 \in \mathbb{A}$,*
3. *H_a is a point of homoclinic tangency for Ω_a.*

Then there exists an arbitrarily small C^∞-perturbation $(\tilde{f}_a)_{a\in\mathbb{A}}$ of $(f_a)_{a\in\mathbb{A}}$ and a smooth family of points $(Q_a)_{a\in\mathbb{A}}$ such that for every $a \in A$ the point Q_a is periodic for $\tilde{f}_a$ close to H_a and with eigenvalues $\{1, e^{2i\pi/3}, e^{-2i\pi/3}\}$.

Actually Gochenko-Meiss-Ovsyannikov gave the normal form (3-dimensional Hénon-like map) for the renormalization nearby this homoclinic tangency; from this Theorem 2.5.6 can be deduced. It seems possible to extend their bounds on derivatives to parameter families in order to obtain Theorem H. Instead we preferred providing a geometric proof of Theorem 2.5.6, which passes easily to the parametric case (without any extra computation). As a preparation for the next step toward Conjecture 2.5.5, we show that the hypotheses of Theorem 2.5.6 are locally dense in the space of diffeomorphisms. This provides the following corollary:

Corollary A. *For every $1 \leq r \leq \infty$, there exists a locally dense set of C^r-diffeomorphism of M formed by maps displaying a periodic point with eigenvalues equal to 1, $e^{2i\pi/3}$ and $e^{-2i\pi/3}$.*

For $r = 1$, this corollary is the principal result of Bonatti-Diáz [BDP03]. For $r = \infty$, this result is also a consequence of a Theorem of Gonchenko-Shilnikov-Turaev [GST09].

Chapter 3

Scales

Scales

Mathieu Helfter [1]

1. IMJ-PRG, CNRS, Sorbonne University, Paris University, partially supported by the ERC project 818737 Emergence of wild differentiable dynamical systems.

Abstract

The notion of *scale* is introduced to obtain geometric information on metric spaces and measures by generalizing part of dimension theory. Several versions of scales are introduced such as *Hausdorff, packing, box, local and quantization*. They are defined for different *growth*, allowing a refined study of infinite dimensional spaces. We prove general theorems comparing the different versions of scales. They are applied to describe geometries of ergodic decompositions, of the Wiener measure and of functional spaces. The first application solves a problem of Berger on the notions of emergence (2020); the second lies in the geometry of the Wiener measure and extends the work of Dereich-Lifshits (2005); the last refines Kolmogorov-Tikhomirov (1958) study on finitely differentiable functions.

3.1 Introduction and results

Dimension theory was popularized by Mandelbrot in the article *How long is the coast of Britain ?* [Man67] and shed light on the general problem of measuring how large a natural object is. The category of objects considered are metric spaces possibly endowed with a measure. Dimension theory encompasses not only smooth spaces such as manifolds, but also wild spaces such as fractals, so that the dimension may be any non-negative real number. There are several notions of dimension: for instance Hausdorff [Hau18], packing [Tri82] or box dimensions [Bou28]. Also, when the space is endowed with a measure, there are moreover the local and the quantization dimensions. These different versions of dimension are bi-Lipschitz invariants. They are in general not equal, so that they reveal different aspects of the underlying space. Seminal works by Hausdorff, Frostman, Tricot, Fan, Tamashiro, Pötzelberger, Graf-Luschgy, and Dereich-Lifshits have described the relationships between these notions and provided conditions under which they coincide.

Obviously, these invariants do not give much information on infinite dimensional spaces. However, such spaces are the subject of many studies. As motivations, Kolmogorov-Tikhomirov in [KT93] gave asymptotics of the covering numbers of some functional spaces. Dereich-Lifshits gave asymptotics of the mass of the small balls for the Wiener measure and exhibited their relationship with the quantization problem, see [DFMS03, DL05, CM44, Chu47, BR92, KL93]. Also, Berger and Bochi [Ber20] gave estimates on the covering number and quantization number of the ergodic decomposition of some smooth dynamical systems. See also [BR92, Klo15, BB21].

These results lead to the following:

Question. *Are there infinite-dimensional counterparts of the various versions of dimension that maintain similar relationships ?*

To address this question, we introduce the notion of *scale*. The key idea involves considering a *scaling*, which is a one-parameter family of gauge functions that satisfies mild assumptions, dictating at which 'scale' the size of the space is examined. For instance the families for the dim for the *dimension* and ord for the *order* given in Example 3.1.1 are scalings. Given a scaling, different versions of scales are defined. In particular, Hausdorff dimension, packing dimension or box dimension are scales.

Given a scaling, we will generalize comparison theorems between the different kinds of dimensions to all the different kinds of scales in Theorem A, B and C. The definition of scaling is crafted so that the proofs for Theorem A and Theorem B are nearly direct extensions of established results from dimension theory (see Section 3.1.2).

The main difficulty will be then to prove Theorem C which enables us to compare the quantization scales with both the local and the box scales. Also even for the specific case of dimension, new inequalities between the quantization dimension of a measure and box dimension of the set of positive mass are proven and generalized for any scaling in Theorem C (inequalities (f) and (h)). Main novelties from Theorem C are reformulated in Theorems 3.3.10 and 3.3.11.

In the next Section 3.1.1 we recall usual definitions of dimension and introduce the notions of scaling and scales. The theorems comparing the different versions of scales are stated in Section 3.1.2. Precise definitions of the involved scales are given in Section 3.2, and in Section 3.3 for measures.

Then, Section 3.1.3 is devoted to applications of the main results. In Section 3.1.3, a first application of Theorem C together with Dereich-Lifshits estimate in [DL05] implies the coincidence of local, Hausdorff, packing, quantization and box orders of the Wiener measure for the L^p-norm, for any $p \in [1, \infty]$. Then in Section 3.1.3, we apply Theorem A to show the coincidence of the box, Hausdorff and packing orders for finitely regular functional spaces; refining the Kolmogorov-Tikhomirov study in [KT93, Thm XV]. Lastly in Section 3.1.3, a consequence of Theorem C is that the local order of the ergodic decomposition is at most its quantization order. This solves a problem set by Berger in [Ber20].

3.1.1 From dimension to scale

Let us first recall some classical definitions of dimension theory and see how they could be naturally extended to define finite invariants for infinite dimensional spaces. The Hausdorff, packing and box dimensions of a totally bounded metric space (X, d) are defined by looking at families of subsets of X. First consider the box dimension. Recall that, given an error $\epsilon > 0$, the covering number $\mathcal{N}_\epsilon(X)$ is the minimal cardinality of a covering of X by balls of radius ϵ. Then the *lower and upper box dimensions* of (X, d) are given by:

$$\underline{\dim}_B X := \sup\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \phi_\alpha(\epsilon) \to +\infty\} \quad \text{and} \quad \overline{\dim}_B X := \inf\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \phi_\alpha(\epsilon) \to 0\} \ ,$$

where $(\phi_\alpha)_{\alpha>0}$ is the family of functions on $(0, 1)$ given for $\alpha > 0$ by $\phi_\alpha : \epsilon \mapsto \epsilon^\alpha$.

Box dimensions, also called box counting dimensions or Minkovski-Bouligand dimensions were introduced by Bouligand in [Bou28]. In general, upper and lower box dimensions do not coincide (see e.g. [FF97, Fal04]). However, when X is a smooth manifold endowed with the Euclidean metric, these two dimensions coincide with the usual definition of dimension. Basic properties of box dimensions are revealed when looking at subsets of a metric space with the induced metric. Notably, box dimensions are non decreasing for the inclusion and are invariant by topological closure.

In general they are not *σ-stable*, i.e. the box dimensions of a countable union of subsets of a metric space are a priori not equal to the suprema of the corresponding dimensions of the subsets. The most popular version of dimension that enjoys the property of σ-stability is *Hausdorff dimension*. Let us recall its definition. Given an error $\epsilon > 0$, consider:

$$\mathcal{H}^\alpha_\epsilon(X) := \inf_{C \in \mathcal{C}_H(\epsilon)} \sum_{B(x,\delta) \in C} \phi_\alpha(\delta) \ ,$$

where $\mathcal{C}_H(\epsilon)$ is the set of countable coverings of X by balls of radius at most ϵ. Then, the *Hausdorff dimension* of (X, d) is given by:

$$\dim_H X = \sup\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon \to 0}{\longrightarrow} +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^\alpha_\epsilon(X) \underset{\epsilon \to 0}{\longrightarrow} 0\right\} \ .$$

Lastly, another interesting dimension that enjoys σ-stability is the *packing dimension*. Its construction is analogous to that of Hausdorff dimension and was introduced by Tricot in his thesis [Tri82]. It is actually linked to upper box dimension by the following characterization that we will use for the moment as a definition:

$$\dim_P X := \inf \sup_{n \geq 1} \overline{\dim}_B E_n \ ,$$

where the infimum is taken over countable coverings $(E_n)_{n\geq 1}$ by subsets of X. These four versions of dimension are bi-Lipschitz invariants; they quantify different aspects of the geometry of the studied metric space since they a priori do not coincide. However, it always holds:

$$\mathsf{dim}_H X \leq \underline{\mathsf{dim}}_B X \leq \overline{\mathsf{dim}}_B X \quad \text{and} \quad \mathsf{dim}_H X \leq \mathsf{dim}_P X \leq \overline{\mathsf{dim}}_B X \ .$$

See e.g. [FF97], [Fal04] for detailed proofs.

Let us now introduce *scales*. A simple observation is that all of the above versions of dimension involve a specific parameterized family $(\phi_\alpha)_{\alpha>0} = (\epsilon \mapsto \epsilon^\alpha)_{\alpha>0}$ of *gauge functions* with polynomial behavior. In dimension theory, gauge functions generalize the measurement of ball diameters, enabling more control over the definition of the Hausdorff measure in the finite dimensional case. For scales, the objective is different. Roughly speaking, we will allow gauge functions to exhibit behaviors that are far from being polynomial. Let us precise the discussion. If a space (X, d) is infinite dimensional then its covering number $\mathcal{N}_\epsilon(X)$ grows faster than any polynomial in ϵ^{-1} as ϵ decreases to 0. In order to define finite invariants for infinite dimensional spaces we must allow other gauge functions that decrease faster than any polynomial when the radius of the involved balls decreases to 0. Consequently, we propose to replace the family $(\phi_\alpha)_{\alpha>0} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$ in all the above definitions of dimensions, by other families of gauge functions that encompass the following examples of growth:

Example 3.1.1.
1. *The family* $\mathsf{dim} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0}$ *which is used in the definitions of dimensions,*
2. *the family* $\mathsf{ord} = (\epsilon \in (0,1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$ *which is called* order. *It fits with the growth of the covering number of spaces of finitely regular functions studied by Kolmogorov-Tikhomirov [KT93], see Theorem 3.1.11, or with the one of the space of ergodic measures spaces by Berger-Bochi [Ber20], as we will see in Theorem F,*
3. *the family* $(\epsilon \in (0,1) \mapsto \exp(-(\log \epsilon^{-1})^\alpha))_{\alpha>0}$ *which fits with the growth of the covering number of holomorphic functions estimated by Kolmogorov-Tikhomirov [KT93], as we will see in Theorem 3.2.5.*

To properly extend definitions and comparison theorems among different scales, i.e. the generalized box, Hausdorff, and packing dimensions; the family of functions $(\phi_\alpha)_{\alpha>0}$ must satisfy certain mild assumptions. This requirement leads us to introduce the notion of *scaling*:

Definition 3.1.2 (Scaling). *A family* $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$ *of positive non-decreasing functions on* $(0,1)$ *is a* scaling *when for every* $\alpha > \beta > 0$ *and any* $\lambda > 1$ *close enough to* 1*, it holds:*

$$\mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \textit{and} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right) \ \textit{as}\ \epsilon \to 0 \ . \tag{$*$}$$

Remark 3.1.3. *The left hand side condition is used in all the proofs of the theorems represented on Fig. 3.1. The right hand side condition is only used to prove the equalities between packing and upper local scales in Theorem B and to compare upper local scales with upper box and upper quantization scales in Theorem C inequalities* $(c)\&(g)$*. It also allows us to characterize packing scale with packing measure.*

Remark 3.1.4. *There are scalings that allow us to study* 0*-dimensional spaces, for instance the family:*

$$\left(\epsilon \mapsto \log(\epsilon^{-1})^{-\alpha}\right)_{\alpha>0} .$$

We will show in Proposition 3.2.4 that the families in Example 3.1.1 are scalings. Scalings allow us to define *scales* that generalize packing dimension, Hausdorff dimension, box dimensions, quantization dimensions and local dimensions. For each scaling, the different kinds of scales do not a priori coincide on a generic space. Nevertheless, in Section 3.1.3, as a direct application of our comparison theorems, we bring examples of metric spaces and measures where all those definitions coincide. In these examples, equalities between the different scales are linked to some underlying "homogeneity" of the space that is provided by the existence of an equilibrium state.

Now for a metric space (X, d), replacing the specific family dim in the definition of box dimensions by a given scaling $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha>0}$ gives the following:

Definition 3.1.5 (Box scales)**.** Lower and upper box scales *of a metric space* (X, d) *are defined by:*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} .$$

Moreover, we will generalize the notion of Hausdorff and packing dimensions to the *Hausdorff scale* denoted $\mathsf{scl}_H X$ (see Definition 3.2.13) and *packing scale* denoted $\mathsf{scl}_P X$ (see Definition 3.2.14). The constructions are fully detailed in the next section. Let us now state the main results on comparison of scales of metric spaces.

3.1.2 Results on comparisons of scales

In this section, we introduce other kinds of scales and Theorems A, B and C that state the inequalities between them as illustrated in Fig. 3.1.

First, we bring the following generalization of classical inequalities comparing dimensions of metric spaces to the frame of scales:

Theorem A. *Let* (X, d) *be a metric space and* scl *a scaling, the following inequalities hold:*

$$\mathsf{scl}_H X \le \mathsf{scl}_P X \le \overline{\mathsf{scl}}_B X \quad \textit{and} \quad \mathsf{scl}_H X \le \underline{\mathsf{scl}}_B X \le \overline{\mathsf{scl}}_B X .$$

In the specific case of dimension, these inequalities are well known and presented for instance by Tricot [Tri82] or Falconer [FF97, Fal04]. The proof of this theorem will be done in Section 3.2.5. The key part is to show that Hausdorff scales and packing scales are well defined quantities. Then we will follow the lines of Falconer's proof to show Theorem A.

The relationship between the dimensions of (X, d) and its measures was first studied by Frostman [Fro35], who used equilibrium states to describe the Hausdorff dimension of the underlying space. Conversely, the dimensions of sets can characterize the asymptotic behavior of the mass of small balls of a measure. This concept was introduced by Fan as *local dimension* [Fan94], leading to seminal studies by Fan, Lau, and Rao [FLR02], Pötzelberger [Pöt99], and Tamashiro [Tam95].

Similarly we introduce *local scales* that extend the notion of local dimensions of a measure:

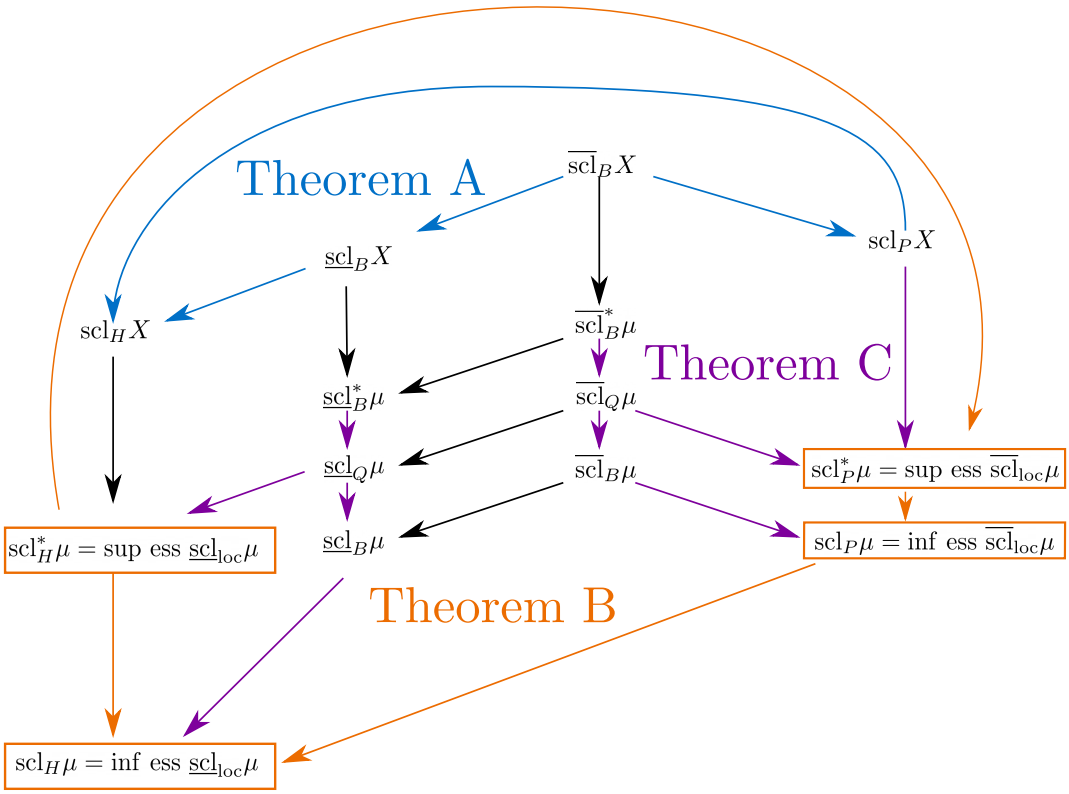

Figure 3.1 – Diagram presenting results of Theorems A, B and C.
Each arrow is an inequality, the scale at the starting point of the arrow is at least the one at its ending point : " $\to$ " = " $\geq$ ".

Definition 3.1.6 (Local scales). *Let μ be a Borel measure on a metric space (X, d) and* scl *a scaling. The* lower and upper scales *of μ are the functions that map a point $x \in X$ to:*

$$\underline{\mathsf{scl}}_{loc}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} 0\right\}$$

and

$$\overline{\mathsf{scl}}_{loc}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} +\infty\right\} .$$

As in dimension theory, we should not compare the local scales with the scales of X but with those of its subsets with positive mass. This observation leads to considering the following:

Definition 3.1.7 (Hausdorff, packing and box scales of a measure). *Let* scl *be a scaling and μ a non-null Borel measure on a metric space (X, d). For any $\mathsf{scl}_\bullet \in \left\{\mathsf{scl}_H, \mathsf{scl}_P, \underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\right\}$ we define lower and upper scales of the measure μ by:*

$$\mathsf{scl}_\bullet\mu = \inf_{E\in\mathcal{B}}\left\{\mathsf{scl}_\bullet E : \mu(E) > 0\right\} \quad \textit{and} \quad \mathsf{scl}^*_\bullet\mu = \inf_{E\in\mathcal{B}}\left\{\mathsf{scl}_\bullet E : \mu(X\backslash E) = 0\right\} ,$$

where $\mathcal{B}$ is the set of Borel subsets of X.

In the case of dimension Fan [Fan94, FLR02] and Tamashiro [Tam95] exhibited the relationships between the Hausdorff and packing dimensions of measures and their local dimensions that we generalize as:

Theorem B. *Let μ be a Borel measure on a metric space (X, d), then for any scaling* scl*, Hausdorff and packing scales of μ are characterized by:*

$$\mathsf{scl}_H\mu = \operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu, \quad \mathsf{scl}^*_H\mu = \operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu, \quad \mathsf{scl}_P\mu = \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu, \quad \mathsf{scl}^*_P\mu = \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \ ,$$

where ess sup *and* ess inf *denote the essential suprema and infima of a function.*

The proof of Theorem B is provided in Section 3.3.1. It is inspired by the proofs of Fan [Fan94] and Tamashiro [Tam95] in the dimensional case.

Let us introduce a last kind of scale, the *quantization scale*. It generalizes the quantization dimension. This definition is motivated by the following works [GL07, Pöt99, DFMS03, DL05, Ber17, BB21, Ber20].

Definition 3.1.8 (Quantization scales)**.** *Let (X, d) be a metric space and μ a Borel measure on X. Given an error $\epsilon > 0$, the* quantization number $\mathcal{Q}_\mu(\epsilon)$ *of μ is the minimal cardinality of a finite set of points that is ϵ-close on average to any point in X:*

$$\mathcal{Q}_\mu(\epsilon) = \inf\left\{N \geq 0 : \exists\, \{c_i\}_{i=1,\dots,N} \subset X, \int_X d(x, \{c_i\}_{1\leq i\leq N})d\mu(x) \leq \epsilon\right\} \ .$$

Then lower and upper quantization scales *of μ for a given scaling* scl *are defined by:*

$$\underline{\mathsf{scl}}_Q\mu = \sup\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon)\cdot \mathcal{Q}_\mu(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_Q\mu = \inf\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon)\cdot \mathcal{Q}_\mu(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} \ .$$

The following gives relationships between the different kinds of scales of measures:

Theorem C (Main)**.** *Let (X, d) be a metric space. Let μ be a Borel measure on X. For any scaling* scl*, the following inequalities on the scales of μ hold:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(a)} \underline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(b)} \underline{\mathsf{scl}}_Q\mu \quad ; \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(c)} \overline{\mathsf{scl}}_B\mu \underbrace{\leq}_{(d)} \overline{\mathsf{scl}}_Q\mu$$

and

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(e)} \underline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(f)} \underline{\mathsf{scl}}^*_B\mu \quad ; \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \underbrace{\leq}_{(g)} \overline{\mathsf{scl}}_Q\mu \underbrace{\leq}_{(h)} \overline{\mathsf{scl}}^*_B\mu \ .$$

Inequalities (b) and (d) are part of Theorem 3.3.10 and relies mainly on the use of the Borel-Cantelli lemma. Even in the specific case of dimension, these inequalities are new, as far as we know. Inequalities (e) and (g) were shown by Pötzelberger in [Pöt99] for dimension and in $[0, 1]^d$. A new approach for the general case is brought in Theorem 3.3.11. We deduce the inequality (a) from (e) and (f) and the inequality (c) from (g) and (h). The proof of inequalities (f) and (h) is straightforward, see Lemma 3.3.8.

As a direct application, inequality (e) allows us to answer a problem set by Berger in [Ber20] (see Section 3.1.3). We will give in Section 3.4.1 examples of spaces for which the different versions of orders do not coincide.

3.1.3 Applications

Let us see how our main theorems easily imply the equality between the different scales of some natural infinite dimensional spaces.

Wiener measure

The first example is the computation of the orders of the Wiener measure W that describes unidimensional standard Brownian motion on $[0,1]$. Recall that W is the law of a continuous process $(B_t)_{t\in[0,1]}$ with independent increments. It is such that for any $t \geq s$ the law of the random variable $B_t - B_s$ is $\mathcal{N}(0, t-s)$. Computation of the local scales of the Wiener measure relies on small ball estimates, which received much interest [CM44, Chu47, BR92, KL93]. These results gave asymptotics on the measure of small balls centered at 0 for L^p norms and Hölder norms. Moreover, for a random ball, Dereich-Lifshits made the following estimate for L^p-norms:

Theorem 3.1.9 (Dereich-Lifshits [DL05][3.2, 5.1, 6.1, 6.3])**.** *For the Wiener measure on $C^0([0,1],\mathbb{R})$ endowed with the L^p-norm, for $p \in [1,\infty]$, there exists*[2] *a constant $\kappa > 0$ such that for W-almost any $\omega \in C^0([0,1],\mathbb{R})$:*

$$-\epsilon^2 \cdot \log W(B(\omega,\epsilon)) \to \kappa, \textit{ when } \epsilon \to 0 \ ,$$

and moreover the quantization number of W verifies:

$$\epsilon^2 \cdot \log \mathcal{Q}_W(\epsilon) \to \kappa, \textit{ when } \epsilon \to 0 \ .$$

As a direct consequence of Theorem B and Theorem C we get that the new invariants we introduced for a measure with growth given by ord all coincide:

Theorem D (Orders of the Wiener measure)**.** *For the Wiener measure on $C^0([0,1],\mathbb{R})$ endowed with the L^p-norm, for $p \in [1,\infty]$, verifies for W almost every $\omega \in C^0([0,1])$:*

$$\begin{aligned} 2 &= \underline{\mathsf{ord}}_{loc}(\omega) = \mathsf{ord}_H W = \mathsf{ord}^*_H W = \underline{\mathsf{ord}}_B W = \underline{\mathsf{ord}}_Q W \\ &= \overline{\mathsf{ord}}_{loc}(\omega) = \mathsf{ord}_P W = \mathsf{ord}^*_P W = \overline{\mathsf{ord}}_B W = \overline{\mathsf{ord}}_Q W \ . \end{aligned}$$

In particular, the framework of scales allows us to define Hausdorff, packing and box orders of a measure that are equal to 2 for the Wiener measure. This indicates some kind of constant geometric property of subsets of maps with positive Wiener measure.

Proof. By Theorem 3.1.9, for W-almost ω and for any $p \in [1,\infty]$, in the L^p-norm it holds:

$$\underline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \overline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = 2 = \underline{\mathsf{ord}}_Q W = \overline{\mathsf{ord}}_Q W \ .$$

Now by Theorem B, it holds:

$$\mathsf{ord}_H W = \underline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}^*_H W \quad \text{and} \quad \mathsf{ord}_P W = \overline{\mathsf{ord}}_{\mathrm{loc}} W(\omega) = \mathsf{ord}^*_P W \ .$$

Finally, since by Theorem C we have:

$$\overline{\mathsf{ord}}_Q W \geq \overline{\mathsf{ord}}_B W \geq \underline{\mathsf{ord}}_B W \geq \mathsf{ord}_H W \ ,$$

the desired result comes by combining the three above lines of equalities and inequalities. □

2. Note that for $p < \infty$, the constant κ does not depend on the value of p.

Remark 3.1.10. *Since $(C^0([0,1],\mathbb{R}), L^p)$ is not (totally) bounded, as well as any of its subsets with total mass, it holds $\underline{\mathsf{ord}}_B^* = +\infty$.*

Functional spaces endowed with the C^0-norm

Let d be a positive integer. For any integer $k \geq 0$ and for any $\alpha \in (0,1]$ denote:

$$\mathcal{F}^{d,k,0} := \left\{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \right\}$$

and

$$\mathcal{F}^{d,k,\alpha} := \left\{ f \in C^k([0,1]^d, [-1,1]) : \|f\|_{C^k} \leq 1 \text{ and } D^k f \text{ is } \alpha\text{-Hölder with constant } 1 \right\} .$$

We endow these spaces with the C^0 uniform norm (see Section 3.4.2 for details).

Kolmogorov and Tikhomirov gave the following asymptotics:

Theorem 3.1.11 (Kolmogorov-Tikhomirov, [KT93][Thm XV])**.** *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, there exist two constants $C_1 > C_2 > 0$ such that for every $\epsilon > 0$, the covering number $\mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha})$ of the space $(\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty)$ verifies:*

$$C_1 \cdot \epsilon^{-\frac{d}{k+\alpha}} \geq \log \mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha}) \geq C_2 \cdot \epsilon^{-\frac{d}{k+\alpha}} .$$

In Section 3.4.2, we will embed an infinite-dimensional Kantor set into $\mathcal{F}^{k,d,\alpha}$ via an expanding map. This embedding will enable us to prove:

Lemma 3.1.12. *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, it holds:*

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \frac{d}{k+\alpha} .$$

The above lemma together with Theorem A gives the following consequence of the theorem of Kolmogorov-Tikhomirov:

Theorem E. *Let d be a positive integer. For any integer k and for any $\alpha \in [0,1]$, it holds:*

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} = \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} = \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} .$$

Proof. First, by Theorem A, it holds:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \quad \text{and} \quad \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} .$$

From there, by Theorem 3.1.11 and Lemma 3.1.12, it holds:

$$\frac{d}{k+\alpha} \leq \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \underline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} ,$$

and

$$\frac{d}{k+\alpha} \leq \mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \leq \mathsf{ord}_P \mathcal{F}^{d,k,\alpha} \leq \overline{\mathsf{ord}}_B \mathcal{F}^{d,k,\alpha} = \frac{d}{k+\alpha} .$$

From there, all of the above inequalities are indeed equalities, which gives the desired result. □

Local and global emergence

The framework of scales moreover allows us to answer a problem set by Berger in [Ber20] on the largeness of ergodic decomposition for wild dynamical systems. We now consider a compact metric space (X, d) and a measurable map $f : X \to X$. We denote $\mathcal{M}$ the set of probability Borel measures on X and $\mathcal{M}_f$ the subset of $\mathcal{M}$ containing f-invariant measures. The space $\mathcal{M}$ is endowed with the Wasserstein distance W_1 defined by:

$$W_1(\nu_1, \nu_2) = \sup_{\phi \in \mathrm{Lip}^1(X)} \int \phi d(\nu_1 - \nu_2) \ ,$$

inducing the weak $*$- topology for which $\mathcal{M}$ is compact. A way to measure the wildness of a dynamical system is to measure how far from being ergodic an invariant measure μ is. Then by Birkhoff's theorem, given a measure $\mu \in \mathcal{M}_f$, for μ-almost every $x \in X$ the following measure is well defined:

$$e^f(x) := \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \ ,$$

and also this limit measure is ergodic. The notion of *emergence*, introduced by Berger, describes the size of the subset of ergodic measures that can be obtained by limits of empirical measures, given an f-invariant probability measure on X.

Definition 3.1.13 (Emergence, [Ber17, BB21])**.** *The* emergence *of a measure $\mu \in \mathcal{M}_f$ at $\epsilon > 0$ is defined by:*

$$\mathcal{E}_\mu(\epsilon) = \min\{N \in \mathbb{N} \ : \ \exists \nu_1, \dots, \nu_N \in \mathcal{M}_f, \ \int_X W_1(e^f(x), \{\nu_i\}_{1 \leq i \leq N}) d\mu(x) \leq \epsilon\} \ .$$

Note that the emergence is the quantization number of the ergodic decomposition of μ. The case of high emergence corresponds to dynamics where the considered measure is far from being ergodic. The following result shows us that the order is an adapted scaling in the study of the ergodic decomposition.

Theorem 3.1.14 ([BGV07, Klo15, BB21])**.** *Let (X, d) be a metric compact space of finite dimension, then:*

$$\underline{\dim}_B X \leq \underline{\mathsf{ord}}_B(\mathcal{M}) \leq \overline{\mathsf{ord}}_B(\mathcal{M}) \leq \overline{\dim}_B X \ .$$

Given measure $\mu \in \mathcal{M}_f$ we define its *emergence order* by:

$$\overline{\mathsf{ord}}\mathcal{E}_\mu := \limsup_{\epsilon \to 0} \frac{\log \log \mathcal{E}_\mu(\epsilon)}{-\log \epsilon} = \inf \left\{ \alpha > 0 : \mathcal{E}_\mu(\epsilon) \cdot \exp(-\epsilon^{-\alpha}) \underset{\epsilon \to 0}{\longrightarrow} 0 \right\} \ .$$

We denote $\mu_{e^f} := {e^f}_* \mu$ the *ergodic decomposition* of μ; i.e. the push forward by e^f of μ. A local analogous local quantity to the emergence order is the local order of the ergodic decomposition of μ, for $\nu \in \mathcal{M}_f$ it is defined by:

$$\overline{\mathsf{ord}}\mathcal{E}^{\mathrm{loc}}_\mu(\nu) := \limsup_{\epsilon \to 0} \frac{\log - \log(\mu_{e^f}(B(\nu, \epsilon))}{-\log \epsilon} \ .$$

Berger asked if the following comparison between the asymptotic behavior of the mass of the balls of the ergodic decomposition of μ and the asymptotic behavior of its quantization holds.

Problem 3.1.15 (Berger, [Ber20, Pbm 4.22])**.** *Let (X,d) be a compact metric space, $f : X \to X$ a measurable map and μ an f-invariant probability measure on X. Does the following hold ?*

$$\int_{\mathcal{M}_f} \overline{\mathsf{ord}}\mathcal{E}_\mu^{loc} d\mu_{e^f} \leq \overline{\mathsf{ord}}\mathcal{E}_\mu \ .$$

We propose here a stronger result that answers the latter problem as a direct application of Theorem C:

Theorem F. *Let (X,d) be a compact metric space, $f : X \to X$ a measurable map and μ an f-invariant probability measure on X. For μ_{e^f}-almost every $\nu \in \mathcal{M}$, it holds:*

$$\overline{\mathsf{ord}}\mathcal{E}_\mu^{loc}(\nu) \leq \overline{\mathsf{ord}}\mathcal{E}_\mu \ .$$

Proof. Note that $\overline{\mathsf{ord}}\mathcal{E}_\mu^{\mathrm{loc}} = \overline{\mathsf{ord}}_{\mathrm{loc}}\mu_{e^f}$ and $\overline{\mathsf{ord}}\mathcal{E}_\mu = \overline{\mathsf{ord}}_Q\mu_{e^f}$. Now by Theorem C, it holds μ_{e^f}-almost surely that $\overline{\mathsf{ord}}_{\mathrm{loc}}\mu_{e^f} \leq \overline{\mathsf{ord}}_Q\mu_{e^f}$ which is the desired result. □

3.2 Metric scales

Metric scales will be bi-Lipschitz invariants generalizing Hausdorff, packing and box dimensions of metric spaces. Before defining and comparing metric scales we show a handful of basic properties of scalings and present some relevant examples.

3.2.1 Scalings

We first recall that a family $\mathsf{scl} = (\mathrm{scl}_\alpha)_{\alpha\geq 0}$ of positive non-decreasing functions on $(0,1)$ is a *scaling* when for any $\alpha > \beta > 0$ and any $\lambda > 1$ close enough to 1, it holds:

$$\mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon^\lambda)\right) \quad \text{and} \quad \mathrm{scl}_\alpha(\epsilon) = o\left(\mathrm{scl}_\beta(\epsilon)^\lambda\right) \ \text{when } \epsilon \to 0 \ . \tag{$*$}$$

An immediate consequence of the latter definition is the following:

Fact 3.2.1. *Let* scl *be a scaling, then for any $\alpha > \beta > 0$ and for any constant $C > 0$, for $\epsilon > 0$ small enough, it holds:*

$$\mathrm{scl}_\alpha(\epsilon) \leq \mathrm{scl}_\beta(C \cdot \epsilon) \ .$$

A consequence of the latter fact is the following which will allow us to compare the different versions of scales:

Lemma 3.2.2. *Let $f, g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be two functions such that $f \leq g$ near 0. For any constant $C > 0$, it holds:*

$$\inf\left\{\alpha > 0 : f(C\cdot\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} \leq \inf\left\{\alpha > 0 : g(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\}$$

and

$$\sup\left\{\alpha > 0 : f(C\cdot\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \leq \sup\left\{\alpha > 0 : g(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\} \ .$$

Proof. It suffices to observe that, by Fact 3.2.1, for any $\alpha > \beta > 0$, and $\epsilon > 0$ small, it holds:

$$f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \leq g(\epsilon) \cdot \mathrm{scl}_\beta(C^{-1} \cdot \epsilon) = g(C \cdot \tilde{\epsilon}) \cdot \mathrm{scl}_\beta(\tilde{\epsilon}) \ ,$$

with $\tilde{\epsilon} = C \cdot \epsilon$. □

The following will provide a sequential characterization of scales:

Lemma 3.2.3 (Sequential characterization of scales)**.** *Let* scl *be a scaling and $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a non increasing function. Let $(r_n)_{n\geq 1}$ be a sequence of positive real numbers decreasing to 0 and such that $\log r_{n+1} \sim \log r_n$ as $n \to +\infty$, then it holds:*

$$\inf\left\{\alpha > 0 : f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} = \inf\left\{\alpha > 0 : f(r_n) \cdot \mathrm{scl}_\alpha(r_n) \xrightarrow[n\to+\infty]{} 0\right\}$$

and

$$\sup\left\{\alpha > 0 : f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\} = \sup\left\{\alpha > 0 : f(r_n) \cdot \mathrm{scl}_\alpha(r_n) \xrightarrow[n\to+\infty]{} +\infty\right\} \ .$$

Proof. Fix $\alpha > 0$ and $\epsilon > 0$. If ϵ is sufficiently small, there exists an integer $n > 0$ such that $r_{n+1} < \epsilon \leq r_n$. Since f is non-increasing and scl_α is increasing, we have the inequalities:

$$f(r_n) \cdot \mathrm{scl}_\alpha(r_{n+1}) \leq f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \leq f(r_{n+1}) \cdot \mathrm{scl}_\alpha(r_n). \tag{3.2.1}$$

Now, let β and γ be positive real numbers such that $0 < \beta < \alpha < \gamma$. For λ sufficiently close to 1 and for sufficiently small ϵ, by Eq. $(*)$ it holds:

$$\mathrm{scl}_\gamma(r_n) \leq \mathrm{scl}_\alpha(r_n^\lambda) \quad \text{and} \quad \mathrm{scl}_\alpha(r_n) \leq \mathrm{scl}_\beta(r_n^\lambda) \ . \tag{3.2.2}$$

As $\lambda \geq \frac{\log r_{n+1}}{\log r_n}$ for large n, it holds:

$$r_n^\lambda \leq r_{n+1} \ .$$

This together with Eq. (3.2.2) implies:

$$\mathrm{scl}_\gamma(r_n) \leq \mathrm{scl}_\alpha(r_{n+1}) \quad \text{and} \quad \mathrm{scl}_\alpha(r_n) \leq \mathrm{scl}_\beta(r_{n+1}). \tag{3.2.3}$$

By combining Eqs. (3.2.1) and (3.2.3), we obtain:

$$f(r_n) \cdot \mathrm{scl}_\gamma(r_n) \leq f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \quad \text{and} \quad f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \leq f(r_{n+1}) \cdot \mathrm{scl}_\beta(r_{n+1}).$$

Thus, it follows that:

$$\limsup_{n\to+\infty} f(r_n) \cdot \mathrm{scl}_\gamma(r_n) \leq \limsup_{\epsilon\to 0} f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \leq \limsup_{n\to+\infty} f(r_n) \cdot \mathrm{scl}_\beta(r_n),$$

and similarly:

$$\liminf_{n\to+\infty} f(r_n) \cdot \mathrm{scl}_\gamma(r_n) \leq \liminf_{\epsilon\to 0} f(\epsilon) \cdot \mathrm{scl}_\alpha(\epsilon) \leq \liminf_{n\to+\infty} f(r_n) \cdot \mathrm{scl}_\beta(r_n) \ .$$

Since this holds for every positive α, and since β and γ can be taken arbitrarily close to α, we obtain the desired result. □

The following provides a whole class of scalings. It shows in particular that the families brought in Example 3.1.1 are indeed scalings.

Proposition 3.2.4. *For any integers $p, q \geq 1$, the family $\mathsf{scl}^{p,q} = (\mathrm{scl}_\alpha^{p,q})_{\alpha>0}$ defined for any $\alpha > 0$ by:*

$$\mathrm{scl}_\alpha^{p,q} : \epsilon \in (0,1) \mapsto \frac{1}{\exp^{\circ p}(\alpha \cdot \log_+^{\circ q}(\epsilon^{-1}))}$$

is a scaling; where $\log_+ : t \in \mathbb{R} \mapsto \log(t) \cdot \mathbb{1}_{t>1}$ is the positive part of the logarithm.

We prove this proposition below. Note in particular that:

$$\mathsf{scl}^{1,1} = \mathsf{dim} = (\epsilon \in (0,1) \mapsto \epsilon^\alpha)_{\alpha>0} \quad \text{and} \quad \mathsf{scl}^{2,1} = \mathsf{ord} = (\epsilon \in (0,1) \mapsto \exp(-\epsilon^{-\alpha}))_{\alpha>0}$$

are both scalings. Let us give an example of a space that has finite box scales for the scaling $\mathsf{scl}^{2,2}$ as defined in Proposition 3.2.4. Consider the space A of holomorphic functions on the disk $\mathbb{D}(R) \subset \mathbb{C}$ of radius $R > 1$ that are bounded by 1:

$$A = \left\{ \phi = \sum_{n\geq 0} a_n z^n \in C^\omega(\mathbb{D}(R), \mathbb{C}) : \sup_{\mathbb{D}(R)} |\phi| \leq 1 \right\} \text{endowed with the norm} \|\phi\|_\infty := \sup_{z\in\mathbb{D}(1)} |\phi(z)| \ .$$

Kolmogorov and Tikhomirov gave the following estimate of its covering number:

Theorem 3.2.5 (Kolmogorov, Tikhomirov [KT93][Equality (129)])**.** *The following estimate on the covering number of $(A, \|\cdot\|_\infty)$ holds:*

$$\log \mathcal{N}_\epsilon(A) = (\log R)^{-1} \cdot |\log \epsilon|^2 + O(\log \epsilon^{-1} \cdot \log\log \epsilon^{-1}), \text{ when } \epsilon \text{ tends to } 0. \ .$$

In the framework of scales, the above translates as:

$$\underline{\mathsf{scl}}_B^{2,2} A = \overline{\mathsf{scl}}_B^{2,2} A = 2 \ .$$

Let us now show:

Proof of Proposition 3.2.4. The proof is based on the following two facts:

Fact 3.2.6. *For every $\nu > 1$, for every $d \geq 1$ and for $y > 0$ large enough, it holds:*

$$\log^{\circ d}(y^\nu) \leq \nu \cdot \log^{\circ d}(y) \ .$$

Fact 3.2.7. *For any $\gamma > 0$ and $\nu > 1$ close to 1, it holds for $\epsilon > 0$ small:*

$$\mathrm{scl}_{\nu\cdot\gamma}^{p,q}(\epsilon) \leq \mathrm{scl}_\gamma^{p,q}(\epsilon^\nu) \quad \textit{and} \quad \mathrm{scl}_{\nu\cdot\gamma}^{p,q}(\epsilon) \leq \mathrm{scl}_\gamma^{p,q}(\epsilon)^\nu \ .$$

Actually Fact 3.2.7 will be proved using Fact 3.2.6. First let us show recursively:

Proof of Fact 3.2.6. We prove this fact by induction on $d \geq 1$. For $d = 1$ note that the inequality is obvious as the equality holds. Then we conclude by induction on d based on the following:

$$\log^{\circ(d+1)}(y^\nu) = \log(\log^{\circ d}(y^\nu)) \leq \log(\nu \log^{\circ d} y) = \log \nu + \log^{\circ(d+1)} y \ ,$$

where the inequality is given by the induction hypothesis. □

We are now ready to show:

Proof of Fact 3.2.7. We apply Fact 3.2.6 with $d = q$ and $y = \epsilon^{-1}$ for sufficiently small values of ϵ to obtain for every $\nu > 1$:

$$\log^{\circ q}(\epsilon^{-\nu}) \leq \nu \cdot \log^{\circ q}(\epsilon^{-1}) \ .$$

Multiplying by γ and composing by $t \mapsto 1/\exp^{\circ p}(t)$ which is decreasing, yields the first inequality in Fact 3.2.7.

To show the second inequality, we apply again Fact 3.2.6 with $d = p$ and $y = \frac{1}{\mathrm{scl}_{\gamma}^{p,q}(\epsilon)}$ which is large for small values of $\epsilon > 0$ to obtain:

$$\log^{\circ p}(y^{\nu}) \leq \nu \cdot \log^{\circ p}(y) \ .$$

As $y = \exp^{\circ p}(\gamma \cdot \log^{\circ q}(\epsilon^{-1}))$, the above inequality translates as $\log^{\circ p}(y^{\nu}) \leq \nu \cdot \gamma \cdot \log^{\circ q}(\epsilon^{-1})$. It follows that:

$$y^{\nu} \leq \exp^{\circ p}(\nu \cdot \gamma \cdot \log^{\circ q}(\epsilon^{-1})) = \frac{1}{\mathrm{scl}_{\nu\cdot\gamma}^{p,q}(\epsilon)} \ .$$

In other words, it holds $\mathrm{scl}_{\nu\cdot\gamma}^{p,q}(\epsilon) \leq y^{-\nu}$ which is exactly the second inequality of Fact 3.2.7. □

We are now ready to prove the two estimates of Eq. $(*)$ in the definition of scaling for the family $\mathsf{scl}^{p,q}$. Let us fix $\alpha > \beta > 0$ and $\lambda > 1$ such that $\alpha > \lambda^2 \cdot \beta$. As $\mathrm{scl}_{\alpha}^{p,q}$ is decreasing with α, it holds:

$$\mathrm{scl}_{\alpha}^{p,q}(\epsilon) \leq \mathrm{scl}_{\lambda^2\cdot\beta}^{p,q}(\epsilon) \ . \tag{3.2.4}$$

On the other hand, by Fact 3.2.7 we have:

$$\mathrm{scl}_{\lambda^2\cdot\beta}^{p,q}(\epsilon) \leq \mathrm{scl}_{\lambda\cdot\beta}^{p,q}(\epsilon^{\lambda}) \leq \left(\mathrm{scl}_{\beta}^{p,q}(\epsilon^{\lambda})\right)^{\lambda} \quad \text{and} \quad \mathrm{scl}_{\lambda^2\cdot\beta}^{p,q}(\epsilon) \leq \left(\mathrm{scl}_{\beta}^{p,q}(\epsilon)\right)^{\lambda^2} \ . \tag{3.2.5}$$

The above Eqs. (3.2.4) and (3.2.5) imply:

$$\frac{\mathrm{scl}_{\alpha}^{p,q}(\epsilon)}{\mathrm{scl}_{\beta}^{p,q}(\epsilon^{\lambda})} \leq \left(\mathrm{scl}_{\beta}^{p,q}(\epsilon^{\lambda})\right)^{\lambda-1} \quad \text{and} \quad \frac{\mathrm{scl}_{\alpha}^{p,q}(\epsilon)}{\mathrm{scl}_{\beta}^{p,q}(\epsilon)^{\lambda}} \leq \left(\mathrm{scl}_{\beta}^{p,q}(\epsilon)\right)^{\lambda\cdot(\lambda-1)} \ ,$$

which both converge to 0 as ϵ goes to 0, providing the desired result. □

3.2.2 Box scales

As introduced in Definition 3.1.5, *lower and upper box scales* of a metric space (X, d) are defined by:

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \mathcal{N}_{\epsilon}(X) \cdot \mathrm{scl}_{\alpha}(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \mathcal{N}_{\epsilon}(X) \cdot \mathrm{scl}_{\alpha}(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} \ ,$$

where the covering number $\mathcal{N}_{\epsilon}(X)$ is the minimal cardinality of a covering of X by balls with radius $\epsilon > 0$.

In general, the upper and lower box scales must not coincide. We will give such examples for the order in Example 3.4.4. Now we list a few properties of box scales that are well known in the specific case of dimension.

Fact 3.2.8. *Let (X,d) be a metric space. The following properties hold true:*

1. *if $\underline{\mathsf{scl}}_B(X) < +\infty$, then (X,d) is totally bounded,*
2. *for every subset $E \subset X$ it holds $\underline{\mathsf{scl}}_B E \leq \underline{\mathsf{scl}}_B X$ and $\overline{\mathsf{scl}}_B E \leq \overline{\mathsf{scl}}_B X$,*
3. *for every subset $E \subset X$ it holds $\underline{\mathsf{scl}}_B E = \underline{\mathsf{scl}}_B \mathrm{cl}(E)$ and $\overline{\mathsf{scl}}_B E = \overline{\mathsf{scl}}_B \mathrm{cl}(E)$.*

Observe that 1. and 2. are direct consequences of the definitions.

To see 3. it is enough to observe that $\mathcal{N}_\epsilon(E) \leq \mathcal{N}_\epsilon(\mathrm{cl}(E)) \leq \mathcal{N}_{\epsilon/2}(E)$ for every $\epsilon > 0$.

As for box dimensions, box scales can also be defined by replacing covering number with packing number:

Definition 3.2.9 (Packing number)**.** *For $\epsilon > 0$ let $\tilde{\mathcal{N}}_\epsilon(X)$ be the* packing *number of the metric space (X,d). It is the maximum cardinality of an ϵ-separated set of points in X for the distance d:*

$$\tilde{\mathcal{N}}_\epsilon(X) = \sup\{N \geq 0 : \exists x_1, \dots x_N \in X, d(x_i,x_j) \geq \epsilon \text{ for every } 1 \leq i < j \leq N\} .$$

A well known comparison between packing and covering numbers is the following:

Lemma 3.2.10. *Let (X,d) be a metric space. For every $\epsilon > 0$, it holds:*

$$\tilde{\mathcal{N}}_{2\epsilon}(X) \leq \mathcal{N}_\epsilon(X) \leq \tilde{\mathcal{N}}_\epsilon(X) .$$

In virtue of the basic properties of scalings, the covering number can be replaced by the packing number in the definitions of box scales:

Lemma 3.2.11. *Let (X,d) be a metric space and scl a scaling, then box scales of X can be written as:*

$$\underline{\mathsf{scl}}_B X = \sup\left\{\alpha > 0 : \tilde{\mathcal{N}}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_B X = \inf\left\{\alpha > 0 : \tilde{\mathcal{N}}_\epsilon(X) \cdot \mathrm{scl}_\alpha(\epsilon) \xrightarrow[\epsilon\to 0]{} 0\right\} .$$

The proof is provided by direct application of Lemmas 3.2.2 and 3.2.10.

Remark 3.2.12. *Another property for the scaling $\mathsf{scl}^{p,q}$ from Proposition 3.2.4, with $p,q \geq 1$, is that the upper and lower box scales for a metric space (X,d) can be written as:*

$$\underline{\mathsf{scl}}_B^{p,q}(X) = \liminf_{\epsilon\to 0} \frac{\log^{\circ p}(\mathcal{N}_\epsilon(X))}{\log^{\circ q}(\epsilon^{-1})}$$

and

$$\overline{\mathsf{scl}}_B^{p,q}(X) = \limsup_{\epsilon\to 0} \frac{\log^{\circ p}(\mathcal{N}_\epsilon(X))}{\log^{\circ q}(\epsilon^{-1})} .$$

In particular, for dimension and order:

$$\underline{\mathsf{dim}}_B(X) = \liminf_{\epsilon\to 0} \frac{\log(\mathcal{N}_\epsilon(X))}{-\log\epsilon} \quad , \quad \overline{\mathsf{dim}}_B(X) = \limsup_{\epsilon\to 0} \frac{\log(\mathcal{N}_\epsilon(X))}{-\log\epsilon}$$

and

$$\underline{\mathsf{ord}}_B(X) = \liminf_{\epsilon\to 0} \frac{\log\log(\mathcal{N}_\epsilon(X))}{-\log\epsilon} \quad , \quad \overline{\mathsf{ord}}_B(X) = \limsup_{\epsilon\to 0} \frac{\log\log(\mathcal{N}_\epsilon(X))}{-\log\epsilon} .$$

The above equalities coincide with the most usual definitions of box dimensions and orders.

3.2.3 Hausdorff scales

The definition of Hausdorff scales, generalizing Hausdorff dimension, is introduced here using the definition of Hausdorff outer measure as given for instance by Tricot in [Tri82]. We still consider a metric space (X, d). Given a non-decreasing function $\phi \in C(\mathbb{R}_+^*, \mathbb{R}_+^*)$, such that $\phi(\epsilon) \to 0$ when $\epsilon \to 0$, we recall:

$$\mathcal{H}_\epsilon^\phi(X) := \inf_{J \text{ countable set}} \left\{ \sum_{j\in J} \phi(|B_j|) : X = \bigcup_{j\in J} B_j,\ \forall j \in J : |B_j| \leq \epsilon \right\} ,$$

where $|B|$ is the radius of a ball $B \subset X$. A countable family $(B_j)_{j\in J}$ of balls with radius at most $\epsilon > 0$ such that $X = \bigcup_{j\in J} B_j$ will be called an *ϵ-cover* of X.[3] Since the set of ϵ-cover is non-decreasing for inclusion as ϵ decreases to 0, the following limit does exist:

$$\mathcal{H}^\phi(X) := \lim_{\epsilon\to 0} \mathcal{H}_\epsilon^\phi(X) .$$

Now replacing (X, d) in the previous definitions by any subset E of X endowed with the same metric d, we observe that $\mathcal{H}^\phi$ defines an outer measure on X. It is usually called the ϕ-Hausdorff measure on X. We now introduce the following:

Definition 3.2.13 (Hausdorff scale)**.** *The* Hausdorff scale *of a metric space (X, d) is defined by:*

$$\mathsf{scl}_H X = \sup\left\{\alpha > 0 : \mathcal{H}^{\mathrm{scl}_\alpha}(X) = +\infty\right\} = \inf\left\{\alpha > 0 : \mathcal{H}^{\mathrm{scl}_\alpha}(X) = 0\right\} .$$

Note that the above definition gives us two quantities that are a priori not equal. However, the mild assumptions in the definition of scaling allow us to verify that they indeed coincide and allow us to use the machinery of Hausdorff outer measure to define metric invariants generalizing Hausdorff dimension. Thus scalings allow us to have some consistent extension of the definition of Hausdorff dimension.

Proof of the equality in Definition 3.2.13. It is clear from its definition that $\alpha \mapsto \mathcal{H}^{\mathrm{scl}_\alpha}(X)$ is non-increasing. It is then enough to check that if there exists $\alpha > 0$ such that $0 < \mathcal{H}^{\mathrm{scl}_\alpha}(X) < +\infty$ then, for any positive $\delta < \alpha$, it holds:

$$\mathcal{H}^{\mathrm{scl}_{\alpha+\delta}}(X) = 0 \quad \text{and} \quad \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) = +\infty .$$

Let us fix $\eta > 0$, by Definition 3.1.2 , for $\epsilon > 0$ small it holds:

$$\mathrm{scl}_{\alpha+\delta}(\epsilon) \leq \eta \cdot \mathrm{scl}_\alpha(\epsilon) \quad \text{and} \quad \mathrm{scl}_\alpha(\epsilon) \leq \eta \cdot \mathrm{scl}_{\alpha-\delta}(\epsilon) .$$

Since ϵ is small, it holds:

$$0 < \frac{1}{2}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_\alpha}(X) < +\infty .$$

Given $(B_j)_{j\in J}$ an ϵ-cover of X, the following holds:

$$\frac{1}{2}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \leq \sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) ,$$

3. Note that the historical construction of the Hausdorff measures uses subsets of X with diameter at most ϵ instead of the balls with radius at most ϵ. However, both these constructions lead to the same definitions of Hausdorff scales.

and then:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \frac{1}{\eta}\sum_{j\in J}\mathrm{scl}_\alpha(|B_j|) \leq \sum_{j\in J}\mathrm{scl}_{\alpha-\delta}(|B_j|) \ .$$

Since this holds for every ϵ-cover, the latter inequality leads to:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}_\epsilon^{\mathrm{scl}_{\alpha-\delta}}(X) \ ,$$

and so:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) \ . \tag{3.2.6}$$

On the other side, there exists an ϵ-cover $(B_j)_{j\in J}$ of E such that:

$$\sum_{j\in J}\mathrm{scl}_\alpha(|B_j|) \leq 2\mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \ .$$

Now since $\mathcal{H}_\epsilon^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_\alpha}(X)$, this leads to:

$$\sum_{j\in J}\mathrm{scl}_{\alpha+\delta}(|B_j|) \leq \eta\cdot\sum_{j}\mathrm{scl}_\alpha(|B_j|) \leq 2\eta\cdot\mathcal{H}^{\mathrm{scl}_\alpha}(X) \ .$$

From there:

$$\mathcal{H}_\epsilon^{\mathrm{scl}_{\alpha+\delta}}(X) \leq 2\eta\cdot\mathcal{H}^{\mathrm{scl}_\alpha}(X) \ , \tag{3.2.7}$$

and this holds for every small ϵ.

Taking the limit as $\epsilon \to 0$ in Eqs. (3.2.6) and (3.2.7) gives:

$$\frac{1}{2\eta}\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) \quad \text{and} \quad \mathcal{H}_\epsilon^{\mathrm{scl}_{\alpha+\delta}}(X) \leq 2\eta\cdot\mathcal{H}^{\mathrm{scl}_\alpha}(X) \ .$$

To conclude, note that as the latter holds for η arbitrarily small, it follows that $\mathcal{H}^{\mathrm{scl}_{\alpha-\delta}}(X) = +\infty$ and $\mathcal{H}^{\mathrm{scl}_{\alpha+\delta}}(X) = 0$. □

As box scales, Hausdorff scales are non-decreasing for inclusion. We will see a stronger property of Hausdorff scales in Lemma 3.2.21.

3.2.4 Packing scales

Packing scales through modified box scales

The original construction of packing dimension relies on the packing measure introduced by Tricot in [Tri82]. We first define packing scales by modifying upper box scales and we show later how they are related to packing measures.

Definition 3.2.14 (Packing scale)**.** *Let (X,d) be a metric space and* scl *a scaling. The* packing scale *of X is defined by:*

$$\mathsf{scl}_P X = \inf\left\{\sup_{n\geq 1}\overline{\mathsf{scl}}_B E_n : (E_n)_{n\geq 1} \in X^{\mathbb{N}^*} \text{ s.t. } \bigcup_{n\geq 1} E_n = X\right\} \ .$$

The following comes directly from the definition of packing scale:

Proposition 3.2.15. *Let (X,d) be a metric space and let* scl *be a scaling. It holds:*

$$\mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \ .$$

Packing measures

In this paragraph we show the relationship between packing measures and packing scales. Let us first recall a few definitions.

Given $\epsilon > 0$, an *ϵ-pack* of a metric space (X, d) is a countable collection of disjoint balls of X with radii at most ϵ. As for Hausdorff outer measure, consider $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ a non-decreasing function such that $\phi(\epsilon) \to 0$ as $\epsilon \to 0$. For $\epsilon > 0$, put:

$$\mathcal{P}_\epsilon^\phi(X) := \sup\left\{\sum_{i\in I}\phi(|B_i|) : (B_i)_{i\in I} \text{ is an } \epsilon\text{-pack of } X\right\}.$$

Since $\mathcal{P}_\epsilon^\phi(X)$ is non-increasing when ϵ decreases to 0, the following quantity is well defined:

$$\mathcal{P}_0^\phi(X) := \lim_{\epsilon\to 0}\mathcal{P}_\epsilon^\phi(X).$$

The idea of Tricot is to build an outer measure from this quantity:

Definition 3.2.16 (Packing measure)**.** *For every subset E of X endowed with the same metric d, the* packing ϕ-measure *of E is defined by:*

$$\mathcal{P}^\phi(E) = \inf\left\{\sum_{n\geq 1}\mathcal{P}_0^\phi(E_n) : E = \bigcup_{n\geq 1} E_n\right\}.$$

Note that $\mathcal{P}^\phi$ is an outer measure on X and can eventually be infinite or null. The following shows the equivalence of Tricot's counterpart definition of the packing scale; this will be useful to show the equality between the upper local scale and the packing scale of a measure given by Theorem C eq. (c&g).

Proposition 3.2.17. *The packing scale of a metric space (X, d) verifies:*

$$\sup\left\{\alpha > 0 : \mathcal{P}^{\mathrm{scl}_\alpha}(X) = +\infty\right\} = \mathsf{scl}_P X = \inf\left\{\alpha > 0 : \mathcal{P}^{\mathrm{scl}_\alpha}(X) = 0\right\}.$$

Proof. Let $(E_n)_{n\geq 1}$ be a family of subsets of X. Since each map $\alpha \mapsto \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)$ is non-increasing and non negative, we have:

$$\inf\left\{\alpha > 0 : \sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = 0\right\} = \sup_{n\geq 1}\inf\left\{\alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n) = 0\right\}. \tag{3.2.8}$$

We decompose the proof into two intermediary steps given by the following lemmas:

Lemma 3.2.18. *Given $\alpha > 0$, if $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)$ is finite, then for every $\delta \in (0, \alpha)$, it holds:*

$$\mathcal{P}_0^{\mathrm{scl}_{\alpha+\delta}}(E) = 0 \quad \textit{and} \quad \mathcal{P}_0^{\mathrm{scl}_{\alpha-\delta}}(E) = +\infty.$$

Lemma 3.2.19. *For every $E \subset X$, it holds:*

$$\sup\left\{\alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E) = +\infty\right\} = \overline{\mathsf{scl}}_B E = \inf\left\{\alpha > 0 : \mathcal{P}_0^{\mathrm{scl}_\alpha}(E) = 0\right\}.$$

Lemma 3.2.18 will allow proving Lemma 3.2.19. Before proving these lemmas let us see how they allow us to conclude the proof of Proposition 3.2.17. First note that the second equality of Lemma 3.2.18 implies:

$$\sup\left\{\alpha>0:\sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=+\infty\right\}=\sup_{n\geq 1}\sup\left\{\alpha>0:\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=+\infty\right\}\ .$$

Consequently by Lemma 3.2.19 it holds:

$$\sup\left\{\alpha>0:\sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=+\infty\right\}=\sup_{n\geq 1}\overline{\mathsf{scl}}_B E_n=\inf\left\{\alpha>0:\sum_{n\geq 1}\mathcal{P}_0^{\mathrm{scl}_\alpha}(E_n)=0\right\}\ .$$

Taking the infimum over families $(E_n)_{n\geq 1}$ that cover X we obtain the desired result. □

It remains to show Lemmas 3.2.18 and 3.2.19.

Proof of Lemma 3.2.18. Given $\eta>0$, by Definition 3.1.2 of scaling, for $\epsilon>0$ small enough, it holds:

$$\mathrm{scl}_{\alpha+\delta}(\epsilon)\leq\eta\cdot\mathrm{scl}_\alpha(\epsilon)\quad\text{and}\quad\mathrm{scl}_\alpha(\epsilon)\leq\eta\cdot\mathrm{scl}_{\alpha-\delta}(\epsilon)\ . \tag{3.2.9}$$

Let $(B_j)_{j\geq 1}$ be an ϵ-pack of E. Then, by the above Eq. (3.2.9) it holds:

$$\eta^{-1}\cdot\sum_{j\geq 1}\mathrm{scl}_{\alpha+\delta}(|B_j|)\leq\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)\leq\eta\cdot\sum_{j\geq 1}\mathrm{scl}_{\alpha-\delta}(|B_j|)\ .$$

As this holds for every δ - pack of E, it follows:

$$\eta^{-1}\cdot\mathcal{P}_\epsilon^{\mathrm{scl}_{\alpha+\delta}}(E)\leq\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)\leq\eta\cdot\mathcal{P}_\epsilon^{\mathrm{scl}_{\alpha-\delta}}(E)\ .$$

Taking the limit as ϵ goes to 0 and η arbitrary small allows us to conclude. □

Proof of Lemma 3.2.19. By Lemma 3.2.18, it suffices to show that:

$$\sup\left\{\alpha>0:\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=+\infty\right\}\leq\overline{\mathsf{scl}}_B E\leq\inf\left\{\alpha>0:\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=0\right\}\ . \tag{3.2.10}$$

We start with the second inequality. If $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)>0$ for every $\alpha>0$, there is nothing to prove. Thus consider $\alpha>0$ such that $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=0$. Then for every $\epsilon>0$ sufficiently small it holds $\mathcal{P}_\epsilon^{\mathrm{scl}_\alpha}(E)<1$. In particular, the packing number (see def. 3.2.9) satisfies $\tilde{\mathcal{N}}_\epsilon(E)\cdot\mathrm{scl}_\alpha(\epsilon)<1$. By Lemma 3.2.11 we obtain $\overline{\mathsf{scl}}_B E\leq\alpha$ which gives the second inequality by taking the infima of such $\alpha>0$.

To prove the first inequality, assume that there exists $\alpha>0$ such that $\mathcal{P}_0^{\mathrm{scl}_\alpha}(E)=+\infty$, otherwise there is nothing to prove. For such an α and for every $\epsilon>0$ there exists an ϵ-pack $(B_j)_{j\geq 1}$ such that:

$$\sum_{j\geq 1}\mathrm{scl}_\alpha(|B_j|)>1\ . \tag{3.2.11}$$

For an integer $k\geq 1$, denote:

$$n_k:=\mathrm{Card}\left\{j\geq 1:2^{-(k+1)}\leq\mathrm{scl}_\alpha(|B_j|)<2^{-k}\right\}\ . \tag{3.2.12}$$

Since scl_α is non-decreasing, it holds:

$$\sum_{k\geq 1} n_k \cdot 2^{-k} \geq \sum_{j\geq 1} \mathrm{scl}_\alpha(|B_j|) \ .$$

Thus, by Eq. (3.2.11), it holds:

$$\sum_{k\geq 1} n_k \cdot 2^{-k} > 1 \ . \tag{3.2.13}$$

Note that, as $|B_j| \leq \delta$ for every $j \geq 1$, it holds $n_k = 0$ for every $k < -\log_2 \mathrm{scl}_\alpha(\delta)$. Then for δ small it holds:

Fact 3.2.20. *There exists an integer $j \geq 2$ such that:*

$$n_j > j^{-2} \cdot 2^j \ .$$

Proof. Otherwise we would have:

$$\sum_{k\geq 1} n_k \cdot 2^{-k} \leq \sum_{k\geq 2} \frac{1}{k^2} < 1 \ ,$$

as $n_0 = n_1 = 0$ for small δ, and this contradicts Eq. (3.2.13). □

This latter fact translates as: there exist at least n_j disjoint balls with radii at least $\mathrm{scl}_\alpha^{-1}(2^{-(j+1)})$. This implies:

$$\tilde{\mathcal{N}}_{\mathrm{scl}_\alpha^{-1}(2^{-(j+1)})}(E) \geq n_j > j^{-2}2^j \ ,$$

and moreover:

$$j \geq -\log_2 \mathrm{scl}_\alpha(\delta) \ .$$

Since these inequalities hold true for δ arbitrarily small, there exists an increasing sequence of integers $(j_n)_{n\geq 1}$ such that:

$$\tilde{\mathcal{N}}_{\epsilon_n}(E) > j_n^{-2}2^{j_n} \quad \text{where } \epsilon_n := \mathrm{scl}_\alpha^{-1}(2^{-(j_n+1)}) \ . \tag{3.2.14}$$

Given a positive $\beta < \alpha$, by Definition 3.1.2 of scaling, for $\lambda > 1$ close to 1, it holds:

$$\mathrm{scl}_\beta(\epsilon) \cdot (\mathrm{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon\to 0]{} +\infty \ . \tag{3.2.15}$$

On the other hand, given such $\lambda > 1$, for n large enough, it holds:

$$j_n^{-2}2^{j_n} \geq 2^{\lambda^{-1}\cdot(j_n+1)} \ .$$

It follows by Eq. (3.2.14):

$$\tilde{\mathcal{N}}_{\epsilon_n}(E) \geq \left(2^{-(j_n+1)}\right)^{-\lambda^{-1}} = (\mathrm{scl}_\alpha(\epsilon_n))^{-\lambda^{-1}} \ .$$

This together with Eq. (3.2.15) gives:

$$\mathrm{scl}_\beta(\epsilon_n) \cdot \tilde{\mathcal{N}}_{\epsilon_n}(E) > \mathrm{scl}_\beta(\epsilon) \cdot (\mathrm{scl}_\alpha(\epsilon))^{-\lambda^{-1}} \xrightarrow[\epsilon\to 0]{} +\infty \ .$$

By Lemma 3.2.11 we deduce $\overline{\mathrm{scl}}_B E \geq \beta$. Taking β close to α ends the proof. □

3.2.5 Properties and comparison of scales of metric spaces

We first give a few basic properties of scales that would allow us to compare them. Since both packing and Hausdorff scales are defined via measures, they are both σ-stable as shown in the following:

Lemma 3.2.21. *Let (X,d) be a metric space. Let I be a countable set and $(E_i)_{i\in I}$ a covering of X, then for any scaling* scl*:*

$$\mathsf{scl}_H X = \sup_{i\in I} \mathsf{scl}_H E_i \quad \textit{and} \quad \mathsf{scl}_P X = \sup_{i\in I} \mathsf{scl}_P E_i \ .$$

Proof. The equality on packing scales is clear by definition. Let us prove the equality on Hausdorff scales. By monotonicity of the Hausdorff measure, it holds $\mathsf{scl}_H X \geq \sup_{i\in I} \mathsf{scl}_H E_i$. For the reverse inequality, consider $\alpha > \sup_{i\in I} \mathsf{scl}_H E_i$, then for any $i \in I$ it holds $\mathcal{H}^{\mathrm{scl}_\alpha}(E_i) = 0$. Thus, it holds:

$$\mathcal{H}^{\mathrm{scl}_\alpha}(X) \leq \sum_{i\in I} \mathcal{H}^{\mathrm{scl}_\alpha}(E_i) = 0 \ ,$$

and then $\mathsf{scl}_H X \leq \alpha$. Since this is true for any $\alpha > \sup_{i\in I} \mathsf{scl}_H E_i$, the desired result comes. □

Note that σ-stability is not a property of box scales. To see that, it suffices to consider a countable dense subset of a metric space (X,d) with positive box scales. This is actually a basic known fact for the specific case of dimension that naturally still holds there.

The following lemma shows in particular that the above scales are bi-Lipschitz invariants.

Lemma 3.2.22. *Let (X,d) and (Y,d) be two metric spaces such that there exists a Lipschitz map $f : (X,d) \to (Y,d)$. Then for any scaling* scl*, the scales of $f(X)$ are at most the ones of X:*

$$\mathsf{scl}_H f(X) \leq \mathsf{scl}_H X; \quad \mathsf{scl}_P f(X) \leq \mathsf{scl}_P X; \quad \underline{\mathsf{scl}}_B f(X) \leq \underline{\mathsf{scl}}_B X; \quad \overline{\mathsf{scl}}_B f(X) \leq \overline{\mathsf{scl}}_B X \ .$$

Proof. Let us fix $\epsilon > 0$. Let $K > 0$ be a Lipschitz constant for f. We first show the inequalities on box and packing scales.

Consider a minimal covering $(B(x_j,\epsilon))_{1\leq j\leq N}$ of ϵ-balls centered in X. it holds:

$$f(X) = f\left(\bigcup_{j=1}^{N} B(x_j,\epsilon_j)\right) \subset \bigcup_{j=1}^{N} B(f(x_j), K\cdot\epsilon_j) \ .$$

Then $(B(f(x_j), K\cdot\epsilon))_{1\leq j\leq N}$ is a covering by $K\cdot\epsilon$-balls of $f(X)$. Then $\mathcal{N}_{K\cdot\epsilon}(f(X)) \leq \mathcal{N}_\epsilon(X)$ and all the inequalities on the box and packing scales are immediately deduced from Eq. (3.2.2). Now for Hausdorff scales, consider a countable set J and $(B(x_j,\epsilon_j))_{j\in J}$ an ϵ-cover of X. Then it holds:

$$f(X) \subset \bigcup_{j\in J} B(f(x_j), K\cdot\epsilon_j) \ .$$

For any $\alpha > \beta > 0$ and $\delta > 0$ small enough, by Fact 3.2.1, it holds:

$$\mathrm{scl}_\alpha(\delta) \leq \mathrm{scl}_\beta(K^{-1}\cdot\delta) \ .$$

Hence for ϵ small, it holds:

$$\mathcal{H}^{\mathrm{scl}_\alpha}_{K\cdot\epsilon}(f(X)) \leq \sum_{j\in J} \mathrm{scl}_\alpha(K\cdot\epsilon_j) \leq \sum_{j\in J} \mathrm{scl}_\beta(\epsilon_j) \ .$$

As $\beta > \mathsf{scl}_H X$, the ϵ-cover $(B(x_j, \epsilon_j))_{j\in J}$ can be chosen such that $\sum_{j\in J} \mathrm{scl}_\beta(\epsilon_j)$ is arbitrarily small. Consequently, it holds $\mathcal{H}^{\mathrm{scl}_\alpha}_{K\cdot\epsilon}(f(X)) = 0$, and so $\mathsf{scl}_H f(X) \leq \alpha$. As α is arbitrarily close to $\mathsf{scl}_H X$, it holds:

$$\mathsf{scl}_H f(X) \leq \mathsf{scl}_H X \ .$$

□

As a direct application, we obtain the following:

Corollary 3.2.23. *Let (X, d) and (Y, d) be two metric spaces. Assume that there exists an embedding $g : (Y, \delta) \to (X, d)$ such that g^{-1} is Lipschitz on $g(Y)$. Then for every scaling* scl*, the scales of Y are at most the ones of X:*

$$\mathsf{scl}_H Y \leq \mathsf{scl}_H X; \quad \mathsf{scl}_P Y \leq \mathsf{scl}_P X; \quad \underline{\mathsf{scl}}_B Y \leq \underline{\mathsf{scl}}_B X; \quad \overline{\mathsf{scl}}_B Y \leq \overline{\mathsf{scl}}_B X \ .$$

Proof. By Lemma 3.2.22 we have $\mathsf{scl}_\bullet Y \leq \mathsf{scl}_\bullet g(Y)$ for any $\mathsf{scl}_\bullet \in \{\mathsf{scl}_H, \mathsf{scl}_P\underline{\mathsf{scl}}_B, \overline{\mathsf{scl}}_B\}$. As $g(Y) \subset X$, we have also $\mathsf{scl}_\bullet g(Y) \leq \mathsf{scl}_\bullet X$. □

Remark that Lemma 3.2.22 and Corollary 3.2.23 hold even for scalings that have sub-polynomial behaviors.

The end of this section consists of comparing the different scales introduced and proving Theorem A. We start by comparing the Hausdorff with lower box scales. The following proposition generalizes well known facts on dimension. See e.g. [Fal04][(3.17)].

Proposition 3.2.24. *Let (X, d) be a metric space and* scl *a scaling, its Hausdorff scale is at most its lower box scale:*

$$\mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \ .$$

Proof. We can assume without any loss that (X, d) is totally bounded. If $\mathsf{scl}_H X = 0$ the inequality obviously holds, thus consider a positive number $\alpha < \mathsf{scl}_H X$. For $\delta > 0$ small enough, $\mathcal{H}^{\mathrm{scl}_\alpha}_\delta(X) > 1$. Also, there exists a δ-cover $(B_j)_{1\leq j\leq \mathcal{N}_\delta(X)}$. It verifies:

$$1 < \sum_{1\leq j\leq \mathcal{N}_\delta(F)} \mathrm{scl}_\alpha(|B_j|) = \mathcal{N}_\delta(X) \cdot \mathrm{scl}_\alpha(\delta) \ .$$

From there, it holds that $\underline{\mathsf{scl}}_B X \geq \alpha$. We conclude by taking α arbitrarily close to $\mathsf{scl}_H X$. □

We have compared Hausdorff and packing scales with their corresponding box scales. It remains to compare each other with the following:

Proposition 3.2.25. *Let (X, d) be a metric space and* scl *a scaling. It holds:*

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \ .$$

Proof. By Lemma 3.2.21, Hausdorff scale is σ-stable:

$$\mathsf{scl}_H X = \inf_{\bigcup_{n\geq 1} E_n = X} \sup_{n\geq 1} \mathsf{scl}_H E_n \ ,$$

where the infimum is taken over countable coverings of X. Moreover, by Proposition 3.2.24, we have:

$$\mathsf{scl}_H E \leq \underline{\mathsf{scl}}_B E \leq \overline{\mathsf{scl}}_B E \ ,$$

for any subset E of X. It follows then:

$$\mathsf{scl}_H X \leq \inf_{\bigcup_{n\geq 1} E_n = X} \sup_{n\geq 1} \overline{\mathsf{scl}}_B E_n = \mathsf{scl}_P X \ .$$

□

For the sake of completeness we will resume:

Proof of Theorem A. Let (X, d) be a metric space and scl a scaling. By Proposition 3.2.24, Proposition 3.2.25 and Proposition 3.2.15, it holds respectively:

$$\mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X, \quad \mathsf{scl}_H X \leq \mathsf{scl}_P X \quad \text{and} \quad \mathsf{scl}_H X \leq \overline{\mathsf{scl}}_B X \ .$$

Now since $\underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X$ obviously holds, we deduce the desired result:

$$\mathsf{scl}_H X \leq \mathsf{scl}_P X \leq \overline{\mathsf{scl}}_B X \quad \text{and} \quad \mathsf{scl}_H X \leq \underline{\mathsf{scl}}_B X \leq \overline{\mathsf{scl}}_B X \ .$$

□

3.3 Scales of measures

In this section we recall the different versions of scales of measures we introduced and show the inequalities and equalities comparing them. In particular, we provide proofs of Theorem B and Theorem C. They generalize known results from dimension theory to any scaling and moreover bring new comparisons (see Theorem 3.3.10) between quantization and box scales that were not shown yet for even the case of dimension.

3.3.1 Hausdorff, packing and local scales of measures

Let us recall the definition of local scales. Let μ be a Borel measure on a metric space (X, d) and scl a scaling. The *lower and upper scales* of μ are the functions that map a point $x \in X$ to:

$$\underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \sup\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} 0\right\}$$

and

$$\overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) = \inf\left\{\alpha > 0 : \frac{\mu\left(B(x,\epsilon)\right)}{\mathrm{scl}_\alpha(\epsilon)} \xrightarrow[\epsilon\to 0]{} +\infty\right\} \ .$$

We shall compare local scales with the followings:

Definition 3.3.1 (Hausdorff scales of a measure)**.** *Let* scl *be a scaling and μ a non-null Borel measure on a metric space (X, d). We define the* Hausdorff and $*$-Hausdorff scales *of the measure μ by:*

$$\mathsf{scl}_H\mu = \inf_{E\in\mathcal{B}}\{\mathsf{scl}_H E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_H\mu = \inf_{E\in\mathcal{B}}\{\mathsf{scl}_H E : \mu(X\backslash E) = 0\} \ ,$$

where $\mathcal{B}$ is the set of Borel subsets of X.

Definition 3.3.2 (Packing scales of a measure)**.** *Let* scl *be a scaling and* μ *a non-null Borel measure on a metric space* (X, d)*. We define the* packing and $*$-packing scales *of* μ *by:*

$$\mathsf{scl}_P\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_P E : \mu(E) > 0\} \quad \textit{and} \quad \mathsf{scl}^*_P\mu = \inf_{E\in\mathcal{B}} \{\mathsf{scl}_P E : \mu(X\backslash E) = 0\} \ .$$

Remark 3.3.3. *In order to avoid excluding the null measure* 0*, we set as a convention:*

$$\mathsf{scl}_H 0 = \mathsf{scl}^*_H 0 = \mathsf{scl}_P 0 = \mathsf{scl}^*_P 0 = 0 \ .$$

The lemma below will allow us to compare local scales with the other scales of measures.

Lemma 3.3.4. *Let* μ *be a Borel measure on* X*. Then for any Borel subset* F *of* X *such that* $\mu(F) > 0$*, the restriction* σ *of* μ *to* F *verifies:*

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\sigma \quad \textit{and} \quad \operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\sigma.$$

Moreover, if there exists $\alpha > 0$ *such that* $F \subset \left\{x \in X : \overline{\mathsf{scl}}_{loc}\mu(x) > \alpha\right\}$*, it holds then:*

$$\operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\sigma \geq \alpha \ ,$$

and similarly if $F \subset \{x \in X : \underline{\mathsf{scl}}_{loc}\mu(x) > \alpha\}$*, it holds:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\sigma \geq \alpha \ .$$

Proof. Consider a point $x \in X$, then for any $\epsilon > 0$, one has $\sigma(B(x,\epsilon)) \leq \mu(B(x,\epsilon))$, thus by definition of local scales:

$$\overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \quad \text{and} \quad \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \ .$$

Now if there exists $\alpha > 0$ such that $F \subset \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\right\}$, as $\overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) \geq \alpha$ for μ-almost every x in F, it comes by the above inequality that $\overline{\mathsf{scl}}_{\mathrm{loc}}\sigma(x) \geq \alpha$ for μ-almost every x in F, and thus for σ-almost every $x \in X$. It follows $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \geq \alpha$. And the same holds for lower local scales. □

The following lemma corresponds to part of the results of Theorem B. We will prove this lemma later in Section 3.3.4. First, we will use it to prove Theorem C in Section 3.3.3. This lemma states that the lower and upper local scales of a measure are, respectively, not greater than the Hausdorff and packing scales of the underlying space:

Lemma 3.3.5. *Let* (X, d) *be a metric space and* μ *a Borel measure on* X*. Let* scl *be a scaling. Then it holds:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}_H X \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}_P X \ .$$

Note that in the above we can replace X by any of its subsets with total mass. This observation directly implies:

Corollary 3.3.6. *Let* (X, d) *be a metric space and* μ *a Borel measure on* X*. Let* scl *be a scaling. It holds:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}^*_H\mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}^*_P\mu \ .$$

To prove Theorem C we need to study quantization scales of measures.

3.3.2 Quantization and box scales of measures

Let us first recall the definition of quantization scales. Let (X, d) be a metric space and μ a Borel measure on X. Given $\epsilon > 0$, the *ϵ-quantization number* $\mathcal{Q}_\mu(\epsilon)$ of μ is the minimal cardinality of a set of points that is on average ϵ-close to any point in X:

$$\mathcal{Q}_\mu(\epsilon) = \inf\left\{N \geq 0 : \exists\, \{c_i\}_{i=1,\dots,N} \subset X, \int_X d(x, \{c_i\}_{1\leq i\leq N})d\mu(x) < \epsilon\right\} .$$

Then *lower and upper quantization scales* of μ for a given scaling scl are defined by:

$$\underline{\mathsf{scl}}_Q\mu = \sup\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty\right\}$$

and

$$\overline{\mathsf{scl}}_Q\mu = \inf\left\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon)\cdot \mathrm{scl}_\alpha(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} 0\right\} .$$

Quantization scales of a measure are compared in Theorem C with box scales of measures:

Definition 3.3.7 (Box scales of a measure)**.** *Let* scl *be a scaling and* μ *a positive Borel measure on a metric space* (X, d)*. We define the* lower box scale *and the* $*$-lower box scale *of* μ *by:*

$$\underline{\mathsf{scl}}_B\mu = \inf_{E\in\mathcal{B}}\{\underline{\mathsf{scl}}_B E : \mu(E) > 0\} \quad \textit{and} \quad \underline{\mathsf{scl}}^*_B\mu = \inf_{E\in\mathcal{B}}\{\underline{\mathsf{scl}}_B E : \mu(X\backslash E) = 0\} ,$$

where $\mathcal{B}$ *is the set of Borel subsets of* X*. Similarly, we define the* upper box scale *and the* $*$-upper box scale *of* μ *by:*

$$\overline{\mathsf{scl}}_B\mu = \inf_{E\in\mathcal{B}}\left\{\overline{\mathsf{scl}}_B E : \mu(E) > 0\right\} \quad \textit{and} \quad \overline{\mathsf{scl}}^*_B\mu = \inf_{E\in\mathcal{B}}\left\{\overline{\mathsf{scl}}_B E : \mu(X\backslash E) = 0\right\} .$$

As for Hausdorff scales of measures, we chose that all box scales of the null measure are equal to 0 as a convention. The following is straightforward:

Lemma 3.3.8. *Let* (X, d) *be a metric space and* μ *a Borel measure on* X*. Given* scl *a scaling, it holds:*

$$\underline{\mathsf{scl}}_Q\mu \leq \underline{\mathsf{scl}}^*_B\mu \quad \textit{and} \quad \overline{\mathsf{scl}}_Q\mu \leq \overline{\mathsf{scl}}^*_B\mu .$$

Proof. We can assume without loss of generality that $\overline{\mathsf{scl}}^*_B\mu$ and $\underline{\mathsf{scl}}^*_B\mu$ are finite. Let E be a Borel set with total mass such that $\overline{\mathsf{scl}}_B E$ is finite, then E is totally bounded by Fact 3.2.8. Now for $\epsilon > 0$, consider a minimal covering by ϵ-balls centered at some points $x_1, ..., x_N$ in E. Since $\mu(X\backslash E) = 0$, it comes:

$$\int_X d(x, \{x_i\}_{1\leq i\leq N})d\mu(x) = \int_E d(x, \{x_i\}_{1\leq i\leq N})d\mu(x) \leq \epsilon < 2\epsilon .$$

It follows that $\mathcal{Q}_\mu(2\epsilon) \leq \mathcal{N}_\epsilon(E)$, and thus by Lemma 3.2.2, we obtain:

$$\underline{\mathsf{scl}}_Q\mu \leq \underline{\mathsf{scl}}_B E \quad \text{and} \quad \overline{\mathsf{scl}}_Q\mu \leq \overline{\mathsf{scl}}_B E .$$

Since this holds true for any Borel set E with total mass, the desired result comes. □

The following lemma will allow us to compare quantization scales with box scales.

Lemma 3.3.9. *Let μ be a Borel measure on (X, d) such that $\mathcal{Q}_\mu(\epsilon) < +\infty$ for any $\epsilon > 0$. Let us fix $\epsilon > 0$ and an integer $N \geq \mathcal{Q}_\mu(\epsilon)$. Thus consider $x_1, \ldots, x_N \in X$ such that:*

$$\int_X d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) < \epsilon \ .$$

For any $r > 0$, with $E_r := \bigcup_{i=1}^N B(x_i, r)$, it holds:

$$\mu(X \backslash E_r) < \frac{\epsilon}{r} \ .$$

Proof. Since $X \backslash E_r$, the complement of E_r in X is the set of points with distance at most r from the set $\{x_1, \ldots, x_n\}$, it holds:

$$r \cdot \mu(X \backslash E_r) \leq \int_{X \backslash E_r} d(x, \{x_i\}_{1 \leq i \leq N}) d\mu(x) < \epsilon \ ,$$

which gives the desired result by dividing both sides by r. □

The following result exhibits the relationship between quantization scales and box scales. As far as we know, this result has not yet been proved, even for the specific case of dimension. It is a key element in the answer to Problem 3.1.15.

Theorem 3.3.10. *Let μ be a non null Borel measure on a metric space (X, d). For any scaling* scl*, there exists a Borel set F with positive mass arbitrarily close to $\mu(X)$ such that:*

$$\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Consequently, it holds:

$$\underline{\mathsf{scl}}_B \mu \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \overline{\mathsf{scl}}_B \mu \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Proof. If $\mathcal{Q}_\mu(\epsilon)$ is not finite for any $\epsilon > 0$ we have $\underline{\mathsf{scl}}_Q \mu = +\infty = \overline{\mathsf{scl}}_Q \mu$ and nothing to prove. So we can assume that $\mathcal{Q}_\mu(\epsilon) < +\infty$ for every $\epsilon > 0$. We consider two sequences of positive numbers $\epsilon_n := \exp(-n)$ and $r_n := n^2 \cdot \exp(-n) = n^2 \cdot \epsilon_n$ for $n \geq 1$. Then for each $n \geq 1$, consider a finite set $C_n \subset X$ with minimal cardinality such that:

$$\int_X d(x, C_n) d\mu(x) < \epsilon_n \ .$$

By Lemma 3.3.9, it holds:

$$\mu(X \backslash E_n) < \frac{\epsilon_n}{r_n} = \frac{1}{n^2} \quad \text{with } E_n := \bigcup_{x \in C_n} B(x, r_n) \ .$$

By the Borell-Cantelli lemma, it holds:

$$\mu(\liminf E_n) = \mu\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} E_n \right) = \mu(X) > 0 \ .$$

Hence, for sufficiently large m, we have $\mu(F) > 0$, where $F := \bigcap_{n \geq m} E_n$. Moreover if μ is finite by taking m even larger we can have $\mu(X \backslash F)$ arbitrarily small; and if μ is infinite, we can take $\mu(F)$ arbitrarily large. We now fix such a value of m. Observe that $F \subset E_n$ for every $n \geq m$. As E_n is a union of balls of radius r_n, we have trivially:

$$\forall n \geq m : \mathcal{N}_{r_n}(F) \leq \operatorname{Card} C_n = \mathcal{Q}_\mu(\epsilon_n) \ .$$

Finally by Lemma 3.2.3, and the fact that $\log r_n \sim \log \epsilon_n \sim \log r_{n+1}$, we obtain:

$$\begin{aligned}\underline{\mathsf{scl}}_B F &= \sup\{\alpha > 0 : \mathcal{N}_{r_n}(F) \cdot \mathrm{scl}_\alpha(r_n) \xrightarrow[n\to+\infty]{} +\infty\} \\ &\leq \sup\{\alpha > 0 : \mathcal{Q}_\mu(\epsilon_n) \cdot \mathrm{scl}_\alpha(\epsilon_n) \xrightarrow[n\to+\infty]{} +\infty\} \\ &= \underline{\mathsf{scl}}_Q \mu \ .\end{aligned}$$

Similarly, we can prove $\overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu$. Then the last two inequalities stated in Theorem 3.3.10 follow directly from the definitions of $\underline{\mathsf{scl}}_B \mu$ and $\overline{\mathsf{scl}}_B \mu$. □

3.3.3 Comparison between local and global scales of measures and proof of Theorem C

To prove Theorem C we will need:

Theorem 3.3.11. *Let (X, d) be a separable metric space and μ a finite Borel measure on X. Let* sc*l be a scaling. It holds:*

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc} \mu \leq \underline{\mathsf{scl}}_Q \mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc} \mu \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Proof. If the lower (respectively upper) local scale of μ is zero at almost every point then obviously it is not greater than the lower (respectively upper) quantization scale of μ. Otherwise, consider $\alpha, \beta > 0$ such that:

$$\alpha < \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}} \mu \quad \text{and} \quad \beta < \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}} \mu \ . \tag{3.3.1}$$

Note that $E := \left\{ x \in X : \underline{\mathsf{scl}}_{\mathrm{loc}} \mu(x) > \alpha \quad \text{and} \quad \overline{\mathsf{scl}}_{\mathrm{loc}} \mu(x) > \beta \right\}$ has positive mass. Applying Theorem 3.3.10 to the restriction of μ to E provides a Borel subset $F \subset E$ with $\mu(F) > 0$ such that:

$$\underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu \quad \text{and} \quad \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Yet by Proposition 3.2.24 and Proposition 3.2.15, it holds respectively:

$$\mathsf{scl}_H F \leq \underline{\mathsf{scl}}_B F \quad \text{and} \quad \mathsf{scl}_P F \leq \overline{\mathsf{scl}}_B F \ . \tag{3.3.2}$$

Now, by setting σ the restriction of μ to F, we obtain by Lemma 3.3.4 and Eq. (3.3.1):

$$\alpha \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \sigma \quad \text{and} \quad \beta \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \sigma \ . \tag{3.3.3}$$

By Lemma 3.3.5, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \mathsf{scl}_H F \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \mathsf{scl}_P F \ . \tag{3.3.4}$$

Finally, combining Eqs. (3.3.2) to (3.3.4) gives:

$$\alpha \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \mathsf{scl}_H F \leq \underline{\mathsf{scl}}_B F \leq \underline{\mathsf{scl}}_Q \mu$$

and

$$\beta \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}} \sigma \leq \mathsf{scl}_P F \leq \overline{\mathsf{scl}}_B F \leq \overline{\mathsf{scl}}_Q \mu \ .$$

Since this holds true for any α and β arbitrarily close to $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}} \mu$ and $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}} \mu$ we obtain the desired result. □

We are now able to show:

Proof of Theorem C. By Theorem 3.3.10 and Lemma 3.3.8, it holds:

$$\underline{\mathsf{scl}}_B\mu \leq \underline{\mathsf{scl}}_Q\mu \leq \underline{\mathsf{scl}}^\star_B\mu \quad \text{and} \quad \overline{\mathsf{scl}}_B\mu \leq \overline{\mathsf{scl}}_Q\mu \leq \overline{\mathsf{scl}}^\star_B\mu \ .$$

By Theorem 3.3.11 it holds:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_Q\mu \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_Q\mu \ .$$

Thus it remains only to show:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_B\mu \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_B\mu \ . \tag{3.3.5}$$

To prove this, consider a subset $E \subset X$ with positive mass. Set σ the restriction of μ to E. By Lemma 3.3.4, it holds:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \ . \tag{3.3.6}$$

By Theorem 3.3.11 it holds:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \underline{\mathsf{scl}}_Q\sigma \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_Q\sigma \ . \tag{3.3.7}$$

Moreover, by Lemma 3.3.8 we have:

$$\underline{\mathsf{scl}}_Q\sigma \leq \underline{\mathsf{scl}}_B E \quad \text{and} \quad \overline{\mathsf{scl}}_Q\sigma \leq \overline{\mathsf{scl}}_B E \ . \tag{3.3.8}$$

Combining Eqs. (3.3.6) to (3.3.8) provides:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \underline{\mathsf{scl}}_B E \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \overline{\mathsf{scl}}_B E \ .$$

Taking the infima over such subsets $E \subset X$ with positive mass gives Eq. (3.3.5). □

3.3.4 Proof of Theorem B

This subsection contains the proof of Theorem B, we recall its statement below. We will use Vitali covering lemma to compare local scales with Hausdorff and packing scales. This lemma was first used for the dimensional case by Tamashiro [Tam95].

Lemma 3.3.12 (Vitali covering lemma)**.** *Let (X, d) be a separable metric space. Given $\delta > 0$, $\mathcal{B}$ a family of open balls in X with radii at most δ and F the union of these balls. There exists a countable set J and a δ-pack $(B(x_j, r_j))_{j\in J} \subset \mathcal{B}$ of F such that:*

$$F \subset \bigcup_j B(x_j, 5r_j) \ .$$

For a proof of this version of the Vitali covering lemma, see [EG15][1.5.1, p. 35]. Despite they stated it for the Euclidean case, their proof adapts straightforwardly for separable metric spaces. We are now ready to prove:

Proof of Lemma 3.3.5. Proof of the first inequality: If $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu = 0$ or $\mathsf{scl}_H X = +\infty$, it obviously holds that $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H X$. Otherwise, X is separable and consider a positive $\alpha < \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$. There exists $r_0 > 0$ such that the set:

$$A := \{x \in X : \mu(B(x,r)) \leq \mathrm{scl}_\alpha(r),\ \forall r \in (0, r_0)\}$$

has positive measure. Consider $\delta \leq r_0$ and a δ-cover $(B_j)_{j\in J}$ of A. It holds:

$$0 < \mu(A) \leq \sum_{j\in J} \mu(B_j) \leq \sum_{j\in J} \mathrm{scl}_\alpha(|B_j|)\ .$$

Taking the infimum over δ-covers provides:

$$0 < \mu(A) \leq \mathcal{H}^{\mathrm{scl}_\alpha}_\delta(A)\ .$$

Now as δ goes to 0 we obtain:

$$0 < \mu(A) \leq \mathcal{H}^{\mathrm{scl}_\alpha}(A)\ .$$

It follows:

$$\mathsf{scl}_H X \geq \mathsf{scl}_H A \geq \alpha\ ,$$

which allows us to conclude the proof of that first inequality.

Proof of the second inequality: If $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu = 0$ or $\mathsf{scl}_P X = +\infty$, it obviously holds $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P X$. Otherwise, X is separable and consider a positive $\alpha < \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Put:

$$F = \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) > \alpha\right\}.$$

Let $(F_N)_{N\geq 1}$ be a covering of X by Borel subsets. Given $0 < \beta < \alpha$, by Fact 3.2.1, there exists $\delta_0 > 0$ such that for any $r \leq \delta_0$, it holds:

$$\mathrm{scl}_\alpha(5r) \leq \mathrm{scl}_\beta(r)\ . \tag{3.3.9}$$

Fix $\delta \in (0, \delta_0)$ and an integer $N \geq 1$. For each x in F_N, by Lemma 3.2.3 there exists a minimal integer $n(x)$ such that:

$$\mu\left(B(x, 5r(x))\right) \leq \mathrm{scl}_\alpha(5r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta\ . \tag{3.3.10}$$

We now set:

$$\mathcal{F} = \{B(x, r(x)) : x \in F_N\}\ .$$

Thus by Vitali covering Lemma 3.3.12 there exists a countable subset $J \subset F_N$ such that $\left(B(x, r(x))\right)_{x\in J}$ is a δ-pack of F_N and $\left(B(x, 5r(x))\right)_{x\in J}$ is a covering of F_N. From there, by Eqs. (3.3.9) and (3.3.10) it holds:

$$\mu(F_N) \leq \sum_{x\in J} \mu(B(x, 5r(x))) \leq \sum_{x\in J} \mathrm{scl}_\alpha(5r(x)) \leq \sum_{x\in J} \mathrm{scl}_\beta(r(x))\ .$$

Since this holds true for any δ-pack, we have:

$$\mathcal{P}^{\mathrm{scl}_\beta}_\delta(F_N) \geq \mu(F_N)\ ,$$

and then taking δ arbitrarily close to 0 leads to:

$$\mathcal{P}^{\mathrm{scl}_\beta}_0(F_N) \geq \mu(F_N)\ .$$

Summing over $N \geq 1$ provides:

$$\sum_{N\geq 1} \mathcal{P}_0^{\mathsf{scl}_\beta}(F_N) \geq \sum_{N\geq 1} \mu(F_N) \geq \mu(F) > 0 \ .$$

Recall that $(F_N)_{N\geq 1}$ is an arbitrary covering by Borel sets of F, thus by definition of the packing measure and the latter equation, it holds:

$$\mathcal{P}^{\mathsf{scl}_\beta}(F) \geq \mu(F) > 0 \ .$$

It holds then $\mathsf{scl}_P F \geq \beta$ for any $\beta < \alpha < \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$, which allows us to conclude the proof. □

We deduce then:

Proposition 3.3.13. *Let (X, d) be a metric space and μ a Borel measure on X, then:*

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}_H\mu \quad \textit{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}_P\mu \ ,$$

and

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}^*_H\mu \quad \textit{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{loc}\mu \leq \mathsf{scl}^*_P\mu \ .$$

Proof. The second line of inequalities is given by Corollary 3.3.6. It remains to show the first line of inequalities. Let E be a Borel subset of X with μ-positive mass. With σ the restriction of μ to E, it holds by Lemma 3.3.5:

$$\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \mathsf{scl}_H E \quad \text{and} \quad \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\sigma \leq \mathsf{scl}_P E \ .$$

Then, by Lemma 3.3.4,it follows:

$$\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_H E \quad \text{and} \quad \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P E \ .$$

Taking the infima over E with positive mass ends the proof. □

Explicit links between packing scales, Hausdorff scales and local scales of measures can now be established by proving Theorem B. Let us first recall its statement: Let (X, d) be a metric space and μ a Borel measure on X, then it holds:

$$\mathsf{scl}_H\mu = \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}_P\mu = \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$$

and

$$\mathsf{scl}^*_H\mu = \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \leq \mathsf{scl}^*_P\mu = \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \ .$$

Proof of Theorem B. By Proposition 3.3.13 it remains only to show four inequalities:

$$\mathsf{scl}_H\mu \underbrace{\leq}_{(i)} \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \quad ; \quad \mathsf{scl}_P\mu \underbrace{\leq}_{(ii)} \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$$

and

$$\mathsf{scl}^*_H\mu \underbrace{\leq}_{(iii)} \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu \quad ; \quad \mathsf{scl}^*_P\mu \underbrace{\leq}_{(iv)} \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu \ .$$

Note that for each of the above inequalities, if the right-hand side quantity is infinite, there is nothing to prove. In each of the following proofs, we will assume that this quantity is finite.

Proof of (i): Fix $\alpha > \operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}$ and $\beta > \alpha$. By definition of scaling, there exists $\delta > 0$ such that for any $r \in (0, \delta)$ it holds

$$\mathrm{scl}_\beta(5r) \leq \mathrm{scl}_\alpha(r) \ . \tag{3.3.11}$$

Denote:

$$F := \{x \in X : \underline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) < \alpha\} \ .$$

Then it holds that $\mu(F) > 0$ and by Lemma 3.2.3 for any x in F there exists a minimal integer $n(x)$ such that:

$$\mu\left(B(x, r(x))\right) \geq \mathrm{scl}_\alpha(r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta \ . \tag{3.3.12}$$

Now set:

$$\mathcal{F} := \{B(x, r(x)) : x \in F\} \, .$$

By Vitali covering Lemma 3.3.12, there exists a countable subset $J \subset F$ such that $(B(x, r(x)))_{x \in J}$ is a δ-pack of F and $F \subset \bigcup_{x \in J} B(x, 5r(x))$. Then, by Eqs. (3.3.11) and (3.3.12) it holds:

$$\sum_{x \in J} \mathrm{scl}_\beta(5r(x)) \leq \sum_{x \in J} \mathrm{scl}_\alpha(r(x)) \leq \sum_{x \in J} \mu(B(x, r(x))) \leq \mu(F) \ .$$

It follows that $\mathcal{H}_\delta^{\mathrm{scl}_\beta}(F) \leq \mu(F)$ and thus $\mathcal{H}^{\mathrm{scl}_\beta}(F) \leq \mu(F)$ as δ goes to 0. It follows that $\mathsf{scl}_H \mu \leq \mathsf{scl}_H F \leq \beta$. Taking $\beta > \alpha$ close to $\operatorname{ess\,inf} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$, we obtain inequality (i).

Proof of (ii): Consider $\alpha > \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Set $E = \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu(x) < \alpha\right\}$. Observe that $\mu(X \backslash E) = 0$ and:

$$E = \bigcup_{i \geq 1} E_i \quad \text{where } E_i := \left\{x \in E : \forall r \leq 2^{-i},\ \mu(B(x, r)) \geq \mathrm{scl}_\alpha(r)\right\} \ .$$

By σ-stability of packing scales provided by Lemma 3.2.21, it holds $\mathsf{scl}_P E = \sup_{i \geq 1} \mathsf{scl}_P E_i$. It is then enough to show that for any $i \geq 1$, we have $\mathsf{scl}_P E_i \leq \alpha$. Indeed, then taking α arbitrarily close to $\operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ allows us to conclude. In that way, let us fix $i \geq 1$. Fix also $\delta \in (0, 2^{-i})$ and consider a countable set J and $(B_j)_{j \in J}$ a δ-pack of E_i. It follows:

$$\sum_{j \in J} \mathrm{scl}_\alpha(|B_j|) \leq \sum_{j \in J} \mu(B_j) \leq 1 \ .$$

Since this holds true for any δ-pack, we have:

$$\mathcal{P}_\delta^{\mathrm{scl}_\alpha}(E_i) \leq 1 \ .$$

As δ goes to 0 we obtain:

$$\mathcal{P}^{\mathrm{scl}_\alpha}(E_i) \leq 1 \ .$$

It follows that we indeed have $\mathsf{scl}_P E_i \leq \alpha$.

Proof of (iii): Fix $\alpha > \operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}$. For $\beta > \alpha$, consider $\delta > 0$ such that for any $r \in (0, \delta)$ it holds:

$$\mathrm{scl}_\beta(5r) \leq \mathrm{scl}_\alpha(r) \ . \tag{3.3.13}$$

Denote:

$$F := \{x \in X : \underline{\mathsf{scl}}_{loc}\mu(x) < \alpha\} \ .$$

Note that F has total mass and by Lemma 3.2.3 for every $x \in F$ there exists a minimal integer $n(x)$ such that:

$$\mu\left(B(x, r(x))\right) \geq \mathrm{scl}_\alpha(r(x)) \quad \text{where } r(x) := \exp(-n(x)) \leq \delta \ . \tag{3.3.14}$$

Now put:

$$\mathcal{F} := \{B(x, r(x)) : x \in F\} \ .$$

By Vitali's Lemma 3.3.12, there exists a countable subset $J \subset F$ such that $(B(x, r(x)))_{x\in J}$ is a δ-pack and $F = \bigcup_{x\in J} B(x, 5r(x))$. Thus, by Eqs. (3.3.13) and (3.3.14) it holds:

$$\sum_{x\in J} \mathrm{scl}_\beta(5r(x)) \leq \sum_{x\in J} \mathrm{scl}_\alpha(r(x)) \leq \sum_{x\in J} \mu(B(x, r(x))) \leq \mu(F) \ .$$

It follows that $\mathcal{H}_\delta^{\mathrm{scl}_\beta}(F) \leq \mu(F)$ and thus $\mathcal{H}^{\mathrm{scl}_\beta}(F) \leq \mu(F)$ as δ goes to 0. It follows that $\mathsf{scl}_H^*\mu \leq \mathsf{scl}_H F \leq \beta$. Taking $\beta > \alpha$ close to $\operatorname{ess\,sup} \underline{\mathsf{scl}}_{\mathrm{loc}}\mu$, we obtain inequality (iii).

Proof of (iv): Fix $\alpha > \operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. Consider the set $E := \left\{x \in X : \overline{\mathsf{scl}}_{\mathrm{loc}}\mu < \alpha\right\}$. Then observe that $\mu(X\backslash E) = 0$ and that we can write:

$$E = \bigcup_{i\geq 1} E_i \qquad \text{where } E_i = \left\{x \in E : \forall r \leq 2^{-i},\ \mu(B(x, r)) \geq \mathrm{scl}_\alpha(r)\right\} \ .$$

By Lemma 3.2.21, we have $\mathsf{scl}_P E = \sup_{i\geq 1} \mathsf{scl}_P E_i$, it is then enough to show that for any $i \geq 1$, we have $\mathsf{scl}_P E_i \leq \alpha$. We indeed can take α arbitrarily close to $\operatorname{ess\,inf} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$. We then fix $i \geq 1$. Fix $\delta \in (0, 2^{-i})$. We consider J a countable set and $(B_j)_{j\in J}$ a δ-pack of E_i. Then it holds:

$$\sum_{j\in J} \mathrm{scl}_\alpha(|B_j|) \leq \sum_{j\in J} \mu(B_j) \leq 1 \ .$$

Since this holds true for any δ-pack, it follows:

$$\mathcal{P}_\delta^{\mathrm{scl}_\alpha}(E_i) \leq 1 \ .$$

When δ tends to 0, the latter inequality leads to:

$$\mathcal{P}^{\mathrm{scl}_\alpha}(E_i) \leq \mathcal{P}_0^{\mathrm{scl}_\alpha}(E_i) \leq 1 \ .$$

From there, we deduce $\mathsf{scl}_P E_i \leq \alpha$, which concludes the proof of $\mathsf{scl}_P^*\mu \leq \operatorname{ess\,sup} \overline{\mathsf{scl}}_{\mathrm{loc}}\mu$ and thus the one of Theorem B. □

3.4 Examples and applications of theorems of comparison of scales

3.4.1 Scales of infinite products of finite sets

Natural toy models in the study of scales are given by products $Z = \prod_{n\geq 1} Z_k$ of finite sets. To define the metric δ on this set, we fix a sequence $(\epsilon_n)_{n\geq 1}$ decreasing to 0 and verifying $\log \epsilon_{n+1} \sim$

$\log \epsilon_n$ as $n \to +\infty$. We then define the distance between $\underline{x} = (x_n)_{n\geq 1} \in Z$ and $\underline{x}' = (x'_n)_{n\geq 1} \in Z$ by:

$$\delta(\underline{x}, \underline{x}') := \epsilon_m \ ,$$

where $m = \nu(\underline{x}, \underline{x}') := \inf\{n \geq 1 : x_n \neq x'_n\}$ is the minimal index such that the sequences $\underline{x}$ and $\underline{x}'$ differ. We endow each Z_n with the discrete topology, thus δ provides the product topology on Z.

A natural measure on Z is the following product measure:

$$\mu := \otimes_{n\geq 1} \mu_n \ ,$$

where μ_n is the equidistributed measure on Z_n for $n \geq 1$. Note that the metric is highly dependent on the choice of the sequence $(\epsilon_n)_{n\geq 1}$. We will tune both values of the sequences $(\epsilon_n)_{n\geq 1}$ and $(\mathrm{Card}\ Z_n)_{n\geq 1}$ to reach different values of scales for Z and μ. First, box scales of Z coincide with local scales of μ according to the following:

Proposition 3.4.1. *For any scaling* scl*, it holds for every* $\underline{x} \in Z$*:*

$$\underline{\mathsf{scl}}_{loc}\mu(\underline{x}) = \underline{\mathsf{scl}}_B Z = \sup\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \cdot \prod_{k=1}^{n} \mathrm{Card}\ Z_k \xrightarrow[n\to+\infty]{} +\infty\right\} \tag{3.4.1}$$

and

$$\overline{\mathsf{scl}}_{loc}\mu(\underline{x}) = \overline{\mathsf{scl}}_B Z = \inf\left\{\alpha > 0 : \mathrm{scl}_\alpha(\epsilon_n) \cdot \prod_{k=1}^{n} \mathrm{Card}\ Z_k \xrightarrow[n\to+\infty]{} 0\right\} \ . \tag{3.4.2}$$

Proof. To prove this, first note that for $\underline{x} \in X$ and $n \geq 1$, it holds:

$$B(\underline{x}, \epsilon_n) = \{\underline{x}' \in Z : x'_1 = x_1, \ldots, x'_n = x_n\} \ .$$

Such a ball is usually called a n-cylinder. By definition of μ it holds:

$$\mu\left(B(\underline{x}, \epsilon_n)\right) = \prod_{k=1}^{n} \mu_k(x_k) = \prod_{k=1}^{n} \frac{1}{\mathrm{Card}\ Z_k} \ .$$

Moreover as two balls with the same radius are either equal or disjoint, there are exactly $\prod_{k=1}^{n} \mathrm{Card}\ Z_k$ different balls of radius ϵ_n and in particular $\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} \mathrm{Card}\ Z_k$.

We just proved:

$$\mathcal{N}_{\epsilon_n}(Z) = \prod_{k=1}^{n} \mathrm{Card}\ Z_k = \mu(B(\underline{z}, \epsilon_n))^{-1} \ . \tag{3.4.3}$$

Since $\log \epsilon_{n+1} \sim \log \epsilon_n$ as $n \to +\infty$, by Lemma 3.2.3 and Eq. (3.4.3), we obtain:

$$\underline{\mathsf{scl}}_B Z = \sup\left\{\alpha > 0 : \prod_{k=1}^{n} \mathrm{Card}\ Z_k \cdot \mathrm{scl}_\alpha(\epsilon_n) \xrightarrow[n\to+\infty]{} 0\right\} = \underline{\mathsf{scl}}_{\mathrm{loc}}\mu(\underline{x}) \ , \tag{3.4.4}$$

which is actually Eq. (3.4.1). We obtain similarly Eq. (3.4.2).

□

Proposition 3.4.1 together with Theorem A and C directly implies:

Corollary 3.4.2. *We have moreover:*

$$\mathsf{scl}_H Z = \underline{\mathsf{scl}}_Q \mu = \underline{\mathsf{scl}}_B Z \quad \textit{and} \quad \mathsf{scl}_P Z = \overline{\mathsf{scl}}_Q \mu = \overline{\mathsf{scl}}_B Z \ .$$

Also, for $\mathsf{scl} = \mathsf{ord}$ and if $\frac{-\log \epsilon_n}{n}$ is moreover converging to some positive finite constant, we have:

Corollary 3.4.3. *Suppose further that $\frac{-\log \epsilon_n}{n}$ converges to $C > 0$ when $n \to +\infty$ (a typical choice is $\epsilon_n = \exp(C \cdot n)$). Then for any scaling scl, it holds:*

$$\mathsf{ord}_H Z = \underline{\mathsf{ord}}_B Z = \liminf_{n\to+\infty} \frac{1}{C \cdot n} \log \left(\sum_{k=1}^{n} \log \left(\operatorname{Card} Z_k \right) \right)$$

and

$$\mathsf{ord}_P Z = \overline{\mathsf{ord}}_B Z = \limsup_{n\to+\infty} \frac{1}{C \cdot n} \log \left(\sum_{k=1}^{n} \log \left(\operatorname{Card} Z_k \right) \right) .$$

Proof. Recall that the scaling defining order is given for $\alpha > 0$ and $\epsilon \in (0, 1)$ by $\mathrm{scl}_\alpha^{2,1} = \exp(-\epsilon^{-\alpha})$. The first equality in Eq. (3.4.4) reads as:

$$\underline{\mathsf{ord}}_B(X) = \sup \left\{ \alpha > 0 : -\epsilon_n^{-\alpha} + \sum_{k=1}^{n} \log \operatorname{Card} Z_k \xrightarrow[n\to+\infty]{} +\infty \right\} .$$

As $\epsilon_n^{-\alpha} = e^{-\alpha \log \epsilon_n}$ and $-\log \epsilon_n \sim Cn$, we obtain the first announced formula. The second formula is deduced similarly.

□

Such examples of products of finite sets allow us to exhibit compact metric spaces with arbitrarily high orders :

Example 3.4.4. *For any $\alpha \geq \beta > 0$, there exists a compact metric probability space (Z, δ, μ) such that for any $z \in Z$:*

$$\beta = \underline{\mathsf{ord}}_{loc}\mu(z) = \mathsf{ord}_H Z = \underline{\mathsf{ord}}_Q \mu = \underline{\mathsf{ord}}_B Z$$

and

$$\alpha = \overline{\mathsf{ord}}_{loc}\mu(z) = \mathsf{ord}_P Z = \overline{\mathsf{ord}}_Q \mu = \overline{\mathsf{ord}}_B Z .$$

In particular, with $\alpha > \beta$ we obtain examples of metric spaces with finite order such that the Hausdorff and packing orders do not coincide. Moreover, for a countable dense subset F of X, it holds:

$$\mathsf{ord}_H F = \mathsf{ord}_P F = 0 \quad \text{and} \quad \underline{\mathsf{ord}}_B F = \beta < \alpha = \overline{\mathsf{ord}}_B F .$$

It follows that none of the inequalities of Theorem A for the case of order in an equality in the general case.

Construction of Example 3.4.4. Let $(u_k)_{k\geq 0}$ be the sequence defined by:

$$u_k = \begin{cases} \lceil \exp(\exp(\beta \cdot k)) \rceil & \text{if} \quad c^{2j} \leq k < c^{2j+1} \\ \lfloor \exp(\exp(\alpha \cdot k)) \rfloor & \text{if} \quad c^{2j+1} \leq k < c^{2j+2} \end{cases}$$

where $c = \lfloor \frac{\alpha}{\beta} \rfloor + 1$. We denote $Z := \prod_{n\geq 1} \mathbb{Z}/u_k\mathbb{Z}$ endowed with the metric δ defined by:

$$\delta(\underline{z}, \underline{w}) := \exp(-\inf \{ n \geq 1 : z_n \neq w_n \})$$

for $\underline{z} = (z_n)_{n\geq 1}$ and $\underline{w} = (w_n)_{n\geq 1}$ in Z. Let us denote $\lambda_n = \frac{1}{n}\log\sum_{k=1}^{n}\log u_k$. Thus by Corollary 3.4.3, it follows:

$$\mathsf{ord}_H Z = \underline{\mathsf{ord}}_B Z = \liminf_{n\to+\infty}\lambda_n \quad \text{and} \quad \mathsf{ord}_P Z = \overline{\mathsf{ord}}_B Z = \limsup_{n\to+\infty}\lambda_n \ .$$

It remains to show that $\lambda^- := \liminf_{n\to+\infty}\lambda_n = \beta$ and $\lambda^+ := \limsup_{n\to+\infty}\lambda_n = \alpha$ in order to show that (Z,δ) satisfies the desired properties. First notice that $\exp(\exp(\beta\cdot n)) \leq u_n \leq \exp(\exp(\alpha\cdot n))$ for every integer n. It follows that $\lambda^- \geq \beta$ and $\lambda^+ \leq \alpha$. Denote $n_j = c^{2j+1}$ and observe that:

$$\lambda_{n_j} \geq \frac{1}{n}\log\log(u_{n_j}) \xrightarrow[j\to+\infty]{} \alpha \ .$$

Thus we have $\lambda^+ \geq \alpha$. Moreover, denote $m_j = c^{2j+1} - 1$. We have the following:

Lemma 3.4.5. *For any $j \geq 1$ and for any $1 \leq k \leq m_j$, it holds:*

$$u_k \leq u_{m_j} \ .$$

Proof. If $c^{2j} \leq k \leq m_j$, then $u_k = \lceil\exp(\exp(\beta\cdot k))\rceil \leq \lceil\exp(\exp(\beta\cdot m_j))\rceil = u_{m_j}$. Otherwise, we have $k < c^{2j}$, and then $u_k < \lfloor\exp(\exp(\alpha\cdot c^{2j}))\rfloor < \lceil\exp(\exp(\beta\cdot c^{2j+1}))\rceil = u_{m_j}$, since $\alpha < \beta\cdot c$. □

From the above lemma, we have:

$$\lambda_{m_j} \leq \frac{1}{m_j}\log m_j \log(u_{m_j}) \xrightarrow[j\to+\infty]{} \beta \ ,$$

and so $\lambda^- \leq \beta$ which ends the construction of Example 3.4.4. □

3.4.2 Functional spaces

This last sub-section consists in showing Lemma 3.1.12 that allows us to show Theorem E.

Denote by $\|\cdot\|_{C^k}$ the C^k-uniform norm on $C^k([0,1]^d,\mathbb{R})$:

$$\|f\|_{C^k} := \sup_{0\leq j\leq k}\|D^j f\|_\infty \ .$$

Definition 3.4.6. *For integers $d \geq 1$ and $k \geq 0$, let us denote:*

$$\mathcal{F}^{d,k,0} := \left\{f \in C^k([0,1]^d,[-1,1]) : \|f\|_{C^k} \leq 1\right\}$$

the C^k-unit ball; and for $\alpha \in (0,1]$ let us define:

$$\mathcal{F}^{d,k,\alpha} := \left\{f \in C^k([0,1]^d,[-1,1]) : \|f\|_{C^k} \leq 1 \text{ and } D^k f \text{ is } \alpha\text{-Hölder with constant } 1\right\} \ .$$

Recall that for $\alpha > 0$, the map $D^k f$ is α-Hölder with constant 1 if for any $x, y \in [0,1]^d$ it holds:

$$\|D^k f(x) - D^k f(y)\|_\infty \leq \|x-y\|^\alpha \ .$$

Let us recall the asymptotic given by Kolmogorov-Tikhomirov [KT93][Thm XV] on the covering number of $(\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty)$ stated in Theorem 3.1.11:

$$C_1 \cdot \epsilon^{-\frac{d}{k+\alpha}} \geq \log \mathcal{N}_\epsilon(\mathcal{F}^{d,k,\alpha}) \geq C_2 \cdot \epsilon^{-\frac{d}{k+\alpha}} \ ,$$

where $C_1 > C_2 > 0$ are two constants depending on d, k and α. In order to prove Theorem E which states that box, packing and Hausdorff scales of $\mathcal{F}^{d,k,\alpha}$ are all equal to $\frac{d}{k+\alpha}$, by Theorem B, it remains to prove Lemma 3.1.12. The latter states:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \frac{d}{k+\alpha} \ .$$

Proof of Lemma 3.1.12. We first assume $\alpha > 0$. The case $\alpha = 0$ will be deduced at the end of the proof. We consider the following product of finite sets:

$$\Lambda = \prod_{n \geq 1} \{0,1\}^{\mathcal{R}_n} \ .$$

where:

$$\mathcal{R}_n := \left\{ \left(\frac{i_1}{R^n}, \dots, \frac{i_d}{R^n} \right) : i_1, \dots, i_d \in \{0, \dots, R^n - 1\} \right\} \text{ with } R := \lfloor 3^{\frac{1}{k+\alpha}} \rfloor + 1 \ . \tag{3.4.5}$$

Observe that $\mathcal{R}_n$ is a meshgrid of step R^{-n} of $[0,1]^d$ with cardinal R^{nd}. We endow Λ with the ultrametric distance δ defined by:

$$\delta(\underline{\lambda}, \underline{\lambda}') = \epsilon_m \ ,$$

with m the minimal index such that the sequences $\underline{\lambda}$ and $\underline{\lambda}'$ differ and $(\epsilon_n)_{n \geq 1}$ a decreasing sequence of positive real numbers. The following will enable us to conclude the proof of Lemma 3.1.12:

Lemma 3.4.7. *We can chose the sequence $(\epsilon_n)_{n \geq 1}$ such that:*

$$-\log \epsilon_n \sim n \cdot \log R^{k+\alpha} \text{ as } n \to +\infty \ . \tag{3.4.6}$$

and such that there exists an expanding embedding $I : (\Lambda, \delta) \to (\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty)$. More precisely, for every $\lambda, \lambda' \in \Lambda$ it holds:

$$\|I(\lambda) - I(\lambda')\|_\infty \geq \frac{1}{2} \delta(\lambda, \lambda') \ . \tag{3.4.7}$$

The choice of $(\epsilon_n)_{n \geq 1}$ will be given in Eq. (3.4.9). Lemma 3.4.7 is proven below. Let us see how it allows us to finish the proof of Lemma 3.1.12. Since $\log \epsilon_{n+1} \sim \log \epsilon_n$, by Corollary 3.4.3 it holds:

$$\begin{aligned} \mathsf{ord}_H \Lambda &= \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left(\sum_{j=1}^{n} \log \mathrm{Card} \left(\{0,1\}^{\mathcal{R}_j} \right) \right) \\ &= \liminf_{n \to +\infty} \frac{1}{n \log R^{k+\alpha}} \log \left(\sum_{j=1}^{n} R^{dj} \cdot \log 2 \right) = \frac{d}{k+\alpha} \ . \end{aligned}$$

Now Lemma 3.4.7 together with Corollary 3.2.23 implies:

$$\mathsf{ord}_H \mathcal{F}^{d,k,\alpha} \geq \mathsf{ord}_H \Lambda = \frac{d}{k+\alpha} \ ,$$

which is the desired result.

It remains to show:

Proof of Lemma 3.4.7. Let us denote $q := k+\alpha$ so that $R = \lfloor 3^{\frac{1}{q}} \rfloor + 1$. For $f \in \mathcal{F}^{d,k,\alpha}$, let $\|f\|_q$ be the α-Hölder constant of $D^k f$, i.e the minimal constant $C > 0$ such that for any $x, y \in [0,1]^d$ it holds:

$$\|D^k f(x) - D^k f(y)\|_\infty \leq C \cdot \|x - y\|^\alpha \ .$$

Note that $\|\cdot\|_q$ is a semi-norm on $\mathcal{F}^{d,k,\alpha}$ and moreover we have:

$$\mathcal{F}^{d,k,\alpha} = \left\{ f \in \mathcal{F}^{d,k,0} : \|f\|_q \leq 1 \right\} \ .$$

We consider the following function defined on $\mathbb{R}$:

$$\phi(t) := 4^q \cdot t^q \cdot (1-t)^q \cdot \mathbb{1}_{0<t<1} \ .$$

It is clear that $\operatorname{supp} \phi = [0,1]$, $\phi(0) = 0 = \phi(1)$, $\|\phi\|_\infty = \phi(1/2) = 1$. Observe also that $D^k\phi(0) = 0 = D^k\phi(1)$.
Let us denote:

$$\Phi : x \in \mathbb{R}^d \mapsto \phi(2\|x\|) \ .$$

Note that Φ is of class C^k with support $(\|x\| \leq 1/2)$ and $\|\Phi\|_\infty = 1$. Also $D^k\Phi$ equals 0 outside $(\|x\| < 1/2)$ and is α-Hölder with constant:

$$0 < \|\Phi\|_q < +\infty \ . \tag{3.4.8}$$

We now proceed to the construction of the embedding. We first fix n and denote $\mathcal{R} := \mathcal{R}_n$. To each $\lambda = (\lambda_r)_{r\in\mathcal{R}} \in \{0,1\}^{\mathcal{R}}$ we associate the following map:

$$f_\lambda : x \in [0,1]^d \mapsto \epsilon_n \cdot \sum_{r\in\mathcal{R}} \lambda_r \cdot \Phi(R^n(x-r)) \ ,$$

with:

$$\epsilon_n := \frac{3}{\pi^2 \cdot n^2 \cdot R^{qn} \cdot \|\Phi\|_q} \ . \tag{3.4.9}$$

First note that $(\epsilon_n)_{n\geq 1}$ obviously verifies Eq. (3.4.6).

We have the following result:

Lemma 3.4.8. *For every $\lambda \neq \lambda'$ in $\{0,1\}^{\mathcal{R}}$, it holds:*

$$\|f_\lambda - f_{\lambda'}\|_\infty = \epsilon_n \quad \textit{and} \quad \|f_\lambda - f_{\lambda'}\|_q \leq \frac{6}{\pi^2 \cdot n^2} \ .$$

Proof. For any $x \in [0,1]^d$, there exists at most one point $r \in \mathcal{R}$ such that $\|x - r\| < R^{-n}/2$. Consequently, the maps $x \mapsto \Phi(R^n(x-r))$ for $r \in \mathcal{R}$ have supports with disjoint interiors. It comes then:

$$\|f_\lambda - f_{\lambda'}\|_\infty = \epsilon_n \cdot \left\| \sum_{r\in\mathcal{R}} (\lambda_r - \lambda'_r) \cdot \Phi(R^n(\cdot - r)) \right\|_\infty = \epsilon_n \cdot \sup_{r\in\mathcal{R}} |\lambda_r - \lambda'_r| \cdot \|\Phi\|_\infty = \epsilon_n \ .$$

This provides the equality of Lemma 3.4.8.

Now note similarly that the maps $x \mapsto D^k\Phi(R^n \cdot (x-r))$ for $r \in \mathcal{R}$ also have supports with disjoint interiors and moreover they are equal to 0 on the boundary of their support. Thus for every $x \in [0,1]^d$ there exists at most one value of $r \in \mathcal{R}$ such that $D^k\Phi(R^n(x-r))$ is not null. It follows:

$$\begin{aligned}
\|f_\lambda - f_{\lambda'}\|_q &= \epsilon_n \cdot \left\| \sum_{r\in\mathcal{R}} (\lambda_r - \lambda'_r) \cdot \Phi(R^n(\cdot - r)) \right\|_q \\
&\leq \epsilon_n \cdot \sup_{r,s\in\mathcal{R}} \sup_{x\neq y\in\mathbb{R}^d} \frac{\left\|(\lambda_r - \lambda'_r)R^{kn}D^k\Phi(R^n(x-r)) - (\lambda_s - \lambda'_s)R^{kn}D^k\Phi(R^n(y-s))\right\|_\infty}{\|x-y\|^\alpha} \\
&= \epsilon_n \cdot R^{nq} \cdot \sup_{r,s\in\mathcal{R}} \sup_{X\neq Y\in\mathbb{R}^d} \frac{\left\|(\lambda_r - \lambda'_r)\cdot D^k\Phi(X - R^n r) - (\lambda_s - \lambda'_s)\cdot D^k\Phi(Y - R^n s)\right\|_\infty}{\|X-Y\|^\alpha}
\end{aligned}$$

where the last equality is obtained using the change of variable $X = R^n x$ and $Y = R^n y$. Using triangular inequality of $\|\cdot\|_q$, it follows:

$$\|f_\lambda - f_{\lambda'}\|_q \leq 2\epsilon_n \cdot R^{qn} \cdot \sup_{r\in\mathcal{R}} \|\Phi(\cdot - R^n r))\|_q = 2\epsilon_n \cdot R^{qn}\|\Phi\|_q \ .$$

This allows us to conclude as the latter term is equal to $\frac{6}{\pi^2 n^2}$ by Eq. (3.4.9).

□

Note in particular, in Lemma 3.4.8, that if $\lambda' = 0$ then $f_{\lambda'} = 0$, thus for $\lambda \neq 0$, it holds:

$$\|f_\lambda\|_\infty = \epsilon_n \quad \text{and} \quad \|f_\lambda\|_q \leq \frac{6}{\pi^2 \cdot n^2} \ . \tag{3.4.10}$$

To embed Λ into $\mathcal{F}^{d,k,\alpha}$ we use the following:

Lemma 3.4.9. *For every $\underline{\lambda} = (\lambda_n)_{n\geq 1} \in \Lambda$ the function series $\sum_{n\geq 1} f_{\lambda_n}$ converges in $C^0([0,1]^d, [-1,1])$ and moreover its limit lies in $\mathcal{F}^{d,k,\alpha}$.*

Proof. By the equality in Eq. (3.4.10) the normal convergence of the series $\sum f_{\lambda_n}$ holds for the C^0-norm as $\sum \epsilon_n < +\infty$. Thus its limit g is continuous. Let us show that g indeed lies in $\mathcal{F}^{d,k,\alpha}$. First note that for any $n \geq 1$ and for any $1 \leq l \leq k$ it holds $D^l f_{\lambda_n}(0) = 0$, thus by Taylor integral formula, it holds:

$$\|D^l f_{\lambda_n}\|_\infty \leq \|D^k f_{\lambda_n}\|_\infty \ . \tag{3.4.11}$$

Still by Eq. (3.4.10), it holds:

$$\sum_{n\geq 1} \|f_{\lambda_n}\|_q \leq \sum_{n\geq 1} \frac{6}{\pi^2 n^2} = 1 \ .$$

Note moreover that as $D^k f_{\lambda_n}(0) = 0$, it holds $\|D^k f_{\lambda_n}\|_\infty \leq \|f_{\lambda_n}\|_q$ thus by Lemma 3.4.9 it follows:

$$\sum_{n\geq 1} \|f_{\lambda_n}\|_{C^k} \leq \sum_{n\geq 1} \|f_{\lambda_n}\|_q \leq 1 \ .$$

Consequently, the partial sums of the series lie in $\mathcal{F}^{d,k,\alpha}$ and so does g as $\mathcal{F}^{d,k,\alpha}$ is closed for the C^0-norm. □

By the above Lemma 3.4.9, the following map is well defined:

$$I : \underline{\lambda} = (\lambda_n)_{n\geq 1} \in (\Lambda, \delta) \mapsto \lim_{n\to+\infty} \sum_{n\geq 1} f_{\lambda_n} \in (\mathcal{F}^{d,k,\alpha}, \|\cdot\|_\infty) \ .$$

To conclude the proof, it remains to show Eq. (3.4.7). Consider $\underline{\lambda} = (\lambda_n)_{n\geq 1}$ and $\underline{\lambda}' = (\lambda'_n)_{n\geq 1} \in \Lambda$. Denote $k := \nu(\underline{\lambda}, \underline{\lambda}')$ the first index such that the sequences λ and λ' differ. Then it holds:

$$\|I(\underline{\lambda}) - I(\underline{\lambda}')\|_\infty = \left\| \sum_{n\geq k} f_{\lambda_n} - f_{\lambda'_n} \right\|_\infty \geq \|f_{\lambda_k} - f_{\lambda'_k}\|_\infty - \sum_{n>k} \|f_{\lambda_n} - f_{\lambda'_n}\|_\infty \ . \tag{3.4.12}$$

Now Lemma 3.4.8 provides:

$$\|f_{\lambda_k} - f_{\lambda'_k}\|_\infty = \epsilon_k \quad \text{and} \quad \sum_{n>k} \|f_{\lambda_n} - f_{\lambda'_n}\|_\infty \leq \sum_{n>k} \epsilon_n \ . \tag{3.4.13}$$

Note that Eq. (3.4.9) implies:

$$\sum_{n>k} \epsilon_n \leq \sum_{n>k} \epsilon_k \cdot R^{-q(n-k)} = \epsilon_k \cdot \frac{1}{R^q - 1} \ . \tag{3.4.14}$$

As $R^q \geq 3$, it holds then $\frac{1}{R^q-1} \leq \frac{1}{2}$ and consequently combining Eqs. (3.4.12) to (3.4.14) leads to:

$$\|I(\underline{\lambda}) - I(\underline{\lambda}')\|_\infty \geq \tfrac{1}{2}\epsilon_{\nu(\underline{\lambda},\underline{\lambda}')} \ .$$

Since $\epsilon_{\nu(\underline{\lambda},\underline{\lambda}')} = \delta(\underline{\lambda}, \underline{\lambda}')$ by definition of δ, the desired result comes. □

It remains to deduce the case $\alpha = 0$ from that previous one. For any $\beta > 0$, it holds:

$$\mathcal{F}^{d,k,\beta} \subset \mathcal{F}^{d,k,0} \ .$$

From there, since Hausdorff scales are non decreasing for inclusion, it holds then:

$$\mathsf{ord}_H \mathcal{F}^{d,k,0} \geq \mathsf{ord}_H \mathcal{F}^{d,k,\beta} \geq \frac{d}{k+\beta} \ .$$

Since we can take $\beta > 0$ arbitrarily small, it indeed holds $\mathsf{ord}_H \mathcal{F}^{d,k,0} \geq \frac{d}{k}$. □

Chapter 4

Total renormalization of close to identity maps

Every diffeomorphism is a total renormalization of a close to identity map

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Abstract

For any $1 \leq r \leq \infty$, we show that every diffeomorphism of a manifold of the form $\mathbb{R}/\mathbb{Z} \times M$ is a total renormalization of a C^r-close to identity map. In other words, for every diffeomorphism f of $\mathbb{R}/\mathbb{Z} \times M$, there exists a map g arbitrarily close to identity such that the first return map of g to a domain is conjugate to f and moreover the orbit of this domain is equal to $\mathbb{R}/\mathbb{Z} \times M$. This enables us to localize near the identity the existence of many properties in dynamical systems, such as being Bernoulli for a smooth volume form.

Introduction

4.0.1 Statements of the main theorems

Let $\mathbb{B}^n$ be the unit closed ball of $\mathbb{R}^n$.

Definition 4.0.1 (Primitive renormalization)**.** *A* primitive renormalization G *of a diffeomorphism* $g \in \mathrm{Diff}(\mathbb{B}^n)$ *is a rescaling of an iteration of* g*. In other words, there exists* $N \geq 2$ *and an embedding* $\psi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ *such that* $g^i(\psi(\mathbb{B}^n)) \cap \psi(\mathbb{B}^n) = \emptyset$ *for every* $0 < i < N$ *and:*

$$G = \psi^{-1} \circ g^N \circ \psi.$$

A long standing open problem of dynamical systems theory is:

Problem 4.0.2 (1971)**.** *Which dynamics can be reached by renormalization of close to identity maps?*

This problem was first studied by Ruelle and Takens in [RT71]. Motivated by the study of turbulence, they proved that for any integer $n \geq 2$, any dynamics on the n-dimensional torus is the renormalization of a C^n-close to identity map. This enabled them to construct perturbations of the identity map of the torus with a strange attractor. Based on this, they conjectured that this appears as well in fluid dynamics and could be used as a mathematical definition of the notion of turbulence [Lor63].

The main mathematical issue with this result is that the regularity is limited by the dimension of the torus. However when considering flows, this problem was solved by Newhouse, Ruelle and Takens in [NRT78]: given any vector field X equal to a rotation on the torus $\mathbb{T}^n$, $n \geq 3$ and any map $F_0 \in \mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ homotopic to the identity, they perturbed X to $\widetilde{X}$ so that its first return map to a global transverse section is F_0. Yet $\mathrm{Diff}^\infty(\mathbb{T}^n)$ is "much larger" than $\mathrm{Diff}^\infty(\mathbb{T}^{n-1})$ and so the mathematical Problem 4.0.2 remains unsolved.

A breakthrough was then performed in the seminal work of Turaev [Tur15b] who proved that a *C^r-dense subset* of C^r-orientation preserving embeddings of $\mathbb{B}^n$ could be obtained after renormalization of an arbitrarily close to identity map, for every $0 \leq r \leq \infty$.

A first main result is a solution to Problem 4.0.2, where we improve Turaev's theorem to obtain, via a self-contained and new proof, *any* C^r-orientation preserving map of $\mathbb{B}^n$ (instead of maps among a dense subset):

Theorem G. *For any* $1 \leq r \leq \infty$ *and any orientation preserving* $G \in \mathrm{Diff}^r(\mathbb{B}^n)$*, in any neighborhood* $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{B}^n)$ *of the identity, there exists* $g \in \mathcal{N}$ *such that a primitive renormalization of* g *is* ***equal*** *to* G*. Moreover the rescaling map of this renormalization can be chosen affine.*

A natural open problem is whether g can be obtained as conservative or symplectic when G is conservative or symplectic. In this direction let us mention the work of Gonchenko-Shilnikov-Turaev [GST07] who proved that, for every $0 \leq r \leq \infty$, a C^r-dense subset of volume preserving embeddings of $\mathbb{B}^2$ could be obtained after renormalization of an arbitrarily close to identity volume preserving map. While in their construction they obtained quasi-periodic renormalization, Fayad and Saprykina in [FS22] showed that any conservative map of the n-dimensional ball can be realized by periodic renormalized iteration of a conservative C^n-perturbation of the identity.

If all these theorems indicate the richness of the possible dynamical behaviors near the identity, one can object to the following. In the setting of Definition 4.0.1, the orbit of $\bigcup_{k=0}^{N-1} \psi(\mathbb{B}^n)$ of the renormalization domain might be extremely small and so experimentally not observable. This objection is lifted completely when the renormalization domain intersects every orbit. This leads us to generalize the notion of renormalization by the following:

Definition 4.0.3 (Renormalization). *Let $r \in \{1, \dots, \infty\} \cup \{\omega\}$ and let V be a manifold (with boundary). A map $g \in \mathrm{Diff}^r(V)$ is* renormalizable *if there exists a strict submanifold with corners $\Delta \subsetneq V$ such that:*

- *there exists a bijective, local C^r-diffeomorphism $H : \Delta \to V$, called the* rescaling map *of the* renormalization domain Δ,
- *the first return time $\tau : \Delta \to \mathbb{N}^*$ into Δ by g is bounded and the* renormalization $G = H \circ g^\tau \circ H^{-1}$ *belongs to* $\mathrm{Diff}^r(V)$.

The map g is totally renormalizable *if the forward orbit of Δ covers V, i.e. $\bigcup_{n\geq 0} g^n(\Delta) = V$. The map G is then a* total renormalization *of g.*

Remark 4.0.4. *Note that if $g \in \mathrm{Diff}^r(\mathbb{B}^n)$ displays a primitive renormalization with embedding $\psi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ and time N, then $\Delta := \psi(\mathbb{B}^n)$ is the renormalization domain of g with constant return time $\tau \equiv N$ and rescaling map $H = \psi^{-1}$. Hence Definition 4.0.3 generalizes Definition 4.0.1. Note that the latter renormalization is never total.*

Moreover Definition 4.0.3 allows us to consider a larger class of manifolds V as we do not ask $H = \psi^{-1}$ to be continuous on the boundary of the renormalization domain. The next example is about a total renormalization on the circle; a renormalization which is not primitive.

Example 4.0.5. *When V is the circle $\mathbb{T}$, a diffeomorphism g is totally renormalizable iff it does not fix a point. Indeed in this case, take any point $0 \in \mathbb{T}$ and consider the interval $\Delta = [0, g(0))$. Then we glue the two endpoints of Δ using g to obtain a circle and we uniformize it to obtain $\mathbb{T}$. This defines a map H. For this setting one easily shows that the mapping g is renormalizable. This construction was intensively used by Yoccoz [Yoc95b].*

Let V be a compact manifold (possibly with corners) and $1 \leq r \leq \infty$. We recall that the *support* $\mathrm{supp}\, f$ of $f \in \mathrm{Diff}^r(V)$ is the closure of the set of points such that $f(x) \neq x$.

Definition 4.0.6. *Let $\mathrm{Diff}^r_0(V)$ be the component of the identity in $\mathrm{Diff}^r(V)$. Let $\mathrm{Diff}^r_c(V)$ be the subset of $\mathrm{Diff}^r_0(V)$ formed by maps isotopic to id through isotopies $(f_t)_{t\in[0,1]}$ whose support $\bigcup_{t\in[0,1]} \mathrm{supp}\, f_t$ is a compact subset of $V \setminus \partial V$.*

Observe that when V is boundaryless, it holds $\mathrm{Diff}^r_0(V) = \mathrm{Diff}^r_c(V)$. A natural question is:

Question 4.0.7. *For which manifold V, any map $F \in \mathrm{Diff}^r_c(V)$ is a* total *renormalization of a close to identity map?*

So far no example of such a manifold V was known. In this work we give a full class of examples:

Theorem H. *Let $0 \leq r \leq \infty$, let M be a compact manifold of dimension ≥ 1 and put $V := \mathbb{T} \times M$. Let $\mathcal{N} \subset \mathrm{Diff}^r(V)$ be a neighborhood of the identity. Then any $G \in \mathrm{Diff}^r_c(V)$ is a total renormalization of some $g \in \mathcal{N}$.*

Theorem H allows us to find new dynamical phenomena near the identity. A new improvement brought by the latter result is that the renormalization domain is larger than in all of the previous extensions of Ruelle-Takens theorems: its orbit coincides with the whole domain of the dynamics. In Theorem I, we will give a precise formula defining the renormalization domain and rescaling map involved in Theorem H. This will enable new applications such as the proof of existence of maps preserving smooth SRB near the identity (see Corollary 4.0.11) or universal maps whose renormalization domains decrease as slowly as we want (see Corollary 4.0.12).

In Proposition 4.0.10, we show that Theorem H is wrong when $\mathbb{T} \times M \approx \mathbb{T}$, hence the set of dimensions of the manifold is optimal. On the other hand, a natural open problem communicated to us by Turaev is:

Problem 4.0.8. *Show that a dense subset of* $\mathrm{Diff}^{\omega}(\mathbb{B}^n)$ *is equal to the renormalization of a close to identity map in* $\mathrm{Diff}^{\omega}(\mathbb{B}^n)$*?*

Another natural question is:

Question 4.0.9. *Is Theorem H correct in the area preserving or symplectic categories?*

An extension of Theorem H concerns the C^r-families $f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}}$ of maps $f_p \in \mathrm{Diff}^r(V)$ indexed by a manifold $\mathscr{P}$. A family $(f_p)_{p\in\mathscr{P}}$ is *of class* C^r if the following is in $\mathrm{Diff}^r(V \times \mathscr{P})$:

$$\widehat{f_{\mathscr{P}}} := (x, p) \mapsto (f_p(x), p) \ . \tag{4.0.1}$$

We denote by $\mathrm{Diff}^r(V)_{\mathscr{P}}$ the space of such families and this space is endowed with the topology induced by $\mathrm{Diff}^r(V \times \mathscr{P})$. Let $\mathrm{Diff}^r_c(V)_{\mathscr{P}}$ be the connected component of the identity in $\mathrm{Diff}^r(V)_{\mathscr{P}}$ via homotopies of the form $(f_{p,t})_{(p,t)\in\mathscr{P}\times[0,1]} \in \mathrm{Diff}^r(V)_{\mathscr{P}\times[0,1]}$ whose support $\bigcup_{(t,p)\in[0,1]\times\mathscr{P}} \mathrm{supp} f_{p,t} \times \{p\}$ is a compact subset of the interior of $V \times \mathscr{P}$. Observe that $\widehat{f_{\mathscr{P}}} \in \mathrm{Diff}^r_c(V \times \mathscr{P})$.

Theorem 1. *Let* $0 \leq r \leq \infty$*, let* M *and* $\mathscr{P}$ *be compact manifolds of dim* ≥ 1 *and set* $V := \mathbb{T} \times M$*. Let* $\mathcal{N} \subset \mathrm{Diff}^r(V)_{\mathscr{P}}$ *be a neighborhood of* $(id)_{p\in\mathscr{P}}$*.*

Then for any $(G_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^r_c(V)_{\mathscr{P}}$*, there exist* $(g_p)_{p\in\mathscr{P}} \in \mathcal{N}$ *and a rescaling map (independent of* $p \in \mathscr{P}$*) of a total renormalization domain that renormalizes each* g_p *to* G_p*.*

This theorem implies that any bifurcation in $\mathrm{Diff}^r_c(V)$ occurs at small unfolding of the identity. In Section 4.1.1, we will state the main general Theorem I which implies Theorems G and H and also its parametric counterpart Theorem 2 which implies Theorem 1. In Section 4.0.2 we will give several applications of them. Now let us discuss the optimality of Theorem H.

Proposition 4.0.10. *When* $r \geq 2$*, Theorem H is wrong if* $V \approx \mathbb{T}$*, i.e. when it is isomorphic to the circle as a smooth manifold (and so* $\dim M = 0$*).*

Proof. Indeed if F is a renormalization of a close to identity map f for a renormalization domain Δ, then we have necessarily $|\tau(\theta) - \tau(\theta')| \leq 1$ for any $\theta, \theta' \in \Delta$. Let $N := \min\{\tau(\theta), \tau(\theta')\}$. We compute the derivative of the $N-1$ first iterates $(\theta_i)_i$ and $(\theta'_i)_i$ of θ and θ':

$$\log\left|\frac{D_\theta F}{D_{\theta'} F}\right| = \log\left|\frac{D_\theta f^N}{D_{\theta'} f^N}\right| + o(1) \quad \text{when } f \to id$$

$$= \sum_{i=0}^{N-1} \log \left| \frac{D_{\theta_i} f}{D_{\theta_i'} f} \right| + o(1) \leq \| \log |Df| \|_{C^1} \cdot \sum_{i=0}^{N-1} |\theta_i - \theta_i'| + o(1) = o(1),$$

where the latter inequality uses that $\| \log |Df| \|_{C^1}$ is small while the segments $[\theta_i, \theta_i']$ are disjoint and so the union of their length is at most 1. Hence this proves that the derivative of F is constant and so that F must be a rotation. □

Also we cannot change $\mathrm{Diff}^r_c(\mathbb{T} \times M)$ by $\mathrm{Diff}^r_0(\mathbb{T} \times M)$ in Theorem H. Indeed the latter proposition applied to the boundary of $[0, 1]$ implies immediately:

Corollary A. *There are $G \in \mathrm{Diff}^\infty_0(\mathbb{T} \times [0, 1])$ which are not total renormalizations of C^2-close to identity map.*

Yet in view of Question 4.0.7, Theorem H seems to be generalizable for a manifold V on which $\mathbb{T}$ acts properly discontinuously without any fixed point.

4.0.2 Applications and open problems

Smooth SRB near the identity In Section 4.1.1 we will state Theorem I which will imply together with Katok's theorem [Kat79], an answer to an open question of Thouvenot:

Corollary 4.0.11. *In any neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ there is a map $g \in \mathcal{N}$ which leaves invariant an ergodic smooth volume form and displays positive Lyapunov exponent at Lebesgue a.e. point.*

This corollary will be proved in Section 4.1.2 from Proposition 4.1.7, and generalized to higher dimension using [DP02].

Universal mappings A map $f \in \mathrm{Diff}^r(V)$ is said to be *universal* if there exists a dense subset of $\mathrm{Diff}^r_0(V)$ such that each of its elements is a renormalization of f. Bonatti and Díaz in [BD03] have shown that universal maps are locally C^1-generic on $\mathbb{B}^3$. Turaev in [Tur15a] has shown that universal maps are locally C^∞-generic on $\mathbb{B}^2$. Yet mathematicians wondered whether we can "see" the universality of such mapping. While there are infinitely many renormalization domains, the above proofs lead to a very small volume for the union of their orbits.

Corollary 4.0.12. *Let $1 \leq r \leq \infty$ and $n \geq 2$. For any sequence $(F_i)_{i \geq 0}$ of maps $F_i \in \mathrm{Diff}^r_0(\mathbb{T} \times [0, 1])$ and any sequence of positive numbers $(\ell_i)_{i \geq 0}$ s.t. $\sum \ell_i < 1$, there exists a C^r-arbitrarily close to identity map $f \in \mathrm{Diff}^r_0(\mathbb{T} \times [0, 1])$ which displays a family of renormalization domains $(\Delta_i)_{i \geq 0}$ such that:*

1. *a renormalization of f associated to Δ_i is F_i for every $i \geq 0$,*
2. *the orbit $\tilde{\Delta}_i := \bigcup_{n \geq 0} f^n(\Delta_i)$ has volume equal to ℓ_i,*
3. *the sets $\tilde{\Delta}_i$ and $\tilde{\Delta}_j$ are disjoint for $i \neq j$.*

This corollary will be proved in Section 4.1.2.

The proof of the main theorem is constructive and it seems to us that, in the case where M is boundaryless, the map g of Theorem H depends smoothly on G in a neighborhood of the identity. This leads us to propose:

Conjecture 4.0.13. *For every compact boundaryless manifold M of dimension ≥ 1, there exists a neighborhood $\mathcal{N}_0$ of $id \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ such that for every neighborhood $\mathcal{N}$ of $id \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$, there is a smooth injective map $\mathcal{I} : G \in \mathcal{N}_0 \mapsto g \in \mathcal{N}$ such that G is a total renormalization of $g = \mathcal{I}(G)$ for every $G \in \mathcal{N}_0$.*

Roughly speaking, this conjecture asserts that modulo total renormalization, a fixed neighborhood $\mathcal{N}_0$ of $id \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ can be smoothly embedded into any smaller neighborhood. This defines infinitely many inverse branches of the renormalization operator with image converging to the identity.

4.0.3 Sketch of proof

Plugins and pluggable dynamics: The framework of the proof of the main theorem relies on a new object called *plugin* and the notion of *pluggable map*. A plugin is a renormalizable map of a special form, so that it has a canonical total renormalization called its *output*. See Def. 4.1.1 and 4.1.3 below and Figs. 4.1 and 4.2. We will say that a map is *pluggable* if it is the output of an arbitrarily close to identity plugin and likewise for its inverse. In particular a pluggable map is a total renormalization of a close to identity map. Most of this work will be dedicated to showing Theorem J stating that:

$$\textit{any map of } \mathrm{Diff}^\infty_c(\mathbb{T} \times M) \textit{ is pluggable.}$$

The finite regularity counterpart of Theorem J is stated as Theorem I in Section 4.1.1 and will be deduced from Theorem J in Section 4.2.3. We will deduce Theorems G and H and Corollaries 4.0.11 and 4.0.12 from Theorem I in Section 4.1.2.

In Section 4.1.3 we will specify the topologies of the involved spaces. Also we will show that the following group is formed by pluggable maps, see Proposition 4.1.16:

$$\mathbb{G}_1 := \{(\theta, y) \in \mathbb{T} \times M \mapsto (\theta + \nu(y), y) \in \mathbb{T} \times M : \nu \in C^\infty_c(M, \mathbb{T})\} .$$

Topological group structure on Plug**.** It is easier to work with pluggable maps rather than directly with the set of total renormalizations of close to identity maps. Indeed, we will show in Proposition 4.2.4 that the set Plug of pluggable maps endowed with the composition rule $\circ$ is a group. To prove this, we will define in Section 4.2.1 a binary operation $\star$ on compatible plugins g_1, g_2 such that the output of $g_1 \star g_2$ is the composition of the outputs of g_1 and g_2. See Fig. 4.5. In Section 4.2.2 we will show that the following group is formed by pluggable maps, see Proposition 4.2.7:

$$\mathbb{G}_2 := \{(\theta, y) \in \mathbb{T} \times M \mapsto (\theta, F(y)) \in \mathbb{T} \times M : F \in \mathrm{Diff}^\infty_c(M)\} .$$

Then in Section 4.2.3 we will show that Plug is closed in $\mathrm{Diff}^\infty_c(\mathbb{T} \times M)$, see Proposition 4.2.9. To prove this, we will construct plugins whose dynamics enlarge an iterate of the renormalization domain and will perform a perturbation therein. Note that the elements of $\mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ generated by compositions of elements of $\mathbb{G}_1$ and $\mathbb{G}_2$ have constant derivatives w.r.t. θ. Consequently we will

need to construct other diffeomorphisms in Plug to prove Theorem J. To do so we will consider the space:

$$\mathfrak{diff}_c^\infty(\mathbb{T}\times M) \text{ of compactly supported vector fields on } \mathbb{T}\times M\ ,$$

and study the space $\mathfrak{plug}$ of vector fields whose flow is pluggable:

$$\mathfrak{plug} := \{X\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M) : \mathrm{Fl}_X^t\in\mathrm{Plug},\quad \forall t\in\mathbb{R}\}\ ,$$

where Fl_X^t denotes the flow of X at time t.

Using that Plug is a closed subgroup, we will deduce in Section 4.2.4 that $\mathfrak{plug}$ is a closed sub-Lie algebra of $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$, see Proposition 4.2.16.

Note that the following Lie algebras are formed by fields whose flows are in $\mathbb{G}_1$ or $\mathbb{G}_2$:

$$\mathfrak{g}_1 = \{X_1 : (\theta,y)\in\mathbb{T}\times M\mapsto (v(y),0)\in\mathbb{R}\times TM\ :\ v\in C^\infty(M,\mathbb{R})\}\subset\mathfrak{plug}\ ,$$

$$\mathfrak{g}_2 = \{X_2 : (\theta,y)\in\mathbb{T}\times M\mapsto (0,f(y))\in\mathbb{R}\times TM\ :\ f\in\mathfrak{diff}_c^\infty(M)\}\subset\mathfrak{plug}\ .$$

Construction of pluggable flows. In Section 4.3.1, using the connectedness of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$ and that Plug is a closed group, we will show that, to prove Theorem J, it suffices to show:

$$\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M)\ .$$

This equality will be stated in Proposition 4.3.3. Its proof relies on two phenomena. The first one is stated in Proposition 4.3.7 as:

$$\{Y\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M) : \exists X\in\mathfrak{plug} \text{ such that } Y=[X,Y]\}\subset\mathfrak{plug}\ ,$$

where $[\cdot,\cdot]$ denotes the Lie brackets. This will be proved in Section 4.3.2 by noting that for any $Y=[X,Y]$, it holds $\mathrm{Fl}_Y^t = \mathrm{Fl}_X^s\circ\mathrm{Fl}_Y^{t\cdot e^{-s}}\circ\mathrm{Fl}_X^{-s}$. Then by using the same technique as for the proof of the closedness of Plug, we will show that arbitrarily close to identity there exists a plugin with output $\mathrm{Fl}_{e^{-s}Y}^t$ for any s large enough. We will conclude the proof by doing two $\star$ products of the latter with plugins with outputs Fl_X^s and Fl_X^{-s}.

The second phenomenon, stated in Proposition 4.3.10, is that for any vector field $T\in\mathfrak{diff}_c^\infty(M)$, there exist finite families $(X_i)_i,(Y_i)_i,(Z_i)_i$ of vector fields in $\mathfrak{diff}_c^\infty(M)$ such that:

$$T=\sum_i [Y_i,Z_i]\quad\text{and}\quad Y_i=[X_i,Y_i]\ .$$

From this we will deduce the same statement for vector fields in $\mathfrak{g}_2$. To show Proposition 4.3.10, we will remark that when $M=\mathbb{R}$, for the vector fields $X : y\in\mathbb{R}\mapsto -y$ and $Y=1$, it holds $Y=[X,Y]$ and for any $T\in\mathfrak{diff}_c^\infty(\mathbb{R})$, with $Z=\int_{-\infty}^y T(t)dt$, it also holds $T=[Y,Z]$. Then we will deduce a compactly supported and parametric version of this property which will enable us to prove Proposition 4.3.10 in the case $M=\mathbb{R}^n$. Finally, we will use a partition of unity to deduce the proposition for any manifold.

These phenomena will enable us to prove Proposition 4.3.1 stating that for any $T\in\mathfrak{g}_2$ and $\phi\in C_c^\infty(\mathbb{T}\times M)$ depending only on θ, the field $\phi\cdot T$ is in $\mathfrak{plug}$. Indeed, by the second phenomenon, there are $X_i,Y_i,Z_i\in\mathfrak{g}_2$ s.t.:

$$\phi\cdot T=\sum_i \phi\cdot[Y_i,Z_i]=\sum_i[\phi\cdot Y_i,Z_i]\quad\text{and}\quad Y_i=[X_i,Y_i]\ .$$

Also a simple computation shows that $[\phi \cdot Y_i, X_i] = \phi \cdot Y_i$. As X_i is in $\mathfrak{g}_2 \subset \mathfrak{plug}$, we deduce by the first phenomenon that $\phi \cdot Y_i$ is in $\mathfrak{plug}$. This gives that $\phi \cdot T \in \mathfrak{plug}$ as stated in Proposition 4.3.1. In Section 4.3.1, we will use Fourier decomposition Theorem and the closedness of $\mathfrak{plug}$ to deduce Proposition 4.3.1 stating that any vector field of the form $(0, Y(\theta, y))$ is in $\mathfrak{plug}$. Finally using this with a Lie bracket with an element of $\mathfrak{g}_1$ will enable us to obtain that any vector field has a pluggable flow ($\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T} \times M)$) as stated by Proposition 4.3.3.

Parametric counterparts. At the end of each subsection, we will prove a parametric generalization of the aforementioned statements. This will enable us to show the parametric counterpart Theorem 3 of Theorem J. It will imply the parametric counterparts Theorem 2 of Theorem I and Theorem 1 of Theorem H.

4.1 Plugins and Pluggable dynamics

4.1.1 Plugins

For the rest of this chapter, we fix $1 \leq r \leq \infty$ and compact connected manifolds M and $\mathcal{P}$ of dim ≥ 1.

For $\sigma > 0$, define the rotation:

$$R_\sigma : (\theta, y) \in \mathbb{T} \times M \to (\theta + \sigma, y) \in \mathbb{T} \times M \ .$$

We are now ready to introduce:

Definition 4.1.1. *A* plugin *with* step $\sigma \in \{2^{-k} : k \geq 1\}$ *is a map $g \in \mathrm{Diff}^r(\mathbb{T} \times M)$ satisfying the following assertions:*

- (i) *g restricted to $\Delta_\sigma := [0, \sigma) \times M$ is equal to R_σ,*
- (ii) *the first return time in Δ_σ of g is a well-defined and bounded function $\tau : \Delta_\sigma \to \mathbb{N}^*$,*
- (iii) *the union of the iterates $\bigcup_{k \geq 0} g^k(\Delta_\sigma)$ equals $\mathbb{T} \times M$.*

Remark 4.1.2. *One can show by compactness of M that, under condition (i), condition (ii) is equivalent to (iii) and that in (ii) the return time is necessarily bounded.*

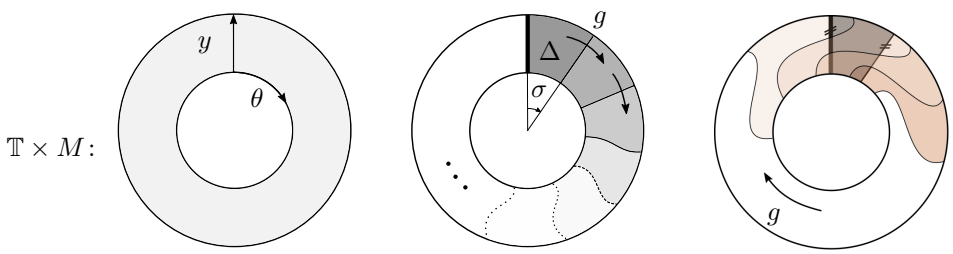


Figure 4.1 – Plugin g of step σ.

Let $H_\sigma := (\theta, y) \in \Delta_\sigma \mapsto (\theta/\sigma, y) \in \mathbb{T} \times M$. It is a bijective local diffeomorphism.

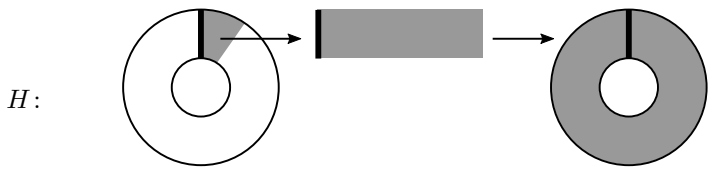


Figure 4.2 – Rescaling map $H_\sigma \colon \Delta_\sigma \to \mathbb{T} \times M$.

Definition 4.1.3. *The* output *of a plugin g of step σ is the following rescaling of the first return map $g^\tau \colon \Delta_\sigma \to \Delta_\sigma$:*

$$G := H_\sigma \circ g^\tau \circ H_\sigma^{-1} \colon \mathbb{T} \times M \to \mathbb{T} \times M.$$

Example 4.1.4. *For every $k \geq 0$, the map $g_k : (\theta, y) \mapsto (\theta + 2^{-k}, y)$ is a plugin of step 2^{-k}, return time 2^k and outputs the identity.*

Actually, we can show that the output of a plugin is always smooth:

Proposition 4.1.5. *Let $1 \leq r \leq \infty$. The output of a plugin $g \in \mathrm{Diff}^r(\mathbb{T} \times M)$ is in $\mathrm{Diff}^r(\mathbb{T} \times M)$ and depends continuously on g. In particular, the output is a total renormalization of the plugin.*

This proposition is a consequence of a classical, yet beautiful, argument that will be recalled in Section 4.4. We are now ready to state the general result:

Theorem I (Main)**.** *Let $1 \leq r \leq \infty$ and a compact manifold M of dimension ≥ 1, and let $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{T} \times M)$ be a neighborhood of the identity. Then any $G \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$ is the output of some plugin $g_G \in \mathcal{N}$.*

And here is its parametric counterpart:

Theorem 2. *Let $1 \leq r \leq \infty$ and a compact manifold M of dimension ≥ 1, fix a compact manifold $\mathcal{P}$ and a compactly supported family $(G_p)_{p \in \mathcal{P}}$ in $\mathrm{Diff}^r_c(\mathbb{T} \times M)_{\mathcal{P}}$. Let $\mathcal{N} \subset \mathrm{Diff}^r(\mathbb{T} \times M)_{\mathcal{P}}$ be a neighborhood of $(id)_{p \in \mathcal{P}}$. Then there exists a C^r-family of plugins $(g_p)_{p \in \mathcal{P}} \in \mathcal{N}$ such that G_p is the output of g_p for every $p \in \mathcal{P}$.*

4.1.2 Proof of the corollaries of the main Theorem I

Observe that Theorem I implies immediately Theorem H and that Theorem 2 implies immediately Theorem 1. In this subsection we show that Theorem I implies furthermore Theorem G and Corollaries 4.0.11 and 4.0.12. To this end, the following will be useful:

Fact 4.1.6. *If the output of a plugin restricted to $\{0\} \times M$ is the identity, then the return time τ of the plugin is constant on the renormalization domain Δ.*

Proof. As f preserves the orientation it suffices to show that τ is constant on the interior of Δ. As τ is integer valued and M is connected, it suffices to show that τ is continuous on $int\,\Delta$ to conclude the proof. We start by showing that τ is lower semi-continuous on Δ. Suppose that it is not the case. Then there exist $N \geq 2$, a point $x \in \Delta$ and a point $x' \in \Delta$ arbitrarily close to x such that $N = \tau(x') < \tau(x)$. By continuity of g, it holds $g^N(x) \in \mathrm{cl}(\Delta)$. By assumption, we have $g^N(x) \notin \Delta$.

Thus $g^N(x) \in \{\sigma\} \times M$ and since g is the translation by σ on Δ it holds then that $g^{N-1}(x) \in \Delta$ and consequently $\tau(x) < N$, which contradicts the assumption.

We now show that τ is upper semi-continuous on $int\,\Delta$. Consider $x \in int\,\Delta$. As g^τ is a bijection of Δ which leaves invariant $\{0\} \times M$, it comes that $g^\tau(x)$ is in the interior of Δ. Then for every x' close to x, the iterate $g^{\tau(x)}(x')$ belongs to Δ and so $\tau(x') \leq \tau(x)$. □

Proof that Theorem I implies Theorem G. Let G be in $\mathrm{Diff}^r(\mathbb{B}^n) = \mathrm{Diff}^r_0(\mathbb{B}^n)$. We observe [4] that G is the restriction to $\mathbb{B}_n$ of a diffeomorphism $\bar{G}$ in $\mathrm{Diff}^r_c(2 \cdot \mathbb{B}^n)$. As $2 \cdot id$ conjugates $\bar{G}$ to a map in $\mathrm{Diff}^r_c(\mathbb{B}^n)$, without any loss of generality we can assume that G belongs to $\mathrm{Diff}^r_c(\mathbb{B}^n)$.

Let $M := \mathbb{B}^{n-1}$ and embed $i : \mathbb{B}^n \hookrightarrow \mathbb{T} \times M$ so that the embedded ball $i(\mathbb{B}^n)$ does not meet $\{0\} \times M$ nor the boundary of $\mathbb{T} \times M$. Extend then G by id to a diffeomorphism $\tilde{G} \in \mathrm{Diff}^r_0(\mathbb{T} \times M)$. By Theorem I, $\tilde{G}$ is the output of a plugin $\tilde{g} \in \mathrm{Diff}^r(\mathbb{T} \times M)$ arbitrarily close to id. Since $\tilde{G}$ leaves $\{0\} \times M$ invariant it holds by Fact 4.1.6 that the first return time τ of the plugin $\tilde{g}$ is a constant N. In particular, the restriction $\tilde{g}^N | i(\mathbb{B}^n)$ is conjugate to G.

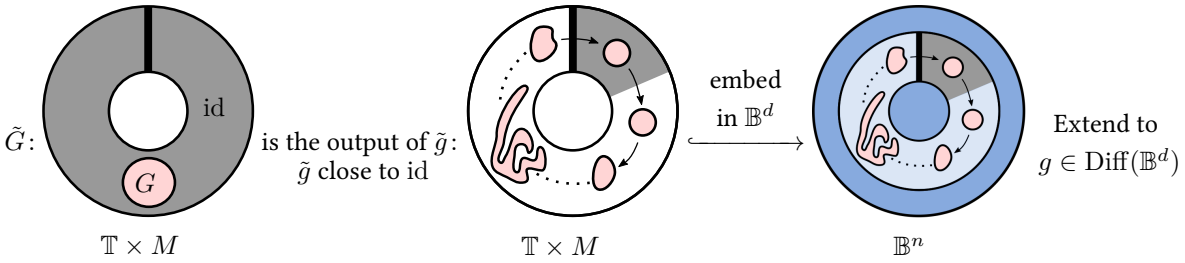


Figure 4.3 – Proof of Theorem G.

To conclude it suffices to embed $\mathbb{T} \times M$ into the interior of $\mathbb{B}^n$ and extend $\tilde{g}$ then the diffeomorphism $\tilde{g}$ to a diffeomorphism g of $\mathbb{B}^n$ that is close to id. □

The following enables us to localize near the identity the existence of some ergodic properties:

Proposition 4.1.7. *Let g be a plugin of step σ and G its output. Then there is a canonical bijection $\mu_g \to \mu_G$ between G-invariant probability measures and g-invariant probability measures:*

$$\mu_g \mapsto \mu_G := H_{\sigma *} \frac{\mu_g | \Delta}{\mu_g(\Delta)} \quad \textit{and} \quad \mu_G \mapsto \mu_g := \sum_{N \geq 0} g^{*N} H_\sigma^* \mu_G | \{\tau > N\} \ .$$

Moreover:

1. *μ_g is ergodic iff μ_G is ergodic,*
2. *μ_g is hyperbolic iff μ_G is hyperbolic.*
3. *μ_g is a smooth volume form iff μ_G is a smooth volume form,*

Proof. We recall that H_σ conjugates G with the first return time g^τ of g in Δ. Then it is sufficient to prove the analogous properties for the following classical bijection between g^τ-invariant probability

4. By connectedness, there exists an isotopy $(h_t)_{t \in [0,1]}$ between $G|\partial\mathbb{B}^n$ and $id_{\partial\mathbb{B}^n}$ that can be chosen C^r-smooth, made of diffeomorphisms and and flat at the endpoints. Note that we can extend G on $2 \cdot \mathbb{B}^n \setminus \mathbb{B}^n$ by $h_{\|z\|-1}\left(\frac{z}{\|z\|}\right)$ to construct an element of $\mathrm{Diff}^r_0(2 \cdot \mathbb{B}^n)$.

measures and g-invariant probability measures:

$$\mu_{g^\tau} \mapsto \mu_g := \sum_{N\geq 0} g^{*N}\mu_{g^\tau}|\{\tau > N\} \quad \text{and} \quad \mu_g \mapsto \mu_{g^\tau} := \frac{\mu_g|\Delta}{\mu_g(\Delta)} .$$

Indeed it is well known that μ_{g^τ} is ergodic iff μ_g is ergodic (see for instance [EW13, Lemma 2.43]), which implies 1. Also it well known that the Lyapunov exponent of μ_G is equal to the mean of τ times the Lyapunov exponent of μ_g, from which we deduce 2.

Let us prove 3. If μ_G is smooth, then $\mu_g|\Delta = \mu_{g^\tau}$ is smooth on Δ. Hence, the measure $\mu_g|\Delta \cup g(\Delta) = \mu_{g^\tau} + g^*\mu_{g^\tau}$ is smooth on $\Delta \cup g(\Delta)$. Now observe that for every $x \in \mathbb{T} \times M$ there exists a neighborhood U and $N \geq 0$ such that $g^{-N}(U) \subset \Delta \cup g(\Delta)$. Then by invariance, the density of μ_g on U is as smooth as $\mu_g|\Delta \cup g(\Delta)$. Conversely, if μ_g is smooth, then $\mu_{g^\tau} := \frac{\mu_g|\Delta}{\mu_g(\Delta)}$ must be smooth. Since g is a plugin, the density μ_g at the left hand side of Δ is equal to the translation by $(\sigma, 0)$ of the density of μ_g at the right hand side of $cl(\Delta)$. Thus μ_{g^τ} is pushed forward by H_σ to a smooth volume form μ_G on $\mathbb{T} \times M$. □

Proof of Corollary 4.0.11. Katok in [Kat79][Thm B] showed the existence of Bernoulli diffeomorphisms homotopic to the identity and preserving a given smooth measure on any surface. Hence there exists $G \in \mathrm{Diff}^\infty_0(\mathbb{T}^2)$ preserving a smooth Bernouilli volume form μ_G. Equivalently by the Pesin Theorem the measure μ_G is a volume form which is ergodic and hyperbolic. By Theorem I, arbitrarily close to the identity, there exists g whose output is G. Then by Proposition 4.1.7, the volume form μ_G induces an ergodic and hyperbolic volume form μ_g for g. □

We now proceed to:

Proof of Corollary 4.0.12. We will construct a sequence $(\Delta_i)_{i\geq 1}$ of domains with disjoint closures and of the form $\Delta_i := [0, \sigma_i) \times (s_i, s_i + l_i) \subset \mathbb{T} \times [0, 1]$ with $0 < \sigma_i < 1$ and $0 < s_i < 1 - l_i$. Then we will be able to construct a map f that renormalizes to F_i on each Δ_i and such that the orbit of the renormalization domain Δ_i is exactly $\mathbb{T} \times (s_i, s_i + l_i)$. Moreover we will be able to take f arbitrarily close to identity.

To do so, consider a sequence $(l'_i)_{i\geq 0}$ such that $\sum_i l'_i < 1$ and $l'_i > l_i$ for any $i \geq 0$. For $i \geq 0$ denote $s_i := \sum_{j<i} l'_j$ and $B_i := \mathbb{T} \times [s_i, s_i + l_i]$. Note that B_i has volume l_i and that $\mathrm{cl}(B_i) \cap \mathrm{cl}(B_j) = \emptyset$ for any $i \neq j \geq 0$. Let us define the map $\psi_i : (\theta, y) \in \mathbb{T} \times [0, 1] \mapsto (\theta, s_i + l_i \cdot y)$ that sends $\mathbb{T} \times [0, 1]$ to $\mathrm{cl}(B_i)$. Now by Theorem I, for any $i \geq 0$, there exists a plugin $\tilde{f}_i : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1]$ arbitrarily close to identity with renormalization domain with output F_i. Let step σ_i be its step. Observe then that the map $f_i := \psi_i \circ \tilde{f}_i \circ \psi_i^{-1} : B_i \to B_i$ is close to identity. Also a renormalization of f_i associated to the domain $\Delta_i := [0, \sigma_i) \times [s_i, s_i + l_i]$ is F_i. Since the supports $\mathrm{supp}\, f_i = \mathrm{cl}(B_j)$ are disjoint and each f_i is close to identity, there exists a close to identity map $f : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1]$ that coincides with f_i on B_i. Such a map verifies the conditions of the Corollary 4.0.12. □

4.1.3 Pluggable dynamics

We will first work in the C^∞-topology. Indeed, the proof relies on the fact that $\mathrm{Diff}^\infty(V)$ is a Fréchet Lie group, which is not the case of $\mathrm{Diff}^r(V)$ for $r < \infty$. The C^r case will be deduced from the C^∞ case in Section 4.2.3.

Topologies on spaces of smooth maps and parameter families

We endow V with a Riemannian metric. Let $\mathfrak{diff}^\infty(V)$ be the space of smooth vector fields on V that are tangent to the boundary (if any). We endow $\mathrm{Diff}^\infty(V)$ and the space $\mathfrak{diff}^\infty(V)$ with the following distances:

$$d_{C^\infty}(f,g) = \max_{x\in V} \sum_{k\geq 1} 2^{-k} \min(1, \|D^k_x f - D^k_x g\|)$$

and

$$d_{C^\infty}(X,Y) = \max_{x\in V} \sum_{k\geq 1} 2^{-k} \min(1, \|D^k_x X - D^k_x Y\|) \ .$$

For these distances $\mathrm{Diff}^\infty(V)$ and $\mathfrak{diff}^\infty(V)$ are complete. Actually $\mathrm{Diff}^\infty(V)$ is a Lie group with algebra the Fréchet space $\mathfrak{diff}^\infty(V)$. We endow the connected component $\mathrm{Diff}^\infty_0(V)$ of the identity with the topology induced by $\mathrm{Diff}^\infty(V)$.

On the other hand, we endow the space $\mathrm{Diff}^\infty_c(V)$ and the space of compactly supported smooth vector fields $\mathfrak{diff}^\infty_c(V)$ with the finer *Whitney topology*. A basis of open sets of these respective topologies is:

$$U_{\eta,f,m} := \{g \in \mathrm{Diff}^\infty_c(V) : d(D^k_x f, D^k_x g) \leq \eta(x), \forall k \leq m\}$$

and

$$U_{\eta,X,m} := \{Y \in \mathfrak{diff}^\infty_c(V) : \|D^k_x X - D^k_x Y\| \leq \eta(x), \forall k \leq m\} \ ,$$

among $m \in \mathbb{N}$, $f \in \mathrm{Diff}^\infty_c(V)$, $X \in \mathfrak{diff}^\infty_c(V)$, and continuous functions $\eta : V \setminus \partial V \to (0,\infty)$. A well known theorem asserts that $f_n \to f$ in $\mathrm{Diff}^\infty_c(V)$ if and only if there exists a compact subset $K \subset V \setminus \partial V$ such that the supports of f and f_n are included in K for n and for every $r \geq 1$, we have $f_n \to f$ in the uniform C^r-topology when $n \to \infty$. The analogous property holds true for $\mathfrak{diff}^\infty_c(V)$.

We endow $\mathrm{Diff}^r(V)_{\mathscr{P}}$ and $\mathrm{Diff}^r_c(V)_{\mathscr{P}}$ with the topologies induced by $\mathrm{Diff}^r(V\times\mathscr{P})$ and $\mathrm{Diff}^r_c(V\times\mathscr{P})$ with the inclusions:

$$\mathrm{Diff}^r(V)_{\mathscr{P}} \hookrightarrow \mathrm{Diff}^r(V\times\mathscr{P}) \quad \text{and} \quad \mathrm{Diff}^r_c(V)_{\mathscr{P}} \hookrightarrow \mathrm{Diff}^r_c(V\times\mathscr{P})$$

via $f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}} \mapsto \widehat{f_{\mathscr{P}}}$ where:

$$\widehat{f_{\mathscr{P}}} := (v,p) \in V\times\mathscr{P} \mapsto (f_p(v),p) \ .$$

The spaces $\mathrm{Diff}^r(V)_{\mathscr{P}}$ and $\mathrm{Diff}^r_c(V)_{\mathscr{P}}$ endowed with the composition law $f_{\mathscr{P}} \circ g_{\mathscr{P}} := (f_p \circ g_p)_{p\in\mathscr{P}}$ are groups.

Note that we have:

Fact 4.1.8. *The following inclusion is a morphism of topological groups:*

$$f_{\mathscr{P}} = (f_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^\infty_0(V)_{\mathscr{P}} \mapsto \widehat{f_{\mathscr{P}}} \in \mathrm{Diff}^\infty_0(V\times\mathscr{P}) \ .$$

The latter fact will be used among some proofs of the parametric counterparts.

From now on we will work with $V = \mathbb{T}\times M$.

The space Plug **of Pluggable maps**

Every time, but when explicitly stated, we will focus on the case $r = \infty$: plugins will be of class C^∞ and their outputs as well. Recall that we endow $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the Euclidean Riemannian metric, M and $\mathscr{P}$ with their Riemannian metric. The product spaces $\mathbb{T} \times M$, $\mathbb{T} \times M \times \mathscr{P}$ and $M \times \mathscr{P}$ are endowed with their product Riemannian metric.

Definition 4.1.9. *A map* $G \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ *is* semi-pluggable *if there is a sequence* $(g_k)_{k\geq 1}$ *of plugins* $g_k \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ *with step* 2^{-k}*, so that for every large integer* k *the output of* g_k *is* G *and* $g_k \to \mathrm{id}$ *for the* C^∞*-topology when* $k \to \infty$*. The map* G *is* pluggable *if* G *and* G^{-1} *are semi-pluggable. Let:*

$$\mathrm{Plug} := \{G \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M) : G \text{ is pluggable}\} .$$

Example 4.1.10. *The identity* id *of* $\mathbb{T} \times M$ *is pluggable since it is the output of every plugin of the sequence* $\Big((\theta, y) \in \mathbb{T} \times M \mapsto (\theta + 2^{-k}, y)\Big)_{k\geq 1}$ *as in Example 4.1.4.*

We give in Section 4.1.3 more sophisticated examples of pluggable dynamics. Note that in Definition 4.1.9, we are only interested by plugins whose output is compactly supported (outside of the boundary). This is because not every mapping of $\mathrm{Diff}^\infty_0(\mathbb{T} \times M)$ is pluggable by Corollary A, while we will show that every map in $\mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ is pluggable:

Theorem J. *We have* $\mathrm{Plug} = \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$.

Observe that the first assertion of Theorem I in the case $r = \infty$ is an immediate consequence of Theorem J.

Parametric counterpart. The proof of Theorem J is basically constructive and depends smoothly on the output. For the sake of completeness, we will verify this by giving the parametric counterpart of each statement. Some of the proofs will be designed to be verbatim the same. For a first reading of the proofs, we advise the reader to skip the all parametric counterpart of the arguments. Here is the parametric counterpart of the notion of plugin:

Definition 4.1.11. *A family* $g_{\mathscr{P}} = (g_p)_{p\in\mathscr{P}} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ *defines a* $\mathscr{P}$-plugin *if the diffeomorphism:*

$$\widehat{g_{\mathscr{P}}} : (z, p) \in M \times \mathscr{P} \mapsto (g_p(x), p)$$

in $\mathrm{Diff}^\infty(\mathbb{T} \times M \times \mathscr{P})$ *is a plugin.*

Similarly we have the parametric counterpart of the output of a $\mathscr{P}$-plugin:

Definition 4.1.12. *The* output *of a* $\mathscr{P}$*-plugin* $g_{\mathscr{P}} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ *of step* σ *is the family* $G_{\mathscr{P}}$ *such that for each* $p \in \mathscr{P}$*, the map* G_p *is the rescaling of the return map* $g_p^{\tau_p} : \Delta_\sigma \to \Delta_\sigma$*:*

$$G_p := H_\sigma \circ g_p^{\tau_p} \circ H_\sigma^{-1} : \mathbb{T} \times M \to \mathbb{T} \times M.$$

By Proposition 4.1.5 the output $\widehat{G_{\mathscr{P}}}$ of the plugin $\widehat{g_{\mathscr{P}}}$ on $\mathbb{T} \times M \times \mathscr{P}$ is smooth and so:

Fact 4.1.13. *The output of a* $\mathscr{P}$*-plugin of* $\mathbb{T} \times M$ *lies in* $\mathrm{Diff}^r_0(\mathbb{T} \times V)_{\mathscr{P}}$.

In the parametric setting, Definition 4.1.9 becomes:

Definition 4.1.14. *A family $G_{\mathscr{P}} \in \mathrm{Diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}}$ is* $\mathscr{P}$-semi-pluggable *if there is a sequence $(g_{k\mathscr{P}})_{k\geq 1}$ of $\mathscr{P}$-plugins with step 2^{-k}, so that for every large k, the output of $g_{k\mathscr{P}}$ is $G_{\mathscr{P}}$ and $\widehat{g_{k\mathscr{P}}}$ converges to $\mathrm{id}_{\mathbb{T}\times M\times\mathscr{P}}$ for the C^∞-topology. The family of maps $G_{\mathscr{P}}$ is* $\mathscr{P}$-semi-pluggable *if both $G_{\mathscr{P}}$ and $G_{\mathscr{P}}^{-1} := (G_p^{-1})_{p\in\mathscr{P}}$ are $\mathscr{P}$-semi-pluggable. Let:*

$$\mathrm{Plug}_{\mathscr{P}} := \{G_{\mathscr{P}} \in \mathrm{Diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : G_{\mathscr{P}} \text{ is } \mathscr{P}\text{-pluggable}\} \ .$$

To obtain Theorem 2 in the case $r=\infty$, it suffices to prove:

Theorem 3. *We have* $\mathrm{Plug}_{\mathscr{P}} = \mathrm{Diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}}$.

Examples of pluggable dynamics

Consider the following subgroup of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$:

$$\mathbb{G}_1 := \{(\theta, y)\in\mathbb{T}\times M \mapsto (\theta+\nu(y), y) : \nu\in C_c^\infty(M,\mathbb{R})\} \ .$$

This gives a first example of a subgroup of pluggable maps:

$$\mathbb{G}_1 \subset \mathrm{Plug} \ .$$

Proposition 4.1.15. *The group $\mathbb{G}_1$ is included in* Plug.

The subgroup $\mathbb{G}_1$ was first studied in [BT24] into a set of generators of symplectomorphisms.

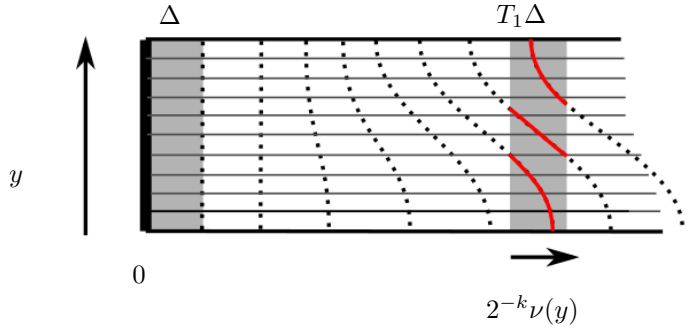


Figure 4.4 – Dynamics of an element of $\mathbb{G}_1$.

Proof. Let $\rho\in C^\infty(\mathbb{T},\mathbb{R}^+)$ be a function with support contained in $[\frac{1}{3},\frac{2}{3}]\subset\mathbb{T}$ and integral 1. Given $\epsilon\in[-2^{-k-2}, 2^{-k-2}]$, we define the smooth vector field:

$$X_\epsilon : \theta\in\mathbb{T}\mapsto 2^{-k}/(1-\epsilon\cdot\rho(\theta)) \ . \tag{4.1.1}$$

Let ϕ_ϵ^t be the flow of X_ϵ. The time taken to go all around the circle equals to:

$$\tau(\epsilon) = \int_{\mathbb{T}} \frac{1}{X_\epsilon(\theta)} d\theta = \int_{\mathbb{T}} 2^k\cdot(1-\epsilon\cdot\rho(\theta))d\theta = 2^k\cdot(1-\epsilon) \ . \tag{4.1.2}$$

If we stop at time 2^k, then the lacking or exceeding time for a complete lap is $2^k\epsilon \in [-\frac{1}{4}, \frac{1}{4}]$. As near 0 the vector field X_ϵ equals 2^{-k}, this implies that the image of 0 by $\phi_\epsilon^{2^k}$ is equal to ϵ. So for every $\nu \in C_c^\infty(M, \mathbb{R})$, for every k large, the map:

$$g_k := (\theta, y) \mapsto (\phi^1_{2^{-k}\nu(y)}(\theta), y) \tag{4.1.3}$$

coincides with $R_{2^{-k}}$ on a set which contains the complement of $S = [\frac{1}{4}, \frac{3}{4}] \times \operatorname{supp} \nu$. Thus g_k is a plugin with step 2^{-k}. Furthermore, by the above discussion, its output equals to:

$$G : (\theta, y) \mapsto (\theta + \nu(y), y) \text{ with } \nu \in C_c^\infty(M, \mathbb{R}) \ . \tag{4.1.4}$$

Finally observe that g_k is C^∞-close to identity. Thus G is semi-pluggable. Hence $\mathbb{G}_1$ is formed by semi-pluggable maps, and as $\mathbb{G}_1$ is a group, it is formed by pluggable maps. □

Parametric counterpart. Let $\mathbb{G}_{1\mathscr{P}}$ be the subset of $\mathrm{Diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p \in \mathbb{G}_1$ for every $p \in \mathscr{P}$.

We have similarly:

Proposition 4.1.16. *The group $\mathbb{G}_{1\mathscr{P}}$ is included in $\mathrm{Plug}_{\mathscr{P}}$.*

Proof. An element $(G_p)_{p\in\mathscr{P}} \in \mathbb{G}_{1\mathscr{P}}$ is formed by mapping of the form $G_p : (\theta, y) \mapsto (\theta + \nu_p(y), y)$ with $(\nu_p)_{p\in\mathscr{P}}$ smooth. Thus the vector field $Z_p : (\theta, y) \mapsto (X_{2^{-k}\nu_p}(y), 0)$ depends smoothly on p. Hence the time one maps g_p form a family of plugins $(g_p)_{p\in\mathscr{P}}$ (with output G_p and step 2^{-k}) which depends smoothly on $p \in \mathscr{P}$. □

4.2 Topological group structure on Plug

4.2.1 Group structure on Plug

In this section we show that Plug endowed with the composition rule is a group. To this end, let $\pi : \bar{\mathbb{T}} \times M := \mathbb{R}/2\mathbb{Z} \times M \to \mathbb{T} \times M = \mathbb{R}/\mathbb{Z} \times M$ be the canonical 2-sheeted covering map and denote by $\psi : (\theta, y) \in \bar{\mathbb{T}} \times M \mapsto (\theta/2, y) \in \mathbb{T} \times M$ the canonical diffeomorphism. The following defines a binary operation $\star$ on the space of plugins:

Definition 4.2.1. *Let g_1 and g_0 be two plugins with same step $\sigma = 2^{-k}$. Let $\bar{g}_1, \bar{g}_0 \in \mathrm{Diff}^\infty(\widehat{\mathbb{T}} \times M)$ be the lifts of g_1 and g_0 such that $\bar{g}_1(0, y) = \bar{g}_0(0, y) = (\sigma, y)$ for every $y \in M$. Let $\overline{g_1 \star g_0}$ be equal to the lift $\bar{g}_0$ on the first half of $\bar{\mathbb{T}} \times M$ and be equal to the lift of $\bar{g}_1$ on the second half of $\bar{\mathbb{T}} \times M$:*

$$\overline{g_1 \star g_0} : (\theta, y) \in \bar{\mathbb{T}} \times M \mapsto \begin{cases} \bar{g}_0(\theta, y) & \text{if } \theta \in [0, 1) + 2\mathbb{Z} \ , \\ \bar{g}_1(\theta, y), & \text{if } \theta \in [1, 2) + 2\mathbb{Z} \ , \end{cases} \quad \text{and put } g_1 \star g_0 := \psi \circ \overline{g_1 \star g_0} \circ \psi^{-1} \ .$$

Fact 4.2.2. *Given a neighborhood V of id in $\mathrm{Diff}_0^r(\mathbb{T} \times M)$, there exists a neighborhood W of id such that for any pair of plugins $f, g \in W$ of same steps, we have $f \star g \in V$.*

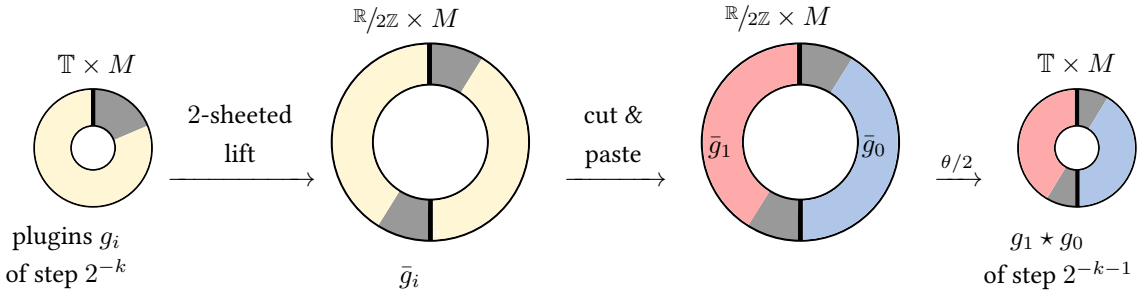


Figure 4.5 – Concatenation of two plugins.

The $\star$-product associates to a pair of plugins of the same step a plugin of half that step and whose output is the composition of the outputs:

Proposition 4.2.3. *If g_0 and g_1 are plugins with the same step σ and outputs G_0 and G_1 then $g_1 \star g_0$ is a plugin with step $\sigma/2$ and output $G_1 \circ G_0$.*

Proof. The set $\Delta = ([0, \sigma) + \mathbb{Z}) \times M$ lifts into the union of the two sets:

$$\bar{\Delta}_0 = ([0, \sigma) + 2\mathbb{Z}) \times M \quad \text{and} \quad \bar{\Delta}_1 = ([1, 1 + \sigma) + 2\mathbb{Z}) \times M.$$

Let π_i be the restriction of π to each $\bar{\Delta}_i$.

Note that $\bar{g}_0$ and $\bar{g}_1$ coincide with the translation by $(\sigma, 0)$ on $\bar{\Delta}_0 \cup \bar{\Delta}_1$. Hence the map $\overline{g_0 \star g_1}$ is a smooth diffeomorphism that also coincides with the translation by $(\sigma, 0)$ on $\bar{\Delta}_0 \cup \bar{\Delta}_1$. Let $\tau = \tau_0 \circ \pi + \tau_1 \circ \pi \circ \bar{g}_0^{\tau_0}$. Then $\overline{g_1 \star g_0}^{\tau}$ sends $\bar{\Delta}_0$ onto $\bar{\Delta}_1$ by $\bar{g}_0^{\tau_0 \circ \pi}$ and then sends $\bar{\Delta}_1$ onto $\bar{\Delta}_0$ by $\bar{g}_1^{\tau_1 \circ \pi}$. Thus we have:

$$\bar{g}_0^{\tau_0 \circ \pi} | \bar{\Delta}_0 = \pi_1^{-1} \circ H_\sigma^{-1} \circ G_0 \circ H_\sigma \circ \pi_0 \quad \text{and} \quad \bar{g}_1^{\tau_1 \circ \pi} | \bar{\Delta}_1 = \pi_0^{-1} \circ H_\sigma^{-1} \circ G_1 \circ H_\sigma \circ \pi_1 \ .$$

Therefore the return time of $\overline{g_1 \star g_0}$ into $\bar{\Delta}_0$ is defined on $\bar{\Delta}_0$ and the first return map is:

$$\pi_0^{-1} \circ H_\sigma^{-1} \circ G_1 \circ G_0 \circ H_\sigma \circ \pi_0.$$

This implies that $g_1 \star g_0$ is a plugin with output $G_1 \circ G_0$ and step $\sigma/2$. □

Proposition 4.2.4. *The set* Plug *is a subgroup of* $\mathrm{Diff}_c^\infty(\mathbb{T} \times M)$ *endowed with the composition rule.*

Proof. By Example 4.1.10, Plug contains the identity and by definition it is stable by inversion. Thus it remains to show that semi-pluggability is stable by composition. Let G_1, G_0 be semi-pluggable. Then for every k large, C^∞-close to identity, there are plugins g_1 and g_0 with step 2^{-k} and with outputs G_1 and G_0 respectively. Then by Fact 4.2.2 , the map $g_1 \star g_0$ is close to identity when k is large. By Proposition 4.2.3, the output of $g_1 \star g_0$ is $G_1 \circ G_0$ and its step is 2^{-k-1}. Thus $G_1 \circ G_0$ is semi-pluggable. The second assertion is proved similarly. □

Parametric counterpart. The proofs of the two latter propositions imply immediately:

Proposition 4.2.5. *If $f_{\mathcal{P}}, g_{\mathcal{P}}$ are two $\mathcal{P}$-plugins with same step σ and output $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$, then the family $(f_p \star g_p)_{p \in \mathcal{P}}$ is a $\mathcal{P}$-plugin with step $\sigma/2$ and output $(F_p \circ G_p)_{p \in \mathcal{P}}$.*

Thus we deduce:

Proposition 4.2.6. *The set* $\mathrm{Plug}_{\mathcal{P}}$ *is a subgroup of* $\mathrm{Diff}_c^\infty(\mathbb{T} \times M)_{\mathcal{P}}$ *endowed with the operation:*

$$(F_p)_{p \in \mathcal{P}} \circ (G_p)_{p \in \mathcal{P}} = (F_p \circ G_p)_{p \in \mathcal{P}} \ .$$

4.2.2 Another subgroup included in Plug

Consider the following sub-group of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$:

$$\mathbb{G}_2 := \{(\theta, y) \in \mathbb{T}\times M \mapsto (\theta, F(y)) : F \in \mathrm{Diff}_c^\infty(M)\}$$

Proposition 4.2.7. *The group $\mathbb{G}_2$ is included in* Plug.

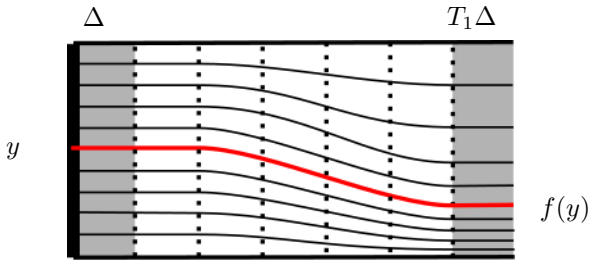


Figure 4.6 – Dynamics of a plugin with output in $\mathbb{G}_2$.

Proof. As $\mathbb{G}_2$ is connected and Plug is a group by Proposition 4.2.4, it suffices to show that a neighborhood W of id in $\mathbb{G}_2$ is included in Plug. Indeed by [Wil84, Prop 3.18], any such neighborhood W generates $\mathbb{G}_2$. Up to replacing W by $W \cap W^{-1}$, it suffices to show that any element of W is semi-pluggable.

In order to do so, we develop an idea which appears in [NRT78]. Take W sufficiently small so that for every $G : (\theta, y) \mapsto (\theta, F(y)) \in W$, the map F is sufficiently close to id to be isotopic to it via a smooth path. In other words, there exists a C^∞-family $(F_t)_{t\in[0,1]}$ of maps $F_t \in \mathrm{Diff}_c^\infty(M)$ such that $F_0 = \mathrm{id}$ and $F_1 = F$. Such a family can be obtained using the exponential map $\exp$ of the Riemannian metric, via the formula $F_t := y \mapsto \exp_y(t \cdot \exp_y^{-1} F(y))$. Define:

$$Y(t, y) := \partial_t F_t \circ F_t^{-1}(y)\ , \tag{4.2.1}$$

and observe that F is the time one map of the (compactly supported) non-autonomous vector field Y. Let $\tau : \mathbb{T} \to [0, 1]$ be a map which is smooth on $\mathbb{T} \setminus \{0\}$ and such that near 0^+ it equals 0 and near 0^- is equal 1. For k large, let:

$$X_k : (\theta, y) \in \mathbb{T}\times M \mapsto (2^{-k}, 2^{-k} \cdot \partial_\theta\tau(\theta) \cdot Y(\tau(\theta), y))\ . \tag{4.2.2}$$

Let g_k be the time one map of this vector field. Observe that for k large enough, g_k is a plugin with step $\sigma = 2^{-k}$ and return time σ^{-1}. Furthermore, its output is G. Indeed the second coordinate of the output is the time σ^{-1} map of the flow of $\partial_\theta\tau(\theta) \cdot Y(\tau(\theta), y)$ which is the time one map F of the flow of Y. Thus we have:

$$g_k^{1/\sigma} \circ H_\sigma(x) = H_\sigma \circ G(x), \quad \forall x \in \Delta_\sigma\ . \tag{4.2.3}$$

Furthermore, when k is large, the plugin g_k is close to identity. Thus G is semi-pluggable. □

Parametric counterpart. Let $\mathbb{G}_{2\mathscr{P}}$ be the subset of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p \in \mathbb{G}_2$ for every $p \in \mathscr{P}$. The following is a counterpart of Proposition 4.2.7:

Proposition 4.2.8. *The group $\mathbb{G}_{2\mathscr{P}}$ is included in* $\mathrm{Plug}_{\mathscr{P}}$.

Proof. Let W be as defined in the proof of Proposition 4.2.7. We then define $W_{\mathscr{P}}$ as the subset of $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}}$ formed by families $(G_p)_{p\in\mathscr{P}}$ such that $G_p \in W$ for every $p\in\mathscr{P}$. For the same reasons it suffices to show that any element of $W_{\mathscr{P}}$ is $\mathscr{P}$-semi-pluggable. Similarly, for any family $(G_p)_{p\in\mathscr{P}} =: (id_{\mathbb{T}}\times F_p)_{p\in\mathscr{P}} \in W_{\mathscr{P}}$, the family $(F_p)_{p\in\mathscr{P}}$ is isotopic to the identity via a smooth path $((F_{pt})_{p\in\mathscr{P}})_{t\in[0,1]}$ where $F_{pt} := y\mapsto \exp_y(t\cdot \exp_y^{-1}F_p(y))$. We define:

$$Y_p(t,y) := \partial_t F_{pt}\circ F_{pt}^{-1}(y)\ . \tag{4.2.4}$$

Note that the family of vector fields $(Y_p)_{p\in\mathscr{P}}$ is smooth. Define then the family of vector fields:

$$X_{kp} : (\theta, y)\in \mathbb{T}\times M \mapsto (2^{-k}, 2^{-k}\cdot \partial_\theta\tau(\theta)\cdot Y_p(\tau(\theta), y))\ , \tag{4.2.5}$$

where τ is the function defined in Proposition 4.2.7. For k large enough, the family of time one maps $(g_{kp})_{p\in\mathscr{P}}$ is a $\mathscr{P}$-plugin with output the family $(G_p)_{p\in\mathscr{P}}$. Moreover, when $k\to\infty$, the $\mathscr{P}$-plugin $(g_{kp})_{p\in\mathscr{P}}$ tends to the identity $\mathrm{id}\in\mathrm{Diff}_c^\infty(M)_{\mathscr{P}}$. □

4.2.3 Closedness of the group Plug

In this section we prove that Plug is closed in $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$.

Proposition 4.2.9. *The subgroup* $\mathrm{Plug}\subset\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$ *is closed.*

This proposition uses the following lemma proved below:

Lemma 4.2.10. *For any* $1\le r\le\infty$*, for any neighborhood* $\mathcal{N}$ *of* $\mathrm{id}\in\mathrm{Diff}^r(\mathbb{T}\times M)$*, there exist* $N\ge 1$ *and a neighborhood* $\mathcal{N}_c$ *of* $\mathrm{id}\in\mathrm{Diff}_c^r(\mathbb{T}\times M)$ *such that for all* $G\in\mathcal{N}_c$ *and* $k\ge N$*, there is a* C^r*-plugin* $g\in\mathcal{N}$ *with output* G *and step* 2^{-k}*.*

Note that the latter lemma is stated for any regularity $1\le r\le\infty$. It will allow us to deduce Theorem I from Theorem J.

Proof of Proposition 4.2.9. It suffices to show that the set of semi-pluggable maps is closed. Indeed, the continuity of the involution $G\mapsto G^{-1}$ implies that the set of maps with semi-pluggable inverse is closed; and so it comes that the intersection Plug of these two sets is closed.

Let $(G_j)_{j\ge 0}$ be a sequence of semi-pluggable maps converging in the C^∞-topology to a diffeomorphism $G\in\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$. Let us show that G is semi-pluggable. In other words, let us show that for every neighborhood $\mathcal{V}$ of $\mathrm{id}\in\mathrm{Diff}^\infty(\mathbb{T}\times M)$, for every k large enough, there exists a plugin $g\in\mathcal{V}$ with output G and step 2^{-k}.

To this end, let us fix a small neighborhood $\mathcal{N}$ of $\mathrm{id}\in\mathrm{Diff}^\infty(\mathbb{T}\times M)$ and j large so that $G_j^{-1}\circ G$ belongs to the open set $\mathcal{N}_c$ given by Lemma 4.2.10. Hence for every $k\ge 0$ large enough, there exists a plugin $g_1\in\mathcal{N}$ with output $G_j^{-1}\circ G$ and step 2^{-k}. As G_j is pluggable, for every k large enough, there exists a plugin $g_0\in\mathcal{N}$ with output G_j and step 2^{-k}. Now we merge the plugins g_1 and g_0 to obtain a plugin $g = g_1\star g_0$ of G of step 2^{-k-1}. By Fact 4.2.2, when $\mathcal{N}$ is small, the map g is close to identity and so in $\mathcal{V}$. □

The idea of the proof of Lemma 4.2.10 is to find, for each fixed small $\delta > 0$, a sequence of close to identity plugins g_k with step 2^{-k} and output the identity such that some iterates of each g_k stretch the 2^{-k}-thin fundamental domain $\Delta_{2^{-k}}$ onto a wider fundamental domain, isometric to $[0, 2\delta[\times M$. The iteration by g_k will produce a horizontal zooming effect on $\Delta_{2^{-k}}$. Then we will be able to perturb the plugin on this stretched fundamental domain, to obtain an open set of outputs independent of k.

The following produces the sequence $(g_k)_k$:

Sublemma 4.2.11. *For every neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$, for every $\delta > 0$ small, there exists $N \geq 1$ and a sequence $(g_k)_{k \geq N}$ of plugins in $\mathcal{N}$ with step 2^{-k}, output id such that:*

- *for all $y \in M$ and $\theta \in [\frac{1}{2}, \frac{2}{3}]$; we have $g_k(\theta, y) = (\theta + \delta, y)$,*
- *g_k is of the form $g_k : (\theta, y) \in \mathbb{T} \times M \mapsto (\phi_k(\theta), y)$ where the map ϕ_k is a flow.*

Proof. Let $A = [\frac{1}{2}, \frac{2}{3}]$ and $B = [\frac{1}{4}, \frac{1}{3}]$. Let ψ_A, $\psi_B \in C^\infty(\mathbb{T}, [0, 1])$ be two non-negative functions with disjoint supports, vanishing at a neighborhood of 0 and such that:

$$\psi_A | A = 1 \quad \text{and} \quad \psi_B | B = 1 \, . \tag{4.2.6}$$

For $\beta \geq 0$ we define the following vector field on the circle $\mathbb{T}$:

$$X_{\beta,k} := \delta \cdot \psi_A + \beta \cdot \psi_B + (1 - \psi_A - \psi_B) \cdot 2^{-k}. \tag{4.2.7}$$

Let $\tau_{\beta,k}$ be the time needed to make one turn around the circle along the flow of X_β. This number is large since δ is small. Note that $\tau_{\beta,k}$ depends smoothly on $\beta > 0$. Also $\tau_{\beta,k} \to \infty$ when $\beta \to 0$ and $\partial_\beta \tau_{\beta,k} < 0$. Let $N \geq 1$ be so that $\delta > 2^{-N}$. Let $k \geq N$. We have $\delta > 2^{-k}$. If $\beta = 2^{-k}$, then the time $\tau_{\beta,k}$ is smaller than 2^k. Thus by the mean value theorem, there exists a unique $\beta = \beta(k, \alpha)$ close to 2^{-k} such that $\tau_{\delta,\beta(k,\delta),k} = 2^k$.

Then the time 1 map g_k of the flow of $(X_{\beta(k),k}, 0)$ satisfies the desired properties. □

We now have the tools to prove the following restricted version of Lemma 4.2.10:

Sublemma 4.2.12. *For any $1 \leq r \leq \infty$, for any neighborhood $\mathcal{V}$ of $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$, there exist $N \geq 1$ and a neighborhood $\mathcal{N}_c$ of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$ such that for any $k \geq N$ and every $G \in \mathcal{N}_c$ whose restriction to a neighborhood of $\{0\} \times M$ or a neighborhood of $\{\frac{1}{2}\} \times M$ is the identity, there is a C^r-plugin $g \in \mathcal{V}$ with output G and step 2^{-k}.*

Proof of Sublemma 4.2.12 . Let $\mathcal{N}$ be a neighborhood of $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$ such that[5] $\mathcal{N} \Subset \mathcal{V}$. We apply Sublemma 4.2.11 which provides $\delta > 0$, $N \geq 1$ and a sequence $(g_k)_{k \geq N}$ of plugins $g_k : (\theta, y) \mapsto (\phi_k(\theta), y)$ with output id and step 2^{-k}. By Sublemma 4.2.11, there exists $n_k > 0$ minimal such that $\theta_k := \phi_k^{n_k}(0) \in [\frac{1}{2}, 1]$. Taking $\mathcal{N}$ small, we have that θ_k is smaller than $\frac{2}{3} - \frac{3}{2}\delta$, and so g_k equals the translation by δ on $(\theta_k, \theta_k + \frac{3}{2}\delta)$. Let $\mathcal{N}_c$ be a small neighborhood of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$. Let $G \in \mathcal{N}_c$ be equal to the identity near $\{0\} \times M$ or $\{\frac{1}{2}\} \times M$. We would like to C^r-perturb g_k so that its output is G.

Case 1: If $G \in \mathcal{N}_c$ coincides with the identity near $\{0\} \times M$, then we perform a perturbation of g_k supported by $(\theta_k, \theta_k + \delta) \times M$ and therein equals to:

$$\begin{array}{rcl} \tilde{g}_k : [\theta_k, \theta_k + \delta] \times M & \to & [\theta_k + \delta, \theta_k + 2\delta] \times M \\ (\theta_k + x, y) & \mapsto & (\theta_k + \delta, 0) + \delta \cdot \tilde{G}(\delta^{-1} x, y) \end{array} \tag{4.2.8}$$

5. in the sense that the distance between $\mathcal{N}$ and the complement of $\mathcal{V}$ is positive.

where $\tilde{G} : \mathbb{R} \times M \to \mathbb{R} \times M$ is a lifting of G which fixes $\{0\} \times M$. Note that $\tilde{g}_k$ is a C^r-plugin with output G and step 2^{-k}. Furthermore, if $\mathcal{N}_c$ is small enough (at δ fixed), then for every $k \geq N$, the map $\tilde{g}_k$ is uniformly close to $g_k \in \mathcal{N}$ in the C^r-topology and so $\tilde{g}_k$ belongs to $\mathcal{V}$.

Case 2: If $G \in \mathcal{N}_c$ coincides with the identity near $\{\frac{1}{2}\} \times M$, then we perform a perturbation of g_k supported by $(\theta_k + \frac{\delta}{2}, \theta_k + \frac{3}{2}\delta) \times M$ and therein equals to:

$$\begin{array}{rcl} \tilde{g}_k : [\theta_k + \frac{\delta}{2}, \theta_k + \frac{3}{2}\delta] \times M & \to & [\theta_k + \frac{3}{2}\delta, \theta_k + \frac{5}{2}\delta] \times M \\ (\theta_k + \frac{\delta}{2} + x, y) & \mapsto & (\theta_k + \frac{\delta}{2} + \delta, 0) + \delta \cdot \tilde{G}(\delta^{-1}x, y) \ . \end{array} \tag{4.2.9}$$

Similarly, this is a C^r-plugin of output G and step 2^{-k}, which is in $\mathcal{V}$ for every k provided that $\mathcal{N}_c$ is small enough. □

Proof that Sublemma 4.2.12 implies Lemma 4.2.10 . First let us 'fragment' any C^r-close to identity map $G \in \mathcal{N}_c$ into the composition of two C^r-maps $G_1 \circ G_0$ such that G_0 coincides with the identity near $\{0\} \times M$ and G_1 coincides with the identity near $\{\frac{1}{2}\} \times M$.

To this end, we use the exponential map $\exp$ associated to the geodesic flow of $\mathbb{T} \times M$ and a function $\rho \in C^\infty(\mathbb{T} \times M, [0, 1])$ such that $\rho|\{0\} \times M = 1$ and $\rho|\{\frac{1}{2}\} \times M = 0$. Let $\mathcal{N}_c$ be a sufficiently small neighborhood of $\mathrm{id} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ such that for every $G \in \mathcal{N}_c$ the following is a smooth diffeomorphism:

$$G_0 := x \mapsto \exp(\rho(x) \cdot \exp_x^{-1} G(x)) \ . \tag{4.2.10}$$

Let $G_1 := G \circ G_0^{-1}$ and note that $G = G_1 \circ G_0$.

Sublemma 4.2.12 states that there are C^r-plugins g_1 and g_0 close to identity with step 2^{-k} and outputs G_1 and G_0 for every k large enough. Then by Fact 4.2.2 and Proposition 4.2.3, the C^r-plugin $g_1 \star g_0$ of step 2^{-k-1} is close to identity and has output $G_1 \circ G_0 = G$. □

Proof that Theorem J implies Theorem I . When $r = \infty$, the result of Theorem I corresponds to the one of Theorem J. Consider now $r < \infty$. Let $G \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$, and $\mathcal{N} \subset \mathcal{V}$ be two neighborhoods of $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)$. We smooth the map G into a map $\tilde{G} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)$ such that the map $\tilde{G}^{-1} \circ G$ belongs to the neighborhood $\mathcal{N}_c$ given by Lemma 4.2.10. Then for every $k \geq 0$ large enough, there exists a plugin $g_0 \in \mathcal{N}$ with output $\tilde{G}^{-1} \circ G$ and step 2^{-k}. By Theorem J, for k large enough, there exists a plugin $g_1 \in \mathcal{N}$ with output $\tilde{G}$. We merge g_0 and g_1 to get a plugin $g = g_1 \star g_0$ with output G. By Fact 4.2.2, when $\mathcal{N}$ is small, the map g is in $\mathcal{V}$. □

Parametric counterpart. Here is the parametric counterpart of Proposition 4.2.9:

Proposition 4.2.13. *The set* $\mathrm{Plug}_{\mathscr{P}}$ *is closed.*

To show this proposition we will use the following counterpart of Lemma 4.2.10 proved below:

Lemma 4.2.14. *For any* $1 \leq r \leq \infty$*, for any neighborhood* $\mathcal{N}$ *of* $\mathrm{id} \in \mathrm{Diff}^r(\mathbb{T} \times M)$*, there exists* $N \geq 1$ *and a neighborhood* $\mathcal{N}_c$ *of* $\mathrm{id} \in \mathrm{Diff}^r_c(\mathbb{T} \times M)_{\mathscr{P}}$ *such that for all* $(G_p)_{p \in \mathscr{P}} \in \mathcal{N}_c$ *and* $k \geq N$*, there is a* $\mathscr{P}$*-*C^r*-plugin* $(g_p)_{p \in \mathscr{P}} \in \mathcal{N}$ *with output* $(G_p)_{p \in \mathscr{P}}$ *and step* 2^{-k}*.*

Proof of Proposition 4.2.13. We proceed literally as in the proof Proposition 4.2.9 up to some minor changes : we apply Lemma 4.2.14 instead of Lemma 4.2.10 and consider families instead of single maps. Note that the continuity of $\star$ given by Fact 4.2.2 is also valid for families since the embedding:

$$(f_p)_p \in \mathrm{Diff}^\infty(\mathbb{T}\times M)_{\mathscr{P}} \mapsto f_{\mathscr{P}} \in \mathrm{Diff}^r(\mathbb{T}\times M\times \mathscr{P})$$

commutes with the $\star$-product. □

Proof of Lemma 4.2.14. The explicit construction of Sublemma 4.2.12 gives directly a parametric counterpart of this lemma. Indeed note that the maps $\tilde{g}_k$ defined in its proof depend smoothly on G. Moreover the maps G_0 and G_1 obtained in the proof of Lemma 4.2.10 by the fragmentation formula depends smoothly on the involved diffeomorphism. □

Similarly, we show:

Proof that Theorem 3 implies Theorem 2. This goes exactly the same as for the proof that Theorem J implies Theorem 2. It suffices to replace maps in $\mathrm{Diff}^r_c(\mathbb{T}\times M)$ by families of maps in $\mathrm{Diff}^r_c(\mathbb{T}\times M)_{\mathscr{P}}$, plugins by $\mathscr{P}$-plugins and Lemma 4.2.10 by Lemma 4.2.14. □

4.2.4 Vector fields whose flow is pluggable

In this section V denotes a manifold. We recall that $\mathrm{Diff}^\infty_c(V)$ endowed with the composition rule $\circ$ is a Lie group. The space $\mathfrak{diff}^\infty_c(V)$ of C^∞-vector fields X on V whose support is a compact subset of $V\setminus \partial V$ is a Lie algebra endowed with the Lie bracket:

$$[X,Y] := DY(X) - DX(Y)\ , \quad \forall X,Y \in \mathfrak{diff}^\infty_c(V).$$

For more detail on properties of infinite dimensional Lie algebra, see for instance [KN63].We will work with the Lie algebra counterpart of the considered subgroups. We recall that a subset of $\mathfrak{diff}^\infty_c(V)$ is a *Lie subalgebra* if it is a vector space stable by Lie Bracket. To define the counterpart, we will use the flow $(\mathrm{Fl}^t_X)_t$ of vector fields $X \in \mathfrak{diff}^\infty_c(V)$.

Definition 4.2.15. *We denote* $\mathfrak{plug}$ *the set of vector fields whose flow is pluggable. In short:*

$$\mathfrak{plug} := \{X \in \mathfrak{diff}^\infty_c(\mathbb{T}\times M) : \mathrm{Fl}^t_X \in \mathrm{Plug},\ \forall t\in\mathbb{R}\}\ .$$

Using that Plug is a closed subgroup, the following is an immediate consequence of Proposition 4.5.1 of Section 4.5 below:

Proposition 4.2.16. *The space* $\mathfrak{plug}$ *is a closed Lie subalgebra of* $\mathfrak{diff}^\infty_c(\mathbb{T}\times M)$.

In the latter proposition, the closure of the group Plug implies directly that $\mathfrak{plug}$ is a closed Lie subalgebra. Note however that if Plug was not closed the set $\mathfrak{plug}$ might not even be a Lie subalgebra. The following are closed Lie algebras of $\mathfrak{diff}^\infty_c(\mathbb{T}\times M)$:

$$\mathfrak{g}_1 = \mathfrak{g}_1(M) := \{X \in \mathfrak{diff}^\infty_c(\mathbb{T}\times M) : \mathrm{Fl}^t_X \in \mathbb{G}_1,\ \forall t\in\mathbb{R}\} = \{X : (\theta,y)\mapsto (v(y),0) : v\in C^\infty(M,\mathbb{R})\}\ .$$

$$\mathfrak{g}_2 = \mathfrak{g}_2(M) := \{X \in \mathfrak{diff}^\infty_c(\mathbb{T}\times M) : \mathrm{Fl}^t_X \in \mathbb{G}_2,\ \forall t\in\mathbb{R}\} = \{X : (\theta,y)\mapsto (0,f(y)) : f\in \mathfrak{diff}^\infty_c(M)\}\ .$$

The subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ are in Plug, so:

$$\mathfrak{g}_1 \subset \mathfrak{plug} \quad \text{and} \quad \mathfrak{g}_2 \subset \mathfrak{plug}\ . \tag{4.2.11}$$

Parametric counterpart. We denote $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$ the subspace of families $X_{\mathscr{P}} = (X_p)_{p\in\mathscr{P}}$ in $\mathfrak{diff}_c^\infty(V)$ such that:

$$\widehat{X_{\mathscr{P}}}\colon (x,p)\mapsto (X_p(x),0) \tag{4.2.12}$$

is smooth and compactly supported, that is, such that $\widehat{X_{\mathscr{P}}}\in\mathfrak{diff}_c^\infty(V\times\mathscr{P})$. By Fact 4.1.8, the space $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$ is a Lie algebra endowed with the Lie bracket $[X_{\mathscr{P}},Y_{\mathscr{P}}]:=([X_p,Y_p])_{p\in\mathscr{P}}$.

Definition 4.2.17. *The $\mathscr{P}$-families of vector fields whose flow is $\mathscr{P}$-pluggable is denoted:*

$$\mathfrak{plug}_{\mathscr{P}}:=\left\{(X_p)_{p\in\mathscr{P}}\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : (\mathrm{Fl}^t_{X_p})_{p\in\mathscr{P}}\in\mathrm{Plug}_{\mathscr{P}},\ \forall t\in\mathbb{R}\right\}.$$

Using that $\mathrm{Plug}_{\mathscr{P}}$ is a closed subgroup, the following is an immediate consequence of Corollary D of Section 4.5:

Proposition 4.2.18. *The space $\mathfrak{plug}_{\mathscr{P}}$ is a closed Lie subalgebra of $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}}$.*

Also note that the space $\mathfrak{plug}_{\mathscr{P}}$ contains:

$$\mathfrak{g}_{1\mathscr{P}}:=\{(X_p)_{p\in\mathscr{P}}\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : X_p\in\mathfrak{g}_1,\ \forall p\in\mathscr{P}\}$$

and

$$\mathfrak{g}_{2\mathscr{P}}:=\{(X_p)_{p\in\mathscr{P}}\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : X_p\in\mathfrak{g}_2,\ \forall p\in\mathscr{P}\}.$$

We define also:

$$\widehat{\mathfrak{g}_{1\mathscr{P}}}:=\{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}}\in\mathfrak{g}_{1\mathscr{P}}\}\quad\text{and}\quad\widehat{\mathfrak{g}_{2\mathscr{P}}}:=\{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}}\in\mathfrak{g}_{2\mathscr{P}}\}.$$

Observe that:

$$\widehat{\mathfrak{g}_{1\mathscr{P}}}=\{(\theta,y,p)\mapsto(\nu(y,p),0,0) : \nu\in C_c^\infty(M\times\mathscr{P},\mathbb{R})\}$$

and

$$\widehat{\mathfrak{g}_{2\mathscr{P}}}=\{(\theta,y,p)\mapsto(0,f(y,p),0) : (f,0)\in\mathfrak{diff}_c^\infty(M\times\mathscr{P})\}.$$

Thus we have $\widehat{\mathfrak{g}_{1\mathscr{P}}}=\mathfrak{g}_1(M\times\mathscr{P})$ and $\widehat{\mathfrak{g}_{2\mathscr{P}}}\subsetneq\mathfrak{g}_2(M\times\mathscr{P})$.

4.3 Construction of Pluggable flows

In this section we show Theorem J by proving that any vector field has a pluggable flow. In order to do so we will show that the following subspace of $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ is in $\mathfrak{plug}$:

$$\mathfrak{g}_3:=\mathfrak{g}_3(M)=\{X\in\colon\mathfrak{diff}_c^\infty(\mathbb{T}\times M) : X \text{ has null } \mathbb{T}\text{-coordinate}\}.$$

The non-trivial remaining part of the proof is to show the following:

Proposition 4.3.1. *Any vector field of $\mathfrak{g}_3$ has pluggable flows:*

$$\mathfrak{g}_3\subset\mathfrak{plug}.$$

The proof of this proposition will occupy Sections 4.3.2 and 4.3.3. In the next section we show that it implies main Theorem J. The proof of several propositions will involve the following notation. If $\mathfrak{g}$ and $\mathfrak{h}$ are two sub-Lie algebras, we denote $[\mathfrak{g},\mathfrak{h}]$ the vector space spanned by Lie brackets of elements of $\mathfrak{g}$ and $\mathfrak{h}$:

$$[\mathfrak{g},\mathfrak{h}]:=\left\{\sum_{\text{i:finite}}[X_i,Y_i] : X_i\in\mathfrak{g},\ Y_i\in\mathfrak{h}\right\}.$$

Parametric counterpart. Similarly we set:

$$\mathfrak{g}_{3\mathscr{P}} := \{(X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathscr{P}} : X_p \in \mathfrak{g}_3,\ \forall p \in \mathscr{P}\}$$

and

$$\widehat{\mathfrak{g}_{3\mathscr{P}}} := \{\widehat{X_{\mathscr{P}}} : X_{\mathscr{P}} \in \mathfrak{g}_{3\mathscr{P}}\} \subsetneq \mathfrak{g}_3(M\times\mathscr{P}) \ .$$

We will prove the following parametric counterpart of Proposition 4.3.1 in Sections 4.3.2 and 4.3.3:

Proposition 4.3.2. *Every smooth family of vector fields in $\mathfrak{g}_3$ has a $\mathscr{P}$-pluggable flow:*

$$\mathfrak{g}_{3\mathscr{P}} \subset \mathfrak{plug}_{\mathscr{P}} \ .$$

4.3.1 Proof of main Theorem J and Theorem 3

The first step of the proof is the following:

Proposition 4.3.3. *Any vector field has a pluggable flow:*

$$\mathfrak{plug} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M) \ .$$

Proof. As $\mathfrak{plug}$ is a Lie algebra by Proposition 4.2.16 and since $\mathfrak{g}_1$ and $\mathfrak{g}_3$ are in $\mathfrak{plug}$ by Eq. (4.2.11) and Proposition 4.3.1, it suffices to show that the Lie algebra generated by $\mathfrak{g}_1$ and $\mathfrak{g}_3$ equals $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$. We first prove the latter for $M=\mathbb{R}^n$. Let $X\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ and $f\in C_c^\infty(\mathbb{T}\times\mathbb{R}^n,\mathbb{R})$ be its $\mathbb{T}$-coordinate. Let $W\in\mathfrak{g}_3$ be:

$$W := (0, f, 0, \dots, 0) \ . \tag{4.3.1}$$

Let $\rho\in C^\infty(\mathbb{T}\times M,\mathbb{R})$ be a compactly supported function which is equal to 1 near the support of f and that is independent of the $\mathbb{T}$ coordinate. Let $Y\in\mathfrak{g}_1$ be:

$$Y(\theta, y_1, \dots, y_n) = (\rho(y,\theta)\cdot y_1, 0, \dots, 0) \ . \tag{4.3.2}$$

A computation gives a Lie bracket of the form:

$$[W,Y](\theta,y) = (f(\theta,y), -\partial_\theta f(\theta,y)\cdot y_1, 0, \dots, 0) \ . \tag{4.3.3}$$

Thus $X-[W,Y]$ is in $\mathfrak{g}_3$ which gives the desired result for $M=\mathbb{R}^n$.

When M is another manifold, we fix a locally finite covering $(U_i)_i$ by balls U_i. Using a partition of the unity, every $X\in\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ can be written as a sum:

$$X = \sum_i X_i \ . \tag{4.3.4}$$

where each X_i is supported by U_i. As X is compactly supported, the X_i are almost all null and thus the above sum is finite. As each U_i is diffeomorphic to $\mathbb{R}^n$, we can apply the case $M=\mathbb{R}^n$ which gives $Y_i, Z_i\in\mathfrak{g}_3$ and $W_i\in\mathfrak{g}_1$ all supported by $\mathbb{T}\times U_i$ such that $[W_i,Y_i]+Z_i=X_i$. We conclude by summing over i. □

Remark 4.3.4. *We proved that* $[\mathfrak{g}_3,\mathfrak{g}_1]+\mathfrak{g}_3 = \mathfrak{diff}_c^\infty(\mathbb{T}\times M)$.

We now have the tools to show:

Proof of Theorem J. Let $G \in \mathrm{Diff}_c^\infty(\mathbb{T}\times M)$ and let $(G_t)_{t\in[0,1]}$ be a compactly supported smooth path from $G_0 = \mathrm{id}$ to $G_1 = G$ in $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$. Derivatives $X_t := \partial_t G_t \circ G_t^{-1}$ define a smooth family $X = (X_t)_t$ of vector fields all supported in a compact subset $K \subset V \setminus \partial V$. In particular, the time $1/N$-map F_i of the vector field $X_{i/N}$ is supported by K, and likewise for $F = F_{N-1}\circ\cdots\circ F_0$. By definition of the Whitney topology, it suffices to show that for every $r \geq 1$, when N is large, the map F is C^r-close to G to obtain that F is close to G in $\mathrm{Diff}_c^\infty(\mathbb{T}\times M)$.

Note that indeed each F_i is $O_{C^r}(1/N^2)$-close to $\tilde{F}_i := G_{(i+1)/N}\circ G_{i/N}^{-1}$ and so it holds:

$$G = \tilde{F}_{N-1}\circ\cdots\circ\tilde{F}_0 = F_{N-1}\circ\cdots\circ F_0 + O_{C^r}(1/N) = F + O_{C^r}(1/N)\ . \tag{4.3.5}$$

As each F_i belongs to Plug, and since Plug is a closed group by Proposition 4.2.13, it comes that F belongs to Plug and its limit G when $N\to\infty$ as well. □

Parametric counterpart. To prove Theorem 3 we will use the following parametric counterpart of Proposition 4.3.3:

Proposition 4.3.5. *The flow of every $\mathcal{P}$-family of vector fields is $\mathcal{P}$-pluggable:*

$$\mathfrak{plug}_{\mathcal{P}} = \mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathcal{P}}\ . \tag{4.3.6}$$

Proof. As in the proof of Proposition 4.3.3, it suffices to show that the Lie algebra generated by $\mathfrak{g}_{1\mathcal{P}}$ and $\mathfrak{g}_{3\mathcal{P}}$ equals $\mathfrak{plug}_{\mathcal{P}}$. We start with the case $M = \mathbb{R}^n$ and $\mathcal{P} = \mathbb{R}^d$. Let $X_{\mathcal{P}} = (X_p)_{p\in\mathcal{P}}$ be a $\mathcal{P}$-family in $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathcal{P}}$ and $f_p \in \mathfrak{diff}_c(\mathbb{T}\times M)$ be the $\mathbb{T}$-coordinate of X_p for each $p\in\mathcal{P}$. Let $W_{\mathcal{P}} \in \mathfrak{g}_{3\mathcal{P}}$ be:

$$W_{\mathcal{P}} := ((0, f_p, 0, \dots, 0))_{p\in\mathcal{P}}\ .$$

With Y as defined in Eq. (4.3.2), it holds by Eq. (4.3.3):

$$[W_p, Y](\theta, y) = (f_p(\theta, y), -\partial_\theta f_p(\theta, y)\cdot y_1, 0, \dots, 0))_{p\in\mathcal{P}}\ .$$

We set $Y_{\mathcal{P}} := (Y)_{p\in\mathcal{P}}\mathfrak{g}_{1\mathcal{P}}$. The latter computation gives:

$$[W_{\mathcal{P}}, Y_{\mathcal{P}}](\theta, y) = ((f_p(\theta, y), -\partial_\theta f_p(\theta, y)\cdot y_1, 0, \dots, 0))_{p\in\mathcal{P}}\ . \tag{4.3.7}$$

Thus $X_{\mathcal{P}} - [W_{\mathcal{P}}, Y_{\mathcal{P}}]$ is in $\mathfrak{g}_{3\mathcal{P}}$. Now for the general case we conclude by using a partition of unity as before. □

It allows us to conclude:

Proof of Theorem 3. Consider $G_{\mathcal{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T}\times M)_{\mathcal{P}}$. As a corollary of Theorem J, for any $r \geq 0$, the map $\widehat{G_{\mathcal{P}}} \in \mathrm{Diff}_c^\infty(\mathbb{T}\times M\times\mathcal{P})$ can be approximated by a composition of flows of vector fields with a zero $\mathcal{P}$-coordinate. Thus the family $G_{\mathcal{P}}$ can be approximated arbitrarily close by a composition of compactly supported families of flows of vector fields in the C^r-norm for any $r \geq 0$. By Proposition 4.3.3, each of the families of flows belongs to $\mathrm{Plug}_{\mathcal{P}}$. Since $\mathrm{Plug}_{\mathcal{P}}$ is a group by Proposition 4.2.13 the above composition belongs to $\mathrm{Plug}_{\mathcal{P}}$ as well and it comes that $(G_p)_{p\in\mathcal{P}}$ is in $\mathrm{Plug}_{\mathcal{P}}$ by closedness of $\mathrm{Plug}_{\mathcal{P}}$. □

4.3.2 Eigenvectors of the adjoint representation

Observe that the Lie algebra generated by $\mathfrak{g}_1 = \{(\theta, y) \mapsto (\nu(y), 0)\}$ and $\mathfrak{g}_2 = \{(\theta, y) \mapsto (0, f(y))\}$ contains the vector fields that do not depend on the $\mathbb{T}$-coordinate. To obtain more elements in $\mathfrak{plug}$ we study the eigenvectors of operators of the form:

$$\mathrm{ad}_X : Y \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M) \mapsto [X, Y] \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M) \ ,$$

for $X \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)$.

In this subsection, we show that if Y is an eigenvector of ad_X for $X \in \mathfrak{plug}$ then Y is in $\mathfrak{plug}$ (see Proposition 4.3.7). This will enable us to prove Proposition 4.3.1 in Section 4.3.3.

Definition 4.3.6. *Let V be a manifold and let $\mathfrak{g}$ be a subspace of $\mathfrak{diff}_c^\infty(V)$. We denote:*

$$\mathfrak{Eig}(\mathrm{ad}_\mathfrak{g}) := \{Y \in \mathfrak{diff}_c^\infty(V) : \exists X \in \mathfrak{g} \text{ such that } Y = [X, Y]\} \ .$$

The following is the key proposition enabling us to construct new examples of pluggable flows:

Proposition 4.3.7. *If $Y \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)$ satisfies $\mathrm{ad}_X Y = Y$ for $X \in \mathfrak{plug}$ then Y is in $\mathfrak{plug}$:*

$$\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{plug}}) \subset \mathfrak{plug} \ .$$

Proof. Given a map $f \in \mathrm{Diff}_c^\infty(M)$ and a vector field $W \in \mathfrak{diff}_c^\infty(M)$ we denote the pushforward of W by f as:

$$\mathrm{Ad}_f W := f_* W := Df \circ W \circ f^{-1} \ .$$

This notation is consistent with the usual composition rules given by the following commuting diagram:

$$\begin{array}{ccccc} & & W & & \\ & M & \longrightarrow & TM & \\ f & \downarrow & & \downarrow & Df \\ & M & \longrightarrow & TM & \\ & & \mathrm{Ad}_f W & & \end{array} \ .$$

The following contains a key idea for the proof of the main theorem:

Lemma 4.3.8. *If Y satisfies $[X, Y] = Y$, then for every $s, t \in \mathbb{R}$ it holds:*

$$\mathrm{Fl}_Y^t = \mathrm{Fl}_X^{-s} \circ \mathrm{Fl}_Y^{t \cdot e^{-s}} \circ \mathrm{Fl}_X^s \ . \tag{4.3.8}$$

Proof. We recall the following well known result on adjunction of vector fields by flows, see e.g. [KN63, Prop 1.9]:

Fact 4.3.9. *Let $U, W \in \mathfrak{diff}_c^\infty(M)$ be two vector fields, then it holds:*

$$\partial_t \mathrm{Ad}_{\mathrm{Fl}_U^t} W_{|t=0} = -[U, W] \ .$$

First observe that by the latter fact it holds $X = \mathrm{Ad}_{\mathrm{Fl}_X^s}(X)$ for every $s \in \mathbb{R}$. Let then $Y_s := \mathrm{Ad}_{\mathrm{Fl}_X^s}(Y) = D\mathrm{Fl}_X^s \circ Y \circ \mathrm{Fl}_X^{-s}$. Observe that since $Y = [X, Y]$, it holds:

$$[X, Y_s] = [\mathrm{Ad}_{\mathrm{Fl}_X^s}(X), \mathrm{Ad}_{\mathrm{Fl}_X^s}(Y)] = \mathrm{Ad}_{\mathrm{Fl}_X^s}([X, Y]) = \mathrm{Ad}_{\mathrm{Fl}_X^s}(Y) = Y_s \ . \tag{4.3.9}$$

Also for every $s \in \mathbb{R}$, we have:

$$\partial_s Y_s = \partial_t (Y_{s+t})_{|t=0} = \partial_t \mathrm{Ad}_{\mathrm{Fl}_X^t}(Y_s)_{|t=0} \ . \tag{4.3.10}$$

Thus by the latter fact it follows:

$$\partial_s Y_s = -[X, Y_s] = -Y_s \ . \tag{4.3.11}$$

Consequently $\partial_s Y_s = -Y_s$ and thus $e^{-s} \cdot Y = Y_s = \mathrm{Ad}_{\mathrm{Fl}_X^s}(Y)$. After integration between 0 and t, we obtain:

$$\mathrm{Fl}_{e^{-s}.Y}^t = \mathrm{Fl}_X^s \circ \mathrm{Fl}_Y^t \circ \mathrm{Fl}_X^{-s} \ . \tag{4.3.12}$$

As $\mathrm{Fl}_{e^{-s}.Y}^t = \mathrm{Fl}_Y^{t.e^{-s}}$ we obtain the desired result by composing the latter equation on the right by Fl_X^s and on the left by Fl_X^{-s}. □

We can now prove Proposition 4.3.7. Fix a neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)$ and let us show the existence of a plugin in $\mathcal{N}$ whose output is Fl_Y^t. Let $\mathcal{N}_c$ be the neighborhood of $\mathrm{id} \in \mathrm{Diff}_c^\infty(\mathbb{T} \times M)$ given by Lemma 4.2.10. Let $s \in \mathbb{R}$ be sufficiently large such that $\mathrm{Fl}_Y^{e^{-s} \cdot t}$ belongs to the neighborhood $\mathcal{N}_c$ of the identity. Then for any $k \geq 1$ large enough $\mathrm{Fl}_Y^{e^{-s} \cdot t}$ is the output of a plugin $g \in \mathcal{N}$ of step 2^{-k}. Since $X \in \mathfrak{plug}$, it holds $\mathrm{Fl}_X^s \in \mathrm{Plug}$ so for any $k \geq 1$ large enough Fl_X^s and Fl_X^{-s} are the outputs of plugins h and f in $\mathcal{N}$ of respective steps 2^{-k} and 2^{-k-1}. So for any large k the map Fl_Y^t is the output of the plugin $(h \star g) \star f$ with step 2^{-k-2}. Since this holds for any neighborhood $\mathcal{N}$ of the identity the plugin $(h \star g) \star f$ can be taken arbitrarily close to identity. Therefore for every t, there exists a plugin with output Fl_Y^t arbitrarily close to identity for any small enough step. Hence Y is in $\mathfrak{plug}$. □

A second main ingredient of the proof of the main theorem is the following of independent interest:

Proposition 4.3.10. *For every $T \in \mathfrak{diff}_c^\infty(V)$, there exist finite families $(X_i)_i, (Y_i)_i, (Z_i)_i$ of vector fields in $\mathfrak{diff}_c^\infty(V)$ such that:*

$$T = \sum_i [Y_i, Z_i] \quad \textit{and} \quad Y_i = [X_i, Y_i] \ .$$

The proof of this proposition and its parametric counterpart will occupy the full Section 4.3.3. Now note that the following is an isomorphism of Lie algebras:

$$\mathfrak{i}_2 : X \in \mathfrak{diff}_c^\infty(M) \mapsto (0, X) \in \mathfrak{g}_2 \ .$$

Thus, by applying this isomorphism to the image of sets involved in the statement of Proposition 4.3.10 for every $T \in \mathfrak{g}_2$, there exist finite families $(X_i)_i, (Y_i)_i, (Z_i)_i$ of vector fields in $\mathfrak{g}_2$ such that:

$$T = \sum_i [Y_i, Z_i] \quad \text{and} \quad Y_i = [X_i, Y_i] \ .$$

In other words, we proved:

Corollary B. *Any element of $\mathfrak{g}_2$ can be written as a sum of Lie brackets of elements of $\mathfrak{g}_2$ with elements in $\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2}) \cap \mathfrak{g}_2$.*

This corollary allows us to deduce:

Proof of Proposition 4.3.1. Let $p_\theta : \mathbb{T} \times M \to \mathbb{T}$ be the projection on the $\mathbb{T}$-coordinate.

First note that by Fourier decomposition theorem, the space $\mathfrak{g}_3$ is the closure of the vector space spanned by elements of the form:

$$\phi \circ p_\theta \cdot Y \quad \text{for } \phi \in C^\infty(\mathbb{T}, \mathbb{R}) \quad \text{and} \quad Y \in \mathfrak{g}_2 \ .$$

Since $\mathfrak{plug}$ is a closed vector space by Proposition 4.2.16, it suffices to show any such $\phi \circ p_\theta \cdot Y$ is in $\mathfrak{plug}$. To do so, we first start with the case where there exists $X \in \mathfrak{g}_2$ such that $Y = [X, Y]$, i.e. we assume that $Y \in \mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2})$. Since the $\mathbb{T}$-coordinate $p_\theta \circ X$ of X is zero, it follows:

$$[X, \phi \circ p_\theta \cdot Y] = \phi \circ p_\theta \cdot DY(X) - DX(\phi \circ p_\theta \cdot Y) = \phi \circ p_\theta \cdot [X, Y] = \phi \circ p_\theta \cdot Y \ . \tag{4.3.13}$$

Thus by Proposition 4.3.7 we have $\phi \circ p_\theta \cdot Y \in \mathfrak{plug}$ since $X \in \mathfrak{g}_2 \subset \mathfrak{plug}$. Now in the general case, for every $Y \in \mathfrak{g}_2$, by Corollary B, there exist an integer $N \geq 1$, $Z_i \in \mathfrak{g}_2$ and $Y_i \in \mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_2}) \cap \mathfrak{g}_2$ for any $1 \leq i \leq N$, such that:

$$Y = \sum_{1 \leq N} [Y_i, Z_i] \ . \tag{4.3.14}$$

By the first case for any $1 \leq i \leq N$ we have $\phi \circ p_\theta \cdot Y_i \in \mathfrak{plug}$. Since $\mathfrak{plug}$ is a Lie-algebra and each Z_i is in $\mathfrak{g}_2 \subset \mathfrak{plug}$, it immediately follows:

$$\phi \circ p_\theta \cdot Y = \sum_{1 \leq i \leq N} \phi \circ p_\theta \cdot [Y_i, Z_i] = \sum_{1 \leq i \leq N} [\phi \circ p_\theta \cdot Y_i, Z_i] \in \mathfrak{plug} \ . \tag{4.3.15}$$

□

Parametric counterpart. The following is the parametric counterpart of Proposition 4.3.7.

Proposition 4.3.11. *If $Y_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{T} \times M)_{\mathscr{P}}$ satisfies $[X_{\mathscr{P}}, Y_{\mathscr{P}}] = Y_{\mathscr{P}}$ for $X_{\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$, then $Y_{\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$.*

Proof. Let $X_{\mathscr{P}} =: (X_p)_{p \in \mathscr{P}}$ and $Y_{\mathscr{P}} =: (Y_p)_{p \in \mathscr{P}}$. By Lemma 4.3.8, it holds:

$$\mathrm{Fl}^t_{Y_p} = \mathrm{Fl}^{-s}_{X_p} \circ \mathrm{Fl}^{t \cdot e^{-s}}_{Y_p} \circ \mathrm{Fl}^s_{X_p} \ , \tag{4.3.16}$$

for any $t, s \in \mathbb{R}$. For any neighborhood $\mathcal{N}$ of $\mathrm{id} \in \mathrm{Diff}^\infty(\mathbb{T} \times M)_{\mathscr{P}}$, we denote $\mathcal{N}_c$ the neighborhood of $\mathrm{id} \in \mathrm{Diff}^\infty_c(\mathbb{T} \times M)_{\mathscr{P}}$ given by Lemma 4.2.14. For all $t \in \mathbb{R}$ and for s large enough it holds $(\mathrm{Fl}^{t \cdot e^{-s}}_{Y_p})_{p \in \mathscr{P}} \subset \mathcal{N}_c$ and thus it is the output of a $\mathscr{P}$-plugin in $\mathcal{N}$ of any small step. We now regard the $\star$ product of the latter plugins with $\mathscr{P}$-plugins of $(\mathrm{Fl}^s_{X_p})_{p \in \mathscr{P}}$ and $(\mathrm{Fl}^{-s}_{X_p})_{p \in \mathscr{P}}$ to obtain a $\mathscr{P}$-plugin with output $(\mathrm{Fl}^t_{Y_p})_{p \in \mathscr{P}}$ of any small step. Moreover, by construction, this $\mathscr{P}$-plugin can be taken arbitrarily close to identity family, which ends the proof. □

For $\mathfrak{g}_{\mathscr{P}} \subset \mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$, we denote:

$$\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_{\mathscr{P}}}) := \{Y_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(V)_{\mathscr{P}} : \exists X_{\mathscr{P}} \in \mathfrak{g}_{\mathscr{P}} \text{ such that } Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}]\} \ .$$

The following parametric counterpart of Proposition 4.3.10 holds:

Proposition 4.3.12. *For every $T_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$, there exist finite families $(X_{i\mathscr{P}})_i, (Y_{i\mathscr{P}})_i, (Z_{i\mathscr{P}})_i$ of vector fields in $\mathfrak{diff}_c^\infty(V)_{\mathscr{P}}$ such that:*

$$T_{\mathscr{P}} = \sum_i [Y_{i\mathscr{P}}, Z_{i\mathscr{P}}] \quad \textit{and} \quad Y_{i\mathscr{P}} = [X_{i\mathscr{P}}, Y_{i\mathscr{P}}] \ .$$

Using the isomorphism:

$$X_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(M)_{\mathscr{P}} \mapsto (0, X_p)_{p\in\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$$

leads as before to:

Corollary C. *Any element of $\mathfrak{g}_{2\mathscr{P}}$ can be written as a sum of Lie brackets of elements of $\mathfrak{g}_{2\mathscr{P}}$ with elements in $\mathfrak{Eig}(\mathrm{ad}_{\mathfrak{g}_{2\mathscr{P}}}) \cap \mathfrak{g}_{2\mathscr{P}}$.*

The latter allows us to deduce:

Proof of Proposition 4.3.2. By the Fourier decomposition theorem, the space $\mathfrak{g}_{3\mathscr{P}}$ is the closure in $\mathfrak{diff}_c^\infty(\mathbb{T}\times M)$ of the vector space spanned by vector fields of the form $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ with $\phi \in C^\infty(\mathbb{T},\mathbb{R})$ and $(Y_p)_{p\in\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$. Since $\mathfrak{plug}_{\mathscr{P}}$ is a closed vector space by Proposition 4.2.18, it suffices then to show that $(\phi\circ p_\theta \cdot Y_p)_{p\in\mathscr{P}}$ is in $\mathfrak{plug}_{\mathscr{P}}$. To do so, we first start with the case where there exists $X_{\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$ such that $Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}]$. By Eqs. (4.3.13) and (4.3.15) of the proof of Proposition 4.3.1, for each $p\in\mathscr{P}$, it holds $\phi\circ p_\theta\cdot Y_p = [X_p, \phi\circ p_\theta\cdot Y_p]$. And thus the family $(\phi\circ p_\theta\cdot Y_p)_{p\in\mathscr{P}}$ is in $\mathfrak{plug}_{\mathscr{P}}$ by Proposition 4.3.11.

Now in the general case, by the latter Corollary C we can decompose:

$$Y_{\mathscr{P}} = \sum [Y_{i\mathscr{P}}, Z_{i\mathscr{P}}] \ , \tag{4.3.17}$$

with $Y_{i\mathscr{P}} \in \mathfrak{Eig}(\mathfrak{g}_{2\mathscr{P}}) \cap \mathfrak{g}_{2\mathscr{P}}$ and $Z_{i\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}}$. Thus Eq. (4.3.15) applied for each $p\in\mathscr{P}$ leads to:

$$(\phi\circ p_\theta\cdot Y_p)_{p\in\mathscr{P}} = \sum [(\phi\circ p_\theta\cdot Y_{ip})_{p\in\mathscr{P}}, Z_{i\mathscr{P}}] \ , \tag{4.3.18}$$

where $(Y_{ip})_{p\in\mathscr{P}} = Y_{i\mathscr{P}}$. Since $Z_{i\mathscr{P}} \in \mathfrak{g}_{2\mathscr{P}} \subset \mathfrak{plug}_{\mathscr{P}}$ and $(\phi\circ p_\theta\cdot Y_{ip})_{p\in\mathscr{P}} \in \mathfrak{plug}_{\mathscr{P}}$ by the first case, it follows that $(\phi\circ p_\theta\cdot Y_p)_{p\in\mathscr{P}}$ lies in the sub Lie algebra $\mathfrak{plug}_{\mathscr{P}}$ as well. □

4.3.3 Decomposition of vector fields

To prove the Theorem J, it remains only to show Proposition 4.3.10. We first prove this proposition in the case $M=\mathbb{R}^n$, which we will deduce from the case $M=\mathbb{R}$ and a parametric version thereof. This whole section is dedicated to this proof. For a manifold M, let us denote:

$$\mathfrak{Eig}(M) := \{Y \in \mathfrak{diff}_c^\infty(M) : \exists X \in \mathfrak{diff}_c^\infty(M) \text{ such that } \mathrm{ad}_X Y = [X,Y] = Y\} \ .$$

We can rephrase Proposition 4.3.10 as:

Proposition 4.3.13. *Every vector field in $T \in \mathfrak{diff}_c^\infty(V)$ is a finite sum of vector fields of the form $[Y_i, Z_i]$ with $Y_i \in \mathfrak{Eig}(M)$. In other words:*

$$[\mathfrak{Eig}(M), \mathfrak{diff}_c^\infty(M)] = \mathfrak{diff}_c^\infty(M).$$

Two key ingredients of the proof of this proposition are the following observations:

Fact 4.3.14. *The vector field $Y : y \mapsto 1$ on $\mathbb{R}$ satisfies $[X, Y] = Y$ with $X : y \in \mathbb{R} \mapsto -y$.*

Fact 4.3.15. *For any diffeomorphism $\psi\colon W \to V$ between manifolds and any pair of vector fields $X, Y \in \mathfrak{diff}_c^\infty(V)$ satisfying $[X, Y] = Y$, it holds $[\psi^* X, \psi^* Y] = \psi^* Y$.*

An important consequence of the latter fact is:

Fact 4.3.16. *For any diffeomorphism $\psi\colon W \to V$ from a manifold W into V, if $X \in [\mathfrak{Eig}(V), \mathfrak{diff}_c^\infty(V)]$, then $\psi^* X \in [\mathfrak{Eig}(W), \mathfrak{diff}_c^\infty(W)]$.*

Let $\mathcal{Q}$ be a manifold. We define:

$$\mathfrak{Eig}(\mathbb{R})_{\mathcal{Q}} = \Big\{ Y_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}} : \text{ there exists a family } X_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}} \text{ such that } [X_{\mathcal{Q}}, Y_{\mathcal{Q}}] = Y_{\mathcal{Q}} \Big\}.$$

We can prove Proposition 4.3.13 in the case $M = \mathbb{R}$:

Lemma 4.3.17. *We have:*

$$[\mathfrak{Eig}(\mathbb{R}), \mathfrak{diff}_c^\infty(\mathbb{R})] = \mathfrak{diff}_c^\infty(\mathbb{R}) \, .$$

Moreover for any manifold $\mathcal{Q}$, every $T_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$ satisfies $T_{\mathcal{Q}} = [Y_{\mathcal{Q}}, Z_{\mathcal{Q}}]$ for some $Y_{\mathcal{Q}} \in \mathfrak{Eig}(\mathbb{R})_{\mathcal{Q}}$ and $Z_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$.

Proof. We first give an intuitive idea of the proof using Fact 4.3.14. First note that for every $T \in \mathfrak{diff}_c^\infty(\mathbb{R})$, there exist $X, Y, Z \in \mathfrak{diff}^\infty(\mathbb{R})$ such that $Y = [X, Y]$ and $T = [Y, Z]$. Indeed it suffices to take $X\colon y \mapsto -y$, $Y\colon y \mapsto 1$ and $Z\colon y \mapsto \int_{-\infty}^{y} T(t)\, dt$. This almost proves the first assertion of the lemma. To have exactly the desired result we shall modify X, Y and Z to make them compactly supported. Let $\widetilde{T} \in \mathfrak{diff}_c^\infty(\mathbb{R})$. Take intervals $[-A, A] \subset (-a, a)$ containing its support. Consider the map $\psi\colon y \mapsto y \cdot e^{e^{(a^2-y^2)^{-1}}}$ from $(-a, a)$ to $\mathbb{R}$. We compute its derivative at $y \in (-a, a)$ by:

$$D\psi(y) = \phi(y) \cdot e^{e^{(a^2-y^2)^{-1}}} \, , \quad \text{where } \phi(y) := 1 + 2y^2 \cdot (a^2 - y^2)^{-2} e^{(a^2-y^2)^{-1}} \, . \tag{4.3.19}$$

Thus the map ψ is a diffeomorphism. This allows us to consider the push-forward:

$$T = \psi_* \widetilde{T} = D\psi \circ \widetilde{T} \circ \psi^{-1} \, . \tag{4.3.20}$$

Now we define as above:

$$X\colon y \mapsto -y \, , \quad Y\colon y \mapsto 1 \quad \text{and} \quad Z\colon y \mapsto \int_{-\infty}^{y} T(t)\, dt \, . \tag{4.3.21}$$

Note that $\tilde{T} = \psi^* T$ and put $\widetilde{X} = \psi^* X$, $\widetilde{Y} = \psi^* Y$ and $\widetilde{Y} = \psi^* Y$. Then by Fact 4.3.15, it holds:

$$\widetilde{Y} = [\widetilde{X}, \widetilde{Y}] \quad \text{and} \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{T} \, . \tag{4.3.22}$$

Let us show that these pulled-back vector fields on $(-a, a)$ extend smoothly by 0 to $\mathbb{R}$. For any vector field $S \in \mathfrak{diff}_c^\infty(\mathbb{R})$, we have $\psi^* S = (D\psi)^{-1} \circ S \circ \psi = \frac{S \circ \psi}{D\psi}$. It follows:

$$\widetilde{X}(y) = \psi^* X(y) = \tfrac{-\psi(y)}{D\psi(y)} = \tfrac{-y}{\phi(y)} \quad \text{and} \quad \widetilde{Y}(y) = \psi^* Y(y) = \tfrac{1}{D\psi(y)}, \tag{4.3.23}$$

Observe that $\phi(y)$ grows exponentially fast to $+\infty$ when $|y| \to a$. Thus $\widetilde{X}$ and $\widetilde{Y}$ have all their derivatives tending to 0 as y tends to $\pm a$. So they extend smoothly by 0 to vector fields in $\mathfrak{diff}_c^\infty(\mathbb{R})$. Also note that $Z(y) = 0$ for $y \leq \psi(-A)$ and $Z(y) = \int_{\mathbb{R}} T(t)\, dt$ for $y \geq \psi(A)$. Thus:

$$\widetilde{Z}(y) = \psi^* Z(y) = 0 \quad \text{for } -a < y \leq -A \quad \text{and} \quad \widetilde{Z}(y) = \frac{1}{D\psi(y)} \cdot \int_{\mathbb{R}} T(t) \cdot dt \quad \text{for } A \leq y < a.$$

Likewise $\widetilde{Z}$ extends smoothly by 0 to form a vector field in $\mathfrak{diff}_c^\infty(\mathbb{R})$. As $\widetilde{Y} = [\widetilde{X}, \widetilde{Y}]$ and $[\widetilde{Y}, \widetilde{Z}] = \widetilde{T}$, this proves the first assertion of the lemma.

For the parametric assertion, observe that ψ, $\widetilde{X}$ and $\widetilde{Y}$ depend only on the segment $[-a, a]$. Hence given $\widetilde{T}_{\mathcal{Q}} = (\widetilde{T}_q)_{q \in \mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$, we set $[-A, A] \subset (-a, a)$ containing the supports of all $\widetilde{T}_q$ and define ψ, $\widetilde{X}$ and $\widetilde{Y}$ as above. Then we observe that $\widetilde{Z}_q = \psi^* \int T_q$ depends smoothly on q and define a family $\widetilde{Z}_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$ which satisfies the desired equalities with $\widetilde{T}_{\mathcal{Q}}$ and the constant families $(\widetilde{X})_{q \in \mathcal{Q}}$ and $(\widetilde{Y})_{q \in \mathcal{Q}}$. □

We are going to use the parametric assertion of the latter lemma to obtain:

Lemma 4.3.18. *We have* $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)] = \mathfrak{diff}_c^\infty(\mathbb{R}^n)$.

Proof. Lemma 4.3.17 corresponds to the case $n = 1$. For $n \geq 2$, given $T \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$, we write its components as $T = (T_1, \ldots, T_n)$. By linearity of the condition, it suffices to show that each vector field $(0, \ldots, 0, T_i, 0, \ldots, 0)$ is in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. Using an adjunction by a permutation of the coordinates and Fact 4.3.16, its suffices to show that each $(T_i, 0, \ldots, 0)$ is in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. In other words, it suffices to prove that the following subalgebra $\mathfrak{h}(\mathbb{R}^n)$ of $\mathfrak{diff}_c^\infty(\mathbb{R}^n)$ is included in $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$:

$$\mathfrak{h}(\mathbb{R}^n) := \{y \in \mathbb{R}^n \mapsto (h(y), 0, \ldots, 0) : h \in C_c^\infty(\mathbb{R}^n, \mathbb{R})\} \ .$$

To this end, note that $X_{\mathbb{R}^{n-1}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathbb{R}^{n-1}} \mapsto \widehat{X_{\mathbb{R}^{n-1}}} \in \mathfrak{h}(\mathbb{R}^n)$ is an isomorphism of Lie algebras. By Lemma 4.3.17 with $\mathcal{Q} = \mathbb{R}^{n-1}$, we have:

$$\mathfrak{h}(\mathbb{R}^n) = [\widehat{\mathfrak{Eig}(\mathbb{R})}_{\mathcal{Q}}, \mathfrak{h}(\mathbb{R}^n)] \ .$$

Finally we note that $\widehat{\mathfrak{Eig}(\mathbb{R})}_{\mathcal{Q}}$ is formed by vector fields of the form $\widehat{Y_{\mathcal{Q}}}$ such that $Y_{\mathcal{Q}} = [X_{\mathcal{Q}}, Y_{\mathcal{Q}}]$ for $X_{\mathcal{Q}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathcal{Q}}$. Thus $\widehat{Y_{\mathcal{Q}}} = [\widehat{X_{\mathcal{Q}}}, \widehat{Y_{\mathcal{Q}}}]$, this proves that $\mathfrak{h}(\mathbb{R}^n) \subset [\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{h}(\mathbb{R}^n)]$.

□

We can now treat the general case:

Proof of Proposition 4.3.13. Let $T \in \mathfrak{diff}_c^\infty(M)$. Then it decomposes in a finite sum $T = \sum_i T_i$ where each T_i is compactly supported in an open set U_i which is diffeomorphic to $\mathbb{R}^n$ via a map $\psi_i \colon U_i \to \mathbb{R}^n$. Consider the push forward $\psi_{i*} T_i \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$. By Lemma 4.3.18 the field $\psi_{i*} T_i | U_i$ belongs to $[\mathfrak{Eig}(\mathbb{R}^n), \mathfrak{diff}_c^\infty(\mathbb{R}^n)]$. Thus by Lemma 4.3.17, it holds $T_i | U_i \in [\mathfrak{Eig}(U_i), \mathfrak{diff}_c^\infty(U_i)]$. This means that $T_i = \sum_{\text{Finite}} [Y_j, Z_j]$ with $Y_j = [X_j, Y_j]$ for some $X_j, Y_j, Z_j \in \mathfrak{diff}_c^\infty(\mathbb{R}^n)$. Extending all these vector fields by 0, we obtain that T_i belongs to $[\mathfrak{Eig}(M), \mathfrak{diff}_c^\infty(M)]$. So does $T = \sum_i T_i$. □

Parametric counterpart. We set:

$$\mathfrak{Eig}(M)_{\mathscr{P}} := \{Y_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(M)_{\mathscr{P}} : Y_{\mathscr{P}} = [X_{\mathscr{P}}, Y_{\mathscr{P}}] \text{ with } X_{\mathscr{P}} \in \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}\} .$$

We shall prove Proposition 4.3.12 that we rephrase as:

Proposition 4.3.19. *It holds:*

$$[\mathfrak{Eig}(M)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}] = \mathfrak{diff}_c^\infty(M)_{\mathscr{P}}.$$

Similarly to the proof of Proposition 4.3.13, Proposition 4.3.19 is an easy consequence of the following:

Lemma 4.3.20. *We have* $[\mathfrak{Eig}(\mathbb{R}^n)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}] = \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}$.

Proof. As in the proof of Proposition 4.3.13, we only need to prove that the following Lie subalgebra $\mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}}$ of $\mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}$ is included in $[\mathfrak{Eig}(\mathbb{R}^n)_{\mathscr{P}}, \mathfrak{diff}_c^\infty(\mathbb{R}^n)_{\mathscr{P}}]$:

$$\mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}} := \{((h_p, 0, \dots, 0))_{p\in\mathscr{P}} : h_{\mathscr{P}} = (h_p)_{p\in\mathscr{P}} \in C_c^\infty(\mathbb{R}^n, \mathbb{R})_{\mathscr{P}}\} .$$

To this end we proceed as in Lemma 4.3.18, by using the isomorphism of Lie algebra:

$$X_{\mathbb{R}^{n-1}\times\mathscr{P}} \in \mathfrak{diff}_c^\infty(\mathbb{R})_{\mathbb{R}^{n-1}\times\mathscr{P}} \mapsto Y_{\mathscr{P}} \in \mathfrak{h}(\mathbb{R}^n)_{\mathscr{P}} \text{ such that } \widehat{X_{\mathbb{R}^{n-1}\times\mathscr{P}}} = \widehat{Y_{\mathscr{P}}} ,$$

and using Lemma 4.3.17. □

4.4 Smoothness of outputs

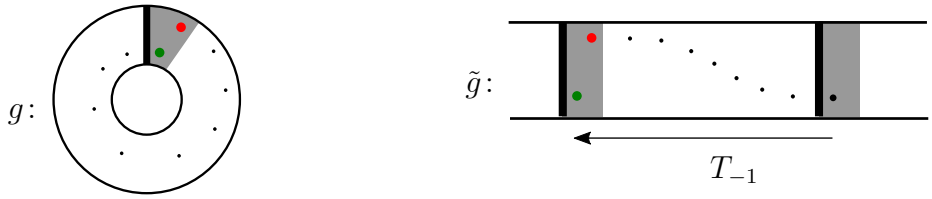
The proof of Proposition 4.1.5 stating that the output of a plugin is necessarily smooth is similar to the classical renormalization performed by Douady-Ghys [Dou87, Ghy84],Yoccoz [Yoc95a] and Shilnikov-Turaev [ST00].

Proof of Proposition 4.1.5. Let g be a plugin with step σ. Let $\pi : \tilde{\mathbb{T}} \times M := \mathbb{R} \times M \to \mathbb{T} \times M$ be the canonical cover. Let $\tilde{g}$ be a lift of g such that $\tilde{g}(0, y) = (\sigma, y)$ for every $y \in M$.

Fact 4.4.1. *The action* $\phi : (k, z) \in \mathbb{Z} \times \tilde{\mathbb{T}} \times M \mapsto \tilde{g}^k(z) \in \tilde{\mathbb{T}} \times M$ *is free, proper and discontinuous.*

Proof. The action is free since no point of $\Delta + \mathbb{Z}$ is fixed by $\tilde{g}$ nor in its complement (every point must come back to $\Delta + \mathbb{Z}$). It is discontinuous since any $x \in \mathbb{R} \times M$ has its orbit which equals the one of a certain $z \in [k, k + \sigma) \times M$ for some $k \in \mathbb{Z}$ by Definition 4.1.1.(iii), and the orbit of z is discrete by Definition 4.1.1.(i). Finally the action is proper since τ is bounded by some $N \geq 1$ by Definition 4.1.1.(ii), and so any $\tilde{g}^{2N}(\theta, y)$ has its $\mathbb{R}$-coordinate greater than $\theta + 1$. □

Thus the quotient $C := \tilde{\mathbb{T}} \times M/\phi$ is a manifold. As $\tilde{g}$ sends the left hand side of Δ to its right hand side, the image of Δ by the group action is both open and closed, hence equal to the connected set C. Therefore, $\Delta_\sigma = [0, \sigma) \times M$ is a fundamental domain of this group action. Also the rescaling map $H_\sigma : \Delta_\sigma \to \mathbb{T} \times M$ induces a diffeomorphism between C and $\mathbb{T} \times M$.



Now observe that $\tilde{T} : (\theta, y) \mapsto (\theta + 1, y)$ and $\tilde{g}$ commutes: $\tilde{T} \circ \tilde{g} = \tilde{g} \circ \tilde{T}$. Thus $\tilde{T}$ defines a smooth diffeomorphism T on C. To determine T, we (abusively) identify Δ_σ to a subset of both $\tilde{\mathbb{T}} \times M$ and $\mathbb{T} \times M$. Given $x \in \Delta_\sigma$, the point $\tilde{T}(x) \in [1, 1+\sigma) \times M$ is equivalent to the point $y \in \Delta_\sigma$ such that there exists $k \geq 0$ satisfying $\tilde{g}^k(y) = \tilde{T}(x)$. Note that $k = \tau(y)$ and so $\tilde{T}(x) = \tilde{g}^\tau(y)$. Composing by π, we obtain that $x = g^\tau(y)$. Hence T is equal to the inverse of the action of g^τ on C. Therefore G is a diffeomorphism. Observe that this construction depends continuously on g and so the output depends continuously on the plugin. As the space of plugins is connected, it comes that the space of outputs is connected to id by Example 4.1.4.

□

4.5 Lie algebras associated to closed subgroups

This section is dedicated to show Proposition 4.2.16 which states that $\mathfrak{plug}$ is a closed Lie algebra. We prove this using general arguments on closed subgroups of $\mathrm{Diff}_c^\infty(V)$ where V is a manifold.

Proposition 4.5.1. *For every closed subgroup $\mathbb{G} \subset \mathrm{Diff}_c^\infty(V)$, the following is a closed Lie subalgebra of $\mathfrak{diff}_c^\infty(V)$:*

$$\mathfrak{g} := \{X \in \mathfrak{diff}_c^\infty(V) : \mathrm{Fl}_X^t \in \mathbb{G},\ \forall t \in \mathbb{R}\}.$$

We immediately deduce the result of Proposition 4.2.16 by applying the latter proposition with $V = \mathbb{T} \times M$ and $\mathbb{G} = \mathrm{Plug}$.

Proof of Proposition 4.5.1. $\mathfrak{g}$ is a vector space. First note that if $X \in \mathbb{G}$ and $\lambda \in \mathbb{R}$, it holds $\lambda \cdot X \in \mathfrak{g}$. Now for $X, Y \in \mathfrak{g}$, for any large integer N and $r \geq 1$, observe that:

$$\mathrm{Fl}_{X+Y}^{1/N} = \mathrm{Fl}_X^{1/N} \circ \mathrm{Fl}_Y^{1/N} + O(N^{-2})$$

for the C^r-norm. Thus, Fl_{X+Y}^t is uniformly C^r-bounded for $t \in [0, 1]$, it follows:

$$Fl_{X+Y}^1 = \left(Fl_X^{1/N} \circ \mathrm{Fl}_Y^{1/N}\right)^N + O_{C^r}(N^{-1}),$$

and the supports of $\left(Fl_X^{1/N} \circ \mathrm{Fl}_Y^{1/N}\right)^N$ are included in the union of those of X and Y. Thus Fl_{X+Y}^1 is the limit of $\left(Fl_X^{1/N} \circ \mathrm{Fl}_Y^{1/N}\right)^N$ when $N \to \infty$ in the topology of $\mathrm{Diff}_c^\infty(V)$. As $\mathbb{G}$ is a group, the map $(Fl_X^{1/N} \circ \mathrm{Fl}_Y^{1/N})^N$ belongs to $\mathbb{G}$, and since $\mathbb{G}$ is closed, the map Fl_{X+Y}^1 also belongs to $\mathbb{G}$. Also for every $t \in \mathbb{R}$, by replacing X, Y by (tX, tY), we obtain that $Fl_{X+Y}^t = Fl_{tX+tY}^1$ belongs to $\mathbb{G}$.

$\mathfrak{g}$ is a Lie algebra. For $X, Y \in \mathfrak{g}$ and $r \geq 1$, we have for the C^r-norm:

$$\mathrm{Fl}_{[X,Y]}^{\tau^2} = [\mathrm{Fl}_X^\tau, \mathrm{Fl}_Y^\tau] + O(\tau^3)\ .$$

Thus by taking $\tau^2 = 1/N$ small we have:

$$\mathrm{Fl}^1_{[X,Y]} = \left([\mathrm{Fl}^{\sqrt{N}}_X, \mathrm{Fl}^{\sqrt{N}}_Y]\right)^N + O(\sqrt{N}^{-1}) \ .$$

So $\mathrm{Fl}^1_{[X,Y]}$ is in $\mathbb{G}$. Similarly, we have for any t that $\mathrm{Fl}^1_{[tX,Y]} \in \mathbb{G}$, and so $\mathrm{Fl}^t_{[X,Y]} = \mathrm{Fl}^1_{[tX,Y]} \in \mathbb{G}$.

$\underline{\mathfrak{g} \text{ is closed.}}$ As for every $t \geq 0$, the map $\mathrm{Fl}^t : X \in \mathrm{Diff}^\infty_c(V) \mapsto \mathrm{Fl}^t_X$ is continuous and $\mathbb{G}$ is closed, the set $\{X \in \mathfrak{diff}^\infty_c(V) : \mathrm{Fl}^t_X \in \mathbb{G}\}$ is closed. Thus the intersection $\mathfrak{g}$ of the latter sets for all t is closed. □

To state the parameteric counterpart of the latter proposition, given a manifold $\mathscr{P}$, we define:

$$\mathrm{Fl}^t : X_{\mathscr{P}} = (X_p)_{p\in\mathscr{P}} \in \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} \mapsto (\mathrm{Fl}^t_{X_p})_{p\in\mathscr{P}} \in \mathrm{Diff}^\infty_c(V)_{\mathscr{P}} \ .$$

Note that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathfrak{inc} & & \\ & \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} & \hookrightarrow & \mathfrak{diff}^\infty_c(V\times\mathscr{P}) & \\ \mathrm{Fl}^t & \downarrow & & \downarrow & \mathrm{Fl}^t \\ & \mathrm{Diff}^\infty_c(V)_{\mathscr{P}} & \hookrightarrow & \mathrm{Diff}^\infty_c(V\times\mathscr{P}) & \\ & & inc & & \end{array}$$

where $\mathfrak{inc}(X_{\mathscr{P}}) := \widehat{X_{\mathscr{P}}}$ and $inc(f_{\mathscr{P}}) := \widehat{f_{\mathscr{P}}}$.

Corollary D. *For every closed subgroup $\mathbb{G}_{\mathscr{P}} \subset \mathrm{Diff}^\infty_c(V)_{\mathscr{P}}$, the following is a closed Lie subalgebra of $\mathfrak{diff}^\infty_c(V)_{\mathscr{P}}$:*

$$\mathfrak{g}_{\mathscr{P}} := \{X_{\mathscr{P}} \in \mathfrak{diff}^\infty_c(V)_{\mathscr{P}} : \mathrm{Fl}^t_X \in \mathbb{G},\ \forall t \in \mathbb{R}\}.$$

Proof. First note that $inc(\mathbb{G}_{\mathscr{P}})$ is a closed Lie sub-group of $\mathrm{Diff}^\infty_c(V\times\mathscr{P})$. Hence it defines via Proposition 4.5.1 a closed Lie algebra $\widehat{\mathfrak{g}_{\mathscr{P}}}$. By commutativity of the diagram, we have:

$$\mathfrak{inc}(\mathfrak{g}_{\mathscr{P}}) := (\widehat{\mathfrak{g}_{\mathscr{P}}}) \ .$$

Hence $\mathfrak{g}_{\mathscr{P}}$ is a closed Lie subalgebra of $\mathfrak{diff}^\infty_c(V)_{\mathscr{P}}$. □

Chapter 5

Birth of families of totally parabolic points

5.1 Introduction and statements of the results

The background and motivation linking typicality in the sense of Komogorov with Baire genericity are fully explained in Section 2.5 Page 32. Turaev has announced that if a diffeomorphism displays a periodic point with Jacobian equal to the identity then an arbitrarily small perturbation allows us to create a periodic spot in C^∞-regularity. This together with theorems from the previous chapter [BGH24] shows that small perturbations creating totally periodic points lead to complex dynamical behaviors. In that direction we show a parametric counterpart Theorem B of a result of [GMO06].

Theorem A (Gochenko-Meiss-Ovsyannikov)**.** *Let f be a smooth diffeomorphism of 3-manifold M. Assume that there exists a hyperbolic saddle periodic point Ω with one complex eigenvalue and Jacobian determinant equal to 1 that displays a homoclinic tangency at some point H. Then for every $\epsilon > 0$, there exists an ϵ-small C^∞-perturbation $\tilde{f}$ of f that coincides with f outside $B(H,\epsilon) \cup B(\Omega,\epsilon)$ and a periodic point $Q \in B(H,\epsilon)$ with multipliers $\{1, j, j^2\}$ where $j = e^{2i\pi/3}$.*

Here is the parametric version of this result:

Theorem B (Main)**.** *Let $(f_a)_{a\in\mathbb{A}}$ be a C^∞-family of smooth diffeomorphisms of a 3-manifold M parameterized by a compact connected smooth manifold $\mathbb{A}$. Assume that there exist an angle $\theta \in \mathbb{R}/$*

$2\pi\mathbb{Z}\backslash\{0\}$, *a smooth family of hyperbolically continued periodic points*[1] $(\Omega_a)_{a\in\mathbb{A}}$ *and a smooth family of points* $(H_a)_{a\in\mathbb{A}}$ *such that for every* $a \in \mathbb{A}$ *it holds:*

1. Ω_a *is a saddle periodic point of* f_a *with one complex multiplier with argument* θ *and Jacobian equal to* 1,
2. H_a *is a point of homoclinic tangency for* Ω_a.

Then, for every $\epsilon > 0$, *there exists an* ϵ*-small* C^∞*-perturbation* $(\tilde{f}_a)_{a\in\mathbb{A}}$ *of* $(f_a)_{a\in\mathbb{A}}$ *and a smooth family of points* $(Q_a)_{a\in\mathbb{A}}$ *such that for every* $a \in A$ *it holds:*

1. Q_a *lies in the* ϵ*-neighborhood of* H_a,
2. Q_a *is periodic for* $\tilde{f}_a$ *with multipliers* $\{1, j, j^2\}$,
3. $\tilde{f}_a$ *coincides with* f_a *outside* $B(H_a, \epsilon) \cup B(\Omega_a, \epsilon)$.

Actually Gochenko-Meiss-Ovsyannikov gave the normal form (3-dimensional Hénon-like map) for the renormalization nearby this homoclinic tangency; from this Theorem A can be deduced. It seems possible to extend their bounds on derivatives to parameter families in order to obtain Theorem B. Instead we preferred providing a geometric proof of Theorem A, which passes easily to the parametric case (without any extra computation).

As a preparation for the next step toward Conjecture 2.5.5 Page 33, we show in Section 5.6 that the hypotheses of Theorem A are locally dense in the space of diffeomorphisms using Bonatti-Díaz's Horseshoe. This provides the following corollary:

Corollary A. *For every* $1 \leq r \leq \infty$, *there exists a locally dense set of* C^r*-diffeomorphisms of* M *formed by maps displaying a periodic point with multiplier equal to* 1, j *and* j^2.

For $r = 1$, this corollary is the main result of Bonatti-Díaz [BDP03]. For $r = \infty$, this result is also a consequence of a theorem of Gonchenko-Shilnikov-Turaev [GST09].

5.2 Preliminaries and notations

Involved spaces and their topology Let M be a three dimensional smooth manifold and denote $\mathrm{Diff}^\infty(M)$ the set of C^∞-diffeomorphisms on M. The space $\mathrm{Diff}^\infty(M)$ is endowed with the Whitney C^∞-topology. We denote by TM the tangent bundle and by T^1M the unit tangent bundle. For a real vector space E and a non trivial vector $v \in E$ we will denote $[v] = \mathbb{R}\cdot v$ the line in E containing v and $[v) = \mathbb{R}^+ \cdot v$ the half line directed by v in E.

Dynamical setting Consider a diffeomorphism $f \in \mathrm{Diff}^\infty(M)$. A fixed point Ω of f is said to be *hyperbolic* if there exists a Df-invariant splitting $E^s \oplus E^u = T_\Omega M$ such that:

$$\|D_\Omega f_{|E^s}\| < 1 \quad \text{and} \quad \|(D_\Omega f_{|E^u})^{-1}\| < 1 \ .$$

It is a sink if $E^u = \{0\}$, a source if $E^s = \{0\}$ and a saddle otherwise. Then the *index* of Ω is the dimension of E^u.

1. see Definition 5.2.2

The *stable and unstable manifolds* of the hyperbolic fixed point Ω are defined by:

$$W^s(\Omega; f) := \{Q \in M : f^n(Q) \xrightarrow[n\to+\infty]{} \Omega\} \quad \text{and} \quad W^u(\Omega; f) := \{Q \in M : f^{-n}(Q) \xrightarrow[n\to+\infty]{} \Omega\} .$$

We recall that the stable and unstable manifolds are sub-manifolds of the same class of regularity as the diffeomorphism f, and that moreover the stable space E^s (resp. the unstable space E^u) is tangent to $W^s(\Omega; f)$ (resp. $W^u(\Omega; f)$) at Ω. Moreover, given a neighborhood $V \subset M$ of Ω, if the connected components of $W^{s/u}(\Omega; f) \cap V$ containing Ω are embedded manifolds. They are called *local stable and unstable manifolds* and denoted $W^{s/u}_{\text{loc}}(\Omega; f; V)$. For a periodic point P, let us denote $\text{per}(P)$ its period. We say that P is hyperbolic for f if it is for $f^{\text{per}(P)}$. Thus we can define the (local) stable and unstable manifolds of P for f to be the one for $f^{\text{per}(P)}$. Moreover, let us denote:

$$\text{Jac}\,(P) := \text{Jac}\,(P; f) := Df^{\text{per}(P)}(P) \quad \text{and} \quad |\text{Jac}\,(P)| := |\text{Jac}\,(P; f)| := |\det(Df^{\text{per}(P)}(P))| .$$

Then, the eigenvalues of $\text{Jac}\,(P; f)$ will be called the *multipliers* of P. A hyperbolic periodic point P displays a *homoclinic tangency* if its stable and unstable manifolds have a tangent intersection.

A key tool in the construction of the proof of Theorem A is the following:

Lemma 5.2.1 (Inclination lemma)**.** *Let $V \subset M$ be an open neighborhood of Ω such that $W^u_{loc}(\Omega; f; V)$ is a well-defined local unstable manifold. Let Γ be a small disk of dimension $\dim E^u$ transverse to $W^s(\Omega; f)$ at a point H. Denote Γ_n the connected component of $f^n(\Gamma) \cap V$ containing $f^n(H)$. Then the disk Γ_n is C^∞-close to $W^u_{loc}(\Omega; f; V)$ when n is large.*

Topologies on the spaces of smooth maps and their parameter families

We endow the three dimensional manifold M with a Riemannian metric g. For a compact subset K of M, let us consider the distance:

$$d_{K,C^\infty}(f, f') = \sup_{P\in K} g(f(P), f'(P)) + \sum_{k\geq 1} 2^{-k} \min(1, \|D^k_P f - D^k_P f'\|)$$

for $f, f' \in C^\infty(M, M)$. Then the following sets form a basis of the Whitney topology on $C^\infty(M, M)$:

$$B(f, K, \epsilon) := \{f' \in C^\infty(M, M) : d_{K,C^\infty}(f, f') \leq \epsilon\} ,$$

for every $f \in C^\infty(M, M)$, for every compact subset K of M and for every $\epsilon > 0$. The space $\text{Diff}^\infty(M)$ of smooth diffeomorphisms is endowed with the induced Whitney topology.

Parametric setting We now consider a compact manifold of parameters $\mathbb{A}$. Let us denote the space of C^∞-families of smooth endomorphisms on M parameterized by $\mathbb{A}$ by:

$$C^\infty(M)_{\mathbb{A}} := \{(f_a)_{a\in\mathbb{A}} : (a, P) \in \mathbb{A} \times M \mapsto (a, f_a(P)) \in C^\infty(\mathbb{A} \times M, \mathbb{A} \times M)\} .$$

Similarly we denote $\text{Diff}^\infty(M)_{\mathbb{A}}$ the space of smooth parameter families of smooth diffeomorphisms. These spaces are endowed with the topology induced by $C^\infty(\mathbb{A} \times M, \mathbb{A} \times M)$. For such a parameter family we will denote $f_{\mathbb{A}} := (f_a)_{a\in\mathbb{A}}$. Also a family of sub-manifolds $(N_a)_{a\in\mathbb{A}}$ is C^∞-smooth if it is

the image by a C^∞-family of smooth embeddings. The topology on the space of smooth families of embeddings induces the one on smooth families of sub-manifolds.

Recall that for every diffeomorphism $f \in \mathrm{Diff}^\infty(M)$ displaying a periodic point Ω with period ω, there exists a neighborhood $\mathcal{V}_f$ of f such that there exists a continuous operator:

$$g \in \mathcal{V}_f \mapsto \Omega_g \in M \ ,$$

where Ω_g is a periodic point of g with period ω. The point Ω_g is called the *hyperbolic continuation* of Ω at g. Thus to lighten the statement of the parametric counterpart Theorem B of Theorem A, we introduce the following definition:

Definition 5.2.2 (Smooth family of hyperbolically continued points)**.** *Let $f_\mathbb{A} = (f_a)_{a\in\mathbb{A}} \in \mathrm{Diff}^\infty(M)_\mathbb{A}$ be a C^∞-family of smooth diffeomorphisms. A family $\Omega_\mathbb{A}$ of points is said to be a smooth* family of hyperbolically continued points *if for every $a \in \mathbb{A}$, the point Ω_a is hyperbolic for f_a and for every $a_0 \in \mathbb{A}$, it is the hyperbolic continuation of Ω_{a_0}.*

And here is the parametric counterpart of Lemma 5.2.1 to prove Theorem B:

Lemma 5.2.3 (Para-inclination lemma [Ber16])**.** *Let $(\Omega_a)_{a\in\mathbb{A}}$ be a smooth family of hyperbolic fixed points of f_a for every $a \in \mathbb{A}$. Let $(V_a)_{a\in\mathbb{A}}$ be a continuous family of open neighborhoods of $(\Omega_a)_{a\in\mathbb{A}}$ such that $(W^u_{loc}(\Omega_a; f_a; V_a))_{a\in\mathbb{A}}$ is a well-defined smooth family of local unstable manifolds. Let $(\Gamma_a)_{a\in\mathbb{A}}$ be a C^∞-family of smooth embedded disks of dimension $\dim E^u$ such that each Γ_a is transverse to $W^s(\Omega_a; f_a)$ at some point H_a depending smoothly on $a \in \mathbb{A}$. Denote Γ^n_a the connected component of $f^n_a(\Gamma_a) \cap V_a$ containing $f^n_a(H_a)$. Then the family of disks $(\Gamma^n_a)_{a\in\mathbb{A}}$ is smooth and C^∞-close to the family $(W^u_{loc}(\Omega_a; f_a; V_a))_{a\in\mathbb{A}}$ when n is large.*

5.3 Protectively totally parabolic points nearby homoclinic tangency: non parametric version

As a warm-up we propose a geometric proof of the following:

Proposition 5.3.1. *Let $f \in \mathrm{Diff}^\infty(M)$ be a diffeomorphism having a hyperbolic periodic point Ω displaying a homoclinic tangency at some point H. Assume that $\mathrm{Spec}(D_\Omega f) = \{\sigma, \lambda \cdot e^{\pm i\theta}\}$ with $0 < |\sigma| < 1 < \lambda$ and $\theta \in (\mathbb{R}\backslash\mathbb{Q})/2\pi\mathbb{Z}$. For every neighborhood $\mathcal{V} \subset \mathrm{Diff}^\infty(M)$ of f, for every neighborhood $U \subset M$ of H and for every neighborhood $\hat{U}$ of the closure of the orbit of H, there exist $\tilde{f} \in \mathcal{V}$ that coincides with f outside of U and a periodic point $Q \in U$ of $\tilde{f}$ whose orbit is in $\hat{U}$ and with multipliers:*

$$\mathrm{Spec}(D_Q\tilde{f}^n) = \det(D_Q\tilde{f}^n) \cdot \{1, j, j^2\} \ ,$$

where n is the period of Q.

Observe that it sufficient to prove the latter proposition for U and $\hat{U}$ small.

Proof. The proof of this proposition is deduced from the three following Lemmas 5.3.2 to 5.3.4 shown below.

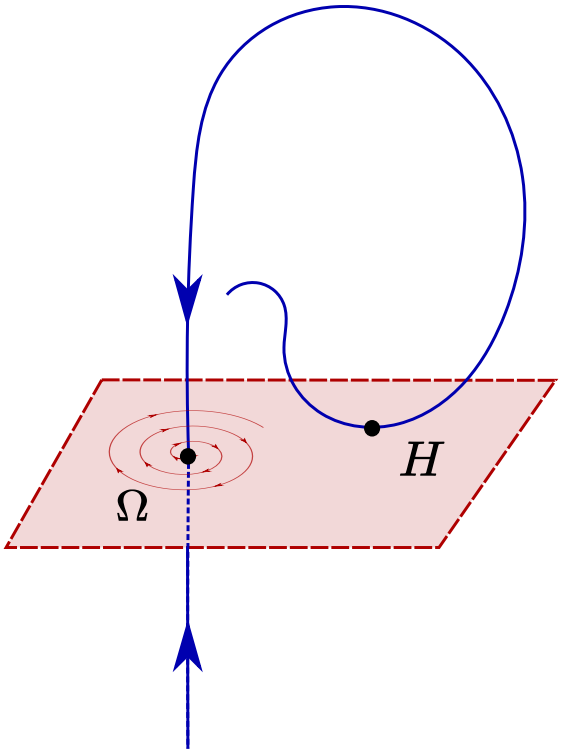


Figure 5.1 – A three dimensional homoclinic tangency of index 2

Lemma 5.3.2. *There exists a neighborhood U_0 of H such that for every large integer n, there exists a point Q close to H such that $f^n(Q)$ is close to H and for any $0 < k < n$ it holds $f^k(Q) \notin U_0$ but $f^k(Q) \in \hat{U}$.*

Without any loss of generality we can assume that $U \subset U_0$. Then for every large n, this lemma defines a point Q so that $\bar{Q} := f^n(Q)$ is close to Q. In Lemma 5.3.7, we will state a stronger version of Lemma 5.3.2 that allows us to choose Q such that the following holds:

Lemma 5.3.3. *There exists $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that Q satisfies moreover the following property.*

For every n large such that $n\theta$ is close to α, there exists a linear map $\ell : T_{\bar{Q}}M \to T_QM$ close to identity[2] *such that:*

$$\mathrm{Spec}(\ell \circ D_Q f^n) = \det(\ell \circ D_Q f^n) \cdot \{1, j, \bar{j}\} \; . \qquad (\star)$$

The following lemma allows us to realize the map $\ell : T_{\bar{Q}}M \to T_QM$ as the differential of a diffeomorphism close to identity supported by U.

Lemma 5.3.4. *For every $Q, \bar{Q} \in M$ close to H and for any linear map $\ell : T_{\bar{Q}}M \to T_QM$ close to identity, there exists a C^∞- small perturbation $\xi \in \mathrm{Diff}^\infty(M)$ of the identity supported by U such that:*

$$\xi(\bar{Q}) = Q \quad \textit{and} \quad D_{\bar{Q}}\xi = \ell \; .$$

Let us show how these lemmas allow us to conclude the proof of the proposition. Let $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ be given by Lemma 5.3.3. Let n be a large integer such that $n \cdot \theta$ is close to α. Thus Lemma 5.3.2 provides Q and $\bar{Q} = f^n(Q)$ close to H. Then Lemma 5.3.3 provides a linear map $\ell : T_{\bar{Q}}M \to T_QM$

2. This means that it is close to the inclusion $T_{\bar{Q}}M \hookrightarrow TM$.

close to identity that verifies Eq. ($\star$). Now Lemma 5.3.4 provides a diffeomorphism $\xi \in \mathrm{Diff}^\infty(M)$ close to identity and supported by U such that $\tilde{f} = \xi \circ f$ satisfies:

$$\tilde{f}^n(Q) = \xi \circ f^n(Q) = \xi(\bar{Q}) = Q$$

and

$$\mathrm{Spec}(D_Q \tilde{f}^n) = \mathrm{Spec}(l \circ D_Q f^n) = \det(l \circ D_Q f^n) \cdot \{1, j, \bar{j}\} = \det(D_Q \tilde{f}^n) \cdot \{1, j, \bar{j}\} \ .$$

This concludes the proof of Proposition 5.3.1. □

A geometric approach to the proof of Proposition 5.3.1

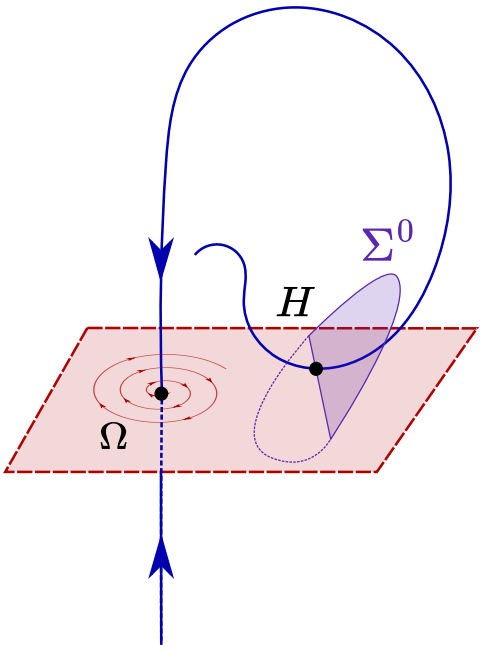


Figure 5.2 – Construction of Σ^0

We first deduce the proofs of Lemmas 5.3.2 and 5.3.3 by a common geometric construction.

By the classical result of Fact 5.8.1, up to replacing f by some iterate, we can assume without any loss of generality that Ω is a fixed point. Indeed, it is shown in Fact 5.8.1 that any small perturbation of $f^{\mathrm{per}}(\Omega)$ nearby Ω can be realized by a small perturbation of f nearby Ω.

Let V be an open neighborhood of Ω such that $W^{s/u}_{\mathrm{loc}}(\Omega; f, V)$ are embedded manifolds containing H. We now denote $W^{s/u} := W^{s/u}(\Omega; f)$ and $W^{s/u}_{\mathrm{loc}} := W^{s/u}_{\mathrm{loc}}(\Omega; f; V)$. Let Σ^0 be a small disk contained in U and intersecting W^s_{loc} transversely at H. Taking Σ^0 small ensures that W^s_{loc} and Σ^0 intersect only once. By the inclination lemma 5.2.1 there exists a sequence $(\Sigma^n)_{n\geq 1}$ of disks in M such that the following properties hold true:

1. $\Sigma^{n+1} \subset f(\Sigma^n)$ for any $n \geq 0$,
2. $f^{-n}(\Sigma^n)$ is close to $\{H\}$ as n goes to $+\infty$,

3. Σ^n is C^∞-close to W^u_{loc} as n goes to $+\infty$.
4. f^{-1} is contracting on Σ^n for $n \geq 0$.

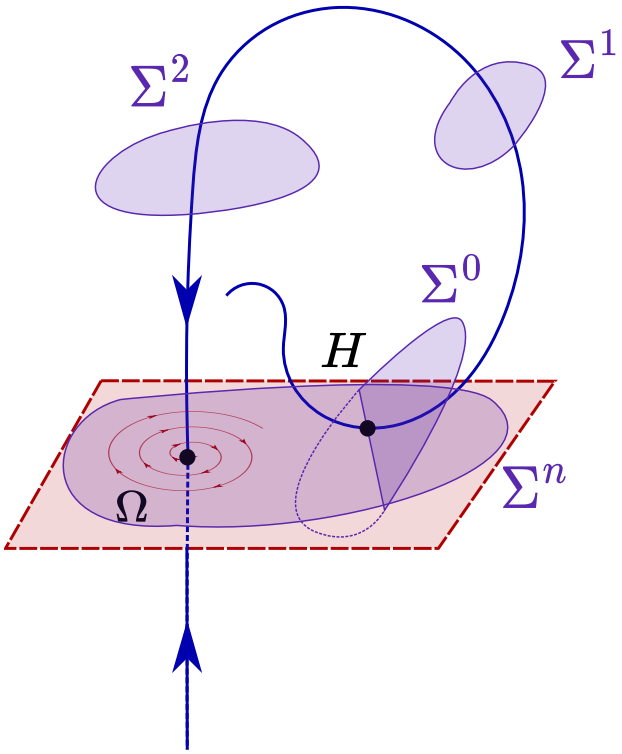


Figure 5.3 – Construction of Σ^n

For $\alpha \in 2\pi\mathbb{R}/\mathbb{Z}$, let us denote R_α the rotation of angle α of $\mathbb{R}^2$. Since $D_\Omega f$ has spectrum $\{\sigma, \lambda{\cdot}e^{\pm i\theta}\}$, its restriction to the unstable tangent space is then conjugate to the linear map $L := \lambda{\cdot}R_\theta$ on $\mathbb{R}^2$.Moreover, the disk Σ^0 can be chosen such that the following holds:

Lemma 5.3.5. *For any $n \geq 0$ there exists a C^∞-smooth map $\varphi^n : \Sigma^n \hookrightarrow \mathbb{R}^2$ such that:*

1. *$\varphi^n \circ f^n = L^n \circ \varphi^0$ on $f^{-n}(\Sigma^n)$ for any $n \geq 0$,*
2. *φ^n is C^∞-converging to some map $\varphi^\infty : W^u_{loc} \to \mathbb{R}^2$.*

To be more precise on 2. in the latter lemma, recall that the sequence Σ^n is C^∞ -converging to W^u. So φ^n to be C^∞-converging means that the map φ defined on $\cup_{n\geq 0}\Sigma^n$ by $\varphi_{|\Sigma^n} = \varphi^n$ extends smoothly to W^u and φ_{W^u} is the limit φ^∞. This lemma is proved in Section 5.7 using Shilnikov coordinates [GST08].

Since Σ^0 intersects W^s_{loc} transversely at H then $T_H W^u \cap T_H \Sigma^0$ is a line. Hence, see Fig. 5.4, there exists $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that:

$$D_H\varphi^\infty(T_H W^s) := R_\alpha \left(D_H\varphi^0(T_H W^u \cap T_H \Sigma^0) \right) . \tag{5.3.1}$$

Let $\Delta^0 \subset \Sigma^0 \cap U$ be a small curve containing H and such that it holds:

$$D_H\varphi^\infty(T_H W^u \cap T_H \Sigma^0) = R_\alpha \cdot D_H\varphi^0(T_H \Delta^0) . \tag{5.3.2}$$

Recall that $T_H\Sigma^0 \cap T_H W^u \neq T_H W^s$ since Σ^0 is transverse to W^s at H. Thus by Eq. (5.3.1), it holds:

$$R_\alpha \cdot D_H\varphi^0(T_H \Delta^0) = D_H\varphi^\infty(T_H W^u \cap T_H \Sigma^0) \neq D_H\varphi^\infty(T_H W^s) .$$

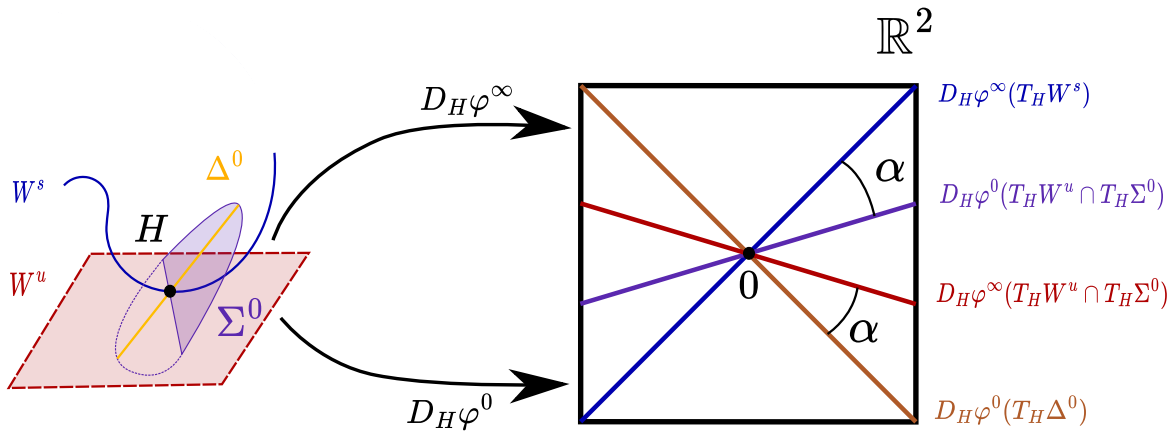


Figure 5.4 – Construction of Δ^0 and α

Now by Eq. (5.3.1) the latter term is equal to $R_\alpha \cdot D_H\varphi^0(T_HW^u \cap T_H\Sigma^0)$. Thus we have:

$$T_H\Delta^0 \neq T_H\Sigma^0 \cap T_HW^u \ .$$

Consequently we have just shown the following:

Fact 5.3.6. *Δ^0 intersects W^s_{loc} transversely at H.*

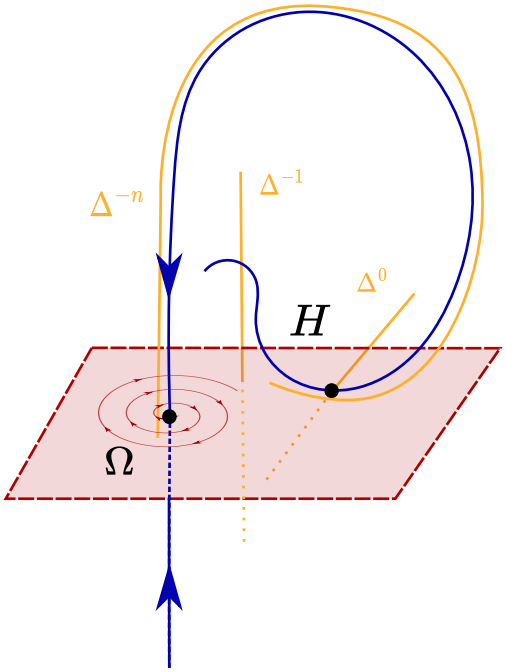


Figure 5.5 – Construction of Δ^{-n}

By the above fact and still by the inclination lemma 5.2.1, there exists a sequence $(\Delta^{-n})_{n\geq 1}$ of embedded curves in M such that the following properties hold true:

1. $f(\Delta^{-(n+1)}) \subset \Delta^{-n}$ for any $n \geq 0$,
2. $f^n(\Delta^{-n})$ is close to $\{H\}$ as n goes to $+\infty$,
3. Δ^{-n} is C^∞-close to W^s_{loc} as n goes to $+\infty$,

4. f is contracting on Δ^{-n} for every $n \geq 0$.

Using this setting, we can prove the following stronger version of Lemma 5.3.2:

Lemma 5.3.7. *For n sufficiently large, there exists a unique point $Q \in \Sigma^0 \cap \Delta^{-n}$. It is close to H and so is $\bar{Q} := f^n(Q) \in \Delta^0$. Moreover, for every $\epsilon > 0$, there exists an integer m such that for any n sufficiently large, it holds:*

1. *$f^l(Q)$ is ϵ-close to $f^l(H)$ for $0 \leq l \leq m$,*
2. *$f^{n-l}(Q)$ is ϵ-close to $f^{-l}(H)$ for $0 \leq l \leq n - m$.*

In particular when ϵ is small, the n first iterates of Q by f lie in $\hat{U}$.

Proof. Since $(\Delta^{-n})_{n\in\mathbb{N}}$ converges to $W^s_{\text{loc}} \ni H$ as n goes to infinity, then for n large, Δ^{-n} intersects Σ^0 uniquely at a point $Q \in \Sigma^0 \cap \Delta^{-n}$. Since Δ^{-n} is close to W^s_{loc} and $\Sigma^0 \cap W^s_{\text{loc}} = \{H\}$ it follows that Q is close to H as n goes to infinity. Observe then that $\bar{Q} := f^n(Q)$ lies in Δ^0 since $Q \in \Delta^{-n}$. Now remark that, by continuity of f, for any fixed integer m and for any n sufficiently large, it holds that $f^l(Q)$ is ϵ-close to $f^l(H)$ for $0 \leq l \leq m$. Furthermore, since f is contracting on W^s_{loc} with fixed point Ω and Δ^{-k} is close to W^s_{loc} when k is large, there exists an integer m such that for any $n - l \geq m$, the map:

$$f^{n-l}_{|\Delta^{-n}} : \Delta^{-n} \hookrightarrow \Delta^{-l} ,$$

is contracting with a ratio smaller than $\epsilon \cdot \operatorname{diam}(M)^{-1}$. Thus for $n - l \geq m$, the point $f^{n-l}(Q)$ is ϵ-close to $f^{n-l}(f^{-n}(H))) = f^{-l}(H)$ for any $0 \leq l \leq n - m$. In particular, with $l = 0$ we indeed have that $\bar{Q}$ is close to H. □

For the proof of Proposition 5.3.1 to be complete, it remains to show:

Proof of Lemma 5.3.3. Let (u, v, w) be a basis of vectors of $T_H M$ as shown in Fig. 5.5 such that:

$$u \in T_H\Delta^0, \quad v \in T_H W^u \cap T_H\Sigma^0 \quad \text{and} \quad w \in T_H W^s .$$

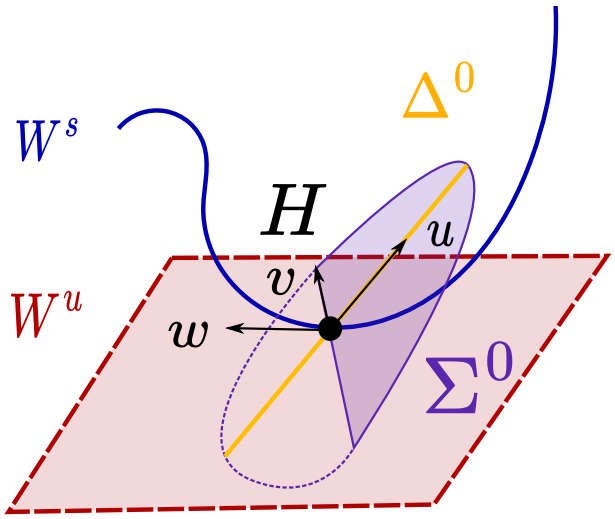


Figure 5.6 – The basis (u, v, w) at H

It is clear that such a basis exists as $T_H\Delta^0$ is transverse to T_HW^u and $T_H\Sigma^0$ is transverse to T_HW^s. Recall that:

$$D_H\varphi^\infty(T_HW^s) = R_\alpha(D_H\varphi^0(T_HW^u \cap T_H\Sigma^0))$$

and

$$D_H\varphi^\infty(T_HW^u \cap T_H\Sigma^0) = R_\alpha(D_H\varphi^0(T_H\Delta^0)) \ ,$$

thus up to multiplying u, v and w by scalars we can assume that $D_H\varphi^\infty(w)$ is a unit vector and moreover that u and v verify:

$$D_H\varphi^\infty(w) = R_\alpha(D_H\varphi^0(v)) \quad \text{and} \quad D_H\varphi^\infty(v) = R_\alpha(D_H\varphi^0(u)) \ . \tag{5.3.3}$$

We moreover impose that $\det(D_H\varphi^\infty(u), D_H\varphi^\infty(v), D_H\varphi^\infty(w)) > 0$.

We also consider the following basis of T_QM shown on Fig. 5.7:

1. $u^n := (D_Q\varphi^0)^{-1} \cdot R_{-\alpha} \cdot D_H\varphi^\infty(v)$,
2. $v^n := (D_Q\varphi^0)^{-1} \cdot R_{-\alpha} \cdot D_H\varphi^\infty(w)$,
3. $w^n \in T_Q\Delta^{-n}$ such that $\|D_Q\varphi^n(w^n)\| = 1$ and $\langle D_Q\varphi^n(w^n), D_H\varphi^\infty(w)\rangle > 0$.

Note that we have:

Fact 5.3.8. *When n is large, the basis (u^n, v^n, w^n) of T_QM is uniquely well defined and close to (u, v, w) in TM.*

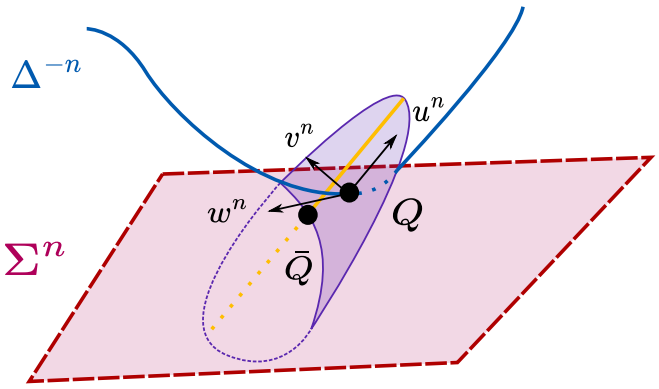


Figure 5.7 – The basis (u^n, v^n, w^n) of T_QM

Proof. Since Q is close to H as n is large it holds that u^n and v^n are close to $D_H\varphi^0(v) \cdot R_{-\alpha} \cdot D_H\varphi^\infty(v)$ and $D_H\varphi^0 \cdot R_{-\alpha} \cdot D_H\varphi^\infty(w)$; which are equal to u and v by Eq. (5.3.3). Now as $(\Delta^{-n})_{n\geq 0}$ converges to W^s_{loc} as n goes to infinity, it holds that $T_Q\Delta^{-n}$ is close to T_HW^s when n is large and thus $w^n \in \Delta^{-n}$ is well defined and close to $w \in T_HW^s$. □

Recall that for a vector $\vec{v}$ the half line and the line directed by $\vec{v}$ are denoted by $[\vec{v})$ and $[\vec{v}]$. We now prove:

Sublemma 5.3.9. *For every large value of n such that $n \cdot \theta$ is close to α, it holds:*

$$D_Qf^n \cdot [w^n] \text{ is close to } [u], \quad D_Qf^n \cdot [u^n) \text{ is close to } [v) \quad \text{and} \quad D_Qf^n \cdot [v^n) \text{ is close to } [w) \ . \tag{5.3.4}$$

Proof. Obviously, by construction, it holds that $D_Q f^n \cdot [w^n] = D_Q f^n(T_Q \Delta^{-n}) = T_{\bar{Q}} \Delta^0$ which is close to $T_H \Delta^0 = [u]$ as $\bar{Q}$ and H are close to each other. Now since $u^n \in T_Q \Sigma^0$ and since $L = \lambda \cdot R_\theta$ with $\lambda > 0$, it holds:

$$D_{\bar{Q}} \varphi^n \cdot D_Q f^n \cdot [u^n) = L^n \cdot D_Q \varphi^0 [u^n) = R_{n \cdot \theta} \cdot D_Q \varphi^0 \cdot [u^n) \ .$$

Then it follows:

$$D_{\bar{Q}} \varphi^n \cdot D_Q f^n \cdot [u^n) = R_{n \cdot \theta - \alpha} \cdot D_Q \varphi^\infty \cdot [v) \ ,$$

which is close to $D_H \varphi^\infty \cdot [v)$ as n is large and $n \cdot \theta$ is close to α. Since $\bar{Q}$ is close to H as well, it follows that $D_Q f^{\cdot}[u^n)$ is close to $[v)$.

Similarly, v^n also lies in $T_Q \Sigma^0$ and thus $D_{\bar{Q}} \varphi^n \cdot D_Q f^n \cdot [v^n) = L^n \cdot D_Q \varphi^0 [v^n) = R_{n \cdot \theta} \cdot D_Q \varphi^0 [v^n)$. Thus it holds:

$$D_{\bar{Q}} \varphi^n \cdot D_Q f^n \cdot [v^n) = R_{n \cdot \theta - \alpha} \cdot D_Q \varphi^\infty [w) \ ,$$

which is close to $D_H \varphi^\infty \cdot [w)$ and consequently $D_Q f \cdot [v^n)$ is close to $[w)$, which allows us to conclude the proof of the sub-lemma. □

By the above discussion and results, there exists a basis:

$$(\bar{u}^n, \bar{v}^n, \bar{w}^n) \subset D_Q f^n \cdot [w^n) \times D_Q f^n \cdot [u^n) \times D_Q f^n \cdot [v^n)$$

of $T_{\bar{Q}} M$ close to (u, v, w). Let then $\ell : T_{\bar{Q}} M \to T_Q M$ be the linear map such that:

$$\ell(\bar{u}^n) = u^n, \quad \ell(\bar{v}^n) = v^n \quad \text{and} \quad \ell(\bar{w}^n) = w^n \ .$$

By construction, ℓ is close to identity and moreover it holds:

$$D_Q f^n \cdot \ell[\bar{u}^n) = [\bar{v}^n), \quad D_Q f^n \cdot \ell[\bar{v}^n) = [\bar{w}^n) \quad \text{and} \quad D_Q f^n \cdot \ell[\bar{w}^n] = [\bar{u}^n] \ .$$

Note then that $D_Q f^n \cdot \ell \cdot [\bar{w}^n) = [\bar{u}^n)$ if f is orientation preserving and $[-\bar{u}^n)$ otherwise. It follows then that:

$$\mathrm{Spec}(\ell \cdot D_Q f^n) = \mathrm{Spec}(D_Q f^n \cdot \ell) = \det(D_Q f^n \cdot \ell) \cdot \{1, j, \bar{j}\} = \det(\ell \circ D_Q f^n) \cdot \{1, j, \bar{j}\} \ .$$

□

5.4 Parametric counterpart

We now provide the parametric counterpart of Proposition 5.3.1:

Proposition 5.4.1. *Let $f_{\mathbb{A}} = (f_a)_{a \in \mathbb{A}} \in \mathrm{Diff}^\infty(M)_{\mathbb{A}}$ be a C^∞-family of smooth diffeomorphisms. Let us assume that there exist a smooth family of hyperbolic points $\Omega_{\mathbb{A}}$ and a smooth family of points $H_{\mathbb{A}}$ in M such that for every parameter $a \in \mathbb{A}$, the point Ω_a is a hyperbolic periodic saddle displaying a homoclinic tangency at H_a. Assume moreover that there exists $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ such that for every $a \in \mathbb{A}$ it holds $\mathrm{Spec}(\mathrm{Jac}\,(\Omega_a; f_a) = \{\sigma_a, \lambda_a \cdot e^{\pm i\theta}\}$ with $0 < |\sigma_a| < 1 < \lambda_a$.*

Then for every $\epsilon > 0$, there exist an arbitrarily large integer n, an ϵ-small C^∞-perturbation $\tilde{f}_{\mathbb{A}}$ of $f_{\mathbb{A}}$, and a family of points $Q_{\mathbb{A}}$ of M such that for every parameter $a \in \mathbb{A}$, it holds that $Q_a \in B(H_a, \epsilon)$ is n-periodic and Q_a has multipliers $\det D_{Q_a} \tilde{f}_a^n \cdot \{1, j, j^2\}$.

Moreover, we have the following properties:

1. *the point* Ω_a *is periodic for* $\tilde{f}_a$ *and* $\det(D_{\Omega_a}\tilde{f}_a^{\mathrm{per}(\Omega_a)}) = \det(D_{\Omega_a}f_a^{\mathrm{per}(\Omega_a)})$,
2. *there exists an integer* N *independent of* n *and the perturbation such that for every integer* k *such that* $N \leq k \leq n - N$, *it holds* $\tilde{f}_a^k(Q_a) \in B(\Omega_a, \epsilon)$.

Proof. Similarly as for the non parametric case, since the period of Ω_a is bounded with a, we can assume by Fact 5.8.2 that Ω_a is a fixed point of f_a. The proof follows the same lines as the proof of Proposition 5.3.1. Let $V_{\mathbb{A}}$ be a C^0- family of open neighborhoods of $\Omega_{\mathbb{A}}$ such that for every $a \in \mathbb{A}$ we have that $W^{s/u}_{\mathrm{loc}}(f_a, \Omega_a, V)$ are embedded manifolds containing H_a. Up to some smooth change of coordinates depending smoothly on the parameter, we can assume that $\Omega_a = \Omega \in M$ and also $H_a = H \in M$ for every $a \in \mathbb{A}$. Moreover, we can assume that the local unstable manifolds do not depend on $a \in \mathbb{A}$ and denote $W^u_{\mathrm{loc}} := W^u_{\mathrm{loc}}(\Omega; f_a; V_a)$. Moreover we can assume that $W^s_{\mathrm{loc}}(\Omega; f_a; V_a)$ contains some smooth curve W^s_{loc} containing 0 in its interior. We will also denote $W^s_a := W^s_{\mathrm{loc}}(\Omega; f_a; V_a)$. Consequently we can write $T_\Omega M = E^u \oplus E^s$ with:

$$D_\Omega f_{a|E^s} = \sigma_a \cdot \mathrm{id}_{E^s} \quad \text{and} \quad D_\Omega f_{a|E^u} = \lambda_a \cdot R_\theta \ ,$$

where E^u is identified to $\mathbb{R}^2$. Note that in the non parametric case we introduced an angle $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ in Lemma 5.3.3. Here with parameter families, this angle will still be defined but may vary according to the parameter. The solution brought to freeze its value is to add an additional parameter β to the family that slightly changes the angle of the unstable multipliers at Ω. Let us denote $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. Let $\kappa > 0$ be a small fixed positive number. Let $\xi_{\mathbb{T}} := (\xi_\beta)_{\beta\in\mathbb{T}}$ be a C^∞-family of C^∞ diffeomorphisms of M such that $\xi_0 = \mathrm{id}_M$ and for every $\beta \in \mathbb{T}$, it holds:

1. $\xi_\beta(\Omega) = \Omega$,
2. $D_\Omega \xi_{\beta|E^s} = \mathrm{id}_{E^s}$ and $D_\Omega \xi_{\beta|E^u} = R_\beta$,
3. ξ_β coincides with the identity outside $B(\Omega, \kappa)$,
4. ξ_β leaves invariant $W^{s/u}_{\mathrm{loc}}$.

Consider then the smooth family $f_{\mathbb{A}\times\mathbb{T}} = (\xi_\beta \circ f_a)_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$. Note that for every $(a, \beta) \in \mathbb{A} \times \mathbb{T}$ it holds:

1. $f_{a,0} = f_a$,
2. $f_{a,\beta}(\Omega) = \Omega$,
3. $D_\Omega f_{a,\beta|E^s} = \sigma_a \cdot \mathrm{id}_{E^s}$ and $D_\Omega f_{a,\beta|E^u} = \lambda_a \cdot R_{\theta+\beta}$,
4. $f_{a,\beta}$ coincides with f_a outside $B(\Omega, \kappa)$,
5. $W^u_{\mathrm{loc}}(\Omega; f_{a,\beta}; V_a) = W^u_{\mathrm{loc}}$ and $W^s_{\mathrm{loc}}(\Omega; f_{a,\beta}; V_a) = W^s_a$.

In particular, since κ is small we can ensure ξ_β leaves invariant W^s_a for every $a \in \mathbb{A}$. Thus H is a point of homoclinic tangency for every $f_{a,\beta}$. Moreover, $f_{a,\beta}$ coincides with f_a nearby H.

For $(a, \beta) \in \mathbb{A} \times \mathbb{T}$ let us denote $L_{a,\beta} := \lambda \cdot R_{\theta+\beta}$. Let $\Sigma^0_{\mathbb{A}\times\mathbb{T}} := (\Sigma_{a,\beta})_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$ be a smooth family of embedded disks of M such that each $\Sigma^0_{a,\beta}$ intersects W^s_{loc} transversally at H. Then, as a direct application of the para-inclination lemma 5.2.3, we have the following:

Lemma 5.4.2. *There exists a sequence of* C^∞*-families of smooth embedded disks* $((\Sigma^n_{\mathbb{A}\times\mathbb{T}})_{n\geq 1}$ *such that it holds:*

1. *$f_{a,\beta}^{-n}(\Sigma_{a,\beta}^n)$ is close to $\{H\}$ as n goes to $+\infty$,*
2. *$\Sigma_{a,\beta}^n$ is C^∞-close to W_{loc}^u as n goes to $+\infty$,*
3. *$f_{a,\beta}(\Sigma_{a,\beta}^{n+1}) \subset \Sigma_{a,\beta}^n$,*
4. *$f_{a,\beta}^{-1}$ is contracting on $\Sigma_{a,\beta}^n$.*

Then the family $\Sigma^0_{\mathbb{A}\times\mathbb{T}}$ can be chosen such that the following parametric counterpart of Lemma 5.3.5 holds:

Lemma 5.4.3. *There exists a sequence of C^∞-families of smooth maps $\varphi^n_{\mathbb{A}\times\mathbb{T}}$ such that for every $(a,\beta) \in \mathbb{A}\times\mathbb{T}$:*

1. *$\varphi^n_{a,\beta} : \Sigma^n_{a,\beta} \hookrightarrow \mathbb{R}^2$ is a smooth embedding,*
2. *$\varphi^n_{a,\beta} \circ f^n_{a,\beta} = L^n_{a,\beta} \circ \varphi^0_{a,\beta}$ on $f^{-n}_{a,\beta}(\Sigma^n_{a,\beta})$,*
3. *the family $\varphi^n_{\mathbb{A}\times\mathbb{T}}$ is C^∞-converging to a smooth family $\varphi^\infty_{\mathbb{A}\times\mathbb{T}}$ such that each $\varphi^\infty_{a,\beta} : W^u_{loc} \hookrightarrow \mathbb{R}^2$ is a smooth embedding,*

Since $\Sigma^0_{a,\beta}$ intersects W^s_a transversely at H then $T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta}$ is a line. It follows that there exists a smooth function $\alpha \in C^\infty(\mathbb{A}\times\mathbb{T},\mathbb{T})$ that verifies Eq. (5.3.1) for anyy parameters, i.e.:

$$D_H\varphi^\infty_{a,\beta}(T_H W^s_a) := R_{\alpha(a,\beta)}\left(D_H\varphi^0_{a,\beta}(T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta})\right) . \tag{5.4.1}$$

Let us now define:

$$\Delta^0_{a,\beta} := (\varphi^0_{a,\beta})^{-1}\left(R_{\alpha(a),\beta}\cdot D_H\varphi^\infty_{a,\beta}(T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta})\right) .$$

As $T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta}$ is a line, $\Delta^0_{a,\beta}$ is an embedded curve in $\Sigma^0_{a,\beta}$. Moreover, by construction the family $\Delta^0_{\mathbb{A}\times\mathbb{T}} := (\Delta^0_{a,\beta})_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$ is C^∞-smooth.

Note that it holds:

$$D_H\varphi^\infty_{a,\beta}(T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta}) = R_{\alpha(a,\beta)}\cdot D_H\varphi^0_{a,\beta}(T_H\Delta^0_{a,\beta}) . \tag{5.4.2}$$

Then by Fact 5.3.6 applied to every parameter, it holds:

Fact 5.4.4. *For every $(a,\beta) \in \mathbb{A}\times\mathbb{T}$, the curve $\Delta^0_{a,\beta}$ intersects W^s_{loc} transversely at H.*

Still by the para-inclination Lemma 5.2.3, there exists a sequence of families $(\Delta^{-n}_{\mathbb{A},\mathbb{T}})_{n\geq 1}$, such that for every $a,\beta \in \mathbb{A}\times\mathbb{R}/2\pi\mathbb{Z}$ it holds:

1. $f_{a,\beta}(\Delta^{-(n+1)}_{a,\beta}) \subset \Delta^{-n}_{a,\beta}$ for every $n \geq 0$,
2. the family $(f^n_{a,\beta}(\Delta^{-n}_{a,\beta}))_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$ is close to $\{H\}$ as n goes to infinity,
3. the family $\Delta^{-n}_{\mathbb{A},\mathbb{T}}$ is close to W^u_{loc} as n goes to infinity,
4. $f_{a,\beta}$ is contracting on $\Delta^{-n}_{a,\beta}$ for every $n \geq 0$.

It follows that for every sufficiently large integer n and for any parameters a,β there exists a unique point $Q_{a,\beta} \in \Sigma^0_{a,\beta} \cap \Delta^{-n}_{a,\beta}$ close to H that verifies properties given by Lemma 5.3.7 for $f_{a,\beta}$. Since $\Delta^{-n}_{a,\beta}$ and $\Sigma^0_{a,\beta}$ depend smoothly on the parameter, so does $Q_{a,\beta}$. As in Eq. (5.3.3), there exists a smooth family of bases $((u_{a,\beta}, v_{a,\beta}, w_{a,\beta}))_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$ of $T_H M$ such that their direction are given by:

$$u_{a,\beta} \in T_H\Delta^0_{a,\beta}, \quad v_{a,\beta} \in T_H W^u_{\text{loc}} \cap T_H\Sigma^0_{a,\beta} \quad \text{and} \quad w_{a,\beta} \in T_H W^s_a .$$

And the norm and orientation are given by:

$$D_H\varphi^\infty_{a,\beta}(v_{a,\beta}) = R_{\alpha(a,\beta)}(D_H\varphi^0_{a,\beta}(u_{a,\beta})), \quad D_H\varphi^\infty_{a,\beta} = (w_{a,\beta}) = R_{\alpha(a,\beta)}(D_H\varphi^0_{a,\beta}(v_{a,\beta}))$$

and

$$\|D_H\varphi^\infty_{a,\beta}(w_{a,\beta})\| = 1, \quad \det(D_H\varphi^\infty_{a,\beta}(u_{a,\beta}), D_H\varphi^\infty_{a,\beta}(v_{a,\beta}), D_H\varphi^\infty_{a,\beta}(w_{a,\beta})) > 0\ .$$

Then for large values of n, there exists a smooth family $(u^n_{a,\beta}, v^n_{a,\beta}, w^n_{a,\beta})_{(a,\beta)\in\mathbb{A}\times\mathbb{T}}$ of bases such that:

1. $u^n_{a,\beta} := (D_{Q_{a,\beta}}\varphi^0_{a,\beta})^{-1} \cdot R_{-\alpha(a,\beta)} \cdot D_H\varphi^\infty_{a,\beta}(v_{a,\beta})$,
2. $v^n_{a,\beta} := (D_{Q_{a,\beta}}\varphi^0_{a,\beta})^{-1} \cdot R_{-\alpha(a,\beta)} \cdot D_H\varphi^\infty_{a,\beta}(w_{a,\beta})$,
3. $w^n_{a,\beta} \in T_{Q_{a,\beta}}\Delta^{-n}_{a,\beta}$ such that $\|D_{Q_{a,\beta}}\varphi^n_{a,\beta}(w^n_{a,\beta})\| = 1$ and $\langle D_{Q_{a,\beta}}\varphi^n_{a,\beta}(w^n), D_H\varphi^\infty_{a,\beta}(w_{a,\beta})\rangle \geq 0$.

The idea is now to have a parametric version of Sublemma 5.3.9. First, by the implicit function theorem, for n sufficiently large, there exists a smooth map $\beta^{(n)} : \mathbb{A} \to \mathbb{T}$ C^∞-close to 0 and such that:

$$n \cdot \theta + n \cdot \beta^{(n)}(a) = \alpha(a, \beta^{(n)}(a)) \tag{5.4.3}$$

for every $a \in \mathbb{A}$. Let us now go back to a family parametrized by $\mathbb{A}$ only, and to simplify notations let us denote:

1. $\hat{f}_a := f_{a,\beta^{(n)}_a}$,
2. $Q_a := Q_{a,\beta^{(n)}(a)}$ and $\bar{Q}_a := \bar{Q}_{a,\beta^{(n)}(a)}$,
3. $\varphi^0_a := \varphi^0_{a,\beta^{(n)}(a)}$ and $\varphi^\infty_a := \varphi^\infty_{a,\beta^{(n)}(a)}$,
4. $(u_a, v_a, w_a) := (u_{a,\beta^{(n)}(a)}, v_{a,\beta^{(n)}(a)}, w_{a,\beta^{(n)}(a)})$,
5. $(u^n_a, v^n_a, w^n_a) := (u^n_{a,\beta^{(n)}(a)}, v^n_{a,\beta^{(n)}(a)}, w^n_{a,\beta^{(n)}(a)})$,

As $\beta^{(n)}$ is C^∞-small for large value of n, it follows that the family $\hat{f}_\mathbb{A}$ is C^∞-close to the initial family $f_\mathbb{A}$. Moreover, the families $Q_\mathbb{A}, \bar{Q}_\mathbb{A}, u_\mathbb{A}, v_\mathbb{A}, w_\mathbb{A}, u^n_\mathbb{A}, v^n_\mathbb{A}$ and $w^n_\mathbb{A}$ are C^∞-smooth.

Then similarly as in Sublemma 5.3.9 we have:

Lemma 5.4.5. *For every sufficiently large n, it holds:*

$$(D_{Q_a}\hat{f}^n_a \cdot [w^n_a])_{a\in\mathbb{A}} \text{ is } C^\infty\text{-close to } ([u_a])_{a\in\mathbb{A}}\ , \tag{5.4.4}$$

$$(D_{Q_a}\hat{f}^n_a \cdot [u^n_a])_{a\in\mathbb{A}} \text{ is } C^\infty\text{-close to } ([v_a])_{a\in\mathbb{A}} \tag{5.4.5}$$

and

$$(D_{Q_a}\hat{f}^n_a \cdot [v^n_a])_{a\in\mathbb{A}} \text{ is } C^\infty\text{-close to } ([w_a])_{a\in\mathbb{A}} \tag{5.4.6}$$

The proof is a straightforward adaptation of the one of Sublemma 5.3.9. Before that, let us just add some further notations for n large and $a \in \mathbb{A}$:

1. $\alpha_a := \alpha(a, \beta^{(n)}(a))$,
2. $\theta_a := \theta + \beta^{(n)}(a)$,
3. $L_a := \lambda_a \cdot R_{\theta_a}$.

Then we have $\alpha_a = n \cdot \theta_a$ for every $a \in \mathbb{A}$ and the above quantities all induce C^∞-parameter families.

Proof. By construction, it holds $(D_{Q_a}\hat{f}_a^n \cdot [w_a^n])_{a\in\mathbb{A}} = (D_{Q_a} f_a^n (T_{Q_a}\Delta_a^{-n}))_{a\in\mathbb{A}} = (T_{\bar{Q}_a}\Delta_a^0)_{a\in\mathbb{A}}$ which is close to $(T_H\Delta_a^0)_{a\in\mathbb{A}} = ([u_a])_{a\in\mathbb{A}}$ as the family $\bar{Q}_{\mathbb{A}}$ is C^∞-close to the constant family equal to H. Now since for every $a \in \mathbb{A}$ it holds $u_a^n \in T_{Q_a}\Sigma_a^0$, then by Lemma 5.4.3 we have:

$$D_{\bar{Q}_a}\varphi_a^n \cdot D_{Q_a}\hat{f}_a^n \cdot [u_a^n) = L_a^n \cdot D_{Q_a}\varphi_a^0[u_a^n) = R_{n\cdot\theta_a} \cdot D_{Q_a}\varphi_a^0 \cdot [u_a^n) \ .$$

Now as $n \cdot \theta_a = \alpha_a$ and by definition of u_a^n, we have:

$$D_{\bar{Q}_a}\varphi_a^n \cdot D_{Q_a}\hat{f}_a^n \cdot [u_a^n) = D_{Q_a}\varphi_a^\infty \cdot [v_a) \ .$$

Similarly, as v_a^n lies in $T_{Q_a}\Sigma_a^0$, it holds by Lemma 5.4.3 and definition of v_n^a:

$$\begin{aligned} D_{\bar{Q}_a}\varphi_a^n \cdot D_{Q_a}\hat{f}_a^n \cdot [v_a^n) &= L_a^n \cdot D_{Q_a}\varphi_a^0[v_a^n) \\ &= R_{\alpha(a)} \cdot D_{Q_a}\varphi_a^0[v_a^n) \\ &= D_{Q_a}\varphi_a^\infty[w_a) \ . \end{aligned}$$

Recall that the families $(D_{\bar{Q}_a}\varphi_a^n)_{a\in\mathbb{A}}$and $(D_{Q_a}\varphi_a^\infty)_{a\in\mathbb{A}}$ are both C^∞-close to the family $(D_H\varphi_a^\infty)_{a\in\mathbb{A}}$ when n is large. It follows that the families $(D_{Q_a}\hat{f}_a^n \cdot [u_a^n))_{a\in\mathbb{A}}$ and $(D_{Q_a}\hat{f}_a^n \cdot [v_a^n))_{a\in\mathbb{A}}$ are indeed C^∞-close to $([v_a))_{a\in\mathbb{A}}$ and $([w_a))_{a\in\mathbb{A}}$. □

By the latter lemma, it follows that there exists a smooth family of bases $((\bar{u}_a^n, \bar{v}_a^n, \bar{w}_a^n))_{a\in\mathbb{A}}$ of $(T_{\bar{Q}_a}M)_{a\in\mathbb{A}}$ that is C^∞-close to the family $(u_{a,\beta^{(n)}(a)}, v_{a,\beta^{(n)}(a)}, w_{a,\beta^{(n)}(a)}))_{a\in\mathbb{A}}$ and such that for $a \in \mathbb{A}$ it holds:

$$(\bar{u}_a^n, \bar{v}_a^n, \bar{w}_a^n) \subset D_{Q_a}\hat{f}_a^n \cdot [w_a^n) \times D_{Q_a}\hat{f}_a^n \cdot [u_a^n) \times D_{Q_a}\hat{f}_a^n \cdot [v_a^n) \ .$$

Let then consider:

$$\ell_a : T_{\bar{Q}_a}M \to T_{Q_a}M \ ,$$

the linear map such that $\ell_a(\bar{u}_a^n) = u_a^n$, $\ell_a(\bar{v}_a^n) = v_a^n$ and $\ell(\bar{w}_a^n) = w_a^n$. By the above discussion, ℓ_a depends smoothly on the parameter $a \in \mathbb{A}$ and the family $\ell_{\mathbb{A}} := (\ell_a)_{a\in\mathbb{A}}$ is C^∞-close to the family $(\mathrm{id}_{T_{\bar{Q}_a}M})_{a\in\mathbb{A}}$. Thus we have:

$$D_{Q_a}\hat{f}_a^n \circ \ell_a[\bar{u}_a^n) = [\bar{v}_a^n), \quad D_{Q_a}\hat{f}_a^n \circ \ell_a[\bar{v}_a^n) = [\bar{w}_a^n) \quad \text{and} \quad D_{Q_a}\hat{f}_a^n \circ \ell_a[\bar{w}_a^n] = [\bar{u}_a^n] \ .$$

As in the non-parametric case, it follows:

$$\mathrm{Spec}(D_{Q_a}\hat{f}_a \circ \ell_a) = \det(D_{Q_a}\hat{f}_a^n) \cdot \{1, j, j^2\} \ .$$

Now we infer the following parametric version of Lemma 5.3.4:

Lemma 5.4.6. *For every $Q_{\mathbb{A}}, \bar{Q}_{\mathbb{A}} \in C^\infty(\mathbb{A}, M)$ close to the constant family equal to H and for any close to identity family $\ell_{\mathbb{A}}$ of linear maps $\ell_a : T_{\bar{Q}_a}M \to T_{Q_a}M$, there exists a C^∞- small perturbation $\xi_{\mathbb{A}} \in \mathrm{Diff}^\infty(M)_{\mathbb{A}}$ of the identity family such that:*

$$\xi_a(\bar{Q}_a) = Q_a \quad \textit{and} \quad D_{\bar{Q}_a}\xi_a = \ell_a \ .$$

This latter lemma is shown in Section 5.9. It provides a family $\xi_{\mathbb{A}} \in \mathrm{Diff}^\infty(M)_{\mathbb{A}}$ C^∞-close to identity such that $\tilde{f}_{\mathbb{A}} = (\xi_a \circ \hat{f}_a)_{a\in\mathbb{A}}$ satisfies the conclusion of the proposition. □

5.5 Proofs of Theorems A and B

Using Proposition 5.4.1 we are now ready to prove Theorem A and its parametric counterpart Theorem B.

Proof of Theorem A. Up to replacing f by f^{-1} we assume without any loss of generality that Ω has index 2. Moreover, by Fact 5.8.1 we can assume without any loss of generality that Ω is fixed by f. The idea is to unfold f into a one parameter family such that the determinant of the Jacobian of f at Ω varies with the parameter. Then we will use Proposition 5.4.1 to show that there exists an arbitrarily small value of the parameter such that the eigenvalues of the provided periodic points all lie on the unit circle. Fix $\epsilon > 0$. We then define a C^∞-family of smooth $(f_\eta)_{\eta\in[-\delta,+\delta]}$ for some small $\delta > 0$ such that each map of the family is C^∞-close to the initial map f and for every $\eta \in [-\delta, +\delta]$ it holds:

1. $f_0 = f$,
2. $f_\eta(\Omega) = \Omega$ and $D_\Omega f_\eta = \exp(\eta/3) \cdot D_\Omega f$,
3. H is a point of homoclinic tangency of Ω for f_η.

This can be done similarly as in the proof of Proposition 5.4.1 by composing f by a one parameter family supported on a neighborhood of Ω with an adapted derivative at Ω and leaving invariant the local stable and unstable manifolds of f. Since $\log|\det(D_\Omega f)| = 0$, for every parameter $\eta \in [-\delta, +\delta]$, it holds:

$$\log|\det(D_\Omega f_\eta)| = \eta \ .$$

Fix a chart $\psi : U \hookrightarrow \mathbb{R}^3$ on U a small neighborhood of Ω in M. Then for every parameter $\eta \in [-\delta, +\delta]$ and every small C^∞-perturbation $\tilde{f}_\eta$ of f_η it holds:

$$|\log|\det D\psi^{-1} \circ D\tilde{f}_\eta \circ D\psi| - \eta| \leq \delta/2 \ , \quad \text{on } U \ . \tag{5.5.1}$$

By Proposition 5.4.1, there exist an arbitrarily large integer n, a C^∞-small perturbation $(\tilde{f}_\eta)_{\eta\in[-\delta,+\delta]}$ of $(f_\eta)_{\eta\in[-\delta,+\delta]}$ and a family of points $(Q_\eta)_{\eta\in[-\delta,+\delta]}$ such that Q_η is n-periodic with multipliers $|\mathrm{Jac}\,(\Omega_\eta, \tilde{f}_\eta)| \cdot \{1, j, j^2\}$. Moreover by Proposition 5.4.1 the number of points of the orbits of Q_η that lie outside U does not depend neither on n nor on η. Consequently, by Eq. (5.5.1), there exist real constants C^- and C^+ such that when n is large it holds:

$$C^- + n \cdot \eta - n \cdot \delta/2 \leq \log|\mathrm{Jac}\,(Q_\eta; \tilde{f}_\eta)| \leq C^+ + n \cdot \eta + n \cdot \delta/2 \ .$$

Now observe that when n is sufficiently large it holds:

$$\log|\mathrm{Jac}\,(Q_{-\delta}; \tilde{f}_{-\delta})| < 0 < \log|\mathrm{Jac}\,(Q_{+\delta}; \tilde{f}_{+\delta})| \ .$$

Consequently, by the mean value theorem, there exists a parameter $\eta \in [-\delta, \delta]$ such that $|\mathrm{Jac}\,(Q_\eta, \tilde{f}_\eta)|$ is equal to 1. Moreover, the map $\tilde{f}_\eta$ is C^∞-close to f as η is small. □

We will use the latter proof to show:

Proof of Theorem B. Similarly as in the latter proof we can assume that Ω has index 2 for every parameter and, by Fact 5.8.2 that Ω_a is fixed by f_a for every $a \in \mathbb{A}$. Moreover, for simplicity, up to a smooth family of change of coordinates we can assume that $\Omega_a = \Omega$ and $H_a = H$ for every $a \in \mathbb{A}$ where Ω and H are some point in M.

Fix $\epsilon > 0$ and $\delta > 0$ and a C^∞-family $(f_{a,\eta})_{a\in\mathbb{A},\eta\in[-\delta,+\delta]}$ of smooth diffeomorphisms such that for every parameters $a \in \mathbb{A}$ and $\eta \in [-\delta, +\delta]$:

1. $f_{a,0} = f_a$,
2. $f_{a,\eta}(\Omega) = \Omega$ and $D_\Omega f_{a,\eta} = \exp(\eta/3) \cdot D_\Omega f_a$,
3. H is a point of homoclinic tangency of Ω for the map $f_{a,\eta}$.

When taking $\delta > 0$ small, each $f_{a,\eta}$ is C^∞-close to f_a. Fix a smooth family of charts $(\psi_a)_{a\in\mathbb{A}}$ such that each $\psi_a : U \hookrightarrow \mathbb{R}^3$ is defined on a neighborhood U of Ω. Since $\mathbb{A}$ is compact, for every parameters $a \in \mathbb{A}$ and $\eta \in [-\delta, +\delta]$ and every small C^∞-perturbation $\tilde{f}_{a,\eta}$ of $f_{a,\eta}$ it holds:

$$|\log|\det D\psi_a^{-1} \circ D\tilde{f}_{a,\eta} \circ D\psi_a| - \eta| \leq \delta/2 \ , \quad \text{on } U \ . \tag{5.5.2}$$

By Proposition 5.4.1, there exist an arbitrarily large integer n, an arbitrarily small C^∞-perturbation $(\tilde{f}_{a,\eta})_{a\in\mathbb{A},\ \eta\in[-\delta,+\delta]}$ of $(f_{a,\eta})_{a\in\mathbb{A},\ \eta\in[-\delta,+\delta]}$ and a family of points $(Q_{a,\eta})_{a\in\mathbb{A},\ \eta\in[-\delta,+\delta]}$ such that $Q_{a,\eta}$ is n-periodic with multipliers $\det(D_\Omega \tilde{f}_{a,\eta}^n)) \cdot \{1, j, j^2\}$. Moreover, there exists an integer N independent of n such that when n is large enough and for every $N \leq k \leq n - N$, it holds $f_{a,\eta}^k(Q_{a,\eta}) \in U$. As well as in the non parametric case, by Eq. (5.5.1) and since $\mathbb{A}$ is compact, there exist real constants C^- and C^+ such that when n is large and for every parameters a, η it holds:

$$C^- + n \cdot \eta - n \cdot \delta/2 \leq \log|\mathrm{Jac}\,(Q_{\eta,a}; \tilde{f}_{\eta,a})| \leq C^+ + n \cdot \eta + n \cdot \delta/2 \ ,$$

and

$$\partial_\eta \log|\mathrm{Jac}\,(Q_{\eta,a}; \tilde{f}_{\eta,a})| > 0 \ . \tag{5.5.3}$$

Moreover, the latter quantity is large as n goes to infinity. Thus for n large, by the two above equations and by the implicit function theorem, there exists a C^0-small smooth map $\eta^* : \mathbb{A} \to \mathbb{R}$ such that $|\mathrm{Jac}\,(Q_{a,\eta(a)}, \tilde{f}_{a,\eta(a)})|$ is constantly equal to 1 on $\mathbb{A}$. To conclude, it suffices to show that η^* is C^∞-small. Denote $\{\sigma_{a,\eta}, \lambda_{a,\eta} \cdot e^{\pm i\theta_{a,\eta}}\}$ the eigenvalues of $D_\Omega \tilde{f}_{a,\eta}$ for $a \in \mathbb{A}$ and $\eta \in [-\delta, +\delta]$.

Up to taking U smaller, we have the following:

Lemma 5.5.1. *We can write:*

$$\det(\mathrm{Jac}\,(Q_{a,\eta}, \tilde{f}_{a,\eta})) = \exp(n \cdot \eta) \cdot \Xi_{a,\eta} \ ,$$

where $(\Xi_{a,\eta})_{(a,\eta)\in\mathbb{A}\times[-\delta,+\delta]}$ *is a smooth family of positive real numbers whose* log *is* C^∞*-bounded as* n *is large.*

Proof. Up to taking U smaller, by Lemma 5.7.4 with $k = n - 2N$ sufficiently large, we have that for every parameters a, η and since $\tilde{f}_{a,\eta}^{l+N}(Q_{a,\eta}) \in U$ for every $0 \leq l \leq k$ and $(\tilde{f}_{a,\eta}^{k+N}(Q_{a,\eta}))_{(a,\eta)\in\mathbb{A}\times[-\delta,+\delta]}$ which is equal to $(\tilde{f}_{a,\eta}^{-N}(Q_{a,\eta}))_{(a,\eta)\in\mathbb{A}\times[-\delta,+\delta]}$ is C^∞-close to Ω as n is large, it holds that:

$$(a, \eta) \in \mathbb{A} \times [-\delta, +\delta] \mapsto \det(D_{Q_{a,\eta}} \tilde{f}_{a,\eta}^k) \cdot \det(D_\Omega \tilde{f}_{a,\eta})^k =: \Psi_{a,\eta}$$

is C^∞-close to 1 in some appropriate system of coordinates independent of k. Recall that by Proposition 5.4.1 we have that $\det(D_\Omega \tilde{f}_{a,\eta}) = \exp(\eta)$. Thus it follows:

$$\det(\mathrm{Jac}\,(Q_{a,\eta}, \tilde{f}_{a,\eta})) = \exp(k\cdot\eta)\cdot\Psi_{a,\eta}\cdot\det(D_{\tilde{f}^{-N}(Q_{a,\eta})}\tilde{f}_{a,\eta}^{2N})\ .$$

Thus $\Xi_{a,\eta} := \exp(-2N\cdot\eta)\cdot\Psi_{a,\eta}\cdot\det(D_{\tilde{f}^{-N}(Q_{a,\eta})}\tilde{f}_{a,\eta}^{2N})$ satisfies the sought properties as N is fixed. □

Then for every $a\in\mathbb{A}$, the map η^* verifies :

$$\eta^*(a) = n^{-1}\cdot\log(\Xi_{a,\eta^*(a)})\ .$$

Thus by the implicit function theorem, η^* is C^∞-small when n is large. Consequently the family $(\tilde{f}_{a,\eta^*(a)})_{a\in\mathbb{A}}$ is C^∞-close to $(f_a)_{a\in\mathbb{A}}$ and each map of this family displays a periodic point with multipliers $\{1, j, j^2\}$. □

5.6 Local density of totally parabolic points

A compact f-invariant subset K of M is *uniformly hyperbolic* if the restriction to K of the tangent bundle TM splits into two continuous Df-invariant sub-bundles $TM_{|K} = E^u\oplus E^s$ that are uniformly expanded and contracted:

$$\|D_P f_{|E^s}\| < 1 \quad\text{and}\quad \|(D_P f_{|E^u})^{-1}\| < 1 \quad \forall P\in K.$$

An f-invariant compact set K is said to be *transitive* if it admits a dense orbit. A *basic set* K is a compact, f-invariant, transitive, uniformly hyperbolic set K which is *locally maximal*. It means that there exists a neighborhood U of K such that $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$

Let $K\subset M$ be a uniformly hyperbolic invariant set. For $P\in K$ we will denote its *stable and unstable* manifolds by:

$$W^s(P;f) := \{P'\in M : d(f^n(P'), f^n(P)) \xrightarrow[n\to+\infty]{} 0\}$$

and

$$W^u(P;f) := \{P'\in M : d(f^{-n}(P'), f^{-n}(P)) \xrightarrow[n\to+\infty]{} 0\}\ .$$

Moreover, given a neighborhood $V\subset M$ of K, if the connected components of $W^{s/u}(P;f)\cap V$ containing P are embedded manifolds, then they are called *local stable and unstable manifolds* and denoted $W^{s/u}_{\mathrm{loc}}(P;f;V)$. Note that if P is periodic, then the above definitions coincide with the previous ones.

The set K displays a *homoclinic tangency* if there is a pair of points $x, y\in K$ such that the stable manifold $W^s(x)$ of x and the unstable manifold $W^u(y)$ of y have some non-transverse intersection. Given a hyperbolic set K of a diffeomorphism f, for $\tilde{f}$ close to f, we denote by $K_{\tilde{f}}$ the hyperbolic set of f which is the continuation of K (i.e., $K_{\tilde{f}}$ is close to K and the dynamics of f on K is conjugated to the dynamics of $\tilde{f}$ on $K_{\tilde{f}}$. Moreover, the hyperbolic set K will be said to display *C^r-robust homoclinic tangency*, for $r\geq 1$, if the map f is of class C^r and there exists a C^r- neighborhood $\mathcal{N}$ of f such that for any $\tilde{f}\in\mathcal{N}$, the hyperbolic continuation $K_{\tilde{f}}$ of K displays a homoclinic tangency.

The following theorem provides examples for the set of maps involved in Theorem A is not empty.

Theorem 5.6.1 (Bonatti-Díaz [BD96]). *There exists a diffeomorphism $f_0 \in \mathrm{Diff}^\infty(M)$ and a mixing hyperbolic basic set $K_0 \subset M$ such that:*

1. *K_0 admits robustly a C^1-robust homoclinic tangency,*
2. *there exist periodic points $\check{P}_0$ and $\hat{P}_0$ in K_0 such that $|\mathrm{Jac}\,(\check{P}_0)| < 1 < |\mathrm{Jac}\,(\hat{P}_0)|$,*
3. *$\check{P}_0$ and $\hat{P}_0$ have complex multipliers.*

Note that $1-3$ are C^1-open conditions. Consequently there exists an open set $\mathcal{V}$ of diffeomorphisms verifying $1-3$. Moreover, we can assume that the hyperbolic continuations K, $\check{P}$ and $\hat{P}$ of K_0, $\check{P}_0$ and $\hat{P}_0$ are well defined for $f \in \mathcal{V}$ and verify the properties of the above lemma for f.

Proposition 5.6.2. *Let $f \in \mathrm{Diff}^\infty(M)$ such that f admits a mixing basic hyperbolic set K that displays* robustly a homoclinic tangency *and such that there exist hyperbolic periodic points $\check{P}$ and $\hat{P}$ in K such that:*

1. *$|\mathrm{Jac}\,(\check{P}; f)| < 1 < |\mathrm{Jac}\,(\hat{P}; f)|$,*
2. *$\check{P}$ and $\hat{P}$ have a non real multiplier.*

Then there exists a C^∞-small perturbation $\tilde{f}$ of f and a periodic hyperbolic point P in the hyperbolic continuation of K displaying a homoclinic tangency *with a non real multiplier and $|\mathrm{Jac}\,(P; \tilde{f})| = 1$.*

The proof of the latter proposition is directly deduced from the following:

Lemma 5.6.3. *Let $f \in \mathrm{Diff}^\infty(M)$ such that f admits a mixing hyperbolic basic set K such that there exist hyperbolic periodic points $\check{P}$ and $\hat{P}$ in K verifying:*

1. *$|\mathrm{Jac}\,(\check{P}; f)| < 1 < |\mathrm{Jac}\,(\hat{P}; f)|$,*
2. *$\check{P}$ and $\hat{P}$ have a non real multiplier.*

Then there exists a C^∞-perturbation $\tilde{f}$ of f and a periodic hyperbolic point P in the hyperbolic continuation of K with a non real multiplier and $|\mathrm{Jac}\,(P; \tilde{f})| = 1$.

Proof that Lemma 5.6.3 implies Proposition 5.6.2. By Lemma 5.6.3 there exists a C^∞-perturbation $\tilde{f}$ of f and a periodic hyperbolic point P in the hyperbolic continuation of K with a non real multiplier and $|\mathrm{Jac}\,(P; \tilde{f})| = 1$. Now since K displays robustly a homoclinic tangency, so does its hyperbolic continuation. Consequently, since K is mixing, up to an arbitrarily small perturbation, we can ensure that P moreover displays a homoclinic tangency. □

We now prove:

Proof of Lemma 5.6.3. First, up to replacing f by f^{-1} we can assume without any loss of generality that K has index 2. By Fact 5.8.1, up to replacing f by some iterate of an arbitrarily small C^∞-perturbation, we can assume without any loss of generality that $\hat{P}$ and $\check{P}$ are fixed by f. Since K is uniformly hyperbolic with index 2, there exists a continuous Df-invariant splitting $E^u \oplus E^s$ of $TM_{|K}$ such that Df is uniformly contracting on E^s and uniformly expanding on E^u.

Note that a periodic point in K has a complex non real multiplier if and only if the restriction of its differential to the unstable space is elliptic. This leads to introducing the following notations. For a periodic point $P \in K$ of f, let $\mathrm{Jac}^u(P; f)$ be the restriction of $\mathrm{Jac}\,(P; f)$ to E^u_P. Then we define:

$$[\mathrm{Jac}^u](P) := [\mathrm{Jac}^u](P; f) := \frac{1}{\sqrt{|\mathrm{Jac}^u(P; f)|}} \cdot \mathrm{Jac}^u(P; f) \in SL(E^u_P)\ .$$

Using the identification of $E^u_{\check{P}}$ and $E^u_{\hat{P}}$ with $\mathbb{R}^2$, there exists angles $\check{\theta}, \hat{\theta} \in \mathbb{T}$ such that $[\mathrm{Jac}^u](\check{P}) = R_{\check{\theta}}$ and $[\mathrm{Jac}^u](\hat{P}) = R_{\hat{\theta}}$.

Let us denote $\check{\mu} = \log|\mathrm{Jac}(\check{P})| < 0$ and $\hat{\mu} = \log|\mathrm{Jac}(\hat{P})| > 0$. Fix $\epsilon > 0$ and let $\hat{U}$ and $\check{U}$ be small neighborhoods of $\hat{P}$ and $\check{P}$. Let ψ be a chart on $\hat{U} \cup \check{U}$ such that for every it holds:

$$|\log|\mathrm{det}D_P(\psi \cdot f \cdot \psi^{-1})| - \check{\mu}| < \epsilon \quad \text{if } P \in \check{U} \cap f^{-1}(\check{U}) \tag{5.6.1}$$

and

$$|\log|\mathrm{det}D_P(\psi \cdot f \cdot \psi^{-1})| - \hat{\mu}| < \epsilon \quad \text{if } P \in \hat{U} \cap f^{-1}(\hat{U})\ . \tag{5.6.2}$$

Let V be a small neighborhood of the closure of $\check{U} \cup \hat{U}$. Then we consider a smooth family $(f_{\eta,\beta})_{\eta \in \mathbb{R},\ \beta \in \mathbb{T}}$ of diffeomorphisms of M such that $f_{0,0} = f$ and for every $(\eta, \beta) \in \mathbb{R} \times \mathbb{T}$, it holds:

1. $f_{\eta,\beta}$ fixes $\check{P}$ and $\hat{P}$,
2. $j(\check{P}; f_{\eta,\beta}) = R_{\check{\theta}}$ and $j(\hat{P}; f_{\eta,\beta}) = R_{\beta+\hat{\theta}}$,
3. $f_{\eta,\beta}$ coincides with f outside V.
4. $\log|D(\psi \cdot f_{\eta,\beta} \cdot \psi^{-1})| = \log|D(\psi \cdot f \cdot \psi^{-1})| + \eta$ on $(\check{U} \cap f^{-1}(\check{U})) \cup (\hat{U} \cap f^{-1}(\hat{U}))$.

We will consider periodic points provided by the following:

Lemma 5.6.4. *There exists $\bar{t} = (\check{t}, \hat{t}) \in \mathbb{N}^2$ such that for every $(\eta, \beta) \in \mathbb{R} \times \mathbb{T}$ close to 0 and for every pair of integers $\bar{n} := (n_{\check{P}}, n_{\hat{P}})$, there exists a periodic point $P = P(\bar{n}, \eta, \beta) \in K$ of $f_{\eta,\beta}$ with period given by $p := p(\bar{n}) := |\bar{t}| + |\bar{n}| = \check{t} + \hat{t} + \check{n} + \hat{n}$, such that for every $0 \leq k \leq p$:*

1. *$f^k_{\eta,\beta}(P) \in \hat{U}$ if $0 \leq k \leq \hat{n}$,*
2. *$f^k_{\eta,\beta}(P) \in \check{U}$ if $\hat{n} + \check{t} \leq k \leq \hat{n} + \check{t} + \check{n}$.*

The proof relies only on the fact that the involved fixed points are homoclinically related.

Proof. Since K is a hyperbolic invariant transitive, it supports a Markov partition (see [Bow75]) with arbitrarily small rectangles. We can assume then that the rectangles containing the hyperbolic fixed points $\check{P}$ and $\hat{P}$ are included in $\check{U}$ and $\hat{U}$. Moreover, for small values of (η, β), the continuation of this Markov partition verifies the same property. Let us consider such rectangles $\check{\mathcal{R}}_{\eta,\beta}$ and $\hat{\mathcal{R}}_{\eta,\beta}$. The transition times $\check{t}$ and $\hat{t}$ are the number of iterations to go from $\check{\mathcal{R}}_0$ to $\hat{\mathcal{R}}_0$ and from $\hat{\mathcal{R}}_0$ to $\check{\mathcal{R}}_0$. Then by continuity of the family, for small values of (η, β), they are also the transition times to go from $\check{R}_{\eta,\beta}$ to $\hat{\mathcal{R}}_{\eta,\beta}$ and from $\hat{\mathcal{R}}_{\eta,\beta}$ to $\check{\mathcal{R}}_{\eta,\beta}$. Consequently, for every $\bar{n} = (\check{n}, \hat{n})$ there exists a periodic point $P \in \hat{R}_{\eta,\beta}$ for $f_{\eta,\beta}$ - thus depending on η, β - whose $\hat{n}$-first iterates stay in $\hat{R}_{\eta,\beta} \subset \hat{U}$, then transitions to $\check{\mathcal{R}}_{\eta,\beta}$ after $\hat{t}$ iterations, then stay in $\check{\mathcal{R}}_\beta \subset \check{U}$ for $\check{n}$ iterations then goes back to P after $\check{t}$ iterations. □

Then for every (η, β) small enough, the latter lemma provides a family $(P(\bar{n}, \eta, \beta))_{\bar{n} \in \mathbb{N}^2}$ of periodic points for which we have control on their itinerary. We exhibit a sub-family of these points with determinant equal to 1 by the following:

Lemma 5.6.5. *There exists a neighborhood $\mathcal{B} \subset \mathbb{T}$ of 0 such that there exists $\bar{n}$ arbitrarily large and a C^1-function $\eta^\star : \mathcal{B} \to \mathbb{R}$ arbitrarily close to the null function such that for every $\beta \in \mathcal{B}$, in the chart ψ it holds:*

$$|\mathrm{Jac}(P(\bar{n}, \eta^\star(\beta), \beta))| = 1\ .$$

Proof. Up to shrinking $\mathcal{B}$, we can ensure that for every large $\bar{n}$ and for every $\beta \in \mathcal{B}$, it holds:

$$\partial_\eta \log|\mathrm{det} Df_{\eta,\beta}| > 1/2 \quad \text{ on } (\check{U} \cap f^{-1}(\check{U})) \cup (\hat{U} \cap f^{-1}(\hat{U})) \ . \tag{5.6.3}$$

Using the itinerary prescribed by Lemma 5.6.4, with $\eta = 0$ and $\beta \in \mathcal{B}$, we deduce that there exists a constant $C^0 > 0$ such that it holds:

$$|\log|\mathrm{Jac}\,(f_{0,\beta}, P(\bar{n}, 0, \beta))|| \leq |\hat{n}\cdot\hat{\mu} + \check{n}\cdot\check{\mu}| + \epsilon\cdot|\bar{n}| + C \ .$$

Now there exists a constant $C > 0$ such that there exists arbitrarily large $\bar{n}$ such that $|\hat{n}\cdot\hat{\mu}+\check{n}\cdot\check{\mu}| \leq C$. For such values of $\bar{n}$ we have and $\beta \in \mathcal{B}$:

$$|\log|\mathrm{Jac}\,(f_{0,\beta}, P(\bar{n}, 0, \beta))|| \leq 2|\bar{n}|\cdot\epsilon \ . \tag{5.6.4}$$

Now, inferring Eq. (5.6.3) together with the itinerary of Lemma 5.6.4, it holds for $\bar{n}$ large enough:

$$|\partial_\eta \log|\mathrm{Jac}\,(f_{\eta,\beta}, P(\bar{n}, \eta, \beta))|| \geq \tfrac{1}{4}|\bar{n}| \ . \tag{5.6.5}$$

Combining the two above inequalities provides for every β a value of η such that:

$$\log|\mathrm{Jac}\,(f_{\eta,\beta}, P(\bar{n}, \eta, \beta))| = 0 \ .$$

Moreover, by the implicit function theorem, the corresponding map $\eta^\star : \mathcal{B} \to \mathbb{R}$ is C^1-small. □

For $k \in \mathbb{N}$ let us denote $j^k_\beta := Q \in K \mapsto [D_{|E^u_Q} f^k_{\eta^\star(\beta),\beta}(Q)] \in PGL(E^u_Q, E^u_{f^k(Q)})$. To simplify a bit the notations, let us put $P := P(\bar{n}, \eta^\star(\beta), \beta)$. To conclude, it suffices now to show that we can pick β arbitrarily small such that P has a non real multiplier, or equivalently that $j^p_\beta(P)$ is elliptic. For that purpose, we provide the following estimate:

Lemma 5.6.6. *For every sufficiently large $\bar{n}$ and for every $\beta \in \mathcal{B}$, $j^p_\beta(P)$ is conjugated to $A_\beta \cdot R_{\hat{n}\cdot\beta}$ where $(A_\beta)_{\beta\in\mathcal{B}}$ is a C^1-family of elements in $SL(E^u_P)$ such that $(\partial_\beta A_\beta(\bar{n}))_{\beta\in\mathcal{B}}$ is C^0- bounded as $\bar{n}$ goes to infinity.*

Proof. By Gochenko-Shilnikov-Turaev Lemma 5.7.1 there exist local coordinates $(u, v) \in \mathbb{R}^2 \times \mathbb{R}$ on $\check{U}$ such that for every parameter η, β close to 0, when writing:

$$f^{\hat{n}+\hat{t}}(P(\bar{n}, \eta, \beta)) = (u^{(0)}, v^{(0)}) \quad \text{and} \quad f^{\check{n}+\hat{n}+\hat{t}}_{\eta,\beta}(P(\bar{n}, \eta, \beta)) = (u^{(\check{n})}, v^{(\check{n})}) \ ,$$

it holds:

$$\begin{cases} u^{(0)} = \lambda^{-\check{n}}\cdot R_{-\check{n}\cdot} u^{(\check{n})} + \rho^{\check{n}}\cdot\lambda^{-\check{n}}\cdot\nu^{(\check{n})}_{\eta,\beta}(u^{(\check{n})}, v^{(0)}) \\ v^{(\check{n})} = \sigma^{\check{n}}\cdot v^{(0)} + \rho^{\check{n}}\cdot\sigma^{\check{n}}\cdot\xi^{(\check{n})}_a(u^{(\check{n})}, v^{(0)}) \end{cases} \tag{5.6.6}$$

where σ and $\lambda\cdot e^{\pm i\check{\theta}}$ are the multipliers of $\check{P}$, $0 < 1 < \rho$ is a constant, and $(\xi^{\check{n}}_{\beta,\eta})_{\beta,\eta}$ and $(\nu^{\check{n}}_{\beta,\eta})_{\beta,\eta}$ are C^∞-bounded uniformly with $\check{n}$. Now as $\eta^\star$ is C^1-smooth relatively to β and as the iterates of P lie in the hyperbolic continuation of K it follows by the C^1-structural stability of K (see [Mos69]) that the iterates of P have uniformly bounded derivatives relatively to β independently of $\bar{n}$. It follows by the above Eq. (5.6.6) and Lemma 5.7.4 with $r = 1$, that the action on $T^1M_{|E^u}$ of $D_{f^{\hat{n}+\hat{t}}_{\beta,\eta^*(\beta)}(P)} f^{\check{n}}_{\beta,\eta^*(\beta)}$ writes in this system of coordinates as:

$$R_{\check{n}\cdot\check{\theta}}\cdot(\mathrm{id} + \check{\xi}^{\bar{n}}_\beta(P)) \ ,$$

where $(\check{\xi}^{\bar{n}}_\beta(P))_{\beta\in\mathcal{B}}$ is a C^1-small family as $\bar{n}$ goes to infinity. By the same argumentation, we have a similar result for $\hat{P}$: there exists local system of coordinates such that the action on $T^1M_{E^u}$ of $D_P f^{\hat{n}}_{\beta,\eta^*(\beta)}$ is equal to:

$$R_{\hat{n}\cdot(\hat{\theta}+\beta)}\cdot(\mathrm{id}+\hat{\xi}^{\bar{n}}_\beta(P))\ ,$$

where $(\hat{\xi}^{\bar{n}}_\beta(P))_{\beta\in\mathcal{B}}$ is a C^1-small family as $\bar{n}$ goes to infinity. Then observe that $j^{\check{t}}_\beta(f^{\hat{n}+\hat{t}+\check{t}}_{\eta^*(\beta),\beta}(P))$ and $j^{\hat{t}}_\beta(f^{\hat{n}}_{\eta^*(\beta),\beta}(P))\circ R_{\hat{n}\cdot\hat{\theta}}$ corresponds to the transitions from $\hat{P}$ to $\check{P}$ and reversely. They have bounded derivatives relatively to β and independently of $\bar{n}$. Finally, by the above discussions, it holds that the family $(A_\beta)_{\beta\in\mathcal{B}}$ defined by:

$$A_\beta := (\mathrm{id}+\check{\xi}^{\bar{n}}_\beta(P))\circ j^{\check{t}}_\beta(f^{\hat{n}+\hat{t}+\check{t}}_{\eta^*(\beta),\beta}(P))\circ R_{\check{n}\cdot\check{\theta}}\circ(\mathrm{id}+\check{\xi}^{(\bar{n})}_\beta(P))\circ j^{\hat{t}}_\beta(f^{\hat{n}}_{\eta^*(\beta),\beta}(P))\circ R_{\hat{n}\cdot\hat{\theta}}\ , \qquad (5.6.7)$$

satisfies the sought properties. □

Note that we have:

Fact 5.6.7. *Either A_0 is elliptic or the image of $[0,\pi]\subset\mathbb{T}$ by the map $\alpha\mapsto\mathrm{Tr}(A_0\circ R_\alpha)$ contains 0 in its interior.*

Proof. Assume that A_0 is not elliptic. Then a simple computation gives:

$$\mathrm{Tr}(A_0\circ R_\alpha)=\cos(\alpha)\cdot\mathrm{Tr}(A_0)+\sin(\alpha)\cdot\mathrm{Tr}(A_0\circ R_{\pi/2})\ .$$

Since A_0 is not elliptic, it is either parabolic or hyperbolic. In both cases it holds $\mathrm{Tr}(A_0)\neq 0$. Thus at $\alpha=0$ and $\alpha=\pi$ the values of $\mathrm{Tr}(A_0\cdot R_\alpha)$ have opposite signs and so vanish at a certain $\alpha\in(0,\pi)$. □

To conclude, if A_0 is elliptic we take the parameters $(0,\eta^*(0))$. Otherwise, by the above fact and since $\partial_\beta A_\beta$ is bounded independently of $\bar{n}$, then when $\bar{n}$ is sufficiently large, there exists a small parameter $\beta\in[0,\pi/\hat{n}]$ such that $\mathrm{Tr}(A_\beta\cdot R_{\hat{n}\cdot\beta})$ is equal to 0. To conclude the proof of the proposition, recall that in both cases the map $f_{\eta^\star(\beta),\beta}$ is close to f and displays a periodic point with determinant equal to 1 and a complex non real multiplier. □

5.7 Shilnikov coordinates

Recall that $\mathbb{A}$ is a smooth compact connected smooth manifold and M a smooth manifold of dimension 3. An important tool to almost linearize the dynamic nearby hyperbolic fixed point is the following result form Gochenko-Shilnikov-Turaev[GST08].

Lemma 5.7.1 (Gochenko-Shilnikov-Turaev, Lemma 6. and 7. [GST08])**.** *Let $f_{\mathbb{A}}$ be a C^∞-parameter family of smooth diffeomorphisms such that there exists a point Ω fixed by every f_a. Assume moreover that Ω_a has index 2 for every $a\in\mathbb{A}$ with expanding multipliers $\lambda_a\cdot e^{\pm i\theta_a}$ where $\lambda_a>1$ and $\theta_a\in\mathbb{T}$ and with contracting multiplier $\sigma_a\in(-1,1)$.*

Then there exists a neighborhood U of Ω and coordinates $u,v\in\mathbb{R}^2\times\mathbb{R}$ on U such that:

$$W^s_{loc}(\Omega;f_a;U)=(u=0)\quad \textit{and}\quad W^u_{loc}(\Omega;f_a;U)=(v=0)\ .$$

Moreover, for every point $(u^{(0)}, v^{(0)}) \in U$ *and for every parameter* $a \in \mathbb{A}$ *such that* $(u^{(l)}, v^{(l)}) := f_a^l(u^{(0)}, v^{(0)}) \in U$ *for every* $l \leq k$*, for some sufficiently large integer* k*, it holds:*

$$\begin{cases} u^{(0)} = \lambda_a^{-k} \cdot R_{-k\cdot\theta_a} \cdot u^{(k)} + \rho^k \cdot \lambda_a^{-k} \cdot \eta_a^{(k)}(u^{(k)}, v^{(0)}) \\ v^{(k)} = \sigma_a^k \cdot v^{(0)} + \rho^k \cdot \sigma_a^k \cdot \xi_a^{(k)}(u^{(k)}, v^{(0)}) \end{cases} \tag{5.7.1}$$

where $0 < \rho < 1$ *is a constant*[3] *and* $(\xi_a^{(k)})_{a\in\mathbb{A}}, (\eta_a^{(k)})_{a\in\mathbb{A}}$ *are* C^∞*-families uniformly bounded with* $k \in \mathbb{N}$*.*

Using coordinates provided by the above lemma, we prove Lemma 5.3.5. Let us first give a precise statement:

Lemma 5.7.2. *In the setting of Proposition 5.3.1, and when* Ω *is a fixed point, there exists a sequence of smooth embedded disks* $(\Sigma^n)_{n\geq 0}$ *and a sequence of smooth coordinates* $(\varphi^n : \Sigma^n \to \mathbb{R}^2)_{n\geq 0}$ *such that:*

1. Σ^0 *intersects* W^s_{loc} *transversely at* H*,*
2. $f^{-n}(\Sigma^n)$ *is close to* $\{H\}$ *as* n *goes to* $+\infty$*,*
3. Σ^n *is* C^∞*-close to* W^u_{loc} *as* n *goes to* $+\infty$*,*
4. $f(\Sigma^{n+1}) \subset \Sigma^n$*,*
5. f^{-1} *is contracting on* Σ^n*,*
6. $\varphi^n : \Sigma^n \hookrightarrow \mathbb{R}^2$ *is a smooth embedding,*
7. $\varphi^n \circ f^n = L^n \circ \varphi^0$ *on* $f^{-n}(\Sigma^n)$*,*
8. φ^n *is* C^∞*-converging to some map* $\varphi^\infty : W^u_{loc} \hookrightarrow \mathbb{R}^2$*.*

Recall that $L = \lambda \cdot R_\theta$ is conjugate to the restriction to the unstable space of $D_\Omega f$.

Proof. Let $\mathcal{C} \subset W^s_{\text{loc}}$ be a small curve containing Ω. Let N be a positive fixed large integer such that $f^N(H) \in \mathcal{C}$. Let U be a small neighborhood of $\mathcal{C} \cup W^u_{\text{loc}}$. Then, as a direct application of Lemma 5.7.1, there exist local coordinates $u, v \in \mathbb{R}^2 \times \mathbb{R}$ on a small neighborhood U of $\mathcal{C} \cup W^u_{\text{loc}}$ such that $\mathcal{C} = (u = 0)$ and $W^u_{\text{loc}} = (v = 0)$. Moreover, for every point $(u^{(0)}, v^{(0)}) \in U$ such that $(u^{(l)}, v^{(l)}) := f^l(u^{(0)}, v^{(0)}) \in U$ for every $l \leq k$, for some sufficiently large integer k, then it holds:

$$\begin{cases} u^{(0)} = L^{-k} \cdot u^{(k)} + \hat{\lambda}^{-k} \cdot \eta^{(k)}(u^{(k)}, v^{(0)}) \\ v^{(k)} = \sigma^n \cdot v^{(0)} + \hat{\sigma}^k \cdot \xi^{(k)}(u^{(k)}, v^{(0)}) \end{cases} \tag{5.7.2}$$

where $\hat{\sigma}, \hat{\lambda}$ are positive constants such that $0 < \hat{\sigma} < |\sigma|$ and $\lambda < \hat{\lambda}$; and $\xi^{(k)}, \eta^{(k)}$ are C^∞-bounded maps uniformly with k.

Since $f^N(H) \in \mathcal{C}$ it has coordinates $(0, v^+)$ for some $v^+ \in \mathbb{R}$. Let Σ^N be a small disk contained in $\{P \in U : P \text{ has } v\text{-coordinate equal to } v^+\}$. Then for every integer $n \neq N$, let us define Σ^n to be the connected component containing $f^n(H)$ in $f^{n-N}(\Sigma^N) \cap U$. Note that Σ^N is not empty and intersects $W^u_{\text{loc}}(\Omega)$ transversely at $f^N(H)$, thus Σ^0 is a small disk intersecting W^s_{loc} transversely at H. Thus by the inclination lemma we have the following:

3. See remark 1 under lemma 6 and the proof of lemma 6 in [GST08] for justification of the smoothness according to the parameter of the coordinates change. Also, since $\mathbb{A}$ is connected and compact, the constant ρ does not depend on the parameter.

1. $\Sigma^{n+1} \subset f(\Sigma^n)$ for any $n \geq 0$,
2. $f^{-n}(\Sigma^n)$ is close to $\{H\}$ as n goes to $+\infty$,
3. Σ^n is C^∞-close to W^u_{loc} as n goes to $+\infty$.
4. f^{-1} is contracting on Σ^n for $n \geq 0$.

Observe that the function $(u^{(k)}, v^{(k)}) \in \Sigma^{N+k} \mapsto v^{(0)}$ is constantly equal to v^+. For $k \geq 0$ let then:

$$\begin{array}{rccc} \varphi^{k+N} & : & \Sigma^{N+k} & \to & \mathbb{R}^2 \\ & & (u^{(k)}, v^{(k)}) & \mapsto & L^k \cdot u^{(0)} \ , \end{array}$$

i.e $\varphi^{k+N} = L^k \circ f^{-k} \circ \varphi^N$ on Σ^{k+N} and φ^N sends $f^N(u,v)$ to u for $f^N(u,v) \in \Sigma^N$. Moreover, for $0 \leq k \leq N-1$, we define:

$$\begin{array}{rccc} \varphi^k & : & \Sigma^k & \to & \mathbb{R}^2 \\ & & u, v & \mapsto & L^{j-N} \circ \varphi^N \circ f^{N-j}(u,v) \ . \end{array}$$

Observe that for every $n \in \ N$ it holds :

$$\varphi^n \circ f^n = L^n \circ \varphi^0 \ .$$

Moreover, for $n \geq N$, we have:

$$\varphi^n(u,v) = u + \left(\frac{\lambda}{\hat{\lambda}}\right)^{N-n} \cdot R_{(n-N)\cdot\theta} \cdot \eta^{(n-N)}(u, v^+) \ .$$

Then the map φ^n is C^∞-converging to the map:

$$\begin{array}{rccc} \varphi^\infty & : & W^u_{\text{loc}} & \to & \mathbb{R}^2 \\ & & (u,v) & \mapsto & u \ . \end{array}$$

□

Following the same lines of the above lemma, we now prove:

Lemma 5.7.3. *Let $f_{\mathbb{A}}$ be a parameter family such that there exist points Ω and H such that for every $a \in \mathbb{A}$ the point Ω is a saddle fixed point of f_a displaying a homoclinic tangency at H. Assume moreover that Ω has index 2 with expanding multipliers $\lambda_a \cdot e^{\pm i\theta_a}$ with $\lambda_a > 1$ and $\theta_a \in \mathbb{T}$ for every $a \in \mathbb{A}$. Then, for every $n \geq 0$ there exists a smooth family of embedded disks $\Sigma^n_{\mathbb{A}}$ and a smooth family of maps $\varphi^n_{\mathbb{A}}$ such that for every $a \in \mathbb{A}$, it holds:*

1. *Σ^0_a intersects the local stable manifold of Ω transversely at H,*
2. *$f_a^{-n}(\Sigma^n_a)$ is close to $\{H\}$ as n goes to $+\infty$,*
3. *Σ^n_a is C^∞-close to W^u_{loc} as n goes to $+\infty$,*
4. *$f_a(\Sigma^{n+1}_a) \subset \Sigma^n_a$,*
5. *f_a^{-1} is contracting on Σ^n_a,*
6. *$\varphi^n_a : \Sigma^n_a \hookrightarrow \mathbb{R}^2$ is a smooth embedding,*
7. *$\varphi^n_a \circ f^n_a = \lambda_a R_{\theta_a} \circ \varphi^0_a$ on $f_a^{-n}(\Sigma^n_a)$,*

8. *φ_a^n is C^∞-converging to some map $\varphi_a^\infty : W^u_{loc} \hookrightarrow \mathbb{R}^2$,*
9. *the family $\varphi_{\mathbb{A}}^\infty := (\varphi_a)_{a\in\mathbb{A}}$ is C^∞-smooth.*

Moreover, 2. *and* 3. *are verified uniformly with the parameters.*

Note that Lemma 5.4.3 is a direct application of the above by considering $\mathbb{A}\times\mathbb{R}/2\pi\mathbb{Z}$ as parameter space.

Proof. For $a \in \mathbb{A}$ let us denote σ_a the contracting multiplier of $D_\Omega f_a$ and recall that the restriction $D_\Omega f_a$ to its unstable manifold is conjugate to $L_a := \lambda_a \cdot R_{\theta_a}$. Let W^u_{loc} be a local unstable manifold containing Ω and let W^s_{loc} be a small stable manifold. By compactness of $\mathbb{A}$, there exists a positive integer N such that $f_a^N(H)$ lies in W^s_{loc} for every $a \in \mathbb{A}$. Let $\bar{H}_a = f_a^N(H)$. Then by Lemma 5.7.1, there exists a neighborhood U of $W^u_{\text{loc}} \cup W^s_{\text{loc}}$ such that there exists a system of local coordinates $(u, v) \in \mathbb{R}^2 \times \mathbb{R}$ such that for every parameter $a \in \mathbb{A}$, the point $\bar{H}_a$ has coordinates $(0, v_a^+)$ and the family $(v_a^+)_{a\in\mathbb{A}}$ is C^∞-smooth. Moreover, for every $(u^{(0)}, v^{(0)}) \in U$ and for every parameter $a \in \mathbb{A}$ for which there exists some large integer k that verifies $(u^{(l)}, v^{(l)}) := f_a^l(u^{(0)}, v^{(0)}) \in U$ for every $l \leq k$, it holds:

$$\begin{cases} u^{(0)} = L_a^{-k} \cdot u^{(k)} + \rho^k \cdot \lambda_a^{-k} \cdot \eta_{a,\beta}^{(k)}(u^{(k)}, v^{(0)}) \\ v^{(k)} = \sigma_a^k \cdot v^{(0)} + \rho^k \cdot \sigma_a^k \cdot \xi_a^{(k)}(u^{(k)}, v^{(0)}) \end{cases} \tag{5.7.3}$$

where $0 < \rho < 1$ is a positive constant and $(\eta_a^{(k)})_{a\in\mathbb{A}}, (\xi_a^{(k)})_{a\in\mathbb{A}}$ are C^∞-bounded maps uniformly with $k \in \mathbb{N}$.

Let us denote $\Sigma_a^N := \{P \in U^+ : P \text{ has } v\text{-coordinate equal to } v_a^+ \}$.

Observe that the map $(u^{(k)}, v^{(k)}) \in f_a^k(\Sigma_a^N) \cap U^- \to v_a^{(0)}$ is constantly equal to v_a^+. Then the family $(\Sigma_a^N)_{a\in\mathbb{A}}$ is a C^∞-smooth family of embedded disks. Then for every integer $n \neq N$, let us define $\Sigma_{\mathbb{A}}^n$ to be the family of connected components containing $f^n(H_a)$ in $f_a^{n-N}(\Sigma_a^N) \cap U$ for every $a \in \mathbb{A}$. For $a \in \mathbb{A}$, note that each Σ_a^N is not empty and intersects $W^u_{\text{loc}}(\Omega)$ transversely at $f^N(H_a)$, thus Σ_a^0 is a small disk intersecting W^s_{loc} transversely at H_a. Thus by the para-inclination lemma 5.2.3 and for every parameter $a \in \mathbb{A}$ we have

1. $\Sigma_a^{n+1} \subset f(\Sigma_a^n)$ for any $n \geq 0$,
2. the family $(f_a^{-n}(\Sigma_a^n))_{a\in\mathbb{A}}$ is close to $\{H\}$ as n goes to $+\infty$,
3. the family $(\Sigma_a^n)_{a\in\mathbb{A}}$ is C^∞-close to the constant family equal to W^u_{loc} as n goes to $+\infty$.
4. f_a^{-1} is contracting on Σ_a^n for $n \geq 0$.

As in the non parametric case, for $k \geq 0$ and $a \in \mathbb{A}$ let:

$$\begin{array}{rccc} \varphi_a^{k+N} & : & \Sigma_a^{N+k} & \to & \mathbb{R}^2 \\ & & (u^{(k)}, v^{(k)}) & \mapsto & L_a^k \cdot u^{(0)} \end{array},$$

Moreover, for $0 \leq k \leq N-1$, we define:

$$\begin{array}{rccc} \varphi_a^k & : & \Sigma_a^k & \to & \mathbb{R}^2 \\ & & u, v & \mapsto & L_a^{j-N} \circ \varphi_a^N \circ f_a^{N-j}(u, v) \end{array}.$$

Then for every $n \in \mathbb{N}$ the family $(\varphi_a^n)_{n\geq 0}$ is C^∞-smooth. By the proof of Lemma 5.7.2, it follows that φ_a^n verifies $5. - 8.$ in the statement of Lemma 5.7.3. And by the above construction, the family of limit maps $(\varphi_a^\infty)_{a\in\mathbb{A}} = (\lim_{n\to\infty} \varphi_a^n)_{a\in\mathbb{A}}$ is indeed C^∞-smooth. □

Another useful application of these systems of coordinates is the following lemma that is used in the proof of Theorem B and Proposition 5.6.2.

Lemma 5.7.4. *Let $f_{\mathbb{A}}$ be a C^∞-parameter family of smooth diffeomorphisms. Assume that there exists a point $\Omega \in M$ such that for every $a \in \mathbb{A}$ it is a fixed saddle for f_a. Then there exists a neighborhood U of Ω such that for every large integer k, for every $P \in U$ such that $f_a^l(P) \in U$ for every $0 \leq l \leq k$ and every $a \in \mathbb{A}$, the map:*

$$a \in \mathbb{A} \mapsto \det(D_P f_a^k) \cdot (\det D_\Omega f_a)^{-k} ,$$

is C^∞-close to the map constantly equal to 1 on $\mathbb{A}$.

Moreover, for any $1 \leq r \leq \infty$ and for any C^r-family of points $(P_a)_{a\in\mathbb{A}}$ such that each $f_a^l(P_a) \in U$ for $l \leq k$ and the family $(f_a^k(P_a))_{a\in\mathbb{A}}$ is C^r-close to Ω, then it holds moreover that the map:

$$a \in \mathbb{A} \mapsto \det(D_{P_a} f_a^k) \cdot (\det D_\Omega f_a)^{-k} ,$$

is C^r-close to the map constantly equal to 1 on $\mathbb{A}$.

Proof. We can assume, up to replacing each f_a by f_a^{-1}, that Ω has index 2. Let us denote its expanding multipliers $\lambda_a \cdot e^{\pm i\theta_a}$ where $\lambda_a > 1$ and $\theta_a \in \mathbb{T}$. As well, let us denote $\sigma_a \in (-1, 1)$ its contracting multiplier. By Lemma 5.7.1, there exists a neighborhood U of Ω such that there exists a system of local coordinates $(u, v) \in \mathbb{R}^2 \times \mathbb{R}$ such that for every parameter $a \in \mathbb{A}$ and for every $(u^{(0)}, v^{(0)}) \in U$ for which there exists some large integer k that verifies $(u^{(l)}, v^{(l)}) := f_a^l(u^{(0)}, v^{(0)}) \in U$ for every $0 \leq l \leq k$, it holds:

$$\begin{cases} u^{(0)} = L_a^{-k} \cdot u^{(k)} + \rho^k \cdot \lambda_a^{-k} \cdot \eta_{a,\beta}^{(k)}(u^{(k)}, v^{(0)}) \\ v^{(k)} = \sigma_a^k \cdot v^{(0)} + \rho^k \cdot \sigma_a^k \cdot \xi_a^{(k)}(u^{(k)}, v^{(0)}) \end{cases} \tag{5.7.4}$$

where $0 < \rho < 1$ is a positive constant and $(\eta_a^{(k)})_{a\in\mathbb{A}}, (\xi_a^{(k)})_{a\in\mathbb{A}}$ are C^∞-bounded maps uniformly with $k \in \mathbb{N}$. Let us fix such a large value of k and for $a \in \mathbb{A}$ denote:

$$A_a : (u^{(k)}, v^{(0)}) \mapsto u^{(0)} \quad \text{and} \quad B_a : (u^{(k)}, v^{(0)}) \mapsto v^{(k)} .$$

This translates as:

$$f_a^k(A_a(u^{(k)}, v^{(0)}), v^{(0)}) = (u^{(k)}, B_a(u^{(k)}, v^{(0)})) .$$

Then with $P = (u^{(0)}, v^{(0)})$, it follows that in these local coordinates we have:

$$\det(D_P f_a^k) = \frac{\partial_v B_a(u^{(k)}, v^{(0)})}{\det\left(\partial_u A_a(u^{(k)}, v^{(0)})\right)} .$$

Using Eq. (5.7.4), it follows:

$$\det(D_P f_a^k) \cdot \sigma_a^{-k} \cdot \lambda_a^{-2k} = \frac{1 + \rho^k \cdot \partial_v \xi_a^{(k)}(u^{(k)}, v^{(0)})}{\det\left(\mathrm{id} + \rho^k \cdot R_{k\cdot\theta_a} \cdot \partial_u \eta_a^{(k)}(u^{(k)}, v^{(0)})\right)} .$$

To conclude for the first statement in the lemma it is enough to observe that the map:

$$a \in \mathbb{A} \mapsto \frac{1 + \rho^k \cdot \partial_v \xi_a^{(k)}(u^{(k)}, v^{(0)})}{\det\left(\mathrm{id} + \rho^k \cdot R_{k\cdot\theta_a} \cdot \partial_u \eta_a^{(k)}(u^{(k)}, v^{(0)})\right)}$$

is C^∞-close to 1 as k goes to infinity since $0 < \rho < 1$ and and $(\eta_a^{(k)})_{a\in\mathbb{A}}, (\xi_a^{(k)})_{a\in\mathbb{A}}$ are C^∞-bounded maps uniformly with $k \in \mathbb{N}$. Then for the second statement let us denote $(u_a^{(0)}, v_a^{(0)}) = P_a$ and $(u_a^{(k)}, v_a^{(k)}) = f_a^k(P_a)$ which are C^r-close to Ω. Then the following map is C^r-close to identity:

$$a \in \mathbb{A} \mapsto \frac{1 + \rho^k \cdot \partial_v \xi_a^{(k)}(u_a^{(k)}, v_a^{(0)})}{\det\left(\mathrm{id} + \rho^k \cdot R_{k\cdot\theta_a} \cdot \partial_u \eta_a^{(k)}(u_a^{(k)}, v^{(0)})\right)},$$

which allows us to conclude. □

5.8 Classical tools for perturbation of diffeomorphisms

This section is dedicated to the proof of technical lemmas that are used along the proofs of this chapter. They rely on classical techniques from perturbation of of smooth diffeomorphisms using bump functions.

The following fact and its parametric counterpart stated below allow us to replace periodic points by fixed point to simplify most of the proofs.

Fact 5.8.1. *Let $f \in \mathrm{Diff}^\infty(M)$. Let $Q \in M$ be a point and let $N \geq 1$ be an integer. Let $\mathcal{N}^N$ be a neighborhood of f^N in $\mathrm{Diff}^\infty(M)$. Assume that there exists $\epsilon > 0$ small such that it holds $f^k(B(Q,\epsilon)) \cap B(Q,\epsilon) = \emptyset$ for every $1 \leq k \leq N$. Then there exists a neighborhood $\mathcal{N}$ of f such that for every $g \in \mathcal{N}^N$ such that g coincides with f^N outside $B(Q,\epsilon)$, there exists a map $\tilde{f} \in \mathcal{N}$ such that $\tilde{f}^N = g$ on $B(Q,\epsilon/2)$ and $\tilde{f}$ coincides with f outside $B(Q,\epsilon)$.*

Proof. Up to taking ϵ smaller, there exists a smooth embedding $\psi : B(Q,\epsilon) \to \mathbb{B}^3$ such that $\psi(Q) = 0$. Let then introduce a non decreasing function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ equal to 0 on $[0, 1/2]$ and equal to 1 on $[1, +\infty)$. Then, on $B(Q,\epsilon)$ we define:

$$\tilde{f} = \psi^{-1}\left(\rho(\epsilon^{-1} \cdot \|\cdot - Q\|) \cdot f + (1 - \rho(\epsilon^{-1} \cdot \|\cdot - Q\|)) \cdot g \circ f^{-(N-1)}\right). \tag{5.8.1}$$

Note that $\tilde{f}$ extends smoothly to f on M. Moreover, since ϵ is fixed, $\tilde{f}$ is C^∞-close to f as g is close to f^N. Since the N-first iterates of $B(Q,\epsilon)$ do not meet $B(Q,\epsilon)$ it holds that $\tilde{f}^N$ coincides with $f^{N-1} \circ \tilde{f}$ on $B(Q,\epsilon/2)$ which is $g_{|B(Q,\epsilon/2)}$. □

As a corollary of the above proof we deduce:

Fact 5.8.2. *Let $f_\mathbb{A} \in \mathrm{Diff}^\infty(M)_\mathbb{A}$ be a C^∞-family of smooth diffeomorphisms. Let $Q_\mathbb{A} \in C^r(\mathbb{A}, M)$ be a smooth family of points and let $N \geq 1$ be an integer. Let $\mathcal{V}$ be a neighborhood of $f_\mathbb{A}^N := (f_a^N)_{a\in\mathbb{A}}$ in $\mathrm{Diff}^\infty(M)_\mathbb{A}$. Assume that there exists $\epsilon > 0$ small such that for every $a \in \mathbb{A}$, it holds $f_a^k(B(Q_a,\epsilon)) \cap B(Q_a,\epsilon) = \emptyset$ for every $1 \leq k \leq N$. Then there exists a neighborhood $\mathcal{N}$ of $f_\mathbb{A}$ such that for every $g_\mathbb{A} \in \mathcal{V}$ with g_a coinciding with f_a^N outside $B(Q_a,\epsilon)$ for every $a \in \mathbb{A}$, then there exists a family $\tilde{f}_\mathbb{A} \in \mathcal{N}$ such that $\tilde{f}_a^N = g_a$ on $B(Q_a,\epsilon/2)$ and $\tilde{f}_a$ coincides with f_a outside $B(Q_a,\epsilon)$.*

Proof. Up to reduce $\epsilon > 0$, consider a smooth family of charts $(\psi_a)_{a\in\mathbb{A}}$ such that for each $a \in \mathbb{A}$, the map $\psi_a : B(Q_a,\epsilon) \to \mathbb{B}^3$ verifies $\psi_a(Q_a) = 0$. For $a \in \mathbb{A}$, let us define $\tilde{f}_a$ as in Eq. (5.8.1) by:

$$\tilde{f}_a = \psi_a^{-1}\left(\rho(\epsilon^{-1} \cdot \|\cdot - Q_a\|) \cdot f_a + (1 - \rho(\epsilon^{-1} \cdot \|\cdot - Q_a\|)) \cdot g_a \circ f_a^{-(N-1)}\right), \tag{5.8.2}$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is the map from the proof of Fact 5.8.1. Then the family $(\tilde{f}_a)_{a\in\mathbb{A}}$ extends smoothly to a C^∞-family of smooth diffeomorphisms on M. Moreover, it is close to the initial family $f_{\mathbb{A}}$ as $g_{\mathbb{A}}$ is close to $f_{\mathbb{A}}$. Finally, as in the non parametric case it verifies that for every $a \in \mathbb{A}$ the map $\tilde{f}_a^N$ coincides with g_a on $B(Q_a, \epsilon)$. □

5.9 Perturbation of the 1-Jet

We shall now prove Lemma 5.3.4 and its parametric counterpart Lemma 5.4.6.

Proof of Lemma 5.3.4. There exists a local C^∞-smooth chart φ defined on U of $f(H)$ such that $\varphi(f(H)) = 0$ and $\varphi(U) = \mathbb{B}$. Since U is small we can moreover assume that it iterates $f(U)$ and U have empty intersection. For n sufficiently large we have $\|\varphi(Q)\| \leq 1/2$ and $\varphi(\bar{Q}) \leq 1/2$. We will first construct a perturbation ξ of the identity supported on U. Let $\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ be the affine map given by:

$$\gamma : p \in \mathbb{R}^3 \mapsto \varphi(Q) + \Gamma \cdot (p - \varphi(\bar{Q})) \,, \tag{5.9.1}$$

where $\Gamma \in GL_3(\mathbb{R})$ is the matrix given by:

$$\Gamma := D_Q\varphi \circ l \circ (D_{\bar{Q}}\varphi)^{-1} \,. \tag{5.9.2}$$

Note that γ is close to identity affine map when n is large. Let

$$\rho : \mathbb{R}^+ \to \mathbb{R}^+ \tag{5.9.3}$$

be a C^∞-function such that equals to 1 on $[0, 1/2]$ and equals 0 on $[1, +\infty)$. Then we define the map $\xi : U \hookrightarrow M$ for $p \in \mathbb{B}$ by:

$$\varphi \circ \xi \circ \varphi^{-1}(p) = \rho\left(\|p\|_2^2\right) \cdot \gamma(p) + \left(1 - \rho\left(\|p\|_2^2\right)\right) \cdot p \,, \tag{5.9.4}$$

coincides with the identity on a neighborhood of ∂U and then extends smoothly to the identity on M. Note that this extension is close to identity on M. To conclude, it indeed holds:

$$\xi(\bar{Q}) = Q \quad \text{and} \quad D_{\bar{Q}}\xi = \ell \,.$$

□

We are now ready to prove its parametric-counterpart:

Proof of Lemma 5.4.6. The proof is a corollary of the proof of Lemma 5.3.4. For $a \in \mathbb{A}$, let Γ_a, γ_a and ξ_a be the maps given respectively by Eq. (5.9.2), Eq. (5.9.1) and Eq. (5.9.4) where ρ is given by Eq. (5.9.3) and the chart ϕ is chosen independently of the parameter. Then Γ_a obviously depends smoothly on the parameter $a \in \mathbb{A}$ and so does γ_a and then ξ_a. □

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