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Anna Roig Sanchis

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École doctorale de sciences mathématiques de Paris centre

THÈSE DE DOCTORAT

Discipline : Mathématiques

présentée par

Anna ROIG SANCHIS

On the geometry of random hyperbolic 3-manifolds

dirigée par Bram PETRI

Soutenue le 25 octobre 2024 devant le jury composé de :

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Per ma mare, mon pare i el meu germà.

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Abstract

In this thesis, we study the geometry of typical hyperbolic 3-manifolds, as their volume goes to infinity. To do so, we consider a model for random hyperbolic 3-manifolds, called random triangulations. These are compact manifolds with boundary constructed by randomly gluing truncated tetrahedra along their faces.

We prove that, as the volume tends to infinity, their length spectrum converge in distribution to a Poisson point process on $\mathbb{R}_{>0}$, with computable intensity λ . We also study the systole, that is, the first element of the length spectrum. More precisely, we show that the limit, as the volume tends to infinity, of the expected value of the systole of these manifolds exists, and we give a closed formula of it. Moreover, we compute a numerical approximation of this value.

The general idea behind the proofs is to understand the combinatorics of the complex made of truncated tetrahedra, and then infer from these its geometric properties. These combinatorial properties are (almost all) contained in the dual graph of the complex. Hence, most of the arguments in these proofs combine results coming from hyperbolic geometry, random graph theory, and probability theory.

Keywords

Hyperbolic geometry, random 3-manifolds, length spectrum, systole, random regular graphs, Poisson point process.

Résumé

Dans cette thèse, nous étudions la géométrie des variétés tridimensionnelles (3-variétés) hyperboliques typiques, lorsque leur volume tend vers l'infini. Pour cela, nous considérons un modèle de 3-variétés hyperboliques aléatoires, appelé triangulations aléatoires. Il s'agit des variétés compactes à bord construites en collant aléatoirement des tétraèdres tronqués le long de leurs faces.

Nous prouvons que, lorsque le volume tend vers l'infini, leur spectre des longueurs converge en distribution vers un processus ponctuel de Poisson sur $\mathbb{R}_{>0}$, d'intensité calculable λ . Nous étudions également la systole, c'est-à-dire, le premier élément du spectre des longueurs. Plus précisément, nous montrons que la limite, lorsque le volume tend vers l'infini, de l'espérance de la systole de ces variétés existe, et nous en donnons une formule fermée. De plus, nous calculons une approximation numérique de cette valeur.

L'idée générale derrière les preuves est de comprendre la combinatoire du complexe composé de tétraèdres tronqués, et ensuite d'en déduire ses propriétés géométriques. Ces propriétés combinatoires sont (presque toutes) contenues dans le graphe dual du complexe. Ainsi, la plupart des arguments dans ces preuves combinent des résultats provenant de la géométrie hyperbolique, de la théorie des graphes aléatoires et de la théorie des probabilités.

Mots-clés

Géométrie hyperbolique, variétés tridimensionnelles aléatoires, spectre des longueurs, systole, graphes réguliers aléatoires, processus de Poisson.

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Introduction

Perelman's proof of Thurston's geometrisation conjecture [Thu82, Per02, Per03], together with other important results such as the surface subgroup theorem [KM12], the virtual Haken theorem [Ago13] or the ending lamination theorem [Min10, BCM12], led to remarkable progress in the study of hyperbolic 3-manifolds.

Even so, there are still plenty of open questions in the field. Many of them are about the behaviour of geometric invariants of these manifolds -such as the lengths of their geodesics or the spectral gap- as their complexity grow (see for instance [MT23, Conjecture 1.5]).

The thing is that, similar to what happens in the two dimensional case, the class of hyperbolic 3-manifolds is the "wildest" class among all geometric 3-manifolds. That is, it contains manifolds with very different features: some with special symmetries, algebraic properties, some highly connected (or disconnected), etc. Therefore, it is very hard to prove results about their geometric properties that are true for all of them. A natural approach, then, is to set aside a small set of "degenerate manifolds", and try to find properties, or behaviours, that are verified for "most" of hyperbolic 3-manifolds. We refer to these as *typical* properties, and to the set of manifolds verifying them, *typical* manifolds.

In this thesis, we aim to (partially) answer the following question:

Q: What is the geometry of a typical hyperbolic 3-manifold of large volume?

More precisely, we are interested in the *typical behaviour* of two geometric invariants: the *length spectrum* and the *systole*. This question is addressed by using a probabilistic approach, that is explained next.

Probabilistic approach

A way of tackling problems concerning asymptotic, extremal or typical behaviours is by using probabilistic techniques. Actually, it is only in a probabilistic context where the latter makes mathematical sense. This method has been proven to be very successful in areas like graph theory, and more recently also in hyperbolic geometry.

In our case, this probabilistic approach consists in the study of *random manifolds*. A way to think about them is to consider a parameterized class of manifolds, and put a probability measure on this parameter space. As one can deduce from this description, there isn't a canonical way of defining random manifolds. Hence, we need to consider a *model*.

Several models of random hyperbolic surfaces were developed by Mirzakhani [Mir13], Brooks-Makover [BM04], Guth-Parlier-Young [GPY11], Budzinski-Curien-Petri [BCP21] and Magee-Naud-Puder [MNP22]. Together with this, there has been an extensive research on the study of typical extremal properties of these surfaces (see for instance [GPY11, LMW16, BCP21, BM04, MNP22, Mir13, HM23, AM23, LW23, WX22, Mon22, Liu22, Pet17, MP19]).

In the case of 3 dimensions, there are three well known models: *random Heegaard splittings*, *random mapping tori* -both introduced by Dunfield and Thurston [DT06]- and *random triangulations* -introduced by Petri and Raimbault [PR22].

The former model is probably the most studied one, and is obtained from a Heegaard splitting, that is, by gluing together two copies of a handlebody H_g of genus g along a "random" orientation preserving diffeomorphism f of the boundary $\partial H_g = \Sigma$ (see Figure 1). Since the manifold depends only on the isotopy class of f , it is well defined for the mapping class $[f] \in \text{Mod}(\Sigma)$. Thus, a random Heegaard splitting is one such that the mapping class is taken at random by doing a random walk in the mapping class group $\text{Mod}(\Sigma)$. Note, that this model samples only manifolds of bounded Heegaard genus.

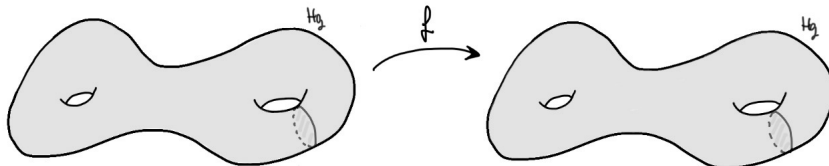


Figure 1: Gluing of two copies of a handlebody along a diffeomorphism f

Considerable work has been done in the study of geometric invariants of this model, such as the spectrum of the Laplacian [HV22], the growth of the diameter and injectivity radius [FSV22], or that of their volume [Via21] -providing an answer to the volume conjecture of Dunfield and Thurston [DT06, Conjecture 2.11]-. These often used combinatorial models similar to those that are behind the proof of the ending lamination conjecture [Min10, BCM12].

The model of random 3-manifolds we work on is the model of *random triangulations* [PR22]. In this model, a random 3-manifold M_n is constructed by randomly gluing together

n truncated tetrahedra along their hexagonal faces, resulting in a compact, oriented 3-manifold with boundary (see Figure 2). By Moise [Moi52], one has that, as n tends to infinity, every compact 3-manifold with boundary gets sampled by this model.

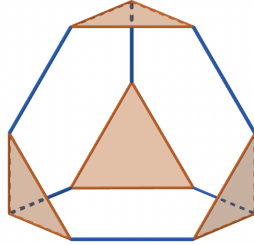


Figure 2: A truncated tetrahedron

Moreover, it turns out that asymptotically almost surely (a.a.s) these manifolds are hyperbolic (Theorem 2.1.3, by Petri-Raimbault), which makes it a suitable model for studying properties of hyperbolic 3-manifolds.

In this manuscript, we study the following two geometric invariants of hyperbolic 3-manifolds built under this model: the primitive length spectrum, and the systole.

The primitive length spectrum

A *geodesic* is a curve that locally minimises the distance. A closed geodesic is called *primitive* if it travels along its image exactly once. Given a hyperbolic manifold M , the *primitive length spectrum* is the (multi-)set of lengths of all primitive closed geodesics.

Related work

The study of lengths of (primitive) closed geodesics on hyperbolic manifolds has been a topic of interest for a long time. Specially in dimension two, there is a large literature on this topic, which can be found, for instance, in [Bus92].

One of the first results regarding counting of curves was due to Huber and Selberg [Hub61], which proved the well known prime geodesic theorem. This result states that the number of primitive closed geodesics of length at most some number L in a hyperbolic surface is asymptotic to e^L/L as L tends to infinity. This result was generalised for finite-volume hyperbolic n -manifolds by Sarnak [Sar83] and compact negatively curved manifolds by Margulis in this thesis [Mar04], constituting one of the firsts results on this topic in higher dimensions. Some time later, Mirzakhani [Mir08, Mir07] made a significant breakthrough on the counting of (simple and non-simple) closed geodesics on hyperbolic surfaces of length up to L , for $L \rightarrow \infty$, work that has continued to be developed by several authors, among them [ES23, ES22].

Another natural question that has been studied is the multiplicity of the elements in the spectrum. The prime geodesic theorem also provided an exponential bound on the multiplicity of a length. On the other hand, Randol [Ran80] proved that the length spectrum of any compact hyperbolic surface has unbounded multiplicities. In dimension 3, an analogous result was proved by Masters [Mas00].

There has also been work done on the marked length spectrum. For instance, it was proved by Otal [Ota90] that this invariant, in contrast with the length spectrum, does determine a surface with any non-constant negatively curved metric up to isometry. In higher dimensions, this question has been addressed, for instance, by [Ham99, GL19]. However, it still remains open for general negatively curved metrics.

Now, in the case of random surfaces, the length spectrum has also been studied. Puder-Zimhoni [PZ24] proved a Poisson approximation theorem for the length spectrum of random surfaces under the random cover model. Petri [Pet17] gave also a similar result in the Brooks-Makover model. In the Weil-Petterson model, various results regarding closed geodesics have been shown in [MP19, Liu22, MT22, NWX23].

For random 3-manifolds, however, much less is known. We highlight the study of short curves in the random mapping tori model by Taylor-Sisto [ST19] and in the Heegaard splitting model by Feller-Sisto-Viaggi [FSV22].

Contribution

In this manuscript we study the typical behaviour of the length spectrum of a random hyperbolic 3-manifold M_n of large volume, constructed with the model of random triangulations. The first main result of this manuscript is:

Theorem A ([RS23]). *As $n \rightarrow \infty$, the primitive length spectrum of a random compact hyperbolic 3-manifold with boundary M_n converges in distribution to a Poisson point process (PPP) on $\mathbb{R}_{>0}$, of computable intensity λ .*

This λ is an atomic measure that is supported on the length spectrum of $\mathrm{PSL}(2, Z[i])$. It will be properly defined later in the text. The theorem, then, gives a concrete characterisation of the behaviour of the length spectrum of these random hyperbolic 3-manifolds M_n . Moreover, since the distribution of a Poisson random variable has an explicit formula, and we have an explicit expression for its mean, this result also tells us that it is possible to compute -with the help of a computer- the limits of the probability that a measurable subset $[a, b] \subset \mathbb{R}_{>0}$ contains k points, for any $k \in \mathbb{N}$ and $a, b > 0$ fixed.

Structure and ideas of the proof

The structure of the proof of Theorem A has some common points with the one of Theorem 2.1.3 by Petri-Raimbault, regarding the hyperbolicity of the manifolds M_n . There,

they first construct a specific model of non-compact hyperbolic manifolds -made out of a gluing of hyperbolic ideal right-angled octahedra- named Y_n , and observe that these can be transformed to the manifolds M_n via Dehn filling. Then, they prove that after this compactification process, the resulting manifolds M_n are still hyperbolic, using as main tools Andreev's theorem [RHD07] and Theorem 3.2.3, by Futer-Purcell-Schleimer.

The point here is that the geometry of these non-compact random manifolds Y_n is much better understood -by construction of the manifolds and their hyperbolic metric- than the one of the manifolds M_n , so it is easier to study their geometric properties.

Therefore, in order to prove Theorem A regarding the length spectrum of M_n , we will follow a similar strategy: first, we will prove the result for these manifolds Y_n , and then we will see that after the Dehn filling, the result is still true for the compactified manifolds, that is, the M_n .

The main idea behind the proof of this first part is to translate the problem of counting closed geodesics in Y_n to the problem of counting certain cycles in the dual graph of this complex, which is a random 4-regular graph (see Figure 3). Then, the asymptotic behaviour of the expected number of these cycles is given by Theorem 3.1.2, using the method of moments. To prove this part it is also essential a technical result concerning the growth of the translation length in terms of the word length of the cycles in the graph (Proposition 3.1.3).

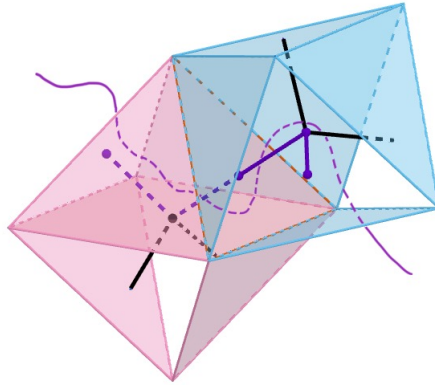


Figure 3: Homotopy of a curve into the dual graph.

On the other hand, the second part of the proof of Theorem A comes down to showing two points: the first and principal, is that the length of the curves after the Dehn fillings of the cusps stays roughly the same (given by Proposition 3.2.1). For this, we'll rely as well on Andreev's theorem and Futer-Purcell-Schleimer, although in a different way as in Petri-Raimbault's proof. And secondly, that closed geodesics don't collapse after the compactification (given by Lemma 3.2.4). Both results -as well as some other statements in the thesis- are proved to hold asymptotically almost surely (a.a.s), which means with

probability tending to 1 as $n \rightarrow \infty$.

Finally, we note that one could also try to carry out a proof for this result using the Chen–Stein method for Poisson approximation [AGG89, BC05] -instead of the method of moments- which might provide explicit errors terms in the approximation.

The systole

The *systole* of a hyperbolic n -manifold M is the length of the shortest geodesic of M . It constitutes one of the simplest geometric invariants of M , yet gives very rich information about the manifold. Indeed, it is related to many other of its geometric properties, like the volume [Gro83], the diameter [Bav96, BDP23], the kissing number [Par13, BP22], or the Cheeger constant [Bro92].

Related work

The study of systolic geometry started in 1949 with Loewner, who proved an inequality for the systole of the 2-torus in terms of its area [Pu52]. This was followed by Pu, who stated a similar inequality for the real projective plane [Pu52]. Some years later, Gromov gave one of the most well known results in the area [Gro83]; he proved an upper-bound on the 1-systole of essential n -manifolds (such as hyperbolic manifolds) in terms only of their volume, for all $n \geq 2$.

Over the years, there has been extensive research on the behavior of the systole of hyperbolic manifolds. It is known, for instance, that the infimum of the systole of any closed hyperbolic n -manifold with $\text{vol}(M) \leq v$ is zero for v large enough if $n = 2, 3$, and tends to zero as $v \rightarrow \infty$ for $n \geq 4$ [Ago06, BT11]. Nonetheless, the behaviour of the maximum of this value as the volume grow, is still unknown for any $n \geq 2$. Thus, many results in the field are about finding sharper lower and upper bounds of it, and examples of manifolds with large systole.

Up to now, the best upper-bound for dimensions 4 and above comes from a simple volume growth argument [Bus92]. For surfaces, a better one coming from the injectivity radius was found by Bavard [Bav96], which was only improved recently by Fourtier Bourque-Petri [BP22] by linear programming techniques. Using this same approach, the upper-bound was also improved in dimension 3 by Bonifacio-Mazac-Pal [BMP23]. In all cases, the bound has logarithmic growth.

As for lower-bounds, Brooks [Bro88] and Buser-Sarnak [BS94] prove the existence of sequences of closed hyperbolic surfaces $(S_k)_k$ with genus increasing in k and $\text{sys}(S_k) \geq \frac{4}{3} \log(g_k) + O(1)$, using arithmetic constructions. This was generalised later on by Katz-Schappas-Vishne [KSV07] for other classes of surfaces and 3-manifolds, and by Murillo [Mur19] for n -manifolds, with similar methods. In dimension 2 there are also combinatorial

constructions which provide examples of surfaces with logarithmic systoles; these are given by Petri-Walker [PW18] and Liu-Petri [LP24], the latter using probabilistic techniques.

Finally, the systole has also been studied in some models of random 3-manifolds. Feller-Sisto-Viaggi [FSV22] found an upper bound to the decay rate of the length of the shortest geodesic of a 3-manifold under the model of random Heegaard splittings. A similar result was proven for the model of random mapping tori by Taylor-Sisto [ST19].

Contribution

The second and third main results of this thesis are about the asymptotic behaviour of the expected value of the systole of random hyperbolic 3-manifolds M_n , built using the model of random triangulations. These answer a question posed in [PR22, Question 3].

The first proves the existence of the limit, as $n \rightarrow \infty$, of the expected value of the systole of the model of compact hyperbolic 3-manifolds with boundary M_n , by giving an explicit (and computable) expression of it.

Theorem B ([RS24]). *Let $\{l_i\}_{i \geq 1}$ be the ordered set of all possible translation lengths coming from (classes of) words $[w] \in \mathcal{W}$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\text{sys}(M_n)] = \sum_{i=1}^{\infty} \left(\prod_{[w] \in \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \cdot l_i.$$

Here \mathcal{W} corresponds to a collection of matrices in $\text{SL}(2, \mathbb{Z}[i])$. The precise definition can be found in Section 4.1.

The third main result, Proposition C, completes the answer to the question by computing a sharp numerical approximation of this value.

Proposition C ([RS24]). *We have:*

$$\lim_{n \rightarrow \infty} \mathbb{E}(\text{sys}(M_n)) = 2.56033312683887522062 \pm 2.95489 \cdot 10^{-16}.$$

Structure and ideas of the proofs

The general strategy of the proof of Theorem B follows the same two big steps as the one for Theorem A.

Thus, first, we compute the limit of the expected value for the model of hyperbolic manifolds Y_n . To get here the expression of the expected value, we again translate the study of the number of closed geodesics of certain lengths in Y_n to the study of their corresponding cycles in the dual graph G_{Y_n} , and rely on our Theorem 3.1.2. Nevertheless, to prove the result, we need also to make sure that we can swap the limit and the infinite sum given by the expected value. For that, we apply the dominated convergence theorem, using mainly graph theory tools such as Corollary 1.3.2, by McKay-Wormald-Wysocka.

The second part of the proof consists, as before, on seeing that the result applies also to the compactified manifolds M_n . For that, we prove that the contribution to the expected value of a set of "bad" manifolds -in which the systole could degenerate- goes to zero as $n \rightarrow \infty$. The study of these potential "bad" manifolds relies heavily on the proof of Proposition 3.2.1, and uses results coming from both graph theory, like Corollary 1.3.2 or Bordenave's Lemma 1.3.4, and hyperbolic geometry, like Theorem 3.2.3 by Futer-Purcell-Schleimer.

The argument for finding the numerical value of Proposition C also goes in two steps. Indeed, we divide the infinite sum corresponding to the expected value in two terms: a computable part and an error term, that we bound. Thus, the first sum is computed with a Sage program, showed later in the text. For the error term, we carefully study the probabilities appearing, and use Theorem 3.1.2 together with some computational data to obtain the bound. The main difficulty that we encounter -which differentiates it from the two dimensional case [Pet17]- is that there is not a natural way of ordering the lengths l_i from the information given by the cycles. This makes the computation much less straightforward.

Organization

The thesis is divided into 5 chapters. Here it is an outline of their content:

- Chapter 1: In this chapter, we give an overview of the mathematical background that is involved in the thesis. It has three sections, containing mathematical notions on hyperbolic geometry, probability theory and random graph theory correspondingly.
- Chapter 2: Here we present the models of random 3-manifolds we use for our results. We start by explaining the main model of interest, the model of random triangulations, and we state some known properties of these random manifolds. Then, we describe another model of random 3-manifolds, that plays an auxiliary role in the proofs of both Theorems A and B, and end by describing the relation between the two.
- Chapter 3: This chapter is based on the article [RS23]. It contains the proof of Theorem A. There are two big sections, corresponding to the two general steps of the proof explained above.
- Chapter 4: This chapter is extracted from the article [RS24]. It has three sections; the firsts two contain (the two steps of) the proof of Theorem B, and the last one the proof of Proposition C.
- Chapter 5: We conclude the thesis by giving some future research directions, which emerge after the work presented in the manuscript.

Chapter 1

Background

The topic of this thesis connects mainly three domains of mathematics: hyperbolic geometry, probability theory and graph theory. Hence, in this chapter, we will recall some basic notions used later in the text, of all three domains.

1.1 Hyperbolic geometry

Much of the content in this section is based on the books of Martelli [Mar16] and Marden [Mar07]. These contain a more complete explanation of all the following notions, in case the reader is interested in more details.

A *hyperbolic n -manifold* M is a complete n -dimensional Riemannian manifold with constant sectional curvature equal to -1 . The first, most elementary example of a hyperbolic manifold is the hyperbolic n -space \mathbb{H}^n . This constitutes the unique (up to isometry) simply connected hyperbolic n -manifold. Thus, another way to define a hyperbolic n -manifold is to say that it is a complete Riemannian manifold that locally looks like (or more precisely, is locally isometric to) \mathbb{H}^n . In this thesis, we will be interested in three-dimensional hyperbolic manifolds, that is, when $n = 3$.

A way of thinking of \mathbb{H}^3 is by considering a *model*. Here we present two models, which are the ones that will appear in the rest of the manuscript.

- The *Upper half-space model*:

$$\mathcal{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

with the metric:

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

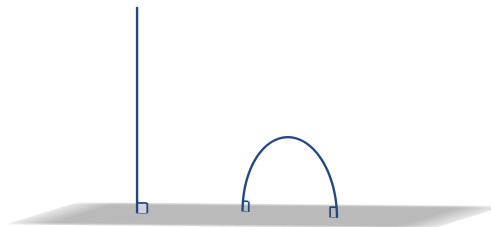


Figure 1.1: Geodesics in the upper half-space model \mathcal{H}^3 (blue).

- The *Poincaré ball model*:

$$\mathcal{B}^3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$$

with the metric:

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - (x^2 + y^2 + z^2))^2}.$$

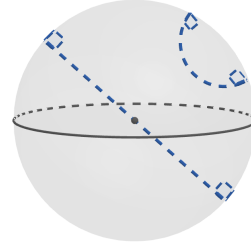


Figure 1.2: Geodesics in the Poincaré ball model \mathcal{B}^3 (blue).

Now that we have introduced \mathbb{H}^3 , we have that any other orientable hyperbolic 3-manifold can be obtained as a quotient \mathbb{H}^3/Γ , where Γ is a discrete, torsion-free subgroup (*Kleinian group*) of $\text{Isom}^+(\mathbb{H}^3)$, the group of orientation-preserving isometries of \mathbb{H}^3 . This group, for $n = 2, 3$ is particularly nice, since it is isomorphic to a well known group of matrices. In our case,

$$\text{Isom}^+(\mathbb{H}^3) \simeq \text{PSL}(2, \mathbb{C}) = \{M \in M_{2 \times 2}(\mathbb{C}) : \det(M) = 1\} / \{\pm Id_2\}.$$

The action of this group on the boundary of \mathbb{H}^3 - $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ - under the upper half-space model \mathcal{H}^3 is given by Möbius transformations [MT98, Theorem 1.8]:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

This action has an extension to an isometry on \mathbb{H}^3 , by decomposing the previous map into composition of two reflections on circles C_1 and C_2 in $\hat{\mathbb{C}}$. Indeed, these circles define two hemispheres H_1 and H_2 in \mathcal{H}^3 . Then, each reflection on C_i has a unique extension to \mathbb{H}^3 as a reflection in the hemisphere spanned by the circle. The Möbius transformation, then, acts on \mathbb{H}^3 by composition of these reflections in the hemispheres.

Moreover, every isometry of \mathbb{H}^3 can be seen as the extension of a conformal map from $\hat{\mathbb{C}}$ to itself, since it must send hemispheres orthogonal to $\hat{\mathbb{C}}$ to hemispheres orthogonal to $\hat{\mathbb{C}}$, and so circles in $\hat{\mathbb{C}}$ to circles in $\hat{\mathbb{C}}$. We can, in fact, classify the types of isometries of \mathbb{H}^3 by looking at the behaviour of this conformal map in the boundary: either it fixes two points in $\partial\mathbb{H}^3$ and none in \mathbb{H}^3 (in which case the isometry is called *loxodromic* or *hyperbolic*), it fixes one point in $\partial\mathbb{H}^3$ and none in \mathbb{H}^3 (*parabolic*), or it fixes no point in $\partial\mathbb{H}^3$ and at least one in \mathbb{H}^3 (*elliptic*). The type of isometry they represent is determined by the trace of the element in $\text{PSL}(2, \mathbb{C})$. For more information, see [Mar07, Chapter 1].

Remark 1. *If we consider hyperbolic manifolds M with totally geodesic boundary, the definition is slightly different: they are quotients of closed convex subsets of \mathbb{H}^3 by discrete,*

torsion-free subgroups $\Gamma < \text{Isom}^+(\mathbb{H}^3)$. Thus, the interior of these manifolds is locally isometric to \mathbb{H}^3 , and the boundary locally isometric to a half-space in \mathbb{H}^3 . This boundary ∂M is a hyperbolic surface without boundary.

1.1.1 Coxeter polyhedra

Interesting examples of hyperbolic manifolds can be built using Coxeter polyhedra. Recall that a finite polyhedron in \mathbb{H}^n is the convex hull of finitely many points in $\mathbb{H}^n \cup \partial\mathbb{H}^n$. Then, a *Coxeter polyhedron* P in \mathbb{H}^n is a polyhedron in \mathbb{H}^n in which the dihedral angle of every codimension-two faces divides π . The dihedral angles can be computed by considering the two planes spanned by the corresponding faces, and taking the normal vectors at the point where the planes intersect. These two vectors define an angle, which equals the dihedral angle between the faces. Some examples of finite Coxeter polyhedra in \mathbb{H}^3 are the ideal dodecahedron (with dihedral angles $\frac{\pi}{3}$), or the ideal regular right-angled octahedron (with dihedral angles $\frac{\pi}{2}$) showed in Figure 1.3.

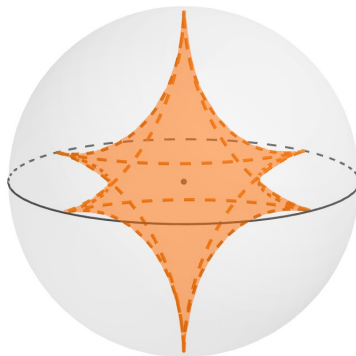


Figure 1.3: Ideal right-angled regular octahedron in \mathcal{B}^3 .

The reflections along the facets of a finite Coxeter polyhedron P generate a discrete isometry group. These groups are called *reflection groups*. Each of these reflection groups Γ contains, by Selberg's Lemma, a torsion-free subgroup Γ' of finite index k . Then, the quotient

$$M = \mathbb{H}^n / \Gamma'$$

is a hyperbolic manifold, tessellated with k copies of P .

1.1.2 Curves

A *closed curve* γ in a manifold M is a smooth map $\gamma : \mathbb{S}^1 \rightarrow M$. It is said to be *simple* if it doesn't intersect itself. We say that two closed curves γ_1, γ_2 on M are *freely homotopic* if there exists a continuous map $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow M$ such that $f|_{\{0\} \times \mathbb{S}^1} = \gamma_1$ and

$f \sim_{\{1\} \times \mathbb{S}^1} \gamma_2$. This is an equivalence relation. In this manuscript, we will generally be looking at free-homotopy classes of these curves, that will be denoted by $[\gamma]$.

Let $M = \mathbb{H}^n/\Gamma$ be now a hyperbolic manifold. We say that a curve is *homotopically trivial* if it is homotopic to a point, and *peripheral* if it is homotopic to a cusp in M - a hyperbolic manifold \mathbb{H}^n/Γ' , where $\Gamma' < \Gamma$ is a subgroup of parabolic isometries of \mathbb{H}^3 (see Section 1.1.5 for more details).

A standard fact in hyperbolic geometry, highly relevant for our purposes, is the following:

Proposition 1.1.1 ([Mar16], Proposition 4.1.13). *Let M be a complete hyperbolic manifold of dimension $n \geq 2$, and γ a non-homotopically trivial and non-peripheral closed curve in M . Then, there exists a unique closed geodesic in the free homotopy class of γ , which minimises the length in the homotopy class.*

We know, on the other hand, that there is a one-to-one correspondence between the free homotopy classes of closed curves in $M = \mathbb{H}^n/\Gamma$, and the conjugacy classes in Γ [Mar16, Section 4.1.5]. Thus, given the nice description of $\text{Isom}^+(\mathbb{H}^3)$, one can obtain information about the (homotopy classes of) curves $[\gamma]$ on a hyperbolic 3-manifold M by studying the matrices lying in the corresponding conjugacy classes $[M_\gamma]$. For instance, we can compute the length of the closed geodesic in the homotopy class by looking the traces of the matrices. This length is exactly the translation length of $[M_\gamma]$, that is, the distance between p and $M_\gamma(p)$ for any $p \in \text{axis}(M_\gamma)$ -the geodesic line in \mathbb{H}^3 preserved by the isometry. This is given by:

$$l_\gamma(M_\gamma) = 2\text{Re} \left[\text{arcosh} \left(\frac{\text{trace}([M_\gamma])}{2} \right) \right]. \quad (1.1)$$

Closed curves are interesting objects to study as they contain important information about the geometry of the manifold. For instance, their lengths determine the Laplacian spectrum or the volume of the manifold.

There are two geometric invariants directly related to them, which constitute the objects of study in this thesis: the length spectrum and the systole.

1.1.3 The length spectrum

The *length spectrum* is the (multi-)set of lengths of all closed geodesics of a hyperbolic manifold M . This set is a discrete and countable set of \mathbb{R}_+ . This follows from Proposition 1.1.1.

A closed geodesic is called *primitive* if it travels along its image exactly once. One can consider then the *primitive length spectrum*, which is the set of lengths of primitive closed geodesics. Since every closed geodesic can be expressed uniquely as a power of a primitive closed geodesic, we see that the latter completely determines the general length spectrum. This is the set that will be studied in Chapter 3.

1.1.4 The systole

The *systole* of a hyperbolic manifold M is length of the shortest closed geodesic in M . Hence, it is the first element of the primitive length spectrum.

The term "systole" is also used to refer to the geodesics realising the shortest length. In this text, however, we'll take the first definition.

We recall here the classical upper bound on the systole coming from the volume growth, as will be used later on.

Lemma 1.1.2. *Let M be a closed hyperbolic manifold of dimension $n \geq 2$. Then, there exists a constant $c_n > 0$ such that*

$$\text{sys}(M) \leq \frac{2}{n-1} \log(\text{vol}(M)) + c_n.$$

Proof. For any point $p \in M$, let $B(p, \text{sys}(M)/2)$ denote the open ball of center p and radius $\text{sys}(M)/2$. We want to show first that this ball is isometric to an open ball in \mathbb{H}^n .

We argue by contradiction: suppose it is not isometric. This means that the supremal radius $r > 0$ for which an open ball of center p and this radius is isometric to an open ball in \mathbb{H}^n is smaller than $\text{sys}(M)/2$.

Consider then two geodesic segments γ, γ' of length this radius $r > 0$, going from the point p to some point $q \in M$. We see that the composition $\gamma^{-1} \circ \gamma'$ is a geodesic loop of length $2r < \text{sys}(M)$. Moreover, it is non-contractible, as otherwise its lift in \mathbb{H}^n would be a geodesic bigon, but these don't exist. We have found then a non-contractible geodesic closed curve in M with length smaller than the systole. This gives us a contradiction, by definition of systole.

Now that we know this, let $\tilde{p} \in \mathbb{H}^n$ be a lift of the point p , and consider the open ball $B(\tilde{p}, \text{sys}(M)/2) \subset \mathbb{H}^n$ of center \tilde{p} and radius $\text{sys}(M)/2$. We have that:

$$\text{vol}(B(\tilde{p}, \text{sys}(M)/2)) = \text{vol}(B(p, \text{sys}(M)/2)).$$

On the other hand, the volume of an open ball in M is smaller than the volume of the whole manifold M . Hence, we get:

$$\text{vol}(B(\tilde{p}, \text{sys}(M)/2)) \leq \text{vol}(M).$$

Finally, if we compute the expression on the left, we obtain:

$$\text{vol}(B(\tilde{p}, \text{sys}(M)/2)) = \text{vol}(\mathbb{S}^{n-1}) \cdot \int_0^{\text{sys}(M)/2} \sinh^{n-1}(t) dt \leq \text{vol}(M),$$

which implies the lemma. □

This invariant will be studied in Chapter 4.

1.1.5 Thick-thin decomposition

In this manuscript, we consider manifolds with finite volume. However, we deal with both compact and non-compact ones. For all of them, we have the so-called *Thick-thin decomposition for hyperbolic manifolds*, a consequence of Margulis' lemma that enables us to understand much better the topology of these manifolds.

In general lines, this decomposition divides the manifolds into two parts: a compact one, and another with very simple topology. To define these more precisely, we need the notion of injectivity radius.

Let M be a hyperbolic manifold, and $x \in M$. The *injectivity radius*, denoted by $\text{injrad}(x)$, is defined as:

$$\begin{aligned} \text{injrad}(x) &= \sup\{r > 0 : B(x, r) \subset M \text{ is isometric to a ball in } \mathbb{H}^n\} \\ &= \frac{1}{2} \inf\{\text{len}(\gamma) : \gamma \text{ is a geodesic loop based at } x\}. \end{aligned}$$

where a *geodesic loop* γ_x in M based at x is a smooth map $\gamma_x : [0, 1] \rightarrow M$ that is a geodesic, and such that $\gamma_x(0) = \gamma_x(1) = x$.

Then, for $\epsilon > 0$, the ϵ -*thick* and ϵ -*thin* parts of M correspond to:

$$M^{\geq \epsilon} = \{x \in M : \text{injrad}(x) \geq \epsilon/2\}, \quad M^{< \epsilon} = \{x \in M : \text{injrad}(x) < \epsilon/2\}.$$

Then, the thick-thin decomposition theorem states the following:

Theorem 1.1.3 (Margulis). *There exists a universal constant $\epsilon_n > 0$ (the Margulis constant), depending only on the dimension n of the manifold, such that for any $0 < \epsilon < \epsilon_n$, the ϵ -thin part of any orientable hyperbolic n -manifold M consists of R -tubes around short geodesics (Margulis tubes), and truncated cusps. On the other hand, the ϵ -thick part is a compact hyperbolic manifold.*

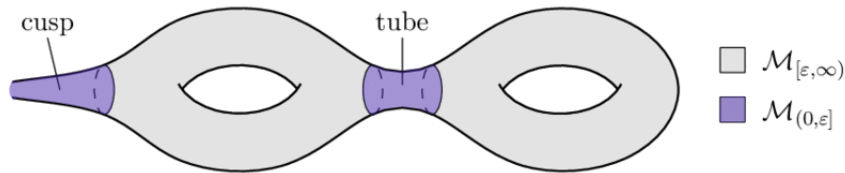


Figure 1.4: Thick-thin decomposition of a non compact hyperbolic surface.

Let's describe more precisely the geometry of these two components forming the ϵ -thin part.

Tubes. An *infinite tube* is a hyperbolic manifold $M = \mathbb{H}^n / \Gamma$, where $\Gamma = \langle \varphi \rangle$ is an infinite cyclic group generated by a hyperbolic element φ on \mathbb{H}^n with axis I and translation length $d > 0$.

Given a point $p \in I$, the Dirichlet domain $D(p)$ of Γ , that is, the set:

$$D(p) = \{q \in \mathbb{H}^n : d_{\mathbb{H}^n}(p, q) \leq d_{\mathbb{H}^n}(p, \varphi(q))\}$$

is the space lying between two hyperplanes orthogonal to I at distance $d/2$ [Mar16, Section 4.1.2]. Thus, the infinite tube M is obtained from $D(p)$ by identifying the two hyperplanes along φ . The axis I projects in M onto a closed geodesic α of length d , called the *core geodesic* of the tube.

Now, an R -tube is obtained by the quotient $N_R(I)/\Gamma$, where $N_R(I)$ is the closed neighbourhood of I of radius R (see Figure 1.5). Thus, it is the R -neighbourhood of the core geodesic α . For $n = 3$ these R -tubes are diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$ (a solid torus).

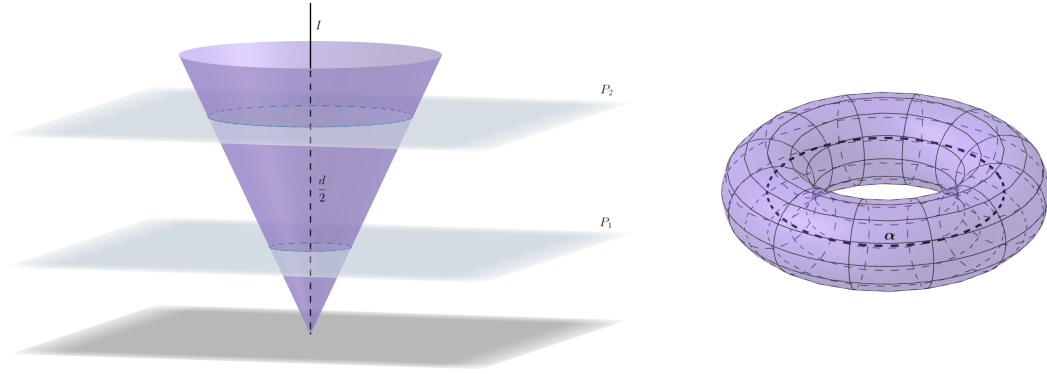


Figure 1.5: Section of a R -neighbourhood of the vertical axis I in \mathcal{H}^3 , intersected by two orthogonal planes at distance $d/2$ (left). R -tube around the core geodesic α (right).

Cusps. A *cusp* is a hyperbolic manifold \mathbb{H}^n/Γ , where Γ is a discrete group generated by parabolic elements of \mathbb{H}^n fixing a point at infinity.

Indeed, let $\Gamma < \text{Isom}(\mathbb{R}^{n-1})$ be a non-trivial discrete group of Euclidean isometries acting freely on \mathbb{R}^{n-1} . Then, the quotient $M = \mathbb{R}^{n-1}/\Gamma$ is a flat $(n-1)$ -manifold. If we consider the upper half-space model of \mathbb{H}^n , every element of Γ acts as a parabolic transformation on \mathbb{H}^n preserving the vertical coordinate. Thus, Γ can be seen as a discrete group of parabolic transformations of \mathbb{H}^n fixing the point ∞ . The quotient \mathbb{H}^n/Γ is then diffeomorphic to $M \times \mathbb{R}_{>0}$ [Mar16, Section 4.1.3]. Every cusp comes from this construction.

Then, a *truncated cusp* is the quotient of a *horoball*, that is, the set

$$\{(z, t) \in \mathbb{H}^n : t \geq a > 0\}$$

in the upper half-space of \mathbb{H}^n , by a discrete group of Euclidean translations Γ preserving the vertical coordinate (see Figure 1.6). This is diffeomorphic to the portion $N = M \times [a, \infty)$, $a > 0$, bounded by the Euclidean manifold $M \times a$.

For our case of interest, $n = 3$ and $\Gamma < \text{Isom}^+(\mathbb{R}^2)$, we have that either $\Gamma \cong \mathbb{Z}$ -in

which case the cusp is called of *rank one*- or $\Gamma \cong \mathbb{Z}^2$ -the cusp being called of *rank two*. In the first case, a truncated cusp is then diffeomorphic to $\mathbb{S}^1 \times \mathbb{R} \times [a, \infty)$, $a > 0$, and in the second it is diffeomorphic to $\mathbb{T}^2 \times [a, \infty)$, $a > 0$.

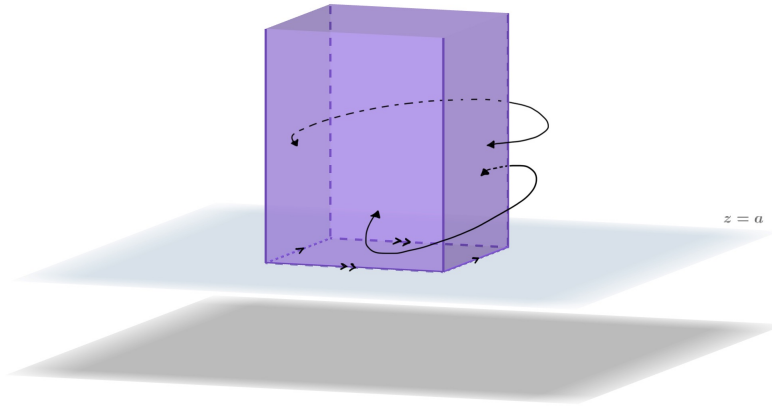


Figure 1.6: A fundamental domain in \mathcal{H}^3 of a rank two truncated cusp.

1.1.6 Hyperbolic Dehn filling

A Dehn filling is an operation particular to three dimensional geometry, used to modify 3-manifolds. Consider first a 3-manifold M with boundary made of tori T_1, \dots, T_k . Then, a *Dehn filling* on M is the operation of attaching solid tori to its torus boundary components T_i , along diffeomorphisms $\psi : \partial D \times S^1 \rightarrow T_i$. Each of the meridians $\partial D \times \{t\}$ of the solid tori is sent to some simple closed curve γ_i in T_i . The choice of these free homotopy classes of curves $[\gamma_i]$ -called the *slopes*- determines, up to homeomorphism, the Dehn filling procedure. Figure 1.7 shows an example.

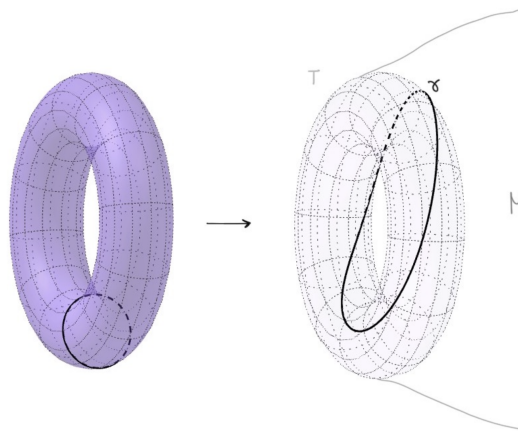


Figure 1.7: Dehn filling

This operation enables us, in particular, to construct compact hyperbolic manifolds from non-compact ones. Indeed, every rank-2 cusped hyperbolic 3-manifold can be thought as the interior of a compact manifold with toroidal boundary. Then, by doing the procedure above, one obtains a new manifold, which "most likely" will be hyperbolic. This was made precise by Thurston, in his Hyperbolic Dehn filling theorem [Thu82, Theorem 2.6]:

Theorem 1.1.4 (Hyperbolic Dehn filling theorem). *Let M be a finite-volume orientable hyperbolic 3-manifold. Then, except for finitely many Dehn fillings on each boundary component, all other Dehn fillings of M admit a finite-volume hyperbolic structure.*

A Dehn filling operation can also be done on manifolds with rank-1 cusps. In this case, the procedure is done with solid cylinders, which are attached, via diffeomorphisms, to cylinders which are part of the boundary components of the 3-manifold. In contrast with the rank-2 case, here there is only one possible slope to take (see Figure 1.8), and so the Dehn filling is already determined.

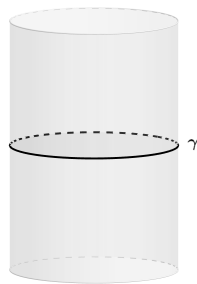


Figure 1.8: The unique slope $[\gamma]$ in a cylinder.

1.1.7 Mostow-Prasad rigidity

We end this section by giving a result that differentiates considerably the geometry of hyperbolic 3-manifolds with respect to dimension 2:

Theorem 1.1.5 (Mostow-Prasad rigidity theorem). *Let M_1 and M_2 be two complete, finite-volume hyperbolic manifolds of dimension $n \geq 3$. Then, any isomorphism $f : \pi_1(M_1) \rightarrow \pi_1(M_2)$ between their fundamental groups is induced by a unique isometry from M_1 to M_2 .*

This theorem tells us essentially that if a manifold of dimension $n \geq 3$ admits a complete finite-volume hyperbolic structure, this is unique up to isometry. Thus, the geometry of a closed hyperbolic 3-manifold is a topological invariant, and so geometric properties such as the ones we've just introduced, depend only on the topology of the manifold.

1.2 Probability theory

The notions on probability theory contained in this section can be found generally in the lectures of any probability course. A good source for more information is also Bollobás' book [Bol01].

A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the sample space -a non-empty set-, \mathcal{F} is a σ -algebra -a set of subsets of Ω that is closed under complements and countable unions and in which $\Omega \in \mathcal{F}$ - and \mathbb{P} is a probability measure -a measure that is countably additive and such that $\mathbb{P}(\Omega) = 1$.

A probability space is *discrete* if the σ -algebra \mathcal{F} is just the set of all subsets of Ω (of cardinal 2^Ω), where Ω is finite or countable. In this case, the probability measure \mathbb{P} is defined by the probability of its singletons $\{w\}$, for $w \in \Omega$. This is the space in which we will work. For simplicity, we will write (Ω, \mathbb{P}) when referring to the probability space. Thus, the rest of the concepts of this section will concern this case.

A *discrete random variable* is a real-valued function $X : \Omega \rightarrow \mathbb{R}$ taking finitely or countably many values a_1, a_2, \dots . In this case, its *expected value* (or first moment) is defined as:

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} a_i \cdot \mathbb{P}[X = a_i],$$

provided that the sum is absolutely convergent in the infinite case. In general for $n \in \mathbb{N}$, we define the *nth moment* of X as $\mathbb{E}[X^n]$. There exists also the *nth-factorial moment* of X , which is defined as:

$$\mathbb{E}[(X)_n] = \mathbb{E}[X(X-1)(X-2)\cdots(X-n+1)].$$

Related to this, there is also the *variance* of X , which can be written as $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

One of the most classical inequalities concerning the expected value is Markov's inequality, which states the following:

Theorem 1.2.1 (Markov's inequality). *Let X be a non-negative random variable such that $\mathbb{E}[X]$ is well defined, and $a > 0$. Then,*

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Despite its simple statement, it is a very powerful tool, as it enables to find a bound on the probability only in terms of the expected value, which is sometimes easier to determine.

1.2.1 Probability distributions

Given a discrete random variable X , its probability distribution is given by the probabilities of each of its values. In this manuscript, we will be speaking about two types of probability distributions: the (discrete) uniform distribution and the Poisson distribution. Let's define them.

The *discrete uniform distribution* is a symmetric probability distribution in which every possible outcome has equal probability. Thus, here Ω is a finite set, and $\mathbb{P} : 2^\Omega \rightarrow [0, 1]$ is defined by:

$$\mathbb{P}[A] = \frac{|A|}{|\Omega|}, \quad \text{for all } A \subset \Omega.$$

We write that $X \sim \mathcal{U}(\Omega)$.

On the other hand, a random variable $X : \Omega \rightarrow \mathbb{N}$ is said to have a *Poisson distribution with parameter $\lambda > 0$* if:

$$\mathbb{P}[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad \text{for all } k \in \mathbb{N}.$$

We note it by $X \sim \text{Pois}(\lambda)$. We recall also some of its properties:

- $\mathbb{E}[X] = \text{Var}(X) = \lambda$.
- $\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)\cdots(X-r+1)] = \lambda^r$.
- If $X_i \sim \text{Pois}(\lambda_i)$ for $i = 1, \dots, n$ are independent r.v, then the r.v $Y = \sum_{i=1}^n X_i \sim \text{Pois}(\sum_{i=1}^n \lambda_i)$.

1.2.2 Convergence

Given a sequence of random variables $(X_n)_{n \in \mathbb{N}}$, we can ask what can we say about its limiting behaviour. There are different notions of convergence of random variables, capturing different properties about the sequence. Here we will present two: convergence in distribution and almost sure convergence.

Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables, and X another random variable. For our purposes, it is enough to consider \mathbb{N} -valued random variables.

Then, we say that $(X_n)_{n \in \mathbb{N}}$ *converges in distribution* to X if:

$$\lim_{n \rightarrow \infty} \mathbb{P}[X_n = a] = \mathbb{P}[X = a],$$

for any $a \in \mathbb{N}$. We denote it by $X_n \xrightarrow{d} X$ as $n \rightarrow \infty$. This definition can be extended analogously to random vectors.

On the other hand, suppose that $(X_n)_{n \in \mathbb{N}}$ and X are both defined in the same probability space (Ω, \mathbb{P}) . Then, we say that $(X_n)_{n \in \mathbb{N}}$ *converges almost surely* to X if the sequence

of numbers $(X_n(w))_{n \in \mathbb{N}}$ converges to $X(w)$ for almost all $w \in \Omega$, that is,

$$\mathbb{P}[\{w \in \Omega : \lim_{n \rightarrow \infty} X_n(w) = X(w)\}] = 1.$$

We denote it by $X_n \xrightarrow{\text{a.s.}} X$ as $n \rightarrow \infty$. This notion of convergence implies convergence in distribution.

1.2.3 Method of moments

The *method of moments* is one of the most classical methods in probability theory to prove convergence in distribution. For this to work, we need to apply it to random variables whose distribution is determined by its moments. We say that the distribution of a random variable X is *determined by its moments* if X has finite moments -that is, $\mathbb{E}[X^n] < \infty$ for all $n \in \mathbb{N}$ -, and any other random variable with the same moments as X has the same distribution. Since this definition is hard to verify, one can alternatively look at the moment generating function $\mathbb{E}[e^{tX}]$. Indeed, a sufficient condition for the distribution of X to be determined by its moments is that $\mathbb{E}[e^{tX}] < \infty$ where $t \in (-a, a)$, for some $a > 0$. This is for instance true if X is Poisson or normally distributed.

Given a sequence $(X_n)_{n \in \mathbb{N}}$ that verifies this, the method of moments asserts that, for every k fixed, we are able to estimate the factorial moments $\mathbb{E}[(X_n)_k]$, then we can conclude that $(X_n)_{n \in \mathbb{N}}$ converges in distribution to some random variable [Bol01, Section 1.4].

We will be interested in applying this method to random vectors, for proving convergence to a Poisson distribution. Note that this method is specially convenient for this, as the factorial moments of a Poisson random variable have a very simple expression. Thus, we state the version of the result particular to this case.

Theorem 1.2.2 ([JLuR00], Theorem 6.10). *Let $\{(X_n^1, \dots, X_n^m)\}_{n \in \mathbb{N}}$ be vectors of random variables, where $m \geq 1$ is fixed. If $\lambda_1, \dots, \lambda_m \geq 0$ are such that, as $n \rightarrow \infty$,*

$$\mathbb{E}[(X_n^1)_{k_1} \cdots (X_n^m)_{k_m}] \rightarrow \lambda_1^{k_1} \cdots \lambda_m^{k_m}$$

for every $k_1, \dots, k_m \geq 0$, then $(X_n^1, \dots, X_n^m) \xrightarrow{d} (Z_1, \dots, Z_m)$, where:

- *For each $i = 1, \dots, m$, Z_i is a Poisson random variable with parameter λ_i .*
- *For all $1 \leq i, j \leq m$, $i \neq j$, the random variables Z_i and Z_j are independent.*

The expectations $\mathbb{E}[(X_n^1)_{k_1} \cdots (X_n^m)_{k_m}]$ are usually called the *joint factorial moments* of the random vector (X_n^1, \dots, X_n^m) , which appear only in the multivariate case of the theorem.

This method is often used for counting variables of the form $S_I = \sum_{a \in A} \mathbb{1}_a$, where $\mathbb{1}_a$ are indicator variables. Here $(S_I)_k$ counts the number of ordered k -tuples of elements such

that $\mathbb{1}_a = 1$, and so the k th-factorial moment of S_I has the expression:

$$\mathbb{E}[(S_I)_k] = \sum_{\substack{\forall a_1, \dots, a_k \\ a_i \neq a_j}} \mathbb{E}[\mathbb{1}_{a_1} \cdots \mathbb{1}_{a_k}] = \sum_{\substack{\forall a_1, \dots, a_k \\ a_i \neq a_j}} \mathbb{P}[\mathbb{1}_{a_1} = \dots = \mathbb{1}_{a_k} = 1].$$

This will be very useful for our case later on in Chapter 3.

1.2.4 Poisson point process

We finish this section by giving the definition of a Poisson point process, a mathematical object that will play an important role when studying the length spectrum. More information about this can be found in [Kin93, Chapter 2].

There exist different mathematical interpretations of a point process -such as a random counting measure or a random set. In our case, we follow the second one.

Let (E, μ) be a measurable space. A *Poisson point process* \mathcal{S} on E with intensity μ is a random countable subset of E which satisfies the following two properties:

- For any A countable subset of E , we define the random variable N_A as the number of points lying in $S \cap A$, that is,

$$N_A = \#\{S \cap A\}.$$

Then, N_A is Poisson distributed with parameter $\mu(A)$.

- For any A_1, \dots, A_t pairwise-disjoint countable subsets of E , the vector

$$(N_{A_1}, \dots, N_{A_t})$$

is a vector of independent random variables.

Such processes exist under the following condition on the measure μ :

Theorem 1.2.3 ([LP18], Theorem 3.6). *Let μ be an s -finite measure on E , that is, a measure that can be written as a countable sum of finite measures μ_n , $n \in \mathbb{N}$:*

$$\mu = \sum_{n=1}^{\infty} \mu_n, \quad \mu_n(E) < \infty.$$

Then there exists a Poisson process on E with intensity measure μ .

The simplest example of Poisson point process is the one where the measure μ is given by the product of a non-negative constant and the area or volume of the region. The constant is usually known as both the *intensity* or *rate*. This is called a *homogeneous Poisson point process*. When considered it on the positive half-line, this can be interpreted

as a counting process. In our case, however, μ will be an atomic measure. This case is an example of what is usually called a *general Poisson point process*.

1.3 Random regular graphs

In this section we will talk about the theory of random graphs. Much of this is extracted from [JLuR00].

Here we're interested in a particular kind of random graphs: random regular graphs. In general, a graph is said to be *regular* if every vertex has the same degree, that is, the same number of edges incident to it. A regular graph with vertices of degree d is usually called a d -regular graph. Let Ω_n^d be the set of all d -regular graphs on n vertices, for $n > d$ and dn even. Then, a *random d -regular graph* $G(n, d)$ is a random graph (Ω_n^d, \mathbb{P}) where \mathbb{P} is the uniform distribution over Ω_n^d .

It is not practical, however, to work with this definition, since we don't know a simple formula for the total number of d -regular graphs on n vertices. Thus, a way to construct such graphs is by using the so-called *configuration model*.

1.3.1 Configuration model

The *configuration model* is a method for generating random graphs of fixed degree sequence, which consists on the following two steps: let $(d_i)_{i \geq 1}$ be a sequence of n positive integers, for $n \in \mathbb{N}$, such that $\sum_i d_i$ is even. Now, let V be a set on n vertices. Then,

1. We assign a degree d_i to each vertex $v_i \in V$, $i = 1, \dots, n$. This is represented by d_i half-edges, incident to v_i .
2. We pair two half-edges uniformly at random and join them with an edge. We repeat the same procedure until all half-edges are paired. We call a *configuration* a partition of the set of half-edges.

In the case where $d_i = d$ for all $i = 1, \dots, n$, one obtains a random d -regular graph. Figure 1.9 shows an example.

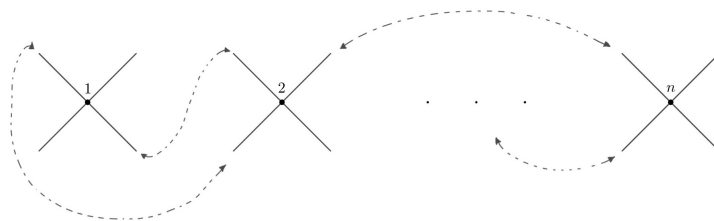


Figure 1.9: Pairing of half-edges ($d = 4$).

The total number of possible configurations in this case is:

$$\mathcal{N}(dn) = \frac{\binom{dn}{2} \binom{dn-2}{2} \cdots \binom{2}{2}}{(2n)!} = (dn-1)!!.$$

On the other hand, if we fix k independent (vertex disjoint) edges, there are:

$$\mathcal{N}_k(dn) = \frac{\binom{dn-2k}{2} \binom{dn-2k-2}{2} \cdots \binom{2}{2}}{(2n-k)!} = (dn-2k-1)!!$$

configurations containing these k edges.

The random d -regular graphs obtained with this method might have loops or multiple edges. We can, however, condition it to be *simple* -that is, without loops or multiple edges. This conditioning won't affect the properties of the graph that are true with asymptotic probability one. In other words, any property P holding asymptotically almost surely (a.a.s) for $G(n, d)$ -that is $\mathbb{P}[G(n, d) \text{ has } P] \rightarrow 1$ as $n \rightarrow \infty$ - will also hold a.a.s for the conditioned graph.

Indeed, the probability that $G(n, d)$ is simple, as $n \rightarrow \infty$, tends to:

$$\mathbb{P}[G(n, d) \text{ is simple}] \rightarrow \exp\left(-\frac{d^2-1}{4}\right) > 0.$$

Since d is fixed, we hence obtain:

$$\begin{aligned} & \mathbb{P}[G(n, d) \text{ has } P \mid G(n, d) \text{ is simple}] \\ & \leq \frac{\mathbb{P}[G(n, d) \text{ has } P]}{\mathbb{P}[G(n, d) \text{ is simple}]} \xrightarrow{n \rightarrow \infty} 1. \end{aligned}$$

Moreover, it turns out that, after conditioning it on being simple, the resulting random d -regular graph is, as $n \rightarrow \infty$, uniformly distributed over the set of all simple random d -regular graphs on n vertices [Wor99, Section 2.7].

1.3.2 Distribution of short cycles

Given a random d -regular graph, a natural question to ask is what is the distribution of the number of different types of sub-graphs on $G(n, d)$, as n grows.

It is known, for instance, that if H is a fixed graph with more edges than vertices, the expected number of subgraphs isomorphic to H tends to 0 as $n \rightarrow \infty$ [Bol80].

We will be interested in *cycles*: the closed paths in which only the first and last vertices are equal. A well known result related to this question was given by Bollobás [Bol80], who showed the following about the distribution of short cycles on $G(n, d)$. Let $X_{k,n}$ be the random variable counting the number of cycles of length k in $G(n, d)$, for $k \in \mathbb{N}$. Then,

Theorem 1.3.1 ([Bol80], Theorem 2). *As $n \rightarrow \infty$, the random variables $X_{1,n}, \dots, X_{k,n}$, converge in distribution to some random variables X_1, \dots, X_k , where:*

- *Each X_i is Poisson distributed with parameter $\lambda_i = \frac{(d-1)^i}{2^i}$, $i = 1, \dots, k$.*
- *The random variables X_1, \dots, X_k are mutually independent.*

Remark 2. *Among all short cycles, there is one that has special interest, called the girth of the graph. This is the cycle with the shortest length. One could think of it as an analogue, in the case of graphs, of the systole in systolic geometry.*

The main tool in the proof of Theorem 1.3.1 is the method of moments (see Section 1.2.3). A more recent tool to study the distribution of subgraphs in the context of random regular graph was introduced by McKay [McK81]. This is called the *switching method*.

The most basic version of the method is given by simple switchings. Given a pairing of vertices P -that is, a subset of pairs of vertices- we define a *simple switching* to be the replacement of two pairs $\{p_1, p_2\}, \{p_3, p_4\}$ by $\{p_1, p_3\}, \{p_2, p_4\}$ or $\{p_1, p_4\}, \{p_2, p_3\}$. This, consequently, induces a switching of the corresponding edges in the graph. Figure 1.10 shows an example.

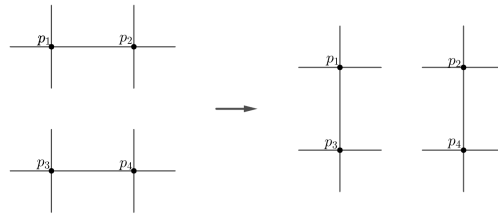


Figure 1.10: Simple switching

This method enabled, for instance, to generalise Theorem 1.3.1 for d (the degree) and k (the size of the cycle) increasing with n [MWW04, Theorem 1]. This result from McKay-Wormald-Wysocka yield, in turn, the following one concerning the girth of these graphs:

Corollary 1.3.2 ([MWW04], Corollary 1). *For $(d-1)^{2g-1} = o(n)$, the probability that a random d -regular graph has girth greater than $g \geq 3$ is*

$$\exp\left(-\sum_{r=3}^g \frac{(d-1)^r}{2^r} + o(1)\right),$$

as $n \rightarrow \infty$.

This result will appear in several arguments on Chapter 4.

Others properties of random regular graphs have been studied by different authors, such as Wormald [Wor99] or Bollobás [Bol01]. We finish by highlighting one regarding its connectivity, due to both independently:

Theorem 1.3.3 ([Bol01], [Wor81]). *Let $d \geq 3$. Then,*

$$\mathbb{P}[G(n, d) \text{ is connected}] \rightarrow 1, \text{ as } n \rightarrow \infty.$$

1.3.3 Tangle-free graphs

To end the section, we present a notion introduced first by Friedman [Fri08], and redefined later by Bordenave [Bor20] to provide a new proof of Friedman's theorem [Fri08] regarding the spectral gap of the Laplacian on large regular graphs. This property will play an important role in several proofs of this manuscript.

We say that a graph G is *tangle-free* if it contains at most one cycle (loops and multiple edges included). Similarly, G is *l -tangle-free* if every neighbourhood of radius l in G contains at most one cycle. Otherwise, we say that G is tangled or *l -tangled*.

Recall that a *neighbourhood of radius l* in a graph G is the subgraph spanned by the vertices at distance (in the graph) at most l from some fixed vertex (see Figure 1.11).

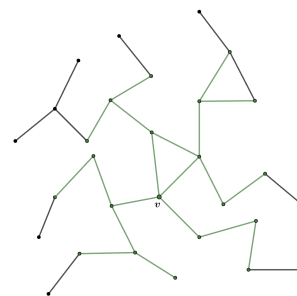


Figure 1.11: Neighbourhood of radius 3 around v (green).

Bordenave [Bor20] proved that a random d -regular graph is l -tangle-free if l is not too large. More precisely,

Theorem 1.3.4 ([Bor20], Lemma 9). *Let $d \geq 3$, and $G(n, d)$ be a random d -regular graph on n vertices generated by the configuration model. Then,*

$$\mathbb{P}[G(n, d) \text{ is } l\text{-tangle-free}] = 1 - O\left(\frac{(d-1)^{4l}}{n}\right).$$

Chapter 2

The model of random 3-manifolds

In this chapter we define the two models of random hyperbolic 3-manifolds used for our results in this thesis, and we state some of their properties. Then, we describe the geometric relation between the two models.

2.1 The model M_n

In this section, we explain the probabilistic model of *random triangulations* for 3-manifolds. The manifolds obtained with this model will be the objects of study in this manuscript, and will be denoted by M_n .

The model is an analogue in three dimensions of Brooks and Makover's model for random surfaces [BM04].

The *general idea* is to construct manifolds by *randomly gluing polyhedra together along their faces*. In dimension two, the polygon taken to do this procedure is the two-simplex, that is, the triangle, which are then glued along their edges. Thus, it seems a priori that the natural choice for creating this complex in dimension 3 would be the 3-simplex, i.e., the tetrahedron. However, as shown by Dunfield and Thurston [DT06], with this polyhedron, the procedure fails: as the number of tetrahedra grows, the result won't typically be a manifold.

Proposition 2.1.1 (Dunfield-Thurston). *Let T_n be the complex made of a random gluing of n tetrahedra along their faces. Then, as $n \rightarrow \infty$,*

$$\mathbb{P}[T_n \text{ is a manifold}] \rightarrow 0.$$

Proof. Suppose that T_n is homeomorphic to a 3-manifold. This means that every point in T_n has a neighbourhood homeomorphic to \mathbb{R}^3 . In particular, the neighbourhoods around the vertices of the complex need to be homeomorphic to balls, hence their links are spheres.

We observe that the triangulation of the complex yields a triangulation of these spheres.

Let (V_s, E_s, F_s) denote the triangulation of a sphere, where V_s , E_s and F_s are the sets of vertices, edges and faces, which satisfy $|F_s| = \frac{2}{3}|E_s|$. We know also that:

$$\chi(\mathbb{S}^2) = |V_s| - |E_s| + |F_s| = |V_s| - \frac{1}{3}|E_s| = 2.$$

Using these two relations, we see that the average degree of a vertex in a triangulation of a sphere needs to be less than 6. Indeed, this can be written as:

$$\frac{1}{|V_s|} \sum_{v \in V_s} \deg(v) = \frac{2|E_s|}{|V_s|} = \frac{6|V_s| - 12}{|V_s|} < 6$$

as $|V_s| > 0$. This, in turn, implies that the number of vertices with degree ≤ 6 is at least $1/7$ the total number of vertices in the sphere. We prove it: let S_s denote the set of vertices in V_s with degree ≤ 6 , and S_s^c its complementary. We argue by contradiction. Suppose that $|S_s| < |V_s|/7$ (and so $|S_s^c| > 6|V_s|/7$). Then,

$$\begin{aligned} 6|V_s| &> \sum_{v \in V_s} \deg(v) = \sum_{v \in S_s} \deg(v) + \sum_{v \in S_s^c} \deg(v) \\ &\geq |S_s| + \frac{6|V_s|}{7} \cdot 7 = |S_s| + 6|V_s|, \end{aligned}$$

which gives a contradiction, since $|S_s|$ needs to be bigger than 0 for the average degree to be less than 6. This remains true when considering the set of all vertices of all triangulated spheres V : the average degree of a vertex in this set is less than 6, and so the number of vertices with degree ≤ 6 is at least $1/7$ the total number of vertices.

Now, since vertices in the triangulation of the spheres correspond (2 to 1) to edges in T_n , we get that the average number of tetrahedra incident to an edge in T_n is equal to the average degree of a vertex in the triangulated spheres, and so in particular, less than 6. This implies then, that the number of edges in T_n with less that 6 tetrahedra around them is at least $|V|/7$.

This set V is a multiple of the number of tetrahedra. Indeed, since T_n is made of n tetrahedra glued along their 4 faces each, we have that the total number of faces of T_n is $2n$. These faces correspond (1 to 3) to the total number of edges E of the triangulated spheres. Hence, using the previous Euler characteristic computation, we obtain then that:

$$|V| > \frac{1}{3}|E| = 2n.$$

Therefore, T_n has at least $\frac{2n}{7}$ edges with fewer than 6 tetrahedra around them.

If we look at the dual graph of the complex, we see that an edge with k tetrahedra around it gives a cycle of length k in this graph. Thus, the previous observation tells us that there are at least $\frac{2n}{7}$ distinct cycles of length less than 6. However, the dual graph

of T_n is a random 4-regular graph. Therefore, by Bollobás' theorem 1.3.1, we obtain that the expected value of the number of cycles of length less than 6 converges to a constant, as $n \rightarrow \infty$. Therefore, by Markov's inequality 1.2.1, the probability that this dual graph has at least $\frac{2n}{7}$ distinct cycles of length less than 6 tends to 0 as $n \rightarrow \infty$. \square

This problem can be solved, however, by truncating the tetrahedra at their vertices (see Figure 2.1). Then, the complex we obtain by gluing n of these polytopes along their hexagonal faces, namely N_n , is a *random compact 3-manifold with boundary*.

The *random* aspect of the construction comes from the *gluing*. The $4n$ hexagonal faces are partitioned into pairs uniformly at random, and for each pair, one of the three cyclic-order-reversing gluings is also chosen uniformly at random.

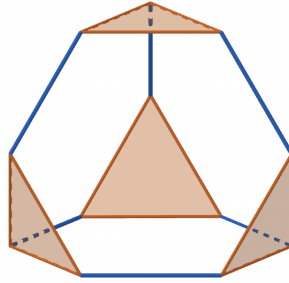


Figure 2.1: A truncated tetrahedron. The orange faces are the boundary faces, and the white ones the interior faces. N_n is obtained by gluing these polyhedra along the interior (hexagonal) faces.

More formally, the combinatorics of this model go as follows. We start by considering n tetrahedra. This will be glued randomly along their faces in pairs, through one of the three orientation reversing euclidean isometries. This gluing is described in the following way:

1. We label the n tetrahedra, that is, we assign to every vertex of these n tetrahedra a unique label in $\{1, 2, \dots, 4n\}$.
We denote the face given by the vertices $v_1, v_2, v_3 \in \{1, 2, \dots, 4n\}$ by a cycle $(v_1 v_2 v_3)$. The order of the vertices in the cycle determines at the same time an orientation on the face.
2. We partition these $4n$ faces into $2n$ pairs, uniformly at random. We denote this partition by $\rho_n = (\rho_n^{(i)})_{i=1}^{2n}$, where $\rho_n^{(i)} = \{(v_1 v_2 v_3), (w_1 w_2 w_3)\}$.
3. For each pair of faces $\rho_n^{(i)}$, we choose, again uniformly at random, one of the three cyclic-order-reversing pairings between the vertices. We denote the pairings by $\vartheta_n = (\vartheta_n^{(i)})_{i=1}^{2n}$, where $\vartheta_n^{(i)} = \{(v_1 v_2 v_3), (\vartheta_n^{(i)}(v_1) = w_1 \vartheta_n^{(i)}(v_2) = w_2 \vartheta_n^{(i)}(v_3) = w_3)\}$.

4. We identify each pair of faces $w\rho_n^{(i)}$ using the pairing of its vertices described by ϑ_n^i , for every $i = 1, \dots, 2n$.

Let T_n denote the resulting tetrahedral complex. Then, the manifold N_n we presented above is obtained by truncating T_n at the vertices.

One can deduce from classical work by Moise [Moi52] that every compact orientable 3-manifold with boundary can be obtained with this construction. In other words, every such manifold gets, as n goes to infinity, sampled by this model.

Furthermore, we can obtain random closed manifolds from N_n : it suffices to take two disjoint copies of N_n and glue their boundaries together using the identity map. Note, however, that these will have additional symmetries, hence do not represent "typical" closed 3-manifolds. The resulting random manifold will be denoted by DN_n .

The *dual graph* of the complex N_n , that is, the graph obtained by considering a vertex in each tetrahedron of N_n and joining them with an edge whenever they have a face in common, is a *random 4-regular graph*. The construction of this graph follows exactly the one described by the configuration model, a model for random regular graphs presented in Section 1.3.1. For any path p of length $k > 0$ in this graph, we define a *labelling of p* , a k -tuple of pairs:

$$((\rho_n^{(1)}, \vartheta_n^{(1)}), \dots, (\rho_n^{(k)}, \vartheta_n^{(k)})) \quad (2.1)$$

where each $\rho_n^{(i)}$ denoted a pairing between two faces (and so two half-edges), and $\vartheta_n^{(i)}$ denoted the pairing between the vertices of the two corresponding faces in the octahedra, describing the orientation on the pairing of these half-edges. We denote by Lab_p the set of all labellings the path p can have as a set of all possible k -tuples of pairs of the form (2.1). These labellings will play a role in the proof of Theorem 3.1.2.

We recall that, in this model, the expected number of loops and multi-edges converge to Poisson random variables with parameters $\frac{3}{2}$ and $\frac{9}{4}$ respectively. Thus, for any property P that holds a.a.s. for N_n (i.e, $\mathbb{P}[N_n \text{ has } P] \rightarrow 1$ as $n \rightarrow \infty$) also does for the manifold conditioned on its dual graph being simple. Therefore, we will condition our manifold N_n on not having any loops or multi-edges in the dual graph of the tetrahedral complex. This manifold will be the one denoted by M_n .

2.1.1 Properties

Several topological and geometric properties of M_n were studied by Petri and Raimbault in [PR22]. These are collected in the following two results.

The first concerns the topology of M_n . Before stating it, we recall that the *Heegaard genus* of a compact oriented 3-manifold is the minimal genus of the splitting surface in a Heegaard splitting of that manifold.

Theorem 2.1.2 ([PR22], Theorem 1.1). (a) We have:

$$\lim_{n \rightarrow \infty} \mathbb{P}[M_n \text{ is connected and has a single boundary component}] = 1.$$

(b) The genus $g(\partial M_n)$ of the boundary of M_n satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}[n - \theta(n) \leq g(\partial M_n) \leq n + 1] = 1,$$

for any function $\theta : \mathbb{N} \rightarrow \mathbb{R}$ that grows super-logarithmically.

(c) Let DM_n denote the double of M_n along its boundary and $g(DM_n)$ its Heegaard genus. Then,

$$\lim_{n \rightarrow \infty} \mathbb{P}[n - \theta(n) \leq g(DM_n) \leq n + \theta(n)] = 1,$$

for any function $\theta : \mathbb{N} \rightarrow \mathbb{R}$ that grows super-logarithmically.

(d) The Betti numbers $b_1(M_n)$ and $b_1(M_n, \partial M_n)$ satisfy:

$$\lim_{n \rightarrow \infty} \mathbb{P}[b_1(M_n, \partial M_n) \leq \theta(n)] = 1, \quad \lim_{n \rightarrow \infty} \mathbb{P}[|b_1(M_n) - n| \leq \theta(n)] = 1,$$

for any function $\theta : \mathbb{N} \rightarrow \mathbb{R}$ that grows super-logarithmically.

The second one contains some geometric properties of these manifolds.

Theorem 2.1.3 ([PR22], Theorem 1.2). We have

$$\lim_{n \rightarrow \infty} \mathbb{P}[M_n \text{ carries a hyperbolic metric with totally geodesic boundary}] = 1.$$

This metric has the following properties:

(a) The hyperbolic volume $\text{vol}(M_n)$ of M_n satisfies:

$$\lim_{n \rightarrow \infty} \mathbb{P}\left[\frac{\text{vol}(M_n)}{n \cdot v_O} \in [1 - \epsilon, 1 + \epsilon]\right] = 1,$$

for any $\epsilon > 0$, where v_O denotes the volume of the regular right-angled ideal hyperbolic octahedron.

(b) There exists a constant $c_\lambda > 0$ so that the first discrete Laplacian eigenvalue $\lambda_1(M_n)$ of M_n satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}[\lambda_1(M_n) > c_\lambda] = 1.$$

(c) There exists a constant $c_d > 0$ such that the diameter $\text{diam}(M_n)$ of M_n satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{diam}(M_n) < c_d \log(\text{vol}(M_n))] = 1.$$

(d) There exists a constant $c_s > 0$ such that the systole $\text{sys}(M_n)$ of M_n satisfies

$$\lim_{n \rightarrow \infty} \mathbb{P}[\text{sys}(M_n) > c_s] = 1.$$

(e) For every $\epsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{1 - \epsilon}{4n} < \text{sys}(DM_n) < \frac{1}{n^{1-\epsilon}} \right] = 1.$$

The same holds for the minimal length among arcs in M_n that are homotopically non-trivial relative to ∂M_n .

The fact that these manifolds admit a hyperbolic metric is a fundamental feature for our further study. Mostow's rigidity theorem (Theorem 1.1.5) tells us then that this metric is unique up to isometry. This implies, in particular, that geometric invariants of M_n like the length spectrum or the systole become topological invariants, and so can be understood from the combinatorics of the gluing.

Remark 3. *The condition that the dual graph of M_n is simple is needed for the proof of the hyperbolicity of M_n .*

2.2 The model Y_n

Here we describe the details of a model of random non-compact hyperbolic 3-manifolds. As mentioned before, this model appears in the proof of hyperbolicity of M_n , and plays an auxiliary but necessary role in the proofs of both Theorem A and Theorem B. The manifolds constructed under this model will be denoted by Y_n .

The building block for the topological random model M_n was a truncated tetrahedron. Now, observe that if we contract the edges joining the triangular faces of a truncated tetrahedron, we get an octahedron (see Figures 2.2 and 2.3). This will be the building block for Y_n .

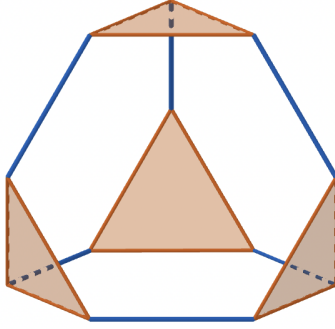


Figure 2.2: A truncated tetrahedron.

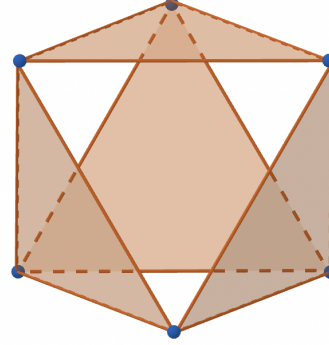


Figure 2.3: The octahedron resulting from contracting the blue edges of Figure 2

Now, the gluing procedure is very similar to the one described above. We consider initially $n \in \mathbb{N}$ copies of a regular octahedron. Then,

1. For each octahedron O_i , $i = 1, \dots, n$, we attribute to its vertices a unique label in $\{a^i, b^i, c^i, d^i, e^i, f^i\}$.
We denote the face given by the vertices $v_1, v_2, v_3 \in \{a^i, \dots, f^i\}$ by a cycle $(v_1 v_2 v_3)$. The order of the vertices in the cycle determines at the same time an orientation on the face.
2. We consider four non-adjacent faces in each O_i , and we partition these $4n$ faces into $2n$ pairs, uniformly at random. We denote this partition by $\rho_n = (\rho_n^{(i)})_{i=1}^{2n}$, where $\rho_n^{(i)} = \{(v_1 v_2 v_3), (w_1 w_2 w_3)\}$.
3. For each pair of faces $\rho_n^{(i)}$, we choose, again uniformly at random, one of the three cyclic-order-reversing pairings between the vertices. We denote the pairings by $\vartheta_n = (\vartheta_n^{(i)})_{i=1}^{2n}$, where $\vartheta_n^{(i)} = \{(v_1 v_2 v_3), (\vartheta_n^{(i)}(v_1) = w_1 \vartheta_n^{(i)}(v_2) = w_2 \vartheta_n^{(i)}(v_3) = w_3)\}$.
4. We identify each pair of faces $\rho_n^{(i)}$ using the pairing of its vertices described by $\vartheta_n^{(i)}$, for every $i = 1, \dots, 2n$.

With this procedure, we obtain an octahedral complex. Now, when taking out the vertices, this complex results into an oriented non-compact 3-manifold with boundary, which admits a hyperbolic metric. Indeed, we can endow each octahedron with the unique (up to isometry) hyperbolic metric of an ideal right-angled regular octahedron (see Figure 1.3). Now, the faces of the octahedra are ideal hyperbolic triangles, and, given a pairing of the vertices of any two faces, there exists a unique isometry between the two ideal hyperbolic triangles respecting this pairing. Thus, by gluing them through these isometries, the hyperbolic metric extends nicely to the whole complex. Therefore, after endowing all octahedra in the complex with this hyperbolic metric, we obtain a *complete finite-volume hyperbolic*

3-manifold with totally geodesic boundary. We will denote this by X_n , following the same notation as in [PR22].

The *dual graph* of the complex X_n , is, as for M_n , a *random 4-regular graph*. In the same way, we can condition X_n on not having loops or bigons in its dual graph. This is the manifold Y_n with which we will work.

By construction, Y_n is hyperbolic. We record this information in the following lemma:

Lemma 2.2.1. *The manifold Y_n carries a complete hyperbolic metric of finite volume with totally geodesic boundary.*

The probability space (Ω_n, \mathbb{P}_n) associated to this model is the following: we define Ω_n to be the finite set of all possibilities of ρ_n and ϑ_n , and we choose \mathbb{P}_n to be the uniform probability measure on Ω_n . We have that:

$$|\Omega_n| = \frac{\binom{4n}{2} \binom{4n-2}{2} \cdots \binom{2}{2}}{(2n)!} \cdot 3^{2n} = (4n-1)!! \cdot 3^{2n},$$

and so the probability of having one specific configuration is:

$$\mathbb{P}(\{\text{a certain partition } \rho_n \text{ and pairing } \vartheta_n\}) = \frac{1}{|\Omega_n|}.$$

2.2.1 From Y_n to M_n

We end the chapter by describing how one can recover the original model M_n from the Y_n just described.

This process is done by Dehn filling. Recall that a *Dehn filling* is a standard operation in three dimensional geometry used to construct compact hyperbolic manifolds from cusped ones. A description of the usual procedure can be found in Section 1.1.6.

In this case, we do a Dehn filling that goes as follows. We have, from [PR22], that the boundary of Y_n is a random hyperbolic surface S with cusps. Now, if we remove a horospherical neighbourhood of each cusp -obtained by intersecting the ideal polyhedron with a small horosphere- we obtain a compact manifold with boundary (see Figure 2.4). This boundary consists on S from which we have removed some linked pairs of disks, forming some open cylinders. Then, we do the following: in each cylinder, we glue a solid cylinder $D \times [0, 1]$ along its boundary. Note that in this case, the slope is determined, and so the Dehn filling as well. The result is a compact manifold M whose boundary is a compact surface \bar{S} .

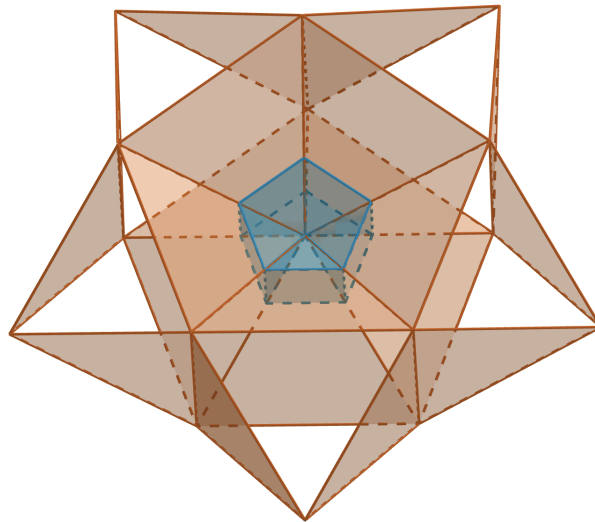


Figure 2.4: Cusp neighbourhood in Y_n (blue).

The manifolds M obtained by this procedure have the same distribution as the M_n presented in Section 2.1. Hence, we can, and will, think of M_n as the Dehn fillings of Y_n .

Remark 4. *This approach has also been worked on in other papers, although for other purposes. See, for instance, [CFMP07].*

Chapter 3

The length spectrum

The content of this chapter is adapted from [RS23]. We will study the length spectrum of random hyperbolic 3-manifolds under the model of random triangulations. More precisely, we prove:

Theorem A. *As $n \rightarrow \infty$, the primitive length spectrum of a random compact hyperbolic 3-manifold with boundary M_n converges in distribution to a Poisson point process (PPP) on $\mathbb{R}_{>0}$, of computable intensity λ .*

As mentioned before, the proof goes in two big steps: first, we prove the result for the model of non-compact random hyperbolic 3-manifolds Y_n -described in Section 2.2-, and then we show that it still holds after the Dehn filling of these manifolds, that is, for the manifolds M_n .

3.1 Proof step 1: the length spectrum of Y_n

In this section, we explain first how the combinatorics of the model Y_n gives us information about its curves. For that, we look at the dual graph of the complex, and study the distribution of its cycles (see Theorem 3.1.2). Using this, we then prove a version of Theorem A for these manifolds, which we reformulate as shown next.

We encode the length spectrum of Y_n by the following counting function: for any $0 \leq a < b$, we define:

$$C_{[a,b]}(Y_n) := \#\{\text{non-oriented primitive closed geodesics of length } \in [a, b] \text{ on } Y_n\},$$

where $C_{[a,b]}(Y_n) : \Omega_n \rightarrow \mathbb{R}$ is a random variable, since Y_n is a random manifold. Then, we prove the following statement, equivalent to the convergence to a PPP.

Theorem 3.1.1. *For any finite collection of disjoint intervals $[a_1, b_1], \dots, [a_t, b_t] \subset \mathbb{R}_{\geq 0}$, the random vector $(C_{[a_1, b_1]}(Y_n), \dots, C_{[a_t, b_t]}(Y_n))$ converges jointly in distribution, as $n \rightarrow$*

∞ , to a vector of independent random variables

$$(\mathcal{C}_{[a_1, b_1]}, \dots, \mathcal{C}_{[a_t, b_t]}),$$

where $\forall i = 1, \dots, t$, $\mathcal{C}_{[a_i, b_i]}$ is Poisson distributed with parameter $\lambda = \lambda([a_i, b_i]) > 0$.

3.1.1 Geometry of Y_n

Recall that, by construction, the manifold Y_n is hyperbolic (Lemma 2.2.1). Thus, we can use the following standard facts from hyperbolic geometry -presented in Section 1.1.2- to help us study its length spectrum.

First, we have that every element in the set of free homotopy classes of closed curves in Y_n that is neither trivial or homotopic to a cusp, is represented by a unique closed geodesic (Proposition 1.1.1).

Moreover, as it has totally geodesic boundary, we can write Y_n as a quotient \mathcal{C}/Γ , where \mathcal{C} is a convex domain of \mathbb{H}^3 and Γ is a discrete torsion-free subgroup of orientation preserving isometries of \mathbb{H}^3 . It is known that $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$, so we can think of Γ as a subgroup of $\text{PSL}(2, \mathbb{C})$. Then, we have that there is a bijection between the conjugacy classes in Γ and the free homotopy classes of closed curves [Mar16, Section 4.1.5]. This means that, given a closed curve γ , it corresponds to a conjugacy class $[M_\gamma]$ in $\text{PSL}(2, \mathbb{C})$.

An important thing about this fact is that from $[M_\gamma]$ one can compute the length of the corresponding geodesic in the homotopy class of γ . This length is exactly the translation length of $[M_\gamma]$, that is, the distance between p and $M_\gamma(p)$ for any $p \in \text{axis}(M_\gamma)$ - the geodesic line in \mathbb{H}^3 preserved by the isometry -. This is given by:

$$l_\gamma(M_\gamma) = 2\text{Re} \left[\text{arcosh} \left(\frac{\text{trace}([M_\gamma])}{2} \right) \right]. \quad (1.1)$$

The first goal is, then, to try to describe precisely this class $[M_\gamma]$ of Γ . For that, we will look at the dual graph of Y_n .

3.1.2 From curves to paths

The dual graph of Y_n encodes part of the combinatorics of the complex. To completely determine it, we need to include in it the orientation-reversing gluings of the pair of faces corresponding to the pair of half-edges. Then, from this "enriched" graph, that will be denoted by G_{Y_n} , we can get information about the length of the curves of the manifold Y_n . Ultimately, we will see that the distribution of the number of closed curves of a fixed length in Y_n can be studied by looking at the distribution of the number of certain closed paths in G_{Y_n} .

Let us start by showing how curves in Y_n and paths in G_{Y_n} are related. A first and essential observation is that any curve in Y_n can be homotoped to a path on the dual graph.

This homotopy can be done as follows: we cut the curve into pieces, each one corresponding to the part of the curve that enters and leaves exactly once some octahedron along two of the four non-adjacent faces. Then, if this part of the curve enters and leaves through the same face, we homotope it to the middle point of the face. Otherwise, we homotope the entry and exit points to the center of the faces, and all the remaining part of the curve to the graph. Figure 3.1 shows an example.

Once this process is done, we remove the possible backtracking from the graph, i.e, we remove the edges through which the path goes and turns back in the opposite direction after reaching some vertex. The resulting path is what we call the *reduced path*. From now on, these are the paths we will consider.

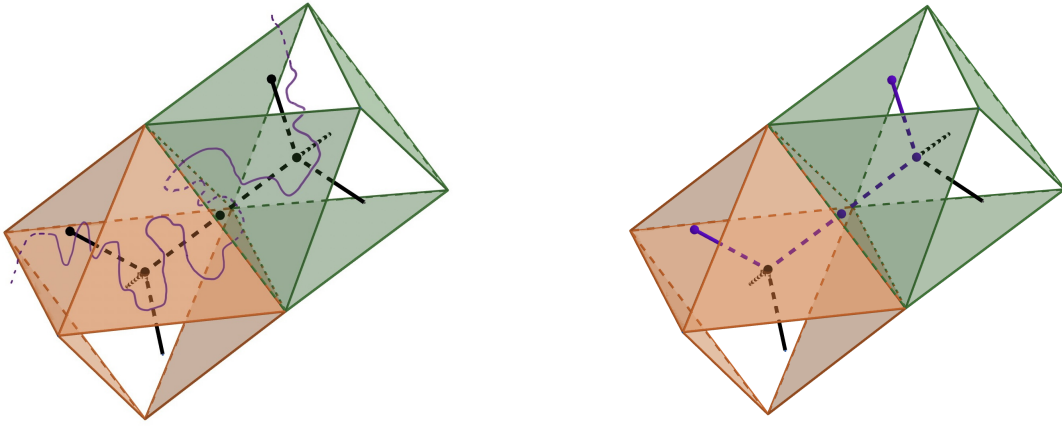


Figure 3.1: Homotopy of a curve into the dual graph.

Thus, in order to describe a closed curve γ , we can analyse its reduced path in the dual graph along the octahedra involved.

3.1.3 From paths to words

A closed path of some length $k \in \mathbb{N}$ on this enriched 4-regular graph mentioned in the previous section, can be described by picking a midpoint of an edge as a starting point, and giving a sequence of "movements"

$$(w_1\Theta_1, w_2\Theta_2, \dots, w_k\Theta_k),$$

that return to this starting point. Each w_i indicates which of the three incoming edges at the i -th vertex the path takes, and Θ_i describes in which of the 3 possible cyclic-order-reversing orientations the two faces of the octahedra are glued. We will call each of the elements of the form $w_i\Theta_i$ a *letter*, and the concatenation of movements describing the path a *word*. This will be denoted by:

$$w = w_1\Theta_1 \cdot w_2\Theta_2 \cdot \dots \cdot w_k\Theta_k.$$

In general, the *length* of a word -that is, the number of letters- will be denoted by $|w|$.

In order to determine the word associated to a path in the graph (and so to a curve in the manifold), we do the following: we fix a vertex of each face where the curve enters an octahedron (intuitively, the reader can think that the fixed vertex is the one lying above us when entering the octahedron through the face the curves traverses). Then, once we enter this octahedron, there are three possible directions the path can take corresponding to the three remaining non-adjacent faces. Among these faces, there is only one which is incident to the fixed vertex. If the path traverses this face, then we will say that it is *going straight*, and we will denote the corresponding w_i by " S ". On the other hand, if the path goes through the other two non-adjacent faces, we will say that it is either *turning left* -in which case w_i is denoted by " L "- or *turning right* -in which case w_i is denoted by " R ". The choice between these two is determined by the fixed vertex: if, when following an inwards orientation of the incoming face, the vertex next to the fixed one is the vertex common to both faces, we will say that the path is turning right. On the other hand, if, when following an outwards orientation of the incoming face, the vertex next to the fixed one is the common vertex, then we will say that the path is turning left.

On the other hand, the corresponding Θ_i , for $i = 1, \dots, |w|$, is determined by the choice of the fixed vertices in the i th and $(i + 1)$ th octahedra the curves traverses, denoted by O_i and O_{i+1} . Suppose first that the face in O_i where the curves exits, which is glued to the face of O_{i+1} where the curve enters, is the one incident to the fixed vertex of O_i -that is, $w_i = S$. Then, we have three possibilities for Θ_i :

- The fixed vertices in O_i and O_{i+1} coincide: then the gluing is said to be the *identity*.
- The fixed vertex of O_{i+1} is the next one starting from the one of O_i , when following an inwards orientation in the face of O_{i+1} : then we say these are glued *with one twist* -which we denote by θ -.
- The fixed vertex of O_{i+1} is second one starting from the one of O_i , when following an inwards orientation in the face in O_{i+1} : then we say that these are glued *with two twists* - which we denote by θ^2 .

In the case where the face of O_{i+1} is glued to a face in O_i non-incident to the fixed point of this octahedron -that is, $w_i = R$ or $w_i = L$ -, we have an almost analogous description. In this case, however, we define the gluing to be the *identity* when the fixed vertex of O_{i+1} is glued to the vertex opposite to the fixed vertex of O_i . Figure 3.2 shows an example of a word.

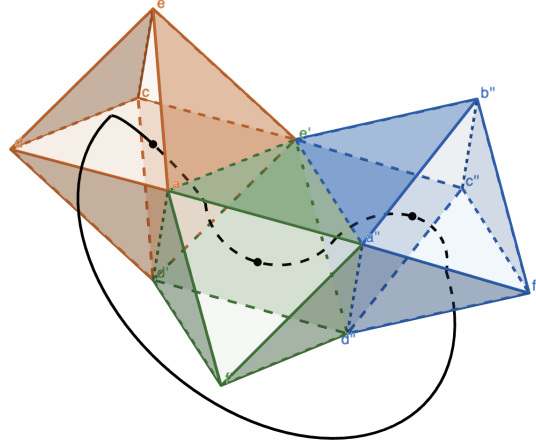


Figure 3.2: A closed curve in the octahedral complex. If we start at the orange octahedron, an element w describing this curve could be $w = R\theta SL\theta^2$.

Note, however, that when assigning a word to a curve according to the above process, we have made some choices. More precisely, we choose:

- A) A fixed vertex in each incoming face of the octahedra the curve traverses.
- B) A starting octahedron, and a direction of travel.

Changing any of these parameters gives rise to a different word, yet it still describes the same curve. Hence, to each path, we actually associate an equivalence class of words defined by these parameters, that we describe next.

Equivalence of words

Let's define more precisely this notion of equivalence class of words, and how these words change when modifying any of the previous choices. Let W denote the words w formed with the letters $\{S, S\theta, S\theta^2, R, R\theta, R\theta^2, L, L\theta, L\theta^2\}$. We will refer to this set as the *alphabet*.

- A) We focus first on the case when we change the fixed vertices in the incoming faces of the octahedra the curve traverses: given a face $(v_1 v_2 v_3)$, we have 3 possible choices for the vertex we want to fix. Hence, for a curve going through k octahedra, there will be 3^k possible choices, each of them yielding a different word, but describing the same curve. In particular, for every change of fix vertex in some octahedron O_i that respects an inwards orientation of the face, the word changes as follows:

- ◇ The orientation Θ_{i-1} of the gluing of O_{i-1} and O_i does one twist, i.e, gets multiplied by θ (here we set $\theta^3 = Id$).

- ◇ The direction of the movement in O_i changes following the cyclic order $(R S L)$.
- ◇ The orientation Θ_i of the gluing of O_i and O_{i+1} does one twist, i.e, gets multiplied by θ .

For example, if consider the word $w = w_1\Theta_1 \cdot w_2\Theta_2 = S \cdot R\theta$, the $3^2 = 9$ words describing the same curve while changing the fixed vertices in O_1 and O_2 (or equivalently the orientation of the gluings Θ_1 and Θ_2) are:

$O_2 \setminus O_1$	0	1	2
0	$SR\theta$	$L\theta R\theta^2$	$R\theta^2 R$
1	$S\theta S\theta^2$	$L\theta^2 S$	$RS\theta$
2	$S\theta^2 L$	$LL\theta$	$R\theta L\theta^2$

where 0, 1 and 2 denote the number of permutations of the fixed vertex with respect to the original word, respecting an inwards orientation of each incoming face.

B) Now, we consider what happens if we change the starting octahedron and/or the direction of travel. Again, changing any of these parameters will gives us new words for the same curve. Thus, we will say that two words $w, w' \in W$ are also equivalent if either:

- ◇ w' is a cyclic permutation of w (noting that $w_i\Theta_i$ is a single block).
For instance, if $w = w_1\Theta_1 \cdot w_2\Theta_2 = SR\theta$, there would be one cyclic permutation $w' = R\theta S$.
- ◇ w' is a cyclic permutation of w^* , where w^* is the word obtained by reading w backwards, and changing the orientation Θ_i of each letter $w_i\Theta_i$ into the orientation Θ_{i-1} of the previous one. In other words, if $w = w_1\Theta_1 \cdot w_2\Theta_2 \cdot \dots \cdot w_{|w|}\Theta_{|w|}$, then

$$w^* = w_{|w|}\Theta_{|w|-1} \cdot w_{|w|-1}\Theta_{|w|-2} \cdot \dots \cdot w_2\Theta_1 \cdot w_1\Theta_{|w|}.$$

Following the previous example, the word w^* of $w = w_1\Theta_1 \cdot w_2\Theta_2 = SR\theta$ is $w^* = RS\theta$, so the w' given by these transformations would be: $\{RS\theta, S\theta R\}$.

With all, we consider the following:

Definition 5. *If a word w' can be obtained from w by any of the transformations described above, then we say that w and w' are equivalent, and write $w' \sim w$. The equivalence class of a word $w \in W$ is then formed by all words $w' \sim w$. We denote it by $[w]$.*

3.1.4 From words to Möbius transformations

In this section, we will see how the (classes of) words that we have just defined can be realised as orientation preserving isometries of \mathbb{H}^3 .

Indeed, these movements constituting a word take on a geometrical meaning when translating our picture to the hyperbolic space. That is, by assigning ideal coordinates in \mathbb{H}^3 to the first octahedron, and placing the next ones with the information given by the word. Each of its elements, then, corresponds to the mapping of one face of an ideal octahedron -that corresponds to an ideal triangle- to some other of its non adjacent faces. These actions are orientation preserving isometries of \mathbb{H}^3 . And, as mentioned before, these can be identified with elements of $\text{PSL}(2, \mathbb{C})$. This group acts on the boundary of the upper half-space model by Möbius transformations [MT98, Theorem 1.8]:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Therefore, these movements on the graph correspond to some Möbius transformations on the hyperbolic 3-space, that send a triple of points -realising a face- to another triple in some cyclic order.

In practise, to simplify the computation of the Möbius transformation corresponding to a word w , we do the following procedure: we put every octahedron the curves goes through in the standard position (see Figure 3.3), and we look for the Möbius transformations that sends the triple of points $(0, i, \infty)$ to the triple of points realising the face of the next octahedron, in a specific order.

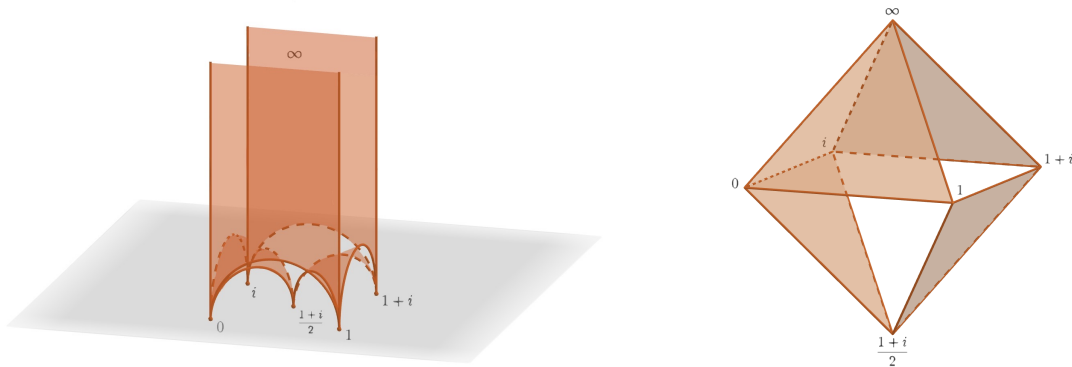


Figure 3.3: Ideal regular octahedron in the upper half-space model \mathcal{H}^3 (left) and its topological image (right).

These Möbius transformations:

$$S\theta^j, R\theta^j, L\theta^j : \mathbb{H}^3 \rightarrow \mathbb{H}^3, \quad j = 0, 1, 2,$$

are described by the following matrices in $PSL(2, \mathbb{C})$:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad R = \begin{pmatrix} -1 & i \\ i-1 & i \end{pmatrix} \quad L = \begin{pmatrix} i & i \\ i+1 & 1 \end{pmatrix} \quad \theta = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}.$$

As there are three possible directions and orientations at each step, we have nine possible isometries:

$$\begin{aligned} S\theta^0 &= S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & S\theta &= \begin{pmatrix} i & i+1 \\ i & 1 \end{pmatrix} & S\theta^2 &= \begin{pmatrix} i-1 & i \\ i & 0 \end{pmatrix} \\ R\theta^0 &= R = \begin{pmatrix} -1 & i \\ i-1 & i \end{pmatrix} & R\theta &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & R\theta^2 &= \begin{pmatrix} 0 & i \\ i & i+1 \end{pmatrix} \\ L\theta^0 &= L = \begin{pmatrix} i & i \\ i+1 & 1 \end{pmatrix} & L\theta &= \begin{pmatrix} -1 & i-1 \\ -i & i \end{pmatrix} & L\theta^2 &= \begin{pmatrix} i+1 & 1 \\ 1 & 1-i \end{pmatrix}. \end{aligned}$$

All these transformations are either parabolic or hyperbolic. More precisely, for each direction, there is one parabolic element - S , $R\theta$ and $L\theta^2$ - and two hyperbolic ones.

The Möbius transformation corresponding to a word w is then the product of the corresponding matrices. This is again an element of $PSL(2, \mathbb{C})$, and so an orientation preserving isometry of \mathbb{H}^3 . The resulting group element w describes the path -and hence the corresponding curve in the manifold- taking the initial face to the last one the curve traverses. These group elements are the M_γ referred to in Section 3.1.1.

The previous formula (1.1) gives us then a precise relation between the length of closed geodesics and their (class of) words. Thus, in order to study $C_{[a,b]}(Y_n)$, we will need to find the (classes of) words that correspond to (homotopy classes of) closed curves on Y_n . Then, by counting the number of homotopy classes of curves corresponding to each of these $[w]$, we will have insight on the number of closed geodesics of bounded length $\in [a, b]$.

We will see next that this argument can be indeed carried out substituting the counting of homotopy classes of curves by the counting of certain paths in G_{Y_n} .

3.1.5 Distribution of cycles

We're interested in counting closed geodesics in Y_n . One can see that these can be contracted into non-homotopic closed paths in the dual graph. This one-to-one correspondence between free homotopy classes of closed curves in the manifold Y_n and closed paths in G_{Y_n} , is due to the fact that the octahedral complex deformation retracts onto the graph. Hence, any two closed curves in Y_n will be homotopic if and only if their representatives in the dual graph are so.

Therefore, in order to count homotopy classes of curves, we can count closed paths in the dual graph. From now on, we will adopt the following terminology: we will call any

closed path a *circuit*, and a simple closed path a *cycle*. Also, we will denote by $[w]$ -*cycle* a cycle that is described by a word $w \in [w]$. On the other hand, we will also say that a curve γ is described by $[w]$ if a word corresponding to its path in the dual graph G_{Y_n} belongs to $[w]$.

It might happen that some simple closed curves in Y_n are homotopic to non-simple circuits in the graph. However, as it is explained later in Remark 6, the number of circuits of bounded length in the graph that are not cycles goes asymptotically almost surely to 0 as n tends to infinity. This tells us, then, that it is enough for our purpose to study the number of cycles in G_{Y_n} . With this in mind, we consider the random variable:

$$Z_{n,[w]} : \Omega_n \rightarrow \mathbb{N}, \quad n \in \mathbb{N}, [w] \in \mathcal{W} := W / \sim,$$

defined as

$$Z_{n,[w]}(\omega) := \#\{\text{cycles } \gamma \text{ on } G_{Y_n} : \gamma \text{ is described by } [w]\}.$$

We prove the following about the asymptotic behaviour of $Z_{n,[w]}$:

Theorem 3.1.2. *Consider a finite set \mathcal{S} of equivalence classes of words in W . Then, as $n \rightarrow \infty$,*

$$Z_{n,[w]} \rightarrow Z_{[w]} \quad \text{in distribution for all } [w] \in \mathcal{S},$$

where:

- $Z_{[w]} : \mathbb{N} \rightarrow \mathbb{N}$ is a Poisson distributed random variable with parameter $\lambda_{[w]} = \frac{|[w]|}{3^{|w|}2^{|w|}}$ for all $[w] \in \mathcal{S}$.
- The random variables $Z_{[w]}$ and $Z_{[w']}$ are independent for all $[w], [w'] \in \mathcal{S}$ with $[w] \neq [w']$.

The proof of this result follows the same structure as the proof of Bollobás' theorem on the asymptotic number of cycles in a random regular graph (Theorem 1.3.1). The argument is based on a version of the *method of moments*, which consists on the following: let $X_{n,1}, \dots, X_{n,k}$ be random variables, for $k \in \mathbb{N}$. We define the variables $(X_{n,i})_m$ as $(X_{n,i})_m = X_{n,i}(X_{n,i} - 1) \cdots (X_{n,i} - m + 1)$. Then, if $\lambda_1, \dots, \lambda_n \geq 0$ are such that, as $n \rightarrow \infty$,

$$\mathbb{E}[(X_{n,1})_{m_1} \cdots (X_{n,k})_{m_k}] \rightarrow \lambda_1^{m_1} \cdots \lambda_k^{m_k},$$

for every $m_1, \dots, m_k \geq 0$, we have that $(X_{n,1}, \dots, X_{n,k}) \xrightarrow{d} (X_1, \dots, X_k)$, where the X_i are independent Poisson random variables of parameter λ_i (for more, see Section 1.2.3).

Proof. Take as random variables the $Z_{n,[w]}$ and consider, for all $n \in \mathbb{N}$ and $[w] \in \mathcal{S}$:

$$(Z_{n,[w]})_{m_{[w]}} = Z_{n,[w]} \cdot (Z_{n,[w]} - 1) \cdots (Z_{n,[w]} - m_{[w]} + 1),$$

where $m_{[w]} \in \mathbb{N}$. Then, we prove that $\exists (\lambda_{[w]})_{[w] \in \mathcal{S}}$, with $\lambda_{[w]} \in \mathbb{R}$, such that as $n \rightarrow \infty$,

$$\mathbb{E} \left[\prod_{[w] \in \mathcal{S}} (Z_{n,[w]})_{m_{[w]}} \right] \rightarrow \prod_{[w] \in \mathcal{S}} \lambda_{[w]}^{m_{[w]}} \quad \text{for all } (m_{[w]})_{[w] \in \mathcal{S}} \in \mathbb{N}^{|\mathcal{S}|},$$

where $\lambda_{[w]} = \frac{|[w]|}{2|w|3^{|w|}}$ for all $[w] \in \mathcal{S}$.

We start with the first moment: $\mathbb{E}[Z_{n,[w]}]$. We denote by $Lab_{[w]}$ the set of all labellings of a $[w]$ -cycle, that is the set of all possible $|w|$ -tuples of pairs of the form (2.1).

Then, we can write this expectation as:

$$\begin{aligned} \mathbb{E}[Z_{n,[w]}] &= \sum_{l \in Lab_{[w]}} \mathbb{E}[\mathbf{1}_{\{\text{the labelling } l \text{ appears in } G_{Y_n}\}}] \\ &= \sum_{l \in Lab_{[w]}} \mathbb{P}[\{\text{the labelling } l \text{ appears in } G_{Y_n}\}] = a_{n,[w]} \cdot p_{n,[w]}, \end{aligned}$$

where $a_{n,[w]}$ denotes the number of possible labellings a $[w]$ -cycle can have, and $p_{n,[w]}$ the probability that an element of Ω_n contains a given set of $|w|$ pairs of half-edges, together with their corresponding orientations. Note that here we've used that $Z_{n,[w]}$ counts only cycles.

To count $a_{n,[w]}$, we will fix first a starting vertex and a direction. Since there are 2 possible directions in a cycle, and $|w|$ possible starting vertices, we will be counting in fact $2|w|a_{n,[w]}$. We will also be using the notation $a_{n,w}$ to refer to the number of labellings of a specific cycle w .

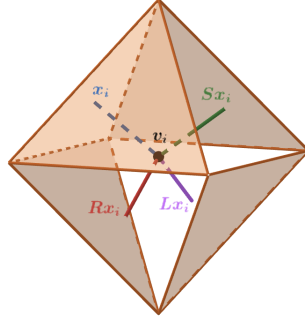
To start, we pick a representative w of a $[w]$ -cycle:

$$w = w_1 \Theta_1 \cdot w_2 \Theta_2 \cdot \dots \cdot w_{|w|} \Theta_{|w|},$$

where each $w_i \Theta_i$, $i = 1, \dots, |w|$, is a letter in the alphabet. Note that the choice of representative fixes the choices of the fix vertices in each octahedron explained in Section 3.1.3. Now, we look at the number of labellings the $[w]$ -cycle can have as a directed cycle described by w , with starting vertex v_1 . This cycle can be described by picking an initial half-edge x_1 adjacent to v_1 , and giving a list:

$$\{((w_1 x_1, x_2), \Theta_1), ((w_2 x_2, x_3), \Theta_2), \dots, ((w_{|w|} x_{|w|}, x_1), \Theta_{|w|})\},$$

where x_i is a half-edge of v_i , $w_i x_i$ is the half-edge on the left of x_i if $w_i = L$, on the right of x_i if $w_i = R$, and in front of x_i if $w_i = S$, and Θ_i is the orientation of the gluing of the half-edges $w_i x_i$ and x_{i+1} , for every $i = 1, \dots, |w|$ (see Figure 3.4).

Figure 3.4: Half-edges of v_i

Taking into account that there are n vertices to choose from (we consider n octahedra), and that in every vertex we have 4 possibilities for x_i , we have, for a specific cycle w :

$$2|w|a_{n,w} = 4^{|w|}n(n-1)(n-2)\dots(n-|w|+1).$$

Note, however, that the previous lists depend on the representative w , and so we get a different set of labellings each time we change representative. Hence, to obtain the total number of possible labellings of a $[w]$ -cycle -that is, $a_{n,[w]}$ - we need to multiply the expression above by the number of elements in the equivalent class $[w]$. Like this, we finally obtain that:

$$a_{n,[w]} = \frac{|[w]|}{2|w|} 4^{|w|}n(n-1)(n-2)\dots(n-|w|+1).$$

Now we compute $p_{n,[w]}$. We have that the total number of possible partitions of the set of half-edges -of cardinal $4n$ - into pairs is:

$$\mathcal{N}(4n) = \frac{\binom{4n}{2}\binom{4n-2}{2}\dots\binom{2}{2}}{(2n)!} = (4n-1)!!.$$

Moreover, if we fix k independent (vertex disjoint) edges, there are:

$$\mathcal{N}_k(4n) = \frac{\binom{4n-2k}{2}\binom{4n-2k-2}{2}\dots\binom{2}{2}}{(2n-k)!} = (4n-2k-1)!!$$

configurations containing these k edges.

However, in our case, we also need to consider the 3 possible orientations in every join of two half edges. With this, $p_{n,[w]}$ is given by:

$$p_{n,[w]} = \frac{3^{2n-|w|}(4n-2|w|-1)!!}{3^{2n}(4n-1)!!} = \frac{1}{3^{|w|}(4n-1)(4n-3)\dots(4n-2|w|+1)}.$$

All together, it yields:

$$\begin{aligned}\mathbb{E}[Z_{n,[w]}] &= \frac{|[w]|}{2|w|} \frac{4^{|w|} n(n-1)(n-2) \dots (n-|w|+1)}{3^{|w|} (4n-1)(4n-3) \dots (4n-2|w|+1)} \\ &\sim \frac{|[w]|}{2|w|} \frac{4^{|w|} n^{|w|}}{3^{|w|} (4n)^{|w|}} \longrightarrow \frac{|[w]|}{2|w| 3^{|w|}} \quad \text{as } n \rightarrow \infty.\end{aligned}$$

Now, we go on to the second factorial moment. $(Z_{n,[w]})_2$ counts the number of ordered pairs of distinct $[w]$ -cycles. These two may or may not intersect, so we can split $(Z_{n,[w]})_2$ as:

$$(Z_{n,[w]})_2 = Y'_{n,[w]} + Y''_{n,[w]},$$

where $Y'_{n,[w]}$ counts the number of ordered pairs of vertex disjoint $[w]$ -cycles, and $Y''_{n,[w]}$ the number of ordered pairs of intersecting $[w]$ -cycles.

The expectation of $Y'_{n,[w]}$ can be computed using a similar argument as for $\mathbb{E}[Z_{n,[w]}]$. As before, we write $\mathbb{E}[Y'_{n,[w]}]$ as:

$$\mathbb{E}[Y'_{n,[w]}] = a'_{n,[w]} \cdot p'_{n,[w]}$$

where $a'_{n,[w]}$ counts the number of label-lings of an ordered pair of distinct non-intersecting $[w]$ -cycles, and $p'_{n,[w]}$ is the probability that an element of Ω_n contains a given pair of disjoint sets, each of them containing $|w|$ pairs of half-edges.

In order to count $a'_{n,[w]}$ we fix again a direction and a starting vertex in each $[w]$ -cycle. Thus, we will be counting $4|w|^2 a_{n,[w]}$. In the same way,

$$4|w|^2 a'_{n,[w]} = 4^{2|w|} n(n-1)(n-2) \dots (n-2|w|+1),$$

and considering all the representatives of $[w]$ for each cycle, we obtain:

$$a'_{n,[w]} = \frac{|[w]|^2}{(2|w|)^2} 4^{2|w|} n(n-1)(n-2) \dots (n-2|w|+1).$$

As for $p'_{n,[w]}$, we observe that:

$$p'_{n,[w]} = p_{n,[w]} \cdot p_{n-|w|,[w]}.$$

Therefore,

$$\mathbb{E}[Y'_{n,[w]}] = a'_{n,[w]} \cdot p'_{n,[w]} \sim \left(\frac{|[w]|}{2|w|} \right)^2 \frac{4^{2|w|} n^{2|w|}}{3^{2|w|} (4n)^{2|w|}} \longrightarrow \left(\frac{|[w]|}{2|w| 3^{|w|}} \right)^2 \quad \text{as } n \rightarrow \infty.$$

Now let's study $\mathbb{E}[Y''_{n,[w]}]$. $Y''_{n,[w]}$ counts the pair of $[w]$ -cycles that have at least one common vertex. Note that these can be seen also as a connected (by the common vertices and edges)

multi-graph P which has more than one cycle. Each of these P , then, will have more edges than vertices by construction. Expressing $\mathbb{E}[Y''_{n,[w]}]$ as in the previous cases, i.e.,

$$\mathbb{E}[Y''_{n,[w]}] = a''_{n,[w]} \cdot p''_{n,[w]},$$

we observe that the number $a''_{n,[w]}$ is of the order $O(n^{\#\text{vertices}})$ and $p''_{n,[w]}$ depends only on the pairs of half-edges, so it is of the order $O(n^{-\#\text{edges}})$. Therefore,

$$\mathbb{E}[Y''_{n,[w]}] = O(n^{\#\text{vertices}} \cdot n^{-\#\text{edges}}) = O(n^{-1}).$$

All in all, we obtain that:

$$\lim_{n \rightarrow \infty} \mathbb{E}[(Z_{n,[w]})_2] = \left(\frac{|[w]|}{2|w|3^{|w|}} \right)^2.$$

As shown in [Bol80, Theorem 2], the same argument applies for any factorial moment $\mathbb{E}[(Z_{n,[w]})_m]$, $m \in \mathbb{N}$, and for any joint factorial moment $\mathbb{E}\left[\prod_{[w] \in \mathcal{S}} (Z_{n,[w]})_{m_{[w]}}\right]$. In this case, we consider the number of sequences of $\sum_{[w] \in \mathcal{S}} m_{[w]}$ distinct $[w]$ -cycles. We split them into the sum of intersecting and non-intersecting, and by the same reasoning as before:

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\prod_{[w] \in \mathcal{S}} (Z_{n,[w]})_{m_{[w]}}\right] = \prod_{[w] \in \mathcal{S}} \left(\frac{|[w]|}{2|w|3^{|w|}} \right)^{m_{[w]}}.$$

By the method of moments, this implies that the random variables $(Z_{n,[w]})_{[w] \in \mathcal{S}}$ converge in distribution, as $n \rightarrow \infty$, to independent Poisson random variables $Z_{[w]}$, with mean $\lambda_{[w]} = \frac{|[w]|}{2|w|3^{|w|}}$ for all $[w] \in \mathcal{S}$. \square

Remark 6. *The same argument that is used to prove that the expected value of $Y''_{n,[w]}$ is of the order $O(n^{-1})$ works to show that asymptotically there won't be any circuits of bounded length that are not cycles in the dual graph: such circuits have more edges than vertices, and so the expected number of copies of them is of the order: $O(n^{\#\text{vertices}} \cdot n^{-\#\text{edges}}) = O(n^{-1})$.*

3.1.6 Proof of Theorem 3.1.1

Now that we have Theorem 3.1.2, we can re-state and prove Theorem 3.1.1 as will follow next.

Recall that W was the set of all words with letters in $\{S, S\theta, S\theta^2, R, R\theta, R\theta^2, L, L\theta, L\theta^2\}$, and $|w|$ denoted the length of a word $w \in W$, being the number of letters in it. On the other hand, we denoted by $\mathcal{W} = W/\sim$ the set of equivalence classes of words in W . From this, we define:

$$\mathcal{W}_{[a,b]} := \{[w] \in \mathcal{W} : |w| > 2, |\text{tr}([w])| > 2 \text{ and } 2\text{Re}\left[\text{arcosh}\left(\frac{\text{tr}([w])}{2}\right)\right] \in [a, b]\}.$$

This is a finite set, consequence of the following result.

Proposition 3.1.3. *Let $w \in W$ be any hyperbolic word formed by the letters $\{S, S\theta, S\theta^2, R, R\theta, R\theta^2, L, L\theta, L\theta^2\}$ of word length $|w| = r > 0$. Then, there exists a constant $J(r) > 0$ satisfying that:*

- ◇ $J(r)$ is strictly increasing,
- ◇ $\frac{J(r)}{\log(r)} \rightarrow 1$ as $r \rightarrow \infty$,

such that the translation length of the geodesic γ corresponding to the equivalence class of w is bounded below by $J(r)$, that is,

$$l_\gamma(w) \geq J(r).$$

Proof. Let P be the plane in \mathbb{H}^3 spanned by the triple $\{0, i, \infty\}$, and $d(w)$ be the distance in the upper half-space of \mathbb{H}^3 between the planes P and $w(P)$, given by:

$$d(w) = \min_{\substack{(x_1, y_1) \in P \\ (x_2, y_2) \in w(P)}} \{d((x_1, y_1), (x_2, y_2))\} \quad (3.1)$$

$$= \min_{\substack{(x_1, y_1) \in P \\ (x_2, y_2) \in w(P)}} \left\{ \operatorname{arcosh} \left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} \right) \right\}. \quad (3.2)$$

We will refer to this function as the w -distance.

We start by considering some number $J(r) > 0$. Then, there are two possibilities, that $d(w) \geq J(r)$ or that $d(w) < J(r)$. From the study of each case, we aim to optimise this constant, while making it a valid lower-bound in any of these two initial cases.

As mentioned above, we have these two possible scenarios:

- $d(w) \geq J(r)$: We show that $l_\gamma(w) \geq d(w)$. For this, it is enough to see that the axis of the isometry runs through the planes spanned by the transformation.

To do so, we first observe the following: Let P denote the plane described by $\{0, i, \infty\}$, and consider the partition $\mathbb{H}^3 \setminus P = H_1 \sqcup H_2$, where H_2 contains all points in \mathbb{H}^3 with positive first component. Then, as described in Figure 3.5, for any of the 9 isometries $M \in \{S\theta^i, R\theta^i, L\theta^i, i = 0, 1, 2\}$, we have

$$M(H_2) \subset H_2.$$

From this, one infers that for any isometry w product of the previous matrices, we get:

$$w(M(H_2)) \subset w(H_2).$$

This means that every time we add a transformation M to our word, the half-space spanned by this new isometry is contained in the previous one. This tells us, in

particular, that the attracting fix point of the axis in any hyperbolic transformation w -sending the initial plane P to $w(P)$ - has to lie inside $w(H_2)$, and the repulsive fix point then, needs to be in H_1 . Therefore, the axis intersects both P and $w(P)$, as well as all the planes spanned by all shorter words of w .

Hence, by definition of the function w -distance, it's clear that:

$$d(w) = d(P, w(P)) \leq l_\gamma(w).$$

Since we had $d(w) \geq J(r)$, we obtain that $l_\gamma(w) \geq J(r)$.

- $d(w) < J(r)$: We can find a representative of $[w]$ in the form:

$$w = S^{s_1}(R\theta)^{s_2}(L\theta^2)^{s_3} \cdots \cdots S^{s_{m-2}}(R\theta)^{s_{m-1}}(L\theta^2)^{s_m}\theta^t, \quad (3.3)$$

where $s_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, m$, $s_1 + \dots + s_m = r$ and $t \in \{0, 1, 2\}$. Indeed, we can choose the fixed vertices in each octahedron in a way that the orientations of the gluings Θ_i , for $i = 1, \dots, |w| - 1$, are the ones appearing in the parabolic letters $\{S, R\theta, L\theta^2\}$. With this, the only gluing that is determined is the one between the last and first faces, which can be any of the three possibilities $\{Id, \theta, \theta^2\}$. We suppose then that w is of this form.

Now, we have the following: any hyperbolic word w of two letters spanning a plane of empty intersection with P is at w -distance $d(w) = \text{arcosh}(3)$ from it. Indeed, one can check by explicit computation that these words can be reduced, by equivalence, to the list:

$$L_0 = \{SR\theta, R\theta S, R\theta L\theta^2, L\theta^2 R\theta, SL\theta^2, L\theta^2 S\},$$

which span all possible different planes (see Figure 3.5), all of them at w -distance $d(w) = \text{arcosh}(3)$.

Since we have, from above, that for any word $w \in W$, $w = w_1 \cdots w_n$:

$$w(H_2 \sqcup P) \subset w_1 \cdots w_{n-1}(H_2 \sqcup P) \subset w_1 \cdots w_{n-2}(H_2 \sqcup P) \subset \cdots \subset H_2 \sqcup P,$$

we get that the geodesic realising the w -distance between P to $w(P)$ intersects the intermediary planes in some point. The segment from P to these points will be then, larger or equal than the distance between the planes, by definition of distance. Therefore, any word w containing at least k words $w_1, \dots, w_k \in L_0$ disjointly, will have w -distance:

$$d(w) \geq \sum_{i=1}^k d(w_i) = k \text{arcosh}(3).$$

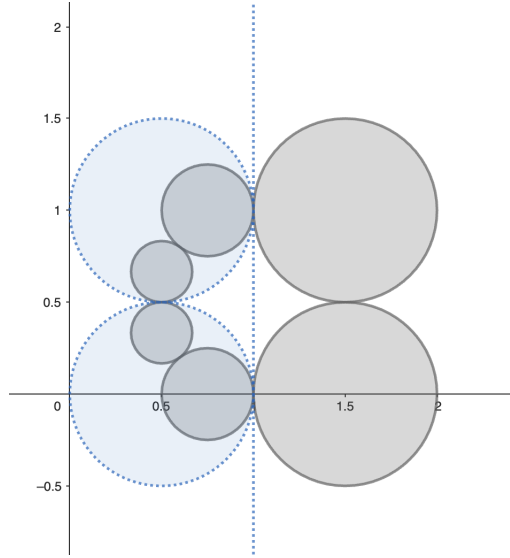


Figure 3.5: View from "infinity" in the upper half-space model of \mathbb{H}^3 : the circles in blue represent the hemispheres spanned by the letters of the alphabet, and the ones in grey the hemispheres spanned by all the words in L_0 .

Equivalently and relying it to our case, if $d(w) \leq J(r) = k \operatorname{arcosh}(3)$, this tells us that w contains at most $\frac{J(r)}{\operatorname{arcosh}(3)}$ disjoint words of the previous type. This fact gives us then an upper bound on the number of s_i 's in w :

$$\begin{aligned} m = n^0 s_i &\leq 2 \cdot n^0 \text{ disjoint hyp. words of two letters} + 1 \\ &\leq \frac{2J(r)}{\operatorname{arcosh}(3)} + 1. \end{aligned}$$

Since w is of length r , by the pigeonhole principle, we obtain that there exists at least a s_i such that

$$s_i \geq \left\lceil \frac{r}{m} \right\rceil \geq \left\lceil \frac{r}{\frac{2J(r)}{\operatorname{arcosh}(3)} + 1} \right\rceil = \left\lceil \frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right\rceil.$$

Let $K_r = \left\lceil \frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right\rceil$. Since there is at least a $s_i \geq K_r$ and the words we consider are of the form (3.3), one can deduce that the word w has as a sub-word one of the followings w_{K_r} :

$$w_{K_r} \in \{S^{K_r} R\theta, (R\theta)^{K_r} S, (R\theta)^{K_r} L\theta^2, (L\theta^2)^{K_r} R\theta, S^{K_r} L\theta^2, (L\theta^2)^{K_r} S\}.$$

All of these words have w -distance $d(w_{K_r}) = \operatorname{arcosh}(2K_r + 1)$ (by direct computation).

Therefore, since $l_\gamma(w) \geq d(w) \geq d(\tilde{w})$, for \tilde{w} in the previous list, we obtain:

$$l_\gamma(w) \geq \operatorname{arcosh}(2K_r + 1) \geq \operatorname{arcosh} \left(2 \left(\frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right) + 1 \right).$$

Hence, in general, we have:

$$l_\gamma(w) \geq d(w) \geq \min \left\{ J(r), \operatorname{arcosh} \left(2 \left(\frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right) + 1 \right) \right\}.$$

This bound is optimised whenever $J(r) = \operatorname{arcosh} \left(2 \left(\frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right) + 1 \right)$. Thus, we define the function

$$\begin{aligned} J : \mathbb{R}_{>0} &\longrightarrow \mathbb{R}_{>0} \\ r &\longmapsto J(r) \end{aligned}$$

$$\text{implicitly by } J(r) = \operatorname{arcosh} \left(2 \left(\frac{\operatorname{arcosh}(3)r}{2J(r) + \operatorname{arcosh}(3)} \right) + 1 \right).$$

It rests to prove that this function is increasing: we can re-write it as

$$(\cosh(J(r)) - 1)(2J(r) + \operatorname{arcosh}(3)) = 2 \operatorname{arcosh}(3)r \quad (3.4)$$

Since $\operatorname{arcosh}(3) > 0$ and $r > 0$, both sides of the equation are positive. Then, for $0 < r_1 < r_2$, we have that $\operatorname{arcosh}(3)r_1 < \operatorname{arcosh}(3)r_2$, which implies:

$$\frac{1}{2}(\cosh(J(r_1)) - 1)(2J(r_1) + \operatorname{arcosh}(3)) < \frac{1}{2}(\cosh(J(r_2)) - 1)(2J(r_2) + \operatorname{arcosh}(3)).$$

Since both components of the product are positive, this means that either:

- ◇ $(2J(r_1) + \operatorname{arcosh}(3)) < (2J(r_2) + \operatorname{arcosh}(3))$: yielding that $J(r_1) < J(r_2)$.
- ◇ $(\cosh(J(r_1)) - 1) < (\cosh(J(r_2)) - 1)$: giving also $J(r_1) < J(r_2)$ as $\cosh(x)$ is an increasing function.

As a final observation, we see that $J(r)$ has a logarithmic behaviour in the limit, that is, $\frac{J(r)}{\log(r)} \rightarrow 1$ as $r \rightarrow \infty$.

Indeed, looking at the equality (3.4), we note that $J(r)$ coincides with a branch W_k of the Lambert W -function for some $k \in \mathbb{Z}$, that is, $J(r) = W_k(2 \operatorname{arcosh}(3)r)$. Recall that the *Lambert W -function* is the multivariate inverse of $f(w) = we^w$, where e^w is the exponential function and w is any complex number. In other words, is the function satisfying the relation $z = W_k(z)e^{W_k(z)}$ for some integer k . In particular, since $r > 0$, and

both r and $J(r)$ are real numbers, $J(r)$ is the principal branch, that is,

$$J(r) = W_0(2 \operatorname{arcosh}(3)r).$$

Now, one has that, for large values of r , this function $W_0(r)$ is asymptotic to $\ln(r) - \ln(\ln(r)) + o(1)$. In our case,

$$\begin{aligned} W_0(2 \operatorname{arcosh}(3)r) &= \ln(2 \operatorname{arcosh}(3)r) - \ln(\ln(2 \operatorname{arcosh}(3)r)) + o(1) \\ &= \ln(r) + \ln(2 \operatorname{arcosh}(3)) - \ln(\ln(2 \operatorname{arcosh}(3)r)) + o(1). \end{aligned}$$

Therefore, we get

$$\lim_{r \rightarrow \infty} \frac{J(r)}{\ln(r)} = \frac{\ln(r) + \ln(2 \operatorname{arcosh}(3)) - \ln(\ln(2 \operatorname{arcosh}(3)r)) + o(1)}{\ln(r)} = 1.$$

□

Note that, although this lower bound may not be the sharpest possible, here we cannot use the growth of the traces or the w -distance to control the growth of the length, as it was possible in the two-dimensional case of this model (see [Pet17]). Indeed, the traces are complex numbers so they don't have a natural ordering, and there exists the possibility that their absolute value decreases whenever we add a letter to a word. In the same way, it is neither true that the translation length grows whenever the w -distance $d(w)$ does so. For example, for $w = RLRR$, we have:

$$d(w) = 2.63 \text{ and } l_\gamma(w) = 3.47,$$

while if we add one more letter $w' = RLRRRL$, we get:

$$d(w') = 3.26 \text{ and } l_\gamma(w) = 3.33.$$

After having proved this, we get to our main point. For $a, b \geq 0$, $C_{[a,b]}(Y_n)$ denoted the number of non-oriented primitive closed geodesics of length $\in [a, b]$ on Y_n . Also,

$$\mathcal{W}_{[a,b]} = \{[w] \in \mathcal{W} : |w| > 2, |\operatorname{tr}([w])| > 2 \text{ and } 2\operatorname{Re}\left[\operatorname{arcosh}\left(\frac{\operatorname{tr}([w])}{2}\right)\right] \in [a, b]\}.$$

When $a = 0$, $\mathcal{W}_{[0,b]}$ will be written as \mathcal{W}_b . Then, we prove:

Theorem 3.1.1. *For any finite collection of disjoint intervals $[a_1, b_1], \dots, [a_t, b_t] \subset \mathbb{R}_{\geq 0}$, the random vector $(C_{[a_1, b_1]}(Y_n), \dots, C_{[a_t, b_t]}(Y_n))$ converges jointly in distribution, as $n \rightarrow$*

∞ , to a vector of independent random variables

$$(\mathcal{C}_{[a_1, b_1]}, \dots, \mathcal{C}_{[a_t, b_t]}),$$

where for all $i = 1, \dots, t$, $\mathcal{C}_{[a_i, b_i]}$ is Poisson distributed with parameter

$$\lambda = \sum_{[w] \in \mathcal{W}_{[a_i, b_i]}} \lambda_{[w]}.$$

Proof. For simplicity, we will consider the interval $[0, l]$, and denote $C_{[0, l]}(Y_n)$ by $C_l(Y_n)$. We can write $C_l(Y_n)$ as:

$$C_l(Y_n) = \sum_{[w] \in \mathcal{W}_l} Z_{n, [w]} + \sum_{[w] \in \mathcal{W}_l} Z'_{n, [w]},$$

where $Z'_{n, [w]}$ denote the number of circuits of G_{Y_n} described by $[w] \in \mathcal{W}_l$ that are not cycles.

Notice that the right-hand side is a sum of two finite sums of independent random variables $Z_{n, [w]}$ and $Z'_{n, [w]}$ respectively. On the one hand, since we have seen that the expected number of copies of any non-simple circuit in G_{Y_n} of bounded length tends to 0 as $n \rightarrow \infty$, the second summand will vanish in the limit. Indeed, we have that:

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{[w] \in \mathcal{W}_l} Z'_{n, [w]} \right] = \lim_{n \rightarrow \infty} \sum_{[w] \in \mathcal{W}_l} \mathbb{E}[Z'_{n, [w]}] = \sum_{[w] \in \mathcal{W}_l} \lim_{n \rightarrow \infty} \mathbb{E}[Z'_{n, [w]}] = 0.$$

For the other summand, we apply Theorem 3.1.2. All in all, we obtain that:

$$C_l(Y_n) \xrightarrow{d} \sum_{[w] \in \mathcal{W}_l} Z_{[w]} = \mathcal{C}_l.$$

Since each $Z_{[w]}$ is Poisson distributed with parameter $\lambda_{[w]} = \frac{|[w]|}{2^{|w|} 3^{|w|}}$, and they are all independent of each other, we obtain that the random variable \mathcal{C}_l is Poisson distributed with parameter $\lambda = \sum_{[w] \in \mathcal{W}_l} \lambda_{[w]}$. Finally, as none of the $[w] \in \mathcal{W}$ can belong to any two $\mathcal{W}_{[a, b]}$, $\mathcal{W}_{[c, d]}$ for $[a, b]$, $[c, d]$ disjoint intervals, one concludes that for any finite collection $(\mathcal{C}_{[a_1, b_1]}(Y_n), \dots, \mathcal{C}_{[a_t, b_t]}(Y_n))$, the corresponding random variables $(\mathcal{C}_{[a_1, b_1]}, \dots, \mathcal{C}_{[a_t, b_t]})$ are independent. \square

3.2 Proof step 2: the length spectrum of M_n

In Section 2.2.1, we explain how M_n is obtained from the intermediary Y_n by Dehn filling. In this section, we finish the proof of Theorem A by showing that, after this compactification procedure, the result obtained in Section 3.1.6 still holds, that is:

Theorem A. *As $n \rightarrow \infty$, the length spectrum of a random compact hyperbolic 3-manifold with boundary M_n converges in distribution to a Poisson point process (PPP) on $\mathbb{R}_{>0}$ of intensity λ , where for any $a, b \geq 0$,*

$$\lambda([a, b]) = \sum_{[w] \in \mathcal{W}_{[a, b]}} \lambda_{[w]}.$$

In order to prove this, we need to see that the number of closed geodesics of some bounded length stays asymptotically the same after the Dehn filling. This comes down to checking mainly two things: first, that the length of the curves doesn't change "too much" after the compactification; and second, that they do not collapse into a point, or into each other -that is, that any two short closed geodesics that weren't homotopic in Y_n , don't become homotopic after the filling of the cusps-. We treat these cases in two separate subsections.

3.2.1 Change in length of closed geodesics

We start proving the first and main point.

Proposition 3.2.1. *Let $l_{max} > 0$. For every $\epsilon > 0$, the following holds a.a.s as $n \rightarrow \infty$: for every closed geodesic γ in M_n with length $l(\gamma) \leq l_{max}$, there exists a closed geodesic γ' in Y_n such that the image of γ' is homotopic to γ , and*

$$\frac{1}{1 + \epsilon} l(\gamma) \leq l(\gamma') \leq (1 + \epsilon) l(\gamma).$$

Like in [PR22], we'll control the change in geometry when doing the Dehn filling of the cusps separately in three steps. First, we will deal with "small" cusps, that is, with cusps made of few octahedra around them, and apply Andreev's theorem [RHD07] to make sure the length doesn't change "too much". Then, we'll treat the "medium" and "large" cusps, in two separate steps, using a result of Futer-Purcell-Schleimer (Theorem 3.2.3), stated later in the proof.

We set some notation beforehand, recalling as well previous objects that will appear. We start by specifying the different types of cusps. We define the combinatorial length of a cusp as the number of octahedra forming it. Then,

<i>Cusps of Y_n</i>	<i>Notation</i>	<i>Description</i>
Small	c_1, \dots, c_s	Of combinatorial length up to $\frac{1}{8} \log_3(n)$
Medium	c_{s+1}, \dots, c_m	Of combinatorial length between $\frac{1}{8} \log_3(n)$ and $n^{1/4}$
Large	c_{m+1}, \dots, c_t	Of combinatorial length bigger than $n^{1/4}$

With this, we distinguish the following manifolds and their parts.

<i>Models</i>	<i>Description</i>
Y_n	Hyperbolic 3-manifold constructed in Section 2.2 conditioned on not having loops or bigons in its dual graph
$Y_{n,1}$	Union of all octahedra of Y_n incident to a small cusp
$Y_{n,2}$	Complement of $Y_{n,1}$ in Y_n
K_n	Manifold obtained from Y_n by filling the small cusps
$K_{n,1}$	Union of all octahedra of K_n incident to a medium cusp
$K_{n,2}$	Complement of $K_{n,1}$ in K_n
DK_n	Double of K_n
Z_n	Manifold obtained from K_n by filling the medium cusps
DZ_n	Double of Z_n
M_n	Manifold obtained from Z_n by filling the large cusps, homeomorphic to the M_n described in Section 2.1
DM_n	Double of M_n

We think of $Y_n \subset K_n \subset Z_n \subset M_n$.

Proof of Proposition 3.2.1.

Filling small cusps. Let O_1, \dots, O_k be the octahedra in contained in $Y_{n,1}$. The Dehn filling procedure for compactifying small cusps consisted on the following: we consider an octahedron on which one vertex has been replaced by an edge. By Andreev's theorem [RHD07], this new polyhedron, denoted by P_l , can be endowed with the structure of an hyperbolic polyhedron with right angles at all edges except the created one, which has angle $2\pi/l$. Then, the Dehn filling consists in changing each of the octahedron incident to a small cusp -that is, the O_1, \dots, O_k - by a copy of the polyhedron P_l , in a way that the egde with angle $2\pi/l$ is the one substituing the ideal vertex shared by the octahedra around the cusp. Figure 3.6 shows the resulting polyhedron Q_l . On the other hand, all the octahedra in $Y_{n,2}$ remain intact. For more details, see [PR22, Section 3.2].

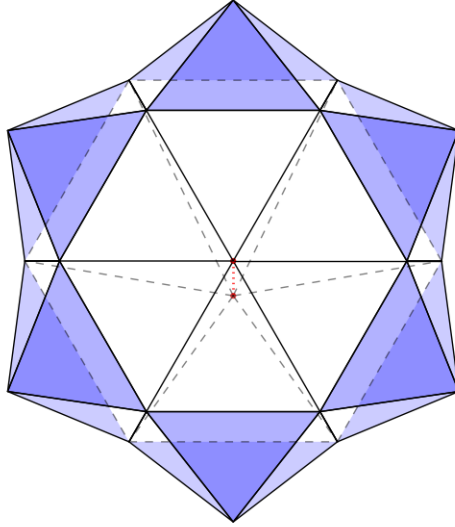


Figure 3.6: The polyhedron Q_l . The red edge is the one with angle $2\pi/l$, where $l = 6$ in this example.

So, to control the change in geometry when doing this procedure, we use, more precisely, a consequence of Andreev's theorem. Let $\phi_1 : Y_n \rightarrow K_n$ denote the inclusion map between those manifolds. As stated in [PR22], this gives the following result:

Lemma 3.2.2 ([PR22], Lemma 3.6). *There exists $J_0 > 0$ such that the following holds for any $\epsilon > 0$ below the Margulis constant for \mathbb{H}^3 , and any $\delta > 0$: with probability at least $1 - \delta$ in the model Y_n for n large enough, we have:*

1. *The ϵ -thick part of the image of $Y_{n,1}$ in K_n under ϕ_1 is J_0 -bilipschitz to that of $Y_{n,1}$,*
2. *The image of $Y_{n,2}$ in K_n under ϕ_1 is isometric to $Y_{n,2}$.*

As $Y_n \subset K_n \subset Z_n$, the same lemma holds in particular for the manifold Z_n , considering the map $\phi_2 : K_n \rightarrow Z_n$.

So, let γ be a closed geodesic in K_n . The goal is to find a closed geodesic γ' in Y_n of length "close" to that of γ , and such that after compactification, it belongs to the same homotopy class as γ .

We look at the preimage $\tilde{\gamma}$ of γ in Y_n under the map ϕ_1 . This curve $\tilde{\gamma}$ is a closed curve, but is not necessarily a geodesic in general. We have the following cases:

- The curve $\tilde{\gamma}$ lies entirely in $Y_{n,2}$: by the second point of Lemma 3.2.2, its length will be exactly the same as the one of γ . Here, then, $\tilde{\gamma}$ is also a geodesic in Y_n . Thus, we take $\tilde{\gamma}$ to be γ' .

- The curve $\tilde{\gamma}$ lies partly in $Y_{n,1}$: in this case, $\tilde{\gamma}$ is a priori not a geodesic. To handle this case, we use the notion of tangle-freeness for graphs. We recall here its definition, for more see Section 1.3.3.

Definition 7. A multigraph G is tangle-free if it contains at most one cycle (including loops and multiple edges). G is l -tangle-free if every neighbourhood of radius l in H contains at most one cycle. Otherwise, we say that G is tangled or l -tangled.

Now, it was proved by Bordenave [Bor20] that random 4-regular graphs -so in particular G_{Y_n} - are l -tangle free, for $l > 0$ not too large. More precisely,

Lemma 1.3.4 (Bordenave). *Let $d \geq 3$, and G_d be a random d -regular graph on n vertices generated by the configuration model. Then,*

$$\mathbb{P}[G_d \text{ is } l\text{-tangle-free}] = 1 - O\left(\frac{(d-1)^{4l}}{n}\right).$$

Using this property, we argue as follows: we have that $\tilde{\gamma}$ is homotopic to a cycle in the dual graph G_{Y_n} . As the curve lies in some part of $Y_{n,1}$, its cycle intersects with the one representing the parabolic element that goes around a small cusp in $Y_{n,1}$. On the other hand, since $l(\gamma) < l_{max}$, by Proposition 3.1.3 we deduce that the part of the cycle in G_{Y_n} corresponding to the part of $\tilde{\gamma}$ lying in $Y_{n,2}$ has length bounded above by $K < l_{max}$. Therefore, we can find two cycles in G_{Y_n} inside a ball of diameter less or equal than $K + \frac{1}{8} \log_3(n)$.

Nevertheless, we know that the dual graph G_{Y_n} is l -tangle free for $l < \frac{1}{4} \log_3(n)$ (Lemma 1.3.4), which means that as $n \rightarrow \infty$, there won't be more than one cycle in any neighbourhood of this radius w.h.p. This tells us, therefore, that a.a.s this case will not occur.

Filling medium cusps. Now, let γ be a closed geodesic in Z_n . A priori Lemma 3.2.2 from [PR22] can also be applied. So, as before, we consider its preimage $\tilde{\gamma}$ in K_n under the inclusion map $\phi_2 : K_n \rightarrow Z_n$, and treat two possible scenarios:

- The curve $\tilde{\gamma}$ lies entirely in $K_{n,2}$: by the second point of Lemma 3.2.2, its length will be exactly the same as the one of γ . Thus, $\tilde{\gamma}$ is a geodesic in K_n , and so we set $\tilde{\gamma}$ to be γ' .
- The curve $\tilde{\gamma}$ lies partly in $K_{n,1}$: in this case, $\tilde{\gamma}$ is a priori not a geodesic. Lemma 3.2.2 tells us that the part lying in $K_{n,2}$ will remain exactly of the same length as in γ . On the other hand, we also have that the ϵ -thick part of the image of $K_{n,1}$ in Z_n is J_0 -bilipschitz to that of $K_{n,1}$, for some $J_0 > 0$. However, this bilipschitz constant can be arbitrarily large, so we don't get explicit control on the change in length.

Therefore, for the case of medium cusps, to find this geodesic γ' we'll use the result of Futer-Purcell-Schleimer stated after the following preliminary notions.

Definition 8. *Let N be a hyperbolic 3-manifold with rank-two cusps C_1, \dots, C_t . Consider a slope s_j for each cusp torus ∂C_j . Then, the normalized length of s_j is*

$$L_j = L(s_j) = \frac{l(s_j)}{\sqrt{\text{area}(\partial C_j)}},$$

where $l(s_j)$ is the length of a geodesic representative of s_j on ∂C_j .

Let $s = (s_1, \dots, s_t)$ be the vector of all slopes. We define the total normalized length $L = L(s)$ via the formula

$$\frac{1}{L^2} = \sum_{j=1}^t \frac{1}{L_j^2}.$$

Remark 9. *Since the cusps in Z_n are of rank 1, there is only one possible slope to choose on each of them to do the Dehn filling. Moreover, in our case, the total normalized length of this slope is proportional to the square root of the combinatorial length of the cusp. Indeed, every cusp is contained in a certain union of octahedra. In each of the cusps, we fix a sufficiently small horosphere -that is, so that it is still isometric to a cylinder-. This horosphere intersects with each of the octahedron incident to the cusp in a square (see Figure 2.4). These squares around the cusp have then a certain fixed area a_0 (the same for all of them). On the other hand, taking the induced euclidean metric on the horosphere, we fix a geodesic representative. The intersection of this geodesic with each square is also of a certain fixed length l_0 . As such, the total area of this cylinder created by the squares around the cusp, is equal to $a_0 \cdot \#\{\text{octahedra incident to the cusp}\}$. On the other hand, the total length l of the geodesic around it is equal to $l_0 \cdot \#\{\text{octahedra incident to the cusp}\}$. Hence, the total normalised length L of the cusp, defined as $L = l/\sqrt{\text{area}(\text{cylinder})}$, is a square root of the number of octahedra around the cusp, its combinatorial length.*

Finally, given a hyperbolic manifold with rank 2 cusps, and its Dehn filling, we recall that a *cone-deformation* between them consists of a family of singular hyperbolic metrics on the filled manifolds, where the singularities lie along the core curves of the attached solid tori. The metrics on discs perpendicular to the core curves have a single cone point at the curve, with a cone angle α , which is constant along each curve. Hyperbolic manifolds with these structures are called *hyperbolic cone-manifolds*. We can think of the complete structure as the case of cone angle 0, and if the deformation reaches cone angles 2π on each curve, we obtain a smooth hyperbolic structure on the filled manifold. For more information, see [HK08, FPS22].

Once all this defined, we have, by Futer-Purcell-Schleimer, the following:

Theorem 3.2.3 ([FPS22], Theorem 9.30). *Fix $0 < \epsilon \leq \log 3$. Let M be a finite-volume hyperbolic 3-manifold and Σ a geodesic link in M . Let $N = M - \Sigma$. Suppose that in the complete structure on $N = M - \Sigma$, the total length of the meridians of Σ satisfies:*

$$L^2 \geq \frac{2\pi \cdot 6771 \cosh(0.6\epsilon + 0.1475)^5}{\epsilon^5} + 11.7.$$

Then there is a cone-deformation M_t connecting the complete hyperbolic metric g_0 on N -with cone-singularities of angle 0 - to the complete hyperbolic metric $g_{4\pi^2}$ on M -of cone-singularities of angle 2π . Furthermore, the cone deformation gives a natural identity map $id : (M - \Sigma, g_0) \rightarrow (M - \Sigma, g_{4\pi^2})$, such that id and id^{-1} restrict to:

$$id \upharpoonright_{N^{\geq \epsilon}} : N^{\geq \epsilon} \hookrightarrow M^{\geq \frac{\epsilon}{1.2}}, \quad id \upharpoonright_{M^{\geq \epsilon}} : M^{\geq \epsilon} \hookrightarrow N^{\geq \frac{\epsilon}{1.2}},$$

that are J -bilipschitz inclusions for

$$J = \exp\left(\frac{11.35l}{\epsilon^{5/2}}\right) \quad \text{and} \quad l \leq \frac{2\pi}{L^2 - 11.7}.$$

For convenience of the following arguments, we consider as our cusped manifold the double of the manifold K_n , -hence $N = DK_n$ - which is a non-compact manifold without boundary. Then, cusp neighbourhoods correspond to (thickened) tori, which get replaced by solid tori after compactification. The resulting manifold will be DZ_n (which corresponds to M in the notation of the theorem). Thus, Σ here corresponds to the cores of the Dehn filling solid tori. The slopes of these Dehn fillings are already determined in our case, since there is only one way of gluing a solid torus in the doubled manifold so that it corresponds to the Dehn fillings of the rank-1 cusps on both sides of the manifold. With this, the cores become geodesics, and so we take Margulis tubes around them. We recall that a Margulis tube of radius $r > 0$ is a tubular neighbourhood of a closed geodesic α . Its radius r denotes the distance between the core geodesic of the tube α and its boundary. In a hyperbolic 3-manifold, it's diffeomorphic to a solid torus, while in \mathbb{H}^3 , it lifts to an infinite solid cylinder around an infinite geodesic (see Section 1.1.5 for more details). We'll denote it by $T_r(\alpha)$.

Another important comment is that, when considering the doubles DK_n , DZ_n new closed geodesics appear. However, our asymptotics only concern closed geodesics lying in one of the copies of the single manifolds, so we will only consider these.

As a final remark, note that in order to satisfy the condition of the theorem, we will compactify the medium cusps individually one by one, and apply the previous result at each step, that is, with $L = L_j$, for $j = s + 1, \dots, m$. Although that could make the bilipschitz constants accumulate, this will not occur as the medium cusps don't intersect a.a.s. The proof of this claim is at the end of the section (Claim 1).

So, let $\epsilon > 0$ be some number arbitrarily small. For the compactification of each medium cusp, we take a Margulis tube $T_{r(\epsilon)}(\alpha)$, where $r(\epsilon)$ is taken to be the radius which separates the ϵ -thin and ϵ -thick part of DZ_n . In this way, after the filling of all of them, we get that $DZ_n^{<\epsilon} = \sqcup_{j=1}^m T_{r(\epsilon)}(\alpha_j)$, corresponding to small and medium cusps.

Note that the only closed geodesics that lie on $DZ_n^{<\epsilon}$ are the core geodesics α_j . Indeed, these Margulis tubes are solid tori, so their fundamental groups are isomorphic to \mathbb{Z} . In each of them, then, there is only one free-homotopy class of closed curves -form by the core curve and its powers- in which the α_j is the geodesic representative. On the other hand, we must recall that we are interested only in closed geodesics lying in one copy of Z_n , in which these core geodesics don't exist. Therefore, they can be dismissed for our argument.

This yields, then, that the only geodesics object of study are either the ones lying in the ϵ -thick part of DZ_n , or the ones having a part inside some Margulis tube corresponding to a medium cusp.

For the first case, we can use directly Theorem 3.2.3. This gives us a bilipschitz inclusion between $DZ_n^{\geq\epsilon}$ and $DK_n^{\geq\frac{\epsilon}{1.2}}$, from which we obtain that the length of the preimage $\tilde{\gamma} \in DK_n$ of the closed geodesic $\gamma \in DZ_n$ under this bilipschitz map might increase at most by a factor $J_{\epsilon,L} \in (1, 1.0005)$. Thus, if take as γ' the geodesic in the homotopy class of $\tilde{\gamma}$ in DK_n , we obtain:

$$l(\gamma') \leq l(\tilde{\gamma}) \leq J_{\epsilon,L} \cdot l(\gamma).$$

On the other hand, if we measure now the length of the image of γ' in the metric DZ_n under the bilipschitz inclusion, we have, by this same result, that its length may increase by at most $J_{\epsilon,L}$. As γ is the geodesic in the homotopy class of the image of γ' in DZ_n , we obtain:

$$\frac{1}{J_{\epsilon,L}} \cdot l(\gamma) \leq l(\gamma'),$$

where, in both inequalities, $J_{\epsilon,L} \rightarrow 1$ as $n \rightarrow \infty$, by definition of the biliptchitz constant given in Theorem 3.2.3.

Finally, let's study the second case. Let $\gamma \in DZ_n$ such that it lies partly in some Margulis tube $T_{r(\epsilon)}(\alpha_j)$, corresponding to a medium cusp. We will see that this curve cannot be in the "very thin" part of DZ_n .

For this, we take $0 < \delta < \epsilon$ such that:

$$\operatorname{arcosh} \left(\frac{\epsilon}{\sqrt{7.256\delta}} \right) - 0.0424 > l_{max}.$$

We consider also the nested Margulis tube $T_{r(\delta)}(\alpha_j)$ of radius $r(\delta) > 0$ around the core curve α_j , which contains the δ -thin part of DZ_n around this core geodesic (see Figure 3.7).

Now, suppose that γ enters the δ -thin part of that manifold. Since it lies only partly in the ϵ -thin part on DZ_n , that means it has to exit both $T_{r(\delta)}(\alpha_j)$ and $T_{r(\epsilon)}(\alpha_j)$ at some point. Therefore, its length needs to be larger than twice the distance between the boundaries of

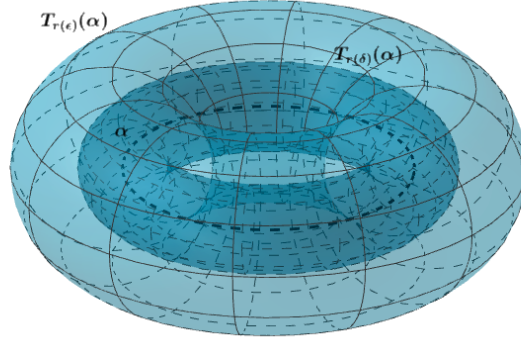


Figure 3.7: Nested Margulis tubes of radius $r(\epsilon)$ and $r(\delta)$ around the core geodesic α .

the two tubes.

Theorem 1.1 from Futer-Purcell-Schleimer [FPS19] gives lower and upper bounds on the distance between these two nested Margulis tubes, provided that the length of the core geodesic is smaller than δ . Thus, in order to be able to apply it, we first check this condition. Using [FPS22, Corollary 6.13], we have that the length of the core curve α_j is bounded by:

$$l(\alpha_j) < \frac{2\pi}{L^2 - 28.78},$$

where L is the total normalized length of the compactified cusp. Since here we're dealing with medium cusps, we have that $L \geq \sqrt{\frac{1}{8} \log_3(n)}$. Hence,

$$l(\alpha_j) \leq \frac{2\pi}{\frac{1}{8} \log_3(n) - 28.78} < \frac{16\pi}{\log_3(n)},$$

which is less than δ for n large enough. Therefore, we can apply [FPS19, Theorem 1.1], that yields:

$$d(\partial T_{r(\delta)}(\alpha_j), \partial T_{r(\epsilon)}(\alpha_j)) \geq \operatorname{arcosh} \left(\frac{\epsilon}{\sqrt{7.256\delta}} \right) - 0.0424 > l_{max}.$$

This leads to a contradiction, since for γ to enter into any of the smallest tubes $T_{r(\delta)}(\alpha_j)$, for $j = s+1, \dots, m$, its length needed to be larger than twice that distance, yet the length of the curve γ is bounded by l_{max} . Therefore, we conclude that asymptotically as $n \rightarrow \infty$, γ won't enter the δ -thin part of DZ_n .

We can apply, consequently, the bilipschitz equivalences from Futer-Purcell-Schleimer (Theorem 3.2.3) to the δ -thick part of DZ_n . In the same way as before, by taking as γ' the geodesic in the homotopy class of the preimage of γ in DK_n under the bilipschitz map, we get:

$$\frac{1}{J_{\delta,L}} \cdot l(\gamma) \leq l(\gamma') \leq J_{\delta,L} \cdot l(\gamma),$$

where $J_{\delta,L} \rightarrow 1$ as $n \rightarrow \infty$.

After the filling of all medium cusps, we obtain then that the length of γ' will be bounded by the length of γ times some product of bilipschitz constants, all tending to 1 as $n \rightarrow \infty$.

Filling large cusps. Finally for this step, we rely entirely on the previously mentioned result of Futer-Purcell-Schleimer.

The structure of the argument is exactly like in the case for medium cusps. Thus, as before, we take as our cusped manifold N the double of Z_n , DZ_n . Equally, the compactification process consists of taking out a horospherical neighbourhood of each cusp, and filling it by gluing a Margulis tube. This time, however, the procedure is done for all cusps at once, as we don't have the certainty that large cusps are not incident to each other. Then, the resulting manifold, made of the ϵ -thick part of DZ_n , for some $\epsilon > 0$ -which is compact- and the Margulis tubes attached along the boundary, is a reasonable model for the compact manifold DM_n (which corresponds to M in the statement of the theorem).

The only condition we need to check to apply Theorem 3.2.3, is that, in this case, the total normalized length is still big enough to verify the condition of the statement. This fails if there are "many" large cusps. However, this will not occur a.s.s. Indeed, by [PR22, Theorem 2.4], we have that the expected number of cusps in Y_n is equal to $\frac{1}{2} \log(n) + O(1)$, as $n \rightarrow \infty$. Hence, if we denote by C_l the number of large cusps in DY_n , using Markov's inequality we have that:

$$\mathbb{P}[C_l \geq Kn^{1/4}, K \in (0, 1)] \leq \frac{\mathbb{E}[C_l]}{Kn^{1/4}} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus, like before, the two same cases for the position of γ in DM_n appear, and following the same argument, we get that there is a closed geodesic γ' in DZ_n such that their lengths differ by a multiplicative constant that tends to 1 as $n \rightarrow \infty$.

All in all, we obtain that for any closed geodesic γ in M_n of uniformly bounded length, we can find a closed geodesic γ' in Y_n such that its length varies from the one of γ by at most some product of bilipschitz constants that tend to 1 as $n \rightarrow \infty$. Hence, this gives Proposition 3.2.1. □

Before moving into the next point, let's prove the remaining claim, stated before:

Claim 1. *Medium cusps don't intersect each other a.a.s. Consequently, when compactifying them one by one, the bilipschitz constants won't accumulate.*

Proof. Following the same notation as in [PR22], we denote by E_{KL} the number of pairs of edges of size $\leq K$ and $\leq L$ -that is, that are incident to less than K and L tetrahedra respectively- and that are both incident to a common tetrahedron in the model M_n . Then,

Theorem 2.4 (d) from [PR22] claims that, for any $K, L = o(n^{1/3})$, the expected value of E_{KL} tends to zero, as $n \rightarrow \infty$. In this setting, having an edge of size k implies having a cusp of length k , as there are k glued tetrahedra around it. Thus, the result tells us that the expected number of pairs of intersecting cusps of lengths $\leq C = o(n^{1/3})$ is asymptotically negligible.

This fact, in particular, assures that the bilipschitz constants resulting at each step will not accumulate for the following: when applying Futer-Purcell-Schleimer's result (Theorem 3.2.3) to each medium cusp, the bilipschitz equivalence will apply only in the corresponding part of K_{n_1} , which is not incident to any other. Then, outside each compactified cusp (i.e. in K_{n_2}) the isometry given by Lemma 3.2.2 from [PR22] will still hold. \square

3.2.2 Non-homotopy of closed geodesics

Once we have this equivalence on the lengths, it rests to rule out the only setback that could arise: the possibility that closed geodesics after compactification are homotopically trivial, or get homotoped into one another. Hence, we prove now our second main point.

Lemma 3.2.4. *Let $l_{max} > 0$, and $\phi : Y_n \rightarrow M_n$ denote the inclusion map between those manifolds. A.a.s as $n \rightarrow \infty$, the images in M_n of any two non-homotopic closed geodesics in Y_n -of lengths bounded by l_{max} - are also non-homotopic, neither homotopically trivial.*

Proof. We show first that, a.a.s, the image in M_n of a closed geodesic in Y_n is not homotopically trivial. Observe that for a short geodesic to be homotopically trivial after the compactification, it needs to go around at least two small cusps in Y_n . Indeed, if a geodesic homotopic to a point in M_n goes around a single cusp, this would imply that, pre-compactification, it is homotopic to the cusp. But since it is a hyperbolic element, this cannot occur. On the other hand, both the cusps and the distance between them have to be of length $< l_{max}$, since the geodesic is so.

If we look, then, at the path in the dual graph of Y_n that this potentially homotopically trivial geodesic does, we see that it is a concatenation of cycles at uniformly bounded distance. Having this would imply that G_{Y_n} is l -tangled, for $l \leq \frac{1}{8} \log_3(n)$. However, we know by Bordenave's Lemma 1.3.4 that this probability tends to 0 as $n \rightarrow \infty$. Therefore, we can conclude that a.a.s as $n \rightarrow \infty$, closed geodesics of this bounded length don't become homotopically trivial after the compactification.

Now, to treat the case of geodesics becoming homotopic to one another, we'll consider the doubled manifolds DY_n and DM_n , and we will denote by $\psi : DY_n \rightarrow DM_n$ the inclusion map between them, satisfying: $\psi|_{Y_n} = \phi$. As before, we argue by contradiction. Suppose there exists a set $A_n \in \Omega_n$ with $\lim_{n \rightarrow \infty} \mathbb{P}(A_n) > 0$, for which the model DY_n verifies the following: there exist two non-homotopic closed geodesics γ_1 and γ_2 in DY_n , such that their images $\psi(\gamma_1)$ and $\psi(\gamma_2)$ in DM_n are homotopic to some other closed geodesic γ , of smaller length $l > 0$ (or analogously, that one of them is homotopic to the other).

We take, then, the cover of DM_n corresponding to that geodesic, that is, we consider:

$$\mathbb{H}^3/\langle\gamma\rangle,$$

where $\langle\gamma\rangle$ is an infinite cyclic subgroup of the fundamental group of DM_n , generated by the loxodromic transformation γ . This is diffeomorphic to $S^1 \times \mathbb{R}^2$, that is, a solid torus (see Figure 3.8).

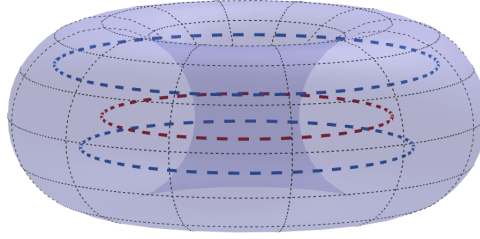


Figure 3.8: The curves $\tilde{\gamma}$ (red) and $\widetilde{\psi(\gamma_1)}$, $\widetilde{\psi(\gamma_2)}$ (blue) in the cover $\mathbb{H}^3/\langle\gamma\rangle$.

We lift now the two curves $\psi(\gamma_1)$ and $\psi(\gamma_2)$ to the cover. Since $\psi(\gamma_1)$ and $\psi(\gamma_2)$ are homotopic to γ , their lifts $\widetilde{\psi(\gamma_1)}$ and $\widetilde{\psi(\gamma_2)}$ in this cover will also contract into $\tilde{\gamma}$. On the other hand, we know, by Proposition 3.2.1, that the lengths of the images in DM_n , and hence also of the lifts, are very close to those in DY_n .

Now, since the geometry of this cover "flares out", that is, the radius of the solid torus grows exponentially fast, this implies that these curves cannot be too far from $\tilde{\gamma}$, or at least, they need to be at bounded distance from it. Thus, if we denote by D_1 the distance from $\tilde{\gamma}$ to $\widetilde{\psi(\gamma_1)}$, and D_2 the distance from $\tilde{\gamma}$ to $\widetilde{\psi(\gamma_2)}$, we have:

$$d(\widetilde{\psi(\gamma_2)}, \widetilde{\psi(\gamma_1)}) \leq d(\widetilde{\psi(\gamma_1)}, \tilde{\gamma}) + d(\tilde{\gamma}, \widetilde{\psi(\gamma_2)}) \leq D_1 + D_2 = D.$$

We have obtained, then, that in the cover, the two curves $\widetilde{\psi(\gamma_1)}$ and $\widetilde{\psi(\gamma_2)}$ are at bounded distance D from each other. But since distances in DM_n are equal to distances in $\mathbb{H}^3/\langle\gamma\rangle$, this yields that $\psi(\gamma_1)$ and $\psi(\gamma_2)$ are also at bounded distance in DM_n . And in particular, so are their paths in the dual graph G_{DM_n} .

However, drawing on graph theory tools, we have that in a random regular graph, the probability that there are two closed paths of lengths l_1 and l_2 at bounded distance $d > 0$ apart, tends to 0 as the number of vertices goes to infinity [BM04, Lemma 5.5]. This, then, leads to a contradiction, proving that a.a.s this homotopy will not occur.

□

3.2.3 Theorem A

Finally, with both Proposition 3.2.1 and Lemma 3.2.4, we are ready to prove our main result. For simplicity of the argument, we will use the same equivalent reformulation of the theorem as for Theorem 3.1.1.

Theorem A. *For any finite collection of disjoint intervals $[a_1, b_1], \dots, [a_t, b_t] \subset \mathbb{R}_{\geq 0}$, the random vector $(C_{[a_1, b_1]}(M_n), \dots, C_{[a_t, b_t]}(M_n))$ converges jointly in distribution, as $n \rightarrow \infty$, to a vector of independent random variables*

$$(C_{[a_1, b_1]}, \dots, C_{[a_t, b_t]}),$$

where for all $i = 1, \dots, t$, $C_{[a_i, b_i]}$ is Poisson distributed with parameter

$$\lambda = \sum_{[w] \in \mathcal{W}_{[a_i, b_i]}} \lambda_{[w]}.$$

Proof. Proposition 3.2.1 states that the lengths in Y_n and M_n are comparable, for n large enough. This, together with Lemma 3.2.4 yields that the number of short closed geodesics in Y_n stays asymptotically the same after compactification. More precisely, given any $a, b \geq 0$, the bounds obtained in the same Proposition 3.2.1 imply that, for every $\epsilon > 0$,

$$C_{[(1+\epsilon)a, \frac{b}{(1+\epsilon)}]}(Y_n) \leq C_{[a, b]}(M_n) \leq C_{[\frac{a}{(1+\epsilon)}, (1+\epsilon)b]}(Y_n).$$

Since by Theorem 3.1.1, $C_{[a, b]}(Y_n)$ converge to a Poisson distributed random variable of parameter $\lambda = \sum_{[w] \in \mathcal{W}_{[a, b]}} \lambda_{[w]}$ as $n \rightarrow \infty$, we can conclude from the inequality above that the same asymptotic distribution of closed geodesics holds for the compact manifold M_n . Moreover, again as a consequence of Theorem 3.1.1, for any finite collection of disjoint intervals, these limiting random variables are independent. \square

Chapter 4

The systole

The content of this chapter is adapted from [RS24]. Here, we address a question posed in [PR22] about the systole of random hyperbolic 3-manifolds under the model of random triangulations.

We prove, on the one hand, the existence of the limit of the expected value of the systole of these manifolds, as their volume goes to infinity. This corresponds to Theorem B. Before stating it, we first recall some notation. Given $l > 0$, let:

$$\mathcal{W}_l = \{[w] \in \mathcal{W} : |w| > 2, |\operatorname{tr}([w])| > 2 \text{ and } 2\operatorname{Re}[\operatorname{arcosh}(\frac{\operatorname{tr}([w])}{2})] \in [0, l]\}.$$

Then,

Theorem B. *Let $\{l_i\}_{i \geq 1}$ be the ordered set of all possible translation lengths coming from (classes of) words $[w] \in \mathcal{W}$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\operatorname{sys}(M_n)] = \sum_{i=1}^{\infty} \left(\prod_{[w] \in \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \cdot l_i.$$

Moreover, we compute a sharp numerical approximation of this value. This is given by Proposition C, the last main result of the manuscript.

Proposition C. *We have:*

$$\lim_{n \rightarrow \infty} \mathbb{E}(\operatorname{sys}(M_n)) = 2.56033312683887522062 \pm 2.95489 \cdot 10^{-16}.$$

For the proof of Theorem B, we follow the same general strategy as before: we compute first the limit of the expected value for the model of hyperbolic manifold Y_n , and we see then that the result applies also to the compactified manifolds M_n . This is done in the two first sections of this chapter. Finally, the last section is devoted to the proof of Proposition C.

4.1 The systole of Y_n

The aim of this section is to prove that we can write:

$$\lim_{n \rightarrow \infty} \mathbb{E}[\text{sys}(Y_n)] = \sum_{i=1}^{\infty} \left(\prod_{[w] \in \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \cdot l_i. \quad (4.1)$$

By definition of systole, if Y_n has systole $l > 0$, it means that there is at least one closed geodesic of length $l > 0$ in Y_n , and no other closed geodesic smaller than this. Hence, in order to have insight on its expected value, we would need to know the number of geodesics of each possible length.

Using the relation (1.1) between the traces of the classes of words and the lengths of their corresponding geodesics, we can translate this counting to the counting of cycles in the dual graph G_{Y_n} corresponding to each class of words $[w]$, as it was done for the study of the length spectrum in the previous chapter. Then, we have Theorem 3.1.2 from Section 3.1.5, which gives us the asymptotic distribution of these random variables. We recall its statement here, as we will refer to it several times: let

$$Z_{n,[w]} : \Omega_n \rightarrow \mathbb{N}, \quad n \in \mathbb{N}, [w] \in \mathcal{W} := W / \sim,$$

defined as

$$Z_{n,[w]}(\omega) := \#\{\text{cycles } \gamma \text{ on } G_{Y_n} : \gamma \text{ is described by } [w]\}.$$

Then,

Theorem 3.1.2. *Consider a finite set \mathcal{S} of equivalence classes of words in W . Then, as $n \rightarrow \infty$,*

$$Z_{n,[w]} \rightarrow Z_{[w]} \quad \text{in distribution for all } [w] \in \mathcal{S},$$

where:

- $Z_{[w]} : \mathbb{N} \rightarrow \mathbb{N}$ is a Poisson distributed random variable with parameter $\lambda_{[w]} = \frac{|[w]|}{3^{|w|}2^{|w|}}$ for all $[w] \in \mathcal{S}$.
- The random variables $Z_{[w]}$ and $Z_{[w']}$ are independent for all $[w], [w'] \in \mathcal{S}$ with $[w] \neq [w']$.

We can now write down an expression for the expected value of the systole of Y_n in terms of these random variables as follows:

$$\mathbb{E}[\text{sys}(Y_n)] = \sum_{i=1}^{\infty} \mathbb{P} \left[\begin{array}{l} \nexists \text{ geodesics of length } l < l_i, \text{ and} \\ \exists \text{ at least one geodesic of length } l_i \text{ in } Y_n \end{array} \right] l_i \quad (4.2)$$

$$= \sum_{i=1}^{\infty} \mathbb{P} \left[\begin{array}{l} Z_{n,[w]} = 0 \text{ for all } [w] \in \mathcal{W}_{i-1}, \text{ and} \\ Z_{n,[w]} > 0 \text{ for some } [w] \in \mathcal{W}_i \setminus \mathcal{W}_{i-1} \end{array} \right] l_i, \quad (4.3)$$

where the sequence $\{l_i\}_{i \geq 1}$ is the ordered set of all possible translation lengths coming from (classes of) words $[w] \in \mathcal{W}$, obtained using (1.1), and $l_0 = 0$.

Thanks to Theorem 3.1.2, we can compute the point-wise limits of these probabilities. Indeed, let A_n^i denote the latter event, that is, $A_n^i = \{Z_{n,[w]} = 0 \text{ for all } [w] \in \mathcal{W}_{l_{i-1}}, \text{ and } Z_{n,[w]} > 0 \text{ for some } [w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}\}$, for $i \geq 1$. We have:

Proposition 4.1.1. *Let $\{l_i\}_{i \geq 1}$ be the ordered set of all possible translation lengths coming from (classes of) words $[w] \in \mathcal{W}$, and $l_0 = 0$. Then, for every $i \geq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{P}[A_n^i] = \left(\prod_{[w] \in \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right).$$

Proof. In Proposition 3.1.3 from Section 3.1.6, we proved that the number of classes of words $[w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}$, for any $i \geq 1$, is finite. Hence, we can use Theorem 3.1.2 which, together with the independence of the random variables, gives:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}[A_n^i] &= \left(\prod_{[w] \in \mathcal{W}_{l_{i-1}}} \mathbb{P}[Z_{[w]} = 0] \right) \left(1 - \prod_{[w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}} \mathbb{P}[Z_{[w]} = 0] \right) \\ &= \left(\prod_{[w] \in \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right). \end{aligned}$$

□

Therefore, up to checking that we can switch the limit and the infinite sum in (4.2), this would allow us to obtain a precise computable expression of the expected value of the systole of the manifolds Y_n .

4.1.1 Convergence

Let us show that we can indeed swap the limit and the infinite sum appearing in the expression (4.2) of the expected systole, so that we can apply Proposition 4.1.1, and obtain the equality (4.1).

For this, we will use the dominated convergence theorem. A first naive observation is that:

$$\mathbb{P}[A_n^i] = \mathbb{P} \left[\begin{array}{l} Z_{n,[w]} = 0 \text{ for all } [w] \in \mathcal{W}_{l_{i-1}}, \text{ and} \\ Z_{n,[w]} > 0 \text{ for some } [w] \in \mathcal{W}_{l_i} \setminus \mathcal{W}_{l_{i-1}} \end{array} \right] \leq \mathbb{P}[Z_{n,[w]} = 0 \text{ for all } [w] \in \mathcal{W}_{l_{i-1}}].$$

We look then for a uniform upper bound for the latter. By definition of the random variable $Z_{n,[w]}$, this tells us that there are no words whose length of the corresponding geodesic is in $[0, l_{i-1}]$. This condition implies, in turn, a lower bound on the combinatorial length of the word. Indeed, if $w = M_1 \cdots M_k$, where $M_i \in \{S, R, L, S\theta, R\theta, L\theta, S\theta^2, R\theta^2, L\theta^2\}$,

using the sub-multiplicity property of the operator norm, we have:

$$\|w\|_\infty = \|M_1 \cdots M_k\|_\infty \leq \|M_1\|_\infty \cdots \|M_k\|_\infty \leq \left(\sup_{1 \leq i \leq k} \|M_i\|_\infty \right)^k = (1 + \sqrt{2})^k.$$

In particular, the absolute value of every coefficient of w is bounded by $(1 + \sqrt{2})^k$. Hence,

$$|\operatorname{tr}(w)| \leq 2(1 + \sqrt{2})^k.$$

Since we have that the length of the geodesic corresponding to w is expressed as:

$$l_\gamma(w) = 2\operatorname{Re} \left[\operatorname{arcosh} \left(\frac{\operatorname{trace}([w])}{2} \right) \right] = 2 \log \left(\left| \frac{\operatorname{tr}(w)}{2} + \sqrt{\left(\frac{\operatorname{tr}(w)}{2} \right)^2 - 1} \right| \right),$$

using the inequality above, we obtain that:

$$l_\gamma(w) \leq k \cdot 2 \log \left(\frac{3}{2}(1 + \sqrt{2}) \right).$$

Hence, if we have that $l_\gamma(w) > l_{i-1}$ for all $w \in G_{Y_n}$, we can bound the probability above by:

$$\mathbb{P} \left[G_{Y_n} \text{ contains no essential cycles of length } \leq \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1 + \sqrt{2}))} \right\rfloor \right],$$

where an essential cycle in G_{Y_n} refers to a cycle representing a curve in the manifold that is non-homotopic to a cusp or a point. As a note, the word essential refers often to curves that are also non-homotopic to a boundary component, but in our case we allow this.

If we denote by τ_n the minimum length of an essential cycle in G_{Y_n} , this is equivalent to:

$$\mathbb{P} \left[\tau_n > \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1 + \sqrt{2}))} \right\rfloor \right].$$

Now, to get the uniform bound on this, we use a version of the following result by McKay-Wormald-Wysocka:

Corollary 1.3.2. *For $(d-1)^{2g-1} = o(n)$, the probability that a random d -regular graph has girth greater than $g \geq 3$ is*

$$\exp \left(- \sum_{r=3}^g \frac{(d-1)^r}{2r} + o(1) \right),$$

as $n \rightarrow \infty$.

This follows from a more general result (see [MWW04, Theorem 1]). The result is very close to what we need, but cannot be applied as is. Note that τ_n does not consider cycles that correspond to parabolic elements in the manifold Y_n . However, these are included

in the result above. Hence, to get a bound for our case, we need to take out a factor corresponding to the possible number of parabolic elements appearing as cycles in the graph.

One of the steps of the proof of Corollary 1.3.2 is to compute the ratio between the set of d -regular graph on n vertices with fixed numbers m_1, \dots, m_t of cycles of certain lengths c_1, \dots, c_t , and the set of those with at most R_1, \dots, R_t cycles of these lengths c_1, \dots, c_t , where $m_i \leq R_i$, $i = 1, \dots, t$ and the R_i are growing in n . For this, they use the switching method (see Section 1.3.2). More precisely, they count the average number of ways of applying a (forward and backward) switching to a graph to get from one set to the other.

By construction of the words w one has that:

$$\frac{\#\{\text{words } w \in W \text{ of } |w| = r \text{ corresponding to parabolic elements}\}}{\#\{\text{words } w \in W \text{ of } |w| = r\}} = \frac{1}{3^r}.$$

Indeed, since there are 9 matrices to choose from at each step, we can create 9^r possible words of length $r > 0$. On the other hand, since classes of words differentiate by trace (up to sign), there exists one class of words representing the parabolic elements, of which $w = S^r$ is a representative. By definition of equivalence class, the cardinal of this class is 3^r .

Hence, we deduce that the average number of cycles of length $r > 0$ in G_{Y_n} corresponding to essential curves is at most $(1 - \frac{1}{3^r})$ the average number of cycles of length r in the graph. This needs to be taken into account in the proof of [MWW04, Theorem 1] when counting the average ways of applying a backward switching, to make sure that the new created cycles are essential. With that in mind, together with some small changes in notation, the rest of the argument of [MWW04, Theorem 1] follows step by step. We record the statement for our particular case as it will be used later on.

Corollary 4.1.2. *For $3^{2g-1} = o(n)$, the probability that G_{Y_n} has no essential cycle of combinatorial length smaller or equal than $g \geq 3$ is less or equal than*

$$\exp\left(-\sum_{r=3}^g \frac{3^r}{2^r} \left(1 - \frac{1}{3^r}\right) + o(1)\right),$$

as $n \rightarrow \infty$.

We use this to bound the probability above. One observation is that $\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor$ is only greater than 3 from some length l_k on. However, the number of terms for which this value is less than 3 is finite. Hence, for these values of l_i , we can bound the term (4.2) by:

$$\mathbb{P}[A_n^i] \cdot l_i \leq \mathbb{P}\left[\tau_n > \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor\right] l_k \leq l_k.$$

On the other hand, we note that Corollary 4.1.2 can be directly applied only when

$g = \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor \leq (\frac{1}{2} + o(1)) \log_3(n)$. For those values of $l_i \geq l_{k+1}$, we have:

$$\begin{aligned} \mathbb{P}[A_n^i] \cdot l_i &\leq \mathbb{P}\left[\tau_n > \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor\right] \cdot l_i \\ &\leq \exp\left(-\sum_{r=3}^{\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \rfloor} \frac{3^r}{2r} \left(1 - \frac{1}{3^r}\right) + K\right) \cdot l_i \\ &= \exp\left(-\sum_{r=3}^{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor} \frac{3^r + K'}{2r}\right) \cdot l_i \\ &\leq \exp\left(-\frac{3^{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor} + K'}{2 \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}\right) \cdot l_i, \end{aligned}$$

for some $K, K' > 0$ independent of n .

Now, for all l_i such that the previous condition is no longer satisfied, we use the following. We refer here to an octahedral hyperbolic manifold a hyperbolic manifold built following the same procedure as Y_n , but in which the gluing is deterministic.

Lemma 4.1.3. *Let X_n be an octahedral hyperbolic manifold made of n octahedra. Then, its dual graph G_{X_n} always has an essential cycle of combinatorial length $\leq 4\lceil \log_2(\frac{n+1}{4}) \rceil + 1$.*

Proof. We pick any vertex v of G_{X_n} , and consider the four vertices neighbours to v . We argue by contradiction: suppose that there doesn't exist any essential cycle starting at any of these four vertices of length up to $k = 4\lceil \log_2(\frac{n+1}{4}) \rceil + 1$.

Now, we consider the paths starting at any of these four vertices that are of the form:

$$w = w_1 \cdots w_t, \quad \text{with } w_i \in \{SR\theta, SL\theta^2\}, \text{ for } i = 1, \dots, t, \quad t \in \mathbb{N}.$$

Note that $SR\theta$ and $SL\theta^2$ are both two-letter words (we don't count θ or θ^2 as letters) that correspond to essential paths, and so their contatenation also form an essential path. Hence, all the paths described by the previous words are essential.

Since, up to combintorial length k , these paths don't form an essential cycle, we have that the number of octahedra they go through after t steps is $4 \cdot 2^t$, for any $t \leq \frac{k-1}{4}$. Equivalently, if $B(v, 2t)$ denotes a ball in G_{X_n} of radius $2t \leq \frac{k-1}{2}$ around v , the number of vertices in $B(v, 2t)$ forming these paths is $4 \cdot 2^t$.

However, if we take $t = \frac{k-1}{4} = \lceil \log_2(\frac{n+1}{4}) \rceil$, the previous fact tells us that the number of vertices forming these paths would be:

$$4 \cdot 2^{\lceil \log_2(\frac{n+1}{4}) \rceil} > n,$$

where n is the total number of vertices. This gives us a contradiction, implying that there is at least two paths w, w' of this form that have the same endpoint.

Now, if w and w' have the same starting vertex, then their contatenation $w \cdot \bar{w}'$ -where \bar{w}' denotes the backwards word of w' - forms a cycle. Since the backwards word \bar{w} of an essential path w is an essential path, and both w and w' are essential ones, we obtain that $w \cdot \bar{w}'$ is an essential cycle of length at most $4\lceil \log_2(\frac{n+1}{4}) \rceil$. On the other hand, if w and w' have different starting vertex, we know that both paths are connected to v . Thus, the path given by $w \cdot \bar{w}' \cdot \tilde{w}$, where $\tilde{w} \in \{S, R\theta, L\theta^2\}$ forms a cycle. Since adding a letter to an essential path gives us an essential path, we conclude that $w \cdot \bar{w}' \cdot \tilde{w}$ forms an essential cycle of length at most $4\lceil \log_2(\frac{n+1}{4}) \rceil + 1$.

□

Using this lemma, it is clear that for the l_i such that $g = \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor \geq 4\lceil \log_2(\frac{n+1}{4}) \rceil + 1$,

$$\mathbb{P} \left[\tau_n > \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor \right] = 0.$$

Finally, for the l_i 's such that $(\frac{1}{2} + o(1)) \log_3(n) \leq \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor \leq 4\lceil \log_2(\frac{n+1}{4}) \rceil + 1$, we observe that the ratio:

$$\frac{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}{(\frac{1}{2} + o(1)) \log_3(n)} \leq 13 + o(1) \leq C \quad \Rightarrow \quad \frac{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}{C} \leq \left(\frac{1}{2} + o(1) \right) \log_3(n).$$

Therefore, using Corollary 4.1.2, we obtain:

$$\begin{aligned} \mathbb{P}[A_n^i] \cdot l_i &\leq \mathbb{P} \left[\tau_n > \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor \right] l_i \\ &\leq \mathbb{P} \left[\tau_n > \frac{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}{C} \right] l_i \\ &\leq \exp \left(- \frac{3 \frac{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}{C} + K''}{\frac{2}{C} \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor} \right) \cdot l_i, \end{aligned}$$

for some $C, K'' > 0$ independent of n .

All in all, we get, for all values of l_i , the following upper-bound:

Lemma 4.1.4. *We have, for all $i \geq 1$:*

$$\mathbb{P}[A_n^i] \leq \exp\left(-\frac{3 \frac{\left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}{C} + K}{\frac{2}{C} \left\lfloor \frac{l_{i-1}}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \right\rfloor}\right),$$

for some $C, K > 0$ independent of n .

To complete the argument, we need now a lower and an upper bound on the lengths. For that, we have the following lemma:

Lemma 4.1.5. *Let l_k , $k \geq 1$, be the k^{th} -entry of $\{l_i\}_{i \geq 1}$, the ordered set of all translation lengths coming from (classes of) words $[w] \in \mathcal{W}$. Then, for k large enough, we have that:*

$$K_1 \cdot \log(k) < l_k < K_2 \cdot \log(k + 3),$$

for some $0 < K_1 < \frac{1}{2}$ and $K_2 \geq 2$.

Proof. We start with the upper bound. Consider the words of the form $w_k = S^{k+1}R\theta$, for $k \geq 1$. They correspond to hyperbolic elements, so the translation lengths related to them are strictly positive. More precisely, one can compute that $\text{tr}(S^{k+1}R\theta) = k + 3$, for any $k \geq 1$. Hence, the translation lengths of the geodesics γ corresponding their equivalence classes $[w_k]$ are given by:

$$l'_k = l_\gamma(w_k) = 2 \text{arcosh}\left(\frac{k+3}{2}\right) < 2 \log(k+3).$$

Since the list of lengths $\{l'_k\}_{k \geq 1}$ derived from them form a subset of $\{l_i\}_{i \geq 1}$, we obtain that:

$$l_k \leq l'_k < 2 \log(k+3), \text{ for all } k \geq 1.$$

Now, the argument for the lower bound relies on the following observation: we have that the group Γ generated by the 9 matrices presented in Section 3.1.2 is a subgroup of $\text{PSL}(2, \mathbb{Z}[i])$. This is a lattice of $\text{PSL}(2, \mathbb{C})$, so by the Prime geodesic theorem for hyperbolic manifolds [Sar83, Theorem 5.1], we know that the number of primitive closed geodesics of length up to some number L in $\mathbb{H}^3/\text{PSL}(2, \mathbb{Z}[i])$ is asymptotic to:

$$\#\{[\gamma] \in \text{PSL}(2, \mathbb{Z}[i]) \text{ primitive} : l(\gamma) < L\} \sim \frac{e^{2L}}{2L}.$$

That implies in particular that there are, asymptotically, at most exponentially many translation lengths up to L in the length spectrum of this manifold.

Now, since the set $\{l_i\}_{i \geq 1}$ is a subset of it -as it contains only the lengths coming from the matrices in Γ -, this tells us also that there are at most exponentially many lengths up

to L in $\{l_i\}_{i \geq 1}$. Let $\{l_1, \dots, l_k\}$ be this set of translations lengths smaller than L . Then, the previous condition translates into:

$$k < e^{2l_k}$$

for k large enough. From this we deduce that $l_k > K_1 \cdot \log(k)$, for some $0 < K_1 < \frac{1}{2}$. \square

All in all, summing over $i \geq 1$, and denoting by $B = \log(\frac{3}{2}(1 + \sqrt{2}))$, we obtain by Lemma 4.1.4 and Lemma 4.1.5:

$$\begin{aligned} \sum_{i=1}^{\infty} \mathbb{P}[A_n^i] \cdot l_i &= \sum_{i=1}^k \mathbb{P}[A_n^i] \cdot l_i + \sum_{i=k+1}^{\infty} \mathbb{P}[A_n^i] \cdot l_i \\ &\leq kl_k + 2 \sum_{i=k+1}^{\infty} \exp\left(-\sum_{r=3}^{\frac{1}{C} \lfloor \frac{l_{i-1}}{2B} \rfloor} \frac{3^r + K}{2^r}\right) \cdot \log(i+3) \\ &\leq kl_k + 2 \sum_{i=k+1}^{\infty} \exp\left(-\frac{3^{\frac{1}{C} \lfloor \frac{l_{i-1}}{2B} \rfloor} + K}{\frac{2}{C} \lfloor \frac{l_{i-1}}{2B} \rfloor}\right) \cdot \log(i+3) \\ &\leq kl_k + 2 \sum_{i=k+1}^{\infty} \exp\left(-\frac{3^{\frac{1}{C} \lfloor \frac{\log(i-1)}{4B} \rfloor} + K}{\frac{2}{C} \lfloor \frac{\log(i+2)}{B} \rfloor}\right) \cdot \log(i+3), \end{aligned}$$

where the latter is a convergent sum. Therefore, we can apply the dominated convergence theorem. This enables us to use Proposition 4.1.1, and so obtain the expression (4.1) for the limit of $\mathbb{E}[\text{sys}(Y_n)]$. The remaining step is then to see that a.a.s, this is also a valid expression for $\mathbb{E}[\text{sys}(M_n)]$. We approach this in the next section.

4.2 The systole of M_n

The goal of this section is to prove that the contribution to the expected value of a set of possible "bad" manifolds B_n arising from the compactification process is asymptotically negligible. In this way, we can conclude that the expected value of the systole computed in the previous section holds a.a.s for the manifolds M_n .

Recall that the manifolds M_n are obtained from a Dehn filling procedure on the manifolds Y_n . This compactification process is done in three steps. The first one deals with the "small" cusps, that is, cusps made of few octahedra around them, and uses a consequence of Andreev's theorem (Lemma 3.2.2 from [PR22]) to control the change in geometry. Then, the "medium" and "large" cusps are treated in two separate steps. In these cases, the main tool that assures enough control is Theorem 3.2.3 by Futer-Purcell-Schleimer. The complete argument can be found in Section 3.2. We recall here, nonetheless, the notation appearing in this chapter.

<i>Cusps of Y_n</i>	<i>Description</i>
Small	Of combinatorial length up to $\frac{1}{8} \log_3(n)$
Medium	Of combinatorial length between $\frac{1}{8} \log_3(n)$ and $n^{1/4}$
Large	Of combinatorial length bigger than $n^{1/4}$

Under these definitions, we consider the following manifolds.

<i>Models</i>	<i>Description</i>
Y_n	Non-compact hyperbolic 3-manifolds with totally geodesic boundary obtained from a gluing of octahedra, and conditioned on not having loops or bigons in its dual graph, described in Section 2.2
K_n	Manifold obtained from Y_n by filling the small cusps
DK_n	Double of K_n
M_n	Manifold obtained from K_n by filling the medium and large cusps homeomorphic to the M_n described in Section 2.1
DM_n	Double of M_n

Finally, recall that $Y_n \subset K_n \subset M_n$, and $\phi : Y_n \rightarrow M_n$ denoted the inclusion map between these manifolds. This is the map we will refer to from now on, so the notation ϕ will be often omitted.

Let us start by defining the set B_n . Informally, this set is formed either by those manifolds whose geometry get distorted in the compactification process, or by the ones whose topological construction yields a degenerated systole. More precisely, this translates into the following subsets:

$$B_n = B_n^{(1)} \cup B_n^{(2)} \cup B_n^{(3)},$$

where:

- $B_n^{(1)} = \left\{ w \in \Omega_n : \begin{array}{l} \exists \text{ a closed geodesic } \gamma \in M_n(w), l(\gamma) < C \log(\log(n)) \\ \text{s.t. } \forall \gamma' \text{ closed geodesic in } Y_n(w), \text{ s.t. } \phi(\gamma') \text{ is homotopic to } \gamma : \\ \frac{l(\gamma)}{l(\gamma')} \notin [1 - \epsilon, 1 + \epsilon], \text{ for some } \epsilon > 0 \end{array} \right\}$.
- $B_n^{(2)} = \left\{ w \in \Omega_n : \begin{array}{l} \exists \text{ a closed geodesic } \gamma \in Y_n(w), l(\gamma) < C \log(\log(n)) \\ \text{s.t. } \gamma \text{ becomes homotopically trivial in } M_n(w) \end{array} \right\}$.
- $B_n^{(3)} = \{w \in \Omega_n : \text{sys}(M_n(w)) \geq C \log(\log(n))\}$.

Here $0 < C < \frac{1}{20}$ is a fixed constant. Now, we can express the expected value of the

systole of M_n in a general way as follows:

$$\begin{aligned} \mathbb{E}[\text{sys}(M_n)] &= \sum_{w \in \Omega_n} \mathbb{P}[w] \text{sys}(M_n(w)) \\ &= \sum_{w \in \Omega_n \setminus B_n} \mathbb{P}[w] \text{sys}(M_n(w)) + \sum_{w \in B_n} \mathbb{P}[w] \text{sys}(M_n(w)) = E_n^{(1)} + E_n^{(2)}. \end{aligned}$$

Remark 10. *Note that there is a positive probability that the manifold M_n is not hyperbolic, even if it's small (Theorem 2.1.3 from Petri-Raimbault). Hence, for these elements $w \in \Omega_n$, we set $\text{sys}(M_n(w)) = 0$.*

Thus, we want to prove that for this set of "bad" manifolds B_n , the following happens:

Proposition 4.2.1.

$$\lim_{n \rightarrow \infty} E_n^{(2)} = \lim_{n \rightarrow \infty} \sum_{w \in B_n} \mathbb{P}[w] \text{sys}(M_n(w)) = 0.$$

For this, we show that the limit of the sum under these sets $B_n^{(1)}$, $B_n^{(2)}$ and $B_n^{(3)}$ vanishes. We separate the proof into four different lemmas, the first studying the term $\text{sys}(M_n(w))$, and the rest the sum for each subset.

Lemma 4.2.2. *Let $w \in \Omega_n$. Then,*

$$\text{sys}(M_n(w)) = O(\log(n)).$$

Proof. By [PR22], we know that the boundary of $M_n(w)$ is a random closed hyperbolic surface $S_n(w)$ of genus $g \geq 2$. By the Gauss-Bonnet theorem, its area is given by:

$$\text{area}(S_n(w)) = -2\pi\chi = 4\pi(g - 1).$$

Now, this surface is built out of $4n$ triangles. Hence, by a simple Euler characteristic computation, we see that its genus has to be less than n . With this, using the inequality for the systole given by the area growth (see Lemma 1.1.2), we get:

$$\text{sys}(S_n(w)) \leq 2 \log(\text{area}(S_n(w))) + K \leq 2 \log(4\pi(n - 1)) = O(\log(n)).$$

Since the curves lying in $S_n(w)$ are part of the length spectrum of M_n , this is in turn an upper bound for $\text{sys}(M_n(w))$, that is,

$$\text{sys}(M_n(w)) \leq \text{sys}(S_n(w)) = O(\log(n)).$$

□

Lemma 4.2.3.

$$\lim_{n \rightarrow \infty} \sum_{w \in B_n(1)} \mathbb{P}[w] \text{ sys}(M_n(w)) = 0.$$

Proof. We bound the probability $\mathbb{P}[B_n(1)]$. As mentioned before, the control on the geometry -and therefore on the lengths of the curves- when doing the Dehn filling of the cusps relies on Lemma 3.2.2 and Theorem 3.2.3.

From Lemma 3.2.2, one has bilipschitz equivalences between the thick parts of Y_n and K_n . However, these bilipschitz constants may not be sufficiently small, or may accumulate if the cusps are very close to each other. On the other hand, Theorem 3.2.3 by Futer-Purcell-Schleimer gives bilipschitz equivalences between the thick parts of K_n and M_n , provided that the total normalized length of the cusps L satisfies:

$$L^2 \geq \frac{2\pi \cdot 6771 \cosh(0.6\delta + 0.1475)^5}{\delta^5} + 11.7. \quad (4.4)$$

We observe that this won't be verified if M_n has many large cusps, or if medium cusps are incident to each other. Moreover, as before, there are some geometric conditions that, even if the theorem is applicable, may cause the bilipschitz constants to degenerate. Indeed, this might occur if geodesics enter the δ -thin parts of the manifold M_n -where the theorem doesn't give any control.

Hence, since we can only assure that the length comparison is good enough when avoiding these cases, we define $B_n(1)$ to be the set of manifolds for which any of the above occurs, that is, in which:

- Small and medium cusps are incident.
- $Y_n(w)$ has many large cusps.
- Geodesics of length $\leq C \log(\log(n))$ enter the δ -thin parts of the manifold $M_n(w)$, for some small $\delta(n) > 0$.
- There exists a closed geodesic $\leq C \log(\log(n))$ in $M_n(w)$ such that every preimage in $Y_n(w)$ goes into octahedra incident to small cusps.

Then, outside this set, using the same arguments as in Proposition 3.2.1 from Section 3.2.1, we can conclude that the lengths pre and post compactification are comparable. In fact, the proof that follows gives an effective version of Proposition 3.2.1 for curves of lengths up to $C \log(\log(n))$, as opposed to curves of uniformly bounded length.

Let's study the first case, that is, the probability that small and medium cusps are incident. Let I_c denote the number of pairs of small or medium incident cusps. By Claim 1 (Section 3.2.1), we know that the expected number pairs of intersecting cusps of lengths exactly $k, l \leq C = o(n^{1/3})$ is $o(n^{-2/3})$. Thus, summing over all possible values of k and l

-that go up to $o(n^{1/4})$ by definition of medium cusps-, gives that $\mathbb{E}[I_c] = o(n^{-1/6})$. Using Markov's inequality (Theorem 1.2.1), we obtain, then:

$$\mathbb{P}[I_c \geq 1] \leq \frac{\mathbb{E}[I_c]}{1} = o\left(\frac{1}{n^{1/6}}\right).$$

We analyze the next point: if $Y_n(w)$ has many large cusps. We have, by [PR22, Theorem 2.4 (a)], that the expected number of cusps is $\frac{1}{2} \log(n) + O(1)$. So, by denoting as C_l the number of large cusps in $Y_n(w)$, and applying Markov's inequality here again, we obtain:

$$\mathbb{P}[C_l \geq Kn^{1/4}, K \in (0, 1)] \leq \frac{\mathbb{E}[C_l]}{Kn^{1/4}} = O\left(\frac{\log(n)}{n^{1/4}}\right).$$

We deal now with the third case: if γ enters the "very thin" part of the manifold M_n . Here, we can suppose that the previous bullet points don't occur a.a.s. Also, note that it is enough to study this event in the thin parts of the compactified medium and large cusps, as we will deal with the components corresponding to the small cusps in the next case.

We will show that closed geodesics of the length we consider don't enter the thin part at all. For this, we consider the double of the manifold resulting from the compactification of the small cusps, DK_n . This new manifold has medium and large cusps, that are then filled with solid cylinders. Then, we consider Margulis tubes of roughly the same area, so that the final manifold models the geometry of the hyperbolic metric in DM_n .

Now, let γ be a closed geodesic in the compactified manifold DM_n , lying only in one copy of M_n , and let $\delta = \frac{1}{\log(n)^{1/10}}$. Consider also the Margulis tubes $T_{r(\delta)}(\alpha)$ of radius $r(\delta) > 0$ around a core curve α , which contains the δ -thin part of DM_n around this core geodesic. Suppose that γ enters the δ -thin part of that manifold. That means, then, that it also enters another nested Margulis tube $T_{r(\epsilon)}(\alpha)$, for some fixed $\epsilon > \delta$ but small enough so that the ϵ -thin part of the manifold is indeed still isometric to a standard Margulis tube. On the other hand, since γ is a geodesic lying in one copy of M_n , we know that it cannot be entirely in the ϵ -thin part. Therefore, its length needs to be at least twice the distance between the boundaries of the two tubes.

Now, we would like to use the bounds on this distance given by Futer-Purcell-Schleimer in [FPS19, Theorem 1.1]. For that, we need to check that the length of the core curve α we're considering is less than δ . Recall that the Margulis tube around α corresponded, pre-compactification, to a cusp neighbourhood around a medium or large cusp. Since these have total normalized length $L \geq \sqrt{\frac{1}{8} \log_3(n)}$, we get, by [FPS22, Corollary 6.13], that the length of the core curve α is bounded by:

$$l(\alpha) < \frac{2\pi}{L^2 - 28.78} \leq \frac{2\pi}{\frac{1}{8} \log_3(n) - 28.78} < \frac{16\pi}{\log_3(n)},$$

which is indeed less than δ when n is large enough. Therefore, using now [FPS19, Theorem

1.1], we obtain that the distance between the boundary torii is bounded below by:

$$\begin{aligned}
d(\partial T_{r(\delta)}(\alpha), \partial T_{r(\epsilon)}(\alpha)) &\geq \operatorname{arcosh} \left(\frac{\epsilon}{\sqrt{7.256\delta}} \right) - 0.0424 \\
&> \operatorname{arcosh} \left(\frac{\epsilon}{\sqrt{7.256 \frac{1}{\log(n)^{1/10}}}} \right) \\
&> \log \left(\frac{\epsilon \log(n)^{1/20}}{\sqrt{7.256}} \right) \\
&> \log \left(\frac{\epsilon}{\sqrt{7.256}} \right) + \frac{1}{20} \log(\log(n)).
\end{aligned}$$

This implies, then, that the length of γ would need to be strictly bigger than $\frac{1}{10} \log(\log(n))$ to be able to enter the δ -thin part around α . However, we are considering geodesics of length less than $C \log(\log(n))$, where $C < \frac{1}{20}$. Therefore, this yields that, for n big enough, γ doesn't enter the δ -thin parts corresponding to filled medium and large cusps.

A last remark is that it is enough to study this case for this value of δ , that is, outside the δ -thin part, the length of the curve is already controlled. Indeed, even if δ is tending to 0 as $n \rightarrow \infty$, the condition (4.4) on the total normalized length is still satisfied. Therefore Futer-Purcell-Schleimer's result (Theorem 3.2.3) applies, and gives a bilipschitz equivalence between the δ -thick parts of both manifolds, with bilipschitz constant tending to 1 as $n \rightarrow \infty$.

Finally, we study the last case. We suppose here again that the previous cases don't occur a.a.s. So, let γ be some closed geodesic in M_n , and $\tilde{\gamma}$ a preimage in Y_n , and suppose that $\tilde{\gamma}$ enters into an octahedron incident to a small cusp. Then, we have that the cycles in the dual graph G_{Y_n} corresponding to this curve and the parabolic element that goes around the cusp intersect. On the other hand, since the δ -thick parts of M_n and K_n are bilipschitz with bilipschitz constant tending to 1, using the isometry given by Lemma 3.2.2, we can deduce that the length of the part of $\tilde{\gamma}$ lying outside the octahedra incident to the small cusp is of length less than $l(\gamma) < C \log(\log(n))$.

Hence, using now Proposition 3.1.3 from Section 3.1.6, we have that the part of the cycle in G_{Y_n} corresponding to the part of $\tilde{\gamma}$ that is outside the octahedra incident to the cusp, has length bounded above by $\log(n)^C < \frac{1}{8} \log_3(n)$ for n large enough. Therefore, we have found two cycles in G_{Y_n} that lie inside a ball of diameter $\leq \frac{1}{8} \log_3(n) + \log(n)^C$ from some common vertex. That would imply then that G_{Y_n} is l -tangled for $l \leq \frac{1}{8} \log_3(n)$, and n large enough. However, by Lemma 1.3.4 from Bordenave, we have:

$$\mathbb{P} \left[G_{Y_n} \text{ is } \frac{1}{8} \log_3(n)\text{-tangled} \right] = O \left(\frac{3^{\frac{1}{2} \log_3(n)}}{n} \right) \approx O \left(\frac{1}{n^{1/2}} \right).$$

A similar argument works if the preimage $\tilde{\gamma}$ enters into octahedra incident to different small cusps (which could happen if the small cusps are close to each other). This would

imply, then, that in the dual graph G_{Y_n} we can find two cycles of combinatorial length $< \frac{1}{8} \log_3(n)$ at distance $< C \log(\log(n))$, yielding that G_{Y_n} is l -tangled for $l \leq \frac{1}{8} \log_3(n)$. However, as shown above, the probability that this happens tends to zero as $n \rightarrow \infty$.

All together, we have that:

$$\begin{aligned} \mathbb{P}[B_n^{(1)}] &\leq \mathbb{P}[G_{Y_n} \text{ is } \frac{1}{8} \log(n)\text{-tangled}] + \mathbb{P}[C_l > Kn^{1/4}] + \mathbb{P}[I_c \geq 1] \\ &\leq O\left(\frac{1}{n^{1/2}}\right) + O\left(\frac{\log(n)}{n^{1/4}}\right) + o\left(\frac{1}{n^{1/6}}\right) \\ &\leq O\left(\frac{1}{n^{1/6}}\right). \end{aligned}$$

Therefore,

$$\sum_{w \in B_n^{(1)}} \mathbb{P}[w] \text{sys}(M_n(w)) \leq \mathbb{P}[B_n^{(1)}] \max_{w \in B_n^{(1)}} \{\text{sys}(M_n(w))\} \leq O\left(\frac{1}{n^{1/6}} \cdot \log(n)\right),$$

which tends to 0 as $n \rightarrow \infty$.

□

Lemma 4.2.4.

$$\lim_{n \rightarrow \infty} \sum_{w \in B_n^{(2)}} \mathbb{P}[w] \text{sys}(M_n(w)) = 0.$$

Proof. Note that for a short geodesic to be homotopically trivial after the compactification, it needs to go around at least two small cusps in Y_n . Indeed, there might exist geodesics -for instance corresponding to words of the form $w = S^k \theta$ - that go around one cusp. However, if they were to be homotopic to a point in M_n , that would imply that, pre-compactification, they are homotopic to the cusp. But they are hyperbolic elements, so this cannot happen. On the other hand, both the cusps and the distance between them have to be of length $< C \log(\log(n))$, since the geodesic is so.

If we look at the paths that these potentially homotopically trivial geodesics do in the dual graph of Y_n , we see that they are concatenations of cycles, such as Figure 4.1.

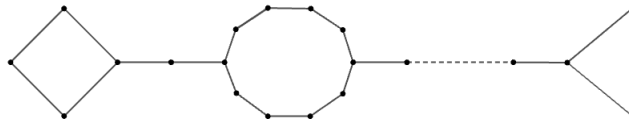


Figure 4.1: Paths in G_{Y_n} of homotopically trivial curves.

But having paths like that would imply that G_{Y_n} is l -tangled, for $l \leq \frac{1}{8} \log_3(n)$. Hence,

again by Bordenave's Lemma 1.3.4, we get that:

$$\mathbb{P}[B_n^{(2)}] \leq \mathbb{P}[G_{Y_n} \text{ is } \frac{1}{8} \log_3(n)\text{-tangled}] = O\left(\frac{1}{n^{1/2}}\right).$$

Therefore,

$$\sum_{w \in B_n^{(2)}} \mathbb{P}[w] \text{sys}(M_n(w)) \leq O\left(\frac{1}{n^{1/2}} \cdot \log(n)\right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

□

Lemma 4.2.5.

$$\lim_{n \rightarrow \infty} \sum_{w \in B_n^{(3)}} \mathbb{P}[w] \text{sys}(M_n(w)) = 0.$$

Proof. For simplicity, here we can suppose that the length of short geodesics don't change when doing the Dehn filling, that is, that $w \notin B_n^{(1)}$ and $w \notin B_n^{(2)}$. Like this, the set $B_n^{(3)}$ can be also defined as:

$$B_n^{(3)} = \{w \in \Omega_n : \text{sys}(Y_n(w)) > C' \log(\log(n)), \text{ for } C' \in (0, 1)\}.$$

This condition, in turn, can be translated into a condition on the length of paths in G_{Y_n} . More precisely, if the translation length of all closed geodesics in Y_n is larger than $C' \log(\log(n))$, this implies that all corresponding closed paths have combinatorial length larger than $\frac{C' \log(\log(n))}{2 \log(\frac{3}{2}(1+\sqrt{2}))} \geq \lfloor C'' \log(\log(n)) \rfloor$, for $C'' \in (0, 1)$. Hence, the probability of the event $B_n^{(3)}$ is bounded by:

$$\mathbb{P}[w \in \Omega_n : G_{Y_n} \text{ contains no essential cycles of lengths } \in \{3, \dots, \lfloor C'' \log(\log(n)) \rfloor\}].$$

For this, we use again Corollary 4.1.2. This gives us:

$$\begin{aligned} \mathbb{P}[B_n^{(3)}] &\leq \exp\left(-\sum_{r=3}^{\lfloor C'' \log(\log(n)) \rfloor} \frac{3^r}{2^r} \left(1 - \frac{1}{3^r}\right) + o(1)\right) \\ &\leq K \exp\left(-\frac{3^{\lfloor C'' \log(\log(n)) \rfloor} - 1}{2^{\lfloor C'' \log(\log(n)) \rfloor}}\right) \text{ for some } K > 0, \\ &\leq O\left(\exp\left(-\frac{3^{\lfloor \log(\log(n)) \rfloor}}{\lfloor \log(\log(n)) \rfloor}\right)\right). \end{aligned}$$

Therefore,

$$\sum_{w \in B_n^{(3)}} \mathbb{P}[w] \text{sys}(M_n(w)) \leq O\left(\frac{1}{e^{\frac{\lfloor \log(\log(n)) \rfloor}{3^{\lfloor \log(\log(n)) \rfloor}}}} \cdot \log(n)\right),$$

which goes to 0 as $n \rightarrow \infty$.

□

With all this, we are ready to prove Proposition 4.2.1.

Proof of Proposition 4.2.1. Having studied the three cases, we obtain:

$$\begin{aligned} \sum_{w \in B_n} \mathbb{P}[w] \operatorname{sys}(M_n(w)) &\leq \mathbb{P}[B_n] \max_{w \in B_n} \{\operatorname{sys}(M_n(w))\} \\ &\leq (\mathbb{P}[B_n^{(1)}] + \mathbb{P}[B_n^{(2)}] + \mathbb{P}[B_n^{(3)}]) \max_{w \in B_n} \{\operatorname{sys}(M_n(w))\} \\ &\leq O\left(\frac{1}{n^{1/6}}\right) O(\log(n)), \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. □

Now, this, together with Proposition 4.1.1, enables to prove what we aimed for:

Theorem B. *Let $\{l_i\}_{i \geq 1}$ be the ordered set of all possible translation lengths coming from (classes of) words $[w] \in \mathcal{W}$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\operatorname{sys}(M_n)] = \sum_{i=1}^{\infty} \left(\prod_{[w] \in \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \cdot l_i.$$

Proof. Using Proposition 4.1.1, we proved in Section 4.1.1 that the right hand side of the equality is a valid expression for the limit of the expected systole of Y_n .

On the other hand, Proposition 4.2.1 implies that, as $n \rightarrow \infty$,

$$\mathbb{E}(\operatorname{sys}(M_n)) = \sum_{w \in \Omega_n \setminus B_n} \mathbb{P}[w] \operatorname{sys}(M_n(w)).$$

Since B_n was exactly the set of manifolds for which the compactification process could degenerate the length of their curves, for all $w \in \Omega_n \setminus B_n$, we have that, $\forall \epsilon > 0$:

$$\frac{1}{1 + \epsilon} \lim_{n \rightarrow \infty} \mathbb{E}(\operatorname{sys}(Y_n)) \leq \lim_{n \rightarrow \infty} \mathbb{E}(\operatorname{sys}(M_n)) \leq (1 + \epsilon) \lim_{n \rightarrow \infty} \mathbb{E}(\operatorname{sys}(Y_n)).$$

Therefore, all combined, we obtain the expression we are looking for. □

4.3 A numerical value

Since we have a formula for the limit of $\mathbb{E}(\operatorname{sys}(M_n))$, we can try to compute a numerical value of it. The problem is, that the list of ordered lengths l_i is hard to determine, and the program for computing the sets $\mathcal{W}_i \setminus \mathcal{W}_{i-1}$ for all $i \geq 1$ is computationally very slow.

Indeed, even though the lengths can be computed using formula (1.1), this equality depends on the trace of some class of words, which corresponds to a complex number. Hence, we cannot order lengths by trace, as these don't have a natural ordering. This fact differentiates it from the two dimensional case [Pet17], making the computation much harder.

One could consider, then, taking the w -distance (defined in the proof of Proposition 3.1.3, and recalled in Section 4.3.1) to get the ordered list. However, that doesn't completely work either, as it is not true that the translation length increases whenever this w -distance does so. Another natural parameter for this is the combinatorial length of the words. In this case, it does exist a coarse comparison between word length and geometric length (given by our Proposition 3.1.3) which would enable us, a priori, to obtain this ordered list of lengths. However, the bound that this gives is too big to make it computationally feasible. These obstacles also make it less evident to get a complete list of words of length less than l_i , for every $i \geq 1$.

On top of it, the complexity of the computation to check whether two words belong to the same equivalent class grows exponentially on the length of the word. Thus, computing the classes of words belonging to $\mathcal{W}_i \setminus \mathcal{W}_{i-1}$ is numerically doable only when i is very small.

Hence, to get an approximated value for the limit, we do the following: we compute the first terms of the sum, for which the lengths and the sets $\mathcal{W}_i \setminus \mathcal{W}_{i-1}$ can be determined, and then we give an upper-bound for the rest of the sum. To overcome the aforementioned constraints related to the ordered list of lengths, here we mix different techniques - using the w -distance in some cases, and the structure of the alphabet matrices in others- that allow us to compute more efficiently an ordered list of sufficient lengths to obtain a good approximation.

In order to simplify the formulas, we define some notation. For all $i \geq 1$, let:

$$p_i = \lim_{n \rightarrow \infty} \mathbb{P}[A_n^i] = \left(\prod_{[w] \in \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right) \left(1 - \prod_{[w] \in \mathcal{W}_i \setminus \mathcal{W}_{i-1}} \exp\left(\frac{|[w]|}{2|w|3^{|w|}}\right) \right).$$

Then, we can write:

$$\lim_{n \rightarrow \infty} \mathbb{E}(\text{sys}(M_n)) = \sum_{i=1}^k p_i \cdot l_i + \sum_{i=k+1}^{\infty} p_i \cdot l_i = S_c + S_e,$$

where S_c represents the computable part of the sum, and S_e the error term. We study them separately in the next two subsections.

4.3.1 The program for S_c

To compute the finite sum $S_c = \sum_{i=1}^k p_i \cdot l_i$, we need to know the first k values of $\{l_i\}_{i \geq 1}$, and the (classes of) words that correspond to each of these lengths.

For that, we start by computing all words of translation length less than some number $D > 0$. As mentioned before, this is not completely straightforward. To do so, we will (partly) use the w -distance, which we recall next.

Let P be the plane determined by the triple $\{0, i, \infty\}$, and $w \in W$. Then, the w -

distance, denoted by $d(w)$, is the distance in the upper half-space of \mathbb{H}^3 between the planes P and $w(P)$. This is given by:

$$\begin{aligned} d(w) &= \min_{\substack{(x_1, y_1) \in P \\ (x_2, y_2) \in w(P)}} \{d((x_1, y_1), (x_2, y_2))\} \\ &= \min_{\substack{(x_1, y_1) \in P \\ (x_2, y_2) \in w(P)}} \left\{ \operatorname{arcosh} \left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} \right) \right\}. \end{aligned} \quad (3.1)$$

Thus, we first compute the list of words of $d(w) < D$. This can be easily done as the w -distance increases whenever the combinatorial length of the word does. The reason is that the plane spanned by the larger word is contained in the half-space spanned by the smaller one (Proposition 3.1.3, from Section 3.1.6). This implies also that the w -distance corresponding to some word w is always less or equal than the translation length of the curve related to it, that is,

$$d(w) \leq l_\gamma(w), \quad \text{for any word } w \in W.$$

From this last observation, we can deduce that the list L of words of $d(w) < D$ contains all words of translation length less than D . Indeed, for any word w such that $d(w) > D$, we would have $D < d(w) \leq l_\gamma(w)$. Hence, the remaining step to get the initial list is to filter the words in L by computing their translation length.

This procedure, however, fails for certain words. More precisely, for those corresponding to parabolic elements with an extra twist when gluing the faces of the octahedra, for instance $w = SSS\theta$. These now correspond to hyperbolic elements, so their translation length is positive, but their w -distance is always zero. Hence, we need another way to see when we need to stop considering them. For that, we compute directly their translation length, and rely on the following lemma:

Lemma 4.3.1. *Let $l_\gamma(w)$ denote the translation length of a closed geodesic $\gamma \in Y_n$ corresponding to a class of words $[w] \in \mathcal{W}$. Then, for all $k \geq 3$, we have:*

$$l_\gamma(S^k\theta) < l_\gamma(S^{k+1}\theta)$$

$$l_\gamma(S^k\theta^2) < l_\gamma(S^{k+1}\theta^2).$$

Proof. One can compute that the traces of the words in $[S^k\theta]$ are equal to $\pm(ki + 1)$, and hence those of $[S^{k+1}\theta]$ are $\pm((k + 1)i + 1)$, for every $k \geq 1$. Now, recall that:

$$\begin{aligned} l_\gamma(S^k\theta) &= 2\operatorname{Re} \left[\operatorname{arcosh} \left(\frac{ki + 1}{2} \right) \right] = 2\operatorname{Re} \left[\log \left(\frac{ki + 1}{2} + \sqrt{\left(\frac{ki + 1}{2} \right)^2 - 1} \right) \right] \\ &= 2 \log \left(\left| \frac{ki + 1}{2} + \sqrt{\left(\frac{ki + 1}{2} \right)^2 - 1} \right| \right). \end{aligned}$$

Thus, to see that the translation length of this class of words increases, it is sufficient to see that the absolute value of this complex number z inside the log does so. We know that $|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$, where:

$$\begin{aligned}\operatorname{Re}(z) &= \frac{1}{2} + \operatorname{Re}\left(\sqrt{2ki - \frac{k^2 + 3}{4}}\right) \\ &= \frac{1}{2} + \left(\frac{1}{2}\sqrt[4]{k^4 + 70k^2 + 9} \cdot \sin\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2 + 3}\right)\right)\right) \\ \operatorname{Im}(z) &= \frac{k}{2} + \operatorname{Im}\left(\sqrt{2ki - \frac{k^2 + 3}{4}}\right) \\ &= \frac{k}{2} + \left(\frac{1}{2}\sqrt[4]{k^4 + 70k^2 + 9} \cdot \cos\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2 + 3}\right)\right)\right).\end{aligned}$$

By computing their squares, and summing them, we get:

$$\begin{aligned}|z|^2 &= \operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 \\ &= \frac{k^2 + 1}{4} + \frac{1}{4}\sqrt{k^4 + 70k^2 + 9} \\ &\quad + \frac{1}{2}\sqrt[4]{k^4 + 70k^2 + 9} \left(\sin\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2 + 3}\right)\right) + k \cos\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2 + 3}\right)\right) \right).\end{aligned}$$

The first line of the expression is clearly increasing in k , for $k \geq 3$. For the second line, consider $f(k) = \sin\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right) + k \cos\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right)$. Since $0 < \sin\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right) < 0.55$ and $0.83 < \cos\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right) < 1$ for all $k \geq 3$, $f(k)$ is a positive function. Now, if we compute its derivative, we have that:

$$f'(k) = \frac{4k(k^2 - 3) \sin\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right) + (k^4 + 66k^2 + 21) \cos\left(\frac{1}{2}\arctan\left(\frac{8k}{k^2+3}\right)\right)}{k^4 + 70k^2 + 9} > 0.$$

As the product and composition of positive increasing functions is positive and increasing, we finally obtain that $|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$ is increasing in k , which is what we wanted.

For the class $[S^k\theta^2]$, an analogous argument works: the traces of words in $[S^k\theta^2]$ are equal to $\pm(ki - 1)$, and hence those of $[S^{k+1}\theta^2]$ are $\pm((k+1)i - 1)$, for every $k \geq 1$. We get, then, the same expressions for the real and imaginary parts, so the rest follows from above. \square

This tells us, then, that once one of these words have translation length bigger than D , we can stop checking the larger ones of that same form.

Joining these two procedures, we obtain the complete list of words of translation length less than D . This process is carried out by a Sage program, of which this is a pseudo code.

Algorithm 1 Computes the list of words of translation length less than D

```

1: procedure TRANS. LENGTH( $D$ )
2:   Set list  $tocheck_{hyp}$            ▷ of hyp. words that we need to check at each step
3:   Set list  $valid_{hyp}$              ▷ of words from  $tocheck_{hyp}$  that have  $w$ -distance  $< D$ 
4:   Set list  $tocheck_{par}$          ▷ of parab. words with twists that we need to check at each
   step
5:   Set list  $valid_{par}$            ▷ of words from  $tocheck_{par}$  that have tr. length  $< D$ 
6:   Set list  $valid$                ▷ of all words of tr. length  $< D$ 
7:   Set list  $trace_w$            ▷ of traces of all words from  $valid$ 
8:
9:   Initial case:
10:   $tocheck_{hyp} \leftarrow$  3-tuples of indices from  $\{0, 1, 2\}$ 
11:  for each tuple in  $tocheck_{hyp}$  do
12:    if tuple is a parabolic word then
13:       $w2, w3 \leftarrow$  Matrices corresponding to tuple, with one and two twists
14:       $l2 \leftarrow$  translation length corresponding to  $w2$ 
15:      if  $l2$  is less than  $D$  then
16:        SAVE PAR( $valid, valid_{par}, trace_w, tuple, w2$ )
17:       $l3 \leftarrow$  translation length corresponding to  $w3$ 
18:      if  $l3$  is less than  $D$  then
19:        SAVE PAR( $valid, valid_{par}, trace_w, tuple, w3$ )
20:    else
21:       $w \leftarrow$  Matrix corresponding to tuple
22:       $d \leftarrow$   $w$ -distance from  $P = \{0, i, \infty\}$  to  $w(P)$ 
23:      if  $d$  is less than  $D$  then
24:         $valid_{hyp} \leftarrow$  tuple
25:         $l \leftarrow$  translation length corresponding to  $w$ 
26:        if  $l$  is less than  $D$  then
27:          SAVE( $valid, trace_w, tuple, w$ )
28:         $w2, w3 \leftarrow$  Matrices corresponding to tuple, with one and two twists
29:         $l2 \leftarrow$  translation length corresponding to  $w2$ 
30:        if  $l2$  is less than  $D$  then
31:          SAVE( $valid, trace_w, tuple, w2$ )
32:         $l3 \leftarrow$  translation length corresponding to  $w3$ 
33:        if  $l3$  is less than  $D$  then
34:          SAVE( $valid, trace_w, tuple, w3$ )

```

Algorithm 1 Computes the list of words of translation length less than D

```

35: procedure TRANS. LENGTH( $D$ )
36:
37: Iterative case:
38:    $n=3$ 
39:   while lists  $valid_{hyp}$  or  $valid_{par}$  are non-empty do
40:      $tocheck_{hyp} \leftarrow$  empty the list
41:      $tocheck_{hyp} \leftarrow$   $(n+1)$ -tuples of indices from  $valid_{hyp}$  and  $\{0, 1, 2\}$ 
42:      $tocheck_{par} \leftarrow$   $(n+1)$ -tuples of indices from  $valid_{par}$  by repeating 1st index
43:      $valid_{hyp} \leftarrow$  empty the list
44:      $valid_{par} \leftarrow$  empty the list
45:
46:     for each tuple in  $tocheck_{hyp}$  do
47:        $w \leftarrow$  Matrix corresponding to tuple
48:        $d \leftarrow$   $w$ -distance from  $P = \{0, i, \infty\}$  to  $w(P)$ 
49:       if  $d$  is less than  $D$  then
50:          $valid_{hyp} \leftarrow$  tuple
51:          $l \leftarrow$  translation length corresponding to  $w$ 
52:         if  $l$  is less than  $D$  then
53:           SAVE( $valid$ ,  $trace_w$ ,  $tuple$ ,  $w$ )
54:            $w_2, w_3 \leftarrow$  Matrices corresponding to tuple, with one and two twists
55:            $l_2 \leftarrow$  translation length corresponding to  $w_2$ 
56:           if  $l_2$  is less than  $D$  then
57:             SAVE( $valid$ ,  $trace_w$ ,  $tuple$ ,  $w_2$ )
58:            $l_3 \leftarrow$  translation length corresponding to  $w_3$ 
59:           if  $l_3$  is less than  $D$  then
60:             SAVE( $valid$ ,  $trace_w$ ,  $tuple$ ,  $w_3$ )
61:
62:     for each tuple in  $tocheck_{par}$  do
63:        $w_p \leftarrow$  Matrix corresponding to tuple
64:        $l_p \leftarrow$  translation length corresponding to  $w_p$ 
65:       if  $l_p$  is less than  $D$  then
66:         SAVE PAR( $valid$ ,  $valid_{par}$   $trace_w$ ,  $tuple$ ,  $w_p$ )
67:      $n \leftarrow n + 1$ 
68:
69:    $length \leftarrow$  length of list  $valid$ 
70:   return  $valid$ ,  $length$ ,  $trace_w$   $\triangleright$  returns the list of all valid words, its length, and
the list of their traces

```

Algorithm 2 Arranges and saves the valid words coming from parabolics

procedure SAVE PAR(*valid*, *valid_{par}*, *trace_w*, *tuple*, *w*)

- 2: $tc \leftarrow$ change indices of tuple to 9 matrix system
 $valid_{par} \leftarrow tc$
- 4: $valid \leftarrow tc$
 $trace_w \leftarrow$ trace of w
-

Algorithm 3 Arranges and saves the valid words

procedure SAVE(*valid*, *trace_w*, *tuple*, *w*)

- 2: $tc \leftarrow$ change indices of tuple to 9 matrix system
 $valid \leftarrow tc$
- 4: $trace_w \leftarrow$ trace of w
-

As a note, we can think of this program in a more geometric way: when computing all words of w -distance less than D , we are constructing a polyhedron made of octahedra which contains all paths w whose w -distance $d(w)$ with respect to some initial face P is less than D .

Once we have this list of words of translation length $< D$, we group them in classes of words -by analysing, for each one, the conditions that define the equivalence class-, and compute the cardinals of each class. On the other hand, we compute the complete list of translation lengths that appear up to the number D . These lists give us, then, all the information we need to compute S_c .

Taking $D = 4.6$, we obtain a list of 31 lengths, and a value of S_c :

$$S_c = \sum_{i=1}^{31} p_i \cdot l_i = 2.56033312683887522062 \dots$$

4.3.2 The error term S_e

In order to bound the sum S_e , we subdivide it into blocks, and bound each of these, that is,

$$S_e = \sum_{i=32}^{\infty} p_i \cdot l_i = \sum_{i:l_i \in (l_{31}, [l_{32}])} p_i \cdot l_i + \sum_{k=[l_{32}]}^{\infty} \sum_{i:l_i \in [k, k+1)} p_i \cdot l_i.$$

Observe that in these sums, the lengths l_i can be bounded by $[l_{32}]$ and $k+1$ respectively, which are natural numbers. To get a sharper bound on the error term, then, we decrease the growth in the sum over k by defining:

$$\tau(l_i) := 2 \cosh\left(\frac{l_i}{2}\right),$$

and re-writing the previous expression as:

$$S_e = \sum_{i:\tau(l_i) \in (\tau(l_{32}), \lceil \tau(l_{32}) \rceil)} p_i \cdot l_i + \sum_{k=\lceil \tau(l_{32}) \rceil}^{\infty} \sum_{i:\tau(l_i) \in [k, k+1)} p_i \cdot l_i =: A + B.$$

The term A can be bounded by:

$$\begin{aligned} A &\leq 2 \operatorname{arcosh} \left(\frac{\lceil \tau(l_{32}) \rceil}{2} \right) \lim_{n \rightarrow \infty} \mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) < \tau(l_{32})] \\ &= 2 \operatorname{arcosh} \left(\frac{\lceil \tau(l_{32}) \rceil}{2} \right) \lim_{n \rightarrow \infty} \mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W}_{l_{31}}] = 2.9220 \cdot 10^{-16}. \end{aligned}$$

where the probability can be found using the computation for S_c .

Now, let's study B. As before, we have that:

$$B \leq \sum_{k=\lceil \tau(l_{32}) \rceil}^{\infty} 2 \operatorname{arcosh} \left(\frac{k+1}{2} \right) \lim_{n \rightarrow \infty} \mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) < k].$$

Denote by $q_{[0, k]} = \mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) < k]$. We can decompose $q_{[0, k]}$ as:

$$q_{[0, k]} = q_{[0, \tau(l_{32})]} \cdot q_{[\tau(l_{32}), k]}.$$

The first factor corresponds to the same probability as in A, so we can compute it. Like this, we get:

$$B \leq e^{-\frac{112}{3}} \sum_{k=\lceil \tau(l_{32}) \rceil}^{\infty} 2 \operatorname{arcosh} \left(\frac{k+1}{2} \right) \lim_{n \rightarrow \infty} \mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) \in [\tau(l_{32}), k]].$$

To get a sharper bound, we study the first term B_1 (when $k = \lceil \tau(l_{32}) \rceil$) separately. Using computational data, we have that the probability:

$$\mathbb{P}[Z_{n, [w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) \in [\tau(l_{32}), \lceil \tau(l_{32}) \rceil]]$$

can be bounded above by:

$$\mathbb{P}[Z_{n, [w]} = 0, \text{ for } [w] \in \Lambda],$$

where:

$$\Lambda = \left\{ \begin{array}{l} [S^{10}\theta], [S^{10}\theta^2], [S^2(L\theta^2)^2S\theta], [S^2(L\theta)^2S\theta], [S^2(L\theta^2)^2R\theta], \\ [S^2(R\theta)^2L\theta^2], [S^3(L\theta^2)^2R\theta^2], [S^3(R\theta)^2L\theta], [S^3(S\theta)^2R], [S^3(S\theta^2)^2L] \end{array} \right\}.$$

Each of these classes of words have $\lambda_{[w]} = \frac{1}{2}$, so by Theorem 4.1.1 we have that, as

$n \rightarrow \infty$, this probability is exactly e^{-5} . Hence, the first term is bounded by:

$$B_1 \leq e^{-\frac{112}{3}} 2 \operatorname{arcosh} \left(\frac{\lceil \tau(l_{32}) \rceil + 1}{2} \right) e^{-5} = e^{-\frac{112}{3}} 2 \operatorname{arcosh}(6) e^{-5} = 2.04164 \cdot 10^{-18}.$$

Finally, we deal with the remaining term, that is:

$$B_2 = e^{-\frac{112}{3}} \sum_{k=\lceil \tau(l_{32}) \rceil + 1}^{\infty} 2 \operatorname{arcosh} \left(\frac{k+1}{2} \right) \lim_{n \rightarrow \infty} \mathbb{P}[Z_{n,[w]} = 0, \forall [w] \in \mathcal{W} : \tau(l_\gamma(w)) \in [\tau(l_{32}), k]].$$

Similarly as before, we argue that this probability above can be bounded by:

$$\mathbb{P}[Z_{n,[w]} = 0, \text{ for } [w] \in \{\Lambda \cup [S^{r-3}R\theta], r \in (\lceil \tau(l_{32}) \rceil, k]\}].$$

Indeed, one can compute that $\tau(l([S^{k-3}R\theta])) = \operatorname{tr}([S^{k-3}R\theta]) = k - 1$, which verifies the condition: $\tau(l_\gamma(w)) \in [\tau(l_{32}), k]$. On the other hand, it is not hard to see that $\|[S^{k-1}R\theta]\| = 3^k 2k$. Therefore, we have that:

$$\mathbb{P}[Z_{n,[w]} = 0, \text{ for } [w] \in \{\Lambda \cup [S^{r-3}R\theta], r \in (\lceil \tau(l_{32}) \rceil, k]\} \leq e^{-5} \cdot e^{\lceil \tau(l_{32}) \rceil - k}.$$

Hence, B_2 can be bounded by:

$$\begin{aligned} B_2 &\leq 2e^{-\frac{112}{3}} e^{-5} e^{\lceil \tau(l_{32}) \rceil} \sum_{k=\lceil \tau(l_{32}) \rceil + 1}^{\infty} \operatorname{arcosh} \left(\frac{k+1}{2} \right) e^{-k} \\ &= 2e^{-\frac{112}{3}} e^6 \sum_{k=12}^{\infty} \operatorname{arcosh} \left(\frac{k+1}{2} \right) e^{-k} \simeq 1.24718 \cdot 10^{-18}. \end{aligned}$$

All together, we have that the error term S_e is bounded by:

$$S_e = A + B \leq A + B_1 + B_2 \leq 2.95489 \cdot 10^{-16}.$$

Joining the values obtained for S_c and S_e , we finally obtain:

Proposition C.

$$2.56033312683887522062 \leq \lim_{n \rightarrow \infty} \mathbb{E}(\operatorname{sys}(M_n)) \leq 2.56033312683887522062 + 2.95489 \cdot 10^{-16}.$$

Chapter 5

Future directions

In this thesis, we've tried to shed some light on the question of what a typical hyperbolic 3-manifold looks like, by giving a characterisation of the typical behaviour of two of its geometric invariants, the length spectrum and the systole. Nonetheless, as mentioned at the beginning of this manuscript, there is still a lot to know about this class of 3-manifolds.

In this last chapter, we present different future directions, some of which arise from the work presented above. We've divided them into two types of questions.

Geometric questions

A natural continuation of my work would be to study the *ortholength spectrum* of the manifolds under this model of random triangulations. The ortholength spectrum of a hyperbolic manifold with totally geodesic boundary is the (multi)-set of lengths of the geodesic arcs whose endpoints are orthogonal to the boundary. This appears in Basmajian's famous identity [Bas93], and has been recently studied, for example, in [MM23, BB22]. In the case of the random 3-manifold M_n , Petri and Raimbault gave a bound on the shortest element of its ortholength spectrum, using Basmajian's tubular neighbourhood theorem [Bas94]. It would be interesting to understand the whole spectrum.

Another geometric invariant, closely related to the length spectrum, is the *Laplacian spectrum*. This contains a lot of geometric information on the given manifold. The first non-zero eigenvalue λ_1 (also known as the spectral gap) is of particular interest as it measures the connectivity of a hyperbolic manifold. Concerning it, we could ask, for instance, what the maximal spectral gap for the doubled manifolds DM_n is. This is planned to be joint work with my PhD advisor Bram Petri, by adapting some ideas from Hide-Magee [HM23] and Bordenave–Collins [BC19]. Another interesting problem to approach would be to try to prove effective convergence of the spectral measure of a random 3-manifold towards that of its Benjamini–Schramm limit, using the relation between the Laplacian and the length spectrum given by the Selberg trace formula.

Finally, a -probably much harder, but- stimulating question is that of finding other

models of hyperbolic 3-manifolds, preferably without boundary.

Combinatorial questions

There are also many interesting questions about the combinatorics of the polyhedral complex.

For instance, as stated in [PR22], one could ask the question of *Poisson-Dirichlet distribution* for edges. This question has already been addressed for random surfaces made out of gluing of polygons [BCP19, Gam06, CP16]. Hence, one could try to follow the same line of argument.

To end, it would be also intriguing to look for *universality properties* of this model of random triangulations. That is, to see whether there are some properties of the resulting random cell complex that stay equal if we change the polyhedron with which we do this combinatorial construction.

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