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► **To cite this version:**

Juan Manuel Santiago Suárez. Infinitary logics and forcing. General Mathematics [math.GM]. Université Paris Cité; Ecole polytechnique Turin (Italie), 2024. English. NNT : 2024UNIP7024 . tel-04919832

HAL Id: tel-04919832

<https://theses.hal.science/tel-04919832v1>

Submitted on 29 Jan 2025

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UNIVERSITÉ PARIS CITÉ
EN COTUTELLE AVEC POLITECNICO DI TORINO

École doctorale 386 Sciences Mathématiques de Paris Centre
Institut de Mathématiques de Jussieu - Paris Rive Gauche

Infinitary logics and forcing

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Thèse de doctorat de MATHÉMATIQUES : LOGIQUE ET FONDEMENTS
DE L'INFORMATIQUE

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Présentée et soutenue publiquement le 16/12/2024

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Résumé

En 1870, George Cantor a prouvé que si la série trigonométrique

$$\sum_{n=0}^{\infty} (c_n \cos n\pi x + d_n \sin n\pi x)$$

est égale à 0 pour tout $x \in \mathbb{R}$, alors tous ses coefficients c_n et d_n doivent être nuls. Ce résultat l'a amené à poser la question suivante:

Pour combien de valeurs $x \in \mathbb{R}$ la série doit-elle s'annuler, afin de garantir que les coefficients s'annulent également ?

Cantor a compris que la réponse reposait sur le concept d'infini. Plus précisément, il a prouvé que s'il existe $S \subset \mathbb{R}$ tel que la dérivée de Cantor-Bendixon de S s'annule à une étape dénombrable, et que la série s'annule pour tout $x \in \mathbb{R} \setminus S$, alors tous ses coefficients doivent également s'annuler. On peut affirmer que cette découverte a marqué le début de la théorie des ensembles, établissant les bases de l'un des développements les plus profonds des mathématiques modernes.

Une fois que les travaux de Cantor, Dedekind, Zermelo, Hausdorff, Fraenkel, Gödel, Tarski entre autres ont établi les fondements de la théorie des ensembles, et que Cohen a prouvé l'indépendance de certains énoncés naturels par rapport à la théorie ZFC - parmi lesquels, en particulier, le problème du Continuum - la question centrale de la théorie des ensembles, connue sous le nom de programme de Gödel, est devenue:

Quels nouveaux axiomes doivent être incorporés à ZFC ?

Des recherches ultérieures ont suggéré qu'une famille intéressante d'axiomes potentiels pouvait être détecté en analysant la méthode du forcing. Deux axiomes clés ont émergé de ces investigations. Tout d'abord, Martin et Solovay [15] ont introduit l'axiome de Martin (MA), un axiome de forcing concernant la classe des forcings satisfaisant la condition de la chaîne dénombrable (c.c.c.). Plus tard, Foreman, Magidor et Shelah [2] ont étendu cette idée en développant une variante maximale appelée Martin's Maximum (MM), qui prend en compte les forcings qui préservent les sous-ensembles stationnaires de ω_1 .

Les principaux résultats de cette thèse sont liés au forcing, mais notre présentation bénéficie de sa mise en relation avec un autre domaine de la logique: la théorie des modèles des logiques infinitaires.

Dans les années 1950, après l'établissement du cadre de base de la théorie des modèles du premier ordre, Carol Karp, suivie par Makkai, Keisler et Mansfield, parmi d'autres (voir par

exemple [4, 7, 10, 14]), a développé la branche de la logique connue sous le nom de logiques infinitaires. Une idée clé de notre travail, qui était plus ou moins implicite dans les recherches de nombreux auteurs (voir par exemple Mansfield [14] ou Keisler [7]), est que le forcing joue un rôle en logique infinitaire similaire à celui joué par le théorème de compacité en logique du premier ordre. Plus précisément, de la même manière que le théorème de compacité est l'outil clé pour produire des modèles de théories du premier ordre, le forcing peut être l'outil clé pour produire les modèles intéressants des théories $\mathcal{L}_{\infty\omega}$.

La première partie de cette thèse explore la relation entre les logiques infinitaires et les modèles à valeurs booléennes.

Les modèles à valeurs booléennes sont apparus pour la première fois dans le livre de Rasiowa-Sikorski [17], où il est démontré que les modèles à valeurs booléennes fournissent une sémantique complète pour la logique du premier ordre. Ils ont rapidement émergé comme une sémantique possible non seulement pour la logique du premier ordre, mais aussi pour les logiques infinitaires, en particulier dans le cadre de $\mathcal{L}_{\omega_1\omega}$ [11, 18]. En s'appuyant sur la traduction du forcing dans la terminologie des modèles à valeurs booléennes développée par Solovay, Scott et Vopěnka, cette partie de la thèse pose les bases reliant les logiques infinitaires au forcing. S'appuyant sur la traduction du forcing dans la terminologie des modèles à valeurs booléennes développée par Solovay, Scott et Vopěnka, on introduit les bases reliant les logiques infinitaires au forcing.

Une propriété de consistance est une famille d'ensembles de formules non contradictoires, fermée sous certaines opérations logiques naturelles (voir Def. 1.3.2). Les propriétés de consistance reproduisent dans le contexte des logiques infinitaires la technique donnée par la méthode de résolution pour produire des modèles d'une formule du premier ordre; elles sont l'outil standard pour produire des modèles de formules infinitaires non contradictoires. Le livre de Keisler [7] est notre référence sur ce sujet.

Le premier résultat majeur que nous établissons dans cette thèse est le Théorème d'Existence des Modèles Booléens 1.4.14, affirmant que toute $\mathcal{L}_{\infty\omega}$ -formule ψ appartenant à un s qui est dans une propriété de consistance S possède un modèle à valeurs booléennes avec la propriété de "mixing", et renforce (mais uniquement dans le contexte de $\mathcal{L}_{\infty\omega}$) le résultat original de Mansfield [14] affirmant que le même résultat vaut pour les formules $\mathcal{L}_{\infty\infty}$ ϕ si l'on affaiblit la conclusion à la demande que ϕ possède un modèle à valeurs booléennes (mais non "mixing").

La version que nous produisons du Théorème d'Existence des Modèles Booléens pour les formules $\mathcal{L}_{\infty\omega}$ est un renforcement propre du résultat de Mansfield, au vu d'un contre-exemple (dû à Ben de Bondt, et que nous présenterons en détail avec sa permission) montrant une $\mathcal{L}_{\infty\infty}$ -formule qui est booléenne consistante mais n'admet pas de modèle à valeurs booléennes avec la propriété de "mixing".

Le Théorème d'Existence des Modèles Booléens nous permet de prouver trois résultats supplémentaires dans la théorie des modèles des logiques infinitaires munis de la sémantique des modèles à valeurs booléennes avec la propriété de "mixing": un théorème de complétude par rapport à un calcul de type Gentzen pour $\mathcal{L}_{\infty\omega}$ (Thm. 2.3.1), un théorème d'interpolation (Thm. 2.4.1) et un théorème d'omission des types (Thm. 2.5.3). Ceux-ci peuvent être montrés comme des généralisations à $\mathcal{L}_{\infty\omega}$ des résultats correspondants pour la logique du premier ordre, étant donné qu'une formule du premier ordre possède un modèle de Tarski si et seulement si elle possède un modèle à valeurs booléennes.

Cependant, nous croyons que le résultat central de cette partie de la thèse est le Théorème de

Compacité Conservative (Thm. 2.2.5). Dans la poursuite d'une généralisation de la compacité du premier ordre pour les logiques infinitaires, nous introduisons le concept de "renforcement conservatif" et de "conservativité finie" (Def. 2.2.2). Nous soutenons que la généralisation appropriée de la consistance finie (relative à la sémantique de Tarski pour la logique du premier ordre) est la conservativité finie (relative à la sémantique donnée par les modèles à valeurs booléennes pour $\mathcal{L}_{\infty\omega}$). Cela est vrai car nous pouvons montrer que la conservativité finie est équivalente à la consistance finie (modulo l'équivalence logique) dans le cadre du premier ordre (Thm. 2.2.6). Le Théorème de Compacité Conservative stipule que toute famille finiment conservative de formules $\mathcal{L}_{\infty\omega}$ admet un modèle à valeurs booléennes avec la propriété de "mixing". Il est bien connu que la généralisation standard à $\mathcal{L}_{\infty\omega}$ du Théorème de Compacité pour la logique du premier ordre est fautive (voir l'Exemple 2.2.1). À notre avis, ces résultats nous permettent de soutenir que:

Les modèles à valeurs booléennes avec la propriété de "mixing" fournissent une sémantique naturelle pour $\mathcal{L}_{\infty\omega}$.

Dans la seconde partie de la thèse, nous nous appuyons sur les résultats de la première partie pour aborder la question suivante :

Pour quelle famille de formules infinitaires peut-on forcer l'existence d'un modèle de Tarski sans détruire les sous-ensembles stationnaires de ω_1 ?

Kasum et Veličković [6] ont introduit une caractérisation des $\mathcal{L}_{\infty\omega}$ -formules pour lesquelles un modèle de Tarski peut être forcé par un forcing préservant les ensembles stationnaires (AS-goodness). Leur travail s'appuie sur le résultat révolutionnaire d'Asperò et Schindler [1] montrant que l'axiome de forcing (*) introduit par Woodin est une conséquence de la forme forte de MM, connue sous le nom de MM^{++} . Nous définissons la propriété ASK - une variante de l'AS-goodness - que nous utilisons également de la même manière que Kasum et Veličković. Il est démontré dans le Thm. 5.4.2 que pour toute formule ψ ayant la propriété ASK, on peut forcer l'existence d'un modèle de Tarski de ψ d'une manière qui préserve les ensembles stationnaires. La preuve de ce résultat s'appuie sur la perspective de la théorie des modèles de forcing présentée dans la première partie de la thèse, tout en introduisant une nouvelle notion de forcing itéré. Cette présentation du forcing itéré est étroitement liée au Théorème de Compacité Conservateur, soulignant à nouveau l'analogie entre les paires (forcing, logiques infinitaires) et (compacité, logique du premier ordre).

L'idée derrière la propriété ASK est la suivante : Tout d'abord, le (preuve du) Théorème d'Existence des Modèles à Valeurs Booléennes montre qu'une propriété de consistance S pour $\mathcal{L}_{\infty\omega}$ est une notion de forcing qui produit dans son extension générique un modèle de Tarski de toute formule ψ telle que $\{\psi\} \in S$. Supposons qu'on cherche à construire un modèle d'une formule infinitaire ψ_0 de manière à préserver les ensembles stationnaires. Pour commencer, si cela est possible, il faut d'abord démontrer que la formule est consistante dans une extension générique où un fragment suffisamment grand de l'univers est effondré. Cela fournit la propriété de consistance/forcing S_0 dans la première étape de l'itération qui produit un modèle à valeurs booléennes de ψ_0 . Aux étapes successives, nous considérons un sous-ensemble stationnaire S de ω_1 et \dot{C} un (nom de P_0 pour un) sous-ensemble club de ω_1 . En nous appuyant sur le

pouvoir expressif de $\mathcal{L}_{\infty\omega}$, nous pouvons écrire une formule $\mathcal{L}_{\infty\omega} \psi_1$, qui est un renforcement conservateur de ψ_0 et affirme que S et \dot{C} ne sont pas disjoints. ψ_1 est naturellement associée à une propriété de consistance/forcing S_1 , qui produira un modèle à valeurs booléennes de ψ_1 . La propriété ASK pour ψ_0 joue ici un rôle crucial, garantissant que la nouvelle formule est un renforcement conservateur de l'originale (encore une fois en s'appuyant sur le Théorème de Compacité Conservateur). Aux étapes limites β , nous nous assurons que tous les progrès réalisés dans la production de $(S_\alpha, \psi_\alpha : \alpha < \beta)$, de sorte que S_α est une propriété de consistance fournissant un modèle de ψ_α , soient préservés en considérant une paire appropriée (S_β, ψ_β) . En continuant cette itération $(S_\alpha, \psi_\alpha : \alpha < \kappa)$ jusqu'à un cardinal κ avec une suite Diamond, on obtient un forcing final qui préserve les ensembles stationnaires et produit un modèle de la formule originale ψ_0 .

Pour conclure cette thèse, nous démontrons que tout forcing préservant les ensembles stationnaires est absorbé par le forcing SSP généré par une formule avec la propriété ASK. En d'autres termes, la méthode que nous avons présentée pour construire des forcings SSP est optimale.

Mots-clés : Logique, forcing, modèles à valeurs booléennes, compacité, forcing itéré, préservation des sous-ensembles stationnaires.

Résumé court

Les principaux résultats de cette thèse sont liés au forcing, mais notre présentation bénéficie de sa mise en relation avec un autre domaine de la logique: la théorie des modèles des logiques infinitaires. Une idée clé de notre travail, qui était plus ou moins implicite dans les recherches de nombreux auteurs, est que le forcing joue un rôle en logique infinitaire similaire à celui joué par le théorème de compacité en logique du premier ordre. Plus précisément, de la même manière que le théorème de compacité est l'outil clé pour produire des modèles de théories du premier ordre, le forcing peut être l'outil clé pour produire les modèles des théories infinitaires. La première partie de cette thèse explore la relation entre les logiques infinitaires et les modèles à valeurs booléennes. Une propriété de consistance est une famille d'ensembles de formules non contradictoires, fermée sous certaines opérations logiques naturelles. Les propriétés de consistance reproduisent dans le contexte des logiques infinitaires la technique donnée par la méthode de résolution pour produire des modèles d'une formule du premier ordre; elles sont l'outil standard pour produire des modèles de formules infinitaires non contradictoires. Le premier résultat majeur que nous établissons dans cette thèse est le Théorème d'Existence des Modèles Booléens, affirmant que toute formule dans un ensemble qui est dans une propriété de consistance possède un modèle à valeurs booléennes avec la propriété de "mixing", et renforce le résultat original de Mansfield. Le Théorème d'Existence des Modèles Booléens nous permet de prouver trois résultats supplémentaires dans la théorie des modèles des logiques infinitaires munis de la sémantique des modèles à valeurs booléennes avec la propriété de "mixing": un théorème de complétude par rapport à un calcul de type Gentzen, un théorème d'interpolation et un théorème d'omission des types. Cependant, nous croyons que le résultat central de cette partie de la thèse est le Théorème de Compacité Conservative. Dans la poursuite d'une généralisation de la compacité du premier ordre pour les logiques infinitaires, nous introduisons le concept de "renforcement conservatif" et de "conservativité finie". Nous soutenons que la généralisation appropriée de la consistance finie (relative à la sémantique de Tarski pour la logique du premier ordre) est la conservativité finie (relative à la sémantique donnée par les modèles à valeurs booléennes). À notre avis, ces résultats nous permettent de soutenir que: Les modèles à valeurs booléennes avec la propriété de "mixing" fournissent une sémantique naturelle pour les logiques infinies. Dans la seconde partie de la thèse, nous nous appuyons sur les résultats de la première partie pour aborder la question suivante: pour quelle famille de formules infinitaires peut-on forcer l'existence d'un modèle de Tarski sans détruire les sous-ensembles stationnaires? Kasum et Velickovic ont introduit une caractérisation des formules pour lesquelles un modèle de Tarski peut être forcé par un forcing préservant les ensembles stationnaires (AS-goodness). Leur travail s'appuie sur le résultat révolutionnaire d'Asperò et Schindler. Nous définissons la propriété ASK - une variante de l'AS-goodness - que nous utilisons également de la même manière que Kasum et Velickovic. Il est démontré que pour toute formule ayant la propriété ASK, on peut forcer l'existence d'un modèle de Tarski d'une

manière qui préserve les ensembles stationnaires. La preuve de ce résultat s'appuie sur la perspective de la théorie des modèles de forcing présentée dans la première partie de la thèse, tout en introduisant une nouvelle notion de forcing itéré. Cette présentation du forcing itéré est étroitement liée au Théorème de Compacité Conservateur, soulignant à nouveau l'analogie entre les paires (forcing, logiques infinitaires) et (compacité, logique du premier ordre).

Mots clés: Logique, forçage, modèles à valeurs booléennes, compacité, forcing itéré, préservation des sous-ensembles stationnaires.

Short abstract

The main results of this thesis are related to forcing, but our presentation benefits from relating them to another domain of logic: the model theory of infinitary logics. In the 1950s, after the basic framework of first-order model theory had been established, Carol Karp, followed by Makkai, Keisler and Mansfield among others, developed the area of logic known as "infinitary logics". One key idea from our work, which was more or less implicit in the research of many, is that forcing plays a role in infinitary logic similar to the role compactness plays in first-order logic. Specifically, much alike compactness is the key tool to produce models of first-order theories, forcing can be the key tool to produce the interesting models of infinitary theories. The first part of this thesis explores the relationship between infinitary logics and Boolean valued models. Leveraging on the translation of forcing in the Boolean valued models terminology, this part lays the foundations connecting infinitary logics to forcing. A consistency property is a family of sets of non-contradictory sentences closed under certain natural logical operations. Consistency properties are the standard tools to produce models of non-contradictory infinitary sentences. The first major result we establish in the thesis is the Boolean Model Existence Theorem, asserting that any sentence which belongs to some set which is in some consistency property has a Boolean valued model with the mixing property, and strengthens Mansfield's original result. The Boolean Model Existence Theorem allows us to prove three additional results in the model theory of Boolean valued models for the semantics induced by Boolean valued models with the mixing property: a completeness theorem, an interpolation theorem, and an omitting types theorem. These can be shown to be generalizations of the corresponding results for first order logic in view of the fact that a first order sentence has a Tarski model if and only if it has a Boolean valued model. However we believe that the central result of this part of the thesis is the Conservative Compactness Theorem. In pursuit of a generalization of first-order compactness for infinitary logics, we introduce the concepts of conservative strengthening and of finite conservativity. We argue that the appropriate generalization of finite consistency (relative to Tarski semantics for first order logic) is finite conservativity (relative to the semantics given by Boolean valued models). The Conservative Compactness Theorem states that any finitely conservative family of sentences admits a Boolean valued model with the mixing property. In our opinion these results support the claim: Boolean-valued models with the mixing property provide a natural semantics for infinitary logics. In the second part of the thesis we leverage on the results of the first part to address the following question: For what family of infinitary formulae can we force the existence of a Tarski model for them without destroying stationary sets? Kasum and Velickovic introduced a characterization of which sentences can be forced by a stationary set preserving forcing (AS-goodness). Their work builds on the groundbreaking result of Asperò and Schindler. We define the ASK property -a variant of AS-goodness- which we also employ to the same effect of Kasum and Velickovic. It is shown that for any formula with the ASK-property, one can force the existence of a Tarski model in a stationary set

preserving way. The proof of this result builds on the model theoretic perspective of forcing presented in the first part of the thesis, and does so introducing a new notion of iterated forcing. This presentation of iterated forcing is strictly intertwined with the Conservative Compactness Theorem, thereby emphasizing again the analogy between the pairs (forcing, infinitary logics) and (compactness, first-order logic).

Keywords Logic, forcing, Boolean valued models, compactness, iterated forcing, stationary set preserving.

Abstract

In 1870, George Cantor proved that if the trigonometric series

$$\sum_{n=0}^{\infty} (c_n \cos n\pi x + d_n \sin n\pi x)$$

equals 0 for all $x \in \mathbb{R}$, then all of its coefficients c_n and d_n must vanish. This result led him to pose the following question:

How many values of $x \in \mathbb{R}$ must the series vanish for, in order to guarantee that the coefficients also vanish?

Cantor realised that the answer was rooted in the concept of infinite. More precisely, he proved that if there exists $S \subset \mathbb{R}$ such that the Cantor-Bendixon derivative of S vanishes at a countable stage, and the series vanishes for all $x \in \mathbb{R} \setminus S$, then all of its coefficients must also vanish. It is fair to claim that this discovery marked the beginning of set theory, laying the foundations for one of the most profound developments in modern mathematics.

Once the works of Cantor, Dedekind, Zermelo, Fraenkel, Hausdorff, Gödel, Tarski and many others had established the foundations of set theory, and Cohen demonstrated the independence of certain natural statements from ZFC -among which, notably, the Continuum problem-, the central question in set theory, known as Gödel's program, became:

What new axioms should be incorporated into ZFC?

Subsequent research suggested that an interesting pattern of potential axioms for set theory could be detected by analyzing the method of forcing. Two key axioms emerged from these investigations. First, Martin and Solovay [15] introduced Martin's Axiom (MA), a forcing axiom dealing with the class of c.c.c. (countable chain condition) forcings. Later, Foreman, Magidor and Shelah [2] extended this idea by developing a maximal variant called Martin's Maximum (MM), which takes into consideration those forcings that preserve stationary subsets of ω_1 .

The main results of this thesis are related to forcing, but our presentation benefits from relating them to another pillar of logic: the model theory of infinitary logics.

In the 1950s, after the basic framework of first-order model theory had been established, Carol Karp, followed by Makkai, Keisler and Mansfield among others (see for example [4, 7, 10, 14]), developed the area of logic known as "infinitary logics". One key idea from our work, which was more or less implicit in the research of many (see for example Mansfield's [14] or Keisler's [7]),

is that forcing plays a role in infinitary logic similar to the role compactness plays in first-order logic. Specifically, much alike compactness is the key tool to produce models of first-order theories, forcing can be the key tool to produce the interesting models of $\mathcal{L}_{\infty\omega}$ -theories.

The first part of this thesis explores the relationship between infinitary logics and Boolean valued models.

Boolean valued models appeared first in Rasiowa-Sikorski's book [17], where it is proved that Boolean valued models provide a complete semantics for first order logic. They rapidly emerged as a possible semantics not only for first order, but for infinitary logics, particularly in the $\mathcal{L}_{\omega_1\omega}$ setting [11, 18]. Leveraging on the translation of forcing in the Boolean valued models terminology developed by Solovay, Scott, and Vopěnka, this part of the thesis lays the foundations connecting infinitary logics to forcing.

A consistency property is a family of sets of non-contradictory sentences closed under certain natural logical operations (see Def. 1.3.2). Consistency properties reproduce in the context of infinitary logics the technique given by the method of resolution for producing models of a first order sentence; they are the standard tools to produce models of non-contradictory infinitary sentences. Keisler's and Väänänen's books [7, 21] are our main references on this topic. The first major result we establish in the thesis is the Boolean Model Existence Theorem 1.4.14, asserting that any $\mathcal{L}_{\infty\omega}$ -sentence ψ which belongs to some s which is in some consistency property S has a Boolean valued model with the mixing property, and strengthens (but only in the context of $\mathcal{L}_{\infty\omega}$) Mansfield's original result [14] stating that the same holds for $\mathcal{L}_{\infty\infty}$ -sentences ϕ if one weakens the conclusion to the request that ϕ has a Boolean valued model. It has to be noted that an equivalent variant of Theorem 1.4.14 has been independently proved by De Bondt and Veličković.

The version we produce of the Boolean Model Existence Theorem for $\mathcal{L}_{\infty\omega}$ -sentences is a proper strengthening of Mansfield's result in view of a counterexample (due to Ben de Bondt, and which we will present in full detail with his kind permission) showing an $\mathcal{L}_{\infty\infty}$ -sentence which is Boolean consistent but does not admit a Boolean valued model with the mixing property.

The Boolean Model Existence Theorem allows us to prove three additional results in the model theory of Boolean valued models for $\mathcal{L}_{\infty\omega}$ for the semantics induced by Boolean valued models with the mixing property: a completeness theorem with respect to a Gentzen type calculus for $\mathcal{L}_{\infty\omega}$ (Thm. 2.3.1), an interpolation theorem (Thm. 2.4.1) and an omitting types theorem (Thm. 2.5.3). These can be shown to be generalizations to $\mathcal{L}_{\infty\omega}$ of the corresponding results for first order logic in view of the fact that a first order sentence has a Tarski model if and only if it has a Boolean valued model.

However we believe that the central result of this part of the thesis is the Conservative Compactness Theorem (Thm. 2.2.5): In pursuit of a generalization of first-order compactness for infinitary logics, we introduce the concepts of conservative strengthening and of finite conservativity (Def. 2.2.2). We argue that the appropriate generalization of finite consistency (relative to Tarski semantics for first order logic) is finite conservativity (relative to the semantics given by Boolean valued models for $\mathcal{L}_{\infty\omega}$). This holds true as we can show that finite conservativity is "equivalent" to finite consistency (modulo logical equivalence) in the first-order setting (see Thm. 2.2.6 for a precise statement). The Conservative Compactness Theorem states that any finitely conservative family of $\mathcal{L}_{\infty\omega}$ -sentences admits a Boolean valued model with the mixing property. It is well known that the usual generalization to $\mathcal{L}_{\infty\omega}$ of the standard formulation

of the Compactness Theorem for first order logic is false (see Example 2.2.1). In our opinion these results support the claim:

Boolean-valued models with the mixing property provide a natural semantics for $\mathcal{L}_{\infty\omega}$.

In the second part of the thesis we leverage on the results of the first part to address the following question:

For what family of infinitary formulae can we force the existence of a Tarski model for them without destroying stationary subsets of ω_1 ?

Kasum and Veličković [6] found an answer to the above question characterizing such $\mathcal{L}_{\infty\omega}$ -sentences as those which are AS-good. Their work builds on the groundbreaking result of Asperò and Schindler [1] showing that the forcing axiom () introduced by Woodin is a consequence of the strong form of Martin's maximum known as MM^{++} . We define the ASK-property -a variant of AS-goodness- which we also employ to the same effect of Kasum and Veličković. It is shown in Thm. 5.4.2 that for any formula ψ with the ASK-property, one can force the existence of a Tarski model of ψ in a stationary set preserving way. The proof of this result builds on the model theoretic perspective of forcing presented in the first part of the thesis, and does so introducing a new notion of iterated forcing. This presentation of iterated forcing is strictly intertwined with the Conservative Compactness Theorem, thereby emphasizing again the analogy between the pairs (forcing, infinitary logics) and (compactness, first-order logic).*

The idea behind the ASK-property is as follows: First of all the (proof of the) Boolean valued models Existence Theorem shows that a consistency property S for $\mathcal{L}_{\infty\omega}$ is a forcing notion which produces in its generic extension a Tarski model of any ψ such that $\{\psi\} \in S$. Suppose one aims to construct a model of an infinitary sentence ψ_0 in a stationary set preserving manner. To begin, should this be possible, one should first be able to demonstrate that the sentence is consistent in some generic extension where a sufficiently large fragment of the universe is collapsed. This provides the consistency property/forcing S_0 in the initial stage of the iteration which produces a Boolean valued model of ψ_0 . At successor stages, we consider a stationary subset S of ω_1 and \dot{C} a (P_0 -name for a) club subset of ω_1 . Leveraging on the expressive power of $\mathcal{L}_{\infty\omega}$, we can write an $\mathcal{L}_{\infty\omega}$ -sentence ψ_1 which is a conservative strengthening of ψ_0 and asserts that S and \dot{C} are not disjoint. ψ_1 is naturally attached to a consistency property/forcing S_1 which will produce Boolean valued model of ψ_1 . The ASK-property for ψ_0 plays a crucial role here, ensuring that the new formula is a conservative strengthening of the original one. At limit stages β , we ensure that all progresses made up in producing $(S_\alpha, \psi_\alpha : \alpha < \beta)$ so that S_α is a consistency property giving a model of ψ_α , are preserved by considering an appropriate pair $(S_\beta, \bigwedge_{\alpha < \beta} \psi_\alpha)$ (the latter pair can be found appealing to the Conservative Compactness theorem, as $\{\psi_\alpha : \alpha < \beta\}$ is a finitely conservative family of $\mathcal{L}_{\infty\omega}$ -sentences). Continuing this iteration $(S_\alpha, \psi_\alpha : \alpha < \kappa)$ up to a cardinal κ with a Diamond sequence yields a final forcing that is stationary set preserving and produces a model of the original sentence ψ_0 .

The last result of the thesis is a converse of Thm. 5.4.2: it is shown that any stationary set preserving forcing is absorbed by a forcing/consistency property generated by a sentence with the ASK-property (Thm. 5.5.1). In other words, the method we have presented for constructing forcings that preserve the stationarity of subsets of ω_1 is optimal.

Key words: Logic, forcing, Boolean valued models, compactness, iterated forcing, stationary set preserving.

- Chapter 1 introduces infinitary logics of the form $\mathcal{L}_{\infty\omega}$, Boolean valued models and consistency properties. These three notions, together with forcing, form the backbone of this thesis. Once the basic facts about Boolean-valued models with the mixing property are established, we address the issue of forcing with an arbitrary consistency property. The main result is the Boolean Model Existence Theorem, which gives a powerful tool for producing models of $\mathcal{L}_{\infty\omega}$ -sentences, and states that for any consistency property S and any $s \in S$, there exists a Boolean valued model with the mixing property satisfying $\bigwedge s$.
- Chapter 2 studies the model theory of Boolean valued models with respect to infinitary logics of the form $\mathcal{L}_{\infty\omega}$. We start by showing that Boolean satisfiability is equivalent to Tarski satisfiability in the first-order setting. Then we prove a completeness theorem for Boolean valued semantics for $\mathcal{L}_{\infty\omega}$ relative to a natural Gentzen calculus for $\mathcal{L}_{\infty\omega}$. By analysing the failure of compactness for $\mathcal{L}_{\omega_1\omega}$, we come to isolate the notion of conservative strengthening. Leveraging on this notion, we prove the Conservative Compactness Theorem and we show that it generalizes the classical Compactness theorem for first-order logic. Later, building on the consistency properties presented in Keisler's book [7], we prove an omitting types theorem and an interpolation theorem. We close the chapter presenting Ben De Bondt's example on why Boolean valued models with the mixing property are not a right semantics for $\mathcal{L}_{\infty\infty}$.
- Chapter 3 presents a number of key examples from set theory of $\mathcal{L}_{\infty\omega}$ -sentences of interest to us. It also arguments why in the remaining part of the thesis it is natural to work exclusively with $\mathcal{L}_{\infty\omega}$ -sentences which are quantifier-free. We present the examples in this chapters with two key objectives: on the one hand that of outlining the expressive power of $\mathcal{L}_{\infty\omega}$ in set theory, on the other hand that of establishing the basic results needed in the final chapters of the thesis.
- Chapter 4 introduces a new form of iterated forcing based on the concept of conservative strengthening and on the Conservative Compactness Theorem. It is shown that if the iteration has length equal to an ineffable cardinal κ , the final forcing is $< \kappa$ -c.c. Additionally, the chapter introduce a key complexity class for quantifier free $\mathcal{L}_{\infty\omega}$ -sentences and examines some model theoretic properties of the sentences in this class.
- Chapter 5 presents the central definition of the second part of the thesis: the ASK-property (which is a slight variant of a notion - AS-goodness - isolated by Kasum and Velickovic [6]). With the aim of building models of infinitary sentences in a stationary set preserving (SSP) manner, the ASK-property emerges as the crucial condition for handling successor stages in our iterations. The chapter begins by introducing the necessary background for defining the ASK-property. The remainder of the chapter is dedicated to the proof of the main theorem: for any $\mathcal{L}_{\infty\omega}$ -sentence with the ASK-property, a Tarski model of it can be forced to exist by an SSP forcing. The main theorem of the chapter gives an alternative account and rielaboration in the language we developed in the previous chapters of the result presented in [6], where the same conclusion is drawn for $\mathcal{L}_{\infty\omega}$ -sentences which are

AS-good according to the terminology of the paper. Finally, we argue the converse of the previous theorem by showing that any stationary set preserving forcing can be absorbed by some forcing/consistency property generated by some $\mathcal{L}_{\infty\omega}$ -sentence with the ASK-property.

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Part I

Infinitary logics, consistency properties and Boolean valued semantics

Chapter 1

Infinitary logics and consistency properties

Logics are determined by their expressive power and their class of models. In first order logic the finitary nature of its sentences is in perfect balance with Tarski semantics, as shown by the completeness theorem.

In the 1950s, the mathematician Carol Karp found herself in a situation where being able to consider a countable disjunction over the natural numbers would make a problem easier (see the introduction of [4]). Infinitary logics arose as a generalization of first order logic with increased expressive power.

The logic $\mathcal{L}_{\omega_1\omega}$ still maintains that Tarski models provide a nice semantics for it, as argued by the Model Existence Theorem 1.3.3. Nonetheless, as soon as one increases the sizes of the disjunctions and conjunctions under consideration (and defines the logic $\mathcal{L}_{\infty\omega}$), Tarski semantics is not anymore a right set-up where to interpret these infinitary formulae, mostly because of the non-absoluteness of the concept of cardinality.

The main aim of the first part of this thesis is to prove that one recovers a natural semantics for the infinitary logic $\mathcal{L}_{\infty\omega}$ when one interprets its formulae not only in Tarski structures but in Boolean valued models. In order to do so we maintain the same path that was historically developed for $\mathcal{L}_{\omega_1\omega}$ and Tarski semantics:

Build the right consistency property + Model Existence Theorem.

In the first chapter we obtain the Boolean valued version 1.4.15 of the Model Existence Theorem through the forcing method. In the second chapter we focus on the model theory of Boolean valued models with respect to infinitary logics of the form $\mathcal{L}_{\infty\omega}$. In the third chapter we introduce set theoretic examples of $\mathcal{L}_{\infty\omega}$ -sentences.

Finally, it is worth noting that the nice balance between infinitary logics and Boolean valued models (or forcing) was already made explicit in the works of Mansfield [14], Stern [19], and others. However, a key advancement is the restriction of Boolean-valued models to those with the mixing property. For example Pierobon and Viale [16] showed that, for a fixed Boolean algebra \mathbf{B} , Boolean valued models over \mathbf{B} correspond to presheaves over the category generated by \mathbf{B}^+ , while Boolean valued models over \mathbf{B} with the mixing property correspond to sheaves with respect to the dense Grothendieck topology. Furthermore, the sheafification process corresponds in the Boolean valued setting to a certain Boolean ultrapower construction introduced by Mansfield in [13].

1.1 The infinitary logics $\mathcal{L}_{\kappa\omega}$

The set of formulae for a language in first order logic is constructed by induction from atomic formulae by taking negations, finite conjunctions and disjunctions and finite quantifications. The logic $\mathcal{L}_{\kappa\omega}$ generalizes the finite conjunction operation to a cardinal κ allowing conjunctions and disjunctions of size less than κ . Our main references on infinitary logics are Keisler's book [7] and Väänänen's book [21].

1.1.1 Syntax

To simplify slightly our notation we confine our attention to finitary relational languages, i.e. languages that do not have function symbols and where all relations symbols have finite arity and are set sized many.¹ From now on by a language or signature we will mean a finitary relational (and set sized) one.

Definition 1.1.1. Let \mathcal{L} be a relational signature. Let κ be a cardinal. Let $\{v_\alpha : \alpha < \kappa\}$ be a set of κ variables. The set of terms and atomic formulae for $\mathcal{L}_{\kappa\omega}$ is constructed in analogy to first order logic using the symbols of $\mathcal{L} \cup \{v_\alpha : \alpha < \kappa\}$. The other $\mathcal{L}_{\kappa\omega}$ -formulae are defined by induction as follows:

- if ϕ is an $\mathcal{L}_{\kappa\omega}$ -formula, then $\neg\phi$ is an $\mathcal{L}_{\kappa\omega}$ -formula;
- if Φ is a set of $\mathcal{L}_{\kappa\omega}$ -formulae of size $< \kappa$ with finitely many free variables, then $\bigwedge \Phi$ and $\bigvee \Phi$ are $\mathcal{L}_{\kappa\omega}$ -formulae;
- if $\phi(v)$ is an $\mathcal{L}_{\kappa\omega}$ -formula, then $\forall v\phi(v)$ and $\exists v\phi(v)$ are $\mathcal{L}_{\kappa\omega}$ -formulae.

We denote by

$$\mathcal{L}_{\infty\omega} = \bigcup_{\kappa \text{ a cardinal}} \mathcal{L}_{\kappa\omega}$$

the set of formulae whose conjunctions and disjunctions are of arbitrary (set-)size.

The restriction on the number of free variables for the clauses \bigwedge and \bigvee is intended to avoid formulae for which there is no quantifier closure. Another common possibility is to call pre-formula any “formula”, and formula the ones that verify this property.

1.1.2 Proof systems for $\mathcal{L}_{\infty\omega}$

We present a proof system for $\mathcal{L}_{\infty\omega}$ that is a direct generalization of the standard Sequent Calculus for first order logic. When dealing with sequents, and in order to make proofs shorter, we will assume that formulae only contain \neg , \bigwedge and \forall as logical symbols; this is not restrictive as all reasonable semantics for these logics (among which all those we consider) should validate the natural logical equivalences $\neg\forall v\neg\phi \equiv \exists v\phi$, $\neg\bigwedge_{i \in I} \neg\phi_i \equiv \bigvee_{i \in I} \phi_i$.

¹With some notational efforts our results transfer to arbitrary signatures.

Definition 1.1.2. Let Γ and Δ be arbitrary sets of $\mathcal{L}_{\infty\omega}$ -formulae. A **proof** of $\Gamma \vdash \Delta$ in $\mathcal{L}_{\infty\omega}$ is a sequence $(s_\alpha)_{\alpha \leq \beta}$ of sequents, where s_β is $\Gamma \vdash \Delta$ and each element s_α is either an axiom or comes from an application of the following rules to (some elements of) $(s_\gamma)_{\gamma < \alpha}$.

| | | | |
|---------------------|---|--|----------------------|
| Axiom rule | $\frac{}{\Gamma, \phi \vdash \phi, \Delta}$ | $\frac{\Gamma, \phi \vdash \Delta \quad \Gamma' \vdash \phi, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$ | Cut Rule |
| Substitution | $\frac{\Gamma \vdash \Delta}{\Gamma(\bar{w}/\bar{v}) \vdash \Delta(\bar{w}/\bar{v})}$ | $\frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$ | Weakening |
| Left Negation | $\frac{\Gamma \vdash \phi, \Delta}{\Gamma, \neg\phi \vdash \Delta}$ | $\frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}$ | Right Negation |
| Left Conjunction | $\frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \bigwedge \Gamma' \vdash \Delta}$ | $\frac{\Gamma \vdash \phi_i, \Delta, i \in I}{\Gamma \vdash \bigwedge_{i \in I} \{\phi_i : i \in I\}, \Delta}$ | Right Conjunction |
| Left Quantification | $\frac{\Gamma, \phi(t/v) \vdash \Delta}{\Gamma, \forall v \phi(v) \vdash \Delta}$ | $\frac{\Gamma \vdash \phi(w/v), \Delta}{\Gamma \vdash \forall v \phi(v), \Delta} *$ | Right Quantification |
| Equality 1 | $\frac{}{v_\alpha = v_\beta \vdash v_\beta = v_\alpha}$ | $\frac{}{\bar{u} = t, \phi(\bar{t}) \vdash \phi(\bar{u})}$ | Equality 2 |

* The Right Quantification rule can only be applied in the case that the variable w does not occur free in formulae from $\Gamma \cup \Delta \cup \{\phi\}$.

Let us argue that with this deduction system the completeness theorem for $\mathcal{L}_{\infty\omega}$ (even for $\mathcal{L}_{\omega_2\omega}$) fails for the usual semantics given by Tarski structures. Remark first that this proof system is forcing invariant: the existence of a proof for a certain sentence is described by a Σ_1 statement in parameter the sequent to be proved; if the proof exists in V , then it exists in any further extension of V .

Consider now a set of κ constants $\{c_\alpha : \alpha < \kappa\}$ for $\kappa > \omega$ and the sentence

$$\psi := \left(\bigwedge_{\omega \leq \alpha \neq \beta < \kappa} c_\alpha \neq c_\beta \right) \Rightarrow \exists v \left(\bigwedge_{n < \omega} v \neq c_n \right).$$

The sentence ψ is valid in the usual Tarski semantics for $\mathcal{L}_{\infty\omega}$ but it cannot be proved (in our deduction system or in any forcing invariant system) since the sentence is no longer valid when moving to $V[G]$ for G a V -generic filter for $\text{Coll}(\omega, \kappa)$.

Malitz [12, Thm. 3.2.4] showed also that the above formula is a counterexample to Craig's interpolation property for Tarski semantics in $\mathcal{L}_{\infty\omega}$.

Our opinion is that a proof system should not depend on the model of set theory in which one is working, which is the case for the proof system presented here.

In contrast with our point of view, one finds a complete proof system for Tarski semantics in Malitz's thesis [12, Thm. 3.3.1]. However, this proof system (which by the way is due to Karp [4, Ch. 11]), is not forcing invariant e.g. a proof of some sequent in some model of set theory may not be anymore a proof of that same sequent in some forcing extension.

1.2 Boolean valued semantics

Definition 1.2.1. Let \mathcal{L} be a signature and \mathbf{B} be a Boolean algebra. A **B-valued structure** \mathcal{M} for \mathcal{L} is given by:

1. a non-empty set M ;
2. the Boolean value of equality,

$$\begin{aligned} M^2 &\rightarrow \mathbf{B} \\ (\tau, \sigma) &\mapsto \llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}}; \end{aligned}$$

3. the interpretation of relation symbols $R \in \mathcal{L}$ of arity n ,

$$\begin{aligned} M^n &\rightarrow \mathbf{B} \\ (\tau_i : i \leq n) &\mapsto \llbracket R(\tau_i : i \leq n) \rrbracket_{\mathbf{B}}^{\mathcal{M}}; \end{aligned}$$

4. the interpretation $c^{\mathcal{M}} \in M$ of constant symbols c in \mathcal{L} .

We require that the following conditions hold:

- (a) For all $\tau, \sigma, \pi \in M$,

$$\begin{aligned} \llbracket \tau = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}} &= 1_{\mathbf{B}}, \\ \llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} &= \llbracket \sigma = \tau \rrbracket_{\mathbf{B}}^{\mathcal{M}}, \\ \llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \wedge \llbracket \sigma = \pi \rrbracket_{\mathbf{B}}^{\mathcal{M}} &\leq \llbracket \tau = \pi \rrbracket_{\mathbf{B}}^{\mathcal{M}}. \end{aligned}$$

- (b) If $R \in \mathcal{L}$ is an n -ary relation symbol, then for all $(\tau_i : i \leq n), (\sigma_i : i \leq n) \in M^n$,

$$\left(\bigwedge_{i \leq n} \llbracket \tau_i = \sigma_i \rrbracket_{\mathbf{B}}^{\mathcal{M}} \right) \wedge \llbracket R(\tau_i : i \leq n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket R(\sigma_i : i \leq n) \rrbracket_{\mathbf{B}}^{\mathcal{M}}.$$

See Appendix 6 for a precise definition of $\text{RO}(P)$.

Definition 1.2.2.

- ✿ Fix a Boolean algebra \mathbf{B} and a \mathbf{B} -valued structure \mathcal{M} for a signature \mathcal{L} . We define the $\text{RO}(\mathbf{B}^+)$ -**value** of an $\mathcal{L}_{\infty\omega}$ -formula $\phi(\bar{v})$ with assignment $\bar{v} \mapsto \bar{m}$ by induction as follows:

$$\llbracket R(t_i : i \leq n)[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \llbracket R(t_i[\bar{v} \mapsto \bar{m}] : i \leq n) \rrbracket_{\mathbf{B}}^{\mathcal{M}} \text{ for } R \in \mathcal{L} \text{ of arity } n,$$

$$\llbracket (\neg\phi)[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \neg \llbracket \phi[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}},$$

$$\llbracket (\bigwedge \Phi)[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \bigwedge_{\phi \in \Phi} \llbracket \phi[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}},$$

$$\llbracket (\bigvee \Phi)[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \bigvee_{\phi \in \Phi} \llbracket \phi[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}},$$

$$\llbracket (\forall v\phi(v))[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \bigwedge_{a \in M} \llbracket \phi[\bar{v} \mapsto \bar{m}, v \mapsto a] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}},$$

$$\llbracket (\exists v\phi(v))[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} = \bigvee_{a \in M} \llbracket \phi[\bar{v} \mapsto \bar{m}, v \mapsto a] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}}.$$

- ✿ A \mathbf{B} -valued structure \mathcal{M} is **well behaved** for $\mathcal{L}_{\kappa\omega}$ if

$$\llbracket \phi(t_i : i \leq n)[\bar{v} \mapsto \bar{m}] \rrbracket_{\text{RO}(\mathbf{B}^+)}^{\mathcal{M}} \in \mathbf{B}$$

for any $\mathcal{L}_{\kappa\omega}$ -formula $\phi(\bar{v})$.

- ✿ Let T be an $\mathcal{L}_{\infty\omega}$ theory and \mathcal{M} be a well behaved \mathbf{B} -valued \mathcal{L} -structure. The relation

$$\mathcal{M} \models T$$

holds if

$$\llbracket \bigwedge T \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}.$$

If \mathbf{B} is complete, then any \mathbf{B} -valued model is well behaved. We write just $\llbracket \phi(\tau_i : i < n) \rrbracket$ or $\llbracket \phi(\tau_i : i < n) \rrbracket^{\mathcal{M}}$ or $\llbracket \phi(\tau_i : i < n) \rrbracket_{\mathbf{B}}$ when no confusion arises on which structure we are considering or in which Boolean algebra we are evaluating the formula ϕ .

1.2.1 The mixing property

Definition 1.2.3. Let \mathbf{B} be a complete Boolean algebra and let \mathcal{M} be a \mathbf{B} -valued \mathcal{L} -structure. \mathcal{M} has the **mixing property** if for any antichain $A \subset \mathbf{B}$ and any subset $\{\tau_a : a \in A\} \subset M$ there is some $\tau \in M$ such that

$$a \leq \llbracket \tau = \tau_a \rrbracket_{\mathbf{B}}^{\mathcal{M}}$$

for all $a \in A$.

Definition 1.2.4. Let κ be an infinite cardinal, \mathbf{B} be a complete Boolean algebra, and \mathcal{M} be a \mathbf{B} -valued \mathcal{L} -structure. \mathcal{M} is **full for the logic** $\mathcal{L}_{\kappa\omega}$ if for every $\mathcal{L}_{\kappa\omega}$ -formula $\phi(v, \bar{w})$ and $\bar{y} \in M^{\bar{w}}$ there exists $x \in M$ such that

$$\llbracket \exists v \phi(v, \bar{y}) \rrbracket_{\mathbf{B}}^{\mathcal{M}} = \llbracket \phi(x, \bar{y}) \rrbracket_{\mathbf{B}}^{\mathcal{M}}.$$

\mathcal{M} is **full** if it is full for the logic $\mathcal{L}_{\infty\omega}$.

Proposition 1.2.5. Let \mathcal{L} be a signature and let \mathbf{B} be a complete Boolean algebra. Any \mathbf{B} -valued \mathcal{L} -structure \mathcal{M} with the mixing property is full.

Proof. Let $\exists v \phi(v)$ be a $\mathcal{L}_{\infty\omega}$ -sentence. Fix a maximal antichain A among

$$\{b \in \mathbf{B} : b \leq \llbracket \phi(x) \rrbracket \text{ for some } x \in M\}.$$

Then we can fix

$$\{x_b : b \in A\}$$

such that $b \leq \llbracket \phi(x_b) \rrbracket$. The mixing property for \mathcal{M} gives x such that $\llbracket x = x_b \rrbracket \geq b$ for all $b \in A$. Then

$$\begin{aligned} \llbracket \exists v \phi(v) \rrbracket &= \bigvee_{b \in A} b = \bigvee_{b \in A} (b \wedge \llbracket \phi(x_b) \rrbracket) \leq \bigvee_{b \in A} (\llbracket x = x_b \rrbracket \wedge \llbracket \phi(x_b) \rrbracket) \leq \\ &\bigvee_{b \in A} \llbracket \phi(x) \rrbracket = \llbracket \phi(x) \rrbracket. \end{aligned}$$

□

1.2.2 Quotients of Boolean valued models

Definition 1.2.6. Let \mathbf{B} be a Boolean algebra, let \mathcal{M} be a \mathbf{B} -valued \mathcal{L} -structure and let $F \subset \mathbf{B}$ be a filter. The quotient of \mathcal{M} by F is the \mathcal{L} -structure $\mathcal{M}/_F$ defined as follows:

1. its domain $M/_F$ is the quotient of M by the equivalence

$$\tau \equiv_F \sigma \leftrightarrow \llbracket \tau = \sigma \rrbracket \in F,$$

2. if $R \in \mathcal{L}$ is an n -ary relation symbol,

$$R^{\mathcal{M}/_F} = \{([\tau_i]_F : i \leq n) \in (M/_F)^n : \llbracket R(\tau_i : i \leq n) \rrbracket \in F\},$$

3. if $c \in \mathcal{L}$ is a constant symbol,

$$c^{\mathcal{M}/_F} = [c^{\mathcal{M}}]_F \in M/_F.$$

Lemma 1.2.7. *Let \mathbf{B} be a Boolean algebra, let \mathcal{M} be a \mathbf{B} -valued \mathcal{L} -structure and let $F \subset \mathbf{B}$ be a filter. The quotient $\mathcal{M}/_F$ is well defined.*

Proof. We need to argue that if $R \in \mathcal{L}$ is an n -ary relation symbol and $\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n$ are such that

$$\tau_i \equiv_F \sigma_i \text{ for every } i \leq n,$$

then $([\tau_i]_F : i \leq n) \in R^{\mathcal{M}/_F}$ if and only if $([\sigma_i]_F : i \leq n) \in R^{\mathcal{M}/_F}$. By definition of $R^{\mathcal{M}/_F}$ we have that if $([\tau_i]_F : i \leq n) \in R^{\mathcal{M}/_F}$, then

$$\llbracket R(\tau_i : i \leq n) \rrbracket \in F.$$

By definition of Boolean algebra (precisely condition (b)),

$$\llbracket R(\tau_i : i \leq n) \rrbracket \wedge \bigwedge_{i \leq n} \llbracket \tau_i = \sigma_i \rrbracket \leq \llbracket R(\sigma_i : i \leq n) \rrbracket.$$

We have $\llbracket R(\tau_i : i \leq n) \rrbracket \in F$ by assumption and $\llbracket \tau_i = \sigma_i \rrbracket \in F$ since $\tau_i \equiv_f \sigma_i$. Then

$$\llbracket R(\tau_i : i \leq n) \rrbracket \wedge \bigwedge_{i \leq n} \llbracket \tau_i = \sigma_i \rrbracket \in F$$

as filters are closed under finite conjunctions. Hence,

$$\llbracket R(\sigma_i : i \leq n) \rrbracket \in F$$

as F is upward closed. □

Definition 1.2.8. Let κ be an infinite cardinal, \mathbf{B} be a $< \kappa$ -complete Boolean algebra, and $U \subset \mathbf{B}$ be an ultrafilter on \mathbf{B} existing in some generic extension of V .

U is $< \kappa$ -complete for V -sequences if $\bigwedge X \in U$ whenever $X \in V$ is such that $X \subseteq U$ and X has size less than κ in V .

Theorem 1.2.9 (Łoś). *Let κ be an infinite cardinal, \mathbf{B} be a $< \kappa$ -complete Boolean algebra, \mathcal{M} be an $\mathcal{L}_{\infty\omega}$ -full \mathbf{B} -valued structure and $U \subset \mathbf{B}$ be a $< \kappa$ -complete ultrafilter for V -sequences existing in some generic extension $V[G]$ of V . Then for every $\mathcal{L}_{\kappa\omega}$ -formula $\phi(\bar{v})$ which is in V and $\bar{\tau} \in M^{|\bar{v}|}$,*

$$\left[\left[\phi(\bar{v})[\bar{v} \mapsto \bar{\tau}] \right]_{\mathbf{B}/U} \right]^{\mathcal{M}/_U} = 1_{\mathbf{B}/U} \text{ if and only if } \left[\left[\phi(\bar{v})[\bar{v} \mapsto \bar{\tau}] \right]_{\mathbf{B}} \right]^{\mathcal{M}} \in U$$

holds in $V[G]$.

Proof. We proceed by induction on the complexity of formulae. For atomic formulae the result holds as the thesis is precisely the definition of the quotient.

Assume the result true for $\phi(\bar{v})$ and let us prove it for $\neg\phi(\bar{v})$. We have that

$$\begin{aligned} \llbracket \neg\phi(\bar{v})[\bar{v} \mapsto \overline{[\tau]_U}] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 1_{\mathbf{B}/U} &\Leftrightarrow \llbracket \phi(\bar{v})[\bar{v} \mapsto \overline{[\tau]_U}] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 0_{\mathbf{B}/U} \Leftrightarrow \\ \llbracket \phi(\bar{v})[\bar{v} \mapsto \bar{\tau}] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \notin U &\Leftrightarrow \llbracket \neg\phi(\bar{v})[\bar{v} \mapsto \bar{\tau}] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \notin U. \end{aligned}$$

Assume the result true for every $\phi_\alpha(\bar{v})$ and let us prove it for $\bigwedge_{\alpha < \gamma} \phi_\alpha(\bar{v})$ (a formula in V with $\gamma < \kappa$ and free variables $\bar{v} = (v_0, \dots, v_n)$). We have that

$$\begin{aligned} \llbracket \bigwedge_{\alpha < \gamma} \phi_\alpha(\bar{v})[(v_0, \dots, v_n) \mapsto ([\tau_0]_U, \dots, [\tau_n]_U)] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 1_{\mathbf{B}/U} &\Leftrightarrow \left(\text{definition of truth for } \bigwedge \right) \\ \bigwedge_{\alpha < \gamma} \llbracket \phi_\alpha(\bar{v})[(v_0, \dots, v_n) \mapsto ([\tau_0]_U, \dots, [\tau_n]_U)] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 1_{\mathbf{B}/U} &\Leftrightarrow \left(\text{induction hypothesis} \right) \\ \forall \alpha < \gamma \left(\llbracket \phi_\alpha(\bar{v})[(v_0, \dots, v_n) \mapsto ([\tau_0]_U, \dots, [\tau_n]_U)] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \in U \right) &\Leftrightarrow \left(U \text{ is } < \kappa\text{-complete for } V\text{-sequences} \right) \\ \llbracket \bigwedge_{\alpha < \gamma} \phi_\alpha(\bar{v})[(v_0, \dots, v_n) \mapsto ([\tau_0]_U, \dots, [\tau_n]_U)] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \in U. & \end{aligned}$$

Assume the result true for $\phi(w, \bar{v})$ and let us prove it for $\exists w\phi(w, \bar{v})$. We have that

$$\begin{aligned} \llbracket \exists w\phi(w, \bar{v})[\bar{v} \mapsto \overline{[\tau]_U}] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 1_{\mathbf{B}/U} &\Leftrightarrow \left(\text{definition of truth for } \exists \right) \\ \bigvee_{\sigma \in M} \llbracket \phi(w, \bar{v})[w \mapsto \sigma, \bar{v} \mapsto \overline{[\tau]_U}] \rrbracket_{\mathbf{B}/U}^{\mathcal{M}/U} = 1_{\mathbf{B}/U} &\Leftrightarrow \left(\text{induction hypothesis} \right) \\ \bigvee_{\sigma \in M} \llbracket \phi(w, \bar{v})[w \mapsto \sigma, \bar{v} \mapsto \bar{\tau}] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \in U &\Leftrightarrow \left(\text{fullness of } \mathcal{M} \right) \\ \llbracket \exists w\phi(w, \bar{v})[\bar{v} \mapsto \bar{\tau}] \rrbracket_{\mathbf{B}}^{\mathcal{M}} \in U. & \end{aligned}$$

□

1.2.3 Boolean satisfiability

Definition 1.2.10.

- ❁ BVM denotes the class of Boolean valued models with values on a complete Boolean algebra.
- ❁ Sh denotes the subclass of Boolean valued models with the mixing property with values on a complete Boolean algebra.

Let Γ and Δ be sets of $\mathcal{L}_{\infty\omega}$ -formulae. If $\Gamma = \emptyset$, then we let

$$\llbracket \bigwedge \Gamma \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}.$$

If $\Delta = \emptyset$, then we let

$$\llbracket \bigvee \Delta \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 0_{\mathbf{B}}.$$

✱ Γ is **Boolean satisfiable** if there is a complete Boolean algebra \mathbf{B} and a \mathbf{B} -valued structure \mathcal{M} such that $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}$ for each $\phi \in \Gamma$.

✱ Γ is **Boolean valid for BVM** if for every complete Boolean algebra \mathbf{B} and every \mathbf{B} -valued structure \mathcal{M} we have that $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}$ for each $\phi \in \Gamma$.

✱ Γ is **Boolean valid for Sh** if for every complete Boolean algebra \mathbf{B} and every \mathbf{B} -valued structure \mathcal{M} with the mixing property we have that $\llbracket \phi \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}$ for each $\phi \in \Gamma$.

✱ $\Gamma \vDash_{\text{BVM}} \Delta$ if

$$\llbracket \bigwedge \Gamma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket \bigvee \Delta \rrbracket_{\mathbf{B}}^{\mathcal{M}}$$

for any complete Boolean algebra \mathbf{B} and \mathbf{B} -valued structure \mathcal{M} .

✱ $\Gamma \vDash_{\text{Sh}} \Delta$ if

$$\llbracket \bigwedge \Gamma \rrbracket_{\mathbf{B}}^{\mathcal{M}} \leq \llbracket \bigvee \Delta \rrbracket_{\mathbf{B}}^{\mathcal{M}}$$

for any complete Boolean algebra \mathbf{B} and \mathbf{B} -valued structure \mathcal{M} with the mixing property.

✱ $\Gamma \equiv_{\text{BVM}} \Delta$ if $\Gamma \vDash_{\text{BVM}} \Delta$ and $\Delta \vDash_{\text{BVM}} \Gamma$.

✱ $\Gamma \equiv_{\text{Sh}} \Delta$ if $\Gamma \vDash_{\text{Sh}} \Delta$ and $\Delta \vDash_{\text{Sh}} \Gamma$.

1.3 Consistency properties

In first order logic the main tool for building Tarski models of a theory is the compactness theorem. However, this technique is not suited for the infinitary logics $\mathcal{L}_{\kappa\omega}$ since it fails even for the weakest non-trivial case given by $\mathcal{L}_{\omega_1\omega}$. Actually, a cardinal κ is (weakly) compact if and only if the (weak) compactness theorem holds for the logic $\mathcal{L}_{\kappa\omega}$. Thus, a new recipe for building models is needed.

*In this section we introduce consistency properties as the canonical tool for building models of infinitary sentences. **Consistency properties** are partial approximations of a model of an infinitary sentence. Theorem 1.4.14 shows that by means of consistency properties one gets a powerful tool to produce Boolean valued models.*

We follow the approach of Keisler's book [7] to consistency properties for $\mathcal{L}_{\omega_1\omega}$.

First of all, it is convenient for technical reasons to reduce the satisfaction problem to formulae where negations occur only in atomic formulae. This operation is used in the proofs of Thm. 1.3.3 and Thm. 1.4.7.

Definition 1.3.1. Let ϕ be a $\mathcal{L}_{\infty\omega}$ -formula. We define **moving a negation inside** $\phi\neg$ by induction on the complexity of formulae:

- If ϕ is an atomic formula, then $\phi\neg$ is $\neg\phi$.
- If ϕ is $\neg\varphi$, then $\phi\neg$ is φ .
- If ϕ is $\bigwedge\Phi$, then $\phi\neg$ is $\bigvee\{\varphi\neg : \varphi \in \Phi\}$.
- If ϕ is $\bigvee\Phi$, then $\phi\neg$ is $\bigwedge\{\varphi\neg : \varphi \in \Phi\}$.
- If ϕ is $\forall v\varphi(v)$, then $\phi\neg$ is $\exists v(\varphi(v)\neg)$.
- If ϕ is $\exists v\varphi(v)$, then $\phi\neg$ is $\forall v(\varphi(v)\neg)$.

The formulas $\neg\phi$ and $\phi\neg$ can be proved to be equivalent by induction on the complexity of formulae.

Definition 1.3.2. Let $\mathcal{L} = \mathcal{R} \cup \mathcal{D}$ be a language where the relation symbols are in \mathcal{R} and the constants symbols are in \mathcal{D} . Given an infinite set of constants \mathcal{C} disjoint from \mathcal{D} , consider $\mathcal{L}(\mathcal{C})$ the signature obtained by extending \mathcal{L} with the constants in \mathcal{C} . A set S whose elements are set sized subsets of $\mathcal{L}(\mathcal{C})_{\infty\omega}$ is a **consistency property** for $\mathcal{L}(\mathcal{C})_{\infty\omega}$ if for each $s \in S$ the following properties hold.

(Con) For any $r \in S$ and any $\mathcal{L}(\mathcal{C})_{\infty\omega}$ -sentence ϕ either $\phi \notin r$ or $\neg\phi \notin r$,

(Ind.1) If $\neg\phi \in s$, then $s \cup \{\phi\neg\} \in S$.

(Ind.2) If $\bigwedge\Phi \in s$, then for any $\phi \in \Phi$, $s \cup \{\phi\} \in S$.

(Ind.3) If $\forall v\phi(v) \in s$, then for any $c \in \mathcal{C} \cup \mathcal{D}$, $s \cup \{\phi(c)\} \in S$.

(Ind.4) If $\bigvee\Phi \in s$, then for some $\phi \in \Phi$, $s \cup \{\phi\} \in S$.

(Ind.5) If $\exists v\phi(v) \in s$, then for some $c \in \mathcal{C}$, $s \cup \{\phi(c)\} \in S$.

(Str.1) If $c, d \in \mathcal{C} \cup \mathcal{D}$ and $c = d \in s$, then $s \cup \{d = c\} \in S$.

(Str.2) If $c, d \in \mathcal{C} \cup \mathcal{D}$ and $\{c = d, \phi(d)\} \subset s$, then $s \cup \{\phi(c)\} \in S$.

(Str.3) If $d \in \mathcal{C} \cup \mathcal{D}$, then for some $c \in \mathcal{C}$, $s \cup \{c = d\} \in S$.

The following result, due to Makkai [10], shows the value of consistency properties for $\mathcal{L}_{\omega_1\omega}$.

Theorem 1.3.3 (Model Existence Theorem). *Let \mathcal{L} be a language, let \mathcal{C} be a countable set of fresh constants and let $S \subset [\mathcal{L}(\mathcal{C})_{\omega_1\omega}]^{\leq\omega}$ a consistency property of countable size. Then any $s \in S$ is realized in some Tarski model.*

Let us present some examples of consistency properties for $\mathcal{L}(\mathcal{C})_{\infty\omega}$.

1. Let κ and μ be cardinals and let \mathcal{K} be a class of Tarski structures for $\mathcal{L}(\mathcal{C})$. The following families are consistency properties for $\mathcal{L}(\mathcal{C})_{\infty\omega}$.

- $S_{\kappa\mu} = \{s \in [\mathcal{L}(\mathcal{C})_{\kappa\omega}]^{\leq\mu} : \exists \mathcal{A} \in \mathcal{K} \mathcal{A} \models \bigwedge s\}$.
- $S_{\kappa<\mu} = \{s \in [\mathcal{L}(\mathcal{C})_{\kappa\omega}]^{<\mu} : \exists \mathcal{A} \in \mathcal{K} \mathcal{A} \models \bigwedge s\}$.
- $S_{\kappa\mu}$ and $S_{\kappa<\mu}$ where only a finite number of constants from \mathcal{C} appear in each $s \in S$.
- Any of the previous cases where the family of Tarski structures \mathcal{K} may exist only in some generic extension of V

2. Let \mathcal{M} be a \mathbf{B} -valued \mathcal{L} -structure with domain M for a signature $\mathcal{L} = \mathcal{R} \cup \mathcal{D}$. Let $\mathcal{C} = M$ and S be the set of finite (less than κ -sized ...) sets r of $\mathcal{L}(M)_{\kappa\omega}$ -sentences such that

$$\llbracket \bigwedge r \rrbracket_{\mathbf{B}}^{\mathcal{M}} > 0_{\mathbf{B}}.$$

Then S is a consistency property.

3. Let ψ be an $\mathcal{L}_{\infty\omega}$ -formula. Denote by S_{ψ} the consistency property given by finite sets $s \cup \{\psi\}$ that are Boolean consistent and s only contains subformulae of ψ .

The example where \mathcal{K} is a family existing in some generic extension is based on the following observation. Let S in V be a consistency property for $\mathcal{L}_{\kappa+\omega}$ of size κ whose elements are all sets of formulae of size at most κ . Let G be a V -generic filter for the forcing $\text{Coll}(\omega, \kappa)$. Then, in the generic extension $V[G]$, S becomes a consistency property of countable size all whose elements are countable and Model Existence Theorem 1.3.3 applied in $V[G]$ provides a Tarski model of any $s \in S$.

1.4 Forcing with consistency properties

In this section \mathcal{L} denotes a language, \mathcal{C} denotes a set of fresh constants and $S \subset \mathcal{P}(\mathcal{L}(\mathcal{C})_{\infty\omega})$ is a set-sized consistency property.

Fact 1.4.1. *If S is a consistency property, then so is $\{s \subset \mathcal{L}(\mathcal{C})_{\infty\omega} : \exists s_0 \in S \ s \subseteq s_0\}$.*

Definition 1.4.2. Let \mathcal{L} be a language, let \mathcal{C} be a set of fresh constants and let S be a consistency property in language $\mathcal{L}(\mathcal{C})_{\infty\omega}$. The **forcing notion** \mathbb{P}_S is given by:

- domain: $\{s \subset \mathcal{L}(\mathcal{C})_{\infty\omega} : \exists s_0 \in S \ (s \subseteq s_0)\}$;
- order: $p \leq q$ if and only if $q \subseteq p$.

Given a filter F on \mathbb{P}_S , denote

$$\Sigma_F = \bigcup F.$$

Let S be a consistency property. The proof of the Model Existence Theorem for $\mathcal{L}_{\omega_1\omega}$ as given in [7] corresponds naturally to the construction of a suitable filter G on \mathbb{P}_S generic over countably many dense sets.

The clauses of a consistency property are naturally attached to dense sets a maximal filter G on \mathbb{P}_S needs to meet in order to produce a Tarski model of the formulae $\phi \in \bigcup G$. For example, suppose $\bigvee \Phi \in s_0 \in S$. Clause 1.3.2 together with Fact 1.4.1 states that the set $\{s \in S : \Phi \cap s \neq \emptyset\}$ is dense below s_0 . In Keisler's case the elements of a consistency property are countable and each $\mathcal{L}(\mathcal{C})_{\omega_1\omega}$ -formula has countably many subformulae. Therefore, one can take an enumeration of all the dense sets at issue and diagonalize.

In the general case $\mathcal{L}_{\infty\omega}$ one deals with many more dense sets. Hence, a filter meeting all the relevant dense sets may not exist. However, we can translate Keisler's argument using forcing and produce a Boolean valued model with the mixing property for the associated consistency property.

For the rest of this section we work with consistency properties made up from finite sets of sentences. The reader familiar with Keisler's book [7] will find this restriction natural.

We split our generalization of Keisler's result in two results. The first shows how far one can go in proving the Model Existence Theorem assuming only the existence of a maximal filter. The second one shows how genericity fills the missing gaps.

1.4.1 The structure \mathcal{A}_F for a maximal filter $F \subset S$

Let S be a consistency property. In this subsection we prove the existence of an \mathcal{L} -structure \mathcal{A}_F for each maximal filter $F \subset S$.

Fact 1.4.3. *Let S be a consistency property for $\mathcal{L}(\mathcal{C})_{\infty\omega}$ whose elements are finite. Let $F \subseteq \mathbb{P}_S$ be a filter. Then $[\Sigma_F]^{<\omega} = F$.*

Proof. The inclusion $F \subset [\Sigma_F]^{<\omega}$ follows from

$$\Sigma_F = \bigcup F$$

and conditions in S being finite.

We now prove $[\Sigma_F]^{<\omega} \subseteq F$. Suppose $p = \{\phi_1, \dots, \phi_n\} \in [\Sigma_F]^{<\omega}$. Then there exist $s_1, \dots, s_n \in F$ such that $\phi_i \in s_i$. Hence $p \subseteq \bigcup_{i \leq n} s_i$. Since F is a filter, we have that

$$\bigcup_{i \leq n} s_i \in F \subseteq \mathbb{P}_S.$$

The set p is a condition in \mathbb{P}_S since \mathbb{P}_S is closed under subsets. Finally, $\bigcup_{i \leq n} s_i \leq p$ and $\bigcup_{i \leq n} s_i \in F$ imply $p \in F$. \square

Definition 1.4.4. Let $\mathcal{L} = \mathcal{R} \cup \mathcal{D}$ be a language, let \mathcal{C} be a set of fresh constants, let S be a consistency property for $\mathcal{L}(\mathcal{C})_{\infty\omega}$ and let F be a maximal filter on \mathbb{P}_S . Denote by

$$\mathcal{A}_F = (A_F, R_F : R \in \mathcal{R}, d_F : d \in \mathcal{D})$$

the following string of symbols.

- A_F is the set of equivalence classes on $\mathcal{C} \cup \mathcal{D}$ for the equivalence relation $c \cong_F d$ if and only if $(c = d) \in \Sigma_F$.
- For $R \in \mathcal{D}$ n -ary relation symbol and $c_1, \dots, c_n \in \mathcal{C} \cup \mathcal{D}$, $R_F([c_1]_F, \dots, [c_n]_F)$ holds if and only if $R(c_1, \dots, c_n) \in \Sigma_F$.
- $d_F = [d]_F$ for any $d \in \mathcal{D} \cup \mathcal{C}$.

Fact 1.4.5. *Let $\mathcal{L} = \mathcal{R} \cup \mathcal{D}$ be a language, \mathcal{C} be a fresh set of constants, S be a consistency property for $\mathcal{L}(\mathcal{C})_{\infty\omega}$ and F be a maximal filter on \mathbb{P}_S . Then \mathcal{A}_F is a Tarski structure in language $\mathcal{L}(\mathcal{C})$.*

Proof. We check that the definitions of A_F and of R_F do not depend on the chosen representatives. Suppose

$$c_1 = d_1, \dots, c_n = d_n, R(c_1 \dots c_n) \in \Sigma_F.$$

By the previous Fact $\{c_1 = d_1, \dots, c_n = d_n, R(c_1 \dots c_n)\} \in F$. Hence, by Clause 1.3.2(Ind2), for any $p \supseteq \{c_1 = d_1, \dots, c_n = d_n, R(c_1 \dots c_n)\}$ in \mathbb{P}_S , $p \cup \{R(d_1, \dots, d_n)\} \in \mathbb{P}_S$. This combined with Clause 1.3.2(Con) gives that no $p \in \mathbb{P}_S$ can contain $\{c_1 = d_1, \dots, c_n = d_n, R(c_1 \dots c_n), \neg R(d_1, \dots, d_n)\}$. By maximality of F

$$\{c_1 = d_1, \dots, c_n = d_n, R(c_1 \dots c_n), R(d_1, \dots, d_n)\} \in \Sigma_F$$

must be the case. □

Lemma 1.4.6. *Let \mathcal{L} be a language, \mathcal{C} be a fresh set of constants, $S \subseteq [\mathcal{L}(\mathcal{C})_{\infty\omega}]^{<\omega}$ be a consistency property and $F \subseteq \mathbb{P}_S$ be a maximal filter on \mathbb{P}_S . Denote by $\Sigma'_F \subset \Sigma_F$ the set of (quantifier free) formulae $\psi \in \Sigma_F$ which are either atomic, negated atomic, or such that any subformula of ψ which is neither atomic nor negated atomic contains just the logical symbol \wedge . Then $\mathcal{A}_F \models \Sigma'_F$.*

Proof. We proceed by induction on the complexity of $\psi \in \Sigma'_F$. For atomic formulae it follows from Def. 1.4.4. \mathbb{P}_S is a consistency property of which S is a dense subset.

\neg Suppose $\psi = \neg\phi \in \Sigma'_F$ with ϕ an atomic formula. Let's see that

$$\mathcal{A}_F \not\models \phi.$$

Since ϕ is atomic it is enough to check $\phi \notin \Sigma'_F$. Suppose otherwise. Then there exists $p \in F$ with $\phi \in p$. Also $\psi \in q$ for some $q \in F$. By compatibility of filters there exists $r \leq p, q$. But $\phi, \neg\phi \in r$ contradicts clause 1.3.2(Con). Therefore,

$$\phi \notin \Sigma'_F.$$

\wedge Suppose $\psi = \bigwedge \Phi$ is in Σ'_F . Since $\bigwedge \Phi \in \Sigma'_F$, we have that $\phi \in \Sigma'_F$ for every $\phi \in \Phi$. We have to check

$$\mathcal{A}_F \models \phi$$

for any $\phi \in \Phi$. Fix $\phi \in \Phi$. Let us first prove that:

For any $q \in \mathbb{P}_S$ with $\bigwedge \Phi \in q$, we have that $q \cup \{\phi\} \in \mathbb{P}_S$ and $q \cup \{\neg\phi\} \notin \mathbb{P}_S$.

Take q in \mathbb{P}_S with $\bigwedge \Phi \in q$. By Clause 1.3.2(Ind.2), $q \cup \{\phi\} \in \mathbb{P}_S$. Assume now that $\{\neg\phi\} \cup q \in \mathbb{P}_S$. Since $\bigwedge \Phi \in q \cup \{\neg\phi\}$, another application of Clause 1.3.2(Ind.2) ensures $q \cup \{\phi, \neg\phi\} \in \mathbb{P}_S$, a contradiction.

By maximality of F , if some $q \in F$ is such that $\bigwedge \Phi \in q$, then $q \cup \{\phi\} \in F$. By Fact 1.4.3 we have that $\phi \in \Sigma_F$. Finally, by induction hypothesis

$$\mathcal{A}_F \models \phi.$$

□

1.4.2 The structure \mathcal{A}_G for a generic filter $G \subset S$

Theorem 1.4.7 (Model Existence Theorem). *Let \mathcal{L} be a language, \mathcal{C} be a set of fresh constants, S be a consistency property consisting of $\mathcal{L}(\mathcal{C})_{\infty\omega}$ -sentences and G be a V -generic filter for \mathbb{P}_S . Then in $V[G]$ it holds that:*

1. The domain of \mathcal{A}_G is exactly $\{[c]_G : c \in \mathcal{C}\}$.
2. For any $\mathcal{L}(\mathcal{C})_{\infty\omega}$ -sentence $\psi \in \Sigma_G$ we have that

$$\mathcal{A}_G \models \psi.$$

Proof. Let (in $V[G]$) \mathcal{A}_G be the structure obtained from G as in Def. 1.4.4. Since S is a dense subset of \mathbb{P}_S , $G \cap S$ is a generic filter for (S, \supseteq) .

❖ Let us prove statement 1 of the Theorem. Fix $d \in \mathcal{D}$. By Clause 1.3.2(Str.3), we have that the set

$$D_d = \{p \in S : \exists c \in \mathcal{C} \ c = d \in p\}$$

is dense in \mathbb{P}_S . Let $p \in G \cap D_d$. Then for some $c \in \mathcal{C}$, $d = c \in p \subset \Sigma_G$ and $[d]_G = [c]_G$.

❖ Let us prove statement 2 of the Theorem. We proceed by induction on the complexity of $\phi \in \Sigma_G$. We have to handle only the cases for \neg , \bigvee , \exists , \forall formulae, since the atomic case and the case \bigwedge follow by the same proof as Lemma 1.4.6.

\bigvee : Suppose $\bigvee \Phi \in \Sigma_G$. Let $p_0 \in G$ be such that $\bigvee \Phi \in p_0$. By Clause 1.3.2(Ind.4) we have that the set

$$D_{\bigvee \Phi} = \{p \in S : \exists \phi \in \Phi \ \phi \in p\}$$

is dense below p_0 . Since G is V -generic over \mathbb{P}_S and $p_0 \in G$, there exists $p \in G \cap D_{\bigvee \Phi}$. Then for some $\phi \in \Phi$, $\phi \in p \subset \Sigma_G$ and

$$\mathcal{A}_G \models \phi,$$

proving

$$\mathcal{A}_G \models \bigvee \Phi.$$

\exists : Suppose $\exists v \phi(v) \in \Sigma_G$. Let $p_0 \in G$ such that $\exists v \phi(v) \in p_0$. By Clause 1.3.2(Ind.5) we have that the set

$$D_{\exists v \phi(v)} = \{p \in S : \exists c \in \mathcal{C} \ \phi(c) \in p\}$$

is dense below p_0 . Since G is V -generic over \mathbb{P}_S and $p_0 \in G$, there exists $p \in G \cap D_{\exists v \phi(v)}$. Then for some $c \in \mathcal{C}$, $\phi(c) \in p \subset \Sigma_G$. Therefore

$$\mathcal{A}_G \models \phi(c),$$

hence

$$\mathcal{A}_G \models \exists v \phi(v).$$

\forall : Suppose $\psi = \forall v \phi(v)$ is in Σ_G . By the first item 1 it is enough to check

$$\mathcal{A}_G \models \phi(v)[v/[c]_G]$$

for all $c \in \mathcal{C}$. By Clause 1.3.2(Ind.3) and $\forall v \phi(v) \in \Sigma_G$, we get that $\phi(c) \in \Sigma_G$ for all $c \in \mathcal{C}$. By induction hypothesis we get that

$$\mathcal{A}_G \models \phi(v)[v/[c]_G]$$

for all $[c]_G \in \mathcal{A}_G$.

\neg : Suppose $\neg\phi \in \Sigma_G$. Clause 1.3.2(Ind.1) ensures that $G' = [\Sigma_G \cup \{\phi\neg\}]$ is a prefilter on \mathbb{P}_S containing G . By maximality of G , $\phi\neg \in G$. Since $\phi\neg$ and $\neg\phi$ are equivalent, it is enough to argue

$$\mathcal{A}_G \models \phi\neg.$$

But $\phi\neg$ starts with a logical symbol among $\bigwedge, \forall, \bigvee$ or \exists , and for these the proof has already been given. □

Note the following apparently trivial corollary of the above Theorem:

Corollary 1.4.8. *Assume S is a consistency property on $\mathcal{L}(\mathcal{C})_{\infty\omega}$ satisfying the assumptions of Thm. 1.4.7. Then for any $s \in S$, we have that*

$$s \not\vdash \emptyset.$$

Proof. Assume $s \vdash \emptyset$ for some $s \in S$. Note that if G is V -generic for \mathbb{P}_S with $s \in G$, the same proof existing in V of $s \vdash \emptyset$, is a proof of the same sequent in $V[G]$. By Theorem 1.4.7

$$\mathcal{A}_G \models \bigwedge s$$

holds in $V[G]$. By the soundness of Tarski semantics for \vdash in $V[G]$, we get that $\mathcal{A}_G \models \psi \wedge \neg\psi$ for some ψ holds in $V[G]$. This is a contradiction. □

Remark 1.4.9. Essentially the same Theorem and Corollary have been proved independently by Ben De Bondt and Boban Velickovic (using the language of forcing via partial orders to formulate them).

1.4.3 Maximal consistency properties

It may occur that for some $\mathcal{L}(\mathcal{C})_{\infty\omega}$ -sentence ψ , neither ψ nor $\neg\psi$ belongs to any $r \in S$. Hence, for some V -generic filter G for \mathbb{P}_S it can be the case that $s \notin G$ while $\mathcal{A}_G \models \psi$. This occurs for consistency properties of the form S_ψ (as in Example 3), since the only sentences in elements of S_ψ are subformulae of ψ . We introduce a strengthening of the notion of consistency property to prove a converse of Thm. 1.4.7.

Definition 1.4.10. Let \mathcal{L} be a language, \mathcal{C} be a fresh set of constants, S be a consistency property as in Def. 1.3.2 and κ be a cardinal. A consistency property S is κ -**maximal** if all its elements consist of $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentences and S satisfies the following clause:

(S-Max) For any $p \in S$ and $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentence ϕ , either $p \cup \{\phi\} \in S$ or $p \cup \{\neg\phi\} \in S$.

Example 2, given by the finite sets of $\mathcal{L}(M)_{\kappa\omega}$ -sentences which have positive value in some fixed Boolean valued model with domain M , gives the standard case of a κ -maximal consistency property.

Proposition 1.4.11. *With the notation of Thm. 1.4.7 assume S is κ -maximal. Then for any $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentence ψ and any V -generic filter G , we have that*

$$\mathcal{A}_G \models \psi \text{ if and only if } \psi \in \Sigma_G.$$

Proof. We need to prove the “only if” part of the implication assuming S is κ -maximal. Suppose ψ is an $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentence not in Σ_G . By κ -maximality of S we get that

$$D_\psi = \{r \in S : \psi \in r \text{ or } \neg\psi \in r\}$$

is dense in \mathbb{P}_S . Since G is V -generic for \mathbb{P}_S , we get that $G \cap D_\psi$ is non-empty. Hence either $\psi \in \Sigma_G$ or $\neg\psi \in \Sigma_G$, but the first is not the case by hypothesis. Then $\neg\psi \in \Sigma_G$ and by Theorem 1.4.7 $\mathcal{A}_G \models \neg\psi$, that is $\mathcal{A}_G \not\models \psi$. \square

1.4.4 Boolean Model Existence Theorem

Given a complete Boolean algebra \mathbf{B} , an \in -formula $\phi(v_1, \dots, v_n)$ for $\mathcal{L}_{\infty\omega}$ (for $\mathcal{L} = \{\in\}$), and any family $\tau_1, \dots, \tau_n \in V^{\mathbf{B}}$, $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}$ denotes the \mathbf{B} -value of $\phi(\tau_1, \dots, \tau_n)$ in the Boolean valued model $V^{\mathbf{B}}$. The definition of $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}$ is by induction on the complexity of ϕ . It is the standard one for the atomic formulae $\llbracket \tau \in \sigma \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}$ and $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}$. We extend it to all $\mathcal{L}_{\infty\omega}$ according to Def. 1.2.2.

Let us recall one result about $< \kappa$ -cc forcing notions. Proposition 4.1.3 appears in [3].

Proposition 1.4.12. *Let κ be a regular cardinal and $\mathbb{P} \subseteq H_\kappa$ be a forcing notion with the $< \kappa$ -cc. Suppose $p \in \mathbb{P}$ and $\dot{\tau}$ is a \mathbb{P} -name such that $p \Vdash_{\mathbb{P}} \dot{\tau} \in H_{\check{\kappa}}$,² then there exists $\dot{\sigma} \in H_\kappa$ such that $p \Vdash_{\mathbb{P}} \dot{\sigma} = \dot{\tau}$.*

Definition 1.4.13. Let \mathcal{L} be a language, \mathcal{C} be a set of fresh constants and S be a consistency property. Denote by

$$\mathcal{A}_S = (A_S, R_S : R \in \mathcal{R}, d_S : d \in \mathcal{D} \cup \mathcal{C})$$

the string defined as follows:

- $A_S = \left\{ \sigma \in V^{\text{RO}(\mathbb{P}_S)} \cap H_\mu : \llbracket \sigma \in A_{\dot{G}} \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = 1_{\text{RO}(\mathbb{P}_S)} \right\}$, where μ is a regular cardinal big enough so that $\mathcal{L} \subseteq H_\mu$ and for any $\sigma \in V^{\text{RO}(\mathbb{P}_S)}$ such that

$$\llbracket \sigma \in A_{\dot{G}} \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = 1_{\text{RO}(\mathbb{P}_S)},$$

one can find $\tau \in V^{\text{RO}(\mathbb{P}_S)} \cap H_\mu$ with

$$\llbracket \tau = \sigma \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = 1_{\text{RO}(\mathbb{P}_S)};$$

²Here $\dot{\tau} \in H_{\check{\kappa}}$ is a short-hand for the \in -formula in parameters $\tau, \check{\kappa}$ asserting that the transitive closure of τ has size less than $\check{\kappa}$.

- $\llbracket R_S(\sigma_1, \dots, \sigma_n) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \llbracket \mathcal{A}_{\dot{G}} \models R_{\dot{G}}(\sigma_1, \dots, \sigma_n) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}}$ for $R \in \mathcal{R}$;
- for $d \in \mathcal{D} \cup \mathcal{C}$, $d_S = \check{d}$.

Theorem 1.4.14. *Let \mathcal{L} be a language, \mathcal{C} be a set of fresh constants and S be a consistency property whose elements are finite. Then \mathcal{A}_S is a $\text{RO}(\mathbb{P}_S)$ -valued model with the mixing property and for every $s \in S$*

$$\llbracket \bigwedge s \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \llbracket \mathcal{A}_{\dot{G}} \models \bigwedge s \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} .$$

Proof.

- We first establish that \mathcal{A}_S has the mixing property. Let $\{\sigma_a : a \in A\}$ be a family of elements of A_S indexed by an antichain A of $\text{RO}(\mathbb{P}_S)$. Apply the mixing property of $V^{\text{RO}(\mathbb{P}_S)}$ to find $\sigma \in V^{\text{RO}(\mathbb{P}_S)}$ such that $\llbracket \sigma = \sigma_a \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} \geq a$ for all $a \in A$. By choice of A_S we can suppose that $\sigma \in A_S$. By definition of \mathcal{A}_S

$$\llbracket \sigma = \sigma_a \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \llbracket \sigma = \sigma_a \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} \geq a$$

for all $a \in A$. Hence σ is a mixing element for the family $\{\sigma_a : a \in A\}$.

- Now we prove the second part of the Theorem. One needs to check that for any $\mathcal{L}_{\kappa\omega}$ -formula $\phi(v)$ and $\sigma_1, \dots, \sigma_n \in A_S$,

$$\llbracket \phi(\vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \llbracket \mathcal{A}_{\dot{G}} \models \phi(\vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} .$$

It is clear that this allows one to prove

$$\llbracket \bigwedge s \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \llbracket \mathcal{A}_{\dot{G}} \models \bigwedge s \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} ,$$

letting $\phi = \bigwedge s$.

We proceed by induction on the complexity of formulae.

- For atomic sentences this follows by definition.
- For \neg ,

$$\llbracket \neg \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \neg \llbracket \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \neg \llbracket \mathcal{A}_{\dot{G}} \models \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \llbracket \mathcal{A}_{\dot{G}} \not\models \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \llbracket \mathcal{A}_{\dot{G}} \models \neg \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} .$$

- For \bigwedge ,

$$\llbracket \bigwedge \Phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \bigwedge_{\phi \in \Phi} \llbracket \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \bigwedge_{\phi \in \Phi} \llbracket \mathcal{A}_{\dot{G}} \models \phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \llbracket \mathcal{A}_{\dot{G}} \models \bigwedge \Phi \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} .$$

- For \exists ,

$$\begin{aligned} \llbracket \exists v \phi(v, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} &= \bigvee_{\tau \in \mathcal{A}_S} \llbracket \phi(\tau, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} = \bigvee_{\tau \in \mathcal{A}_S} \llbracket \mathcal{A}_{\dot{G}} \models \phi(\tau, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} \leq \\ &\bigvee_{\tau \in V^{\text{RO}(\mathbb{P}_S)}} \llbracket \mathcal{A}_{\dot{G}} \models \phi(\tau, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \llbracket \mathcal{A}_{\dot{G}} \models \exists v \phi(v, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \\ &\llbracket \mathcal{A}_{\dot{G}} \models \phi(\tau_0, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{V^{\text{RO}(\mathbb{P}_S)}} = \llbracket \phi(\tau_0, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} \leq \llbracket \exists v \phi(v, \vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S}, \end{aligned}$$

where $\tau_0 \in \mathcal{A}_S$ is obtained by fullness of $V^{\text{RO}(\mathbb{P}_S)}$ and can be supposed in H_μ by Proposition 4.1.3; while the equality in the last line holds by inductive assumptions. □

Corollary 1.4.15 (Boolean Model Existence Theorem). *Let \mathcal{L} be a language, \mathcal{C} be a set of fresh constants and S be a consistency property whose elements are finite. Then for any $s \in S$ there is a \mathbf{B} -Boolean valued model \mathcal{M} with the mixing property in which*

$$\llbracket \bigwedge s \rrbracket_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}.$$

Proof. Given $s \in S$, we let $\mathbf{B} = \text{RO}(\mathbb{P}_S) \upharpoonright \text{Reg}(N_s)$. Since

$$s \Vdash_{\mathbb{P}_S} (\mathcal{A}_{\dot{G}} \models \bigwedge s),$$

we get that $\text{Reg}(N_s) \leq \llbracket \bigwedge s \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S}$. In particular, if we consider \mathcal{A}_S as a \mathbf{B} -valued model by evaluating all atomic formulae $R(\vec{\sigma})$ by

$$\llbracket R(\vec{\sigma}) \rrbracket_{\text{RO}(\mathbb{P}_S)}^{\mathcal{A}_S} \wedge \text{Reg}(N_s),$$

then we get that

$$\llbracket \bigwedge s \rrbracket_{\mathbf{B}}^{\mathcal{A}_S} = 1_{\mathbf{B}}.$$

\mathbf{B} is a non-trivial complete Boolean algebra, since $\text{Reg}(N_s) \neq \emptyset = 0_{\text{RO}(\mathbb{P}_S)}$ for all $s \in S$. □

Remark 1.4.16. When working with a consistency property S for $\mathcal{L}(\mathcal{C})_{\kappa\omega}$, there is a canonical way of extending it to a κ -maximal one. Consider the Boolean valued model \mathcal{A}_S of Def. 1.4.13, let also $\mathbf{B} = \text{RO}(\mathbb{P}_S)$. Then

$$S \subset M_S = \{t \in [\mathcal{L}(\mathcal{C} \cup \mathcal{A}_S)_{\kappa\omega}]^{<\omega} : \llbracket t \rrbracket_{\mathbf{B}}^{\mathcal{A}_S} > 0_{\mathbf{B}}\}$$

and M_S is a κ -maximal consistency property for $\mathcal{L}(\mathcal{C} \cup \mathcal{A}_S)_{\kappa\omega}$.

Chapter 2

Model theory of Boolean valued models

This chapter explores the model theory of Boolean valued models in the context of infinitary logics of the form $\mathcal{L}_{\infty\omega}$. Our analysis is inspired by the classical model theory of first order logic. To set the stage, let us start by recalling the original version of the results we will deal with.

- ❁ *Compactness Theorem. A first order theory T is consistent if and only if it is finitely consistent.*
- ❁ *Completeness Theorem. A first order sentence ψ holds in all models of a first order theory T if and only if there is a proof of ψ from T .*
- ❁ *Interpolation Theorem. Let ψ_0 and ψ_1 be first order sentences in languages \mathcal{L}_0 and \mathcal{L}_1 . If $\psi_0 \vdash \psi_1$, then there exists θ in language $\mathcal{L}_0 \cap \mathcal{L}_1$ such that $\psi_0 \vdash \theta$ and $\theta \vdash \psi_1$.*
- ❁ *Omitting Types Theorem. Let Φ be a first order type not isolated by a first order theory T . Then there exists a model of T in which the type Φ is not realized.*

We claim that Boolean valued models with the mixing property are a natural semantics for infinitary logics of the form $\mathcal{L}_{\infty\omega}$ since all four theorems have canonical translations in this setting. Let us summarize the main differences and similarities from the first order version to the infinitary one.

- ❁ *Compactness Theorem. This result requires significant adjustments. The main issue for infinitary logics occurs when we have two $\mathcal{L}_{\infty\omega}$ -sentences ψ and θ with ψ of logical complexity (for example set theoretic rank) much higher than that of θ . In first order logic this issue does not arise, since both sentences will contain at most finitely many conjunctions and disjunctions and their "logical distance" is "negligible". But for infinitary logics, we might have that ψ is an $\mathcal{L}_{\kappa+\omega}$ -sentence with κ supercompact while θ is an atomic sentence of very low set-theoretic rank; the expressive power of σ might be in such case "far too strong" compared to that of θ . Once the notion of conservative strengthening is introduced to control the "distance in logical complexity" between formulae, the natural form of compactness can be proved.*
- ❁ *Completeness Theorem. This result translates literally by replacing first order logic for $\mathcal{L}_{\infty\omega}$ and Tarski semantics for Boolean valued models with the mixing property. The reader*

should take into account that we need to use the relation $T \vdash S$ where T and S are both sets of sentences. This relation has been presented in Definition 1.2.10 and asserts that $\bigwedge T \vdash \bigvee S$, which in particular gives the usual definition when $S = \{\psi\}$.

- ❖ *Interpolation Theorem.* As for completeness this result also translates literally by replacing first order logic with $\mathcal{L}_{\infty\omega}$ and Tarski semantics with the semantics given by Boolean valued models with the mixing property.
- ❖ *Omitting Types Theorem.* This theorem cannot be literally translated to $\mathcal{L}_{\infty\omega}$ since any type is isolated by the existential closure of its conjunction. Nonetheless, once the right hypothesis about the complexity of the type and the theory are introduced, the same statement as in first order logic can be proved.

All proofs in this chapter are based on the following recipe, which has been developed in details in Chapter 1:

Build the right consistency property + Boolean Model Existence Theorem 1.4.15.

A natural question at this point would be why not work with the more general version of infinitary logics given by $\mathcal{L}_{\infty\infty}$ instead of $\mathcal{L}_{\infty\omega}$. Historically, one of the first results in this subject, and actually one of the main sources of motivation for all our work, is Mansfield Completeness Theorem [14] for Boolean valued models (without the mixing property) with respect to $\mathcal{L}_{\infty\infty}$. In the final section we present an example due to Ben De Bondt arguing that mixing Boolean valued models are not a correct semantics for $\mathcal{L}_{\infty\infty}$. This counterexample, together with the work of Viale and Pierobon [?] and Monro [?] establishing that in category theory mixing \mathbf{B} -valued models correspond with sheaves on \mathbf{B}^+ with respect to the dense Grothendieck topology, while general \mathbf{B} -valued models correspond with \mathbf{B}^+ -presheaves, are the two main reasons motivating our focus on the infinitary logic $\mathcal{L}_{\infty\omega}$.

2.1 Boolean satisfiability generalizes Tarski satisfiability

As a first step, we argue that Boolean satisfiability correctly generalizes Tarski satisfiability.

Lemma 1. *Assume T is a first order theory. Then T is Boolean satisfiable if and only if T is Tarski satisfiable.*

Proof. If T is Tarski satisfiable, then T is Boolean satisfiable as any Tarski model is a Boolean valued model for the Boolean algebra $\mathbf{B} = \{0, 1\}$.

Let T be Boolean satisfiable, \mathcal{M} be a \mathbf{B} -valued model of T and κ be the size of T . Consider S the consistency property from Example 2 given by the sets s of size κ of $\mathcal{L}(M)_{\infty\omega}$ -sentences such that

$$\left[\bigwedge s \right]_{\mathbf{B}}^{\mathcal{M}} > 0_{\mathbf{B}}.$$

We have that T belongs to S since

$$\left[\bigwedge T \right]_{\mathbf{B}}^{\mathcal{M}} = 1_{\mathbf{B}}.$$

By the Boolean Model Existence Theorem 1.4.15 there exists \mathcal{N} a \mathbf{B}' -valued model with the mixing property such that

$$\llbracket \bigwedge T \rrbracket_{\mathbf{B}'}^{\mathcal{N}} = 1_{\mathbf{B}}.$$

Let G be an ultrafilter on \mathbf{B}' . Then $\mathcal{M}/_G$ is a Tarski model of T by Proposition 1.2.5 and Theorem 1.2.9. \square

Hence, when dealing with first order theories we need not distinguish between Boolean or Tarski satisfiability.

2.2 Conservative Compactness

Suppose $\{\psi_i : i \in I\}$ is a set of first order sentences. Compactness for first order logic says that every finite subset of $\{\psi_i : i \in I\}$ has a Tarski model if and only if $\bigwedge_{i \in I} \psi_i$ has a Tarski model. With this formulation the result does not generalize to $\mathcal{L}_{\infty\omega}$, even if one replaces Tarski models with Boolean valued models (see Example 2.2.1).

We produce a generalization to $\mathcal{L}_{\infty\omega}$ of the compactness theorem; toward this aim we introduce the key concept of conservative strengthening and the corresponding notion of being a finitely conservative set of $\mathcal{L}_{\infty\omega}$ -sentences. We show that (with minor twists) being a finitely conservative set of sentences is a natural refinement of being finitely consistent (and in a precise sense an equivalent reformulation of this concept).

Replacing finitely consistent with finitely conservative, compactness generalizes naturally to $\mathcal{L}_{\infty\omega}$ logics (see the Conservative Compactness Theorem 2.2.5).

2.2.1 The failure of the simplistic notion of compactness for $\mathcal{L}_{\infty\omega}$

Example 2.2.1 (The failure of compactness). Let \mathcal{L} be a language containing constants $\{c_n : n \in \omega\} \cup \{c_\omega\}$ and the equality relation symbol. Denote by T the theory

$$\{c_n \neq c_\omega : n \in \omega\} \cup \left\{ \bigvee_{n \in \omega} c_\omega = c_n \right\}.$$

✿ Failure of compactness for Tarski semantics. Let us argue that this theory has no Tarski model, yet it has models for every finite subset. T has no Tarski model since any realization $c_n = c_\omega$ of the axiom

$$\bigvee_{n \in \omega} c_\omega = c_n$$

in a Tarski model would contradict axiom $c_n \neq c_\omega$ from the second family of sentences. Nonetheless, if we consider $t \subset T$ finite subset, then by interpreting in a Tarski structure with infinite domain c_ω the same way as c_n for n bigger than the highest index appearing inside t , we can produce a model of t . Hence, we have an inconsistent $\mathcal{L}_{\infty\omega}$ -theory for Tarski semantics all

whose finite subsets are Tarski consistent. That is, compactness fails for the logic $\mathcal{L}_{\omega_1\omega}$ with respect to Tarski semantics.

❖ **Failure of compactness for Boolean valued semantics.** Let us argue that the same theory shows the failure of compactness for the semantics given by Boolean valued models. Assume \mathcal{M} is a \mathbf{B} -valued model of T . Then

$$\llbracket c_n \neq c_\omega \rrbracket = 1_{\mathbf{B}}$$

for every $n \in \omega$ and

$$\llbracket \bigwedge_{n \in \omega} c_n \neq c_\omega \rrbracket = \bigwedge_{n \in \omega} \llbracket c_n \neq c_\omega \rrbracket = 1_{\mathbf{B}}.$$

But at the same time

$$\llbracket \neg \bigwedge_{n \in \omega} c_n \neq c_\omega \rrbracket = \llbracket \bigvee_{n \in \omega} c_n = c_\omega \rrbracket = 1_{\mathbf{B}},$$

a contradiction.

2.2.2 Conservative strengthening

Definition 2.2.2 (Conservative strengthening).

❖ Let ψ_0 and ψ_1 be $\mathcal{L}_{\infty\omega}$ -sentences. We say that ψ_1 is a **conservative strengthening** of ψ_0 if:

1. $\psi_1 \vdash \psi_0$ and
2. for any finite set s of subformulae of ψ_0 , $s \cup \{\psi_0\}$ is Boolean consistent if and only if $s \cup \{\psi_1\}$ is Boolean consistent.¹

❖ A family $\{\psi_i : i \in I\}$ of formulae is **finitely conservative** if at least one ψ_i is consistent and for any finite $s \subset \{\psi_i : i \in I\}$, $\bigwedge s$ is a conservative strengthening of every $\psi_i \in s$.

❖ A family $\{\psi_i : i \in I\}$ of formulae is **conservative** if

$$\bigwedge_{i \in I} \psi_i$$

is a conservative strengthening of ψ_i for every $i \in I$ and at least one ψ_i is consistent.

❖ An $\mathcal{L}_{\infty\omega}$ -formula ψ is **strongly conservative** over ϕ if any model of ϕ can be expanded to a model of ψ .

The following is clear:

Fact 2.2.3. *If ψ is strongly conservative over ϕ and $\psi \vdash \phi$, then ψ is a conservative strengthening of ϕ .*

¹Note that ψ_1 and ψ_0 might be in distinct signatures, in several applications ψ_1 will be in a richer signature than ψ_0 .

There can be ϕ which are conservative strengthening of ψ but not strongly conservative over ϕ . We will see an example with $\phi := \psi \wedge \text{AS}(\beta, \dot{X}, X^*)$.

Let us now argue that finitely conservative generalizes finitely consistent.

Fact 2.2.4. *Any finitely conservative theory $\{\psi_i : i \in I\}$ is finitely consistent.*

Proof. Let $J \subset I$ be finite. We need to argue $\{\psi_i : i \in J\}$ is consistent. By hypothesis we can fix $i_0 \in I$ such that ψ_{i_0} is consistent. By assumption $\{\psi_i : i \in J\} \cup \{\psi_{i_0}\}$ is finitely conservative. Then, since ψ_{i_0} is consistent, so is $\{\psi_i : i \in J\} \cup \{\psi_{i_0}\}$. In particular $\{\psi_i : i \in J\}$ is consistent. \square

Consider the setting from Example 2.2.1. Let us argue that T is not finitely conservative. Let $t \subset T$ be the finite subset given by the two sentences

$$\bigvee_{n \in \omega} c_n = c_n \quad \text{and} \quad c_\omega \neq c_0.$$

Let us argue that $\bigwedge t$ is not a conservative strengthening of $\bigvee_{n \in \omega} c_n = c_n$. In order to do so, we need to find a subsentence of $\bigvee_{n \in \omega} c_n = c_n$ that is consistent with $\bigvee_{n \in \omega} c_n = c_n$, but is not consistent with $\bigwedge t$.

We have that $c_\omega = c_0$ is a consistent subsentence of $\bigvee_{n \in \omega} c_n = c_n$. Nonetheless, $c_\omega = c_0$ is not consistent with $\bigwedge t$ since $c_\omega \neq c_0 \in t$. Thus, $\bigwedge t$ is not a conservative strengthening of $\bigvee_{n \in \omega} c_n = c_n$ and T is not finitely conservative. The next result shows that this is the reason for which T admits no Boolean valued model.

2.2.3 Conservative Compactness

Theorem 2.2.5 (Conservative Compactness). *A family of $\mathcal{L}_{\infty\omega}$ -formulae is finitely conservative if and only if it is conservative.*

Proof. Only the direction *finitely conservative* \Rightarrow *conservative* requires a detailed argument.

Assume $\{\psi_i : i \in I\}$ is finitely conservative (and thus also finitely consistent). Without loss of generality (enlarging our family by adding all the conjunctions of its finite subsets), we may assume that $\{\psi_i : i \in I\}$ is closed under finite conjunctions and is still finitely conservative. Let

$$\Psi = \bigwedge_{i \in I} \psi_i.$$

1. Consider the family S of sets $\{\Psi\} \cup t$ such that:

- t is finite,
- t is Boolean consistent,
- there exists $i_t \in I$ such that:
 - $\psi_{i_t} \in t$,
 - θ is a subformula of ψ_{i_t} for each $\theta \in t$.

2. Let us first show that if S is a consistency property, then Ψ is a conservative strengthening of ψ_i for any $i \in I$.

- Let s be a finite set of subformulae of ψ_i . We need to show that if $s \cup \{\psi_i\}$ is Boolean consistent, then so is $s \cup \{\Psi\}$. Note that if $s \cup \{\psi_i\}$ is Boolean consistent, $\{\Psi\} \cup s \cup \{\psi_i\} \in S$, as i is exactly the i_t for $t = s \cup \{\psi_i\}$ witnessing that $\{\Psi\} \cup t \in S$. By the Boolean Model Existence Theorem 1.4.15, this implies the existence of a Boolean valued model of $\{\Psi\} \cup t$, which is also a model of $\{\Psi\} \cup s$.

3. Now we show that S is a consistency property.

- Suppose $\{\Psi\} \cup t \in S$. We need to argue that for each clause in the definition of consistency property the relevant formula required by the clause belongs to an extension of $\{\Psi\} \cup t$.
- Since t is Boolean consistent, all clauses in the definition of a consistency property will automatically be met for formulas **inside** of t .
- Therefore, we only need to deal with the case of Ψ , i.e. a formula that is not inside of t .
- Fix $j \in I$. We need to argue that there is $r \in S$ such that $\{\Psi\} \cup t \cup \{\psi_j\} \subset r$.
- If $\psi_j \in t$, then we are done.
- Suppose otherwise. By the definition of S , there is $p = i_t \in I$ such that $\psi_p \in t$ and all θ in t are subformulae of ψ_p .
- Since the family $\{\psi_i : i \in I\}$ is finitely conservative, we have that $\psi_p \wedge \psi_j$ is a conservative extension of ψ_p .
- Since the family $\{\psi_i : i \in I\}$ is closed under finite conjunctions, we have that $\psi_p \wedge \psi_j = \psi_k$ for some $k \in I$.
- By definition of conservative strengthening, for any s finite subset of ψ_p -subformulae, $\psi_k \wedge \bigwedge s = \psi_p \wedge \psi_j \wedge \bigwedge s$ is Boolean consistent if and only if $\psi_p \wedge \bigwedge s$ is Boolean consistent.
- Since t is Boolean consistent, $\psi_p \in t$ and all formulas in t are subformulae of ψ_p ,

$$\bigwedge t = \bigwedge (\{\psi_p\} \cup t) = \psi_p \wedge \bigwedge t$$

is Boolean consistent. Therefore, $s = \{\psi_j, \psi_k\} \cup t$ is Boolean consistent.

- Finally, as all formulae in s are subformulae of ψ_k , $t \cup \{\psi_j, \psi_k, \Psi\}$ belongs to S .

□

2.2.4 Conservative compactness generalizes first order compactness

The goal of this subsection is to prove that Conservative Compactness Theorem 2.2.5 is truly a generalization of compactness from first order logic. This would follow immediately if one could show that a first order theory is finitely conservative if and only if it is finitely consistent. This is almost true. Before dealing with the precise relation between these two concepts, let us mention the issue one might face. Suppose T is a finitely consistent theory. Fix ϕ a sentence of T such that $\neg\phi$ is consistent. We have that

$$T' = \{\theta \vee \neg\phi : \theta \in T\} \cup \{\phi\}$$

is consistent since $T \vdash \psi$ for all $\psi \in T'$. In particular T' is finitely consistent. Nonetheless, T' is not finitely conservative: for $\theta \in T$ and $t = \{\phi, \theta \vee \neg\phi\}$, $\bigwedge t$ is not a conservative strengthening of $\theta \vee \neg\phi$. Indeed, since $\neg\phi$ is consistent, we have that $\neg\phi$ is a subsentence of $\theta \vee \neg\phi$ consistent with $\theta \vee \neg\phi$, but not consistent with $\bigwedge t$.

This is essentially the unique type of obstruction impeding the literal equivalence of the two concepts.

Theorem 2.2.6. *Let T be a finitely consistent first order theory. Then there exists T^* a first order theory such that*

- T^* is logically equivalent to T and
- T^* is finitely conservative.

In particular, this result shows that the Conservative Compactness Theorem is a strengthening (and thus a generalization to $\mathcal{L}_{\infty\omega}$) of the usual compactness theorem for first order logic: if T is a first order finitely consistent theory, then the Conservative Compactness Theorem 2.2.5 applied to T^* together with Lemma 1 produces a Tarski model of T .

Proof. For each $\phi \in T$, let $\{\theta_i : i \in I_\phi\}$ be the family of subformulae of ϕ and consider

$$\phi^* = \phi \wedge \bigwedge \left\{ \bigvee_{i \in J_0} \neg\theta_i \vee \bigvee_{i \in J_1} \theta_i : J_0, J_1 \subseteq I_\phi, J_0 \cap J_1 = \emptyset, \text{ and } T \vdash \bigvee_{i \in J_0} \neg\theta_i \vee \bigvee_{i \in J_1} \theta_i \right\}.$$

Let us argue that

$$T^* = \{\phi^* : \phi \in T\}$$

is the theory we are searching for.

- First, we prove that T^* is logically equivalent to T . We have that $T^* \vdash \phi$ for any $\phi \in T$, since $\phi^* \vdash \phi$ and $\phi^* \in T^*$ for every $\phi \in T$. We also have that $T \vdash \phi^*$ for every $\phi \in T$, by the very definition of ϕ^* .
- Before proving that T^* is finitely conservative, let us show the following characterization of the subformulae of ϕ^* .

For any $\phi \in T$, every subformula ψ of ϕ^* is:

- either a subformula of ϕ ,
- or such that for some $J_0, J_1 \subseteq I_\phi$ with $J_0 \cap J_1 = \emptyset$ and such that

$$T \vdash \bigvee_{i \in J_0} \neg \theta_i \vee \bigvee_{i \in J_1} \theta_i,$$

ψ is either of the form $\neg \theta_i$ with $i \in J_0$ or exactly the formula

$$\bigvee_{i \in J_0} \neg \theta_i \vee \bigvee_{i \in J_1} \theta_i.$$

Proof. If ψ is not a subformula of ϕ , then it is a subformula of

$$\bigvee_{i \in J_0} \neg \theta_i \vee \bigvee_{i \in J_1} \theta_i.$$

Since each θ_i is a subformula of ϕ , ψ is not a subformula of any θ_i . Then the only possibilities are

$$\bigvee_{i \in J_0} \neg \theta_i \vee \bigvee_{i \in J_1} \theta_i$$

or $\neg \theta_i$ for some $i \in J_0$. □

- Let us argue T^* is finitely conservative. First, since T is finitely consistent and T^* is logically equivalent to it, then T^* is finitely consistent and there is at least one sentence from T^* that is consistent.

Now assume by contradiction that there exist $t \subset T^*$ a finite subset, $\phi^* \in t$ and η_1, \dots, η_k subformulae of ϕ^* whose conjunction is consistent with ϕ^* but not with $\bigwedge t$. Then

$$\bigwedge t \vdash \bigvee_{i=1}^k \neg \eta_i$$

and since $T \vdash \eta^*$ for all $\eta \in T$ and $t \subseteq T^* = \{\eta^* : \eta \in T\}$, we have that

$$T \vdash \bigvee_{i=1}^k \neg \eta_i.$$

Fix k minimal for which subformulae η_1, \dots, η_k of ϕ^* can be found so that:

$$T \vdash \bigvee_{i=1}^k \neg \eta_i \quad \text{and}$$

$$(\phi^* \wedge \bigwedge_{i=1}^k \eta_i) \text{ is Boolean consistent.}$$

Our analysis of the subformulae of ϕ^* shows that for every $i = 1, \dots, k$

- (i) either η_i is a subformula of ϕ ,
- (ii) or for some $J_0^i, J_1^i \subseteq I_\phi$ such that $J_0^i \cap J_1^i = \emptyset$ and

$$T \vdash \bigvee_{l \in J_0^i} \neg \theta_l \vee \bigvee_{l \in J_1^i} \theta_l$$

$$\eta_i \text{ is } \bigvee_{l \in J_0^i} \neg \theta_l \vee \bigvee_{l \in J_1^i} \theta_l,$$

- (iii) or for some $J_0^i, J_1^i \subseteq I_\phi$ such that $J_0^i \cap J_1^i = \emptyset$ and

$$T \vdash \bigvee_{l \in J_0^i} \neg \theta_l \vee \bigvee_{l \in J_1^i} \theta_l$$

$$\eta_i \text{ is } \neg \theta_{l_i} \text{ for some } l_i \in J_0^i.$$

✿ First, we prove that the second case cannot occur for any $i \in \{1, \dots, k\}$. Indeed, if for some $i \in \{1, \dots, k\}$ the formula η_i is

$$\bigvee_{l \in J_0^i} \neg \theta_l \vee \bigvee_{l \in J_1^i} \theta_l,$$

then $T \vdash \eta_i$, and therefore

$$T \vdash \bigvee_{j=1, j \neq i}^k \neg \eta_j.$$

This contradicts the minimality of k .

✿ Hence, only the first and the third case can occur for every $i \in \{1, \dots, k\}$. Let

$$J_1 = \{l_i : i \in J_0^i, i \in I\}$$

and J_0 be such that the set $\{\theta_m : m \in J_0\}$ is exactly the set of η_i for which the first case above occurs. Let us argue that $\bigvee_{i=1}^k \neg \eta_i$ is logically equivalent to the formula

$$\bigvee_{m \in J_0} \neg \theta_m \vee \bigvee_{m \in J_1} \theta_m.$$

- For η_i satisfying the third case, η_i is $\neg \theta_{l_i}$ with $l_i \in J_0^i$ for some $i \in I$, hence $\neg \eta_i$ is logically equivalent to θ_m with $m \in J_1$, by the very definition of J_1 .
- For η_i satisfying the first case, η_i is θ_m for some $m \in J_0$, hence $\neg \eta_i$ is $\neg \theta_m$ for exactly that m .

Since

$$T \vdash \bigvee_{i=1}^k \neg \eta_i,$$

we have that

$$T \vdash \bigvee_{m \in J_0} \neg \theta_m \vee \bigvee_{m \in J_1} \theta_m.$$

But now each θ_m for $m \in J_0 \cup J_1$ is a subformula of ϕ . Since $\eta_1 \wedge \cdots \wedge \eta_k$ is consistent, we must have that $J_0 \cap J_1$ must be empty, otherwise $\bigvee_{m \in J_0} \neg \theta_m \vee \bigvee_{m \in J_1} \theta_m$ would be trivially provable, making $\eta_1 \wedge \cdots \wedge \eta_k$ inconsistent. Hence

$$\bigvee_{k \in J_0} \neg \theta_k \vee \bigvee_{k \in J_1} \theta_k$$

is one of the conjuncts of ϕ^* . This entails that

$$\phi^* \vdash \bigvee_{i=1}^k \neg \eta_i,$$

giving that ϕ^* cannot be consistent with the conjunction of the various η_i . We reached the desired contradiction.

□

2.3 Completeness

Recall from Definition 1.2.10 that \models_{Sh} refers to model theoretic implication with respect to the class of Boolean valued models with the mixing property, \models_{BVM} refers to model theoretic implication with respect to the class of all Boolean valued models and \vdash means there exists a proof of the sequent.

Theorem 2.3.1 (Boolean Completeness). *Let \mathcal{L} be a language. The following are equivalent for T, S sets of $\mathcal{L}_{\infty\omega}$ -formulae.*

1. $T \models_{\text{Sh}} S$,
2. $T \models_{\text{BVM}} S$,
3. $T \vdash S$.

Proof. We have that 3 implies 2 since the deduction system we presented is sound. We have that 2 implies 1 since the class of all Boolean valued models contains the class of Boolean valued models with the mixing property.

Assume 3 fails, we show that 1 fails as well. Assume $T \not\vdash S$ with T, S sets of $\mathcal{L}_{\infty\omega}$ -formulae. Let \mathcal{C} be the infinite set of fresh constants for which T proves the quantifier elimination axiom and let R be the family of finite sets $r \subseteq \mathcal{L}(\mathcal{C})$ such that

- $r \cup T \not\vdash S$,
- any $\phi \in r$ contains only finitely many constants from \mathcal{C} .

✱ Let us first argue that if R is a consistency property, then the result follows. Consider \mathcal{A}_R the Boolean valued model with the mixing property from Theorem 1.4.15 derived from R . We have \mathcal{A}_R ensures $T \not\vdash_{\text{Sh}} S$ as:

- $\llbracket \psi \rrbracket^{\mathcal{A}_R} = 1_{\text{RO}(\mathbb{P}_R)}$ for all $\psi \in T$, since for any $\psi \in T$ the set

$$E_\psi = \{r \in R : \psi \in r\}$$

is dense in \mathbb{P}_R ;

- $\llbracket \phi \rrbracket^{\mathcal{A}_R} = 0_{\text{RO}(\mathbb{P}_R)}$ for all $\phi \in S$, since for any such ϕ the set

$$F_\phi = \{r \in R : \neg\phi \in r\}$$

is dense in \mathbb{P}_R : note that $r \cup \{\neg\phi\} \cup T \vdash S$ if and only if $r \cup T \vdash S \cup \{\phi\}$, which, if $\phi \in S$, amounts to say that $r \notin R$.

✱ Now we show that R is a consistency property:

(Con) If $\{\phi, \neg\phi\} \in r$, then $r \cup T \vdash S$ and we have a contradiction with the definition of R .

(Ind.1) Let $\neg\phi \in r \in R$. Then $r \cup \{\phi\} \cup T \not\vdash S$, since it can be proved by induction on the complexity of formulae that $r \vdash r \cup \{\phi\}$. Hence, $r \cup \{\phi\} \in R$.

(Ind.2) Let $\phi \in \Phi$ and $\bigwedge \Phi \in r \in R$. Then $r \cup \{\phi\} \cup T \not\vdash S$ since $r \vdash r \cup \{\phi\}$ by the left \bigwedge -rule of the calculus together with axiom rule. Hence, $r \cup \{\phi\} \in R$.

(Ind.3) Let $\forall v\phi(v) \in r \in R$ and $c \in \mathcal{C}$. Since $\forall v\phi(v) \vdash \phi(c)$, any proof from $r \cup T \cup \{\phi(c)\}$ induces a proof from $r \cup T$. Hence, $r \cup T \cup \{\phi(c)\} \not\vdash S$ and $r \cup \{\phi(c)\} \in R$.

(Ind.4) Let $\bigvee \Sigma \in r \in R$. Since $r \in R$, $r \cup T \not\vdash S$. By contradiction suppose that for all $\sigma \in \Sigma$, $r \cup \{\sigma\} \cup T \vdash S$. Then, by the left \bigvee -rule of the calculus $r \cup \{\bigvee \Sigma\} \cup T \vdash S$. This contradicts $r \in R$, since $r = r \cup \{\bigvee \Sigma\}$.

(Ind.5) Suppose $\exists v\varphi(v) \in r$. Pick $c \in \mathcal{C}$ which does not appear in any formula in r . It exists by definition of R . Suppose $r \cup \{\varphi(c)\} \cup T \vdash S$. Since c does not appear in any formula of $r \cup S$, $r \cup \{\exists x\varphi(x)\} \vdash S$ (applying the rules of the calculus). This contradicts $r \in R$, since $r = r \cup \{\exists x\varphi(x)\}$.

(Str.1,2,3) All three cases follow from the rules of the calculus for equality.

□

2.4 Interpolation

Let us recall a model theoretic interpretation of interpolation theorems. Let ψ_0 be a sentence in the language of ordered groups and ψ_1 be a sentence in the language of groups. If ψ_0 implies ψ_1 , then interpolation guarantees that we can find a sentence θ in the common language, that is the language of groups, such that ψ_0 implies θ and θ implies ψ_1 . Since θ is a property about groups, this results is telling us that the reason for which ψ_1 implies ψ_0 is rooted in the properties of groups and is not dependant on the order symbol.

Recall from Definition 1.2.10 that \models_{Sh} refers to model theoretic implication with respect to the class of Boolean valued models with the mixing property.

Theorem 2.4.1 (Boolean Craig Interpolation). *Assume $\models_{\text{Sh}} \phi \Rightarrow \psi$ with $\phi, \psi \in \mathcal{L}_{\kappa\omega}$. Then there exists a sentence θ in $\mathcal{L}_{\kappa\omega}$ such that*

- $\models_{\text{Sh}} \phi \Rightarrow \theta$,
- $\models_{\text{Sh}} \theta \Rightarrow \psi$,
- all non logical symbols appearing in θ appear both in ϕ and ψ .

Proof. Fix a set \mathcal{C} of fresh constants for \mathcal{L} of size κ . Consider X_ϕ the set of all $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentences χ such that all non logical symbols from \mathcal{L} appearing in χ also appear in ϕ . Define X_ψ similarly. Consider S the set of finite sets of $\mathcal{L}(\mathcal{C})_{\kappa\omega}$ -sentences s such that:

- (i) $s = s_1 \cup s_2$,
- (ii) $s_1 \subset X_\phi$,
- (iii) $s_2 \subset X_\psi$,
- (iv) if $\theta, \sigma \in X_\phi \cap X_\psi$ are such that
 - no constant symbols of \mathcal{C} appears in θ and σ and
 - $\models_{\text{Sh}} \bigwedge s_1 \Rightarrow \theta$ and $\models_{\text{Sh}} \bigwedge s_2 \Rightarrow \sigma$,

then $\theta \wedge \sigma$ is Boolean consistent.

✱ Assuming that S is a consistency property, we now show how to obtain the interpolant. The Boolean Model Existence Theorem 1.4.15 grants that any $s \in S$ has a Boolean valued model with the mixing property. By hypothesis $\models_{\text{Sh}} \phi \Rightarrow \psi$, thus the set $\{\phi, \neg\psi\}$ is not consistent and it cannot belong to S .

Then the set $\{\phi, \neg\psi\}$ has to miss at least one property from the definition of S . We have that $s_1 = \{\phi\} \subset X_\phi$, $s_2 = \{\neg\psi\} \subset X_\psi$ and $s = s_1 \cup s_2$ is finite. Therefore, the last property (iv) must fail. This means that there exist $\theta, \sigma \in X_\phi \cap X_\psi$ such that θ has no constant symbols of \mathcal{C} and such that

$$\models_{\text{Sh}} \phi \Rightarrow \theta,$$

$\models_{\text{Sh}} \neg\psi \Rightarrow \sigma$ and $\theta \wedge \sigma$ is not consistent. The last assertion gives

$$\models_{\text{Sh}} \theta \Rightarrow \neg\sigma.$$

This together with

$$\models_{\text{Sh}} \neg\sigma \Rightarrow \psi$$

implies

$$\models_{\text{Sh}} \theta \Rightarrow \psi.$$

Recall that θ has no constant symbol from \mathcal{C} . Hence, an interpolant is given by the $\mathcal{L}_{\kappa\omega}$ -sentence θ .

✱ It remains to check that S is a consistency property.

- (Con) By the definitions of S and X_ϕ we have that if some $s \in S$ is such that $\theta, \neg\theta \in S$, then $\theta, \neg\theta \in s_1 \subseteq X_\phi$ or $\theta, \neg\theta \in s_2 \subseteq X_\psi$. Towards a contradiction we can suppose w.l.o.g. that $\theta, \neg\theta \in X_\phi$. Consider any sentence $\chi' \in X_\phi \cap X_\psi$ such that $\models_{\text{Sh}} \bigwedge s_2 \Rightarrow \chi'$. Because s_1 is contradictory we have $\models_{\text{Sh}} \bigwedge s_1 \Rightarrow \neg\chi'$. But $\chi' \wedge \neg\chi'$ is not Boolean consistent, a contradiction with item (iv).
- (Ind.1) Suppose $\neg\chi \in s_1 \subseteq s$. Because $s_1 \cup \{\chi\}$ and s_1 are equivalent, any sentence χ' such that $\models_{\text{Sh}} \bigwedge s_1 \cup \{\chi\} \Rightarrow \chi'$ also verifies $\models_{\text{Sh}} \bigwedge s_1 \Rightarrow \chi'$. Then, $s \cup \{\chi\} \in S$ since $s \in S$ ensures all four conditions are met.
- (Ind.2) Suppose $\chi \in \Phi$ and $\bigwedge \Phi \in s_1 \subseteq s$. Because $\bigwedge s_1$ and $\bigwedge s_1 \cup \{\chi\}$ are equivalent we have that $s \cup \{\chi\} \in S$.
- (Ind.3) Suppose $\forall v\chi(v) \in s_1 \subseteq s$ and $c \in \mathcal{C}$. Because $\bigwedge s_1$ and $\bigwedge s_1 \cup \{\chi(c)\}$ are equivalent, $s \cup \{\chi(c)\} \in S$.
- (Ind.4) Let $\bigvee \Sigma \in s_1 \subseteq s$. By contradiction we suppose that for no $\sigma \in \Sigma$, $s \cup \{\sigma\} \in S$. This means that for each $\sigma \in \Sigma$ there exist $\chi_\sigma^1, \chi_\sigma^2 \in X_\phi \cap X_\psi$ such that

$$\models_{\text{Sh}} \bigwedge (s_1 \cup \{\sigma\}) \Rightarrow \chi_\sigma^1 \text{ and } \models_{\text{Sh}} \bigwedge s_2 \Rightarrow \chi_\sigma^2,$$

but $\chi_\sigma^1 \wedge \chi_\sigma^2$ is inconsistent. Then

$$\begin{aligned} \models_{\text{Sh}} \bigwedge (s_1 \cup \{\bigvee \Sigma\}) &\Rightarrow \bigvee \{\chi_\sigma^1 : \sigma \in \Sigma\} \text{ and} \\ \models_{\text{Sh}} \bigwedge s_2 &\Rightarrow \bigwedge \{\chi_\sigma^2 : \sigma \in \Sigma\}. \end{aligned}$$

Note that $s_1 \cup \{\bigvee \Sigma\} = s_1$. Because $\chi_\sigma^1 \wedge \chi_\sigma^2$ is Boolean inconsistent for each $\sigma \in \Sigma$, so is

$$\begin{aligned} & \left(\bigvee_{\sigma \in \Sigma} \chi_\sigma^1 \right) \wedge \left(\bigwedge_{\sigma' \in \Sigma} \chi_{\sigma'}^2 \right) \equiv_{\text{Sh}} \\ & \bigvee_{\sigma \in \Sigma} \left(\chi_\sigma^1 \wedge \left(\bigwedge_{\sigma' \in \Sigma} \chi_{\sigma'}^2 \right) \right) \equiv_{\text{Sh}} \\ & \bigvee_{\sigma \in \Sigma} \bigwedge_{\sigma' \in \Sigma} \chi_\sigma^1 \wedge \chi_{\sigma'}^2 \models_{\text{Sh}} \\ & \bigvee_{\sigma \in \Sigma} \chi_\sigma^1 \wedge \chi_\sigma^2. \end{aligned}$$

Then θ being $\bigvee \{\chi_\sigma^1 : \sigma \in \Sigma\}$ and σ being $\bigwedge \{\chi_\sigma^2 : \sigma \in \Sigma\}$ witness that $s = s_1 \cup s_2 \notin S$, contradiction.

(Ind.5) Suppose $\exists v \chi(v) \in s_1 \subseteq s$ and consider $c \in \mathcal{C}$ a constant not appearing in s , which exists by the clause on the number of constants from \mathcal{C} in sentences in X_ϕ . Let us check $s \cup \{\chi(c)\} \in S$. For this take $\theta, \sigma \in X_\phi \cap X_\psi$ such that $\models_{\text{Sh}} \bigwedge s_1 \cup \{\chi(c)\} \rightarrow \theta$ and $\models_{\text{Sh}} \bigwedge s_2 \rightarrow \sigma$ with no constants from \mathcal{C} either in θ or in σ . We must show that $\theta \wedge \sigma$ is Boolean satisfiable. It is enough to prove $\models_{\text{Sh}} s_1 \rightarrow \theta$. Consider \mathcal{M} a Boolean valued model for $\mathcal{L} \cup \{c\}$ with the mixing property such that $\mathcal{M} \models s_1$. Since $\exists v \chi(v) \in s_1$, $\llbracket \exists v \chi(v) \rrbracket_{\mathbb{B}}^{\mathcal{M}} = 1_{\mathbb{B}}$; since \mathcal{M} is full, we can find $\tau \in M$ such that

$$\llbracket \exists v \chi(v) \rrbracket_{\mathbb{B}}^{\mathcal{M}} = \llbracket \chi(\tau) \rrbracket_{\mathbb{B}}^{\mathcal{M}} = 1_{\mathbb{B}}.$$

Consider \mathcal{M}' to be the model obtained from \mathcal{M} reinterpreting all symbols of \mathcal{L} the same way, but mapping now c to τ . Then $\mathcal{M}' \models \bigwedge s_1 \cup \{\phi(c)\}$, hence $\llbracket \theta \rrbracket_{\mathbb{B}}^{\mathcal{M}'} = 1_{\mathbb{B}}$ as well. Since c does not appear in θ we get that $\llbracket \theta \rrbracket_{\mathbb{B}}^{\mathcal{M}} = \llbracket \theta \rrbracket_{\mathbb{B}}^{\mathcal{M}'} = 1_{\mathbb{B}}$.

(Str.1,2,3) All three cases follow from $\bigwedge s_1$ and $\bigwedge s_1 \cup \{\chi\}$ being Sh-equivalent when χ is the relevant formula of each clause.

□

2.5 Omitting types

Let us recall some notions about omitting types to make the statement of the theorem intelligible. Suppose $\Sigma(v_1, \dots, v_n)$ is a set of $\mathcal{L}_{\infty\omega}$ -formulae in free variables v_1, \dots, v_n . We say that a model \mathcal{M} realizes $\Sigma(v_1, \dots, v_n)$ if there exist $m_1, \dots, m_n \in M$ such that

$$\mathcal{M} \models \bigwedge \Sigma(m_1, \dots, m_n).$$

\mathcal{M} omits the type Σ amounts to say that for any $m_1, \dots, m_n \in M$,

$$\mathcal{M} \models \bigvee_{\phi \in \Sigma} \neg \phi(m_1, \dots, m_n).$$

Thus, a model \mathcal{M} omits the family of types $\mathcal{F} = \{\Sigma(v_1, \dots, v_{n_\Sigma}) : \Sigma \in \mathcal{F}\}$ if it models the sentence

$$\bigwedge_{\Sigma \in \mathcal{F}} \forall \bar{v}_\Sigma \bigvee \{\neg \phi(\bar{v}_\Sigma) : \phi \in \Sigma\}.$$

In the statement of the following theorem the sets Φ will be playing the roles of $\{\neg \phi : \phi \in \Sigma\}$, where Σ is the type we wish to omit. In this context, the type Σ is not isolated by a sentence θ if whenever there is a model of θ , there is also a model of $\theta \wedge \neg \phi$ for some $\phi \in \Sigma$.

Another essential ingredient to translate omitting types to infinitary logics is given by the notion of fragment. Since any type is isolated by its conjunction, we need to introduce restrictions on what formulae might appear inside the theory T .

Definition 2.5.1. Suppose κ is an infinite cardinal and let \mathcal{L} be a signature. A **fragment** $\mathcal{L}_A \subset \mathcal{L}_{\kappa\omega}$ consists of a set of $\mathcal{L}_{\kappa\omega}$ -formulae such that:

- \mathcal{L}_A is closed under \neg , \wedge and \vee ,
- if $\phi \in \mathcal{L}_A$ and v is a variable appearing in some \mathcal{L}_A -formula, $\forall v \phi$ and $\exists v \phi$ belong to \mathcal{L}_A ,
- \mathcal{L}_A is closed under subformulae,
- if $\phi \in \mathcal{L}_A$, then $\phi \neg \in \mathcal{L}_A$,
- if $\phi \in \mathcal{L}_A$, then there is a variable appearing in \mathcal{L}_A which does not occur in ϕ ,
- if $\phi(v) \in \mathcal{L}_A$ and t is any \mathcal{L} -term, $\phi(t) \in \mathcal{L}_A$,
- if $\phi(v_1, \dots, v_n) \in \mathcal{L}_A$ and w_1, \dots, w_n are variable appearing in \mathcal{L}_A , $\phi(w_1, \dots, w_n) \in \mathcal{L}_A$.

Let κ be an infinite cardinal, let \mathcal{L} be a signature and let T be a set of $\mathcal{L}_{\kappa\omega}$ -formulae. Then there exists a smallest fragment \mathcal{L}_A such that $T \subset \mathcal{L}_A$ and

$$|\mathcal{L}_A| = |L| + |T| + \kappa.$$

Definition 2.5.2. Let T be a theory, $\Phi(v_0, \dots, v_{n_\Phi})$ be a type and $\mathcal{L}_{T, \Phi}$ be the smallest fragment containing T and Φ . We say that Φ is not Boolean isolated in $\mathcal{L}_{T, \Phi}$ if for all $\mathcal{L}_{T, \Phi}$ -formulae θ in free variables v_0, \dots, v_{n_θ} , the theory

$$T + \exists v_0 \dots v_{n_\theta} \theta$$

is Boolean satisfiable if and only if so is

$$T + \exists v_0 \dots v_{\max\{n_\theta, n_\Phi\}} [\theta \wedge \phi]$$

for some $\phi \in \Phi$.

Theorem 2.5.3 (Boolean Omitting Types Theorem).

- Let T be a Boolean consistent $\mathcal{L}_{\infty\omega}$ -theory.
- Assume \mathcal{F} is a family of sets such that every $\Phi \in \mathcal{F}$ is a set of $\mathcal{L}_{\infty\omega}$ -formulae with free variables among v_0, \dots, v_{n_Φ} .
- Let $\mathcal{L}_{T,\mathcal{F}}$ be the smallest fragment of $\mathcal{L}_{\infty\omega}$ such that $T, \Phi \subset \mathcal{L}_{T,\mathcal{F}}$ for all $\Phi \in \mathcal{F}$.
- Suppose that no Φ is Boolean isolated in $\mathcal{L}_{T,\mathcal{F}}$.

Then there exists a Boolean valued model \mathcal{M} with the mixing property such that

$$\mathcal{M} \models T + \bigwedge_{\Phi \in \mathcal{F}} \forall v_0 \dots v_{n_\Phi-1} \bigvee \Phi.$$

Proof. Fix a set of constants $\mathcal{C} = \{c_\alpha : \alpha < \kappa\}$. Consider $\mathcal{L}(\mathcal{C})_{T,\mathcal{F}}$ the set of all sentences obtained by replacing in the $\mathcal{L}_{T,\mathcal{F}}$ -formulae with free variables in $\{v_i : i \in \omega\}$ all occurrences of these finitely many free variables by constants from \mathcal{C} . The consistency property S has as elements the sets

$$s = s_0 \cup \left\{ \bigvee \{ \phi[c_{\sigma_\Phi(0)}, \dots, c_{\sigma_\Phi(n_\Phi)}] : \phi \in \Phi \} : \Phi \in \mathcal{F}_0 \right\},$$

where:

- s_0 is a finite set of $\mathcal{L}(\mathcal{C})_{T,\mathcal{F}}$ sentences,
- only finitely many constants from \mathcal{C} appear in s_0 ,
- \mathcal{F}_0 is a finite subset of \mathcal{F} ,
- $\sigma_\Phi : \omega \rightarrow \mathcal{C}$ for all $\Phi \in \mathcal{F}_0$, and
- $T \cup s_0$ is Boolean consistent.

✱ Let us first argue that if S is a consistency property, then the thesis follows. We have that for all $\Phi \in \mathcal{F}$ and $\sigma : \omega \rightarrow \mathcal{C}$ the set

$$D_{\Phi,\sigma} = \left\{ s \in S : \bigvee \{ \phi[c_{\sigma(0)}, \dots, c_{\sigma(n_\Phi)}] : \phi \in \Phi \} \in s \right\}$$

is dense in \mathbb{P}_S . By the Model Existence Theorem there is a model \mathcal{M} of

$$T \cup \left\{ \bigvee \{ \phi[c_{\sigma(0)}, \dots, c_{\sigma(n_\Phi)}] : \phi \in \Phi \} : \Phi \in \mathcal{F}, \sigma : \omega \rightarrow \mathcal{C} \right\}$$

in which all the elements are the interpretation of some constant from \mathcal{C} . Thus \mathcal{M} models the theory

$$T \cup \left\{ \bigwedge_{\Phi \in \mathcal{F}} \forall v_0 \dots v_{n_\Phi} \bigvee \Phi(v_0, \dots, v_{n_\Phi}) \right\},$$

as required.

✱ Let us argue now that S is a consistency property. Consider $s \in S$ and $\psi \in s$. First of all, by definition of S and the Completeness Thm. 2.3.1 we can fix a mixing model \mathcal{M} of $s_0 \cup T$. We deal with two cases.

- If $\psi \in s_0 \cup T$, then $\mathcal{M} \models \psi$ allows to find the correspondent formula. Here one also uses that only finitely many constants from \mathcal{C} occur in s_0 .
- If

$$\psi = \bigvee \{ \phi [c_{\sigma_{\Phi}(0)}, \dots, c_{\sigma_{\Phi}(n_{\Phi})}] : \phi \in \Phi \}$$

for some $\Phi \in \mathcal{F}$ and $\sigma_{\Phi} : \omega \rightarrow \mathcal{C}$, then we need to find some $\phi \in \Phi$ such that $s \cup \{ \phi \} \in S$. Denote $d_0, \dots, d_m \in \mathcal{C}$ the constants in s_0 from \mathcal{C} that are not $c_{\sigma(0)}, \dots, c_{\sigma(n_{\Phi})}$ and write s_0 as

$$s_0 [c_{\sigma_{\Phi}(0)}, \dots, c_{\sigma_{\Phi}(n_{\Phi})}, d_0, \dots, d_m]$$

with all its constant symbols displayed. Since

$$\mathcal{M} \models T \cup s_0,$$

we have that

$$\mathcal{M} \models \exists v_0 \dots \exists v_{n_{\Phi}} \exists w_0 \dots \exists w_m \bigwedge s_0 [v_0, \dots, v_{n_{\Phi}}, w_0, \dots, w_m].$$

All sentences in s_0 belong to the fragment which is closed under finite conjunctions and quantifications, hence

$$\exists v_0 \dots \exists v_{n_{\Phi}} \exists w_0 \dots \exists w_m \bigwedge s_0 [v_0, \dots, v_{n_{\Phi}}, w_0, \dots, w_m]$$

is an $\mathcal{L}_{T, \mathcal{F}}$ -formula. By the Theorem assumptions, since the type is not isolated, we get that for some $\phi \in \Phi$,

$$T \cup \left\{ \exists v_0 \dots \exists v_{n_{\Phi}} \exists w_0 \dots \exists w_m \bigwedge s_0 [v_0, \dots, v_{n_{\Phi}}, w_0, \dots, w_m] \wedge \phi [v_0, \dots, v_{n_{\Phi}}] \right\}$$

has an $\mathcal{L}_{T, \mathcal{F}}$ -model \mathcal{N} , which again by completeness can be supposed to be mixing. Make \mathcal{N} an $\mathcal{L}(\mathcal{C})_{T, \mathcal{F}}$ -structure by choosing an interpretation of the constants from \mathcal{C} such that $c_{\sigma_{\Phi}(0)}, \dots, c_{\sigma_{\Phi}(n_{\Phi})}$ are assigned to the witnesses of $v_0, \dots, v_{n_{\Phi}}$ and d_0, \dots, d_m are assigned to the witnesses of w_0, \dots, w_m . Then

$$s_0 \cup \{\phi\} \cup T \cup \left\{ \bigvee \{ \phi [c_{\sigma_{\Phi}(0)}, \dots, c_{\sigma_{\Phi}(n_{\Phi})}] : \phi \in \Phi \} : \Phi \in \mathcal{F}_0 \right\} \in S.$$

This concludes the proof that S is a consistency property. □

2.6 Counterexamples for $\mathcal{L}_{\infty\infty}$

This section reports a crucial counterexample to the good behaviour of Boolean valued semantics for $\mathcal{L}_{\infty\infty}$ due to Ben De Bondt. Since it is not available elsewhere and it's one of the reasons behind our choice for $\mathcal{L}_{\infty\omega}$ instead of $\mathcal{L}_{\infty\infty}$, we present it in details (with the author's permission).

The logic $\mathcal{L}_{\infty\infty}$ extends $\mathcal{L}_{\infty\omega}$ by removing the restriction that formulae have only finitely many variables and allowing the quantification over infinite strings of variables. The notion of consistency properties given in Def. 1.3.2, the Boolean valued semantics given in Def. 1.2.1 and the proof system given in Def. 1.1.2 for $\mathcal{L}_{\infty\omega}$ can naturally be extended to $\mathcal{L}_{\infty\infty}$. For details see [20]. Mansfield established in [14] that an $\mathcal{L}_{\infty\infty}$ -sentence ψ is provably consistent if and only if it is Boolean consistent if and only if there is a consistency property S such that $\{\psi\} \in S$.

However, the following counterexample brings to light a significant limitation in the application of Boolean valued semantics to $\mathcal{L}_{\infty\infty}$. Specifically, the counterexample below produces an $\mathcal{L}_{\infty\infty}$ -sentence ψ which is boolean consistent but cannot hold in any Boolean valued model with the mixing property. Given that the Boolean valued models produced by forcing in Theorem 1.4.14 are mixing, this suggests that forcing is a powerful tool for building models of $\mathcal{L}_{\infty\omega}$ -sentences, but not that helpful in building models for $\mathcal{L}_{\infty\infty}$ -sentences.

Definition 2.6.1.

✱ Let \mathcal{L} be the language containing:

- a binary predicate symbol $<$ and
- countably many unary predicate symbols $\{C_n : n < \omega\}$.

✱ Let T be the $\mathcal{L}_{\infty\infty}$ -theory asserting the following:

- (i) the intersection of all C_n is empty: $\forall x \left(\bigvee_{n < \omega} x \notin C_n \right)$,
- (ii) C_n is unbounded for each n : $\forall x \exists y \left(C_n(y) \wedge y > x \right)$,
- (iii) C_n is closed for each n : $\forall x \left(\neg C_n(x) \Rightarrow \exists y \forall z (x > z \geq y \Rightarrow \neg C_n(z)) \right)$,
- (iv) $<$ is well founded: $\neg \exists (v_n : n < \omega) \left(\bigwedge_{n < \omega} v_{n+1} < v_n \right)$,
- (v) $<$ has uncountable cofinality: $\forall (v_n : n < \omega) \exists x \left(\bigwedge_{n < \omega} v_n < x \right)$,

(vi) $<$ is a linear order.

Theorem 2.6.2. *The theory T belongs to a consistency property but there is no forcing extension containing a Tarski model of T . Hence, there is no Boolean valued model of T with the mixing property.*

The Theorem is a consequence of the following two Lemmata:

Lemma 2.6.3. *The theory T has no Tarski model in any forcing extension.*

Proof. Suppose there is a model \mathcal{M} . Since T ensures $<$ is a well order, we can identify it with some ordinal α , which -by definition of T - has to be of uncountable cofinality in $V[G]$. But then the interpretations of the C_n are club subsets of α and their intersection is empty, a contradiction. \square

Lemma 2.6.4. *There is a consistency property containing T as an element.*

Proof. Let $\{c_\alpha : \alpha < \omega_1\} = \mathcal{C}$ be a set of fresh constants. Consider S the consistency property on $\mathcal{L}(\mathcal{C})$ made of sets

$$\{\bigwedge T\} \cup s$$

where s is a finite set contained in the fragment generated by T such that $\bigwedge s$ holds in the Tarski model

$$\mathcal{M}_s = (\omega_1, \in, D_n^s : n \in \omega, \nu_s)$$

where $\nu_s : \mathcal{C}_s \rightarrow \omega_1$ has as domain the set of all constants c_α appearing in formulae from s and satisfies the following requests:

- the constants c_α for $\alpha \in \omega_1$ are interpreted by $\nu_s(c_\alpha) \in \omega_1$,
- the order $<$ is interpreted as \in ,
- the set $\{D_n^s : n \in \omega\}$ is a finite set of clubs on ω_1 (hence many of the predicates C_n get the same interpretation),
- if c_α is some constant appearing in $\bigwedge s$, there is i such that $\nu_s(c_\alpha) \notin D_i^s$.

Let us argue that S is a consistency property.

Note that any subformula of axioms (ii)—(vi) contains at most one predicate symbol C_n , and no subformula of axiom (i) (other than the atomic subformulae) is a subformula of any of the axioms (ii)—(vi) (and conversely). Remark that if θ belongs to some s with $s \cup \{\bigwedge T\} \in S$, this is the case because θ is obtained by a subformula of axioms (i)—(vi) replacing the free variables by appropriate constants in \mathcal{C} .

Let for any $s \in S$ n_s be such that $\{D_1^s, \dots, D_{n_s}^s\} = \{D_n^s : n \in \omega\}$ and all predicates C_n occurring in some $\phi \in s$ with ϕ a subformula of axioms (ii)—(vi) are indexed by some $n < n_s$.

Note that no $s \in S$ is such that \mathcal{M}_s can satisfy Axiom (i) of T , since $\bigcap \{D_n^s : n \in \omega\}$ is a club on ω_1 . Note also that \mathcal{M}_s satisfies axioms (ii) to (vi) of T . Also it is well possible that $\bigvee_{n \in \omega} (c_\alpha \notin C_n)$ is a formula occurring in some $s \in S$.

Since the quantifications appearing in the theory T are over countable sequences, only countably many new constants are introduced by the formula θ generated by some clause of a consistency property applied to some $\psi \in s$ coming from axioms (i)-(vi) of T . Consider such a formula θ and let us argue $\{\bigwedge T\} \cup s \cup \{\theta\} \in S$.

✱ If θ comes from a subformula of axioms (ii)-(vi), then we have canonical choices to realize some clause applied to any of them, as \mathcal{M}_s is a model of these axioms. Let $\nu_{s \cup \{\theta\}} : \mathcal{C}_{s \cup \{\theta\}} \rightarrow \omega_1$ be such that

$$(\omega_1, \in, D_1^s, \dots, D_n^s, D_{n+1}^s, D_{n+2}^s, \dots, \nu_{s \cup \{\theta\}}) \models (\bigwedge s) \wedge \theta.$$

Let α be the supremum of the ordinals in the range of $\nu_{s \cup \{\theta\}}$ and consider now the model

$$\mathcal{M}_{s \cup \{\theta\}} = (\omega_1, \in, D_1, \dots, D_n, [\alpha + 1, \omega_1), D_{n+2}^s, \dots, \nu_{s \cup \{\theta\}}).$$

This model witnesses that $s \cup \{\theta\} \cup \{\bigwedge T\} \in S$, as all the new constants are interpreted by ordinals not in $[\alpha + 1, \omega_1)$.

✱ Regarding a consistency clause applied in $s \cup \{\bigwedge T\}$ to some sentence $\psi \in S$ obtained by some subformula of axiom (i), ψ must either be

$$\left(\bigvee_{n < \omega} c_\alpha \notin C_n \right)$$

(as ψ cannot be axiom (i) itself, and this is the only possible subsentence obtained by a subformula of axiom (i) which is not atomic), or $c_\alpha \notin C_n$ for some α and n .

Therefore c_α is a constant appearing in $\bigwedge s$. Furthermore

$$\mathcal{M}_s \models \psi,$$

as $\psi \in S$ and $s \cup \{\bigwedge T\} \in S$.

It is clear now that the unique cases in which a clause of a consistency property can be applied to ψ are either given by ψ being

$$\left(\bigvee_{n < \omega} c_\alpha \notin C_n \right)$$

and the clause being that associated to a disjunction, or ψ being $c_\alpha \notin C_n$ and the clause being one associated to the equality relation. In either cases we know what to do to satisfy these clauses, since \mathcal{M}_s models ψ .

□

Chapter 3

Examples

3.1 Quantifier elimination

Let us argue why consistency properties naturally induce quantifier elimination.

Definition 3.1.1. Let \mathcal{L} be a language and let \mathcal{C} be a set of fresh constants. The **quantifier elimination axiom** for $\mathcal{L}(\mathcal{C})$ is

$$\forall v \bigvee_{c \in \mathcal{C}} v = c.$$

Fact 3.1.2. Let \mathcal{L} be a language and let \mathcal{C} be a set of fresh constants. Under the quantifier elimination axiom existential quantifiers can be replaced by disjunctions

$$\exists v \phi(v, \bar{w}) \equiv \bigvee_{c \in \mathcal{C}} \phi(c, \bar{w})$$

and universal quantifiers can be replaced by conjunctions

$$\forall v \phi(v, \bar{w}) \equiv \bigwedge_{c \in \mathcal{C}} \phi(c, \bar{w}).$$

The reader should be careful at this point. Our choice does not simply imply that we can forget about the quantifier clauses from the definition of consistency property. From now on, whenever we prove that something is a consistency property, we will need to make sure that the quantifier elimination axiom is a sentence appearing densely often. This will generally be achieved by working with theories T such that

$$T \vdash \forall v \bigvee_{c \in \mathcal{C}} v = c.$$

More precisely, clauses (Ind.3) and (Ind.5) from the definition of consistency property 1.3.2 have switched places with a new condition

$$(QE) \text{ if } s \in S, \text{ then } s \cup \{\forall v \bigvee_{c \in \mathcal{C}} v = c\} \in S.$$

Since this new condition still contains a universal sentence, a priori one should check that condition (Ind.3) for the universal quantifier is realized for the formula

$$\forall v \bigvee_{c \in \mathcal{C}} v = c.$$

But notice that this would require checking that if

$$\forall v \bigvee_{c \in \mathcal{C}} v = c \in s \in S,$$

then

$$s \cup \left\{ \bigvee_{c \in \mathcal{C}} d = c \right\} \in S$$

for any $d \in \mathcal{C}$. Then the clause for the disjunction would require checking that for some $c \in \mathcal{C}$,

$$s \cup \left\{ \bigvee_{c \in \mathcal{C}} d = c \right\} \cup \{d = c\} \in S.$$

But this will always be the case since, if S is a consistency property, then

$$\{s \in S : (d = d) \in s\}$$

is dense for every $d \in \mathcal{C}$.

Summarizing, this argument shows that in practice,

- if the quantifier elimination clause (QE) is met by S and
- S realizes all conditions to be a consistency property apart from those related to the quantifier elimination axiom,

then S is a consistency property and we don't need to deal with the quantifier clauses (Ind.3) and (Ind.5).

3.2 Examples from set theory

From now on the quantifier elimination axiom is always implicit. This means that every structure (or sort in a structure) corresponds with the set of interpretations of the constants (of that sort).

3.2.1 Coll(ω, ω_1)

The first example shows how to describe Coll(ω, ω_1) as a consistency property.

Example 3.2.1 ($\text{Coll}(\omega, \omega_1)$). In Definition 1.3.2 there is no constraint on the size of \mathcal{C} . While Theorem 1.4.7 holds for any size of \mathcal{C} , some sizes automatically collapse cardinals. Consider the language $\mathcal{L} = \{d_\alpha : \alpha < \omega_1\}$ and the set of constants $\mathcal{C} = \{c_n : n < \omega\}$. Let S denote the set whose elements are the conditions $s \in [\mathcal{L}(\mathcal{C})_{\omega_2\omega}]^{<\omega}$ such that for some injective interpretation

$$c_{i_1} \mapsto \alpha_{i_1}, \dots, c_{i_n} \mapsto \alpha_{i_n}, \alpha_{i_j} < \omega_1,$$

of the constants from \mathcal{C} appearing in s ,

$$(\omega_1, =, c_{i_k} \mapsto \alpha_{i_k}, d_\alpha \mapsto \alpha) \models s.$$

S is a consistency property. Consider $\mathcal{A}_G \in V[G]$ for G a filter V -generic for \mathbb{P}_S . It is a model of $\bigwedge_{\alpha \neq \beta \in \omega_1^V} d_\alpha \neq d_\beta$. Furthermore, the interpretation maps

$$\begin{aligned} f : \omega_1^V &\rightarrow \{[d]_G : d \in \mathcal{D}\} \\ \alpha &\mapsto d_\alpha^{A_G} \end{aligned}$$

$$\begin{aligned} g : \omega &\rightarrow \{[c_n]_G : n < \omega\} \\ n &\mapsto c_n^{A_G} \end{aligned}$$

are both injective. This entails that the map $\alpha \mapsto n$ if $\{d_\alpha = c_n\} \in G$ is also injective. Therefore ω_1^V is collapsed.

3.2.2 \in -graph of a transitive set

Example 3.2.2 (Axiom for the \in -graph of a transitive set). Let X be a transitive set, let \mathcal{L} be a language with one predicate symbol \in and let $\mathcal{C} \supset \{\tilde{x} : x \in X\}$ be a set of fresh constants. The axiom ψ_X in language $\mathcal{L}(\mathcal{C})$ is the conjunction of the following axioms:

- For all $x \in y$ in X

$$\tilde{x} \in \tilde{y}.$$

- For all if $x \notin y$ in X

$$\tilde{x} \notin \tilde{y}.$$

- Transitivity of X

$$\bigwedge_{x \in X} \bigwedge_{c \in \mathcal{C}} \left(c \in \tilde{x} \Rightarrow \bigvee_{y \in x} c = \tilde{y} \right).$$

Let $\mathcal{M} \models \psi_X$ be a Tarski model. The axiom ψ_X ensures that

$$(X, \in)$$

is isomorphic to

$$(\{\check{x}^{\mathcal{M}} : x \in X\}, \in^{\mathcal{M}})$$

and $\{\check{x}^{\mathcal{M}} : x \in X\}$ sits as a transitive subclass inside \mathcal{M} .

Lemma 3.2.3. *Let X be a transitive set. Then*

- any model of ψ_X contains a transitive subclass isomorphic to (X, \in) ,
- any set containing (X, \in) as a transitive substructure induces a model of ψ_X ,
- ψ_X is a conjunction of size $|\mathcal{C}|$ of sentences of complexity $\left(\bigwedge_{\omega} \bigvee_{\infty}\right)$.

3.2.3 Iterations

Example 3.2.4 (Axiom for an iteration of length γ). Denote by $\mathcal{L}_{\text{ite}(\gamma)}$ the language with the following sorts, constants and predicates.

Sorts

- Sorts \dot{N}_α for $\alpha \leq \gamma$.

Constants

- Constants $\{c_{n\alpha} : n < \omega\}$ of sort \dot{N}_α for each $\alpha < \gamma$.
- Constants $\{c_{\beta\gamma} : \beta < \gamma\}$ of sort \dot{N}_γ .
- A constant $\omega_1^{N_\alpha}(N_\beta)$ of sort \dot{N}_β for each $\alpha \leq \beta < \gamma$.

Predicates

1. Unary predicate symbols $\{\dot{G}_\alpha : \alpha < \gamma\}$ of sort \dot{N}_α for $\alpha < \gamma$ to denote the N_α -generic filter for $(\mathcal{P}(\omega_1) / \mathbf{NS})^{N_\alpha}$.
2. Binary predicate symbols $\dot{j}_{\alpha\beta}$ of sort $(\dot{N}_\alpha, \dot{N}_\beta)$ to denote the elementary embeddings $j_{\alpha\beta}$ for $\alpha \leq \beta \leq \gamma$.
3. Binary predicates \in_α for each $\alpha \leq \gamma$ of sort $(\dot{N}_\alpha, \dot{N}_\alpha)$ to denote the \in -relation restricted to N_α .
4. For each $\alpha < \gamma$ a satisfaction predicate

$$\text{Sat}_{N_\alpha}(\{\psi(x_1, \dots, x_m)\})$$

of sort¹ $(\dot{N}_\alpha, \dot{N}_\alpha)$ for each $\alpha < \omega_1$ to be represented by the pairs

$$\langle \{\psi\}, (a_1, \dots, a_m) \rangle$$

with $\psi(x_1, \dots, x_m)$ an \in -formula in displayed free variables and $(a_1, \dots, a_m) \in N_\alpha^m$ such that

$$(N_\alpha, \in) \models \psi(x_1, \dots, x_m)[x_1/a_1, \dots, x_m/a_m].$$

For the sake of convenience Form_\in denotes the set of \in -formulae. Denote by $\psi_{\text{ite}(\gamma)}$ the axiom given by the conjunction of the following sentences.

Satisfaction predicates

1. For all $\alpha < \gamma$ the predicate Sat_{N_α} is correctly computed.

Generic filters

1. For all $\alpha < \gamma$ the filter G_α is N_α -generic

$$\bigwedge_{n < \omega} \left(\text{Sat}_{N_\alpha}(\{c_{n\alpha} \text{ is dense for } \mathcal{P}(\omega_1) / \mathbf{NS}_{\omega_1}\}) \rightarrow \bigvee_{m < \omega} (\dot{G}_\alpha(c_{m\alpha}) \wedge c_{m\alpha} \in c_{n\alpha}) \right).$$

2. For all $\alpha < \gamma$ the filter G_α is the filter derived from the map $j_{\alpha\alpha+1}$

$$\bigwedge_{n < \omega} \bigwedge_{m < \omega} \left(j_{\alpha\alpha+1}(c_{n\alpha}) = c_{m\alpha+1} \Rightarrow \left(\dot{G}_\alpha(c_{n\alpha}) \Leftrightarrow \left(\omega_1^{N_\alpha}(N_{\alpha+1}) \in_{\alpha+1} c_{m\alpha+1} \wedge \text{Sat}_{N_\alpha}(\{c_{n\alpha} \in \mathbf{NS}_{\omega_1}^+\}) \right) \right) \right).$$

Elementary maps

1. For all $\alpha < \gamma$ the map $j_{\alpha\alpha+1}$ is elementary

$$\bigwedge_{\psi \in \text{Form}_\in} \bigwedge_{n < \omega} \bigwedge_{m < \omega} \left(j_{\alpha\alpha+1}(c_{n\alpha}) = c_{m\alpha+1} \Rightarrow (\text{Sat}_{N_\alpha}(\{\psi(c_{n\alpha})\}) \Leftrightarrow \text{Sat}_{N_{\alpha+1}}(\{\psi(c_{m\alpha+1})\})) \right).$$

2. For all $\alpha \leq \beta \leq \eta < \gamma < \omega_1$ the maps $j_{\alpha\beta}$, $j_{\beta\eta}$ and $j_{\alpha\eta}$ commute

$$\bigwedge_{n < \omega} \bigwedge_{m < \omega} \bigwedge_{l < \omega} \left((j_{\alpha\beta}(c_{n\alpha}) = c_{m\beta} \wedge j_{\beta\eta}(c_{m\beta}) = c_{l\eta}) \Rightarrow j_{\alpha\eta}(c_{n\alpha}) = c_{l\eta} \right).$$

3. For all $\alpha \leq \eta < \gamma$ the maps $j_{\alpha\eta}$, $j_{\eta\gamma}$ and $j_{\alpha\gamma}$ commute

$$\bigwedge_{n < \omega} \bigwedge_{m < \omega} \bigwedge_{\beta < \gamma} \left((j_{\alpha\eta}(c_{n\alpha}) = c_{m\eta} \wedge j_{\eta\gamma}(c_{m\eta}) = c_{\beta\gamma}) \Rightarrow j_{\alpha\gamma}(c_{n\alpha}) = c_{\beta\gamma} \right).$$

¹The interpretation of this predicate symbol will subsume the interpretation of \in_α by considering the formula $\text{Sat}_{N_\alpha}(\{x \in y\})$. However, it is convenient (just for notational simplicity) to have a special symbol to denote $\in \upharpoonright N_\alpha$.

Constants for critical points

1. For all $\alpha \leq \beta < \gamma$ the constant $\omega_1^{N_\alpha}(N_\beta)$ gives the ordinal $\omega_1^{N_\alpha}$ inside N_β

$$\left(\bigwedge_{n < \omega} \dot{j}_{\alpha\beta}(c_{n\alpha}) \neq \omega_1^{N_\alpha}(N_\beta) \right) \wedge \left(\bigwedge_{m < \omega} (c_{m\beta} \in \omega_1^{N_\alpha}(N_\beta) \Rightarrow \bigvee_{n < \omega} \dot{j}_{\alpha\beta}(c_{n\alpha}) = c_{m\beta}) \right) \\ \wedge \text{Sat}_{N_\beta}(\{\omega_1^{N_\alpha}(N_\beta) \in \text{Ord}\}).$$

Equality of sets

1. For all $\alpha < \gamma$ the set $\{j_{\alpha\alpha+1}(f)(\omega_1^{N_\alpha}) : f \in N_\alpha \wedge \text{dom } f = \omega_1^{N_\alpha}\}$ equals the sort $N_{\alpha+1}$

$$\bigwedge_{n < \omega} \bigvee_{m < \omega} \left(\text{Sat}_{N_\alpha}(\{c_{m\alpha} \text{ is a function with domain } \omega_1\}) \wedge \right. \\ \left. c_{n\alpha+1} = j_{\alpha\alpha+1}(c_{m\alpha})(\omega_1^{N_\alpha}(N_{\alpha+1})) \right).$$

Isomorphisms

1. The sort $N_{\alpha+1}$ is isomorphic to the ultrapower of N_α by G_α for all $\alpha < \gamma$

$$\bigwedge_{n < \omega} \bigwedge_{m < \omega} \left(c_{n\alpha+1} \in_{\alpha+1} c_{m\alpha+1} \Leftrightarrow \right. \\ \left(\bigvee_{p < \omega} \bigvee_{q < \omega} (j_{\alpha\alpha+1}(c_{p\alpha})(\omega_1^{N_\alpha}(N_{\alpha+1})) = c_{n\alpha+1} \wedge j_{\alpha\alpha+1}(c_{q\alpha})(\omega_1^{N_\alpha}(N_{\alpha+1})) = c_{m\alpha+1}) \right) \\ \wedge \bigvee_{l < \omega} \left(\dot{G}_\alpha(c_{l\alpha}) \wedge \bigwedge_{r < \omega} (c_{p\alpha}(c_{r\alpha}) = c_{q\alpha}(c_{r\alpha}) \Leftrightarrow c_{r\alpha} \in_\alpha c_{l\alpha}) \right) \Bigg).$$

2. The sort N_η is isomorphic to the direct limit of $(N_\alpha, j_{\alpha\beta} : \alpha \leq \beta < \eta)$ for all limit ordinals $\eta < \gamma$

$$\bigwedge_{n < \omega} \bigwedge_{m < \omega} \bigvee_{\alpha < \eta} \bigvee_{r < \omega} \bigvee_{s < \omega} \\ \left(c_{n\eta} = j_{\alpha\eta}(c_{r\alpha}) \wedge c_{m\eta} = j_{\alpha\eta}(c_{s\alpha}) \wedge (c_{n\eta} \in_\eta c_{m\eta} \Leftrightarrow c_{r\alpha} \in_\alpha c_{s\alpha}) \right).$$

3. The sort N_γ is isomorphic to the direct limit of $(N_\alpha, j_{\alpha\beta} : \alpha \leq \beta < \gamma)$

$$\bigwedge_{\beta < \gamma} \bigwedge_{\eta < \gamma} \bigvee_{\alpha < \gamma} \bigvee_{r < \omega} \bigvee_{s < \omega} \\ \left(c_{\beta\gamma} = j_{\alpha\gamma}(c_{r\alpha}) \wedge c_{\eta\gamma} = j_{\alpha\gamma}(c_{s\alpha}) \wedge (c_{\beta\gamma} \in_{\omega_1} c_{\eta\gamma} \Leftrightarrow c_{r\alpha} \in_\alpha c_{s\alpha}) \right).$$

Let $\mathcal{M} \models \psi_{\text{ite}(\gamma)}$. Let us argue the existence of an iteration $\mathcal{J}_\mathcal{M}$ of length γ derived from \mathcal{M} . For a precise definition of iteration see 6.4.1.

- Axiom 1 for the satisfaction predicate ensures Sat_{N_α} is correctly computed for all $\alpha < \gamma$.
- Axiom 1 for the generic filter ensures that $\dot{G}_\alpha^{\mathcal{M}}$ meets all dense sets in N_α for $\mathbf{NS}_{\omega_1}^{N_\alpha}$. Axiom 2 for the generic filter ensures it is a filter (this axiom requires of the constants given by axiom 1 for the critical points to be formulated).
- Axioms 1 for the elementary maps ensures that the maps $j_{\alpha\alpha+1}^{\mathcal{M}}$ are elementary for all $\alpha < \gamma$. Axiom 2 for the generic filter ensures $\dot{G}_\alpha^{\mathcal{M}}$ is the filter derived from $j_{\alpha\alpha+1}^{\mathcal{M}}$ by setting for $S \in (\mathbf{NS}_{\omega_1}^+)^{N_\alpha}$

$$S \in G_\alpha \Leftrightarrow \omega_1^{N_\alpha} \in j_{\alpha\alpha+1}(S).$$

Axiom 1 for the isomorphisms ensures $N_{\alpha+1}^{\mathcal{M}}$ is isomorphic to $\text{Ult}(N_\alpha, G_\alpha)$ (this axiom requires of axiom 1 for the equality of sets and the constants given by axiom 1 to be formulated).

- Axioms 2 and 3 for the isomorphisms together with axioms 2 and 3 for the commutativity of the maps ensure at all limit stages $\lambda \leq \gamma$ the sort N_λ is the direct limit of $(N_\alpha^{\mathcal{M}}, j_{\alpha\beta}^{\mathcal{M}} : \alpha \leq \beta < \lambda)$.

Let \mathcal{J} be an iteration of length γ of a countable model N . Then, modulo choosing a bijection for each iterate with ω (and with γ for the last sort), there is a unique $\mathcal{L}_{\text{ite}(\gamma)}$ -structure $\mathcal{M}_{\mathcal{J}}$ such that

$$\mathcal{M}_{\mathcal{J}} \models \psi_{\text{ite}(\gamma)}.$$

Furthermore, the iteration derived from $\mathcal{M}_{\mathcal{J}}$ is \mathcal{J} .

- ❖ Suppose $\gamma = \omega_1$. Let us discuss the satisfaction predicate for \dot{N}_{ω_1} . We have not introduced it in the above axiomatization for two reasons. First, its logical complexity does not match the one we will use in Definition 5.1.13. Second, it can be defined from the satisfaction predicates for \dot{N}_α , $\alpha < \omega_1$, since \dot{N}_{ω_1} is the direct limit of $(N_\alpha, j_{\alpha\beta} : \alpha \leq \beta < \omega_1)$. We denote by

$$\text{Sat}_{N_{\omega_1}}(\{\psi(c_{\beta\omega_1})\})$$

the sentence

$$\bigvee_{\alpha < \omega_1} \bigvee_{n < \omega} \left(j_{\alpha\omega_1}(c_{n\alpha}) = c_{\beta\omega_1} \wedge \text{Sat}_{N_\alpha}(\{\psi(c_{n\alpha})\}) \right).$$

Let us summarize the relevant information.

Lemma 3.2.5.

- Any model of $\psi_{\text{ite}(\gamma)}$ induces an iteration of length γ where the sorts correspond with the iterates,

- any iteration of length γ induces a model of $\psi_{ite(\gamma)}$ where the sorts correspond with the iterates,
- $\psi_{ite(\gamma)}$ is a conjunction of size $|\gamma|$ of sentences of complexity $\left(\bigwedge_{\infty}^{\omega}\right)$.

3.2.4 Models coded by trees

Example 3.2.6 (Models coded by the branches of a tree). Let T be a tree on $\omega \times \kappa$ for $\kappa \geq \omega_2$ a regular cardinal and $\langle (k_n, j_n) : n \in \omega \rangle$ be a bijection between ω and $\omega \times \omega$. Denote by \mathcal{L}_T the language with the following sorts, constants and predicates.

Sorts

- One sort \dot{N}_0 .
- One sort \dot{N}_{ω_1} .
- One sort $\{r, f\}$.

Constants

- Constants $\{c_{n0} : n < \omega\}$ of sort \dot{N}_0 .
- Constants $\{c_{\beta\omega_1} : \beta < \omega_1\}$ of sort \dot{N}_{ω_1} .
- Constants $\{\check{x} : x \in H_\kappa\}$ of sort \dot{N}_{ω_1} .

Predicates

1. Binary predicates \in_0 and \in_{ω_1} for the sorts \dot{N}_0 and \dot{N}_{ω_1} .
2. A binary predicate br_T of sort $(\dot{N}_{\omega_1}, \{r, f\}, \{r, f\})$.
3. A unary predicate \dot{T} of sort \dot{N}_{ω_1} for the tree T .
4. A satisfaction predicate

$$\text{Sat}_{N_0}(\{\psi(x_1, \dots, x_m)\})$$

as before.

5. A binary predicate symbol \sqsubseteq^2 of sort $(\dot{N}_{\omega_1}, \dot{N}_{\omega_1})$ to interpret the order relation on T .
6. A binary predicate \in^* of sort $(\dot{N}_{\omega_1}, \{r, f\})$ to interpret the extensions of r and f .

Denote by ψ_T the axiom given by the conjunction of the following axioms.

1. An axiom to ensure that H_κ^V sits as a transitive subclass of \dot{N}_{ω_1}

$$\psi_{H_\kappa} \text{ (from the previous example 3.2.2).}$$

2. An axiom to ensure that \dot{T} is interpreted by T :

$$\left(\bigwedge_{x \in T} \check{x} \in \dot{T} \right) \wedge \left(\bigwedge_{x \in H_\kappa \setminus T} \check{x} \notin \dot{T} \right).$$

3. (r, f) is a pair of infinite sequences:

$$\bigwedge_{x \in H_\kappa} [\check{x} \in^* \dot{r} \Leftrightarrow \bigvee_{n \in \omega, i \in 2} (\check{x} = \langle \check{n}, i \rangle \wedge \check{n} \in \check{\omega} \wedge \check{i} \in \check{2})]$$

$$\bigwedge_{x \in H_\kappa} [\check{x} \in^* \dot{f} \Leftrightarrow \bigvee_{y \in H_\kappa, n \in \omega} (\check{x} = \langle \check{n}, y \rangle \wedge \check{n} \in \check{\omega})]$$

$$\bigwedge_{n \in \omega, i, j \in 2} [(\langle \check{n}, i \rangle \in^* \dot{r} \wedge \langle \check{n}, j \rangle \in^* \dot{r}) \rightarrow \check{i} = \check{j}]$$

$$\bigwedge_{n \in \omega} (\langle \check{n}, 0 \rangle \in^* \dot{r} \vee \langle \check{n}, 1 \rangle \in^* \dot{r})$$

$$\bigwedge_{n \in \omega, i, j \in H_\kappa} [(\langle \check{n}, i \rangle \in^* \dot{f} \wedge \langle \check{n}, j \rangle \in^* \dot{f}) \rightarrow \check{i} = \check{j}]$$

$$\bigwedge_{n \in \omega} \bigvee_{y \in H_\kappa} (\langle \check{n}, y \rangle \in^* \dot{f})$$

$$\bigwedge_{\alpha < \omega_1} (c_\alpha \in \dot{r} \rightarrow \bigvee_{x \in H_\kappa} c_\alpha = \check{x})$$

$$\bigwedge_{\alpha < \omega_1} (c_\alpha \in \dot{f} \rightarrow \bigvee_{x \in H_\kappa} c_\alpha = \check{x}).$$

4. The pair (r, f) is a branch trough T

$$\bigwedge_{\langle s, t \rangle, \langle u, v \rangle \in T} \left((\text{br}_T(\langle \check{s}, t \rangle, \dot{r}, \dot{f}) \wedge \text{br}_T(\langle \check{u}, v \rangle, \dot{r}, \dot{f})) \Rightarrow (\langle \check{s}, t \rangle \sqsubseteq^2 \langle \check{u}, v \rangle \vee \langle \check{u}, v \rangle \sqsubseteq^2 \langle \check{s}, t \rangle) \right)$$

$$\bigwedge_{n < \omega} \bigvee_{\langle s, t \rangle \in T \cap (\omega \times \kappa)^n} \text{br}_T(\langle \check{s}, t \rangle, \dot{r}, \dot{f}).$$

5. The sort N_0 is isomorphic to the model coded by the real r

$$\bigwedge_{\langle s,t \rangle \in T} \bigwedge_{n \in \omega} \left(\text{br}_T(\{s, t\}, \dot{r}, \dot{f}) \wedge (s(\check{n}) = \check{1}) \right) \Rightarrow c_{k_n 0} \in_0 c_{j_n 0},$$

$$\bigwedge_{\langle s,t \rangle \in T} \bigwedge_{n \in \omega} \left(\text{br}_T(\{s, t\}, \dot{r}, \dot{f}) \wedge (\{s(\check{n}) = \check{0}\}) \right) \Rightarrow c_{k_n 0} \notin_0 c_{j_n 0}.$$

Let $\mathcal{M} \models \psi_T$. By the example 3.2.2 we have that H_κ sits as a transitive subclass of the interpretation of the sort \dot{N}_{ω_1} in \mathcal{M} . The axioms 3 and 4 ensure that the pair (r, f) given by the sort $\{r, f\}$ gives a branch through T . The two axioms 5 for the sort \dot{N}_0 ensure that the structure

$$(\{c_{n0}^{\mathcal{M}} : n < \omega\}, \in_0^{\mathcal{M}})$$

is isomorphic to the structure

$$(\omega, E_r)$$

where $E_r = \{(k_n, j_n) \in \omega \times \omega : n < \omega \wedge r(n) = 1\}$.

Conversely, whenever we have a tree $T \in H_\kappa$ and a branch $(r, f) \in [T]$, we obtain a model of ψ_T by interpreting the sort \dot{N}_0 as the structure coded by r , interpreting $\{r, f\}$ as $\{r, f\}$ and interpreting \dot{N}_{ω_1} as a transitive superset of H_κ^V .

Part II

The ASK-property and iterations of consistency properties

Chapter 4

Iterations of consistency properties

4.1 Forcing notions as consistency properties

By the results of Section 1.4 a consistency property S for $\mathcal{L}_{\kappa\omega}$ can be naturally seen as a forcing notion \mathbb{P}_S ; then, using the forcing machinery on \mathbb{P}_S , we can produce a Boolean valued model with the mixing property of $\bigwedge p$ for any $p \in S$. In this section we show that it is possible to go the other way round: we prove that any forcing notion \mathbb{P} has a consistency property $S_{\mathbb{P}}$ associated to it, so that it is equivalent to force with \mathbb{P} or with $\mathbb{P}_{S_{\mathbb{P}}}$.

Given a complete Boolean algebra \mathbf{B} , we show that for some regular κ large enough in V , \mathbf{B} is forcing equivalent to a consistency property describing the \in -theory of H_κ as computed in a V -generic extension by \mathbf{B} .

First, Lemma 4.1.1 establishes that given a \mathbf{B} -name in H_κ^V its interpretation in the generic extension will remain in $H_\kappa^{V[G]}$ as long as κ is not collapsed and G is V -generic for \mathbf{B} . Second, Lemma 4.1.3 shows the converse. Under some conditions, we prove that if the interpretation of a \mathbf{B} -name $\dot{\sigma}$ is in $H_\kappa^{V[G]}$, then we can suppose that $\dot{\sigma}$ is already in H_κ^V .

Let \mathbb{P} be a forcing notion and let $\dot{\kappa} \in V^{\mathbb{P}}$ be a \mathbb{P} -name such that

$$\Vdash_{\mathbb{P}} \dot{\kappa} \text{ is a cardinal,}$$

$H_{\dot{\kappa}}$ is some \mathbb{P} -name such that

$$\Vdash_{\mathbb{P}} H_{\dot{\kappa}} \text{ is the set of all elements whose transitive closure has size less than } \dot{\kappa}.$$

We now recall that we have a canonical name for $H_{\dot{\kappa}}$ whenever $\kappa > |\mathbf{B}|$.

Lemma 4.1.1. *Let κ be a regular cardinal and let \mathbb{P} be a forcing notion that does not collapse κ . Then for any \mathbb{P} -name $\dot{\sigma} \in H_\kappa \cap V^{\mathbb{P}}$ we have that*

$$\Vdash_{\mathbb{P}} \dot{\sigma} \in H_{\dot{\kappa}}.$$

Proof. We proceed by induction on the rank of $\dot{\sigma}$. The result holds true for $\text{rank}(\dot{\sigma}) = 0$ since \emptyset belongs to H_κ . Suppose the result holds for any \mathbb{P} -name $\dot{\tau}$ of rank strictly less than $\text{rank}(\dot{\sigma})$. Then for any $\dot{\tau} \in \text{dom}(\dot{\sigma})$, the induction hypothesis ensures

$$\Vdash_{\mathbb{P}} \dot{\tau} \in H_{\check{\kappa}}.$$

This proves $\Vdash_{\mathbb{P}} \dot{\sigma} \subset H_{\check{\kappa}}$. Because κ is not collapsed and $|\dot{\sigma}| < \kappa$, we have that $\Vdash_{\mathbb{P}} |\dot{\sigma}| < \check{\kappa}$. Recall that for regular cardinals, $x \in H_\kappa$ if and only if $x \subset H_\kappa$ and $|x| < \kappa$. Then by stability of $\Vdash_{\mathbb{P}}$ under logical deduction

$$\Vdash_{\mathbb{P}} \dot{\sigma} \subset H_{\check{\kappa}} \quad \text{and} \quad \Vdash_{\mathbb{P}} |\dot{\sigma}| < \check{\kappa}$$

imply

$$\Vdash_{\mathbb{P}} \dot{\sigma} \in H_{\check{\kappa}}.$$

□

We need a way to approximate the transitive closure of any $x \in H_\kappa$ with a sequence of length $< \kappa$.

Fact 4.1.2. *Let $x \in H_\kappa$ with κ a regular cardinal. There exist $\lambda < \kappa$ and $(x_\alpha)_{\alpha \leq \lambda} \subset H_\kappa$ such that $x_\alpha \subset \{x_\beta : \beta < \alpha\}$ and $x_\lambda = x$.*

Proof. We proceed by induction on the rank of x . The base case holds by setting $x_0 = \emptyset$. Let $x \in H_\kappa$. The induction hypothesis ensures the result holds for any element of rank less than $\text{rank}(x)$. In particular it holds for each $y \in x$. Then there exist $(y_\alpha)_{\alpha \leq \lambda_y}$ for each $y \in x$ verifying all the conditions from the statement. We have $\lambda = \sum_{y \in x} \lambda_y < \kappa$ by regularity of κ . Consider $(x_\alpha)_{\alpha < \lambda}$ the concatenation of all these sequences (no matter the order in which they are glued) and set $x_\lambda = x$. □

The following appears in [3].

Proposition 4.1.3. *Let κ be a regular cardinal and let $\mathbb{P} \subset H_\kappa$ be a forcing notion with the $< \kappa$ -cc. If $p \in \mathbb{P}$ and $\dot{\tau}$ is a \mathbb{P} -name such that $p \Vdash \dot{\tau} \in H_{\check{\kappa}}$, then there exists $\dot{\sigma} \in H_\kappa$ such that $p \Vdash \dot{\sigma} = \dot{\tau}$.*

Proof. First, by the $< \kappa$ -c.c., κ is not collapsed and it makes sense to talk about H_κ in the generic extension by \mathbb{P} .

By Fact 4.1.2 and stability under logical deduction,

$$p \Vdash \exists \lambda \exists \pi (\lambda < \check{\kappa} \wedge \pi : \lambda + 1 \rightarrow H_{\check{\kappa}} \wedge \pi(\lambda) = \dot{\tau} \wedge \forall \alpha < \lambda \pi(\alpha) \subset \{\pi(\beta) : \beta < \alpha\}).$$

By the Maximality Lemma we can find \mathbb{P} -names $\dot{\lambda}$ and $\dot{\pi}$ such that

$$p \Vdash \dot{\lambda} < \check{\kappa} \wedge \dot{\pi} : \dot{\lambda} + 1 \rightarrow H_{\check{\kappa}} \wedge \dot{\pi}(\dot{\lambda}) = \dot{\tau} \wedge \forall \alpha < \dot{\lambda} \dot{\pi}(\alpha) \subset \{\dot{\pi}(\beta) : \beta < \dot{\lambda}\}.$$

Let us argue that by the $< \kappa$ -c.c. we can replace $\dot{\lambda}$ by a canonical \mathbb{P} -name $\check{\lambda}$ for some ordinal $\lambda < \kappa$. It is enough to see that for some $\lambda < \kappa$, we have that $p \Vdash \dot{\lambda} < \check{\lambda}$, since we can then define $\dot{\pi}(\alpha) = \dot{\pi}(\check{\lambda})$ for any $\dot{\lambda} \leq \alpha < \check{\lambda}$. Suppose that for no $\lambda < \kappa$, $p \Vdash \dot{\lambda} < \check{\lambda}$. Then for each $\lambda < \kappa$ there exists $\delta > \lambda$ and $q_\lambda \leq p$ such that $q_\lambda \Vdash \dot{\lambda} = \check{\delta}$. Then $\{q_\lambda : \lambda < \kappa\}$ is an antichain of size κ , a contradiction. Therefore we can fix $\lambda < \kappa$ such that

$$p \Vdash \check{\lambda} < \check{\kappa} \wedge \dot{\pi} : \check{\lambda} + 1 \rightarrow H_{\check{\kappa}} \wedge \dot{\pi}(\check{\lambda}) = \dot{\tau} \wedge \forall \alpha < \check{\lambda} \dot{\pi}(\alpha) \subset \{\dot{\pi}(\beta) : \beta < \check{\lambda}\}.$$

We define a sequence $(\dot{\sigma}_\alpha)_{\alpha \leq \lambda}$ of \mathbb{P} -names by induction on α . It will be contained in H_κ and satisfy $p \Vdash \dot{\sigma}_\alpha = \dot{\pi}(\check{\alpha})$ for all $\alpha \leq \lambda$. Since $p \Vdash \dot{\pi}(\check{\lambda}) = \dot{\tau}$, $\dot{\sigma}_\lambda$ will be the desired \mathbb{P} -name. The definition is by induction on α . Suppose the sequence has been defined for any $\beta < \alpha$ and set

$$\dot{\sigma}_\alpha = \{(\dot{\sigma}_\beta, q) : \beta < \lambda \wedge q \leq p \wedge q \Vdash \dot{\pi}(\beta) \in \dot{\pi}(\alpha)\}.$$

- ❖ We first check $\dot{\sigma}_\alpha \in H_\kappa$. Suppose the result holds for all $\beta < \alpha$. Consider $(\dot{\sigma}_\beta, q) \in \dot{\sigma}_\alpha$. By hypothesis, $\dot{\sigma}_\beta \in H_\kappa$ and since $q \in \mathbb{P} \subset H_\kappa$, $(\dot{\sigma}_\beta, q) \in H_\kappa$ and $\dot{\sigma}_\alpha \subset H_\kappa$. We also have $\dot{\sigma}_\alpha \subset \bigcup_{\beta < \lambda} \{\dot{\sigma}_\beta\} \times \mathbb{P}$, this yields $|\dot{\sigma}_\alpha| < \kappa$. Hence, $\dot{\sigma}_\alpha \in H_\kappa$.
- ❖ We now prove $p \Vdash \dot{\pi}(\alpha) = \dot{\sigma}_\alpha$ for all $\alpha \leq \lambda$. Suppose by induction that the result has been proven for all $\beta < \alpha$ and let's do it for α . Let G be a V -generic filter containing p . By hypothesis $(\dot{\sigma}_\beta)_G = \dot{\pi}_G(\beta)$ for all $\beta < \alpha$.

- We first prove $(\dot{\sigma}_\alpha)_G \subset \dot{\pi}_G(\alpha)$. Consider $y \in (\dot{\sigma}_\alpha)_G$. There exists $(\dot{\sigma}_\beta, q) \in \dot{\sigma}_\alpha$ and $q \in G$ such that $y = (\dot{\sigma}_\beta)_G$, $\beta < \alpha$, and $q \Vdash \dot{\pi}(\beta) \in \dot{\pi}(\alpha)$. By hypothesis $(\dot{\sigma}_\beta)_G = \dot{\pi}_G(\beta)$. Since $q \in G$ and $q \Vdash \dot{\pi}(\beta) \in \dot{\pi}(\alpha)$, we have that

$$y = (\dot{\sigma}_\beta)_G = \dot{\pi}_G(\beta) \in \dot{\pi}_G(\alpha).$$

- It remains to check $\dot{\pi}_G(\alpha) \subset (\dot{\sigma}_\alpha)_G$. Consider $y \in \dot{\pi}_G(\alpha)$. Since $p \Vdash \dot{\pi}(\alpha) \subset \{\dot{\pi}(\beta) : \beta < \alpha\}$ and $p \in G$, we can find $\beta < \alpha$ such that $y = \dot{\pi}_G(\beta)$. Since $\dot{\pi}_G(\alpha) \in \dot{\pi}_G(\beta)$, there exists $q \leq p$ in G such that $q \Vdash \dot{\pi}(\beta) \in \dot{\pi}(\alpha)$. By definition of $\dot{\sigma}_\alpha$, $(\dot{\sigma}_\beta, q) \in \dot{\sigma}_\alpha$. By induction hypothesis $\dot{\pi}_G(\beta) = (\dot{\sigma}_\beta)_G$ and we have that

$$y = \dot{\pi}_G(\beta) = (\dot{\sigma}_\beta)_G \in (\dot{\sigma}_\alpha)_G.$$

□

We are now ready to define the consistency property associated to an arbitrary forcing notion. Fix \mathbf{B} a cba and let κ denote its cardinality. In order to ensure $\mathbf{B} \in H_{\kappa^+}$ it remains to check $\mathbf{B} \subset H_{\kappa^+}$. Wlog this holds assuming the domain of \mathbf{B} is κ . By means of Proposition 4.1.3 we now restrict our attention to \mathbf{B} -names in H_{κ^+} in order to define a consistency property which we will show to be equivalent to \mathbf{B} . We need beforehand to extend the forcing relation to formulae of infinitary logic.

Definition 4.1.4. Let \mathbf{B} be a complete Boolean algebra, let $\phi(v_1, \dots, v_n)$ be an \in -formula for $\mathcal{L}_{\infty\omega}$ and let $\tau_1, \dots, \tau_n \in V^{\mathbf{B}}$. Denote by

$$\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}$$

the \mathbf{B} -value of $\phi(\tau_1, \dots, \tau_n)$ in the Boolean valued model $V^{\mathbf{B}}$ for first order formulae. The definition of $\llbracket \phi(\tau_1, \dots, \tau_n) \rrbracket_{\mathbf{B}}$ for $\mathcal{L}_{\infty\omega}$ is by induction on the complexity of ϕ . For first order formulae we maintain the same definition and we extend it to all $\mathcal{L}_{\infty\omega}$ according to Definition 1.2.2.

- Let $\bigwedge \Phi$ be an infinite formula with parameters in $V^{\mathbf{B}}$. Then $\llbracket \bigwedge \Phi \rrbracket_{\mathbf{B}} = \bigwedge_{\phi \in \Phi} \llbracket \phi \rrbracket_{\mathbf{B}}$ and $c \Vdash \bigwedge \Phi$ if and only if $c \Vdash \phi$ for every $\phi \in \Phi$.
- Let $\bigvee \Phi$ be an infinite formula with parameters in $V^{\mathbf{B}}$. Then $\llbracket \bigvee \Phi \rrbracket_{\mathbf{B}} = \bigvee_{\phi \in \Phi} \llbracket \phi \rrbracket_{\mathbf{B}}$ and $c \Vdash \bigvee \Phi$ if and only if the set $\{b \in \mathbf{B} : b \Vdash \phi \text{ for some } \phi \in \Phi\}$ is dense below c .

Let \mathbf{B} be a complete Boolean algebra of cardinality κ . let \mathcal{L} be the language $\{\in\}$ and \mathcal{C} be the set of constants $V^{\mathbf{B}} \cap H_{\kappa^+}$. We use $\phi^{H_{\kappa^+}}$ to denote that all quantifiers from ϕ are restricted to H_{κ^+} .

Theorem 4.1.5. *For any complete Boolean algebra \mathbf{B} of size less or equal than κ and any regular cardinal λ the following holds:*

- (i) $S_{\mathbf{B}} = \{s \in [(\mathcal{C})_{\lambda\omega}]^{<\omega} : \llbracket (\bigwedge s)^{H_{\kappa^+}} \rrbracket_{\mathbf{B}} > 0_{\mathbf{B}}\}$ is a consistency property,
- (ii) the map

$$\begin{aligned} \pi_{\mathbf{B}} : (S_{\mathbf{B}}, \leq) &\rightarrow (\mathbf{B}^+, \leq_{\mathbf{B}}) \\ s &\mapsto \llbracket (\bigwedge s)^{H_{\kappa^+}} \rrbracket_{\mathbf{B}} \end{aligned}$$

is a dense embedding. In particular \mathbf{B} and $S_{\mathbf{B}}$ are equivalent forcing notions.

Proof. We first prove (ii).

- If $p \leq q$, then $q \subseteq p$ and $\pi(p) = \llbracket \bigwedge p \rrbracket_{\mathbf{B}} \leq \llbracket \bigwedge q \rrbracket_{\mathbf{B}} = \pi(q)$.
- We have $p \perp q \Leftrightarrow p \cup q \notin S_{\mathbf{B}} \Leftrightarrow \llbracket \bigwedge (p \cup q) \rrbracket_{\mathbf{B}} = 0_{\mathbf{B}} \Leftrightarrow \llbracket \bigwedge p \rrbracket_{\mathbf{B}} \wedge \llbracket \bigwedge q \rrbracket_{\mathbf{B}} = 0_{\mathbf{B}} \Leftrightarrow \pi(p) \perp \pi(q)$.
- Let $\dot{G} = \{(\check{b}, b) : b \in \mathbf{B}\}$ be the canonical \mathbf{B} -name for a V -generic filter. Since for any $b \in \mathbf{B}^+$

$$\pi(\{\check{b} \in \dot{G}\}) = \llbracket \check{b} \in \dot{G} \rrbracket_{\mathbf{B}} = b,$$

the map π is surjective; in particular $\pi[S_{\mathbf{B}}]$ is dense in \mathbf{B}^+ .

Now we prove (i). We have to check that $S_{\mathbf{B}}$ satisfies the clauses of Definition 1.3.2. By choice of κ we have that

$$\llbracket \forall v \in H_{\kappa^+} \phi^{H_{\kappa^+}}(v) \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}} = \bigwedge_{\tau \in \mathcal{C}} \llbracket \phi^{H_{\kappa^+}}(v) \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}.$$

For notational simplicity we use $\llbracket \phi \rrbracket_{\mathbf{B}}$ instead of $\llbracket \phi^{H_{\kappa^+}} \rrbracket_{\mathbf{B}}^{V^{\mathbf{B}}}$. Note also that in the proof below we will only be interested in formulae where quantifiers range over (and constants belong to) $H_{\kappa^+} \cap V^{\mathbf{B}}$.

(Con) Consider $s \in S_{\mathbb{B}}$ and $\phi \in \mathcal{L}(\mathcal{C})_{\infty\omega}$. If ϕ and $\neg\phi$ are both in s , $\llbracket \bigwedge s \rrbracket_{\mathbb{B}} \leq \llbracket \phi \wedge \neg\phi \rrbracket_{\mathbb{B}} = 0_{\mathbb{B}}$, a contradiction since $\llbracket \bigwedge s \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$. Then for any ϕ , either $\phi \notin s$ or $\neg\phi \notin s$.

(Ind.1) Consider $s \in S_{\mathbb{B}}$ and $\neg\phi \in s$. Since $\llbracket \neg\phi \rrbracket_{\mathbb{B}} = \llbracket \phi \neg \rrbracket_{\mathbb{B}}$, $\llbracket \bigwedge (s \cup \{\phi \neg\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$ and $s \cup \{\phi \neg\} \in S_{\mathbb{B}}$.

(Ind.2) Consider $s \in S_{\mathbb{B}}$ and $\bigwedge \Phi \in s$. For any $\phi \in \Phi$, $\llbracket \bigwedge (s \cup \{\phi\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$ and $s \cup \{\phi\} \in S_{\mathbb{B}}$.

(Ind.3) Consider $s \in S_{\mathbb{B}}$, $\forall v\phi(v) \in s$ and $\tau \in \mathcal{C}$. We have

$$\llbracket \forall v\phi(v) \rrbracket_{\mathbb{B}} = \bigwedge_{\sigma \in V^{\mathbb{B}} \cap H_{\kappa^+}} \llbracket \phi(\sigma) \rrbracket_{\mathbb{B}} \leq \llbracket \phi(\tau) \rrbracket_{\mathbb{B}}.$$

Therefore

$$\llbracket \bigwedge (s \cup \{\phi(\tau)\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}},$$

and $s \cup \{\phi(\tau)\} \in S_{\mathbb{B}}$.

(Ind.4) Consider $s \in S_{\mathbb{B}}$ and $\bigvee \Phi \in s$. Suppose that for no $\phi \in \Phi$, $s \cup \{\phi\} \in S_{\mathbb{B}}$. Then for any $\phi \in \Phi$,

$$\llbracket \bigwedge (s \cup \{\phi\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} \wedge \llbracket \phi \rrbracket_{\mathbb{B}} = 0_{\mathbb{B}}.$$

Therefore $\llbracket \bigwedge s \rrbracket_{\mathbb{B}} \leq \llbracket \neg\phi \rrbracket_{\mathbb{B}}$ for any $\phi \in \Phi$. Since $\bigwedge_{\phi \in \Phi} \llbracket \neg\phi \rrbracket_{\mathbb{B}}$ is the greatest lower bound of $\{\llbracket \neg\phi \rrbracket_{\mathbb{B}} : \phi \in \Phi\}$, we have

$$\llbracket \bigwedge s \rrbracket_{\mathbb{B}} \leq \bigwedge_{\phi \in \Phi} \llbracket \neg\phi \rrbracket_{\mathbb{B}} = \llbracket \neg \bigvee \Phi \rrbracket_{\mathbb{B}}.$$

Then $\llbracket \bigwedge (s \cup \{\neg \bigvee \Phi\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} > 0_{\mathbb{B}}$, but since $\bigvee \Phi$ and $\neg \bigvee \Phi$ are both in $s \cup \{\neg \bigvee \Phi\}$, we have that $\llbracket \bigwedge (s \cup \{\neg \bigvee \Phi\}) \rrbracket_{\mathbb{B}} = 0_{\mathbb{B}}$, a contradiction.

(Ind.5) Consider $s \in S_{\mathbb{B}}$ and $\exists v\phi(v) \in s$. Suppose that for no $\tau \in \mathcal{C}$, $s \cup \{\phi(\tau)\} \in S_{\mathbb{B}}$. Then for any $\tau \in \mathcal{C}$,

$$\llbracket \bigwedge (s \cup \{\phi(\tau)\}) \rrbracket_{\mathbb{B}} = \llbracket \bigwedge s \rrbracket_{\mathbb{B}} \wedge \llbracket \phi(\tau) \rrbracket_{\mathbb{B}} = 0_{\mathbb{B}}.$$

This gives that

$$\llbracket \bigwedge s \rrbracket_{\mathbb{B}} \leq \llbracket \neg\phi(\tau) \rrbracket_{\mathbb{B}}$$

for any $\tau \in V^{\mathbb{B}} \cap H_{\kappa^+}$. Therefore

$$\llbracket \bigwedge s \rrbracket_{\mathbf{B}} \leq \bigwedge_{\tau \in V^{\mathbf{B}} \cap H_{\kappa^+}} \llbracket \neg\phi(\tau) \rrbracket_{\mathbf{B}} = \llbracket \forall v \neg\phi(v) \rrbracket_{\mathbf{B}} = \llbracket \neg\exists v \phi(v) \rrbracket_{\mathbf{B}}.$$

The equality

$$\bigwedge_{\tau \in V^{\mathbf{B}} \cap H_{\kappa^+}} \llbracket \neg\phi(\tau) \rrbracket_{\mathbf{B}} = \llbracket \forall v \neg\phi(v) \rrbracket_{\mathbf{B}}$$

holds since the quantifiers from ϕ are restricted to H_{κ^+} . Therefore

$$\llbracket \bigwedge (s \cup \{\neg\exists v \phi(v)\}) \rrbracket_{\mathbf{B}} = \llbracket \bigwedge s \rrbracket_{\mathbf{B}} > 0_{\mathbf{B}}.$$

But $\exists v \phi(v)$ and $\neg\exists v \phi(v)$ are both in $s \cup \{\neg\exists v \phi(v)\}$, hence

$$\llbracket \bigwedge (s \cup \{\neg\exists v \phi(v)\}) \rrbracket_{\mathbf{B}} = 0_{\mathbf{B}}.$$

We reached a contradiction.

- (Str.1)** Suppose $s \in S_{\mathbf{B}}$ and $\tau = \sigma \in s$; since \mathbf{B} -valued models for set theory verify $\llbracket \tau = \sigma \rrbracket_{\mathbf{B}} = \llbracket \sigma = \tau \rrbracket_{\mathbf{B}}$, $\llbracket \bigwedge (s \cup \{\sigma = \tau\}) \rrbracket_{\mathbf{B}} = \llbracket \bigwedge s \rrbracket_{\mathbf{B}} > 0_{\mathbf{B}}$ and $s \cup \{\sigma = \tau\} \in S_{\mathbf{B}}$.
- (Str.2)** Suppose $s \in S_{\mathbf{B}}$ and $\{\sigma = \tau, \phi(\tau)\} \subset s$. We have $\llbracket \sigma = \tau \rrbracket_{\mathbf{B}} \wedge \llbracket \phi(\tau) \rrbracket_{\mathbf{B}} \leq \llbracket \phi(\sigma) \rrbracket_{\mathbf{B}}$; therefore $\llbracket \bigwedge (s \cup \{\phi(\sigma)\}) \rrbracket_{\mathbf{B}} > 0_{\mathbf{B}}$ and $s \cup \{\phi(\sigma)\} \in S_{\mathbf{B}}$.
- (Str.3)** This condition is a vacuous since $\mathcal{L} = \{\in\}$ has no constant symbol.

□

Note that the only formulae one needs to keep in $S_{\mathbf{B}}$ in order to ensure that there is a dense embedding between both forcing notions are $\check{b} \in \dot{G}$. This is because in order to prove that the embedding has dense image, one only uses that the \mathbf{B} -value of $\check{b} \in \dot{G}$ is b . In particular one can consider various choices of constants \mathcal{C} to produce the desired consistency property $S_{\mathbf{B}}$, other than the one we made.

4.2 Complexity classes for infinitary logics

In this section we present two classes of $\mathcal{L}_{\infty\omega}$ -formulae based on their complexity

$$\left(\bigwedge_{\omega} \right) \text{ and } \bigwedge \bigvee.$$

The first one contains $\mathcal{L}_{\infty\omega}$ -formulae whose conjunctions are of countable size and whose disjunctions are of arbitrary size. The second one contains $\mathcal{L}_{\infty\omega}$ -formulae of complexity $\bigwedge \bigvee$. By this we mean formulae starting with one infinitary conjunction followed by one infinitary disjunction of quantifier free $\mathcal{L}_{\omega_1\omega}$ -formulae. All of this is formalized in Definition 4.2.1.

✱ The definition of the class $(\bigwedge_{\infty}^{\omega})$ is motivated by the study of realizability of infinitary formulae in the context of elementary embeddings

$$j : V \rightarrow W$$

with uncountable critical point existing in some generic extension $V[G]$ of V containing W as a transitive subclass. Why such elementary embeddings are important for our purposes will become transparent in Section 5.1.13 when we introduce the ASK-property. Let us argue why this class of formulae is the canonical one for dealing with such embeddings.

Assume $\Psi \in V$ is an infinitary formula in V and in $V[G]$ there is a model for Ψ . Now let $j : V \rightarrow W$ be elementary and consider the formula $j(\Psi) \in W$. Can we find a model for $j(\Psi)$ in $V[G]$?

Realizing a conjunction $\Psi := \bigwedge_{i \in I} \psi_i$ requires realizing **every** conjunct ψ_i . If I is countable in V , then $j(I) = j[I]$ remains countable in W as the critical point of j is uncountable. Working in $V[G]$, if we wish to find a model of $j(\Psi)$ assuming there is a model of Ψ , it will be easier to find it if we need to consider only the countably many conjuncts indexed by $j(I) = j[I]$, rather than if we had to realize a conjunction indexed by $j(I)$ of size strictly bigger than $j[I]$.

For disjunctions the situation is opposite. Realizing a disjunction $\Psi := \bigvee_{i \in I} \psi_i$ requires realizing just **one** disjunct. If the disjunction is realized in V , by pushing the realization of the disjunct ψ_k witnessing the consistency of Ψ to the realization of $j(\psi_k)$, one is able to realize the formula $j(\Psi)$ no matter how big $j(I)$ becomes in W . This idea is formalized in the Preservation Lemma 4.2.3.

✱ The class $\bigwedge \bigvee$ is of interest because of its nice syntactic properties. Suppose S is a consistency property and $\{\psi\}$ belongs to S . If G is V -generic for S and contains $\{\psi\}$, then in $V[G]$ there exists a model of ψ . For ψ of arbitrary complexity the consistency property S needs to contain sets of formulae with (almost) the same complexity as ψ . However, in case ψ has complexity $\bigwedge \bigvee$, one only needs to deal with consistency properties containing $\mathcal{L}_{\omega_1\omega}$ -sentences. This idea is formalized in the Realization Lemma 4.2.4.

Through the rest of this chapter a particular family of consistency properties will be considered. Let ψ be an $\mathcal{L}_{\infty\omega}$ -sentence. The consistency property S_ψ contains finite sets of sub-sentences of ψ that are Boolean consistent with ψ . For this set not to be empty, one already needs ψ to be Boolean consistent. This will not be an issue since the goal here is not to produce models of ψ , but to **omit or realize certain types** in models of ψ .

Definition 4.2.1 (Complexity).

✱ An $\mathcal{L}_{\infty\omega}$ -formula ψ has complexity $(\bigwedge_{\infty}^{\omega})$ if:

- negation symbols occur only in its subformulas which are negation of atomic formulas,
- its subsentences of the form

$$\bigwedge_{i \in I} \phi_i$$

are such that I is at most countable.

✱ An $\mathcal{L}_{\infty\omega}$ -formula ψ has complexity $\bigwedge \bigvee$ if it is of the form

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{ij}$$

with each ψ_{ij} in $\mathcal{L}_{\omega_1\omega}$.

$(\bigwedge_{\infty}^{\omega})$ contains $\mathcal{L}_{\omega_1\omega}$, is closed under countable conjunctions and arbitrary disjunctions, but is not closed under negation. Formulas of complexity $\bigwedge \bigvee$ are infinitary conjunctions of formulas of type $(\bigwedge_{\infty}^{\omega})$. Nonetheless, we maintain a distinct notation for both types of formulas, since their *raison d'être* is not the same.

The following will play a crucial role in arguments to follow.

Fact 4.2.2. *The sentences of Examples 3.2.2, 3.2.4, 3.2.6 are all infinitary conjunctions of sentences of type $(\bigwedge_{\infty}^{\omega})$.*

4.2.1 Complexity with respect to elementary embeddings

Lemma 4.2.3 (Preservation). *Suppose that*

- $V[G]$ is a forcing extension of V ,
- in $V[G]$ there exists an elementary embedding $j : V \rightarrow W$ with critical point ω_1 ,
- \mathcal{L} is a language in V ,
- $\mathcal{M} \in V[G]$ is an \mathcal{L} -structure,
- $\mathcal{N} \in V[G]$ is a $j(\mathcal{L})$ -structure and
- for atomic \mathcal{L} -sentences ϕ , we have that

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models j(\phi).$$

Then

- for $\mathcal{L}_{\omega_1\omega}$ -formulae ϕ , we have that

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models j(\phi);$$

- for $\mathcal{L}_{\infty\omega}$ -formulae ϕ of complexity $(\bigwedge_{\infty}^{\omega})$, we have that

$$\mathcal{M} \models \phi \Rightarrow \mathcal{N} \models j(\phi).$$

Proof. We prove the both statements by induction on the complexity of formulae.

ϕ is an $\mathcal{L}_{\omega_1\omega}$ -formula:

ϕ **atomic or negated atomic:** Apply the hypothesis.

\neg : Suppose the result true for an $\mathcal{L}_{\omega_1\omega}$ -formula ϕ and let us prove it for $\neg\phi$:

$$\begin{aligned} \mathcal{M} \models \neg\phi &\Leftrightarrow \\ \mathcal{M} \not\models \phi &\Leftrightarrow \\ \mathcal{N} \not\models j(\phi) &\Leftrightarrow \\ \mathcal{N} \models \neg j(\phi) &\Leftrightarrow \\ \mathcal{N} \models j(\neg\phi). & \end{aligned}$$

$\Phi = \bigwedge_{n<\omega} \phi_n$: Suppose by induction the result true for each ϕ_n . Then

$$\begin{aligned} \mathcal{M} \models \Phi &\Leftrightarrow \\ \forall n < \omega \ \mathcal{M} \models \phi_n &\Leftrightarrow \\ \forall n < \omega \ \mathcal{N} \models j(\phi_n) &\Leftrightarrow \\ \mathcal{N} \models \bigwedge_{n<\omega} j(\phi_n) &\Leftrightarrow \\ \mathcal{N} \models j(\Phi). & \end{aligned}$$

The last equivalence requires that $j(\Phi) = \bigwedge_{n<\omega} j(\phi_n)$ holds. This is the case since the critical point of j is ω_1 .

ϕ is of complexity $(\bigwedge_{\infty}^{\omega})$:

$\mathcal{L}_{\omega_1\omega}$ -formulas: We just proved something stronger.

\neg : We do not need to consider the case of negations, since formulas of complexity $(\bigwedge_{\infty}^{\omega})$ only have negations occurring inside $\mathcal{L}_{\omega_1\omega}$ -formulas, which is a case already covered.

$\Phi = \bigwedge_{n<\omega} \phi_n$: Assume the result holds true for each ϕ_n . Then we can replicate the proof from 4.2.1 above for $\mathcal{L}_{\omega_1\omega}$ -formulae ϕ . In this case we only need the left to right implication in moving from the second to the third line, and this implication holds true by the inductive assumptions.

$\Phi = \bigvee_{i \in I} \phi_i$: Assume the result holds true for each ϕ_i . Then

$$\begin{aligned} \mathcal{M} \models \Phi &\Rightarrow \\ \exists i \in I \ \mathcal{M} \models \phi_i &\Rightarrow \\ \exists i \in I \ \mathcal{N} \models j(\phi_i) &\Rightarrow \\ \mathcal{N} \models j(\Phi). & \end{aligned}$$

□

4.2.2 Complexity with respect to consistency properties

Given a consistency property S in signature \mathcal{L} and a maximal filter H on S , let

$$\Sigma_H = \{\psi \in \mathcal{L}_{\infty\omega} : \exists s \in H \psi \in s\}$$

and \mathcal{M}_H denote the $\mathcal{L}_{\infty\omega}$ -structure given by the term model induced by the atomic $\phi \in H$ as in Section 1.4.

Suppose ψ is a sentence, S is a consistency property and $G \subset S$ is a V -generic filter. The easiest way to ensure $\mathcal{M}_G \models \psi$ is to have $\{\psi\} \in G$ as argued in the Model Existence Theorem. Nonetheless, this much is not needed. We now address the question:

How far can the complexity of a formula ψ be with respect to that of formulae appearing in the elements of S in order to ensure that a V -generic filter for S produces a model of ψ ?

The next result ensures that whenever we wish to force a formula of the form

$$\psi := \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{i,j}$$

with each $\psi_{i,j}$ an $\mathcal{L}_{\omega_1\omega}$ -formula, whether or not S forces ψ is decided by the finite subsets of $\{\psi_{i,j} : i \in I, j \in J_i\}$ that are in S . This means **two levels of complexity** below that of ψ . Actually, the next lemma shows that we can take any fragment of $\mathcal{L}_{\infty\omega}$ as building blocks instead of $\mathcal{L}_{\omega_1\omega}$.

Lemma 4.2.4. *Let \mathcal{L} be a language and*

$$\psi := \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{i,j}$$

be an $\mathcal{L}_{\infty\omega}$ -sentence with the $\psi_{i,j}$ being arbitrary $\mathcal{L}_{\infty\omega}$ -sentences. Suppose S is a consistency property such that for every $i \in I$ the set

$$D_i = \{s \in S : \exists j \in J_i \psi_{i,j} \in s\}$$

is predense in S . Then for any V -generic filter $G \subset S$, we have that

$$\mathcal{M}_G \models \psi.$$

Proof. Fix $G \subset S$ a filter V -generic. We need to argue that

$$\mathcal{M}_G \models \psi.$$

Fix $i \in I$. We need to prove that

$$\mathcal{M}_G \models \bigvee_{j \in J_i} \psi_{i,j}.$$

Since

$$D_i = \{s \in S : \exists j \in J_i \psi_{i,j} \in s\}$$

is predense in S , there exists $s \in G$ and $j \in J_i$ such that

$$\psi_{i,j} \in s \in G.$$

By the Model Existence Theorem 1 we know that $s \in G$ implies

$$\mathcal{M} \models \bigwedge s.$$

Hence,

$$\mathcal{M} \models \bigvee_{j \in J_i} \psi_{i,j},$$

as wanted. □

The converse holds if we consider maximal consistency properties.

Lemma 4.2.5. *Let \mathcal{L} be a language and*

$$\psi := \bigwedge_{i \in I} \bigvee_{j \in J_i} \psi_{i,j}$$

be an $\mathcal{L}_{\infty\omega}$ -sentence with the $\psi_{i,j}$ sentences in $\mathcal{L}_{\kappa\omega}$. Let S be a consistency property maximal for $\mathcal{L}_{\kappa\omega}$ as in Definition 1.4.10. If for every V -generic filter $G \subset S$ we have that

$$\mathcal{M}_G \models \psi,$$

then for every $i \in I$ the set

$$D_i = \{s \in S : \exists j \in J_i \psi_{i,j} \in s\}$$

is predense in S .

Proof. Fix $i \in I$. We need to argue that

$$D_i = \{s \in S : \exists j \in J_i \psi_{i,j} \in s\}$$

is predense in S . Let G be a V -generic filter for S containing s . Since

$$\mathcal{M}_G \models \bigvee_{j \in J_i} \psi_{i,j},$$

Proposition 1.4.11 ensures that for some $j \in J_i$ we have that $\psi_{i,j} \in \Sigma_G$. Hence,

$$s \cup \{\psi_{i,j}\} \in G \subset S$$

and D_i is predense. □

4.2.3 The consistency properties S_ψ induced by a boolean consistent ψ

Now we present the family of consistency properties we will deal with in the remainder of this chapter. They come in pairs with the notion of Conservative Strengthening 2.2.2 and with the Conservative Compactness Theorem 2.2.5.

Definition 4.2.6. Let \mathcal{L} be a language and let ψ be an $\mathcal{L}_{\infty\omega}$ -sentence. S_ψ is the set of s such that:

1. $|s| < \omega$,
2. all elements in s are proper subsentences of ψ or (negated) atomic \mathcal{L} -sentences and
3. $\bigwedge(s \cup \{\psi\})$ is Boolean consistent.

The following relates the Conservative Compactness Theorem with our new family of consistency properties.

Corollary 4.2.7. Let $\{\psi_i : i \in I\}$ be a family of $\mathcal{L}_{\infty\omega}$ -sentences closed under finite conjunctions and finitely conservative. Let Ψ be $\bigwedge_{i \in I} \psi_i$. Then for each $i \in I$ we have that

$$S_{\psi_i} \subset S_\Psi.$$

Furthermore, the inclusion map is order and incompatibility preserving and has a predense target.

Note that we are not asserting that the inclusion map of S_{ψ_i} into S_Ψ is a complete embedding. This is in general false.

Proof. First, we check that $S_{\psi_i} \subset S_\Psi$. Consider $s \in S_{\psi_i}$. We need to prove that $s \in S_\Psi$. Since s is a finite set of subsentences of ψ_i , s is also a finite set of subsentences of Ψ . It remains to prove that $\Psi \wedge \bigwedge s$ is Boolean consistent. Since $s \in S_{\psi_i}$, $\psi_i \wedge \bigwedge s$ is Boolean consistent. As $\{\psi_i : i \in I\}$ is finitely conservative, the Conservative Compactness Theorem 2.2.5 ensures Ψ is a conservative strengthening of ψ_i . Hence, $\Psi \wedge \bigwedge s$ is Boolean consistent and $s \in S_\Psi$.

The inclusion map preserves order and incompatibility, so we only need to check it has a predense target. Let $t \in S_\Psi$. We need to argue t is compatible with a condition from S_{ψ_i} . The set s of formulae in t that are subformulae of ψ_i is in S_{ψ_i} and is compatible with t in S_Ψ . □

Remark 4.2.8. The inclusion map embedding S_{ψ_i} into S_{Ψ} is most likely not a complete embedding. The map preserves the predense subsets defined by the conditions in a consistency property which ensure the realizations of finite sets of subformulae of ψ_i . There can be however predense subsets of S_{ψ_i} which are possibly not expressed by conditions on subformulae of ψ_i inherent to the fact that S_{ψ_i} is a consistency property. These predense sets are possibly not preserved by the above map. This issue will be addressed in the following sections.

4.3 Iterations which are $< \kappa$ -CC

Suppose we want to build a sequence of formulas $(\psi_\alpha)_{\alpha \leq \kappa}$ such that ψ_β is a conservative strengthening of ψ_α for every $\alpha \leq \beta \leq \kappa$. If at all successor stages $\psi_{\alpha+1}$ is a conservative strengthening of ψ_α , then Conservative Compactness Theorem 2.2.5 ensures that at limit stages we can define

$$\psi_\lambda = \bigwedge_{\alpha < \lambda} \psi_\alpha.$$

However, as we argued before, it is likely not the case that S_{ψ_α} is a complete subforcing of S_{ψ_β} for $\alpha < \beta$. We can amend this obstacle if we consider “long enough” iterations. If κ is an ineffable cardinal, then at stationarily many stages, we will have that S_{ψ_α} is a complete subforcing of S_{ψ_κ} .

This result, combined with Baumgartner’s theorem on the preservation of $< \kappa$ -CC iterations where direct limits are taken stationarily often, ensures S_{ψ_κ} is $< \kappa$ -CC.

4.3.1 Ineffable cardinals

Definition 4.3.1. A cardinal κ is **ineffable** if for every sequence $(A_\alpha : \alpha < \kappa)$ with $A_\alpha \subset \alpha$ there exists a set $X \subset \kappa$ such that the set

$$\{\alpha < \kappa : X \cap \alpha = A_\alpha\}$$

is stationary.

The version of ineffability we will apply is the following.

Lemma 4.3.2. *Suppose κ is ineffable. Then for every sequence $(A_\alpha : \alpha < \kappa)$ with $A_\alpha \subset V_\alpha$ there exists a set $X \subset V_\kappa$ such that the set*

$$\{\alpha < \kappa : X \cap V_\alpha = A_\alpha\}$$

is stationary.

An important consequence for us is that ineffable cardinals produce diamond sequences.

Lemma 4.3.3. *Suppose κ is ineffable. Then there is a diamond sequence on κ , i.e. there exists a sequence $(A_\alpha : \alpha < \kappa)$ such that:*

- $A_\alpha \subseteq V_\alpha$ for all $\alpha < \kappa$ and

- for each $X \subset V_\kappa$ the set

$$S_X = \{\alpha < \kappa : X \cap V_\alpha = A_\alpha\}$$

is stationary.

4.3.2 Complete subforcings and $< \kappa$ -CC

In order to prove the main result we first need to present a canonical example of conservative strengthening. It is derived from Proposition 4.2.4 and based on the following idea. Let ψ be a Boolean consistent $\mathcal{L}_{\infty\omega}$ -sentence. All conditions in the forcing S_ψ are made up of $\mathcal{L}_{\infty\omega}$ -sentences. Hence, if we consider $D \subset S_\psi$ a dense set, we can define the $\mathcal{L}_{\infty\omega}$ -formula

$$\bigvee_{s \in D} \bigwedge s$$

to ensure that the dense set D is met.

Lemma 4.3.4. *Let ψ be a Boolean consistent infinitary formula in signature \mathcal{L} and E a predense subset of S_ψ . Then*

$$\psi \wedge \bigvee_{s \in E} \bigwedge s$$

is a conservative strengthening of ψ .

Proof. Let t be a finite set of subformulae of ψ such that $\psi \wedge \bigwedge t$ is Boolean consistent. We need to argue

$$\left(\psi \wedge \bigvee_{s \in E} \bigwedge s \right) \wedge \bigwedge t$$

is Boolean consistent. By definition of S_ψ 4.2.6 we have that $t \in S_\psi$. Let G be a filter V -generic for S_ψ containing t . By Model Existence Theorem 1 we have that

$$\mathcal{M}_G \models \psi \wedge \bigwedge t.$$

Since the set E is predense in S_ψ , the set

$$\{\{\bigwedge s\} : s \in E\}$$

is also predense in S_ψ . Then by Proposition 4.2.4 we have that

$$\mathcal{M}_G \models \bigvee_{s \in E} \bigwedge s.$$

□

Theorem 4.3.5. *Let κ be an ineffable cardinal and $(A_\alpha : \alpha < \kappa)$ be a diamond sequence on κ . Let $\{(\psi_\alpha, \phi_\alpha) : \alpha \in \kappa\}$ be such that:*

- ψ_0 is Boolean consistent,
- ϕ_α is an $\mathcal{L}_{\kappa\omega}$ -formula,
- $\psi_\beta = \bigwedge_{\alpha < \beta} \phi_\alpha$ for all $\beta < \kappa$,
- $\{\phi_\alpha : \alpha \in \kappa\}$ is finitely conservative,
- for all α for which A_α is a predense subset of S_{ψ_α} , ϕ_α is

$$\bigvee_{s \in A_\alpha} \bigwedge s.$$

Let ψ_κ be $\bigwedge_{\alpha < \kappa} \phi_\alpha$. Then

1. S_{ψ_α} is a complete subforcing of S_{ψ_κ} for stationarily many $\alpha < \kappa$;
2. S_{ψ_κ} is $< \kappa$ -CC.

Proof. Since $\{\phi_\alpha : \alpha \in \kappa\}$ is finitely conservative and ψ_0 is Boolean consistent, we have that ψ_κ is Boolean consistent by Conservative Compactness Theorem 2.2.5. Hence, S_{ψ_κ} is non empty.

1:

- Assume the first item fails as witnessed by a club C .
- For each $\alpha \in C$ let D_α be an open dense subset of S_{ψ_α} such that D_α is not predense in S_{ψ_κ} .
- Since κ is ineffable there exists $D \subseteq V_\kappa$ such that

$$S_D = \{\alpha < \kappa : D \cap V_\alpha = D_\alpha\}$$

is stationary.

- Let us argue D is open dense in S_{ψ_κ} .
 - Dense: pick $s \in S_{\psi_\kappa}$, find $\alpha \in S_D$ such that $s \in S_{\psi_\alpha}$, then $s \subset r$ for some $r \in D_\alpha \subseteq D$.
 - Open: if $r \in D$ and $s \supseteq r$ is in S_{ψ_κ} , there is α such that $r, s \in S_{\psi_\alpha}$ and $D \cap V_\alpha = D_\alpha$; then $s \in D_\alpha \subset D$.
- For each $\alpha \in C$ (since D_α is not predense in S_{ψ_κ}) we can find $s_\alpha \in D$ such that $s_\alpha \cup r \notin S_{\psi_\kappa}$ for all $r \in D_\alpha$.
- Aiming for a contradiction let us find $\beta \in C$ and $s \in D_\beta$ such that $s \cup s_\beta \in S_{\psi_\kappa}$.
- Since $(A_\alpha : \alpha < \kappa)$ is a Diamond sequence on κ , we can find $\alpha \in C$ such that $D \cap V_\alpha = A_\alpha$ and

$$(V_\alpha, A_\alpha, S_{\psi_\alpha}) \prec (V_\kappa, D, S_{\psi_\kappa}).$$

- By the assumptions of the theorem,

$$\phi_\alpha = \bigvee_{s \in D \cap V_\alpha} \bigwedge s$$

and ψ_κ is a conservative strengthening of $\psi_{\alpha+1} := \psi_\alpha \wedge \phi_\alpha$.

- Pick any $\beta \in S \setminus \alpha$. Then $s_\beta \in S_{\psi_\kappa}$ entails that

$$\{\phi_\alpha\} \cup s_\beta$$

is in S_{ψ_κ} (any model of $\psi_\kappa \wedge \bigwedge s_\beta$ is also a model of ϕ_α).

- Therefore there must be an $s \in D_\alpha = D \cap V_\alpha \subseteq D \cap V_\beta = D_\beta$ such that

$$s_\beta \cup \{\phi_\alpha\} \cup \left\{ \bigwedge s \right\} \in S_{\psi_\kappa}.$$

This gives that $s_\beta \cup s$ must also be in S_{ψ_κ} , which is a contradiction.

2:

Each S_{ψ_α} has size less than κ . By 1 for stationarily many $\alpha < \kappa$, we have that S_{ψ_α} is a complete subforcing of S_{ψ_κ} .

Let us argue that if $\mathbb{P} \subset \mathbb{Q} \subset \mathbb{R}$ are three posets such that \mathbb{P}, \mathbb{Q} are both complete subforcings of \mathbb{R} , then \mathbb{P} is a complete subforcing of \mathbb{Q} . Let D be a predense set for \mathbb{P} . We need to argue D is predense for \mathbb{Q} . Since \mathbb{P} is a complete subforcing of \mathbb{R} , we have that D is predense for \mathbb{R} . Since $D \subset \mathbb{Q} \subset \mathbb{R}$ and D is predense for \mathbb{R} , we have that D is predense for \mathbb{Q} .

Then, letting S be the set of α such that S_{ψ_α} is a complete subforcing of S_{ψ_κ} , we obtain that

$$\{S_{\psi_\alpha} : \alpha \in S\}$$

is such that S_{ψ_α} is a complete subforcing of S_{ψ_β} for $\alpha \leq \beta$ both in S . The above is an iteration of posets in the ordinary sense such that direct limits are taken at all limit stages. Since every ϕ_α is an $\mathcal{L}_{\kappa\omega}$ -sentence, S_{ψ_α} has size less than κ and in particular is $< \kappa$ -CC. Then, S_{ψ_κ} is $< \kappa$ -CC by Baumgartner's theorem (see for example [22, Thm. 3.13]).

□

Chapter 5

ASK-property

5.1 The ASK-property

Suppose ψ is an \mathcal{L} -sentence and forcing with S_ψ produces a model of ψ but kills a stationary set S on ω_1 of V . Then there exists \dot{X} an S_ψ -name for a club on ω_1^V disjoint from S . The problem with our setting so far is that the language \mathcal{L} may not be able to talk about \dot{X} .

Subsection **Languages** 5.1.1 introduces the minimal requirements on languages and formulae that will be assumed all over this chapter.

Subsection **Names for subsets of ω_1** 5.1.2 addresses the issue of “talking about \dot{X} ”. First, we discuss how to move from \mathcal{L} to $\mathcal{L}^* = \mathcal{L} \cup \{X^*\}$ so that \dot{X} “becomes a member” of \mathcal{L}^* . Second, we introduce an \mathcal{L}^* -axiom called **Name** of complexity $(\bigwedge_\infty^\omega)$ ensuring \dot{X} and X^* agree (up to some countable ordinal). Third, we show that $\psi \wedge \mathbf{Name}$ is conservative over ψ . Hence, moving to the new setting is safe and allows to state properties about \dot{X} .

Now that we can talk about the S_ψ -name \dot{X} for a subset of ω_1^V , Subsection **AS condition** 5.1.3 addresses the problem of removing the possibility that \dot{X} denotes a club disjoint from a stationary set $S \in V$. We introduce an axiom **AS** in language \mathcal{L}^* ensuring \dot{X} and S are not disjoint as witnessed by some countable ordinal. We don't know if $\psi \wedge \mathbf{AS}$ is a conservative strengthening of ψ for arbitrary ψ . Nonetheless, building on work of Kasum and Veličković [6], we isolate a rather broad family of formulae for which this is the case.

Subsection **ASK-property** 5.1.13 introduces new requirements on the language \mathcal{L} in order to present a family of formulas ψ for which $\psi \wedge \mathbf{AS}$ is a conservative strengthening of ψ . The **ASK condition** is the most important definition in this whole chapter. It is a technical and rough (at least on a first sight) statement isolated with one precise goal in mind: showing that for formulae ψ with the **ASK-property**, $\psi \wedge \mathbf{AS}$ is a conservative strengthening of ψ . A key syntactic requirements (to be paired with others) to ensure the **ASK-property** for ψ will be that of having ψ an infinitary conjunction of formulas of type $(\bigwedge_\infty^\omega)$.

5.1.1 Languages

From this section onward, we restrict our attention to languages and theories satisfying the following requirements.

Languages: We only consider multi-sorted languages containing at least the following symbols:

- A sort denoted \dot{N}_{ω_1} (see Example 3.2.4).
- Constants $\{c_{\beta\omega_1} : \beta < \omega_1\}$ of sort \dot{N}_{ω_1} (see Example 3.2.4).
- Constants $\{\tilde{x} : x \in H_\kappa\}$ of sort \dot{N}_{ω_1} (see Example 3.2.2).

Theories: We only consider logical theories implying at least the following axioms:

- Quantifier elimination for the sort \dot{N}_{ω_1}

$$\forall x \bigvee_{\beta < \omega_1^V} x = c_{\beta\omega_1^V}.$$

- Axiom ψ_{H_κ} from Example 3.2.2 to ensure H_κ^V sits as a transitive substructure of \dot{N}_{ω_1} .
- \dot{N}_{ω_1} models $\text{ZFC}^- + \text{there exists an uncountable cardinal}$.

To be precise, we will always need the quantifier elimination axiom and (fragments of) axiom ψ_{H_κ} . We will only need that the interpretation of sort \dot{N}_{ω_1} contains ω_1^V up to and including Section 5.1.2, from Section 5.1.3 onwards we will also need that $\mathcal{P}(\omega_1)^V$ is contained in the interpretation of sort \dot{N}_{ω_1} , from Section 5.1.4 onwards all of the above listed axioms and symbols and more.

Notation 5.1.1. All over this chapter \mathcal{L}^* denotes a language expanding the above language and we shall only consider quantifier free formulae in languages expanding \mathcal{L}^* .

5.1.2 Names for subsets of ω_1

Let ψ be a Boolean consistent $\mathcal{L}_{\infty\omega}$ -sentence.

$$\text{up}(\tau, \sigma) = \{\langle \tau, \{\top\} \rangle, \langle \sigma, \{\top\} \rangle\}$$

and

$$\text{op}(\tau, \sigma) = \{\langle \text{up}(\tau, \tau), \{\top\} \rangle, \langle \text{up}(\tau, \sigma), \{\top\} \rangle\}$$

are the canonical S_ψ -names for (un)ordered pairs.

Definition 5.1.2. Let ψ be an $\mathcal{L}_{\infty\omega}$ -sentence. Consider \dot{X} an S_ψ -name such that

$$\Vdash_{S_\psi} \dot{X} : \omega_1 \rightarrow 2 \text{ is a partial function.}$$

For each $\alpha < \omega_1$ let

$$D_{\dot{X}, \psi, \alpha} = \{s \in S_\psi : \langle \text{op}(\check{\alpha}, \check{1}), s \rangle \in \dot{X}\}$$

and

$$E_{\dot{X},\psi,\alpha} = \{s \in S_\psi : \langle \text{op}(\check{\alpha}, \check{0}), s \rangle \in \dot{X}\}.$$

We say that \dot{X} is **canonical for** ψ if $D_{\dot{X},\psi,\alpha} \cup E_{\dot{X},\psi,\alpha}$ is predense in S_ψ for all $\alpha < \omega_1$.

\dot{X} is canonical for ψ if and only if

$$\Vdash_{S_\psi} \dot{X} : \omega_1 \rightarrow 2 \text{ is a total function.}$$

Fact 5.1.3. *Assume ψ is a Boolean consistent infinitary sentence. Sticking to the notation of Def. 5.1.2, assume $s \in D_{\dot{X},\psi,\alpha}$ and $t \in E_{\dot{X},\psi,\alpha}$. Then $u = \{\psi\} \cup s \cup t$ is not Boolean consistent.*

Proof. If u were Boolean consistent, then $s \cup t$ would belong to S_ψ . If G were V -generic for S_ψ with $s \cup t \in G$, then we would have $\dot{X}_G(\alpha) = 1$ as witnessed by s and $\dot{X}_G(\alpha) = 0$ as witnessed by t . A contradiction. \square

If $S \subset T$ are consistency properties, then any S -name is also a T -name, even if S is not a complete subforcing of T .

Recall that given any filter G on T one can always recursively define the map

$$\begin{aligned} \text{val}_G : V^T &\rightarrow V[G] \\ \dot{X} &\mapsto \dot{X}_G = \left\{ \dot{Y}_G : \exists t \in T \langle \dot{Y}, t \rangle \in \dot{X} \right\}. \end{aligned}$$

We are used to consider this map only when G is V -generic for T , but it can be defined always. We are interested in considering what happens for this map also in cases when G is not V -generic for T .

Fact 5.1.4. *Let ψ be a Boolean consistent infinitary sentence. Assume $T \supseteq S_\psi$ is any consistency property, \dot{X} is canonical for ψ and G is a maximal filter on T .¹*

Then:

- $\dot{X}_G(\alpha) = 1$ if and only if $G \cap D_{\dot{X},\psi,\alpha}$ is non-empty,
- $\dot{X}_G(\alpha) = 0$ if and only if $G \cap E_{\dot{X},\psi,\alpha}$ is non-empty,
- \dot{X}_G is undefined on α if and only if $G \cap (E_{\dot{X},\psi,\alpha} \cup D_{\dot{X},\psi,\alpha})$ is empty.

Proof. Suppose $\dot{X}_G(\alpha) = 1$. By definition of \dot{X}_G there exists $s \in G$ such that $\langle \text{op}(\check{\alpha}, \check{1}), s \rangle \in \dot{X}$. Then $s \in G \cap D_{\dot{X},\psi,\alpha}$.

Consider $s \in G \cap D_{\dot{X},\psi,\alpha} \neq \emptyset$. By definition of $D_{\dot{X},\psi,\alpha}$ we have that $\langle \text{op}(\check{\alpha}, \check{1}), s \rangle \in \dot{X}$. Since $s \in G$, $\dot{X}_G(\alpha) = 1$.

The proof of the second item is analogous to the first one. The third item follows from the other two. \square

¹Note that S_ψ might not be a complete subforcing of T .

Definition 5.1.5. Let ψ be a Boolean consistent sentence. Let \dot{X} be an S_ψ -name for a subset of ω_1 as described by its characteristic function. Let \mathcal{L}^* be the language $\mathcal{L} \cup \{X^*\}$ where X^* is a unary predicate of sort \dot{N}_{ω_1} .

For each ordinal γ we define axiom $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ in language \mathcal{L}^* as the conjunction of

$$\bigwedge_{\alpha < \gamma} \left(X^*(\check{\alpha}) \Rightarrow \bigvee \{ \bigwedge s : s \in D_{\dot{X}, \psi, \alpha} \} \right)$$

and

$$\bigwedge_{\alpha < \gamma} \left(\neg X^*(\check{\alpha}) \Rightarrow \bigvee \{ \bigwedge s : s \in E_{\dot{X}, \psi, \alpha} \} \right).$$

Lemma 5.1.6. Let γ a countable ordinal and ψ be an infinite conjunction of formulae of complexity $(\bigwedge_{\infty}^{\omega})$, then $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ (as in Def. 5.1.5) is of complexity $(\bigwedge_{\infty}^{\omega})$.

Proof. $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ is equivalent to the conjunction of

$$\bigwedge_{\alpha < \gamma} \left(\neg X^*(\check{\alpha}) \vee \bigvee \{ \bigwedge s : s \in D_{\dot{X}, \psi, \alpha} \} \right)$$

and

$$\bigwedge_{\alpha < \gamma} \left(X^*(\check{\alpha}) \vee \bigvee \{ \bigwedge s : s \in E_{\dot{X}, \psi, \alpha} \} \right).$$

These sentences have at least the conjunctions

$$\bigwedge_{\alpha < \gamma} \text{ and } \bigwedge s \text{ for } s \in D_{\dot{X}, \psi, \alpha} \cup E_{\dot{X}, \psi, \alpha}$$

plus the ones inside any of the above s . The ones listed above are countable since γ is countable and each of the s under consideration is finite.

It remains to deal with the ones that might appear inside some $s \in D_{\dot{X}, \psi, \alpha} \cup E_{\dot{X}, \psi, \alpha}$. By hypothesis ψ is an infinite conjunction of formulae of complexity $(\bigwedge_{\infty}^{\omega})$. Therefore the proper subformulas of ψ are of complexity $(\bigwedge_{\infty}^{\omega})$. Since conditions in S_ψ only contain proper subformulas of ψ , we have that any $s \in D_{\dot{X}, \psi, \alpha} \cup E_{\dot{X}, \psi, \alpha} \subset S_\psi$ only contains formulas of complexity $(\bigwedge_{\infty}^{\omega})$. Thus, all conjunctions appearing inside s are countable. \square

We outline below how $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ allows to control the interpretation of a canonical S_ψ -name \dot{X} in generic extensions by some $T \supset S_\psi$.

Lemma 5.1.7. Let ψ be a Boolean consistent infinitary sentence, let \dot{X} be an S_ψ -name for a subset of ω_1 canonical for ψ , let γ be a countable ordinal, and let $T \supset S_\psi$ be a consistency property in language $\mathcal{L}_1 \supset \mathcal{L}^*$. Suppose $t_0 \in T$ is such that

$$\{t \in T : \mathbf{Name}(\psi, \gamma, \dot{X}, X^*) \in t\}$$

is dense below t_0 . Then for any V -generic filter G for T containing t_0 and $\alpha < \gamma$, we have that

$$\dot{X}_G(\alpha) = 1 \Leftrightarrow X^*(\check{\alpha}) \in \Sigma_G.$$

Proof. Consider α such that $\dot{X}_G(\alpha) = 1$. Fact 5.1.4 ensures the existence of some $s \in G \cap D_{\dot{X}, \psi, \alpha}$. Since $t_0 \in G$ and the set of conditions containing axiom $\mathbf{Name}(\psi, \gamma, \dot{X}, X^*)$ is dense below t_0 , we have that

$$t = s \cup \{\mathbf{Name}(\psi, \gamma, \dot{X}, X^*)\} \in G.$$

Then axiom $\mathbf{Name}(\psi, \gamma, \dot{X}, X^*)$ and $\bigwedge s \in \Sigma_G$ ensure $X^*(\check{\alpha}) \in \Sigma_G$: otherwise $\neg X^*(\check{\alpha}) \in \Sigma_G$ entails that there is some $t \in E_{\dot{X}, \psi, \alpha} \cap G$. This is impossible because $s \cup t \cup \{\psi\}$ is not Boolean consistent, while $s \cup t \in G$ makes it Boolean consistent.

The proof for $\dot{X}_G(\alpha) = 0$ is symmetric. □

Corollary 5.1.8. *Let ψ be a Boolean consistent infinitary sentence in language \mathcal{L}^* , let \dot{X} be a canonical name for ψ , let γ be an ordinal and let $T \supset S_\psi$ be a consistency property in language $\mathcal{L}_1 \supset \mathcal{L}^*$. Suppose $t_0 \in T$ is such that*

$$\{t \in T : \mathbf{Name}(\psi, \gamma, \dot{X}, X^*) \in t\}$$

is dense below t_0 . Then for any $\alpha < \gamma$, $D_{\dot{X}, \psi, \alpha} \cup E_{\dot{X}, \psi, \alpha}$ is predense below t_0 in T .

Proof. Fix $\alpha < \omega_1$. Consider $s \leq t_0 \in T$ and G a V -generic filter for T containing s . Since

$$X^*(\check{\alpha})$$

is an atomic formula, we have that either $s \cup \{X^*(\check{\alpha})\} \in G$ or $s \cup \{\neg X^*(\check{\alpha})\} \in G$. The previous Lemma ensures

$$\dot{X}_G(\alpha) = 1 \Leftrightarrow X^*(\check{\alpha}) \in \Sigma_G$$

and

$$\dot{X}_G(\alpha) = 0 \Leftrightarrow \neg X^*(\check{\alpha}) \in \Sigma_G.$$

Since \dot{X}_G is a canonical name for ψ , $\dot{X}_G(\alpha) = 1$ if and only if some $s \in D_{\dot{X}, \psi, \alpha}$ is in G and $\dot{X}_G(\alpha) = 0$ if and only if some $s \in E_{\dot{X}, \psi, \alpha}$ is in G . □

Lemma 5.1.9. *Let ψ be a Boolean consistent infinitary sentence, let \dot{X} be a canonical name for ψ and let γ be a countable ordinal. Then $\psi \wedge \mathbf{Name}(\psi, \gamma, \dot{X}, X^*)$ is conservative over ψ .*

Proof. Let $s \in S_\psi$. We need to argue that

$$\left(\psi \wedge \text{Name}(\psi, \gamma, \dot{X}, X^*) \right) \wedge \bigwedge s$$

is Boolean consistent. Let G be a V -generic filter for S_ψ containing s . Then

$$\mathcal{M}_G \models \psi \wedge \bigwedge s.$$

Define

$$(X^*)^{\mathcal{M}_G} = \{\alpha < \gamma : \dot{X}_G(\alpha) = 1\}.$$

Consider $\alpha < \lambda$ such that $\mathcal{M} \models X^*(\check{\alpha})$. In order to realize $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ we need to ensure

$$\mathcal{M}_G \models \bigvee \{ \bigwedge s : s \in D_{\dot{X}, \psi, \alpha} \}.$$

By definition of $(X^*)^{\mathcal{M}_G}$ we have that $\dot{X}_G(\alpha) = 1$. By Fact 5.1.4 there exists $s \in G \cap D_{\dot{X}, \psi, \alpha}$. By the Model Existence Theorem 1.4.7 we have that

$$\mathcal{M}_G \models \bigwedge s.$$

The proof for $X^*(\alpha) = 0$ is completely symmetric. □

5.1.3 AS condition

In the previous subsection we introduced axiom Name to ensure \dot{X} and X^ describe the same set up to some ordinal γ . Now, for each stationary set S and each name for a club \dot{X} , we introduce an axiom denoted by AS to ensure X^* and S are not disjoint.*

We need more expressive power than that given by the languages introduced in Section 5.1.1.

Notation 5.1.10. We expand the language introduced in Section 5.1.1 with a satisfaction predicate $\text{Sat}_{N_{\omega_1}}$ for the sort \dot{N}_{ω_1} and with constants $\{d_\beta : \beta < \omega_1\}$ of sort \dot{N}_{ω_1} .

The satisfaction predicate $\text{Sat}_{N_{\omega_1}}$ is a unary predicate of sort \dot{N}_{ω_1} which takes as inputs quantifier free infinitary sentences ϕ in V in the language

$$\{\in, c_{\gamma\omega_1} : \gamma < \omega_1, d_\beta : \beta < \omega_1\}.$$

$\text{Sat}_{N_{\omega_1}}(\phi)$ holds or not in an \mathcal{L}^* -structure \mathcal{M} according to the evaluation of ϕ in the structure

$$\langle \dot{N}_{\omega_1}^{\mathcal{M}}, \in_{\omega_1}^{\mathcal{M}}, c_{\gamma\omega_1}^{\mathcal{M}} : \gamma < \omega_1, d_\beta^{\mathcal{M}} : \beta < \omega_1 \rangle.$$

We denote by \mathcal{L}^* this expanded language.

We shall only consider quantifier free \mathcal{L}^* -sentences ψ implying that the satisfaction predicate² is correctly interpreted and the following axioms to ensure that the constants $\{d_\beta : \beta < \omega_1\}$ give all the stationary sets of \dot{N}_{ω_1} :

$$\bigwedge_{\beta < \omega_1} \bigwedge_{\gamma < \omega_1} \left(\text{Sat}_{N_{\omega_1}}(c_{\gamma\omega_1} \text{ is club on } \omega_1) \Rightarrow d_\beta \cap c_{\gamma\omega_1} \neq \emptyset \right)$$

$$\bigwedge_{\gamma < \omega_1} \left(\text{Sat}_{N_{\omega_1}}(c_{\gamma\omega_1} \text{ is stationary on } \omega_1) \Rightarrow \bigvee_{\beta < \omega_1} d_\beta = c_{\gamma\omega_1} \right).$$

Definition 5.1.11 (The Asperò-Schindler condition). Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$.

Let \dot{X} be a canonical S_ψ -name, X^* be a unary predicate symbol of sort \dot{N}_{ω_1} , and d_β be a constant for a stationary set. Define the Asperò-Schindler condition

$$\text{AS}(\beta, \dot{X}, X^*)$$

as the conjunction of

$$(i) \bigvee_{\gamma \in \omega_1} \left(\text{Name}(\psi, \gamma, \dot{X}, X^*) \wedge \text{Unbounded}(X^*, \gamma) \wedge (\text{Closed}(\gamma, X^*) \Rightarrow \check{\gamma} \in d_\beta) \right)$$

and

$$(ii) \bigwedge_{\delta < \omega_1} (X^*(c_{\delta\omega_1}) \Rightarrow \bigvee_{\alpha < \omega_1} c_{\delta\omega_1} = \check{\alpha})$$

where

$$\text{Unbounded}(\gamma, X^*) := \bigwedge_{\alpha < \gamma} \bigvee_{\alpha < \beta < \gamma} X^*(\check{\beta})$$

$$\text{Closed}(\gamma, X^*) := \bigwedge_{\alpha < \gamma} \left(\neg X^*(\check{\alpha}) \Rightarrow \bigvee_{\beta < \alpha} \bigwedge_{\beta < \delta < \alpha} \neg X^*(\check{\delta}) \right).$$

Each formula is introduced with the following purpose:

- (I) $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ is intended to ensure \dot{X} and X^* describe the same set up to γ as argued in Lemma 5.1.7.
- (II) $\text{Unbounded}(\gamma, X^*)$ ensures the interpretation of $X^* \upharpoonright \gamma$ is unbounded on γ .
- (III) $\text{Closed}(\gamma, X^*)$ forces the interpretation of $X^* \cap \gamma$ to be a closed subset of γ .
- (IV) Axiom (ii) ensures X^* is a subset of ω_1 .

²This is not trivial since the satisfaction predicate for N_{ω_1} is not expressible by an infinitary conjunction of formulae of type $(\bigwedge_{\infty}^\omega)$. However we will set up ψ so that a coherent interpretation of the satisfaction predicate for N_{ω_1} is provably definable in models of ψ , and ψ is an infinitary conjunction of $(\bigwedge_{\infty}^\omega)$ -formulas.

When (II) and the premise of (III) holds, $X^* \upharpoonright \gamma$ is a closed unbounded subset of γ .

Fact 5.1.12. *Sticking to the notation of the previous definition*

- Axiom (i) from $\text{AS}(\beta, \dot{X}, X^*)$ has complexity $(\bigwedge_{\infty}^{\omega})$ in language $\mathcal{L}^* = \mathcal{L} \cup \{X^*\}$.
- Axiom (ii) from $\text{AS}(\beta, \dot{X}, X^*)$ is a conjunction of size ω_1 of sentences of complexity $(\bigwedge_{\infty}^{\omega})$ in language $\mathcal{L}^* = \mathcal{L} \cup \{X^*\}$.

Proof. We have that $X^*(\check{\alpha})$ and $\check{\gamma} \in d_{\beta}$ are atomic formulae and $\text{Closed}(\gamma, X^*)$ is an $\mathcal{L}_{\omega_1\omega}^*$ -formula. We have seen in Lemma 5.1.6 that $\text{Name}(\psi, \gamma, \dot{X}, X^*)$ is of complexity $(\bigwedge_{\infty}^{\omega})$. Thus, so is axiom (i) from $\text{AS}(\beta, \dot{X}, X^*)$ being an infinite disjunction of sentences of complexity $(\bigwedge_{\infty}^{\omega})$.

We have that (ii) is given by a conjunction of size ω_1 of sentences of complexity $(\bigwedge_{\infty}^{\omega})$. \square

5.1.4 ASK-property

Suppose ψ is a Boolean consistent formula which is an infinitary conjunction of sentences of complexity $(\bigwedge_{\infty}^{\omega})$. Our goal is to show that for any S_{ψ} -name for a club and any constant $c_{\beta\omega_1}$ for a stationary set in the sort \dot{N}_{ω_1} , $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is a conservative strengthening of ψ . We will prove this for formulae ψ satisfying a property, which is a refined version of a similar concept isolated by Kasum and Veličković. With respect to the original concept of Kasum and Veličković [6] the refinement takes into account the complexity of the formula to which it applies.

Definition 5.1.13 (The Asperò-Schindler-Kasum property). Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$.

ψ has the ASK-property if:

1. it is Boolean consistent,
2. it is an **infinitary conjunction** of sentences of complexity $(\bigwedge_{\infty}^{\omega})$ and
3. the following holds.
 - For any λ large enough and any V -generic filter $G \subset \text{Coll}(\omega, < \lambda)$,
 - for any $\mathcal{M} \models \psi$ in $V[G]$ with $M_{\omega_1^V}$ the domain for the sort $\dot{N}_{\omega_1^V}$ being a transitive set and
 - for any constant d_{β} for a stationary set,

in $V[G]$ it holds that:

- (I) $\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{M_{\omega_1^V}} \cap V$ and $(M_{\omega_1^V}, \in)$ is a model of ZFC^- containing $H_{\omega_2}^V$,
- (II) there exists an elementary embedding $j : V \rightarrow W$ with critical point ω_1^V and target W transitive,
- (III) there exists \mathcal{N} such that:

- (i) \mathcal{N} is a $j(\mathcal{L})$ -structure,
- (ii) $\mathcal{N} \models \check{\omega}_1^V \in j(d_\beta)$,
- (iii) for all **atomic** \mathcal{L} -sentences ϕ , we have that

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models j(\phi),$$

- (iv) \mathcal{N} models $j(\psi)$.

5.2 Preservation of the ASK-property through successor stages

Theorem 5.2.1 (Successor stage). *Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$.*

Assume ψ has the ASK-property, \dot{X} is a canonical S_ψ -name for an unbounded subset of ω_1 , and d_β is a constant in \mathcal{L}^ for a stationary set. Then,*

- $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is a conservative strengthening of ψ in signature $\mathcal{L}^* = \mathcal{L} \cup \{X^*\}$ and
- $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ has the ASK-property.

The following proof is technical and long. Its content is divided in two steps whose proofs are (almost) independent: checking that $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is a conservative strengthening of ψ and checking that the ASK-property propagates to $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$. The proof is divided in micro-steps which are itemized: each bullet contains just one logical step. Whenever different paths open inside the proof, and this happens twice in each proof, we use bullet symbols different from the typical ones to indicate the beginning of each path.

Proof.

Conservative strengthening:

- Suppose s is a finite set of subsentences of ψ such that $\psi \wedge \bigwedge s$ is Boolean consistent. We need to show that $\psi \wedge \text{AS}(\beta, \dot{X}, X^*) \wedge \bigwedge s$ is Boolean consistent.
- Let λ be such that $|\mathcal{P}(S_\psi)| < \lambda$ holds in V . Fix G a filter V -generic for $\text{Coll}(\omega, < \lambda)$.
- Since $\psi \wedge \bigwedge s$ is Boolean consistent and $\mathcal{P}(S_\psi)$ is countable in $V[G]$, we can find $H \in V[G]$ a filter V -generic for S_ψ containing s .
- By the Model Existence Theorem 1.4.7 there exists $\mathcal{M}_H \in V[H]$ an \mathcal{L} -model of $\psi \wedge \bigwedge s$.
- Since \dot{X} is a canonical S_ψ -name and H is V -generic for S_ψ , \dot{X}_H is a characteristic function with domain ω_1^V .
- The argument now depends on whether \dot{X}_H is a club subset of ω_1^V or not.

- ✿ Assume \dot{X}_H is not a club subset of ω_1^V .
- In order to prove that \mathcal{M}_H ensures the Boolean consistency of $\psi \wedge \text{AS}(\beta, \dot{X}, X^*) \wedge \bigwedge s$, it remains to argue $\mathcal{M}_H \models \text{AS}(\beta, \dot{X}, X^*)$.
- Since \dot{X}_H is not a club subset of ω_1^V , it is unbounded and not closed. Let γ be a limit point of ordinals in \dot{X}_H such that $\dot{X}_H \upharpoonright \gamma$ is not closed.
- Extend \mathcal{M}_H to an $\mathcal{L} \cup \{X^*\}$ -structure by setting

$$(X^*)^{\mathcal{M}_H} = \dot{X}_H \subset \omega_1^V.$$

- Axiom (ii) of $\text{AS}(\beta, \dot{X}, X^*)$ requires that X^* is interpreted by a subset of ω_1^V . Then our choice of $(X^*)^{\mathcal{M}_H}$ ensures it holds.
- Let us argue that \mathcal{M}_H realizes axiom (i) from $\text{AS}(\beta, \dot{X}, X^*)$ as witnessed by γ :
 - $\mathcal{M}_H \models \text{Name}(\psi, \gamma, \dot{X}, X^*)$ since $(X^*)^{\mathcal{M}_H} = \dot{X}_H$,
 - $\mathcal{M}_H \models \text{Unbounded}(X^*, \gamma)$ since γ is a limit of $(X^*)^{\mathcal{M}_H}$ and
 - $\mathcal{M}_H \models \text{Closed}(X^*, \gamma) \Rightarrow \check{\gamma} \in d_\beta$ since $(X^*)^{\mathcal{M}_H}$ is not closed.

- ✿ Assume \dot{X}_H is a club subset of ω_1 . Let us argue how to obtain the Boolean consistency of $\psi \wedge \text{AS}(\beta, \dot{X}, X^*) \wedge \bigwedge s$ from the ASK-property.
- In $V[G]$, fix $j : V \rightarrow W$ and \mathcal{N} the witnesses for the ASK-property given by conditions (II) and (III) with respect to $G \subset \text{Coll}(\omega, < \lambda)$, $\mathcal{M}_H \models \psi$ and d_β .
- By the ASK-property (i) we have that \mathcal{N} is a $j(\mathcal{L})$ -structure. Extend it to a $j(\mathcal{L}) \cup \{j(X^*)\}$ -structure by setting

$$j(X^*)^{\mathcal{N}} = \left\{ \alpha < \omega_1 : \dot{X}_H(\alpha) = 1 \right\}.$$

- The rest of the proof is divided in two arguments:

1. Show that

$$j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)) \wedge \bigwedge j(s)$$

holds in \mathcal{N} .

2. Argue why this is enough to ensure that $\psi \wedge \text{AS}(\beta, \dot{X}, X^*) \wedge \bigwedge s$ is Boolean consistent in V .

✿ We start addressing 1.

- By condition (iv) in the definition of the ASK-property, \mathcal{N} realizes $j(\psi)$.

- By condition (iii) in the definition of the ASK-property, \mathcal{M}_H and \mathcal{N} realize the hypothesis of the Preservation Lemma 4.2.3. Since s is a finite set of formulae of complexity $(\bigwedge_{\infty}^{\omega})$ and $\mathcal{M}_H \models \bigwedge s$, Preservation Lemma 4.2.3 ensures

$$\mathcal{N} \models \bigwedge j(s).$$

- Axiom (ii) of $\text{AS}(j(\beta), j(\dot{X}), j(X^*))$ requires that $j(X^*)$ is interpreted by a subset of $j(\omega_1^V)$. This holds by definition of $j(X^*)^{\mathcal{N}}$ since it is a subset of $\omega_1^V \subseteq j(\omega_1^V)$.
- Let us argue that

$$\mathcal{N} \models \text{Name}(j(\psi), \omega_1^V, j(\dot{X}), j(X^*)) \wedge \text{Unbounded}(X^*, \omega_1) \wedge \text{Closed}(\omega_1^V, j(X^*)) \wedge \omega_1^V \in j(d_{\beta}).$$

If this is done, axiom (i) of

$$\text{AS}(j(\beta), j(\dot{X}), j(X^*))$$

holds in \mathcal{N} as witnessed by the ω_1^V -th disjunct.

- We have that $\mathcal{N} \models \check{\omega}_1^V \in j(d_{\beta})$ by condition (ii) of the ASK-property.
- $\text{Closed}(\omega_1^V, j(X^*))$ holds in \mathcal{N} since $j(X^*)^{\mathcal{N}}$ is a closed subset of ω_1^V .
- $\text{Unbounded}(X^*, \omega_1^V)$ holds in \mathcal{N} since $j(X^*)^{\mathcal{N}}$ is unbounded in ω_1^V .
- Let us argue that $\mathcal{N} \models \text{Name}(j(\psi), \omega_1^V, j(\dot{X}), j(X^*))$. Fix $\alpha < \omega_1^V$. We need to show that $\mathcal{N} \models j(X^*)(\alpha)$ implies

$$\mathcal{N} \models \bigvee \{ \bigwedge r : r \in D_{j(\dot{X}), j(\psi), j(\alpha)} \}.$$

- Since $j(X^*)^{\mathcal{N}}(\alpha)$ holds, we have that $\dot{X}_H(\alpha) = 1$.
- By definition of evaluation for a name there exists $t \in H$ such that $\langle \text{op}(\check{\alpha}, \check{1}), t \rangle \in \dot{X}$, which is precisely the definition of $t \in D_{\dot{X}, \psi, \alpha}$. Hence, $t \in D_{\dot{X}, \psi, \alpha} \cap H$.
- By elementarity $j(t) \in j(D_{\dot{X}, \psi, \alpha}) = D_{j(\dot{X}), j(\psi), j(\alpha)}$.
- Since $t \in H$, the Model Existence theorem 1.4.7 ensures that $\mathcal{M}_H \models \bigwedge t$.
- Since $t \in S_{\psi}$ is a finite set of proper subsentences of ψ , $\bigwedge t$ has complexity $(\bigwedge_{\infty}^{\omega})$.
- The Preservation Lemma 4.2.3 ensures $\mathcal{N} \models \bigwedge j(t)$.
- Hence, $\mathcal{N} \models \bigvee \{ \bigwedge r : r \in D_{j(\dot{X}), j(\psi), j(\alpha)} \}$ as witnessed by $\bigwedge j(t)$.
- The same argument proves that if $\mathcal{N} \models \neg j(X^*)(\alpha)$, then $\mathcal{N} \models \bigvee \{ \bigwedge r : r \in E_{j(\dot{X}), j(\psi), j(\alpha)} \}$.

✿ Now, we argue for 2. The main complication being that $V[G]$ is not a forcing extension of W .

- Consider G_1 a filter $V[G]$ -generic for $\text{Coll}(\omega, \kappa)$ for some κ big enough so that

$$\mathcal{N} \in H_{\omega_1}^{V[G_1]}$$

and

$$j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)) \wedge \bigwedge j(s) \in H_{\omega_1}^{W[G_1]},$$

which is a subset of $H_{\omega_1}^{V[G_1]} = H_{\omega_1}^{V[G][G_1]}$ since $W \subset V[G]$.

- By Shoenfield's Absoluteness Lemma ?? between $H_{\omega_1}^{V[G_1]}$ and $H_{\omega_1}^{W[G_1]}$, there exists $\mathcal{N}' \in W[G_1]$ a model of $j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)) \wedge \bigwedge j(s)$. Therefore, W models that $j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)) \wedge \bigwedge j(s)$ is Boolean consistent.
- Since W models $j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)) \wedge \bigwedge j(s)$ is Boolean consistent, V models $\psi \wedge \text{AS}(\beta, \dot{X}, X^*) \wedge \bigwedge s$ is Boolean consistent by elementarity of $j : V \rightarrow W$.

ASK-property:

- We follow the indexes from Definition 5.1.13.
1. Since $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is a conservative strengthening of ψ and ψ is Boolean consistent, $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is Boolean consistent.
 2. $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is an infinitary conjunction of $(\bigwedge_{\infty}^{\omega})$ -sentences:
 - ψ is an infinitary conjunction of $(\bigwedge_{\infty}^{\omega})$ -sentences since it has the ASK-property,
 - $\text{AS}(\beta, \dot{X}, X^*)$ is an infinite conjunction of sentences of complexity $(\bigwedge_{\infty}^{\omega})$ by Fact 5.1.12.
 3. Let G be a filter V -generic for $\text{Coll}(\omega, \lambda)$ for λ big enough, let \mathcal{M} be an $\mathcal{L} \cup \{X^*\}$ -model of $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ and let $d_{\beta'}$ be a constant for a stationary set. We need to ensure conditions (I), (II) and (III) are met.

(I) We have that

$$\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{M_{\omega_1}^V} \cap V$$

since ψ has the ASK-property.

- For conditions (II) and (III) we need to produce $j : V \rightarrow W$ and \mathcal{N} witnesses for the ASK-property relative to G , $\mathcal{M} \models \psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ and $d_{\beta'}$.
- Since ψ has the ASK-property, we can find $j : V \rightarrow W$ and \mathcal{N} witnesses for the ASK-property relative to G , $\mathcal{M} \upharpoonright \mathcal{L} \models \psi$ and $d_{\beta'}$.
- Enrich \mathcal{N} to a $j(\mathcal{L}^*) = j(\mathcal{L}) \cup \{j(X^*)\}$ -structure denoted by \mathcal{N}^* by setting

$$j(X^*)^{\mathcal{N}} = (X^*)^{\mathcal{M}}.$$

- Let us show that $j : V \rightarrow W$ and \mathcal{N}^* are witnesses for the ASK-property relative to G , $\mathcal{M} \models \psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ and $d_{\beta'}$.
- (i) \mathcal{N}^* is a $j(\mathcal{L}^*)$ -structure by definition.
- (ii) We have that $\mathcal{N}^* \models \check{\omega}_1^V \in j(d_{\beta'})$ since $\mathcal{N} \models \check{\omega}_1^V \in j(d_{\beta'})$ by condition (ii) of the ASK-property relative to ψ .
- (iii) Let us argue that for \mathcal{L}^* -atomic formulae ϕ we have that

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N}^* \models j(\phi).$$

An atomic \mathcal{L}^* -formula might come from two different places:

- for those in language \mathcal{L} the ASK-property ensures

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N}^* \models j(\phi),$$

- the remaining atomic formulas are of the form $X^*(c)$ for c some constant of sort $\dot{N}_{\omega_1^V}$. Then the desired conclusion follows from the fact that $j(X^*)^{\mathcal{N}^*} = (X^*)^{\mathcal{M}} \subset \omega_1^V$ together with $j(\alpha) = \alpha$ for $\alpha < \omega_1^V$.

- (iv) Finally, we argue

$$\mathcal{N}^* \models j(\psi) \wedge \text{AS}(j(\beta), j(\dot{X}), j(X^*)).$$

- We have that

$$\mathcal{N}^* \models j(\psi)$$

by the ASK-property for ψ .

- Axiom (i) from $\text{AS}(\beta, \dot{X}, X^*)$ is of complexity $(\bigwedge_{\omega}^{\omega})$ by Fact 5.1.12, and

$$\mathcal{M} \models \text{AS}(\beta, \dot{X}, X^*);$$

hence, by the Preservation Lemma 4.2.3,

$$\mathcal{N}^* \models \text{AS}(j(\beta), j(\dot{X}), j(X^*)).$$

- Axiom (ii) of $\text{AS}(j(\beta), j(\dot{X}), j(X^*))$ requires that $j(X^*)$ is interpreted by a subset of $j(\omega_1^V)$. This holds by definition of $j(X^*)^{\mathcal{N}^*}$ since it is a subset of ω_1^V .

The proof of the Theorem is completed. □

5.3 Preservation of the ASK-property through limit stages

In the previous section we argued that $\psi \wedge \text{AS}(\beta, \dot{X}, X^*)$ is a conservative strengthening of ψ for any ψ with the ASK-property. If we iterate this construction, the Conservative Compactness Theorem 2.2.5 ensures the formula obtained at stage ω is a conservative strengthening of ψ . Nonetheless, we don't know if the resulting formula maintains the ASK-property. This issue is addressed and solved in this section.

We introduce a weaker version of the above iteration where constants e_η are introduced to decide randomly if axiom $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ is taken into account at stage η or not. For this type of iterations we are able to ensure the ASK-property is maintained at all limit stages.

In the next section we will argue why the Successor Stage Theorem 5.2.1 ensures $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ is taken into account generically often as long as we index the names \dot{X}_η to be taken into account according to a diamond sequence. This will occur when in the generic term model for ψ_κ the formula $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ is realized for stationarily many $\eta < \kappa$.

Lemma 5.3.1 (Limit stage). *Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$ with the ASK-property. Let also:*

- $0^*, 1^*$ be two fresh constants symbols of sort -1 ,
- $\{e_\eta : \eta < \kappa\}$ be fresh constants symbols of sort -1 and
- $\{X_\eta^* : \eta < \kappa\}$ be fresh unary predicate symbols of sort $\dot{N}_{\omega_1^V}$.

Suppose $(\psi_\eta)_{\eta \leq \kappa}$ is a sequence of sentences such that:

- ψ_0 is ψ ,
- $\psi_{\eta+1}$ is $\psi_\eta \wedge \phi_\eta$, with ϕ_η being

$$(e_\eta = 0^*) \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*),$$

where:

- \dot{X}_η is a canonical S_{ψ_η} -name,

$$- \beta_\eta < \omega_1^V;$$

- $\psi_\gamma = \psi_0 \wedge \bigwedge_{\eta < \gamma} \phi_\eta$ for all $\gamma \leq \kappa$ limit ordinal.

Then ψ_κ is a conservative strengthening of ψ and has the ASK-property with respect to the language $\mathcal{L} \cup \{e_\eta, X_\eta^* : \eta < \kappa\} \cup \{0^*, 1^*\}$.

Proof. By induction on $\eta \leq \kappa$ we prove that ψ_η has the ASK-property and is a conservative strengthening of ψ .

Successor stage: Suppose the result true for $\eta < \kappa$ and let us prove it for $\eta + 1$.

- ❁ First, we show that $\psi_{\eta+1}$ is a conservative extension of ψ_η . Consider s a finite set of subsentences of ψ_η such that $\{\psi_\eta\} \cup s$ is Boolean consistent. We need to argue the Boolean consistency of $\psi_{\eta+1} \wedge \bigwedge s$.
- By the Successor Theorem 5.2.1 we have that $\psi_\eta \wedge \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ is a conservative strengthening of ψ_η . Therefore, there exists (in some generic extension of V) \mathcal{M} a model of $\psi_\eta \wedge \bigwedge s \wedge \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$. Expand \mathcal{M} to \mathcal{M}^* by letting $e_\eta^{\mathcal{M}^*} = 0$. Then $\mathcal{M}^* \models \psi_{\eta+1} \wedge \bigwedge s$.
- ❁ Second, we show that $\psi_{\eta+1}$ has the ASK-property. Let λ be large enough, let $G \subset \text{Coll}(\omega, < \lambda)$ be a filter V -generic, let \mathcal{M} be a model of $\psi_{\eta+1}$ and let $d_\beta \in \mathcal{L}$ be a constant for a stationary set. Let us argue conditions (I),(II) and (III) from the ASK-property 5.1.13.

(I) We have that

$$\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{M_{\omega_1}^V} \cap V$$

since any model of $\psi_{\eta+1}$ is a model of ψ_η and ψ_η has the ASK-property.

- For items (II) and (III) we need to build $j : V \rightarrow W$ and \mathcal{N} witnesses for the ASK-property with respect to G , $\mathcal{M} \models \psi_{\eta+1}$ and d_β .
 - Consider $j : V \rightarrow W$ and \mathcal{N} witnesses for the ASK-property for ψ_η with respect to G , $\mathcal{M} \models \psi_\eta$ and d_β . Let us show that j and an expansion of \mathcal{N} are the witnesses we are looking for.
- (i) By the ASK-property for ψ_η we have that \mathcal{N} is a $j(\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha < \eta\} \cup \{0^*, 1^*\})$ -structure. Expand \mathcal{N} to a $j(\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha < \eta\} \cup \{0^*, 1^*\}) \cup \{e_{j(\eta)}, X_{j(\eta)}^*\}$ denoted by \mathcal{N}^* by interpreting $(X_{j(\eta)}^*)^{\mathcal{N}}$ as $(X_\eta^*)^{\mathcal{M}}$ and $(e_\eta)^{\mathcal{N}}$ as $(e_\eta)^{\mathcal{M}}$.
- (ii) We have that $\mathcal{N}^* \models \check{\omega}_1^V \in j(d_\beta)$ since $\mathcal{N} \models \check{\omega}_1^V \in j(d_\beta)$ hold by the ASK-property relative to ψ_η .

(iii) Let ϕ be an atomic $\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha \leq \eta\} \cup \{0^*, 1^*\}$ -formula. Let us argue $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models j(\phi)$.

- For formulae in language $\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha < \eta\} \cup \{0^*, 1^*\}$ it holds by the ASK-property for ψ_η .
- For the new formulae it is a consequence of the definition we chose for the new symbols e_η and X_η^* together with $j(\alpha) = \alpha$ for $\alpha < \omega_1^V$ and $(X_\eta^*)^{\mathcal{N}^*} \subseteq \omega_1^V$.

(iv) We need to argue that

$$\mathcal{N}^* \models \psi_\eta \wedge (e_\eta = 0^* \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)).$$

- We have that $\mathcal{N}^* \models \psi_\eta$ by the ASK-property for ψ_η .
- The only subformula of $e_\eta = 0^* \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ whose complexity is a conjunction of formulae of complexity $(\bigwedge_\infty^\omega)$ is conjunct (ii) from $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$. But this is realized by \mathcal{N}^* since

$$(X_{j(\eta)}^*)^{\mathcal{N}} = (X_\eta^*)^{\mathcal{M}} \subseteq \omega_1^V \subseteq j(\omega_1^V).$$

Since $\mathcal{M} \models (e_\eta = 0^*) \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ and conjunct (i) of $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ is of complexity $(\bigwedge_\infty^\omega)$, the Preservation Lemma 4.2.3 ensures that if ϕ is conjunct (i) of $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)$ and it holds in \mathcal{M} , then $j(\phi)$ holds in \mathcal{N}^* . Now:

- if \mathcal{M} models $e_\eta = 1^*$, then the premise of $j(e_\eta = 0^* \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*))$ is false in \mathcal{N}^* ;
- if \mathcal{M} models $e_\eta = 0^*$, then ϕ holds in \mathcal{M} , hence $j(\phi)$ holds in \mathcal{N}^* .

In either cases

$$\mathcal{N}^* \models j(e_\eta = 0^* \Rightarrow \text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^*)),$$

as required.

Limit stage: Suppose the result true for $\alpha < \eta$ and let us prove it for η .

- ❖ First, by the Conservative Compactness Theorem 2.2.5 we have that ψ_η is a conservative strengthening of ψ_0 .
 - ❖ Second, let us argue the ASK-property for ψ_η .
1. We have that ψ_η is Boolean consistent since it is a conservative strengthening of ψ_0 and ψ_0 is Boolean consistent.

2. By definition ψ_η is the conjunction of ψ_0 with ϕ_α , $\alpha < \eta$. Since all of them are conjunctions of sentences of complexity $(\bigwedge_\infty^\omega)$ we have that ψ_η is a conjunction of sentences of complexity $(\bigwedge_\infty^\omega)$.
3. Let λ be large enough, let G be a filter V -generic for $\text{Coll}(\omega, < \lambda)$, let \mathcal{M} be a model of ψ_η and let $d_\beta \in \mathcal{L}$ be a constant for a stationary set.

(I) We have that

$$\mathbf{NS}_{\omega_1}^V = \mathbf{NS}_{\omega_1}^{M_{\omega_1^V}} \cap V$$

since any model of ψ_η is a model of ψ_0 and ψ_0 has the ASK-property.

- For items (II) and (III) we need to build $j : V \rightarrow W$ and \mathcal{N} the witnesses for the ASK-property.
- Fix $j : V \rightarrow W$ and \mathcal{N} witnesses for the ASK-property for ψ_0 with respect to G , $\mathcal{M} \upharpoonright \mathcal{L} \models \psi_0$ and d_β . Let us show $j : V \rightarrow W$ and an expansion of \mathcal{N} are the witnesses we are looking for.
- Expand \mathcal{N} to a structure denoted by \mathcal{N}^* for the language $j(\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha < \eta\} \cup \{0^*, 1^*\})$ as follows:

– for $\alpha < \eta$ let

$$(j(X_\alpha^*))^{\mathcal{N}^*} = (X_{j(\alpha)}^*)^{\mathcal{N}^*} = (X_\alpha^*)^{\mathcal{M}},$$

and

$$(j(e_\alpha))^{\mathcal{N}^*} = (e_{j(\alpha)})^{\mathcal{N}^*} = (e_\alpha)^{\mathcal{M}},$$

– for $\alpha \notin j[\eta]$, let $e_\alpha^{\mathcal{N}^*} = 1^*$ and $(X_\alpha^*)^{\mathcal{N}^*} = \emptyset$.

- We have that $\mathcal{N}^* \models \check{\omega}_1^V \in j(d_\beta)$ since $\mathcal{N} \models \check{\omega}_1^V \in j(d_\beta)$ holds by the ASK-property relative to ψ_η .
- (iii) Let ϕ be an atomic $\mathcal{L} \cup \{e_\alpha, X_\alpha^* : \alpha < \eta\} \cup \{0^*, 1^*\}$ -formula. Let us argue $\mathcal{M} \models \phi$ if and only if $\mathcal{N}^* \models j(\phi)$.
- For formulae in language \mathcal{L} it holds by the ASK-property for ψ_0 .
 - For the new formulae it is a consequence of the definition we chose for the new symbols e_α and X_α^* together with $j(\beta) = \beta$ for $\beta < \omega_1^V$ and $(X_\alpha^*)^{\mathcal{N}^*} \subset \omega_1^V$.

(iv) Let us argue $\mathcal{N}^* \models j(\psi_\eta)$. We have that $\mathcal{N}^* \models j(\psi_0)$ by the ASK-property for ψ_0 . We need to ensure

$$\mathcal{N} \models \bigwedge_{\alpha < j(\eta)} ((e_\alpha^* = 0^*) \Rightarrow \text{AS}(\beta_\alpha, \dot{X}_\alpha, X_\alpha^*)).$$

We have that axiom (ii) in $\text{AS}(\beta_\alpha, \dot{X}_\alpha, X_\alpha^*)$ holds in \mathcal{N}^* since $(X_\alpha^*)^{\mathcal{N}^*} \subset \omega_1^V$. The rest of the formulae appearing in $(e_\alpha^* = 0^*) \Rightarrow \text{AS}(\beta_\alpha, \dot{X}_\alpha, X_\alpha^*)$ are of complexity (\bigwedge_∞) for all $\alpha < j(\eta)$.

– For $\alpha < \eta$, since

$$\mathcal{M} \models (e_\alpha^* = 0^*) \Rightarrow \text{AS}(\beta_\alpha, \dot{X}_\alpha, X_\alpha^*),$$

we can repeat mutatis mutandis the argument of the case for $\psi_{\eta+1}$ of this proof to conclude

$$\mathcal{N} \models (e_{j(\alpha)}^* = 0^*) \Rightarrow \text{AS}(j(\beta_\alpha), j(\dot{X}_\alpha), j(X_\alpha^*))$$

by the Preservation Lemma 4.2.3.

– For $\alpha \notin j[\eta]$, since we set $e_\alpha^* = 1^*$, the implication is true as the premise is false.

• We have finished the proof of the theorem. □

5.4 The ASK property and stationary set preservation

In this section we argue that the forcing S_{ψ_κ} of the previous section is stationary set preserving whenever κ is a cardinal for which there exists a diamond sequence.

As said, we cannot ensure that the iteration $(\psi_\eta)_{\eta < \kappa}$ built in Limit Lemma 5.3.1 takes the formula $\text{AS}(\beta_\eta, \dot{X}_\eta, X_\eta^)$ into account at all stages. Lemma 5.4.1 below ensures that, whenever κ is a cardinal with a diamond sequence, this occurs stationarily often for each name for a club and each constant for a stationary set. We will use the results of Section 4.2, on $\bigwedge \bigvee$ formulas to ensure S_{ψ_κ} deals with all names for clubs and all constants for stationary sets. Theorem 5.4.2 will use the condition*

$$\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{M_{\omega_1}} \cap V$$

given by (I) from the ASK-property to grant that S_{ψ_κ} is stationary set preserving.

Lemma 5.4.1. *Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$ with the ASK-property.*

Let also:

- $\kappa > |\psi|$ be a regular cardinal with $\{(N_\eta, A_\eta) : \eta < \kappa\}$ a Diamond sequence on κ .³
- $\{0^*, 1^*\} \cup \{e_\eta, X_\eta^* : \eta < \kappa\}$ be fresh symbols for \mathcal{L} with $e_\eta, 0^*, 1^*$ constant symbols of sort -1 and X_η^* unary predicate symbols of sort \dot{N}_{ω_1} .
- ψ_κ be the $\mathcal{L}_{\infty\omega}$ -sentence of the preceding Limit Lemma 5.3.1, where $\dot{X}_\eta = \pi_0[A_\eta]$ and $\beta_\eta = \pi_2[A_\eta] < \omega_1$ whenever
 - $\pi_0[A_\eta]$ is a canonical ψ_η -name and
 - $A_\eta = \dot{X}_\eta \times B \times \beta_\eta$

and each π_i denotes the projection map on coordinate i .

Let \dot{X} be a canonical S_{ψ_κ} -name and let $\beta < \omega_1$. $S_{\dot{X}, \beta}$ consists of those η such that:

- $A_\eta = (\dot{X} \times \psi_\kappa \times \beta) \cap N_\eta$ and
- there is a map $k : N_\eta \rightarrow H_\lambda$ with N_η transitive and k with critical point η witnessing $(N_\eta, \dot{X} \cap N_\eta, \psi_\eta, \in_{\Delta_0}) \prec (H_\lambda, \dot{X}, \psi_\kappa, \in_{\Delta_0})$.

Then for any canonical ψ_κ -name \dot{X} and $\beta < \omega_1$ we have that $S_{\dot{X}, \beta}$ is stationary in κ and

$$\{\{e_\eta = 0^*\} : \eta \in S_{\dot{X}, \beta}\}$$

is predense in S_{ψ_κ} . In particular, forcing with S_{ψ_κ} realizes the sentence

$$\psi_\kappa \wedge \bigwedge \left\{ \bigvee \{e_\eta = 0^* : \eta \in S_{\dot{X}, \beta}\} : \dot{X} \text{ is a canonical } \psi_\kappa\text{-name and } \beta < \omega_1 \right\}.$$

Proof.

- The fact that

$$\mathcal{M}_G \models \psi_\kappa \wedge \bigwedge \left\{ \bigvee \{e_\eta = 0^* : \eta \in S_{\dot{X}, \beta}\} : \dot{X} \text{ is a canonical } \psi_\kappa\text{-name and } \beta < \omega_1 \right\}.$$

holds for all V -generic $G \subset S_{\psi_\kappa}$ is a consequence of Proposition 4.2.4 once we check

$$\{\{e_\eta = 0^*\} : \eta \in S_{\dot{X}, \beta}\}$$

is predense in S_{ψ_κ} .

- The Diamond property ensures $S_{\dot{X}, \beta}$ is stationary whenever \dot{X} is a canonical ψ_κ -name and $\beta < \omega_1$.

³I.e. a sequence such that $N_\eta \in H_\kappa$ is transitive for all η , and for some fixed $\lambda > \kappa$ and all $X \subseteq H_\kappa$ there are stationarily many η such that some $k : N_\eta \rightarrow H_\lambda$ defines an elementary embedding of (N_η, A_η, \dots) into (H_λ, X, \dots) .

- Let \dot{X} be a canonical S_{ψ_κ} -name and $\beta < \omega_1$. Let $s \in S_{\psi_\kappa}$ be any condition. We need to find $\eta \in S_{\dot{X},\beta}$ such that $s \cup \{e_\eta = 0^*\} \in S_{\psi_\kappa}$. This means finding $\eta \in S_{\dot{X},\beta}$ such that

$$\psi_\kappa \wedge \bigwedge s \wedge (e_\eta = 0^*)$$

is Boolean consistent.

- Find $\eta \in S_{\dot{X},\beta}$ such that :

- $s \in N_\eta$,
- $(\dot{X} \times \psi_\kappa \times \beta) \cap N_\eta = A_\eta$ and
- $(N_\eta, \dot{X} \cap N_\eta, \psi_\eta, \in_{\Delta_0}) \prec (H_\lambda, \dot{X}, \psi_\kappa, \in_{\Delta_0})$ via some $k : N_\eta \rightarrow H_\kappa$.

Then $\beta_\eta = \beta$.

- Since H_λ models $s \in S_{\psi_\kappa}$ we have that N_η models $s \in S_{\psi_\eta}$. In particular, s is a finite set of subsentences of ψ_η and does not contain any sentence using the constant e_η^* (in other words, s does not decide the value of e_η^*).
- $(N_\eta, \dot{X} \cap N_\eta, \psi_\eta, \in_{\Delta_0})$ models that $\dot{X} \cap N_\eta$ defines a canonical ψ_η -name since

$$(N_\eta, \dot{X} \cap N_\eta, \psi_\eta, \in_{\Delta_0}) \prec (H_\lambda, \dot{X}, \psi_\kappa, \in_{\Delta_0}).$$

- The statement $\dot{X} \cap N_\eta$ defines a canonical ψ_η -name is a Δ_1 -property in parameters $\dot{X} \cap N_\eta, \psi_\eta$, as:
 - the statement

$$\langle \text{op}(\check{\alpha}, \check{i}), s \rangle \in A_\eta$$

is Δ_0 in parameters $s, \dot{X} \cap N_\eta, \alpha, i$;

- the statement

$$D_{A_\eta, \psi_\eta, \alpha} \cup E_{A_\eta, \psi_\eta, \alpha} \text{ is predense in } S_{\psi_\eta}$$

is Δ_0 in parameters $\dot{X} \cap N_\eta, \psi_\eta$ for all $\alpha < \omega_1$.

Therefore both statements are computed the same way in $(N_\eta, \dot{X} \cap N_\eta, \psi_\eta, \in_{\Delta_0})$ and in V for all relevant s, i and α .

- Recall that ψ_η has the ASK-property by the Limit Lemma 5.3.1. Hence the Successor Theorem 5.2.1 ensures

$$\psi_{\eta+1}^* := \psi_\eta \wedge \text{AS}(d_\beta, A_\eta, X_\eta^*)$$

has the ASK-property and is a conservative extension of ψ_η .

- Since e_η^* does not appear in $\psi_{\eta+1}^*$,

$$\psi_{\eta+1}^{**} := \psi_{\eta+1}^* \wedge (e_\eta = 0^*)$$

is strongly conservative over $\psi_{\eta+1}^*$ and has the ASK-property.

- Consider a new iteration of same length κ where the first formula is $\psi_{\eta+1}^{**}$ and the following elements follow the original iteration from $\psi_{\eta+2}$ onwards. Denote as ψ_κ^* the last formula of this iteration. Then ψ_κ^* is (equivalent to) $\psi_\kappa \wedge (e_\eta^* = 0^*)$. Since the first formula $\psi_{\eta+1}^{**}$ of this new iteration has the ASK-property, the Limit Lemma ensures ψ_κ^* is conservative over $\psi_{\eta+1}^{**}$.
- Since $s \in S_{\psi_\kappa}$, we have that $\bigwedge s \wedge \psi_\kappa$ is Boolean consistent. In particular, so is $\bigwedge s \wedge \psi_\eta$.
- Since $\psi_{\eta+1}^{**}$ is a conservative strengthening of ψ_η and $\bigwedge s \wedge \psi_\eta$ is Boolean consistent, $\bigwedge s \wedge \psi_{\eta+1}^{**}$ is Boolean consistent.
- Since ψ_κ^* is a conservative strengthening of $\psi_{\eta+1}^{**}$ and $\bigwedge s \wedge \psi_{\eta+1}^{**}$ is Boolean consistent,

$$\bigwedge s \wedge \psi_\kappa^* = \bigwedge s \wedge \psi_\kappa \wedge (e_\eta^* = 0^*)$$

is Boolean consistent.

The proof is completed. □

Theorem 5.4.2. *Let \mathcal{L}^* be the signature of Notation 5.1.10 and ψ be an $\mathcal{L}_{\infty\omega}$ -sentence in some signature $\mathcal{L} \supseteq \mathcal{L}^*$ with the ASK-property.*

Let $\kappa > |\psi|$ be a cardinal with a diamond sequence and let ψ_κ be the formula defined in the preceding Lemma 5.4.1. Then S_{ψ_κ} is stationary set preserving.

Furthermore, for any constant for a stationary set d_β and any V -generic filter H for S_{ψ_κ} we have that $d_\beta^{M^H}$ is stationary in $V[H]$.

Proof.

- First of all, the ASK-property for ψ_κ ensures that

$$\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{M_{\omega_1}^V} \cap V$$

where $M_{\omega_1^V}$ is the interpretation of the sort \dot{N}_{ω_1} in \mathcal{M}_H and H is any filter V -generic for S_{ψ_κ} . Hence, any stationary subset S of ω_1 in V is in $V[H]$ of the form $d_\beta^{\mathcal{M}_H}$. It is then enough to argue that all sets of the form $d_\beta^{\mathcal{M}_H}$ are stationary in $V[H]$.

- Let \dot{X} be a canonical S_{ψ_κ} -name for a club and let d_β be a constant for a stationary set. By the preceding lemma we can find $\eta \in S_{\dot{X}, \beta}$ with $e_\eta = 0^* \in \Sigma_H$.
- Then $\text{AS}(\beta, \dot{X} \cap N_\eta, X_\eta^*)$ holds in \mathcal{M}_H and we can find $\gamma < \omega_1$ such that

$$\mathcal{M}_H \models \text{Name}(\psi, \gamma, \dot{X}, X^*) \wedge \text{Unbounded}(X^*, \gamma) \wedge (\text{Closed}(\gamma, X^*) \Rightarrow \check{\gamma} \in d_\beta)$$

- The proof will be completed once we show that

$$\gamma \in d_\beta^{\mathcal{M}_H} \cap \dot{X}_H.$$

- ✿ In order to prove $\gamma \in d_\beta^{\mathcal{M}_H}$ it is enough to check that

$$\mathcal{M}_H \models \text{Closed}(\gamma, X_\eta^*)$$

since

$$\mathcal{M}_H \models \text{Closed}(\gamma, X^*) \Rightarrow \check{\gamma} \in d_\beta.$$

- ✿ In order to prove $\gamma \in \dot{X}_H$ we proceed as follows:

- we prove that $(\dot{X} \cap N_\eta)_H \upharpoonright \gamma = \dot{X}_H \upharpoonright \gamma$,
- we prove that γ is a limit point of $(X_\eta^*)^{\mathcal{M}_H}$,
- using $\text{Name}(\gamma, \dot{X} \cap N_\eta, X_\eta^*)$ we get that $(\dot{X} \cap N_\eta)_H = (X_\eta^*)^{\mathcal{M}_H} \upharpoonright \gamma$,
- then γ is a limit point of $(\dot{X} \cap N_\eta)_H = \dot{X}_H \upharpoonright \gamma$ and
- since \dot{X}_H is club on ω_1 we have that $\gamma \in \dot{X}_H$.

- First, let us argue that $\dot{X} \cap N_\eta$ is in V an S_{ψ_η} -name for the characteristic function of a club subset of ω_1 . By the elementarity of N_η into H_λ we have that $\dot{X} \cap N_\eta$ is in N_η an S_{ψ_η} -name for the characteristic function of a club subset of ω_1 . Since this is expressible by a Δ_1 -property in parameters $\omega_1, \dot{X} \cap N_\eta$ and ψ_η , it is also in V a S_{ψ_η} -name for the characteristic function of a club subset of ω_1 .

- ✿ Let us argue that

$$(\dot{X} \cap N_\eta)_H \upharpoonright \gamma = \dot{X}_H \upharpoonright \gamma.$$

- $\text{Name}(\gamma, \dot{X} \cap N_\eta, X_\eta^*) \in \Sigma_H$ gives that for all $\alpha < \gamma$:

$$X_\eta^*(\check{\alpha}) \in \Sigma_H$$

if and only if there is some $s \in D_{\alpha, \dot{X} \cap N_\eta, \psi_\eta} \cap H$; and

$$\neg X_\eta^*(\check{\alpha}) \in \Sigma_H$$

if and only if there is some $s \in E_{\alpha, \dot{X} \cap N_\eta, \psi_\eta} \cap H$.

- By elementarity of N_η into H_λ we have that

$$D_{\alpha, \dot{X} \cap N_\eta, \psi_\eta} \subseteq D_{\alpha, \dot{X}, \psi_\kappa}$$

and

$$E_{\alpha, \dot{X} \cap N_\eta, \psi_\eta} \subseteq E_{\alpha, \dot{X}, \psi_\kappa}$$

for all $\alpha < \omega_1$.

- This gives that for all $\alpha < \gamma$ we have that

$$\begin{aligned} (\dot{X} \cap N_\eta)_H(\alpha) &= 1 \\ \Downarrow \\ X_\eta^*(\check{\alpha}) &\in \Sigma_H \\ \Downarrow \\ \text{there is some } s &\in D_{\alpha, \dot{X} \cap N_\eta, \psi_\eta} \cap H \\ \Downarrow \\ \text{there is some } s &\in D_{\alpha, \dot{X}, \psi_\kappa} \cap H \\ \Downarrow \\ \dot{X}_H(\alpha) &= 1. \end{aligned}$$

Similarly, we get that

$$\begin{aligned} (\dot{X} \cap N_\eta)_H(\alpha) &= 0 \\ \Downarrow \\ \dot{X}_H(\alpha) &= 0. \end{aligned}$$

✱ We have that γ is a limit point of $(X^*)^{\mathcal{M}_H}$ since $\mathcal{M}_H \models \text{Unbounded}(\gamma, X^*)$.

✱ Since \dot{X}_H is the characteristic function of a club subset of ω_1^V we have that $\dot{X}_H \upharpoonright \gamma = (\dot{X} \cap N_\eta)_H \upharpoonright \gamma$ is the characteristic function of a closed subset of γ . Therefore

$$\mathcal{M}_H \models \text{Closed}(\gamma, X_\eta^*)$$

and γ is a limit point of $(X_\eta^*)^{\mathcal{M}_H}$.

- Hence, we have proved

$$\gamma \in d_\beta^{\mathcal{M}_H} \cap \dot{X}_H$$

and $d_\beta^{\mathcal{M}_H}$ is stationary in $V[H]$.

□

5.5 SSP forcings are absorbed by the ASK-property

This section is devoted to the proof of the following result.

Theorem 5.5.1. *Let κ be inaccessible and $\mathbb{P} \in H_\kappa$ be SSP and forcing \mathbf{NS}_{ω_1} to be saturated. Then there exists ψ an $\mathcal{L}_{\infty\omega}$ -sentence with the ASK-property such that any model of ψ induces a V -generic filter for \mathbb{P} .*

Since Theorem 5.4.2 ensures that for such a sentence ψ there exists a conservative strengthening θ such that S_θ is SSP, we have that \mathbb{P} is a complete subforcing of S_θ . The proof is divided in three steps:

1. build a sentence $\psi_{\mathbb{P}}$ such that its models are precisely iterations of length ω_1^V whose last iterate is $H_\kappa[G]$ for some V -generic filter $G \subset \mathbb{P}$,
2. show that the sentence is Boolean consistent,
3. prove that $\psi_{\mathbb{P}}$ has the ASK-property.

In this section we will use repeatedly the following theorem.

Theorem 5.5.2 (Laver [9]). *There is an \in -formula $\phi_{\text{GenExt}}(x, y, z, w)$ with the following properties. If*

- N models ZFC and is transitive,
- $M^* \in N$ is a transitive model of ZFC^- ,
- $P \in M^*$ is a forcing notion and
- $G \in N$ is filter on P which is M^* -generic,

then there is at most one transitive class M which is the extension in N of $\phi_{\text{GenExt}}(x, M^*, G, P)$ such that:

- ✿ M is a model of ZFC,
- ✿ $M^* = H_\kappa^M$ for some κ cardinal in M .

Furthermore, whenever $M \subseteq N$ is a transitive model of ZFC, the following are equivalent:

- ✿ $N = M[G]$ is a generic extension of M by G which is an M -generic filter for $P \in M$;
- ✿ M is the extension in N in parameters P, G, H_κ^M for some (any) κ such that $P \in H_\kappa^M$ of $\phi_{\text{GenExt}}(x, H_\kappa^M, G, P)$.

Definition 5.5.3. Let $\mathbb{P} \in H_\kappa$ be a forcing notion with κ inaccessible and $\mathbb{P} \in H_\lambda$ for some $\lambda < \kappa$. The language $\mathcal{L}_\mathbb{P}$ extends the languages \mathcal{L}_{ite} and \mathcal{L}_{H_κ} with one constant symbol \dot{G} of sort \dot{N}_{ω_1} and unary predicate symbols H_κ^α of sort \dot{N}_α for each $\alpha < \omega_1$. Axiom $\psi_\mathbb{P}$ is the conjunction of the following sentences.

- (i) Axiom ψ_{ite} from example 3.2.4 to ensure any model produces an iteration of length ω_1^V where the iterates correspond with the sorts.
- (ii) Axiom ψ_{H_κ} from example 3.2.2 to ensure H_κ sits as a transitive subclass of the sort $\dot{N}_{\omega_1^V}$.
- (iii) Axioms to ensure \dot{G} is a filter H_κ -generic for \mathbb{P} .
 - (a) \dot{G} is a subset of \mathbb{P}

$$\bigwedge_{x \in H_\kappa} \check{x} \in_{\omega_1} \dot{G} \Rightarrow \check{x} \in_{\omega_1} \check{\mathbb{P}}.$$

- (b) \dot{G} is a filter.
- (c) \dot{G} meets all dense sets in V

$$\bigwedge_{D \subset \mathbb{P} \text{ dense}, D \in H_\kappa} \bigvee_{x \in D} \check{x} \in_{\omega_1} \dot{G}.$$

- (iv) Axioms to ensure the sort $\dot{N}_{\omega_1^V}$ equals $H_\kappa[G]$, such as:

$$j_{0\omega_1}(c_{00}) = \check{\mathbb{P}} \wedge j_{0\omega_1}(c_{10}) = \dot{G} \wedge j_{0\omega_1}(c_{20}) = \check{H}_\lambda$$

$$\text{Sat}_{N_0} \left(\forall x \exists \sigma (\sigma \text{ is a } c_{00}\text{-name} \wedge x = \sigma_{c_{10}} \wedge \phi_{\text{GenExt}}(\sigma, c_{20}, c_{10}, c_{00})) \right)$$

$$\bigwedge_{\alpha < \omega_1} \bigwedge_{n \in \omega} \left(H_\kappa^\alpha(c_{n\alpha}) \Leftrightarrow \text{Sat}_{N_\alpha}(\phi_{\text{GenExt}}(c_{n\alpha}, j_{0\alpha}(c_{20}), j_{0\alpha}(c_{10}), j_{0\alpha}(c_{00}))) \right)$$

$$\bigwedge_{\alpha < \omega_1} \left[H_\kappa^{\omega_1}(c_{\alpha\omega_1}) \Leftrightarrow \bigvee_{\beta < \omega_1, n < \omega} [(c_{\alpha\omega_1} = j_{\beta\omega_1}(c_{n\beta})) \wedge H_\kappa^\alpha(c_{n\beta})] \right]$$

$$\bigwedge_{\alpha < \omega_1} \left[H_\kappa^{\omega_1}(c_{\alpha\omega_1}) \Leftrightarrow \bigvee_{x \in H_\kappa^V} (c_{\alpha\omega_1} = \check{x}) \right].$$

Lemma 5.5.4. *Let $\mathbb{P} \in H_\kappa$ force \mathbf{NS}_{ω_1} to be saturated. Then $\psi_{\mathbb{P}}$ is Boolean consistent.*

Proof.

- Let H be a V -generic filter for $\text{Coll}(\omega, \kappa)$. Since $\mathcal{P}(\mathbb{P})^V$ is countable in $V[H]$, there exists G a filter V -generic for \mathbb{P} in $V[H]$.
- Since \mathbb{P} forces \mathbf{NS}_{ω_1} to be saturated, we have that $H_\kappa[G]$ is iterable in $V[H]$ (see Lemma 6.4.5 in Appendix 6.4). Denote by \mathcal{J}^* an \mathbf{NS}_{ω_1} -correct iteration of $H_\kappa[G]$ of length $\eta = \omega_1^{V[H]}$ in $V[H]$.
- Consider

$$\mathcal{J} = \{j_{\alpha\beta} : \alpha \leq \beta \leq \eta\}$$

the extension of \mathcal{J}^* given by Lemma 6.4.2 (see Appendix 6.4) to an iteration of $V[G]$ of length η . Denote by

$$j : V[G] \rightarrow M$$

the map $j_{0\eta}$. We have that $j(H_\kappa^{V[G]})$ is the last iterate of \mathcal{J}^* and by elementarity of j ,

$$j(H_\kappa^{V[G]}) = H_{j(\kappa)}^M.$$

- Consider $K \subset \text{Coll}(\omega, j(\lambda))$ a filter V -generic for λ large enough with K chosen such that $H \in V[K]$ and $H_{j(\kappa)}^M$ is countable in $M[K]$.
- Then \mathcal{J} witnesses that

$$H_{\omega_1}^{V[K]} \models \text{there exists an iteration of length } \omega_1^{V[H]} \text{ such that the last sort is } H_{j(\kappa)}^M.$$

- Since $\omega_1^M = \omega_1^{V[H]} = j(\omega_1^V)$ and $H_{j(\kappa)}^M = j(H_\kappa^V[G])$ is in $H_{\omega_1}^{M[K]}$, Shoenfield's absoluteness ensures that

$H_{\omega_1}^{M[K]} \models$ there exists an iteration of length $j(\omega_1^V)$ such that the last sort is $j(H_\kappa^V[G])$.

- By homogeneity of $\text{Coll}(\omega, j(\lambda))$ we have that

$M^{\text{Coll}(\omega, j(\lambda))} \models$ there exists an iteration of length $j(\omega_1^V)$ with last sort $j(H_\kappa^V[G])$.

- By elementarity of j we have that in the generic extension of V given by $\text{Coll}(\omega, \lambda)$ there exists an iteration of length ω_1^V whose last iterate is $H_\kappa[G]$. This iteration naturally gives rise to a model of $\psi_{\mathbb{P}}$. Therefore, the sentence $\psi_{\mathbb{P}}$ is Boolean consistent.

□

Lemma 5.5.5. *Let $\mathbb{P} \in H_\kappa$ be SSP and forcing NS_{ω_1} to be saturated. Then $\psi_{\mathbb{P}}$ has the ASK property.*

Proof. We check the conditions in the definition of the ASK-property 5.1.13. We follow closely the indexes from the definition.

- 1 The sentence is Boolean consistent by the previous Lemma ??.
- 2 The axiomatization of $\psi_{\mathbb{P}}$ we gave shows that it is a conjunction of sentences of complexity $\left(\bigwedge_{\omega} \bigvee_{\infty}\right)$.
- 3 Let $H \subset \text{Coll}(\omega, < \lambda)$ be a filter V -generic, let \mathcal{M} be a model of $\psi_{\mathbb{P}}$ in $V[H]$ with $M_{\omega_1^V}$ the domain for the sort $\dot{M}_{\omega_1^V}$ and let d_β be such that

$$M_{\omega_1^V} \models d_\beta \text{ is stationary.}$$

We need to argue all the conditions are met.

- (I) If $\mathcal{M} \models \psi_{\mathbb{P}}$, then the sort $M_{\omega_1^V}$ is $H_\kappa[G]$ for G a filter V -generic for \mathbb{P} . Since \mathbb{P} is SSP we have that

$$\text{NS}_{\omega_1}^V = \text{NS}_{\omega_1}^{V[G]} \cap V = \text{NS}_{\omega_1}^{M_{\omega_1^V}} \cap V.$$

- (II) Since \mathbb{P} forces \mathbf{NS}_{ω_1} to be saturated we have that $H_\kappa[G]$ is iterable (see Lemma 6.4.5 in Appendix 6.4). Denote by

$$\mathcal{J} = (M_\alpha, j_{\alpha\beta} : \alpha \leq \beta \leq \omega_1^{V[H]})$$

an iteration of length $\omega_1^{V[H]}$ that continues the iteration \mathcal{J}_M and is such that

$$d_\beta^M \in G_{\omega_1^V}.$$

Enlarge the iteration

$$\mathcal{J} \upharpoonright [\omega_1^V, \omega_1^{V[H]}) = (M_\alpha, j_{\alpha\beta} : \omega_1^V \leq \alpha \leq \beta \leq \omega_1^{V[H]})$$

using Lemma 6.4.2 (see Appendix 6.4) in $V[G]$ so that it becomes an iteration

$$\mathcal{J}^* = \left\{ j_{\alpha\beta}^* : \omega_1^V \leq \alpha \leq \beta \leq \omega_1^{V[H]} \right\}$$

of $V[G]$ and denote by

$$j : V[G] \rightarrow M$$

the map $j_{\omega_1^V \omega_1^{V[H]}}^*$ so that

$$j(H_\kappa[G]) = M_{\omega_1^{V[H]}} = H_{j(\kappa)}^M.$$

We have our witnesses for condition (II) since j has critical point ω_1^V and M is transitive.

- (III) Let us build \mathcal{N} realizing conditions (i)-(iv).

- (i) – Denote by \mathcal{N} the $j(\mathcal{L}_{ite})$ -structure generated by the iteration \mathcal{J} continued with the iteration \mathcal{J}^* .
- * We interpret the constants $c_{n\alpha}$ for $\alpha \in \omega_1^V$ in the same way as in the structure \mathcal{M} .
 - * The interpretation of the constants $c_{n\beta}$ for $\omega_1^V \leq \beta < \omega_1^{V[H]}$ is arbitrary.
 - * For the constants $c_{\alpha\omega_1^{V[H]}}$ we need to make sure that $c_{j(\alpha)j(\omega_1^V)}$ is interpreted by $j(c_{\alpha\omega_1^V}^M)$. For the other constants we can pick any interpretation.
- We have that $j(H_\kappa[G])$ is the domain of the sort $\dot{N}_{\omega_1^{V[H]}}$. Hence, every constant \tilde{x} with $x \in j(H_\kappa)$ can be interpreted as x .

- We have that $j(H_\kappa[G])$ is the domain of the sort $\dot{N}_{\omega_1^{V[H]}}$. Hence, we can interpret $j(\dot{G})$ as $j(G)$.
- We interpret the predicates H_κ^α for $\alpha < \omega_1^{V[H]}$ in the sort N_α as the extension in M_α of $\phi_{\text{GenExt}}(x, j_{0\alpha}(c_{20}), j_{0\alpha}(c_{10}), j_{0\alpha}(c_{00}))$. That is, since for each α the α -th iterate M_α is of the form $M_\alpha^*[j_{0\alpha}(c_{10})]$, we interpret H_κ^α to be M_α^* .
- We interpret $H_\kappa^{\omega_1^{V[H]}}$ as $j(H_\kappa)$.

(ii) We have that $\mathcal{N} \models \check{\omega}_1^V \in j(d_\beta)$ since $d_\beta^{\mathcal{M}} = d_\beta^{\mathcal{N}}$ and

$$d_\beta^{\mathcal{M}} \in G_{\omega_1^V}.$$

(iii) Let ϕ be an $\mathcal{L}_{\mathbb{P}}$ -atomic formula.

- If ϕ comes from \mathcal{L}_{ite} the equivalence comes from our choice of interpretations for the constants depending on the sort.
- Atomic formulae from \mathcal{L}_{H_κ} are of the form $\check{x} \in \check{y}$ with $x, y \in H_\kappa$. The equivalence comes from j being defined on all of $V[G]$.
- Assume ϕ is either $H_\kappa^\alpha(c_{n\alpha})$ for some $\alpha < \omega_1^V$. Then the equivalence follows from j being the identity below ω_1^V and \mathcal{N} copying the interpretations from \mathcal{M} for $\mathcal{J}_{\mathcal{M}}$.
- Assume ϕ is $c_{\beta\omega_1^V} \in \dot{G}$. Then the equivalence comes from the fact that we interpreted $c_{j(\alpha)j(\omega_1^V)}$ by $j(c_{\alpha\omega_1^V}^{\mathcal{M}})$ and $j(\dot{G})$ by $j(G)$.
- Assume ϕ is $H_\kappa^{\omega_1^V}(c_{n\omega_1^V})$. Then the equivalence comes from the fact that we interpreted $c_{j(\alpha)j(\omega_1^V)}$ by $j(c_{\alpha\omega_1^V}^{\mathcal{M}})$ and $H_\kappa^{\omega_1^{V[H]}}$ as $j(H_\kappa^V)$.

(iv) Finally,

$$\mathcal{N} \models j(\psi_{\mathbb{P}})$$

since

- being an iteration of length ω_1^M ensures $j(\psi_{ite})$ holds,
- the last iterate being $j(H_\kappa[G])$ ensures $j(\psi_{H_\kappa})$ holds,
- the last iterate being $j(H_\kappa[G])$ ensures $j(G)$ is a filter $j(H_\kappa^V)$ -generic for $j(\mathbb{P})$,
- the axioms concerning the predicates H_κ^α hold by elementarity of j .

□

Part III
Appendix

Chapter 6

Appendix

6.1 Boolean algebras

Let us recall the following basic facts about partial orders and their Boolean completions:

Definition 6.1.1. Given a Boolean algebra \mathbf{B} and a partial order $\mathbb{P} = (P, \leq)$:

- \mathbf{B}^+ denotes the partial order given by its positive elements and ordered by $a \leq_{\mathbf{B}} b$ if $a \wedge b = a$.
- \mathbf{B} is $< \lambda$ -complete if any subset of \mathbf{B} of size less than λ has an infimum and a supremum according to $\leq_{\mathbf{B}}$.
- A set $G \subset P$ is a prefilter if for any $a_1, \dots, a_n \in G$ we can find $b \in G$, $b \leq a_1, \dots, a_n$.
- A set $F \subset P$ is a filter if it is a prefilter and is upward close:

$$(a \in F \wedge a \leq b) \Rightarrow b \in F.$$

Remark 6.1.2. Given a partial order $\mathbb{P} = (P, \leq)$:

- The order topology on P is the one whose open sets are given by the downward closed subsets of P ; the sets $N_p = \{q \in P : q \leq p\}$ form a basis for this topology.
- $\text{RO}(P)$ is the complete Boolean algebra given by the regular open sets of the order topology on P .
- The map $p \mapsto \text{Reg}(N_p)$ defines an order and incompatibility preserving map of P into a dense subset of $(\text{RO}(P)^+, \subseteq)$; hence (P, \leq) and $(\text{RO}(P)^+, \subseteq)$ are equivalent forcing notions.

If \mathbf{B} is a Boolean algebra, \mathbf{B}^+ sits inside its Boolean completion $\text{RO}(\mathbf{B}^+)$ as a dense subset via the map $b \mapsto N_b$ (e.g. for all $A \in \text{RO}(\mathbf{B}^+)$ there is $b \in \mathbf{B}$ such that $N_b \subseteq A$). From now on we identify \mathbf{B} with its image in $\text{RO}(\mathbf{B}^+)$ via the above map.

6.2 Direct limits

Definition 6.2.1. Let \mathcal{L} be a language, (I, \leq) be a directed set, let $(\mathcal{M}_i : i \in I)$ be a family of \mathcal{L} -structures and let $\pi_{ij} : \mathcal{M}_i \rightarrow \mathcal{M}_j$ be an \mathcal{L} -morphism for every $i \leq j \in I$. The **direct limit** of the system $(\mathcal{M}_i, \pi_{ij} : i \leq j \in I)$ is denoted by

$$\lim_{i \in I} \mathcal{M}_i$$

and is the \mathcal{L} -structure given by the following.

- Let $\bigsqcup_{i \in I} M_i$ be the disjoint union of the grounds M_i . For $x_i \in M_i$ and $x_j \in M_j$ we say that

$$x_i \sim x_j$$

if for some $k \geq i, j$ we have that

$$\pi_{ik}(x_i) = \pi_{jk}(x_j).$$

The domain of $\lim_{i \in I} \mathcal{M}_i$ is given the equivalence classes of this equivalence relation.

- Let R be an n -ary relation symbol in \mathcal{L} . Define

$$\lim_{i \in I} \mathcal{M}_i \models R([x_{i_1}], \dots, [x_{i_n}])$$

if for some $j \geq i_1, \dots, i_n$ we have that

$$\mathcal{M}_j \models R(\pi_{i_1 j}(x_{i_1}), \dots, \pi_{i_n j}(x_{i_n})).$$

- Let $c \in \mathcal{L}$ be a constant symbol. Define

$$c^{\lim_{i \in I} \mathcal{M}_i} = [c_i^{\mathcal{M}_i}]$$

for some $i \in I$.

- For $i \in I$ define $\pi_i : \mathcal{M}_i \rightarrow \lim_{j \in I} \mathcal{M}_j$ by $\pi_i(x) = [x]$.

Lemma 6.2.2. Let \mathcal{L} be a language, let (I, \leq) be a directed set, let $(\mathcal{M}_i : i \in I)$ be a family of \mathcal{L} -structures and let $\pi_{ij} : \mathcal{M}_i \rightarrow \mathcal{M}_j$ be an \mathcal{L} -morphism for every $i \leq j \in I$.

- The direct limit $\lim_{i \in I} \mathcal{M}_i$ is an \mathcal{L} -structure.

- The induced maps π_i are \mathcal{L} -morphisms.
- The following commutativity property holds

$$\pi_{j \circ \pi_{ij}} = \pi_i.$$

- If the maps π_{ij} are elementary for every $i \leq j$ in I , then the maps π_i are elementary.

6.3 Stationary set preserving forcings

The goal of this section is to argue that assuming Woodin cardinals, SSP forcings are absorbed by SSP forcings that force \mathbf{NS}_{ω_1} to be saturated.

Definition 6.3.1. Let \mathbb{P} be a forcing notion. We say that \mathbb{P} is **stationary set preserving**, or simply **SSP**, if for any S stationary subset of ω_1^V , S remains stationary in $V^{\mathbb{P}}$.

Definition 6.3.2. The non stationary ideal is **saturated** if every maximal antichain in

$$\mathcal{P}(\omega_1) / \mathbf{NS}_{\omega_1}$$

has size at most ω_1 .

Suppose $\mathbb{P} \in H_\kappa$ is SSP and δ is a Woodin cardinal above κ . Let G be a V -generic filter for \mathbb{P} . Then δ remains Woodin in $V[G]$. Consider the following result which appears in [5] as Corollary 3.3.7.

Theorem 6.3.3. *Let δ be a Woodin cardinal. Then there exists \mathbb{Q} an SSP forcing notion such that in $V^{\mathbb{Q}}$ the non stationary ideal is saturated and $\delta = \omega_2^{V^{\mathbb{Q}}}$.*

If we let H be $V[G]$ -generic for the forcing stated in the previous theorem, then $V[G][H]$ models \mathbf{NS}_{ω_1} is saturated and $\delta = \omega_2^{V[G][H]}$. Hence, for any SSP forcing \mathbb{P} , if we assume the existence of a Woodin cardinal above $|\mathbb{P}|$, there exists \mathbb{P}' (given by $\mathbb{P} * \mathbb{Q}$) such that:

- \mathbb{P} is a complete subforcing of \mathbb{P}' ,
- \mathbb{P}' is SSP and
- \mathbf{NS}_{ω_1} is saturated in the generic extension given by \mathbb{P}' .

This might be of interest if one wishes to apply Theorem 5.5.1.

6.4 Iterable models

Definition 6.4.1. Let

$$M \models \text{ZFC}^-$$

be a countable transitive set. An **iteration** \mathcal{J} of M of length $\gamma \leq \omega_1$ is a sequence of models $\langle (M_\alpha, E_\alpha) : \alpha \leq \gamma \rangle$, a sequence of sets $\langle G_\alpha : \alpha < \gamma \rangle$ and a commuting family of embeddings

$$\langle j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : \alpha \leq \beta \leq \gamma \rangle$$

such that:

1. $(M_0, E_0) = (M, \in)$,
2. G_α is M_α -generic for $(\mathcal{P}(\omega_1) / \mathbf{NS}_{\omega_1})^{M_\alpha}$ for each $\alpha < \gamma$,
3. $j_{\alpha\alpha}$ is the identity map for each $\alpha < \gamma$,
4. $(M_{\alpha+1}, E_{\alpha+1}) \cong (\text{Ult}(M_\alpha, G_\alpha), \in_{G_\alpha})$ and $j_{\alpha\alpha+1}$ is (modulo isomorphism) the ultrapower embedding for each $\alpha < \gamma$,
5. (M_δ, E_δ) is (isomorphic to) the direct limit of the system $\{(M_\alpha, E_\alpha), j_{\alpha\beta} : \alpha \leq \beta < \delta\}$ for every limit ordinal $\delta \leq \gamma$, and $j_{\alpha\delta}$ is (modulo isomorphism) the induced embedding for each $\alpha < \delta$ (see Appendix 6.2 for a precise definition).

The following result appears in [23] as Lemma 3.8 and in [8] as Lemma 1.5. Since both omit the proof, let us present it as it will be used in the proofs of Theorem 5.5.1 and Lemmas 5.5.4, 5.5.5.

Lemma 6.4.2 (Upward extension of iterations). *Let $M \in N$ be transitive sets such that*

- M is iterable,
- $M, N \models \text{ZFC}^-$,
- $\omega_1^M = \omega_1^N$,
- $N \models \mathbf{NS}_{\omega_1}$ is saturated and
- M is closed under sequences of length ω_1^M existing in N , that is for any function

$$f : \omega_1^M \rightarrow M$$

belonging to N , we have that $f \in M$.

Then for any generic iteration

$$\mathcal{J} = \{M_\alpha, G_\alpha, j_{\alpha\beta} : \alpha \leq \beta \leq \gamma\}$$

of length γ of M there exists

$$\mathcal{J}' = \{N_\alpha, G'_\alpha, j'_{\alpha\beta} : \alpha \leq \beta \leq \gamma\}$$

a generic iteration of length γ of N such that

- (i) $G_\alpha = G'_\alpha$,
- (ii) $j'_{\alpha\beta}(M_\alpha) = M_\beta$ and as a consequence $M_\alpha \in N_\alpha$ and M_α is transitive inside N_α ,
- (iii) $j'_{\alpha\beta}(x) = j_{\alpha\beta}(x)$ for $x \in M_\alpha$,
- (iv) $\omega_1^{M_\alpha} = \omega_1^{N_\alpha}$ and
- (v) M_α is closed under $\omega_1^{M_\alpha}$ -sequences in N_α .

For the proof we will need the following result.

Lemma 6.4.3 (Upward extension of filters). *Let $M \subset N$ be countable transitive models such that*

- $M, N \models \text{ZFC}^-$,
- $\omega_1^M = \omega_1^N$,
- $N \models \mathbf{NS}_{\omega_1}$ is saturated and
- M is closed under sequences of length ω_1^M existing in N , that is for any function

$$f : \omega_1^M \rightarrow M$$

belonging to N , we have that $f \in M$.

Then every filter M -generic for $\mathbf{NS}_{\omega_1}^M$ is also a filter N -generic for $\mathbf{NS}_{\omega_1}^N$.

Proof. Since M and N have the same first uncountable ordinal and M is closed under ω_1^M -sequences in N , both models have the same stationary sets. Then G is a filter on $\mathbf{NS}_{\omega_1}^N$. Let us argue that it is N -generic. Consider \mathcal{A} a maximal antichain in N . Since

$$N \models \mathbf{NS}_{\omega_1} \text{ is saturated}$$

there exists $f : \omega_1^M \rightarrow \mathcal{A} \subset M$ an enumeration of \mathcal{A} in order type ω_1^M . Since M is closed under ω_1^M -sequences in N , this enumeration is also in M and \mathcal{A} belongs to M . Then $G \cap \mathcal{A}$ is non empty since G meets all maximal antichains in M . \square

Proof. Any iteration is determined by the first model and the sequence of generic filters: at successor stages the definition of iteration forces to consider the ultrapower given by the generic filter and at limit stages the definition of iteration forces to consider the direct limit. Hence, in order to argue the existence of a generic iteration for N , it is enough to argue the existence of a sequence of N_α -generic filters $G'_\alpha \subset (\mathcal{P}(\omega_1)/\mathbf{NS}_{\omega_1})^{N_\alpha}$.

Since we claim that $G_\alpha = G'_\alpha$, building the iteration comes down to check that G_α is N_α -generic for $(\mathcal{P}(\omega_1)/\mathbf{NS}_{\omega_1})^{N_\alpha}$ for all $\alpha \in \gamma$.

Since Lemma 6.4.3 ensures that whenever conditions (ii), (iv) and (v) are met G_α is N_α -generic for $(\mathcal{P}(\omega_1)/\mathbf{NS}_{\omega_1})^{N_\alpha}$, we only need to deal with conditions (ii)-(v).

The proof is by induction on the length γ of the iteration.

- ✱ Suppose $\gamma = 0$. Then all conditions that are not vacuous are met by hypothesis.
- ✱ Suppose $\gamma = \alpha + 1$. By induction hypothesis the result holds true for the iteration restricted to length α . Let us extend it to $\alpha + 1$.

(ii) Proceed by double inclusion. Suppose

$$\text{Ult}(N_\alpha, G_\alpha) \models [f]_{G_\alpha} \in j'_{\alpha\alpha+1}(M_\alpha).$$

We need to argue that $[f]_{G_\alpha} \in M_{\alpha+1} = \text{Ult}(M_\alpha, G_\alpha)$. Since

$$j'_{\alpha\alpha+1}(M_\alpha) = [c_{M_\alpha}]_{G_\alpha}$$

there exists $S \in G_\alpha$ such that for all $\beta \in S$ we have that $f(\beta) \in c_{M_\alpha}(\beta) = M_\alpha$. Define f' by setting $f'(\alpha) = f(\alpha)$ for $\alpha \in S$ and $f'(\alpha) = 0$ for $\alpha \notin S$. Since M_α is closed under $\omega_1^{M_\alpha}$ -sequences in N_α and $f(\alpha) \in M_\alpha$ for $\alpha \in S$, we have that $f' \in M_\alpha$. Since $S \in G_\alpha$, we have that

$$[f]_{G_\alpha} = [f']_{G_\alpha} \in \text{Ult}(M_\alpha, G_\alpha).$$

For the other inclusion suppose

$$[f]_{G_\alpha} \in M_{\alpha+1} = \text{Ult}(M_\alpha, G_\alpha).$$

By definition of $\text{Ult}(M_\alpha, G_\alpha)$ we have that $f \in M_\alpha$. Then for all $\beta \in \omega_1^{M_\alpha}$ we have that $f(\beta) \in M_\alpha = c_{M_\alpha}(\beta)$ and

$$\text{Ult}(N_\alpha, G_\alpha) \models [f]_{G_\alpha} \in j'_{\alpha\alpha+1}(M_\alpha).$$

(iii) Let $x \in M_\alpha$. We have that

$$j_{\alpha\alpha+1}(x) = [c_x]_{G_\alpha} = j'_{\alpha\alpha+1}(x).$$

(iv) We have that

$$\omega_1^{M_{\alpha+1}} = j_{\alpha\alpha+1}(\omega_1^{M_\alpha}) = j'_{\alpha\alpha+1}(\omega_1^{N_\alpha}) = \omega_1^{N_{\alpha+1}}.$$

(v) Let us argue that $M_{\alpha+1}$ is closed under $\omega_1^{M_\alpha}$ -sequences in $N_{\alpha+1}$. Let

$$N_{\alpha+1} \models [f]_{G_\alpha} \text{ is a function with domain } \omega_1 \text{ and range } j'_{\alpha\alpha+1}(M_\alpha).$$

Then for some $S \in G_\alpha$ and for every $\beta \in S$ we have that

$$N_\alpha \models f(\beta) \text{ is a function with domain } \omega_1 \text{ and range } M_\alpha.$$

Define

$$\begin{aligned} g : \omega_1^{M_\alpha} &\rightarrow M_\alpha \\ \beta &\mapsto f(\beta) \quad \text{for } \beta \in S \\ \beta &\mapsto 0 \quad \text{for } \beta \notin S. \end{aligned}$$

We have that $g \in M_\alpha$ since M_α is closed under $\omega_1^{M_\alpha}$ -sequences in N_α . Finally,

$$[g]_{G_\alpha} = [f]_{G_\alpha} \quad \text{and} \quad [g]_{G_\alpha} \in M_{\alpha+1}.$$

✿ Suppose γ is a limit ordinal. In the following arguments we will assume $\alpha \leq \beta < \gamma$ for notational simplicity but the same arguments work for any other arrangement of the indexes.

(ii) We proceed by double inclusion. Assume $[x] \in j'_{\alpha\gamma}(M_\alpha)$. Then for some $\beta \geq \alpha$

$$N_\beta \models x \in j'_{\alpha\beta}(M_\alpha) = M_\beta.$$

Since $j'_{\alpha\beta}(M_\alpha) = M_\beta$ and M_β is transitive inside N_β we have that $x \in M_\beta$ and $[x] \in M_\gamma$. For the other inclusion assume $[x] \in M_\gamma$. Then $x \in M_\alpha$ for some $\alpha < \gamma$ and

$$[x] = j_{\alpha\gamma}(x) \in j'_{\alpha\gamma}(M_\alpha).$$

(iii) Let $x \in M_\alpha$. We have that

$$j_{\alpha\beta}(x) = j'_{\alpha\beta}(x)$$

for any β . Then

$$j_{\alpha\gamma}(x) = [x] = [j_{\alpha\beta}(x)] = [j'_{\alpha\beta}(x)] = [x] = j'_{\alpha\gamma}(x).$$

(iv) We have that

$$\omega_1^{M_\gamma} = j_{0\gamma}(\omega_1^{M_\gamma}) = j'_{0\gamma}(\omega_1^{N_\gamma}) = \omega_1^{N_\gamma}.$$

(v) Let $[f] : \omega_1^{M_\gamma} \rightarrow M_\gamma$ be in N_γ . Then for some $\alpha < \gamma$ we have that

$$N_\alpha \models f : \omega_1^{M_\alpha} \rightarrow M_\alpha.$$

Then $f \in M_\alpha$ since M_α is closed under $\omega_1^{M_\alpha}$ -sequences in N_α . Hence, $[f] \in M_\gamma$.

□

Remark 6.4.4. Let $G \subset \text{Coll}(\omega, \delta)$ be a filter V -generic such that H_κ^V is countable in $V[G]$ for $\kappa \geq \omega_2^V$. Let \mathcal{J} be an iteration of H_κ^V of length $\gamma \leq \omega_1^{V[H]}$. Let us argue why the previous lemma allows to enlarge the domain of the iterates so that \mathcal{J} becomes “an iteration of V ”. Let λ be a regular cardinal above κ . Assume \mathbf{NS}_{ω_1} is saturated in V . We have that H_κ^V is closed under ω_1^V -sequences in H_λ^V since $\kappa \geq \omega_1^V$. Hence, all conditions from the previous lemma are met by the couple $(H_\kappa^V, H_\lambda^V)$ and the iteration can be enlarged from H_κ^V to H_λ^V in a coherent way. Repeating this argument for all cardinals λ in the class of regular cardinals above κ produces a class iteration of V since all the extensions are coherent by condition (iii).

Lemma 6.4.5. *Assume the non stationary ideal is saturated in V and κ is a cardinal in V larger than ω_1^V . If G is a filter V -generic such that in $V[G]$ the set H_κ^V is countable, then H_κ^V is iterable in $V[G]$.*

The proof of the above result is based on Lemma 6.4.2, the fact that being an iterable model is Π_2^1 in the codes and the following Lemma which appears in [8] as Lemma 1.6.

Lemma 6.4.6. *Suppose M is a countable transitive model of ZFC^- , $\mathbf{NS}_{\omega_1}^M$ is precipitous and \mathcal{J} is an iteration of M of length $\eta \in M$. Then M_η is well founded.*

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