Constraints on the action of effective theories in quantum gravity

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PhD dissertation

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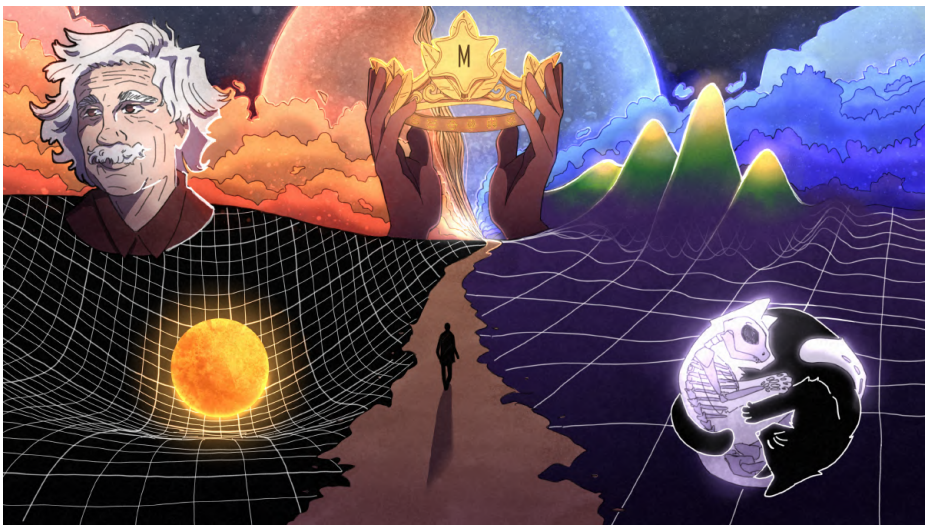

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supervised by
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Probably a great number of PhD dissertations in theoretical physics, mathematics or other abstract subjects start with the quote from Cyrano de Bergerac's death scene: "*c'est bien plus beau lorsque c'est inutile*". Indeed over the course of the last three years I have been able to appreciate the beauty in things I would never have imagined before. Over this great adventure I have been blessed to have been accompanied by many people who helped me throughout this time. I would like to thank them now.

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After three years of PhD I feel like I have learned more than in any other period of my life and I would therefore like to complete my initial quote from Cyrano de Bergerac with another equally famous quote from the Petit Prince: "*c'est véritablement utile puisque c'est joli*".

Louise Dupuis, *M-theory is the solution* (2024), from an original draft by Diego Ruiz. This artwork was created as the thumbnail of a YouTube video titled "*Le plan pour finir la physique*" hosted on the channel "Stream theory" and loosely inspired by the work in this thesis.

À mon grand-père,

Abstract

String theory constitutes one of the most popular and studied framework to approach quantum gravity. It is well known that the low energy limit of string theory gives a wide range of effective field theories. One recent and promising way to extract information about quantum gravity from this string landscape has been the swampland program. The string lamppost principle postulates that the quantum gravity landscape and the string landscape coincide. In this thesis we propose to study this claim for the case of higher order Wilson couplings in the very restricted case of maximal supersymmetry.

We first study the low energy limit of genus 0, 1 and 2 string amplitudes for type II string theory compactified on a torus and compare them to tree level, 1-loop and 2-loop maximal supergravity amplitudes respectively. This allows us to compute the perturbative contributions to the leading Wilson coefficients of maximally supersymmetric string theory. We also show that in dimension 8 logarithmic divergences of the supergravity amplitudes can be linked to divergences of Wilson couplings. We give a prescription to properly regularise the divergence by using the finite string amplitude.

We then use the differential equations entailed by the supersymmetric Ward identities as well as the constraints imposed by U-duality to derive the full non perturbative Wilson coefficients for maximally supersymmetric string theory in dimensions higher or equal to 6. These are given for the leading and next to leading Wilson coefficients by Eisenstein and Epstein series for the relevant U-duality group. The parabolic Fourier expansions of these series can then be used to check the different degeneration limits of the Wilson coefficients.

Finally we study the minima of these functions on moduli space to give lower bounds on Wilson coefficients coming from maximally supersymmetric string theory. This implies finding the minima of Epstein series for special values of the s parameter. We first extend Grenier's recursive construction of a fundamental domain to almost any simple Lie group which allows us to properly define the domain of study

of our functions. We then show that symmetric points are necessarily extrema of automorphic forms and give precise criteria for them to be minima. We also identify these symmetric points as corners of fundamental domains. We study relevant symmetric points for the case of $SL(n)$ and $SO(n,n)$ groups and give additional density arguments regarding the global minima for large s parameter. We then checked our conjecture numerically for the cases $n = 5$ relevant for dimensions 7 and 6.

These lower bounds should then be compared to lower bounds coming from unitarity constraints using S-matrix bootstrap methods. As far as we know this analysis still needs to be performed in dimensions lower or equal to 8. We have shown that in dimension 6 factorisation properties of maximally supersymmetric amplitudes imply that the unitarity properties of superamplitudes reduce to the unitarity properties of scalar amplitudes. These kind of factorisations also exist in other dimensions but do not always lead to such drastic simplifications. However one can always restrict to elastic scattering to make the numerics bearable. This kind of analysis, if successful, would be a strong argument for the validity of the string lamppost principle in the case of maximal supersymmetry.

Résumé en français

La théorie des cordes constitue l'un des cadres les plus populaires et étudiés pour aborder la gravitation quantique. Il est bien connu que la limite basse énergie de la théorie des cordes donne une large gamme de théories des champs effectives. Une manière récente et prometteuse d'extraire des informations sur la gravitation quantique à partir de ce paysage de cordes est le programme du "marais". Le principe du "lampadaire des cordes" postule que le paysage de la gravité quantique et le paysage des cordes coïncident. Dans cette thèse, nous proposons d'étudier cette affirmation dans le cas des couplages de Wilson d'ordre supérieur dans le cadre très restreint de la supersymétrie maximale.

Nous étudions d'abord la limite basse énergie des amplitudes de genre 0, 1 et 2 pour la théorie des cordes de type II compactifiée sur un tore et les comparons respectivement aux amplitudes de supergravité maximale à l'ordre des arbres, à 1 boucle et à 2 boucles. Cela nous permet de calculer les contributions perturbatives aux coefficients de Wilson dominants de la théorie des cordes maximalement supersymétrique. Nous montrons également qu'en dimension 8, les divergences logarithmiques des amplitudes de supergravité peuvent être liées aux divergences des couplages de Wilson. Nous donnons une prescription pour régulariser correctement la divergence en utilisant l'amplitude de corde finie.

Nous utilisons ensuite les équations différentielles induites par les identités de Ward supersymétriques ainsi que les contraintes imposées par la U-dualité pour dériver les coefficients de Wilson non perturbatifs complets pour la théorie des cordes maximalement supersymétrique en dimensions supérieures ou égales à 6. Ceux-ci sont donnés pour les coefficients de Wilson dominant et subdominants par des séries d'Eisenstein et d'Epstein pour le groupe de U-dualité pertinent. Les développements de Fourier paraboliques de ces séries peuvent alors être utilisés pour vérifier les différentes limites de dégénérescence des coefficients de Wilson.

Enfin, nous étudions les minima de ces fonctions sur l'espace des modules pour

donner des bornes inférieures sur les coefficients de Wilson provenant de la théorie des cordes maximalement supersymétrique. Cela implique de trouver les minima des séries d'Epstein pour des valeurs spéciales du paramètre s. Nous étendons d'abord la construction récursive de Grenier d'un domaine fondamental à presque tous les groupes de Lie simples, ce qui nous permet de définir correctement le domaine d'étude de nos fonctions. Nous montrons ensuite que les points symétriques sont nécessairement des extrêma des formes automorphes et donnons des critères précis pour qu'ils soient des minima. Nous identifions aussi ces points symétriques comme des coins de domaines fondamentaux. Nous étudions des points symétriques pertinents pour le cas des groupes $SL(n)$ et $SO(n,n)$ et donnons des arguments supplémentaires portant sur la densité concernant les minima globaux pour des valeurs élevées du paramètre s. Nous avons ensuite vérifié notre conjecture numériquement pour les cas $n = 5$ pertinents pour les dimensions 7 et 6.

Ces bornes inférieures devraient ensuite être comparées aux bornes inférieures provenant des contraintes d'unitarité en utilisant des méthodes de bootstrap de la matrice S. À notre connaissance, cette analyse reste encore à être effectuée en dimensions inférieures ou égales à 8. Nous avons montré qu'en dimension 6, les propriétés de factorisation des amplitudes maximalement supersymétriques impliquent que les propriétés d'unitarité des super-amplitudes se réduisent aux propriétés d'unitarité des amplitudes scalaires. Ces types de factorisations existent également dans d'autres dimensions mais ne conduisent pas toujours à de telles simplifications drastiques. Cependant, on peut toujours se restreindre à la diffusion élastique pour rendre les calculs numériques plus abordables. Ce type d'analyse, s'il réussit, constituerait un argument en faveur de la validité du principe du "lampadaire des cordes" dans le cas de la supersymétrie maximale.

Contents

Publications

This manuscript is based on the following publication

- G. Bossard and A. Loty, "Saturating unitarity bounds at U-duality symmetric points," *JHEP* **10** (2023), 110, arXiv:2308.02847 [hep-th].

The two following works were in preparation at the time of writing of this manuscript[1]

- G. Bossard, and A. Loty, " Bounds on the next-to-leading Wilson coefficient in maximal supergravity," CPHT-RR001-012025.
- G. Bossard, G. Casagrande, E. Dudas and A. Loty, "A unique coupling of the massive spin-2 field to supergravity," CPHT-RR002-012025.

[1]The titles may not be definitive.

Chapter 1

Introduction

Throughout this thesis we will use standard natural units where $c = \hbar = 1$. We will also use the mostly plus metric signature $(-, +, \ldots, +)$.[1]

1.1 Quantum gravity

Of the four known fundamental forces: gravity, electromagnetism, the strong interaction and the weak interaction, gravity is certainly the one that has been known for the longest time. Owing to its strictly attractive nature and the absence of negative masses, any screening effect that would result in neutral objects at the macroscopic level, like in the case of electromagnetism, is absent for gravity.[2] This is the reason why ancient philosophers were always able to observe the mysterious effects of gravity such as the fall of objects or the orbits of heavenly bodies. Some even tried to come up with models and explanations for gravity such as Aristotle who tried to explain gravitational attraction using his four elements theory. Of course it was Newton who first linked the fall of objects on earth and orbits of celestial bodies to a single phenomenon, gravity. It was also Newton who gave us our first universal theory of gravity via his famous equation

$$\vec{g} = -\frac{Gm}{r^2}\vec{e}_r\,, \tag{1.1.1}$$

where $\vec{g}$ is the gravitational field generated by a particle of mass m. G is the gravitational constant, r is the radial distance from the particle and $\vec{e}_r$ is a radially oriented,

[1] In any case, it was made abundantly clear by my supervisor Guillaume that, had I used the opposite convention, he could not have worked with me.

[2] the reasons for the other two fundamental forces to be invisible at the macroscopic level are a bit more subtle, they are linked to the property of confinement in the case of the strong interaction and the masses of the W and Z bosons in the case of the weak interaction.

outward pointing unit vector. In analogy to the link between Coulomb's law and Gauss's law in electromagnetism equation (1.1.1) can be made to look more like a field equation using Poisson's equation

$$\Delta\phi = 4\pi G\rho\,, \tag{1.1.2}$$

where ϕ is the gravitational potential related to the gravitational field by $\vec{g} = -\vec{\nabla}\phi$, Δ is the three dimensional Laplacian and ρ is the mass distribution acting as the source of gravity. Equations (1.1.1) and (1.1.2) have a spectacular range of applicability allowing such diverse computations as the orbital mechanics of the solar system to the free fall of objects on earth. The precision of these laws is such that it allowed to predict the existence the planet Neptune from perturbations in the orbit of Uranus as well as the landing of the first man on the moon almost three hundred years after their publication.

There was, however, a conceptual issue with this theory which was first raised by Newton himself but was not resolved for two centuries. This is the fact that equation (1.1.2) lacks a time derivative, therefore changes in the sources on the right hand side of the equation result in instantaneous modification of the resulting gravitational field on the left hand side of the equation throughout all space. This is in contrast to the case of electromagnetism where the presence of time derivatives make it so that perturbations in the electromagnetic field travel at finite speed through space. These sort of instantaneous actions at a distance became especially troubling in the beginning of the twentieth century when it appeared that this was manifestly in contradiction with the theory of special relativity developed by Einstein. Motivated by this, as well as a mismatch in the theoretical calculation of Mercury's perihelion shift, Einstein set out to find proper field equations of gravity in analogy with Maxwell's equations for electromagnetism which were compatible with special relativity. Einstein came up with the general theory of relativity whose field equations are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.1.3}$$

where $g_{\mu\nu}$ is the metric tensor describing the gravitational field, $R_{\mu\nu}$ and R are the Ricci and scalar curvatures respectively and $T_{\mu\nu}$ is the stress energy tensor replacing the mass density as the source of gravity.

Why should gravity be quantized? Obviously, since its inception in 1915 general relativity has enjoyed great experimental success. In the years following immediately after its publication several experimental tests validated the theory such as the calculation of Mercury's perihelion shift and the deviation of light from the sun's

gravitational field. Since then, further experimental confirmations have been abundant such as gravitational time dilation, gravitational lensing, gravitational redshift of light, the Shapiro time delay effect, the discovery of black holes, the detection of gravitational waves [1] etc. On top of that, general relativity provides the right framework to explain cosmological discoveries such as the expansion of the universe and the cosmic microwave background. Enumerating all the successes of general relativity would take up more space than this entire thesis, therefore we will not risk ourselves.

So far no experimental test of general relativity has revealed any discrepancy with experimental data. One would therefore be entitled to ask why would anyone ever need another theory of gravity, much less a quantum one. These doubts are further justified by the fact that the gravitational interaction is orders of magnitude weaker than the other three fundamental interactions. A simple computation shows us that the ratio of the fine structure constant $\alpha = e^2/4\pi$ to the gravitational structure constant $\alpha_G = Gm_p^2$ is given by[3]

$$\frac{\alpha}{\alpha_G} \sim 10^{36}\,, \tag{1.1.4}$$

where e and m_p are the charge and the mass of the proton respectively. This shows that there is absolutely no hope of measuring any gravitational effects in matter interactions within colliders where quantum effects are usually observed.

Another insightful computation is the one loop quantum correction to the gravitational potential. For the scattering of two massive particles of mass m_1 and m_2 this is given by [2]

$$V(r) = -G\frac{m_1 m_2}{r}\left(1 + 3G\frac{m_1+m_2}{r} + \frac{41}{10\pi}\frac{l_P^2}{r^2} + \dots\right)\,, \tag{1.1.5}$$

where $l_P = \sqrt{G}$ is the Planck length. The first term comes from the newtonian theory, the second term is the general relativistic correction and the third term is the leading quantum correction. If we take $m_1 = M_S$ the mass of the sun and neglect m_2 then even for $r = R_S = 2GM_S$ the Schwarzschild radius of the sun, when the relativistic correction is of order $GM_S/R_S \sim 1$ then the quantum correction is still of order

$$\frac{l_P^2}{R_S^2} \sim 10^{-76}\,. \tag{1.1.6}$$

This shows that the effects of quantum gravity should be completely suppressed at any scale available for experiments. Hence there is at the present no empirical imperative

[3] the ratio with the other fundamental forces is approximately the same, even bigger in the case of the strong interaction

to find a quantum theory of gravity as our current theories well exceed the existing experimental bounds.

However there are limits to the applicability of general relativity even if those limits are well outside the current experimental bounds. For example, it is well known that many solutions of general relativity exhibit curvature singularities. This is even the case for phenomenologically realistic solutions such as black hole solutions or FLRW solutions. These solutions predict that matter density should be infinite at these curvature singularities, however one should expect quantum effects of matter to be relevant at some scale before this point is reached. In such a situation equation (1.1.3) relates the stress energy tensor of quantized matter to the classical geometry of spacetime which is conceptually incoherent. Hence we would need a quantum theory of gravity in order to consistently couple gravity to quantum matter to describe these situations. Equation (1.1.5) tells us that quantum effects of gravity are expected to appear at energy scales $1/l_P \sim 10^{19}$ GeV.

Problems with quantum gravity The formalism of quantum field theory was developed over the course of the twentieth century by such people as Feynman, Schwinger, Dirac and many others with equal success to that of general relativity. The initial confusions about UV divergences were solved by the introduction of renormalisation and the renormalisation group equations. This was even applied to non abelian gauge theories such as Yang-Mills theory which was proved to be renormalisable by 'tHooft in 1971 [3]. The same techniques can be applied to general relativity which can also be brought into a form compatible with the formalism of perturbative quantum field theory by linearizing around a flat background

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \,, \tag{1.1.7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is the graviton field and $\kappa^2 = 32\pi G$. This type of linearized gravity was already studied in the 30s by Fierz, Pauli and Rosenfeld and in the 60s Feynman and DeWitt used this formalism to extract the Feynman rules for gravity. It was even shown by 'tHooft and Veltman that pure general relativity was finite at the one loop order [4]. However trouble knocked at the door when it was realised in the 70s that this naive quantization of general relativity was perturbatively non renormalisable at higher loop orders [5] as was already realised by 'tHooft and Veltman at the one loop level in presence of matter [4]. This can be seen from the negative mass dimension of the gravitational coupling constant G which suggests that new divergent diagrams naively appear at every order of perturbation. Thus regularisation of the effective action would require introducing an infinite number of counter terms making the theory ill defined. An example of such a divergence was

found by Goroff and Sagnotti who showed that UV divergences appeared at the two loop level in pure general relativity [6]. This comes from three graviton diagrams with ghost loops such as given in figure 1.1.

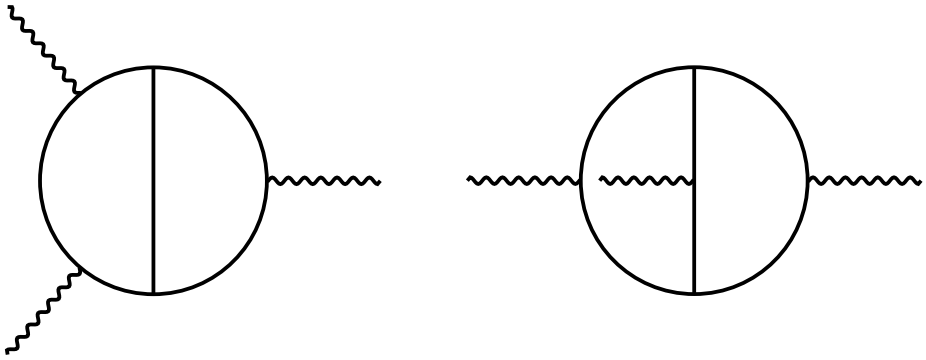


Figure 1.1: Example of diagrams contributing to the two loop divergence of pure general relativity. Gravitons are represented by wiggly lines while ghosts are represented by straight lines.

The divergence of the two loop effective action is then given by

$$\Gamma_{\infty}^{\text{2-loop}} = \frac{209}{2880}\frac{\kappa^2}{(4\pi)^4\epsilon}\int d^4x\sqrt{-g}\, R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\tau\lambda} R_{\tau\lambda}{}^{\mu\nu}\,, \tag{1.1.8}$$

where $R_{\mu\nu}{}^{\rho\sigma}$ is the Riemann curvature tensor and $\epsilon \to 0$ is the dimensional regularisation parameter. On top of restricting the predictive power of the theory, such UV divergences are a sign that the theory doesn't capture the right degrees of freedom at high energies. Hence another theory is needed to understand the high energy behavior of quantum gravity.

1.2 Effective field theories

Unfortunately modifying general relativity is not so simple as there are a lot of uniqueness results that constrain our choices. For example Lovelock's theorem states that, in four dimensions, the only symmetric divergence-free tensor which depends at most on the second derivative of the metric (and is linear in the second derivative of the metric) is necessarily a linear combination of the Einstein tensor and the metric itself [7]. This means that the left hand side of the Einstein field equations (1.1.3) is more or less unique up to the inclusion of a cosmological constant. In order to go around that theorem one can either:

1. Add other fields on top of the metric tensor,

2. Work in more than four dimensions of spacetime,

3. Add terms which are more than second order in derivatives of the metric or non linear in the second derivative of the metric.

The approach that we will follow in this thesis, string theory, follows all three different directions. But for now we will focus on the last one. Higher derivative terms or terms non linear in the curvature are called higher order in curvature. These terms are especially desirable because their effects typically only become visible at higher energy scales while they don't modify the IR behavior of the theory. Such terms are called irrelevant.[4] This means that the inclusion of such terms is, at first glance, compatible with current experimental observations.

As we have seen in the previous section the presence of UV divergences forces one to introduce such higher curvature terms as counter terms anyways. In fact perturbative non renormalisibility forces us to introduce an infinity of ever higher order terms. A finite number of irrelevant terms cannot lead to a renormalisable theory by definition. But the fact that a given field theory is not renormalisable is not such a big problem if this theory is seen, in its domain of validity, as the low energy expansion of some unknown UV complete theory. This is the idea behind the effective field theory approach. In an effective field theory perspective one fixes the field content and the symmetries of the theory and writes down all the allowed terms with growing number of derivatives and arbitrary coefficients. These coefficients will then be fixed by consistency or experimental constraints.

For example the Lagrangian of general relativity, including a cosmological constant, is given by the Einstein-Hilbert Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\frac{2}{\kappa^2}\left(R - 2\Lambda\right)\right], \tag{1.2.1}$$

where Λ is the cosmological constant and g is the determinant of the metric. From the effective theory point of view this is only the first terms in a series expansion of higher order curvature terms. In some sense this is the minimal theory, i.e. the universal, experimentally verified part of the theory which is leading at low energies. To get the full effective theory we would have to write all linearly independent diffeomorphism invariant terms containing the metric and its derivatives in order of increasing derivatives with some arbitrary coefficients. Schematically the first couple of terms of an effective Lagrangian for pure general relativity would look like

[4] This terminology comes from statistical physics where terms whose effects become important in the IR are called relevant and terms whose effects become important in the UV are called irrelevant.

$$\mathcal{L} = \sqrt{-g}\left[-\frac{4\Lambda}{\kappa^2} + \frac{2}{\kappa^2}R + c_1 R^2 + c_2 R^{\mu\nu}R_{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + c_4\kappa^2 R^3 + \dots\right], \tag{1.2.2}$$

where $c_1, c_2, c_3, c_4, \dots$ are arbitrary coupling constants, called Wilson coefficients, to be determined empirically. The higher order terms in the expansion are also sometimes called Wilson operators.[5] For such an effective field theory UV divergences are not a problem as they can always be reabsorbed by a suitable redefinition of some higher order coupling constants. One can see that because the gravitational coupling constant κ has a negative mass dimension then higher powers of the coupling constant will have to appear in front of higher order curvature terms to keep them dimensionless. Therefore divergent higher order loop diagrams which involve higher powers of κ^2 can only be reabsorbed in higher order curvature terms, this is the reason why minimal pure general relativity is not perturbatively renormalisable and why all terms in the infinite series expansion (1.2.2) are needed. Here the minimal theory only appears as the leading terms in a low energy expansion of the full effective theory. Since higher curvature terms are suppressed by powers of the Planck length we can see that such a low energy expansion is only valid until energy scales $1/l_P$. Therefore as mentioned in the previous section the Planck length acts as a natural cutoff where quantum gravity effects are expected to appear. Seen as such an effective field theory, quantum gravity still holds useful predictive power. This is the modern way to see quantum gravity and in some ways it is also the modern way to see the standard model of particle physics as well.

In many ways gravity is particularly well suited to the effective field theory point of view. First of all, as seen in the previous section the Planck length is exceptionally small compared to any length scale relevant to us. This implies that the first few terms in the series expansion of the full effective field theory hold very accurately up to a very high energy scale. Therefore, in some sense, quantum gravity is by far the best effective field theory we have. Moreover there has been a variety of results by Weinberg, Boulware, Deser and others [8–11] which have shown that the low energy behavior of gravity is uniquely captured by the effective theory of a spin two field of the form (1.2.2). This makes the effective field theory approach very promising. However the effective field theory formalism is agnostic about the full UV complete theory. This theory can be more exotic and in particular violate more of Lovelock's

[5]Note that up to field redefinition one can always add to an effective Lagrangian higher order terms that vanish modulo the equations of motion of the theory defined by the lowest order terms. Here this allows us to fix c_1 and c_2 so that the first three Wilson operators are proportional to the Gauss-Bonnet term $R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ which is topological in four dimensions. In the presence of a cosmological constant however this may come with a redefinition of the coupling constant and one has to beware strong coupling effects.

assumptions, this is the case with string theory which will be presented in the next chapter. The nature of the relationship between the effective field theory approach and the full UV complete theory, in particular string theory, is the essence of this thesis.

Analytic and non analytic contribution When we will look at the amplitudes associated with such gravitational effective field theories it will be useful to decompose them into so-called analytic and non analytic parts. By (non-)analytic we usually mean in the sense of a complex function of the transferred momentum squared p^2 (which can be seen as a complex variable) or, equivalently, of the Mandelstam variables. At low energies, i.e. for low transferred momentum $p^2 \to 0$, non-analytic contributions come from the propagation of massless particles as can be seen from the massless propagator for a spinless particle

$$G(p) = \frac{1}{p^2}\,, \tag{1.2.3}$$

which is indeed non analytic at $p^2 = 0$.[6] On the other hand the analytic contributions come from the propagation of massive particles as can be seen from the massive propagator for a spinless particle of mass m

$$G(p) = \frac{1}{p^2 + m^2}\,, \tag{1.2.4}$$

which is analytic at $p^2 = 0$. Therefore non analytic contributions dominate in the low energy limit of the effective field theory. This interplay between analytic and non analytic contributions to the amplitude will be crucial in the analysis carried out in this thesis. It will notably be very relevant to the discussion of unitarity of the S-matrix.

1.3 S-matrix bootstrap

In the 60s two rival approaches to quantum field theory coexisted. The first one was the S-matrix approach defended by Wheeler while the second one was the perturbative approach developed by Feynman. A brief look at quantum field theory books today, which are usually covered with Feynman diagrams, will reveal that Feynman's approach won out. However even though it eventually fell out of favor the S-matrix program did manage to achieve some relevant results and was even recently revived in the form of the very promising S-matrix bootstrap program.

[6] This observation are essentially the same in the case of particles with spin

The basic idea of the S-matrix approach is that instead of looking at the microscopic degrees of freedom of the theory one can instead focus on very general and robust principles that the theory must obey such as unitarity and causality. Causality typically implies other principles including relativistic invariance and the fact that at the microscopic level the theory is described by local propagating degrees of freedom (the precise nature of which is irrelevant), all of which are principles that must be obeyed by any reasonable quantum field theory. One can then use these very general principles to constrain amplitudes of scattering particles. At the time this looked like a very robust alternative to the perturbative approach which was plagued with the issues of UV divergences and the poorly understood problem of renormalisation. This very formal, non perturbative approach was not as fertile as initially thought however and though this program was (temporarily) abandoned it was very influential to the historic development of string theory. Today however it is getting back the attention it deserves [12–15].

S-matrix If we suppose that interactions in our theory are bounded in time then we can assume that asymptotic states in the far past and in the far future are that of the free theory, i.e. Fock states. Indeed we can define a different Fock space of states in the asymptotic past and future. If we write $|\chi\rangle^{\text{in}}$ for some state in the far past then $|\chi\rangle^{\text{out}}$ corresponds to the state in the far future obtained from time evolution of $|\chi\rangle^{\text{in}}$. The scattering matrix, or S-matrix, then relates these two different Fock spaces. It is defined by its action on such asymptotic states by

$$S\,|\chi\rangle^{\text{out}} = |\chi\rangle^{\text{in}}\,, \quad {}^{\text{in}}\langle\chi|\,S = {}^{\text{out}}\langle\chi|\,. \tag{1.3.1}$$

Correspondingly the matrix elements of S are given by

$${}^{\text{out}}\langle\chi|\xi\rangle^{\text{in}} = {}^{\text{in}}\langle\chi|S\,|\xi\rangle^{\text{in}} = {}^{\text{out}}\langle\chi|S\,|\xi\rangle^{\text{out}}\,. \tag{1.3.2}$$

In particular for the vacuum states we impose

$$S\,|0\rangle^{\text{out}} = |0\rangle^{\text{in}}\,, \quad {}^{\text{in}}\langle 0|\,S = {}^{\text{out}}\langle 0|\,, \tag{1.3.3}$$

which is just the statement that the vacuum state is unique and invariant under time translation. (1.3.2) shows that the matrix elements of S are the probability amplitudes of finding any given state in the far future for a given state in the far past. For definite particle states $|i\rangle^{\text{in}}$, $|j\rangle^{\text{in}}$ these are scattering amplitudes[7]

$${}^{\text{in}}\langle j|\,S\,|i\rangle^{\text{in}} = i\mathcal{A}_{i\to j}\,. \tag{1.3.4}$$

[7]Here and in the rest of this thesis we omit delta factors related to momentum conservation and assume that all exterior momenta are conserved.

In any reasonable quantum theory conservation of probability implies that time evolution should be unitary, this translates into unitarity of the S-matrix

$$S^\dagger S = I\,. \tag{1.3.5}$$

In fact in most quantum theories the S-matrix is simply given in the interaction picture by

$$S = T\exp\left(-i\int_{-\infty}^{+\infty} dt\, H_{\text{int}}\right) = T\exp\left(i\int d^4x\, \mathcal{L}_{\text{int}}\right), \tag{1.3.6}$$

where H_{int} is the interaction Hamiltonian, $\mathcal{L}_{\text{int}}$ is the interaction Lagrangian density and T is the time ordering operator. Therefore unitarity of the S-matrix simply translates into hermiticity of the Hamiltonian operator.

Axioms of S-matrix bootstrap Unitarity is one of the most important principle that the S-matrix obeys and it is the one that will preoccupy us the most but there are other properties that the S-matrix should obey, these are the so-called axioms of S-matrix bootstrap. Although there isn't a definitive list of axioms, most of them include:

- Relativistic invariance: the S-matrix should be a representation of the Poincaré group,
- Unitarity: defined by equation (1.3.5),
- Crossing symmetry: this implies that the amplitudes of incoming (resp. outgoing) particles are the same as the amplitudes of outgoing (resp. incoming) antiparticles,
- Analyticity: roughly speaking the amplitudes should be analytic in the transferred momentum or, equivalently, in the Mandelstam variables (up to crossing symmetry) everywhere except for cuts and poles wherever physical particles can appear as intermediate states.[8]

There are other axioms that one can add in order to constrain further the form of the S-matrix but they will not be explored here. The properties enumerated above are expected of any reasonable quantum field theory which obeys both the principles of quantum theory and relativity. Note that the last two axioms are not typically

[8] Actually this is typically referred to as maximal analyticity. Often the claim is much weaker than that and only requires a bounded domain of analyticity below the first physical mass. In the case of gapped theories this can be proved. In this thesis maximal analyticity will be more relevant.

referred to as axioms in the literature as they can actually be proved in the case of theories exhibiting a mass gap [16, 17]. However as gravity, our theory of interest, is not a gapped theory and much less is proved in this case we will have to take these properties as assumptions and thus we denote them as axioms.

Optical theorem and positivity bounds One can separate the trivial part of the S-matrix from the interaction part as follows

$$S = I + iT\,, \tag{1.3.7}$$

where the identity represents the non interacting part of the S-matrix while T represents the interacting part. Injecting this into (1.3.5) one gets

$$2\,\mathrm{Im}\,T = T^\dagger T\,. \tag{1.3.8}$$

This is known as the optical theorem. For any non trivial interacting theory the interacting part of the S-matrix must therefore have a non zero imaginary part. Since amplitudes are often defined via real coefficient integrals that are analytically continued in the Mandelstam variables to the physical domain then imaginary parts can only come from functions which are non analytic in the domain of convergence. In the low energy limit these are precisely the non analytic contributions coming from the propagations of massless particles. This explains the previous statement that non analytic contributions are crucial to discussing the unitarity of the S-matrix. If we look at matrix elements of (1.3.8) and insert a complete basis of definite particle states we get

$$2\,\mathrm{Im}\,\mathcal{A}_{i\to j} = \sum_n \int \prod_{a=1}^{n} d^3k_a\, \mathcal{A}^*_{j\to n}\mathcal{A}_{i\to n}\,, \tag{1.3.9}$$

where k_a are the momenta of the n outgoing particles on the right hand side. Importantly, since the right hand side of (1.3.8) is positive definite one gets the following positivity bound on amplitudes

$$\mathrm{Im}\,\mathcal{A}_{i\to j} \geq 0\,. \tag{1.3.10}$$

In the case where $i = j = 2$, if we only restrict the right hand side of (1.3.9) to elastic scattering, i.e. $n = 2$, then since this side of the equation is positive definite the optical theorem reduces to the following inequality

$$2\,\mathrm{Im}\,\mathcal{A}_{2\to 2'} \geq \int d^3k_1 d^3k_2\, \mathcal{A}^*_{2'\to 2''}\mathcal{A}_{2\to 2''}\,. \tag{1.3.11}$$

These two inequalities can be used to give positivity bounds on Wilson coefficients of effective field theories as we will see later.

Today the old S-matrix ideas have been revived with some slightly more realistic goals. Instead of trying to fully understand a given theory from its S-matrix the modern S-matrix bootstrap program uses the axioms above to put constraints on the space of allowed S-matrices describing all consistent quantum field theories including theories of quantum gravity [18]. This is also in the spirit of what this thesis aims to do, which is constraining the S-matrices of effective field theories compatible with string theory.

1.4 String theory

As we have seen above, one of the main advantage of the effective field theory approach is that it is agnostic about the full UV complete theory. But over the 70 years of interest over quantum gravity there have been several proposals for a UV complete theory of quantum gravity, one of the most promising of which is string theory. As mentioned in the previous section the S-matrix program had a large influence on the early developments of string theory. Indeed string theory was initially born out of a necessity to describe the strong interactions and particularly meson interactions in a way consistent with the axioms of the S-matrix. In 1968 Veneziano wrote down an amplitude satisfying precisely these criteria [19]

$$\mathcal{A}_{2\to 2'} = -ig_s^3 \alpha'^{11} \left(\frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha'(s+t) - 2)} + \frac{\Gamma(-\alpha' s - 1)\Gamma(-\alpha' u - 1)}{\Gamma(-\alpha'(s+u) - 2)} + \frac{\Gamma(-\alpha' u - 1)\Gamma(-\alpha' t - 1)}{\Gamma(-\alpha'(u+t) - 2)} \right) , \quad (1.4.1)$$

where g_s is called the string coupling, α' is called the Regge slope and s, t, u are the usual Mandelstam variables defined by $s = -(k_1 + k_2)^2$, $t = -(k_1 + k_4)^2$ and $u = -(k_1 + k_3)^2$ with k_i being the exterior momenta (all taken as incoming).[9] One can clearly see that the above amplitude satisfies crossing symmetry as required by the axioms of the S-matrix and is in fact totally symmetric in the Mandelstam variables. One can also see that it is analytic in the Mandelstam variables excepts for some points $s, t, u = n/\alpha'$ for $n = -1, \ldots, \infty$ corresponding to the poles and zeros of the Gamma function and which can be interpreted as coming from particles

[9] They are not independent as they are related by the relation $s + t + u = \sum_{i=1}^{4} m_i^2$ where m_i is the mass of the ith particle.

of ever higher masses $m^2 = n/\alpha'$, we call these Regge trajectories. It was then by looking for a model that would give rise to such an amplitude from first principles that physicists stumbled upon string theory. Indeed the correct model was found by quantizing a classical string of length $l_s = \sqrt{\alpha'}$ and tension $T = 1/(2\pi\alpha')$. Quantum constraints having to do with unitarity and Lorentz invariance impose that this string propagates in 26 dimensional space-time. The infinite tower of massive particles was then understood to come from the infinite modes of the string. However one of these modes corresponded to a massless spin two particle which, as we have seen, can only correspond to gravity, at least at low energy. This made string theory into the theory of quantum gravity that we know it today.

As seen with the infinite spectrum of massive particles string theory is not an ordinary quantum field theory. In some sense it corresponds to an infinite number of quantum fields. However one can see that the tower of masses is controlled by the Regge parameter α' and when one takes the limit $\alpha' \to 0$ the infinite tower decouples leaving only the lowest lying levels for $n = -1$ and $n = 0$ therefore reducing to a proper quantum field theory. Indeed all the stringy effects of string theory are contained in this dimensionful parameter α' the only free parameter of the theory. Therefore, at low energy, one can expect string theory to reduce to an effective field theory of quantum gravity where the series expansion has ever higher powers of α' in front of ever more irrelevant Wilson operators. However it turns out that this effective theory is far from unique. Indeed the Wilson coefficients, instead of being constants, are actually functions of massless fields called moduli fields. Therefore there are actually many low energy effective field theories which are compatible with string theory as a UV completion. It is the purpose of this thesis to explore which effective field theories are compatible with string theory. The methods of S-matrix bootstrap can also serve as an independent method to reduce the space of allowed S-matrices for consistent theories of quantum gravity.

Superstring theory In the 70s as bosonic string theory was developed [20–22] there were many great features that made it a serious candidate for a theory of everything : as mentioned before it contained a graviton in its spectrum, its amplitudes were free of the UV divergences that plagued quantum field theory and in particular quantum gravity, it also had the potential to include other gauge interactions and finally it was a very constrained theory with only one dimensionful parameter. The finiteness of its amplitudes could be superficially understood as coming from the finite length of the string which "smears out" the interaction instead of being localised to a point. This alone, having a non divergent quantum theory containing a massless spin two particle is no small feat and cannot be understated. However three prob-

lems loomed in the face of anyone wanting to take this theory seriously as a realistic theory of nature. First of all, as the name would suggest, there were only bosons in the spectrum of the theory, secondly as mentioned before the theory could only exist in 26 dimensional spacetime, but perhaps worst of all is that the spectrum also contained a tachyon as its lowest lying string mode. A tachyon usually indicates a pathology regarding the causal structure and unitarity of the theory.

A straightforward fix to the problem of absence of fermions in the theory came when Ramond, Neveu and Schwarz added fermionic fields to the worldsheet theory of the string [23, 24]. The fields obeyed a very particular type of symmetry mixing bosonic and fermionic fields called supersymmetry. In fact this two dimensional version of supersymmetry was actually the first instance of supersymmetric theory to be developed. It was then showed that a certain truncation of the spectrum of the theory (now called GSO projection after Gliozzi, Scherk, and Olive) led to a consistent, supersymmetric and tachyon-free theory [25]. As an added bonus the required number of dimensions for the theory to live in is reduced from 26 to 10. The resulting class of theories is called superstring theory. It is worth mentioning that the fact that the final theory is supersymmetric is non trivial. Indeed the supersymmetry was initially introduced on the two dimensional worldsheet whereas the supersymmetry of the spectrum corresponds to ten dimensional space time supersymmetry. An equivalent formulation of superstring theory with space-time supersymmetry introduced from the beginning was developed by Green and Schwarz in 1981 [26–28].

Actually superstring theory is not one theory. In fact there are five different superstring theories corresponding to different coherent ways to introduce supersymmetry and get rid of the tachyon: there are two maximally supersymmetric theories dubbed type IIA and type IIB superstring theory and three half maximally supersymmetric theories called type I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ superstring theory (in the last two cases the name heterotic refers to the asymmetric roles played by right and left moving modes on the closed string). A key result was the discovery in 1984 by Green and Schwarz of the miraculous anomaly cancellation in all five superstring theories [29].[10] This marked the point when a large number of the theoretical physics community turned to string theory and was called the first superstring revolution.

The fact that the main candidate for the theory of everything was actually five different theories remained unsatisfactory. However in the 90s, mainly under the impulsion of Witten, it was discovered that all five superstring theories were actually

[10] Actually the Green-Schwarz mechanism only concerns the type I and heterotic string theories. Anomaly cancellation in type IIA string theory is trivial as the theory is non-chiral whereas anomaly cancellation in type IIB string theory is non-trivial but does not require the Green-Schwarz mechanism.

linked by dualities, thereby suggesting that none was more fundamental than the others but instead that they were all different sides of the same coin. In addition, strong evidence was found that all five superstring theories were different limits of some unique 11 dimensional theory called M-theory [30]. This sparked the beginning of the second superstring revolution when the field gained even more momentum. At the same time Polchinski discovered the existence of dynamical extended objects other than strings within string theory which were named D-branes [31]. These D-branes were required by the web of duality and opened up a window into the non-perturbative structure of string theory. Finally in 1997 Maldacena discovered another duality between five dimensional D-branes in type IIB string theory and another type of gauge theory called $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions [32]. This type of duality between a d dimensional gravitational theory and a $d-1$-dimensional gauge theory sparked the beginning of the AdS/CFT correspondence and holography.

String compactifications and the vacuum problem Acute readers may have noticed that we have brushed under the carpet one of the three initial problems presented by bosonic string theory: that of higher dimensions. Indeed the problems of the tachyon and the absence of fermions were straightforwardly solved by introducing supersymmetry on the string worldsheet but the problem of higher dimensions was only reduced by going from 26 dimensions to 10. It is clear that these are six dimensions too many in order to be a viable description of our universe. It turns out however that this is not as serious a problem as one might first think. Indeed suppose that our universe is described by a spacetime which can be split into a cartesian product $\mathbb{R}^4 \times K$ where K is a compact space. Then, in theory, if the hypervolume of K is sufficiently small these extra dimensions would be unobservable and it would be effectively as if we lived in $\mathbb{R}^4$. We call these extra dimensions wrapped up in K compactified dimensions.

Hence there is a priori no way to know if our universe is really four dimensional or if it contains extra compactified dimensions. This may seem like a very unscientific assumption that should be disregarded in virtue of Occam's razor but actually compactified dimensions have a number of phenomenologically interesting consequences. This goes back to the insight of Kaluza and Klein in the 1920s. Suppose that there is only one extra dimension compactified on a circle $\mathbb{R}^4 \times S^1$. We can therefore split the 5-dimensional coordinates $\hat{x}$ into 4-dimensional coordinates x on $\mathbb{R}^4$ and compact coordinate θ on S^1. All 5 dimensional field $\hat{\phi}$ have to obey the boundary condition $\hat{\phi}(x, \theta + 2\pi R) = \hat{\phi}(x, \theta)$ where R is the radius of the circle S^1. Therefore we can Fourier-expand the fields on S^1

$$\hat{\phi}(x,\theta) = \sum_n \phi_n(x) e^{\frac{in\theta}{R}} \,. \tag{1.4.2}$$

Let us suppose that $\hat{\phi}$ is a massless scalar field in 5 dimensions. Therefore it obeys the massless Klein-Gordon equation (the wave equation)

$$\hat{\Box}\hat{\phi} = 0\,, \tag{1.4.3}$$

where $\hat{\Box}$ is the 5-dimensional d'Alembertian operator. When we insert (1.4.2) into the massless Klein-Gordon equation we get the following equation of motion for each Fourier mode

$$\Box\phi_n = \left(\frac{n}{R}\right)^2 \phi_n\,, \tag{1.4.4}$$

which is a massive Klein-Gordon equation. Therefore we obtain an infinite tower of fields with mass $m = |\frac{n}{R}|$ propagating in $\mathbb{R}^4$. Taking the limit where the radius of the circle goes to zero $R \to 0$, these KK-modes become infinitely massive and decouple from the theory except for the 0-mode which remains massless. These massive modes can be neglected in an effective field theory limit. The process of compactification combined with consistently truncating away the massive modes is called Kaluza-Klein reduction.

Another interesting property of compactified dimensions comes specifically when one considers the compactification of a higher dimensional theory containing gravity. This is what Kaluza and Klein where initially interested about. The gravitational part of the 5-dimensional theory is given by the Einstein Hilbert action

$$S_5 = \frac{2}{\kappa_5^2}\int d^5\hat{x}\sqrt{-\hat{g}}\,\hat{R}\,. \tag{1.4.5}$$

The 5 dimensional metric can be decomposed into

$$ds_5^2 = e^{\frac{2}{\sqrt{3}}\phi}(d\theta + A_\mu dx^\mu)^2 + e^{-\frac{\phi}{\sqrt{3}}}ds_4^2\,, \tag{1.4.6}$$

where ds_4^2 describes the 4 dimensional metric on $\mathbb{R}^4$, $e^{\frac{2}{\sqrt{3}}\phi}$ describes the metric of S^1, i.e. its radius, and A_μ describes the fibration of S^1 over $\mathbb{R}^4$. We call ϕ a dilaton and A_μ a Kaluza-Klein vector. We require that all fields are independent of θ which is equivalent to only keeping the 0-modes of the Fourier expansion. After Kaluza-Klein reduction the 5-dimensional action becomes

$$S_4 = \frac{2}{\kappa_4^2}\int d^4x\sqrt{-g}\left(R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}e^{\sqrt{3}\phi}|F|^2\right)\,, \tag{1.4.7}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and the 4-dimensional coupling constant is related to the 5-dimensional one by the volume of S^1 i.e. $\kappa_4^2 = \kappa_5^2/2\pi R$. We have also defined $|\omega|^2 = \frac{1}{n!}\omega_{\mu_1\dots\mu_n}\omega^{\mu_1\dots\mu_n}$ for some n-form $\omega_{\mu_1\dots\mu_n}$. We can show that the diffeomorphism symmetry in 5 dimensions corresponds exactly to 4 dimensional diffeomorphism symmetry of the 4 dimensional metric plus $U(1)$ gauge symmetry of the Kaluza-Klein vector.

It is an incredible surprise that we find a $U(1)$ gauge field behaving exactly like electromagnetism simply by compactifying pure gravity on a circle. This was actually called the Kaluza-Klein miracle. This is less surprising when we realise that we have constructed an S^1 fiber bundle which behaves exactly as a $U(1)$ principal bundle.

The $e^{-\frac{\phi}{\sqrt{3}}}$ factor in front of the 4 dimensional metric in the ansatz (1.4.6) corresponds to a conformal rescaling which is necessary if we want the Einstein-Hilbert term in the 4 dimensional action (1.4.7) to be canonically normalised. Without this field redefinition we would have non minimal gravitational couplings with the dilaton. Such non minimal couplings are however present between the dilaton and the Kaluza-Klein vector, this is an example of so-called scalar-tensor gravity. This is historically the reason why Kaluza-Klein theory was abandoned.

Even if compactification of pure gravity on a circle is ruled out, compactifications of richer theories on more general manifolds can still be phenomenologically realistic. With more compactified dimensions one can even get non abelian gauge theories after Kaluza-Klein reduction. Let us consider some theory involving gravity in D dimensions and suppose that the space-time manifold splits topologically into a 4-dimensional non compact manifold times a $(D-4)$-dimensional compact manifold i.e. $\mathbb{R}^4 \times K$. The fields can again be decomposed on a basis of harmonic functions of K and the masses of the KK tower are given by eigenvalues of the Laplacian on K. The D dimensional metric can be decomposed into

$$ds_D^2 = g_{ij}(d\bar{x}^i + K_a^i A_\mu^a dx^\mu)(d\bar{x}^j + K_b^j A_\mu^b dx^\mu) + ds_4^2\,, \tag{1.4.8}$$

where g_{ij} describes the metric of the compact space K, the A_μ^a describe the fibration of K over $\mathbb{R}^4$ and the K_a^i are Killing vectors of the compact metric on K. All the components of the internal metric g_{ij} are dilatons. The Killing vectors $K_a = K_a^i\partial_i$ generate the Lie algebra of the isometry group G of K. One can show that under D dimensional Killing isometry of K the vectors A_μ^a transform as non abelian connections of a G principal bundle. This shows that we can indeed interpret the A_μ^a as non abelian gauge fields associated to the gauge group G. We can furthermore show that the Kaluza-Klein reduction of the D-dimensional Einstein-Hilbert action will give rise to the 4-dimensional Yang-Mills action for the gauge group G (modulo some non minimal couplings). This construction is due to Scherk and Schwarz [33,34] and is thus sometimes called a Scherk–Schwarz reduction

We can see that there are several ways to get the same gauge group G as there are many manifolds that can share the same isometry group. Two of the most simple ways are to compactify on the gauge group manifold itself i.e. $K = G$. This is pretty much the standard construction of Yang-Mills theory for some generic gauge group G. Another way would be to compactify on some coset space $K = G/H$ where H is a maximal compact subgroup of G, the most common examples are the sphere compactifications $K = S^n \simeq SO(n+1)/SO(n)$. However contrary to the case of S^1 there are obstructions to reductions on general manifolds. The only exceptions where a consistent reduction can always be performed are the circle, the torus and the group-manifold reductions. Note that, when possible, coset reductions are preferred to group manifold reductions, since less extra dimensions are needed to obtain the same gauge group. For example, in order to obtain the gauge group $SO(8)$ one could use the $SO(8)$ group manifold which requires $\dim(K) = 28$ extra dimensions or the coset $SO(8)/SO(7)$ corresponding to the seven-sphere S^7 which only requires $\dim(K) = 7$ extra dimensions.

Hence Kaluza-Klein reduction is a very attractive mechanism for string theorists as it can simultaneously get rid of extra unwanted dimensions and explain geometrically where gauge symmetries come from. This is why KK reductions are used throughout string theory. In particular they are used to get from the ten dimensional world described by superstring theory to the four dimensional world we inhabit. Even though the presence of extra dimensions is a common criticism of string theory the Kaluza Klein mechanism actually makes it a strength as the complexity and arbitrariness of the field content of the standard model is simply contained in the geometry of the theory. We call the process of finding a 6 dimensional compact manifold over which to perform a Kaluza-Klein reduction of string theory finding a "geometric vacuum of the theory". Indeed such a manifold can be thought of as a vacuum expectation value for the 10 dimensional metric around which to expand the theory. It is interesting to note that the minimum number of compact dimensions needed to recover the standard model gauge group from KK reduction of a purely gravitational theory is 7 [35], therefore the minimum number of dimension for a realistic Kaluza-Klein theory of everything would be 11 putting 11-dimensional M-theory in a uniquely good position. While there are many manifolds with the right isometry group[11] Witten showed in 1981 [35] that none of them reproduces the rest of the field content of the standard model (particularly the chiral fermions). However more exotic compactifications such as G_2-manifolds with orbifolds are not ruled out.

Other string theories which already possess large gauge groups like type I and

[11]The simplest 7-dimensional compact manifold with isometry group $U(1) \times SU(2) \times SU(3)$ is $S^1 \times S^2 \times \mathbb{CP}^2$

heterotic string theories appear as more viable candidates to recover the standard model of particle physics from Kaluza-Klein reduction. Until recently and the negative results at the LHC the minimal supersymmetric extension of the standard model was considered a very phenomenologically viable extension of the standard model and for other reasons minimal $\mathcal{N} = 1$ supersymmetry was seen as very attractive. Therefore physicists looked for compactification manifolds which would preserve exactly this amount of supersymmetry. In 1985, Candelas, Horowitz, Strominger and Witten showed that this imposed that the compactification manifold be a Calabi-Yau manifold, i.e. a Ricci flat Kähler manifold [36]. While it was initially thought that there would be a very small number of such manifolds obeying additional topological constraints in order to be consistent with the particle content of the standard model, it turns out that there is in fact a very large number. The high number of such realistic vacua puts into question the ability of string theory to predict anything at our energy scales as for any prediction given by some vacuum there may be another vacuum that makes another prediction. This is called the vacuum problem.

We saw that Kaluza-Klein reduction usually gives some massless scalar fields in the dimensionally reduced theory like for examples the dilatons coming from the reduction of the metric. Dilatons typically control the geometry of the compact space. If in addition one reduces a theory with p-forms this will also give massless scalar fields called axions. Dilatons and axions are called moduli fields. Moduli fields are typically problematic in any realistic theory because they are massless and there are no obvious mechanisms to give them mass, at least in pure gravity Kaluza-Klein reduction. This is called the moduli stabilisation problem. However in supergravity and string theory there are mechanisms which give mass to the moduli fields, this is done by imposing that the p-form fields acquire non trivial vacuum expectation values, which are called fluxes. This also allows to compactify on non flat compact spaces which was also impossible in pure gravity Kaluza-Klein reduction. More stringy constructions for Kaluza-Klein reductions include non geometric constructions such as orbifolds and orientifolds. These allowed to return to compactifications of type II string theory and M-theory. We should mention that Kaluza-Klein reduction is also important for the web of dualities underlying string theory as some theories are related to each other by KK reduction. This is the case with type IIA string theory which is a KK reduction of M-theory on a circle.

U-duality and Wilson coefficients String theory has a worldsheet formulation as a two dimensional supersymmetric sigma model. However, as we have seen, in the limit $\alpha' \to 0$ we should recover a field theory in the low energy limit. The leading terms of this expansion is a ten dimensional supergravity theory. This can be seen

either from the expansion of the amplitudes themselves or from the vanishing of the beta functionals of the worldsheet theory which guarantee conformal invariance on the worldsheet and which, to first loop order, give the equations of motion for ten dimensional supergravity. One can then integrate out the massive string modes to get the higher order terms of the effective field theory. As explained before these higher order terms will be suppressed by powers of α' which is proportional to the square of the Planck constant of the supergravity theory. One important difference with effective field theories we have talked about so far is that string theory contains a certain number of massless scalar fields, the axion-dilaton fields. These should not be integrated out as they lie on the same level as the graviton. As scalar fields they are by definition Lorentz invariant and so are any functions of these fields. Therefore in general nothing stops non trivial functions of the moduli fields from appearing in front of Wilson operators. They should not appear in front of the leading terms however as this would correspond to non minimal couplings which can be removed by conformal field redefinition. Hence, if we restrict ourselves to the gravitational sector of our theory the effective field theory expansion looks like

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{-g}\left[R + l_P^2\mathcal{E}_1 R^2 + l_P^2\mathcal{E}_2 R^{\mu\nu}R_{\mu\nu} + l_P^2\mathcal{E}_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + l_P^4\mathcal{E}_4 R^3 + \dots\right], \tag{1.4.9}$$

where the Wilson coefficients $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \dots$ are functions of the axion dilaton fields.[12] In a general dimensions D the Planck length is related to the D-dimensional gravitational constant by $l_P = G^{\frac{1}{D-2}}$. When the theory is compactified to a lower dimensional theory, other moduli fields will appear which make the Wilson coefficients even more complicated.

The Wilson coefficients can in principle be very general functions of the moduli. Luckily the structure of these moduli fields and hence of the Wilson coefficients is severely restricted by supersymmetry. Indeed supersymmetry imposes that the moduli fields parametrise some manifold with a large amount of structure, this is called the moduli space. For example, in all dimensions, in theories with more than 1/4th maximal supersymmetry the moduli fields parametrise a symmetric space [37]. Lower amounts of supersymmetry also constrain the structure of the moduli space, for example in four dimensions, in theories with 1/8th maximal supersymmetry the moduli fields must parametrise a Kähler manifold while for 1/4th maximal supersymmetry the moduli fields parametrise either special Kähler, hyperkähler or quaternionic Kähler manifolds. In addition the effective field theory must respect the various du-

[12] In a general dimension using field redefinitions to bring the first three Wilson couplings to a form proportional to the Gauss-Bonnet term is also useful as it avoids modifying the propagator.

alities obeyed by the full string theory. These are for example T-duality which corresponds to inverting some compactification radius and S-duality which is a strong-weak duality corresponding to inverting some coupling constant. These dualities are not always completely distinct as some coupling constants can correspond to non minimal couplings with some moduli field coming from some compactification radius. In fact one can often combine T and S-duality into a single unified duality group called U-duality. This is a discrete group of symmetries of the full effective field theory. In general the moduli fields transform non trivially under U-duality, mixing together. The most famous example being the $SL(2,\mathbb{Z})$ symmetry of type IIB string theory, in this case the axion C_0 and dilaton ϕ mix to give $\tau = C_0 + ie^{-\phi}$ which then transforms non linearly under $SL(2,\mathbb{Z})$ as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}\,, \tag{1.4.10}$$

where a, b, c, d are integer such that $ad - bc = 1$. Therefore the Wilson coefficients must respect this symmetry. This constrains very much the form of $\mathcal{E}_n$. In fact, along with other constrains coming from supersymmetry and internal consistency, this allows us to calculate exactly, non perturbatively, the leading Wilson coefficients for maximally supersymmetric string theory.

1.5 The swampland program

It is clear from (1.4.9) that, depending of the range of the Wilson coefficients as functions, many low energy effective field theories may be compatible with string theory as a UV completion. Indeed even though the structure of string theory is very constraining, the high number of fields and dimensions means there are a lot of degrees of freedom which can play a role in the IR limit. For example we have seen that the choice of vacuum around which to expand the theory and reduce the number of dimensions is very large. Even after choosing a vacuum the extra degrees of freedom coming from the internal dimensions come back in the form of a large number of moduli fields. This means that the low energy limit of string theory spans a wide range of effective field theories, we call this the "landscape" [38]. It was initially feared that just about any effective field theory might sit inside this spectrum, this is called the "landscape problem". The realisation that this was not the case and that, in fact, it was possible that many more theories sit outside this spectrum is the main idea behind what is called the "swampland program" [39]. Indeed any effective field theory that cannot be UV completed into string theory or any theory of quantum gravity, is said to be in the "swampland". Hence the swampland program is a bottom-up approach aiming to come up with precise criteria for an effective field theory to be

compatible with string theory as a UV completion.

Since the inception of the program in 2005 by Vafa several conjectures have emerged with diverse degrees of acceptance. Some of the most popular include the weak gravity conjecture: this conjecture says that any theory in the landscape must include at least one particle whose charge is larger than its mass (in the proper units). In other words gravity must be the weakest of all gauge interactions in the theory. This conjecture emerged from black hole physics and particularly Hawking decay of extremal black holes. Indeed the argument is that if we consider a slightly subextremal charged black hole, this black hole must decay since it is not protected by supersymmetry. If it emits a particle then this particle must have a mass m and charge q such that $|q| > m$ in order for the black hole to remain above the BPS bound.

Another popular conjecture is the no global symmetry conjecture. This states that in any theory in the landscape all global symmetries must be either gauged or broken. Once again this conjecture originates from black hole physics since it is believed that black hole evaporation breaks any charge conservation that does not originate from a gauge symmetry. This conjecture can also be checked in the context of the AdS/CFT conjecture.

Finally the infinite distance conjecture states that when the vacuum expectation value of a moduli field is taken to infinity there must exist an infinite tower of light and weakly coupled states whose mass goes to 0. The name of the conjecture comes from the fact that taking the value of a moduli field to infinity corresponds to an infinite distance on the moduli space measured with the metric appearing in the kinetic term of the scalar fields. This conjecture builds on the intuition of Kaluza-Klein theory where taking the compactification radius to infinity one recovers the limit of the higher dimensional theory and the states of the KK tower become light.

These conjectures are meant to chart out the landscape of string theory and if proven to be correct they help throw out a number of effective field theories from the landscape to the swampland. See [40] for a review. In order for string theory to be predictive one might hope for the swampland to be as large as possible and for the landscape to be small but still large enough to contain the standard models of particle physics and cosmology. This last requirement to include big bang and inflationary cosmology has proved to be very restrictive.

The string lamppost principle/String unity For someone agnostic about the UV completion of quantum gravity the swampland program can still be useful as looking at the set of all effective field theories compatible with string theory as a UV

completion might tell us interesting things about effective field theories of quantum gravity in general. However, for this to be the case the string landscape must be representative of the, in principle larger, set of theories that can be UV completed into a consistent theory of quantum gravity.[13] In fact if the string landscape is only a very small and very special subset of quantum gravity effective theories then the swampland program might even be misleading. The string lamppost principle or string universality [41, 42] is the claim that the opposite is true and that actually the landscape contains all theories consistent with any quantum gravity theory in the UV. In other words effective field theories of string theory saturate the space of effective field theories of quantum gravity. If this were proven to be true this would be a strong indication that string theory is in fact the unique consistent theory of quantum gravity or at least that it is general enough that all other ones can be realised as special cases of string theories.

For a large number of dimensions and for a large number of supersymmetries (which go hand in hand as one can cannot have a low number of supersymmetry in large dimensions apart from 0 supersymmetry) there are good arguments to believe that this string lamppost principle should hold. Indeed if one restricts to at most two derivatives then in eleven dimensions there is a unique theory of gravity compatible with supersymmetry and this theory is believed to be the low energy limit of M-theory. Similarly in ten dimensions there are only two consistent theories of gravity with maximal supersymmetry and at most two derivatives[14] and they are realised as UV complete quantum theories as type IIA and type IIB string theory respectively. For half maximal supersymmetry in ten dimensions the Green-Schwarz anomaly cancellation mechanism [29] restricts the possible gauge groups of the theory to only four options and two of them are realised in heterotic and type I string theory. It was later shown that the two other gauge groups were in fact in the swampland [41, 43]. Note however that if one asks about higher derivatives operators, as is natural in the context of effective field theories, things may not be so unique, we will be coming back to this in more details. Other arguments like this have been applied to theories with higher dimensions or higher supersymmetry in order to prove the string lamppost conjecture [44].

[13] Sometimes in the literature the terms "landscape" and "swampland" are used to talk about theories compatible/incompatible with any consistent theory of quantum gravity and not just string theory. Other than the fact that this inclusive definition is of questionable interest as consistent theories of quantum gravity outside of string theory are not well known we will not use these definitions in this thesis because this would make discussion of the string lamppost principle, which is at the center of this thesis, obscured.

[14] This is true if we only consider ungauged supergravity theories which we will do throughout this thesis.

Maximally supersymmetric string theory One can also study what happens if we look at higher order terms in the effective field theory expansion. If one still focuses on large numbers of dimensions and large amounts of supersymmetry then interesting things can be said. In fact we will focus on maximally supersymmetric theories. This set of theories can serve as a kind of playground where concrete computations can be performed more easily. Indeed maximal supersymmetry entails considerable simplifications that can be taken advantage of. For example it allows to discriminate against certain terms in the effective field theory expansion which do not respect maximal supersymmetry. Hence we can write schematically the first few terms of the effective Lagrangian of maximally supersymmetric gravity as

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{-g}\left[R + c_{(0,0)} l_P^6 \mathcal{R}^4 + c_{(1,0)} l_P^{10} \nabla^4 \mathcal{R}^4 + c_{(0,1)} l_P^{12} \nabla^6 \mathcal{R}^4 + \dots\right], \tag{1.5.1}$$

where $\mathcal{R}^k$ represents some contraction of the Riemann tensor of degree k. For example by $\mathcal{R}^4$ (one of the terms we will be most interested in) we mean $t_8 t_8 R^4$ where t_8 is a well known contraction of the Riemann tensor defined in chapter 3. Terms in $l_P^2\mathcal{R}^2$, $l_P^4\mathcal{R}^3$ and $l_P^8\nabla^2\mathcal{R}^4$ are theoretically possible but maximal supersymmetry imposes that their Wilson coefficients vanish identically.[15]

It is also known that in maximally supersymmetric gravity the 2 to 2 graviton scattering amplitude can be related to the much simpler scattering amplitude of its scalar superpartner. This has been used in [45] in order to extract unitarity bounds on the amplitudes coming from the first allowed corrections to gravity (represented as $c_{(0,0)} l_P^6 \mathcal{R}^4$ in (1.5.1)) using bootstrap methods. Indeed the full non perturbative 4-point graviton amplitude contains contributions from the Wilson operator which are analytic in the Mandelstam variables because they come from integrating out massive modes, as well as contributions from Einstein-Hilbert term which are non analytic because they come from the propagation of the massless graviton

$$\begin{aligned}\mathcal{A}_{2\to 2'} = -i\frac{\kappa^2}{2^{12}}\mathcal{R}^4(k_i,\epsilon_i)\Bigg(\frac{64}{stu} + c_{(0,0)} l_P^6 + 16\kappa^2\left(I_4^{(1)}(s,t) + I_4^{(1)}(t,u) + I_4^{(1)}(u,s)\right)\\ + c_{(1,0)}\frac{l_P^{10}}{16}(s^2+t^2+u^2) + \dots\Bigg),\end{aligned} \tag{1.5.2}$$

where $I_4^{(1)}$ is the one-loop supergravity integral and $\mathcal{R}^4(k_i,\epsilon_i)$ is a linearised version of the operator $\mathcal{R}^4$ containing the momenta and polarisations of the four gravitons.

[15] More specifically the Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is forbidden by supersymmetry while the Wilson coefficients of R^2 and $R_{\mu\nu}R^{\mu\nu}$ can be set to zero by field redefinitions and the $\nabla^2\mathcal{R}^4$ term is technically allowed but is zero on shell due to the fact that there is no non trivial linear invariant which is totally symmetric under the Mandelstam variables.

The D dimensional gravitational coupling constant and Planck lengths are related by $\kappa^2 = 32\pi l_P^{D-2}$. Therefore in dimensions $9 \leq D \leq 11$ the first Wilson term is leading in the IR with respect to the one loop supergravity amplitude while in dimensions $D \leq 8$ the one loop supergravity amplitude is leading. Actually in dimension $D = 8$ the two contribution are of the same order, we call this the critical dimension, we will come back to this in chapter 3. One can use a partial wave expansion and the optical theorem on the amplitude to extract unitarity bounds involving both the leading Wilson coefficient and the one-loop supergravity amplitude. This has been done in [45,46] for dimensions higher than 9 where the one-loop supergravity amplitude can be neglected and bounds on the Wilson coefficient $c_{(0,0)}$ in (1.5.1) have been derived. These bounds can then be compared with the IR limit of maximally supersymmetric versions of string theory, which means type II string theory or M-theory and their respective toroidal compactifications. This is because supersymmetry again offers a high degree of control over the higher order corrections entailed in the low energy limit as well as over the moduli space. As we saw the effective field theory expansion of the gravitational sector of string theory should look like

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{-g}\left[R + l_P^6\mathcal{E}_{(0,0)}\mathcal{R}^4 + l_P^{10}\mathcal{E}_{(1,0)}\nabla^4\mathcal{R}^4 + l_P^{12}\mathcal{E}_{(0,1)}\nabla^6\mathcal{R}^4 + \dots\right], \tag{1.5.3}$$

where the Wilson coefficients $\mathcal{E}_{(p,q)}$ are functions of the moduli space. We also saw that in the maximal supersymmetric case the moduli space is constrained to be a symmetric space and that further constraints coming from supersymmetry and the symmetry under U-duality allow to compute exactly the first Wilson coefficients including all perturbative and non perturbative corrections in g_s [47]. For example $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$, the coefficients that we will be most interested in, are specific types of automorphic forms called Eisenstein series [48–51]. We can then look at the minima of $\mathcal{E}_{(0,0)}$ (something which is not that easy in lower dimensions as we will see in this thesis) and compare it to the bound on $c_{(0,0)}$. If the minimum of $\mathcal{E}_{(0,0)}$ does not saturate the bound on $c_{(0,0)}$ this may go against the string lamppost principle and may leave room for other UV completions of maximally supersymmetric theories. For example it was found in [46] that in dimension 9, 10 and 11 the bounds coming from superstring theory were close to but not saturating the unitarity bounds, although it must be noted that their bounds are not sharp. In this thesis we will present works to find the bounds on scattering amplitudes in maximally supersymmetric string theory in dimensions 6, 7 and 8.

1.6 Plan of this thesis

In chapter 2 we give a quick review of superstring theory. We show how to derive the massless spectrum in covariant quantization with special emphasis on maximally supersymmetric type II string theory. We give the field contents and bosonic actions of the supergravity limits and we discuss the anomaly cancelations. Finally we review the web of dualities underlying superstring theories.

In chapter 3 we review superstring amplitudes. After discussing the construction of general string amplitudes we again take special interest in the maximally supersymmetric type II case where we review the moduli space of Riemann surfaces for genus zero, one and two. We then talk about factorisation properties of maximally supersymmetric amplitudes. In particular we show using the formalism of spinor helicities that unitarity properties of maximally R-symmetry violating superamplitudes in 6 dimensions reduce to the unitarity properties of scalar amplitudes. Finally we talk about the low energy limit of type II string amplitudes in relation with supergravity amplitudes using tropical limits and tropical geometry. We also discuss the ambiguity coming from the logarithmic divergence of the one loop supergravity amplitude in 8 dimensions and show how to regularise it properly using the finite string amplitude.

In chapter 4 we talk about U-duality and automorphic forms. We give a short reminder on automorphic forms with particular interest in Eisenstein series and their parabolic decompositions. We review U duality and give the moduli spaces and U-duality groups of maximally supersymmetric string theories. We then discuss the leading Wilson coefficients, their automorphic properties and the interpretations of the different decompactification limits.

Finally in chapter 5 we discuss our results regarding the minima of the leading Wilson coefficients of string theory. We extend Grenier's construction of a fundamental domain to more general Lie groups. We then show that symmetric points of moduli space are natural candidates for minima of Eisenstein series, in some cases we prove that they are local minima. We conjecture with the help of numerical checks the global minima of leading and next to leading Wilson coefficients.

Chapter 2

Superstring theory

String theory is generally accepted to be the only known consistent perturbative theory of quantum gravity. It is UV complete and reduces to known theories of supergravity at low energies. As such it allows us to gain insight into the structure of perturbative, and sometimes non perturbative, quantum gravity. At present it has proved a very difficult task to find which vacuum of string theory describes our universe at low energies. However if the swampland program is successful then string theory can still tell us interesting things about the coupling to quantum gravity at energy scales relevant to us.

In this chapter we review basic constructions of string theory using covariant quantization and we derive the massless superstring spectrum. We give a rundown of the different superstring theories and their supergravity limits as well as the web of dualities linking them together. The first parts of this chapter follows [52] and [53]. More details can be found in [54–57]. The reader already familiar with this material should feel free to skip this chapter.

2.1 Relativistic particle

The classical theory of relativistic strings is completely analogous with the classical theory of relativistic particles. We therefore start with the simpler case of the relativistic particle. The equation of motion of a massive particle in D-dimensional space-time is given by extremising the proper length of its path through space-time. Therefore the action is given by

$$S = -m \int ds \,. \tag{2.1.1}$$

Let $g_{\mu\nu}$ be the spacetime metric and let x^μ be the embedding coordinates of the particle in spacetime. We can define the induced metric on the worldline as $u =$

$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ where $\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}$ with σ being the coordinate on the worldline. Therefore we can rewrite the action (2.1.1) as

$$S = -m \int d\sigma \sqrt{-u}\,. \tag{2.1.2}$$

It is easy to check that this action is invariant under reparametrisation of the worldline as well as under D-dimensional spacetime isometries of the metric $g_{\mu\nu}$. However the action (2.1.2) contains a square root which makes it very complicated to quantize. Luckily there exists an equivalent classical action which is easier to work with. This action given by

$$S = -\frac{m}{2} \int d\sigma \sqrt{-h} \left(\frac{1}{h} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} + 1 \right) , \tag{2.1.3}$$

where h is the metric on the worldline. The equations of motion coming from this action are

$$\sqrt{-h} \frac{\mathrm{d}}{\mathrm{d}\sigma} \left(\frac{1}{\sqrt{-h}} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \right) + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0\,, \tag{2.1.4}$$

$$\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} h = 0\,, \tag{2.1.5}$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols for the metric $g_{\mu\nu}$. The first equation comes from the variation with respect to x^μ and the second equation comes from the variation with respect to h and can thus be interpreted as the vanishing of the worldline stress energy tensor of the theory. This can be rewritten as $h = u$ which says that the metric on the worldline is equal to the induced metric. If we plug this equation back into the action this yields

$$S = -\frac{m}{2} \int d\sigma \sqrt{-u} \left(\frac{1}{u} u + 1 \right) = -m \int d\sigma \sqrt{-u}\,, \tag{2.1.6}$$

proving that the two actions are classically equivalent and that the worldline metric h is simply an auxiliary field.

Diffeomorphism invariance of the action (2.1.3) allows us to fix the worldline metric h to any value we want (because all 1 dimensional manifolds are locally diffeomorphic), a common choice is $h = -1$ which gives us the well known action for relativistic particles

$$S = \frac{m}{2} \int d\sigma \ \left(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - 1 \right) . \tag{2.1.7}$$

2.2 Relativistic string

Let us now look at the theory describing strings. The equation of motion of a tensionful string in D-dimensional space-time is given by extremising the proper area of its path through space-time. Therefore the action is given by

$$S = -T \int d\mathcal{A} \,. \tag{2.2.1}$$

Similarly we can define the induced metric on the worldsheet as $\gamma_{ab} = g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu$ where $g_{\mu\nu}$ is the spacetime metric, x^μ are the embedding coordinates of the string and where $\partial_a x^\mu = \frac{\partial x^\mu}{\partial \sigma^a}$ with σ^a being the coordinates on the worldsheet. Therefore we can rewrite (2.2.1) as

$$S = -T \int d^2\sigma \sqrt{-\gamma} \,. \tag{2.2.2}$$

In the context of string theory this is known as the Nambu-Goto action. It is straightforward to check that this action is invariant under diffeomorphisms of the worldsheet and under D-dimensional spacetime isometries. Again, to get rid of the square root we introduce the equivalent action which is called the Polyakov action [58–60]

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h}\, h^{ab} \partial_a x^\mu \partial_b x^\nu g_{\mu\nu} \,, \tag{2.2.3}$$

where h_{ab} is the metric on the worldsheet. The equations of motion coming from this action are

$$\begin{aligned} \frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b x^\mu \right) + \Gamma^\mu_{\nu\rho} h^{ab} \partial_a x^\nu \partial_b x^\rho = 0 \,, \\ g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu - \frac{1}{2} h_{ab} h^{cd} g_{\mu\nu} \partial_c x^\mu \partial_d x^\mu = 0 \,. \end{aligned} \tag{2.2.4}$$

The second equation comes from the variation with respect to h_{ab} and can thus be interpreted as the vanishing of the worldsheet stress energy tensor of the theory which is a constraint coming from diffeomorphism invariance. This can be rewritten as $\gamma_{ab} = \frac{1}{2} h_{ab} h^{cd} \gamma_{cd}$. Taking the square root of the determinant of this equation gives us $\sqrt{-\gamma} = \frac{1}{2}\sqrt{-h} h^{cd} \gamma_{cd}$. Plugging this back into (2.2.3) yields

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h}\, h^{ab} \gamma_{ab} = -T \int d^2\sigma \sqrt{-\gamma} \,, \tag{2.2.5}$$

proving that the two actions are indeed classically equivalent.

Notice that the equation of motion (2.2.4) for h_{ab} is solved for any metric conformally equal to the induced metric $h_{ab} = e^{2\lambda}\gamma_{ab}$. In fact this signals an extra gauge symmetry of the string action that the particle action does not have: Weyl invariance

$$h_{ab} \to h'_{ab} = e^{2\lambda} h_{ab} \,, \tag{2.2.6}$$

for some function $\lambda(\sigma)$. This is unique to strings, it does not occur for particles, membranes and higher-dimensional branes. We will see that requiring that the Weyl symmetry remains unbroken at the quantum level recovers the Einstein equations for the classical background. This shows that strings are somehow special and uniquely adapted for a theory of quantum gravity.

We can use the diffeomorphism and Weyl invariance of the action to fix the worldline metric h_{ab} to anything we want (because all 2 dimensional manifolds are conformally equivalent), a common choice is the flat metric $h_{ab} = \eta_{ab}$ which gives us the action

$$S = -\frac{T}{2} \int d^2\sigma \, g_{\mu\nu} \partial_a x^\mu \partial^a x^\nu \,. \tag{2.2.7}$$

We see that Weyl invariance is crucial to fixing the worldsheet metric and in fact for branes in higher dimension than two the worldvolume metric cannot be gauged away, this again points to something special about strings. Note that fixing the world sheet metric doesn't use up all our gauge freedom. Indeed the action (2.2.7) is still invariant residual symmetries which correspond to diffeomorphisms of the worldsheet that are also Weyl transformations of the flat metric i.e. conformal transformations. Therefore we call this choice of gauge the conformal gauge.

The action (2.2.7) is sometimes called a non linear sigma model. It describes the classical theory of the bosonic string in curved spacetime, so called because there are no fermionic degrees of freedom and because if we were to quantize this string we would find exclusively bosons in the spectrum. Actually quantization in curved spacetime is extremely hard and not generally well understood therefore we will stick to flat spacetime. We will not quantize the bosonic string but for the purpose of completeness we rewrite the action (2.2.7) in flat space-time. We also bring back the string Regge parameter $\alpha' = 1/(2\pi T)$. Therefore the Polyakov action takes the form

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_a x^\mu \partial^a x_\mu \,, \tag{2.2.8}$$

and the stress energy tensor reduces to

$$T_{ab} = \partial_a x^\mu \partial_b x_\mu - \frac{1}{2}\eta_{ab}\partial_c x^\mu \partial^c x_\mu \,, \tag{2.2.9}$$

with the constraint $T_{ab} = 0$. Note that Weyl symmetry doesn't lead to a constraint but instead makes the stress energy tensor trace free.

2.3 Type II superstrings

Let us now show how to add supersymmetry to the picture and swiftly move on to superstrings. In this section we present the covariant quantization of the type II superstring in the RNS formalism. More detail about the bosonic string and its quantization can be found in [52–54]

2.3.1 The classical action and equations of motion

The action. In order to move on to superstrings we need to implement local supersymmetry on the world sheet in a way that doesn't break diffeomorphism or Weyl invariance. The action for type II superstrings is given by [54, 59]

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, e\left(h^{ab}\partial_a x^\mu \partial_b x_\mu + \bar{\psi}^\mu \rho^a \nabla_a \psi_\mu - 2\bar{\chi}_a \rho^b \rho^a \psi^\mu \partial_b x_\mu - \frac{1}{2}\bar{\psi}^\mu \psi_\mu \bar{\chi}_a \rho^b \rho^a \chi_b\right), \tag{2.3.1}$$

where the fields x^μ are 2-dimensional scalars on the worldsheet but D-dimensional vectors in space-time, the ψ^μ are 2-dimensional Majorana spinors on the worldsheet but also D-dimensional vectors in space-time, χ_a is the Rarita-Schwinger field and e_a^m is the zweibein. The zweibein is related to the worldsheet metric by $h_{ab} = e_a^m e_b^n \eta_{mn}$. We have also defined $\rho^a = e_m^a \rho^m$ where the ρ^m are the two dimensional Dirac matrices which form a representation of the two dimensional Clifford algebra $\{\rho^m, \rho^n\} = 2\eta^{mn}$. We can write these explicitly in a Majorana-Weyl representation as

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.3.2}$$

We also define the Dirac conjugate as $\bar{\lambda} = i\lambda^\dagger \rho^0$ for any spinor λ. Since the worldsheet is 2 dimensional the Majorana spinors can be decomposed into two 1-dimensional Majorana-Weyl spinors

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}, \tag{2.3.3}$$

where the Majorana condition imposes that $\psi_\pm^* = \psi_\pm$. Finally the covariant derivative is defined as

$$\nabla_a \psi^\mu = \partial_a \psi^\mu + \frac{1}{4}\omega_a^{mn}\rho_{mn}\psi^\mu \,, \tag{2.3.4}$$

with $\rho^{mn} = \frac{1}{2}[\rho^m, \rho^n]$ and where ω_a^{mn} is the spin connection but this term actually drops out of the action so that ∇_a can be replaced by ∂_a.

The action (2.3.1) still has diffeomorphism invariance and Weyl invariance. It also has local Lorentz invariance because of the zweibein and the spinors. But it has also two additional local fermionic symmetries with respect to the bosonic action. The first one is local supersymmetry under which x^μ and ψ^μ form scalar multiplets and e_a^m and χ_a form a graviton multiplet. The local supersymmetry transformations are given by

$$\begin{aligned}
\delta_\epsilon x^\mu &= \bar{\epsilon}\psi^\mu \,, \\
\delta_\epsilon \psi^\mu &= \rho^a(\partial_a x^\mu - \bar{\psi}^\mu \chi_a)\epsilon \,, \\
\delta_\epsilon e_a^m &= 2\bar{\epsilon}\rho^m \chi_a \,, \\
\delta_\epsilon \chi_a &= \nabla_a \epsilon \,,
\end{aligned} \tag{2.3.5}$$

and the second one is a supersymmetric extension of Weyl symmetry.

The Euler-Lagrange equations for the zweibein and the Rarita-Schwinger field lead to the constraints

$$\begin{aligned}
J_a &\equiv -\frac{\pi\alpha'}{e}\frac{\delta S}{\delta \chi^\alpha} = 0 \,, \\
T_a^m &\equiv -\frac{4\pi\alpha'}{e}\frac{\delta S}{\delta e_m^a} = 0 \,,
\end{aligned} \tag{2.3.6}$$

where we have defined the fermionic supercurrent J_a and the stress energy tensor T_{ab}. This is a consequence of superdiffeomorphism invariance. Just like in the bosonic case superWeyl invariance doesn't lead to a constraint but instead makes the stress energy tensor trace free and the supercurrent gamma trace free. In two dimensions the superdiffeomorphism and superWeyl invariance have the required amount of degrees of freedom to fix the zweibein and the Rarita-Schwinger field to $e_a^m = \delta_a^m$ and $\chi_a = 0$ respectively which again fixes the worldsheet metric to be the flat metric $h_{ab} = \eta_{ab}$. This is called the superconformal gauge and in this gauge the action for type II superstrings becomes

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\partial_a x^\mu \partial^a x_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu\right) , \tag{2.3.7}$$

with constraints $T_{ab} = 0$ and $J_a = 0$ where the expression for T_{ab} and J_a are given by

$$\begin{aligned} T_{ab} &= \partial_a x^\mu \partial_b x_\mu - \frac{1}{2}\eta_{ab}\partial_c x^\mu \partial^c x_\mu + \frac{1}{2}\bar{\psi}^\mu \rho_{\{a}\partial_{b\}}\psi_\mu - \frac{1}{4}\eta_{ab}\bar{\psi}^\mu \rho^c \partial_c \psi_\mu \,, \\ J_a &= -\frac{1}{2}\rho^b \rho_a \psi^\mu \partial_b x_\mu \,. \end{aligned} \tag{2.3.8}$$

Just like the conformal gauge the superconformal gauge doesn't use up all our gauge freedom and the residual symmetries are global conformal invariance as well as global supersymmetry

$$\begin{aligned} \delta_\epsilon x^\mu &= \bar{\epsilon}\psi^\mu \,, \\ \delta_\epsilon \psi^\mu &= \rho^a \partial_a x^\mu \epsilon \,. \end{aligned} \tag{2.3.9}$$

Therefore the symmetries that preserve the conformal gauge condition form global superconformal invariance. The action also has manifest global space-time Poincaré invariance.

Equations of motion and mode expansion Applying the variational principle to the action with respect to the fields x^μ and ψ^μ we find the massless wave equations

$$\begin{aligned} \partial_a \partial^a x^\mu &= 0 \,, \\ \rho^a \partial_a \psi^\mu &= 0 \,, \end{aligned} \tag{2.3.10}$$

as well as boundary terms. In order for the variation of the action to vanish we need to cancel these boundary terms by implementing boundary conditions for our strings. If we write our coordinates $(\sigma^0, \sigma^1) = (\tau, \sigma)$ then, up to conformal transformation, we can choose without loss of generality $\sigma \in [0, \pi[$. Therefore the endpoints of the string are $\sigma = 0, \pi$. We can either ask that the boundary terms vanish independently at each endpoint or that they cancel each other out, this corresponds to open and closed strings respectively.

For open strings, in order for the boundary terms to cancel independently we need that $\partial_1 x^\mu = 0$ and $\psi_+^\mu = \pm\psi_-^\mu$ hold at each endpoint. But since the overall relative sign between ψ_+^μ and ψ_-^μ is a matter of convention, we may always redefine one of

them so that $\psi_+^\mu(\tau,0) = \psi_-^\mu(\tau,0)$ holds. Hence we only need to worry about the $\sigma = \pi$ endpoint. Therefore we only have two possibilities

$$\begin{aligned}
&\underline{\text{Open strings}}\text{: } \partial_1 x^\mu(\tau,0) = \partial_1 x^\mu(\tau,\pi) = 0\,, \\
&\qquad\qquad \text{Ramond: } \psi_+\big(\tau,\pi\big) = \psi_-(\tau,\pi)\,, \quad \text{(R)} \\
&\qquad\qquad \text{Neveu-Schwarz: } \psi_+\big(\tau,\pi\big) = -\psi_-(\tau,\pi)\,. \quad \text{(NS)}
\end{aligned} \tag{2.3.11}$$

The condition on x^μ is known as the Neumann boundary condition, it can be interpreted as saying the endpoints of the string are free. There is actually a different condition we could impose called the Dirichlet boundary condition where the endpoints are fixed but we do not consider it in this section.

For closed strings we need that the x^μ obey periodic boundary conditions and that ψ_+^μ and ψ_-^μ obey separately periodic or anti-periodic boundary conditions. This leads to four possibilities.

$$\begin{aligned}
&\underline{\text{Closed strings}}\text{: } x^\mu\big(\tau,0) = x^\mu(\tau,\pi)\,, \\
&\qquad \text{Ramond-Ramond: } \psi_+\big(\tau,0) = \psi_+(\tau,\pi)\,, \\
&\qquad\qquad\qquad\qquad \psi_-\big(\tau,0) = \psi_-(\tau,\pi)\,, \quad \text{(R-R)} \\
&\qquad \text{Ramond-Neveu-Schwarz: } \psi_+\big(\tau,0) = \psi_+(\tau,\pi)\,, \\
&\qquad\qquad\qquad\qquad \psi_-\big(\tau,0) = -\psi_-(\tau,\pi)\,, \quad \text{(R-NS)} \\
&\qquad \text{Neveu-Schwarz-Ramond: } \psi_+\big(\tau,0) = -\psi_+(\tau,\pi)\,, \\
&\qquad\qquad\qquad\qquad \psi_-\big(\tau,0) = \psi_-(\tau,\pi)\,, \quad \text{(NS-R)} \\
&\qquad \text{Neveu-Schwarz-Neveu-Schwarz: } \psi_+\big(\tau,0) = -\psi_+(\tau,\pi)\,, \\
&\qquad\qquad\qquad\qquad \psi_-\big(\tau,0) = -\psi_-(\tau,\pi)\,. \quad \text{(NS-NS)}
\end{aligned} \tag{2.3.12}$$

Using these boundary conditions we can solve the equations of motion and express the solutions in a mode expansion. We introduce lightcone coordinates $\sigma^\pm = \tau \pm \sigma$

Closed string For the closed string the general solution can be decomposed into independent left and right moving parts $x^\mu(\tau,\sigma) = x_L^\mu(\sigma^+) + x_R^\mu(\sigma^-)$. These are given explicitly by

$$x_L^\mu(\sigma^+) = \frac{1}{2}x_0^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\bar{\alpha}_n^\mu}{n} e^{-2in\sigma^+} ,$$
$$x_R^\mu(\sigma^-) = \frac{1}{2}x_0^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2in\sigma^-} , \tag{2.3.13}$$

where reality requires that x_0^μ and p^μ be real and that $(\bar{\alpha}_n^\mu)^* = \bar{\alpha}_{-n}^\mu$ and $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$. We can show that x_0^μ and p^μ represent the center of mass position and momentum of the string respectively. For the spinors we have

$$\begin{aligned}
\psi_+^\mu(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} \bar{d}_n^\mu \, e^{-2in\sigma^+} \quad \text{or} \quad \psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{b}_r^\mu \, e^{-2in\sigma^+} , \\
\psi_-^\mu(\tau,\sigma) &= \sum_{n\in\mathbb{Z}} d_n^\mu \, e^{-2in\sigma^-} \quad \text{or} \quad \psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu \, e^{-2in\sigma^-} , \\
&\text{(R)} \qquad\qquad\qquad\qquad \text{(NS)}
\end{aligned} \tag{2.3.14}$$

where the Majorana condition requires that $(\bar{d}_n^\mu)^* = \bar{d}_{-n}^\mu$, $(d_n^\mu)^* = d_{-n}^\mu$, $(\bar{b}_r^\mu)^* = \bar{b}_{-r}^\mu$ and $(b_r^\mu)^* = b_{-r}^\mu$. The $\bar{\alpha}_n^\mu$, $\bar{d}_n^\mu$ and $\bar{b}_r^\mu$ represent left moving oscillatory modes while the α_n^μ, d_n^μ and b_r^μ represent the right moving oscillatory modes.

If we now express the stress energy tensor in light cone coordinates we find that the T_{+-} and T_{-+} components vanish identically. Therefore the only non trivial components of the constraints are $T_{\pm\pm} = 0$ and $J_\pm = 0$. We can also expand the stress energy tensor and the supercurrent into modes

$$\begin{aligned}
T_{++} &= \sum_{n\in\mathbb{Z}} \bar{L}_n \, e^{-2in\sigma^+} , \\
T_{--} &= \sum_{n\in\mathbb{Z}} L_n \, e^{-2in\sigma^-} , \\
J_+ = \sum_{n\in\mathbb{Z}} \bar{F}_n \, e^{-2in\sigma^+} \quad &\text{or} \quad J_+ = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{G}_r \, e^{-2ir\sigma^+} , \\
J_- = \sum_{n\in\mathbb{Z}} F_n \, e^{-2in\sigma^-} \quad &\text{or} \quad J_- = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r \, e^{-2ir\sigma^+} , \\
&\text{(R)} \qquad\qquad\qquad\qquad \text{(NS)}
\end{aligned} \tag{2.3.15}$$

where the stress energy tensor modes receive contributions from the bosonic sector and from the fermionic sector $\bar{L}_n = \bar{L}_n^{(b)} + \bar{L}_n^{(f)}$ and $L_n = L_n^{(b)} + L_n^{(f)}$. The bosonic contributions are given by

$$\begin{aligned} \bar{L}_n^{(b)} &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m\,, \\ L_n^{(b)} &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m\,, \end{aligned} \tag{2.3.16}$$

while the fermionic contributions are given by

$$\begin{aligned} \bar{L}_n^{(f)} &= \frac{1}{2}\sum_{m\in\mathbb{Z}}(m-\frac{n}{2})\bar{d}_{n-m}\cdot\bar{d}_m \quad \text{or} \quad \bar{L}_n^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}(r-\frac{n}{2})\bar{b}_{n-r}\cdot\bar{b}_r\,, \\ L_n^{(f)} &= \frac{1}{2}\sum_{m\in\mathbb{Z}}(m-\frac{n}{2})d_{n-m}\cdot d_m \quad \text{or} \quad L_n^{(f)} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}(r-\frac{n}{2})b_{n-r}\cdot b_r\,, \\ &\qquad\text{(R)} \qquad\qquad\qquad\qquad\qquad\qquad\text{(NS)} \end{aligned} \tag{2.3.17}$$

and the supercurrent modes are given by

$$\begin{aligned} \bar{F}_n &= \sum_{m\in\mathbb{Z}}\bar{\alpha}_m\cdot\bar{d}_{n-m} \quad \text{or} \quad \bar{G}_r = \sum_{m\in\mathbb{Z}}\bar{\alpha}_m\cdot\bar{b}_{r-m}\,, \\ F_n &= \sum_{m\in\mathbb{Z}}\alpha_m\cdot d_{n-m} \quad \text{or} \quad G_r = \sum_{m\in\mathbb{Z}}\alpha_m\cdot b_{r-m}\,, \\ &\text{(R)} \qquad\qquad\qquad\qquad\text{(NS)} \end{aligned} \tag{2.3.18}$$

where we have set $\bar{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. We can check that $\bar{L}_n^* = \bar{L}_{-n}$, $\bar{F}_n^* = \bar{F}_{-n}$, $\bar{G}_r^* = \bar{G}_{-r}$ and same for the unbarred modes. For the constraint to be satisfied classically all theses modes must vanish. They correspond to an infinite number of trivial conserved charges associated with the residual superconformal symmetry of the worldsheet theory. Only in 2 dimension is the (super)conformal algebra infinite dimensional.

Open string For the open string the boundary conditions impose that the right and left movers must combine into one single standing wave solution which is given explicitly by

$$x^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}\, e^{-in\tau}\cos(n\sigma)\,, \tag{2.3.19}$$

where reality requires again that x_0^μ and p^μ be real and that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$. For the spinors we distinguish between the two possible sectors

$$\underset{\text{(R)}}{\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu\, e^{-in\sigma^\pm}} \quad \text{or} \quad \underset{\text{(NS)}}{\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu\, e^{-in\sigma^\pm}}\,, \tag{2.3.20}$$

and the Majorana condition requires that $(d_n^\mu)^* = d_{-n}^\mu$ and $(b_r^\mu)^* = b_{-r}^\mu$. Now since the modes of the right and left moving components are the same then we have $T_{++} = T_{--}$ and $J_+ = J_-$. Therefore the modes of the stress energy tensor and the supercurrent are simply given by the unbarred modes of the closed string case where the definition of the 0 mode changes to $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$.

2.3.2 Quantization

As we can see from the action (2.3.7), from the worldsheet point of view, type II string theory simply corresponds to a free theory of scalars x^μ and spinors ψ^μ. As such, canonical quantization of the theory should be relatively easy. In this section we use the covariant quantization procedure as opposed to the light-cone quantization. These are old quantization schemes but they provide a fast way to derive the physical spectrum of the theory

Closed string Covariant quantization is a canonical quantization scheme which therefore relies on the Hamiltonian. Starting from the type II superstring action in conformal gauge one can show that the total classical worldsheet Hamiltonian is given by $H = \bar{L}_0 + L_0$. We also use the classical action to define the canonical conjugate momenta. As usual with canonical quantization we then promote all functions to operators on the Hilbert space of states and we impose the canonical commutation relations

$$[x^\mu(\tau,\sigma), \pi_1^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma'), \quad [\psi^\mu(\tau,\sigma')\,, \pi_2^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\,. \tag{2.3.21}$$

Using the mode expansions this gives us the following commutation relations on the modes

$$
\begin{gathered}
[x_0^\mu, p_0^\nu] = i\eta^{\mu\nu}\,, \\
[\alpha_m^\mu, \alpha_n^\nu] = [\bar\alpha_m^\mu\,, \bar\alpha_n^\nu] = m\,\eta^{\mu\nu}\delta_{m+n,0}\,, \\
\underset{\text{(R)}}{\{d_m^\mu, d_n^\nu\} = \{\bar d_m^\mu, \bar d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}} \quad \text{or} \quad \underset{\text{(NS)}}{\{b_s^\mu, b_r^\nu\} = \{\bar b_s^\mu, \bar b_r^\nu\} = \eta^{\mu\nu}\delta_{s+r,0}}\,,
\end{gathered}
\tag{2.3.22}
$$

all other (anti)commutators vanishing. We can see that for each component $\mu \neq 0$ and for each $m, r > 0$ these define standard bosonic and fermionic ladder operators $(\bar\alpha_m^\mu, \bar\alpha_m^{\mu\,\dagger}) \equiv (\frac{1}{\sqrt{m}}\bar\alpha_m^\mu, \frac{1}{\sqrt{m}}\bar\alpha_{-m}^\mu)$, $(\bar d_m^\mu, \bar d_m^{\mu\,\dagger}) \equiv (\bar d_m^\mu, \bar d_{-m}^\mu)$, $(\bar b_r^\mu, \bar b_r^{\mu\dagger}) \equiv (\bar b_r^\mu, \bar b_{-r}^\mu)$ and same for the unbarred modes

$$
\begin{gathered}
[a_m^\mu, a_m^{\mu\,\dagger}] = [\bar a_m^\mu, \bar a_m^{\mu\,\dagger}] = 1\,, \\
\underset{\text{(R)}}{\{d_m^\mu, d_m^{\mu\,\dagger}\} = \{\bar d_m^\mu, \bar d_m^{\mu\,\dagger}\} = 1} \quad \text{or} \quad \underset{\text{(NS)}}{\{b_r^\mu, b_r^{\mu\dagger}\} = \{\bar b_r^\mu, \bar b_r^{\mu\dagger}\} = 1}\,,
\end{gathered}
\tag{2.3.23}
$$

all of which are independent. Since all of them commute with $\alpha_0^\mu \sim p^\mu$ we can define a ground state $|k^\mu; 0\rangle$ which is an eigenvector of p^μ of eigenvalue k^μ and which is annihilated by all lowering operators a_m^μ, $\bar a_m^\mu$, d_m^μ, $\bar d_m^\mu$, b_r^μ and $\bar b_r^\mu$, for all $m, r > 0$ and for all $\mu = 0, \ldots, d-1$. Therefore the Hilbert space of states is a Fock space generated by all the possible products of raising operators $a_m^{\mu\,\dagger}$, $\bar a_m^{\mu\,\dagger}$, $d_m^{\mu\,\dagger}$, $\bar d_m^{\mu\,\dagger}$, $b_r^{\mu\dagger}$ and $\bar b_r^{\mu\dagger}$ applied to the ground state for $m, r > 0$. However our Hilbert space is not unitary, indeed because of the Lorentzian signature of space-time the time components of the oscillators give oscillators with the wrong sign

$$
\begin{gathered}
[a_m^0, a_m^{0\,\dagger}] = [\bar a_m^0, \bar a_m^{0\,\dagger}] = -1\,, \\
\underset{\text{(R)}}{\{d_m^0, d_m^{0\,\dagger}\} = \{\bar d_m^0, \bar d_m^{0\,\dagger}\} = -1} \quad \text{or} \quad \underset{\text{(NS)}}{\{b_r^0, b_r^{0\dagger}\} = \{\bar b_r^0, \bar b_r^{0\dagger}\} = -1}\,,
\end{gathered}
\tag{2.3.24}
$$

We can use these to construct negative norm states, this indicates that our Hilbert space is not positive definite. However we have not imposed the constraints on our Hilbert space, we will show that this will help make our theory unitary.

First, let us remark that, contrary to the classical theory, not all our quantities commute, therefore there can be an ambiguity in the definition of our operators. We choose the usual normal ordering convention where all lowering operators are placed to the right and all raising operators to the left. Almost all of the modes of the stress

energy tensor and supercurrent only involve expressions of commuting operators and hence all orderings are equal to the normal ordered expression. The only exceptions are L_0 and $\bar{L}_0$ which have expressions involving non commuting operators. Hence their ordering is ambiguous as not all orders are equivalent. Since we cannot know which ordering is the one appearing in the constraint we leave the possibility for an ordering constant.

We now want impose the constraints in the quantum theory. It turns out that imposing that all the stress energy tensor and supercurrent modes vanish on all physical states is too strong and doesn't lead to a consistent theory. Therefore we define instead the space of physical states as the subspace of states $|\psi\rangle$ such that

$$\begin{gathered} \langle\psi| L_n |\psi\rangle = \langle\psi| \bar{L}_n |\psi\rangle = 0\,, \\ \langle\psi| F_n |\psi\rangle = \langle\psi| \bar{F}_n |\psi\rangle = 0 \quad \text{or} \quad \langle\psi| G_r |\psi\rangle = \langle\psi| \bar{G}_r |\psi\rangle = 0\,, \\ \text{(R)} \qquad\qquad\qquad\qquad\qquad\qquad \text{(NS)} \end{gathered} \tag{2.3.25}$$

for all $n, r \neq 0$ and

$$\langle\psi| (L_0 - a) |\psi\rangle = \langle\psi| (\bar{L}_0 - \bar{a}) |\psi\rangle = 0\,, \tag{2.3.26}$$

where L_0 and $\bar{L}_0$ are the normal ordered versions and where a and $\bar{a}$ are normal ordering constants. But since we have that $\bar{L}_n^\dagger = \bar{L}_{-n}$, $\bar{F}_n^\dagger = \bar{F}_{-n}$, $\bar{G}_r^\dagger = \bar{G}_{-r}$ (and same for the unbarred modes) this means we can restrict the physical state condition to the positive indices $n, r \geq 0$. Hence the physical state condition is

$$\begin{aligned} &(\bar{L}_0 - \bar{a}_R) |\psi\rangle = 0 \quad \text{or} \quad (\bar{L}_0 - \bar{a}_{NS}) |\psi\rangle = 0\,, \\ &(L_0 - a_R) |\psi\rangle = 0 \quad \text{or} \quad (L_0 - a_{NS}) |\psi\rangle = 0\,, \\ &\bar{F}_n |\psi\rangle = 0 \qquad \text{or} \quad \bar{G}_r |\psi\rangle = 0\,, \\ &F_n |\psi\rangle = 0 \qquad \text{or} \quad G_r |\psi\rangle = 0\,, \\ &\qquad \text{(R)} \qquad\qquad\quad \text{(NS)} \end{aligned} \tag{2.3.27}$$

for all $n, r > 0$ and for all physical state $|\psi\rangle$. Actually one can show that $F_0^2 = L_0$ and $\bar{F}_0^2 = \bar{L}_0$. This already implies that $a_R = \bar{a}_R = 0$. We can also show that the commutation relations imply that the modes follow the super-Virasoro algebra. These are

$$
\begin{aligned}
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{D}{8} n(n^2-1)\delta_{n+m,0}\,, \\
[L_n, L_m] &= (n-m)L_{n+m} + \frac{D}{8} n(n^2-1)\delta_{n+m,0}\,, \\
[\bar{L}_n, \bar{F}_m] &= \left(\frac{n}{2} - m\right)\bar{F}_{n+m} \quad \text{or} \quad [\bar{L}_n, \bar{G}_r] = \left(\frac{n}{2} - r\right)\bar{G}_{n+r}\,, \\
[L_n, F_m] &= \left(\frac{n}{2} - m\right)F_{n+m} \quad \text{or} \quad [L_n, G_r] = \left(\frac{n}{2} - r\right)G_{n+r}\,, \\
\{\bar{F}_n, \bar{F}_m\} &= 2\bar{L}_{n+m} + \frac{D}{2} n^2 \delta_{n+m,0} \quad \text{or} \quad \{\bar{G}_r, \bar{G}_s\} = 2\bar{L}_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,, \\
\{F_n, F_m\} &= 2L_{n+m} + \frac{D}{2} n^2 \delta_{n+m,0} \quad \text{or} \quad \{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,, \\
&\qquad \text{(R)} \qquad\qquad\qquad\qquad \text{(NS)}
\end{aligned}
\tag{2.3.28}
$$

where D is the space-time dimension. All other commutators vanish. The even part of this superalgebra is called the Virasoro algebra. It is the unique central extension of the Witt algebra, the algebra of the 2-dimensional conformal group. Therefore the central charge in the Virasoro algebra represents a conformal anomaly. The odd parts of this algebra represent the two minimal $\mathcal{N} = 1$ supersymmetric extension of the Virasoro algebra. We notice that the charges with $-1 \leq n, r \leq 1$ generate closed subalgebras insensitive to the central charge, these represent global super-conformal transformations which remain unbroken. We also note that the barred and unbarred generators form two independent algebras.

We can come back to our initial concern that our theory wasn't unitary. Does imposing the physical state condition allow us to build unitary representations of the Poincaré algebra? There is a theorem called the no-ghost theorem (because negative norm states are called ghosts, not to confuse with ghost fields of the BRST cohomology used to gauge fix path integrals which are fields obeying the wrong spin statistics and which decouple from every amplitude) which states that the subspace of physical states is unitary if and only if

$$
D = 10 \tag{2.3.29}
$$

and

$$
\begin{cases}
a_R &= \bar{a}_R = 0\,, \\
a_{NS} &= \bar{a}_{NS} = \dfrac{1}{2}\,,
\end{cases}
\tag{2.3.30}
$$

Therefore, if we restrict to the subspace of physical states our theory is indeed unitary and we have unitary representations of the Poincaré algebra that we can interpret as particles. The fact that the space time dimension of the theory is constrained is a typical feature of supersymmetric theories but that it is uniquely fixed is pretty much unique to string theory.

In the covariant quantization procedure we have manifest Poincaré covariance and it is imposing unitarity which leads to fixing the space-time dimension and normal ordering constant. It is interesting to note that in the light cone quantization procedure it is unitarity that is manifest and imposing Poincaré covariance fixes the dimension and normal ordering constants to these same values. This means that it is the conflict between unitarity and causality which uniquely fixes string theory, this is of course reminiscent to anyone practicing S-matrix bootstrap.

Open string In the open string case the total classical worldsheet hamiltonian is given by $H = L_0$. Then everything else is given by just the unbarred sector of the closed string. Indeed one can think of the right and left moving sectors of the closed string as forming two independent copies of the open string.

2.3.3 Mass shell formula

Closed string If we add and subtract the first two constraints in (2.3.27) we get

$$(L_0 + \bar{L}_0 - a - \bar{a})\,|\psi\rangle = 0\,, \tag{2.3.31}$$
$$(L_0 - \bar{L}_0 - a + \bar{a})\,|\psi\rangle = 0\,. \tag{2.3.32}$$

Since $H = L_0 + \bar{L}_0 - a - \bar{a}$ is the quantum worldsheet Hamiltonian which generates time translations on the worldsheet the first constraint can therefore be interpreted as an equivalent of the Wheeler-DeWitt equation on the worldsheet. It says that there is no physical significance to the worldsheet time coordinate because the non gauge fixed theory has worldsheet time diffeomorphism invariance. Similarly $P = L_0 - \bar{L}_0 - a + \bar{a}$ is the worldsheet momentum which generates space translations on the worldsheet. Therefore the second constraint says that there is no physical significance to the worldsheet space coordinate because the non gauge fixed theory has worldsheet space diffeomorphism invariance. Let us unpack these constraints, we can write

$$\bar{L}_n^{(b)} = \frac{1}{2}\bar{\alpha}_0 \cdot \bar{\alpha}_0 + \sum_{m\geq 1} \bar{\alpha}_{-m} \cdot \bar{\alpha}_m \,,$$
$$L_n^{(b)} = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{m\geq 1} \alpha_{-m} \cdot \alpha_m \,. \tag{2.3.33}$$

Let us define

$$\bar{N}^{(b)} \equiv \sum_{m\geq 1} \bar{\alpha}_{-m} \cdot \bar{\alpha}_m = \sum_{m\geq 1} m\, \bar{a}_m^\dagger \cdot \bar{a}_m \,,$$
$$N^{(b)} \equiv \sum_{m\geq 1} \alpha_{-m} \cdot \alpha_m = \sum_{m\geq 1} m\, a_m^\dagger \cdot a_m \,, \tag{2.3.34}$$

where we have introduced the ladder operators. Similarly we can define

$$\bar{N}^{(f)} \equiv \bar{L}_0^{(f)} = \sum_{m\geq 1} m\, \bar{d}_m^\dagger \cdot \bar{d}_m \quad \text{or} \quad \bar{N}^{(f)} \equiv \bar{L}_0^{(f)} = \sum_{r+\frac{1}{2}\geq 1} r\, \bar{b}_r^\dagger \cdot \bar{b}_r \,,$$
$$N^{(f)} \equiv L_0^{(f)} = \sum_{m\geq 1} m\, d_m^\dagger \cdot d_m \quad \text{or} \quad N^{(f)} \equiv L_0^{(f)} = \sum_{r+\frac{1}{2}\geq 1} r\, b_r^\dagger \cdot b_r \,, \tag{2.3.35}$$
$$\text{(R)} \qquad\qquad\qquad\qquad \text{(NS)}$$

Therefore we see that $\bar{N} \equiv \bar{N}^{(b)} + \bar{N}^{(f)}$ and $N \equiv N^{(b)} + N^{(f)}$ correspond to the number operators in the left and right moving sectors respectively. We can see that in the NS sector the number operator can have half integer eigenvalues. Recalling that $\bar{\alpha}_0^\mu = \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$ the second constraint (2.3.32) is just the matching condition

$$\left(\bar{N} - \bar{a}\right) |\psi\rangle = (N - a)\, |\psi\rangle \,, \tag{2.3.36}$$

and if we define the mass operator $M^2 \equiv -p^\mu p_\mu$ the first constraint (2.3.31) is the mass-shell condition which gives the string spectrum

$$M^2\, |\psi\rangle = \frac{4}{\alpha'} \left(\bar{N} - \bar{a}\right) |\psi\rangle = \frac{4}{\alpha'} (N - a)\, |\psi\rangle \,. \tag{2.3.37}$$

We see that for all sectors except (NS-NS) the matching conditions prevent the mass operator from being negative. For the (NS-NS) sector however the ground state has negative mass $M^2 = -2/\alpha'$, it is a tachyon. This usually suggests some inconsistency having to do with causality in our theory or at least that the vacuum vacuum is unstable. We will see how we can resolve this by introducing a consistent projection of the spectrum of our theory.

Open string For the open string there are no independent left and right movers therefore there is no matching condition. Furthermore the mass-shell condition is slightly different because we have $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ therefore

$$M^2 |\psi\rangle = \frac{1}{\alpha'} (N - a) |\psi\rangle \ . \tag{2.3.38}$$

Again in the (NS) sector nothing prevents the mass operator from having negative eigenvalues and in fact the ground state is again tachyonic $M^2 = -1/2\alpha'$.

2.3.4 Spectrum

We will now find the particle spectrum of type II superstring theory, that is, find irreducible unitary representations of the 10-dimensional Poincaré group that make up the space of physical states. We will want to focus on the massless spectrum i.e. the particles of eigenvalue $M^2 = 0$ because in the field theory limit, when $\alpha' \to 0$, all other modes get infinitely high masses and decouple from the theory. Therefore the effective field theory of type II superstring theory will only capture the massless spectrum. Since we have seen that the closed string can be obtained from the tensor product of two copies of the open string we start by deriving the spectrum of the open string.

Open string Let us start by considering the spectrum of the (R) sector. We can easily see that

$$[d_0^\mu, N_R] = 0 \, , \tag{2.3.39}$$

and therefore

$$[d_0^\mu, M^2] = 0 \, , \tag{2.3.40}$$

for $\mu = 0, \ldots, 9$. Therefore the d_0^μ stabilize the eigenspaces of M^2. Let $|\psi, a\rangle$ be a basis of some eigenspace of M^2. This means that each d_0^μ maps basis states to some linear combination of themselves

$$d_0^\mu |\psi, a\rangle = (A^\mu)_a{}^b |\psi, b\rangle \ , \tag{2.3.41}$$

for some matrix A^μ. However we can see from the commutation relations (2.3.22) that the 0-modes d_0^μ satisfy

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \, . \tag{2.3.42}$$

Therefore we can see that the $\sqrt{2}\, d_0^\mu$ define a representation of the 10-dimensional space-time Clifford algebra on the Hilbert space. Hence the $\sqrt{2}\, A^\mu$ must also form a matrix representation of the 10-dimensional space-time Clifford algebra, we can therefore see that $A^\mu = \frac{1}{\sqrt{2}}\gamma^\mu$ where γ^μ are the ten dimensional gamma matrices and

$$d_0^\mu \left|\psi, a\right\rangle = \frac{1}{\sqrt{2}} (\gamma^\mu)_a{}^b \left|\psi, b\right\rangle \ . \tag{2.3.43}$$

Hence the $|\psi, a\rangle$ must form a representation on which the Clifford algebra acts, they are 10-dimensional spacetime spinors. By the spin-statistics theorem this also means that they are space-time fermions. We have therefore shown that ll states in the (R) sector are spacetime fermions.

In particular the ground state extends into a Dirac spinor representation $|a\rangle_R \equiv |k^\mu; 0, a\rangle_R$. However we are looking for irreducible representations and we know that Dirac spinors are not irreducible in 10 dimensions where we can impose both a Majorana and Weyl condition simultaneously. Therefore we can decompose the ground state into two inequivalent irreducible representations $|+, a\rangle_R$ and $|-, a\rangle_R$ corresponding to the right and left chirality representations. These finally form irreducible unitary representations of the Poincaré group and can thus be interpreted as particles. All other states of the (R) sector will be obtained by acting on the ground state with the (R) raising operators ${a_m^\mu}^\dagger$ and $d_n^{\mu\dagger}$ and since they are 10 dimensional space-time vectors then all physical states in the (R) spectrum will be tensor products of vector representations with a Majorana-Weyl spinor representation which can then be decomposed into direct sums of irreducible representations.

In the (NS) sector the ground state $|0\rangle_{NS} \equiv |k^\mu; 0\rangle_{NS}$ isn't degenerate. This means that it forms the trivial scalar representation of the Poincaré group. All other states of the (NS) sector will be obtained by acting with the (NS) raising operators ${a^\mu}_m^\dagger$ and ${b^\mu}_r^\dagger$ and since they are 10 dimensional space-time vectors then all physical states in the (NS) spectrum will be tensor products of vector representations which can then be decomposed into direct sums of irreducible representations. This means that there will only be tensor representations and no spinor representations. By the spin-statistics theorem this also means that they will all be space-time bosons. We have therefore shown that all states in the (NS) sector are spacetime bosons.

However we have seen that the ground state of the (NS) sector is actually a tachyon which indicates that the theory is still not fully consistent. We will now see how we can remedy that by applying the so called GSO projection [25]. Indeed superstring theories admit a consistent truncation of their spectrum which is necessary for consistency of the interacting theory. Let us define a projector P_{GSO} which is given

in the (NS) sector by

$$P_{GSO} \equiv \frac{1}{2}\left(1-(-1)^{N^{(f)}}\right) , \quad \text{(NS)} \tag{2.3.44}$$

where $N^{(f)}$ is the fermionic number operator. This means that we only keep states with an odd number of $b_r^{\mu\dagger}$ oscillator excitations and we remove those with an even number. This automatically gets rid of the tachyonic ground state $|0\rangle_{NS}$. Now the new ground state is the spacetime vector $b_{\frac{1}{2}}^{\mu\,\dagger}|0\rangle_{NS}$ which has mass $M^2=0$. In fact the GSO projection forces the number operator N to only take half integer values, hence from the mass shell formula we can see that the spectrum of allowed physical masses will be multiples of $1/\alpha'$, i.e.

$$M^2 = 0, \frac{1}{\alpha'}, \frac{2}{\alpha'} \dots \quad \text{(NS)} \tag{2.3.45}$$

In the (R) sector the GSO projector is defined as

$$P^{\pm}_{GSO} \equiv \frac{1}{2}\left(1 \pm \gamma_* (-1)^{N^{(f)}}\right) , \quad \text{(R)} \tag{2.3.46}$$

where γ_* is the 10 dimensional chirality operator such that

$$(\gamma_*)_a^{\ b}\,|\pm, b\rangle_R = \pm\,|\pm, a\rangle_R \ . \tag{2.3.47}$$

In particular, since $N^{(f)}\,|\pm, a\rangle_R = 0$, this only keeps one of the chirality ground states. Here this is merely a matter of conventions and hence we choose P^{+}_{GSO} in order to keep $|+, a\rangle_R$. From the mass shell formula we can see that the spectrum of allowed physical masses will also be multiples of $1/\alpha'$, i.e.

$$M^2 = 0, \frac{1}{\alpha'}, \frac{2}{\alpha'} \dots \quad \text{(R)} \tag{2.3.48}$$

We can now give the massless spectrum of the type II open superstring. In the (R) sector we have seen that the ground state $|+, a\rangle_R$ has mass $M^2=0$ and that it is a Majorana-Weyl spinor which is already in an irreducible unitary representation of the Poincaré group (actually it is a finite dimensional non-unitary representation of the Lorentz group but one can show that it indeed leads to an irreducible unitary representation of the Poincaré group on the Hilbert space with little group $SO(8)$ as for all massless 10 dimensional particles, the same is true for all other particles that follow). Therefore it describes one particle and cannot be further decomposed. Similarly in the (NS) sector the ground state $b_{\frac{1}{2}}^{\mu\,\dagger}|0\rangle_{NS}$ has mass $M^2=0$ and it is a vector which is also an irreducible unitary representation of the Poincaré group. Therefore the whole massless spectrum of the open string is made up of two particles

Sector	State	Boson/Fermion	Field limit
(R)	$\|+, a\rangle_R$	fermion	λ^+
(NS)	$b^\mu_{-\frac{1}{2}} \|0\rangle_{NS}$	boson	A_μ

Table 2.1: Massless spectrum of the open type II superstring

One can see that there is an equal number of bosonic and fermionic degrees of freedom and hence that the massless spectrum forms a spacetime $\mathcal{N} = 1$ massless gauge supermultiplet. One can actually show that after GSO projection the entire spectrum, including the massive spectrum, of the open string theory is supersymmetric. This is remarkable as the theory we started from only has worldsheet supersymmetry but it turns out that it also has space-time supersymmetry as well.

Closed string We can now recover the closed string spectrum by taking the tensor product of two copies of the open string, one for the right moving sector and one for the left moving sector and then imposing the suitable matching condition. The tensor products of two Poincaré representations in the open string sector then further decomposes into irreducible representations in the closed string sector. One can also show analogously to the case of the open string that all states in the (R-R) and (NS-NS) sectors are spacetime bosons and all states in the (R-NS) and (NS-R) sectors are spacetime fermions.

The GSO projection is also performed independently on the right and left movers by applying $P_{GSO} \otimes P_{GSO}$. However we can see that in the (R-R) sector we now have two inequivalent choices. Indeed, depending on if we keep the same chirality ground states or opposite chirality ground states for the left and right movers we get physically inequivalent theories. In other words we can apply the $P^+_{GSO} \otimes P^-_{GSO}$ or $P^+_{GSO} \otimes P^+_{GSO}$ projectors (the other two choices being equivalent to one of those two). The two possible string theories one can construct this way are called type IIA and type IIB. Let's construct their spectrum. One can show that in both theories and in every sector the mass shell formula states that the spectrum of allowed physical masses will also be multiples of $4/\alpha'$, i.e.

$$M^2 = 0, \frac{4}{\alpha'}, \frac{8}{\alpha'} \dots \tag{2.3.49}$$

Type IIA superstring: This is the theory one gets by keeping ground states of opposite chirality in the (R-R) sector, in other words, by applying $P^+_{GSO} \otimes P^-_{GSO}$. In the (NS-NS) sector the ground state $b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS} \otimes b^{\nu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS}$ has mass $M^2 = 0$. However it is not irreducible, it decomposes into a spin 2 (symmetric traceless) representation, a 2-form (antisymmetric) representation and a scalar (trace) representation. In the (R-NS) sector the massless ground state is $|+, a\rangle_R \otimes b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS}$ it decomposes into a spin 3/2 and a spin 1/2 Majorana-Weyl representation. In the (NS-R) sector the ground state is $b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS} \otimes |-, a\rangle_R$ it also decomposes into a spin 3/2 and a spin 1/2 Majorana-Weyl representation but of opposite chirality to the (R-NS) sector. Finally in the (R-R) sector the ground state is $|+, a\rangle_R \otimes |-, b\rangle_R$, it decomposes into a three form and a vector representation.

Sector	State	Boson/Fermion	Field limit
(NS-NS)	$b^{\mu\,\dagger}_{\frac{1}{2}} \|0\rangle_{NS} \otimes b^{\nu\,\dagger}_{\frac{1}{2}} \|0\rangle_{NS}$	boson	$g_{\mu\nu},\ B_{\mu\nu},\ \phi$
(R-NS)	$\|+, a\rangle_R \otimes b^{\mu\,\dagger}_{\frac{1}{2}} \|0\rangle_{NS}$	fermion	$\psi^+_\mu,\ \lambda^-$
(NS-R)	$b^{\mu\,\dagger}_{\frac{1}{2}} \|0\rangle_{NS} \otimes \|-, a\rangle_R$	fermion	$\psi^-_\mu,\ \lambda^+$
(R-R)	$\|+, a\rangle_R \otimes \|-, b\rangle_R$	boson	$C_{\mu\nu\rho},\ C_\mu$

Table 2.2: Massless spectrum of the closed type IIA superstring

Type IIB superstring: This is the theory one gets by keeping ground states of same chirality in the (R-R) sector, in other words, by applying $P^+_{GSO} \otimes P^+_{GSO}$. The (NS-NS) sector is the same as for type IIA, the ground state $b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS} \otimes b^{\nu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS}$ has mass $M^2 = 0$ and it decomposes into a spin 2 representation, a 2-form representation and a scalar representation. In the (R-NS) sector the massless ground state is $|+, a\rangle_R \otimes b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS}$, it decomposes into a spin 3/2 and a spin 1/2 Majorana-Weyl representation. In the (NS-R) sector the ground state is $b^{\mu\,\dagger}_{\frac{1}{2}} |0\rangle_{NS} \otimes |+, a\rangle_R$, it also decomposes into a spin 3/2 and a spin 1/2 Majorana-Weyl representation and of same chirality as the (R-NS) sector so that those two sectors are actually the same. Finally in the (R-R) sector the ground state is $|+, a\rangle_R \otimes |+, b\rangle_R$, it decomposes into a self dual four form, a two form and a scalar representation.

Sector	State	Boson/Fermion	Field limit
(NS-NS)	$b^{\mu\,\dagger}_{\frac{1}{2}} \left\|0\right\rangle_{NS} \otimes b^{\nu\,\dagger}_{\frac{1}{2}} \left\|0\right\rangle_{NS}$	boson	$g_{\mu\nu},\ B_{\mu\nu},\ \phi$
(R-NS)	$\left\|+,a\right\rangle_R \otimes b^{\mu\,\dagger}_{\frac{1}{2}} \left\|0\right\rangle_{NS}$	fermion	$\psi^+_\mu,\ \lambda^-$
(NS-R)	$b^{\mu\,\dagger}_{\frac{1}{2}} \left\|0\right\rangle_{NS} \otimes \left\|+,a\right\rangle_R$	fermion	$\psi'^+_\mu,\ \lambda'^-$
(R-R)	$\left\|+,a\right\rangle_R \otimes \left\|+,b\right\rangle_R$	boson	$C_{\mu\nu\rho\sigma},\ C_{\mu\nu},\ C_0$

Table 2.3: Massless spectrum of the closed type IIB superstring

In both cases one can show that the massless field content of the closed string forms a spacetime $\mathcal{N} = 2$ massless graviton supermultiplet. One can also show that the massive spectrum is supersymmetric (even though their ground state differ the massive states of type IIA and type IIB string theory are actually all the same). Note that because type II string theory possesses $\mathcal{N} = 2$ supersymmetry this actually prevents the theory from containing open strings as these only have $\mathcal{N} = 1$ supersymmetry. Therefore type II superstring theory only contains closed strings. One might think that there must also exist a theory containing only type II open strings and no closed strings however this theory would be inconsistent because open strings can always close to make closed strings. Actually if one includes D-branes in type II string theory then there can be open strings whose endpoints are attached to the D-branes, therefore these have Dirichlet boundary conditions instead of Neumann boundary conditions. This is possible because BPS D-branes typically break half of the supersymmetry of the theory. However D-branes are non perturbative objects and therefore don't appear in the perturbative spectrum of the theory.

Here we have used the RNS formalism to construct the superstring where we impose supersymmetry on the worldsheet and we recover space-time supersymmetry after GSO projection. There is another formalism that can be used which is called the GS formalism where space-time supersymmetry is imposed from the beginning, one can show that we recover the same theory without having to impose GSO projection, the two formalisms are thus equivalent. However the GS formalism is much harder to quantize and to use in practice.

2.4 Other superstring theories

We have seen that type II superstrings leads to two consistent superstring theories. These are the only two maximally supersymmetric string theories. However they are not the only consistent string theories.

Type I superstring The type I superstring is derived by applying a so-called orientifold projection on the type IIB superstring. This means quotienting the theory by the $\mathbb{Z}_2$ symmetry that exchanges the orientation of the string. This breaks half of the supersymmetry and therefore therefore allows open strings. One thus gets a theory of unoriented closed and open strings which only has $\mathcal{N} = 1$ spacetime supersymmetry. In the open string sector, anomaly cancelation singles out $SO(32)$ as the only possible gauge symmetry group.

Heterotic superstring The heterotic superstring is constructed by exploiting the independence between left and right movers in the closed string sector. Here the left movers are given a bosonic string structure and the right movers are given a type II superstring structure. The spacetime dimension is still 10 because we only add 10 x^μ coordinate fields ($\mu = 0, \ldots, 9$). But since the left moving sector is bosonic the central charge of the Virasoro algebra needs to be 26 and not 10. In order to remedy this we add 32 fermionic left movers so that $c = 10 + 32/2 = 26$. This is known as the fermionic construction of the heterotic superstring. Note that there is an equivalent bosonic construction which uses 26 left moving coordinates where it appears that the spacetime dimension is different in the right moving and left moving sector, however this formalism leads to the same theory as the fermionic construction which clearly has 10 spacetime dimensions. We can see that heterotic string theory only contains closed strings because there are no independent right and left movers for open strings. The action for the heterotic string in conformal gauge is given by

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left(2\partial_+ x^\mu \partial_- x_\mu + i\psi^\mu \partial_+ \psi_\mu + i\lambda^A \partial_- \lambda^A\right) , \tag{2.4.1}$$

where λ^A are 1-dimensional worldsheet Majorana-Weyl spinors but are spacetime scalars, $A = 1, \ldots, 32$. The additional 16 coordinates in the left moving sector are compactified on a torus defined by a 16-dimensional even self dual lattice. There are only two such lattices which lead to two different theories

Heterotic $SO(32)$ superstring The first choice is to use the root lattice of the Lie group $\mathrm{Spin}(32)/\mathbb{Z}_2$ which is often just abbreviated to $SO(32)$ because $\mathrm{Spin}(32)$ is the four-fold cover of $SO(32)$. The resulting theory has $\mathcal{N} = 1$ supersymmetry. The compactification lattice give the theory an $SO(32)$ gauge symmetry.

Heterotic $E_8 \times E_8$ superstring The second choice is to use the root lattice of the Lie group $E_8 \times E_8$. The resulting theory also has $\mathcal{N} = 1$ supersymmetry. Its compactification lattice give it an $E_8 \times E_8$ gauge symmetry. Historically this was phenomenologically very interesting because the gauge group of the standard model fits inside E_8 through a nice chain of embeddings

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8 \,. \tag{2.4.2}$$

The various Lie groups that appear in this sequence are precisely the ones that have been the most studied as candidates for grand unification symmetry groups. This is why initially many Calabi-Yau compactifications were aimed at reducing the heterotic $E_8 \times E_8$ superstring down to 4 dimensions. Today however this motivation is seen as less important as grand unification theories have been mostly abandoned and we have now understood how to get gauge groups from type II theories via singularities.

2.5 Supergravity limit

It is a fact that all string theories contain a massless spin 2 field. Therefore, as we have seen, it can be shown from very general arguments that the low energy limit must be that of general relativity [8–11] and therefore this field must be the graviton. One can also compute that the low energy limit of the graviton amplitudes calculated within string theory and check that they reproduce exactly scattering amplitudes calculated with general relativity. There is a third way to see that string theory is indeed a theory of quantum gravity that reproduces general relativity at low energies. Indeed, from the beginning we have quantized our strings around flat space-time. Even though this is much harder we can also choose to quantize them in a curved background, at least formally. If we focus on the bosonic sector of the theory this is achieved by using the form (2.2.7) of the Polyakov action where the flat metric is replaced by a generic metric $g_{\mu\nu}$. This is like a non-linear sigma model with the metric $g_{\mu\nu}$ acting as a coupling function. And the condition that this new action is still invariant under worldsheet conformal transformation at the quantum level is that the beta functional for the metric vanishes. One can show that at the one loop level this condition is exactly equivalent to the metric obeying the Einstein equation for general relativity

$$0 = \beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} \,. \tag{2.5.1}$$

One can even add other background field corresponding to the other massless bosonic fields of the spectrum (these are the only fields which can form a coherent condensate and propagate at long distances and therefore have a classical background). The particle content of the (NS-NS) sector is actually universal to all string theories,

these are called the graviton, the Kalb-Ramond form and the dilaton. Therefore we can couple the action to the dilaton and the Kalb-Ramond background fields to get

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \left(\sqrt{-h}\, h^{ab}\partial_a x^\mu \partial_b x^\nu g_{\mu\nu} + \varepsilon^{ab} B_{\mu\nu}\partial_a x^\mu \partial_b x^\nu + \sqrt{-h}\,\alpha'\phi R\right) , \quad (2.5.2)$$

where R is the worldsheet Ricci scalar. Therefore we can see that The Kalb-Ramond form can be interpreted as some sort of generalised gauge field coupled to the strings. Now the condition that the one loop beta functionals vanish is given by [61–64]

$$\begin{aligned} 0 &= \beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{4}H_{\mu\sigma\rho}H_\nu{}^{\sigma\rho}\,, \\ 0 &= \beta_{\mu\nu}(B) = -\frac{\alpha'}{2}\nabla^\rho H_{\rho\mu\nu} + \alpha'\nabla^\rho\phi H_{\rho\mu\nu}\,, \\ 0 &= \beta(\phi) = -\frac{\alpha'}{2}\nabla_\mu\nabla^\mu\phi + \alpha'\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\,, \end{aligned} \quad (2.5.3)$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength associated to the Kalb Ramond field. These correspond to the simplest equations of motion one can write involving a graviton, a two form and a scalar. If one repeats the same procedure with the other massless bosonic fields of the (R-R) sectors one would find appearing the equations of motion of the bosonic sector of various supergravity theories in ten dimensions. These are the low energy field theory limit of string theory. Actually these equations of motion give an action where the dilaton is non minimally coupled to gravity, this is the so-called string frame. One can perform a conformal field redefinition to recover a canonically normalised Einstein-Hilbert term, this is the so-called Einstein frame.

For the rest of this thesis and contrary to the previous chapter we define the D-dimensional the gravitational coupling constant by $\kappa_D^2 = 8\pi G$ and we define the D-dimensional Planck length by $2\kappa_D^2 = (2\pi)^{D-3} l_P^{D-2}$.

2.5.1 Classification of supergravity theories

Supergravity is, at its core, a supersymmetric version of Einstein's theory of general relativity. However it is more than that. If general relativity can be seen as the gauge theory of local Poincaré symmetry, then supergravity can be seen as the gauge theory of local superPoincaré symmetry. Therefore, just like the graviton is the gauge field for Poincaré invariance the gravitino field, the graviton's supersymmetric partner, can be seen as the gauge field for supersymmetry invariance. See [65] for a comprehensive

introduction to supergravity.

Supergravity theories are invariant under local supersymmetry. Therefore their field content must form a supermultiplet. Since this is a theory of gravity we require that the theory contains at least the graviton supermultiplet, that is, a supermultiplet containing a massless spin 2 particle. And since no consistent theory containing a massless particle with spin higher than 2 is known we require that the theory contains no such particles in its supermultiplet. One can show that this implies that our theory can have no more that 32 real supercharges components.

This result holds in any dimension but can easily be checked in 4 dimensions. Indeed, in 4 dimensions, we know that for each supercharge we add a particle with helicity higher by 1/2 to the massless supermultiplet. Therefore starting from a particle with helicity -2 it takes 8 supercharges to get to a particle with helicity 2. This forms a massless supermultiplet that is automatically CPT invariant, contains the graviton and no particle with higher spin. Anymore supercharges would necessarily add particles of spin higher than 2 to the supermultiplet. Therefore 8 supercharges is the maximum one can have in 4 dimensions. However, one is always required to use minimal spinor representations for supercharges, in 4 dimensions we can use Majorana or Weyl spinors, both of which have 4 real components. Therefore the maximum number of real supercharges components is indeed $8 \times 4 = 32$.

This also justifies why the result holds in other dimensions. Indeed if we could find a consistent supergravity theory with more that 32 real supercharges components in some dimension higher than 4 we could always dimensionally reduce the theory to 4 dimensions by compactifying on a torus. This doesn't break any supersymmetry, therefore our dimensionally reduced theory would define a consistent four dimensional theory with more that 32 supercharge components, but this would necessarily include some particle with spin higher than 2 and we know that is not possible. Hence, since for any dimension greater than 11 the minimal spinor representation has more than 32 components then the maximal dimension in which supergravity can exist is 11.

Up to the non gravitational sector, it is possible to classify the supergravity theories by the local supersymmetry algebra they are based on. For each number of supercharge $\mathcal{N}$ we will get a different supergravity theory. When the only minimal spinor representation is Weyl or Majorana-Weyl (in dimensions 6 and 10 for example) we have to distinguish the possible chiralities of the supercharges, hence we write $\mathcal{N} = (p, q)$ where we have p supercharges of one chirality and q supercharges of the other chirality. Note however that the SUSY algebra $\mathcal{N} = (p, q)$ is equivalent to the SUSY algebra $\mathcal{N} = (q, p)$. When both Majorana and Weyl represenations are avail-

able then both descriptions are equivalent and hence we do not have to distinguish by distribution of chiralities since this distinction doesn't apply for Majorana spinors.

However not all supersymmetry algebras with less that 32 supercharge components lead to independent supergravity theories. For example in 6 dimensions the $\mathcal{N} = (3,1)$, $\mathcal{N} = (4,0)$ and $\mathcal{N} = (3,0)$ theories don't contain a graviton but instead a more complicated tensor field, no non-linear action is known for them and thus they are generally not considered to exist as supergravity theories. In 4 dimensions the $\mathcal{N} = 7$ is automatically equivalent to the $\mathcal{N} = 8$ theory, thus it is not included as an independent supergravity theory. The independent supergravity theories are given in table 2.4 [65].

Dim	32		24	20	16		12	8	4
11	M $\mathcal{N} = 1$								
10	IIA $(1,1)$	IIB $(2,0)$			I $(1,0)$				
9	$\mathcal{N} = 2$				$\mathcal{N} = 1$				
8	$\mathcal{N} = 2$				$\mathcal{N} = 1$				
7	$\mathcal{N} = 2$				$\mathcal{N} = 1$				
6	$(2,2)$		$(2,1)$		$(1,1)$	$(2,0)$		$(1,0)$	
5	$\mathcal{N} = 4$		$\mathcal{N} = 3$		$\mathcal{N} = 2$			$\mathcal{N} = 1$	
4	$\mathcal{N} = 8$		$\mathcal{N} = 6$	$\mathcal{N} = 5$	$\mathcal{N} = 4$		$\mathcal{N} = 3$	$\mathcal{N} = 2$	$\mathcal{N} = 1$

Table 2.4: Classification of supergravity theories in dimensions greater than 4. Each entry represents the possibility to have supergravity theories in a specific dimension with the number of real supersymmetries indicated in the top row.

Note that in 7 and 5 dimensions, where symplectic-Majorana spinors are used the supergravity theories are often referred to as $\mathcal{N} = 2, 4, 6, 8$ for 5 dimensions and $\mathcal{N} = 2, 4$ for 7 dimensions. Thus effectively counting twice as less components per

supercharge. The supergravity theories in dimensions 10 and 11 have special names because they are the low energy limits of string theories. Indeed $D = 11$ supergravity is the low energy limit of M-theory, type IIA and type IIB supergravity are the low energy limits of type IIA and type IIB string theory respectively while type I supergravity is the low energy limit of the gravitational sector of type I and heterotic string theories. Let us quickly review their field contents and the bosonic parts of their actions.

2.5.2 $D = 11$ Supergravity

Eleven dimensional supergravity was discovered in 1978 by Cremmer, Julia and Scherk [66]. The field content of 11 dimensional supergravity theory can be found by matching bosonic and fermionic degrees of freedom, it is surprisingly simple, we have

$$g_{\mu\nu}\,, \quad \psi_\mu\,, \quad A_{\mu\nu\rho}\,, \tag{2.5.4}$$

where $g_{\mu\nu}$ is the graviton, ψ_μ is the gravitino and $A_{\mu\nu\rho}$ is a three form. This forms a massless $\mathcal{N} = 1$ graviton multiplet. The bosonic part of the action is given by

$$S = \frac{1}{2\kappa_{11}^2}\int d^{11}x\,\sqrt{-g}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}^2}\int A_3 \wedge F_4 \wedge F_4\,, \tag{2.5.5}$$

where $F_4 = dA_3$ is the field strength tensor associated with the three form. The eleven dimensional coupling constant is given in terms of the eleven dimensional Planck length by $2\kappa_{11}^2 = (2\pi)^8 l_P^9$.

Other than the fact that this is the maximal dimension in which we can have supergravity eleven dimensional supergravity is also special because of its uniqueness. Nothing can be added and it cannot be modified. The only tunable parameter of the theory is the eleven dimensional gravitational constant. Since its discovery eleven dimensional supergravity has sparked a lot of interest, indeed many interesting maximally supersymmetric supergravity theories can be obtained from it by dimensional reduction such as $\mathcal{N} = 8$ supergravity and $SO(8)$ gauged supergravity in four dimensions and type IIA supergravity in ten dimensions. Eleven dimensional supergravity is also the low energy limit of the massless sector of M-theory which is also believed to be unique and containing no free dimensionless parameters [30].

2.5.3 Type IIA supergravity

The field content of type IIA supergravity is

$$\begin{aligned} &g_{\mu\nu}\,, \quad \psi^+_\mu\,, \quad B_{\mu\nu}\,, \quad \lambda^-\,, \quad \phi\,, \\ &\psi^-_\mu\,, \quad C_{\mu\nu\rho}\,, \quad C_\mu\,, \quad \lambda^+\,, \end{aligned} \tag{2.5.6}$$

where ψ^+_μ and ψ^-_μ are gravitini of opposite chirality, $B_{\mu\nu}$ is the Kalb-Ramond two form, λ^- and λ^+ are spinors of opposite chirality called the dilatini, ϕ is a scalar called the dilaton, $C_{\mu\nu\rho}$ is a three form and C_μ is a one form. This is exactly the massless field content of type IIA string theory, it forms a massless $\mathcal{N} = (1,1)$ graviton multiplet. Type IIA supergravity can be found as the dimensional reduction of 11-dimensional supergravity on a circle.

String frame The bosonic part of the action in string frame is given by

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(e^{-2\phi}\left(R + 4|\nabla\phi|^2 - \frac{1}{2}|H_3|^2\right) - \frac{1}{2}|F_2|^2 - \frac{1}{2}|\tilde{F}_4|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4\,, \end{aligned} \tag{2.5.7}$$

where $H_3 = dB_2$, $F_2 = dC_1$, $F_4 = dC_3$, $\tilde{F}_4 = F_4 + C_1 \wedge H_3$. This is the action which reproduces the vanishing of the one loop beta functionals (2.5.3) as its equations of motion if one also includes the (R-R) fields of type IIA string theory, therefore it is indeed the low energy limit of type IIA string theory. This allows us to relate the gravitational coupling constant to the string length by $2\kappa_{10}^2 = (2\pi)^7 l_s^8 g_s^2$ where we have assumed the we have absorbed the vacuum expectation value ϕ_0 of the dilaton into the coupling constant and defined the string coupling $g_s = e^{\phi_0}$. By definition this means that the ten dimensional Planck length is related to the string length by $l_P^8 = l_s^8 g_s^2$.

Einstein frame The action (2.5.7) reproduces the right low energy limit of type IIA string amplitudes but we can see that it contains non minimal gravitational couplings with the dilaton and the dilaton kinetic term has the wrong sign. We can recover a canonically normalised action by the following conformal field redefinition $g^{(E)}_{\mu\nu} = e^{-\frac{\phi}{2}} g^{(s)}_{\mu\nu}$. Therefore the bosonic part of the action in Einstein frame is given by

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}e^{-\phi}|H_3|^2 - \frac{1}{2}e^{\frac{3}{2}\phi}|F_2|^2 - \frac{1}{2}e^{\frac{\phi}{2}}|\tilde{F}_4|^2\right) - \frac{1}{4\kappa_{10}^2}\int B_2 \wedge F_4 \wedge F_4\,. \tag{2.5.8}$$

This action can be found by dimensional reduction of eleven dimensional supergravity with the ansatz

$$ds_{11}^2 = e^{\frac{4}{3}\phi}(d\theta + C_\mu dx^\mu)^2 + e^{-\frac{\phi}{6}}ds_{10}^2\,. \tag{2.5.9}$$

Therefore the coupling constants of 11-dimensional supergravity and type IIA supergravity are related by $\kappa_{11}^2 = 2\pi R\,\kappa_{10}^2$ where R is the radius of the compactified circle. This explains why the field content of type IIA supergravity is non chiral because dimensional reduction always gives achiral theories.

2.5.4 Type IIB supergravity

The field content of type IIB supergravity is

$$\begin{gathered} g_{\mu\nu}\,,\quad \psi_\mu^+\,,\quad B_{\mu\nu}\,,\quad \lambda^-\,,\quad \phi\,,\\ \psi_\mu^{'+}\,,\quad C_{\mu\nu\rho\sigma}\,,\quad C_{\mu\nu}\,,\quad \lambda^{'-}\,,\quad C_0\,, \end{gathered} \tag{2.5.10}$$

where ψ_μ^+ and $\psi_\mu^{'+}$ are gravitini of the same chirality, λ^+ and $\lambda^{'+}$ are dilatini of the same chirality, $C_{\mu\nu\rho\sigma}$ is a self dual four form, $C_{\mu\nu}$ is a two form and C_0 is a scalar. This is exactly the massless field content of type IIB string theory, it forms a massless $\mathcal{N} = (2,0)$ graviton multiplet.

String frame Because of the self duality of the four form an action for type IIB supergravity is not strictly well defined as the kinetic term for the four form would identically vanish. This would be a problem for a path integral formulation of type IIB supergravity however if the purpose is only to find a classical action which reproduces the vanishing of the one loop beta functionals (2.5.3) as its equations of motion (if one also includes the (R-R) fields of type IIB string theory) then it is not a problem to simply supplement the equations of motion with the self duality of the four form as a constraint. Therefore the bosonic part of the action in string frame is given by

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(e^{-2\phi}\left(R + 4|\nabla\phi|^2 - \frac{1}{2}|H_3|^2\right) - \frac{1}{2}|F_1|^2 - \frac{1}{2}|\tilde{F}_3|^2 - \frac{1}{4}|\tilde{F}_5|^2\right) - \frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3\,, \tag{2.5.11}$$

where $H_3 = dB_2$, $F_1 = dC_0$, $F_3 = dC_2$, $F_5 = dC_4$, $\tilde{F}_3 = F_3 - C_0 H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. The self duality condition is imposed as a constraint separately

$$\tilde{F}_5 = \star \tilde{F}_5 \,. \tag{2.5.12}$$

Without this self duality constraint the number of bosonic and fermionic degrees of freedom in the action do not match meaning that it is not properly supersymmetric.

One can check that type IIB supergravity dimensionally reduced on a circle and type IIA supergravity dimensionally reduced on a circle give the same 9 dimensional theory. This is because there is only one maximally supersymmetric theory in 9 dimensions. In the string theory limit this comes from T-duality.

Einstein frame The bosonic part of the action in Einstein frame is given by

$$\begin{aligned} S = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} e^{-\phi} |H_3|^2 - \frac{1}{2} e^{2\phi} |F_1|^2 - \frac{1}{2} e^{\phi} |\tilde{F}_3|^2 - \frac{1}{4} |\tilde{F}_5|^2 \right) \\ & - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \,. \end{aligned} \tag{2.5.13}$$

If we define the axio-dilaton as $\tau = C_0 + i e^{-\phi}$ then one can check that the transformation $\tau \to -1/\tau$ and $B_{\mu\nu} \leftrightarrow -C_{\mu\nu}$ leaves the action invariant. In the string theory limit this is called S-duality. If $C_0 = 0$ then the dilaton transforms as $\phi \to -\phi$ and therefore the string coupling transforms as $g_s \to 1/g_s$ which shows that S-duality is a non perturbative strong-weak duality.

2.5.5 Type I supergravity

The field content of type I supergravity is

$$g_{\mu\nu} \,, \quad \psi_\mu \,, \quad C_{\mu\nu} \,, \quad \lambda \,, \quad \phi \,. \tag{2.5.14}$$

This forms a massless $\mathcal{N} = 1$ graviton multiplet. However unlike the maximally supersymmetric case there is another $\mathcal{N} = 1$ multiplet that the theory can be consistently coupled to. Indeed it turns out that ten dimensions is the maximal dimension in which we can have super Yang Mills theory. Since it also has $\mathcal{N} = 1$ supersymmetry it can be consistently coupled to type I supergravity. For reasons of anomaly cancellations only certain gauge groups are allowed, we select $SO(32)$. The field content of $SO(32)$ super Yang-Mills theory is

$$A^a_\mu\,, \quad \lambda^a\,, \tag{2.5.15}$$

where A^a_μ gauge connection one form in the adjoint representation of $SO(32)$ and λ^a is a spinor also in the adjoint representation of $SO(32)$ called the gaugino. This forms a massless $\mathcal{N}=1$ gauge multiplet.

String frame The bosonic part of the action in string frame is given by

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(e^{-2\phi}\left(R+4|\nabla\phi|^2\right)-\frac{1}{2}|\tilde{F}_3|^2\right) \\ & -\frac{1}{2g_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{-\phi}\mathrm{tr}\,|F_2|^2\,, \end{aligned} \tag{2.5.16}$$

where $F_3 = dC_2$, $F_2 = dA_1 + A_1\wedge A_1$ and $\tilde{F}_3 = F_3 - \frac{l_s^2 g_s}{4}(\omega_3^{\mathrm{YM}} - \omega_3^{\mathrm{L}})$. ω_3^{YM} and ω_3^{L} are the non abelian Chern-Simons three forms

$$\begin{aligned} \omega_3^{\mathrm{YM}} &= \mathrm{tr}\left(A_1\wedge dA_1 + \frac{2}{3}A_1\wedge A_1\wedge A_1\right)\,, \\ \omega_3^{\mathrm{L}} &= \mathrm{tr}\left(\omega_1\wedge d\omega_1 + \frac{2}{3}\omega_1\wedge\omega_1\wedge\omega_1\right)\,, \end{aligned} \tag{2.5.17}$$

where ω_1 is the spin connection. This is the low energy limit of type I string theory provided the Yang-Mills coupling constant is given by $2g_{10}^2 = 4(2\pi)^7 l_s^6 g_s$.

Einstein frame The bosonic part of the action in Einstein frame is given by

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(R-\frac{1}{2}|\nabla\phi|^2-\frac{1}{2}e^{\phi}|\tilde{F}_3|^2\right) \\ & -\frac{1}{2g_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{\frac{\phi}{2}}\mathrm{tr}\,|F_2|^2\,. \end{aligned} \tag{2.5.18}$$

2.5.6 Heterotic supergravity

The field content of heterotic supergravity is

$$\begin{gathered} g_{\mu\nu}\,, \quad \psi_\mu\,, \quad B_{\mu\nu}\,, \quad \lambda\,, \quad \phi\,, \\ A^a_\mu\,, \quad \lambda^a\,, \end{gathered} \tag{2.5.19}$$

where A^a_μ gauge connection one form in the adjoint representation of $SO(32)$ or $E_8\times E_8$ depending on the version of the theory and λ^a is the gaugino also in the adjoint representation of $SO(32)$ or $E_8\times E_8$.

String frame The bosonic part of the action in string frame is given by

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\phi}\left(R + 4|\nabla\phi|^2 - \frac{1}{2}|\tilde{H}_3|^2\right) - \frac{1}{2g_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\phi}\mathrm{tr}\,|F_2|^2\,, \tag{2.5.20}$$

where $H_3 = dB_2$, $F_2 = dA_1 + A_1 \wedge A_1$ and $\tilde{H}_3 = H_3 - \frac{l_s^2}{4}(\omega_3^{\mathrm{YM}} - \omega_3^{\mathrm{L}})$. ω_3^{YM} and ω_3^{L} are the non abelian Chern-Simons three forms associated to the non abelian Yang-Mills connection and the spin connection respectively.

The $SO(32)$ theory and the $E_8 \times E_8$ theory give the same 9 dimensional theory when dimensionally reduced on a circle provided one gives a vacuum expectation value to the gauge field A_1 along the compactified dimension which breaks down the gauge group to $SO(16) \times SO(16)$.[1] This is another example of T-duality in the string theory limit. Also the $E_8 \times E_8$ theory arises as 11-dimensional supergravity dimensionally reduced on an interval $I^1 = S^1/\mathbb{Z}_2$.

Einstein frame The bosonic part of the action in Einstein frame is given by

$$S = \frac{1}{2\kappa_{10}^2}\int d^{10}x\,\sqrt{-g}\,\left(R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}e^{-\phi}|H_3|^2\right) - \frac{1}{2g_{10}^2}\int d^{10}x\,\sqrt{-g}\,e^{-\frac{\phi}{2}}\mathrm{tr}\,|F_2|^2\,. \tag{2.5.21}$$

One can see that if the gauge group is $SO(32)$ the transformation $\phi \to -\phi$ and $B_{\mu\nu} \leftrightarrow C_{\mu\nu}$ maps heterotic supergravity to type I supergravity. This is another example of S-duality in the string theory limit.

2.5.7 Anomalies

One can show that all five supergravity limits of string theory are anomaly free. This is quite a miraculous result which led historically to the first superstring revolution. Indeed the fields that can contribute to anomalies are chiral fermions in $D = 0 \mod 2$ dimensions, i.e. spinors and gravitini, as well as (anti) self dual forms in $d = 2 \mod 4$ dimensions. Gravitini and higher gauge forms can't couple to gauge interactions,[2]

[1] This can be though of as giving a non trivial vacuum expectation value to a Wilson loop.

[2] This is only true because we are not considering gauged supergravities

only to gravity, therefore they are only subject to pure gravitational anomalies which only occur for dimensions $D = 2 \mod 4$. In $D = 10$ dimensions all of these anomalies are possible and they are indeed present in all supergravity limits of string theory.

Type II supergravity If we assume no gauge symmetry in our theory then the cancellation of the gravitational anomaly polynomial constitutes a critically determined homogeneous linear system. In general such a system only has the trivial solution which corresponds to the field content of type IIA supergravity which is trivially anomaly free because it is achiral. Amazingly, however the system is degenerate and admits non-trivial solutions. The simplest and only solution with less that two gravitini (which is necessary in order to have less than $\mathcal{N} = 2$ supersymmetry which is the maximum allowed in ten dimensions) corresponds to the field content of type IIB supergravity. Therefore, not only is type IIB supergravity miraculously free of gravitational anomalies, it is also the simplest chiral theory in ten dimensions in which gravitational anomaly cancellation occurs. Hence the field contents of type IIA and type IIB supergravity are the only two consistent field contents in 10 dimensions without gauge symmetry.

Green-Schwarz mechanism If one allows for a gauge group one cannot have more than $\mathcal{N} = 1$ supersymmetry. Hence the theory will necessarily have the field content of type I supergravity coupled to super Yang-Mills theory. One can show that if the anomaly polynomial factorises as

$$I_{12} = \left(\operatorname{Tr} R_2^2 - \operatorname{Tr} F_2^2\right) \wedge X_8 \,, \tag{2.5.22}$$

for some for some gauge invariant and local Lorentz invariant closed form X_8, where $R_2 = d\omega_1 + \omega_1 \wedge \omega_1$ is the Riemann curvature two form and $F_2 = dA_1 + A_1 \wedge A_1$ is the Yang-Mills gauge field strength. Then the ensuing anomaly can be cancelled by introducing the following non gauge invariant and non Lorentz invariant counter-term

$$S_{\text{counter-term}} = \int B_2 \wedge X_8 \,, \tag{2.5.23}$$

where $B_{\mu\nu}$ is the two form in the massless $\mathcal{N} = 1$ graviton multiplet of type I supergravity (it is the Kalb-Ramond two form in heterotic supergravity and the (R-R) two form $C_{\mu\nu}$ in type I supergravity). One can check that such a factorisation of the anomaly polynomial is highly non trivial and happens for only four groups : $SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$ and $U(1)^{496}$ [29]. The fact that such groups even exist is nothing short of a miracle. This anomaly cancellation mechanism is called the Green-Schwarz

mechanism. It is only possible because the theory contains a 2-form which transforms under both gauge and local Lorentz symmetry and from which we can build a non invariant counterterm.

Therefore anomaly cancellation severely constrains the possible supergravity theories in ten dimensions. What is more incredible is that these consistent supergravity theories turned out to be exactly the low energy limits of the various superstring theories. For type I superstring theory, the gauge group $SO(32)$ is singled out by the fact that open strings can only carry charges associated with orthogonal or symplectic groups while for the heterotic superstring we have seen that the gauge groups $SO(32)$ and $E_8 \times E_8$ are singled out by the fact that their weight lattices are the only 16 dimensional even self dual lattices. On the other hand, the somewhat more trivial solutions to the anomaly cancellation conditions, $E_8 \times U(1)^{248}$ and $U(1)^{496}$ do not seem to correspond to any consistent superstring theory [41, 43]. Since only the massless spectrum can contribute to anomalies this means that the full string theories, including the massive spectrum, are also anomaly free.[3]

2.6 Dualities and M-theory

We therefore have five consistent superstring theories : type IIA, type IIB, type I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. These have as low energy field theory limit: type IIA, type IIB and type I supergravity along with $SO(32)$ and $E_8 \times E_8$ super Yang Mills.

However it was realised in the 90s that all superstring theories were related by dualities [30,67,68]. First of which is called T-duality which relates a theory compactified on a circle S^1 of radius R to another theory compactified on a circle S^1 of radius α'/R. One can show that type IIA and type IIB string theory are related to each other by T-duality. This is not surprising as type IIA and type IIB supergravity give the same unique 9 dimensional supergravity theory when dimensionally reduced on a circle S^1. Similarly one can show that heterotic $SO(32)$ and heterotic $E_8 \times E_8$ string theory are also related by T-duality. For a review of T-duality in string theory see [69].

The second type of duality is called S-duality which relates a theory with coupling constant g_s to another theory with coupling constant $1/g_s$. It is the analog of electric-magnetic duality (or strong weak coupling duality) in gauge theories. One

[3] Actually this only applies to the perturbative spectrum, in the non perturbative sector string theory can have D-branes whose coupling to supergravity can induce anomalies on their worldvolume, one also needs to check that these anomalies cancel.

can show that type I and heterotic $SO(32)$ string theories are related by S-duality and that type IIB string theory is related to itself by S-duality [67, 68].

Since we already know that type I string theory is related to type IIB string theory by orientifold projection this shows that all superstring theories are related to each other by some sort of duality. But this is not all, one can also show that type IIA and heterotic $E_8 \times E_8$ grow an eleventh dimension in the strong coupling limit : a circle S^1 of radius $R = g_s l_s$ for type IIA and an interval I^1 of length $L = g_s l_s$ for heterotic $E_8 \times E_8$. This leads us to see that there exists an 11 dimensional theory which is the strong coupling limit of either type IIA or heterotic $E_8 \times E_8$ string theory, this is M-theory [30]. Since type IIA supergravity results from the dimensional reduction of eleven dimensional supergravity on a circle S^1 it is natural to expect that the low energy field theory limit of M-theory is eleven dimensional supergravity.

<table>
<tr><th>Theory</th><th colspan="2">Low energy limit</th></tr>
<tr><td>M-theory</td><td colspan="2">Eleven dimensional supergravity</td></tr>
<tr><td>Type IIA string theory</td><td colspan="2">Type IIA supergravity</td></tr>
<tr><td>Type IIB string theory</td><td colspan="2">Type IIB supergravity</td></tr>
<tr><td>Type I string theory</td><td rowspan="2">Type I supergravity</td><td rowspan="2">$SO(32)$ super Yang-Mills</td></tr>
<tr><td>Heterotic $SO(32)$ string theory</td></tr>
<tr><td>Heterotic $E_8 \times E_8$ string theory</td><td></td><td>$E_8 \times E_8$ super Yang-Mills</td></tr>
</table>

Table 2.5: Low energy field theory limit of M-theory and the five superstring theories.

The full formulation of M-theory is still not very well understood because it is fundamentally a non perturbative theory. However all five superstring theories can be thought of as originating from M-theory: first by dimensional reduction to either type IIA or heterotic $E_8 \times E_8$ string theory and then by the application of S-duality, T-duality or orientifold projection to any other type of string theory. This is summarized in figure 2.1.

The fact that all of these theories are related by dualities seems to indicate that they are all description of some single underlying theory. In fact it was conjectured that all five superstring theories and M-theory are just different solutions of some

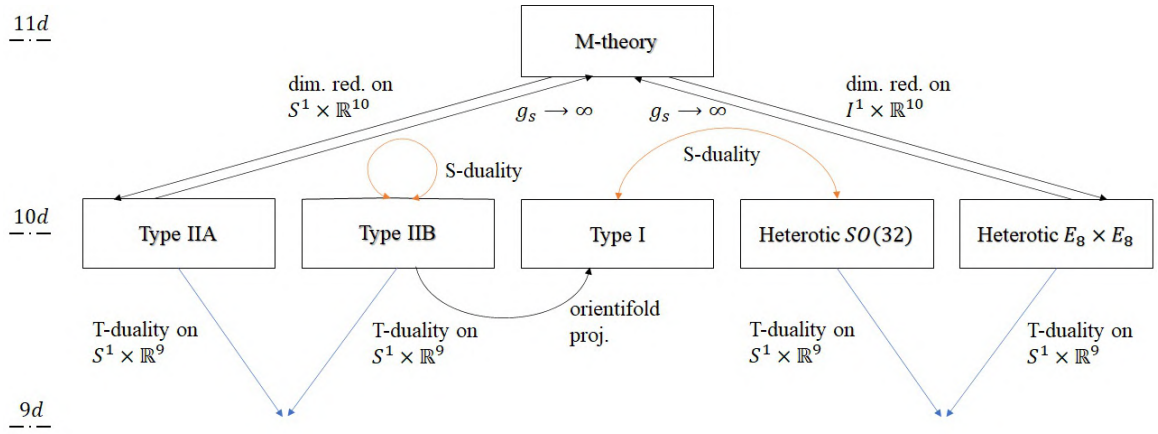


Figure 2.1: The various dualities connecting the superstring theories and M-theory in 11, 10 and 9 dimensions.

unique underlying theory $\mathcal{U}$ around different consistent quantum vacua.[4] Thus this would suggest a completely unique theory of nature, whose equation of motion admits many vacua. This is the realisation that sparked the second superstring revolution.

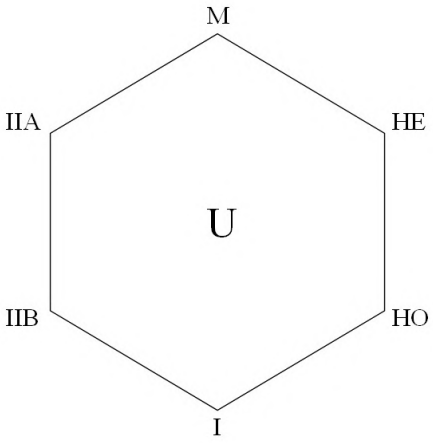


Figure 2.2: The space $\mathcal{U}$ of consistent quantum vacua. M-theory, type IIA, type IIB, type I, heterotic $SO(32)$ (HO) and heterotic $E_8 \times E_8$ (HE) string theories all arise as solutions of the same underlying theory $\mathcal{U}$.

[4]Most authors in the literature do not make the distinction between $\mathcal{U}$ and M-theory as very little is known about the full formulation of this unifying theory anyways. This nomenclature is due to [53].

Chapter 3

String amplitudes

Contrary to other attempts at quantum gravity, string theory is a perturbative approach. This may only give us a partially obstructed insight into the phenomenology of quantum gravity (although there are many non perturbative results which have come out of string theory, some of which we will see) but on the other hand this allows us to make actual calculations of interactions. Indeed, after more than 80 years of quantum field theory it seems like the best understood way to handle non trivial interactions is still through perturbation theory. In fact we will see that the perturbative structure of string theory can even be compared to the perturbative structure of quantum field theory, in particular for graviton interactions.

In this section we begin by reviewing the subject of string amplitudes and string interactions. We then specialise to the maximally supersymmetric case where we give a review of the moduli space of Riemann surfaces. We then discuss the factorisation properties of maximally supersymmetric amplitudes. Using the formalism of spinor helicities we show that the unitarity properties of superamplitudes in 6 dimensions reduce to unitarity properties of scalar amplitudes. Finally we review the field theory limit of string amplitudes through the lens of tropical geometry. We apply these tools to the one loop string amplitudes in 8 dimensions to address the logarithmic divergences in the corresponding supergravity amplitudes.

3.1 Vertex operators

We have seen that from the worldsheet perspective string theory simply corresponds to a free theory. Before gauge fixing it can be argued to contain gravity but two dimensional gravity is locally trivial. Furthermore the fields of the theory, the bosonic coordinates x^μ and the fermionic ψ^μ do not directly correspond to the states we want to study. One might then wonder how one goes from an almost trivial free theory

of scalars and spinors to an interacting theory of gravitons and other fields. This comes from the fact that string theory can be seen as a first quantized theory where spacetime corresponds to the target space and not the base space. In fact it is precisely because the geometry of the base space, the worldsheet, can be non trivial which is responsible for much of the non triviality of string theory.

Indeed, in the quantum theory, before gauge fixing, one will have to sum over all worldsheet geometries. We have seen that locally they are all conformally equivalent, however globally they may differ by their topology. This sum over worldsheet topologies can be interpreted as a sum over histories of the strings. One can see that strings merging and splitting will correspond to worldsheets with different numbers of holes, see figure 3.1. Even the number of initial and final strings will change the topology of the worldsheet. Furthermore the initial and final strings can have different states corresponding to their vibration modes. Therefore there must be quantum numbers associated with asymptotic legs of the worldsheet. Since by conformal rescaling these external legs can be reduced to points or punctures on the worldsheet, see figure 3.2, this means that these quantum numbers have to be realised by local operators in the quantum theory. For open strings these operators must act on the boundary of the worldsheet while for closed strings they act on the interior of the worldsheet. We call these vertex operators.

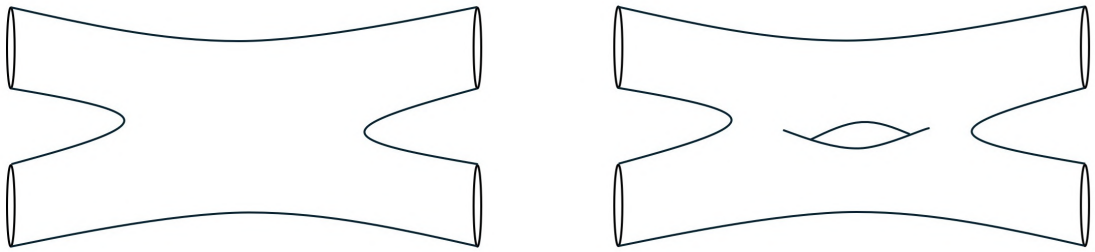


Figure 3.1: Worldsheets of closed strings merging and splitting. Different closed string interactions lead to different worldsheet topologies.

In order to find these vertex operators we first perform a Wick rotation of the worldsheet time coordinate $\tau \to -i\tau$ so that the metric becomes positive definite. This allows us to go from Lorentzian geometry to the much simpler Riemannian geometry. We then define holomorphic and anti-holomorphic coordinates $\bar{\zeta}, \zeta = \tau \pm i\sigma$ which turns the worldsheet into a Riemann surface. Finally we make the coordinate transformation $\bar{z} = e^{2\bar{\zeta}}$, $z = e^{2\zeta}$ for closed string and $\bar{z} = e^{\bar{\zeta}}$, $z = e^{\zeta}$ for open strings. This maps the closed string worldsheet, which is a cylinder, to the plane and it maps

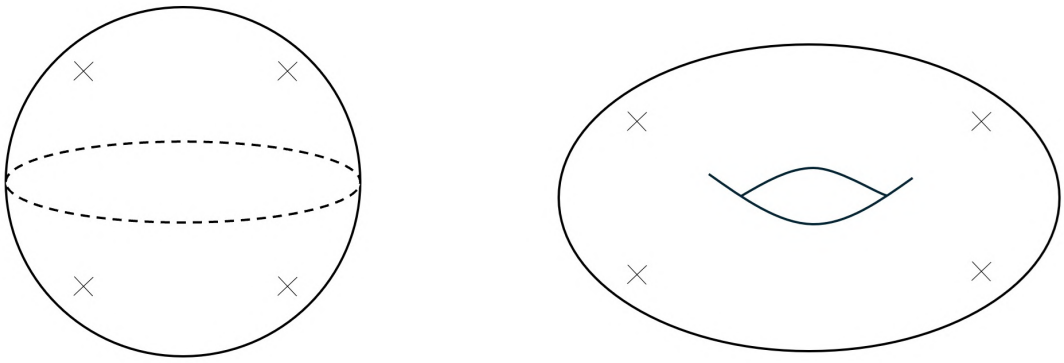


Figure 3.2: Conformally rescaled closed string worldsheets. The external legs have been rescaled to punctures represented by crosses on the Riemann surfaces.

the open string world sheet, which is a strip, to the upper half plane. In both cases the infinite past $\tau = -\infty$ is mapped to the origin $z = 0$ while the infinite future $\tau = +\infty$ is mapped to infinity $z = +\infty$.

Under this coordinate change the gauge fixed Polyakov action (2.3.7) becomes

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left(\partial x^\mu \bar{\partial} x_\mu - \psi^\mu \bar{\partial} \psi_\mu - \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) , \tag{3.1.1}$$

where ψ^μ and $\tilde{\psi}^\mu$ are holomorphic and antiholomorphic fields corresponding to ψ_+^μ and ψ_-^μ respectively in the Lorentzian description. Any local and unitary quantum field theory such as the one we have built in the previous chapter should have an appropriate state-operator correspondence, i.e. a way to associate to each state $|\psi\rangle$ a field $V_\psi(z)$ such that

$$|\psi\rangle = \lim_{z \to 0} V_\psi(z) \, |0;0\rangle \ , \tag{3.1.2}$$

where $|0;0\rangle$ is the ground state of zero momentum $p^\mu \, |0;0\rangle = 0$. This state operator correspondence is especially important in conformal field theory of which the world sheet theory of string theory is an example[1]. We can see that these local operators are natural candidates for vertex operators as they generate states from the vacuum, much like quantum fields do. If we require that they obey certain mathematical conditions related to the fact that they generate physical and on shell states we can

[1] Actually it is even an example of a superconformal field theory

find their explicit expressions. For the universal (NS-NS) massless spectrum of the type II string this is given by [57][2]

$$V(k^\mu, \epsilon^{\mu\nu}) = g_s e^{-\varphi-\tilde{\varphi}} \epsilon_{\mu\nu} \psi^\mu \tilde{\psi}^\nu e^{ik_\rho x^\rho} \,, \tag{3.1.3}$$

where k^μ is the momentum of the state, $\epsilon^{\mu\nu}$ is its polarisation and φ, $\tilde{\varphi}$ are holomorphic and antiholomorphic fields belonging to the bosonised representation of the superghosts. For a graviton we have $\epsilon^{\mu\nu} = \epsilon^{\nu\mu}$, $\epsilon^\mu{}_\mu = 0$ and $k_\mu \epsilon^{\mu\nu} = 0$ while for the Kalb-Ramond field we have $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ and $k_\mu \epsilon^{\mu\nu} = 0$ and finally for the dilaton we have $\epsilon^{\mu\nu} = \eta^{\mu\nu} - k^\mu \bar{k}^\nu - k^\nu \bar{k}^\mu$ with $\bar{k}^\mu$ an arbitrary vector satisfying $\bar{k}_\mu \bar{k}^\mu = 0$ and $k_\mu \bar{k}^\mu = 1$ so that $k_\mu \epsilon^{\mu\nu} = 0$ [54]. Since we are looking at the massless spectrum we also have $k_\mu k^\mu = 0$. Vertex operators are the quantum fields associated to the physical states in the worldsheet theory.

Note that vertex operators always contain a piece $\exp(ik \cdot x)$. This is because when we expand x^ρ into modes this in turn contains a factor $\exp(ik \cdot x_0)$ which, by the commutation relations (2.3.22), generates the ground state of momentum k^μ from the ground state of zero momentum

$$|k^\mu; 0\rangle = \exp(ik \cdot x_0) \, |0; 0\rangle \,. \tag{3.1.4}$$

3.2 String perturbation theory

Until now we have used an operator based formulation of the quantum theory. We have done that because the canonical quantization scheme was the simplest way to quantize the free theory and build the space of physical states as well as the spectrum of the theory. However we have now seen that our theory does in fact contain interactions, string interactions. Let us therefore move on to a path integral formulation which is better suited to handle interactions.

We start again with the non gauge fixed action (2.3.1). As we have seen, amplitudes in the quantum theory must include a sum over all possible worldsheets in order to account for string interactions. Furthermore we have seen that the quantum fields

[2]This is not unique and in general depends on the superghost picture. We have given here the vertex operator in the canonical $(-1,-1)$ superghost picture where the state operator correspondence (3.1.2) is most easily checked. For loop calculations the non canonical $(0,0)$ superghost picture may be more appropriate [57]

$$V(k^\mu, \epsilon^{\mu\nu}) = g_s \epsilon_{\mu\nu} (\partial x^\mu + ik_\rho \psi^\rho \psi^\mu)(\bar{\partial} x^\nu + ik_\rho \tilde{\psi}^\rho \tilde{\psi}^\nu) e^{ik_\rho x^\rho} \,.$$

that carry the boundary data representing the ingoing and outgoing states on the worldsheet are the vertex operators. They must therefore be inserted in the worldsheet path integral in order to specify the scattering process. Therefore, schematically, the string theory path integral will look like

$$\mathcal{A} = \sum_{\Sigma} \int \mathcal{D}h\, \mathcal{D}\chi\, \mathcal{D}x\, \mathcal{D}\psi\, V_1 \dots V_n\, e^{-S}\,, \tag{3.2.1}$$

where Σ are all the allowed worldsheets and $V_1 \dots V_n$ are vertex operators. Let us specify this sum.

We can extend our action by coupling it to background fields fields as was done in (2.5.2) for the (NS-NS) fields. Let us specifically look at the coupling to the dilaton

$$S_{\text{dil}} = \frac{1}{4\pi} \int d^2z \sqrt{-h}\, \phi R\,. \tag{3.2.2}$$

Let us suppose that the dilaton is constant, or at least that it has a vacuum expectation value ϕ_0 that can be separated from the rest of ϕ. Then the Gauss Bonnet theorem tells us that

$$S_{\text{dil}} = \phi_0\, \chi(\Sigma)\,, \tag{3.2.3}$$

where $\chi(\Sigma)$ is the Euler characteristic of the manifold Σ. It is a topological invariant which is given explicitly for two dimensional surfaces by $\chi(\Sigma) = 2 - 2n_{\text{h}} - n_{\text{b}} - n_{\text{c}}$ with n_{h} the number of handles, n_{b} the number of boundaries and n_{c} the number of cross caps of Σ. Therefore the path integral will include a factor

$$e^{-S_{\text{dil}}} = e^{-\phi_0\, \chi(\Sigma)} = g_s^{-\chi(\Sigma)}\,, \tag{3.2.4}$$

where we have defined the string coupling $g_s = e^{\phi_0}$. Hence the we can classify the allowed worldsheets Σ by their Euler characteristics and we can see that the path integral (3.2.1) will have the structure of a perturbative series in the string coupling where worldsheets of lowest Euler characteristic will contribute more.

$$\mathcal{A} = \sum_{\chi=2}^{-\infty} g_s^{-\chi} \sum_{\Sigma_\chi} \int \mathcal{D}h\, \mathcal{D}\chi\, \mathcal{D}x\, \mathcal{D}\psi\, V_1 \dots V_n\, e^{-S}\,, \tag{3.2.5}$$

where Σ_χ are worldsheets of Euler characteristic χ.

We saw that the action (2.3.1) has superdiffeomorphism and superWeyl invariance, it is a gauge theory. Therefore the path integral (3.2.5) formally diverges as we are summing over an uncountable infinity of equivalent configurations. We must

therefore gauge fix our theory. In the covariant quantization scheme this was done at the classical level at the cost of introducing constraints and then restricting the subspace of physical states to the states satisfying these constraints. In the path integral formulation this is done by introducing Faddeev-Popov ghost fields b_{ab}, c^a and superghost fields β_a and γ which take care of the Jacobian factor in the path integral. For further information one may see [54], we will not develop this further in this thesis. One may then locally fix the gauge to the superconformal gauge. However for topologically non trivial Riemann surfaces this can only be done locally and not globally. A given choice of local patches and superdiffeomorphism and superWeyl transformations which gauge away the zweibein and the Rarita Schwinger field corresponds to choosing a complex structure on the super-Riemann surface. Therefore superconformal equivalence classes of super-Riemann surfaces are in one to one correspondence with complex structures on these super-Riemann surfaces. We call the space of complex structures on a super-Riemann surface its super-moduli space[3]. Topologically inequivalent Riemann surfaces are certainly not in the same superconformal class, therefore this classification can be done at each order in Euler characteristic. This is very good news because contrary to the set of all super-Riemann surfaces (Σ, h_{ab}, χ_a) the super moduli space of super Riemann surfaces of a given topology is generally a finite dimensional integral

$$\mathcal{A} = \sum_{\chi=2}^{-\infty} g_s^{-\chi} \sum_{\Sigma_\chi} \int_{s\mathcal{M}(\Sigma_\chi)} d\mu \int \mathcal{D}b\, \mathcal{D}c\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}x\, \mathcal{D}\psi\, V_1 \dots V_n\, e^{-S}\,, \tag{3.2.6}$$

where $s\mathcal{M}(\Sigma_\chi)$ is the super moduli space of the worldsheet Σ_χ of Euler characteristic χ and $d\mu$ is a suitable measure on that space which is discussed in the next section. S is now the gauge fixed action including the Faddeev-Popov ghosts.

Let us now specify the allowed worldsheets. For closed strings the periodicity condition implies that the worldsheet is a Riemann surface without boundary, on the other hand for open strings the endpoints of the string create a boundary therefore the worldsheet will be a Riemann surfaces with boundaries. Oriented strings create worldsheets which are oriented Riemann surfaces, this implies that there can be no cross-caps. On the other hand unoriented strings create worldsheets which are unoriented Riemann surfaces. Finally we saw that the in and out states were given by operator insertions on the worldsheet. Therefore the number of in and out states is

[3]One should not confuse the (super-)moduli space of (super-)Riemann surfaces discussed here, which is supposed to be integrated over in the path integral, with the moduli space parametrised by axion and dilaton fields in a supersymmetric theory as was presented in chapter 1 and will later be expanded upon in chapter 4.

represented by the number of punctures on that Riemann surface. For closed strings the punctures will be in the interior of the Riemann surface while for open strings they will be on the boundary. Note that in general the super moduli space will depend on the punctures.

Since we have spinors on the worldsheet one might think that we can also specify a spin structure on the worldsheet. This is equivalent to specifying the boundary conditions of the spinors. We saw that in the open string case, when the worldsheet was a strip, there were only two options for the spin structure while in the closed string case, when the worldsheet was a cylinder, there were four. But in the case where the worldsheet admits more boundaries or more cycles there can be more choices of spin structures which are always either odd or even. However we are not free to choose a spin structure for our worldsheet, indeed in the path integral formulation the GSO projection is implemented by requiring we sum over the different spin structure with a weighted prefactor. Actually the GSO projection ensures that the integrand is invariant under so called large diffeomorphisms, i.e. diffeomorphisms which are not connected to the identity. Indeed large diffeomorphisms do not preserve the spin structure on the worldsheet, therefore the GSO projection is a summation on the different spin structures which is invariant under large diffeomorphisms. Hence the GSO projection is actually necessary for the theory to be invariant under the full group of diffeomorphisms.

$$\mathcal{A}_{i\to i'} = \sum_{\chi=2}^{-\infty} g_s^{-\chi} \sum_{\Sigma_\chi^{(n)}} \int_{s\mathcal{M}(\Sigma_\chi^{(n)})} d\mu \sum_{\nu} w(\nu) \int \mathcal{D}b\,\mathcal{D}c\,\mathcal{D}\beta\,\mathcal{D}\gamma\,\mathcal{D}x\,\mathcal{D}\psi\, V_1 \dots V_n\, e^{-S}\,, \tag{3.2.7}$$

where ν are spin structures on the worldsheet $\Sigma_\chi^{(n)}$, $w(\nu)$ is a weight factor and the superscript on $\Sigma_\chi^{(n)}$ indicates that the worldsheet has been punctured n times with $n = i + i'$ where i and i' are the numbers of in and out states respectively.

A nice feature of these amplitudes is that at fixed Euler characteristic, i.e. at each order in string coupling, the string amplitudes have no UV divergences. It is in this sense that string theory is a UV finite theory of quantum gravity. This is a key point which will be further elaborated in sections 3.5.2 and 3.5.3. In particular we will see an explicit example of how UV divergences cancel each other out in order to give a finite string amplitude in 8 dimensions.

Let us now specialise to the case of the type II superstring which consists of closed oriented strings and no open strings. Therefore the allowed worldsheets are oriented Riemann surfaces without boundaries such as depicted in figure 3.2. Hence

the Euler characteristic of such surfaces is only determined by the number of handles $\chi(\Sigma) = 2-2n_{\rm h}$, this is also called the genus g. This is also good news because it turns out that topologically inequivalent oriented Riemann surfaces without boundaries are uniquely determined by their genus. This means that there is only one string worldsheet per order of string coupling. Hence the type II string amplitude can be written as

$$\mathcal{A}_{i\to i'}^{\text{type II}} = \sum_{g=0}^{\infty} g_s^{2g-2} \int_{s\mathcal{M}_{g,n}} d\mu \sum_{\nu} w(\nu) \int \mathcal{D}b\, \mathcal{D}c\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}x\, \mathcal{D}\psi\, V_1 \dots V_n\, e^{-S}\,, \quad (3.2.8)$$

where the difference between type IIA and type IIB string amplitudes coming from the GSO projection is encoded in the different weight factors $w(\nu)$ for the spin structures. We can rewrite this more concisely as

$$\mathcal{A}_{i\to i'}^{\text{type II}} = \sum_{g=0}^{\infty} g_s^{2g-2} \int_{s\mathcal{M}_{g,n}} d\mu \sum_{\nu} w(\nu) \left\langle V_1 \dots V_n \right\rangle , \quad (3.2.9)$$

where the angle brackets denote a functional integration over the worldsheet fields.

Therefore we have seen that type II string amplitudes are given by a perturbative expansion in worldsheet genus. This can be seen in analogy with the perturbative expansion in loop number of quantum field theory. We will see later in this chapter that this is more than an analogy and that there is a deep connection between the genus expansion of string theory and the loop expansion of quantum field theory. For now let us notice that a profound consequence of the smoothness of string worldsheets is that the structure of the interactions are completely determined by the free theory. Indeed the interactions are the result of worldsheet topology rather than worldline junctions. There are no arbitrary interactions to be chosen.

Finally let us massage a little more the term $\langle V_1 \dots V_n \rangle$. The integrand $V_1 \dots V_n\, e^{-S}$ of the functional integral will contain a factor

$$\prod_{i=1}^{n} \exp(ik_i \cdot x(z_i)) \exp(-S) = \exp\left(\sum_{i=1}^{n} ik_i \cdot x(z_i) - S \right) . \quad (3.2.10)$$

Hence the argument of the exponential will contain a term

$$\frac{1}{2\pi\alpha'}\int d^2z\left(-\partial x^\mu\bar{\partial}x_\mu+2\pi i\alpha'\sum_{i=1}^{n}k_\rho^i x^\rho\delta^2(z-z_i)\right)$$
$$=\frac{1}{2\pi\alpha'}\int d^2z\left(x^\mu\Delta x_\mu+2\pi i\alpha'\sum_{i=1}^{n}k_\rho^i x^\rho\delta^2(z-z_i)\right), \tag{3.2.11}$$

where we have integrated by parts the first term and introduced the Laplacian $\Delta=\partial\bar{\partial}$ on the Riemann surface. We can perform the usual trick of completing the square by performing the change of variable $\tilde{x}^\mu(z)=x^\mu(z)+\frac{i\alpha'}{2}\sum_{i=1}^{n}G(z,z_i)k_i^\mu$ where $G(z,z')$ is the Green function for the Riemann surface Laplacian

$$\Delta G(z,z')=2\pi\delta^2(z-z') \tag{3.2.12}$$

(the Laplacian acts on the variable z). It can be seen as the inverse of the Laplacian. We therefore get

$$\frac{1}{2\pi\alpha'}\int d^2z\left(\tilde{x}^\mu\Delta\tilde{x}_\mu+\frac{\pi\alpha'^2}{2}\sum_{i=1}^{n}k_\rho^i G(z,z_i)k_i^\rho\delta^2(z-z_i)\right.$$
$$\left.-\frac{\pi\alpha'^2}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}k_\rho^i G(z,z_i)k_j^\rho\delta^2(z-z_j)\right). \tag{3.2.13}$$

Furthermore this does not change the functional integration measure $\mathcal{D}x=\mathcal{D}\tilde{x}$ so we can come back to usual Polyakov action by a simple relabeling. We can see that the terms with $i=j$ are pathological, they can be seen as tadpole divergences which can be disregarded through renormalisation. In the operator formalism one can explain their disappearance by normal ordering. Note that for massless particles they do not contribute anyways because $k_i\cdot k_i=0$.

Actually, since the Laplacian annihilates the zero modes x_0^μ this makes its inverse, the Green function, ill defined. But because the Laplacian doesn't see the zero modes this can be taken care of by treating the zero modes separately. Indeed from the expression of the vertex operators and the string action we can see that the only place the fields x^μ appear without a derivative is in the factors $\exp(ik_i\cdot x)$. Therefore we can perform the integration over the zero modes in the path integral to get

$$\int d^{10}x_0\prod_{i=1}^{n}e^{ik_i\cdot x_0}=(2\pi)^{10}\delta^{10}\left(\sum_{i=1}^{n}k_i\right). \tag{3.2.14}$$

This is the momentum conservation factor. Throughout this thesis we shall omit these terms and assume that the momenta are indeed conserved. Hence we have shown that the amplitude will always contain a term

$$\exp\left(-\frac{\alpha'}{2}\sum_{1\leq i<j\leq n} k_i\cdot k_j\, G(z_i,z_j)\right). \tag{3.2.15}$$

This is called a Koba-Nielsen factor [70] and it is part of the universal structure of string amplitudes. Hence we see that we can decompose

$$\sum_{\nu} w(\nu)\,\langle V_1\dots V_n\rangle = g_s^n W_{g,n}\exp\left(-\frac{\alpha'}{2}\sum_{1\leq i<j\leq n} k_i\cdot k_j\, G(z_i,z_j)\right), \tag{3.2.16}$$

where $W_{g,n}$ contains all the information about the scattering process such as the particle types and their polarization vectors. We have also explicitly factorized the factors of g_s appearing in the vertex operator (3.1.3). Therefore the final form of the type II string amplitudes is

$$\mathcal{A}^{\text{type II}}_{i\to i'} = \sum_{g=0}^{\infty} g_s^{2g-2+n}\int_{s\mathcal{M}_{g,n}} d\mu\, W_{g,n}\exp\left(-\frac{\alpha'}{2}\sum_{1\leq i<j\leq n} k_i\cdot k_j\, G(z_i,z_j)\right). \tag{3.2.17}$$

3.3 Moduli space of Riemann surfaces

In order to further specify the path integral (3.2.17) we will have to talk about the super moduli space of super Riemann surfaces. We have seen that for type II strings these only depend on the genus and the number of punctures, we will therefore focus on a type II genus-g, n-point amplitude

$$\mathcal{A}_{g,n} = g_s^{2g-2+n}\int_{s\mathcal{M}_{g,n}} d\mu\, W_{g,n}\exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right). \tag{3.3.1}$$

Let us now specify the super moduli spaces $s\mathcal{M}_{g,n}$.

3.3.1 Genus 0

In analogy with quantum field theory we call the case $g=0$ the tree level. It was shown that at genus 0 there exists a so called global holomorphic section of the super moduli space $s\mathcal{M}_{0,n}$. This allows us to integrate out the fermionic moduli and

project the super moduli space $s\mathcal{M}_{0,n}$ onto its bosonic base $\mathcal{M}_{0,n}$ the moduli space of Riemann surfaces. Therefore we can consider that $W_{0,n}$ contains the integrated fermionic moduli and the integral (3.3.1) is performed on $\mathcal{M}_{0,n}$.

The only oriented Riemann surface without boundary of genus 0 is the two sphere S^2. Therefore the moduli space $\mathcal{M}_{0,n}$ is the space of conformal classes of the n-punctured two sphere. The sphere can be conformally mapped to the complex plane plus a point at infinity (the so called Riemann sphere). The globally defined conformal isometries of this space are given by the conformal group $PSL(2,\mathbb{C})$ which has complex dimension 3. These can be used to fix the positions of three of the punctures while the position of the remaining $n-3$ punctures define inequivalent conformal classes of the n-punctured two sphere. Therefore the moduli space $\mathcal{M}_{0,n}$ is the space of positions of $n-3$ of the punctures, it has complex dimension $n-3$. The measure on that space is

$$d\mu = \prod_{i=1}^{n} d^2 z_i \, \delta^2(z_A - z_A^0)\delta^2(z_B - z_B^0)\delta^2(z_C - z_C^0)|(z_A - z_B)(z_B - z_C)(z_C - z_A)|^2 \,, \quad (3.3.2)$$

where the last term in a conformally covariant function of weight 2 that ensures that the measure is invariant under conformal mappings of the points z_A^0, z_B^0 and z_B^0 for $1 \leq A, B, C \leq n$. (it can be seen as a mini Faddeev-Popov determinant).

On the Riemann sphere the propagator is given by

$$G(z, z') = -\ln|z - z'|^2 \,. \quad (3.3.3)$$

Therefore the full tree level n-point type II string amplitude is given by

$$\mathcal{A}_{0,n} = g_s^{n-2} \prod_{i=4}^{n} \int_{\Sigma} d^2 z_i \, \tilde{W}_{g,n} \prod_{j=4}^{n} |z_j|^{\alpha' k_1 \cdot k_j} |1 - z_j|^{\alpha' k_2 \cdot k_j} \prod_{4 \leq k < l \leq n} |z_k - z_l|^{\alpha' k_k \cdot k_l} \,, \quad (3.3.4)$$

where for convenience we have fixed $z_1^0 = 0$, $z_2^0 = 1$ and $z_3^0 = \infty$ and we have also defined $\tilde{W}_{g,n} = \lim_{z_3 \to \infty} W_{g,n} |z_3|^{\alpha' M_3^2 + 4}$ where M_3^2 is the mass of the state inserted at z_3, one can show that this limit is finite.

As an example we give the tree level type II graviton four point amplitude [28, 71], it was shown that the four-graviton amplitudes in type IIA and type IIB theories are equal up to genus 2 [72][4] therefore we need not distinguish between them

[4] This is expected not to be true beyond genus 2. For five external gravitons this is not true at genus 2 and for more external gravitons this is not true at genus 1 or 2 [72].

$$\mathcal{A}_{0,4}^{\text{graviton}} = i\frac{(2\pi)^7\alpha'^7 g_s^2}{2^{11}}\mathcal{R}^4(k_i,\epsilon_i)\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}\,, \tag{3.3.5}$$

where $\mathcal{R}^4(k_i,\epsilon_i)$ is a particular tensorial combination of four powers of the linearized Riemann tensor $R(k_i,\epsilon_i)_{\mu\nu}{}^{\rho\sigma} = -4k_{i[\mu}k_i^{[\sigma}\epsilon_{i\nu]}^{\rho]}$ written in term the contraction tensor t_8 as $\mathcal{R}^4 = t_8 t_8 R^4$ where k_i^μ and $\epsilon_i^{\mu\nu}$ are the momentum and polarisation of the ith graviton. The t_8 contraction tensor is defined by its action on four antisymmetric tensors $F_i^{\mu\nu}$ as

$$\begin{aligned} t_8F^4 =& 4F_1^\mu{}_\nu F_2^\nu{}_\rho F_3^\rho{}_\sigma F_4^\sigma{}_\mu + 4F_3^\mu{}_\nu F_2^\nu{}_\rho F_1^\rho{}_\sigma F_4^\sigma{}_\mu + 4F_2^\mu{}_\nu F_3^\nu{}_\rho F_1^\rho{}_\sigma F_4^\sigma{}_\mu \\ &+ 4F_1^\mu{}_\nu F_3^\nu{}_\rho F_2^\rho{}_\sigma F_4^\sigma{}_\mu + 4F_3^\mu{}_\nu F_1^\nu{}_\rho F_2^\rho{}_\sigma F_4^\sigma{}_\mu + 4F_2^\mu{}_\nu F_1^\nu{}_\rho F_3^\rho{}_\sigma F_4^\sigma{}_\mu \\ &- 2(F_1^\mu{}_\nu F_2^\nu{}_\mu)(F_3^\rho{}_\sigma F_4^\sigma{}_\rho) - 2(F_2^\mu{}_\nu F_3^\nu{}_\mu)(F_1^\rho{}_\sigma F_4^\sigma{}_\rho) - 2(F_3^\mu{}_\nu F_1^\nu{}_\mu)(F_2^\rho{}_\sigma F_4^\sigma{}_\rho)\,. \end{aligned} \tag{3.3.6}$$

3.3.2 Genus 1

Again in analogy with quantum field theory we call the case $g = 1$ the one loop level (and the genus g case the g-loop level). At genus 1 the super moduli space $s\mathcal{M}_{1,n}$ is also holomorphically projected which allows us to integrate out the fermionic moduli and focus on the moduli space $\mathcal{M}_{1,n}$.

The only oriented Riemann surface without boundary of genus 1 is the two torus T^2. Therefore the moduli space $\mathcal{M}_{1,n}$ is the space of conformal classes of the n-punctured two torus. Using Weyl transformations and diffeomorphisms connected to the identity the torus can be conformally mapped to the complex plane identified up to two periods $z \sim z+1$ and $z \sim z+\tau$ for τ in the upper complex half plane $\mathcal{H}_1 = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ which is also called the Poincaré plane. τ is called the modulus of the torus. Note that the upper complex half plane is isomorphic to the symmetric space $SO(2)\backslash SL(2,\mathbb{R})$. However we still have large diffeomorphisms which are diffeomorphisms modulo diffeomorphisms connected to the identity. This is a discrete group which acts on τ by

$$\tau \to \frac{a\tau+b}{c\tau+d}\,, \tag{3.3.7}$$

with $a,b,c,d \in \mathbb{Z}$ such that $ad-bc=1$. This is the so called modular group $PSL(2,\mathbb{Z})$ which is generated by the transformations $\tau \to \tau+1$ and $\tau \to -1/\tau$. Therefore we actually have that $\tau \in SO(2)\backslash SL(2,\mathbb{R})/PSL(2,\mathbb{Z})$. One natural parametrisation for this space is

$$\mathcal{F}_1 = \{\tau \in \mathbb{C} \mid |\mathrm{Re}\,\tau| \leq 1/2,\ \mathrm{Im}\,\tau > 0,\ |\tau| \geq 1\}\,. \tag{3.3.8}$$

The globally defined conformal isometries of the complex plane identified up to two periods are the translations $\mathbb{C}$ which have complex dimension 1. These can be used to fix the positions of one of the puncture while the position of the remaining $n-1$ punctures define inequivalent conformal classes of the n-punctured two torus. Therefore the moduli space $\mathcal{M}_{1,n}$ is the space of positions of $n-1$ of the punctures times $SO(2)\backslash SL(2,\mathbb{R})/PSL(2,\mathbb{Z})$, it has complex dimension n. The measure on that space is

$$d\mu = \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \prod_{i=1}^{n} \frac{d^2 z_i}{\mathrm{Im}\,\tau}\ \mathrm{Im}\,\tau\, \delta^2(z_A - z_A^0)\,, \tag{3.3.9}$$

where the denominator in the first term ensures that the measure is invariant under modular transformations of the representative $\mathcal{F}_1$ of $SO(2)\backslash SL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ and the denominator in the second term is just the dimensionless volume of the torus $\mathrm{Im}\,\tau$ so that the integral is normalised to 1. The measure is also invariant under translations of z_A^0 for $1 \leq A \leq n$.

On the torus the propagator is given by

$$G(z,z') = -\ln\left|\frac{\vartheta_1(\tau, z-z')}{\vartheta_1'(\tau)}\right|^2 + \frac{2\pi}{\mathrm{Im}\,\tau}\left(\mathrm{Im}(z-z')\right)^2\,, \tag{3.3.10}$$

where ϑ_1 is the first modular theta series in Jacobi's notation, i.e. $\vartheta_1 = \vartheta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ with

$$\vartheta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau, z) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n+\frac{a}{2})^2 + 2\pi i(n+\frac{a}{2})(z+\frac{b}{2})}\,, \tag{3.3.11}$$

where $a, b \in \{0,1\}$ and $\vartheta_1(\tau) = \vartheta_1(\tau, 0)$. Therefore the full one loop n-point type II string amplitude is given by

$$\mathcal{A}_{1,n} = g_s^n \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \prod_{i=1}^{n} \int_\Sigma \frac{d^2 z_i}{\mathrm{Im}\,\tau}\ \mathrm{Im}\,\tau\, \delta^2(z_1) W_{g,n} \exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i \cdot k_j\, G(z_i, z_j)\right), \tag{3.3.12}$$

where for convenience we have fixed $z_1^0 = 0$.

As an example we give the one loop type II graviton four point amplitude [73]

$$\mathcal{A}_{1,4}^{\text{graviton}} = -i\frac{(2\pi)^8\alpha'^7 g_s^4}{2^{11}}\mathcal{R}^4(k_i,\epsilon_i)\int_{\mathcal{F}_1}\frac{d^2\tau}{(\text{Im}\,\tau)^2}\prod_{i=1}^{4}\int_\Sigma\frac{d^2z_i}{\text{Im}\,\tau}\ \text{Im}\,\tau\,\delta^2(z_1)$$
$$\exp\left(-\frac{\alpha'}{2}\sum_{i<j}k_i\cdot k_j\,G(z_i,z_j)\right). \tag{3.3.13}$$

3.3.3 Higher genus

For higher genus the super moduli space $s\mathcal{M}_{g,n}$ is much harder so specify. Indeed it was shown in [74] that for genus $g \geq 5$ the super moduli space is not holomorphically projected. Therefore in general we cannot integrate out the fermionic moduli. At genus g and n punctures the super moduli space has complex bosonic dimension $3g-3-n$ and complex fermionic dimension $2g-2-n$. In the absence of punctures the super moduli space is holomorphically projected for genus 2 the while for genus 3 and 4 the question remains open. Here we will assume that the super moduli space is holomorphically projected and focus on the moduli space $\mathcal{M}_{g,n}$.

It is also difficult to describe explicitly the moduli space of higher genus Riemann surfaces. For genus $g \geq 2$ there are no globally defined conformal isometries for oriented Riemann surfaces without boundaries. Therefore the position of no puncture can be fixed and they all define inequivalent conformal classes of the n-punctured Riemann surface.

The first homology group of a genus g oriented Riemann surface without boundaries has $2g$ generators. It is convenient to introduce a canonical basis of g a-cycles $\{A_i\}$ and g b-cycles $\{B^i\}$ such that $A_i \cap A_j = B^i \cap B^j = \emptyset$ and $A_i \cap B^j = \delta_i^j$ as depicted in figure 3.3.

According to de Rham's theorem the first cohomology group also has $2g$ generators. We can use the complex structure of the Riemann surface to divide these into g holomorphic and g anti-holomorphic one forms. One can choose a basis ω^i of holomorphic one forms such that

$$\oint_{A_i}\omega^j = \delta_i^j\,. \tag{3.3.14}$$

The integral along the b-cycles gives a complex matrix

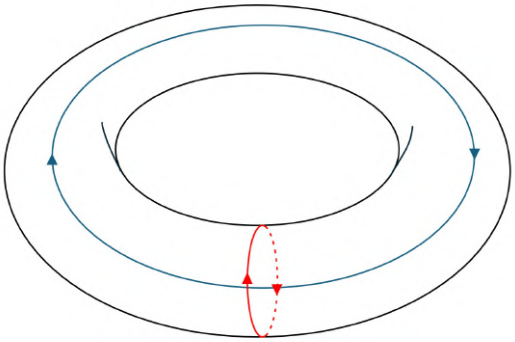


Figure 3.3: Canonical homology basis on a genus 1 surface. The a-cycle is depicted in red while the b-cycle is depicted in blue.

$$\oint_{B^i} \omega^j = \Omega^{ij} \,, \tag{3.3.15}$$

called the period matrix. Ω is symmetric and its imaginary part is positive definite, we say that Ω is in the Siegel upper half plane $\mathcal{H}_g = \{\Omega \in \mathrm{Mat}(g, \mathbb{C}) \mid \Omega^T = \Omega,\, \mathrm{Im}\,\Omega > 0\}$. For the genus 1 case we have $\omega = dz$ and $\Omega = \tau$. Note that the Siegel upper half plane is isomorphic to the symmetric space $U(g)\backslash Sp(2g, \mathbb{R})$. The period matrices are equivalent up to a transformation of the form

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1} \,, \tag{3.3.16}$$

with $A, B, C, D \in \mathrm{Mat}(g, \mathbb{Z})$ such that $AB^T = BA^T$, $CD^T = DC^T$ and $AD^T - BC^T = I$ with I the identity matrix. These conditions can be rewritten as

$$\begin{pmatrix} D & B \\ C & A \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} D & B \\ C & A \end{pmatrix}^T = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} . \tag{3.3.17}$$

This defines the action of the so called symplectic modular group $PSp(2g, \mathbb{Z})$. Therefore we actually have that $\Omega \in U(g)\backslash Sp(2g, \mathbb{R})/PSp(2g, \mathbb{Z})$. The space of positions of the n punctures times $U(g)\backslash Sp(2g, \mathbb{R})/PSp(2g, \mathbb{Z})$ has complex dimension $\frac{1}{2}g(g+1)+n$ whereas the moduli space has complex dimension $3g-3+n$. Therefore in general there are too many dimensions for this to be the moduli space, it will be a subspace. Finding this subspace is in general a very difficult problem called the Riemann-Schottky problem. However we see that for genus 2 and 3 the dimensions match. Indeed this is the moduli space up to excision of some critical locus. Let us focus on the genus 2 case. If we parametrise Ω as

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} , \tag{3.3.18}$$

with $\rho, v, \sigma \in \mathbb{C}$ then one natural parametrisation for $U(2)\backslash Sp(4,\mathbb{R})/PSp(4,\mathbb{Z})$ is given by [75]

$$\mathcal{F}_2 = \{\rho, v, \sigma \in \mathbb{C} \mid |\mathrm{Re}\,\rho|, |\mathrm{Re}\,v|, |\mathrm{Re}\,\sigma| \leq 1/2,\ 0 < \mathrm{Im}\,v \leq \mathrm{Im}\,\rho \leq \mathrm{Im}\,\sigma, \\ |\det(C\Omega + D)|^2 \geq 1\} , \tag{3.3.19}$$

for all $C, D \in \mathrm{Mat}(2,\mathbb{Z})$ such that

$$\begin{pmatrix} D & B \\ C & A \end{pmatrix} \in Sp(4,\mathbb{Z}) . \tag{3.3.20}$$

This last condition only needs to be checked for a finite number of matrices C, D. $v = 0$ is actually a regular point in $U(2)\backslash Sp(4,\mathbb{R})/PSp(4,\mathbb{Z})$ however it corresponds to a singular genus two Riemann surface made of two regular genus one surfaces connected by an infinitely thin cylinder, this is called the separating degeneration locus. Therefore it must be removed from the moduli space, this is why we have not included it in $\mathcal{F}_2$. Note that in the limit $v \to 0$ ρ and σ can be interpreted as the moduli of each of the tori.

If we also focus on 4 punctures the string amplitude gives a measure on $\mathcal{M}_{2,4}$ [76]

$$d\mu = \frac{d^6\Omega}{(\det \mathrm{Im}\,\Omega)^5} \mathcal{Y}_S \wedge \overline{\mathcal{Y}}_S , \tag{3.3.21}$$

which is invariant under symplectic modular transformations of the representative $\mathcal{F}_2$ of $U(2)\backslash Sp(4,\mathbb{R})/PSp(4,\mathbb{Z})$. $\mathcal{Y}_S$ is the holomorphic 4-form on four copies of the Riemann surface defined as

$$\mathcal{Y}_S = \frac{1}{3}((t-u)\varepsilon_{ij}\varepsilon_{kl} + (s-t)\varepsilon_{ik}\varepsilon_{lj} + (u-s)\varepsilon_{il}\varepsilon_{jk})\omega^i(z_1)\omega^j(z_2)\omega^k(z_3)\omega^l(z_4) . \tag{3.3.22}$$

On an oriented Riemann surfaces without boundaries the propagator is given by

$$G(z,z') = -\ln|E(z,z')|^2 + 2\pi(\mathrm{Im}\,\Omega)^{-1}_{ij} \left(\mathrm{Im}\int_z^{z'} \omega^i\right)\left(\mathrm{Im}\int_z^{z'} \omega^j\right) , \tag{3.3.23}$$

where E is the prime form defined by

$$E(z,z') = \frac{\vartheta[\nu](\Omega, \int_z^{z'} \omega)}{\sqrt{\partial_i \vartheta[\nu](\Omega,0)\omega^i(z)\partial_j\vartheta[\nu](\Omega,0)\omega^j(z')}}\,, \tag{3.3.24}$$

where the symplectic modular theta series ϑ is defined as

$$\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\Omega, Z) = \sum_{n\in\mathbb{Z}^2} e^{i\pi\Omega^{ij}(n_i+\frac{a_i}{2})(n_j+\frac{a_j}{2})+2\pi i(n_i+\frac{a_i}{2})(Z^i+\frac{b^i}{2})}\,, \tag{3.3.25}$$

where $a, b \in (\mathbb{Z}/2\mathbb{Z})^2$ and we have defined $\partial_i = \frac{\partial}{\partial Z^i}$. $\nu = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ parametrises the spin structures on the worldsheet Σ. We call $\nu = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ an odd (resp. even) spin structure if it satisfies $a_i b^i = 1 \mod 2$ (resp. $a_i b^i = 0 \mod 2$). The prime form (3.3.24) is defined for any odd spin structure and one can even show that it is independent of the particular choice of odd spin structure. It is however multivalued, indeed it is invariant up to sign when the path of integration is changed by a cycle A_i but it picks up a multiplicative factor when changing the path of integration by a cycle B^i. However this is compensated by the second term in (3.3.23) so that the propagator is single valued and well defined. Therefore the full one loop 4-point type II string amplitude is given by

$$\mathcal{A}_{2,4} = g_s^{n+2} \int_{\mathcal{F}_2} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} \mathcal{Y}_S \wedge \overline{\mathcal{Y}}_S W_{g,n} \exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right). \tag{3.3.26}$$

As an example we give the two loop type II graviton four point amplitude [76, 77]

$$\mathcal{A}_{2,4}^{\text{graviton}} = -i\frac{(2\pi)^8 {\alpha'}^7 g_s^6}{2^{19}} \mathcal{R}^4(k_i,\epsilon_i) \int_{\mathcal{F}_2} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^5} \int_{\Sigma^4} \mathcal{Y}_S \wedge \overline{\mathcal{Y}}_S \exp\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right). \tag{3.3.27}$$

For more details on the two loop calculation see [76–83].

3.4 Factorisation of maximally supersymmetric amplitudes

Notice that the tree level, one loop and two loop four point amplitudes (3.3.5), (3.3.13) and (3.3.27) are all of the form

$$\mathcal{A}_{g,4}^{\text{graviton}} = -i\frac{(2\pi)^7\alpha'^4 g_s^2}{2^{11}}\mathcal{R}^4(k_i,\epsilon_i)f_g(s,t,u)\,, \tag{3.4.1}$$

for some function $f_g(s,t,u)$ which is different for each genus but is invariant under permutation of the Mandelstam variables. This is actually a common feature of any maximally supersymmetric theory in any dimension. In fact maximal supersymmetry implies that the four point amplitude for any field ϕ is of the form [84]

$$\mathcal{A}_4^{\phi} = t^{\phi}(k_i,\epsilon_i)f(s,t,u;\varphi)\,, \tag{3.4.2}$$

where $t^{\phi}(k_i,\epsilon_i)$ is a kinematic factor containing the field polarisations which only depends on the field type ϕ and $f(s,t,u;\varphi)$ is a function invariant under permutation of the Mandelstam variables and is a function of the moduli fields φ parametrising some symmetric space but is independent of the field type. For example, if we reabsorb the relevant constants into $f(s,t,u;\varphi)$ we have for the graviton $t^{\text{graviton}}(k_i,\epsilon_i) = \mathcal{R}^4(k_i,\epsilon_i)$ while for a scalar field $t^{\text{scalar}}(k_i) = s^4$. $f(s,t,u;\varphi)$ can then be expanded into a perturbative series

$$f(s,t,u;\varphi) = \sum_{n=0}^{+\infty} f^{n\text{-loop}}(s,t,u;\varphi) + \text{non perturbative terms}\,. \tag{3.4.3}$$

The fact that the kinematic prefactor t^{ϕ} only depends on the field type means that for any field ϕ the full amplitude is related to the tree amplitude by

$$\mathcal{A}_4^{\phi} = \left(\frac{f}{f^{\text{tree}}}\right)\mathcal{A}_4^{\phi,\,\text{tree}}\,. \tag{3.4.4}$$

These factorisation properties of maximally supersymmetric amplitudes can be used to reduce the unitarity properties of superamplitudes down to the unitarity properties of the amplitudes of the lowest lying field in the chiral superfield containing the supermultiplet. The case that we will be interested in is that of dimension 6 where the spinor helicity formalism is available and the lowest lying field in the chiral superfield is a scalar field thus considerably simplifying calculations.

3.4.1 Maximally R-symmetry violating amplitudes

Let us first present the spinor helicity formalism in 6 dimensions. For an extensive review of the spinor helicity formalism and on shell amplitudes see [85]. For more details on the spinor helicity formalism in 6 dimensions see [86, 87]. For a discussion of spinor helicity formalism in general even dimensions see [88].

Spinor helicity variables The Lorentz group in 6 dimensional Minkowski space is $SO(1,5)$ whose universal covering group is $SU^*(4)$ while the massless little group is $SO(4)$ whose universal covering group is $SU(2) \times SU(2)$. The vector representation V_μ of $SO(1,5)$ corresponds to the antisymmetric representation $V_{[ab]}$ of $SU^*(4)$ and the invariant scalar product $\eta^{\mu\nu}$ corresponds to the invariant tensor $\frac{1}{2}\varepsilon^{abcd}$. Therefore the null vector condition obeyed by the momentum p_μ of an on shell massless particle is written as

$$\frac{1}{2}\varepsilon^{abcd}\, p_{ab} p_{cd} = 0\,. \tag{3.4.5}$$

This is easily solved by introducing so-called spinor helicity variables λ_a^α and writing

$$p_{ab} = \varepsilon_{\alpha\beta}\lambda_a^\alpha \lambda_b^\beta\,, \tag{3.4.6}$$

where a, b are $SU^*(4)$ spinor indices and α, β are $SU(2)$ fundamental spinor indices which can be raised and lowered using the invariant tensor $\varepsilon_{\alpha\beta}$. The $SU(2)$ invariance of (3.4.6) simply corresponds to the action of the little group which by definition leaves the momentum invariant. For the momentum p_{ab} to be real the spinor helicity variables need to be pseudo-majorana. One can also introduce spinor helicity variables $\tilde{\lambda}_{\hat{\alpha}}^a$ such that

$$p^{ab} = \frac{1}{2}\varepsilon^{abcd}\, p_{cd} = \varepsilon^{\hat{\alpha}\hat{\beta}}\tilde{\lambda}_{\hat{\alpha}}^a \tilde{\lambda}_{\hat{\beta}}^b\,, \tag{3.4.7}$$

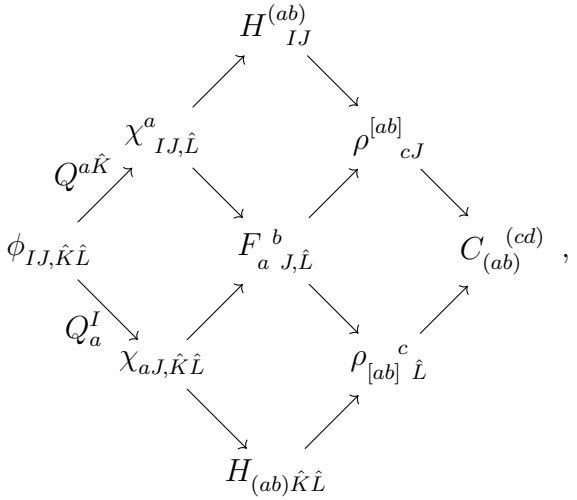
where $\hat{\alpha}, \hat{\beta}$ are $SU(2)$ antifundamental spinor indices which can be raised and lowered using the invariant tensor $\varepsilon^{\hat{\alpha}\hat{\beta}}$. The null vector condition then imposes that $\lambda_a^\alpha \tilde{\lambda}_{\hat{\beta}}^a = 0$. We can see that we have another $SU(2)$ group that leaves the momentum invariant, this indeed corresponds to the 6 dimensional massless little group. We can convince ourselves that the bispinor form of the momentum solves the on-shell constraint by counting the degrees of freedom: a null vector in 6 dimensions has 5 independent components while λ_a^α has $4 \times 2 = 8$ components and the $SU(2)$ invariance removes 3 of them. This matching between massless degrees of freedom and the spinor helicity components moded by the little group only exists in 3,4 and 6 dimensions.

6 dimensions $\mathcal{N} = (2,2)$ superspace According to table 2.4 the maximal amount of supersymmetry in 6 dimensions containing a graviton multiplet is $\mathcal{N} = (2,2)$. The R-symmetry group in this case is $Sp(2) \times Sp(2)$[5] which each carry an invariant symplectic form Ω^{IJ} and $\Omega^{\hat{I}\hat{J}}$ where $I, J, \hat{I}, \hat{J}$ are $Sp(2)$ indices. The supercharges Q_a^I, $Q^{a\hat{I}}$ are symplectic Majorana $SU^*(4)$ spinors

[5]Here $Sp(2)$ is the compact symplectic group, sometimes written as $USp(4)$, which is not to be confused with the symplectic group $Sp(4,\mathbb{R})$ introduced earlier

$$\{Q^I_a, Q^J_b\} = \Omega^{IJ} P_{ab}, \qquad \{Q^{a\hat{I}}, Q^{b\hat{J}}\} = \Omega^{\hat{I}\hat{J}} P^{ab}\,. \tag{3.4.8}$$

The graviton multiplet can be found by starting with the lowest lying field in the multiplet and successively applying the supercharges Q^I_a and $Q^{a\hat{I}}$ as follows



(3.4.9)

where ϕ represents scalar fields, χ^a represents spinor fields, $H_{(ab)}$ represents the self dual field strengths of two forms, $H^{(ab)}$ represents the anti self dual field strengths of two forms, $F_a^{\ b}$ represents the field strengths of one forms, $\rho_{[ab]}^{\ \ c}$ represents the field strength of a gravitino and $C_{(ab)}^{\ \ (cd)}$ represent the Weyl tensor which is the field strength of the graviton. All pairs of $Sp(2)$ indices IJ and $\hat{I}\hat{J}$ represent antisymmetric symplectic traceless indices. The usual superspace construction would then imply introducing Grassmann variables θ^I_a, $\theta^{a\hat{I}}$ and fitting the whole multiplet into a superfield. However the spinor helicity variables permit the construction of an on shell superspace by writing the fields of the multiplet in terms of spinor helicity variables and some polarization constants

$$\begin{aligned}
C_{(ab)}^{\ \ (cd)} &= \lambda^\alpha_a \lambda^\beta_b \tilde{\lambda}^c_{\hat{\gamma}} \tilde{\lambda}^d_{\hat{\delta}}\ h_{(\alpha\beta)}^{\ \ (\hat{\gamma}\hat{\delta})}\,,\\
\rho^{[ab]}_{\ \ cJ} &= \lambda^\gamma_c \tilde{\lambda}^a_{\hat{\alpha}} \tilde{\lambda}^b_{\hat{\beta}}\ \psi^{[\hat{\alpha}\hat{\beta}]}_{\ \ \gamma J}\,,\\
&\vdots\\
\chi_{aJ,\hat{K}\hat{L}} &= \lambda^\alpha_a\ \theta_{\alpha J,\hat{K}\hat{L}}\,,\\
\phi_{IJ,\hat{K}\hat{L}} &= \phi_{IJ,\hat{K}\hat{L}}\,.
\end{aligned} \tag{3.4.10}$$

The on shell superspace is then constructed by introducing Grassmann variables η^I_α, $\eta^{\hat{\alpha}\hat{I}}$ and fitting the polarisation constants $h_{(\alpha\beta)}^{\ \ (\hat{\gamma}\hat{\delta})}$, $\psi^{[\hat{\alpha}\hat{\beta}]}_{\ \ \gamma J}$, $\ldots$, $\theta_{\alpha J,\hat{K}\hat{L}}$, $\phi_{IJ,\hat{K}\hat{L}}$ into a

superfield. Due to the self-CPT conjugate nature of the physical spectrum one can contain the full multiplet using either chiral or anti-chiral superspace, i.e. only half of the full superspace is required to contain all physical degrees of freedom.

In order to split the on shell superspace we break each of the R-symmetry group into $U(1) \times SU(2) \simeq U(2) \subset Sp(2)$. Therefore the $Sp(2)$ indices I, $\hat{I}$ break into $\pm i$, $\hat{\pm}\hat{\imath}$ where $\pm$, $\hat{\pm}$ refers to the $U(1)$ chirality and i, $\hat{\imath}$ are $SU(2)$ indices. Hence the on shell superspace coordinates η^I_α, $\eta^{\hat{\alpha}\hat{I}}$ split into η^{+i}_α, $\eta^{\hat{\alpha}\hat{+}\hat{\imath}}$, $\bar{\eta}^{-i}_\alpha$, $\bar{\eta}^{\hat{\alpha}\hat{-}\hat{\imath}}$. Antisymmetric symplectic traceless indices IJ and $\hat{I}\hat{J}$ break into $(\pm\pm, ij)$ and $(\hat{\pm}\hat{\pm}, \hat{\imath}\hat{\jmath})$ where pairs of $SU(2)$ indices ij and $\hat{\imath}\hat{\jmath}$ are symmetric. Therefore polarisation constants of the fields of the graviton multiplet split into

$$\begin{aligned}\phi_{\pm\pm,\hat{\pm}\hat{\pm}}\,, \quad \phi_{\pm\pm,\hat{+}\hat{+}}\,, \quad \phi_{ij,\hat{\pm}\hat{\pm}}\,, \quad \phi_{\pm\pm,\hat{\imath}\hat{\jmath}}\,, \quad \phi_{ij,\hat{k}\hat{l}}\,, \quad \theta_{\alpha\pm i,\hat{\pm}\hat{\pm}}\,,\\ \theta_{\alpha\pm i,\hat{\imath}\hat{\jmath}}\,, \quad \dots\,, \quad h_{(\alpha\beta)}^{(\hat{\gamma}\hat{\delta})}\,.\end{aligned} \tag{3.4.11}$$

We can now use the $U(1)$ factors in the R-symmetry group to split the on shell superspace into chiral and antichiral parts. We choose the chiral part of the superspace to be the subspace generated only by the η^{+i}_α, $\eta^{\hat{\alpha}\hat{+}\hat{\imath}}$ coordinates which corresponds to the positive $U(1)$ chiralities. In 4 dimensions this splitting is done similarly except that the $U(1)$ used is that of the little group instead of the R-symmetry group. We can therefore fit the polarisation constants inside a chiral superfield

$$\begin{aligned}W_{++,\hat{+}\hat{+}}(\eta^{+i}_\alpha, \eta^{\hat{\alpha}\hat{+}\hat{\imath}}) =& \phi_{++,\hat{+}\hat{+}} + \eta^{\alpha+i}\theta_{\alpha+i,\hat{+}\hat{+}} + \dots + \eta^{\alpha+i}\eta^{\beta}_{+i}\eta^{\hat{+}\hat{\imath}}_{\hat{\gamma}}\eta_{\hat{\delta}\hat{+}\hat{\imath}}h_{(\alpha\beta)}^{(\hat{\gamma}\hat{\delta})}\\ &+ \dots + \eta^4_+\eta^4_{\hat{+}}\phi_{--,\hat{-}\hat{-}}\,.\end{aligned} \tag{3.4.12}$$

Therefore we can see that the lowest lying field in the chiral superfield is indeed the scalar. In 4 dimensions this would have been the graviton. The supercharges Q^I_a, $Q^{a\hat{I}}$ also split into $Q^{\pm i}_a$, $Q^{a\hat{\pm}\hat{\imath}}$ whereas the susy algebra relation (3.4.8) splits into

$$\begin{aligned}&\{Q^{+i}_a, Q^{-j}_b\} = \delta^{ij}P_{ab}\,, \quad \{Q^{+i}_a, Q^{+j}_b\} = 0\,, \quad \{Q^{-i}_a, Q^{-j}_b\} = 0\,,\\ &\{Q^{a\hat{+}\hat{\imath}}, Q^{b\hat{-}\hat{\jmath}}\} = \delta^{\hat{\imath}\hat{\jmath}}P^{ab}\,, \quad \{Q^{a\hat{+}\hat{\imath}}, Q^{b\hat{+}\hat{\jmath}}\} = 0\,, \quad \{Q^{a\hat{-}\hat{\imath}}, Q^{b\hat{-}\hat{\jmath}}\} = 0\,.\end{aligned} \tag{3.4.13}$$

Therefore the on shell superspace representation of our supercharges is given by

$$\begin{aligned}Q^{+i}_a &= \lambda^\alpha_a\eta^{+i}_\alpha\,, \qquad Q^{-i}_a = \lambda^\alpha_a\frac{\partial}{\partial\eta^\alpha_{+i}}\,,\\ Q^{a\hat{+}\hat{\imath}} &= \tilde{\lambda}^a_{\hat{\alpha}}\eta^{\hat{\alpha}\hat{+}\hat{\imath}}\,, \qquad Q^{a\hat{-}\hat{\imath}} = \tilde{\lambda}^a_{\hat{\alpha}}\frac{\partial}{\partial\eta_{\hat{\alpha}\hat{+}\hat{\imath}}}\,.\end{aligned} \tag{3.4.14}$$

N point superamplitudes In the case we want to write scattering superamplitudes for N incoming and outgoing particles we have N momenta $p_{n[ab]}$ and therefore we have N spinor helicity variables λ^{α}_{na}, $\tilde{\lambda}^{a}_{n\hat{\beta}}$. The on shell superspace is also the cartesian product of all the on shell superspaces for each particle, hence we have N copies of the superspace coordinates $\eta^{+i}_{n\alpha}$, $\eta^{\hat{\alpha}\hat{+}\hat{\imath}}_{n}$. Therefore the positive chirality supercharges are given by

$$Q^{+i}_{na} = \lambda^{\alpha}_{na}\eta^{+i}_{n\alpha}\,, \qquad Q^{a\hat{+}\hat{\imath}}_{n} = \tilde{\lambda}^{a}_{n\hat{\alpha}}\eta^{\hat{\alpha}\hat{+}\hat{\imath}}_{n}\,. \tag{3.4.15}$$

Translation and susy invariance imply conservation of the total momentum $\sum_{n=1}^{N} p_{nab}$ and of the total supercharge $\sum_{n=1}^{N} Q^{+i}_{na}$ and $\sum_{n=1}^{N} Q^{a\hat{+}\hat{\imath}}_{n}$, the conservation of the negative chirality supercharges $\sum_{n=1}^{N} Q^{-i}_{na}$, $\sum_{n=1}^{N} Q^{a\hat{-}\hat{\imath}}_{n}$ is redundant. Therefore superamplitudes will always contain factors $\delta^{6}\left(\sum_{n=1}^{N} p_{n}\right)$ and $\delta^{16}\left(\sum_{n=1}^{N} Q^{+}_{n}\right)$. As always we do not write the momentum conservation factors. In superspace the supercharge conservation factors are written as

$$\delta^{16}\left(\sum_{n=1}^{N} Q^{+}_{n}\right) = \prod_{i=1}^{2}\prod_{a=1}^{4}\left(\sum_{n=1}^{N}\lambda^{\alpha}_{na}\eta^{+i}_{n\alpha}\right)\prod_{\hat{\imath}=1}^{2}\prod_{b=1}^{4}\left(\sum_{n=1}^{N}\tilde{\lambda}^{b}_{n\hat{\alpha}}\eta^{\hat{\alpha}\hat{+}\hat{\imath}}_{n}\right)\,. \tag{3.4.16}$$

One can show that all amplitudes involving the scattering of $N \geq 4$ graviton multiplets with $N-2$ scalars of polarisation $\phi_{++,\hat{+}\hat{+}}$ and 2 scalars of polarisation $\phi_{--,\hat{-}\hat{-}}$ (all the particles are counted as incoming) are contained in a superamplitude of the form [87]

$$\mathbf{A}^{\mathrm{MRV}}_{N\geq 4} = \frac{\delta^{16}\left(\sum_{n=1}^{N} Q^{+}_{n}\right)}{16(p_{Nab}p^{ab}_{N-1})^{4}}\mathcal{A}^{\phi}_{N\geq 4}(\phi_{++,\hat{+}\hat{+}}{}^{N-2}, \phi_{--,\hat{-}\hat{-}}{}^{2})\,, \tag{3.4.17}$$

where $\mathcal{A}^{\phi}_{N\geq 4}$ is the regular N point amplitude for the scalar field. Note that this contains the amplitudes for the whole graviton multiplet. Indeed if this amplitude is expanded in superspace coordinates then the term proportional to $\prod_{n=1}^{N}\eta^{\alpha+i}_{n}\eta^{\beta}_{n+i}\eta^{\hat{+}\hat{\imath}}_{n\hat{\gamma}}\eta_{n\hat{\delta}\hat{+}\hat{\imath}}$ will be the graviton amplitude.

In 4 dimensions the amplitude appearing on the right hand side of (3.4.17) would have been the graviton because it is the lowest lying field in the chiral superfield. We can see that this is the equivalent of the maximally helicity violating (MHV) amplitudes in 4 dimensions but since here the role of the helicity is played by the $U(1)$ factors in the R-symmetry group we call these maximally R-symmetry violating amplitudes (MRV). For $N = 4$ all on shell amplitudes are MRV amplitudes, therefore the four point superamplitude

$$\mathbf{A}_4 = \frac{\delta^{16}\left(\sum_{n=1}^4 Q_n^+\right)}{16(p_{Nab}p_{N-1}^{ab})^4}\mathcal{A}_4^\phi\,, \tag{3.4.18}$$

contains all four point amplitudes for fields of the graviton multiplet.

3.4.2 Sewing relations and unitarity

Let us now show that the positivity bounds on the superamplitude (3.4.18) reduce to positivity bounds on the regular four point scalar amplitude which is much simpler. Let us write the optical theorem (1.3.11) for the four point superamplitude

$$2\,\mathrm{Im}\,\mathbf{A}_{2\to 2'} > \int d^5k_{1''}d^5k_{2''}\int d^8\eta_{1''}d^8\eta_{2''}\,\mathbf{A}^*_{2'\to 2''}\mathbf{A}_{2\to 2''}\,, \tag{3.4.19}$$

where $k_{1''}$, $k_{2''}$ are the momenta associated to the two outgoing particles in the amplitudes on the right hand side of the equation (denoted by $2''$) while $\eta_{1''\alpha}^{+i}$, $\eta_{1''}^{\hat\alpha\hat+\hat\imath}$ and $\eta_{2''\alpha}^{+i}$, $\eta_{2''}^{\hat\alpha\hat+\hat\imath}$ are the superspace coordinates associated to these particles. The $d^8\eta_{1''}d^8\eta_{2''}$ integrals are required to sum over all intermediate particle states in the graviton multiplet. We also define $s=-(k_1+k_2)^2=-(k_{1'}+k_{2'})^2=-(k_{1''}+k_{2''})^2$. Using (3.4.18) and (3.4.2) (along with the fact that $t^\phi = s^4$) we get

$$\begin{aligned}
&\delta^{16}\left(Q_1^+ + Q_2^+ + Q_{1'}^+ + Q_{2'}^+\right) 2\,\mathrm{Im}\,f(1,2,1',2')\\
&> \int d^5k_{1''}d^5k_{2''}\int d^8\eta_{1''}d^8\eta_{2''}\,\frac{\delta^{16}\left(Q_{1'}^+ + Q_{2'}^+ + Q_{1''}^+ + Q_{2''}^+\right)}{s^4}\mathcal{A}^{*\phi}_{2'\to 2''}\\
&\qquad\qquad\times\frac{\delta^{16}\left(Q_1^+ + Q_2^+ + Q_{1''}^+ + Q_{2''}^+\right)}{s^4}\mathcal{A}^{\phi}_{2\to 2''}\,,
\end{aligned} \tag{3.4.20}$$

where the different indices n, n' and n'' are to indicate the different incoming and outgoing particles. We can then use (3.4.4) to make the tree level amplitudes appear

$$\begin{aligned}
&\delta^{16}\left(Q_1^+ + Q_2^+ + Q_{1'}^+ + Q_{2'}^+\right) 2\,\mathrm{Im}\,f(1,2,1',2')\\
&> \int d^5k_{1''}d^5k_{2''}\left(\frac{f^*(1',2',1'',2'')f(1,2,1'',2'')}{f^{*\mathrm{tree}}(1',2',1'',2'')f^{\mathrm{tree}}(1,2,1'',2'')}\right)\\
&\times\int d^8\eta_{1''}d^8\eta_{2''}\,\frac{\delta^{16}\left(Q_{1'}^+ + Q_{2'}^+ + Q_{1''}^+ + Q_{2''}^+\right)}{s^4}\mathcal{A}^{*\phi\mathrm{tree}}_{2'\to 2''}\frac{\delta^{16}\left(Q_1^+ + Q_2^+ + Q_{1''}^+ + Q_{2''}^+\right)}{s^4}\mathcal{A}^{\phi\mathrm{tree}}_{2\to 2''}\\
&= \int d^5k_{1''}d^5k_{2''}\left(\frac{f^*(1',2',1'',2'')f(1,2,1'',2'')}{f^{*\mathrm{tree}}(1',2',1'',2'')f^{\mathrm{tree}}(1,2,1'',2'')}\right)\int d^8\eta_{1''}d^8\eta_{2''}\,\mathbf{A}^{*\mathrm{tree}}_{2'\to 2''}\mathbf{A}^{\mathrm{tree}}_{2\to 2''}\,.
\end{aligned} \tag{3.4.21}$$

Sewing relations The point of making the tree level amplitudes appear is that we can use so called sewing relations which allow us to glue two tree amplitudes by summing over intermediate states of the graviton multiplet to get another tree amplitude. Indeed using (3.4.2) and (3.4.3) we have

$$\begin{aligned}
&\int d^8\eta_{1''}d^8\eta_{2''}\,\mathbf{A}^{\text{tree}}_{2'\to 2''}\mathbf{A}^{\text{tree}}_{2\to 2''}\\
&=\int d^8\eta_{1''}d^8\eta_{2''}\,\frac{\delta^{16}\left(Q^+_{1'}+Q^+_{2'}+Q^+_{1''}+Q^+_{2''}\right)}{s^4}\mathcal{A}^{*\phi\text{tree}}_{2'\to 2''}\frac{\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1''}+Q^+_{2''}\right)}{s^4}\mathcal{A}^{\phi\text{tree}}_{2\to 2''}\\
&=f^{\text{tree}}(1',2',1'',2'')f^{\text{tree}}(1,2,1'',2'')\\
&\qquad\int d^8\eta_{1''}d^8\eta_{2''}\,\delta^{16}\left(Q^+_{1'}+Q^+_{2'}+Q^+_{1''}+Q^+_{2''}\right)\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1''}+Q^+_{2''}\right)\,.
\end{aligned}\tag{3.4.22}$$

If we notice that

$$\begin{aligned}
&\int d^8\eta_{1''}d^8\eta_{2''}\,\delta^{16}\left(Q^+_{1'}+Q^+_{2'}+Q^+_{1''}+Q^+_{2''}\right)\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1''}+Q^+_{2''}\right)\\
&=s^4\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1'}+Q^+_{2'}\right)\,,
\end{aligned}\tag{3.4.23}$$

then we get the following sewing relation

$$\begin{aligned}
&\int d^8\eta_{1''}d^8\eta_{2''}\,\mathbf{A}^{\text{tree}}_{2'\to 2''}\mathbf{A}^{\text{tree}}_{2\to 2''}\\
&=s^4 f^{\text{tree}}(1',2',1'',2'')f^{\text{tree}}(1,2,1'',2'')\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1'}+Q^+_{2'}\right)\\
&=\left(\frac{f^{\text{tree}}(1',2',1'',2'')f^{\text{tree}}(1,2,1'',2'')}{f^{\text{tree}}(1,2,1',2')}\right)\mathbf{A}^{\text{tree}}_{2\to 2'}\,.
\end{aligned}\tag{3.4.24}$$

As is evident from the derivation this relation can be generalised to any loop order. This was originally proven in all dimensions using the double copy relation between Yang-Mills amplitudes and gravity amplitudes via the KLT relations [89–91]. A superspace derivation for type IIB supergravity can be found in the appendix A of [88].

Using the fact that tree level amplitudes are real we can insert this into (3.4.21) to get

$$\begin{aligned}
&\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1'}+Q^+_{2'}\right)2\,\text{Im}\,f(1,2,1',2')\\
&>\delta^{16}\left(Q^+_{1}+Q^+_{2}+Q^+_{1'}+Q^+_{2'}\right)\int d^5k_{1''}d^5k_{2''}\,s^4 f^*(1',2',1'',2'')f(1,2,1'',2'')\,.
\end{aligned}\tag{3.4.25}$$

If we divide by the delta functions and multiply by s^4 on both sides we get

$$2\,\mathrm{Im}\,\mathcal{A}^{\phi}_{2\to 2'} > \int d^5k_{1''}d^5k_{2''}\,\mathcal{A}^{*\phi}_{2'\to 2''}\mathcal{A}^{\phi}_{2\to 2''}\,, \tag{3.4.26}$$

which is the optical theorem for the scalar field. Notice that now we only sum over intermediate scalar states instead of all particle of the multiplet. This not trivial because in a supersymmetric theory all field of the supermultiplet can appear as intermediate states. What this shows is that their contributions cancel out so that one can look at the theory containing only the scalar part of the multiplet.

3.5 Low energy limits and tropical amplitudes

This section closely follows section 2 of [92]. Let us now generalise the results of section 3.3 to lower dimensions by compactifying type II superstring theory on a d-dimensional torus T^d. The spacetime metric is given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + G_{IJ}dx^I dx^J\,, \tag{3.5.1}$$

where the indices IJ indicate compactified dimensions and G_{IJ} is the metric on T^d. When we also include a background field for the Kalb-Ramond form on the torus the relevant part of the Euclidean action will decompose as

$$\begin{aligned} S + S_{\text{comp}} =& \frac{1}{2\pi\alpha'}\int d^2z\,\partial x^\mu \bar\partial x_\mu + \frac{1}{2\pi\alpha'}\int d^2z\,\left(G_{IJ}\partial x^I\bar\partial x^J - iB_{IJ}\partial x^I\bar\partial x^J\right) \\ &+ \text{other terms}\,. \end{aligned} \tag{3.5.2}$$

Where the non compact string coordinates still obey the closed string boundary condition $x^\mu(\tau,\sigma+\pi) = x^\mu(\tau,\sigma)$ but the compact string coordinates can now obey generalised versions of this boundary condition $x^I(\tau,\sigma+\pi) = x^I(\tau,\sigma) + 2\pi l_s n^I$. The fact that x^I is now a multivalued function represents the fact that strings can now wrap around the compact dimensions (which we assume to be of radius l_s), therefore we call $n^I \in \mathbb{Z}$ a winding number. This can be generalised to other cycles of the worldsheet. If we assume the a-cycles to be the space-like cycles then in form notation we get

$$\oint_{A_i} dx^I = 2\pi l_s n_i^I\,, \tag{3.5.3}$$

where $d = dz\,\partial + d\bar z\,\bar\partial$ is the exterior derivative on the worldsheet. In fact this can also be extended to timelike cycles which we assume to be the b-cycles

$$\oint_{B^i} dx^I = 2\pi l_s m^{Ii}\,, \tag{3.5.4}$$

where this time the $m^I \in \mathbb{Z}$ are interpreted as quantized internal momenta, or Kaluza-Klein modes.[6]. If we now rewrite the compactified part of the action in form notation

$$S_{\text{comp}} = \frac{1}{4\pi\alpha'}\int_\Sigma \left(G_{IJ}\, dx^I \wedge \star dx^J - iB_{IJ} dx^I \wedge dx^J\right) + \text{other terms}\,, \tag{3.5.5}$$

we can decompose the one forms dx^I on the basis of holomorphic forms ω^i defined in section 3.3.3 and use Riemann's bilinear relations to see that the expression above becomes

$$S_{\text{comp}} = \pi G_{IJ}(\operatorname{Im}\Omega)^{-1}_{ij}\left(m^{Ii} + \Omega^{ik}n^I_k\right)\left(m^{Jj} + \bar{\Omega}^{jl}n^J_l\right) - 2\pi i B_{IJ} m^{Ii} n^J_i + \text{other terms}\,, \tag{3.5.6}$$

where Ω is the period matrix defined in (3.3.15). At every order in genus, one must sum over all the possible topological sectors of the worldsheet in the path integral. Therefore we must sum over all the possible winding and Kaluza-Klein modes. If we write $q^I = (m^I, n^I)^T \in \mathbb{Z}^{2g}$ we can define

$$H_\Omega[q^I, q^J] = (\operatorname{Im}\Omega)^{-1}[m^I + (\operatorname{Re}\Omega)n^I, m^J + (\operatorname{Re}\Omega)n^J] + (\operatorname{Im}\Omega)[n^I, n^J]\,, \tag{3.5.7}$$

where we have taken the convention $M[x,y] = x^T M y$ for M a symmetric matrix. This is the standard bilinear form on $\mathbb{Z}^{2g}$ induced by elements in $U(g)\backslash Sp(2g,\mathbb{R})$. In matrix notation we have

$$H_\Omega = \begin{pmatrix} (\operatorname{Im}\Omega)^{-1} & 0 \\ 0 & \operatorname{Im}\Omega \end{pmatrix}\begin{bmatrix} I & \operatorname{Re}\Omega \\ 0 & I \end{bmatrix}\,. \tag{3.5.8}$$

where we take the convention $M[x] = M[x,x] = x^T M x$ for M a symmetric matrix. Let us also define

$$\langle q^I, q^J\rangle = m^I \cdot n^J - m^J \cdot n^I\,, \tag{3.5.9}$$

where we have taken the convention $x \cdot y = x^T y$. This is the standard $Sp(2g,\mathbb{Z})$ invariant scalar product on $\mathbb{Z}^{2g}$ in a Darboux basis. Then the path integral will include a sum

[6]Since we are in euclidean space there are no spacelike and timelike cycles on the worldsheet. Nevertheless this gives us a physical interpretation for the modes n^I and m^I.

$$(\det G)^{\frac{g}{2}} \sum_{q^I \in \mathbb{Z}^{2g}} e^{-\pi G_{IJ} H_\Omega[q^I,q^J] + i\pi B_{IJ}\langle q^I, q^J\rangle}\,, \tag{3.5.10}$$

where at every order in genus we also divide and multiply by $(\det G)^{\frac{g}{2}}$. We can use the Poisson resumation formula

$$\sum_{q\in\mathbb{Z}^n} e^{-\pi M^{-1}[q+x]} = (\det M)^{\frac{1}{2}} \sum_{p\in\mathbb{Z}^n} e^{-\pi M[p]+2\pi i p\cdot x}\,, \tag{3.5.11}$$

to put this in a more familiar form

$$\begin{aligned}
&(\det G)^{\frac{g}{2}} \sum_{m^{Ii},n^I_i\in\mathbb{Z}} e^{-\pi G_{IJ}\left((\operatorname{Im}\Omega)^{-1}_{ij}\left(m^{Ii}+(\operatorname{Re}\Omega)^{ik}n^I_k\right)\left(m^{Jj}+(\operatorname{Re}\Omega)^{jl}n^J_l\right)+(\operatorname{Im}\Omega)^{ij}n^I_i n^J_j\right)+2\pi i B_{IJ}m^{Ii}n^J_i} \\
&= (\det \operatorname{Im}\Omega)^{\frac{d}{2}} \sum_{m_{Ii},n^I_i\in\mathbb{Z}} e^{-\pi(\operatorname{Im}\Omega)^{ij}\left(G^{IJ}\left(m_{Ii}+B_{IK}n^K_i\right)\left(m_{Jj}+B_{JL}n^L_j\right)+G_{IJ}n^I_i n^J_j\right)+2\pi i(\operatorname{Re}\Omega)^{ij}m_{Ii}n^I_j}\,.
\end{aligned} \tag{3.5.12}$$

If we write $q_i = (m_i, n_i)^T \in \mathbb{Z}^{d,d}$ we can define

$$H_{G,B}[q_i,q_j] = G^{-1}[m_i + Bn_i, m_j + Bn_j] + G[n_i,n_j]\,. \tag{3.5.13}$$

This is the standard bilinear form on $\mathbb{Z}^{d,d}$ induced by elements in $(O(d)\times O(d))\backslash O(d,d,\mathbb{R})$. In matrix notation we have

$$H_{G,B} = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}\,. \tag{3.5.14}$$

Let us also define

$$(q_i,q_j) = m_i\cdot n_j + m_j\cdot n_j\,. \tag{3.5.15}$$

This is the standard $O(d,d,\mathbb{Z})$ invariant scalar product on $\mathbb{Z}^{d,d}$ in a null basis. Then the path integral will include the sum

$$\Gamma^{(g)}_{I\!I_{d,d}}(\Omega) = (\det \operatorname{Im}\Omega)^{\frac{d}{2}} \sum_{q_i\in\mathbb{Z}^{d,d}} e^{-\pi(\operatorname{Im}\Omega)^{ij}H_{G,B}[q_i,q_j]+i\pi(\operatorname{Re}\Omega)^{ij}(q_i,q_j)}\,, \tag{3.5.16}$$

where $I\!I_{d,d} = \mathbb{Z}^{d,d}$ stands for the even self-dual lattice of split signature. This is called the Narain partition function or the Siegel theta series. Note that while the form (3.5.10) had manifest symplectic modular invariance under $Sp(2g,\mathbb{Z})$ this form

has manifest $O(d,d,\mathbb{Z})$ invariance. In particular this proves that this function has both. Whereas the first form was found directly from the action and can therefore be thought of as a Lagrangian partition function the second form (3.5.16) can be thought of as its Hamiltonian counterpart. Indeed the Poisson resummation is equivalent to a Legendre transform of the exponent. The $O(d,d,\mathbb{Z})$ symmetry corresponds to the invariance under exchanges and shifts of the winding and Kaluza-Klein modes, it is the generalisation of T duality for toroidal compactifications.

Since the torus T^d has no curvature the compactification only modifies the 0 modes of the worldsheet fields by changing their boundary conditions. Therefore the rest of the path integral goes pretty much unchanged.

Toroidal compactification also preserves maximal supersymmetry which allows us to keep the nice factorisation properties we have discussed in the previous section. Indeed if we write $V_d = (2\pi l_s)^d\sqrt{\det G}$ the volume of T^d we have that the type II superstring graviton four point amplitude in $D = 10 - d$ dimensions is of the form

$$\mathcal{A}_{4,D}^{\text{string}} = -i\frac{(2\pi)^7\alpha'^4 g_s^2}{2^{11}V_d}\mathcal{R}^4(k_i,\epsilon_i)f_D^{\text{string}}(s,t,u;\varphi)\,, \tag{3.5.17}$$

where $f_D^{\text{string}}(s,t,u;\varphi)$ is a function dependent on the dimension which is invariant under permutation of the Mandelstam variables and is a function of the moduli fields φ parametrising some symmetric space.[7] If we define the D dimensional string coupling constant as $g_D^2 = g_s^2/\sqrt{\det G}$ then we get

$$\mathcal{A}_{4,D}^{\text{string}} = -i\frac{(2\pi)^{7-d}\alpha'^{\frac{8-d}{2}} g_D^2}{2^{11}}\mathcal{R}^4(k_i,\epsilon_i)f_D^{\text{string}}(s,t,u;\varphi)\,. \tag{3.5.18}$$

$f_D^{\text{string}}(s,t,u;\varphi)$ can then be expanded into a perturbative series

$$f_D^{\text{string}}(s,t,u;\varphi) = \alpha'^3\sum_{g=0}^{+\infty} g_D^{2g} f_{D,g}^{\text{string}}(s,t,u;\varphi) + \mathcal{O}\left(e^{-\frac{2\pi}{g_D}}\right)\,, \tag{3.5.19}$$

where we have used the fact that we have divided by $(\det G)^{\frac{g}{2}}$ at every order in genus to replace g_s by g_D in our expansion. Our aim will be to look at the low energy limit of such amplitudes to see the corrections type II string theory entails to gravity in the field theory limit. To this end we will compare the low energy limit to type II supergravity which means the Kaluza-Klein reduction of type IIA or type IIB supergravity on a torus. Both theories have the same four point amplitudes up to two loops in 10 dimensions and lead to the same theory in lower dimensions hence

[7] Note that we have put the back the constants as in (3.4.1) instead of absorbing them in f such as in the previous section.

we will not distinguish between them. The gravitational sector of this theory is given by the following action in string frame

$$S^{\text{sugra}} = \frac{1}{2\kappa_D^2}\int d^D x\,\sqrt{-g}\,e^{-2\phi}R\,, \tag{3.5.20}$$

where we have assumed the we have absorbed the vacuum expectation value ϕ_0 of the dilaton into the coupling constant κ_D. One can show that the type II supergravity graviton four point amplitude in D dimensions is of the form

$$\mathcal{A}_{4,D}^{\text{sugra}} = -i\frac{\kappa_D^2}{2^{10}}\mathcal{R}^4(k_i,\epsilon_i)f_D^{\text{sugra}}(s,t,u;\varphi)\,, \tag{3.5.21}$$

where $f_D^{\text{sugra}}(s,t,u;\varphi)$ is a function dependent on the dimension which is invariant under permutation of the Mandelstam variables and is a function of the moduli fields φ parametrising some symmetric space. $f_D^{\text{sugra}}(s,t,u;\varphi)$ can then be expanded into a perturbative series

$$f_D^{\text{sugra}}(s,t,u;\varphi) = 64\sum_{n=0}^{+\infty}\kappa_D^{2n}f_{D,n-\text{loops}}^{\text{sugra}}(s,t,u;\varphi) + \text{non perturbative terms}\,. \tag{3.5.22}$$

3.5.1 Genus zero amplitude low energy limit

Let us first look at the tree level where on the supergravity side we have

$$f_{D,\text{tree}}^{\text{sugra}}(s,t,u) = \frac{1}{stu}\,. \tag{3.5.23}$$

Whereas at genus 0 on the string side we have [28, 71]

$$\begin{aligned} f_{D,0}^{\text{string}}(s,t,u) &= -\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)} \\ &= \frac{64}{\alpha'^3 stu}\exp\left(\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}\left(\frac{\alpha'}{4}\right)^{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})\right) \\ &= \frac{64}{\alpha'^3 stu} + \sum_{p,q\geq 0}\left(\frac{\alpha'}{4}\right)^{2p+3q} c_{(p,q)}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q\,, \end{aligned} \tag{3.5.24}$$

where $c_{(p,q)}$ are constants that are polynomials in the odd zeta values $\zeta(2n+1)$. The first ones are given by

$$c_{(0,0)} = 2\zeta(3)\,, \quad c_{(1,0)} = \zeta(5)\,, \quad c_{(0,1)} = \frac{2\zeta(3)^2}{3}\,. \tag{3.5.25}$$

At the genus 0 order the low energy limit is simply obtained by taking $\alpha' s, \alpha' t, \alpha' u \to 0$. Therefore we can see from (3.5.23) and (3.5.24) that the low energy limit of the genus 0 string amplitude and the tree level supergravity amplitude coincide if we make the identification $2\kappa_D^2 = (2\pi)^{7-d}\alpha'^{\frac{8-d}{2}} g_D^2$. This means that the D dimensional Planck length and string length are related by $l_P = g_D^{\frac{2}{8-d}} l_s$. As we have seen we can decompose the amplitude into analytic and non analytic parts where the non analytic part is associated with the propagation of the massless graviton while the analytic parts are to be interpreted as coming from the propagation of the massive string states.

This result justifies the claim we have made in chapter 2 that string theory reproduces gravity at low energy and it also justifies the claim we have made earlier in chapter 3 that the genus expansion of string theory corresponds to the loop expansion of supergravity in the quantum field theory limit. However we can see that string amplitudes add corrections to the supergravity amplitudes. These corrections can be seen directly from the effective field theory point of view as higher order corrections to the supergravity Lagrangian. In fact, this is the usual method used to find the Wilsonian effective action of string theory [93–95]. Schematically the gravitational sector of the effective action giving rise to such corrections has the form in string frame

$$S^{\text{string}} = \frac{1}{2\kappa_D^2}\int d^D x\,\sqrt{-g}\,e^{-2\phi}\left(R + \frac{l_s^6}{48}\left(\sum_{p,q\geq 0} l_s^{4p+6q} c_{(p,q)} \nabla^{4p+6q}\mathcal{R}^4\right) + \dots\right)\,, \tag{3.5.26}$$

where again we have assumed the we have absorbed the vacuum expectation value ϕ_0 of the dilaton into the coupling constant. These corrections can be thought of as arising from integrating out the massive string states in the effective path integral. Of course this action is not supersymmetric, in principle one must add similar corrections for other fields of the multiplet (which can be found by expanding the corresponding string tree amplitudes for these fields) as well as higher order terms in the Riemann tensor (which can be found by expanding higher point graviton string amplitudes). The Wilson couplings also receive further contributions from the genus 1 and 2 amplitudes. In addition we will see in the next chapter that supersymmetry and U-duality requires one to add terms to the Wilson couplings which are seen as non perturbative contributions coming from instanton corrections.

3.5.2 Genus one amplitude low energy limit

At the one loop level on the supergravity side the only contributing diagram is given by figure 3.4 [72, 84][8].

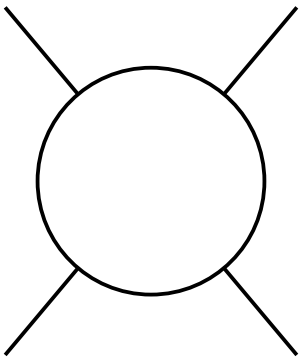


Figure 3.4: Four graviton one loop diagram in maximal supergravity. Gravitons are represented by straight lines.

This gives the integral

$$f^{\text{sugra}}_{D,1}(s,t,u) = I^{(1)}_4(s,t) + I^{(1)}_4(t,u) + I^{(1)}_4(u,s)\,, \tag{3.5.27}$$

with

$$I^{(1)}_4(s,t) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}\,, \tag{3.5.28}$$

where by convention the loop momenta are assumed to be euclidean. In $D \leq 4$ this suffers from IR divergences while in $D \geq 8$ it suffers from UV divergences. Therefore the integral must be dimensionally regularised, i.e. we perform an analytic continuation in D such that D is replaced by $D - 2\epsilon$ and the function is analytically continued around $\epsilon = 0$. However $D = 4$ and $D = 8$ are the critical dimensions where the IR and UV divergences respectively are logarithmic, hence one must define a prescription to remove the pole in $\epsilon = 0$. Also note that in any dimension the integral only converges on the so called domain of convergence where $s, t, u > 0$, therefore the integral is actually defined by analytic continuation of the Mandelstam variable to the upper complex half plane containing the physical domain where $s + t + u = 0$. This amplitude can be put into Schwinger parameter space

[8] In particular there are no diagrams with vertices corresponding to gravitational contact interactions. The fact that the only contributing diagram corresponds to the box diagram of ϕ^3 scalar field theory is a very special feature of the maximally supersymmetric four-graviton amplitude [84].

$$I_4^{(1)}(s,t) = \frac{1}{16(2\pi)^{D-4}} \int_0^\infty dL\, L^{3-\frac{D}{2}} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1\, e^{\pi L[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} \,. \tag{3.5.29}$$

At genus 1 on the string side we have

$$f_{D,1}^{\text{string}}(s,t,u;\varphi) = 2\pi \int_{\mathcal{F}_1} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \Gamma^{(1)}_{I\!I_{d,d}} \prod_{i=1}^4 \int_\Sigma \frac{d^2 z_i}{\operatorname{Im}\tau}\, \operatorname{Im}\tau\, \delta^2(z_1) \times \exp\left(-\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j\, G(z_i,z_j) \right) , \tag{3.5.30}$$

where

$$\Gamma^{(1)}_{I\!I_{d,d}}(\tau) = (\operatorname{Im}\tau)^{\frac{d}{2}} \sum_{q \in I\!I_{d,d}} e^{-\pi \operatorname{Im}\tau\, H_{G,B}[q] + i\pi \operatorname{Re}\tau\,(q,q)} \,, \tag{3.5.31}$$

with the convention that $\Gamma^{(g)}_{I\!I_{0,0}} = 1$. The moduli fields φ in (3.5.30) are the torus metric G_{IJ} and Kalb-Ramond two form on the torus B_{IJ} parametrising the Narain partition function. Following [96] it is convenient to introduce a cutoff parameter $\Lambda > 1$ to split the moduli space into two pieces

$$\mathcal{F}_1 = \mathcal{F}_{1,\Lambda} \cup \{ |\operatorname{Re}\tau| \leq 1/2,\, \operatorname{Im}\tau \geq \Lambda \} \,, \tag{3.5.32}$$

where we have defined the truncated fundamental domain

$$\mathcal{F}_{1,\Lambda} = \{ |\operatorname{Re}\tau| \leq 1/2,\, 0 < \operatorname{Im}\tau < \Lambda,\, |\tau| \geq 1 \} \,. \tag{3.5.33}$$

This also splits the moduli space integral in (3.5.30) into $f^{\text{string}}_{D,1,<\Lambda}$ and $f^{\text{string}}_{D,1,>\Lambda}$. Let us first focus on the first part

$$f_{D,1,<\Lambda}^{\text{string}}(s,t,u;\varphi) = 2\pi \int_{\mathcal{F}_{1,\Lambda}} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \Gamma^{(1)}_{I\!I_{d,d}} \prod_{i=1}^4 \int_\Sigma \frac{d^2 z_i}{\operatorname{Im}\tau}\, \operatorname{Im}\tau\, \delta^2(z_1) \times \sum_{n=1}^\infty \frac{1}{n!} \left(-\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j\, G(z_i,z_j) \right)^n . \tag{3.5.34}$$

We can expand the Koba–Nielsen factor to get

$$f^{\text{string}}_{D,1,<\Lambda}(s,t,u;\varphi) = \sum_{p,q\geq 0}\left(\frac{\alpha'}{4}\right)^{2p+3q}\mathcal{E}^{(1)}_{(p,q)\Lambda}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q\,, \tag{3.5.35}$$

with

$$\mathcal{E}^{(1)}_{(p,q)\Lambda} = 2\pi\int_{\mathcal{F}_{1,\Lambda}}\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\Gamma^{(1)}_{I\!I_{d,d}}\mathcal{B}^{(1)}_{(p,q)}\,, \tag{3.5.36}$$

where $\mathcal{B}^{(1)}_{(p,q)}$ are so called graph functions introduced in [97]. It was shown in [96] that the first graph functions are Eisenstein series

$$\mathcal{B}^{(1)}_{(0,0)} = 1\,,\quad \mathcal{B}^{(1)}_{(1,0)}(\tau) = \frac{\pi^2}{45}E^{SL(2)}_{2\Lambda_1}(\tau)\,,\quad \mathcal{B}^{(1)}_{(0,1)}(\tau) = \frac{2\pi^3}{567}E^{SL(2)}_{3\Lambda_1}(\tau)+\frac{\zeta(3)}{3}\,, \tag{3.5.37}$$

where Λ_1 refers to the fundamental weight of $SL(2,\mathbb{R})$. The notations and precise normalisations for Eisenstein series are defined in chapter 4. For now just notice that they are well defined functions on $SO(2)\backslash SL(2,\mathbb{R})/PSL(2,\mathbb{Z})$, the moduli space of the torus. These graph functions are studied further in [98–103]. It was shown in [97] that for $\operatorname{Im}\tau \gg 1$ they behave as

$$\mathcal{B}^{(1)}_{(p,q)}(\tau) = \sum_{n=0}^{2p+3q-1} b_{(p,q),n}(\pi\operatorname{Im}\tau)^{2p+3q-n} + \mathcal{O}\left(e^{-2\pi\operatorname{Im}\tau}\right)\,, \tag{3.5.38}$$

for constants $b_{(p,q),n}$ which are linear combinations over $\mathbb{Q}$ of single-valued multi-zeta values of weight n. Therefore if we choose $\Lambda \gg 1$ we have that

$$\mathcal{E}^{(1)}_{(p,q)\Lambda} = \mathcal{E}^{(1)}_{(p,q)\epsilon} + 2\pi\sum_{n=0}^{2p+3q-1}\pi^{2p+3q-n}b_{(p,q),n}\frac{\Lambda^{\frac{d-2}{2}+2p+3q-n}}{\frac{d-2}{2}+2p+3q-n} + \mathcal{O}\left(e^{-\Lambda}\right)\,, \tag{3.5.39}$$

where to avoid having to replace the power of Λ by a $\ln\Lambda$ whenever $\frac{d-2}{2}+2p+3q-n=0$ (which happens for $d=2$ and $p=q=0$ and for $d=0$ for $p,q\geq 1$) we assume that the second term is defined by analytic continuation in d such that d is replaced by $d+2\epsilon$ and the function is analytically continued around $\epsilon=0$. We have also defined

$$\mathcal{E}^{(1)}_{(p,q)\epsilon} = 2\pi\int_{\mathcal{F}_1}\frac{d^2\tau}{(\operatorname{Im}\tau)^2}(\operatorname{Im}\tau)^{\epsilon}\Gamma^{(1)}_{I\!I_{d,d}}\mathcal{B}^{(1)}_{(p,q)}\,, \tag{3.5.40}$$

which converges for $d+2\epsilon<2-4p-6q$ and is also defined by analytic continuation in ϵ near $\epsilon=0$.[9] Note that the invariance of $\mathcal{B}^{(1)}_{(p,q)}$ under $PSL(2,\mathbb{Z})$ ensures that the amplitude does not depend on the representative $\mathcal{F}_1$ of $SO(2)\backslash SL(2,\mathbb{R})/PSL(2,\mathbb{Z})$. In the limit $\epsilon\to 0$ we have that the first couplings are given by [51, 104]

$$\begin{aligned}
\mathcal{E}^{(1)}_{(0,0)\epsilon} &= 4\pi\xi(d-2)E^{SO(d,d)}_{\frac{d-2}{2}\Lambda_1}\,, \qquad (d\neq 2)\\
\mathcal{E}^{(1)}_{(1,0)\epsilon} &= \frac{4\pi^3}{45}\xi(d+2)E^{SO(d,d)}_{\frac{d+2}{2}\Lambda_1}\,, \qquad (d\neq 4)\\
\mathcal{E}^{(1)}_{(0,1)\epsilon} &= \frac{\zeta(3)}{3}\mathcal{E}^{(1)}_{(0,0)\epsilon}+\frac{8\pi^4}{567}\xi(d+4)E^{SO(d,d)}_{\frac{d+4}{2}\Lambda_1}\,, \qquad (d\neq 6)
\end{aligned} \tag{3.5.41}$$

where we have used (4.1.60). It is understood that if the limit is regular, as is the case above, ϵ is taken to be 0. We have also defined the Riemann xi function $\xi(s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, also called the completed Riemann function. Again Eisenstein series are defined later in chapter 4 but for now one can notice that they are invariant under $O(d,d,\mathbb{Z})$, which reflects the invariance under T-duality of the amplitude. For $d=0$ the Eisenstein series are not defined but we can see that $\mathcal{E}^{(1)}_{(0,0)\epsilon}=2\pi\mu(\mathcal{F}_1)=\frac{2\pi^2}{3}$ where $\mu(\mathcal{F}_1)=\frac{\pi}{3}$ is the volume of the fundamental domain $\mathcal{F}_1$. Furthermore, in the limit $\epsilon\to 0$, one can show that $\mathcal{E}^{(1)}_{(1,0)\epsilon}=0$ and $\mathcal{E}^{(1)}_{(0,1)\epsilon}=\frac{2\pi^2}{9}\zeta(3)$ because the regularised integral of the Eisenstein series $E^{SL(2)}_{s\Lambda_1}$ on the fundamental domain $\mathcal{F}_1$ vanishes for all s. We have written on the right of (3.5.41) the dimension where the limit $\epsilon\to 0$ diverges, for $\mathcal{E}^{(1)}_{(0,0)\epsilon}$ at $d=2$ this is simply due to the logarithmic divergence in (3.5.39). We will show at the end of the section how to regularise this and calculate $\mathcal{E}^{(1)}_{(0,0)\epsilon}$.

We can see from (3.5.35) that $f^{\text{string}}_{D,1,<\Lambda}$ is analytic in the Mandelstam variables, therefore it is associated with the propagation of massive string states.

Let us now turn on the second part of the amplitude

$$\begin{aligned}
f^{\text{string}}_{D,1,>\Lambda}(s,t,u;\varphi) = 2\pi\int_{\substack{\operatorname{Im}\tau\geq\Lambda\\ |\operatorname{Re}\tau|\leq\frac{1}{2}}}\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\Gamma^{(1)}_{I\!I_{d,d}}\prod_{i=1}^{4}\int_\Sigma\frac{d^2z_i}{\operatorname{Im}\tau}\,\operatorname{Im}\tau\,\delta^2(z_1)\\
\times\exp\left(-\frac{\alpha'}{2}\sum_{i<j}k_i\cdot k_j\,G(z_i,z_j)\right).
\end{aligned} \tag{3.5.42}$$

[9] By abuse of notation we keep the same notation for two different types of regularisations. It is nevertheless understood that $\mathcal{E}^{(1)}_{(p,q)\Lambda}$ and $\mathcal{E}^{(1)}_{(p,q)\epsilon}$ are not equal for $\Lambda=\epsilon$.

Since we have $\Lambda \gg 1$ then this part of the moduli space integral is restricted to the tori with large dimensionless volume $\operatorname{Im}\tau \gg 1$. The natural setting for studying such worldsheets is that of tropical geometry and tropical limits. The tropical limit is a singular limit which takes algebraic varieties to piecewise linear objects. Thus one can understand intuitively how it can take string worldsheets to Feynman diagrams eventually with some lower genus string worldsheet insertions, see figure 3.5. A comprehensive review of the tropical limit of string amplitudes can be found in [105].

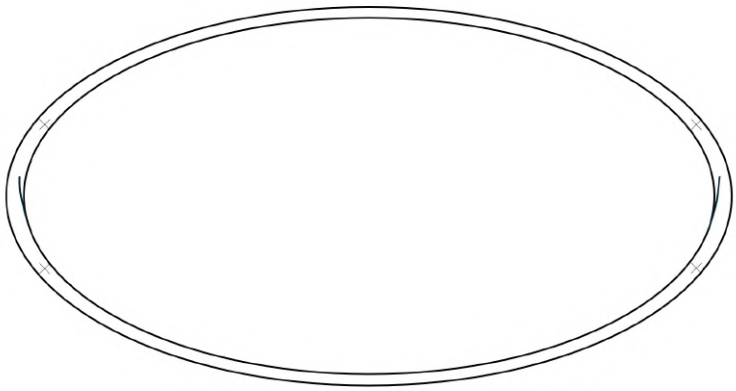


Figure 3.5: Tropical limit of a genus 1 surface where the degenerate surface looks like a one loop supergravity Feynman diagram.

If we define $z_i = x_i + \tau y_i$ for $x_i, y_i \in \mathbb{R}$ then the tropical limit of the propagator in the Koba-Nielsen factor is given by [106]

$$\begin{aligned} G(z,z') =& 2\pi \operatorname{Im}\tau(-|y-y'|+(y-y')^2) - \ln\left|1-e^{-2\pi \operatorname{Im}\tau|y-y'|+2\pi i(x-x'+\operatorname{Re}\tau(y-y'))}\right|^2 \\ &+ \mathcal{O}\left(e^{-\pi \operatorname{Im}\tau}\right) , \end{aligned} \tag{3.5.43}$$

for $|y-y'| < 1/2$. Since we have $\Lambda \gg 1$ then the terms in $\mathcal{O}\left(e^{-\pi \operatorname{Im}\tau}\right)$ can be neglected. We can also focus on a generic point in moduli space where all worldsheet punctures are far from each other and the distances $|y_i - y_j|$ are far from 0, we call the moduli space integral over such points $f^{\text{string}}_{D,1,\bigcirc>\Lambda}$. In this case the logarithmic term can also be neglected. Finally from (3.5.31) we can see that for $\operatorname{Im}\tau \gg 1$ we have $\Gamma^{(1)}_{I\!I_{d,d}}(\tau) \sim (\operatorname{Im}\tau)^{\frac{d}{2}}$ therefore one gets

$$
\begin{aligned}
f^{\text{string}}_{D,1,\bigcirc>\Lambda}(s,t,u) = 2\pi \int_{\substack{\text{Im}\,\tau\geq\Lambda \\ |\text{Re}\,\tau|\leq\frac{1}{2}}} \frac{d^2\tau}{(\text{Im}\,\tau)^2} (\text{Im}\,\tau)^{\frac{d}{2}} \prod_{i=1}^{4} \int_0^1 d^2y_i\, \delta(y_1) \\
\times \exp\left(-\alpha'\pi\,\text{Im}\,\tau \sum_{i<j} k_i\cdot k_j \, \left(-|y_i-y_j| + (y_i-y_j)^2\right)\right) \\
= 4\pi{\alpha'}^{\frac{2-d}{2}} \int_{\alpha'\Lambda}^{\infty} dL\, L^{\frac{d}{2}-2} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1 \left(e^{\pi L[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} \right. \\
\left. + e^{\pi L[(x_2-x_1)(1-x_3)t + x_1(x_3-x_2)u]} + e^{\pi L[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)u]} \right) ,
\end{aligned}
\tag{3.5.44}
$$

where we have decomposed the integral over y_2, y_3 and y_4 into three integrals with orderings $0 \leq y_2 \leq y_3 \leq y_4 \leq 1$, $0 \leq y_4 \leq y_2 \leq y_3 \leq 1$ and $0 \leq y_3 \leq y_4 \leq y_2 \leq 1$ and performed the respective change of variables

$$
\begin{aligned}
y_2 &= 1-x_3\,, \qquad y_3 = 1-x_2\,, \qquad y_4 = 1-x_1\,; \\
y_2 &= 1-x_2\,, \qquad y_3 = 1-x_1\,, \qquad y_4 = 1-x_3\,; \\
y_2 &= x_3\,, \qquad y_3 = x_1\,, \qquad y_4 = x_2\,,
\end{aligned}
\tag{3.5.45}
$$

so as to end up with a unique integral with ordering $0 \leq x_1 \leq x_2 \leq x_3 \leq 1$. We have also performed the change of variable $\text{Im}\,\tau = L/\alpha'$. Notice that (3.5.44) is basically the supergravity integral in Schwinger parameter space with a UV cutoff $\alpha'\Lambda$, therefore this integral is well defined for all space-time dimensions where there are no IR divergences, which means for $D \geq 5$ which corresponds to $d \leq 5$. For $D = 4$ corresponding to $d = 6$ one must introduce some IR regularisation which we will not do here.

Since in the low energy limit we have $\alpha' s, \alpha' t, \alpha' u \to 0$ then we can also choose $\Lambda \ll 1/\alpha' s, 1/\alpha' t, 1/, \alpha' u$ in which case one can show that

$$
\begin{aligned}
& 4\pi{\alpha'}^{\frac{2-d}{2}} \int_{\alpha'\Lambda}^{\infty} dL\, L^{\frac{d}{2}-2} \int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1\, e^{\pi L[(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} \\
=& 32(2\pi)^{7-d}{\alpha'}^{\frac{2-d}{2}} I^{(1)}_{4,\epsilon}(s,t) \\
& - \sum_{n=0}^{\infty} \frac{(4\pi)^{n+1}(n+1)!}{(2n+3)!} \frac{\Lambda^{\frac{d}{2}-1+n}}{\frac{d}{2}-1+n} \int_0^1 dx \frac{\left((1-x)\frac{\alpha' s}{4}\right)^{n+1} - \left(x\frac{\alpha' t}{4}\right)^{n+1}}{(1-x)\frac{\alpha' s}{4} - x\frac{\alpha' t}{4}} ,
\end{aligned}
\tag{3.5.46}
$$

where again to avoid having to replace the power of Λ by a $\ln\Lambda$ whenever $\frac{d}{2}-1+n=0$ (which happen for $d=0$ and $d=2$) we assume that the second term is defined by analytic continuation in d such that d is replaced by $d+2\epsilon$ and the function is analytically continued around $\epsilon=0$. We have also defined

$$I^{(1)}_{4,\epsilon}(s,t)=\int\frac{d^{10-d}p}{(2\pi)^{10-d}}\frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}\,, \tag{3.5.47}$$

which is also defined by analytic continuation in ϵ near $\epsilon=0$. The Λ dependent terms can be expanded as

$$\begin{aligned} f^{\text{string}}_{D,1,\bigcirc>\Lambda}(s,t,u) =&32(2\pi)^{7-d}{\alpha'}^{\frac{2-d}{2}}(I^{(1)}_{4,\epsilon}(s,t)+I^{(1)}_{4,\epsilon}(t,u)+I^{(1)}_{4,\epsilon}(u,s))\\ &-\sum_{p,q\geq 0}\left(\frac{\alpha'}{4}\right)^{2p+3q}b_{(p,q)\bigcirc}\frac{\Lambda^{\frac{d-2}{2}+2p+3q}}{\frac{d-2}{2}+2p+3q}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q\,, \end{aligned} \tag{3.5.48}$$

where the $b_{(p,q)\bigcirc}$ are also linear combinations over $\mathbb{Q}$ of single-valued multi-zeta values.

Other contributions to $f^{\text{string}}_{D,1,>\Lambda}$ come from points in moduli space where two or more punctures are close to each other. Therefore there are four more contributions corresponding to integrals over points in moduli space where one, two, three or four distances $|y_i-y_j|$ are close to 0. We will not calculate them here but when all of these are included it turns out that all Λ dependent terms cancel each other out between $f^{\text{string}}_{D,1,<\Lambda}$ and $f^{\text{string}}_{D,1,>\Lambda}$. This was checked explicitly up to $(p,q)=(2,0)$ in [92] using [107]. Therefore we can forget about the cutoff parameter Λ and the complete integral can be written for $D\geq 5$ as the limit $\epsilon\to 0$

$$\begin{aligned} f^{\text{string}}_{D,1}(s,t,u;\varphi) =&32(2\pi)^{7-d}{\alpha'}^{\frac{2-d}{2}}(I^{(1)}_{4,\epsilon}(s,t)+I^{(1)}_{4,\epsilon}(t,u)+I^{(1)}_{4,\epsilon}(u,s))\\ &+\sum_{p,q\geq 0}\left(\frac{\alpha'}{4}\right)^{2p+3q}\mathcal{E}^{(1)}_{(p,q)\epsilon}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q+\text{other terms}\,, \end{aligned} \tag{3.5.49}$$

where the limit is well defined for each term individually for all dimensions except $d=2$ corresponding to spacetime dimensions $D=8$ (for $d=0$ each integral $I^{(1)}_{4,\epsilon}$ is not well defined individually but the sum over permutations of the Mandelstam variables is, so that the $D=10$ supergravity dimensionally regularised amplitude is well defined on its own). This result also holds in $D=4$ provided one introduces an IR regulator.

We can see from (3.5.27) and (3.5.49) that, again, the low energy limit of the genus 1 string amplitude reproduces the dimensionally regularised one loop supergravity amplitude if we make the identification $2\kappa_D^2 = (2\pi)^{7-d}\alpha'^{\frac{8-d}{2}} g_D^2$. Intuitively this corresponds to the tropical limit of the string worldsheet that looks like a one loop Feynman diagram as depicted in figure 3.5. The other terms coming from the contributions to $f^{\text{string}}_{D,1,>\Lambda}$ with coincident points include massive string mode propagators. They correspond schematically to the tropical limit of the worldsheet that looks like a Feynman diagram with the insertion of a genus 0 string world sheet. They are suppressed by powers of α' with respect to the one loop supergravity integral, the leading $\mathcal{E}^{(1)}_{(0,0)\epsilon}$ term for $D \geq 3$ and the next to leading $\mathcal{E}^{(1)}_{(1,0)\epsilon}$ term for $D \geq 7$.

Again these corrections can be seen as arising from the effective field theory action in string frame

$$S^{\text{string}} = \frac{1}{2\kappa_D^2}\int d^D x\,\sqrt{-g}\,e^{-2\phi} \times \left(R + \frac{l_s^6}{48}\sum_{p,q\geq 0} l_s^{4p+6q}\left(c_{(p,q)} + g_D^2 e^{2\phi}\mathcal{E}^{(1)}_{(p,q)\epsilon}\right)\nabla^{4p+6q}\mathcal{R}^4 + \dots\right), \tag{3.5.50}$$

where again we have assumed the we have absorbed the vacuum expectation value ϕ_0 of the dilaton into the string coupling constant.

Logarithmic ambiguity in 8 dimensions We saw that in dimension $D = 8$ the two terms in (3.5.49) are not individually well defined but their sum is analytic in 0. From the supergravity point of view this is because $D = 8$ is the critical dimension and thus the dimensionally regularised one loop amplitude contains a pole in $\epsilon = 0$ corresponding to the logarithmic divergence. The only term in the analytic part of the amplitude which has the same power in α' and thus can cancel this pole is $\mathcal{E}^{(1)}_{(0,0)\epsilon}$. Indeed this term also diverges. Let us write $d = 2 + 2\epsilon$. We have on one side

$$
\begin{aligned}
&32(2\pi)^{7-d}\alpha'^{\frac{2-d}{2}} I^{(1)}_{4,\epsilon}(s,t) = 32(2\pi)^{5-2\epsilon}\alpha'^{-\epsilon}\int \frac{d^{8-2\epsilon}p}{(2\pi)^{8-2\epsilon}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2} \\
&= 4\pi\alpha'^{-\epsilon}\int_0^\infty dL\, L^{\epsilon-1}\int_0^1 dx_3\int_0^{x_3} dx_2\int_0^{x_2} dx_1\, e^{\pi L[(x_2-x_1)(1-x_3)s+x_1(x_3-x_2)t]} \\
&= 4\pi\int_0^\infty \frac{d\tau_2}{\tau_2}\tau_2^\epsilon\int_0^1 dx_3\int_0^{x_3} dx_2\int_0^{x_2} dx_1\, e^{\pi\tau_2\alpha'[(x_2-x_1)(1-x_3)s+x_1(x_3-x_2)t]} \\
&= -8\pi^{1-\epsilon}\frac{\Gamma(\epsilon-2)\Gamma(3-\epsilon)^2}{\Gamma(5-2\epsilon)}\int_0^1 dx\left((-\alpha' s)^{-\epsilon}\frac{(1-x)^{1-\epsilon}}{(1+\frac{t}{s})-1}+(-\alpha' t)^{-\epsilon}\frac{(1-x)^{1-\epsilon}}{(1+\frac{s}{t})-1}\right) \\
&= \frac{2\pi}{3}\left(\frac{1}{\epsilon}+\frac{11}{3}-\gamma_E-\ln\pi-\frac{s}{s+t}\ln(-\alpha' s)\right. \\
&\qquad\left.-\frac{t}{s+t}\ln(-\alpha' t)-\frac{1}{2}\frac{st}{(s+t)^2}\left(\ln\left(\frac{t}{s}\right)^2+\pi^2\right)\right)+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{3.5.51}
$$

We therefore define the renormalised function

$$
\begin{aligned}
\hat{I}^{(1)}_{4,\mu}(s,t) = -\frac{1}{6(4\pi)^4}\left(\frac{s}{s+t}\ln\big(-s/\mu^2\big)+\frac{t}{s+t}\ln\big(-t/\mu^2\big)\right. \\
\left.+\frac{1}{2}\frac{st}{(s+t)^2}\left(\ln\left(\frac{t}{s}\right)^2+\pi^2\right)\right),
\end{aligned}
\tag{3.5.52}
$$

where we have introduced the renormalisation scale μ. Notice that $\hat{I}^{(1)}_{4,\mu}(s,t) = \hat{I}^{(1)}_{4,1}(s/\mu^2, t/\mu^2)$.

On the other side we write explicitly the $SO(2,2)$ invariant bilinear form $H_{G,B}[q,q]$ appearing in the Narain partition function as

$$
H_{G,B}[q,q] = \frac{1}{y_1^2}(m+x_1q_2+x_2q_1+x_1x_2n)^2+\frac{1}{y_2^2}(q_1+x_1n)^2+y_2^2(q_2+x_2n)^2+y_1^2n^2\,,
\tag{3.5.53}
$$

where y_1, y_2, x_2 parametrise the torus metric G_{IJ} and x_1 parametrises the torus Kalb-Ramond two form B_{IJ} and $q = (m, q_1, q_2, n)^T$. We also have $(q,q) = 2mn+2q_1q_2$. We can then replace the $(\mathrm{Im}\,\tau)^\epsilon$ factor in (3.5.40) by the real analytic Eisenstein series $E^{SL(2)}_{\epsilon\Lambda_1}(\tau)$ without changing the limit. Therefore we compute as in [48]

$$\begin{aligned}\mathcal{E}^{(1)}_{(0,0)\epsilon} &= 2\pi \int_{\mathcal{F}_1} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} E^{SL(2)}_{\epsilon\Lambda_1} \Gamma^{(1)}_{I\!I_{d,d}} = 4\pi\xi(2\epsilon) E^{SL(2)}_{\epsilon\Lambda_1}(T) E^{SL(2)}_{\epsilon\Lambda_1}(U) \\ &= -\frac{2\pi}{\epsilon} + 2\pi(\gamma_E - \ln 4\pi) - 2\pi \ln\left(\operatorname{Im} U |\eta(U)|^4\right) - 2\pi \ln\left(\operatorname{Im} T |\eta(T)|^4\right) + \mathcal{O}(\epsilon)\,,\end{aligned} \tag{3.5.54}$$

where we have used (4.1.60) along with the isomorphism $SO(2,2) = SL(2) \times SL(2)$.[10] We have also defined $U = x_2 + iy_1/y_2 \in \mathcal{H}_1$ and $T = x_1 + iy_1y_2 \in \mathcal{H}_1$ and $\eta(\tau)$ is the Dedekind eta function defined by

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2n\pi i\tau}\right) = \sum_{n\in\mathbb{Z}} e^{3\pi i\tau(n+\frac{1}{6})^2 + i\pi n}\,, \tag{3.5.55}$$

for any $\tau \in \mathcal{H}_1$ in the upper complex half plane. If we consider that we have obtained the 8 dimensional theory by compactifying type IIA string theory on a torus T^2 then U can be interpreted as the modulus of T^2. $y_1 = R_2/l_s$ and $y_2 = R_1/l_s$ are then the radii of both circles of the type IIA torus in string length units. Therefore we can see that the poles in $\epsilon = 0$ indeed cancel out in the total amplitude. In total we have

$$\begin{aligned}f^{\text{string}}_{8,1}(s,t,u;\varphi) =& (4\pi)^5 (\hat{I}^{(1)}_{4,1}(\alpha' s, \alpha' t) + \hat{I}^{(1)}_{4,1}(\alpha' t, \alpha' u) + \hat{I}^{(1)}_{4,1}(\alpha' u, \alpha' s)) \\ & - 2\pi \ln\left(\operatorname{Im} U |\eta(U)|^4\right) - 2\pi \ln\left(\operatorname{Im} T |\eta(T)|^4\right) + \frac{22\pi}{3} - 4\pi \ln 2\pi \\ & + \text{other terms} \\ =& (4\pi)^5 (\hat{I}^{(1)}_{4,1/l_s}(s,t) + \hat{I}^{(1)}_{4,1/l_s}(t,u) + \hat{I}^{(1)}_{4,1/l_s}(u,s)) \\ & - 2\pi \ln\left(\operatorname{Im} U |\eta(U)|^4\right) - 2\pi \ln\left(\operatorname{Im} T |\eta(T)|^4\right) \\ & + \frac{4\pi}{3} \ln g_8 + \frac{22\pi}{3} - 4\pi \ln\left(2\pi \frac{l_P}{l_s}\right) + \text{other terms}\,,\end{aligned} \tag{3.5.56}$$

where in the last step we have used that $l_P = g_8^{\frac{1}{3}} l_s$. Hence we define the renormalised function

$$\mathcal{E}^{(1)}_{(0,0)\mu} = -2\pi \ln\left(\operatorname{Im} U |\eta(U)|^4\right) - 2\pi \ln\left(\operatorname{Im} T |\eta(T)|^4\right) + \frac{4\pi}{3} \ln g_8 + \frac{22\pi}{3} - 4\pi \ln(2\pi l_P \mu)\,. \tag{3.5.57}$$

[10] This is not strictly correct but in this thesis we will mostly neglect factors of $\mathbb{Z}_2$ and topological issues relative to universal covers and connected components unless stated otherwise.

Note that this has modular invariance under $SL(2,\mathbb{Z})$ on both arguments U and T. The isomorphism $SO(2,2) = SL(2) \times SL(2)$ tells us that the function actually has $SO(2,2,\mathbb{Z})$ invariance. If we further add the invariance under the exchange of T and U, which corresponds to $R_1 \longleftrightarrow l_s^2/R_1$, we get the full $O(2,2,\mathbb{Z})$ invariance which again corresponds to the invariance under T-duality of the amplitude. Therefore we can interpret T as the modulus of the type IIB torus. Finally the full renormalised function is

$$\begin{aligned} f_{8,1}^{\text{string}}(s,t,u;\varphi) =& (4\pi)^5(\hat{I}_{4,\mu}^{(1)}(s,t) + \hat{I}_{4,\mu}^{(1)}(t,u) + \hat{I}_{4,\mu}^{(1)}(u,s)) + \mathcal{E}_{(0,0)\mu}^{(1)} \\ &+ \text{other terms}\,. \end{aligned} \tag{3.5.58}$$

Hence we have seen that even though supergravity amplitudes suffer from divergences, string amplitudes offer a completion of these amplitudes which are finite at all orders. Indeed the analytic contributions coming from the infinite tower of massive string states cancel the non renormalisable loop divergences. Since the divergent supergravity amplitude come from the singular tropical limit of the worldsheet, we can interpret the finiteness of the total string amplitudes as resulting from the smoothness of the string worldsheet. Furthermore we have seen that the finite string amplitudes offer a non ambiguous way to define renormalised quantities in the low energy limit.

3.5.3 Genus two amplitude low energy limit

This subsection closely follows subsection 2.3 of [92]. At the two loop level on the supergravity side we actually have two Feynman graphs contributing, see figure 3.6 [72, 89].[11]

There is the planar diagram (depicted on the left in figure 3.6) which gives the integral

$$\begin{aligned} f_{D,2,P}^{\text{sugra}}(s,t,u) = s^2(I_{4,P}^{(2)}(s,t) + I_{4,P}^{(2)}(s,u)) + t^2(I_{4,P}^{(2)}(t,u) + I_{4,P}^{(2)}(t,s)) \\ + u^2(I_{4,P}^{(2)}(u,s) + I_{4,P}^{(2)}(u,t))\,, \end{aligned} \tag{3.5.59}$$

with

[11] Again the two-loop four-graviton amplitude in maximally supersymmetric supergravity continues to have the feature that it can be written in terms of ϕ^3 scalar field theory diagrams [89].

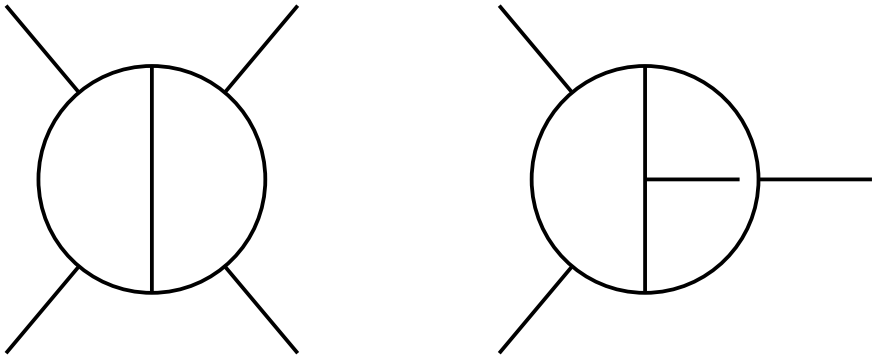


Figure 3.6: Four graviton two loop planar diagram (left) and non planar diagram (right) in maximal supergravity. Gravitons are represented by straight lines.

$$I^{(2)}_{4,P}(s,t) = \int \frac{d^D p d^D q}{(2\pi)^{2D}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_4)^2(q-k_3-k_4)^2}\,, \tag{3.5.60}$$

where again, by convention, the loop momenta are assumed to be euclidean. In Schwinger parameter space this is

$$I^{(2)}_{4,P}(s,t) = \frac{1}{128(2\pi)^{2D-7}} \int_0^\infty dL_1 dL_2 dL_3 \, \frac{L_1^2 L_2^2}{\Delta^{\frac{D}{2}}} \int_0^1 dx_2 dx_4 \int_0^{x_2} dx_1 \int_0^{x_4} dx_3 \\ e^{\pi \frac{L_1 L_2 L_3}{\Delta}[(1-x_1-x_4)(1-x_2-x_3)s + (x_2-x_1)(x_4-x_3)t] + \pi L_1 x_1 (1-x_2)s + \pi L_2 x_3 (1-x_4)s}\,, \tag{3.5.61}$$

where $\Delta = L_1L_2 + L_1L_3 + L_2L_3$. And there is the non planar diagram (depicted on the right in figure 3.6) which gives the integral

$$f^{\text{sugra}}_{D,2,NP}(s,t,u) = s^2(I^{(2)}_{4,NP}(s,t) + I^{(2)}_{4,NP}(s,u)) + t^2(I^{(2)}_{4,NP}(t,u) + I^{(2)}_{4,NP}(t,s)) \\ + u^2(I^{(2)}_{4,NP}(u,s) + I^{(2)}_{4,NP}(u,t))\,, \tag{3.5.62}$$

with

$$I^{(2)}_{4,NP}(s,t) = \int \frac{d^D p d^D q}{(2\pi)^{2D}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2(p+q+k_3)^2q^2(q-k_4)^2}\,. \tag{3.5.63}$$

In Schwinger parameter space this is

$$I^{(2)}_{4,NP}(s,t) = \frac{1}{128(2\pi)^{2D-7}} \int_0^\infty dL_1 dL_2 dL_3 \, \frac{L_1^2 L_2 L_3}{\Delta^{\frac{D}{2}}} \int_0^1 dx_2 dx_3 dx_4 \int_0^{x_2} dx_1 \\ e^{\pi \frac{L_1 L_2 L_3}{\Delta}[(1-x_1-x_4)(1-x_2-x_3)s + (x_2-x_1)(x_4-x_3)t] + \pi L_1 x_1 (1-x_2)s} \,. \tag{3.5.64}$$

The full one loop function is given by

$$f^{\text{sugra}}_{D,2}(s,t,u) = f^{\text{sugra}}_{D,2,P}(s,t,u) + f^{\text{sugra}}_{D,2,NP}(s,t,u) \,. \tag{3.5.65}$$

The integrals suffer from IR divergences in $D \leq 4$ and from UV divergences for $D \geq 7$, therefore they must be dimensionally regularised. $D = 4$ and $D = 7$ are the critical dimensions where the logarithmic divergences introduce poles in $\epsilon = 0$.

At genus 2 on the string side we have

$$f^{\text{string}}_{D,2}(s,t,u;\varphi) = \frac{\pi}{128} \int_{\mathcal{F}_2} \frac{d^6\Omega}{(\det \operatorname{Im} \Omega)^5} \Gamma^{(2)}_{I\!I_{d,d}} \int_{\Sigma^4} \mathcal{Y}_S \wedge \overline{\mathcal{Y}}_S \exp\left(-\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j \, G(z_i, z_j) \right) , \tag{3.5.66}$$

where

$$\Gamma^{(2)}_{I\!I_{d,d}}(\Omega) = (\det \operatorname{Im} \Omega)^{\frac{d}{2}} \sum_{q_i \in I\!I_{d,d}} e^{-\pi (\operatorname{Im}\Omega)^{ij} H_{G,B}[q_i,q_j] + i\pi (\operatorname{Re}\Omega)^{ij}(q_i,q_j)} \,. \tag{3.5.67}$$

Let us parametrise Ω as

$$\Omega = \begin{pmatrix} \rho & u_1 + \rho u_2 \\ u_1 + \rho u_2 & \varsigma + \rho u_2^2 \end{pmatrix} , \tag{3.5.68}$$

with $\rho, \varsigma \in \mathbb{C}$ and $u_1, u_2 \in \mathbb{R}$. The dimensionless volume of the genus 2 worldsheet is then given by $\det \operatorname{Im} \Omega = (\operatorname{Im} \rho)(\operatorname{Im} \varsigma)$. It is convenient to introduce the cutoff parameters $\Lambda_1 \leq \Lambda$ to split the moduli space into three pieces: the truncated fundamental domain $\mathcal{F}_{2,\Lambda}$ where $\operatorname{Im} \varsigma < \Lambda$, the non separating degeneration intermediate region where $\operatorname{Im} \varsigma \geq \Lambda$ and $\operatorname{Im} \rho < \Lambda_1$ and the tropical region where $\operatorname{Im} \varsigma \geq \Lambda$ and $\operatorname{Im} \rho \geq \Lambda_1$ [108]. This also splits the moduli space integral in (3.5.66) into $f^{\text{string}}_{D,2,<\Lambda}$, $f^{\text{string}}_{D,2,<\Lambda_1>\Lambda}$ and $f^{\text{string}}_{D,2,>\Lambda_1}$. Let us first focus on the first part

$$f^{\text{string}}_{D,2,<\Lambda}(s,t,u;\varphi) = \frac{\pi}{128}\int_{\mathcal{F}_{2,\Lambda}} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^5}\Gamma^{(2)}_{I\!I_{d,d}} \int_{\Sigma^4} \mathcal{Y}_S \wedge \overline{\mathcal{Y}}_S \times \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{\alpha'}{2}\sum_{i<j} k_i\cdot k_j\, G(z_i,z_j)\right)^n . \tag{3.5.69}$$

We can expand the Koba–Nielsen factor to get

$$f^{\text{string}}_{D,2,<\Lambda}(s,t,u;\varphi) = \sum_{p,q\geq 0}\left(\frac{\alpha'}{4}\right)^{2p+3q} \mathcal{E}^{(2)}_{(p,q)\Lambda}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q\,, \tag{3.5.70}$$

with

$$\mathcal{E}^{(2)}_{(p,q)\Lambda} = 2\pi\int_{\mathcal{F}_{2,\Lambda}} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^3}\Gamma^{(2)}_{I\!I_{d,d}}\mathcal{B}^{(2)}_{(p,q)}\,, \tag{3.5.71}$$

where the first graph functions are [72, 76, 77]

$$\mathcal{B}^{(2)}_{(0,0)} = 0\,, \quad \mathcal{B}^{(2)}_{(1,0)} = 2\,. \tag{3.5.72}$$

The next graph functions are studied in [72, 104, 109, 110]. As in the genus 1 case we take $\Lambda \gg 1$ and by introducing an analytic continuation $d \to d+2\epsilon$ we can write the divergent part of the integral (3.5.71) as a sum of powers of Λ whereas the cut-off independent part is given by

$$\mathcal{E}^{(2)}_{(p,q)\epsilon} = 2\pi\int_{\mathcal{F}_2} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^3}(\det \operatorname{Im}\Omega)^{\epsilon}\Gamma^{(2)}_{I\!I_{d,d}}\mathcal{B}^{(2)}_{(p,q)}\,, \tag{3.5.73}$$

which is also defined by analytic continuation in ϵ around $\epsilon = 0$. The two parts are individually well defined for finite Λ in every dimension except $d = 3$. Note that the graph functions are invariant under $PSp(4,\mathbb{Z})$ which ensures that the amplitude does not depend on the representative $\mathcal{F}_2$ of $U(2)\backslash Sp(4,\mathbb{R})/PSp(4,\mathbb{Z})$. In the limit $\epsilon \to 0$ we have that the first couplings are given by

$$\begin{aligned} &\mathcal{E}^{(2)}_{(0,0)\epsilon} = 0\,, \\ &\mathcal{E}^{(2)}_{(1,0)\epsilon} = \frac{2\pi^4}{135}\left(E^{SO(d,d)}_{2\Lambda_d\epsilon} + E^{SO(d,d)}_{2\Lambda_{d-1}\epsilon}\right)\,. \qquad (d\leq 4 \text{ and } d\neq 3) \end{aligned} \tag{3.5.74}$$

Again Eisenstein series are defined later in chapter 4 but for now one can notice that they are invariant under $O(d,d,\mathbb{Z})$, which reflects the invariance under T-duality of the amplitude. For $d \leq 4$ the Eisenstein series have poles but in the limit $\epsilon \to 0$ the poles in ϵ cancel between the two functions (for $d \neq 3$). For $d = 0$ the Eisenstein series are not defined but we can see that $\mathcal{E}^{(2)}_{(1,0)\epsilon} = 4\pi\mu(\mathcal{F}_2) = \frac{2\pi^4}{135}$ where $\mu(\mathcal{F}_2) = \frac{\pi^3}{270}$ is the volume of the fundamental domain $\mathcal{F}_2$. For $d = 3$ the limit $\epsilon \to 0$ diverges for $\mathcal{E}^{(2)}_{(1,0)\epsilon}$, this is simply due to the logarithmic divergence in (3.5.73). We will show at the end of the section how to regularise this and calculate $\mathcal{E}^{(1)}_{(1,0)\epsilon}$.

Let us now turn on the integral on the tropical region.

$$f^{\text{string}}_{D,2,>\Lambda_1}(s,t,u;\varphi) = \frac{\pi}{128}\int_{\substack{\mathcal{F}_2 \\ \operatorname{Im}\varsigma\geq\Lambda \\ \operatorname{Im}\rho\geq\Lambda_1}} \frac{d^6\Omega}{(\det \operatorname{Im}\Omega)^5}\Gamma^{(2)}_{I\!I_{d,d}}\int_{\Sigma^4}\mathcal{Y}_S\wedge\overline{\mathcal{Y}}_S \times \exp\left(-\frac{\alpha'}{2}\sum_{i<j}k_i\cdot k_j\,G(z_i,z_j)\right). \tag{3.5.75}$$

We have $\det \operatorname{Im}\Omega \geq \Lambda\Lambda_1 \gg 1$ therefore this part of the moduli space integral is restricted to the Riemann surfaces with large dimensionless volume. This justifies using the tropical limit and tropical geometry. Let us parametrise $\operatorname{Im}\Omega$ as

$$\operatorname{Im}\Omega = \frac{1}{\alpha'}\begin{pmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{pmatrix}. \tag{3.5.76}$$

Then in the tropical limit L_1, L_2 and L_3 can be interpreted as the lengths of the three lines in the degenerate Riemann surface representing the two loop vacuum Feynman diagram given in figure 3.7. We choose the ordering $0 \leq L_3 \leq L_1 \leq L_2$.

If we focus on the generic case where the punctures are far away from the branching points we also have that the one forms ω^i are locally constant and have support on the b-cycle B^i. For a path γ from z to z' one can define the dimensionless geometric length $d(\gamma)$. For example, for the closed path B^1 one has by definition $d(B^1) = \alpha' L_1 + \alpha' L_3$ and for B^2 one has $d(B^2) = \alpha' L_2 + \alpha' L_3$. This distance depends on the chosen path, however in the tropical limit this choice is limited, up to closed loops, to a finite number. The choice basically amounts to choosing around which line to go at the branching points. Therefore, for a given path that does not include closed loops there always exists $\alpha_i \in \{-1,0,1\}$ such that

$$\alpha_i \operatorname{Im}\int_\gamma \omega^i = d(\gamma). \tag{3.5.77}$$

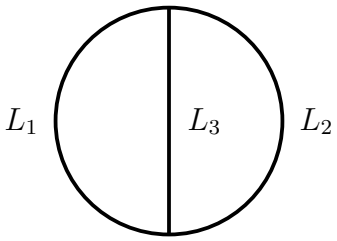


Figure 3.7: Tropical limit of the genus 2 Riemann surface representing the two loop vacuum diagram of maximal supergravity. L_1, L_2 and L_3 represent the lengths of each line.

If γ is a shortest path from z to z', it is convenient to use an odd spin structure $\nu = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ such that $a_i = |\alpha_i|$ and $a_i b^i = 1$ to compute the prime form (3.3.24). Up to exponentially suppressed contributions, the tropical limit of the prime form is given by

$$E(z,z') = \frac{2i \sin\left(\pi \alpha_i \int_\gamma \omega^i\right)}{2\pi \sqrt{\alpha_i \omega^i(z) \alpha_j \omega^j(z')}} + \dots \tag{3.5.78}$$

Then the tropical limit of the propagator in the Koba-Nielsen factor is given by [106]

$$\begin{aligned} G(z,z') =& 2\pi \left(-d(\gamma) + (\operatorname{Im}\Omega)^{-1}_{ij} \left(\operatorname{Im} \int_\gamma \omega^i \right) \left(\operatorname{Im} \int_\gamma \omega^i \right) \right) \\ & + 2\ln 2\pi - \ln \left| 1 - e^{-2\pi d(\gamma) + 2\pi i x_\gamma} \right|^2 + \dots \end{aligned} \tag{3.5.79}$$

up to exponentially suppressed terms, where $x_\gamma \in [0,1]$ represents a phase which is irrelevant here. Indeed we can also focus on a generic point in moduli space where all worldsheet punctures are far from each other and the distances $d(\gamma)$ for paths γ from z_i to z_j are far from 0, we call the moduli space integral over such points $f^{\text{string}}_{D,2,\mathbb{O}>\Lambda_1}$. In this case the logarithmic term can also be neglected.

The properties of the holomorphic 4-form (3.3.22) imply that the only contributions in the tropical limit are when at least two punctures are on each b-cycle and at most two on $B^1 \cap B^2$. This implies that there must be two punctures on one of the three lines and the two others can be either on one or two other lines. This gives us permutations of the planar and non planar diagrams of supergravity. there are 18 choices of planar diagrams and for each we may choose four different orderings of punctures on two lines, so 72 ordered choices in total. There are 36 choices of

non-planar diagrams and for each one may choose two different orderings of punctures on one line, so 72 ordered choices in total. It is convenient to absorb the six permutations of the lines in an unfolding of $\mathcal{F}_2$ to the Siegel upper half plane $L_I > 0$ corresponding to the Schwinger parameter space of the vacuum diagram. Therefore L_1, L_2 and L_3 are not ordered anymore.

The tropical region is now determined by $\alpha'^2 \det \operatorname{Im} \Omega = L_1L_2 + L_1L_3 + L_2L_3 = \Delta \geq \alpha'^2\Lambda\Lambda_1$ and $L_I + L_J \geq \alpha'\Lambda_1$ for all pairs I, J. There is now a single integral for the planar and the non-planar diagrams that we must sum over the six permutations of the Mandelstam variables.

$$f^{\text{string}}_{D,2,\mathbb{O}>\Lambda_1}(s,t,u;\varphi) = \frac{\pi}{4}\alpha'^{5-d} \int_{\substack{L_I>0 \\ \Delta\geq\alpha'^2\Lambda\Lambda_1 \\ L_I+L_J\geq\alpha'\Lambda_1}} dL_1dL_2dL_3 \, \frac{L_1^2}{\Delta^{5-\frac{d}{2}}} \int_0^1 dx_2 \int_0^{x_2} dx_1$$
$$s^2 e^{\pi L_1 x_1(1-x_2)s} \left(\int_0^1 dx_4 \int_0^{x_4} dx_3 \, L_2^2 e^{\pi \frac{L_1L_2L_3}{\Delta}[(1-x_1-x_4)(1-x_2-x_3)s+(x_2-x_1)(x_4-x_3)t]+\pi L_2x_3(1-x_4)s} \right.$$
$$\left. + \int_0^1 dx_4dx_3 \, L_2L_3 e^{\pi \frac{L_1L_2L_3}{\Delta}[(1-x_1-x_4)(1-x_2-x_3)s+(x_2-x_1)(x_4-x_3)t]} \right) + \text{perm.} \tag{3.5.80}$$

This is simply the supergravity integral in Schwinger parameter space with a UV cutoffs $\alpha'\Lambda$ and $\alpha'\Lambda_1$. Therefore this integral is well defined for all space-time dimensions $D \geq 5$ which corresponds to $d \leq 5$ whereas for $D = 4$ corresponding to $d = 6$ one must introduce some IR regularisation. Again as in the genus 1 case we take $\Lambda, \Lambda_1 \ll 1/\alpha' s, 1/\alpha' t, 1/, \alpha' u$ and by introducing an analytic continuation $d \to d + 2\epsilon$ we can write the divergent part of the integral as a sum of powers of Λ and Λ_1 whereas the cut-off independent part is given by

$$16(2\pi)^{14-2d}\alpha'^{5-d} \left(s^2(I^{(2)}_{4,P,\epsilon}(s,t) + I^{(2)}_{4,P,\epsilon}(s,u) + I^{(2)}_{4,NP,\epsilon}(s,t) + I^{(2)}_{4,NP,\epsilon}(s,u)) \right.$$
$$+ t^2(I^{(2)}_{4,P,\epsilon}(t,u) + I^{(2)}_{4,P,\epsilon}(t,s) + I^{(2)}_{4,NP,\epsilon}(t,u) + I^{(2)}_{4,NP,\epsilon}(t,s))$$
$$\left. + u^2(I^{(2)}_{4,P,\epsilon}(u,s) + I^{(2)}_{4,P,\epsilon}(u,t) + I^{(2)}_{4,NP,\epsilon}(u,s) + I^{(2)}_{4,NP,\epsilon}(u,t)) \right), \tag{3.5.81}$$

where we have defined

$$I^{(2)}_{4,P,\epsilon}(s,t) = \int \frac{d^{10-d}p d^{10-d}q}{(2\pi)^{20-2d}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2q^2(q-k_4)^2(q-k_3-k_4)^2}$$
$$I^{(2)}_{4,NP,\epsilon}(s,t) = \int \frac{d^{10-d}p d^{10-d}q}{(2\pi)^{20-2d}} \frac{1}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+q)^2(p+q+k_3)^2q^2(q-k_4)^2}\,, \tag{3.5.82}$$

by analytic continuation in ϵ around $\epsilon = 0$.

When all contributions from $f^{\text{string}}_{D,2,<\Lambda}$, $f^{\text{string}}_{D,2,<\Lambda>\Lambda_1}$ and $f^{\text{string}}_{D,2,>\Lambda_1}$ are included on can show that all Λ and Λ_1 dependent terms cancel out. Therefore we can forget about the cutoff parameters and the complete integral can be written for $D \geq 5$ as the limit $\epsilon \to 0$

$$\begin{aligned} f^{\text{string}}_{D,2}(s,t,u;\varphi) &= 16(2\pi)^{14-2d}\alpha'^{5-d} \\ &\times \Big(s^2(I^{(2)}_{4,P,\epsilon}(s,t) + I^{(2)}_{4,P,\epsilon}(s,u) + I^{(2)}_{4,NP,\epsilon}(s,t) + I^{(2)}_{4,NP,\epsilon}(s,u)) \\ &+ t^2(I^{(2)}_{4,P,\epsilon}(t,u) + I^{(2)}_{4,P,\epsilon}(t,s) + I^{(2)}_{4,NP,\epsilon}(t,u) + I^{(2)}_{4,NP,\epsilon}(t,s)) \\ &+ u^2(I^{(2)}_{4,P,\epsilon}(u,s) + I^{(2)}_{4,P,\epsilon}(u,t) + I^{(2)}_{4,NP,\epsilon}(u,s) + I^{(2)}_{4,NP,\epsilon}(u,t)) \Big) \\ &+ \sum_{p,q\geq 0} \left(\frac{\alpha'}{4}\right)^{2p+3q} \mathcal{E}^{(2)}_{(p,q)\epsilon}(s^2+t^2+u^2)^p(s^3+t^3+u^3)^q + \text{other terms}\,, \end{aligned} \tag{3.5.83}$$

where the limit is well defined for each term individually for all dimensions except $d = 3$ corresponding to spacetime dimensions $D = 7$. This result also holds in $D = 4$ provided one introduces an IR regulator.

We can see from (3.5.59), (3.5.62), (3.5.65) and (3.5.83) that, again, the low energy limit of the genus 2 string amplitude reproduces the dimensionally regularised two loop supergravity amplitude if we make the identification $2\kappa_D^2 = (2\pi)^{7-d}\alpha'^{\frac{8-d}{2}} g_D^2$. This corresponds to the tropical limit of the string worldsheet that looks like the two loop supergravity Feynman diagrams. The other terms coming from $f^{\text{string}}_{D,2,>\Lambda_1}$ with coincident points include massive string mode propagators and correspond to tropical limits of the worldsheet looking like a two loop Feynman diagram with the insertion of a genus 0 string world sheet. Finally the terms coming from the integral on the non separating degeneration intermediate region $f^{\text{string}}_{D,2,<\Lambda>\Lambda_1}$ correspond to tropical limits

of the worldsheet looking like a one loop Feynman diagram with the insertion of a genus 1 string world sheet. They are suppressed by powers of α' with respect to the two loop supergravity integral, the leading $\mathcal{E}^{(2)}_{(0,0)\epsilon}$ term for $D \geq 3$ and the next to leading $\mathcal{E}^{(2)}_{(1,0)\epsilon}$ term for $D \geq 7$.

Again these corrections can be seen as arising from the effective field theory action in string frame

$$S^{\text{string}} = \frac{1}{2\kappa_D^2} \int d^D x \, \sqrt{-g} \, e^{-2\phi} \times \left(R + \frac{l_s^6}{48} \sum_{p,q \geq 0} l_s^{4p+6q} \left(c_{(p,q)} + g_D^2 e^{2\phi} \mathcal{E}^{(1)}_{(p,q)\epsilon} + g_D^4 e^{4\phi} \mathcal{E}^{(2)}_{(p,q)\epsilon} \right) \nabla^{4p+6q} \mathcal{R}^4 + \dots \right) , \tag{3.5.84}$$

where again we have assumed that we have absorbed the vacuum expectation value ϕ_0 of the dilaton into the string coupling constant.

Chapter 4

U-duality and automorphic forms

Even though perturbation theory is very useful and allows us to deal with non trivial interactions it can only give us an incomplete picture. Indeed by definition one can only compute observables by truncating the perturbation series to a finite order. An example includes the Wilson coefficients of (3.5.84) which could potentially receive contributions from all orders of perturbation. One tool which may help us to go beyond perturbation theory is symmetry. Indeed when the symmetry group of a theory is large enough[1] it may allow us to bootstrap observables, sometimes to all orders in perturbation theory and even non perturbatively. This idea has enjoyed great success in the conformal bootstrap program for example where the symmetry used is conformal invariance [111]. Supersymmetry is another type of symmetry that entails strong constraints. For example it can protect some quantities from renormalisation, as we will see this is actually the case for the first few Wilson couplings in the effective action type II string theory [47, 76].

This justifies why we have been mainly interested in maximal supersymmetry. Indeed with such a large amount of symmetry at our disposal one can hope to constrain the effective field theory expansion perhaps even uniquely. On the string theory side maximal supersymmetry also comes with added structure: U-duality.[2] U-duality is a non perturbative discrete symmetry of string theory which acts on the moduli fields. In particular it is a very strong tool to constraint and study the Wilson coefficient of the effective field theory.

We begin this chapter with a quick reminder of automorphic forms. We take special interest in Eisenstein series, especially minimal and next to minimal Eisenstein series for $SL(n)$ and $SO(n,n)$ groups. We give their Fourier expansions on parabolic

[1] Usually Poincaré symmetry is insufficient

[2] U-duality is not restricted to maximal or even extended supersymmetry but it is the case where the U-duality group is the most extended. We will focus exclusively on the maximally supersymmetric case in this thesis.

subgroups. We then review U-duality for maximally supersymmetric string theory with the specific example of type IIB string theory. Finally we show how U-duality and supersymmetry allows us compute the exact leading Wilson coefficients in the effective field theory limit.

4.1 Automorphic forms

Automorphic forms and automorphic representations are a whole subject of study in the mathematical community [112–125] most of which is much beyond the scope of this thesis. In this thesis we will only need a few basic notions and therefore we will not develop the subject in full generality. Note that our conventions, notations and terminology do not always necessarily match that of the literature in the mathematical community, nevertheless we will try to be coherent and self contained with our exposition. See [126] for a short mathematical review.

Roughly speaking an automorphic form is a well behaved function on a coset space G/Γ where G is some Lie group manifold and Γ is typically an infinite discrete subgroup of G. More specifically it is a smooth function f from G to some finite dimensional complex vector space V which is invariant under the action of Γ, i.e.

$$f(g\gamma) = f(g)\,, \tag{4.1.1}$$

for all $\gamma \in \Gamma$. Automorphic forms can also transform covariantly under the action of some compact subgroup K of G. Indeed let $\rho : K \to GL(V)$ be a representation of K acting on V then we say that f transforms in the representation ρ of K if

$$f(kg) = \rho(k)f(g)\,, \tag{4.1.2}$$

for all $k \in K$. We will be particularly interested in the case where K is a maximal compact subgroup of G.

Symmetric spaces Let G be a semi-simple connected Lie group and K be a maximal compact subgroup of G such that $K\backslash G$ is a Riemannian symmetric space of noncompact type. This means that the Lie algebra $\mathfrak{g}$ of G can be decomposed into the direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is its orthogonal complement and that

$$[\mathfrak{k},\mathfrak{k}] \subseteq \mathfrak{k}\,, \qquad [\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}\,, \qquad [\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}\,. \tag{4.1.3}$$

This is called a Cartan decomposition. The first condition simply says that K is a Lie subgroup of G and is automatic for any homogeneous space. The second condition

makes $K\backslash G$ into a reductive homogeneous space. The third condition however really characterises symmetric spaces.

The Maurer-Cartan form on G defines a principal connection on G seen as principal bundle over a trivial base manifold, it also defines a Cartan connection on the symmetric space $K\backslash G$. Indeed as a $\mathfrak{g}$-valued one form it can also be decomposed into

$$dg\, g^{-1} = Q + P\,, \tag{4.1.4}$$

where Q is a $\mathfrak{k}$-valued one form and P is a $\mathfrak{p}$-valued one form. Seeing $K\backslash G$ as a smooth manifold of dimension $\dim G - \dim K$ one can interpret $\mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ as the tangent space of $K\backslash G$ at the identity e and also at any other point by left-right translation invariance. Therefore P can be seen as a vector valued one form. Actually, one can show that P defines a vielbein $E = P$ on $K\backslash G$.

Taking the exterior derivative of the Maurer-Cartan form we get

$$d(dg\, g^{-1}) = -dg \wedge dg^{-1} = dg\, g^{-1} \wedge dg\, g^{-1}\,. \tag{4.1.5}$$

This is known as the Maurer-Cartan equation. Therefore we have

$$d(P+Q) = (P+Q) \wedge (P+Q)\,. \tag{4.1.6}$$

Decomposing this last equation on $\mathfrak{k}$ and $\mathfrak{p}$ this gives us the two equations

$$\begin{aligned} dQ &= Q \wedge Q + P \wedge P\,, \\ dP &= Q \wedge P + P \wedge Q\,. \end{aligned} \tag{4.1.7}$$

The second equation can be interpreted as a Cartan structure equation for torsion

$$dE + [-Q, E] = 0\,. \tag{4.1.8}$$

Therefore $-Q$ can be interpreted as a spin connection $\omega = -Q$ on $K\backslash G$. Hence the first equation can be interpreted as a Cartan structure equation for curvature

$$d\omega + \frac{1}{2}[\omega, \omega] = -P \wedge P\,. \tag{4.1.9}$$

Therefore $-P \wedge P$ can be interpreted as a curvature form $R = -P \wedge P$ on $K\backslash G$. Hence, in the basis of the vielbeins the curvature form $R = -E \wedge E$ is constant.

For some automorphic form f we can define the action of the Lie algebra $\mathfrak{g}$ on f through the left-action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}$ through

$$(X \cdot f)(g) = \left(\frac{d}{dt} f(e^{tX} g) \right) \Big|_{t=0} , \tag{4.1.10}$$

for all $X \in \mathfrak{g}$. For $X \in \mathfrak{p}$ this coincides with the covariant derivative on $K\backslash G$

$$(X \cdot f)(g) = \mathcal{D}_X f(g) , \tag{4.1.11}$$

whereas for $X \in \mathfrak{k}$ this is simply the action of $\mathfrak{k}$ on f induced by the representation ρ of K

$$(X \cdot f)(g) = D\rho(X) f(g) . \tag{4.1.12}$$

Let $\{t_\alpha\}$ and $\{t_a\}$ be a basis of $\mathfrak{k}$ and $\mathfrak{p}$ respectively. We can decompose P and Q as $P = P^a_\mu t_a dx^\mu$ and $Q = Q^\alpha_\mu t_\alpha dx^\mu$ so that the vielbein and the spin connection are written in components as $E^a_\mu = P^a_\mu$ and $\omega^a_{\mu\, b} = Q^\gamma_\mu f^a{}_{b\gamma}$ respectively where $f^a{}_{b\gamma}$ are structure constants of $\mathfrak{g}$. We can therefore write the metric on $K\backslash G$ as

$$G_{\mu\nu} = \kappa_{ab} P^a_\mu P^b_\nu , \tag{4.1.13}$$

where κ_{ab} are coordinates of the Killing form on $\mathfrak{g}$. The curvature tensor on $K\backslash G$ is written in the vielbein basis as

$$R_{cd}{}^a{}_b = f^\gamma{}_{cd} f^a{}_{b\gamma} . \tag{4.1.14}$$

Therefore the curvature tensor is indeed constant in the vielbein basis. By the vielbein postulate which states that the vielbein are covariantly constant this implies that the curvature tensor in coordinate basis is covariantly constant as well. This also characterises symmetric spaces.

We can also give explicitly the action of the covariant derivative on some automorphic form f in components

$$\mathcal{D}_a f(g) = \kappa_{ab} G^{\mu\nu} P^b_\mu \left(\partial_\nu + \rho(\omega_\nu) \right) f(g) . \tag{4.1.15}$$

For any element in the centre $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}$ one can define a left-right invariant differential operator. The canonical example is the quadratic Casimir, which defines the Laplace–Beltrami operator

$$\Delta = \kappa^{ab} \mathcal{D}_a \mathcal{D}_b . \tag{4.1.16}$$

More generally, one can define the centre $\mathcal{Z}(\mathfrak{g})$ as the set of K-invariant polynomials in the covariant derivative $\mathcal{D}_a$ generated by the Casimir operators of $\mathfrak{g}$.

Parabolic subgroups Let G be a semi-simple Lie group of rank r with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Let $\alpha \in \mathfrak{h}^*$ be some element in the dual of the Cartan subalgebra, we define $\mathfrak{g}_\alpha$ as the subspace of $\mathfrak{g}$ given by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h},\ \mathrm{ad}\, H\, X = \alpha(H)X\}\,. \tag{4.1.17}$$

The $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha$ is non trivial are called roots of the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the root space associated to α. Root spaces are all of dimension 1. Let R be the set of all roots which, one can show, forms a lattice and let Δ be the set of all simple roots, i.e. generators of the root lattice. We define R^+ the subset of positive roots spanned by Δ, i.e. positive integer combinations of the simple roots (the set of negative roots is then $-R^+$). We then define the Borel subalgebra $\mathfrak{b}$ by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha\,. \tag{4.1.18}$$

The Borel subgroup B of G is the subgroup generated by the Borel subalgebra $\mathfrak{b}$. Consider a partition of the set of simple roots Δ into disjoint sets Δ_1 and Δ_2 such that $\Delta = \Delta_1 \sqcup \Delta_2$. Let R_1^+ be the set of positive roots spanned by Δ_1 and R_2^+ the set of positive roots spanned by Δ_2. We define the parabolic subalgebra $\mathfrak{p}_{\Delta_2}$ associated with the set of positive roots R_1^+ as

$$\mathfrak{p}_{\Delta_2} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \cup (-R_1^+)} \mathfrak{g}_\alpha\,. \tag{4.1.19}$$

Clearly if $\Delta_2 \subset \tilde{\Delta}_2$ then $\mathfrak{p}_{\tilde{\Delta}_2} \subset \mathfrak{p}_{\Delta_2}$. A parabolic subgroup P_{Δ_2} of G is a subgroup generated by the parabolic subalgebra $\mathfrak{p}_{\Delta_2}$.

Any parabolic subgroup P_{Δ_2} admits a Levi decomposition $P_{\Delta_2} = L_{\Delta_2} U_{\Delta_2}$ into a reductive subgroup L_{Δ_2} called a Levi subgroup and a unipotent radical U_{Δ_2}.[3] In Lie algebra terms we have $\mathfrak{p}_{\Delta_2} = \mathfrak{l}_{\Delta_2} \oplus \mathfrak{n}_{\Delta_2}$ where $\mathfrak{l}_{\Delta_2}$ is Lie algebra of the Levi factor and $\mathfrak{n}_{\Delta_2}$ is the Lie algebra of the unipotent radical (which is nilpotent as a Lie subalgebra). Their expressions are given explicitly by

$$\mathfrak{l}_{\Delta_2} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_1^+ \cup (-R_1^+)} \mathfrak{g}_\alpha\,, \qquad \text{and} \qquad \mathfrak{n}_{\Delta_2} = \bigoplus_{\alpha \in R_2^+} \mathfrak{g}_\alpha\,. \tag{4.1.20}$$

[3] Strictly speaking this is known as a Langlands decomposition. Here we will still use the term Levi subrgoup even though it is not necessarily semi-simple.

When $\Delta_2 = \Delta$ and $\Delta_1 = \emptyset$ then $R_1^+ = \emptyset$ and we have $\mathfrak{p}_\Delta = \mathfrak{b}$. Hence the Borel subalgebra is the minimal parabolic subalgebra. Therefore the Levi factor of the Borel subalgebra is $\mathfrak{l}_\Delta = \mathfrak{h}$ and its nilpotent radical is the sum of all the positive root spaces. When $\Delta_2 = \emptyset$ and $\Delta_1 = \Delta$ then $R_1^+ = R^+$ and we have $\mathfrak{p}_\emptyset = \mathfrak{g}$. Therefore all parabolic subalgebras are contained between $\mathfrak{b} \subset \mathfrak{p}_{\Delta_2} \subset \mathfrak{g}$. We define maximal parabolic subalgebras different from $\mathfrak{g}$ by singling out one simple root α_i and taking $\Delta_2 = \{\alpha_i\}$. We denote this maximal parabolic subalgebra by $\mathfrak{p}_i \equiv \mathfrak{p}_{\{\alpha_i\}}$ and the associated maximal parabolic subgroup as $P_i \equiv P_{\{\alpha_i\}}$. It has rank $r - 1$.

We can also define parabolic subgroups through weights. We define fundamental weights $\{\Lambda_i\}$ as a basis of $\mathfrak{h}^*$ such that $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all simple roots $\alpha_j \in \Delta$ where $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ is the coroot associated to α and $\langle \, . \, , \, . \, \rangle$ is the bilinear form on $\mathfrak{h}^*$ induced from the restriction of the Killing form on $\mathfrak{h}$. All dominant weights are of the form $\Lambda = \sum_i \lambda_i \Lambda_i$ for $\lambda_i \in \mathbb{N}$ and all combinations of this form define a dominant weight. Therefore the fundamental weights generate the whole fundamental Weyl chamber. In particular we define the Weyl vector as $\rho = \sum_i \Lambda_i$. For any root $\alpha \in R$, one can show that it is a positive root, i.e. $\alpha \in R^+$ if and only if $\langle \rho, \alpha \rangle \geq 0$.

Let Λ be a dominant weight we define the set R_Λ by

$$R_\Lambda = \{\alpha \in R \mid \langle \Lambda, \alpha \rangle \geq 0\} \, . \tag{4.1.21}$$

We can see that $R_\rho = R^+$ and if α_i is a simple root we can easily see that if $\Delta_2 = \{\alpha_i\}$, $\Delta_1 = \Delta \setminus \{\alpha_i\}$ then $R_{\Lambda_i} = R^+ \cup (-R_1^+)$. Then we can also define parabolic subalgebras as

$$\mathfrak{p}_\Lambda = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_\Lambda} \mathfrak{g}_\alpha \, . \tag{4.1.22}$$

We write the parabolic subgroup of G generated by the parabolic subalgebra $\mathfrak{p}_\Lambda$ as P_Λ. We can see that $\mathfrak{p}_{\Lambda_i} = \mathfrak{p}_i$ and $\mathfrak{p}_\rho = \mathfrak{b}$.

Finally, if we define

$$R_\Lambda^0 = \{\alpha \in R \mid \langle \Lambda, \alpha \rangle = 0\} \qquad \text{and} \qquad R_\Lambda^+ = \{\alpha \in R \mid \langle \Lambda, \alpha \rangle > 0\} \, , \tag{4.1.23}$$

then the Lie algebras of the Levi factor L_Λ and the unipotent radical U_Λ are given explicitly by

$$\mathfrak{l}_\Lambda = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_\Lambda^0} \mathfrak{g}_\alpha \, , \qquad \text{and} \qquad \mathfrak{n}_\Lambda = \bigoplus_{\alpha \in R_\Lambda^+} \mathfrak{g}_\alpha \, . \tag{4.1.24}$$

Iwasawa decomposition Let G be a semi-simple connected Lie group, let K be a maximal compact subgroup of G and let B be the Borel subgroup of G. The Iwasawa decomposition states that G can be decomposed into $G = KB$. This means that any element $g \in G$ can be uniquely written as $g = kb$ with $k \in K$ and $b \in B$.

The Borel subgroup can be seen as the minimal parabolic subgroup but we can actually extend this decomposition, although non uniquely, to any parabolic subgroup P_Λ as $G = KP_\Lambda$. By the Levi decomposition any parabolic subgroup P_Λ can also be decomposed into its unipotent radical U_Λ and its Levi factor L_Λ so that we have $G = KL_\Lambda U_\Lambda$. This means that any element $g \in G$ can be non uniquely written as $g = klu$ with $k \in K$, $l \in L_\Lambda$ and $u \in U_\Lambda$.

The Langlands decomposition also states that the Levi factor L_Λ of any parabolic subgroup P_Λ can be decomposed into a semi-simple part M_Λ and an abelian part A_Λ. Therefore any element $g \in G$ can be written as $g = kamu$ with $k \in K$, $a \in A_\Lambda$, $m \in M_\Lambda$ and $u \in U_\Lambda$.

We say that a function $f : G \to \mathbb{C}$ has moderate growth if for some norm $\|\,.\,\|$ on G there exists coefficients $C > 0$ and $s \geq 0$ such that

$$|f(g)| \leq C\|g\|^s\,, \tag{4.1.25}$$

for all $g \in G$. We also say that the function f is polynomially bounded. We say that a function $f : G \to \mathbb{C}$ has uniform moderate growth if all its covariant derivatives are polynomially bounded with the same exponent s (although the coefficient C can be different).

One can show that if G, or equivalently G/Γ for some discrete arithmetic subgroup Γ, is not compact then the cusps of G/Γ correspond to taking some abelian factor of A_Λ associated to some parabolic subgroup P_Λ of G to infinity. Therefore moderate growth implies that for each parabolic P_Λ one can find a coefficient C and exponents $s = \{s_1, \ldots, s_l\}$ such that $|f(g)| \geq C|a|^s$ for large $a \in A_\Lambda$ where $l = \dim A_\Lambda$.

We finally have all the tools necessary to properly define automorphic forms. Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$, let Γ be some discrete subgroup of G and let K be a maximal compact subgroup of G. We say that a smooth function f from G to some finite dimensional complex vector space V is an automorphic form for Γ if it satisfies the following four conditions:

1. f is invariant under the action of Γ as in (4.1.1),

2. f transforms covariantly under the action of K according to (4.1.2),

3. f defines a finite dimensional representation of the centre $\mathcal{Z}(\mathfrak{g})$,

4. Each component of f is of moderate growth as in (4.1.25).

Condition 1. makes f into an automorphic representation of G, condition 2. is why we call f a "form", condition 3. gives f good analytic properties while condition 4. ensures f is well behaved in the case G/Γ is non compact. When $G = SL(2,\mathbb{R})$, $K = SO(2)$ and $\Gamma = PSL(2,\mathbb{Z})$ we call f a modular form.

If f transforms in the trivial representation of K we call f an automorphic function. By definition we have $V = \mathbb{C}$ and since f is invariant under the action of K we can see f as a function $f : K\backslash G/\Gamma \to \mathbb{C}$. In the simplest case f is a one dimensional representation of the center $\mathcal{Z}(\mathfrak{g})$, which means that it is an eigenfunction of all Casimir operators. Finally one can show that the components of all automorphic forms automatically have uniform moderate growth [126].

In the rest of this thesis we will focus on automorphic functions seen as smooth complex functions on $K\backslash G/\Gamma$ which are eigenfunctions of all Casimirs and have uniform moderate growth.

4.1.1 Eisenstein series

A prototypical example of a modular form is given by Eisenstein series. This is also the object we will be most interested in throughout the rest of this thesis. Again Eisenstein series constitute a very large subject of mathematical study [127–140] which goes much beyond the scope of this thesis and we will not develop the topic in full generality. For a more extensive review see [141, 142].

Let G be a simple connected group of rank r with Lie algebra $\mathfrak{g}$. Let us denote $\alpha \in R$ the roots and $\alpha_i \in \Delta$ the simple roots of the Lie algebra $\mathfrak{g}$. Let $\{H_{\alpha_i}, E_\alpha\}$ be a Chevalley basis of $\mathfrak{g}$ where $\{H_{\alpha_i}\}$ is a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $H_\alpha \in \mathfrak{h}$ is the dual of the coroot $\alpha^\vee \in \mathfrak{h}^*$ by the restriction of the Killing form $(\,.\,,\,.\,)$ on $\mathfrak{h}$ defined by $(H_\alpha, H) = \alpha^\vee(H)$ for all $H \in \mathfrak{h}$ and $\{E_\alpha\}$ are root generators associated to roots $\alpha \in R$ such that

$$
\begin{aligned}
[H_{\alpha_i}, H_{\alpha_j}] &= 0\,, \\
[H_{\alpha_i}, E_{\pm\alpha_j}] &= \pm C_{ji} E_{\alpha_j}\,, \\
[E_{\alpha_i}, E_{-\alpha_j}] &= -\delta_{ij} H_{\alpha_i} \\
[E_\alpha, E_\beta] &= \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in R\,, \\ -H_\alpha & \text{if } \alpha+\beta = 0\,, \\ 0 & \text{otherwise}\,, \end{cases}
\end{aligned}
\tag{4.1.26}
$$

where $C_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ are the components of the Cartan matrix. The root space $\mathfrak{g}_\alpha$ associated to some root α is given by $\mathrm{Span}(E_\alpha)$.

Let K be a maximal compact subgroup of G. According to the Iwasawa decomposition, for any element $v \in K\backslash G$ we can find a unique coset representative $v \sim b$ with $b \in B$. Therefore, by definition of the Borel subalgebra (4.1.18) we can decompose v as

$$
v = \prod_{i=1}^{r} e^{-\frac{1}{2}\ln y_i H_{\alpha_i}} \prod_{\alpha \in R^+} e^{x_\alpha E_\alpha}\,, \tag{4.1.27}
$$

with $x_\alpha \in \mathbb{R}$ and $y_i \in \mathbb{R}_+^*$. Let us define $\lambda = \sum_{i=1}^r s_i \Lambda_i$ for $s_i \in \mathbb{C}$ where $\{\Lambda_i\}$ are the fundamental weights of $\mathfrak{g}$. According to the commutation relations (4.1.26) the following function

$$
\chi_\lambda(v) = \prod_{i=1}^{r} y_i^{s_i} \tag{4.1.28}
$$

is invariant under the action of the Borel subgroup B. χ_λ is called a multiplicative parabolic character of weight λ. We define the Eisenstein series of G as the Poincaré sum [141]

$$
E_\lambda^G(v) = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_\lambda(v\gamma)\,, \tag{4.1.29}
$$

where the arithmetic subgroup $G(\mathbb{Z})$ is defined as the Chevalley group associated to the weight lattice of G. This automatically defines a smooth complex function on $K\backslash G/G(\mathbb{Z})$ which is absolutely convergent if $\mathrm{Re}(\langle \Lambda_i, \lambda \rangle) > \langle \Lambda_i, \rho \rangle$ for all fundamental weights Λ_i and admits an analytic continuation in $s = \{s_1, \ldots, s_r\}$ to a meromorphic function on $\mathbb{C}^r$ [141]. However it is not well defined for all values of λ. Indeed if for some λ the symmetry group of χ_λ is larger than B then the sum becomes divergent.

One can see using the commutation relations (4.1.26) and the definition of a parabolic subalgebra (4.1.19) that when $s_i = 0$ the symmetry group of χ_λ is enlarged to the parabolic subgroup P_{Δ_2} with $\Delta_2 = \Delta \setminus \{\alpha_i\}$. In fact if $\lambda = \sum_{i=1}^{l\leq r} s_{i_j}\Lambda_{i_j}$ where $s_{i_1}, \ldots, s_{i_l} \neq 0$ are the non zero coefficients then χ_λ is invariant under the parabolic subgroup P_{Δ_2} with $\Delta_2 = \{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$. Therefore the limit of the Eisenstein series to those singular points by analytic continuation is given by

$$E^G_\lambda(v) = \sum_{\gamma \in P_{\Delta_2}(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_\lambda(v\gamma)\,. \tag{4.1.30}$$

The resulting sum is absolutely convergent if $\mathrm{Re}(\langle \Lambda_{i_j}, \lambda\rangle) > \langle \Lambda_{i_j}, \rho\rangle$ for all fundamental weights $\Lambda_{i_1}, \ldots, \Lambda_{i_l}$ and also admits an meromorphic continuation in $s = \{s_{i_1}, \ldots, s_{i_l}\}$ to $\mathbb{C}^l$. The meromorphic continuation is independent of the choice of parabolic subgroup. In particular we can see that we have

$$E^G_0(v) = \sum_{\gamma \in G(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_0(v\gamma) = \sum_{\gamma \in \{Id\}} 1 = 1\,. \tag{4.1.31}$$

One can show that in the domain of absolute convergence, away from the poles in $\mathbb{C}^r$, Eisenstein series are eigenfunctions of all K-invariant operators and in particular of the Laplace operator on $K\backslash G$ (4.1.16)

$$\Delta E^G_\lambda(v) = 2\langle \lambda, \lambda - \rho\rangle E^G_\lambda(v)\,, \tag{4.1.32}$$

where ρ is the Weyl vector. Finally one can also show that Eisenstein series have moderate growth. Therefore they are indeed automorphic forms. It is easy to see that both E^G_λ and ΔE^G_λ are real and strictly positive for $s = \{s_1, \ldots, s_r\}$ real, positive and in the domain of absolute convergence.

Let $w \in W$ be an element in the group of Weyl reflections on the weight lattice of $\mathfrak{g}$. We define $R^w \subset R^+$ to be the set of positive roots which are reflected to negative ones by w, i.e. the $\alpha \in R^+$ such that $w\alpha \in -R^+$. It has been proved that Eisenstein series satisfy the Langlands functional relation [141]

$$E^G_\lambda(v) = \prod_{\alpha \in R^w} \frac{\xi(\langle 2\lambda - \rho, \alpha\rangle)}{\xi(\langle 2\lambda - \rho, \alpha\rangle + 1)} E^G_{w\lambda + \frac{1}{2}(1-w)\rho}(v)\,. \tag{4.1.33}$$

Note that $\xi(s)$ is a meromorphic function of $\mathbb{C}$ which has poles at $s = 0$ and $s = 1$ and obeys the relation $\xi(s) = \xi(1 - s)$.

Maximal parabolic Eisenstein series Let us now specialise to the case where $\lambda = s\Lambda_i$ which will be the case we will focus on for the rest of this thesis. We have

$$\chi_{s\Lambda_i}(v) = y_i^s \,. \tag{4.1.34}$$

Let $e_{\Lambda_i} \in R(\Lambda_i)$ be the highest weight vector of the representation of $\mathfrak{g}$ associated to the fundamental weight Λ_i which we assume to be normalised to one. By definition of the Chevalley basis (4.1.26) and of a highest weight vector if we assume that the generators H_{α_j} and E_{α_j} are in the representation $R(\Lambda_i)$ we have $H_{\alpha_j} e_{\Lambda_i} = \delta_{ij} e_{\Lambda_i}$ and $E_\alpha e_{\Lambda_i} = 0$ for all $\alpha \in R^+$. Therefore if we assume that v is in the representation $R(\Lambda_i)$, by (4.1.27) we have

$$v e_{\Lambda_i} = \frac{1}{\sqrt{y_i}} e_{\Lambda_i} \,. \tag{4.1.35}$$

Hence

$$\chi_{s\Lambda_i}(v) = \frac{1}{\|v e_{\Lambda_i}\|^{2s}} \,, \tag{4.1.36}$$

where $\|\,.\,\|$ is the K-invariant norm of the representation $R(\Lambda_i)$. Therefore we can rewrite the expression of the Eisenstein series as

$$E^G_{s\Lambda_i}(v) = \sum_{\gamma \in P_i(\mathbb{Z})\backslash G(\mathbb{Z})} \frac{1}{\|v\gamma e_{\Lambda_i}\|^{2s}} \,, \tag{4.1.37}$$

where P_i is the maximal parabolic subgroup associated to the fundamental weight Λ_i.

Let $q \in R(\Lambda_i, \mathbb{Z})$ be a vector in the representation of $\mathfrak{g}$ associated with the weight Λ_i over $\mathbb{Z}$.[4] The tensor product $q \otimes q$ belongs to the representation $R(2\Lambda_i, \mathbb{Z})$ of $\mathfrak{g}$ associated with the weight $2\Lambda_i$ over $\mathbb{Z}$ if and only if it satisfies the Casimir eigenvalue equation

$$\kappa^{AB} t_A q \otimes t_B q = \langle \Lambda_i, \Lambda_i \rangle \, q \otimes q \,, \tag{4.1.38}$$

where t_A are generators of $\mathfrak{g}$ and κ_{AB} is the Killing form on $\mathfrak{g}$. One can show that this property is satisfied for any vector $q \in R(\Lambda_i, \mathbb{Z})$ of the form $q = n\gamma e_{\Lambda_i}$ for all $\gamma \in G(\mathbb{Z})/P_i(\mathbb{Z})$ and for all $n \in \mathbb{Z}$ and any vector that satisfies this constraint must be of this form. In fact we have an isomorphism

$$G(\mathbb{Z})/P_i(\mathbb{Z}) = \{q \in R(\Lambda_i, \mathbb{Z}) \,|\, q \otimes q \in R(2\Lambda_i, \mathbb{Z}),\, \gcd q = 1\} \,, \tag{4.1.39}$$

[4] More precisely $R(\Lambda_i, \mathbb{Z}) \subset R(\Lambda_i)$ is a Chevalley lattice which is preserved by the action of $G(\mathbb{Z})$.

where here the gcd includes a notion of sign. Therefore we can rewrite the expression for the Eisenstein series as

$$E^G_{s\Lambda_i}(v) = \frac{1}{2\zeta(2s)} \sum_{\substack{q\in R(\Lambda_i,\mathbb{Z}) \\ q\otimes q\in R(2\Lambda_i,\mathbb{Z})}}{}' \frac{1}{\|vq\|^{2s}}\,, \tag{4.1.40}$$

where the factor of $1/2\zeta(2s)$ compensates the sum over signed gcds and the prime over the sum indicates that we don't include $q = 0$. These series are called maximal parabolic Eisenstein series because they associated to some maximal parabolic subgroup. Their domain of absolute convergence is $\mathrm{Re}\, s > \frac{\langle\Lambda_i,\rho\rangle}{\langle\Lambda_i,\Lambda_i\rangle}$ and we have both $E^G_{s\Lambda_i} > 0$ and $\Delta E^G_{s\Lambda_i} > 0$ for $s > \frac{\langle\Lambda_i,\rho\rangle}{\langle\Lambda_i,\Lambda_i\rangle}$. We define the critical strip as $0 < s < \frac{\langle\Lambda_i,\rho\rangle}{\langle\Lambda_i,\Lambda_i\rangle}$. One can show that in the critical strip the Eisenstein series is integrable. In particular we have

$$0 = \int_{K\backslash G/G(\mathbb{Z})} d\mu\, \Delta E^G_{s\Lambda_i} = 2s\left(s\langle\Lambda_i,\Lambda_i\rangle - \langle\Lambda_i,\rho\rangle\right)\int_{K\backslash G/G(\mathbb{Z})} d\mu\, E^G_{s\Lambda_i}\,. \tag{4.1.41}$$

Therefore if the eigenvalue of the Laplacian isn't zero then the sign of $E^G_{s\Lambda_i}$ cannot be constant. We end by stating without proof that for any group G and any fundamental weight Λ_i we have $E^G_{\frac{1}{2}\Lambda_i}(v) = 0$.

$SL(n)$ maximal parabolic Eisenstein series Let us now look at the particular case when $G = SL(n)$ which will be relevant to us. In this case $K = SO(n)$. Let q be a vector in the representation $R(\Lambda_i)$, the standard $SO(n)$-invariant norm of the representation $R(\Lambda_i)$ is given by $\|q\|^2 = q^T q$. Therefore the maximal parabolic Eisenstein series for $SL(n)$ is given by

$$E^{SL(n)}_{s\Lambda_i}(v) = \frac{1}{2\zeta(2s)} \sum_{\substack{q\in R(\Lambda_i,\mathbb{Z}) \\ q\otimes q\in R(2\Lambda_i,\mathbb{Z})}}{}' \frac{1}{(q^T v^T v q)^s}\,. \tag{4.1.42}$$

The domain of absolute convergence is $\mathrm{Re}\, s > \frac{n}{2}$ and the function admits a meromorphic continuation to $s \in \mathbb{C}$ with a single pole at $s = \frac{n}{2}$. Since for $SL(n)$ the representations $R(\Lambda_i)$ and $R(\Lambda_{n-i})$ are dual to each other we can relate their Eisenstein series.[5] Indeed let v be in the representation $R(\Lambda_i)$, its dual element in $R(\Lambda_{n-i})$ will be given by v^{-1^T}. Since the two representations are dual, we have

[5] In this thesis we use the Bourbaki labelling of Dynkin diagrams such that the exceptional node of the $\mathfrak{e}_n$ family is Λ_2 and the Weyl spinor nodes of the $\mathfrak{d}_n$ family are Λ_{n-1} and Λ_n.

$$E^{SL(n)}_{s\Lambda_{n-i}}(v) = \frac{1}{2\zeta(2s)} \sum_{\substack{q\in R(\Lambda_{n-i},\mathbb{Z}) \\ q\otimes q\in R(2\Lambda_{n-i},\mathbb{Z})}}{}' \frac{1}{(q^T v^{-1} v^{-1^T} q)^s}$$
$$= \frac{1}{2\zeta(2s)} \sum_{\substack{q\in R(\Lambda_{i},\mathbb{Z}) \\ q\otimes q\in R(2\Lambda_{i},\mathbb{Z})}}{}' \frac{1}{(q^T v^{-1} v^{-1^T} q)^s} = E^{SL(n)}_{s\Lambda_i}(v^{-1^T}) \,. \tag{4.1.43}$$

The fundamental and antifundamental series associated to the weights Λ_1 and Λ_{n-1} respectively are also related by the Langlands functional relation (4.1.33) such that

$$E^{SL(n)}_{s\Lambda_1}(v) = \frac{\xi(n-2s)}{\xi(2s)} E^{SL(n)}_{(\frac{n}{2}-s)\Lambda_{n-1}}(v) \,. \tag{4.1.44}$$

They follow the Laplace equation

$$\Delta E^{SL(n)}_{s\Lambda_1}(v) = \frac{2(n-1)}{n} s\left(s-\frac{n}{2}\right) E^{SL(n)}_{s\Lambda_1}(v) \,,$$
$$\Delta E^{SL(n)}_{s\Lambda_{n-1}}(v) = \frac{2(n-1)}{n} s\left(s-\frac{n}{2}\right) E^{SL(n)}_{s\Lambda_{n-1}}(v) \,. \tag{4.1.45}$$

Using the functional relation (4.1.44) one can see that the functions $\Delta E^{SL(n)}_{s\Lambda_1}$ and $\Delta E^{SL(n)}_{s\Lambda_{n-1}}$ are regular at $s=\frac{n}{2}$ and the limit gives the differential equations

$$\Delta E^{SL(n)}_{\frac{n}{2}\Lambda_1}(v) = \frac{1}{2}\frac{n-1}{\xi(n)} \,,$$
$$\Delta E^{SL(n)}_{\frac{n}{2}\Lambda_{n-1}}(v) = \frac{1}{2}\frac{n-1}{\xi(n)} \,. \tag{4.1.46}$$

The Eisenstein series associated to the weights Λ_2 and Λ_{n-2} are also related by the Langlands functional relation (4.1.33) as

$$E^{SL(n)}_{s\Lambda_2}(v) = \frac{\xi(n-2s)\xi(n-2s-1)}{\xi(2s)\xi(2s-1)} E^{SL(n)}_{(\frac{n}{2}-s)\Lambda_{n-2}}(v) \,. \tag{4.1.47}$$

They follow the Laplace equation

$$\Delta E^{SL(n)}_{s\Lambda_2}(v) = \frac{4(n-2)}{n} s\left(s-\frac{n}{2}\right) E^{SL(n)}_{s\Lambda_2}(v) \,,$$
$$\Delta E^{SL(n)}_{s\Lambda_{n-2}}(v) = \frac{4(n-2)}{n} s\left(s-\frac{n}{2}\right) E^{SL(n)}_{s\Lambda_{n-2}}(v) \,. \tag{4.1.48}$$

Using the functional relation (4.1.47) one can see that the functions $\Delta E^{SL(n)}_{s\Lambda_2}$ and $\Delta E^{SL(n)}_{s\Lambda_{n-2}}$ are regular at $s=\frac{n}{2}$ and the limit gives the differential equation

$$\begin{aligned}\Delta E^{SL(n)}_{\frac{n}{2}\Lambda_2}(v) &= \frac{\pi}{6}\frac{n-2}{\xi(n-1)\xi(n)}\,,\\ \Delta E^{SL(n)}_{\frac{n}{2}\Lambda_{n-2}}(v) &= \frac{\pi}{6}\frac{n-2}{\xi(n-1)\xi(n)}\,.\end{aligned} \tag{4.1.49}$$

In general the Eisenstein series associated to the weights Λ_k and Λ_{n-k} are also related by the Langlands functional relation (4.1.33) as

$$E^{SL(n)}_{s\Lambda_k}(v) = \left(\prod_{i=0}^{k-1}\frac{\xi(n-2s-i)}{\xi(2s-i)}\right) E^{SL(n)}_{(\frac{n}{2}-s)\Lambda_{n-k}}(v)\,. \tag{4.1.50}$$

Epstein series Let us now look at a few maximal parabolic Eisenstein series for the fundamental representation Λ_1, we will call these minimal Eisenstein series. For $G=SL(n)$ notice that the symmetric space $SO(n)\backslash SL(n,\mathbb{R})$ is isomorphic to the space of symmetric positive definite unimodular real matrices $\mathcal{H}_S=\{H\in \mathrm{Mat}(n,\mathbb{R})\,|\,H^T=H,\,\det H=1,\,H>0\}$ where the isomorphism is given by $H=v^Tv$ with v is in the fundamental representation $R(\Lambda_1)$ of $SL(n)$. We can use this to write the fundamental maximal parabolic Eisenstein series as

$$E^{SL(n)}_{s\Lambda_1}(H) = \frac{1}{2\zeta(2s)}{\sum_{q\in\mathbb{Z}^n}}' \frac{1}{H[q]^s}\,, \tag{4.1.51}$$

where $H[q]=q^THq$ is the standard $SO(n)$ invariant bilinear form. We have also used the fact that for $q\in R(\Lambda_1,\mathbb{Z})=\mathbb{Z}^n$ the condition $q\otimes q\in R(2\Lambda_1,\mathbb{Z})$ is trivially satisfied.

In the case where $G=O(n,n)$ we have $K=O(n)\times O(n)$. We can use the fact that the symmetric space $O(n)\times O(n)\backslash O(n,n,\mathbb{R})$ is isomorphic to the space of real matrices whose symmetric part is positive definite $\mathcal{H}=\{G+B\in \mathrm{Mat}(n,\mathbb{R})\,|\,G^T=G,\,B^T=-B,\,G>0\}$. Also notice that for $q\in R(\Lambda_1,\mathbb{Z})=I\!I_{n,n}$ the condition $q\otimes q\in R(2\Lambda_1,\mathbb{Z})$ is equivalent to q being light-like. We can use this to write the first maximal parabolic Eisenstein as

$$E^{SO(n,n)}_{s\Lambda_1}(G,B) = \frac{1}{2\zeta(2s)}{\sum_{\substack{q\in I\!I_{n,n}\\(q,q)=0}}}' \frac{1}{H_{G,B}[q]^s}\,, \tag{4.1.52}$$

where $H_{G,B}[q]$ is defined as in (3.5.13) and (q,q) is defined as in (3.5.15). The domain of absolute convergence is $\mathrm{Re}\, s > n-1$ and the function admits a meromorphic continuation of to $s \in \mathbb{C}$. It satisfies the Laplace equation

$$\Delta E_{s\Lambda_1}^{SO(n,n)}(v) = 2s\,(s-n+1)\,E_{s\Lambda_1}^{SO(n,n)}(v)\,. \tag{4.1.53}$$

In the case where $G = Sp(2n)$ we have $K = U(n)$. We know that $U(n)\backslash Sp(2n,\mathbb{R})$ is isomorphic to the Siegel upper half plane $\mathcal{H}_n$ and the first maximal parabolic Eisenstein series is given by

$$E_{s\Lambda_1}^{Sp(2n)}(\Omega) = \frac{1}{2\zeta(2s)}\sum_{q\in\mathbb{Z}^{2n}}{}' \frac{1}{H_\Omega[q]^s}\,, \tag{4.1.54}$$

where $H_\Omega[q]$ is defined as in (3.5.7). The domain of absolute convergence is $\mathrm{Re}\, s > \frac{n+1}{2}$ and the function admits a meromorphic continuation of to $s \in \mathbb{C}$. It satisfies the Laplace equation

$$\Delta E_{s\Lambda_1}^{Sp(2n)}(v) = ns\left(s - \frac{n+1}{2}\right) E_{s\Lambda_1}^{Sp(2n)}(v)\,. \tag{4.1.55}$$

For $n = 1$ we have $Sp(2,\mathbb{R}) = SL(2)$ and we can give the $SL(2)$ Eisenstein series in the more familiar form

$$E_{s\Lambda_1}^{SL(2)}(\tau) = \frac{1}{2\zeta(2s)}\sum_{m,n\in\mathbb{Z}}{}' \frac{(\mathrm{Im}\,\tau)^s}{|m+\tau n|^{2s}}\,. \tag{4.1.56}$$

This is sometimes called the real analytic Eisenstein series or non holomorphic Eisenstein series and it is a modular form. We can think of these functions as a generalisation of the Riemann zeta function, we define the Epstein zeta function or Epstein series as $\mathrm{Ep}_s^G = 2\zeta(2s)E_{s\Lambda_1}^G$ [143]. We can see that $\mathrm{Ep}_s^{SL(1)}(H) = 2\zeta(2s)$.

We end this section by showing an interesting relation. We can easily see that for $\mathrm{Re}\, s > 0$ and $a \neq 0$ we have

$$\int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{a}{t}} = \frac{1}{a^s}\int_0^\infty dx\; x^{s-1}e^{-x} = \frac{\Gamma(s)}{a^s}\,, \tag{4.1.57}$$

where we have performed the change of variable $t = a/x$ and recognised the expression for the Euler Gamma function. We can us this to express the sum (4.1.52) in the domain of absolute convergence as an integral

$$
\begin{aligned}
E^{SO(n,n)}_{s\Lambda_1}(G,B) &= \frac{1}{2\zeta(2s)}\frac{\pi^s}{\Gamma(s)}\sum_{\substack{q\in I\!I_{n,n}\\(q,q)=0}}{}'\int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t}H_{G,B}[q]} \\
&= \frac{1}{2\xi(2s)}\int_{-\frac12}^{\frac12} dl \int_0^\infty \frac{dt}{t^{s+1}} \sum_{q\in I\!I_{n,n}}{}' e^{-\frac{\pi}{t}H_{G,B}[q]+i\pi l\,(q,q)} \\
&= \frac{1}{2\xi(2s)}\int_{\substack{\mathrm{Im}\,\tau>0\\|\mathrm{Re}\,\tau|\le\frac12}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}(\mathrm{Im}\,\tau)^{s-\frac{n-2}{2}}(\mathrm{Im}\,\tau)^{\frac{n}{2}} \sum_{q\in I\!I_{n,n}}{}' e^{-\pi\,\mathrm{Im}\,\tau\, H_{G,B}[q]+i\pi\,\mathrm{Re}\,\tau\,(q,q)}\,,
\end{aligned}
\tag{4.1.58}
$$

where we performed the change of variable $l = \mathrm{Re}\,\tau$ and $t = 1/\,\mathrm{Im}\,\tau$. We can then notice that the domain of integration is made up of several copies of $\mathcal{F}_1$ by the action elements of $SL(2,\mathbb{Z})$ where $\mathcal{F}_1$ is defined as in (3.3.8). Specifically we have

$$
\{\tau\in\mathbb{C}\,|\,\mathrm{Im}\,\tau>0,\ |\mathrm{Re}\,\tau|\le 1/2\} = \bigcup_{\gamma\in\Gamma\backslash SL(2,\mathbb{Z})}\gamma\mathcal{F}_1\,,
\tag{4.1.59}
$$

where Γ is the subgroup of $SL(2,\mathbb{Z})$ made up of upper diagonal elements. We can therefore fold the domain of integration to $\mathcal{F}_1$ at the cost of introducing a sum over $\Gamma\backslash SL(2,\mathbb{Z})$

$$
\begin{aligned}
E^{SO(n,n)}_{s\Lambda_1}(G,B) &= \frac{1}{2\xi(2s)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\sum_{\gamma\in\Gamma\backslash SL(2,\mathbb{Z})}(\mathrm{Im}\,\tau|_\gamma)^{s-\frac{n-2}{2}}\Gamma^{(1)}_{I\!I_{n,n}} \\
&= \frac{1}{2\xi(2s)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\Gamma^{(1)}_{I\!I_{n,n}}E^{SL(2)}_{(s-\frac{n-2}{2})\Lambda_1}\,,
\end{aligned}
\tag{4.1.60}
$$

where in the first line we have introduced the Narain partition function (3.5.16) and used the fact that it and the measure are invariant under the action of $SL(2,\mathbb{Z})$. The resulting formally diverging integral is understood to be regularised with the regularisation prescription amounting to exclude the term $q=0$ from the sum in the Narain partition function. In the next line we have used the fact that upper diagonal elements form the Borel subgroup of $SL(2,\mathbb{Z})$ and we have therefore recognised the expression for the $SL(2)$ Eisenstein series in $Sp(2,\mathbb{R})$ form (4.1.56). This is called a theta lift and the operation on the right had side of (4.1.60) is called the Rankin-Selberg transform of the Narain partition function. One can find a generalisation of this relation to $SO(n,n)$ Eisenstein series of fundamental weights associated with antisymmetric powers of the fundamental representation in appendix B of [144]. In

this case the $E^{SL(2)}_{(s-\frac{n-2}{2})\Lambda_1}$ Eisenstein series appearing in the theta lift generalises to $E^{Sp(2n)}_{(s-\frac{n-k-1}{2})\Lambda_n}$.

We have a similar relation for the symplectic series. Indeed we can rewrite (4.1.54) in the domain of absolute convergence as an integral

$$\begin{aligned} E^{Sp(2n)}_{s\Lambda_1}(\Omega) &= \frac{1}{2\zeta(2s)}\frac{\pi^s}{\Gamma(s)}\sum_{q\in\mathbb{Z}^{2n}}{}' \int_0^\infty \frac{dt}{t^{s+1}} e^{-\frac{\pi}{t}H_\Omega[q]} = \frac{1}{2\xi(2s)}\int_0^\infty \frac{dt}{t^{s+1}} \sum_{q\in\mathbb{Z}^{2n}}{}' e^{-\frac{\pi}{t}H_\Omega[q]} \\ &= \frac{1}{2\xi(2s)}\int_{G>0}\frac{dG}{G} G^{s-\frac{n}{2}} G^{\frac{n}{2}} \sum_{q\in\mathbb{Z}^{2n}}{}' e^{-\pi G\, H_\Omega[q]}\,, \end{aligned} \tag{4.1.61}$$

where we have made the change of variable $t = 1/G$. We can then notice that the domain of integration is made up of two copies of the domain $\{G > 1\}$ related by the action $G' = 1/G$. Specifically we have

$$\{G > 0\} = \{G > 1\} \cup \{1/G' > 1\}\,. \tag{4.1.62}$$

Therefore we can write

$$\begin{aligned} E^{Sp(2n)}_{s\Lambda_1}(\Omega) &= \frac{1}{2\xi(2s)}\int_{G>1}\frac{dG}{G}\left(G^{s-\frac{n}{2}} + G^{\frac{n}{2}-s}\right)\Gamma^{(n)}_{I\!I_{1,1}} \\ &= \frac{1}{2\xi(2s)}\int_{G>1}\frac{dG}{G}\Gamma^{(n)}_{I\!I_{1,1}} E^{SO(1,1)}_{(s-\frac{n}{2})\Lambda_1}\,, \end{aligned} \tag{4.1.63}$$

where in the first line we have recognised the Lagrangian form of the Narain partition function (3.5.10) and used the fact that it and the measure are invariant under the action $G' = 1/G$ of $SO(1,1,\mathbb{Z})$. As previously we understand the formally diverging integral to be regularised with the regularisation prescription amounting to exclude the term $q = 0$ from the sum in the Narain partition function. In the next line we have recognised the expression for the $SO(1,1)$ Eisenstein series (4.1.82) (where there are no non constant Fourier terms for $n = 1$). If we notice that the domain of integration $\{G > 1\}$ can be thought of as a representative of $(SO(1)\times SO(1))\backslash SO(1,1)/SO(1,1,\mathbb{Z})$ we can see that this is a somewhat trivial example of theta lift but which can also be generalised to more general Eisenstein series [145].

4.1.2 Parabolic Fourier expansion

Let G be a connected semi-simple Lie group and let K be a maximal compact subgroup. Let $f : K\backslash G \to \mathbb{C}$ be some smooth function over the symmetric space $K\backslash G$ that is invariant under the right action of $G(\mathbb{Z})$. Let P_Λ be a parabolic subgroup of G, by the Iwasawa decomposition for any element $v \in K\backslash G$ we can find a coset representative $v \sim p$ with $p \in P_\Lambda$, therefore we can see f as a function on P_Λ. Let L_Λ and U_Λ be the Levi factor and unipotent radical of P_Λ, by the Levi decomposition we can write $p = le^a$ with $l \in L_\Lambda$ and $a \in \mathfrak{n}_\Lambda$ the Lie algebra of U_Λ. Let us now suppose that the unipotent radical is abelian. Since our function is invariant under the right action of $G(\mathbb{Z})$ then it is also invariant under the right action of its subgroup $U_\Lambda(\mathbb{Z})$. Hence our function is invariant under the generalised translation $f(le^a e^n) = f(le^{a+n}) = f(le^a)$ for all $n \in \mathfrak{n}_\Lambda(\mathbb{Z})$. Therefore f can be expanded into a convergent Fourier series in this variable

$$f(le^a) = \sum_{q \in \mathfrak{n}_\Lambda^*(\mathbb{Z})} f_q(l) e^{2\pi i q(a)}\,, \tag{4.1.64}$$

with

$$f_q(l) = \int_{\mathfrak{n}_\Lambda/\mathfrak{n}_\Lambda(\mathbb{Z})} da\, e^{-2\pi i \bar{q}(\bar{a})} f(le^a)\,. \tag{4.1.65}$$

This generalised Fourier expansion allows us to restrict the analysis from a function on $K\backslash G$ to functions on L_Λ. This procedure can also be extended to the case where the unipotent radical U_Λ is non abelian but it is more complicated. Let us compute these Fourier expansions for some Eisenstein series.

Minimal Eisenstein series Let $G = SL(n)$, we consider the maximal parabolic subgroup P_k associated to the fundamental weight Λ_k, it is given by

$$P_k = (GL(1) \times SL(k) \times SL(n-k)) \ltimes (\mathbb{R}^k \otimes \mathbb{R}^{n-k})\,. \tag{4.1.66}$$

Therefore we have the Levi subgroup $L_k = GL(1) \times SL(k) \times SL(n-k)$ with abelian factor $A_k = GL(1)$ and semi-simple factor $M_k = SL(k) \times SL(n-k)$ and the unipotent radical is given by $U_k = \mathbb{R}^k \otimes \mathbb{R}^{n-k}$, it is indeed abelian therefore we can perform the Fourier expansion. We can decompose

$$H_n[q_n] = r^{k-n} H_k[q_k + x_{k,n-k} q_{n-k}] + r^k H_{n-k}[q_{n-k}]\,, \tag{4.1.67}$$

where r, H_k, H_{n-k} and $x_{k,n-k}$ parametrise H_n and $q_n = (q_k, q_{n-k})^T$. We have $r \in \mathbb{R}_+^*$,[6] $H_k \in SO(k)\backslash SL(k,\mathbb{R})$, $H_{n-k} \in SO(n-k)\backslash SL(n-k,\mathbb{R})$, $x_{k,n-k} \in \mathbb{R}^k \otimes \mathbb{R}^{n-k}$, $q_k \in \mathbb{Z}^k$

[6] Note that we can see $\mathbb{R}_+^*$ as the symmetric space $O(1)\backslash GL(1) = \mathbb{Z}_2\backslash GL(1) = GL^+(1)$

and $q_{n-k} \in \mathbb{Z}^{n-k}$. In matrix notation this corresponds to a decomposition

$$H_n = \begin{pmatrix} r^{k-n} H_k & 0 \\ 0 & r^k H_{n-k} \end{pmatrix} \begin{bmatrix} I_k & x_{k,n-k} \\ 0 & I_{n-k} \end{bmatrix} . \tag{4.1.68}$$

Therefore using (4.1.57) we can rewrite (4.1.51) as

$$E^{SL(n)}_{s\Lambda_1}(H_n) = \frac{1}{2\xi(2s)} \int_0^\infty \frac{dt}{t^{s+1}} {\sum_{q_k \in \mathbb{Z}^k}}' {\sum_{q_{n-k} \in \mathbb{Z}^{n-k}}}' e^{-\frac{\pi}{t}\left(r^{k-n} H_k[q_k + x_{k,n-k} q_{n-k}] + r^k H_{n-k}[q_{n-k}]\right)} . \tag{4.1.69}$$

We can evaluate the Fourier decomposition by performing a Poisson resummation

$$\begin{aligned} E^{SL(n)}_{s\Lambda_1}(H_n) &= r^{(n-k)s} E^{SL(k)}_{s\Lambda_1}(H_k) + \frac{\xi(2s-k)}{\xi(2s)} r^{k\left(\frac{n}{2}-s\right)} E^{SL(n-k)}_{(s-\frac{k}{2})\Lambda_1}(H_{n-k}) \\ &+ \frac{2}{\xi(2s)} {\sum_{\substack{\Gamma_{n-k,k} \\ \in \mathbb{Z}^{n-k} \otimes \mathbb{Z}^k \\ \mathrm{rank}(\Gamma_{n-k,k})=1}}}' \sum_{\substack{d_k \in \mathbb{Z}^k \\ d_k | \Gamma}} \left(H_k^{-1}[d_k]^{s-\frac{k}{2}}\right) \frac{r^{(\frac{n}{2}-k)s+\frac{kn}{4}}}{|Z(\Gamma)|^{s-\frac{k}{2}}} K_{s-\frac{k}{2}}(2\pi r^{\frac{n}{2}} |Z(\Gamma)|) e^{2\pi i \Gamma(x)} , \end{aligned} \tag{4.1.70}$$

where K_s are modified Bessel functions of the second kind. We have defined $|Z(\Gamma)|^2 = \mathrm{Tr}\left(H_{n-k} \Gamma_{n-k,k} H_k^{-1} \Gamma^T_{n-k,k}\right)$, $\Gamma(x) = \mathrm{Tr}(\Gamma_{n-k,k} x_{k,n-k})$ and the condition that $d_k | \Gamma$ is that there exists $p_{n-k} \in \mathbb{Z}^{n-k}$ such that $\Gamma_{n-k,k} = p_{n-k} d_k^T$. Note that for $k = 1$ we have

$$\sum_{\substack{d \in \mathbb{Z} \\ d | \Gamma}} d^{2s-1} = \sigma_{2s-1}(\Gamma) . \tag{4.1.71}$$

Hence in this case the parabolic decomposition is given by

$$\begin{aligned} E^{SL(n)}_{s\Lambda_1}(H_n) =& r^{(n-1)s} + \frac{\xi(2s-1)}{\xi(2s)} r^{\frac{n}{2}-s} E^{SL(n-1)}_{(s-\frac{1}{2})\Lambda_1}(H_{n-1}) \\ &+ \frac{2}{\xi(2s)} {\sum_{\Gamma_{n-1} \in \mathbb{Z}^{n-1}}}' \sigma_{2s-1}(\Gamma) \frac{r^{(\frac{n}{2}-1)s+\frac{n}{4}}}{|Z(\Gamma)|^{s-\frac{1}{2}}} K_{s-\frac{1}{2}}(2\pi r^{\frac{n}{2}} |Z(\Gamma)|) e^{2\pi i \Gamma(x)} . \end{aligned} \tag{4.1.72}$$

For the special case of $SL(2)$ the identification between the upper complex half plane $\mathcal{H}_1 = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ and the space of symmetric positive definite unimodular 2×2 matrices $\mathcal{H}_S = \{H_2 \in \operatorname{Mat}(2,\mathbb{R}) \mid H_2^T = H_2,\, \det H_2 = 1,\, H_2 > 0\}$ is given by

$$\tau \to H_2 = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} 1 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & |\tau|^2 \end{pmatrix} = \begin{pmatrix} (\operatorname{Im}\tau)^{-1} & 0 \\ 0 & \operatorname{Im}\tau \end{pmatrix}\begin{bmatrix} 1 & \operatorname{Re}\tau \\ 0 & 1 \end{bmatrix} = H_\tau\,. \tag{4.1.73}$$

Therefore we can use (4.1.72) to give the parabolic decomposition of the real analytic Eisenstein series in the $Sp(2,\mathbb{R})$ form (4.1.56)

$$\begin{aligned} E^{SL(2)}_{s\Lambda_1}(\tau) = (\operatorname{Im}\tau)^s + \frac{\xi(2s-1)}{\xi(2s)}(\operatorname{Im}\tau)^{1-s} \\ + \frac{2}{\xi(2s)}{\sum_{\Gamma\in\mathbb{Z}}}' \sigma_{2s-1}(\Gamma)\frac{(\operatorname{Im}\tau)^{\frac12}}{|\Gamma|^{s-\frac12}}K_{s-\frac12}(2\pi\operatorname{Im}\tau|\Gamma|)e^{2\pi i\Gamma\operatorname{Re}\tau}\,. \end{aligned} \tag{4.1.74}$$

We can also give the parabolic decomposition of the antifundamental series associated to the fundamental weight Λ_{n-1} on some maximal parabolic subgroup P_k associated to some fundamental weight Λ_k

$$\begin{aligned} E^{SL(n)}_{s\Lambda_{n-1}}(H_n) = r^{ks}E^{SL(n-k)}_{s\Lambda_{n-k-1}}(H_{n-k}) + \frac{\xi(2s-n+k)}{\xi(2s)}r^{(n-k)(\frac{n}{2}-s)}E^{SL(k)}_{(s-\frac{n-k}{2})\Lambda_{k-1}}(H_k) \\ + \frac{2}{\xi(2s)}{\sum_{\substack{\Gamma_{n-k,k}\\ \in\mathbb{Z}^{n-k}\otimes\mathbb{Z}^k\\ \operatorname{rank}(\Gamma_{n-k,k})=1}}}' \sum_{\substack{d_{n-k}\in\mathbb{Z}^{n-k}\\ d_{n-k}|\Gamma}} \left(H_{n-k}[d_k]^{s-\frac{n-k}{2}}\right)\frac{r^{(k-\frac{n}{2})s+\frac{n(n-k)}{4}}}{|Z(\Gamma)|^{s-\frac{n-k}{2}}}K_{s-\frac{n-k}{2}}(2\pi r^{\frac{n}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,. \end{aligned} \tag{4.1.75}$$

For $k=1$ we have

$$\begin{aligned} E^{SL(n)}_{s\Lambda_{n-1}}(H_n) =& r^s E^{SL(n-1)}_{s\Lambda_{n-2}}(H_{n-1}) + \frac{\xi(2s-n+1)}{\xi(2s)}r^{(n-1)(\frac{n}{2}-s)} \\ &+ \frac{2}{\xi(2s)}{\sum_{\Gamma_{n-1}\in\mathbb{Z}^{n-1}}}' \sigma_{n-1-2s}(\Gamma)\frac{r^{(1-\frac{n}{2})s+\frac{n(n-1)}{4}}}{|Z(\Gamma)|^{\frac{n-1}{2}-s}}K_{s-\frac{n-1}{2}}(2\pi r^{\frac{n}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,. \end{aligned} \tag{4.1.76}$$

Let us now look at the case $G = SO(n,n)$, we consider the maximal parabolic subgroup P_n associated to the fundamental weight Λ_n, it is given by

$$P_n = (GL(1) \times SL(n)) \ltimes \bigwedge^2 \mathbb{R}^n \,. \tag{4.1.77}$$

Therefore we have the Levi subgroup $L_n = GL(1) \times SL(n)$ with abelian factor $A_n = GL(1)$ and semi-simple factor $M_n = SL(n)$ and the unipotent radical is given by $U_n = \bigwedge^2 \mathbb{R}^n$ which is indeed abelian therefore we can perform the Fourier expansion. We can decompose

$$H_{G_n B_n}[q_{2n}] = r^{-1} H_n^{-1}[q_n + B_n p_n] + r H_n[p_n] \tag{4.1.78}$$

and

$$(q_{2n}, q_{2n}) = 2 q_n \cdot p_n \,, \tag{4.1.79}$$

where $rH_n = G_n$ and $q_{2n} = (q_n, p_n)^T$. We have $r \in \mathbb{R}_+^*$, $H_n \in SO(n)\backslash SL(n,\mathbb{R})$, $B_n \in \bigwedge^2 \mathbb{R}^n$ and $q_n, p_n \in \mathbb{Z}^n$. In matrix notation this corresponds to a decomposition

$$H_{G_n B_n} = \begin{pmatrix} r^{-1} H_n^{-1} & 0 \\ 0 & r H_n \end{pmatrix} \begin{bmatrix} I_n & B_n \\ 0 & I_n \end{bmatrix} \,. \tag{4.1.80}$$

Therefore we can rewrite (4.1.58) as

$$E_{s\Lambda_1}^{SO(n,n)}(G_n, B_n) = \frac{1}{2\xi(2s)} \int_{-\frac{1}{2}}^{\frac{1}{2}} dl \int_0^\infty \frac{dt}{t^{s+1}} {\sum_{q_n, p_n \in \mathbb{Z}^n}}' e^{-\frac{\pi}{t}\left(r^{-1} H_n^{-1}[q_n + B_n p_n] + r H_n[p_n]\right) + 2\pi i\, l\, q_n \cdot p_n} \,. \tag{4.1.81}$$

We can evaluate the Fourier decomposition by performing a Poisson resummation and a theta lift unfolding

$$\begin{aligned} E_{s\Lambda_1}^{SO(n,n)}(G_n, B_n) &= r^s E_{s\Lambda_{n-1}}^{SL(n)}(H_n) + \frac{\xi(2s-n+1)}{\xi(2s)} r^{n-1-s} E_{(s-\frac{n-2}{2})\Lambda_1}^{SL(n)}(H_n) \\ &+ \frac{2}{\xi(2s)} {\sum_{\substack{\Gamma_n \in \bigwedge^2 \mathbb{Z}^n \\ \Gamma \wedge \Gamma = 0}}}' \frac{\sigma_{n-1-2s}(\Gamma)}{\gcd(\Gamma)^{\frac{n-2}{2}-s}} E_{(s-\frac{n-2}{2})\Lambda_1}^{SL(2)}(U_\Gamma) \frac{r^{\frac{n-1}{2}}}{|Z(\Gamma)|^{\frac{1}{2}}} K_{s-\frac{n-1}{2}}(2\pi r |Z(\Gamma)|) e^{2\pi i \Gamma(B)} \,, \end{aligned} \tag{4.1.82}$$

where we have defined $|Z(\Gamma)|^2 = \frac{1}{2}\operatorname{Tr}\left(H_n \Gamma_n H_n \Gamma_n^T\right)$ and $\Gamma(B) = \frac{1}{2}\operatorname{Tr}(\Gamma_n B_n)$. The condition $\Gamma \wedge \Gamma = 0$ means that Γ_n is of rank 1 and U_Γ is the $SL(2)$ subgroup of the stabiliser of Γ. For the value $s = \frac{n-2}{2}$ this expression simplifies drastically to

$$E^{SO(n,n)}_{\frac{n-2}{2}\Lambda_1}(G_n,B_n) = r^{\frac{n-2}{2}} E^{SL(n)}_{\frac{n-2}{2}\Lambda_{n-1}}(H_n) + \frac{\pi}{6\xi(n-2)} r^{\frac{n}{2}} + \frac{1}{\xi(n-2)} {\sum_{\substack{\Gamma_n \in \bigwedge^2 \mathbb{Z}^n \\ \Gamma \wedge \Gamma = 0}}}' \sigma_1(\Gamma) \frac{r^{\frac{n-2}{2}}}{|Z(\Gamma)|} e^{-2\pi r |Z(\Gamma)|} e^{2\pi i \Gamma(B)} . \tag{4.1.83}$$

This can be understood from the Langlands functional identity

$$E^{SO(n,n)}_{\frac{n-2}{2}\Lambda_1}(v) = \frac{\xi(2)}{\xi(n-2)} E^{SO(n,n)}_{\Lambda_n}(v) = \frac{\xi(2)}{\xi(n-2)} E^{SO(n,n)}_{\Lambda_{n-1}}(v) , \tag{4.1.84}$$

for $n \geq 3$. Which shows that this function can be realised in the vector representation or in the Weyl spinor representations.

Next to minimal Eisenstein series We call next to minimal Eisenstein series maximal parabolic Eisenstein series for the fundamental weight Λ_2. For $G = SL(n)$ note that we have $R(\Lambda_2) \otimes_S R(\Lambda_2) = R(2\Lambda_2) \oplus R(\Lambda_4)$ where $R(\Lambda_4)$ represents the wedge product. Therefore the condition $q \otimes q \in R(2\Lambda_i, \mathbb{Z})$ can be rewritten as $q \wedge q = 0$. Hence we have in the domain of absolute convergence

$$E^{SL(n)}_{s\Lambda_2}(v) = \frac{1}{2\zeta(2s)} {\sum_{\substack{q \in R(\Lambda_2,\mathbb{Z}) \\ q \wedge q = 0}}}' \frac{1}{(q^T v^T v q)^s} , \tag{4.1.85}$$

where $v \in R(\Lambda_2)$ is in the representation associated to Λ_2. We want to decompose this series on the maximal parabolic P_1. He have can decompose

$$q^T v^T v q = r^{2-n} H_{n-1}[q_{n-1} + x^T_{n-1} p_{n-1}] + r^2 H_{n-1} \wedge H_{n-1}[p_{n-1}] , \tag{4.1.86}$$

where $r \in \mathbb{R}^*_+$, $H_{n-1} \in SO(n-1) \backslash SL(n-1,\mathbb{R})$, $x_{n-1} \in \mathbb{R}^{n-1}$, $q_{n-1} \in \mathbb{Z}^{n-1}$, $p_{n-1} \in \bigwedge^2 \mathbb{Z}^{n-1}$ and we have $(v \wedge v)\, q_1 \wedge q_1 = v q_1 \wedge v q_2$. Therefore using (4.1.57) we can rewrite (4.1.85) as

$$E^{SL(n)}_{s\Lambda_2}(v) = \frac{1}{2\xi(2s)} \int_0^\infty \frac{dt}{t^{s+1}} {\sum_{q_{n-1} \in \mathbb{Z}^{n-1}}}' {\sum_{\substack{p_{n-1} \in \bigwedge^2 \mathbb{Z}^{n-1} \\ p_{n-1} \wedge q_{n-1} = 0 \\ p_{n-1} \wedge p_{n-1} = 0}}}' e^{-\frac{\pi}{t}\left(r^{2-n} H_{n-1}[q_{n-1} + x^T_{n-1} p_{n-1}] + r^2 H_{n-1} \wedge H_{n-1}[p_{n-1}]\right)} . \tag{4.1.87}$$

We can evaluate the Fourier decomposition by performing a Poisson resummation and an auxiliary theta lift unfolding

$$E^{SL(n)}_{s\Lambda_2}(v_n) = r^{(n-2)s}E^{SL(n-1)}_{s\Lambda_1}(H_{n-1}) + \frac{\xi(2s-2)}{\xi(2s)}r^{n-2s}E^{SL(n-1)}_{(s-\frac{1}{2})\Lambda_2}(v_{n-1})$$
$$+ \frac{2}{\xi(2s)}\sum_{\Gamma_{n-1}\in\mathbb{Z}^{n-1}}{}' \frac{\sigma_{2-2s}(\Gamma)}{\gcd(\Gamma)^{\frac{n-3}{n-2}(1-2s)}}E^{SL(n-2)}_{(s-\frac{1}{2})\Lambda_1}(U_\Gamma)\frac{r^{\frac{(n-4)s+n}{2}}}{|Z(\Gamma)|^{\frac{(n-4)s+1}{n-2}}}K_{s-1}(2\pi r^{\frac{n}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,, \tag{4.1.88}$$

where U_Γ is the representative of $SL(n-2)$ in the Levi stabilizer of Γ. We can also give the parabolic decomposition of the anti next to minimal series associated to the fundamental weight Λ_{n-2} on the maximal parabolic subgroup P_1

$$E^{SL(n)}_{s\Lambda_{n-2}}(v_n) = r^{2s}E^{SL(n-1)}_{s\Lambda_{n-3}}(v_{n-1}) + \frac{\xi(2s-n+2)}{\xi(2s)}r^{(n-2)(\frac{n}{2}-s)}E^{SL(n-1)}_{(s-\frac{1}{2})\Lambda_{n-2}}(v_{n-1})$$
$$+ \frac{2}{\xi(2s)}\sum_{\Gamma_{n-1}\in\mathbb{Z}^{n-1}}{}' \frac{\sigma_{n-2-2s}(\Gamma)}{\gcd(\Gamma)^{\frac{1-2s}{n-2}}}E^{SL(n-2)}_{(s-\frac{1}{2})\Lambda_{n-3}}(U_\Gamma)\frac{r^{\frac{(n-4)(\frac{n}{2}-s)+n}{2}}}{|Z(\Gamma)|^{\frac{(n-4)(\frac{n}{2}-s)+1}{n-2}}}K_{s-\frac{n-2}{2}}(2\pi r^{\frac{n}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,. \tag{4.1.89}$$

All of the Fourier expansions above are absolutely convergent and in fact they are the basis on which to obtain the meromorphic continuations of the respective Eisenstein series.

4.2 U duality

As we have seen before U duality is an infinite discrete non perturbative symmetry obeyed by string theory. It is an extension of the strong-weak S-duality which includes T-duality. S-duality itself has its origins in the famous electromagnetic duality. We find enlightening to first review the electromagnetic duality.

Electromagnetic duality S-duality first arose in abelian gauge theory. In this case the duality isn't necessarily visible at the action level, it is a duality of the equations of motion. We will see that at the quantum level the duality swaps electrically charged objects with magnetically charged objects. The action for pure p-form electromagnetism is given by

$$S = -\frac{1}{2e^2}\int F_{p+1}\wedge\star F_{p+1}\,, \tag{4.2.1}$$

where $F_{p+1} = dA_p$. This theory is dual to $d-p-2$-form electromagnetism whose action is given by

$$S = -\frac{(-1)^s}{2m^2}\int \tilde{F}_{d-p-1} \wedge \star \tilde{F}_{d-p-1}\,, \tag{4.2.2}$$

where $\tilde{F}_{d-p-1} = d\tilde{A}_{d-p-2}$ and where s is the parity of the signature of the metric. The two theories seem to be related by the transformation $\tilde{F}_{d-p-1} \longleftrightarrow -\frac{2\pi}{e^2} \star F_{p+1}$, $m \longleftrightarrow \frac{2\pi}{e}$ (we will explain later why we chose these specific factors), however this does not relate the two gauge fields A_p and $\tilde{A}_{d-p-2}$ directly and so it is not clear that extremising the action will give the same theory. Let us therefore look at the equations of motion.

The equations of motion and Bianchi identity for p-form electromagnetism are respectively

$$d \star F_{p+1} = 0\,, \qquad dF_{p+1} = 0\,. \tag{4.2.3}$$

While the equations of motion and Bianchi identity for $d-p-2$-form electromagnetism are respectively

$$(-1)^s d \star \tilde{F}_{d-p-1} = 0\,, \qquad d\tilde{F}_{d-p-1} = 0\,. \tag{4.2.4}$$

Therefore the two theories are indeed related by the transformation

$$\begin{aligned} \tilde{F}_{d-p-1} &\longleftrightarrow -\frac{2\pi}{e^2} \star F_{p+1}\,, \\ m &\longleftrightarrow \frac{2\pi}{e}\,. \end{aligned} \tag{4.2.5}$$

This a $\mathbb{Z}_2$ duality. We see that the coupling constant is mapped to its inverse therefore this is indeed an example of a strong-weak duality. This is called the electromagnetic duality because it exchanges electric and magnetic charges. Indeed in $d-p-2$-form electromagnetism electric and magnetic charges are defined respectively as

$$q = \frac{1}{e^2}\int_{S^{d-p-1}} \star F_{p+1}\,, \qquad g = \int_{S^{p+1}} F_{p+1}\,. \tag{4.2.6}$$

While in $d-p-2$-form electromagnetism electric and magnetic charges are defined respectively as

$$\tilde{q} = \frac{1}{m^2}\int_{S^{p+1}} \star \tilde{F}_{d-p-1}\,, \qquad \tilde{g} = \int_{S^{d-p-1}} \tilde{F}_{d-p-1}\,. \tag{4.2.7}$$

Therefore we find that the duality maps

$$\begin{aligned} \tilde{q} &\longleftrightarrow -(-1)^{(p+1)(d-p-1)+s}\frac{g}{2\pi}\,, \\ \tilde{g} &\longleftrightarrow -2\pi q\,. \end{aligned} \tag{4.2.8}$$

As we have seen, in string theory it is postulated that there are quantum objects called $p-1$-branes which couple electrically to p-form gauge potentials and $d-p-3$-branes which couple magnetically to the same p-form gauge potentials. Quantum consistency conditions related to the existence of Wilson lines imply that their electric and magnetic charges have to obey the Dirac quantization relation $qg \in 2\pi\mathbb{Z}$, this is preserved by the duality. More precisely their electric charges are given by weights of the $U(1)$ algebra $\Lambda_{\text{weight}}(\mathfrak{u}(1)) \simeq \mathbb{Z}$ while their magnetic charges are given by co-roots of the $U(1)$ algebra $2\pi\Lambda_{\text{co-root}}(\mathfrak{u}(1)) \simeq 2\pi\mathbb{Z}$. We have chosen the specific coefficients in the duality transformation to preserve this charge lattice even though it is irrelevant at the classical level. Therefore electromagnetic duality maps electrically charged fundamental $p-1$-branes to magnetically charged solitonic $d-p-3$-branes and vice versa.

Notice that for even dimensions $d = 2k$ we have that $(k-1)$-form electromagnetism is dual to the same theory with a different coupling. In such a theory $(k-2)$-branes can have both electric and magnetic charge, they are called dyons. It is easy to see that duality maps dyonic charges as

$$\begin{pmatrix} q \\ g \end{pmatrix} \longleftrightarrow \begin{pmatrix} -(-1)^{k^2+s}\frac{g}{2\pi} \\ -2\pi q \end{pmatrix}\,. \tag{4.2.9}$$

For $m = e = \sqrt{2\pi}$ the theory is actually dual to itself. This is a $\mathbb{Z}_2$ symmetry (in $d = 4$ dimensions this can be enhanced to an $SO(2)$ symmetry for the classical theory).

Four dimensional theta electromagnetism Let us work in Lorentzian signature. We have seen that in $d = 4$ dimensions 1-form electromagnetism is dual to itself. We can actually enhance this $\mathbb{Z}_2$ duality to a full $SL(2,\mathbb{R})$ group of dualities. We start by adding the so-called theta term to the action

$$S = -\int \left(\frac{1}{2e^2}F_2 \wedge \star F_2 + \frac{\theta}{8\pi^2}F_2 \wedge F_2\right)\,, \tag{4.2.10}$$

where θ is sometimes called an axion, we suppose that it is non constant otherwise the term would be a topological invariant and would not contribute to the equations

of motion. The new equations of motion and Bianchi identity are given respectively by

$$d\star F_2 = -\frac{e^2}{4\pi^2} d\theta \wedge F_2\,, \qquad dF_2 = 0\,. \tag{4.2.11}$$

We can see that if θ is constant then the equations of motion are indeed unchanged by the theta term. Let us define $G_2 = \frac{2\pi}{e^2}\star F_2 + \frac{\theta}{2\pi}F_2$, the equations of motion can be rewritten as $dG_2 = 0$ and $dF_2 = 0$. Let us also define $\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e^2}$. We can show that the following transformation leaves the equations of motion invariant

$$\begin{aligned} \tau &\longleftrightarrow \frac{a\tau + b}{c\tau + d}\,, \\ \begin{pmatrix} F_2 \\ G_2 \end{pmatrix} &\longleftrightarrow \begin{pmatrix} d & b \\ c & a \end{pmatrix}^T \begin{pmatrix} F_2 \\ G_2 \end{pmatrix}\,, \end{aligned} \tag{4.2.12}$$

for $ad - bc = 1$. Hence F_2 and G_2 transform in the transpose doublet representation of $SL(2,\mathbb{R})$ and τ transforms under non linear Möbius transformation. It is important to perform these two transformations simultaneously because G_2 is not independent of τ and only the combined transformation leaves the equations of motion invariant. Notice that for $\theta = 0$ the transformation with $a = d = 0$, $b = -c = 1$ reproduces exactly the usual electromagnetic duality. Let us show the duality more explicitly by rewriting the equations of motion in a manifestly $SL(2,\mathbb{R})$ invariant way. Let us write

$$H = \frac{e^2}{2\pi}\begin{pmatrix} 1 & \frac{\theta}{2\pi} \\ \frac{\theta}{2\pi} & \frac{\theta^2}{4\pi^2} + \frac{4\pi^2}{e^4} \end{pmatrix}\,, \qquad \mathcal{F}_2 = \begin{pmatrix} F_2 \\ G_2 \end{pmatrix}\,. \tag{4.2.13}$$

We can check that we have $H\Omega\mathcal{F}_2 = \star\mathcal{F}_2$ where Ω is the standard symplectic matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{4.2.14}$$

Notice that since $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ we have $\Lambda\Omega\Lambda^T = \Omega$ for all $\Lambda \in SL(2,\mathbb{R})$. The equations of motion can be written as $d\mathcal{F}_2 = 0$ which is manifestly invariant under the transformation

$$\begin{aligned} \mathcal{F}_2 &\longleftrightarrow \Lambda^T\mathcal{F}_2\,, \\ H &\longleftrightarrow \Lambda^T H\Lambda\,, \end{aligned} \tag{4.2.15}$$

for all $\Lambda \in SL(2,\mathbb{R})$. Note that this is only a symmetry of the equations of motion and not of the action. In the presence of the theta term the electric and magnetic charges are defined respectively as

$$q = \int_{S^2} \left(\frac{1}{e^2} \star F_2 + \frac{\theta}{4\pi^2} F_2 \right) = \frac{1}{2\pi} \int_{S^2} G_2, \qquad g = \int_{S^2} F_2 \,. \tag{4.2.16}$$

Therefore we see that that $2\pi q$ and g transform as a doublet of $SL(2,\mathbb{R})$

$$\begin{pmatrix} g \\ 2\pi q \end{pmatrix} \longleftrightarrow \begin{pmatrix} d & b \\ c & a \end{pmatrix}^T \begin{pmatrix} g \\ 2\pi q \end{pmatrix} . \tag{4.2.17}$$

In the quantum theory it is postulated that there are particles which couple to the gauge potential. Since we are in a self dual theory we have dyons, in this case they are 0-branes, i.e. particles. They are mixed together by the $SL(2,\mathbb{R})$ duality. The electric and magnetic charges for any two dyons have to obey the Schwinger-Zwanzinger quantization relation $p_1 g_2 - p_2 g_1 \in 2\pi\mathbb{Z}$, this is preserved by the $SL(2,\mathbb{R})$ duality. Furthermore one can show that their electric charges are given by weights of the $U(1)$ algebra $\Lambda_{\text{weight}}(\mathfrak{u}(1)) \simeq \mathbb{Z}$ while their magnetic charges are given by co-roots of the $U(1)$ algebra $2\pi\Lambda_{\text{co-root}}(\mathfrak{u}(1)) \simeq 2\pi\mathbb{Z}$. This shows that in the quantum theory the duality group is actually broken down to $SL(2,\mathbb{Z})$. For the complexified field τ the generators of the $SL(2,\mathbb{Z})$ duality are $\tau \longleftrightarrow \tau + 1$ and $\tau \longleftrightarrow -\frac{1}{\tau}$. This last transformation at $\theta = 0$ is precisely the usual electromagnetic duality transformation of electromagnetism. In general any non trivial transformation (4.2.12) with $c \neq 0$ represents a strong weak duality and should be considered non perturbative.

For any theory of the form

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} \left((\operatorname{Im}\tau) F_{\mu\nu} F^{\mu\nu} + (\operatorname{Re}\tau) \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right) , \tag{4.2.18}$$

one can check that the symmetric gauge invariant stress energy tensor of the theory is given by

$$T^{\mu\nu} = (\operatorname{Im}\tau) \left(F^{\mu\sigma} F^{\nu}_{\ \rho} - \frac{1}{4} g^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \right) . \tag{4.2.19}$$

In the presence of gravity this is the source of the gravitational field. Therefore it is important that the energy density T^{00} be positive which requires that $(\operatorname{Im}\tau) > 0$. This is preserved by the $SL(2,\mathbb{R})$ transformation (4.2.12). Hence τ must be in the upper complex half plane $\mathcal{H}_1 = SO(2)\backslash SL(2,\mathbb{R})$.

Type IIB supergravity Let us now go back to string theory. S-duality also plays an important role here. We illustrate this with the prototypical example of type IIB string theory. Actually this duality can also be seen at the level of the supergravity limit. Therefore we focus on the case of type IIB supergravity where the duality is actually visible at the level of the action. We will see that at the quantum level the duality swaps objects which are charged under different gauge potentials. However unlike electromagnetic duality it doesn't swap electrically and magnetically charged objects. The gauge sector of type IIB supergravity is given in Einstein frame by

$$\begin{aligned} S =& \frac{1}{2\kappa^2}\int\left(-\frac{1}{2}d\phi\wedge\star d\phi-\frac{1}{2}e^{-\phi}H_3\wedge\star H_3-\frac{1}{2}e^{2\phi}F_1\wedge\star F_1-\frac{1}{2}e^{\phi}\tilde{F}_3\wedge\star\tilde{F}_3-\frac{1}{4}\tilde{F}_5\wedge\tilde{F}_5\right)\\ &-\frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3\,, \end{aligned} \tag{4.2.20}$$

where $H_3 = dB_2$, $F_1 = dC_0$, $F_3 = dC_2$, $F_5 = dC_4$, $\tilde{F}_3 = F_3 - C_0H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3$. One also has to impose as a constraint the self duality condition

$$\tilde{F}_5 = \star\tilde{F}_5\,. \tag{4.2.21}$$

The moduli fields of type IIB supergravity/string theory are the dilaton ϕ and the axion C_0. One can see that if $C_0 = 0$ then type IIB supergravity is dual to itself under the transformation[7]

$$\begin{aligned} \phi &\longleftrightarrow -\phi\,,\\ B_2 &\longleftrightarrow -C_2\,,\\ C_4 &\longleftrightarrow C_4\,. \end{aligned} \tag{4.2.22}$$

Since e^{ϕ} is interpreted again as the coupling constant of the theory this is also a strong weak duality. In type IIB string theory, it is postulated that there exists a 1-brane called the D-string that couples electrically to the gauge potential C_2 and a 5-brane called the D5-brane that couples magnetically to C_2. It is also postulated that there exists a 1-brane called the fundamental string that couples electrically to the gauge potential B_2 and a 5-brane called the NS5-brane that couples magnetically. This duality maps the perturbative fundamental string to the non perturbative D-string and the NS5-branes to the D5-brane [67]. Actually there even exist 1-branes that are

[7]Note that these transformations are only valid in Einstein frame. Indeed, since the string frame and Einstein frame metrics are related by a Weyl rescaling involving the dilaton, at most only one of them can be invariant under S-duality, this is the Einstein frame metric.

charged electrically under both under C_2 and B_2, they are called (p,q)-strings and they generalise D-strings and fundamental strings. Similarly there are 5-branes that are charged magnetically under both under C_2 and B_2, they are called (p,q)-5-branes and they generalise D5-branes and NS5-branes. S-duality maps (p,q)-strings to (q,p)-strings and (p,q)-5-branes to (q,p)-5-branes. This a $\mathbb{Z}_2$ symmetry. If we define the axio-dilaton $\tau = C_0 + ie^{-\phi}$ one can show that the duality extends to a full $SL(2,\mathbb{R})$ symmetry where B_2 and C_2 transform as a doublet, C_4 transforms as a singlet and the axio-dilaton transforms under non linear Möbius transformation

$$\begin{aligned} \tau &\longleftrightarrow \frac{a\tau + b}{c\tau + d}\,, \\ \begin{pmatrix} B_2 \\ C_2 \end{pmatrix} &\longleftrightarrow \begin{pmatrix} d & b \\ c & a \end{pmatrix}^T \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}\,, \\ C_4 &\longleftrightarrow C_4\,. \end{aligned} \tag{4.2.23}$$

Let us show the duality more explicitly by rewriting the action in a manifestly $SL(2,\mathbb{R})$ invariant way. Let us write

$$H = e^{\phi} \begin{pmatrix} 1 & C_0 \\ C_0 & C_0^2 + e^{-2\phi} \end{pmatrix}\,, \qquad \mathcal{B}_2 = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}\,, \tag{4.2.24}$$

then the action can be written as

$$\begin{aligned} S =& \frac{1}{2\kappa^2} \int \left(\frac{1}{4} \operatorname{Tr}\big(dH^{-1} \wedge \star dH\big) - \frac{1}{2} \mathcal{H}_3^T \wedge \star H^{-1} \mathcal{H}_3 - \frac{1}{4} \tilde{F}_5 \wedge \tilde{F}_5 \right) \\ & - \frac{1}{8\kappa^2} \int C_4 \wedge \mathcal{H}_3^T \wedge \Omega \mathcal{H}_3\,, \end{aligned} \tag{4.2.25}$$

where $\mathcal{H}_3 = d\mathcal{B}_2$, $\tilde{F}_5 = F_5 + \frac{1}{2}\mathcal{B}_2^T \wedge \Omega \mathcal{H}_3$ and Ω is the standard symplectic matrix. This is manifestly invariant under the transformation

$$\begin{aligned} \mathcal{B}_2 &\longleftrightarrow \Lambda^T \mathcal{B}_2\,, \\ H &\longleftrightarrow \Lambda^T H \Lambda\,, \\ C_4 &\longleftrightarrow C_4\,, \end{aligned} \tag{4.2.26}$$

for all $\Lambda \in SL(2,\mathbb{R})$. The self duality condition is also $SL(2,\mathbb{R})$ invariant.

This duality mixes (p,q)-strings together and it mixes (p,q)-5-branes together such that

$$\begin{pmatrix} p \\ q \end{pmatrix} \longleftrightarrow \begin{pmatrix} d & b \\ c & a \end{pmatrix}^T \begin{pmatrix} p \\ q \end{pmatrix} . \tag{4.2.27}$$

Again electrically charged branes are mixed together while magnetically charged branes are mixed together, unlike the electromagnetic duality. In string theory the charges p and q have to be integers, this breaks down the duality to $SL(2,\mathbb{Z})$ which is the U-duality group of type IIB string theory. For the axio-dilaton the generators of U-duality are $\tau \longleftrightarrow \tau + 1$ and $\tau \longleftrightarrow -\frac{1}{\tau}$. This last transformation at $C_0 = 0$ is precisely the S-duality transformation.

U-duality actually generalises to all maximally supersymmetric string theories. As we have seen before the moduli fields of string and supergravity theories with more than 1/4th maximal supersymmetry parametrise a symmetric space $K\backslash G$. Furthermore, if the only massless supermultiplet is a gravitational multiplet K is the R-symmetry group (if there are other massless matter multiplets the R-symmetry group will be a factor of K). This is the case for maximal supersymmetry. For a $D = 11 - d$ dimensional maximally supersymmetric theory, which we can see as type IIA or type IIB string theory/supergravity compactified on a torus T^{d-1} or alternatively as M-theory/eleven dimensional supergravity compactified on a torus T^d we have $G = E_{d(d)}$ [146] where $E_{d(d)}$ is the Lie group associated to the split real form of the exceptional Lie algebra $\mathfrak{e}_d$ of rank d (for $d \leq 8$) where we have assumed that we have continued the exceptional series for $d < 6$ by starting from the Dynkin diagram of $\mathfrak{e}_8$ and defining successively the Lie algebra $\mathfrak{e}_n$ from $\mathfrak{e}_{n+1}$ by deleting the node Λ_{n+1}, see figure 4.1.

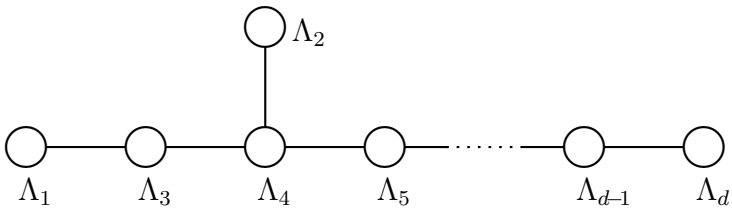


Figure 4.1: Dynkin diagram of $\mathfrak{e}_d$.

The $D = 11 - d$ dimensional maximal supergravity theory then has a global $E_{d(d)}$ symmetry which acts on the gauge and moduli fields such that the moduli fields don't appear in the gravitational amplitudes. But just like for the type IIB case the symmetry is broken down in the string theory to the discrete group $E_{d(d)}(\mathbb{Z})$ which can be defined as the set of matrices in the representation $R(\Lambda_d)$ that are integer

valued in the Chevalley basis (for $d \leq 7$). This is the U duality group. U duality was anticipated in [147] and is still conjectural but there is now overwhelming evidence of its validity, see [148] for a review. The relevant groups are given explicitly in the following table 4.1.

Dimension	G	K	$G(\mathbb{Z})$
10A	$GL(1,\mathbb{R})$	1	1
10B	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$GL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$	$SO(5,5,\mathbb{Z})$
5	$E_{6(6)}(\mathbb{R})$	$Sp(4)$	$E_{6(6)}(\mathbb{Z})$
4	$E_{7(7)}(\mathbb{R})$	$SU(8)$	$E_{7(7)}(\mathbb{Z})$
3	$E_{8(8)}(\mathbb{R})$	$Spin(16)$	$E_{8(8)}(\mathbb{Z})$

Table 4.1: Duality group G and R-symmetry group K of maximal supergravity in $D = 11 - d$ dimensions. The moduli space parametrised by the axion dilaton fields is given by $K\backslash G$. In string theory the U duality group is broken down to $G(\mathbb{Z})$.

U-duality can be interpreted in several ways according to different degeneration limits. The maximal parabolic subgroups P_1, P_2 and P_d of the group $E_{d(d)}$ are all of the form

$$P_i = (GL(1) \times M_i) \ltimes U_i \,, \tag{4.2.28}$$

with Levi subgroup $L_i = GL(1) \times M_i$, abelian factor $A_i = GL(1)$, semi-simple factor M_i and unipotent radical U_i. M_i is a semi-simple group of rank $d-1$ corresponding to the non compact form of the Lie algebra $\mathfrak{e}_d$ where the node Λ_i is deleted, see figure 4.2. If we use the Iwasawa decomposition to identify $K\backslash G \sim P_i$ then when we take the moduli field associated to the abelian factor $A_i = GL(1)$ to some cusp of the moduli space one can show that the moduli space locally looks like $K_i\backslash M_i$ where K_i is the maximal compact subgroup of M_i and the U-duality group reduces to $M_i(\mathbb{Z})$. We distinguish three cases:

1. For the maximal parabolic subgroup P_d associated to the fundamental weight Λ_d the semi-simple factor is $M_d = E_{d-1(d-1)}$. This corresponds to taking the limit of large value $R \to \infty$ for the dilaton corresponding to the radius of the circle S^1 associated with the compactification from $11-d+1$ dimensions to $11-d$. Therefore this degeneration limit corresponds to the limit of decompactification from D to $D+1$ dimensions. $K_d \backslash E_{d-1(d-1)}$ and $E_{d-1(d-1)}(\mathbb{Z})$ simply correspond to the moduli space and U-duality group of string theory in $D = 11-d+1$ dimensions.

2. For the maximal parabolic subgroup P_1 associated to the fundamental weight Λ_1 the semi-simple factor is $M_1 = SO(d-1, d-1)$. This corresponds to taking the limit of large value $1/g_s \to \infty$ for the inverse of the dilaton corresponding to the string coupling. Therefore this degeneration limit corresponds to the limit of perturbative string theory. $SO(d-1) \times SO(d-1) \backslash SO(d-1, d-1)$ and $SO(d-1, d-1, \mathbb{Z})$ simply correspond to the Narain moduli space and the T-duality group of perturbative string theory compactified on a torus T^{d-1}.[8]

3. For the maximal parabolic subgroup P_2 associated to the fundamental weight Λ_2 the semi-simple factor is $M_2 = SL(d)$. This corresponds to taking the limit of large value $V \to \infty$ for the dilaton corresponding to the volume of the torus T^d associated with the compactification from $D = 11$ dimensions to $D = 11-d$. Therefore this degeneration limit corresponds to the limit of decompactification from D to 11 dimensions. $SO(d) \backslash SL(d)$ and $SL(d, \mathbb{Z})$ simply correspond to the moduli space and the group of large diffeomorphisms of the torus T^d.

Therefore we can interpret the U-duality group as being generated from the global diffeomorphisms of the torus in eleven dimensions and the T-duality group.

The unipotent factor $U_i(\mathbb{Z})$ corresponds to symmetries of solitonic objects which cannot be seen as internal symmetries in the degeneration limit either because they are unwrapped or because they cannot be seen in the perturbative limit. These are: black holes and Taub-Nuts for the decompactification limit from D to $D+1$ dimensions, Dp-branes and NS5-branes for the perturbative string theory limit and M2 and M5 branes for the decompactification limit to 11 dimensions.

[8] The full T-duality group of perturbative string theory compactified on a torus is $O(d-1, d-1, \mathbb{Z})$ but this includes a factor $O(1, 1, \mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ which maps type IIA to type IIB string theory. This is not a symmetry of the theory although it is a symmetry of the four-graviton scattering amplitude up to two loops because they are identical in both theories. Nevertheless only the subgroup $SO(d-1, d-1, \mathbb{Z})$ is a symmetry of the effective action.

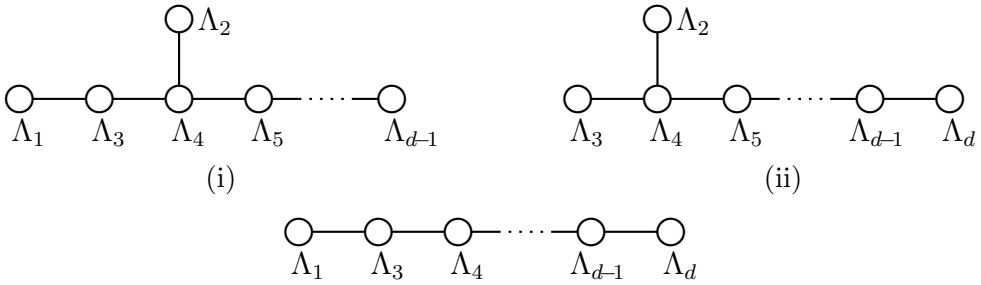


Figure 4.2: Dynkin diagrams corresponding to the three relevant degeneration limits. (i) is obtained by deleting the Λ_d node from 4.1 and corresponds to the limit of decompactification from D to $D+1$ dimensions, (ii) is obtained by deleting the Λ_1 node from 4.1 and corresponds to the limit of perturbative string theory, finally (iii) is obtained by deleting the Λ_2 node from 4.1 and corresponds to the limit of decompactification from D to 11 dimensions. Note that the labelling of nodes is inherited from the labelling of the $\mathfrak{e}_d$ Dynkin diagram and is not standard for the algebra the resulting diagram represents.

4.3 Wilson couplings of maximally supersymmetric string theory

As we have seen in section 3.5 the analytic contributions to the low energy limit of maximally supersymmetric string amplitudes can be reproduced by higher order terms in the effective field theory action. In Einstein frame these are given by

$$S^{\text{string}} = \frac{1}{2\kappa_D^2}\int d^Dx\,\sqrt{-g}\left(R + \frac{l_P^6}{48}\left(\sum_{p,q\geq 0} l_P^{4p+6q}\mathcal{E}_{(p,q)}\nabla^{4p+6q}\mathcal{R}^4\right) + \dots\right), \quad (4.3.1)$$

where we have performed the Weyl rescaling $g^{(E)}_{\mu\nu} = e^{-\frac{4}{D-2}\phi}g^{(s)}_{\mu\nu}$ to get to Einstein frame and we have then restored the vacuum expectation value ϕ_0 into the dilaton. Therefore the Wilson coefficients $\mathcal{E}_{(p,q)}$ are given by

$$\mathcal{E}_{(p,q)}(\varphi) = e^{-\frac{12+8p+12q}{D-2}\phi}\left(c_{(p,q)} + e^{2\phi}\mathcal{E}^{(1)}_{(p,q)_\epsilon}(\varphi) + e^{4\phi}\mathcal{E}^{(2)}_{(p,q)_\epsilon}(\varphi) + \dots\right), \quad (4.3.2)$$

where ϕ is the full dilaton including vacuum expectation value, φ designates the moduli fields parametrising the symmetric space $K\backslash E_{11-D(11-D)}$ and where the "$\dots$" indicate further perturbative and non perturbative contributions.

We know that the full effective action (4.3.1) must be invariant under the action of the U-duality group $E_{11-D(11-D)}(\mathbb{Z})$. Since the Einstein frame metric is U-duality invariant then the $\nabla^{4p+6q}\mathcal{R}^4$ factors are separately invariant, therefore the Wilson couplings $\mathcal{E}_{(p,q)}$ must also be invariant under the action of $E_{11-D(11-D)}(\mathbb{Z})$. Hence they can be seen as well defined functions of $K\backslash E_{11-D(11-D)}/E_{11-D(11-D)}(\mathbb{Z})$. Furthermore the effective action (4.3.1) must also be supersymmetric. One can show that invariance under maximal supersymmetry imposes that the Wilson couplings $\mathcal{E}_{(p,q)}$ satisfy certain differential equations. Already we can see why automorphic forms are natural candidates for Wilson couplings.

Calculating the full Wilson couplings from first principle would require knowing the full perturbative expansion of the maximally supersymmetric string theory as well as the non perturbative theory. This is clearly impossible. However the combination of maximal supersymmetry and U-duality is very constraining and in certain cases it allows us to compute the Wilson coefficients exactly [47, 73, 96, 149–164]. For example the first three Wilson couplings $\mathcal{E}_{(0,0)}$, $\mathcal{E}_{(1,0)}$ and $\mathcal{E}_{(0,1)}$ are protected by supersymmetry which implies that they only receive contributions from perturbative string amplitudes up to finite order in string coupling [47, 76]. $\mathcal{E}_{(0,0)}$ only receives perturbative contributions up to genus 1, i.e. $\mathcal{E}^{(g)}_{(0,0)_\epsilon} = 0$ for $g > 1$. $\mathcal{E}_{(1,0)}$ only receives perturbative contributions up to genus 2, i.e. $\mathcal{E}^{(g)}_{(1,0)_\epsilon} = 0$ for $g > 2$. And $\mathcal{E}_{(0,1)}$ only receives perturbative contributions up to genus 3, i.e. $\mathcal{E}^{(g)}_{(0,1)_\epsilon} = 0$ for $g > 3$. In addition supersymmetry implies that they satisfy the differential equations [48, 51, 165]

$$\begin{aligned}
\Delta\mathcal{E}_{(0,0)} =& \frac{3(11-D)(D-8)}{D-2}\mathcal{E}_{(0,0)} + 6\pi\delta_{D,8}\,, \\
\Delta\mathcal{E}_{(1,0)} =& \frac{5(12-D)(D-7)}{D-2}\mathcal{E}_{(1,0)} + \frac{20\pi^2}{3}\delta_{D,7} + 7\mathcal{E}_{(0,0)}\delta_{D,6}\,, \\
\Delta\mathcal{E}_{(0,1)} =& \frac{6(14-D)(D-6)}{D-2}\mathcal{E}_{(0,1)} - \left(\mathcal{E}_{(0,0)}\right)^2 \\
&+ 40\zeta(3)\delta_{D,6} + \frac{55}{3}\mathcal{E}_{(0,0)}\delta_{D,5} + \frac{85}{2\pi}\mathcal{E}_{(1,0)}\delta_{D,4}\,.
\end{aligned} \tag{4.3.3}$$

We can see that these equations indicate that the Wilson couplings are automorphic forms, perhaps evaluated at some pole as indicated by the inhomogeneous Kronecker delta terms. The non linear inhomogeneous term on the right hand side of $\Delta\mathcal{E}_{(0,1)}$ may indicate that the corresponding Wilson coupling contains some term which is not an automorphic form. We will only focus on the first two Wilson couplings $\mathcal{E}_{(0,0)}$ and $\mathcal{E}_{(1,0)}$ which we will now find.

4.3.1 10 dimensions: type IIB string theory

Let us find the first two Wilson couplings in $D = 10$ dimensions for type IIB string theory. The moduli space is $SO(2)\backslash SL(2,\mathbb{R})$ and the U-duality group is $SL(2,\mathbb{Z})$. The moduli fields of type IIB string theory are the axion dilaton field $\tau = C_0 + ie^{-\phi} \in \mathcal{H}_1$ such that H_τ defined as in (4.1.73) is in the moduli space.

Minimal Wilson coupling In $D = 10$ dimensions the minimal Wilson coupling satisfies the Laplace equation

$$\Delta \mathcal{E}_{(0,0)} = \frac{3}{4}\mathcal{E}_{(0,0)} \,. \tag{4.3.4}$$

The only perturbative contribution to $\mathcal{E}_{(0,0)}$ come from the genus 0 and genus 1 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(0,0)} = e^{-\frac{3}{2}\phi} c_{(0,0)} + e^{\frac{\phi}{2}} \mathcal{E}^{(1)}_{(0,0)_\epsilon} \,. \tag{4.3.5}$$

Recall that for $D = 10$ dimensions we have $c_{(0,0)} = 2\zeta(3)$ and $\mathcal{E}^{(1)}_{(0,0)_\epsilon} = \frac{2\pi^2}{3}$, therefore the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(0,0)} = 2\zeta(3) e^{-\frac{3}{2}\phi} + \frac{2\pi^2}{3} e^{\frac{\phi}{2}} \,. \tag{4.3.6}$$

Note that, because the amplitudes are the same up to two loops, this is also the perturbative part of the Wilson coupling for type IIA string theory. Actually it turns out that it is even the full non perturbative coupling for type IIA string theory.

Let us now consider the large $\operatorname{Im}\tau$ limit of the real analytic Eisenstein series $E^{SL(2)}_{s\Lambda_1}$ in the $Sp(2,\mathbb{R})$ form (4.1.56). This is most easily computed using the Fourier decomposition on the maximal parabolic subgroup P_1 (4.1.74) and using the fact that the Bessel K function behaves asymptotically as $2(\operatorname{Im}\tau)^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi \operatorname{Im}\tau) \sim e^{-2\pi \operatorname{Im}\tau}$, we have

$$E^{SL(2)}_{s\Lambda_1}(\tau) \sim (\operatorname{Im}\tau)^s + \frac{\xi(2s-1)}{\xi(2s)} (\operatorname{Im}\tau)^{1-s} \,. \tag{4.3.7}$$

Therefore we can see that

$$\mathcal{E}_{(0,0)} = 2\zeta(3) E^{SL(2)}_{\frac{3}{2}\Lambda_1} = \text{Ep}^{SL(2)}_{\frac{3}{2}} \tag{4.3.8}$$

has the right degeneration limit for $1/g_s = e^{-\phi} = \operatorname{Im}\tau \to \infty$ and obeys the right Laplace equation, see (4.1.45). This is the full non perturbative Wilson coupling [47].

Therefore the constant terms in the Fourier expansion on the maximal parabolic subgroup P_1, which is associated with the perturbative string theory limit, correspond to the perturbative contributions.

We can also give an interpretation to the non constant terms in the Fourier expansion of the Wilson couplings in the perturbative string theory parabolic decomposition. Indeed type IIB string theory has $D(-1)$-branes which can be charged under the R-R field C_0. These are non perturbative objects called D-instantons. One can show using T-duality with type IIA string theory [47] that the action for a multiply-charged D-instanton of charge Γ is $S = -2\pi i(\Gamma C_0 + i|\Gamma|e^{-\phi})$ which is equal to $-2\pi i|\Gamma|\tau$ for positive charge Γ and $2\pi i|\Gamma|\bar{\tau}$ for negative charge Γ (which is also called an anti-instanton). Note that $|\Gamma|$ can be interpreted as the mass of the (anti)-instanton as it is a BPS state. The degeneracy number of such D-instantons is given by the number of partitions of Γ into two integers, or in other words the number of divisors of Γ. Using the asymptotic expansion for the Bessel K function we can rewrite the Fourier expansion of the Wilson coupling $\mathcal{E}_{(0,0)}$ in the perturbative string theory parabolic decomposition as

$$\begin{aligned}\mathcal{E}_{(0,0)}(C_0,\phi) \sim\ & 2\zeta(3)e^{-\frac{3}{2}\phi} + \frac{2\pi^2}{3}e^{\frac{\phi}{2}} \\ & + 4\pi \sum_{\Gamma\in\mathbb{Z}}{}' |\Gamma|^{\frac{1}{2}}\sigma_{-2}(\Gamma)e^{2\pi i(\Gamma C_0 + i|\Gamma|e^{-\phi})}\left(1+\sum_{k=1}^{\infty}\left(\frac{e^{\phi}}{4\pi|\Gamma|}\right)^k \frac{\Gamma\left(k+\frac{3}{2}\right)}{k!\,\Gamma\left(-k+\frac{3}{2}\right)}\right).\end{aligned} \tag{4.3.9}$$

Hence the non constant Fourier terms have the form of a sum over multiply-charged D-instanton contributions [47]. The terms in parenthesis on the right hand side represent the infinite sum of perturbative corrections around the instantons of charge Γ. The fact that there are no D-instantons in type IIA string theory explains the fact that the perturbative coupling (4.3.6) does not receive any non perturbative corrections.

Next to minimal Wilson coupling In $D = 10$ dimensions the next to minimal Wilson coupling satisfies the Laplace equation

$$\Delta\mathcal{E}_{(1,0)} = \frac{15}{4}\mathcal{E}_{(1,0)}\,. \tag{4.3.10}$$

The only perturbative contribution to $\mathcal{E}_{(1,0)}$ come from the genus 0, genus 1 and genus 2 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(1,0)} = e^{-\frac{5}{2}\phi}c_{(1,0)} + e^{-\frac{\phi}{2}}\mathcal{E}^{(1)}_{(1,0)\epsilon} + e^{\frac{3\phi}{2}}\mathcal{E}^{(2)}_{(1,0)\epsilon}\,. \tag{4.3.11}$$

Recall that for $D = 10$ dimensions we have $c_{(1,0)} = \zeta(5)$, $\mathcal{E}^{(1)}_{(1,0)\epsilon} = 0$ and $\mathcal{E}^{(2)}_{(1,0)\epsilon} = \frac{2\pi^4}{135}$, therefore the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(1,0)} = \zeta(5)e^{-\frac{5}{2}\phi} + \frac{2\pi^4}{135}e^{\frac{3\phi}{2}} . \tag{4.3.12}$$

Just like for the leading Wilson coupling it turns out that this is the full non perturbative coupling for type IIA string theory. Therefore from (4.3.7) we can see that

$$\mathcal{E}_{(1,0)} = \zeta(5)E^{SL(2)}_{\frac{5}{2}\Lambda_1} = \frac{1}{2}\text{Ep}^{SL(2)}_{\frac{5}{2}} \tag{4.3.13}$$

has the right degeneration limit for $1/g_s = e^{-\phi} = \text{Im}\,\tau \to \infty$ and obeys the right Laplace equation. This is the full non perturbative Wilson coupling [72].

4.3.2 9 dimensions

In $D = 9$ dimensions the moduli space is $GL^+(1) \times SO(2) \backslash SL(2, \mathbb{R})$ and the U-duality group is $SL(2, \mathbb{Z})$. The moduli fields in this dimension are $r \in \mathbb{R}_+$ and $\tau \in \mathcal{H}_1$ such that rH_τ is in the moduli space where H_τ is defined as in (4.1.73).

Minimal Wilson coupling In $D = 9$ dimensions the minimal Wilson coupling satisfies the Laplace equation

$$\Delta\mathcal{E}_{(0,0)} = \frac{6}{7}\mathcal{E}_{(0,0)} . \tag{4.3.14}$$

The only perturbative contribution to $\mathcal{E}_{(0,0)}$ come from the genus 0 and genus 1 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(0,0)} = e^{-\frac{12}{7}\phi}c_{(0,0)} + e^{\frac{2}{7}\phi}\mathcal{E}^{(1)}_{(0,0)\epsilon} . \tag{4.3.15}$$

For $D = 9$ dimensions we still have $c_{(0,0)} = 2\zeta(3)$ and the moduli field parametrising the Narain partition function is the metric $G = \left(\frac{l_s}{R}\right)^2$ on the torus T^1 of compactification from 10 to 9 dimensions where R is the radius of the circle. Therefore we have

$$\mathcal{E}^{(1)}_{(0,0)\epsilon} = \frac{2\pi^2}{3}E^{SO(1,1)}_{-\frac{1}{2}\Lambda_1} = \frac{2\pi^2}{3}\left(\frac{R}{l_s} + \frac{l_s}{R}\right) , \tag{4.3.16}$$

where we have used the parabolic Fourier expansion (4.1.83) and the fact that $\bigwedge^2\mathbb{Z} = 0$ so that there are no non constant Fourier coefficients in $E^{SO(1,1)}_{-\frac{1}{2}\Lambda_1}$. Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(0,0)} = 2\zeta(3)e^{-\frac{12}{7}\phi} + \frac{2\pi^2}{3}\left(\frac{R}{l_s} + \frac{l_s}{R}\right)e^{\frac{2}{7}\phi}\,. \tag{4.3.17}$$

This is indeed invariant under T-duality $R \longleftrightarrow l_s^2/R$. If we consider that we have obtained the 9 dimensional theory by compactifying from type IIB string theory then we have $r = e^{\frac{\phi}{2}}\left(\frac{l_s}{R}\right)^{\frac{7}{4}}$ and $\operatorname{Im}\tau = e^{-\phi}\left(\frac{l_s}{R}\right)^{\frac{1}{2}}$ where R is the radius of compactification from type IIB string theory. Then from (4.3.7) we can see that

$$\mathcal{E}_{(0,0)} = 2\zeta(3)r^{-\frac{3}{7}}E^{SL(2)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{3}r^{\frac{4}{7}} = r^{-\frac{3}{7}}\text{Ep}^{SL(2)}_{\frac{3}{2}} + \frac{2\pi^2}{3}r^{\frac{4}{7}} \tag{4.3.18}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$. One can also check that it obeys the right Laplace equation. This is the full non perturbative Wilson coupling [72]. Therefore again the constant terms in the Fourier expansion on the maximal parabolic subgroup P_1, which is associated with the perturbative string theory limit, correspond to the perturbative contributions. The non constant terms correspond to instanton contributions.

Also note that if we take the limit $R \to \infty$ we get

$$\mathcal{E}_{(0,0)} \sim 2\zeta(3)E^{SL(2)}_{\frac{3}{2}\Lambda_1}r^{-\frac{3}{7}}\,. \tag{4.3.19}$$

Which corresponds to the minimal Wilson coupling for type IIB string theory where τ is the type IIB axion dilaton field. Similarly, if we take the limit $R' = l_s^2/R \to \infty$, where R' is the radius of compactification from type IIA string theory, we get

$$\mathcal{E}_{(0,0)} \sim \left(2\zeta(3)(\operatorname{Im}\tau)^{\frac{3}{2}} + \frac{2\pi^2}{3}r\right)r^{-\frac{3}{7}}\,, \tag{4.3.20}$$

which corresponds to the minimal Wilson coupling for type IIA string theory. Therefore $\mathcal{E}_{(0,0)}$ also has the right degeneration limit for the decompactification from 9 to 10 dimensions.

Next to minimal Wilson coupling In $D = 9$ dimensions the next to minimal Wilson coupling satisfies the Laplace equation

$$\Delta\mathcal{E}_{(1,0)} = \frac{30}{7}\mathcal{E}_{(1,0)}\,. \tag{4.3.21}$$

The only perturbative contribution to $\mathcal{E}_{(1,0)}$ come from the genus 0, genus 1 and genus 2 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(1,0)} = e^{-\frac{20}{7}\phi}c_{(1,0)} + e^{-\frac{6}{7}\phi}\mathcal{E}^{(1)}_{(1,0)_\epsilon} + e^{\frac{8}{7}\phi}\mathcal{E}^{(2)}_{(1,0)_\epsilon}\,. \tag{4.3.22}$$

For $D = 9$ dimensions we still have $c_{(1,0)} = \zeta(5)$ and we have

$$\begin{aligned}\mathcal{E}^{(1)}_{(1,0)\epsilon} &= \frac{2\pi^2}{45}\zeta(3)E^{SO(1,1)}_{\frac{3}{2}\Lambda_1} = \frac{2\pi^2}{45}\zeta(3)\left(\frac{R^3}{l_s^3}+\frac{l_s^3}{R^3}\right),\\ \mathcal{E}^{(2)}_{(1,0)\epsilon} &= \frac{2\pi^4}{135}E^{SO(1,1)}_{2\Lambda_1} = \frac{2\pi^4}{135}\left(\frac{R^2}{l_s^2}+\frac{l_s^2}{R^2}\right),\end{aligned} \tag{4.3.23}$$

where we have used the parabolic Fourier expansion (4.1.82) and the fact that $\bigwedge^2\mathbb{Z} = 0$ so that there are no non constant Fourier coefficients in $E^{SO(1,1)}_{\frac{3}{2}\Lambda_1}$ and $E^{SO(1,1)}_{2\Lambda_1}$. Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(1,0)} = \zeta(5)e^{-\frac{20}{7}\phi} + \frac{2\pi^2}{45}\zeta(3)\left(\frac{R^3}{l_s^3}+\frac{l_s^3}{R^3}\right)e^{-\frac{6}{7}\phi} + \frac{2\pi^4}{135}\left(\frac{R^2}{l_s^2}+\frac{l_s^2}{R^2}\right)e^{\frac{8}{7}\phi}. \tag{4.3.24}$$

This is also manifestly invariant under T-duality. Therefore from (4.3.7) we can see that

$$\begin{aligned}\mathcal{E}_{(1,0)} &= \zeta(5)r^{-\frac{5}{7}}E^{SL(2)}_{\frac{5}{2}\Lambda_1} + \frac{2\pi^2}{45}\zeta(3)r^{\frac{9}{7}}E^{SL(2)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{45}\zeta(3)r^{-\frac{12}{7}}\\ &= \frac{1}{2}r^{-\frac{5}{7}}\text{Ep}^{SL(2)}_{\frac{5}{2}} + \frac{\pi^2}{45}r^{\frac{9}{7}}\text{Ep}^{SL(2)}_{\frac{3}{2}} + \frac{2\pi^2}{45}\zeta(3)r^{-\frac{12}{7}},\end{aligned} \tag{4.3.25}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$. One can also check that it obeys the right Laplace equation. This is the full non perturbative Wilson coupling [72].

Also note that if we take the limit $R \to \infty$ we get

$$\mathcal{E}_{(1,0)} \sim \zeta(5)E^{SL(2)}_{\frac{5}{2}\Lambda_1}r^{-\frac{5}{7}} + \frac{2\pi^2}{45}\zeta(3)r^{-\frac{12}{7}}. \tag{4.3.26}$$

The first term corresponds to the next to minimal Wilson coupling for type IIB string theory. Therefore $\mathcal{E}_{(1,0)}$ also has the right degeneration limit for the decompactification from 9 to 10 dimensions. The second term is known to be necessary in order to account for the ten-dimensional normal threshold [54, 94, 95, 147].

4.3.3 8 dimensions

In $D = 8$ dimensions the moduli space is $SO(3)\backslash SL(3,\mathbb{R}) \times SO(2)\backslash SL(2,\mathbb{R})$ and the U-duality group is $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$. The moduli fields in this dimension are $H_3 \in \mathcal{H}_S$ and $U \in \mathcal{H}_1$ such that $H_3 \times H_U$ is in the moduli space where H_U is defined as in (4.1.73).

Minimal Wilson coupling In $D = 8$ dimensions the minimal Wilson coupling satisfies the differential equation

$$\Delta \mathcal{E}_{(0,0)} = 6\pi \,. \tag{4.3.27}$$

This indicates that we may need to consider Eisenstein series evaluated at some pole. The only perturbative contribution to $\mathcal{E}_{(0,0)}$ come from the genus 0 and genus 1 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(0,0)} = e^{-2\phi} c_{(0,0)} + \mathcal{E}^{(1)}_{(0,0)_\epsilon} \,. \tag{4.3.28}$$

For $D = 8$ dimensions we still have $c_{(0,0)} = 2\zeta(3)$ and we saw in section 3.5.2 that $\mathcal{E}^{(1)}_{(0,0)_\epsilon}$ was divergent. This also indicates that we need to consider Eisenstein series evaluated at some pole. We showed how to regularise and calculate the renormalised genus 1 contribution in (3.5.57). Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(0,0)_\mu} = 2\zeta(3)e^{-2\phi} - 2\pi \ln\big(\text{Im}\, U |\eta(U)|^4\big) - 2\pi \ln\big(\text{Im}\, T |\eta(T)|^4\big) + \frac{4\pi}{3}\phi + \frac{22\pi}{3} - 4\pi \ln(2\pi l_P \mu) \,. \tag{4.3.29}$$

This is indeed invariant under T-duality. The $\frac{4\pi}{3}\phi$ term can be thought of as arising from the Weyl rescaling in passing from the string frame to the Einstein frame in the presence of infrared thresholds.

Let us consider the following regularisation of the real analytic Eisenstein series $E^{SL(2)}_{s\Lambda_1}$ in the $Sp(2,\mathbb{R})$ form (4.1.56) at the pole $s = 1$

$$\begin{aligned} \hat{E}^{SL(2)}_{\Lambda_1}(U) &= \lim_{\epsilon \to 0} \left(E^{SL(2)}_{(1+\epsilon)\Lambda_1}(U) - \frac{3}{\pi\epsilon} - \frac{1}{\pi}(72 \ln A - 6 \ln 4\pi) \right) \\ &= -\frac{3}{\pi} \ln\big(\text{Im}\, U |\eta(U)|^4\big) \,, \end{aligned} \tag{4.3.30}$$

where A is the Glaisher–Kinkelin constant. We can also decompose H_3 into r, H_2 and x_2 as in (4.1.68) for the maximal parabolic subgroup P_1 of $SL(3)$. We have $H_T = H_2$ where H_T is defined as in (4.1.73) and $r = e^{-\frac{2}{3}\phi}$. Let us now consider the following regularisation of the Eisenstein series $E^{SL(3)}_{s\Lambda_1}$ at the pole $s = \frac{3}{2}$

$$\begin{aligned}\hat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1}(H_3) &= \lim_{\epsilon\to 0}\left(E^{SL(3)}_{(\frac{3}{2}+\epsilon)\Lambda_1}(H_3) - \frac{\pi}{\zeta(3)\epsilon} - \frac{2\pi}{\zeta(3)}(\gamma_E - 1) + 2\pi\frac{\zeta'(3)}{\zeta(3)^2}\right)\\ &= r^3 - \frac{\pi}{\zeta(3)}\ln\left(\operatorname{Im} T|\eta(T)|^4\right) - \frac{\pi}{\zeta(3)}\ln r\\ &+ \frac{4\pi}{\zeta(3)}{\sum_{\Gamma_2\in\mathbb{Z}^2}}' \sigma_2(\Gamma)\frac{r^{\frac{3}{2}}}{|Z(\Gamma)|}K_1(2\pi r^{\frac{3}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,, \end{aligned} \tag{4.3.31}$$

where we have used the Fourier decomposition on the maximal parabolic subgroup P_1 (4.1.72). Therefore using the asymptotic behavior of the Bessel K function we can see that

$$\begin{aligned}\mathcal{E}_{(0,0)\mu} &= 2\zeta(3)\hat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{3}\hat{E}^{SL(2)}_{\Lambda_1} + \frac{22\pi}{3} - 4\pi\ln(2\pi l_P\mu)\\ &= \hat{\mathrm{Ep}}^{SL(3)}_{\frac{3}{2}} + 2\hat{\mathrm{Ep}}^{SL(2)}_{1} + \frac{22\pi}{3} - 4\pi\ln(2\pi l_P\mu)\end{aligned} \tag{4.3.32}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$ and obeys the right Laplace equation, see (4.1.46). This is the full non perturbative Wilson coupling [166]. Therefore the constant terms in the Fourier expansion on the maximal parabolic subgroup P_1, which is associated with the perturbative string theory limit, correspond again to the perturbative contributions.

If we consider that we have obtained the 8 dimensional theory by compactifying from type IIB string theory then we have $\operatorname{Im} U = R_2/R_1$ and $\operatorname{Im} T = R_1R_2/l_s^2$ where R_1 is the radius of compactification from type IIB string theory to 9 dimensions and R_2 is the radius of compactification from 9 to 8 dimensions. We can now decompose H_3 into r', H_2' and x_2' as in (4.1.68) for the maximal parabolic subgroup P_2 of $SL(3)$. Then if we take the limit $R_2 \to \infty$, which corresponds to only keeping the constant terms in the Fourier decomposition (4.1.70) for the maximal parabolic subgroup P_2, we get

$$\mathcal{E}_{(0,0)\mu} \sim 2\zeta(3)r^{\frac{3}{2}}E^{SL(2)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{3}\operatorname{Im} U + \frac{22\pi}{3} - 4\pi\ln\left(2\pi r(\operatorname{Im} U)^{\frac{1}{2}}l_P\mu\right). \tag{4.3.33}$$

The first two terms correspond to the minimal Wilson coupling in dimension 9. Therefore $\mathcal{E}_{(0,0)\mu}$ also has the right degeneration limit for the decompactification from 8 to 9 dimensions. The last term is an important contribution which resums into the nine-dimensional massless threshold [94] while the $\ln\mu$ term is a scale contribution.

From (4.3.30) and (4.3.31) we can deduce a general way to regularise minimal $SL(n)$ Eisenstein series at the pole $\frac{n}{2}$

$$\hat{E}^{SL(n)}_{\frac{n}{2}\Lambda_1} = \lim_{\epsilon\to 0}\left(E^{SL(n)}_{(\frac{n}{2}+\epsilon)\Lambda_1} - \frac{\xi(1+2\epsilon)}{\xi(n+2\epsilon)}\right) . \tag{4.3.34}$$

Similarly for antifundamental $SL(n)$ series

$$\hat{E}^{SL(n)}_{\frac{n}{2}\Lambda_{n-1}} = \lim_{\epsilon\to 0}\left(E^{SL(n)}_{(\frac{n}{2}+\epsilon)\Lambda_{n-1}} - \frac{\xi(1+2\epsilon)}{\xi(n+2\epsilon)}\right) . \tag{4.3.35}$$

The link between divergences in supergravity and Wilson couplings was further studied in [165, 167].

Next to minimal Wilson coupling In $D = 8$ dimensions the next to minimal Wilson coupling satisfies the Laplace equation

$$\Delta\mathcal{E}_{(1,0)} = \frac{10}{3}\mathcal{E}_{(1,0)} . \tag{4.3.36}$$

The only perturbative contribution to $\mathcal{E}_{(1,0)}$ come from the genus 0, genus 1 and genus 2 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(1,0)} = e^{-\frac{10}{3}\phi}c_{(1,0)} + e^{-\frac{4}{3}\phi}\mathcal{E}^{(1)}_{(1,0)_\epsilon} + e^{\frac{2}{3}\phi}\mathcal{E}^{(2)}_{(1,0)_\epsilon} . \tag{4.3.37}$$

For $D = 8$ dimensions we still have $c_{(1,0)} = \zeta(5)$ and we have

$$\begin{aligned}
\mathcal{E}^{(1)}_{(1,0)_\epsilon} &= \frac{2\pi^5}{2025}E^{SO(2,2)}_{2\Lambda_1} = \frac{2\pi^5}{2025}E^{SL(2)}_{2\Lambda_1}E^{SL(2)}_{2\Lambda_1} , \\
\mathcal{E}^{(2)}_{(1,0)_\epsilon} &= \frac{2\pi^4}{135}\left(E^{SO(2,2)}_{2\Lambda_2} + E^{SO(2,2)}_{2\Lambda_1}\right) = \frac{2\pi^4}{135}\left(E^{SL(2)}_{2\Lambda_1} + E^{SL(2)}_{2\Lambda_1}\right) ,
\end{aligned} \tag{4.3.38}$$

where we have used the isomorphism $SO(2,2) = SL(2)\times SL(2)$. Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(1,0)} = \zeta(5)e^{-\frac{10}{3}\phi} + \frac{2\pi^5}{2025}E^{SL(2)}_{2\Lambda_1}E^{SL(2)}_{2\Lambda_1}e^{-\frac{4}{3}\phi} + \frac{2\pi^4}{135}\left(E^{SL(2)}_{2\Lambda_1} + E^{SL(2)}_{2\Lambda_1}\right)e^{\frac{2}{3}\phi} . \tag{4.3.39}$$

This is also manifestly invariant under T-duality. Therefore by looking at the constant terms in the Fourier decomposition (4.1.72) on the maximal parabolic subgroup P_1 we can see that

$$\begin{aligned}\mathcal{E}_{(1,0)} &= \zeta(5)E^{SL(3)}_{\frac{5}{2}\Lambda_1} + \frac{2\pi^4}{135}E^{SL(3)}_{-\frac{1}{2}\Lambda_1}E^{SL(2)}_{2\Lambda_1} \\ &= \frac{1}{2}\mathrm{Ep}^{SL(3)}_{\frac{5}{2}} - 4\mathrm{Ep}^{SL(3)}_{-\frac{1}{2}}\mathrm{Ep}^{SL(2)}_{2}\,, \end{aligned} \tag{4.3.40}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$. One can also check that it obeys the right Laplace equation. This is the full non perturbative Wilson coupling [48].

Also note that if we take the limit $R_2 \to \infty$, which corresponds to only keeping the constant terms in the Fourier decomposition (4.1.70) for the maximal parabolic subgroup $P_2 \times P_1$ of $SL(3) \times SL(2)$, we get

$$\begin{aligned}\mathcal{E}_{(1,0)} \sim\ & \zeta(5)r'^{\frac{5}{2}}E^{SL(2)}_{\frac{5}{2}\Lambda_1} + \frac{2\pi^2}{45}\zeta(3)(\mathrm{Im}\,\tau)^2 r'^{-\frac{1}{2}}E^{SL(2)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{45}\zeta(3)(\mathrm{Im}\,\tau)^{-1}r'^4 \\ & + \frac{\zeta(3)}{\pi}\left(2\zeta(3)(\mathrm{Im}\,\tau)^{-1}r'^{-\frac{1}{2}}E^{SL(2)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{3}r'^{-2}\right) + \frac{2\pi^5}{2025}(\mathrm{Im}\,\tau)^2 r'^4\,. \end{aligned} \tag{4.3.41}$$

The first line corresponds to the next to minimal Wilson coupling in dimension 9. Therefore $\mathcal{E}_{(1,0)}$ also has the right degeneration limit for the decompactification from 8 to 9 dimensions. The first term of the second line corresponds to the minimal Wilson coupling in dimension 9. The last term comes from a threshold term.

4.3.4 7 dimensions

In $D = 7$ dimensions the moduli space is $SO(5)\backslash SL(5,\mathbb{R})$ and the U-duality group is $SL(5,\mathbb{Z})$. The moduli fields in this dimension are $H_5 \in \mathcal{H}_S$.

Minimal Wilson coupling In $D = 7$ dimensions the minimal Wilson coupling satisfies the Laplace equation

$$\Delta\mathcal{E}_{(0,0)} = -\frac{12}{5}\mathcal{E}_{(0,0)}\,. \tag{4.3.42}$$

The only perturbative contribution to $\mathcal{E}_{(0,0)}$ come from the genus 0 and genus 1 string amplitudes

$$\mathcal{E}^{\mathrm{pert}}_{(0,0)} = e^{-\frac{12}{5}\phi}c_{(0,0)} + e^{-\frac{2}{5}\phi}\mathcal{E}^{(1)}_{(0,0)_\epsilon}\,. \tag{4.3.43}$$

For $D = 7$ dimensions we still have $c_{(0,0)} = 2\zeta(3)$ and we have

$$\mathcal{E}^{(1)}_{(0,0)_\epsilon} = 4\pi\xi(1)E^{SO(3,3)}_{\frac{1}{2}\Lambda_1} = \frac{2\pi^2}{3}E^{SL(4)}_{\Lambda_1}\,. \tag{4.3.44}$$

Where we have used the isomorphism $SO(3,3) = SL(4)$. Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(0,0)} = 2\zeta(3)e^{-\frac{12}{5}\phi} + \frac{2\pi^2}{3}E^{SL(4)}_{\Lambda_1}e^{-\frac{2}{5}\phi}\,. \tag{4.3.45}$$

We can decompose H_5 into r, H_4 and x_4 as in (4.1.68) for the maximal parabolic subgroup P_1 of $SL(5)$. We have $r = e^{-\frac{2}{5}\phi}$. Therefore using the Fourier decomposition on the maximal parabolic subgroup P_1 (4.1.72) and the asymptotic behavior of the Bessel K function we can see that

$$\mathcal{E}_{(0,0)} = 2\zeta(3)E^{SL(5)}_{\frac{3}{2}\Lambda_1} = \text{Ep}^{SL(5)}_{\frac{3}{2}} \tag{4.3.46}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$ and obeys the right Laplace equation. This is the full non perturbative Wilson coupling [48]. Therefore the constant terms in the Fourier expansion on the maximal parabolic subgroup P_1, which is associated with the perturbative string theory limit, correspond again to the perturbative contributions.

We can now decompose H_5 into r', H'_3, H'_2 and $x'_3x'^T_2$ as in (4.1.68) for the maximal parabolic subgroup P_3 of $SL(5)$ which corresponds to the maximal parabolic subgroup P_4 of $E_{4(4)}$. Let R be the radius of compactification from 8 to 7 dimensions. Then if we take the limit $R \to \infty$, which corresponds to only keeping the constant terms in the Fourier decomposition (4.1.70) for the maximal parabolic subgroup P_3 of $SL(5)$, we get

$$\begin{aligned}\mathcal{E}_{(0,0)} \sim \Bigg(2\zeta(3)\hat{E}^{SL(3)}_{\frac{3}{2}\Lambda_1} + \frac{2\pi^2}{3}\hat{E}^{SL(2)}_{\Lambda_1} + \frac{22\pi}{3} - 4\pi\ln\left(2\pi\frac{l_P}{l_s}\right) \\ +2\pi\left(2\gamma_E - \frac{11}{3}\right) + 4\pi\ln\left(\frac{l_P}{2l_s}r'^{\frac{5}{2}}\right)\Bigg)r'^3\,,\end{aligned} \tag{4.3.47}$$

where we have used the functional relation (4.1.44). The first line corresponds to the minimal Wilson coupling in dimension 8. Therefore $\mathcal{E}_{(0,0)}$ also has the right degeneration limit for the decompactification from 7 to 8 dimensions. The second line results from a threshold in 8 dimensions.

Next to minimal Wilson coupling In $D = 7$ dimensions the next to minimal Wilson coupling satisfies the differential equation

$$\Delta\mathcal{E}_{(1,0)} = \frac{20\pi^2}{3}\,. \tag{4.3.48}$$

This indicates again that we may need to consider Eisenstein series evaluated at some pole. The only perturbative contribution to $\mathcal{E}_{(1,0)}$ come from the genus 0, genus 1 and genus 2 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(1,0)} = e^{-4\phi}c_{(1,0)} + e^{-2\phi}\mathcal{E}^{(1)}_{(1,0)\epsilon} + \mathcal{E}^{(2)}_{(1,0)\epsilon}\,. \tag{4.3.49}$$

For $D = 7$ dimensions we still have $c_{(1,0)} = \zeta(5)$ and we have

$$\begin{aligned}
\mathcal{E}^{(1)}_{(1,0)\epsilon} &= \frac{\pi}{15}\zeta(5)E^{SO(3,3)}_{\frac{5}{2}\Lambda_1} = \frac{\pi}{15}\zeta(5)E^{SL(4)}_{\frac{5}{2}\Lambda_2}\,,\\
\mathcal{E}^{(2)}_{(1,0)\epsilon} &= \frac{2\pi^4}{135}\left(E^{SO(3,3)}_{2\Lambda_3\epsilon} + E^{SO(3,3)}_{2\Lambda_2\epsilon}\right) = \frac{2\pi^4}{135}\left(\hat{E}^{SL(4)}_{2\Lambda_1} + \hat{E}^{SL(4)}_{2\Lambda_3}\right) + \frac{16\pi^2}{15}\phi\,,
\end{aligned} \tag{4.3.50}$$

where we have used the isomorphism $SO(3,3) = SL(4)$ and we have used the regularisations (4.3.34) and (4.3.35). The genus 2 contribution is divergent which also indicates that we need to consider Eisenstein series evaluated at some pole. Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(1,0)} = \zeta(5)e^{-4\phi} + \frac{\pi}{15}\zeta(5)E^{SL(4)}_{\frac{5}{2}\Lambda_2}e^{-2\phi} + \frac{2\pi^4}{135}\left(\hat{E}^{SL(4)}_{2\Lambda_1} + \hat{E}^{SL(4)}_{2\Lambda_3}\right) + \frac{16\pi^2}{15}\phi\,. \tag{4.3.51}$$

Note that with the regularisation (4.3.34) the Fourier decomposition (4.1.72) on the maximal parabolic subgroup P_1 of the Eisenstein series $E^{SL(5)}_{\frac{5}{2}\Lambda_1}$ is given by

$$\begin{aligned}
\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_1} = {}& r^{10} + \frac{2\pi^4}{135\zeta(5)}\hat{E}^{SL(4)}_{2\Lambda_1} - \frac{2\pi^2}{3\zeta(5)}\ln r\\
&+ \frac{8\pi^2}{3\zeta(5)}{\sum_{\Gamma_4\in\mathbb{Z}^4}}' \sigma_4(\Gamma)\frac{r^5}{|Z(\Gamma)|^2}K_2(2\pi r^{\frac{5}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,.
\end{aligned} \tag{4.3.52}$$

Let us now consider the following regularisation of the Eisenstein series $E^{SL(5)}_{s\Lambda_3}$ at the pole $s = \frac{5}{2}$

$$\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3}(H_5) = \lim_{\epsilon\to 0}\left(E^{SL(5)}_{(\frac{5}{2}+\epsilon)\Lambda_3}(H_5) - \frac{10\pi}{\zeta(5)\epsilon} - \frac{10\pi}{3\zeta(5)}(12\gamma_E + 6\ln\pi + \ln 64 - 11 - 72\ln A) \right.$$
$$\left. +20\pi\frac{\zeta'(5)}{\zeta(5)^2} + \frac{1800}{\pi^3}\frac{\zeta'(4)}{\zeta(5)}\right)$$
$$= r^5 E^{SL(4)}_{\frac{5}{2}\Lambda_2} + \frac{2\pi^3}{9\zeta(5)}\hat{E}^{SL(4)}_{2\Lambda_3} - \frac{30\pi}{\zeta(5)}\ln r$$
$$+ \frac{8\pi^2}{3\zeta(5)}{\sum_{\Gamma_4\in\mathbb{Z}^4}}' \frac{\sigma_{-2}(\Gamma)}{\gcd(\Gamma)^{-\frac{4}{3}}} E^{SL(3)}_{2\Lambda_2}(U_\Gamma)\frac{r^{\frac{5}{2}}}{|Z(\Gamma)|^{\frac{1}{3}}}K_2(2\pi r^{\frac{5}{2}}|Z(\Gamma)|)e^{2\pi i\Gamma(x)}\,, \tag{4.3.53}$$

where we have used the Fourier decomposition on the maximal parabolic subgroup P_1 (4.1.89). Therefore we can see that

$$\mathcal{E}_{(1,0)} = \zeta(5)\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_1} + \frac{\pi}{15}\zeta(5)\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3} = \frac{1}{2}\hat{\mathrm{Ep}}^{SL(5)}_{\frac{5}{2}} + \frac{\pi}{15}\zeta(5)\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3} \tag{4.3.54}$$

has the right degeneration limit for $1/g_s = e^{-\phi} \to \infty$. One can also check that it obeys the right Laplace equation, see (4.1.46) and (4.1.49). This is the full non perturbative Wilson coupling [48].

One can also check by looking at the constant terms in the Fourier decomposition for the maximal parabolic subgroup P_3 of $SL(5)$ that we get the right degeneration limit for the decompactification from 7 to 8 dimensions.

From (4.3.53) we can deduce a general way to regularise next to minimal Eisenstein series at the pole $s = \frac{n}{2}$

$$\hat{E}^{SL(n)}_{\frac{n}{2}\Lambda_2} = \lim_{\epsilon\to 0}\left(E^{SL(n)}_{(\frac{n}{2}+\epsilon)\Lambda_2} - \frac{\xi(1+2\epsilon)\xi(2+2\epsilon)}{\xi(n-1+2\epsilon)\xi(n+2\epsilon)}\right)\,,$$
$$\hat{E}^{SL(n)}_{\frac{n}{2}\Lambda_{n-2}} = \lim_{\epsilon\to 0}\left(E^{SL(n)}_{(\frac{n}{2}+\epsilon)\Lambda_{n-2}} - \frac{\xi(1+2\epsilon)\xi(2+2\epsilon)}{\xi(n-1+2\epsilon)\xi(n+2\epsilon)}\right)\,. \tag{4.3.55}$$

4.3.5 6 dimensions

In $D = 6$ dimensions the moduli space is $(SO(5)\times SO(5))\backslash SO(5,5,\mathbb{R})$ and the U-duality group is $SO(5,5,\mathbb{Z})$. The moduli fields in this dimension are $G_5 + B_5 \in \mathcal{H}$ such that H_{G_5,B_5} as defined in (3.5.14) is in the moduli space.

Minimal Wilson coupling In $D = 6$ dimensions the minimal Wilson coupling satisfies the Laplace equation

$$\Delta \mathcal{E}_{(0,0)} = -\frac{15}{2}\mathcal{E}_{(0,0)} \,. \tag{4.3.56}$$

The only perturbative contribution to $\mathcal{E}_{(0,0)}$ come from the genus 0 and genus 1 string amplitudes

$$\mathcal{E}^{\text{pert}}_{(0,0)} = e^{-3\phi} c_{(0,0)} + e^{-\phi}\mathcal{E}^{(1)}_{(0,0)_\epsilon} \,. \tag{4.3.57}$$

For $D = 8$ dimensions we still have $c_{(0,0)} = 2\zeta(3)$ and we have

$$\mathcal{E}^{(1)}_{(0,0)_\epsilon} = \frac{2\pi^2}{3} E^{SO(4,4)}_{\Lambda_1} \,. \tag{4.3.58}$$

Hence the perturbative part of the Wilson coupling is given by

$$\mathcal{E}^{\text{pert}}_{(0,0)} = 2\zeta(3)e^{-3\phi} + \frac{2\pi^2}{3} E^{SO(4,4)}_{\Lambda_1} e^{-\phi} \,. \tag{4.3.59}$$

One can check by looking at the constant terms in the Fourier decomposition for the maximal parabolic subgroup P_1 of $SO(5,5)$ that

$$\mathcal{E}_{(0,0)} = 2\zeta(3) E^{SO(5,5)}_{\frac{3}{2}\Lambda_1} = \text{Ep}^{SO(5,5)}_{\frac{3}{2}} \tag{4.3.60}$$

has the right degeneration limit for perturbative string theory and obeys the right Laplace equation, see (4.1.53). This is the full non perturbative Wilson coupling [48].

We can also decompose H_{G_5,B_5} into r, H_5 and B_5 as in (4.1.80) for the maximal parabolic subgroup P_5 of $SO(5,5)$. Then if we look at the constant terms in the Fourier decomposition (4.1.83) for the maximal parabolic subgroup P_5 of $SO(5,5)$, we get

$$\mathcal{E}_{(0,0)} \sim 2\zeta(3)\hat{E}^{SL(5)}_{\frac{3}{2}\Lambda_4} r^{\frac{3}{2}} + \frac{2\pi^2}{3} r^{\frac{5}{2}} \,, \tag{4.3.61}$$

The first term corresponds to the minimal Wilson coupling in dimension 7 where H_5^{-1} are the moduli fields in 7 dimension. Therefore $\mathcal{E}_{(0,0)}$ also has the right degeneration limit for the decompactification from 6 to 7 dimensions. The second term comes from a threshold in 7 dimensions.

We can see a pattern emerge for low dimensions that the full non perturbative minimal Wilson coupling $\mathcal{E}_{(0,0)}$ is given by an Epstein series for the U-duality group

evaluated at $s = \frac{3}{2}$. Similarly one can show that for low dimensions the full non perturbative next to minimal Wilson coupling $\mathcal{E}_{(1,0)}$ is given by half of an Epstein series for the U-duality group evaluated at $s = \frac{5}{2}$. Indeed in general we have in dimension $D = 11 - d$ [47, 48, 153, 157, 158, 166]

$$\begin{aligned}
\mathcal{E}_{(0,0)} &= 2\zeta(3)E^{E_{d(d)}}_{\frac{3}{2}\Lambda_1} = \mathrm{Ep}^{E_{d(d)}}_{\frac{3}{2}}\,, \qquad (D \leq 7) \\
\mathcal{E}_{(1,0)} &= \zeta(5)E^{E_{d(d)}}_{\frac{5}{2}\Lambda_1} = \frac{1}{2}\mathrm{Ep}^{E_{d(d)}}_{\frac{5}{2}}\,. \qquad (D \leq 5)
\end{aligned} \tag{4.3.62}$$

These two relations also hold in ten dimensions for type IIB string theory.

Therefore we have seen that the general strategy to find the full non perturbative Wilson couplings in any dimension $D = 11 - d$ is to start from the perturbative string theory expansion and then find a function invariant under the U-duality group that satisfies the differential equation imposed by supersymmetry and whose constant terms in the Fourier expansion on the maximal parabolic subgroup P_1 of $E_{d(d)}$ agrees with the perturbative expansion. The non constant terms in the Fourier expansion on P_1 then correspond to non perturbative instanton contributions. Important sanity checks involve checking that the constant terms in the Fourier expansion on the maximal parabolic subgroup P_d of $E_{d(d)}$ agree with the full non perturbative Wilson coupling in dimension $D + 1$ and the constant terms in the Fourier expansion on the maximal parabolic subgroup P_2 of $E_{d(d)}$ reproduce the results of eleven dimensional supergravity compactified on a torus T^d.[9] This strategy works best for the first three Wilson couplings which are protected by supersymmetry and hence whose perturbative expansion is finite, see [48, 49, 51]. A natural guess for a non perturbative completion of Wilson coefficients is an automorphic form for the U-duality group because they already satisfy the conditions of invariance under the U-duality group as well as interesting differential equations. However it is not always the case that the non pertrubative Wilson coefficient is constructed from automorphic forms, for example the next to next to leading Wilson coupling in dimension $D \geq 8$ belongs to a class of functions on $K\backslash E_{d(d)}/E_{d(d)}(\mathbb{Z})$ that generalises the notion of automorphic forms. Indeed in general Wilson couplings fail to satisfy condition 3. of the definition of automorphic forms in that they do not define a finite dimensional representation of the centre $\mathcal{Z}(\mathfrak{e}_{d(d)})$, i.e. they are not eigenfunctions of all differential operators.

We can end this section by extending this exposition to 11 dimensions and giving the non perturbative M-theory leading Wilson coupling. As we have seen, there are no massless scalar fields in M-theory/eleven dimensional supergravity and hence no

[9] We have not looked at this limit in our exposition.

moduli fields. Therefore there is no U-duality group in eleven dimensions and the Wilson coupling is a constant which can be found by anomaly cancellation required for consistent coupling with the M5 brane [168, 169]. This is given by

$$\mathcal{E}_{(0,0)} = \frac{2\pi^2}{3}\,, \qquad (D = 11) \tag{4.3.63}$$

for the leading Wilson coefficient and by

$$\mathcal{E}_{(1,0)} = 0\,, \qquad (D = 11) \tag{4.3.64}$$

for the next to leading Wilson coefficient. If one recalls that the type IIA dilaton is related to the radius of compactification R from 11 to 10 dimensions by $e^{\phi} = \frac{R}{l_s}$ one can check that in the limit $R \to \infty$ the type IIA Wilson coefficients given by (4.3.6) and (4.3.12) reproduce the M-theory Wilson couplings.[10]

[10] Although there is a term in (4.3.12) that survives the limit it does not have the right power of the dilaton to come from the $D = 11$ next to minimal Wilson coupling. In fact this term comes from Kaluza-Klein modes contributions.

Chapter 5

Minima of leading Wilson couplings

As we saw in the introduction, maximally supersymmetric gravitational theories provide a natural setting where one might say precise things about the string lamppost principle. We already saw that if we look at the leading two derivatives operators in the effective action then the string lamppost principle is pretty much verified in all dimensions as maximal supergravity is unique[1] and coincides with the low energy limit of maximally supersymmetric string theory compactified on a torus. In this section we will look at the leading higher order operators in the effective field theory expansion which constitutes the core of this thesis. For a general maximally supersymmetric gravitational theory we can write the first few Wilson coefficients of the gravitational sector of the effective action in Einstein frame as

$$S = \frac{1}{2\kappa_D^2}\int d^D x\,\sqrt{-g}\left(R + \frac{l_P^6}{48}c_{(0,0)}\mathcal{R}^4 + \frac{l_P^{10}}{48}c_{(1,0)}\nabla^4\mathcal{R}^4 + \frac{l_P^{12}}{48}c_{(0,1)}\nabla^6\mathcal{R}^4 + \dots\right), \tag{5.0.1}$$

where here $c_{p,q}$ are generic Wilson coefficients which have a priori nothing to do with string theory. We saw in chapter 1 that S-matrix bootstrap methods can be used to constrain the amplitudes of such a theory and therefore the Wilson coefficients as well. Indeed the low energy limit of the amplitude is given by

$$\mathcal{A}_{2\to 2'} = -i\frac{\kappa_D^2}{2^{10}}\mathcal{R}^4(k_i,\epsilon_i)\left(\frac{64}{stu} + c_{(0,0)}l_P^6 + 32(2\pi)^{D-3}l_P^{D-2}\left(I_4^{(1)}(s,t) + I_4^{(1)}(t,u) + I_4^{(1)}(u,s)\right)\right.$$
$$\left. + c_{(1,0)}\frac{l_P^{10}}{16}(s^2+t^2+u^2) + \dots\right), \tag{5.0.2}$$

[1] Except in ten dimensions of course.

where $I_4^{(1)}$ is the one loop supergravity integral given in (3.5.28). One can then use techniques similar to section 3.4 to relate the unitarity bounds on this amplitude to unitarity bounds on the scalar amplitude using maximal supersymmetry and factorisation properties of amplitudes. We can see that for dimensions $D > 8$ the contribution from the leading Wilson coefficient $c_{(0,0)}$ in the amplitude (5.0.2) is leading at low energy with respect to the one loop supergravity amplitude. In these dimensions one can therefore translate the bounds on the amplitudes to bounds on the leading Wilson coefficient $c_{(0,0)}$. For example it is easy enough using the positivity bound (1.3.10) on the scalar amplitude and usual contour integration arguments [170–174] to show that the leading Wilson coefficients in these dimensions must be positive [45].

$$c_{(0,0)} \geq 0\,. \qquad (D > 8) \tag{5.0.3}$$

Of course this bound can be refined using the elastic positivity bound (1.3.11) as was done in [45, 46] although the bounds are not sharp because they are missing inelastic effects in the optical theorem.

In dimensions $D \leq 8$ it is the one loop supergravity amplitude that is leading at low energy with respect to leading Wilson coefficient $c_{(0,0)}$. In this dimensions it is much more complicated to extract bounds on $c_{(0,0)}$ and in particular no such positivity bound can be derived. For dimensions $D \leq 5$ the two loop supergravity amplitudes also becomes leading with respect to the leading Wilson coefficient which further complicates matters.

An interesting question with respect to the string lamppost principle is whether the range of the exact leading Wilson coefficients $\mathcal{E}_{(0,0)}$ of string theory as found in chapter 4 saturate the range allowed by S-matrix unitarity bounds. The Wilson coefficients of string theory can take arbitrary large values which is coherent with the unitarity range which is not bounded from above. Therefore the key question is whether the lower bounds of the Wilson coefficients coincide with the lower bounds imposed by unitarity. It was found in [45, 46] that the leading Wilson coefficients in dimensions $D > 8$ nearly saturate unitarity bounds. Due to non sharpness of these unitarity bounds and the fact that taking into account particle production is expected to raise the unitarity bounds [175] one can conclude that string theory does indeed seem to saturate the sharp unitarity bounds in dimensions $D > 8$. One of the main efforts of this PhD thesis was to derive lower bounds for the leading and next to leading Wilson coefficients in dimensions $6 \leq D \leq 8$. As shown in section 4.3 this involves finding the global minima of Epstein series and Eisenstein series for the groups $SL(n)$ and $SO(n,n)$ for small n at special values of the parameter s.

In this section we begin by finding suitable fundamental domains for the mod-

uli spaces relevant to us. We then give the main results leading to the conjectured minima of Epstein series at symmetric points and in particular prove they are local minima. Finally we give our results regarding the lower bounds on exact leading Wilson coefficients of maximally supersymmetric string theory. More details regarding this section can be found in [176] as well as in an upcoming article.

5.1 Fundamental domain of $K\backslash G/G(\mathbb{Z})$

Let $\mathcal{M} = K\backslash G$ be some symmetric space and let $f : \mathcal{M} \to \mathbb{R}$ be some real automorphic function for some discrete subgroup Γ of G. Instead of studying the minima of f on the whole symmetric space $\mathcal{M}$ one may restrict to considering $\mathcal{M}$ up to transformations by Γ, this leads us to the concept of fundamental domain for $K\backslash G/\Gamma$. A fundamental domain $\mathcal{F}$ for $\mathcal{M}/\Gamma$ is a subset $\mathcal{F} \subset \mathcal{M}$ from which the whole space $\mathcal{M}$ can be recovered by the action of Γ and which is in some sense the smallest possible. More precisely one defines a free regular set $\mathring{\mathcal{F}} \subset \mathcal{M}$ for Γ as an open set in $\mathcal{M}$ such that any point in $\mathcal{M}$ can be mapped to its closure $\mathcal{F} = \overline{\mathring{\mathcal{F}}}$ in $\mathcal{M}$ by the action of Γ, and for any element $\gamma \in \Gamma$ acting non-trivially on $\mathcal{M}$ one has $\gamma\mathcal{F} \cap \mathring{\mathcal{F}} = \emptyset$. It is also required that $\partial\mathcal{F} = \mathcal{F} \setminus \mathring{\mathcal{F}}$ is of measure zero in $\mathcal{M}$. We then call $\mathcal{F}$ a fundamental domain of Γ in $\mathcal{M}$. Equivalently one can define $\mathcal{F}$ to satisfy

$$\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F} = \mathcal{M}\,, \qquad \gamma\mathcal{F} \cap \mathcal{F} \subset \partial\mathcal{F}\,, \tag{5.1.1}$$

for all $\gamma \in \Gamma$ acting non trivially.

Our biggest motivation for studying fundamental domains comes from this second condition. Indeed, as explained in the next section, we have good reasons to believe that potential minima of automorphic forms will lie at self dual points, i.e. points which are mapped to themselves by some subgroup of Γ. By definition these have to lie on the boundary of any fundamental domain. In fact the more symmetries these points have, i.e. the bigger the subgroup of Γ stabilizing the point, the likelier they are to be minima. And the points with the biggest amount of symmetries are typically point which lie on boundaries of low dimension of $\mathcal{F}$, the best candidates being dimension 0 boundary points (sometimes called corners).

It is important to realise however that the study of boundaries of fundamental domains depends on the choice of the latter so this analysis can only take us so far. Indeed there is no canonical fundamental domain. A first fundamental domain was introduced by Minkowski for $GL(n,\mathbb{Z})$ acting on $SO(n)\backslash SL(n)$ [177]. Grenier then defined a different fundamental domain for $GL(n,\mathbb{Z})$ that is easier to generalise to

arbitrary simple groups [178]. Let us first review Grenier's construction in the case of $SL(n,\mathbb{Z})$.

5.1.1 Fundamental domain of $SL(n,\mathbb{Z})$

Let us first review the case of the standard fundamental domain for $SO(2)\backslash SL(2)/SL(2,\mathbb{Z})$. As we saw in section 3.3 the symmetric space $SO(2)\backslash SL(2)$ is isomorphic to the upper complex half plane $\mathcal{H}_1$ and a standard fundamental domain for $SO(2)\backslash SL(2)/SL(2,\mathbb{Z})$ was given by (3.3.8). We can give another equivalent formulation for this fundamental domain which makes it look even more analogous to the fundamental domain for the Siegel upper half plane (3.3.19).

$$\mathcal{F}_1 = \{\tau \in \mathbb{C} \mid |\mathrm{Re}\,\tau| \leq 1/2,\ \mathrm{Im}\,\tau > 0, |n + m\tau|^2 \geq 1\}\,, \tag{5.1.2}$$

for all $n, m \in \mathbb{Z}$ such that

$$\begin{pmatrix} n & r \\ m & s \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{5.1.3}$$

Just like for the Siegel fundamental domain the last condition only needs to be checked for a finite number of n, m. In fact the only doublet for which this condition is non-trivial is $n = 0$, $m = 1$ for which one gets back the usual fundamental domain (3.3.8). If we define $H_\tau \in \mathcal{H}_S$ as in (4.1.73) it is easy to see that $\tau \in \mathcal{H}_1$ is in the fundamental domain $\mathcal{F}_1$ if and only if

1. $H_\tau[q] \geq 1/\,\mathrm{Im}\,\tau$ for all $q \in \mathbb{Z}^2$ such that $\gcd q = 1$,
2. $|\mathrm{Re}\,\tau| \leq \frac{1}{2}$.

This form of the fundamental domain is useful to us because this can be generalised by recursion.

Indeed the idea behind Grenier's fundamental domain is to extend this recusively to $SO(n)\backslash SL(n)$ which we identify with the space of symmetric positive definite unimodular matrices $\mathcal{H}_S$. We can decompose $H_n \in \mathcal{M}_n = SO(n)\backslash SL(n)$ into r, H_{n-1} and x_{n-1} as in (4.1.68) for the parabolic subgroup P_1 of $SL(n)$. Grenier's fundamental domain $\mathcal{F}^{(n)}$ of $\mathcal{M}_n$ under the action of $SL(n,\mathbb{Z})$ is then given in terms of the fundamental domain $\mathcal{F}^{(n-1)}$ of $\mathcal{M}_{n-1}$ under the action of $GL(n-1,\mathbb{Z})$[2] by the conditions

[2]This is a slight subtlety but it is not a complication because the fundamental domain for $GL(n,\mathbb{Z})$ is the same as that of $SL(n,\mathbb{Z})$ except $0 \leq (x_{n-1})_1 \leq \frac{1}{2}$ for all n and the condition $\gcd q = 1$ is replaced by $q \neq 0$ for $n = 2$.

1. $H_{n-1} \in \mathcal{F}^{(n-1)}$,
2. $H_n[q] \geq 1/r^{n-1}$ for all $q \in \mathbb{Z}^n$ such that $\gcd q = 1$,
3. $|(x_{n-1})_i| \leq \frac{1}{2}$ for $2 \leq i \leq n-1$ and $|(x_{n-1})_1| \leq \frac{1}{2}$ for n even or $0 \leq (x_{n-1})_1 \leq \frac{1}{2}$ for n odd.

Again one can actually show that condition 2. only implies a finite number of inequalities. The proof that this is the case and that this indeed defines a fundamental domain was given by Grenier in [178]. The resulting Grenier fundamental domain for the case $n = 3$ is given in [176].

5.1.2 Generalisation to $G(\mathbb{Z})$

It appears that Grenier's construction of a fundamental domain is based on a sequence of maximal abelian parabolic subgroups, i.e. maximal parabolic subgroups for which the unipotent radical is abelian. In this section we therefore consider G to be the split real form of a simple Lie group of rank r admitting an abelian parabolic subgroup. This last condition excludes only $E_{8(8)}$, $F_{4(4)}$ and $G_{2(2)}$. It includes in particular all the relevant moduli spaces and U-duality groups for maximally supersymmetric string theory in dimensions greater or equal to four, see table 4.1.

Let $P_i^{(1)}$ be a maximal abelian parabolic subgroup associated to the fundamental weight Λ_i. We assume it is of the form

$$P_i^{(1)} = (GL(1) \times G_i^{(1)}) \ltimes U_i^{(1)}, \tag{5.1.4}$$

where the semi-simple part of the Levi subgroup $G_i^{(1)}$ is itself the split real form of a semi-simple group and the unipotent radical $U_i^{(1)}$ is abelian. According to the Iwasawa decomposition, for any element $v \in \mathcal{M} = K\backslash G$ we can find a coset representative $v \sim p$ with $p \in P_i^{(1)}$. Therefore, by the Levi decomposition we can decompose v as

$$v = e^{-\frac{1}{2}\ln y_i^{(1)} H_{\alpha_i}} v^{(1)} e^{a^{(1)}}, \tag{5.1.5}$$

with $y_i^{(1)} \in \mathbb{R}_+^*$, $a^{(1)} \in \mathfrak{n}_i^{(1)}$ the Lie algebra of the unipotent radical $U_i^{(1)}$ and $v^{(1)} \in \mathcal{M}^{(1)} = K_i^{(1)}\backslash G_i^{(1)}$ where $K_i^{(1)}$ is the maximal compact subgroup of $G_i^{(1)}$. Also note that H_{α_i} is the generator of the Chevalley basis (4.1.26) associated to the root α_i such that $\langle \Lambda_i, \alpha_i^\vee \rangle = 1$. Therefore if $e_{\Lambda_i} \in R(\Lambda_i)$ is the highest weight vector of the representation $R(\Lambda_i)$ then if we assume that v is in the representation $R(\Lambda_i)$ we have

$$v e_{\Lambda_i} = \frac{1}{\sqrt{y_i^{(1)}}} e_{\Lambda_i}. \tag{5.1.6}$$

For each $v \in \mathcal{M}$ we define the symmetric bilinear form $H[q]$ with $H = v^T v$ where v and q are in the representation $R(\Lambda_i)$. In the following we shall identify a point $v \in \mathcal{M}$ with the symmetric matrix H, we will also use the fact that according to the Iwasawa decomposition v can be parametrised by the coordinates $y_i^{(1)}$, $H^{(1)} = (v^{(1)})^T v^{(1)}$ and $a^{(1)}$. Before generalising Grenier's construction, let us give some definitions. We define

$$\begin{aligned} G_i^{(1)}(\mathbb{Z}) &= G(\mathbb{Z}) \cap (GL(1) \times G_i^{(1)}) \,, \\ U_i^{(1)}(\mathbb{Z}) &= G(\mathbb{Z}) \cap U_i^{(1)} \,, \\ P_i^{(1)}(\mathbb{Z}) &= G(\mathbb{Z}) \cap P_i^{(1)} = G_i^{(1)}(\mathbb{Z}) \ltimes U_i^{(1)}(\mathbb{Z}) \,. \end{aligned} \tag{5.1.7}$$

Note that with this definition $G_i^{(1)}(\mathbb{Z})$ may not be a subgroup of $G_i^{(1)}$ but includes the entire discrete Levi subgroup.[3]

The construction of the fundamental domain is defined recursively. We first consider the fundamental domain of $\mathcal{M}$ for the action of the parabolic subgroup $P_i^{(1)}(\mathbb{Z})$. Let p be an element of $P_i^{(1)}(\mathbb{Z})$, we can decompose it as $p = le^n$ with $l \in G_i^{(1)}(\mathbb{Z})$ and $e^n \in U_i^{(1)}(\mathbb{Z})$. $P_i^{(1)}(\mathbb{Z})$ acts on $\mathcal{M}$ by

$$p^T H p = (y_i^{(1)}, l^T H^{(1)} l, l^{-1} a^{(1)} + n) \,. \tag{5.1.8}$$

Therefore we can see that the action of the adjoint Levi subgroup $G_i^{(1)}(\mathbb{Z})/\mathcal{Z}(G_i^{(1)}(\mathbb{Z}))$, with $\mathcal{Z}(G_i^{(1)}(\mathbb{Z}))$ the centre of $G_i^{(1)}(\mathbb{Z})$, on $\mathcal{M}^{(1)}$ determines the fundamental domain

$$\mathcal{F}^{(1)} = \mathcal{M}^{(1)} / (G_i^{(1)}(\mathbb{Z})/\mathcal{Z}(G_i^{(1)}(\mathbb{Z}))) \,. \tag{5.1.9}$$

To find a fundamental domain of $P_i^{(1)}(\mathbb{Z})$ in $\mathcal{M}$, it remains to act with $\mathcal{Z}(G_i^{(1)}(\mathbb{Z})) \ltimes U_i^{(1)}(\mathbb{Z})$ on $\mathfrak{n}_i^{(1)}$. By construction $U_i^{(1)}(\mathbb{Z})$ acts by translation and $\mathcal{Z}(G_i^{(1)}(\mathbb{Z}))$ acts either trivially or by multiplying $a^{(1)}$ by -1. One gets therefore

$$\mathcal{F}_P^{(1)} = \mathcal{M}/P_i^{(1)}(\mathbb{Z}) = \mathbb{R}_+ \times \mathcal{F}^1 \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^{d^{(1)}} / \mathbb{Z}_\mu \,, \tag{5.1.10}$$

where $d^{(1)}$ is the dimension of $U_i^{(1)}$ and $\mu = 2$ if $Z(G_i^{(1)}(\mathbb{Z}))$ acts as $\mathbb{Z}_2$ on $\mathfrak{n}_i^{(1)}$ and $\mu = 1$ if it acts trivially.

To find a fundamental domain of $\mathcal{M}$ for $G(\mathbb{Z})$, it remains to consider the action of $G(\mathbb{Z})/P_i^{(1)}(\mathbb{Z})$. Let us again consider the isomorphism (4.1.39)

[3] As an example, for $SL(n,\mathbb{Z})$ we would have $G_1^{(1)}(\mathbb{Z}) = GL(n-1,\mathbb{Z})$ which explains the previous footnote.

$$G(\mathbb{Z})/P_i^{(1)}(\mathbb{Z}) = S^{(1)} = \{q \in R(\Lambda_i, \mathbb{Z}) \,|\, q \otimes q \in R(2\Lambda_i, \mathbb{Z}),\, \gcd q = 1\}\,, \tag{5.1.11}$$

where the condition $q \otimes q \in R(2\Lambda_i, \mathbb{Z})$ is again characterised by (4.1.38) and the gcd may include a notion of sign when the only elements in $G(\mathbb{Z})$ that change the sign of q are trivial in $G(\mathbb{Z})/G_i^{(1)}(\mathbb{Z})$. If we define the positive set $S_>^{(1)} \subset \mathcal{M}$ as

$$S_>^{(1)} = \{H \in \mathcal{M} \,|\, H[q] \geq 1/y_i^{(1)},\, \forall q \in S^{(1)}\}\,, \tag{5.1.12}$$

then we can show that a fundamental domain $\mathcal{F} = \mathcal{M}/G(\mathbb{Z})$ can be defined as the intersection

$$\mathcal{F} = \mathcal{F}_P^{(1)} \cap S_>^{(1)}\,. \tag{5.1.13}$$

The proof that this indeed defines a fundamental domain is given in [176].

This can then be extended recursively by applying the same reasoning to $G_i^{(1)}$ and all $G_{i^{(k)}}^{(k)}$ semi-simple subgroups successively with maximal parabolic subgroups

$$P_{i^{(k+1)}}^{(k+1)} = (GL(1) \times G_{i^{(k+1)}}^{(k+1)}) \ltimes U_{i^{(k+1)}}^{(k+1)}\,, \tag{5.1.14}$$

such that $U_{i^{(k)}}^{(k)}$ is an abelian unipotent group for all $k = 1$ to r. By construction the recursion ends with $P_{i^{(r)}}^{(r)} = GL(1) \ltimes U_{i^{(r)}}^{(r)}$. Also by construction the Borel subgroup is given by

$$B = GL(1)^r \ltimes U_{i^{(r)}}^{(r)} \ltimes \cdots \ltimes U_{i^{(1)}}^{(1)}\,, \tag{5.1.15}$$

and the Iwasawa decomposition of G is compatible with this succession of abelian parabolic subgroups

$$B \subset GL(1)^{r-2} \times P_{i^{(r-1)}}^{(r-1)} \ltimes U_{i^{(r-2)}}^{(r-2)} \cdots \ltimes U_{i^{(1)}}^{(1)} \subset \cdots \subset GL(1) \times P_{i^{(2)}}^{(2)} \ltimes U_{i^{(1)}}^{(1)} \subset P_{i^{(1)}}^{(1)} \subset G\,. \tag{5.1.16}$$

5.1.3 Fundamental domain of $SO(n,n,\mathbb{Z})$

Let us now see how this construction applies to the symmetric space $\mathcal{M}_n = (SO(n) \times SO(n))\backslash SO(n,n)$ under the action of $SO(n,n,\mathbb{Z})$ preserving the even self-dual lattice of split signature $I\!I_{n,n}$. As we have seen, in string theory the group of T-duality on the torus T^d is $O(d,d,\mathbb{Z})$ whereas the group of U-duality in dimension 6 is $SO_0(5,5,\mathbb{Z})$. The difference with respect to the different groups $SO_0(n,n,\mathbb{Z})$, $SO(n,n,\mathbb{Z})$ and $O(n,n,\mathbb{Z})$ only appear in the first step because $SO_0(1,1,\mathbb{Z}) = \{1\}$, $SO(1,1,\mathbb{Z}) = \mathbb{Z}_2$

and $O(1,1,\mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

We will use the convention that the split signature metric is

$$\eta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} . \tag{5.1.17}$$

Notice that the symmetric space $(SO(n) \times SO(n))\backslash SO(n,n)$ is isomorphic to the space of invariant symmetric positive definite unimodular real matrices $\mathcal{H} = \{H \in \mathrm{Mat}(2n,\mathbb{R}) \,|\, H^T = H,\, \det H = 1,\, H > 0,\, \eta^T H \eta = H\}$ where the isomorphism is given by $H = v^T v$ with v is in the fundamental representation $R(\Lambda_1)$ of $SO(n,n)$. In the following we shall identify points in $\mathcal{M}_n$ with symmetric matrices $H_n \in \mathcal{H}$.

We consider the maximal parabolic subgroup P_1 associated to the fundamental weight Λ_1, it is given by

$$P_1 = (GL(1) \times SO(n-1,n-1)) \ltimes \mathbb{R}^{n-1,n-1} . \tag{5.1.18}$$

Therefore the semi-simple part of the Levi subgroup is $SO(n-1,n-1)$ which is indeed the split real form of a simple group and the unipotent radical is given by $\mathbb{R}^{n-1,n-1}$ which is indeed abelian. We can decompose

$$H_n[q_n] = r^{-1}\left(q + (x_{n-1}, q_{n-1}) + \frac{1}{2}(x_{n-1}, x_{n-1})p\right)^2 + H_{n-1}[q_{n-1} + x_{n-1}p] + rp^2 \tag{5.1.19}$$

and

$$(q_n, q_n) = (q_{n-1}, q_{n-1}) + 2qp\,, \tag{5.1.20}$$

where $q_n = (q, q_{n-1}, p)^T$. We have $r \in \mathbb{R}_+^*$, $H_{n-1} \in \mathcal{M}_{n-1} = (SO(n-1) \times SO(n-1))\backslash SO(n-1,n-1)$, $x_{n-1} \in \mathbb{R}^{n-1,n-1}$, $q_{n-1} \in I\!I_{n-1,n-1}$ and $q, p \in \mathbb{Z}$. In matrix notation this corresponds to a decomposition

$$H_n = \begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & H_{n-1} & 0 \\ 0 & 0 & r \end{pmatrix} \begin{bmatrix} 1 & x_{n-1}^T \eta & \frac{1}{2}(x_{n-1}, x_{n-1}) \\ 0 & I_{2n-2} & x_{n-1} \\ 0 & 0 & 1 \end{bmatrix} . \tag{5.1.21}$$

With the definitions of the previous paragraph we would have $G_1^{(1)}(\mathbb{Z}) = O(n-1,n-1,\mathbb{Z})$. Also recall that for $q \in R(\Lambda_1,\mathbb{Z}) = I\!I_{n,n}$ the condition $q \otimes q \in R(2\Lambda_1,\mathbb{Z})$ is equivalent to q being light-like. Therefore Grenier's fundamental domain $\mathcal{F}^{(n)}$ of $\mathcal{M}_n$

under the action of $SO(n,n,\mathbb{Z})$ is given in terms of the fundamental domain $\mathcal{F}^{(n-1)}$ of $\mathcal{M}_{n-1}$ under the action of $O(n-1,n-1,\mathbb{Z})$[4] by the conditions

1. $H_{n-1} \in \mathcal{F}^{(n-1)}$,
2. $H_n[q] \geq 1/r$ for all $q \in II_{n,n}$ such that $\gcd q = 1$ and $(q,q) = 0$,
3. $|(x_{n-1})_i| \leq \frac{1}{2}$ for $2 \leq i \leq 2n-2$ and $0 \leq (x_{n-1})_1 \leq \frac{1}{2}$.

Conditions 1. and 3. ensure that $H_n \in \mathcal{F}_P^{(1)}$ while condition 3. implies that $H_n \in S_>^{(1)}$. The proof that this is a fundamental domain as well as the resulting Grenier fundamental domain for the case $n = 2$ is given in [176].

5.2 Minima at symmetric points

Automorphic forms are in general complicated functions on complicated spaces. Analysing these functions even restricted to some fundamental domain is at best difficult and at worst impossible. However we will see that symmetric points, i.e. points which are mapped to themselves by a subgroup of the discrete symmetry group, define special points which are natural candidates for minima. This allows us to restrict the analysis of automorphic forms to a few points lying on the boundaries of the fundamental domain.

5.2.1 Taylor expansion at symmetric points

Let $K\backslash G$ be some symmetric space and let $\mathcal{F}$ be some fundamental domain of $K\backslash G$ for the action of $G(\mathbb{Z})$. The hypersurfaces defining the boundary $\partial\mathcal{F}$ of the fundamental domain are mapped to each other under the action of $G(\mathbb{Z})$ and their intersections are invariant under non-trivial finite subgroups of $G(\mathbb{Z})$. The maximal intersections define isolated points $v_0 \in K\backslash G$ that are invariant under maximal finite subgroups $G^{v_0}(\mathbb{Z}) \subset G(\mathbb{Z})$ given by

$$G^{v_0}(\mathbb{Z}) = \{\gamma \in G(\mathbb{Z}) \,|\, v_0\gamma \sim v_0\}\,. \tag{5.2.1}$$

We call isolated $G^{v_0}(\mathbb{Z})$-fixed points in $K\backslash G$ symmetric points. All the symmetric points are maximal intersections but some maximal intersections may not be symmetric points. By construction for all $\gamma \in G^{v_0}(\mathbb{Z})$ there exists a $k_\gamma = v_0\gamma v_0^{-1} \in K$ such that

$$v_0\gamma = k_\gamma v_0 \sim v_0\,. \tag{5.2.2}$$

[4]The fundamental domain for $O(n,n,\mathbb{Z})$ is the same as that of $SO(n,n,\mathbb{Z})$ except that the condition $\gcd q = 1$ is replaced by $q \neq 0$ for $n = 2$. Similarly the fundamental domain for $SO_0(n,n,\mathbb{Z})$ is the same as that of $SO(n,n,\mathbb{Z})$ except condition $0 \leq (x_1)_1 \leq \frac{1}{2}$ is replaced by $|(x_1)_1| \leq \frac{1}{2}$ for $n = 2$.

Therefore we can rewrite $G^{v_0}(\mathbb{Z})$ as

$$G^{v_0}(\mathbb{Z}) = \{\gamma \in G(\mathbb{Z}) \,|\, v_0 \gamma v_0^{-1} \in K\}\,. \tag{5.2.3}$$

Let f be an automorphic function on $K\backslash G/G(\mathbb{Z})$. We define the Taylor expansion of f at $v = v_0$ from the left-action of $X \in \mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ as

$$f(\exp(X)v_0) \equiv f_{v_0}(X) = f(v_0) + X^a \mathcal{D}_a f(v_0) + \frac{1}{2} X^a X^b \mathcal{D}_a \mathcal{D}_b f(v_0) + \mathcal{O}(X^3)\,, \tag{5.2.4}$$

where a labels the components of X in $\mathfrak{p}$. Note that this can be thought of as a equivalent definition to (4.1.11) for the covariant derivative $\mathcal{D}_a$.

If $G^{v_0}(\mathbb{Z})$ is non-trivial then for any $k_\gamma = v_0 \gamma v_0^{-1} \in K$ such that $\gamma \in G^{v_0}(\mathbb{Z})$ we have

$$f_{v_0}(k_\gamma^{-1} X k_\gamma) = f(k_\gamma^{-1} \exp(X) k_\gamma v_0) = f(\exp(X) v_0 \gamma) = f_{v_0}(X)\,. \tag{5.2.5}$$

Therefore the Taylor expansion of an automorphic function at a symmetric point $v = v_0$ is highly constrained by the symmetry group $G^{v_0}(\mathbb{Z})$. At order n in the Taylor expansion, one can classify the order n polynomials in X that are invariant under all such elements $k_\gamma = v_0 \gamma v_0^{-1} \in K$. If v_0 is a symmetric point, there is no invariant linear polynomial and v_0 is necessarily an extremum of any automorphic function.

For any symmetric point v_0 one must then check if the extremum is a minimum by looking at the Hessian matrix of the function at $v = v_0$. In practice we shall find that symmetric points admit a small number of $G^{v_0}(\mathbb{Z})$-invariant quadratic polynomials, and that it is sufficient to evaluate the automorphic function on a small dimension hypersurface to determine if it is a minimum.

If there is a single invariant quadratic polynomial, by construction it must be proportional to the quadratic Casimir such that

$$f_{v_0}(X) = f(v_0) + \frac{1}{2 \dim \mathfrak{p}} \kappa_{ab} X^a X^b \Delta f(v_0) + \mathcal{O}(X^3)\,. \tag{5.2.6}$$

The Hessian of the function at $v = v_0$ is then completely determined, and the symmetric point is a local minimum provided $\Delta f(v_0) > 0$. Recall that by definition f is an eigenfunction of the Laplacian, therefore in particular the function must be negative at a local minimum if the eigenvalue of the Laplace operator is negative.

Maximal parabolic Eisenstein series We will be interested in maximal parabolic Eisenstein series $E^G_{s\Lambda_i}$ associated to a fundamental weight Λ_i which are defined in section 4.1.1. They satisfy the Laplace equation (4.1.32). We saw that we had $\Delta E^G_{s\Lambda_i} > 0$ in the domain of absolute convergence $s > \frac{\langle \Lambda_i, \rho\rangle}{\langle \Lambda_i, \Lambda_i\rangle}$. Therefore we find that when the symmetric space $K\backslash G$ admits a symmetric point v_0 with a unique invariant quadratic polynomial in X, the symmetric point v_0 is a local minimum of any absolutely convergent Eisenstein series. On the other hand on the critical strip $0 < s < \frac{\langle \Lambda_i, \rho\rangle}{\langle \Lambda_i, \Lambda_i\rangle}$, where the Eisenstein series is integrable, the eigenvalue of the Laplacian is negative. Therefore v_0 is a local minimum of any integrable automorphic function that is negative at v_0. This is consistent with the fact that because of (4.1.41) $E^G_{s\Lambda_i}$ must be negative at its global minimum in the domain of integrability.

When there are two $G^{v_0}(\mathbb{Z})$-invariant quadratic polynomials we need to compute the two independent eigenvalues of the Hessian matrix at $v = v_0$. This can be done by defining a two-dimensional subspace of $K\backslash G$ that includes the two independent eigenvectors of the Hessian matrix at $v = v_0$. There also exists symmetric points with more invariant polynomials, but in our case we shall find that they never correspond to global minima. We will now discuss the cases of $SL(n)$ and $SO(n,n)$.

5.2.2 $SL(n)$ symmetric points

The $SL(2)$ real analytic Eisenstein series $E^{SL(2)}_{s\Lambda_1}$ is known to have a global minimum at the symmetric point $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, [179, 180] for any value of the parameter $s > 0$, with the regularisation (4.3.34) at $s = 1$. There is no general result for the global minimum of the Epstein series $E^{SL(n)}_{s\Lambda_1}$ for $n \geq 4$ and generic $s > 0$. It would be very difficult to carry out a complete numerical analysis of the Epstein series for $n \geq 4$. Based on the results above we shall therefore concentrate on symmetric points in moduli space for which the bilinear form H_n is invariant under a finite subgroup of $SL(n,\mathbb{Z})$.

The symmetric points in $SO(n)\backslash SL(n)$ are invariant under maximal finite irreducible subgroups of $SL(n,\mathbb{Z})$. The maximal finite irreducible subgroups of $GL(n,\mathbb{Z})$ have been classified for all $n \leq 10$ [181–183]. For $n \leq 4$ all the maximal finite subgroups of $SL(n,\mathbb{Z})$ are stabilisers of an even bilinear form C_L of Cartan type, i.e.

$$SL^{C_L}(n,\mathbb{Z}) = \{\gamma \in SL(n,\mathbb{Z}) \,|\, \gamma^T C_L \gamma = C_L\}\,, \tag{5.2.7}$$

where C_L is proportional to the Gram matrix of an even lattice L. The stabilisers of C_L can then be thought of as the group of automorphisms of L preserving its orientation,

i.e. $SL^{C_L}(n,\mathbb{Z}) = \mathrm{Aut}_+(L)$. The irreducible maximal subgroups of $SL(n,\mathbb{Z})$ are associated with the following lattices

$$\begin{aligned}
&n=1: && A_1\,,\\
&n=2: && A_2\,,\quad 2A_1\,,\\
&n=3: && A_3\,,\quad A_3^*\,,\quad 3A_1\,,\\
&n=4: && D_4\,,\quad A_4\,,\quad A_4^*\,,\quad 4A_1\,,\quad 2A_2\,,\quad A_2\times A_2\,,
\end{aligned} \tag{5.2.8}$$

where A_n and D_n are the root lattices of the Lie algebras $\mathfrak{a}_n$ and $\mathfrak{d}_n$ respectively and C_{A_n} and C_{D_n} can be though of as the associated Cartan matrices.[5] By L^* we mean the dual lattice of L. A_n^* and D_n^* can be thought of as the weight lattices of the Lie algebras $\mathfrak{a}_n$ and $\mathfrak{d}_n$ respectively. The reducible maximal subgroups are associated with the reducible lattices A_1+A_2, A_1+A_3 and $A_1+A_3^*$. For Cartan type lattices A_n, D_n and E_n, the automorphism group is the product of the outer automorphisms of the Dynkin diagram and the Weyl group of the corresponding Lie algebra

$$\mathrm{Aut}(L) = \mathrm{Out}(L) \ltimes W(L)\,. \tag{5.2.9}$$

The automorphisms of the dual lattices A_n^*, D_n^* and E_n^* are the same as for A_n, D_n and E_n respectively. In fact by definition we have $C_{L^*} = C_L^{-1}$ and therefore $SL^{C_{L^*}}(n,\mathbb{Z}) = SL^{C_L}(n,\mathbb{Z})$.

We can easily define symmetric points in $SO(n)\backslash SL(n)$ which we identify with the space of symmetric positive definite unimodular matrices $\mathcal{H}_S$ by $H_{C_L} = (\det C_L)^{-\frac{1}{n}} C_L$ which has as stabiliser group $SL^{H_{C_L}}(n,\mathbb{Z}) = SL^{C_L}(n,\mathbb{Z}) = \mathrm{Aut}_+(L)$. We also have $H_{C_{L^*}} = H_{C_L}^{-1}$. Note that since points $H \in SO(n)\backslash SL(n)/SL(n,\mathbb{Z})$ are identified by $\gamma^T H \gamma \sim H$ for all $\gamma \in SL(n,\mathbb{Z})$ we will also consider points up to conjugation by $\gamma \in SL(n,\mathbb{Z})$. In fact we shall identify points with their representatives in the Grenier fundamental domain defined in section (5.1.1).

One easily checks that points H_{C_L} associated with reducible lattices L that are extrema are always saddle points of $SL(n)$ minimal Eisenstein series. Indeed let $L = L_1 + L_2$ be a reducible lattice. H_{C_L} will be of the form

$$H_{C_L} = \begin{pmatrix} H_{C_{L_1}} & 0 \\ 0 & H_{C_{L_2}} \end{pmatrix}\,. \tag{5.2.10}$$

[5] Note that only the root systems of simply laced Lie algebras define even lattices, these are given by the $\mathfrak{a}_n$, $\mathfrak{d}_n$ and $\mathfrak{e}_n$ families, the so called ADE root systems. In any case the Cartan matrices of non simply laced Lie algebras are not even symmetric and thus can't even be thought of as Gram matrices.

If we assume that L_1 is a k dimensional lattice and L_2 is an $n-k$ dimensional lattice we can see that this corresponds to a parabolic decomposition (4.1.68) on the maximal parabolic subgroup P_k (4.1.66) with $r=1$ and $x_{k,n-k}=0$. However if we compute the second derivative with respect to $x_{k,n-k}$ at $x_{k,n-k}=0$ of $E^{SL(n)}_{s\Lambda_1}$ in the Fourier expansion (4.1.70) on P_k we get

$$D^2 E^{SL(n)}_{s\Lambda_1}(H_n)|_{x_{k,n-k}=0}$$
$$= -\frac{2}{\xi(2s)} \sum_{\substack{\Gamma_{n-k,k} \\ \in \mathbb{Z}^{n-k}\otimes\mathbb{Z}^k \\ \mathrm{rank}(\Gamma_{n-k,k})=1}}{}' \sum_{\substack{d_k\in\mathbb{Z}^k \\ d_k|\Gamma}} \left(H_k^{-1}[d_k]^{s-\frac{k}{2}}\right) \frac{r^{(\frac{n}{2}-k)s+\frac{kn}{4}}}{|Z(\Gamma)|^{s-\frac{k}{2}}} K_{s-\frac{k}{2}}(2\pi r^{\frac{n}{2}}|Z(\Gamma)|)(2\pi\Gamma(dx))^2\,, \tag{5.2.11}$$

where $\Gamma(dx)=\mathrm{Tr}(\Gamma_{n-k,k}dx_{k,n-k})$. As minus the absolutely convergent sum of positive terms we can see that this is negative definite. Therefore any extremum at $x_{k,n-k}=0$ in a maximal parabolic decomposition P_k is necessarily a saddle point. This is the case with all points H_{C_L} associated with reducible lattices L. We will therefore only analyse symmetric points H_{C_L} associated with irreducible lattices L.

For $n=5$ the irreducible lattices leading to a maximal finite irreducible subgroup of $SL(5,\mathbb{Z})$ are [182]

$$n=5: \qquad D_5\,,\quad D_5^*\,,\quad A_5\,,\quad A_5^*\,,\quad A_5^{+2}\,,\quad (A_5^{+2})^*=A_5^{+3}\,, \tag{5.2.12}$$

where A_5^{+2} is the S_6[6]-invariant lattice $A_5\oplus(A_5{+}\Lambda_2)\oplus(A_5{+}\Lambda_4)$ that includes the second fundamental weight Λ_2, and $A_5^{+3}=A_5\oplus(A_5{+}\Lambda_3)$. As expected, we have checked in Appendix A of [176] that all these points lie on dimension zero boundaries of Grenier's fundamental domain as defined in section (5.1.1).

There is another argument for studying points corresponding to bilinear forms of Cartan type. In some ways it is even more compelling because of three reasons: first, it relies on specific properties of the Epstein series rather than simply automorphic invariance. Second, it applies to global minima rather that simply local minima. Finally, one can also show that this argument holds at large n where lattices of A, D and E types are no longer global minima of the Epstein series [184]. Indeed, it was proven for $s\gg 1$ that the global minimum of the $SL(n)$ Epstein series is given by the solution to the densest lattice sphere packing in n dimensions [185]. The argument is the following: any point $H\in\mathcal{H}_S$ can be thought of as the Gram matrix of some unimodular lattice and $H[q]$ can be interpreted as the distance of the point

[6] S_n meaning here the group of permutations of order n.

of coordinate $q \in \mathbb{Z}^n$ on the lattice defined by H. The $SL(n)$ Epstein series is given by

$$\mathrm{Ep}_s^{SL(n)}(H) = \sum_{q\in\mathbb{Z}^n}{}' \frac{1}{H[q]^s}\,. \tag{5.2.13}$$

For large s this reduces to

$$\mathrm{Ep}_s^{SL(n)}(H) \sim \sum_{\substack{\text{nearest}\\ \text{neighbors}}}{}' \frac{1}{H[q]^s}\,. \tag{5.2.14}$$

Therefore we can see that the global minimum of the Epstein series will be given by the lattice that maximises the distance between nearest neighbors. This corresponds to the densest sphere packing on the lattice. For $2 \leq n \leq 8$, the densest lattice sphere packings are known to be the following rank n root lattices [186]

$$A_2\,,\quad A_3\,,\quad D_4\,,\quad D_5\,,\quad E_6\,,\quad E_7\,,\quad E_8\,. \tag{5.2.15}$$

However, for the Wilson coefficients appearing in string theory we are interested in small values of s, in particular $s \leq \frac{n}{2}$. It was observed in [184] that the densest lattice packing H_{dlp} cannot be the global minimum of the Epstein series for all $s > 0$ when the lattice and its dual do not define the same point in $SO(n)\backslash SL(n)/SL(n,\mathbb{Z})$, i.e. when $H_{\text{dlp}}^{-1} \not\sim H_{\text{dlp}}$ modulo the action of $SL(n,\mathbb{Z})$. This is a direct consequence of the functional relations (4.1.44) and (4.1.43) which we give for Epstein series as

$$\mathrm{Ep}_s^{SL(n)}(H) = \pi^{2s-\frac{n}{2}}\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s)}\mathrm{Ep}_{\frac{n}{2}-s}^{SL(n)}(H^{-1})\,. \tag{5.2.16}$$

One may therefore argue at most that the densest lattice packing H_{dlp} is the global minimum for $s \geq \frac{n}{4}$. This is not an issue for us as the values of s appearing in the Wilson coefficients of string theory are always bigger than $\frac{n}{4}$.

As was suggested in the previous section, it was proved in [184] that the global minimum of $\mathrm{Ep}_s^{SL(n)}$ must be strictly negative in the critical strip $0 < s < \frac{n}{2}$. However it was proved in [187] that if the density of the lattice defined by H as well as the density of the dual lattice defined by H^{-1} are below a certain critical value then $\mathrm{Ep}_{\frac{n}{4}}^{SL(n)}(H) > 0$. As stated earlier, this excludes the possibility that the minimum at $s = \frac{n}{4}$ be at H_{C_L} for L an ADE lattice for $n > 24$ and suggests instead that the densest lattice sphere packing will be the minimum [184]. The densest lattice sphere packing has been proved to be the global minimum for $n = 8$ and 24 and all values

of $s > \frac{n}{2}$ [188].[7]

The density $\rho(L)$ of some lattice L is defined as the ratio of the volume covered by non overlapping spheres at each point of the lattice over the total volume. If we apply the densest lattice sphere packing criterion to the points (5.2.8) and (5.2.12) we get

$$\rho(A_n) \underset{n\geq 4}{<} \rho(D_n) \underset{n=6,7,8}{<} \rho(E_n)\,, \tag{5.2.17}$$

in agreement with the list (5.2.15). The dual lattices D_n^* and A_n^* are always less dense than the lattices D_n and A_n, but E_n^* is denser than D_n and A_n for $n = 6, 7, 8$. For $n = 5$ one finds

$$\rho(A_5^*) < \rho(D_5^*) < \rho(A_5^{+3}) < \rho(A_5) < \rho(A_5^{+2}) < \rho(D_5)\,, \tag{5.2.18}$$

so the densest sphere lattice packing is indeed the lattice D_5, and A_5^{+2} is denser than A_5. The fact that the D_n series is a better candidate for Epstein minimum than the A_n series for $n = 4, 5$ is also supported by the fact that the Weyl group for D_n is larger that that of A_n therefore it is a more symmetric point.[8]

In the following it will be useful to introduce the following parametrisation of $H_n \in SO(n)\backslash SL(n)$

$$H_n = y^{-\frac{2}{n}} \begin{pmatrix} n^{\frac{1}{n-1}} H_{n-1} & 0 \\ 0 & \frac{y^2}{n} \end{pmatrix} \begin{bmatrix} I_{n-1} & x_{n-1} \\ 0 & 1 \end{bmatrix}\,, \tag{5.2.19}$$

which basically amounts to the parabolic decomposition (4.1.68) on the maximal parabolic subgroup P_{n-1} where we have rescaled $r = y^{\frac{2}{n}}/n^{\frac{1}{n-1}}$. When we take $x_{n-1} = (x, \dots, x)^T$ and $H_{n-1} = H_{C_{A_{n-1}}}$ with C_{A_n} given in the fundamental domain by

$$C_{A_n} = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}\,, \tag{5.2.20}$$

then $H_n(y, x)$ defines a two dimensional hypersurface in $SO(n)\backslash SL(n)$ with coordinates (y, x). This hypersurface includes all the relevant symmetric points. In fact one

[7] These correspond to the root lattice E_8 and the Leech lattice Λ_{24} respectively.

[8] In principle one should also take into account the outer automorphisms but we can show that they do not affect any of our results so we leave them aside.

gets representatives for the A_n, D_n and E_n points by

$$\begin{aligned} H_{C_{A_n}} \quad & \text{for} \quad x = \frac{1}{n}\,,\ y = \sqrt{n+1}\,, \\ H_{C_{D_n}} \quad & \text{for} \quad x = \frac{2}{n}\,,\ y = 2\,, \\ H_{C_{E_n}} \quad & \text{for} \quad x = \frac{3}{n}\,,\ y = \sqrt{9-n}\,. \end{aligned} \tag{5.2.21}$$

Moreover for $n = 5$ one checks that representatives of the A_5^{+2} and A_5^{+3} points are given by

$$\begin{aligned} H_{C_{A_5^{+2}}} \quad & \text{for} \quad x = \frac{2}{5}\,,\ y = \sqrt{\frac{2}{3}}\,, \\ H_{C_{A_5^{+3}}} \quad & \text{for} \quad x = \frac{2}{5}\,,\ y = \sqrt{\frac{3}{2}}\,. \end{aligned} \tag{5.2.22}$$

The dual lattice points can be studied in the same way with the inverse matrix or using the functional relation (5.2.16).

We define the pull-back of an automorphic function on $SO(n)\backslash SL(n)$ to the two-dimensional hypersurface parametrised by y and x by $f_*(y,x) = f(H_n(y,x))$. The second differential of the pull-back function is given by

$$D^2 f_*(y,x) = f_{yy}\, dy^2 + f_{xx}\, dx^2 + 2 f_{yx}\, dy dx\,, \tag{5.2.23}$$

where f_{yy}, f_{xx} and f_{xy} are the second derivatives of the automorphic function at the point (y,x).

A_n symmetric points In the appendix A of [176] we have classified the quadratic polynomials on $\mathfrak{p} = \mathfrak{sl}(n) \ominus \mathfrak{so}(n)$ which are invariant under the Weyl group $W(A_n) = S_{n+1}$ and we have found two independent invariant polynomials. The two independent eigenvalues of the Hessian matrix at $H_{C_{A_n}}$ can be computed on the two-dimensional hypersurface $H_n(y,x)$ at $y = \sqrt{n+1}$ and $x = \frac{1}{n}$.

The second differential of the pull-back function (5.2.23) can be decomposed on the two independent S_{n+1} invariant polynomials to give

$$D^2 f_*(y,x)|_{y=\sqrt{n+1}, x=\frac{1}{n}} = a_+ \frac{2}{n+1}\left(\frac{n-1}{n} dy^2 + n\, dx^2\right) + a_- \frac{(n^2(n-1)-2n)}{n+1} dx^2\,, \tag{5.2.24}$$

for two coefficients a_+ and a_- determined by the second derivatives of the automorphic function at the symmetric point. In particular

$$f_{yy} = 2\frac{n-1}{n(n+1)}a_+, \quad f_{xx} = \frac{2n}{n+1}a_+ + \frac{(n^2(n-1)-2n)}{n+1}a_-, \quad f_{yx} = 0\,. \tag{5.2.25}$$

Hence the condition for the symmetric point A_n to be a local minimum of the automorphic function f is that $a_+ > 0$ and $a_- > 0$, which is equivalent to the condition that

$$f_{xx} > \frac{n^2}{n-1}f_{yy} > 0\,, \tag{5.2.26}$$

for $n \geq 3$. It is therefore sufficient to study the pull-back function $f_*(y,x)$ to determine if the symmetric point A_n is a local minimum of the automorphic function $f(H_n)$ on $SO(n)\backslash SL(n)$. The complete proof can be found in [176].

We have carried out this computation numerically for the Epstein series and found that the A_n symmetric point is always a local minimum for s large enough, but the eigenvalue a_- becomes negative for small s for $n \geq 4$, giving a saddle unstable along the corresponding $\frac{(n+1)(n-2)}{2}$-dimensional hypersurface.[9] It is always a local minimum for $n = 2, 3$. For $n = 2$ the point A_2 is equal to $H_{C_{A_2}} = H_\tau$ (see (4.1.73)) with $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ the global minimum for all $s > 0$. For $n = 3$ the point A_3 is also the global minimum for $s > \frac{3}{4}$ [184], including for the regularised series $\hat{\text{Ep}}^{SL(3)}_{\frac{3}{2}}$ defined in (4.3.34). We find that it is a local minimum for $s > 1$ for $SL(4)$, and for $s \gtrsim 3.16603$ for $SL(5)$. For $SL(6)$ and $SL(7)$ one finds similarly that it is not a local minimum for $s < s_{A_n}$ with s_{A_n} slightly above the critical value $\frac{n}{2}$. Most importantly for us, it is not a local minimum of the Epstein function $\text{Ep}^{SL(5)}_{\frac{3}{2}}$ that defines the leading Wilson coefficient in 7 dimensions.

D_n symmetric points In section 4.2.2 of [176] we have classified the quadratic polynomials on $\mathfrak{p} = \mathfrak{sl}(n) \ominus \mathfrak{so}(n)$ which are invariant under the Weyl group $W(D_n) = S_n \ltimes \mathbb{Z}_2^{n-1}$ and we have found two independent invariant polynomials.[10] The two independent eigenvalues of the Hessian matrix at $H_{C_{D_n}}$ can again be computed on the

[9] For $n = 2$ this hypersurface does not exists as we can see from (5.2.24) where the coefficient of a_- vanishes. In this case (5.2.26) reduces to $f_{xx} = 4f_{yy} > 0$.

[10] Except for D_4 where the triality symmetry implies that there is a unique invariant quadratic polynomial.

two-dimensional hypersurface $H_n(y,x)$ this time at $y=2$ and $x=\frac{2}{n}$.

The second differential of the pull-back function (5.2.23) can be decomposed on the two independent $S_n \ltimes \mathbb{Z}_2^{n-1}$ invariant polynomials to give

$$D^2 f_*(y,x)|_{y=2,x=\frac{2}{n}} = a_+ \frac{n-1}{n} dy^2 + a_- \frac{n^2(n-1)}{2} dx^2 \,, \tag{5.2.27}$$

for a_+ and a_- the two eigenvalues of the Hessian of f at the symmetric point. In particular

$$f_{yy} = \frac{n-1}{n} a_+, \quad f_{xx} = \frac{n^2(n-1)}{2} a_-, \quad f_{yx} = 0 \,. \tag{5.2.28}$$

Hence the condition for the symmetric point D_n to be a local minimum of the automorphic function f is that $a_+ > 0$ and $a_- > 0$, which is equivalent to the condition that

$$f_{xx} > 0, \qquad f_{yy} > 0 \,. \tag{5.2.29}$$

It is therefore also sufficient to study the pull-back function $f_*(y,x)$ to determine if the symmetric point D_n is a local minimum of the automorphic function $f(H_n)$ on $SO(n)\backslash SL(n)$. The complete proof can be found in [176].

One finds by numerical evaluation that the D_n symmetric point is indeed a local minimum of the Epstein series $\mathrm{Ep}_s^{SL(n)}$ for all s and $n \leq 7$. We do not expect this to be true for arbitrary large n, but this is not relevant for the study of Wilson coefficients of string theory. We find in particular that among all symmetric points, the D_n symmetric point gives the minimum value of the Epstein series $\mathrm{Ep}_s^{SL(n)}$ for $n = 4, 5$ and $s \geq \frac{n}{4}$,[11] leading to the conjecture that D_n is the global minimum.

A_n^* and D_n^* symmetric points The analysis of the polynomials in $\mathfrak{p}$ invariant under the automorphisms of the lattices A_n^* and D_n^* gives by construction the same number of invariant polynomials as A_n and D_n respectively. In fact one easily extends all the results of the previous two paragraphs to A_n^* and D_n^* using the functional relation (5.2.16).

For large values of s, the Epstein series evaluated at the dual symmetric points D_n^* and A_n^* is larger, as expected from the sphere packing density argument. By

[11] Note that the root lattice D_4 is self dual so that $H_{C_{D_4^*}}$ and $H_{C_{D_4}}$ define the same point. Therefore the D_4 symmetric point is the lowest value for all $s > 0$.

the functional relation (5.2.16) this must be the opposite for $s < \frac{n}{4}$, and in particular a symmetric point and its dual give the same value at $s = \frac{n}{4}$. Accordingly, there is always a value of $s_c > \frac{n}{4}$ where $\mathrm{Ep}_{s_c}^{SL(n)}(H_{C_{D_n^*}}) = \mathrm{Ep}_{s_c}^{SL(n)}(H_{C_{A_n}})$ and $\mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n^*}}) < \mathrm{Ep}_{s}^{SL(n)}(H_{C_{A_n}})$ for $s < s_c$. The numerical evaluation for $n \leq 5$ shows that there is only one such transition, and that $\mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n^*}}) - \mathrm{Ep}_{s}^{SL(n)}(H_{C_{A_n}})$ and $\mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n}}) - \mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n^*}})$ have only one zero at $s = s_c$ and $s = \frac{n}{4}$, respectively.

A_5^{+2} symmetric point The symmetric point A_5^{+2} admits the same symmetry S_6 as the point A_5 and its dual A_5^*. One obtains that the condition for the Hessian of the function f to be positive definite is (5.2.26) for $n = 5$, as for the A_5 symmetric point. We have checked these conditions numerically and found that the symmetric point A_5^{+2} is a local minimum of the Epstein series $\mathrm{Ep}_{s}^{SL(5)}$ for $s \gtrsim 2.8849$. Below this value the second derivative with respect to y is negative and the saddle is unstable along 5 directions.

In the critical strip $\frac{5}{4} < s < \frac{5}{2}$ we find that

$$\begin{aligned}\mathrm{Ep}_{s}^{SL(5)}(H_{C_{D_5}}) &< \mathrm{Ep}_{s}^{SL(5)}(H_{C_{D_5^*}}) < \mathrm{Ep}_{s}^{SL(5)}(H_{C_{A_5^{+2}}}) \\ &< \mathrm{Ep}_{s}^{SL(5)}(H_{C_{A_5^{+3}}}) < \mathrm{Ep}_{s}^{SL(5)}(H_{C_{A_5}}) < \mathrm{Ep}_{s}^{SL(5)}(H_{C_{A_5^*}})\,, \end{aligned} \tag{5.2.30}$$

and only D_5 and D_5^* are local minima.

E_6, E_7 and E_8 symmetric points We shall be very brief about these cases since they are not relevant to the analysis of the Wilson coefficients of string theory in the dimensions $D \geq 6$. One finds in these three cases that the Weyl groups $W(E_n)$ is large enough to impose that there is a unique invariant quadratic polynomial on $\mathfrak{p}$. It is therefore sufficient to check that the Epstein series is negative in the critical strip to ensure that the E_n symmetric point is a local minimum for all values of s. One finds indeed by numerical evaluation that the E_n symmetric points are local minima for all values of s, and among all symmetric points we have checked they give the smallest value of the Epstein series $\mathrm{Ep}_{s}^{SL(n)}$ for all values of $s \geq \frac{n}{4}$.[12]

In the critical strip $\frac{n}{4} < s < \frac{n}{2}$ we find that

$$\begin{aligned}\mathrm{Ep}_{s}^{SL(n)}(H_{C_{E_n}}) &\underset{n=6,7}{<} \mathrm{Ep}_{s}^{SL(n)}(H_{C_{E_n^*}}) < \mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n}}) \\ &\underset{n=6,7}{<} \mathrm{Ep}_{s}^{SL(n)}(H_{C_{D_n^*}}) < \mathrm{Ep}_{s}^{SL(n)}(H_{C_{A_n}}) < \mathrm{Ep}_{s}^{SL(n)}(H_{C_{A_n^*}})\,. \end{aligned} \tag{5.2.31}$$

[12] We have only checked symmetric points of Cartan type A_n, D_n, E_n and their dual. This exhausts all maximal irreducible symmetry groups for $n = 7$, but not for $n = 6, 8$ [182].

$A_d \times A_{\frac{n}{d}}$ **symmetric points for** $d|n$ For n not prime one can also have symmetric points associated to irreducible bilinear form that are tensor products of lower dimensional bilinear forms. For example for $n = 4$ one has $A_2 \times A_2$. More generally one can have $A_d \times A_{\frac{n}{d}}$ for d dividing n. For $n \leq 8$ the possible lattices are less dense than E_n, D_n and A_n, but one checks that in the critical strip

$$\mathrm{Ep}_s^{SL(4)}(H_{C_{D_4}}) < \mathrm{Ep}_s^{SL(4)}(H_{C_{A_2\times A_2}}) < \mathrm{Ep}_s^{SL(4)}(H_{C_{A_4}})\,. \tag{5.2.32}$$

So the ordering is reversed with the A_4 lattice at low values of s. Nevertheless, the symmetric point D_4 remain the global minimum of $\mathrm{Ep}_s^{SL(4)}$ for all value of $s > 0$.

5.2.3 $SO(n,n)$ symmetric points

We identify $(SO(n) \times SO(n))\backslash SO(n,n)$ with the space of matrices whose symmetric part is positive definite $\mathcal{H}$. To define symmetric points in $(SO(n) \times SO(n))\backslash SO(n,n)$ we start with the assumption that the symmetric part G is itself a symmetric point of $SO(n)\backslash GL(n)$. From the previous section we find that G is then proportional to the Gram matrix C_L of an even lattice L, and the relevant solution will turn out to be

$$G = \frac{1}{2}C_L\,, \qquad G + B = 0 \bmod \mathbb{Z}\,. \tag{5.2.33}$$

One finds in this case that the vectors $q_\pm$

$$q_\pm = \frac{1}{2}G^{-1}\left(m + (B \pm G)n\right)\,, \tag{5.2.34}$$

for $m, n \in \mathbb{Z}^n$ satisfy

$$q_+ - q_- \in \mathbb{Z}^n\,. \tag{5.2.35}$$

On the other hand the $q_\pm$ belong to the dual lattice $L^* = C_L^{-1}\mathbb{Z}^n$. By construction we can decompose any element $r \in L^*$ of the dual lattice as $r = l + \mu$ with $l \in L$ and $\mu \in L^*/L$. It follows that G, B describes the isomorphism

$$I\!I_{n,n} = \bigoplus_{\mu\in L^*/L} \left((L+\mu) \oplus (L+\mu)[-1]\right)\,. \tag{5.2.36}$$

If L is the root lattice of some simply laced lie algebra $\mathfrak{g}$ then L^*/L is the weight lattice of $\mathfrak{g}$ modulo roots of $\mathfrak{g}$. This is therefore a subset of the set of dominant weights. For A_n we have $A_n^*/A_n = \{0, \Lambda_1, \ldots, \Lambda_n\}$ and for D_n we have $D_n^*/D_n = \{0, \Lambda_1, \Lambda_2, \Lambda_{n-1}, \Lambda_n\}$.

We can identify the space of matrices whose symmetric part is positive definite with the space of invariant symmetric positive definite unimodular matrices where the isomorphism is given by (3.5.14). We now consider a point (5.2.33) for L of Cartan type, that we write

$$H_{C_L,B} = \begin{pmatrix} C_L & 0 \\ 0 & C_L \end{pmatrix} \begin{bmatrix} C_L^{-1} & C_L^{-1}(G+B) \\ C_L^{-1} & C_L^{-1}(-G+B) \end{bmatrix} . \tag{5.2.37}$$

Therefore we have

$$H_{C_L,B}[q] = \begin{pmatrix} C_L & 0 \\ 0 & C_L \end{pmatrix} \begin{bmatrix} q_+ \\ q_- \end{bmatrix} , \tag{5.2.38}$$

with $q = (m,n)^T$ for $m,n \in \mathbb{Z}^n$ and with $q_\pm$ defined in (5.2.34). Hence for any pair of elements $\gamma_\pm \in \mathrm{Aut}(L)$ such that

$$\gamma_+ \mu = \gamma_- \mu \bmod L \,, \tag{5.2.39}$$

for all $\mu \in L^*/L$, one has an automorphism $\gamma \in O(n,n,\mathbb{Z})$ of the split signature lattice $I\!I_{n,n}$ that preserves the symmetric matrix $H_{C_L,B}$

$$\gamma^T H_{C_L,B} \gamma = \begin{pmatrix} \gamma_+^T C_L \gamma_+ & 0 \\ 0 & \gamma_-^T C_L \gamma_- \end{pmatrix} \begin{bmatrix} C_L^{-1} & C_L^{-1}(G+B) \\ C_L^{-1} & C_L^{-1}(-G+B) \end{bmatrix} = H_{C_L,B} \,. \tag{5.2.40}$$

For Cartan type lattices A_n, D_n and E_n the Weyl group preserves all weights $\mu \in L^*/L$, so one has the automorphism group

$$\mathrm{Aut}(I\!I^L_{n,n}) = \mathrm{Out}(L) \ltimes W(L) \times W(L) \,. \tag{5.2.41}$$

In particular, the $\gamma_\pm$ can be two independent elements of the Weyl group of the Cartan type lattice L. Therefore $H_{C_L,B}$ defines a symmetric point of $(SO(n) \times SO(n))\backslash SO(n,n)$ with stabiliser group $SO^{H_{C_L,B}}(n,n,\mathbb{Z}) = \mathrm{Aut}_+(I\!I^L_{n,n})$. The different representatives of B simply correspond to different representatives in different fundamental domains, as before we consider $H_{C_L,B}$ to be in the Grenier fundamental domain defined in section (5.1.3).

The density argument also has its analogue here. Let us write $H_{G,B}[q]$ and (q,q) as defined in (3.5.13) and (3.5.15) as

$$H_{G,B}[q] = p_L(q)^2 + p_R(q)^2 \tag{5.2.42}$$

and

$$(q,q) = p_L(q)^2 - p_R(q)^2\,, \tag{5.2.43}$$

where $p_L(q)^2$ (resp. $p_R(q)^2$) is the euclidean scalar product of $p_L(q)$ (resp. $p_R(q)$) with itself. We can think of p_L and p_R as projectors onto the n dimensional hyperspaces of positive and negative definite signature respectively in $\mathbb{R}^{n,n}$. Indeed, by definition the group $SO(n,n)$ preserves this splitting of the signature. In particular it rotates maximal (i.e. n-dimensional) definite signature hyperspaces into each other and one can get all maximal definite signature hyperspaces from the action of $SO(n,n)$ on any maximal definite signature hyperspace. In other words the group $SO(n,n)$ acts transitively on the space of maximal definite signature hyperspaces. On the other hand the group of transformations preserving any single maximal definite signature hyperspace is $SO(n) \times SO(n)$ which corresponds to rotations inside the hyperspace plus rotations orthogonal to the hyperspace, both of which preserve the hyperspace. Therefore $(SO(n) \times SO(n))\backslash SO(n,n)$ can be identified with the space of all maximal definite signature hyperspaces. Therefore a point in $SO(n) \times SO(n)\backslash SO(n,n)$ is equivalent to a choice of projectors p_L and p_R onto the two n dimensional definite signature hyperspaces. We can use (3.5.13) and (3.5.15) to solve explicitly for p_L and p_R as

$$\begin{aligned} p_L(q)^2 &= \frac{1}{2}G^{-1}[m + (B+G)n] = 2G[q_+]\,,\\ p_R(q)^2 &= \frac{1}{2}G^{-1}[m + (B-G)n] = 2G[q_-]\,, \end{aligned} \tag{5.2.44}$$

with $q = (m,n)^T$ for $m, n \in \mathbb{Z}^n$ and with $q_\pm$ defined in (5.2.34). Therefore we can write the Narain theta series (3.5.16) as

$$\sum_{q \in I\!I_{n,n}} e^{-\pi \operatorname{Im}\tau\, H_{G,B}[q] + i\pi \operatorname{Re}\,\tau(q,q)} = \sum_{q \in I\!I_{n,n}} e^{i\pi\tau\, p_L(q)^2 - i\pi\bar{\tau}\, p_R(q)^2}\,. \tag{5.2.45}$$

If we choose G, B of the form (5.2.33) we can decompose $q_\pm = l_\pm + \mu$ with $l_\pm \in L$ and $\mu \in L^*/L$. Therefore we have

$$\sum_{q \in I\!I_{n,n}} e^{i\pi\tau\, p_L(q)^2 - i\pi\bar{\tau}\, p_R(q)^2} = \sum_{\mu \in L^*/L} \sum_{l_\pm \in L} e^{i\pi\tau\, C_L[l_+ + \mu] - i\pi\bar{\tau}\, C_L[l_- + \mu]}\,. \tag{5.2.46}$$

Hence the isomorphism (5.2.36) is expressed for the Narain theta series as

$$\sum_{q \in I\!I_{n,n}} e^{-\pi \operatorname{Im}\tau\, H_{C_L,B}[q] + i\pi \operatorname{Re}\tau\,(q,q)} = \sum_{\mu \in L^*/L} \left| \sum_{l \in L} e^{i\pi\tau C_L[l+\mu]} \right|^2\,. \tag{5.2.47}$$

Therefore using (4.1.58) we can write

$$\begin{aligned}\mathrm{Ep}_s^{SO(n,n)}(H_{C_L,B}) &= \frac{\pi^s}{\Gamma(s)}\int_{\substack{\mathrm{Im}\,\tau>0\\ |\mathrm{Re}\,\tau|\leq\frac{1}{2}}}\frac{d^2\tau}{\mathrm{Im}\,\tau}(\mathrm{Im}\,\tau)^s\sum_{q\in I\!I_{n,n}}{}' e^{-\pi\,\mathrm{Im}\,\tau\, H_{C_L,B}[q]+i\pi\,\mathrm{Re}\,\tau\,(q,q)}\\ &= \sum_{\mu\in L^*/L}\sum_{\substack{l_\pm\in L\\ C_L[l_+ +\mu]=C_L[l_-+\mu]}}{}' \frac{1}{(2C_L[l_+ +\mu])^s}\,, \end{aligned}\tag{5.2.48}$$

where the prime removes the point $l_\pm = 0$ for $\mu = 0$ only. At large s, the leading term in $\mathrm{Ep}_s^{SO(n,n)}(H_{C_L,B})$ is proportional to the smallest length of a vector in L^* to the power $-2s$, and one obtains a minimum for the even lattice with the largest possible minimal length of a vector in L^*. Although this is not exactly the same criterion as for the densest lattice sphere packing, it gives the same ADE classification (5.2.15) for $n \leq 8$.

One then finds for all irreducible Cartan type lattices L, that the unique invariant quadratic polynomial is the quadratic Casimir. The proof for A_n and D_n is given in [176]. According to the discussion of section 5.2.1, we conclude that the $H_{C_{A_n},B}$ and $H_{C_{D_n},B}$ symmetric points are local minima of any $SO(n,n)$ Eisenstein series in the domain of absolute convergence. We have checked numerically that $\mathrm{Ep}_s^{SO(5,5)}$ is negative for both $H_{C_{A_5},B}$ and $H_{C_{D_5},B}$ at the critical value $s = \frac{3}{2}$ relevant in the string theory Wilson coefficient. Therefore the symmetric points $H_{C_{A_5},B}$ and $H_{C_{D_5},B}$ are local minima. The point $H_{C_{D_5},B}$ is a lower value than the $H_{C_{A_5},B}$ symmetric point and we conjecture that it is the global minimum of the Epstein series.

$SO(3,3)$ and $SL(4)$ Because of the isomorphism $SO(3,3) = SL(4)$ it is relevant to compare the results we have obtained for $SO(n,n)$ and $SL(n)$ in this case. The Epstein series of $SO(3,3)$ and $SL(4)$ are related at the special s value by $\pi\mathrm{Ep}_{\frac{1}{2}}^{SO(3,3)} = \mathrm{Ep}_1^{SL(4)}$ with the identification of the $SL(4)$ matrix

$$H_4 = y^{-\frac{1}{2}}\begin{pmatrix} 4^{\frac{1}{3}}H_3 & 0\\ 0 & \frac{y^2}{4}\end{pmatrix}\begin{bmatrix} I_3 & x_3\\ 0 & 1\end{bmatrix} = \frac{1}{\det G}\begin{pmatrix} G & 0\\ 0 & (\det G)^2\end{pmatrix}\begin{bmatrix} I_3 & \star B\\ 0 & 1\end{bmatrix}. \tag{5.2.49}$$

Therefore we have $H_3 = (\det G)^{-\frac{1}{3}}G$, $r = y^{\frac{1}{2}}/4^{\frac{1}{3}} = (\det G)^{\frac{2}{3}}$ and $x_3 = \star B$. The two dimensional hypersurface $H_4(y,x)$ in $SO(4)\backslash SL(4)$ is given by $G(y), B(x)$ such that

$$G = \frac{y}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = x\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \tag{5.2.50}$$

so that the $SL(4)$ D_4 symmetric point of coordinates $y = 2$, $x = 1/2$ corresponds to the $SO(3,3)$ A_3 symmetric point,

$$G = \frac{1}{2}C_{A_3}, \quad B + G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \tag{5.2.51}$$

while the $SL(4)$ A_4 symmetric point of coordinates $y = \sqrt{5}$, $x = 1/4$ gives

$$G = \frac{\sqrt{5}}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = \frac{1}{4}\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{5.2.52}$$

We find consistently that the conjectured global minimum of the $SL(4)$ Epstein series at the D_4 symmetric point agrees with the conjectured global minimum of the $SO(3,3)$ Epstein series at the A_3 symmetric point. One finds that they have the same automorphism groups

$$\mathrm{Aut}(I\!I^{A_3}_{n,n}) = \mathbb{Z}_2 \ltimes (S_4 \times S_4) = S_3 \ltimes (S_4 \ltimes \mathbb{Z}_2^3) = \mathrm{Aut}(D_4), \tag{5.2.53}$$

because of the triality automorphism of D_4. This also explains that there is a unique invariant quadratic polynomial in the case of D_4 for $SL(4)$ as described in the previous section.

We believe this is evidence that the Epstein series $\mathrm{Ep}_s^{SO(n,n)}$ admits its global minimum for all $s > 0$ at the symmetric point $H_{C_L,B}$ for the Cartan type best packing lattices (5.2.15).

5.3 Results

Let us now give our result for the minima of the leading and next to leading Wilson coefficients of maximally supersymmetric string theory as given in section 4.3.

11 dimensions In $D = 11$ dimensions there are no moduli and the Wilson coefficients (4.3.63) and (4.3.64) are constants. They are given by

$$\mathcal{E}_{(0,0)} = \frac{2\pi^2}{3} \approx 6.57974, \tag{5.3.1}$$

for the leading Wilson coefficient and by

$$\mathcal{E}_{(1,0)} = 0\,, \tag{5.3.2}$$

for the next to leading Wilson coefficient.

10A dimensions The type IIA minimal Wilson coefficient (4.3.6) reaches its minimum value at $e^{\phi_{\min}} = \frac{3}{\pi}\zeta(3)^{\frac{1}{2}} \approx 1.04697$. At this point the minimum value is given by

$$\mathcal{E}_{(0,0)}(\phi_{\min}) = \frac{8\pi^{\frac{3}{2}}}{3\sqrt{3}}\zeta(3)^{\frac{1}{4}} \approx 8.97665\,. \tag{5.3.3}$$

The next to minimal Wilson coefficient (4.3.12) reaches its minimum value at a very similar point $e^{\phi_{\min}} = \frac{\sqrt{15}}{2^{\frac{1}{4}}\pi}\zeta(5)^{\frac{1}{4}} \approx 1.04611$. At this point the minimum value is given by

$$\mathcal{E}_{(1,0)}(\phi_{\min}) = \frac{8\times 2^{\frac{5}{8}}\pi^{\frac{5}{2}}}{45\times 15^{\frac{1}{4}}}\zeta(5)^{\frac{3}{8}} \approx 2.47047\,. \tag{5.3.4}$$

10B dimensions The type IIB Wilson coefficients (4.3.8) and (4.3.13) both reach their minimum value at the same point $H_{C_{A_2}} = H_\tau$ for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. At this point the Epstein series $\mathrm{Ep}_s^{SL(2)}$ can be written as the zeta function over $\mathbb{Q}(\sqrt{-3})$ [179]

$$\mathrm{Ep}_s^{SL(2)}(H_{C_{A_2}}) = {\sum_{z\in\mathbb{Z}(e^{\frac{i\pi}{3}})}}' \frac{\left(\frac{\sqrt{3}}{2}\right)^s}{|z|^{2s}} = \frac{6}{12^{\frac{s}{2}}}\zeta(s)\left(\zeta\left(s,\frac{1}{3}\right) - \zeta\left(s,\frac{2}{3}\right)\right)\,, \tag{5.3.5}$$

where $\zeta(s,a)$ is the Hurwitz zeta function. Therefore the minimum value of the Wilson couplings is given by

$$\mathcal{E}_{(0,0)}(\varphi_{\min}) = \frac{3^{\frac{1}{4}}}{\sqrt{2}}\zeta\left(\frac{3}{2}\right)\left(\zeta\left(\frac{3}{2},\frac{1}{3}\right) - \zeta\left(\frac{3}{2},\frac{2}{3}\right)\right) \approx 8.89275\,, \tag{5.3.6}$$

for the minimal Wilson coupling and

$$\mathcal{E}_{(1,0)}(\varphi_{\min}) = \frac{1}{4\times 3^{\frac{1}{4}}\sqrt{2}}\zeta\left(\frac{5}{2}\right)\left(\zeta\left(\frac{5}{2},\frac{1}{3}\right) - \zeta\left(\frac{5}{2},\frac{2}{3}\right)\right) \approx 2.35975\,, \tag{5.3.7}$$

for the next to minimal Wilson coupling.

9 dimensions In $D = 9$ dimensions the minimal Wilson coefficient (4.3.18) reaches its minimum value at $r_{\min} = \frac{9\times 3^{\frac{1}{4}}}{8\sqrt{2}\,\pi^2}\zeta\left(\frac{3}{2}\right)\left(\zeta\left(\frac{3}{2},\frac{1}{3}\right)-\zeta\left(\frac{3}{2},\frac{2}{3}\right)\right) \approx 1.01365$ and $H_{C_{A_2}} = H_\tau$ for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. At this point the minimum value is given by

$$\mathcal{E}_{(0,0)}(\varphi_{\min}) = \frac{7\pi^{\frac{6}{7}}}{2\times 3^{\frac{5}{7}}}\left(\zeta\left(\frac{3}{2}\right)\left(\zeta\left(\frac{3}{2},\frac{1}{3}\right)-\zeta\left(\frac{3}{2},\frac{2}{3}\right)\right)\right)^{\frac{4}{7}} \approx 15.4721\,. \tag{5.3.8}$$

The next to minimal Wilson coefficient (4.3.25) reaches its minimum value at a very similar point $r_{\min} \approx 1.01377$[13] and $H_{C_{A_2}} = H_\tau$ for $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. At this point the minimum value is given by

$$\mathcal{E}_{(1,0)}(\varphi_{\min}) \approx 4.83687\,. \tag{5.3.9}$$

We can see that the leading Wilson coefficients are indeed always positive for $D \geq 9$ because the Eisenstein series are in the domain of absolute convergence. This is consistent with the positivity bound imposed by unitarity (5.0.3).

8 dimensions $D = 8$ is the critical dimension where the one loop supergravity amplitude and the leading Wilson coefficient are of the same order therefore no positivity bound can be imposed by unitarity. Actually we saw in section 3.5.2 that in this dimension both are separately diverging and they can only be made sense of together. We also saw in sections 3.5.2 and 4.3.3 how to properly define the regularised Wilson coupling whose specific value depends on some renormalisation scale μ which cancels in the complete amplitude.

In $D = 8$ dimensions the Wilson coefficients (4.3.32) and (4.3.40) both reach their minimum value at the same point $H_{C_{A_3}} \times H_{C_{A_2}}$. If we choose $\mu = 1/l_P$ the minimum value of the Wilson couplings is given by

$$\mathcal{E}_{(0,0)1/l_P}(\varphi_{\min}) = \hat{\mathrm{E}}\mathrm{p}_{\frac{3}{2}}^{SL(3)}(H_{C_{A_3}}) + 2\hat{\mathrm{E}}\mathrm{p}_{1}^{SL(2)}(H_{C_{A_2}}) + \frac{22\pi}{3} - 4\pi\ln(2\pi) \approx 15.2363\,, \tag{5.3.10}$$

for the minimal Wilson coupling [176][14] and

$$\mathcal{E}_{(1,0)}(\varphi_{\min}) = \frac{1}{2}\mathrm{Ep}_{\frac{5}{2}}^{SL(3)}(H_{C_{A_3}}) - 4\mathrm{Ep}_{-\frac{1}{2}}^{SL(3)}(H_{C_{A_3}})\mathrm{Ep}_{2}^{SL(2)}(H_{C_{A_2}}) \approx 10.7196\,, \tag{5.3.11}$$

[13] The exact values are too cumbersome to give here but are not hard to find by extremising (4.3.25) with respect to r.

[14] There is an error in our article [176] and the correct value is indeed the one given above.

for the next to minimal Wilson coupling [176].[15]

7 dimensions In $D = 7$ dimensions the Wilson coefficients (4.3.46) and (4.3.54) both reach their minimum value at the same point $H_{C_{D_5}}$. At this point the minimum value of the Wilson couplings is given by

$$\mathcal{E}_{(0,0)}(\varphi_{\min}) = \mathrm{Ep}^{SL(5)}_{\frac{3}{2}}(H_{C_{D_5}}) \approx -9.50663\,, \tag{5.3.12}$$

for the minimal Wilson coupling [176] and

$$\mathcal{E}_{(1,0)}(\varphi_{\min}) = \frac{1}{2}\hat{\mathrm{E}}\mathrm{p}^{SL(5)}_{\frac{5}{2}}(H_{C_{D_5}}) + \frac{\pi}{15}\zeta(5)\hat{E}^{SL(5)}_{\frac{5}{2}\Lambda_3}(H_{C_{D_5}}) \approx 26.2315\,, \tag{5.3.13}$$

for the next to minimal Wilson coupling.[16]

6 dimensions In $D = 6$ dimensions the leading Wilson coefficient (4.3.60) reaches its minimum value at the point $H_{C_{D_5},B}$. At this point the minimum value of the leading Wilson coupling is given by [176]

$$\mathcal{E}_{(0,0)}(\varphi_{\min}) = \mathrm{Ep}^{SO(5,5)}_{\frac{3}{2}}(H_{C_{D_5},B}) \approx -3.445\,. \tag{5.3.14}$$

We can see that the global minima of the leading Wilson coefficients are indeed always negative $D \leq 7$ because the Eisenstein series are in the critical strip. This is also consistent with unitarity as the contribution of the leading Wilson coefficient to the amplitude is subleading with respect to the one loop supergravity amplitude, see [189] for example.

[15] The fact that the minimum for the next to minimal coupling is also reached at $H_{C_{A_3}} \times H_{C_{A_2}}$ instead of $H_{C_{A_3^*}} \times H_{C_{A_2}}$ is non trivial and was checked numerically.

[16] The fact that the minimum for the next to minimal coupling is also reached at $H_{C_{D_5}}$ instead of $H_{C_{D_5^*}}$ is non trivial and was checked numerically. This will be the subject of an upcoming article.

Chapter 6

Conclusion and outlook

In this thesis we have tried to present in a coherent and mostly self contained way the context and results of this PhD. We have first presented in the introduction the wider context in which this PhD fits in: that of quantum gravity. We have introduced the concept of effective field theory which is the modern way to view quantum field theory from the high energy standpoint and in our opinion is the also the ideal framework to talk about quantum gravity. We have then talked about the S-matrix bootstrap program as a way to put constraints on effective field theories via bounds on the scattering amplitudes using only very general and widely accepted physical principles such as causality and unitarity. Finally we have presented the string theory approach to quantum gravity and its effective field theory formulation: the swampland program. We argued that symmetry may in fact uniquely constrain the effective field theory of maximally supersymmetric quantum gravity leading to a potential verification of the string lamppost principle.

In the second chapter we gave a brief review of superstring theory. We derived the massless spectrum of type II string theory in the canonical covariant quantization procedure. We then talked about the various supergravity limits of string theory and M-theory and their anomaly cancellations. Finally we reviewed the web of dualities linking the different critical superstring theories and M-theory together.

In chapter 3 we talked about string amplitudes. We reviewed the structure of string perturbation theory and constructed the moduli space of genus 0, 1 and 2 string amplitudes. We then talked about factorisation properties specific to maximally supersymmetric amplitudes. Using the formalism of spinor helicities and superspace we proved the sewing relations in 6 dimensions and showed that the unitarity properties of the full superamplitude reduce to the unitarity properties of the scalar amplitude. We then derived the low energy limit of genus 0, 1 and 2 string amplitudes of type II superstring theory compactified on a torus and compared the result to maximal supergravity amplitudes. We showed using tropical geometry and tropical limits that

the low energy limit of maximally supersymmetric string theory reproduces the supergravity amplitudes with the addition of some higher order Wilson operators. We also saw that UV divergences in dimension 8 supergravity can be linked to divergences in Wilson couplings of the same order and we showed how to properly regularise each term in the finite string amplitude.

In the fourth chapter we gave a mostly self contained review of automorphic forms and Eisenstein series. We also talked about parabolic subgroups and gave the parabolic Fourier decompositions of relevant Eisenstein series. We then reviewed the notion of U-duality for the case of maximally supersymmetric string theory. We showed that maximal supersymmetry along with U-duality entail a certain set of constraints for Wilson coupling that is uniquely solved by Eisenstein series in some case. We gave these couplings in the leading and next to leading case for dimensions 11, 10, 9, 8, 7 and 6 and showed that they obeyed the right degeneration limits.

Finally in chapter 5 we talked about our main result for the minima of these Wilson couplings in dimension 8, 7 and 6. We first reviewed the notion of a fundamental domain and Grenier's construction for an $SL(n)$ fundamental domain which we extended to a more general setting. We then gave arguments to consider so-called symmetric points located at corners of fundamental domains as potential candidates for minima of automorphic forms. We analysed special symmetric points for the relevant cases of $SL(n)$ and $SO(n,n)$ and gave additional density arguments specific to Epstein series to conjecture the global minima. Finally we gave our results for the minima and minimum values of leading and next to leading Wilson coefficients in dimension 11, 10, 9, 8, 7 and 6.

The natural continuation of this work is to compute the lower bounds on the Wilson coefficients coming from the S-matrix bootstrap side and compare the result to see if the string lamppost principle appears to be verified or not. This remains to be done for the leading Wilson coefficients in dimension less or equal to 8 and for the next to leading coefficient in any dimension as far as we know. In both cases the complication is that the Wilson coefficient is subleading with respect to some other term but it is a priori doable by modifying the ansatz for the amplitude [190].[1]

Another natural extension of this work is to dimensions lower or equal to 5 with dimension 4 being of natural phenomenological interest. Apart from the obvious complication that the moduli space is larger and thus the numerical analysis is harder to perform there is also the theoretical complication that the groups involved are actual exceptional groups rather than the much simpler special linear or special orthogo-

[1]The next to leading Wilson coupling is obviously subleading with respect to the leading Wilson coupling and the one loop supergravity amplitude in all dimensions but also with respect to the two loop supergravity amplitude in dimension less or equal to 7 which makes unitarity bounds for this coefficient very hard to extract in general.

nal groups. On the S-matrix bootstrap side the ansatz would also need to take into account that the two-loop amplitude also becomes leading with respect even to the leading Wilson coupling and one also has to take care of IR divergences in 4 dimensions.

Finally on the subject of supersymmetry and string unity we would like to mention a final project which we have worked on during the course of this PhD but which we have decided not to include in this thesis for the sake of brevity and coherence. This project involves looking at causality constraints coming from massive gravitons in supersymmetric theories. Indeed there have been some results suggesting that massive gravitons generically have large polynomial growth in their amplitudes which are incompatible with unitarity and therefore put very restrictive bounds on theories containing them [18]. However it seems that those large growths can cancel each other out in very specific configurations involving an infinite tower of massive gravitons with specific selection rules. For example this is the case with Kaluza Klein compactification where the growths are not more severe than in the higher dimensional massless theory [191–193]. We therefore looked at $\mathcal{N} = 1$ supersymmetric theories in four dimensions involving massive gravitons for which we derived the susy multiplet and linearised susy transformation rules by Kaluza-Klein reduction. We then computed the stress energy tensor multiplet truncated at the quadratic order and found that there is a unique theory with consistent couplings to gravity. It then remains to study the growths appearing in the amplitudes defined by such couplings and see if the cancellation of those growths implies an infinite tower of massive gravitons.

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Saturating unitarity bounds at U-duality symmetric points

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It has recently been shown that the leading Wilson coefficient in type II string theory can take (almost) all values allowed by unitarity, crossing symmetry and maximal supersymmetry in $D = 10$ and $D = 9$ dimensions. This suggests that string theory might define the unique consistent quantum theory of gravity with maximal supersymmetry. We study the minima of the leading Wilson coefficient in $D = 6$, 7 and 8 dimensions and find the global minimum at the point in moduli space with maximal symmetry. The minimum value turns out to always be negative for $D \leq 7$.

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Contents

1 Introduction

Superstring theory on asymptotically flat spacetimes defines a large number of scattering theories including gravitons. In the low energy limit, the scattering amplitudes behave as quantum field theory amplitudes. In spacetime dimensions larger than four, the corresponding S-matrices are well defined and must therefore satisfy the usual quantum field theory conditions of analyticity, crossing symmetry and unitarity. A string theorist may wish that all consistent quantum gravity S-matrices could be obtained in superstring theory. Assuming this is the case, one can in principle derive constraints on consistent effective field theories that would not be visible in perturbative quantum field theory. This is often used in the swampland conjectures, see [1] for a review. A more humble conjecture, and maybe more realistic, is that all consistent quantum gravity theories with extended supersymmetry could be formulated as superstring theories. In this paper, we wish to study this question for maximally supersymmetric theories.

Type II superstring theory on the Cartesian product of D-dimensional Minkowski spacetime and a compact torus admits the maximal number of supersymmetries. The low energy effective theory is then maximal supergravity in D spacetime dimensions and the scattering amplitudes of massless states can in principle be described in supergravity using the Wilsonian effective action obtained by integrating out massive string states. In practice one first computes the perturbative

string amplitudes and compare them with the supergravity amplitudes with operator insertions to deduce the Wilsonian effective action [2–4]. The effective action is highly constrained from supersymmetry and U-duality [5–22]. The leading Wilson coefficient is completely determined by supersymmetry, U-duality and anomaly cancelations, provided one requires consistency in the decompactification limits.

It is commonly believed that type II superstring theory on a torus is the unique consistent quantum theory with maximal supersymmetry. One may therefore expect that the superstring S-matrix of massless states in a maximally supersymmetric vacuum covers all possible S-matrices satisfying analyticity, crossing symmetry, unitarity and maximal supersymmetry, as well as all the required anomaly cancelations. These consistency conditions can be analysed within the S-matrix bootstrap initiated in [23–25]. A lower bound on the leading Wilson coefficient was computed in [26, 27] in maximal supergravity in $D = 9, 10, 11$ dimensions, using the constraints from the two-to-two S-matrix. The unitarity bound was found to be close below the minimal value these Wilson coefficients can take in string theory and eleven-dimensional supergravity. The unitary bounds derived in [26, 27] are not sharp since they neglect non-elastic contributions to the optical theorem. Only integrability in two dimensions provides non-trivial examples of purely elastic S-matrices [28, 29], and taking particle production into account is expected to raise the bound on the leading Wilson coefficient [30]. So one may conclude from [26, 27] that string theory does indeed seem to saturate the sharp unitarity bound.

The case of $D = 11$ is particular because there is no moduli and the leading Wilson coefficient is a fixed number in M-theory. It is determined by the cancelation of the M5-brane anomaly [31, 32]. One may argue that this anomaly inflow argument is independent of string theory [33], so that there should not be any consistent theory in eleven dimensions with a different value of the leading Wilson coefficient. Supersymmetry also fixes the next-to-leading Wilson coefficient and the first Wilson coefficient to be determined by unitarity and crossing symmetry multiplies $\nabla^8 R^4$.

In $D \leq 10$ the leading Wilson coefficient is a function of the moduli and can take arbitrary large values, such that all the values consistent with the S-matrix unitarity bound seem to be covered by the string theory amplitude in $D = 9$ and $D = 10$ dimensions.

The analysis of [27] can in principle be generalised to all spacetime dimensions $5 \leq D \leq 11$. However, one must modify the amplitude ansatz in dimension $D \leq 8$ to include the contribution from the supergravity one-loop amplitude in the low energy limit. In $D = 5$ one would furthermore need to include the contribution from the two-loop amplitude. Although it is technically challenging to include the one-loop correction in the S-matrix bootstrap method, it is a priori doable, see for example [34].

On the string theory side one needs to find the minimum value of the leading Wilson coefficient. It is a maximal parabolic Eisenstein series of the U-duality group in D dimensions [6, 11, 17, 35, 36]. In particular for the type IIB superstring amplitude in ten dimensions it is a real analytic Eisenstein series $2\zeta(3)E_{3/2}(S)$ on the upper complex half-plane [6]. The minimum is known to be at the $\mathbb{Z}_3$-symmetric point $S = \frac{1+i\sqrt{3}}{2}$ [37]. In dimension $D \leq 8$ the

leading Wilson coefficient is again a specific Eisenstein series of higher rank groups associated to their minimal automorphic representation [16, 17]. The $SL(3)$ Epstein series relevant for the leading Wilson coefficient in $D = 8$ dimensions has been studied numerically in [38]. The global minimum was found to be at the point in moduli space defined by the unimodular symmetric matrix proportional to the Gram matrix of the lattice A_3. Finding minima of Eisenstein series is generally an open problem, and is the subject of this paper.

We provide strong evidence that the global minimum of the $SL(N)$ Epstein series $\mathrm{Ep}_s^N(H)$ is obtained at the unimodular symmetric matrix $H = H_{\mathrm{dlp}}$ proportional to the Gram matrix of the densest lattice sphere packing in N dimensions for all $s > \frac{N}{4}$. It is proved for asymptotically large s in [39]. We prove that the densest lattice sphere packing Gram matrix H_{dlp} is a local minimum of the Epstein series for all s and $N \leq 8$. We identify candidates for local minima as symmetric points and we checked that the lowest minimum is at H_{dlp} numerically. For $N = 5$ we study the leading Wilson coefficient in $D = 7$ dimensions on several surfaces containing H_{dlp} in moduli space and find each time that it is a global minimum on these surfaces.

We generalise this analysis to the $\mathrm{Spin}(5, 5)$ Eisenstein series appearing as the leading Wilson coefficient in $D = 6$ dimensions and find strong evidence that the global minimum is at the $W(D_5) \times W(D_5)$ symmetric point. This point is the analogue of the point of enhanced $\mathrm{Spin}(10)$ symmetry in perturbative heterotic string theory on T^5.

One striking feature is that the leading Wilson coefficient can always be negative in dimension $D < 8$. This is not in contradiction with unitarity because the leading Wilson coefficient is subleading with respect to the supergravity one-loop correction for $D \leq 8$. One therefore expects the Wilson coefficient to possibly be negative, and comparable to the one-loop correction at Planck scale [40].

The Eisenstein series appearing in the Wilson coefficients are absolutely convergent when they are dominant compare to the loop corrections, and defined by analytic continuation when they are subleading [41, 42]. It follows that the Wilson coefficient are necessary positive when they are dominant, and can always be possibly negative when they are subleading, consistently with unitarity. We will show indeed that the global minimum of an Eisenstein series is always negative in the critical strip, where it cannot be defined as an absolutely convergent sum.

The paper is organised as follows. In the second section we give some notations and summarise our results. We define a fundamental domain for the various moduli spaces of interest in Section 3. In Section 4 we give the main results leading to the conjectured minimum of Epstein series at symmetric points and in particular prove they are local minima. We discuss the specific case of dimension 8 in Section 5, where the splitting of the string amplitude into analytic and non-analytic pieces is ambiguous due to a logarithmic divergence. In Section 6 we expose numerical checks of our conjecture.

2 Notations and summary of the results

The four-graviton superstring amplitude on $\mathbb{R}^{1,9-d} \times T^d$ factorises in the form

$$\mathcal{M}_4 = -i\frac{\kappa_D^2}{2^{10}} t_8 t_8 \prod_{a=1}^{4} R(k_a, \epsilon_a) \mathcal{A}(s,t,u,\varphi) \tag{2.1}$$

where $\mathcal{A}(s,t,u,\varphi)$ is invariant under permutations of the three Mandelstam variables and is a function of the moduli φ in $K(E_{d+1})\backslash E_{d+1(d+1)}/E_{d+1}(\mathbb{Z})$ [43]. We define the Planck length in $D = 10-d$ spacetime dimensions as

$$\kappa_D^2 = \frac{1}{2}(2\pi)^{7-d} \ell_{\rm P}^{8-d} \,, \tag{2.2}$$

and the D-dimensional effective string coupling $g_D = e^{\phi}/\sqrt{v_d}$ in terms of the ten-dimensional dilaton ϕ and the volume $V_d = (2\pi\sqrt{\alpha'})^d v_d$ of T^d, such that

$$\alpha' = g_D^{-\frac{4}{8-d}} \ell_{\rm P}^2 \,. \tag{2.3}$$

In the low energy limit, one can write

$$\mathcal{A}(s,t,u,\varphi) = \frac{64}{stu} + 32(2\pi)^{7-d}\ell_{\rm P}^{8-d}\big(I_4(s,t) + I_4(t,u) + I_4(u,s)\big) + \ell_{\rm P}^6 \mathcal{E}_{(0,0)}(\varphi) + \frac{\ell_{\rm P}^{10}}{16}\mathcal{E}_{(1,0)}(\varphi)(s^2+t^2+u^2) + o(\ell_{\rm P}^{15-2d}) \tag{2.4}$$

so that the leading Wilson coefficient $\ell_{\rm P}^6 \mathcal{E}_{(0,0)}(\varphi)$ is between the one-loop and the two-loop supergravity corrections for $D = 6, 7, 8$. It is equal to the maximal parabolic Eisenstein series

$$\mathcal{E}_{(0,0)}(\varphi) = 4\pi\xi(d-2) E^{E_{d+1}}_{\frac{d-2}{2}\Lambda_{d+1}} \tag{2.5}$$

where Λ_{d+1} is the fundamental weight associated to the electric charges representation in D dimensions and $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta function. In $D \geq 7$ it can be written in terms of $SL(N)$ Epstein series defined by analytic continuation of the sum [2]

$$\mathrm{Ep}_s^N(H) = \sum_{n \in \mathbb{Z}^N \backslash \{0\}} \frac{1}{H[n]^s} = 2\zeta(2s) E^{SL(N)}_{s\Lambda_1}(H) \,. \tag{2.6}$$

With this definition one has

$$\mathcal{E}_{(0,0)}(\varphi) = \mathrm{Ep}^5_{\frac{3}{2}}(H) \,, \tag{2.7}$$

in seven dimensions [11]. We conjecture that the global minimum is at the densest sphere lattice packing point D_5

$$\mathrm{Ep}^5_{\frac{3}{2}}(H_{\rm dlp}) \approx -9.50663 \,. \tag{2.8}$$

[2] In this paper we use Ep_s^N to distinguish the Epstein series normalisation used in the original papers [6, 11, 35] from the Langlands Eisenstein series normalisation $E^{SL(N)}_{s\Lambda_1}$ more commonly used since [17, 36].

In eight dimensions one must take into account that the one-loop supergravity amplitude is logarithmically divergent, and so are the Epstein series appearing in $\mathcal{E}_{(0,0)}(\varphi)$. One must therefore introduce an appropriate renormalisation

$$\mathcal{E}_{(0,0),\mu}(\varphi) = \widehat{\mathrm{Ep}}^{\,3}_{\frac{3}{2}}(H) + 2\widehat{\mathrm{Ep}}^{\,2}_{1}(U) + \frac{22\pi}{3} - 4\pi \ln(2\pi\ell_{\mathrm{P}}\mu) \ , \tag{2.9}$$

where the renormalised Epstein series $\widehat{\mathrm{Ep}}^{\,3}_{\frac{3}{2}}$ and $\widehat{\mathrm{Ep}}^{\,2}_{1}$ are defined as in [36]. The renormalisation scale μ cancels in the complete amplitude, and the specific finite number is determined by our definition of the 1-loop box integral, as we explain in Section 5. Choosing $\mu = 1/\ell_{\mathrm{P}}$, we find the minimum value

$$\mathcal{E}_{(0,0),\frac{1}{\ell_{\mathrm{P}}}}(\varphi_{\mathrm{dlp}}) \approx 15.2363 \ . \tag{2.10}$$

In eight dimensions it also makes sense to include the next-to-leading Wilson coefficient

$$\mathcal{E}_{(1,0)}(\varphi) = \frac{1}{2}\mathrm{Ep}^{\,3}_{\frac{5}{2}}(H) - 4\mathrm{Ep}^{\,3}_{-\frac{1}{2}}(H)\mathrm{Ep}^{\,2}_{2}(U) \ , \tag{2.11}$$

while neglecting the two-loop supergravity integral. One computes the value at the minimum

$$\mathcal{E}_{(1,0)}(\varphi_{\mathrm{dlp}}) \approx 10.7196 \ . \tag{2.12}$$

In six dimensions the leading Wilson coefficient is written similarly as the vector representation Eisenstein series of $SO(5,5)$ [11]

$$\mathcal{E}_{(0,0)}(\varphi) = \mathrm{Ep}^{\,5,5}_{\frac{3}{2}}(H) \ . \tag{2.13}$$

We conjecture the global minimum to be at the point of enhancement Spin(10) symmetry

$$\mathrm{Ep}^{\,5,5}_{\frac{3}{2}}(H_{D_5}) \approx -3.445 \ . \tag{2.14}$$

3 Fundamental domain of $K\backslash G/G(\mathbb{Z})$

In order to determine the minium of an automorphic function on $K\backslash G/G(\mathbb{Z})$, it is useful to find an appropriate fundamental domain $\mathcal{F}$ for the action of the arithmetic subgroup $G(\mathbb{Z})$ on the symmetric space $\mathcal{M} = K\backslash G$. Here we assume that G is a simple group of real split form different from E_8, F_4 or G_2, and $G(\mathbb{Z})$ is its Chevalley subgroup associated to the weight lattice. This includes in particular the locally symmetric spaces relevant for the type II string theory effective action in dimensions greater than four.

Let us first recall the definition of a fundamental domain. One defines a free regular set $\mathring{\mathcal{F}} \subset \mathcal{M}$ as an open set in $\mathcal{M}$ such that any point in $\mathcal{M}$ can be mapped under the action of $G(\mathbb{Z})$ to its closure $\mathcal{F}$ in $\mathcal{M}$

$$\mathcal{F} = \overline{\mathring{\mathcal{F}}} \subset \mathcal{M} \ , \tag{3.1}$$

and for any element $\gamma \in G(\mathbb{Z})$ acting non-trivially on $\mathcal{M}$ one has

$$\gamma\mathcal{F} \cap \mathring{\mathcal{F}} = \emptyset \ . \tag{3.2}$$

It is required that $\partial\mathcal{F} = \mathcal{F} \setminus \mathring{\mathcal{F}}$ is of measure zero in $\mathcal{M}$. We then call $\mathcal{F}$ a fundamental domain of $G(\mathbb{Z})$ in $\mathcal{M}$. Equivalently one can define $\mathcal{F}$ to satisfy

$$\bigcup_{\gamma\in G(\mathbb{Z})} \gamma\mathcal{F} = \mathcal{M}\,, \qquad \gamma\mathcal{F}\cap\mathcal{F} \subset \partial\mathcal{F}\,. \tag{3.3}$$

There is generally no canonical fundamental domain. A first fundamental domain was introduced by Minkowski for $G(\mathbb{Z}) = GL(N,\mathbb{Z})$ acting on $SO(N)\backslash SL(N)$ [44]. Grenier then defined a different fundamental domain for $GL(N,\mathbb{Z})$ that is easier to generalise to arbitrary simple groups [45]. In this section we shall first review Grenier's construction in the case of $SL(N,\mathbb{Z})$. Then we will show that there is a natural generalisation of Grenier's fundamental domain for all simple groups of split real form but E_8, F_4 or G_2. We will finally discuss $SO(N,N,\mathbb{Z})$ in more detail as an example.

3.1 Fundamental domain of $SL(N,\mathbb{Z})$

Following Grenier, we parametrise the symmetric space $\mathcal{M} = SO(N)\backslash SL(N)$ in the Iwasawa decomposition with $N-1$ variables $y_i \in \mathbb{R}_+$ with $1 \le i \le N-1$ and $\frac{N(N-1)}{2}$ variables $x_{ij} \in \mathbb{R}$ with $1 \le i < j \le N$. The group representative in $SL(N)$ can then be written as the upper triangular matrix

$$\mathcal{V} = \frac{1}{y}\begin{pmatrix} 1 & & & & \\ & y_1 & & & \\ & & y_1y_2 & & \\ & & & \ddots & \\ & & & & y_1y_2\dots y_{N-1} \end{pmatrix}\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix}\,, \tag{3.4}$$

with

$$y = \left(\prod_{i=1}^{N-1} y_i^{N-i}\right)^{\frac{1}{N}}\,. \tag{3.5}$$

For short we do not distinguish the group representative from the corresponding point in $\mathcal{M}$. To any element $\mathcal{V}\in\mathcal{M}$ we can associate a positive definite symmetric bilinear form over $\mathbb{Z}^N$

$$H[n] = n^{\intercal}\mathcal{V}^{\intercal}\mathcal{V}n\,. \tag{3.6}$$

The Grenier fundamental domain is defined by induction by writing first $\mathcal{V}$ in the maximal parabolic subgroup $(GL(1)\times SL(N-1))\ltimes\mathbb{R}^{N-1}\subset SL(N)$ as

$$\mathcal{V} = \frac{1}{y}\begin{pmatrix} 1 & 0 \\ 0 & y^{\frac{N}{N-1}}\mathcal{V}_1 \end{pmatrix}\begin{pmatrix} 1 & x^{\intercal} \\ 0 & \mathbb{1} \end{pmatrix} \tag{3.7}$$

where $x^{\intercal} = (x_{12},\dots,x_{1N})$ and $\mathcal{V}_1 \in SL(N-1)$, and then recursively for each $\mathcal{V}_i \in SL(N-i)$. To avoid taking care of the factors of y, Grenier introduces the non-unimodular symmetric matrix $Y = y^2\mathcal{V}^{\intercal}\mathcal{V}$

$$Y[n] \equiv n^{\intercal}Yn = \left(n_1 + \sum_{k=2}^{N} x_{1k}n_k\right)^2 + y_1^2 Y_1[n] \tag{3.8}$$

where $Y_1 = y_1^{-2} y^{\frac{2N}{N-1}} \mathcal{V}_1^\intercal \mathcal{V}_1$ is a symmetric bilinear form over $\mathbb{Z}^{N-1}$, so its argument n is implicitly $(n_2, \dots, n_N)$. One defines recursively the bilinear forms $Y_i[n] \equiv (n_{i+1}, \dots n_n)^\intercal Y_i (n_{i+1}, \dots n_n)$ for $1 \leq k \leq N$ as

$$Y_i[n] = \left(n_{i+1} + \sum_{k=i+2}^{N} x_{i+1,k} n_k \right)^2 + y_{i+1}^2 Y_{i+1}[n] \ , \tag{3.9}$$

with

$$Y_{N-1}[n] = n_N^2 \ , \tag{3.10}$$

and by convention $Y_0[n] = Y[n]$. The condition for $\mathcal{V}$ to be in the Grenier fundamental domain $\mathcal{F}$ of $\mathcal{M}$ is defined recursively by the two conditions:

1. $Y_i[n] \geq 1$, for all non-vanishing $n \in \mathbb{Z}^{N-i}$ of greatest common divisor $\gcd(n) = 1$.
2. $x_{i,i+j} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for all $i \in \{1, \dots, N-1\}$ and $j \in \{2, \dots, N-i\}$, $x_{i,i+1} \in \left[0, \frac{1}{2}\right]$ for all $i \in \{2, \dots, N-1\}$, whereas $x_{1,2} \in \left[0, \frac{1}{2}\right]$ if N is odd and $x_{1,2} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ if N is even.

Grenier proved that one can always restrict the first condition to a finite set of vectors such that this gives a finite number of inequalities. This follows from the property that $Y_i[n]$ is positive definite and there is a finite number of lattice points in the ball $Y_i[n] \leq 1$.

Fundamental domain of $SL(2, \mathbb{Z})$

For the case $N = 2$ the Grenier fundamental domain agrees with the standard $SL(2, \mathbb{Z})$ fundamental domain. The only vector for which the first condition is non-trivial is $n = (0, 1)^\intercal$ and together with the second condition one gets

$$\text{i) } x_{12}^2 + y_1^2 \geq 1 \ , \qquad \text{ii) } -\tfrac{1}{2} \leq x_{12} \leq \tfrac{1}{2} \ .$$

Fundamental domain of $SL(3, \mathbb{Z})$

Another case of interest for us is $N = 3$. The fundamental domain of $SO(3) \backslash SL(3)$ is given by

i) $x_{23}^2 + y_2^2 \geq 1$

ii) $x_{12}^2 + y_1^2 \geq 1$

iii) $x_{13}^2 + y_1^2 (x_{23}^2 + y_2^2) \geq 1$

iv) $(x_{12} - x_{13})^2 + y_1^2 ((1 - x_{23})^2 + y_2^2) \geq 1$

v) $(1 - x_{12} + x_{13})^2 + y_1^2 ((1 - x_{23})^2 + y_2^2) \geq 1$

vi) $0 \leq x_{12} \leq \frac{1}{2}$

vii) $0 \leq x_{23} \leq \frac{1}{2}$

viii) $-\frac{1}{2} \leq x_{13} \leq \frac{1}{2}$

The condition i) comes from the condition 1) for $Y_1[n]$ applied to $n = (0,1)^\intercal$ while the conditions ii)-v) come from the condition 1) for $Y_0[n]$ applied to $n = (0,1,0)^\intercal$, $n = (0,0,1)^\intercal$, $n = (0,-1,1)^\intercal$, $n = (1,-1,1)^\intercal$ respectively. Again vi)-viii) are a direct consequence of condition 2).

3.2 Generalisation to $G(\mathbb{Z})$

It appears that Grenier's construction of a fundamental domain is based on a sequence of maximal abelian parabolic subgroups. In this section we therefore consider G to be the split real form of a simple group of rank r admitting an abelian parabolic subgroup, which excludes E_8, F_4 and G_2. We shall discuss the explicit example of $SO(N,N)$ in the next section. One can probably generalise this construction to Heisenberg parabolic subgroups to encompass all exceptional groups, but this requires further analysis and will not be relevant for us. In what follows the group G is defined from its fundamental representations and $G(\mathbb{Z}) \subset G$ is the Chevalley subgroup associated to the weight lattice.

Let us first describe an appropriate coordinate system on $\mathcal{M} = K\backslash G$. Let P_1 be a maximal abelian parabolic subgroup

$$P_1 = (GL(1) \times G_1) \ltimes U_1 \subset G \,, \tag{3.11}$$

where G_1 is itself the split real form of a semi-simple group and U_1 is an abelian unipotent subgroup. One calls respectively $GL(1) \times G_1$ the Levi subgroup and U_1 the unipotent radical of the parabolic subgroup P_1. In the last section, for $G = SL(N)$, we had $G_1 = SL(N-1)$ and $U_1 = \mathbb{R}^{N-1}$ the additive group for example.

We can then apply the same decomposition to G_1 and all G_i semi-simple subgroups successively

$$G_i \supset P_{i+1} = (GL(1) \times G_{i+1}) \ltimes U_{i+1} \,, \tag{3.12}$$

with U_i an abelian unipotent group for all $i = 1$ to r. By construction $P_r = GL(1) \ltimes U_r$ and the Borel subgroup can be defined as

$$B = GL(1)^r \ltimes U_r \ltimes U_{r-1} \cdots \ltimes U_1 \,, \tag{3.13}$$

and the Iwasawa decomposition of G is compatible with this succession of abelian parabolic subgroups

$$B \subset GL(1)^{r-2} \times P_{r-1} \ltimes U_{r-2} \cdots \ltimes U_1 \subset \cdots \subset GL(1) \times P_2 \ltimes U_1 \subset P_1 \subset G \,. \tag{3.14}$$

The Iwasawa decompositions provides coordinates on $\mathcal{M}$, with $y_i \in \mathbb{R}_+$ for $i = 1$ to r, parametrising each $GL(1)$ factors of the Cartan torus and a vector $\mathbf{x}_i \in \mathfrak{u}_i$ the Lie algebra of U_i for $i = 1$

to r parametrising each U_i unipotent group. By construction the Borel subgroup $B_i \subset G_i$ can be decomposed accordingly into

$$B_i = GL(1)^{r-i} \ltimes U_r \ltimes U_{r-1} \cdots \ltimes U_{i+1} \,. \tag{3.15}$$

Therefore $\{y_j, \mathbf{x}_j\}$ for $i+1 \leq j \leq r$ provide coordinates on the symmetric space $\mathcal{M}_i = K_i \backslash G_i$, where K_i is the maximal compact subgroup of G_i.

By definition each maximal parabolic subgroup $P_i \subset G_{i-1}$ is determined by a fundamental weight $\Lambda^{(i)}$ of the subgroup G_{i-1}. The associated set of roots of G is called an abelian enumeration in [22]. We write $R^{(i)} = R(\Lambda^{(i)}, \mathbb{R})$ the associated highest weight representation, which highest weight vector $\mathbf{e}^{(i)} \in R^{(i)}$ admits as stabiliser the subgroup $G_i \ltimes U_i \subset G_{i-1}$. The G_{i-1}-orbit of $\mathbf{e}^{(i)}$ is therefore the coset space

$$G_{i-1} \cdot \mathbf{e}^{(i)} = G_{i-1}/(G_i \ltimes U_i) \,. \tag{3.16}$$

For example in the case of $SL(N)$, the maximal parabolic subgroup $(GL(1) \times SL(N-1)) \ltimes \mathbb{R}^{N-1}$ is associated to the fundamental weight Λ_1 of $SL(N)$ and the corresponding highest weight representation is the fundamental representation $\mathbb{R}^N$. The orbit of the highest weight vector is then dense in the module

$$\mathbb{R}^N \smallsetminus \{0\} = SL(N)/(SL(N-1) \ltimes \mathbb{R}^{N-1}) \,. \tag{3.17}$$

By convention we will call $R^{(i)}$ the fundamental representation. In general the orbit of the highest weight vector is not necessarily dense in the module $R^{(i)}$, but is defined instead as the set of non-zero elements $\mathbf{v}^{(i)}$ satisfying the quadratic constraint [46]

$$\mathbf{v}^{(i)} \times \mathbf{v}^{(i)} \equiv \kappa_{\alpha\beta} T^\alpha \mathbf{v}^{(i)} \otimes T^\beta \mathbf{v}^{(i)} - \big(\Lambda^{(i)}, \Lambda^{(i)}\big) \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)} = 0 \,, \tag{3.18}$$

where $\kappa_{\alpha\beta}$ denotes the Killing–Cartan form of $\mathfrak{g}_{i-1}$ and T^α its representation matrices in $R^{(i)}$. One can then write (3.16) as

$$G_{i-1}/(G_i \ltimes U_i) = \big\{ \mathbf{v}^{(i)} \in R^{(i)} \,|\, \mathbf{v}^{(i)} \times \mathbf{v}^{(i)} = 0 \,,\; L(\mathbf{v}^{(i)}) > 0 \big\} \,, \tag{3.19}$$

where some further positivity condition $L(\mathbf{v}^{(i)}) > 0$ may be necessary for specific groups G_{i-1} and representations $R^{(i)}$. For example for $SO(N,N)$ one can take the fundamental representation as the vector representation, and $\mathbf{v}^{(i)} \times \mathbf{v}^{(i)} = (\mathbf{v}^{(i)}, \mathbf{v}^{(i)}) = 0$ is the condition that $\mathbf{v}^{(i)}$ is light-like. For the connected component $SO_0(1,1)$ one would moreover demand that the non-zero lightcone coordinate is positive.

For each point in $\mathcal{M}_i$ we define the associated matrix $\mathcal{V}_i \in P_{i+1}$ in the representation $R^{(i+1)}$ of G_i. For short we will write $\mathcal{V}_i = \rho_{\Lambda^{(i+1)}}[\mathcal{V}_i]$ for both the point $\mathcal{V}_i \in \mathcal{M}_i$ and the matrix $\rho_{\Lambda^{(i+1)}}[\mathcal{V}_i]$ in the representation $R^{(i+1)}$. We can then define the symmetric bilinear form $H_i = \mathcal{V}_i^\intercal \mathcal{V}_i$ in the representation $R^{(i+1)}$

$$H_i[\mathbf{v}^{(i+1)}] \equiv \mathbf{v}^{(i+1)\intercal} H_i \mathbf{v}^{(i+1)} \,. \tag{3.20}$$

The matrix $\mathcal{V}_i$ can be written as

$$\mathcal{V}_i = \rho_{\Lambda^{(i+1)}}\big[e^{-\ln y_{i+1} \mathbf{h}^{(i+1)}} \mathcal{V}_{i+1}(y_{j>i+1}, \mathbf{x}_{j>i+1}) e^{\mathbf{x}_{i+1}} \big] \,, \tag{3.21}$$

where $\rho_{\Lambda^{(i+1)}}$ is the representation homomorphism and $\mathbf{h}^{(i+1)}$ the Cartan generator of G_i associated to the weight $\Lambda^{(i+1)}$ with the normalisation[3]

$$\rho_{\Lambda^{(i+1)}}[\mathbf{h}^{(i+1)}]\mathbf{e}^{(i+1)} = \mathbf{e}^{(i+1)} \ , \tag{3.22}$$

such that

$$\mathcal{V}_i\mathbf{e}^{(i+1)} = \frac{1}{y_{i+1}}\mathbf{e}^{(i+1)} \ . \tag{3.23}$$

Note that the coordinates y_i defined above are not the same as the coordinates introduced in (3.4).[4]

In the following we shall use that a point in $\mathcal{M}_i$ can be parametrised equivalently by all $\{y_j, \mathbf{x}_j\}$ for $j \geq i+1$ or by the symmetric matrix H_i. One can also equivalently use the coordinates $\{y_{i+1}, \mathbf{x}_{i+1}\}$ and the symmetric matrix H_{i+1}, etc...

The idea behind Grenier's construction of a fundamental domain is that one can construct recursively the fundamental domains $\mathcal{F}_i$ for $\mathcal{M}_i$ under the action of $G_i(\mathbb{Z})$ from $i = r-1$ to 0. At each step, one can use the bilinear form (3.20) to define a set of inequalities that determines $\mathcal{F}_i$.

Before generalising Grenier's construction, let us give some definitions. We define for $1 \leq i \leq r-1$

$$P_i(\mathbb{Z}) = G(\mathbb{Z}) \cap P_i = G_i(\mathbb{Z}) \ltimes U_i(\mathbb{Z}) \ , \quad G_i(\mathbb{Z}) = G(\mathbb{Z}) \cap GL(1) \times G_i \ , \quad U_i(\mathbb{Z}) = G(\mathbb{Z}) \cap U_i \ . \tag{3.24}$$

Note that with this definition $G_i(\mathbb{Z})$ may not be a subgroup of G_i but includes the entire discrete Levi subgroup. For $SL(N,\mathbb{Z})$ we would then define $G_i(\mathbb{Z}) = GL(N-i,\mathbb{Z})$ for example.

The construction of the fundamental domain is defined by induction. In order to show that $\mathcal{F}_i$ is a fundamental domain of $\mathcal{M}_i$ under the action of $G_i(\mathbb{Z})$ we have to show that for any element $H_i \in \mathcal{M}_i$ there exists an element $\gamma \in G_i(\mathbb{Z})$ such that $\gamma^\intercal H_i \gamma \in \mathcal{F}_i$ and conversely that for any non-trivial element $\gamma \in G_i(\mathbb{Z})$ and any element $H_i \in \mathcal{F}_i$, either $\gamma^\intercal H_i \gamma \notin \mathcal{F}_i$ or both H_i and $\gamma^\intercal H_i \gamma$ are on the boundary of $\mathcal{F}_i$.

We first consider the fundamental domain in $\mathcal{M}_i$ for the action of the parabolic subgroup $P_{i+1}(\mathbb{Z})$. Let p be an element of $P_{i+1}(\mathbb{Z}) = G_{i+1}(\mathbb{Z}) \ltimes U_{i+1}(\mathbb{Z})$, we can decompose it as

$$p = l \, \exp(\mathbf{b}) \tag{3.25}$$

with $l \in G_{i+1}(\mathbb{Z})$ and $\exp(\mathbf{b}) \in U_{i+1}(\mathbb{Z})$. $P_{i+1}(\mathbb{Z})$ acts on $\mathcal{M}_i$ by

$$p^\intercal H_i p = (y_{i+1}, l^\intercal H_{i+1} l, l^{-1}\mathbf{x}_{i+1} + \mathbf{b}) \ . \tag{3.26}$$

[3] $\mathbf{h}^{(i)}$ are related to the Cartan–Weyl basis generators of G_{i-1} by $\mathbf{h}^{(i)} = \frac{1}{(\Lambda^{(i)},\Lambda^{(i)})}\mathbf{h}_{\Lambda^{(i)}}$ and $\mathbf{x}_i$ decomposes in the basis of root generators $\mathbf{e}_\alpha$ satisfying $(\Lambda^{(i)}, \alpha) = 1$.

[4] Writing $\tilde{y}_i$ the Grenier coordinates in (3.4), the coordinates y_i used in this section are $y_i = \left(\prod_{j=i}^{N-1} \tilde{y}_j^{N-j}\right)^{\frac{1}{N+1-i}}$.

The adjoint Levi subgroup $G_{i+1}(\mathbb{Z})/Z(G_{i+1}(\mathbb{Z}))$, with $Z(G_{i+1}(\mathbb{Z}))$ the centre of $G_{i+1}(\mathbb{Z})$, acts freely on a dense open set in $\mathcal{M}_{i+1}$, and determines the fundamental domain

$$\mathcal{F}_{i+1} = \mathcal{M}_{i+1}/(G_{i+1}(\mathbb{Z})/Z(G_{i+1}(\mathbb{Z}))) \ . \tag{3.27}$$

To find a fundamental domain of $P_{i+1}(\mathbb{Z})$ in $\mathcal{M}_i$, it remains to act with $Z(G_{i+1}(\mathbb{Z})) \ltimes U_{i+1}(\mathbb{Z})$ on $\mathfrak{u}_{i+1}$. By construction $U_{i+1}(\mathbb{Z})$ acts by translation and $Z(G_{i+1}(\mathbb{Z}))$ acts either trivially or by multiplying $\mathbf{x}^{(i+1)} \in \mathfrak{u}_{i+1}$ by -1. One gets therefore

$$\mathcal{F}_i^{\mathrm{P}} = \mathcal{M}_i/P_{i+1}(\mathbb{Z}) = \mathbb{R}_+ \times \mathcal{F}_{i+1} \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^{d_{i+1}}/\mathbb{Z}_\mu \ , \tag{3.28}$$

where d_{i+1} is the dimension of U_{i+1} and $\mu = 2$ if $Z(G_{i+1}(\mathbb{Z}))$ acts as $\mathbb{Z}_2$ on $\mathfrak{u}_{i+1}$ and $\mu = 1$ if it acts trivially.

To define the fundamental domain of $G_i(\mathbb{Z})$, we introduce the Chevalley lattice $R^{(i+1)}(\mathbb{Z}) \subset R^{(i+1)}$, which is preserved by the action of $G_i(\mathbb{Z})$.

The equivalent of (3.19) for the discrete subgroup $G_i(\mathbb{Z})$ gives

$$G_i(\mathbb{Z})/P_{i+1}(\mathbb{Z}) = S_i = \{n \in R^{(i)}(\mathbb{Z}) \mid n \times n = 0,\ L(n) > 0,\ \gcd n = 1\} \tag{3.29}$$

where $n \times n$ is defined as in (3.18) and an additional positivity condition $L(n)$ may be required when the only elements in $G_i(\mathbb{Z})$ that change the sign of n are trivial in $G_i(\mathbb{Z})/G_{i+1}(\mathbb{Z})$.

One defines the positive set $S_{i>} \subset \mathcal{M}_i$ as

$$S_{i>} = \left\{H_i \in \mathcal{M}_i \mid H_i[n] \geq \frac{1}{y_{i+1}^2},\ \forall n \in S_i\right\} \tag{3.30}$$

Where $H_i[n]$ is the positive definite bilinear form (3.20). This domain is non-empty. For fixed $(H_{i+1}, \mathbf{x}_{i+1})$ in $\mathcal{F}_i^P$, one finds that $H_i \in S_{i>}$ for $y_{i+1} > L_{i+1}(H_{i+1}, \mathbf{x}_{i+1})$ sufficiently large.

We will show that a fundamental domain $\mathcal{F}_i = \mathcal{M}_i/G_i(\mathbb{Z})$ can be defined as the intersection

$$\mathcal{F}_i = \mathcal{F}_i^P \cap S_{i>} \ . \tag{3.31}$$

By construction, any element $\gamma \in G_i(\mathbb{Z})$ can be decomposed as

$$\gamma = h(n)p = h(n)\, l \exp(b) \ , \tag{3.32}$$

with $p \in P_{i+1}(\mathbb{Z})$, $l \in G_{i+1}(\mathbb{Z})$ and $\exp(b) \in U_{i+1}(\mathbb{Z})$. For each point in $G_i(\mathbb{Z})/P_{i+1}(\mathbb{Z})$ one can find a representative $h \in G_i(\mathbb{Z})$ that preserves $\mathcal{F}_i^P$. Using the isomorphism (3.29), one has therefore for each $n \in S_i$ a unique $h(n) \in G_i(\mathbb{Z})$ such that

$$h(n)\mathbf{e}^{(i)} = n \ , \qquad h(n)\mathcal{F}_i^P = \mathcal{F}_i^P \ . \tag{3.33}$$

We have already checked that for any point $H_i \in \mathcal{M}_i$, there is an element $p \in P_{i+1}(\mathbb{Z})$ such that $p^{\intercal} H_i p \in \mathcal{F}_i^P$. For any such $H_i' \in \mathcal{F}_i^P$ there is a smallest norm element $n_0 \in S_i$ such that

$$H_i'[n] \geq H_i'[n_0], \qquad \forall n \in S_i \ . \tag{3.34}$$

It may not be unique, but because $R^{(i)}(\mathbb{Z})$ is discrete in $R^{(i)}$, there is a finite number of discrete points in any ball $H'_i[\mathbf{x}] \leq H'_i[n]$ and therefore a finite number of n_0 satisfying (3.34). Choosing $h(n_0)$ one obtains

$$\frac{1}{y^2_{i+1}(\gamma)} = h^\intercal(n_0) H'_i h(n_0)[\mathbf{e}^{(i)}] = H'_i[n_0] \, , \tag{3.35}$$

and $\gamma^\intercal H_i \gamma \in \mathcal{F}_i^P \cap S_{i>}$ for $\gamma = ph(n_0)$. We have therefore proved that for all $H_i \in \mathcal{M}_i$, there exists $\gamma \in G_i(\mathbb{Z})$ such that $\gamma^\intercal H_i \gamma \in \mathcal{F}_i$ so that $\mathcal{F}_i$ includes a fundamental domain.

Now we need to prove that for any point in the interior of $\mathcal{F}_i$

$$\mathring{\mathcal{F}}_i = \mathring{\mathcal{F}}_i^P \cap \mathring{S}_{i>} \, , \tag{3.36}$$

and any non-trivial element $\gamma \in G(\mathbb{Z})/Z(G(\mathbb{Z}))$ we have $\gamma \mathring{\mathcal{F}}_i \cap \mathcal{F}_i = \{0\}$. First of all let us show that one can always determine $S_{i>}$ with a finite number of vectors $n \in S_i$. For $(y_{i+1}, H_{i+1}, \mathbf{x}_{i+1})$ in $\mathcal{F}_i^P$, the set $S_{i>}$ includes the condition $H_i[\mathbf{e}_{-\alpha_{i+1}} \mathbf{e}^{(i+1)}] \geq \frac{1}{y^2_{i+1}}$ for the simple root α_{i+1} of G_i satisfying $(\Lambda^{(i+1)}, \alpha_{i+1}) = 1$. In the appropriate coordinate system this reads

$$\frac{1}{y^2_{i+1}} x(\alpha_{i+1})^2 + \frac{1}{y_{i+1}^{2-\frac{2}{(\Lambda^{(i+1)}, \Lambda^{(i+1)})}}} \frac{1}{y(\alpha_{i+1})^2} \geq \frac{1}{y^2_{i+1}} \quad \Rightarrow \quad y_{i+1}^{\frac{2}{(\Lambda^{(i+1)}, \Lambda^{(i+1)})}} \geq \frac{3}{4} y(\alpha_{i+1})^2 \, , \tag{3.37}$$

for $y(\alpha_{i+1})$ the associated Cartan torus coordinate in the Levi subgroup G_{i+1}. In particular one can always choose the succession of parabolic subgroups (3.14) such that $y(\alpha_{i+1}) = y_{i+2}$. It follows by induction that there exists $l_{i+1} > 0$ independent of $(H_{i+1}, \mathbf{x}_{i+1})$ such that any point $(y_{i+1}, H_{i+1}, \mathbf{x}_{i+1}) \in \mathring{\mathcal{F}}_i^P \cap \mathring{S}_{i>}$ satisfies $y_{i+1} \geq l_{i+1}$. One can therefore restrict the conditions defining $S_{i>}$ to the vectors $n \in S_i$ in the ball $H_i[n] \leq \frac{1}{{l_{i+1}}^2}$ and there is only a finite number of those.

As a consequence $S_{i>}$ is defined by a finite intersection of closed sets and its interior $\mathring{S}_{i>}$ is defined by the finite intersection of open sets

$$\mathring{S}_{i>} = \left\{ H_i \in \mathcal{M}_i \mid H_i[n] > \frac{1}{y^2_{i+1}}, \forall n \in S_i, \; n \neq \mathbf{e}^{(i+1)} \right\} . \tag{3.38}$$

It follows directly that for any point $H_i \in \mathring{\mathcal{F}}_i$, the action of a non-trivial $\gamma = h(n)p \in G_i(\mathbb{Z})/Z(G(\mathbb{Z}))$ gives $\gamma^\intercal H_i \gamma \notin \mathcal{F}_i$. To see this, note that any non-trivial $p \in P_{i+1}(\mathbb{Z})$ moves $\gamma^\intercal H_i \gamma \notin \mathcal{F}_i^P \supset \mathcal{F}_i$. If $p = 1$ and $h(n)$ is non-trivial, we have by definition $h(n)\mathbf{e}^{(i+1)} = n$ and therefore

$$\gamma^\intercal H_i[n] \gamma < \frac{1}{y_{i+1}(\gamma)^2} \tag{3.39}$$

for this specific n, which shows that $\gamma^\intercal H_i \gamma \notin \mathcal{F}_i$. Note that if $H_i \in \partial \mathcal{F}_i$, either it is in the boundary of $\mathcal{F}_i^P$ or in the boundary of $S_{i>}$. In the second case it means that there is a finite set of vectors $n \in S_i$ not equal to $\mathbf{e}^{(i+1)}$ such that $H_i[n] = \frac{1}{y^2_{i+1}}$, and the corresponding $h(n)$ map $\partial S_{i>} \cap \mathcal{F}_i^P$ to itself.

This concludes the proof that $\mathcal{F}_i$ is a fundamental domain of $\mathcal{M}_i/G_i(\mathbb{Z})$, and by induction that $\mathcal{F}$ is a fundamental domain of $\mathcal{M}/G(\mathbb{Z})$.

3.3 Fundamental domain of $SO(N,N,\mathbb{Z})$

In this section we shall illustrate how this construction applies to the symmetric space $\mathcal{M} = \big(SO(N)\times SO(N)\big)\backslash SO(N,N)$ and the arithmetic subgroup $SO(N,N,\mathbb{Z})$ preserving the even self-dual lattice of split signature $I\!I_{N,N}$. In string theory, the group of T-duality on the torus T^d is $O(d,d,\mathbb{Z})$ and the relevant moduli space is $\mathcal{M}/O(d,d,\mathbb{Z})$. The group of U-duality of type II on T^4 is $\mathrm{Spin}(5,5,\mathbb{Z})$ instead and the moduli space is $\mathcal{M}/SO_0(5,5,\mathbb{Z})$. The difference with respect to the different possible duality groups $SO_0(N,N,\mathbb{Z})$, $SO(N,N,\mathbb{Z})$ and $O(N,N,\mathbb{Z})$ only appear in the first step because $SO_0(1,1,\mathbb{Z}) = \{1\}$, $SO(1,1,\mathbb{Z}) = \mathbb{Z}_2$ and $O(1,1,\mathbb{Z}) = \mathbb{Z}_2\times\mathbb{Z}_2$.

We will use the convention that the split signature metric is

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{3.40}$$

and we parametrise points in $\mathcal{M}$ by a symmetric matrix H in the vector representation of $SO(N,N)$, which satisfies

$$\eta^{\intercal} H \eta = H \ . \tag{3.41}$$

For short we will use the symbol H for the point in moduli space and for the symmetric matrix in $SO(N,N)$ that represents it. In this case it is convenient to choose the succession of parabolic subgroups (3.14) such that

$$P_i = GL(1)\times SO(N-i,N-i)\ltimes \mathbb{R}^{N-i,N-i} \ . \tag{3.42}$$

We decompose accordingly $H = H_0$ into each H_i for $1\le i\le N$ as

$$H_{i-1} = \begin{pmatrix} 1 & 0 & 0 \\ \eta x_i & \mathbb{1} & 0 \\ \frac{1}{2}(x_i,x_i) & x_i^{\intercal} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{y_i^2} & 0 & 0 \\ 0 & H_i & 0 \\ 0 & 0 & y_i^2 \end{pmatrix} \begin{pmatrix} 1 & x_i^{\intercal}\eta & \frac{1}{2}(x_i,x_i) \\ 0 & \mathbb{1} & x_i \\ 0 & 0 & 1 \end{pmatrix} \tag{3.43}$$

where $y_i\in\mathbb{R}_+$, $x_i\in\mathbb{R}^{N-i,N-i}$ is a vector of $SO(N-i,N-i)$, $\mathbb{1}$ is the identity matrix in $SO(N-i,N-i)$ and H_i is a symmetric element in $SO(N-i,N-i)$.

For the quotient by $O(N,N,\mathbb{Z})$, one derives that

$$P_i(\mathbb{Z}) = O(N-i,N-i,\mathbb{Z})\ltimes I\!I_{N-i,N-i} \ . \tag{3.44}$$

For $SO_0(N,N,\mathbb{Z})$, one simply gets $SO_0(N-i,N-i,\mathbb{Z})$. The sets S_i in (3.29) can be defined as

$$S_i = O(N-i,N-i,\mathbb{Z})/P_{i+1}(\mathbb{Z}) = \{Q\in I\!I_{N-i,N-i} \mid (Q,Q) = 0\,,\ \gcd Q = 1\} \tag{3.45}$$

for $0\le i\le N-1$. For $SO_0(N,N,\mathbb{Z})$ one must simply replace $S_{N-1} = \{(\pm1,0),(0,\pm1)\}$ by the trivial set to take into account that $SO_0(1,1,\mathbb{Z})$ is trivial.

At each step we decompose $Q\in I\!I_{N+1-i,N+1-i}$ as $Q = (m,q,n)$ with $m,n\in\mathbb{Z}$ and $q\in I\!I_{N-i,N-i}$, such that

$$\eta[Q] = (q,q) + 2mn \ , \tag{3.46}$$

and the bilinear form $H_{i-1}[Q]$ decomposes as

$$H_{i-1}[Q] = \frac{1}{y_i^2}\big(m + (x_i, q) + \tfrac{1}{2}(x_i, x_i)n\big)^2 + H_i[q + x_i n] + y_i^2 n^2 \,. \tag{3.47}$$

The fundamental domain $\mathcal{M}/O(N, N, \mathbb{Z})$ is defined iteratively such that $\mathcal{F}_{i-1}$ is defined as

1. $H_i \in \mathcal{F}_i$.
2. $(x_i)_1 \in \left[0, \frac{1}{2}\right]$ and $(x_i)_k \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for all $2 \leq k \leq 2N - 2i$.
3. For any $m, n \in \mathbb{Z},\ q \in I\!I_{N-i,N-i}$ such that $\gcd(m, q, n) = 1$ and $(q, q) + 2mn = 0$ we have:

$$\big(m + (x_i, q) + \tfrac{1}{2}(x_i, x_i)n\big)^2 + y_i^2 H_i[q + x_i n] + y_i^4 n^2 \geq 1 \tag{3.48}$$

The first two conditions ensure that $H_{i-1} \in \mathcal{F}_{i-1}^P$ while the third imposes $H_{i-1} \in S_{i-1>}$. For the fundamental domain $\mathcal{M}/SO_0(N, N, \mathbb{Z})$ one must further take into account that $SO_0(1, 1)$ is trivial and $(x_{N-1})_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ instead of $\left[0, \frac{1}{2}\right]$ and the last condition $y_N > 1$ is not imposed.

3.3.1 Inductive proof

For simplicity we consider the fundamental domain $\mathcal{M}/O(N, N, \mathbb{Z})$ in this section. Let us first show that for any $H_i \in \mathcal{M}_i$ there exists $\gamma \in O(N - i, N - i, \mathbb{Z})$ such that $\gamma^\intercal H_i \gamma \in \mathcal{M}_i/O(N - i, N - i, \mathbb{Z})$ assuming it is true for $i + 1$. We can write an element of the discrete parabolic subgroup $P_{i+1}(\mathbb{Z}) = O(N - 1 - i, N - 1 - i, \mathbb{Z}) \times \mathbb{Z}^{2N-2-2i}$ as

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b^\intercal \eta & \frac{1}{2}(b, b) \\ 0 & \mathbb{1} & b \\ 0 & 0 & 1 \end{pmatrix} \tag{3.49}$$

with $l \in O(N - 1 - i, N - 1 - i, \mathbb{Z})$ and $b \in \mathbb{Z}^{2N-2-2i}$. Using the decomposition of H_i in (3.43), one obtains that such transformation gives

$$H_{i+1} \to l^\intercal H_{i+1} l \,, \qquad x_{i+1} \to l^{-1} x_{i+1} + b \,. \tag{3.50}$$

By assumption, there exists $l \in O(N - 1 - i, N - 1 - i, \mathbb{Z})$ such that $l^\intercal H_{i+1} l \in \mathcal{F}_{i+1}$. This only fixes l up to sign, and there is therefore enough symmetry together with the shift in b to fix $0 \leq (x_{i+1})_1 \leq 1/2$ and $-1/2 \leq (x_{i+1})_k \leq 1/2$ for all k between 2 and $2N - 2 - 2i$. This shows that there exists p such that $p^\intercal H_i p \in \mathcal{F}_i^P$.

One can write an element h of $O(N - i, N - i, \mathbb{Z})$ as

$$h = \begin{pmatrix} m & * & * \\ q & * & * \\ n & * & * \end{pmatrix} \tag{3.51}$$

for any $m, n \in \mathbb{Z},\ q \in I\!I_{N-1-i,N-1-i}$ with $\gcd(m, n, q) = 1$ and $2mn + (q, q) = 0$. These components are determined by the action on a vector $Q = (1, 0, 0)$ up to right multiplication by

an arbitrary element in $P_{i+1}(\mathbb{Z})$. Using the result above, one can always determine h such that $h^\intercal p^\intercal H_i p h \in \mathcal{F}_i^P$. Writing $p^\intercal H_i p$ in terms of $y_{i+1}, H_{i+1}, x_{i+1}$ satisfying 1 and 2 in the definition of $\mathcal{F}_i$, one obtains that $y_{i+1}(h)$ of $h^\intercal p^\intercal H_i p h$ as

$$y_{i+1}(h) = \Big(\frac{1}{y_{i+1}^2} \big(m + (x_{i+1}, q) + \tfrac{1}{2}(x_{i+1}, x_{i+1}) n \big)^2 + H_{i+1}[q + x_{i+1} n] + y_{i+1}^2 n^2 \Big)^{-\frac{1}{2}} . \tag{3.52}$$

We then simply choose $Q = (m, q, n)$ such that $y_{i+1}(h)$ is maximal. By construction $\gamma = ph$ is now such that $\gamma^\intercal H_i \gamma \in \mathcal{F}_i$.

Let us now show that any non trivial $\gamma = ph \in O(N-i, N-i, \mathbb{Z})$ up to the centre $-\mathbb{1} \in O(N-i, N-i, \mathbb{Z})$ moves an element $H_i \in \mathring{\mathcal{F}}_i$ outside of the fundamental domain. By induction we assume that $\gamma = ph = lbh$ and any non-trivial $l \in O(N-1-i, N-1-i, \mathbb{Z})$ up to sign moves $H_{i+1} \in \mathring{\mathcal{F}}_{i+1}$ out of $\mathring{\mathcal{F}}_{i+1}$, and it is clear that any non-trivial element b or $\ell = -\mathbb{1}$ moves x_{i+1} outside of the domain $0 \leq (x_{i+1})_1 \leq 1/2$ and $-1/2 \leq (x_{i+1})_k \leq 1/2$.

Choosing y_{i+1} sufficiently large compared to all eigenvalues of $H_{i+1},$, it is clear that

$$\big(m + (x_i, q) + \tfrac{1}{2}(x_i, x_i) n \big)^2 + y_i^2 H_i[q + x_i n] + y_i^4 n^2 > 1 , \qquad \forall (m, q, n) \neq (\pm 1, 0, 0) . \tag{3.53}$$

The open set $\mathring{\mathcal{F}}_i$ is therefore defined by the strict inequality and it follows that any non-trivial h moves $H_i \in \mathring{\mathcal{F}}_i$ outside the fundamental domain.

Because the action is continuous it follows that any non-trivial element $\gamma \in O(N-i, N-i, \mathbb{Z})$ acts on a point in the boundary of the fundamental domain $\mathcal{F}_i$ to give another point in the boundary. It follows by induction if H_{i+1} is in the boundary of $\mathcal{F}_{i+1}$, and it is rather obvious if one of the $(x_{i+1})_k = \pm\frac{1}{2}$. If there exists a non-trivial vector $Q \neq (\pm 1, 0, 0)$ such that the inequality (3.48) is saturated, the corresponding element h also preserves the boundary.

This establishes the definition of the fundamental domain described in this section for $O(N, N, \mathbb{Z})$. To prove the result for $SO_0(N, N, \mathbb{Z})$ only require to study the case of $SO(2, 2)$, which we describe now.

3.3.2 The example of $SO(2, 2)$

In this case we write explicitly the bilinear form as

$$H[Q] = \frac{1}{y_1^2} \big(m + x_1 q_2 + x_2 q_1 + x_1 x_2 n \big)^2 + \frac{1}{y_2^2} (q_1 + x_1 n)^2 + y_2^2 (q_2 + x_2 n)^2 + y_1^2 n^2 , \tag{3.54}$$

with

$$\eta[Q] = 2mn + 2 q_1 q_2 . \tag{3.55}$$

The induction starts with the condition that $q_1^2 + y_2^4 q_2 \geq 1$ for any (q_1, q_2) in the orbit of $(1, 0)$. For $O(1, 1, \mathbb{Z})$ one gets the four vectors $(q_1, q_2) = (\pm 1, \pm 1)$ and so one obtains the condition $y_2 \geq 1$. For $SO(1, 1, \mathbb{Z})$ or $SO_0(1, 1, \mathbb{Z})$ one does not get $q_2 = \pm 1$ and there is no further condition on $y_2 > 0$.

At the next step the action of the unipotent subgroup allows to fix both $x_i \in [-\frac{1}{2}, \frac{1}{2}]$. For $O(1, 1, \mathbb{Z})$ and $SO(1, 1, \mathbb{Z})$ one can use the element $-\mathbb{1}$ to constrain $x_1 \geq 0$. For $SO_0(1, 1, \mathbb{Z})$ the trivial group, we do not get this further restriction.

Using now the third condition, one obtains for $m = n = 0$ the two conditions

$$x_1^2 + y_1^2 y_2^2 \geq 1 \,, \qquad x_2^2 + \frac{y_1^2}{y_2^2} \geq 1 \,, \tag{3.56}$$

and all the other conditions one obtains are consequences of these two.

We conclude that a fundamental domain of $O(2,2,\mathbb{Z})$ is defined by the conditions

$$\text{i)}\ x_1^2 + y_1^2 y_2^2 \geq 1 \,, \quad \text{ii)}\ x_2^2 + \frac{y_1^2}{y_2^2} \geq 1 \,, \quad \text{iii)}\ y_2 \geq 1 \,, \quad \text{iv)}\ 0 \leq x_1 \leq \frac{1}{2} \,, \quad \text{v)}\ -1/2 \leq x_2 \leq 1/2 \,, \tag{3.57}$$

a fundamental domain of $SO(2,2,\mathbb{Z})$ by

$$\text{i)}\ x_1^2 + y_1^2 y_2^2 \geq 1 \,, \quad \text{ii)}\ x_2^2 + \frac{y_1^2}{y_2^2} \geq 1 \,, \quad \text{iii)}\ 0 \leq x_1 \leq \frac{1}{2} \,, \quad \text{iv)}\ -1/2 \leq x_2 \leq 1/2 \,, \tag{3.58}$$

and a fundamental domain of $SO_0(2,2,\mathbb{Z})$ by

$$\text{i)}\ x_1^2 + y_1^2 y_2^2 \geq 1 \,, \quad \text{ii)}\ x_2^2 + \frac{y_1^2}{y_2^2} \geq 1 \,, \quad \text{iii)}\ -\frac{1}{2} \leq x_1 \leq \frac{1}{2} \,, \quad \text{iv)}\ -1/2 \leq x_2 \leq 1/2 \,. \tag{3.59}$$

This is consistent with the isomorphism

$$\big(SO(2) \times SO(2)\big) \backslash SO(2,2) / SO_0(2,2,\mathbb{Z}) \cong SO(2) \backslash SL(2) / SL(2,\mathbb{Z}) \times SO(2) \backslash SL(2) / SL(2,\mathbb{Z}) \,, \tag{3.60}$$

and $T = x_1 + i y_1 y_2$ and $U = x_2 + i \frac{y_1}{y_2}$ in the standard fundamental domain of $SL(2,\mathbb{Z})$. The further condition $x_1 \geq 0$ appears for the fundamental domain of $SO(2,2,\mathbb{Z})$ which includes the further generator $SO(2,2,\mathbb{Z}) = S(GL(2,\mathbb{Z}) \times_{\mathbb{Z}_2} GL(2,\mathbb{Z}))$ that changes the signs of x_1 and x_2. The additional condition $y_1 \geq 1$ appears for the fundamental domain of $O(2,2,\mathbb{Z})$ which further includes the generator that exchanges T and U.

4 Minima at symmetric points

The main purpose of this paper is to find the global minimum of Eisenstein series. The $SL(2)$ real analytic Eisenstein series $2\zeta(2s)E_s$ is known to have a global minimum at the symmetric point $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, [37,47] for any value of the parameter $s > 0$, with the appropriate regularisation at $s = 1$. There is no general result for the global minimum of the Epstein series $\mathrm{Ep}_s^N(H)$ for $N \geq 4$ and generic $s > 0$. The sum (2.6) is absolutely convergent for $s > \frac{N}{2}$, but admits an analytic continuation to a meromorphic function of $s \in \mathbb{C}$ with a single pole at $s = \frac{N}{2}$ [48]. One can define the regularised value of the Epstein function at the simple pole by minimal subtraction. For $s \to \infty$, the global minimum is the solution to the densest lattice sphere packing in N dimensions [39]. One can understand this result intuitively using an expansion of Ep_s^N in the length of the lattice vectors. Writing $n_{\min}^H$ a vector of minimal length for the associated quadratic form H and $d_{\min}^H$ the number of vectors of minimal length in the lattice, one gets for $s \to \infty$

$$\mathrm{Ep}_s^N \underset{s\to\infty}{\gtrsim} \frac{d_{\min}^H}{H[n_{\min}^H]^s} \,. \tag{4.1}$$

Therefore the minimum of Ep_s^N is obtained when $H[n^H_{\min}]$ is maximal for large $\mathrm{Re}[s]$. The density of spheres of radius $\sqrt{H[n^H_{\min}]}/2$ in the lattice is

$$\rho(H) = \frac{\left(\frac{\pi H[n^H_{\min}]}{4}\right)^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)} \,, \tag{4.2}$$

and so $H[n^H_{\min}]$ maximal also gives the densest sphere packing in the lattice. For $2 \leq N \leq 8$, the densest lattice sphere packings are known to be the following rank N root lattices [49]

$$A_2 \,, \quad A_3 \,, \quad D_4 \,, \quad D_5 \,, \quad E_6 \,, \quad E_7 \,, \quad E_8 \,, \tag{4.3}$$

such that the minimum of the Epstein series at large s is at $H_L = \det C_L^{-\frac{1}{N}} C_L$ for C_L the even bilinear form associated to the lattice L (with $L = A_N$, D_N, E_N).

However, we are interested in small values of s for the string theory couplings, in particular $s \leq \frac{N}{2}$. It was observed in [38] that the densest lattice sphere packing bilinear form H_{dlp} cannot be the global minimum of the Epstein series for all $s > 0$ when the lattice and its dual do not define the same point in $SO(N)\backslash SL(N)/SL(N,\mathbb{Z})$, i.e. when $H_{\mathrm{dlp}}^{-1} \not\approx H_{\mathrm{dlp}}$ modulo the action of $SL(N,\mathbb{Z})$. This is a direct consequence of the functional relation

$$\mathrm{Ep}_s^N(H) = \pi^{2s-\frac{N}{2}} \frac{\Gamma(\frac{N}{2}-s)}{\Gamma(s)} \mathrm{Ep}^N_{\frac{N}{2}-s}(H^{-1}) \,. \tag{4.4}$$

In particular the Epstein series is the same for the two dual points at $s = \frac{N}{4}$

$$\mathrm{Ep}^N_{\frac{N}{4}}(H_{\mathrm{dlp}}^{-1}) = \mathrm{Ep}^N_{\frac{N}{4}}(H_{\mathrm{dlp}}) \,. \tag{4.5}$$

One may therefore argue at most that the densest lattice packing H_{dlp} is the global minimum for $s \geq \frac{N}{4}$, which implies by the functional relation above that its dual lattice H_{dlp}^{-1} is the global minimum for $0 < s < \frac{N}{4}$.

The case of $N = 3$ was analysed numerically in [38]. It was shown that the densest lattice sphere packing point H_{A_3} is a global minimum for $s > \frac{3}{4}$, and in particular for the regularised series

$$\widehat{\mathrm{Ep}}^3_{\frac{3}{2}} = \lim_{s \to \frac{3}{2}} \left(\mathrm{Ep}^3_s - \frac{2\pi}{s-\frac{3}{2}} \right) . \tag{4.6}$$

They also demonstrate that the minimum of $\widehat{\mathrm{Ep}}^N_s$ must be strictly negative in the critical strip $0 < s < \frac{N}{2}$. However, it was proved in [50] that $\widehat{\mathrm{Ep}}^N_{\frac{N}{4}} > 0$ if the minimal length vector $n^H_{\min}$ of the associated quadratic form H has length bounded from above as

$$\sqrt{H[n^H_{\min}]} < \frac{N}{2\pi e} \quad \text{or} \quad \sqrt{H^{-1}[n^{H^{-1}}_{\min}]} < \frac{N}{2\pi e} \,. \tag{4.7}$$

This excludes the possibility that the minimum be at $H_{\mathfrak{g}}$ for an ADE lattice for $N > 24$ and suggests instead that the densest lattice sphere packing will be the minimum [38]. The densest lattice sphere packing has been proved to be the global minimum for $N = 8$ and 24 and all values of $s > \frac{N}{2}$ [51].

It would be very difficult to carry out a complete numerical analysis of the Epstein series for $N \geq 4$. Based on the results above we shall therefore concentrate on symmetric points in moduli space for which the bilinear form H is invariant under a finite subgroup of $SL(N,\mathbb{Z})$. In this section we define these symmetric points and determine a criterion for them to be local minima of Eisenstein series.

We then generalise the same analysis to $SO(N,N,\mathbb{Z})$ and prove that symmetric points are local minima of Eisenstein series provided they are negative in the critical strip.

4.1 Taylor expansion at symmetric points

The hyperplanes defining the boundary of the fundamental domain are mapped to themselves under elements of $G(\mathbb{Z})$, and their intersections are invariant under non-trivial finite subgroups of $G(\mathbb{Z})$. The maximal intersections define isolated points $\mathcal{V}_0 \in \mathcal{M}$ that are invariant under maximal finite subgroups $G^{\mathcal{V}_0}(\mathbb{Z}) \subset G(\mathbb{Z})$

$$G^{\mathcal{V}_0}(\mathbb{Z}) = \left\{\gamma \in G(\mathbb{Z}) \mid \gamma^{\intercal}\mathcal{V}_0^{\intercal}\mathcal{V}_0\gamma = \mathcal{V}_0^{\intercal}\mathcal{V}_0\right\} = \left\{\gamma \in G(\mathbb{Z}) \mid \mathcal{V}_0\gamma\mathcal{V}_0^{-1} \in K\right\} . \tag{4.8}$$

When the $G^{\mathcal{V}_0}(\mathbb{Z})$-fixed points in $\mathcal{M}$ are isolated, we call them symmetric points. All the symmetric points are maximal intersections but some maximal intersections may not be symmetric points.

By construction for all $\gamma \in G^{\mathcal{V}_0}(\mathbb{Z})$ there exists a $k_\gamma = \mathcal{V}_0\gamma\mathcal{V}_0^{-1} \in K$ such that

$$\mathcal{V}_0\gamma = k_\gamma\mathcal{V}_0 . \tag{4.9}$$

An automorphic form Φ on $\mathcal{M}$ in a representation ρ of K transforms as

$$\Phi(k\mathcal{V}\gamma) = \rho_\Phi(k)\Phi(\mathcal{V}) . \tag{4.10}$$

Here ρ generalises the weight for $SL(2)$. By definition, the automorphic form is invariant under the linear action of $\rho_\Phi(k_\gamma)$ at $\mathcal{V} = \mathcal{V}_0$ for any $\gamma \in G^{\mathcal{V}_0}(\mathbb{Z})$ and k_γ satisfying (4.9). Indeed, one checks that

$$\rho_\Phi(k_\gamma)\Phi(\mathcal{V}_0) = \Phi(k_\gamma\mathcal{V}_0) = \Phi(\mathcal{V}_0\gamma) = \Phi(\mathcal{V}_0) . \tag{4.11}$$

We can use this property to constrain the covariant derivatives of an automorphic function at a symmetric point. We define the covariant derivative from the left-action of $p \in \mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ as

$$f(\exp(p)\mathcal{V}_0) \equiv f_{\mathcal{V}_0}(p) = f(\mathcal{V}_0) + p^a D_a f(\mathcal{V}_0) + \frac{1}{2}p^a p^b D_a D_b f(\mathcal{V}_0) + \mathcal{O}(p^3) , \tag{4.12}$$

where a labels the components of p in $\mathfrak{p}$. Note that the differential operators D_a can equivalently be defined as the covariant derivative in tangent frame on the symmetric space

$$D_a = e_a{}^\mu\big(\partial_\mu + \rho(\omega_\mu)\big) . \tag{4.13}$$

If $G^{\mathcal{V}_0}(\mathbb{Z})$ is non-trivial, we have

$$f_{\mathcal{V}_0}(p) = f_{\mathcal{V}_0}(k_\gamma^{-1}pk_\gamma) \tag{4.14}$$

for any $k_\gamma = \mathcal{V}_0\gamma\mathcal{V}_0^{-1} \in K$ such that $\gamma \in G^{\mathcal{V}_0}(\mathbb{Z})$, because

$$f_{\mathcal{V}_0}(k_\gamma^{-1}pk_\gamma) = f(k_\gamma^{-1}\exp(p)k_\gamma\mathcal{V}_0) = f(\exp(p)\mathcal{V}_0\gamma) = f_{\mathcal{V}_0}(p)\ . \tag{4.15}$$

The Taylor expansion of an automorphic function is therefore highly constrained by the symmetry group $G^{\mathcal{V}_0}(\mathbb{Z})$. At order n in the Taylor expansion, one can classify the order n polynomials in p that are invariant under all such elements $k_\gamma = \mathcal{V}_0\gamma\mathcal{V}_0^{-1} \in K$. If $\mathcal{V}_0$ is a symmetric point as we define above, there is no invariant linear polynomial and $\mathcal{V}_0$ is an extremum of any automorphic function.

For any symmetric point $\mathcal{V}_0$ we must check if the extremum is a minimum. In practice we shall find that the symmetric points admit a small number of $G^{\mathcal{V}_0}(\mathbb{Z})$-invariant quadratic polynomials, and that it is sufficient to evaluate the automorphic function on a small dimension hypersurface to determine if it is a minimum.

If there is a single invariant quadratic polynomial, by construction it must be proportional to the quadratic Casimir such that

$$f_{\mathcal{V}_0}(p) = f(\mathcal{V}_0) + \frac{1}{2\dim\mathfrak{p}}\kappa_{ab}p^a p^b \Delta f(\mathcal{V}_0) + \mathcal{O}(p^3) \tag{4.16}$$

where κ_{ab} is the restriction to $\mathfrak{p}$ of the Killing–Cartan form on $\mathfrak{g}$. The Hessian of the function at $\mathcal{V}_0$ is then completely determined, and the symmetric point is a local minimum provided $\Delta f(\mathcal{V}_0) > 0$. In particular the function must be negative at a local minimum if the eigenvalue of the Laplace operator is negative.

We will be interested in maximal parabolic Eisenstein series $E^G_{s\Lambda_i}$ associated to a fundamental weight Λ_i,

$$E^G_{s\Lambda_i} = \sum_{\gamma\in G(\mathbb{Z})/P_i(\mathbb{Z})} \chi(2s\Lambda_i - \varrho, \mathcal{V}\gamma)\ , \tag{4.17}$$

which satisfy the Laplace equation

$$\Delta E^G_{s\Lambda_i} = 2(s\Lambda_i - \varrho, s\Lambda_i)E^G_{s\Lambda_i}\ , \tag{4.18}$$

with $\varrho = \sum_i \Lambda_i$ the Weyl vector and $\chi(\lambda)$ the multiplicative parabolic character of weight λ. Physically, maximal parabolic Eisenstein series are the sum over the U-duality images of a fixed dilaton to the power s. For maximal parabolics, the dilaton is either the string coupling constant or the volume of a torus T^r inside the M-theory torus or the type IIB torus.

Maximal parabolic Eisenstein series are absolutely convergent for $\mathrm{Re}[s] > \frac{(\Lambda_i,\varrho)}{(\Lambda_i,\Lambda_i)}$, and both $E^G_{s\Lambda_i} > 0$ and $\Delta E^G_{s\Lambda_i} > 0$ for $s > \frac{(\Lambda_i,\varrho)}{(\Lambda_i,\Lambda_i)}$. On the critical strip $0 < s < \frac{(\Lambda_i,\varrho)}{(\Lambda_i,\Lambda_i)}$, the Eisenstein series is integrable and therefore

$$0 = \int_{\mathcal{M}} d\mu\, \Delta E^G_{s\Lambda_i}(\mathcal{V}) = 2(s\Lambda_i - \varrho, s\Lambda_i)\int_{\mathcal{M}} d\mu\, E^G_{s\Lambda_i}(\mathcal{V})\ , \tag{4.19}$$

such that $E^G_{s\Lambda_i}(\mathcal{V})$ must be negative at its global minimum.

We find therefore that when the moduli space $K\backslash G$ admits a symmetric point $\mathcal{V}_0$ with a unique invariant quadratic polynomial in p, the symmetric point $\mathcal{V}_0$ is a local minimum of any

absolutely convergent Eisenstein series. Moreover, $\mathcal{V}_0$ is a local minimum of any integrable automorphic function that is negative at $\mathcal{V}_0$.

When there are two $G^{\mathcal{V}_0}(\mathbb{Z})$-invariant quadratic polynomials we need to compute the two independent eigenvalues of the Hessian matrix at $\mathcal{V}_0$. For this purpose we shall define a two-dimensional subspace of $SO(N)\backslash SL(N)$ that includes the relevant symmetric points. There also exist symmetric points with more invariant polynomials, but we find that they never correspond to global mimima and shall disregard them.

We will now discuss the cases of $SL(N)$ and $SO(N,N)$.

4.2 $SL(N)$ symmetric points

The symmetric points in $SO(N)\backslash SL(N)$ are invariant under maximal finite irreducible subgroups of $PSL(N,\mathbb{Z})$. The maximal finite irreducible subgroups of $GL(N,\mathbb{Z})$ have been classified for all $N \leq 10$ [52–54]. For $N \leq 4$ all the maximal finite subgroups of $PSL(N,\mathbb{Z})$ are stabilisers of an even-bilinear form C_L of Cartan type. One gets for irreducible groups

$$N=1: \qquad A_1\ , \tag{4.20}$$

$$N=2: \qquad A_2\ , \quad 2A_1\ , \tag{4.21}$$

$$N=3: \qquad A_3\ , \quad A_3^*\ , \quad 3A_1\ , \tag{4.22}$$

$$N=4: \qquad D_4\ , \quad A_4\ , \quad A_4^*\ , \quad 4A_1\ , \quad 2A_2\ , \quad A_2\times A_2\ . \tag{4.23}$$

The reducible maximal subgroups stabilise the reducible bilinear forms $A_1{+}A_2$, $A_1{+}A_3$, $A_1{+}A_3^*$. One easily checks that reducible bilinear forms that are extrema are always saddle points. To prove this, one considers coordinates associated to the maximal parabolic subgroup $P_k = \mathbb{R}_+ \times SL(k)\times SL(N-k)\ltimes \mathbb{R}^{k\times(N-k)} \subset SL(N)$ in which

$$H[n,m] = y^{2\frac{k-N}{N}} H_k[n+Xm] + y^{2\frac{k}{N}} H_{N-k}[m] \tag{4.24}$$

with $n\in\mathbb{Z}^k$, $m\in\mathbb{Z}^{N-k}$ and $X\in\mathbb{R}^{k\times(N-k)}$. The Fourier expansion of the Epstein series in the maximal parabolic P_k can be obtained by Poisson summation and reads

$$\begin{aligned}\mathrm{Ep}_s^N(H) &= y^{2\frac{N-k}{N}s}\mathrm{Ep}_s^k(H_k) + \pi^{\frac{k}{2}}\frac{\Gamma(s-\frac{k}{2})}{\Gamma(s)} y^{2\frac{k}{N}(\frac{N}{2}-s)}\mathrm{Ep}_{s-\frac{k}{2}}^{N-k}(H_{N-k}) \\ &+\frac{2\pi^s}{\Gamma(s)}{\sum_{n\in\mathbb{Z}^k}}' {\sum_{m\in\mathbb{Z}^{N-k}}}' y^{\frac{N-2k}{N}s+\frac{k}{2}} \frac{H_k^{-1}[n]^{\frac{s}{2}-\frac{k}{4}}}{H_{N-k}[m]^{\frac{s}{2}-\frac{k}{4}}} K_{s-\frac{k}{2}}\Big(2\pi y\sqrt{H_k^{-1}[n]H_{N-k}[m]}\Big)\cos(2\pi n^{\intercal}Xm)\ .\end{aligned} \tag{4.25}$$

The second derivative with respect to X at $X=0$ is negative definite

$$\begin{aligned}&\mathrm{d}^2\mathrm{Ep}_s^N(H)|_{X=0} \\ &=-\frac{2\pi^s}{\Gamma(s)}{\sum_{n\in\mathbb{Z}^k}}' {\sum_{m\in\mathbb{Z}^{N-k}}}' y^{\frac{N-2k}{N}s+\frac{k}{2}} \frac{H_k^{-1}[n]^{\frac{s}{2}-\frac{k}{4}}}{H_{N-k}[m]^{\frac{s}{2}-\frac{k}{4}}} K_{s-\frac{k}{2}}\Big(2\pi y\sqrt{H_k^{-1}[n]H_{N-k}[m]}\Big)(2\pi n^{\intercal}\mathrm{d}Xm)^2\ ,\end{aligned} \tag{4.26}$$

as the absolutely convergent sum of positive terms. Therefore any extremum at $X=0$ is a saddle point. We will therefore only analyse symmetric points for which the bilinear form H is irreducible.

For $N = 5$ the irreducible bilinear forms invariant under a maximal finite subgroup of $PSL(5,\mathbb{Z})$ are [53]

$$N = 5: \qquad D_5\,, \quad D_5^*\,, \quad A_5\,, \quad A_5^*\,, \quad A_5^{+2}\,, \quad (A_5^{+2})^* = A_5^{+3}\,, \tag{4.27}$$

where A_5^{+2} is the S_6-invariant lattice $A_5 \oplus (A_5{+}\Lambda_2) \oplus (A_5{+}\Lambda_4)$ that includes the second fundamental weight Λ_2, and $A_5^{+3} = A_5 \oplus (A_5{+}\Lambda_3)$. All these points lie on dimension zero boundaries of Grenier's fundamental domain, as we show in Appendix A.

It will be convenient to introduce the following parametrisation of $\mathcal{V} \in SO(N)\backslash SL(N)$ in the maximal parabolic subgroup P_{N-1}

$$\mathcal{V} = y^{-\frac{1}{N}} \begin{pmatrix} N^{\frac{1}{2(N-1)}}\mathcal{V}_1 & N^{\frac{1}{2(N-1)}}\mathcal{V}_1\mathbf{x} \\ 0 & \frac{y}{\sqrt{N}} \end{pmatrix}\,, \tag{4.28}$$

where $\mathcal{V}_1 \in SO(N-1)\backslash SL(N-1)$, $\mathbf{x} \in \mathbb{R}^{N-1}$, and $y \in \mathbb{R}_+$ has been rescaled. When $\mathcal{V}_1$ is at the A_{N-1} symmetric point

$$N^{\frac{1}{N-1}}\mathcal{V}_1^{\intercal}\mathcal{V}_1 = C_{A_{N-1}}\,. \tag{4.29}$$

In this paper we define the A_N point in the fundamental domain as the symmetric matrix

$$C_{A_N} = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}\,, \tag{4.30}$$

or equivalently

$$C_{A_N}[n] = 2\sum_{i=1}^{N} n_i^2 + \sum_{i\neq j} n_i n_j\,. \tag{4.31}$$

In this way the symmetry under the subgroup $\mathrm{S}_N \subset \mathrm{S}_{N+1}$ is manifest. One gets the A_N, D_N, E_N representatives by fixing $\mathcal{V}_1$ at the A_{N-1} lattice point, and all x_i components equal such that

$$y^{\frac{2}{N}}H = C_{A_N} \qquad \text{for} \qquad x_i = \frac{1}{N}\,,\ y = \sqrt{N+1}\,, \tag{4.32}$$

$$y^{\frac{2}{N}}H = C_{D_N} \qquad \text{for} \qquad x_i = \frac{2}{N}\,,\ y = 2\,, \tag{4.33}$$

$$y^{\frac{2}{N}}H = C_{E_N} \qquad \text{for} \qquad x_i = \frac{3}{N}\,,\ y = \sqrt{9-N}\,. \tag{4.34}$$

One moreover checks that the Gram matrix $C_{A_5^{+2}}$ and $C_{A_5^{+3}}$ of A_5^{+2} and A_5^{+3} are given by [5]

$$y^{\frac{2}{5}}H = C_{A_5^{+2}} \qquad \text{for} \qquad x_i = \frac{2}{5}\,,\ y = \sqrt{\frac{2}{3}}\,, \tag{4.35}$$

$$y^{\frac{2}{5}}H = C_{A_5^{+3}} \qquad \text{for} \qquad x_i = \frac{2}{5}\,,\ y = \sqrt{\frac{3}{2}}\,. \tag{4.36}$$

[5] This is not the bilinear form given in [53], the latter is equivalent to $3C_{A_5^{+2}}$ up to $SL(5,\mathbb{Z})$.

This ansatz for $y > 0$ and a single $x \in [0, \frac{1}{2}]$ therefore describes many symmetric points. In particular it covers all the symmetric points (up to inversion) for $SL(3,\mathbb{Z})$ and $SL(5,\mathbb{Z})$, the groups relevant in string theory. The dual lattice points can be studied in the same way with the inverse matrix or using the functional relation (4.4). This representative of the A_N symmetric point is in the Grenier domain, but not the others, see Appendix A.

The densest sphere packing criterion can be checked using the density (4.2)

$$\rho(C_{A_N}) = \frac{\left(\frac{\pi}{2}\right)^{\frac{N}{2}}}{\sqrt{N+1}\Gamma(\frac{N}{2}+1)} \underset{N\geq 4}{<} \rho(C_{D_N}) = \frac{\left(\frac{\pi}{2}\right)^{\frac{N}{2}}}{2\Gamma(\frac{N}{2}+1)} \underset{N=6,7,8}{<} \rho(C_{E_N}) = \frac{\left(\frac{\pi}{2}\right)^{\frac{N}{2}}}{\sqrt{9-N}\Gamma(\frac{N}{2}+1)} \tag{4.37}$$

in agreement with the list (4.3). The dual lattices D_N^* and A_N^* are always less dense than the lattices D_N and A_N, but E_N^* is denser than D_N and A_N for $N = 6, 7, 8$. For $N = 5$ one finds

$$\rho(C_{A_5}) = \frac{\pi^2}{15\sqrt{3}} < \rho(C_{A_5^{+2}}) = \frac{4\sqrt{2}\pi^2}{135} < \rho(C_{D_5}) = \frac{\pi^2}{15\sqrt{2}} \,, \tag{4.38}$$

so the densest sphere lattice packing is indeed the lattice D_5, and A_5^{+2} is denser than A_5. The dual lattices are less dense, with

$$\rho(C_{A_5^*}) = \frac{5\sqrt{5}\pi^2}{432} < \rho(C_{D_5^*}) = \frac{\pi^2}{30} < \rho(C_{A_5^{+3}}) = \frac{3\pi^2}{80} \,. \tag{4.39}$$

On the surface where $H(y, x, C_{A_{N-1}})$ is parametrised by $y > 0$ and $-\frac{1}{2} \leq x \leq \frac{1}{2}$ as in (4.28), one writes the explicit formula for the $SL(N)$ Epstein series as

$$\begin{aligned}\mathrm{Ep}_s^N(y, x, C_{A_{N-1}}) &= \frac{y^{\frac{2s}{N}}}{N^{\frac{s}{N-1}}} \mathrm{Ep}_s^{N-1}(C_{A_{N-1}}) + \frac{2\pi^{\frac{N-1}{2}}\Gamma(s-\frac{N-1}{2})\zeta(2s-N+1)}{\Gamma(s)} \frac{y^{(N-1)(1-\frac{2s}{N})}}{N^{\frac{N}{2}-s}} \\ &+ \frac{4\pi^s}{\Gamma(s)} \frac{y^{\frac{N-1}{2}-\frac{N-2}{N}s}}{\sqrt{N}} {\sum_{n\in\mathbb{Z}^{N-1}}}' \sigma_{N-1-2s}(n) \frac{K_{s-\frac{N-1}{2}}\left(2\pi y\sqrt{C^{-1}_{A_{N-1}}[n]/N}\right)}{\left(NC^{-1}_{A_{N-1}}[n]\right)^{\frac{N-1}{4}-\frac{s}{2}}} \cos\Big(2\pi x \sum_{i=1}^{N-1} n_i\Big) \,. \end{aligned} \tag{4.40}$$

It is defined by induction in N. For $N = 2$ the Epstein series evaluated at $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ can be written as the zeta function over $\mathbb{Q}(\sqrt{-3})$ [37][6]

$$\mathrm{Ep}_s^2(C_{A_2}) = {\sum_{z\in\mathbb{Z}(e^{\frac{2\pi i}{3}})}}' \frac{(\frac{\sqrt{3}}{2})^s}{|z|^{2s}} = \frac{6}{12^{\frac{s}{2}}}\zeta(s)\big(\zeta(s,\tfrac{1}{3}) - \zeta(s,\tfrac{2}{3})\big) \,. \tag{4.41}$$

For $SL(N)$ the elements $p \in \mathfrak{p} = \mathfrak{g} \ominus \mathfrak{k}$ are symmetric-traceless $N \times N$ matrices, and the Killing–Cartan form is normalised such that

$$\kappa_{ab}p^a p^b = 2\,\mathrm{tr}\,p^2 = 2\sum_{i,j=1}^{N} p_{ij}^2 \,. \tag{4.42}$$

[6]The expression using Hurwitz zeta function is obtained as the Euler product

$${\sum_{z\in\mathbb{Z}(e^{\frac{2\pi i}{3}})}}' \frac{1}{|z|^{2s}} = 6\frac{1}{1-3^{-s}} \prod_{p=1 \bmod 3} \left(\frac{1}{1-p^{-s}}\right)^2 \prod_{p=2 \bmod 3} \frac{1}{1-p^{-2s}} \,.$$

We will now study the different symmetric points.

4.2.1 The A_N symmetric point

The automorphism group of the A_N lattice is $\mathbb{Z}_2 \ltimes \mathrm{S}_{N+1}$, but only the alternating group Alt_{N+1} embeds inside $PSL(N,\mathbb{Z})$ for N even and $\mathrm{S}_{N+1} \subset PSL(N,\mathbb{Z})$ for N odd. The Eisenstein series are nonetheless invariant under the action of $PGL(N,\mathbb{Z})$, so their symmetry group is always S_{N+1}. The automorphism group $\mathbb{Z}_2 \ltimes \mathrm{S}_{N+1}$ is realised on the lattice vectors such that $\mathbb{Z}_2$ acts as

$$n_i \mapsto -n_i \ , \tag{4.43}$$

and S_{N+1} acts as the permutation of the N elements n_i and a fictitious N+1th element $-\sum_{j=1}^{N} n_j$, i.e. $\sigma \in \mathrm{S}_N$ acts as

$$\sigma : n_i \mapsto n_{\sigma(i)} \ , \tag{4.44}$$

and

$$\sigma_{i,N+1} : n_i \mapsto -\sum_{j=1}^{N} n_j \ , \quad n_{j\neq 1} \mapsto n_j \ . \tag{4.45}$$

It is useful to split this group into S_N and

$$\mathrm{S}_{N+1}/\mathrm{S}_N = \mathbb{Z}_{N+1} \tag{4.46}$$

where $\mathbb{Z}_{N+1}$ is the cyclic group generated by the transformation

$$\sigma_f : n_{i\neq N} \mapsto n_{i+1} \ , \quad n_N \mapsto -\sum_{i=1}^{N} n_i \ . \tag{4.47}$$

To determine the S_{N+1}-invariant quadratic polynomials, it is convenient to define

$$\tilde{p} = (N+1)^{\frac{1}{N}} \mathcal{V}_0^{\intercal} p \mathcal{V}_0 \ , \tag{4.48}$$

for which the traceless condition reads

$$\mathrm{tr}\, C_{A_N}^{-1} \tilde{p} = \frac{1}{N+1}\left(N \sum_{i=1}^{N} \tilde{p}_{ii} - \sum_{i\neq j} \tilde{p}_{ij} \right) = 0 \ . \tag{4.49}$$

For any invariant polynomial $f_{\mathcal{V}_0}(p)$, one defines the polynomial

$$\tilde{f}_{\mathcal{V}_0}(\tilde{p}) = f_{\mathcal{V}_0}\big((N+1)^{-\frac{1}{N}} \mathcal{V}_0^{-1\,\intercal} \tilde{p} \mathcal{V}_0^{-1} \big) \ , \tag{4.50}$$

which is invariant under

$$\tilde{f}_{\mathcal{V}_0}(\tilde{p}) = \tilde{f}_{\mathcal{V}_0}(\gamma^{\intercal} \tilde{p} \gamma) \ , \tag{4.51}$$

for any $\gamma \in G^{A_N}(\mathbb{Z})$ representing S_{N+1}. The symmetric matrix $\tilde{p}$ simply transforms under S_{N+1} as a traceless bilinear form in the n_i's, i.e. under $\sigma \in \mathrm{S}_N$ as

$$\sigma : \tilde{p}_{ij} \mapsto \tilde{p}_{\sigma^{-1}(i)\sigma^{-1}(j)} \ , \tag{4.52}$$

and under σ_f as

$$\begin{aligned}
&\sigma_f : \tilde{p}_{11} \mapsto \tilde{p}_{NN}, && \tilde{p}_{i\neq 1 i\neq 1} \mapsto \tilde{p}_{NN} + \tilde{p}_{i-1i-1} - 2\tilde{p}_{i-1N} \ , \\
&\sigma_f : \tilde{p}_{1j} \mapsto \tilde{p}_{NN} - \tilde{p}_{j-1N}, && \tilde{p}_{i\neq 1 j\neq 1} \mapsto \tilde{p}_{NN} + \tilde{p}_{i-1j-1} - \tilde{p}_{i-1N} - \tilde{p}_{j-1N} \ .
\end{aligned} \tag{4.53}$$

The vector n transforms in the standard representation of S_{N+1} associated to the partition $(N,1)$. The symmetric tensor product decomposes into irreducible representations as [7]

$$(N,1) \otimes_S (N,1) = (N) \oplus (N,1) \oplus (N-1,2) \ , \tag{4.54}$$

such that $\tilde{p}$ splits into the irreducible components $q \in (N,1)$ and $\tilde{p}^\perp \in (N-1,2)$. The vector q in the standard representation is defined as

$$q_i \equiv \tilde{p}_{ii} - \frac{2}{N-1} \sum_{\substack{j=1 \\ j\neq i}}^{N} \tilde{p}_{ij} \ , \tag{4.55}$$

and transforms by permutation under S_N and as

$$\sigma_f : q_1 \mapsto -q_N \ , \quad q_{i\neq 1} \mapsto q_{i-1} - q_N \ , \tag{4.56}$$

under σ_f. The $(N-1,2)$ tensor $\tilde{p}^\perp$ is defined as

$$\tilde{p}^\perp_{ii} = \tilde{p}_{ij} - q_i + \frac{2}{N+1} \sum_{j=1}^{N} q_j \ , \quad \tilde{p}^\perp_{i\neq j} = \tilde{p}_{ij} + \frac{1}{N+1} \sum_{k=1}^{N} q_k \ , \tag{4.57}$$

and transforms under S_{N+1} as $\tilde{p}$ in (4.53). It follows that there are two independent invariant quadratic polynomials in $\tilde{p}$, one quadratic in q and one quadratic in $\tilde{p}^\perp$, such that the Killing–Cartan form splits into

$$\frac{1}{2}\kappa_{ab}p^a p^b = \operatorname{tr} C_{A_N}^{-1}\, \tilde{p}\, C_{A_N}^{-1}\, \tilde{p} = \frac{N-1}{N+1} C_{A_N}^{-1}[q] + \operatorname{tr} C_{A_N}^{-1} \tilde{p}^\perp C_{A_N}^{-1} \tilde{p}^\perp \ . \tag{4.58}$$

For completeness, we give the explicit proof that there are only two invariant quadratic polynomials in Appendix B.

To study the Taylor expansion of Epstein series at the symmetric point A_N, we use the coset representative (4.28) and its vielbeins

$$\mathcal{P} = \frac{1}{2}\mathrm{d}\mathcal{V}\mathcal{V}^{-1} + \frac{1}{2}(\mathrm{d}\mathcal{V}\mathcal{V}^{-1})^\intercal = \begin{pmatrix} \mathcal{P}_1 - \frac{1}{N}\frac{\mathrm{d}y}{y} & \frac{N^{\frac{N}{2(N-1)}}}{2y}\mathcal{V}_1\,\mathrm{d}x \\ \frac{N^{\frac{N}{2(N-1)}}}{2y}\mathcal{V}_1\,\mathrm{d}x & \frac{N-1}{N}\frac{\mathrm{d}y}{y} \end{pmatrix} \ . \tag{4.59}$$

At the symmetric point we take $\mathcal{V}_1$ constant and such that $N^{\frac{1}{N-1}}\mathcal{V}_1^\intercal\mathcal{V}_1 = C_{A_{N-1}}$ and we define the pull-back momentum such that $\mathcal{P}_{1*} = 0$. One computes then

$$\begin{aligned}
\widetilde{\mathcal{P}}_* &\equiv (N+1)^{\frac{1}{N}}\mathcal{V}_0^\intercal \mathcal{P}_* \mathcal{V}_0 \\
&= \begin{pmatrix} -\frac{1}{N}C_{A_{N-1}}\frac{\mathrm{d}y}{y} & \frac{\sqrt{N+1}}{2y}C_{A_{N-1}}\mathrm{d}x - \frac{1}{N}C_{A_{N-1}}x_*\frac{\mathrm{d}y}{y} \\ \frac{\sqrt{N+1}}{2y}\mathrm{d}x^\intercal C_{A_{N-1}} - \frac{1}{N}x_*^\intercal C_{A_{N-1}}\frac{\mathrm{d}y}{y} & \frac{N-1}{N}\frac{\mathrm{d}y}{y} + \frac{\sqrt{N+1}}{y}x_*^\intercal C_{A_{N-1}}\mathrm{d}x \end{pmatrix} \ ,
\end{aligned} \tag{4.60}$$

[7] Recall that (N) is the trivial representation, $(N,1)$ is the basic representation of dimension N, and $(N-1,2)$ is the symmetric tensor irreducible representation of dimension $\frac{(N+1)(N-2)}{2}$.

where $x_{*i} = \frac{1}{N}$. This gives the standard representation components of $\widetilde{\mathcal{P}}_*$ that are defined from $\widetilde{\mathcal{P}}$ as in (4.55) and take the explicit form

$$\mathcal{Q}_{i\neq N*} = -\frac{\sqrt{N+1}}{N-1}\Big(C_{A_{N-1}}\frac{\mathrm{d}x}{y}\Big)_i\,, \quad \mathcal{Q}_{N*} = \Big(\frac{N+1}{N}\frac{\mathrm{d}y}{y} - \frac{\sqrt{N+1}}{N-1}x_*^{\intercal}C_{A_{N-1}}\frac{\mathrm{d}x}{y}\Big)\,, \tag{4.61}$$

and one checks that the $(N-1,2)$ components $\widetilde{\mathcal{P}}_*^{\perp}$ only depend on $\mathrm{d}x$

$$\frac{y}{\sqrt{N+1}}\widetilde{\mathcal{P}}^{\perp} = \begin{pmatrix} -\frac{1}{N-1}\mathrm{diag}[C_{A_{N-1}}\mathrm{d}x] & \frac{1}{2}C_{A_{N-1}}\mathrm{d}x \\ \frac{1}{2}\mathrm{d}x^{\intercal}C_{A_{N-1}} & -\frac{N}{N-1}x_*^{\intercal}C_{A_{N-1}}\mathrm{d}x \end{pmatrix} - \frac{1}{N-1}C_{A_N}(x_*^{\intercal}C_{A_{N-1}}\mathrm{d}x)\,. \tag{4.62}$$

A direct computation gives

$$\begin{aligned} C_{A_N}^{-1}[\mathcal{Q}_*] &= \frac{N+1}{N-1}\Big(\frac{N-1}{N}\frac{\mathrm{d}y^2}{y^2} + \frac{1}{N-1}\frac{C_{A_{N-1}}[\mathrm{d}x]}{y^2}\Big)\,, \\ 2\,\mathrm{tr}\,C_{A_N}^{-1}\widetilde{\mathcal{P}}_*^{\perp}C_{A_N}^{-1}\widetilde{\mathcal{P}}_*^{\perp} &= \Big(N - \frac{2}{N-1}\Big)\frac{C_{A_{N-1}}[\mathrm{d}x]}{y^2}\,. \end{aligned} \tag{4.63}$$

We define the pull-back of an automorphic function

$$f_*(y,x) = f(y, \mathcal{V}_1 = \mathcal{V}_{1*}, x_i = x)\,, \tag{4.64}$$

to the two-dimensional subspace parametrised by y and x. The second differential of the pull-back function must be consistent with the S_{N+1} symmetry, and therefore

$$\begin{aligned} \mathrm{d}^2 f_*(y,x)\Big|_{y=\sqrt{N+1},x=\frac{1}{N}} &= f_{yy}\mathrm{d}y^2 + f_{xx}\mathrm{d}x^2 + 2f_{yx}\mathrm{d}y\mathrm{d}x \\ &= a_+\frac{2}{N+1}\left(\frac{N-1}{N}\mathrm{d}y^2 + N\mathrm{d}x^2\right) + a_-\frac{(N^2(N-1)-2N)}{N+1}\mathrm{d}x^2\,, \end{aligned} \tag{4.65}$$

for two coefficients a_+ and a_- determined by the second derivatives of the automorphic function at the symmetric point. In particular

$$f_{yy} = 2\frac{N-1}{(N+1)N}a_+, \quad f_{xx} = \frac{2N}{N+1}a_+ + \frac{(N^2(N-1)-2N)}{N+1}a_-, \quad f_{yx} = 0 \tag{4.66}$$

Hence the condition for the symmetric point A_N to be a local minimum of the automorphic function f is that $a_+ > 0$ and $a_- > 0$, which is equivalent to the condition that

$$f_{xx} > \frac{N^2}{N-1}f_{yy} > 0\,. \tag{4.67}$$

It is therefore sufficient to study the pull-back function $f_*(y,x)$ to determine if the symmetric point A_N is a local minimum of the automorphic function $f(y,\mathcal{V}_1,x_i)$ in $\mathcal{M}$.

We have carried out this computation numerically for the Epstein series and found that the A_N symmetric point is always a local minimum for s large enough, but the eigenvalue a_- becomes negative for small s for $N \geq 4$, giving a saddle unstable along the corresponding $\frac{(N+1)(N-2)}{2}$-dimensional hypersurface. It is always a local minimum for $N = 3$, and is in fact the global minimum for $s > \frac{3}{4}$ [38]. We find that it is a local minimum for $s > 1$ for $SL(4)$, and for $s \gtrsim 3.16603$ for $SL(5)$. For $SL(6)$ and $SL(7)$ one finds similarly that it is not a local minimum for $s < s_{A_N}$ with s_{A_N} slightly above the critical value $\frac{N}{2}$. Most importantly for us, it is not a local minimum of the Epstein function $\mathrm{Ep}^{5}_{\frac{3}{2}}$ that defines the leading Wilson coefficient in type II string theory on T^3.

4.2.2 The D_N symmetric point

The D_N symmetric point is realised with the ansatz (4.34) as the bilinear form

$$C_{D_N}[n] = 2\sum_{i=1}^{N-1} n_i^2 + 4n_N^2 + 2\sum_{i<j<N} n_i n_j + 4n_N \sum_{i=1}^{N-1} n_i \ . \tag{4.68}$$

It is easy to show that it is related to the identity matrix by a $GL(N,\mathbb{Q})$ transformation that determines

$$n_{i\neq N} = m_i \ , \qquad n_N = -\frac{1}{2}\sum_{i=1}^{N} m_i \ , \tag{4.69}$$

for N integers m_i such that $\sum_i m_i = 0 \bmod 2$. One then gets

$$C_{D_N}[m] = \sum_{i=1}^{N} m_i^2 \ , \tag{4.70}$$

which reproduces the standard construction of the D_N root lattice in Euclidean space. In this basis the automorphisms $\mathrm{S}_N \ltimes \mathbb{Z}_2^N$ of the lattice D_N are simply realised as the $SL(N,\mathbb{Z})$ transformations

$$\begin{aligned} &\sigma : m_i \mapsto m_{\sigma(i)} \ , \\ &\varpi_i : m_i \mapsto -m_i \ , \quad m_{j\neq i} \mapsto m_j \ , \end{aligned} \tag{4.71}$$

that preserve the condition that $\sum_i m_i = 0 \bmod 2$. For $N = 4$, one has moreover the triality automorphisms. This determines the action of the automorphism group on the momentum $p \in \mathfrak{p}$ for generic N as

$$\begin{aligned} &\sigma : p_{ij} \mapsto p_{\sigma(i)\sigma(j)} \ , \\ &\varpi_i : p_{ij} \mapsto -p_{ij} \ , \quad p_{ii} \mapsto p_{ii} \ , \quad p_{jk} \mapsto p_{jk} \ , \quad \forall j,k \neq i \ . \end{aligned} \tag{4.72}$$

Note that the transformation $\prod_{i=1}^N \varpi_i$ flips the sign of the whole vector $\vec{m}$ and therefore acts trivially on p, so that only $\mathrm{S}_N \ltimes \mathbb{Z}_2^{N-1} \subset PGL(N,\mathbb{Z})$ acts on the moduli space.

The components p_{ij} split into the $N-1$ independent p_{ii} that transform in the standard representation of S_N and are invariant under $\mathbb{Z}_2^{N-1}$, and the p_{ij} for $i \neq j$ that transform under the reducible representation $N \oplus (N-1,1) \oplus (N-2,2)$ of S_N, but mix together under $\mathbb{Z}_2^{N-1}$. One straightforwardly checks that the only invariant quadratic polynomials are

$$\sum_{i=1}^{N} p_{ii}^2 \ , \quad \sum_{i\neq j} p_{ij}^2 \ . \tag{4.73}$$

For $N = 4$ the triality symmetry implies that there is a unique invariant quadratic polynomial, while for generic N one straightforwardly computes that

$$\sum_{i=1}^{N} \mathcal{P}_{ii*}^2 = \mathrm{tr} \begin{pmatrix} -\frac{1}{N}\frac{\mathrm{d}y}{y_*} & 0 \\ 0 & \frac{N-1}{N}\frac{\mathrm{d}y}{{}_*y} \end{pmatrix}^2 = \frac{N-1}{N}\frac{\mathrm{d}y^2}{y_*^2} \ ,$$
$$\sum_{i\neq j} \mathcal{P}_{ij}^2 = \mathrm{tr} \begin{pmatrix} 0 & \frac{N^{\frac{N}{2(N-1)}}}{2y_*}\mathcal{V}_{1*}\,\mathrm{d}x \\ \frac{N^{\frac{N}{2(N-1)}}}{2y_*}\mathcal{V}_{1*}\,\mathrm{d}x & 0 \end{pmatrix}^2 = \frac{N}{2y_*^2} C_{A_{N-1}}[\mathrm{d}x] = \frac{N^2(N-1)}{2y_*^2}\mathrm{d}x^2 \ . \quad (4.74)$$

Hence one has a local minimum of an automorphic function f at the D_N symmetric point if and only if

$$f_{xx} > 0 \ , \qquad f_{yy} > 0 \ , \quad (4.75)$$

with

$$\begin{aligned} \mathrm{d}^2 f_*(y,x)\Big|_{y=2,x=\frac{2}{N}} &= f_{yy}\mathrm{d}y^2 + f_{xx}\mathrm{d}x^2 + 2f_{yx}\mathrm{d}y\mathrm{d}x \\ &= a_+ \frac{N-1}{N}\mathrm{d}y^2 + a_- \frac{N^2(N-1)}{2}\mathrm{d}x^2 \ , \end{aligned} \quad (4.76)$$

and a_+ and a_- the two eigenvalues of the Hessian of f at the symmetric point. One finds by numerical evaluation that the D_N symmetric point is indeed a local minimum of the Epstein series Ep_s^N for all s and $N \leq 7$. We do not expect this to be true for arbitrary large N, but this is not relevant for the study of the type II string theory effective action. We find in particular that among all symmetric points, the D_N symmetric point gives the minimum value of the Epstein series Ep_s^N for $N = 4, 5$ and $s \geq \frac{N}{4}$,[8] leading to the conjecture that D_N is the global minimum.

4.2.3 Other symmetric points

In this subsection we shall briefly describe the other symmetric points.

• The dual symmetric points

Let us first observe that we do not need to describe the dual symmetric point for the lattices A_N^* and D_N^* separately. By construction $C_{L^*} = C_L^{-1}$, and the inverse of $H = \mathcal{V}^{\intercal}\mathcal{V}$ with the ansatz (4.28) allows to describe all dual lattices. One easily extends all the results using the functional relation (4.4). As a consistency check we have evaluated numerically the function for both the lattice and its dual. The analysis of the polynomials in $p \in \mathfrak{p}$ invariant under the automorphisms of the lattice proceeds in the same way for the dual lattices, and gives by construction the same number of invariant polynomials.

For example for A_N^* one defines

$$\tilde{p} = (N+1)^{-\frac{1}{N}} \mathcal{V}_0^{\intercal} p \mathcal{V}_0 \ , \quad (4.77)$$

[8] Note that D_4^* and D_4 define the same unimodular matrix H, so the D_4 symmetric point is the lowest value for all $s > 0$.

for which the traceless condition reads

$$\operatorname{tr} C_{A_N}\tilde{p} = 0 \ . \tag{4.78}$$

The action of the Weyl group is then such that S_N acts by permutation of the indices and σ_f acts on $\tilde{p}$ as

$$\sigma_f : \tilde{p}_{i\neq N j\neq N} \mapsto \tilde{p}_{i+1\,j+1} \ , \quad \sigma_f : \tilde{p}_{Nj\neq N} \mapsto -\sum_{k=1}^{N} \tilde{p}_{Nk} \ , \quad \sigma_f : \tilde{p}_{NN} \mapsto -\sum_{k=1}^{N} \tilde{p}_{kk} \ . \tag{4.79}$$

It decomposes into the vector q in the irreducible representation $(N,1)$

$$q_i = \tilde{p}_{ii} \ , \tag{4.80}$$

and the orthogonal component $\tilde{p}^{\perp}$ in $(N-1,2)$ that has zero diagonal entries $\tilde{p}^{\perp}_{ii} = 0$ and

$$\tilde{p}^{\perp}_{ij} = \tilde{p}_{ij} + \frac{1}{N-1}\big(\tilde{p}_{ii} + \tilde{p}_{jj}\big) \ , \tag{4.81}$$

for $i \neq j$. One obtains then two independent quadratic polynomials in q and $\tilde{p}^{\perp}$ respectively, such that

$$\frac{1}{2}\kappa_{ab}p^a p^b = \operatorname{tr} C_{A_N}\,\tilde{p}\,C_{A_N}\,\tilde{p} = \frac{N-1}{N+1}C_{A_N}[q] + \operatorname{tr} C_{A_N}\tilde{p}^{\perp}C_{A_N}\tilde{p}^{\perp} \ . \tag{4.82}$$

For large values of s, the Epstein series evaluated at the dual symmetric points D_N^* and A_N^* is larger, as expected from the sphere packing density. By the functional relation (4.4) this must be the opposite for $s < \frac{N}{4}$, and in particular a symmetric point and its dual give the same value at $s = \frac{N}{4}$. Accordingly, there is always a value of $s_{\rm c} > \frac{N}{4}$ where $\mathrm{Ep}^N_{s_{\rm c}}(D_N^*) = \mathrm{Ep}^N_{s_{\rm c}}(A_N)$ and $\mathrm{Ep}^N_s(D_N^*) < \mathrm{Ep}^N_s(A_N)$ for $s < s_{\rm c}$. The numerical evaluation for $N \leq 5$ shows that there is only one such transition, and that $\mathrm{Ep}^N_s(D_N^*) - \mathrm{Ep}^N_s(A_N)$ and $\mathrm{Ep}^N_s(D_N) - \mathrm{Ep}^N_s(D_N^*)$ have only one zero at $s = s_{\rm c}$ and $s = \frac{N}{4}$, respectively.

• **The symmetric point A_5^{+2}**

The symmetric point A_5^{+2} admits the same symmetry $\mathrm{S}_6 \subset SL(5,\mathbb{Z})$ as the point A_5 and its dual A_5^*. This is easy to prove using the relation

$$C_{A_5^{+2}} = \gamma^{\intercal} C_{A_5}^{-1}\gamma \tag{4.83}$$

where γ the $GL(5,\mathbb{Q})$ matrix defined such that

$$\vec{m} = \gamma\,\vec{n} \tag{4.84}$$

is the change of basis (4.69). In particular $C_{D_5} = \gamma^{\intercal}\gamma$. One checks that for any $g \in \mathrm{S}_6 \subset SL(5,\mathbb{Z})$, $\gamma g \gamma^{-1} \in SL(5,\mathbb{Z})$, such that the automorphism group of A_5^{+2} is conjugate to the one of A_5^* in $SL(5,\mathbb{Q})$ [53]. It follows that the coset $p \in \mathfrak{p}$ decomposes into the two irreducible representations $(5,1)$ and $(4,2)$ of S_6. To finds the explicit polynomial it is convenient to introduce

$$\widetilde{\mathcal{P}}_* = \big(\tfrac{2}{3}\big)^{\frac{1}{5}}\gamma^{-1}\mathcal{V}_*^{\intercal}\mathcal{P}_*\mathcal{V}_*\gamma^{-1\intercal} \ , \tag{4.85}$$

which transforms under S_6 as $\tilde{p}$ in (4.79). Repeating the computations of Section 4.2.1 for the ansatz (4.28),(4.35) one obtains the following invariants polynomials

$$\frac{4}{3}C_{A_5}[\mathcal{Q}_*] = \frac{12}{5}\frac{\mathrm{d}y^2}{y^2} + 15\mathrm{d}x^2 \ , \qquad 2\,\mathrm{tr}\, C_{A_5}\widetilde{\mathcal{P}}_*^{\perp} C_{A_5}\widetilde{\mathcal{P}}_*^{\perp} = 135\mathrm{d}x^2 \ . \tag{4.86}$$

One obtains that the condition for the Hessian of the function f to be positive definite is (4.67) for $N = 5$, as for the A_5 symmetric point. We have checked these conditions numerically and found that the symmetric point A_5^{+2} is a local minimum of the Epstein series Ep_s^5 for $s \gtrsim 2.8849$. Below this value the second derivative with respect to y is negative and the saddle is unstable along 5 directions.

In the critical strip $\frac{5}{4} < s < \frac{5}{2}$ we find that

$$\mathrm{Ep}_s^5(D_5) < \mathrm{Ep}_s^5(D_5^*) < \mathrm{Ep}_s^5(A_5^{+2}) < \mathrm{Ep}_s^5(A_5^{+3}) < \mathrm{Ep}_s^5(A_5) < \mathrm{Ep}_s^5(A_5^*) \ , \tag{4.87}$$

and only D_5 and D_5^* are local minima.

• **The E_6, E_7, E_8 symmetric points**

We shall be very brief about these cases since they are not relevant to the analysis of the type II string theory low energy effective action. One finds in these three cases that the Weyl group of E_N is large enough to impose that there is a unique invariant quadratic polynomial in p. It is therefore sufficient to check that the Epstein series is negative in the critical strip to ensure that the E_N symmetric point is a local minimum for all values of s. One finds indeed by numerical evaluation that the E_N symmetric points are local minima for all values of s, and among all symmetric points we have checked they give the smallest value of the Epstein series Ep_s^N for all values of $s \geq \frac{N}{4}$.[9] In the critical strip $\frac{N}{4} < s < \frac{N}{2}$ we find that

$$\mathrm{Ep}_s^N(E_N) \underset{N=6,7}{<} \mathrm{Ep}_s^N(E_N^*) < \mathrm{Ep}_s^N(D_N) \underset{N=6,7}{<} \mathrm{Ep}_s^N(D_N^*) < \mathrm{Ep}_s^N(A_N) < \mathrm{Ep}_s^N(A_N^*) \ . \tag{4.88}$$

• **The $A_d \times A_{\frac{N}{d}}$ symmetric points for $d|N$**

For N not prime one can also have symmetric points associated to irreducible bilinear form that are tensor products of lower dimensional bilinear forms. For example for $N = 4$ one has

$$C_{A_2\times A_2} = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix} \ . \tag{4.89}$$

More generally the density of the sphere packing of $A_d \times A_{\frac{N}{d}}$ for d dividing N is

$$\rho(C_{A_d\times A_{\frac{N}{d}}}) = \frac{\pi^{\frac{N}{2}}}{(1+d)^{\frac{N}{2d}}(1+\frac{N}{d})^{\frac{d}{2}}\Gamma(\frac{N}{2}+1)} \ . \tag{4.90}$$

[9] We have only checked symmetric points of Cartan type A_N, D_N, E_N and their dual. This exhausts all maximal irreducible symmetry groups for $N = 7$, but not for $N = 6, 8$ [53].

For $N \leq 8$ the possible lattices are less dense than E_N, D_N and A_N, but one checks that in the critical strip

$$\mathrm{Ep}^4_s(D_4) < \mathrm{Ep}^4_s(A_2 \times A_2) < \mathrm{Ep}^4_s(A_4) \ . \tag{4.91}$$

So the ordering is reversed with the A_4 lattice at low values of s. Nevertheless, the symmetric point D_4 remain the global minimum of Ep^4_s for all value of $s > 0$.

4.3 $SO(N,N)$ symmetric points

We now consider the symmetric space $(SO(N) \times SO(N))\backslash SO(N,N)$ with the discrete group $SO(N,N,\mathbb{Z})$ preserving the split signature even lattice $I\!I_{N,N}$. We introduce the vector representation Epstein series

$$\mathrm{Ep}^{N,N}_s(H) = \sum_{\substack{q \in I\!I_{N,N} \\ (q,q)=0}}{}' \frac{1}{H[q]^s} = 2\zeta(2s) E^{SO(N,N)}_{s\Lambda_1} \ , \tag{4.92}$$

which is proportional to the maximal parabolic Eisenstein series of infinitesimal character $\lambda = 2s\Lambda_1 - \rho$. It is absolutely convergent for $\mathrm{Re}[s] > N-1$, and admits an analytic continuation to a meromorphic function of $s \in \mathbb{C}$. We will be interested in particular in the value $s = \frac{N-2}{2}$, for which the Epstein series is in the minimal representation.[10] It will be convenient to consider the representation of the symmetric $SO(N,N)$ matrix H in the $P_N = GL(N,\mathbb{R}) \ltimes \wedge^2\mathbb{R}^N$ parabolic subgroup such that for any vector $q = (m,n)$ with $m, n \in \mathbb{Z}^N$,

$$H[q] = G^{-1}[m + Bn] + G[n] \ , \tag{4.93}$$

and the split signature bilinear form is

$$(q,q) = 2m \cdot n \ . \tag{4.94}$$

In type II string theory on T^d the $SO(d,d)$ symmetric matrix H is parametrised by the Narain moduli, with the torus metric G and the Kalb–Ramond two-form B in string length. Then m is the vector of Kaluza–Klein modes and n the vector of winding numbers. For the U-duality group $Spin(5,5,\mathbb{Z})$ one can view G as the M-theory metric on T^5 and B as the Hodge dual of the three-form potential on T^5.

To define symmetric points in $SO(N,N)$ we start with the assumption that G is itself a symmetric point of $SO(N)\backslash GL(N)$. From the previous section we find that G is then proportional to the Gram matrix C_L of an even lattice L, and the relevant solution will turn out to be

$$G = \frac{1}{2} C_L \ , \qquad G + B = 0 \bmod \mathbb{Z} \ . \tag{4.95}$$

One finds in this case that the charges $q_\pm$

$$q_\pm = \frac{1}{2} G^{-1}\big(m + (B \pm G)n\big) \ , \tag{4.96}$$

[10] This can be understood from the Langlands functional identity $\xi(N-2)E^{SO(N,N)}_{\frac{N-2}{2}\Lambda_1} = \xi(2)E^{SO(N,N)}_{\Lambda_N} = \xi(2)E^{SO(N,N)}_{\Lambda_{N-1}}$ for $N \geq 3$ which shows that this function can be realised either as a vector representation Epstein series or a spinor representation Epstein series.

satisfy

$$q_+ - q_- \in \mathbb{Z}^N \ , \tag{4.97}$$

while $q_\pm$ are by construction in the dual lattice $L^* = C_L^{-1}\mathbb{Z}^N$. It follows that this bilinear form describes the isomorphism

$$I\!I_{N,N} \cong \bigoplus_{\mu\in L^*/L} \Big((L+\mu) \oplus (L+\mu)[-1] \Big) \ , \tag{4.98}$$

or equivalently for the Narain theta series

$$\sum_{q\in I\!I_{N,N}} e^{-\pi\tau_2 H[q] + i\pi\tau_1(q,q)} = \sum_{\mu\in L^*/L} \Big| \sum_{n\in\mathbb{Z}^N} e^{i\pi\tau C_L[n+\mu]} \Big|^2 \ . \tag{4.99}$$

For any pair of elements $\gamma_\pm \in \mathrm{Aut}(L)$ such that

$$\gamma_+\mu = \gamma_-\mu \bmod L \ , \tag{4.100}$$

one has an automorphism $O(N,N,\mathbb{Z})$ of the split signature lattice $I\!I_{N,N}$ that preserves the symmetric matrix H. For Cartan type lattices A_N, D_N, E_N, the automorphism group is the product of the outer automorphisms of the Dynkin diagram and the Weyl group

$$\mathrm{Aut}(L) = \mathrm{Out}(L) \ltimes W(L) \ , \tag{4.101}$$

and the Weyl group $W(L)$ preserves all weights $\mu \in L^*/L$, so one has the automorphism group

$$\mathrm{Aut}(H) = \mathrm{Out}(L) \ltimes W(L) \times W(L) \ . \tag{4.102}$$

The simplest example of a symmetric point is for $N = 1$, in which case $G = y^2$ and

$$\mathrm{Ep}_s^{1,1}(H) = 2\zeta(2s)\big(y^{2s} + y^{-2s}\big) \ . \tag{4.103}$$

Its global minimum at $y = 1$ can be interpreted as the $SU(2)$ self-dual radius in string theory on S^1. The A, D, E points described above correspond more generally to the points of enhanced gauge symmetry in string theory. For $N = 2$ one recovers the minimum of the product of two $SL(2)$ Epstein series at $T = U = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ for the symmetric matrix H associated to the lattice $L = A_2$. We will see below that for $N = 3$ the symmetric matrix associated to $L = A_3$ also reproduces the minimum of the $SL(4)$ Epstein series at the symmetric point D_4.

Assuming H to be determined by a lattice Gram matrix as in (4.95), one can compute the limit at large s from the theta series

$$\begin{aligned} \mathrm{Ep}_s^{N,N}(H) &= \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{d\tau_2}{\tau_2} \tau_2^s \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \sum_{q\in I\!I_{N,N}}{}' e^{-\pi\tau_2 H[q] + i\pi\tau_1(q,q)} \\ &= \sum_{\mu\in L^*/L} \sum_{\substack{m,n\in\mathbb{Z}^N \\ C_L[n+\mu]=C_L[m+\mu]}}{}' \frac{1}{(2C_L[n+\mu])^s} \ , \end{aligned} \tag{4.104}$$

where the prime removes the point $m = n = 0$ for $\mu = 0$ only. At large s, the leading term in $\mathrm{Ep}_s^{N,N}(H)$ is proportional to the smallest length of a vector in L^* to the power $-2s$, and one obtains a minimum for the even lattice with the largest possible minimal length of a vector in L^*. Although this is not exactly the same criterion as for the densest lattice sphere packing, it gives the same ADE classification (4.3) for $N \leq 8$.

For small values of s we need instead to use the Fourier expansion of the Epstein series [11]

$$\mathrm{Ep}_s^{N,N}(H) = \det G^{\frac{s}{N}} \mathrm{Ep}_s^N(G^{-1}) + \frac{\pi^{\frac{N-1}{2}}\Gamma(s-\frac{N-1}{2})}{\Gamma(s)} \frac{\zeta(2s-N+1)}{\zeta(2s-N+2)} \det G^{\frac{N-1-s}{N}} \mathrm{Ep}^N_{s-\frac{N-2}{2}}(G)$$
$$+ \frac{4\pi^s}{\Gamma(s)} \sqrt{\det G} \sum_{\substack{Q\in\wedge^2\mathbb{Z}^N \\ Q\wedge Q=0}}{}' \frac{\sigma_{N-1-2s}(Q)}{\gcd(Q)^{\frac{N-2}{2}-s}} E^{SL(2)}_{s-\frac{N-2}{2}}(U_Q) \frac{K_{s-\frac{N-1}{2}}(2\pi\sqrt{G[Q]})}{\sqrt[4]{G[Q]}} e^{\pi i \operatorname{tr} BQ} \tag{4.105}$$

where the sum is over all non-zero rank one antisymmetric integer matrices Q, the bilinear form is defined as

$$G[Q] = -\frac{1}{2} \operatorname{tr} GQGQ \ , \tag{4.106}$$

and $E_s^{SL(2)}(U_Q)$ is the $SL(2)$ real analytic Eisenstein series evaluated on the $SL(2)$ subgroup of the stabiliser of Q.[12] Note that the stabiliser of Q in $SL(N)$ is $SL(2) \times SL(N-2) \ltimes \mathbb{R}^{2\times(N-2)}$.

For the value $s = \frac{N-2}{2}$, this expression simplifies drastically to

$$\mathrm{Ep}^{N,N}_{\frac{N-2}{2}}(H) = \det G^{\frac{N-2}{2N}} \mathrm{Ep}^N_{\frac{N-2}{2}}(G^{-1}) + \frac{\pi^{\frac{N}{2}}\sqrt{\det G}}{\Gamma(\frac{N-2}{2})} \left(\frac{1}{3} + 2 \sum_{\substack{Q\in\wedge^2\mathbb{Z}^N \\ Q\wedge Q=0}}{}' \sigma_1(Q) \frac{e^{-2\pi\sqrt{G[Q]}}}{\pi\sqrt{G[Q]}} e^{\pi i \operatorname{tr} BQ} \right) . \tag{4.107}$$

We will use this expression to evaluate numerically the Epstein series for $N = 5$.

We find evidence that the Epstein series in the vector representation (4.92) admits its global minimum for all $s > 0$ at the symmetric point where $G = \frac{1}{2}C_L$ and $G + B = 0 \bmod \mathbb{Z}$ for the Cartan type best packing lattices (4.3).

4.3.1 Symmetric points as minima

In order to describe the symmetries of the polynomials in the coset derivatives we need to define a coset representative $\mathcal{V} \in SO(N,N)$. We introduce two vielbeins basis $V_\pm$ for the same metric $G = V_\pm^\intercal V_\pm$ such that $\mathcal{V}$ reads

$$\mathcal{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} V_+^{-1\intercal} & \frac{1}{\sqrt{2}} V_+^{-1\intercal}(G+B) \\ \frac{1}{\sqrt{2}} V_-^{-1\intercal} & \frac{1}{\sqrt{2}} V_-^{-1\intercal}(-G+B) \end{pmatrix} . \tag{4.108}$$

It transforms under left action of $k_\pm \in SO(N)$ and right action of $\gamma \in O(N,N,\mathbb{Z})$ as

$$\mathcal{V} \to \begin{pmatrix} k_+ & 0 \\ 0 & k_- \end{pmatrix} \mathcal{V}\gamma \ . \tag{4.109}$$

[11] In our convention the $SL(N)$ Epstein series only depend on the unimodular bilinear form so that $\mathrm{Ep}_s^N(G) = \mathrm{Ep}_s^N(\det G^{-\frac{1}{N}} G))$, and in particular $\det G^{\frac{s}{N}} \mathrm{Ep}_s^N(G^{-1}) = \sum'_{n\in\mathbb{Z}^N} \frac{1}{G^{-1}[n]^s}$.

[12] Normalised such that $\mathrm{Ep}_s^2(U) = 2\zeta(2s) E_s^{SL(2)}(U)$.

One can of course set $V_+ = V_- = V$, but it is convenient to keep them different to make manifest the covariance under $SO(N) \times SO(N)$. The coset differential is then

$$\frac{1}{2}\mathrm{d}\mathcal{V}\mathcal{V}^{-1} + \frac{1}{2}\mathcal{V}^{-1\intercal}\mathrm{d}\mathcal{V}^{\intercal} = \begin{pmatrix} 0 & \mathcal{P}^{\intercal} \\ \mathcal{P} & 0 \end{pmatrix} \tag{4.110}$$

with

$$\mathcal{P} = -\frac{1}{2}V_-^{-1\intercal}\big(\mathrm{d}G + \mathrm{d}B\big)V_+^{-1} \,. \tag{4.111}$$

We now consider a point (4.95) for L of Cartan type, that we write

$$\mathcal{V}_0 = \begin{pmatrix} \sqrt{2}V_{0+} & 0 \\ 0 & \sqrt{2}V_{0-} \end{pmatrix} \begin{pmatrix} C_L^{-1} & C_L^{-1}(G+B)_0 \\ C_L^{-1} & C_L^{-1}(-G+B)_0 \end{pmatrix} \,. \tag{4.112}$$

Using the charges $q_\pm$ that transform under the two copies of the Weyl group at a symmetric point (4.96), one obtains that

$$\mathcal{V}_0 Q = \begin{pmatrix} \sqrt{2}V_{0+}q_+ \\ \sqrt{2}V_{0-}q_- \end{pmatrix} \,. \tag{4.113}$$

It follows that under an automorphism in (4.102) realised as a $\gamma \in O(N,N,\mathbb{Z})$ one has

$$\mathcal{V}_0\gamma = \begin{pmatrix} k_+(\gamma) & 0 \\ 0 & k_-(\gamma) \end{pmatrix} \mathcal{V}_0 = \begin{pmatrix} \sqrt{2}V_{0+}\gamma_+ & 0 \\ 0 & \sqrt{2}V_{0-}\gamma_- \end{pmatrix} \begin{pmatrix} C_L^{-1} & C_L^{-1}(G+B)_0 \\ C_L^{-1} & C_L^{-1}(-G+B)_0 \end{pmatrix} \tag{4.114}$$

where the $\gamma_\pm$ satisfy (4.100). In particular, the $\gamma_\pm$ can be two independent elements of the Weyl group of the Cartan type lattice L.

It appears therefore convenient to expand the automorphic function of interest with respect to the pull-back momentum

$$\mathcal{P}_* = -\big(\mathrm{d}G + \mathrm{d}B\big)_* \,. \tag{4.115}$$

Similarly as in section 4.2.1, we introduce the variable

$$\tilde{p} = 2V_{0-}^{\intercal} p V_{0+} \,. \tag{4.116}$$

For any invariant polynomial $f_{H_0}(p)$ of $p \in \mathfrak{so}(N,N) \ominus (\mathfrak{so}(N) \oplus \mathfrak{so}(N))$, one defines the polynomial

$$\tilde{f}_{\mathcal{V}_0}(\tilde{p}) = f_{\mathcal{V}_0}\big(2V_{0-}^{\intercal} p V_{0+}\big) \,, \tag{4.117}$$

that is invariant under

$$\tilde{f}_{\mathcal{V}_0}(\gamma_- \tilde{p} \gamma_+) = \tilde{f}_{\mathcal{V}_0}(\tilde{p}) \,. \tag{4.118}$$

One then finds for all irreducible Cartan type lattices L, that the unique invariant quadratic polynomial is the Killing–Cartan form

$$2\,\mathrm{tr}\, p p^{\intercal} = 2\,\mathrm{tr}\, C_L^{-1}\tilde{p}\, C_L^{-1}\tilde{p}^{\intercal} \,. \tag{4.119}$$

Let us prove this for A_N and D_N.

• **A_N symmetric point**

For A_N, the momentum $\tilde{p}$ transforms in the tensor product of two standard representations $(N,1)$ associated to the two copies of S_{N+1}. It is therefore in an irreducible representation of $\mathrm{S}_{N+1}\times\,\mathrm{S}_{N+1}$ and there is obviously no linear invariant. Introducing the indices $i=1$ to N for one side and $\hat{\imath}=1$ to N for the other, one obtains the action of $\mathrm{S}_N\times\,\mathrm{S}_N$

$$\sigma\times\hat{\sigma}:\tilde{p}_{i\hat{\jmath}}\mapsto\tilde{p}_{\sigma^{-1}(i)\hat{\sigma}^{-1}(\hat{\jmath})}\ , \tag{4.120}$$

while the left σ_f acts as

$$\sigma_f\times 1:\tilde{p}_{1\hat{\jmath}}\mapsto\tilde{p}_{N\hat{\jmath}}\ ,\qquad \sigma_f\times 1:\tilde{p}_{i\neq 1\hat{\jmath}}\mapsto\tilde{p}_{N\hat{\jmath}}-\tilde{p}_{i-1\hat{\jmath}}\ , \tag{4.121}$$

and identically for the right $\hat{\sigma}_f$. For the quadratic polynomial one can use the decomposition into irreducible representations of the tensor product of two standard representations

$$(N,1)\otimes(N,1)=(N)\oplus(N,1)\oplus(N-1,2)\oplus(N-1,1,1)\ , \tag{4.122}$$

to conclude that the quadratic polynomials in $\tilde{p}$ decompose into the irreducible representations of $\mathrm{S}_{N+1}\times\,\mathrm{S}_{N+1}$

$$\big[(N{+}1)\oplus(N,1)\oplus(N{-}1,2)\big]\otimes\big[(N{+}1)\oplus(N,1)\oplus(N{-}1,2)\big]\oplus(N{-}1,1,1)\otimes(N{-}1,1,1)\ . \tag{4.123}$$

It follows that there is single quadratic invariant polynomial (4.119). According to the discussion of section 4.1, we conclude that the A_N symmetric point is a local minimum of any $SO(N,N)$ Eisenstein series in the domain of absolute convergence. We have checked numerically that $\mathrm{Ep}_s^{5,5}(H)$ is negative at the critical value $s=\frac{3}{2}$ relevant in the string theory effective action, and the symmetric point A_5 is therefore a local minimum.

• **D_N symmetric point**

For D_N, the momentum p transforms in the tensor product of two vector representations of $\mathrm{S}_N\ltimes\mathbb{Z}_2^{N-1}$. It is convenient to define the action of the two copies of $\mathrm{S}_N\ltimes\mathbb{Z}_2^N$

$$\begin{aligned}&\sigma\times\hat{\sigma}:p_{i\hat{\jmath}}\mapsto p_{\sigma(i)\hat{\sigma}(\hat{\jmath})}\ ,\\ &\varpi_i\times 1:p_{i\hat{\jmath}}\mapsto -p_{i\hat{\jmath}}\ ,\quad p_{j\hat{\jmath}}\mapsto p_{j\hat{\jmath}}\ ,\quad \forall j\neq i\ ,\end{aligned} \tag{4.124}$$

and identically for the right $\hat{\varpi}_{\hat{\jmath}}$, and then take the subgroup of elements with an even number of ϖ_i and $\hat{\varpi}_{\hat{\jmath}}$. It follows directly that the only invariant quadratic polynomial is

$$\sum_{i=1}^{N}\sum_{\hat{\jmath}=1}^{N}p_{i\hat{\jmath}}p_{i\hat{\jmath}}=\ \mathrm{tr}\,pp^{\mathsf{T}}\ , \tag{4.125}$$

which is the Killing–Cartan form (4.119). One finds therefore that the D_N symmetric point is a local minimum of any $SO(N,N)$ Eisenstein series in the domain of absolute convergence. We have checked numerically that $\mathrm{Ep}_s^{5,5}(H)$ is negative at the critical value $s=\frac{3}{2}$ relevant in the string theory effective action, and the symmetric point D_5 is therefore a local minimum. It is a lower value than the A_5 symmetric point and we conjecture that it is the global minimum of the minimal Epstein series.

4.3.2 $SO(3,3)$ and $SL(4)$

Because of the homomorphism $\mathrm{Spin}_0(3,3) = SL(4)$ it is relevant to compare the results we have obtained for $SO(N,N)$ and $SL(N)$ in this case. It appears that the vector representation Epstein series of $SO(3,3)$ and the Epstein series of $SL(4)$ are related at the special s value

$$\pi \mathrm{Ep}^{3,3}_{\frac{1}{2}}(H) = \mathrm{Ep}^{4}_{1}(H) \ , \tag{4.126}$$

with the identification of the $SL(4)$ matrix

$$\mathcal{V} = y^{-\frac{1}{4}} \begin{pmatrix} 2^{\frac{1}{3}}\mathcal{V}_1 & 2^{\frac{1}{3}}\mathcal{V}_1 x \\ 0 & \frac{y}{2} \end{pmatrix} = \frac{1}{\sqrt{\det V}} \begin{pmatrix} V & Vx \\ 0 & \det V \end{pmatrix} \ , \tag{4.127}$$

and $V_+ = V_- = V$ and $B_{ij} = \varepsilon_{ijk}x_k$ in (4.108). In particular for (4.34) one has $G = \frac{y}{4}C_{A_3}$, and one can check the functional relation (4.126) using (4.107) and (4.40)

$$\begin{aligned} \pi \mathrm{Ep}^{3,3}_{\frac{1}{2}}(H) &= \pi \det G^{\frac{1}{6}} \mathrm{Ep}^{3}_{\frac{1}{2}}(G^{-1}) + \pi^2 \sqrt{\det G} \left(\frac{1}{3} + 2 \sum_{Q \in \wedge^2 \mathbb{Z}^3}{}' \sigma_1(Q) \frac{e^{-2\pi\sqrt{G[Q]}}}{\pi\sqrt{G[Q]}} e^{\pi i \,\mathrm{tr}\, BQ} \right) \\ &= \frac{y^{\frac{1}{2}}}{4^{\frac{1}{3}}} \mathrm{Ep}^{3}_{1}(C_{A_3}) + \frac{\pi^2}{3}\frac{y^{\frac{3}{2}}}{4} + \pi\sqrt{y} \sum_{n \in \mathbb{Z}^3}{}' \sigma_1(n) \frac{e^{-\pi y\sqrt{C^{-1}_{A_3}[n]}}}{\sqrt{C^{-1}_{A_3}[n]}} \cos\Big(2\pi x \cdot n\Big) \ . \end{aligned} \tag{4.128}$$

For the special points (4.34) of the $SL(4)$ Epstein series we have

$$G = \frac{y}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \ , \quad B = x \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \ , \tag{4.129}$$

so that the $SL(4)$ D_4 symmetric point corresponds to the $SO(3,3)$ A_3 symmetric point,

$$G = \frac{1}{2} C_{A_3} \ , \quad B + G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \ , \tag{4.130}$$

while the $SL(4)$ A_4 symmetric point gives

$$G = \frac{\sqrt{5}}{4} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \ , \quad B = \frac{1}{4} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \ . \tag{4.131}$$

We find consistently that the conjectured global minimum of the $SL(4)$ Epstein series at the D_4 symmetric point agrees with the conjectured global minimum of the $SO(3,3)$ Epstein series at the A_3 symmetric point. One finds that they have the same automorphism groups

$$\mathrm{Aut}(\mathit{\Pi}^{A_3}_{3,3}) = \mathbb{Z}_2 \ltimes (\mathrm{S}_4 \times \mathrm{S}_4) \cong \mathrm{Aut}(D_4) = \mathrm{S}_3 \ltimes (\mathrm{S}_4 \ltimes \mathbb{Z}_2^3) \ , \tag{4.132}$$

because of the triality automorphism of D_4, explaining that there is a unique invariant quadratic polynomial in this case.

5 Fixing the logarithmic ambiguity in eight dimensions

The different terms in the low energy expansion of the two-graviton amplitude (2.4) are not well defined individually in eight dimensions because the 1-loop box integral

$$I_4(s,t) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+k_4)^2} \tag{5.1}$$

diverges logarithmically and so do the Epstein series defining $\mathcal{E}_{(0,0)}(\varphi)$. To determine unambiguously the amplitude we must analyse the low energy limit of the one-loop string theory amplitude [5]

$$\mathcal{A}^{\text{1-loop}} = 2\pi {\alpha'}^3 {g_8}^2 \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \Gamma_{I\!I_{2,2}} \prod_{a=1}^{4} \int_{\Sigma} \frac{\mathrm{d}^2 z_a}{\tau_2} \tau_2 \delta^{(2)}(z_4) e^{-\frac{\alpha'}{2}\sum_{a>b} G(z_a - z_b) k_a \cdot k_b} \,, \tag{5.2}$$

where

$$\Gamma_{I\!I_{d,d}}(\tau) = \tau_2^{\frac{d}{2}} \sum_{q \in I\!I_{d,d}} e^{-\pi\tau_2 H[q] + i\pi\tau_1 (q,q)} \,, \tag{5.3}$$

and $G(x)$ is the torus Green function. Following [12], we split the $SL(2)$ fundamental domain into the truncated fundamental domain

$$\mathcal{F}_L = \left\{ \tau_2 < L,\, -\tfrac{1}{2} \le \tau_1 \le \tfrac{1}{2},\, |\tau| > 1 \right\} \tag{5.4}$$

and the complementary region for which $\tau_2 > L$ and $-\frac{1}{2} \le \tau_1 \le \frac{1}{2}$. At leading order one obtains

$$\begin{aligned} \frac{\mathcal{A}^{\text{1-loop}}}{\ell_{\text{P}}^6} = 2\pi \int_{\mathcal{F}_L} \frac{\mathrm{d}^2\tau}{\tau_2^2} \Gamma_{I\!I_{2,2}} + 4\pi \int_L^\infty \frac{\mathrm{d}\tau_2}{\tau_2} \int_0^1 \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 e^{\pi\tau_2 \alpha' [(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} + \circlearrowleft \\ + \mathcal{O}(s^2) \,, \end{aligned} \tag{5.5}$$

where $\circlearrowleft$ represents the two cyclic permutations of the Mandelstam variables. To compute the two terms separately it is more convenient to introduce a dimensional regularisation $d = 2 + 2\epsilon$. This can be achieved with the insertion of τ_2^ϵ in both terms. One can then remove the L-dependent terms that cancel out and use instead

$$\lim_{\epsilon \to 0_+} \left(2\pi \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} \tau_2^\epsilon \Gamma_{I\!I_{2,2}} + 4\pi \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2} \tau_2^\epsilon \int_0^1 \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 e^{\pi\tau_2 \alpha' [(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} + \circlearrowleft \right) \tag{5.6}$$

For the one-loop supergravity amplitude one gets the dimensional regularisation

$$\begin{aligned} & (4\pi)^5 (4\pi^2 \alpha')^{-\epsilon} \int \frac{d^{8-2\epsilon} p}{(2\pi)^{8-2\epsilon}} \frac{1}{p^2 (p-k_1)^2 (p-k_1-k_2)^2 (p+k_4)^2} \\ &= 4\pi \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2} \tau_2^\epsilon \int_0^1 \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 e^{\pi\tau_2 \alpha' [(x_2-x_1)(1-x_3)s + x_1(x_3-x_2)t]} \\ &= -8\pi^{1-\epsilon} \frac{\Gamma(\epsilon-2)\Gamma(3-\epsilon)^2}{\Gamma(5-2\epsilon)} \int_0^1 dx \left((-\alpha' s)^{-\epsilon} \frac{(1-x)^{1-\epsilon}}{(1+\frac{t}{s})-1} + (-\alpha' t)^{-\epsilon} \frac{(1-x)^{1-\epsilon}}{(1+\frac{s}{t})-1} \right) \\ &= \frac{2\pi}{3} \left(\frac{1}{\epsilon} + \frac{11}{3} - \gamma_{\text{E}} - \ln\pi - \frac{s}{s+t} \ln(-\alpha' s) - \frac{t}{s+t} \ln(-\alpha' t) - \frac{1}{2} \frac{st}{(s+t)^2} \left(\ln\left(\frac{t}{s}\right)^2 + \pi^2 \right) \right) \\ &\quad + \mathcal{O}(\epsilon) \,. \end{aligned} \tag{5.7}$$

We define accordingly the renormalised 1-loop box integral in $D = 8$ dimensions

$$\hat{I}_{4,\mu}(s,t) = -\frac{1}{6(4\pi)^4}\left(\frac{s}{s+t}\ln\left(-s/\mu^2\right) + \frac{t}{s+t}\ln\left(-t/\mu^2\right) + \frac{1}{2}\frac{st}{(s+t)^2}\left(\ln\left(\frac{t}{s}\right)^2 + \pi^2\right)\right) , \tag{5.8}$$

where we introduced a renomalisation scale μ. Note that $\hat{I}_{4,\mu}(s,t) = \hat{I}_{4,1}(s/\mu^2, t/\mu^2)$.

For the perturbative Wilson coefficient one can replace τ_2^ϵ by the real analytic Eisenstein series $E_\epsilon(\tau)$ without modifying the limit. One computes as in [36] that

$$\begin{aligned} &2\pi\int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2^2}E_\epsilon(\tau)\Gamma_{I\!I_{2,2}} \\ &= 4\pi\xi(2\epsilon)E_\epsilon(T)E_\epsilon(U) \\ &= -\frac{2\pi}{\epsilon} + 2\pi\big(\gamma_{\mathrm{E}} - \ln(4\pi)\big) - 2\pi\ln(U_2|\eta(U)|^4) - 2\pi\ln(T_2|\eta(T)|^4) + \mathcal{O}(\epsilon) , \end{aligned} \tag{5.9}$$

where η is the Dedekind eta function. One obtains in total

$$\begin{aligned} \frac{\mathcal{A}^{\text{1-loop}}}{\ell_{\mathrm{P}}^6} &= -2\pi\ln(U_2|\eta(U)|^4) - 2\pi\ln(T_2|\eta(T)|^4) + \frac{22\pi}{3} - 4\pi\ln(2\pi) \\ &\quad +(4\pi)^5(\hat{I}_{4,1}(\alpha' s,\alpha' t) + \hat{I}_{4,1}(\alpha' t,\alpha' u) + \hat{I}_{4,1}(\alpha' u,\alpha' s)) + \mathcal{O}(s^2) \\ &= -2\pi\ln(U_2|\eta(U)|^4) - 2\pi\ln(T_2|\eta(T)|^4) + \frac{4\pi}{3}\ln g_8 + \frac{22\pi}{3} - 4\pi\ln(2\pi\ell_{\mathrm{P}}\mu) \\ &\quad +(4\pi)^5\big(\hat{I}_{4,\mu}(s,t) + \hat{I}_{4,\mu}(t,u) + \hat{I}_{4,\mu}(u,s)\big) + \mathcal{O}(s^2) \end{aligned} \tag{5.10}$$

where we used ${\alpha'}^3 {g_8}^2 = \ell_{\mathrm{P}}^6$ in the last step.

We use the same convention as in [36] for the renormalised Eisenstein series, such that we define the $SL(2)$ Epstein series

$$\widehat{\mathrm{Ep}}_1^{\,2}(U) = \lim_{\epsilon\to 0}\left(\mathrm{Ep}_{1+\epsilon}^{\,2}(U) - \frac{\pi}{\epsilon} - 2\pi(\gamma_{\mathrm{E}} - \ln 2)\right) = -\pi\ln\left(U_2|\eta(U)|^4\right) , \tag{5.11}$$

and the $SL(3)$ Epstein series

$$\begin{aligned} \widehat{\mathrm{Ep}}_{\frac{3}{2}}^{\,3}(H) &= \lim_{\epsilon\to 0}\left(\mathrm{Ep}_{\frac{3}{2}+\epsilon}^{\,3}(H) - \frac{2\pi}{\epsilon} - 4\pi(\gamma_{\mathrm{E}} - 1)\right) \\ &= \frac{2\zeta(3)}{{g_8}^2} - 2\pi\ln\left(T_2|\eta(T)|^4\right) + \frac{4\pi}{3}\ln g_8 \\ &\quad +\frac{8\pi}{g_8}{\sum_{m,n\in\mathbb{Z}}}' \frac{\sqrt{T_2}\,\sigma_2(m,n)}{|m+Tn|}K_1\Big(\tfrac{2\pi}{g_8\sqrt{T_2}}|m+Tn|\Big)e^{2\pi i(c_0 m + c_2 n)} . \end{aligned} \tag{5.12}$$

With these definitions we have

$$\mathcal{A}(s,t,u,\varphi) = \frac{64}{stu} + (4\pi)^5\ell_{\mathrm{P}}^6\big(\hat{I}_{4,\mu}(s,t) + \hat{I}_{4,\mu}(t,u) + \hat{I}_{4,\mu}(u,s)\big) + \ell_{\mathrm{P}}^6\mathcal{E}_{(0,0),\mu}(\varphi) + \mathcal{O}(s^2) \tag{5.13}$$

and

$$\mathcal{E}_{(0,0),\mu}(\varphi) = \widehat{\mathrm{Ep}}_{\frac{3}{2}}^{\,3}(H) + 2\widehat{\mathrm{Ep}}_1^{\,2}(U) + \frac{22\pi}{3} - 4\pi\ln(2\pi\ell_{\mathrm{P}}\mu) . \tag{5.14}$$

Note that the logarithm $4\pi\ln(2\pi\ell_{\mathrm{P}})$ can be absorbed in the logarithm of the type IIB torus volume $V_2 = (2\pi\ell_{\mathrm{P}})^2 T_2 {g_8}^{-2/3}$, equivalently $V_2 = (2\pi\ell_{\mathrm{P}})^2 U_2 {g_8}^{-2/3}$ in type IIA.

6 Numerical approximations

In this section we explain the numerical checks we have carried out to establish the conjectured global minima of the Epstein series. We concentrate on the two cases of $SL(5)$ and $SO(5,5)$ that constitute the main new results. The global minimum of the $SL(3)$ Epstein series was obtained as the symmetric point A_3 for all $s > \frac{3}{4}$, including the renormalised value at $s = \frac{3}{2}$ by minimal subtraction, in [38].

The moduli space $SO(5)\backslash SL(5)$ has 14 dimensions and $\big(SO(5) \times SO(5)\big)\backslash SO(5,5)$ 25 dimensions, and it is rather difficult to study the value of the Epstein series systematically over the whole moduli space in these two cases. Nonetheless, the Epstein series are very regular functions, that decrease somewhat monotonically from the cusps at infinity to the symmetric points in the interior.

6.1 The $SL(5)$ Epstein series

We consider several $SL(5,\mathbb{Z})$ equivalent realisations of the D_5 symmetric point $H = 2^{-\frac{2}{5}} C_{D_5}$ with

$$C_{D_5} = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 4 \end{pmatrix} , \quad C'_{D_5} = \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 4 & 2 \\ 1 & 1 & 1 & 2 & 2 \end{pmatrix} , \quad C''_{D_5} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} . \tag{6.1}$$

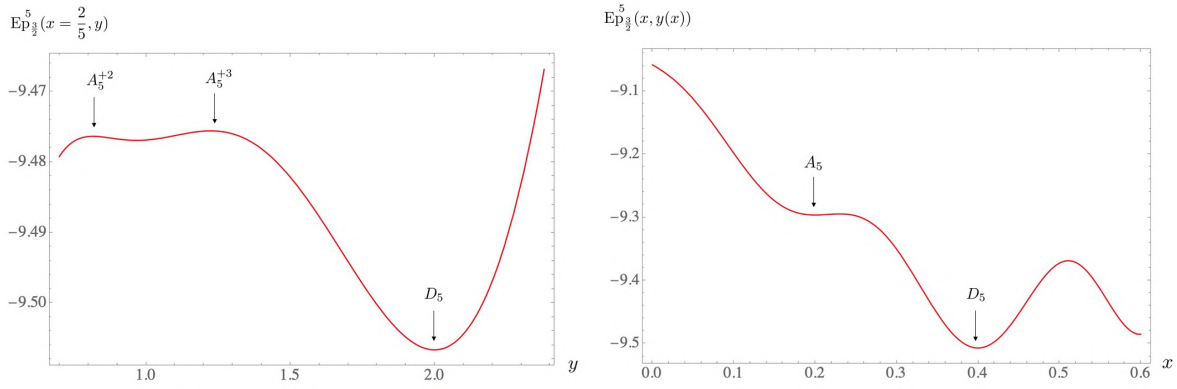
They are determined by iterated inclusions of lattices

$$\begin{aligned} &A_2 \subset A_3 \subset A_4 \subset D_5 \ , \\ &A_2 \subset A_3 \subset D_4 \subset D_5 \ , \\ &A_2 \subset A_2 + A_1 \subset A_2 + 2A_1 \subset D_5 \ . \end{aligned} \tag{6.2}$$

The first case is described with the ansatz (4.28) and the expansion of the Epstein series (4.40) with $N = 5$ as a function of y and x. We use (4.40) iteratively and truncate the sum over $n_i \in \mathbb{Z}$ for $N = 3$ between -50 and 50, the sum over $n_i \in \mathbb{Z}$ for $N = 4$ between -20 and 20 and the sum over $n_i \in \mathbb{Z}$ for $N = 5$ between -10 and 10. We checked that increasing the ranges of the Fourier modes n_i does not change the result within the approximation. We analyse the values of s between $\frac{5}{4}$ and 5. For large values of s the Fourier expansion converges more slowly because the Bessel function does not decrease fast enough for the truncated expansion (4.40) to give a good approximation.

The obtained function of y and x admits only three local minima, corresponding to the symmetric points A_5, A_5^{+2} and D_5, with D_5 being the global minimum. Only D_5 is a local minimum of the function of $H \in SO(5)\backslash SL(5)$ for low values of s.

One also checks the pull-back of the function on different surfaces associated to the parametrisation (4.40) for $N = 4$ as $\mathrm{Ep}^5_s(2, \frac{2}{5}, H) = \frac{2^{\frac{2s}{5}}}{5^{\frac{s}{4}}} \mathrm{Ep}^4_s(y, x, A_3) + \dots$ and similarly for $N = 3$ and find again that the D_5 point is the global minimum on each surface.



(a) (b)

Figure 1: Pullback of the $SL(5)$ Epstein series along the two dimensional (x, y) surface defined by (4.28) for $s = \frac{3}{2}$. (a) Slice $x = \frac{2}{5}$ containing A_5^{+2} , A_5^{+3} and D_5, only D_5 is a local minimum. (b) Slice $y = -5(\sqrt{6} - 2)x + 2(\sqrt{6} - 1)$ containing A_5 and D_5, along this slice A_5 looks like a local minimum but it is a saddle point in $SO(5)\backslash SL(5)$. D_5 is the global minimum on the surface.

In the second case we consider the $SL(N)$ coset representative

$$\mathcal{V} = y^{-\frac{1}{N}} \begin{pmatrix} 2^{\frac{1}{N-1}}\mathcal{V}_1 & 2^{\frac{1}{N-1}}\mathcal{V}_1\mathbf{x} \\ 0 & y/2 \end{pmatrix} , \tag{6.3}$$

with $\mathcal{V}_1$ at the D_{N-1} symmetric point and $\mathbf{x}$ restricted to the last component x_{N-1}. One then obtains the Fourier expansion of the Epstein series

$$\begin{aligned} \mathrm{Ep}_s^N(y, x, C_{D_{N-1}}) = {} & \frac{y^{\frac{2s}{N}}}{4^{\frac{s}{N-1}}}\mathrm{Ep}_s^{N-1}(C_{D_{N-1}}) + \frac{2\pi^{\frac{N-1}{2}}\Gamma(s-\frac{N-1}{2})\zeta(2s-N+1)}{\Gamma(s)}\frac{y^{(N-1)(1-\frac{2s}{N})}}{4^{\frac{N}{2}-s}} \\ & + \frac{2\pi^s}{\Gamma(s)} y^{\frac{N-1}{2}-\frac{N-2}{N}s} {\sum_{n\in\mathbb{Z}^{N-1}}}' \sigma_{N-1-2s}(n) \frac{K_{s-\frac{N-1}{2}}\left(\pi y\sqrt{C^{-1}_{D_{N-1}}[n]}\right)}{\left(4C^{-1}_{D_{N-1}}[n]\right)^{\frac{N-1}{4}-\frac{s}{2}}} \cos\left(2\pi x n_{N-1}\right) \end{aligned} \tag{6.4}$$

which defines another surface parametrised by (x, y) in the moduli space. In this case one obtains for $N = 5$ that the only local minimum is at $y = 2$ and $x = \frac{1}{2}$ corresponding to the D_5 symmetric point.

In the third case we consider the $SL(N)$ coset representative

$$\mathcal{V} = y^{-\frac{1}{N}} \begin{pmatrix} (4N-8)^{\frac{1}{2(N-1)}}\mathcal{V}_1 & (4N-8)^{\frac{1}{2(N-1)}}\mathcal{V}_1\mathbf{x} \\ 0 & \frac{y}{2\sqrt{N-2}} \end{pmatrix} , \tag{6.5}$$

with $\mathcal{V}_1$ at the $A_{N-3}+2A_1$ symmetric point and $\mathbf{x} = (-\frac{2}{N-2}, \ldots, 2\frac{N-3}{N-2}, 1, 1)x$. One obtains again for $N = 5$ that the only local minimum is at $y = 2$ and $x = \frac{1}{2}$ corresponding to the D_5 symmetric point.

We obtain numerically the value of the Epstein series

$$\mathrm{Ep}^{5}_{\frac{3}{2}}(D_5) \approx -9.50663\ . \tag{6.6}$$

The numerical value is stable under modification of the truncation up to fifteen digits.

6.2 The $SO(5,5)$ vector representation Epstein series

The vector representation Epstein series at generic s is much harder to approximate because of the Eisenstein series appearing in the Fourier coefficients in (4.105). Therefore we shall only analyse the minimal Epstein series (4.107) at $s = \frac{3}{2}$

$$\mathrm{Ep}^{5,5}_{\frac{3}{2}}(H) = \det G^{\frac{3}{10}}\,\mathrm{Ep}^{5}_{\frac{3}{2}}(G^{-1}) + \pi\sqrt{\det G}\left(\frac{\pi}{6} + {\sum_{\substack{Q\in\wedge^2\mathbb{Z}^5\\ Q\wedge Q=0}}}' \sigma_1(Q)\frac{e^{-2\pi\sqrt{G[Q]}}}{\sqrt{G[Q]}}e^{\pi i\,\mathrm{tr}\,BQ}\right)\ , \tag{6.7}$$

that defines the leading Wilson coefficient in the string theory effective action.

One difficulty is to generate an appropriate large set of rank two antisymmetric integer matrices $Q \in \wedge^2\mathbb{Z}^5$. To obtain a good approximation, we must include all the M2-instanton charges Q with a Euclidean action below some threshold

$$S[Q] = 2\pi\sqrt{-\tfrac{1}{2}\,\mathrm{tr}\,GQGQ} \leq \Lambda\ , \tag{6.8}$$

without including two many charges Q with a strictly greater Euclidean action. The incomplete sets of charges for a given action greater than Λ do not a priori spoil the approximation, but their inclusion increases the evaluation time of the function without providing a better approximation.

To compute a sample of charges we generate the set of rank two matrices with entries $|Q_{ij}| \leq 6$ and only keep the charges with a Euclidean action evaluated at the D_5 symmetric point bounded by $\Lambda = 2\pi\sqrt{6}$. The values of the action is $S = \pi\sqrt{1+4n}$ or $S = 2\pi\sqrt{n}$ for all integers $n \geq 1$. We have checked up to $S = 2\pi\sqrt{5}$ that the set of charges we have obtained define complete orbits of the Weyl group action $W(D_5)$. They all satisfy $|Q_{ij}| \leq 4$, so we believe they are the complete set of charges for $\Lambda = 2\pi\sqrt{5}$. One also checks that this set of charges gives all charges of action bounded by $\Lambda = 2\pi\sqrt{6}$ at the A_5 symmetric point with entries $|Q_{ij}| \leq 5$. The values of the action is then $S = 2\pi\sqrt{n}$ or $S = \pi\sqrt{3+4n}$ for all integers $n \geq 1$. We expect therefore this set of charges to provide a good approximation of the Epstein series on the fundamental domain.

We consider two surfaces, the first parametrised by

$$G = \frac{y}{2}\begin{pmatrix} 2&1&1&1&2\\ 1&2&1&1&2\\ 1&1&2&1&2\\ 1&1&1&2&2\\ 2&2&2&2&4 \end{pmatrix}\ , \qquad B = x\begin{pmatrix} 0&1&1&1&2\\ -1&0&1&1&2\\ -1&-1&0&1&2\\ -1&-1&-1&0&2\\ -2&-2&-2&-2&0 \end{pmatrix}\ , \tag{6.9}$$

only includes a global minimum at the D_5 symmetric point at $y = 1$ and $x = \frac{1}{2}$.

The second is parametrised by

$$G = \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 & 1 & 5x \\ 1 & 2 & 1 & 1 & 5x \\ 1 & 1 & 2 & 1 & 5x \\ 1 & 1 & 1 & 2 & 5x \\ 5x & 5x & 5x & 5x & \frac{y^2}{5}+20x^2 \end{pmatrix} , \qquad B = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & 5x \\ -1 & 0 & 1 & 1 & 5x \\ -1 & -1 & 0 & 1 & 5x \\ -1 & -1 & -1 & 0 & 5x \\ -5x & -5x & -5x & -5x & 0 \end{pmatrix} . \tag{6.10}$$

On this surface we find that the D_5 symmetric point is a global minimum at $y = 2$ and $x = \frac{2}{5}$. One also finds the local minimum at the A_5 symmetric point at $y = \sqrt{6}$ and $x = \frac{1}{5}$.

The numerical approximation gives the minimum of the Epstein series

$$\mathrm{Ep}^{5,5}_{\frac{3}{2}}(D_5) \approx -3.4447 \pm 0.0002 \, . \tag{6.11}$$

A Grenier domain boundaries

Consider the A_N symmetric point in $SO(N)\backslash SL(N)$ where $H = (N+1)^{-\frac{1}{N}} C_{A_N}$ is proportional to the A_N Cartan matrix. Its representative (4.31) in $SO(N)\backslash SL(N)/SL(N,\mathbb{Z})$ is in the Grenier domain. To exhibit that this is the case, one writes C_{A_N} as

$$C_{A_N}[n] = \sum_{k=0}^{N-1} \frac{k+2}{k+1}\left(n_{k+1} + \frac{1}{k+1}\sum_{i=k+2}^{N} n_i \right)^2 \tag{A.1}$$

and identifies the bilinear forms

$$Y_k[n] = \sum_{i=k+1}^{N} n_i^2 + \frac{2}{k+2}\sum_{k+1\le i<j\le N} n_i n_j \tag{A.2}$$

in the notation of Section 3.1. One finds from this formula that $x_{k+1,i} = \frac{1}{k+1} \in [0, \frac{1}{2}]$ and for each $Y_k[n]$, the minimal length vectors are length one, consistently with the definition of the Grenier domain. This point is on the boundary of the Grenier domain since the inequality

$$Y_0[n] \ge 1 \tag{A.3}$$

is saturated for all A_N root, i.e.

$$n_i = \pm\delta_{ij} \, , \qquad n_i = \delta_{ij} - \delta_{ik} \, , \tag{A.4}$$

for any j and k. For $Y_k[n]$ with $k \ge 1$, the only vectors of minimal length 1 are $n_i = \delta_{ij}$ for all $j \ge k+1$. In fact it is a boundary of dimension zero. To prove this one can write a generic point in $SO(N)\backslash SL(N)$ using the matrix [13]

$$Y_0[n] = n_1^2 + \sum_{i=2}^{N} y_i^2 n_i^2 + \sum_{i<j} x_{ij} n_i n_j \, . \tag{A.5}$$

[13] Take care that the coordinates y_i and x_{ij} are not the same as in (3.4).

Now, solving the condition that $Y_0[n] = 1$ for all roots of A_N, one obtains

$$y_i^2 = 1 \ , \qquad y_i^2 + y_j^2 - x_{ij} = 1 \ , \tag{A.6}$$

which determines $y_i = x_{ij} = 1$ and fixes the A_N point $H = (N+1)^{-\frac{1}{N}} C_{A_N}$.

One can consider similarly the symmetric point D_{N+1} in $SO(N)\backslash SL(N)$. Starting form (4.33) and shifting $n_2 \to n_2 - n_{N+1}$ one obtain the equivalent representative of the D_{N+1} Cartan matrix

$$C'_{D_{N+1}} = 2\sum_{i=1}^{N+1} n_i^2 + 2 \sum_{\substack{i<j \\ (i,j)\neq(2,N+1)}} n_i n_j \ , \tag{A.7}$$

where the second sum is over all pairs except $n_2 n_{N+1}$. One checks that $H = 2^{-\frac{2}{N+1}} C'_{D_{N+1}}$ is in the Grenier domain with the triangular form

$$\begin{aligned} Y_0 = &\Big(n_1 + \frac{1}{2}\sum_{i=2}^{N+1} n_i\Big)^2 + \frac{3}{4}\Big(n_2 + \frac{1}{3}\sum_{i=3}^{N} n_i - \frac{1}{3} n_{N+1}\Big)^2 \\ &+ \frac{1}{2}\sum_{k=2}^{N-1} \frac{k+2}{k+1}\Big(n_{k+1} + \frac{1}{k+1}\sum_{i=k+2}^{N} n_i + \frac{2}{k+2} n_{N+1}\Big)^2 + \frac{4}{N+1} n_{N+1}^2 \ , \end{aligned} \tag{A.8}$$

and the sub-components

$$\begin{aligned} Y_0 &= \sum_{i=1}^{N+1} n_i^2 + \sum_{\substack{i<j \\ (i,j)\neq(2,N+1)}} n_i n_j \\ Y_1 &= \sum_{i=2}^{N+1} n_i^2 + \frac{2}{3} \sum_{\substack{2\leq i<j \\ (i,j)\neq(2,N+1)}} n_i n_j - \frac{2}{3} n_2 n_{N+1} \ , \\ Y_{k\geq 2} &= \sum_{i=k+1}^{N+1} n_i^2 + \frac{2}{k+2} \sum_{k+1\leq i<j\leq N} n_i n_j + \frac{4}{k+2} n_{N+1} \sum_{i=k+1}^{N} n_i \ . \end{aligned} \tag{A.9}$$

The triangular form of Y_0 exhibits that all $x_{ij} \in [0, \frac{1}{2}]$ but $x_{2,N+1} = -\frac{1}{3} \in [-\frac{1}{2}, \frac{1}{2}]$. The sub-components all have minimum length vectors of length one, and H is therefore in the Grenier domain. This point sits on the boundary because the inequalities $Y_0[n] \geq 1$ are saturated for all D_{N+1} roots, i.e.

$$\begin{aligned} &n_i = \pm\delta_{i,j} \ , \qquad n_i = \delta_{i,j} - \delta_{i,k} \quad \text{for } (j,k) \neq (2, N+1) \text{ or } (N+1, 2) \ , \\ &n_i = \pm(\delta_{i,j} - \delta_{i,2} - \delta_{i,N+1}) \quad \text{for } j \neq 2, N+1 \ , \\ &n_i = \pm(\delta_{i,j} + \delta_{i,k} - \delta_{i,2} - \delta_{i,N+1}) \quad \text{for } j,k \neq 2, N+1 \ . \end{aligned} \tag{A.10}$$

For Y_k with $k = 1$ and $k \geq 3$, the only vectors of length 1 are $n_i = \delta_{i,j}$ for all $j \geq k+1$. For $k = 2$ one gets all vectors $n_i = \delta_{i,j}$ with $j \geq 3$ and the vectors $n_i = \pm\delta_{i,j} \mp \delta_{i,N+1}$ for $3 \leq i \leq N$.

Writing the equalities $Y_0[n] = 1$ for all D_{N+1} roots for the generic matrix

$$Y_0[n] = n_1^2 + \sum_{i=2}^{N} y_i^2 n_i^2 + \sum_{i<j} x_{ij} n_i n_j \tag{A.11}$$

one obtains

$$\begin{aligned} &y_i^2 = 1\ , \qquad y_i^2 + y_j^2 - x_{ij} = 1 \quad \text{for } (i,j) \neq (2, N+1)\ , \\ &y_i^2 + y_2^2 + y_{N+1}^2 - x_{i2} - x_{i\,N+1} + x_{2\,N+1} = 1 \quad \text{for } i \neq 2, N+1\ , \\ &y_i^2 + y_j^2 + y_2^2 + y_{N+1}^2 + x_{ij} - x_{i2} - x_{i\,N+1} - x_{j2} - x_{j\,N+1} + x_{2\,N+1} = 1 \quad \text{for } i,j \neq 2, N+1\ . \end{aligned} \tag{A.12}$$

This implies that $y_i = 1$ and $x_{ij} = 1$ except for $x_{2\,N+1} = 0$ and therefore $H = 2^{-\frac{2}{N+1}} C'_{D_{N+1}}$.

In general one does not expect reducible matrices to correspond to a zero dimensional boundary of the Grenier domain. Take as an example $H = (2N+2)^{-\frac{1}{N+1}} C_{A_N+A_1}$ with

$$C_{A_N+A_1}[n] = 2\sum_{i=1}^{N+1} n_i + 2\sum_{i<j\leq N} n_i n_j\ . \tag{A.13}$$

One checks similarly that H is in the Grenier domain, but the saturated inequalities still allow for a dimension N boundary parametrised by

$$Y_0[n] = \sum_{i=1}^{N+1} n_i + \sum_{i<j\leq N} n_i n_j + \sum_{i=1}^{N} x_i n_{N+1} n_i\ , \tag{A.14}$$

provided they satisfy

$$\sum_{i=1}^{k} x_i^2 - \frac{2}{k+1} \sum_{i<j\leq k} x_i x_j \leq 1\ , \tag{A.15}$$

for all k between 1 and N.

For the lattice A_5^{+2}, the matrix (4.35)

$$C_{A_5^{+2}} = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & \frac{10}{3} \end{pmatrix} \tag{A.16}$$

is not in the Grenier domain. But using the $SL(5,\mathbb{Z})$ matrix

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & -1 & -2 \end{pmatrix}\ , \tag{A.17}$$

one obtains that

$$C'_{A_5^{+2}} = \gamma^{\intercal} C_{A_5^{+2}} \gamma \tag{A.18}$$

is. This is manifest in the upper triangular form

$$C'_{A_5^{+2}}[n] = \frac{4}{3}\Bigg(\left(n_1 + \tfrac{1}{2}n_2 + \tfrac{1}{4}n_3 + \tfrac{1}{4}n_4 + \tfrac{1}{2}n_5\right)^2 + \frac{3}{4}\Bigg(\left(n_2 + \tfrac{1}{2}n_3 - \tfrac{1}{2}n_4\right)^2 + \left(n_3 + \tfrac{1}{2}n_4 + \tfrac{1}{2}n_5\right)^2 + \frac{3}{4}\Big(\left(n_4 + \tfrac{1}{3}n_5\right)^2 + \tfrac{8}{9}n_5^2 \Big) \Bigg) \Bigg) . \tag{A.19}$$

This bilinear form admits 30 vectors of norm square $\frac{4}{3}$ in $\mathbb{Z}^5$, the 15 weights in the Weyl orbit of the highest weights Λ_2 and the 15 in the Weyl orbit of Λ_4. The corresponding conditions $Y_0[n] = 1$ determine $C'_{A_5^{+2}}$, and the lattice A_5^{+2} is therefore at a dimension-zero boundary of the Grenier domain.

B Invariant polynomials at the A_N symmetric point

We prove here that there are indeed only two quadratic invariant polynomials at the A_N symmetric point by constructing them explicitly. We can easily see that for any polynomial $F(\tilde{p})$, its average on S_{N+1} defined by

$$\langle F(\tilde{p}) \rangle_{\mathrm{S}_{N+1}} = \frac{1}{|\mathrm{S}_{N+1}|} \sum_{\gamma \in \mathrm{S}_{N+1}} F(\gamma^{\intercal} \tilde{p} \gamma) \, , \tag{B.1}$$

is an S_{N+1}-invariant polynomial. Therefore by calculating the $\langle \tilde{p}_{ij} \rangle_{\mathrm{S}_{N+1}}$ and $\langle \tilde{p}_{ij} \tilde{p}_{rs} \rangle_{\mathrm{S}_{N+1}}$ we can generate all independent S_{N+1}-invariant linear and quadratic polynomials. We start with the linear polynomials where the calculations are less tedious but are completely analogous to the quadratic case. We can see that[14]

$$\langle \tilde{p}_{ij} \rangle_{\mathrm{S}_{N+1}} = \langle \langle \tilde{p}_{ij} \rangle_{\mathrm{S}_N} \rangle_{\mathbb{Z}_{N+1}} \tag{B.2}$$

It is easy to see that the average on S_N doesnâ€™t depend on i, j explicitly however the result will differ if $i = j$ or $i \neq j$. Therefore there are naively only two possible linear invariant polynomials. We can easily compute the two independent averages on S_N.

$$\langle \tilde{p}_{ii} \rangle_{\mathrm{S}_N} = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \tilde{p}_{\sigma^{-1}(i)\sigma^{-1}(i)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{p}_{ii} \tag{B.3}$$

$$\langle \tilde{p}_{ij} \rangle_{\mathrm{S}_N} = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \tilde{p}_{\sigma^{-1}(i)\sigma^{-1}(j)} = \frac{2}{N(N-1)} \sum_{i<j} \tilde{p}_{ij} \tag{B.4}$$

[14] Actually it is not obvious that we don't need to take another average on S_N in the end but it turns out that this is indeed not the case

where $i \neq j$ in the second line. Hence all that remains is to calculate $\langle \tilde{p}_{ij} \rangle_{\mathbb{Z}_{N+1}}$. If we use the convention that the indices i, j of $\tilde{p}_{ij}$ are taken modulo $N+1$ we can write an arbitrary number of iterations of the transformation σ_f in the simplified form

$$\sigma_f^k . \tilde{p}_{ij} = \tilde{p}_{N-k+1N-k+1} + \tilde{p}_{i-kj-k} - \tilde{p}_{i-kN-k+1} - \tilde{p}_{j-kN-k+1} \tag{B.5}$$

for all $k \geq 0$. We find that

$$\langle \tilde{p}_{ij} \rangle_{\mathbb{Z}_{N+1}} = \frac{1}{N+1} \left(\sum_{k=1}^{N} \tilde{p}_{kk} + \sum_{k=1}^{N+1} \tilde{p}_{k+ik+j} - \sum_{k=1}^{N} \tilde{p}_{k+ik} - \sum_{k=1}^{N} \tilde{p}_{k+jk} \right) \tag{B.6}$$

for any i, j. Hence we find that

$$\langle \tilde{p}_{ii} \rangle_{\mathrm{S}_{N+1}} = \frac{1}{N} \sum_{i=1}^{N} \langle \tilde{p}_{ii} \rangle_{\mathbb{Z}_{N+1}} = \frac{2}{N(N+1)} \left(N \sum_{i=1}^{N} \tilde{p}_{ii} - 2 \sum_{i<j} \tilde{p}_{ij} \right) = 0 \tag{B.7}$$

$$\langle \tilde{p}_{ij} \rangle_{\mathrm{S}_{N+1}} = \frac{2}{N(N-1)} \sum_{i<j} \langle \tilde{p}_{ij} \rangle_{\mathbb{Z}_{N+1}} = \frac{1}{N(N+1)} \left(N \sum_{i=1}^{N} \tilde{p}_{ii} - 2 \sum_{i<j} \tilde{p}_{ij} \right) = 0 \tag{B.8}$$

where we have used the traceless condition and where $i \neq j$ in the second line. Therefore there are no linear S_{N+1}-invariant polynomials of $\tilde{p}$. This means that C_{A_N} as defined in (4.30) is an extremum of any automorphic function on $SO(N) \backslash SL(N)$.

We will now apply the same reasoning to show that there are only two independent quadratic S_{N+1}-invariant polynomials of $\tilde{p}$. We can show that

$$\begin{aligned}
\sigma_f^k . \tilde{p}_{ij} \tilde{p}_{rs} =& \sigma_f^k . \tilde{p}_{ij} \, \sigma_f^k . \tilde{p}_{rs} \\
=& (\tilde{p}_{N-k+1N-k+1})^2 \\
& - \tilde{p}_{N-k+1N-k+1} (\tilde{p}_{i-kN-k+1} + \tilde{p}_{j-kN-k+1} + \tilde{p}_{r-kN-k+1} + \tilde{p}_{s-kN-k+1}) \\
& + \tilde{p}_{N-k+1N-k+1} (\tilde{p}_{i-kj-k} + \tilde{p}_{r-ks-k}) + (\tilde{p}_{i-kN-k+1} + \tilde{p}_{j-kN-k+1})(\tilde{p}_{r-kN-k+1} + \tilde{p}_{s-kN-k+1}) \\
& - \tilde{p}_{i-kj-k} (\tilde{p}_{r-kN-k+1} + \tilde{p}_{s-kN-k+1}) - \tilde{p}_{r-ks-k} (\tilde{p}_{i-kN-k+1} + \tilde{p}_{j-kN-k+1}) \\
& + \tilde{p}_{i-kj-k} \tilde{p}_{r-ks-k}
\end{aligned} \tag{B.9}$$

For all i, j, r, s. Let us now calculate $\langle \tilde{p}_{ij} \tilde{p}_{rs} \rangle_{\mathbb{Z}_{N+1}}$

$$\begin{aligned}\langle\tilde{p}_{ij}\tilde{p}_{rs}\rangle_{\mathbb{Z}_{N+1}} =&\frac{1}{N+1}\left(\sum_{k=1}^{N}(\tilde{p}_{kk})^2-\sum_{k=1}^{N}\tilde{p}_{kk}(\tilde{p}_{ik+ik}+\tilde{p}_{k+jk}+\tilde{p}_{k+rk}+\tilde{p}_{k+sk})\right.\\&+\sum_{k=1}^{N}\tilde{p}_{kk}(\tilde{p}_{k+ik+j}+\tilde{p}_{k+rk+s})+\sum_{k=1}^{N}(\tilde{p}_{k+ik}+\tilde{p}_{k+jk})(\tilde{p}_{k+rk}+\tilde{p}_{k+sk})\\&-\sum_{k=1}^{N}\tilde{p}_{k+ik+j}(\tilde{p}_{k+rk}+\tilde{p}_{k+sk})-\sum_{k=1}^{N}\tilde{p}_{k+rk+s}(\tilde{p}_{k+ik}+\tilde{p}_{k+jk})\\&\left.+\sum_{k=1}^{N+1}\tilde{p}_{k+ik+j}\tilde{p}_{k+rk+s}\right)\end{aligned}\tag{B.10}$$

for all i,j,r,s. We can now calculate the average on S_{N+1} in an analogous way to the linear case. Just like the linear case we can see that the average on S_N will not depend on i,j,r,s explicitly, however the result will differ depending on if some indices are equal to each other. One can see that there are 7 different averages to compute: $\langle(\tilde{p}_{ii})^2\rangle_{\mathrm{S}_{N+1}}$, $\langle\tilde{p}_{ii}\tilde{p}_{ij}\rangle_{\mathrm{S}_{N+1}}$, $\langle\tilde{p}_{ii}\tilde{p}_{jj}\rangle_{\mathrm{S}_{N+1}}$, $\langle(\tilde{p}_{ij})^2\rangle_{\mathrm{S}_{N+1}}$, $\langle\tilde{p}_{ii}\tilde{p}_{jr}\rangle_{\mathrm{S}_{N+1}}$, $\langle\tilde{p}_{ij}\tilde{p}_{ir}\rangle_{\mathrm{S}_{N+1}}$ and $\langle\tilde{p}_{ij}\tilde{p}_{rs}\rangle_{\mathrm{S}_{N+1}}$ where all indices are assumed to be different. We can decompose each of them on a basis of S_N invariant polynomials as

$$\begin{aligned}&a_1\sum_{i=1}^{N}(\tilde{p}_{ii})^2+a_2\sum_{i<j}\tilde{p}_{ii}\tilde{p}_{jj}+a_3\sum_{i\neq j}\tilde{p}_{ii}\tilde{p}_{ij}+a_4\sum_{i<j}(\tilde{p}_{ij})^2\\&+a_5\sum_{i=1}^{N}\sum_{\substack{j<l\\j\neq i\\l\neq i}}\tilde{p}_{ii}\tilde{p}_{jl}+a_6\sum_{i=1}^{N}\sum_{\substack{j<l\\j\neq i\\l\neq i}}\tilde{p}_{ij}\tilde{p}_{il}+a_7\sum_{\substack{i\neq j\neq m\neq l\\i<j\\m<l\\i<m}}\tilde{p}_{ij}\tilde{p}_{ml}\end{aligned}\tag{B.11}$$

For $N>2$ we can check that only three out of the seven averages are actually independent and they are linearly dependent with the Killing-Cartan form

$$\begin{aligned}\operatorname{tr} C_{A_N}^{-1}\,\tilde{p}\,C_{A_N}^{-1}\,\tilde{p}=&\frac{4}{(N+1)^2}\left(N^2\sum_{i=1}^{N}(\tilde{p}_{ii})^2+2\sum_{i<j}p_{ii}\tilde{p}_{jj}\right.\\&-4N\sum_{i\neq j}\tilde{p}_{ii}\tilde{p}_{ij}+2(N^2+1)\sum_{i<j}(\tilde{p}_{ij})^2\\&\left.+4\sum_{i=1}^{N}\sum_{\substack{j<l\\j\neq i\\l\neq i}}\tilde{p}_{ii}\tilde{p}_{jl}-4(N-1)\sum_{i=1}^{N}\sum_{\substack{j<l\\j\neq i\\l\neq i}}\tilde{p}_{ij}\tilde{p}_{il}+8\sum_{\substack{i\neq j\neq m\neq l\\i<j\\m<l\\i<m}}\tilde{p}_{ij}\tilde{p}_{ml}\right)\end{aligned}\tag{B.12}$$

as well as

$$\mathrm{tr}(C_{A_N}^{-1}\tilde{p})^2 = \frac{4}{(N+1)^2}\Bigg(N^2\sum_{i=1}^{N}(\tilde{p}_{ii})^2 + 2N^2\sum_{i<j}\tilde{p}_{ii}\tilde{p}_{jj} - 4N\sum_{i\neq j}\tilde{p}_{ii}\tilde{p}_{ij}$$
$$+4\sum_{i<j}(\tilde{p}_{ij})^2 - 4N\sum_{i=1}^{N}\sum_{\substack{j<l\\ j\neq i\\ l\neq i}}\tilde{p}_{ii}\tilde{p}_{jl} + 8\sum_{i=1}^{N}\sum_{\substack{j<l\\ j\neq i\\ l\neq i}}\tilde{p}_{ij}\tilde{p}_{il} + 8\sum_{\substack{i\neq j\neq m\neq l\\ i<j\\ m<l\\ i<m}}\tilde{p}_{ij}\tilde{p}_{ml}\Bigg) = 0 \qquad \text{(B.13)}$$

which vanishes once the traceless condition is imposed. This proves that there are two independent non trivial quadratic S_{N+1}-invariant polynomials of $\tilde{p}$. The other non trivial independent polynomial is given by one of the averages, we give the simplest one

$$\langle(\tilde{p}_{ii})^2\rangle_{\mathrm{S}_{N+1}} = \frac{2}{(N+1)N}\left(N\sum_{i=1}^{N}(\tilde{p}_{ii})^2 - 4\sum_{i\neq j}\tilde{p}_{ii}\tilde{p}_{ij} + 2\sum_{i<j}\tilde{p}_{ii}\tilde{p}_{jj} + 4\sum_{i<j}(\tilde{p}_{ij})^2\right) \qquad \text{(B.14)}$$

In the case $N = 2$ only the first four averages exist and they are all proportional to the Killing-Cartan form once the traceless condition is imposed. Therefore we can recover the well known result that A_2 is a local minimum (and indeed the global minimum) for the $SL(2)$ Eisenstein series.

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ECOLE DOCTORALE

Titre : Contraintes sur l'action des théories effectives en gravitation quantique

Mots clés : Gravitation quantique, Théorie des cordes, Supergravité

Résumé : La théorie des cordes constitue l'un des cadres les plus populaires et étudiés pour aborder la gravitation quantique. Il est bien connu que la limite basse énergie de la théorie des cordes donne une large gamme de théories des champs effectives. Une manière récente et prometteuse d'extraire des informations sur la gravitation quantique à partir de ce paysage de cordes est le programme du "marais". Le principe du "lampadaire des cordes" postule que le paysage de la gravité quantique et le paysage des cordes coïncident. Dans cette thèse, nous proposons d'étudier cette affirmation dans le cas des couplages de Wilson d'ordre supérieur dans le cadre très restreint de la supersymétrie maximale.
Nous étudions d'abord la limite basse énergie des amplitudes de genre 0, 1 et 2 pour la théorie des cordes de type II compactifiée sur un tore et les comparons respectivement aux amplitudes de supergravité maximale à l'ordre des arbres, à 1 boucle et à 2 boucles. Cela nous permet de calculer les contributions perturbatives aux coefficients de Wilson dominants de la théorie des cordes maximalement supersymétrique. Nous montrons également qu'en dimension 8, les divergences logarithmiques des amplitudes de supergravité peuvent être liées aux divergences des couplages de Wilson. Nous donnons une prescription pour régulariser correctement la divergence en utilisant l'amplitude de corde finie.
Nous utilisons ensuite les équations différentielles induites par les identités de Ward supersymétriques ainsi que les contraintes imposées par la U-dualité pour dériver les coefficients de Wilson non perturbatifs complets pour la théorie des cordes maximalement supersymétrique en dimensions supérieures ou égales à 6. Ceux-ci sont donnés pour les coefficients de Wilson dominant et subdominants par des séries d'Eisenstein et d'Epstein pour le groupe de U-dualité pertinent. Les développements de Fourier paraboliques de ces séries peuvent alors être utilisés pour vérifier les différentes limites de dégénérescence des coefficients de Wilson.
Enfin, nous étudions les minima de ces fonctions sur l'espace des modules pour donner des bornes inférieures sur les coefficients de Wilson provenant de la théorie des cordes maximalement supersymétrique. Cela implique de trouver les minima des séries d'Epstein pour des valeurs spéciales du paramètre s. Nous étendons d'abord la construction récursive de Grenier d'un domaine fondamental à presque tous les groupes de Lie simples, ce qui nous permet de définir correctement le domaine d'étude de nos fonctions. Nous montrons ensuite que les points symétriques sont nécessairement des extrêma des formes automorphes et donnons des critères précis pour qu'ils soient des minima. Nous identifions aussi ces points symétriques comme des coins de domaines fondamentaux. Nous étudions des points symétriques pertinents pour le cas des groupes $SL(n)$ et $SO(n,n)$ et donnons des arguments supplémentaires portant sur la densité concernant les minima globaux pour des valeurs élevées du paramètre s. Nous avons ensuite vérifié notre conjecture numériquement pour les cas $n = 5$ pertinents pour les dimensions 7 et 6.
Ces bornes inférieures devraient ensuite être comparées aux bornes inférieures provenant des contraintes d'unitarité en utilisant des méthodes de bootstrap de la matrice S. À notre connaissance, cette analyse reste encore à être effectuée en dimensions inférieures ou égales à 8. Nous avons montré qu'en dimension 6, les propriétés de factorisation des amplitudes maximalement supersymétriques impliquent que les propriétés d'unitarité des super-amplitudes se réduisent aux propriétés d'unitarité des amplitudes scalaires. Ces types de factorisations existent également dans d'autres dimensions mais ne conduisent pas toujours à de telles simplifications drastiques. Cependant, on peut toujours se restreindre à la diffusion élastique pour rendre les calculs numériques plus abordables. Ce type d'analyse, s'il réussit, constituerait un argument en faveur de la validité du principe du "lampadaire des cordes" dans le cas de la supersymétrie maximale.

Title : Constraints on the action of effective theories in quantum gravity

Keywords : Quantum gravity, String theory, Supergravity

Abstract : String theory constitutes one of the most popular and studied framework to approach quantum gravity. It is well known that the low energy limit of string theory gives a wide range of effective field theories. One recent and promising way to extract information about quantum gravity from this string landscape has been the swampland program. The string lamppost principle postulates that the quantum gravity landscape and the string landscape coincide. In this thesis we propose to study this claim for the case of higher order Wilson couplings in the very restricted case of maximal supersymmetry.
We first study the low energy limit of genus 0, 1 and 2 string amplitudes for type II string theory compactified on a torus and compare them to tree level, 1-loop and 2-loop maximal supergravity amplitudes respectively. This allows us to compute the perturbative contributions to the leading Wilson coefficients of maximally supersymmetric string theory. We also show that in dimension 8 logarithmic divergences of the supergravity amplitudes can be linked to divergences of Wilson couplings. We give a prescription to properly regularise the divergence by using the finite string amplitude.
We then use the differential equations entailed by the supersymmetric Ward identities as well as the constraints imposed by U-duality to derive the full non perturbative Wilson coefficients for maximally supersymmetric string theory in dimensions higher or equal to 6. These are given for the leading and next to leading Wilson coefficients by Eisenstein and Epstein series for the relevant U-duality group. The parabolic Fourier expansions of these series can then be used to check the different degeneration limits of the Wilson coefficients.
Finally we study the minima of these functions on moduli space to give lower bounds on Wilson coefficients coming from maximally supersymmetric string theory. This implies finding the minima of Epstein series for special values of the s parameter. We first extend Grenier's recursive construction of a fundamental domain to almost any simple Lie group which allows us to properly define the domain of study of our functions. We then show that symmetric points are necessarily extrema of automorphic forms and give precise criteria for them to be minima. We also identify these symmetric points as corners of fundamental domains. We study relevant symmetric points for the case of $SL(n)$ and $SO(n,n)$ groups and give additional density arguments regarding the global minima for large s parameter. We then checked our conjecture numerically for the cases $n = 5$ relevant for dimensions 7 and 6.
These lower bounds should then be compared to lower bounds coming from unitarity constraints using S-matrix bootstrap methods. As far as we know this analysis still needs to be performed in dimensions lower or equal to 8. We have shown that in dimension 6 factorisation properties of maximally supersymmetric amplitudes imply that the unitarity properties of superamplitudes reduce to the unitarity properties of scalar amplitudes. These kind of factorisations also exist in other dimensions but do not always lead to such drastic simplifications. However one can always restrict to elastic scattering to make the numerics bearable. This kind of analysis, if successful, would be a strong argument for the validity of the string lamppost principle in the case of maximal supersymmetry.

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