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Thèse de doctorat



Generic Vanishing for Holonomic \mathcal{D} -Modules: A Study via Cartier Duality

Thèse de doctorat de l'Institut polytechnique de Paris
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Thèse présentée et soutenue à Palaiseau, le 08 novembre 2024, par

GABRIEL RIBEIRO

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¹This concept appears in Definition 3.3.1.

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And just like the rain, you cast the dust into nothing.

This thesis is dedicated to Pierre Cartier (1932–2024),
whose life's work showed us that mathematics is more
than logic—it is the art of finding harmony in contrast.

Rien n'est plus fécond,
tous les mathématiciens le savent,
que ces obscures analogies,
ces troubles reflets d'une théorie à une autre,
ces furtives caresses,
ces brouilleries inexplicables ;
rien aussi ne donne plus de plaisir au chercheur.

—André Weil

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INTRODUCTION

MOTIVATIONS AND MAIN RESULTS

In this chapter, we outline the historical context that led to the development of the results presented in this thesis. Furthermore, we provide a brief overview of our main findings, situating them within the broader mathematical landscape.

EXPONENTIAL SUMS OVER FINITE FIELDS

Exponential sums over finite fields are ubiquitous in modern analytic number theory. Much of their study revolves around two fundamental questions:

- (1) For an exponential sum S , what are *good* upper bounds for $|S|$?
- (2) For a family $\{S_i\}_{i \in I}$ of exponential sums, how do the complex numbers S_i vary as i ranges through the set I ?

As we will soon see, these questions are closely intertwined. We will illustrate this theory with a particularly enlightening example, which we learned from our advisor Fresán, who also presented it in his first talk at the Séminaire Bourbaki [Fre19].

Let p be a prime number, q a power of p , and fix a non-trivial additive character $\psi_p: \mathbb{F}_p \rightarrow \mathbb{C}^\times$. By precomposing with the trace map $\mathbb{F}_q \rightarrow \mathbb{F}_p$, we obtain the additive character $\psi_q: \mathbb{F}_q \rightarrow \mathbb{C}^\times$. For each multiplicative character $\chi: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$, the *Gauss sum* $g(\psi_q, \chi)$ is defined as

$$g(\psi_q, \chi) := \sum_{x \in \mathbb{F}_q^\times} \psi_q(x) \chi(x).$$

Question (1) has a straightforward answer for Gauss sums. Due to the orthogonality of characters, the absolute value of $g(\psi_q, \chi)$ is 1 when χ is the trivial character, and \sqrt{q} otherwise. In contrast, Question (2) is far more intriguing. There are $q - 2$ angles $\theta_{q, \chi} := g(\psi_q, \chi) / \sqrt{q}$ on the unit circle S^1 , one for each non-trivial character χ .

How are these angles distributed on S^1 as q tends towards infinity?

Although Gauss sums are some of the simplest examples of exponential sums, this question is profound. Its solution only became feasible after Deligne proved the Riemann hypothesis for varieties over finite fields.

THEOREM (Deligne, Katz). As q tends to infinity, the angles $\theta_{q,\chi}$ for $\chi \neq 1$ become equidistributed on S^1 with respect to its normalized Haar measure. In other words, the equation

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = \lim_{q \rightarrow \infty} \frac{1}{q-2} \sum_{\chi \neq 1} f(\theta_{q,\chi})$$

holds for all continuous functions $f: S^1 \rightarrow \mathbb{C}$.

Beginning of the proof. As Laurent polynomials are dense in $\mathcal{C}(S^1)$, it suffices to consider functions of the form $f(z) = z^n$ for $n \in \mathbb{Z}$. The case $n = 0$ is trivial, and the relation $g(\psi_q, \chi)g(\bar{\psi}_q, \bar{\chi}) = q$ allows us to assume $n \geq 1$. In this case, the integral always vanishes, so it remains to prove that the sequence

$$\frac{1}{q-2} \sum_{\chi \neq 1} f(\theta_{q,\chi}) = \frac{1}{q^{n/2}(q-2)} \sum_{\chi \neq 1} g(\psi_q, \chi)^n$$

tends to zero as q goes to infinity. Expanding the expression for $g(\psi_q, \chi)^n$, we obtain

$$\begin{aligned} g(\psi_q, \chi)^n &= \sum_{x_1, \dots, x_n \in \mathbb{F}_q^\times} \psi_q(x_1 + \dots + x_n) \chi(x_1 \dots x_n) \\ &= \sum_{a \in \mathbb{F}_q^\times} \chi(a) \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q^\times \\ x_1 \dots x_n = a}} \psi_q(x_1 + \dots + x_n). \end{aligned}$$

That is, the map $\chi \mapsto g(\psi_q, \chi)^n$ is the Fourier transform of the Kloosterman sums

$$a \mapsto \text{Kl}_n(a, q) := \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q^\times \\ x_1 \dots x_n = a}} \psi_q(x_1 + \dots + x_n).$$

Thus, the solution to Question (2) for Gauss sums depends on resolving Question (1) for Kloosterman sums. *To be continued...*

Kloosterman himself needed to estimate the sums that now bear his name in [Klo27], but specifically for the case $n = 2$. By calculating the fourth moment,

$$\sum_{a \in \mathbb{F}_q^\times} \text{Kl}_2(a, q)^4 = 2q^3 - 3q^2 - 3q - 1,$$

he was able to conclude that $|\text{Kl}_2(a, q)| < 2q^{3/4}$. However, determining the optimal upper bound for $|\text{Kl}_n(a, q)|$, which is necessary to complete the proof above, required the development of étale cohomology and Deligne's subsequent work—advancements that took another fifty years.

To elucidate the role of étale cohomology in this story, let us first review some of its key features. For a connected algebraic variety X over $k = \mathbb{F}_q$, Grothendieck defined

the *étale fundamental group* $\pi_1(X)$, a profinite group that classifies finite étale covers of X . Given a finite extension E/k , the étale fundamental group of $X = \text{Spec } E$ is simply the absolute Galois group $\text{Gal}(E)$ —the free profinite group $\widehat{\mathbb{Z}}$, with a generator given by

$$\bar{E} \rightarrow \bar{E}, \quad x \mapsto x^{|\bar{E}|}.$$

This is known as the *arithmetic Frobenius*. Its inverse, denoted by Frob_E and referred to as the *geometric Frobenius*, plays a crucial role in the theory.

Henceforth, we fix a prime number $\ell \neq p$, an algebraic closure of \mathbb{Q}_ℓ , and an isomorphism between $\overline{\mathbb{Q}_\ell}$ and \mathbb{C} . Consider an ℓ -adic local system \mathcal{L} of rank r on X , characterized as a continuous representation $\rho: \pi_1(X) \rightarrow \text{GL}_r(\overline{\mathbb{Q}_\ell})$. For each point $x \in X(E)$, the natural functoriality of the étale fundamental group induces a map

$$\text{Gal}(E) \simeq \pi_1(\text{Spec } E) \rightarrow \pi_1(X).$$

We denote by $\text{Frob}_{E,x}$ the image of Frob_E under this morphism. Since this construction is canonical only up to the choice of a base point, $\text{Frob}_{E,x}$ represents a conjugacy class in $\pi_1(X)$. Consequently, $\rho(\text{Frob}_{E,x})$ forms a conjugacy class in $\text{GL}_r(\overline{\mathbb{Q}_\ell})$, and we can take its trace, denoted by $\text{tr}(\text{Frob}_{E,x} | \mathcal{L})$.

Grothendieck also introduced the cohomology groups $H^i(X_{\bar{k}}, \mathcal{L})$ and the compactly supported cohomology groups $H_c^i(X_{\bar{k}}, \mathcal{L})$. These are finite-dimensional $\overline{\mathbb{Q}_\ell}$ -vector spaces, equipped with an action of $\text{Gal}(k)$, and vanish outside the range $0 \leq i \leq 2d$, where d is the dimension of X . The action of the geometric Frobenius on an ℓ -adic local system and on its compactly supported cohomology are related by the *Grothendieck trace formula*:

$$\sum_{x \in X(E)} \text{tr}(\text{Frob}_{E,x} | \mathcal{L}) = \sum_{i=0}^{2d} (-1)^i \text{tr}(\text{Frob}_E | H_c^i(X_{\bar{k}}, \mathcal{L})).$$

This formula forms a key component of a powerful method for estimating exponential sums over finite fields. First, an ℓ -adic local system \mathcal{L} is constructed so that the left-hand side of the Grothendieck trace formula corresponds to the given exponential sum. Next, the eigenvalues of the geometric Frobenius acting on $H_c^i(X_{\bar{k}}, \mathcal{L})$ are estimated. This estimation can be accomplished using Deligne's theory of weights.

DEFINITION. We say that \mathcal{L} is *pure* if there exists an integer w , its *weight*, such that for every finite extension E/k and every $x \in X(E)$, the eigenvalues of $\text{Frob}_{E,x}$ acting on \mathcal{L} have absolute value $|E|^{w/2}$.

The ℓ -adic local system \mathcal{L} is called *mixed* if it admits a finite filtration where each successive quotient is pure. The weights of these non-zero quotients are said to be the *weights* of \mathcal{L} .

The same terminology extends to the cohomology groups $H^i(X_{\bar{k}}, \mathcal{L})$ and $H_c^i(X_{\bar{k}}, \mathcal{L})$. The main result in Deligne's seminal paper [Del80] compares the weights of an ℓ -adic local system with those of its cohomology groups.

THEOREM (Deligne). *If \mathcal{L} is mixed of weights $\leq w$, then the compactly supported cohomology group $H_c^i(X_{\bar{k}}, \mathcal{L})$ is mixed of weights $\leq w + i$.*

For smooth X , Poincaré duality implies that if \mathcal{L} is mixed of weights $\geq w$, then $H^i(X_{\bar{k}}, \mathcal{L})$ is mixed of weights $\geq w + i$. This estimation is particularly useful when the forget-supports map $H_c^i(X_{\bar{k}}, \mathcal{L}) \rightarrow H^i(X_{\bar{k}}, \mathcal{L})$ is an isomorphism, in which case these cohomology groups are pure of weight $w + i$.

We now apply the aforementioned method to estimate Kloosterman sums. Consider the morphism of k -schemes $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ defined by $x \mapsto x - x^q$. This is a finite étale Galois cover with Galois group \mathbb{F}_q . Since $\pi_1(\mathbb{A}_k^1)$ is the limit of the Galois groups of all finite étale Galois covers, we obtain a natural surjection $\pi_1(\mathbb{A}_k^1) \twoheadrightarrow \mathbb{F}_q$. As a result, the composition

$$\pi_1(\mathbb{A}_k^1) \twoheadrightarrow \mathbb{F}_q \xrightarrow{\psi_q} \overline{\mathbb{Q}}_\ell^\times$$

corresponds to a rank-one ℓ -adic local system \mathcal{L}_{ψ_q} on \mathbb{A}_k^1 , whose trace at a point $x \in \mathbb{A}_k^1(k) = \mathbb{F}_q$ is simply $\psi_q(x)$.

Fix $\mathfrak{a} \in \mathbb{F}_q^\times$, and let X be the zero locus of $x_1 \dots x_n - \mathfrak{a}$ in \mathbb{A}_k^n . Define $f: X \rightarrow \mathbb{A}_k^1$ as the map that sends (x_1, \dots, x_n) to their sum, so that

$$\text{Kl}_n(\mathfrak{a}, q) = \sum_{x \in X(k)} \text{tr}(\text{Frob}_{k,x} | f^* \mathcal{L}_{\psi_q}) = \sum_{i=0}^{2n} (-1)^i \text{tr}(\text{Frob}_k | H_c^i(X_{\bar{k}}, f^* \mathcal{L}_{\psi_q})).$$

In [SGA 4 $\frac{1}{2}$, Thm. 7.4], Deligne computed these cohomology groups, showing that H_c^i vanishes for $i \neq n - 1$ and that $H_c^{n-1} \simeq H^{n-1}$ is n -dimensional. Moreover, since the values of ψ_q are p -th roots of unity, the ℓ -adic local system $f^* \mathcal{L}_{\psi_q}$ is pure of weight 0. Deligne's theorem then implies that

$$|\text{Kl}_n(\mathfrak{a}, q)| = |\text{tr}(\text{Frob}_k | H_c^{n-1}(X_{\bar{k}}, f^* \mathcal{L}_{\psi_q}))| \leq nq^{(n-1)/2},$$

which is the optimal bound.

We are now in a position to complete the proof of the equidistribution theorem for Gauss sums.

End of the proof. By summing the obtained expression for $g(\psi_q, \chi)^n$ over the non-trivial multiplicative characters χ , we get

$$\begin{aligned} \sum_{\chi \neq 1} g(\psi_q, \chi)^n &= -g(\psi_q, 1)^n + \sum_{\mathfrak{a} \in \mathbb{F}_q^\times} \text{Kl}_n(\mathfrak{a}, q) \sum_{\chi} \chi(\mathfrak{a}) \\ &= (-1)^{n+1} + (q-1) \text{Kl}_n(1, q). \end{aligned}$$

Deligne's bound leads to the estimate

$$\left| \frac{1}{q^{n/2}(q-2)} \sum_{\chi \neq 1} g(\psi_q, \chi)^n \right| \leq \frac{2n+1}{\sqrt{q}},$$

which tends to zero as q approaches infinity. \square

The key property of Gauss sums that enabled the preceding proof is that $\chi \mapsto g(\psi_q, \chi)$ is the Fourier transform of the trace function of an ℓ -adic local system. This property can be generalized in two directions. First, instead of considering characters χ of $\mathbb{F}_q^\times = \mathbb{G}_m(k)$, one can extend the scope to characters of $G(k)$, where G is a commutative connected algebraic group over k . Similar to the construction above, for each such character $\chi \in \widehat{G}(k)$, there exists a rank-one ℓ -adic local system \mathcal{L}_χ on G whose trace at any point $x \in G(k)$ is equal to $\chi(x)$.

Local systems can also be generalized to a broader class of ℓ -adic sheaves known as *perverse sheaves*. Given a finite extension E/k and a perverse sheaf M on G that is pure of weight zero, we define the exponential sum

$$S(M, E, \chi) := \sum_{x \in G(E)} \text{tr}(\text{Frob}_{E,x} | M) \chi(x) = \sum_{x \in G(E)} \text{tr}(\text{Frob}_{E,x} | M \otimes \mathcal{L}_\chi).$$

As shown in [FFK23, Thm. 3.28], there exists a linear algebraic group $\text{Conv}(M)$ over $\overline{\mathbb{Q}}_\ell$ that encapsulates the symmetries of this exponential sum. This group is defined as a tannakian group associated with a certain quotient category of perverse sheaves.

Let K be a maximal compact subgroup of $\text{Conv}(M)_{\text{an}}$. Since $\text{Conv}(M)$ is linear, there exists an integer $r \geq 0$ and a closed embedding $\text{Conv}(M) \hookrightarrow \text{GL}_r$. Denote by μ the normalized Haar measure on K , and by ν its direct image by the trace function. We now present the main theorem from the recent monograph [FFK23].

THEOREM (Forey, Fresán, Kowalski). *As the degree of E/k tends to infinity, the exponential sums $S(M, E, \chi)$ for $\chi \in \widehat{G}(E)$ become ν -equidistributed on average. In other words, the equation*

$$\int_K f(\text{tr}(x)) d\mu(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{[E:k] \leq n} \frac{1}{|G(E)|} \sum_{\chi \in \widehat{G}(E)} f(S(M, E, \chi))$$

holds for all bounded continuous functions $f: \mathbb{C} \rightarrow \mathbb{C}$.

This result not only encompasses the equidistribution theorem for Gauss sums but also extends many other related theorems. Recall that, according to the Grothendieck trace formula, the exponential sum $S(M, E, \chi)$ is given by an alternating sum of traces of Frobenii acting on cohomology groups:

$$\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(\text{Frob}_E | H_c^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)).$$

As before, the proof of this theorem relies on the fact that, for most choices of χ , all but one of these cohomology groups vanish, significantly simplifying the sum. Moreover, the only non-vanishing compactly supported cohomology group coincides with its non-compactly supported counterpart, enabling the application of Deligne's weight estimates.

THEOREM (Forey, Fresán, Kowalski). *Let G be a commutative connected algebraic group over k of dimension d , and let M be a perverse sheaf on G . The set S of characters $\chi \in \widehat{G}$ that satisfy*

$$\begin{aligned} H_c^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) &= H^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) = 0 \text{ for } i \neq 0; \\ H_c^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) &\simeq H^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) \end{aligned}$$

is generic, meaning there exists a real number $c \geq 0$ such that $|\widehat{G}(E) - S(E)| \leq c|E|^{d-1}$ for every finite extension E/k .

EXPONENTIAL SUMS AND DIFFERENTIAL EQUATIONS

Having applied ℓ -adic methods to solve Question (1) for the Kloosterman sums $\text{Kl}_n(a, q)$, we now turn to the distribution of these sums as $a \in \mathbb{F}_q^\times$ varies and q tends to infinity. For simplicity, we consider the case $n = 2$. Utilizing the previously established bound, and noting that the sums $\text{Kl}_2(a, q)$ are real numbers, it follows that there exists a unique angle $\theta_{q,a} \in [0, \pi]$ such that $\text{Kl}_2(a, q) = 2\sqrt{q} \cos \theta_{q,a}$.

How are these angles distributed on $[0, \pi]$ as q tends towards infinity?

THEOREM (Katz). *As q tends to infinity, the angles $\theta_{q,a}$ for $a \in \mathbb{F}_q^\times$ become equidistributed on $[0, \pi]$ with respect to the Sato–Tate measure $(2/\pi) \sin^2 \theta \, d\theta$. In other words, the equation*

$$\frac{2}{\pi} \int_0^\pi f(\theta) \sin^2 \theta \, d\theta = \lim_{q \rightarrow \infty} \frac{1}{q-1} \sum_{a \in \mathbb{F}_q^\times} f(\theta_{q,a})$$

holds for all continuous functions $f: [0, \pi] \rightarrow \mathbb{C}$.

Sketch of proof. We explain how this result fits within the framework of the equidistribution theorem established by Forey, Fresán, and Kowalski. Recall that each character $\chi \in \widehat{\mathbb{F}_q}$ can be expressed as $x \mapsto \psi_q(ax)$, for a unique $a \in \mathbb{F}_q$. This correspondence leads to the following equality:

$$\left\{ \text{Kl}_2(a, q) = \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(ax + \frac{1}{x}\right) \mid a \in \mathbb{F}_q \right\} = \left\{ \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(\frac{1}{x}\right) \chi(x) \mid \chi \in \widehat{\mathbb{F}_q} \right\}.$$

Let $j: \mathbb{G}_m \rightarrow \mathbb{G}_a$ denote the natural inclusion, and $\iota: \mathbb{G}_m \rightarrow \mathbb{G}_m$ be the inversion map. The object $M = j_! \iota^* j^* \mathcal{L}_{\psi_q}[1](1/2)$ is a pure perverse sheaf of weight zero on the additive group \mathbb{G}_a , whose trace function satisfies

$$\mathrm{tr}(\mathrm{Frob}_{k,x} | M) = \begin{cases} -\psi_q(1/x)/\sqrt{q} & \text{if } x \in \mathbb{G}_m(k) = \mathbb{F}_q^\times \\ 0 & \text{otherwise} \end{cases}$$

for x in $\mathbb{G}_a(k) = \mathbb{F}_q$. The main theorem from [FFK23] then provides an equidistribution result, though it remains to determine the associated measure.

The symmetry groups of perverse sheaves on \mathbb{G}_a have a particularly simple description. There exists a dense open subset U of \mathbb{G}_a such that Deligne's Fourier transform $\mathrm{FT}_{\psi_q}(M)$, described in the introduction of Chapter 4, becomes a shifted ℓ -adic local system on U . Denote by $\rho: \pi_1(U) \rightarrow \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ the corresponding representation. Then the symmetry group $\mathrm{Conv}(M)$ is isomorphic to the *monodromy group* of this ℓ -adic local system, which is the Zariski closure of $\rho(\pi_1(U))$ in $\mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$ [FFK23, Prop. 3.32].

For this specific perverse sheaf M , standard arguments [Kat90, Prop. 8.1.12] show that $j^* \mathrm{FT}_{\psi_q}(M) \simeq \mathrm{KL}_2[1](1/2)$, where KL_2 is the *Kloosterman sheaf* described in [Kat88, Thm. 4.1.1]. Katz computed the monodromy group of the ℓ -adic local system $\mathrm{KL}_2(1/2)$ to be the special linear group SL_2 [Kat88, Thm. 11.1]. His computation further implies that we achieve equidistribution, not merely equidistribution on average. (See [FFK23, Thm. 4.15] and [Kat88, Cor. 11.3].)

The maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$ is the special unitary group $\mathrm{SU}(2)$. The theorem then follows by observing that the direct image of its normalized Haar measure by the trace function is the Sato–Tate measure. \square

This story has a counterpart in characteristic zero. By replacing characters of \mathbb{F}_q with those of \mathbb{R} and sums with integrals, the *Bessel function* $\mathrm{Be}(z)$ emerges as a formal analogue of the Kloosterman sum $\mathrm{Kl}_2(a, q)$:

$$\mathrm{Kl}_2(a, q) = \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(x + \frac{a}{x}\right) \longleftrightarrow \mathrm{Be}(z) = \int_{S^1} \exp\left(x + \frac{z}{x}\right) \frac{dx}{x}.$$

The function $\mathrm{Be}(z)$ satisfies the differential equation $(z\partial_z)^2 - z = 0$, whose *differential Galois group* can be shown to be SL_2 [Kol68]. It is no coincidence that this matches the monodromy group obtained in the proof of the preceding theorem! Katz has documented numerous instances of this phenomenon in [Kat90].

A general comparison theorem between monodromy groups and differential Galois groups represents the crowning achievement of Katz' book [Kat90], which we now outline. Let R be a subring of \mathbb{C} that is a finitely generated $\mathbb{Z}[1/\ell]$ -algebra, and let M be an object of $D_c^b(\mathbb{A}_R^1, \overline{\mathbb{Q}}_\ell)$. Suppose that M is adapted to a stratification $\{\mathbb{A}_R^1 \setminus D, D\}$,

where D is a finite étale divisor defined by a monic polynomial whose discriminant is a unit in R .

Assume that M is relatively perverse in the sense of [HS23], meaning that for any morphism of rings $\varphi: R \rightarrow K$, where K is an algebraically closed field, the inverse image M_φ of M in $D_c^b(\mathbb{A}_K^1, \overline{\mathbb{Q}}_\ell)$ is perverse. Let M_0 be the regular holonomic \mathcal{D} -module on $\mathbb{A}_\mathbb{C}^1$ corresponding to the complex fiber of M under the Riemann–Hilbert correspondence. The Fourier transform $\text{FT}(M_0)$ restricts to a connection on $\mathbb{G}_{m,\mathbb{C}}$, thereby giving rise to a differential Galois group $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$.

According to [Kat90, Prop. 14.1.5], for each morphism of rings $\varphi: R \rightarrow K$, where K is an algebraically closed field of characteristic $p > 0$, the perverse sheaf $\text{FT}_{\psi_q}(M_\varphi)|_{\mathbb{G}_m}$ is of the form $\mathcal{L}_\varphi[1]$ for some ℓ -adic local system \mathcal{L}_φ . The monodromy group of this local system is denoted by $\text{Mon}(\mathcal{L}_\varphi)$.

THEOREM (Gabber, Katz). *If $\text{Mon}(\mathcal{L}_\varphi)$ is reductive and $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$ is semisimple, then there exists a dense open subset \mathcal{U} of $\text{Spec } R$ such that for any φ lying over a point of \mathcal{U} , the monodromy group $\text{Mon}(\mathcal{L}_\varphi)$ is isomorphic to $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$.*

As breathtaking as this theorem is, it does not seem to account for the isomorphism between the monodromy group for Kloosterman sums and the differential Galois group for the Bessel function. Nor does it apply to other commutative connected algebraic groups beyond \mathbb{G}_a . Nonetheless, this phenomenon is certainly not confined to the additive group.

By repeating the same formal procedure, we observe that the characteristic-zero analogue of Gauss sums is the *Gamma function* $\Gamma(s)$:

$$g(\psi_q, \chi) = \sum_{x \in \mathbb{F}_q^\times} \psi_q(x)\chi(x) \rightsquigarrow \Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

The Gamma function satisfies the difference equation $\tau - s = 0$, where τ is the shift operator that maps $f(s)$ to $f(s + 1)$. The *difference Galois group* associated with this equation is GL_1 , reflecting the fact that Gauss sums are equidistributed on S^1 . Indeed, S^1 is the maximal compact subgroup of $GL_1(\mathbb{C})$.

In the context of the multiplicative group, there appear to be numerous examples of this phenomenon, yet no general comparison theorem has been established. We refer to [Kat12, Chap. 29] for a related discussion. As Katz so aptly puts it, *much remains to be done*.

SUMMARY OF RESULTS

While a complete understanding of the relation between exponential sums and their characteristic-zero counterparts remains elusive, this thesis marks a step toward that

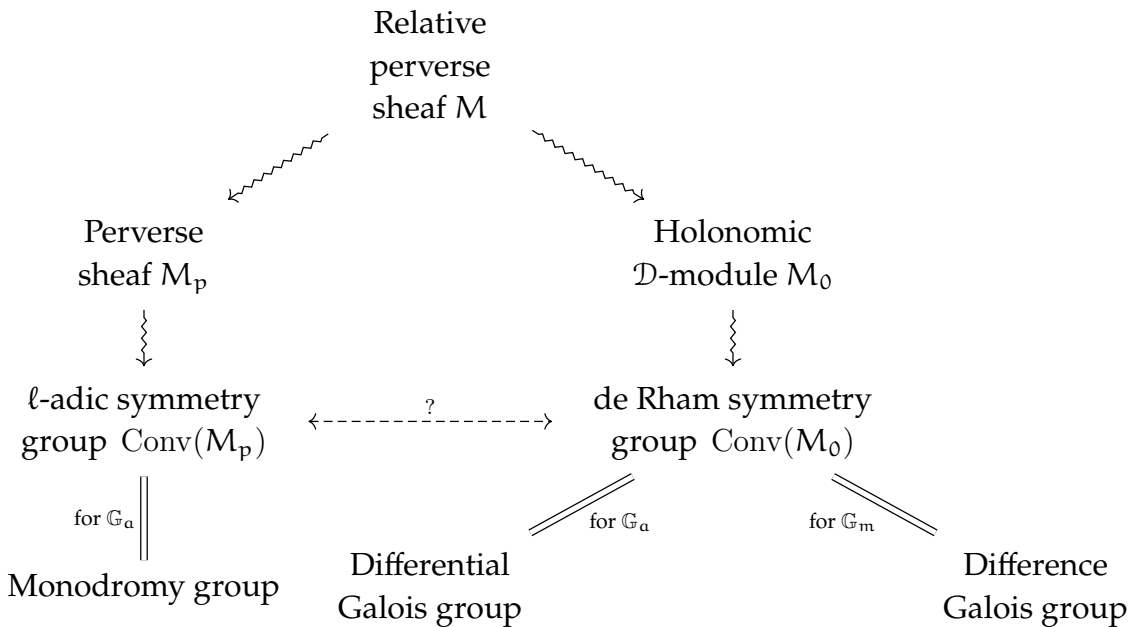
goal. To begin, let us place Gabber and Katz' result in perspective.

Consider a subring R of \mathbb{C} that is a finitely generated $\mathbb{Z}[1/\ell]$ -algebra. Let G be a commutative connected algebraic group over R , and M be a relatively perverse sheaf on G . Fix a sufficiently generic morphism from R to \mathbb{F}_q , and denote by M_p the inverse image of M . Additionally, let M_0 denote the holonomic \mathcal{D} -module corresponding to the complex fiber of M .

According to the aforementioned results from [FFK23], the perverse sheaf M_p gives rise to a family of exponential sums, whose equidistribution is governed by a symmetry group $\text{Conv}(M_p)$. In the special case where G is the additive group \mathbb{G}_a , the symmetry group $\text{Conv}(M_p)$ can be described as a monodromy group, as seen in the proof of equidistribution for Kloosterman sums.

Rather than comparing the monodromy group associated with M_p to the differential Galois group associated with M_0 , this thesis focuses on studying a certain *de Rham symmetry group*, denoted $\text{Conv}(M_0)$, whose definition closely parallels that of $\text{Conv}(M_p)$. Moreover, when G is the additive group, $\text{Conv}(M_0)$ can be identified with a differential Galois group, and when G is the multiplicative group, $\text{Conv}(M_0)$ can be described as a difference Galois group.

Just as the definition of $\text{Conv}(M_p)$ required several conceptual advances, so too does the definition of $\text{Conv}(M_0)$, which will be developed throughout this monograph. A precise comparison theorem relating $\text{Conv}(M_p)$ and $\text{Conv}(M_0)$ under reasonable hypotheses will be the subject of future work.



From this point forward, we discard the previous notation and consider G to be a commutative connected algebraic group over \mathbb{C} . As noted by Krämer in [Krä14, §5],

and subsequently utilized in [FFK23, §3.5], the existence of such a symmetry group is, in essence, a formal consequence of a *generic vanishing theorem*. In the context of de Rham cohomology, this result should follow the template below.

PROTO-THEOREM. *Let G be a commutative connected algebraic group over \mathbb{C} , and let M be a holonomic \mathcal{D} -module on G . The following holds*

$$\begin{aligned} H_c^i(G, M \otimes \mathcal{L}_\chi) &= H^i(G, M \otimes \mathcal{L}_\chi) = 0 \text{ for } i \neq 0; \\ H_c^0(G, M \otimes \mathcal{L}_\chi) &\simeq H^0(G, M \otimes \mathcal{L}_\chi) \end{aligned}$$

for almost every character χ .

As it stands, the proto-theorem above does not make much sense. To refine it into a precise and meaningful statement, we must proceed with the following steps:

- (1) Define *character sheaves*: line bundles with integrable connection corresponding to the rank-one ℓ -adic local systems \mathcal{L}_χ in positive characteristic.
- (2) Construct a moduli space G^\flat of character sheaves and analyse its geometric properties.
- (3) Study the geometry of the loci within G^\flat constituted of the character sheaves satisfying the proto-theorem above.

The initial step is straightforward. For a commutative connected algebraic group G over \mathbb{F}_q , with group law $m: G \times G \rightarrow G$, there exists a clear characterization of the ℓ -adic local systems of the form \mathcal{L}_χ for some character χ . Specifically, these are the rank-one ℓ -adic local systems \mathcal{L} that satisfy the condition $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$ —an isomorphism reflecting the equation $\chi(xy) = \chi(x)\chi(y)$. More precisely, the group morphisms

$$\left\{ \begin{array}{l} \text{Characters} \\ \chi: G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times \end{array} \right\} \begin{array}{c} \xrightarrow{\chi \mapsto \mathcal{L}_\chi} \\ \xleftarrow{\text{tr. } \mathcal{L} \mapsto \chi} \end{array} \left\{ \begin{array}{l} \text{Isomorphism classes of rank-one } \ell\text{-adic local} \\ \text{systems } \mathcal{L} \text{ on } G \text{ satisfying } m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L} \end{array} \right\}$$

are inverses of each other [ST21, Lem. 2.16]. This observation leads to the following definition.

DEFINITION (Character sheaf). A *character sheaf* is a line bundle with integrable connection (\mathcal{L}, ∇) on G satisfying $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$.

To better understand these objects, we recall the Barsotti–Chevalley theorem, which states that the algebraic group G fits in a short exact sequence

$$0 \rightarrow L \rightarrow G \rightarrow A \rightarrow 0,$$

where L is linear group and A is an abelian variety. The linear group L further decomposes into a product of a torus T and a unipotent group U . For reasons that will soon become apparent¹, we denote by $H_m^1(G_{dR}, \mathbb{G}_m)$ the group of isomorphism classes of character sheaves on G . These groups have straightforward descriptions for unipotent groups, tori, and abelian varieties.

■ **EXAMPLE.** In characteristic zero, a unipotent group U is always of the form \mathbb{G}_a^n for some n . The group $H_m^1(U_{dR}, \mathbb{G}_m)$ is then naturally isomorphic to \mathbb{C}^n through the map

$$\begin{aligned} \mathbb{C}^n &\rightarrow H_m^1(U_{dR}, \mathbb{G}_m) \\ (\chi_1, \dots, \chi_n) &\mapsto (\mathcal{O}_U, d - \chi_1 dt_1 - \dots - \chi_n dt_n). \end{aligned}$$

Similarly, over \mathbb{C} , a torus T is isomorphic to a power of the multiplicative group \mathbb{G}_m^r . Here, the group $H_m^1(T_{dR}, \mathbb{G}_m)$ is naturally isomorphic to $(\mathbb{C}/\mathbb{Z})^r$ via the map

$$\begin{aligned} (\mathbb{C}/\mathbb{Z})^r &\rightarrow H_m^1(T_{dR}, \mathbb{G}_m) \\ (\chi_1, \dots, \chi_r) &\mapsto \left(\mathcal{O}_T, d - \chi_1 \frac{dt_1}{t_1} - \dots - \chi_r \frac{dt_r}{t_r} \right). \end{aligned}$$

For abelian varieties A , every line bundle with integrable connection is a character sheaf. This is because a line bundle equipped with an integrable connection has a vanishing first Chern class, implying that it lies in $\text{Pic}^0(A)$. ■

In the context of non-abelian Hodge theory, Simpson constructed a moduli space of line bundles with flat connection on a smooth projective variety [Sim94]. In particular, there exists a moduli space of character sheaves on abelian varieties. For affine groups, however, the group of line bundles with flat connection is too large², but the character condition effectively resolves this issue.

THEOREM A (3.2.6). *There exists a smooth commutative connected group algebraic space G^b satisfying $\dim G \leq \dim G^b \leq 2 \dim G$, whose \mathbb{C} -points parametrize character sheaves.*

The techniques employed in the proof of this theorem represent the main novelty of this thesis. These methods are distinct from other approaches to generic vanishing and lead to a new construction of Simpson's moduli space for abelian varieties.

Sketch of the proof of Theorem A. The inspiration for this proof is the Barsotti–Weil formula, which we generalize in Theorem F. For an abelian variety A , this formula asserts that the extension sheaf $\underline{\text{Ext}}^1(A, \mathbb{G}_m)$, computed in the category of abelian sheaves

¹The proof of Theorem A introduces the de Rham space G_{dR} , and $H^1(G_{dR}, \mathbb{G}_m)$ is its fppf cohomology with coefficients in \mathbb{G}_m . The group $H_m^1(G_{dR}, \mathbb{G}_m)$ is the subgroup consisting of the line bundles satisfying the character condition.

²For instance, the "moduli space" of line bundles with flat connection on \mathbb{A}^1 is the ind-scheme $\mathbb{A}^\infty := \text{colim}_n \mathbb{A}^n$, whereas the moduli space of character sheaves on \mathbb{G}_a is \mathbb{G}_a itself.

on the fppf site $(\text{Sch}/\mathbb{C})_{\text{fppf}}$, is representable by the dual abelian variety A' . This is a moduli space of line bundles \mathcal{L} on A satisfying $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$.

The next ingredient is the so-called *de Rham space*. For an algebraic variety X , line bundles on the de Rham space X_{dR} correspond to line bundles with flat connection on X . Consequently, it can be shown that the abelian sheaf $A^\flat := \underline{\text{Ext}}^1(A_{\text{dR}}, \mathbb{G}_m)$ parametrizes character sheaves on A and is representable by a connected algebraic group. This algebraic group coincides with Simpson's moduli space.

For a general commutative connected algebraic group G , Theorem 2.3.21 provides an isomorphism between the \mathbb{C} -points of the sheaf $G^\flat := \underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)$ and the group $H_m^1(G_{\text{dR}}, \mathbb{G}_m)$. If G is semiabelian (i.e., an extension of an abelian variety by a torus), then this abelian sheaf is representable by a smooth commutative connected group algebraic space, thereby establishing the theorem.

However, when G contains a unipotent subgroup, the sheaf G^\flat is no longer representable by an algebraic space. In Chapter 3, we computed G^\flat in as explicit a manner as possible and identified the problematic constituents. By excluding those problematic parts, we obtain the desired moduli space G^\flat . \square

As a result of the explicit computations required for the preceding proof, we establish a normal form for character sheaves. Let $\psi: G \rightarrow A$ be the map arising from the Barsotti–Chevalley decomposition, and let Ω_G denote the vector group of *invariant* differentials on G . Each $\omega \in \Omega_G$ induces a character sheaf $\mathcal{L}_\omega := (\mathcal{O}_G, d + \omega)$ on G . Finally, let \mathcal{L}_χ denote the character sheaf on G corresponding to a point $\chi \in G^\flat$.

COROLLARY (3.1.10). *Every character sheaf on G can be expressed as $\mathcal{L}_\omega \otimes \psi^*\mathcal{L}_\alpha$, for some $\omega \in \Omega_G$ and $\alpha \in A^\flat$.*

While the decomposition above is not unique, Corollary 3.1.10 provides a precise characterization of the various possible decompositions of a character sheaf. After examining Steps (1) and (2), we now proceed to Step (3), starting with the case of unipotent groups.

THEOREM B (5.3.1). *Let U be a commutative unipotent group over \mathbb{C} , and let M be a holonomic \mathcal{D} -module on U . There exists an open dense subset V of U^\flat such that*

$$\begin{aligned} H_c^i(U, M \otimes \mathcal{L}_\chi) &= H^i(U, M \otimes \mathcal{L}_\chi) = 0 \text{ for } i \neq 0; \\ H_c^0(U, M \otimes \mathcal{L}_\chi) &\simeq H^0(U, M \otimes \mathcal{L}_\chi) \end{aligned}$$

holds for every $\chi \in V$.

A unipotent group U over \mathbb{C} is necessarily a vector group, with the moduli space U^\flat being isomorphic to the dual vector group U^* . For a given holonomic \mathcal{D} -module on U , we suspect that locus of *good* characters is far more structured than merely being an

open dense subset of U^* . We conjecture that its complement consists of a finite union of affine subspaces with positive codimension. This leads to the following definition.

DEFINITION (3.3.1, 3.3.4). For an epimorphism $\rho: G \twoheadrightarrow \tilde{G}$ with connected kernel, the image of $\rho^b: \tilde{G}^b \hookrightarrow G^b$ is said to be a *linear subspace* of G^b . A *generic subspace* of G^b is the complement of a finite union of translates of linear subspaces of G^b with positive codimension.

These subspaces are straightforward to describe for linear algebraic groups and can be connected to established concepts in the literature on abelian varieties.

■ **EXAMPLE — 3.3.2.** Consider a torus T with character group X . A linear subspace of $T^b \simeq \Omega_T/X$ is of the form V/Y , where Y is a subgroup of X and V is the linear subspace of Ω_T generated by Y , in the sense of linear algebra. For a unipotent group U , a linear subspace of $U^b = U^*$ corresponds to a linear subspace of the underlying vector space of U^* , in the sense of linear algebra.

For an abelian variety A , linear subspaces of $A^b \simeq A^\flat$ were first studied by Simpson, who termed them *triple tori* [Sim93, p. 365]. Schnell refers to translates of linear subspaces of A^b as *linear subvarieties* [Sch15, Def. 2.3]. ■

While we have yet to prove that the subspace of good characters in U^b is generic³, we are able to establish a similar result for semiabelian varieties. Here, we let $\varphi: L \rightarrow G$ denote the map from the Barsotti–Chevalley decomposition.

THEOREM C (5.2.1, 5.4.3). *Let G be a semiabelian variety over \mathbb{C} , and let M be an object of $D_h^b(\mathcal{D}_G)$. Then there exists a generic subspace V of L^b such that the forget-supports map*

$$H_c^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha) \rightarrow H^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha)$$

is an isomorphism for every $\alpha \in A^b$ and every $\omega \in \Omega_G$ such that $\varphi^ \mathcal{L}_\omega$ lies in V . Furthermore, if M is concentrated in degree zero, for each ω as above, there exists a generic subspace W of A^b such that*

$$H_c^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha) \simeq H^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha)$$

is concentrated in degree zero for every $\alpha \in W$.

The particular case of abelian varieties in the theorem above was proved by Schnell [Sch15, Thms. 2.2 and 4.1]. Moreover, Krämer proves a related result for regular M in [Krä14, Thm. 2.1]. The former result serves as an ingredient in our proof, which was inspired by the latter.

We are now in a position to introduce the construction of the *de Rham symmetry group*, as promised earlier. The key idea is to apply the formalism of tannakian

³However, Remark 5.3.5 presents a promising idea for such a proof.

categories, which provides criteria for a category to be equivalent to the category of finite-dimensional representations of an affine group scheme.

The fundamental requirement for tannakian categories is that they are symmetric monoidal, which means they possess a binary operation corresponding to the tensor product of representations. In our context, this operation is provided by convolution, which manifests in two forms:

$$\begin{aligned} D_h^b(\mathcal{D}_G) \times D_h^b(\mathcal{D}_G) &\longrightarrow D_h^b(\mathcal{D}_G) & D_h^b(\mathcal{D}_G) \times D_h^b(\mathcal{D}_G) &\longrightarrow D_h^b(\mathcal{D}_G) \\ (M, N) &\longmapsto m_l(M \boxtimes N) & (M, N) &\longmapsto m_+(M \boxtimes N). \end{aligned}$$

While these operations do not agree and do not preserve holonomic \mathcal{D} -modules in degree zero, they do so *up to negligible modules*.

DEFINITION (6.2.1). Let M be a holonomic \mathcal{D} -module on G . We say that M is *negligible* if there exists a generic subspace V of G^b such that

$$H_c^*(G, M \otimes \mathcal{L}_\chi) = 0$$

for all $\chi \in V$. We denote by $\text{Neg}(\mathcal{D}_G)$ the full subcategory of $\text{Hol}(\mathcal{D}_G)$ constituted of the negligible holonomic \mathcal{D} -modules.

THEOREM D (6.2.12). Let G be the additive group, a torus, or an abelian variety. The category $\text{Neg}(\mathcal{D}_G)$ is a thick subcategory of $\text{Hol}(\mathcal{D}_G)$, and the quotient $\text{Hol}(\mathcal{D}_G)/\text{Neg}(\mathcal{D}_G)$ is tannakian under convolution. In particular, every holonomic \mathcal{D} -module M on G has an associated symmetry group $\text{Conv}(M)$.

These symmetry groups can be described quite concretely when G is either the additive group \mathbb{G}_a or the multiplicative group \mathbb{G}_m .

THEOREM E (6.3.11, 6.3.11). Let M be a holonomic \mathcal{D} -module on \mathbb{G}_a , and let $\text{FT}(M)$ denote its Fourier transform. The generic fiber $\text{FT}(M)_\eta$ is a differential module whose differential Galois group is isomorphic to $\text{Conv}(M)$.

Similarly, let N be a holonomic \mathcal{D} -module on \mathbb{G}_m , and let $\text{MT}(N)$ denote its Mellin transform. The generic fiber $\text{MT}(N)_\eta$ is a difference module whose difference Galois group is isomorphic to $\text{Conv}(N)$.

In [Krä22, Thm. 3.2], Krämer established a related result for the Fourier–Mukai transform of holonomic \mathcal{D} -modules on abelian varieties. This characterization was later used to show that the associated symmetry groups are almost connected. The same conclusion holds for \mathbb{G}_m in our setting, despite the different methods employed, and we show that the analogous result for \mathbb{G}_a is false.

EXTENSIONS OF ABELIAN SHEAVES

The proof of Theorem A required a detailed and nuanced analysis of fppf extension sheaves. Along the way, we uncovered several results that are noteworthy in their own right. To our surprise, we found that the well-known Barsotti–Weil formula appears to lack a complete published proof in the literature, alongside other inconsistencies.

Unable to find a reference for the Barsotti–Weil formula in the level of generality we required, we established the following generalization.

THEOREM F (2.3.10). *Let A be an abelian scheme and B an affine commutative group scheme over a base scheme S . For an S -scheme T , the natural maps*

$$\underline{\mathrm{Ext}}^1(A, B)(T) \leftarrow \mathrm{Ext}_T^1(A, B) \rightarrow H_m^1(A_T, B_T)$$

are isomorphisms. (See 2.3.6 for the definition of $H_m^1(A_T, B_T)$.) In particular, the abelian sheaf $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ is representable by the dual abelian scheme A' .

For an abelian scheme A over a regular base S , the vanishing of $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ has been asserted in [Bre75, Rem. 6], [Jos09, Thm. 1.2.5], and [Bro21, Cor. 11.5]. Unfortunately, in each instance, this result is attributed to [Bre69, §7], but additional arguments appear to be lacking. The following theorem, developed in collaboration with Zev Rosengarten, aims to fill this gap, at least in characteristic zero.

THEOREM G (2.4.4). *Let A be an abelian scheme over a characteristic zero scheme S . Then the abelian sheaf $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ vanishes.*

Another abelian sheaf that plays a significant role in this thesis is $\underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$, for a commutative unipotent group U over a characteristic zero field. In [Ros23, Rem. 2.2.16], Rosengarten details Gabber’s construction of a non-zero section of this sheaf. In Example 2.3.17, we provide what may be the simplest such section.

Regrettably, the vanishing of this sheaf has been asserted in the literature. This claim can be found in [Pol11, Lem. 1.3.6], was utilized in the proofs of [BB09, Lem. A.4.5] and [Ber14, Lem. 10], and can be derived from the computations in [Lau85, Ex. p. 25]. Our final theorem demonstrates that, although this sheaf is non-zero, its images often vanish.

THEOREM H (2.3.16). *Let U be a commutative unipotent group over a characteristic zero field k . Then the group $\underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)(T)$ vanishes for all seminormal k -schemes T .*

We note that a similar result was announced by Gabber in a recent conference talk [Gab23]. Our arguments are independent, and we were unaware of Gabber’s proof during the development of this thesis.

MOTIVATIONS ET PRINCIPAUX RÉSULTATS

Dans ce chapitre, nous exposons le contexte historique qui a conduit au développement des résultats présentés dans cette thèse. De plus, nous fournissons un bref aperçu de nos principales découvertes, en les situant dans le paysage mathématique plus large.

SOMMES EXPONENTIELLES ET ÉQUATIONS DIFFÉRENTIELLES

Les sommes exponentielles sur les corps finis sont omniprésentes dans la théorie analytique des nombres moderne. Une grande partie de leur étude tourne autour de deux questions fondamentales :

- (1) Pour une somme exponentielle S , quelle est une *bonne* borne supérieure pour $|S|$?
- (2) Pour une famille $\{S_i\}_{i \in I}$ de sommes exponentielles, comment les nombres complexes S_i varient-ils lorsque i parcourt l'ensemble I ?

Comme nous allons le voir bientôt, ces questions sont étroitement liées. Nous illustrerons cette théorie avec un exemple particulièrement éclairant, que nous avons appris de notre directeur de thèse Fresán, qui l'a également présenté dans son premier exposé au Séminaire Bourbaki [Fre19].

Soit p un nombre premier, q une puissance de p , et fixons un caractère additif non trivial $\psi_p: \mathbb{F}_p \rightarrow \mathbb{C}^\times$. En précomposant avec l'application trace $\mathbb{F}_q \rightarrow \mathbb{F}_p$, nous obtenons le caractère additif $\psi_q: \mathbb{F}_q \rightarrow \mathbb{C}^\times$. Pour chaque caractère multiplicatif $\chi: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$, la *somme de Gauss* $g(\psi_q, \chi)$ est définie par

$$g(\psi_q, \chi) := \sum_{x \in \mathbb{F}_q^\times} \psi_q(x) \chi(x).$$

La Question (1) a une réponse simple pour les sommes de Gauss. En raison de l'orthogonalité des caractères, la valeur absolue de $g(\psi_q, \chi)$ est 1 lorsque χ est le caractère trivial, et \sqrt{q} sinon. En revanche, la Question (2) est bien plus intrigante. Il y a $q - 2$ angles $\theta_{q, \chi} := g(\psi_q, \chi) / \sqrt{q}$ sur le cercle unité S^1 , un pour chaque caractère non trivial χ .

Comment ces angles se répartissent-ils sur S^1 lorsque q tend vers l'infini ?

Bien que les sommes de Gauss soient quelques-uns des exemples les plus simples de sommes exponentielles, cette question est profonde. Sa solution n'est devenue possible qu'après que Deligne a démontré l'hypothèse de Riemann pour les variétés sur les corps finis.

THÉORÈME (Deligne, Katz). *À mesure que q tend vers l'infini, les angles $\theta_{q,\chi}$ pour $\chi \neq 1$ deviennent équirépartis sur S^1 par rapport à sa mesure de Haar normalisée. En d'autres termes, l'équation*

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = \lim_{q \rightarrow \infty} \frac{1}{q-2} \sum_{\chi \neq 1} f(\theta_{q,\chi})$$

est vérifiée pour toutes les fonctions continues $f: S^1 \rightarrow \mathbb{C}$.

Début de la démonstration. Comme les polynômes de Laurent sont denses dans $\mathcal{C}(S^1)$, il suffit de considérer des fonctions de la forme $f(z) = z^n$ pour $n \in \mathbb{Z}$. Le cas $n = 0$ est trivial, et la relation $g(\psi_q, \chi)g(\overline{\psi}_q, \overline{\chi}) = q$ nous permet de supposer que $n \geq 1$. Dans ce cas, l'intégrale s'annule toujours, il reste donc à prouver que la séquence

$$\frac{1}{q-2} \sum_{\chi \neq 1} f(\theta_{q,\chi}) = \frac{1}{q^{n/2}(q-2)} \sum_{\chi \neq 1} g(\psi_q, \chi)^n$$

tend vers zéro lorsque q tend vers l'infini. En développant l'expression de $g(\psi_q, \chi)^n$, on obtient

$$\begin{aligned} g(\psi_q, \chi)^n &= \sum_{x_1, \dots, x_n \in \mathbb{F}_q^\times} \psi_q(x_1 + \dots + x_n) \chi(x_1 \dots x_n) \\ &= \sum_{a \in \mathbb{F}_q^\times} \chi(a) \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q^\times \\ x_1 \dots x_n = a}} \psi_q(x_1 + \dots + x_n). \end{aligned}$$

C'est-à-dire que l'application $\chi \mapsto g(\psi_q, \chi)^n$ est la transformée de Fourier des *sommes de Kloosterman*

$$a \mapsto \text{Kl}_n(a, q) := \sum_{\substack{x_1, \dots, x_n \in \mathbb{F}_q^\times \\ x_1 \dots x_n = a}} \psi_q(x_1 + \dots + x_n).$$

Ainsi, la solution à la Question (2) pour les sommes de Gauss dépend de la résolution de la Question (1) pour les sommes de Kloosterman. *À suivre...*

Kloosterman lui-même a dû estimer les sommes qui portent désormais son nom dans [Klo27], mais spécifiquement pour le cas $n = 2$. En calculant le quatrième moment,

$$\sum_{a \in \mathbb{F}_q^\times} \text{Kl}_2(a, q)^4 = 2q^3 - 3q^2 - 3q - 1,$$

il a pu conclure que $|Kl_2(a, q)| < 2q^{3/4}$. Cependant, déterminer la borne supérieure optimale pour $|Kl_n(a, q)|$, ce qui est nécessaire pour compléter la démonstration ci-dessus, a nécessité le développement de la cohomologie étale et les travaux ultérieurs de Deligne—des avancées qui ont pris cinquante ans de plus.

Pour éclaircir le rôle de la cohomologie étale dans cette histoire, examinons d’abord quelques-unes de ses caractéristiques clés. Pour une variété algébrique connexe X sur $k = \mathbb{F}_q$, Grothendieck a défini le *groupe fondamentale étale* $\pi_1(X)$, un groupe profini qui classe les revêtements étales finis de X . Étant donné une extension finie E/k , le groupe fondamental étale de $X = \text{Spec } E$ est simplement le groupe de Galois absolu $\text{Gal}(E)$ —le groupe profini libre $\widehat{\mathbb{Z}}$, avec un générateur donné par

$$\bar{E} \rightarrow \bar{E}, \quad x \mapsto x^{|E|}.$$

Ceci est connu sous le nom de *Frobenius arithmétique*. Son inverse, noté Frob_E et appelé *Frobenius géométrique*, joue un rôle crucial dans la théorie.

Désormais, fixons un nombre premier $\ell \neq p$, une clôture algébrique de \mathbb{Q}_ℓ , et un isomorphisme entre $\overline{\mathbb{Q}_\ell}$ et \mathbb{C} . Considérons un système local ℓ -adique \mathcal{L} de rang r sur X , caractérisé comme une représentation continue $\rho: \pi_1(X) \rightarrow \text{GL}_r(\overline{\mathbb{Q}_\ell})$. Pour chaque point $x \in X(E)$, la functorialité naturelle du groupe fondamental étale induit une application

$$\text{Gal}(E) \simeq \pi_1(\text{Spec } E) \rightarrow \pi_1(X).$$

Nous désignons par $\text{Frob}_{E,x}$ l’image de Frob_E sous ce morphisme. Comme cette construction n’est canonique qu’à choix de point de base près, $\text{Frob}_{E,x}$ représente une classe de conjugaison dans $\pi_1(X)$. Par conséquent, $\rho(\text{Frob}_{E,x})$ forme une classe de conjugaison dans $\text{GL}_r(\overline{\mathbb{Q}_\ell})$, et nous pouvons en prendre la trace, notée $\text{tr}(\text{Frob}_{E,x} | \mathcal{L})$.

Grothendieck a également introduit les groupes de cohomologie $H^i(X_{\bar{k}}, \mathcal{L})$ et les groupes de cohomologie à support compact $H_c^i(X_{\bar{k}}, \mathcal{L})$. Ce sont des espaces vectoriels de dimension finie sur $\overline{\mathbb{Q}_\ell}$, équipés d’une action de $\text{Gal}(k)$, et ils s’annulent en dehors de l’intervalle $0 \leq i \leq 2d$, où d est la dimension de X . L’action du Frobenius géométrique sur un système local ℓ -adique et sur sa cohomologie à support compact sont liées par la *formule des traces de Grothendieck* :

$$\sum_{x \in X(E)} \text{tr}(\text{Frob}_{E,x} | \mathcal{L}) = \sum_{i=0}^{2d} (-1)^i \text{tr}(\text{Frob}_E | H_c^i(X_{\bar{k}}, \mathcal{L})).$$

Cette formule constitue un élément clé d’une méthode puissante pour estimer les sommes exponentielles sur les corps finis. Tout d’abord, un système local ℓ -adique \mathcal{L} est construit de manière à ce que le côté gauche de la formule des traces de Grothendieck corresponde à la somme exponentielle donnée. Ensuite, les valeurs propres du Frobenius géométrique agissant sur $H_c^i(X_{\bar{k}}, \mathcal{L})$ sont estimées. Cette estimation peut être réalisée en utilisant la théorie des poids de Deligne.

DÉFINITION. Nous disons que \mathcal{L} est *pur* s'il existe un entier w , son *poinds*, tel que pour toute extension finie E/k et tout $x \in X(E)$, les valeurs propres de $\text{Frob}_{E,x}$ agissant sur \mathcal{L} ont valeur absolue $|E|^{w/2}$.

Le système local ℓ -adique \mathcal{L} est dit *mixte* s'il admet une filtration finie où chaque quotient successif est pur. Les poinds de ces quotients non nuls sont les *poinds* de \mathcal{L} .

La même terminologie s'applique aux groupes de cohomologie $H^i(X_{\bar{k}}, \mathcal{L})$ et $H_c^i(X_{\bar{k}}, \mathcal{L})$. Le résultat principal dans l'article fondateur de Deligne [Del80] compare les poinds d'un système local ℓ -adique avec ceux de ses groupes de cohomologie.

THÉORÈME (Deligne). Si \mathcal{L} est mixte de poinds $\leq w$, alors le groupe de cohomologie à support compact $H_c^i(X_{\bar{k}}, \mathcal{L})$ est mixte de poinds $\leq w + i$.

Pour X lisse, la dualité de Poincaré implique que si \mathcal{L} est mixte de poinds $\geq w$, alors $H^i(X_{\bar{k}}, \mathcal{L})$ est mixte de poinds $\geq w + i$. Cette estimation est particulièrement utile lorsque l'application d'oubli de supports $H_c^i(X_{\bar{k}}, \mathcal{L}) \rightarrow H^i(X_{\bar{k}}, \mathcal{L})$ est un isomorphisme, auquel cas ces groupes de cohomologie sont purs de poinds $w + i$.

Nous appliquons maintenant la méthode mentionnée précédemment pour estimer les sommes de Kloosterman. Considérons le morphisme de k -schémas $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ défini par $x \mapsto x - x^q$. Il s'agit d'un revêtement galoisien étale fini avec groupe de Galois \mathbb{F}_q . Étant donné que $\pi_1(\mathbb{A}_k^1)$ est la limite des groupes de Galois de tous les revêtements galoisiens étales finis, nous obtenons une surjection naturelle $\pi_1(\mathbb{A}_k^1) \twoheadrightarrow \mathbb{F}_q$. En conséquence, la composition

$$\pi_1(\mathbb{A}_k^1) \twoheadrightarrow \mathbb{F}_q \xrightarrow{\psi_q} \overline{\mathbb{Q}}_\ell^\times$$

correspond à un système local ℓ -adique de rang un \mathcal{L}_{ψ_q} sur \mathbb{A}_k^1 , dont la trace en un point $x \in \mathbb{A}_k^1(k) = \mathbb{F}_q$ est simplement $\psi_q(x)$.

Fixons $a \in \mathbb{F}_q^\times$, et soit X le lieu des zéros de $x_1 \dots x_n - a$ dans \mathbb{A}_k^n . Définissons $f: X \rightarrow \mathbb{A}_k^1$ comme l'application qui envoie (x_1, \dots, x_n) sur leur somme, de sorte que

$$\text{Kl}_n(a, q) = \sum_{x \in X(k)} \text{tr}(\text{Frob}_{k,x} | f^* \mathcal{L}_{\psi_q}) = \sum_{i=0}^{2n} (-1)^i \text{tr}(\text{Frob}_k | H_c^i(X_{\bar{k}}, f^* \mathcal{L}_{\psi_q})).$$

Dans [SGA 4 $\frac{1}{2}$, Thm. 7.4], Deligne a calculé ces groupes de cohomologie, montrant que H_c^i s'annule pour $i \neq n - 1$ et que $H_c^{n-1} \simeq H^{n-1}$ est de dimension n . De plus, puisque les valeurs de ψ_q sont des racines p -ièmes de l'unité, le système local ℓ -adique $f^* \mathcal{L}_{\psi_q}$ est pur de poinds 0. Le théorème de Deligne implique alors que

$$|\text{Kl}_n(a, q)| = |\text{tr}(\text{Frob}_k | H_c^{n-1}(X_{\bar{k}}, f^* \mathcal{L}_{\psi_q}))| \leq nq^{(n-1)/2},$$

ce qui est la borne optimale.

Nous sommes maintenant en mesure de compléter la démonstration du théorème d'équidistribution pour les sommes de Gauss.

Fin de la démonstration. En sommant l'expression obtenue pour $g(\psi_q, \chi)^n$ sur les caractères multiplicatifs non triviaux χ , nous obtenons

$$\begin{aligned} \sum_{\chi \neq 1} g(\psi_q, \chi)^n &= -g(\psi_q, 1)^n + \sum_{a \in \mathbb{F}_q^\times} \text{Kl}_n(a, q) \sum_{\chi} \chi(a) \\ &= (-1)^{n+1} + (q-1) \text{Kl}_n(1, q). \end{aligned}$$

La borne de Deligne nous mène à l'estimation

$$\left| \frac{1}{q^{n/2}(q-2)} \sum_{\chi \neq 1} g(\psi_q, \chi)^n \right| \leq \frac{2n+1}{\sqrt{q}},$$

qui tend vers zéro lorsque q tend vers l'infini. \square

La propriété clé des sommes de Gauss qui a permis la démonstration précédente est que l'application $\chi \mapsto g(\psi_q, \chi)$ est la transformée de Fourier de la fonction trace d'un système local ℓ -adique. Cette propriété peut être généralisée dans deux directions. Premièrement, au lieu de considérer des caractères χ de $\mathbb{F}_q^\times = \mathbb{G}_m(k)$, on peut élargir la portée aux caractères de $G(k)$, où G est un groupe algébrique commutatif connexe sur k . De manière similaire à la construction ci-dessus, pour chaque caractère $\chi \in \widehat{G}(k)$, il existe un système local ℓ -adique de rang un \mathcal{L}_χ sur G dont la trace en tout point $x \in G(k)$ est égale à $\chi(x)$.

Les systèmes locaux peuvent également être généralisés à une classe plus large de faisceaux ℓ -adiques connus sous le nom de *faisceaux pervers*. Étant donné une extension finie E/k et un faisceau pervers semi-simple M sur G qui est pur de poids zéro, nous définissons la somme exponentielle

$$S(M, E, \chi) := \sum_{x \in G(E)} \text{tr}(\text{Frob}_{E,x} | M) \chi(x) = \sum_{x \in G(E)} \text{tr}(\text{Frob}_{E,x} | M \otimes \mathcal{L}_\chi).$$

Comme montré dans [FFK23, Thm. 3.28], il existe un groupe algébrique linéaire $\text{Conv}(M)$ sur $\overline{\mathbb{Q}}_\ell$ qui encapsule les symétries de cette somme exponentielle. Ce groupe est défini comme un groupe tannakien associé à une certaine catégorie quotient de faisceaux pervers.

Soit K un sous-groupe compact maximal de $\text{Conv}(M)_{\text{an}}$. Puisque $\text{Conv}(M)$ est linéaire, il existe un entier $r \geq 0$ et une immersion fermée $\text{Conv}(M) \hookrightarrow \text{GL}_r$. Notons par μ la mesure de Haar normalisée sur K , et par ν son image directe par la fonction trace. Nous présentons maintenant le théorème principal du récent ouvrage [FFK23].

THÉORÈME (Forey, Fresán, Kowalski). *À mesure que le degré de E/k tend vers l'infini, les sommes exponentielles $S(M, E, \chi)$ pour $\chi \in \widehat{G}(E)$ deviennent ν -équiréparties en moyenne. En d'autres termes, l'équation*

$$\int_{\mathbb{K}} f(\mathrm{tr}(x)) \, d\mu(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{[E:k] \leq n} \frac{1}{|G(E)|} \sum_{\chi \in \widehat{G}(E)} f(S(M, E, \chi))$$

est vérifiée pour toutes les fonctions continues bornées $f: \mathbb{C} \rightarrow \mathbb{C}$.

Ce résultat englobe non seulement le théorème d'équidistribution pour les sommes de Gauss, mais étend également de nombreux autres théorèmes connexes. Rappelons que, selon la formule des traces de Grothendieck, la somme exponentielle $S(M, E, \chi)$ est donnée par une somme alternée de traces des Frobenius agissant sur les groupes de cohomologie :

$$\sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}(\mathrm{Frob}_E | H_c^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi)).$$

Comme précédemment, la démonstration de ce théorème repose sur le fait que, pour la plupart des choix de χ , tous ces groupes de cohomologie sauf un sont nuls, simplifiant ainsi considérablement la somme. De plus, le seul groupe de cohomologie à support compact non nul coïncide avec son homologue sans support compact, permettant ainsi l'utilisation des estimations de poids de Deligne.

THÉORÈME (Forey, Fresán, Kowalski). *Soit G un groupe algébrique commutatif connexe sur k de dimension d , et soit M un faisceau pervers sur G . L'ensemble S des caractères $\chi \in \widehat{G}$ qui satisfont*

$$\begin{aligned} H_c^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) &= H^i(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) = 0 \text{ pour } i \neq 0; \\ H_c^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) &\simeq H^0(G_{\bar{k}}, M \otimes \mathcal{L}_\chi) \end{aligned}$$

est générique, ce qui signifie qu'il existe un nombre réel $c \geq 0$ tel que $|\widehat{G}(E) \setminus S(E)| \leq c|E|^{d-1}$ pour toute extension finie E/k .

SOMMES EXPONENTIELLES ET ÉQUATIONS DIFFÉRENTIELLES

Après avoir appliqué les méthodes ℓ -adiques pour résoudre la Question (1) concernant les sommes de Kloosterman $\mathrm{Kl}_n(a, q)$, nous nous tournons maintenant vers la distribution de ces sommes lorsque $a \in \mathbb{F}_q^\times$ varie et que q tend vers l'infini. Pour simplifier, nous considérons le cas $n = 2$. En utilisant la borne établie précédemment, et en notant que les sommes $\mathrm{Kl}_2(a, q)$ sont des nombres réels, il s'ensuit qu'il existe un angle unique $\theta_{q,a} \in [0, \pi]$ tel que $\mathrm{Kl}_2(a, q) = 2\sqrt{q} \cos \theta_{q,a}$.

Comment ces angles se répartissent-ils sur $[0, \pi]$ lorsque q tend vers l'infini ?

THÉORÈME (Katz). À mesure que q tend vers l'infini, les angles $\theta_{q,a}$ pour $a \in \mathbb{F}_q^\times$ deviennent équirépartis sur $[0, \pi]$ par rapport à la mesure de Sato–Tate $(2/\pi) \sin^2 \theta d\theta$. En d'autres termes, l'équation

$$\frac{2}{\pi} \int_0^\pi f(\theta) \sin^2 \theta d\theta = \lim_{q \rightarrow \infty} \frac{1}{q-1} \sum_{a \in \mathbb{F}_q^\times} f(\theta_{q,a})$$

est vérifiée pour toutes les fonctions continues $f: [0, \pi] \rightarrow \mathbb{C}$.

Esquisse de la démonstration. Nous expliquons comment ce résultat s'inscrit dans le cadre du théorème d'équidistribution établi par Forey, Fresán, et Kowalski. Rappelons que chaque caractère $\chi \in \widehat{\mathbb{F}_q}$ peut être exprimé comme $x \mapsto \psi_q(ax)$, pour un $a \in \mathbb{F}_q$ unique. Cette correspondance conduit à l'égalité suivante :

$$\left\{ \text{KL}_2(a, q) = \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(ax + \frac{1}{x}\right) \mid a \in \mathbb{F}_q \right\} = \left\{ \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(\frac{1}{x}\right) \chi(x) \mid \chi \in \widehat{\mathbb{F}_q} \right\}.$$

Soit $j: \mathbb{G}_m \rightarrow \mathbb{G}_a$ l'inclusion naturelle, et $\iota: \mathbb{G}_m \rightarrow \mathbb{G}_m$ l'application d'inversion. L'objet $M = j_! \iota^* j^* \mathcal{L}_{\psi_q}[1](1/2)$ est un faisceau pervers pur de poids zéro sur le groupe additif \mathbb{G}_a , dont la fonction trace satisfait

$$\text{tr}(\text{Frob}_{k,x} \mid M) = \begin{cases} -\psi_q(1/x)/\sqrt{q} & \text{si } x \in \mathbb{G}_m(k) = \mathbb{F}_q^\times \\ 0 & \text{sinon} \end{cases}$$

pour x dans $\mathbb{G}_a(k) = \mathbb{F}_q$. Le théorème principal de [FFK23] fournit alors un résultat d'équidistribution, bien qu'il reste à déterminer la mesure associée.

Les groupes de symétrie des faisceaux pervers sur \mathbb{G}_a ont une description particulièrement simple. Il existe un sous-ensemble ouvert dense U de \mathbb{G}_a tel que la transformée de Fourier de Deligne $\text{FT}_{\psi_q}(M)$, décrite dans l'introduction du Chapitre 4, devienne un système local ℓ -adique décalé sur U . Notons par $\rho: \pi_1(U) \rightarrow \text{GL}_r(\overline{\mathbb{Q}_\ell})$ la représentation correspondante. Le groupe de symétrie $\text{Conv}(M)$ est alors isomorphe au groupe de monodromie de ce système local ℓ -adique, qui est la clôture de Zariski de $\rho(\pi_1(U))$ dans $\text{GL}_r(\overline{\mathbb{Q}_\ell})$ [FFK23, Prop. 3.32].

Pour ce faisceau pervers spécifique M , des arguments standard [Kat90, Prop. 8.1.12] montrent que $j^* \text{FT}_{\psi_q}(M) \simeq \text{KL}_2[1](1/2)$, où KL_2 est le faisceau de Kloosterman décrit dans [Kat88, Thm. 4.1.1]. Katz a calculé que le groupe de monodromie du système local ℓ -adique $\text{KL}_2(1/2)$ est le groupe spécial linéaire SL_2 [Kat88, Thm. 11.1]. Son calcul montre également que l'on obtient une équidistribution, et non simplement une équidistribution en moyenne. (Voir [FFK23, Thm. 4.15] et [Kat88, Cor. 11.3].)

Le sous-groupe compact maximal de $\text{SL}_2(\mathbb{C})$ est le groupe unitaire spécial $\text{SU}(2)$. Le théorème découle alors du fait que l'image directe de sa mesure de Haar normalisée par la fonction trace est la mesure de Sato–Tate. \square

Cette histoire a un pendant en caractéristique zéro. En remplaçant les caractères de \mathbb{F}_q par ceux de \mathbb{R} et les sommes par des intégrales, la *fonction de Bessel* $\text{Be}(z)$ apparaît comme un analogue formel de la somme de Kloosterman $\text{Kl}_2(a, q)$:

$$\text{Kl}_2(a, q) = \sum_{x \in \mathbb{F}_q^\times} \psi_q\left(x + \frac{a}{x}\right) \rightsquigarrow \text{Be}(z) = \int_{S^1} \exp\left(x + \frac{z}{x}\right) \frac{dx}{x}.$$

La fonction $\text{Be}(z)$ satisfait l'équation différentielle $(z\partial_z)^2 - z = 0$, dont le *groupe de Galois différentiel* peut être montré qu'il est SL_2 [Kol68]. Ce n'est pas une coïncidence que cela corresponde au groupe de monodromie obtenu dans la démonstration du théorème précédent ! Katz a documenté de nombreux exemples de ce phénomène dans [Kat90].

Un théorème de comparaison général entre les groupes de monodromie et les groupes de Galois différentiels représente l'aboutissement du livre de Katz [Kat90], que nous allons maintenant résumer. Soit R un sous-anneau de \mathbb{C} qui est aussi une $\mathbb{Z}[1/\ell]$ -algèbre de type fini, et soit M un objet de $D_c^b(\mathbb{A}_R^1, \overline{\mathbb{Q}}_\ell)$. Supposons que M soit adapté à une stratification $\{\mathbb{A}_R^1 \setminus D, D\}$, où D est un diviseur étale fini défini par un polynôme unitaire dont le discriminant est une unité dans R .

Supposons que M soit relativement pervers au sens de [HS23], c'est-à-dire que pour tout morphisme d'anneaux $\varphi: R \rightarrow K$, où K est un corps algébriquement clos, l'image réciproque M_φ de M dans $D_c^b(\mathbb{A}_K^1, \overline{\mathbb{Q}}_\ell)$ soit perverse. Soit M_0 le \mathcal{D} -module holonome régulier sur $\mathbb{A}_{\mathbb{C}}^1$ correspondant à la fibre complexe de M sous la correspondance de Riemann–Hilbert. La transformée de Fourier $\text{FT}(M_0)$ se restreint à une connexion sur $\mathbb{G}_{m, \mathbb{C}}$, donnant ainsi naissance à un groupe de Galois différentiel $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$.

Selon [Kat90, Prop. 14.1.5], pour chaque morphisme d'anneaux $\varphi: R \rightarrow K$, où K est un corps algébriquement clos de caractéristique $p > 0$, le faisceau pervers $\text{FT}_{\psi_q}(M_\varphi)|_{\mathbb{G}_m}$ est de la forme $\mathcal{L}_\varphi[1]$ pour un certain système local ℓ -adique \mathcal{L}_φ . Le groupe de monodromie de ce système local est noté $\text{Mon}(\mathcal{L}_\varphi)$.

THÉORÈME (Gabber, Katz). *Si $\text{Mon}(\mathcal{L}_\varphi)$ est réductif et $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$ est semi-simple, alors il existe un sous-ensemble ouvert dense \mathcal{U} de $\text{Spec } R$ tel que pour tout φ se situant sur un point de \mathcal{U} , le groupe de monodromie $\text{Mon}(\mathcal{L}_\varphi)$ soit isomorphe à $\text{Gal}(\text{FT}(M_0)|_{\mathbb{G}_m})$.*

Aussi remarquable que soit ce théorème, il ne semble pas rendre compte de l'isomorphisme entre le groupe de monodromie pour les sommes de Kloosterman et le groupe de Galois différentiel pour la fonction de Bessel. Il ne s'applique pas non plus à d'autres groupes algébriques commutatifs connexes au-delà de \mathbb{G}_a . Néanmoins, ce phénomène n'est certainement pas confiné au groupe additif.

En répétant la même procédure formelle, nous observons que l'analogue en carac-

téristique zéro des sommes de Gauss est la *fonction Gamma* $\Gamma(s)$:

$$g(\psi_q, \chi) = \sum_{x \in \mathbb{F}_q^\times} \psi_q(x) \chi(x) \longleftrightarrow \Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

La fonction Gamma satisfait l'équation aux différences $\tau - s = 0$, où τ est l'opérateur de décalage qui associe $f(s)$ à $f(s + 1)$. Le *groupe de Galois aux différences* associé à cette équation est GL_1 , reflétant le fait que les sommes de Gauss sont équiréparties sur S^1 . En effet, S^1 est le sous-groupe compact maximal de $GL_1(\mathbb{C})$.

Dans le contexte du groupe multiplicatif, il semble y avoir de nombreux exemples de ce phénomène, bien qu'aucun théorème de comparaison général n'ait été établi. Nous renvoyons à [Kat12, Chap. 29] pour une discussion connexe. Comme le dit si bien Katz, *much remains to be done*.

RÉSUMÉ DES RÉSULTATS

Bien qu'une compréhension complète de la relation entre les sommes exponentielles et leurs analogues en caractéristique zéro reste insaisissable, cette thèse constitue une étape vers cet objectif. Pour commencer, plaçons le résultat de Gabber et Katz dans son contexte.

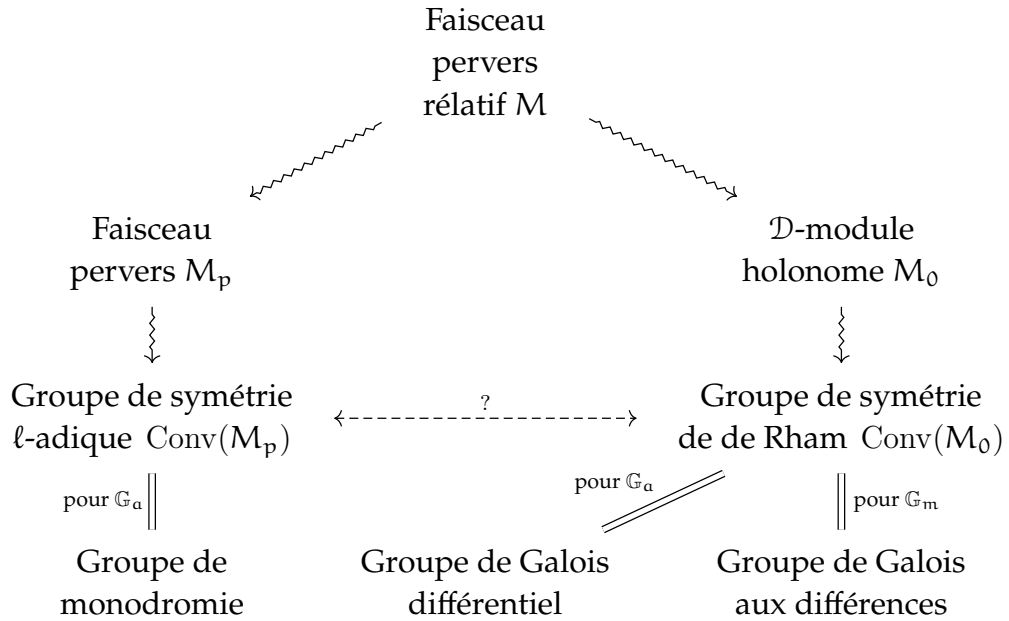
Considérons un sous-anneau R de \mathbb{C} qui est une $\mathbb{Z}[1/\ell]$ -algèbre de type fini. Soit G un groupe algébrique commutatif connexe sur R , et M un faisceau pervers relatif sur G . Fixons un morphisme suffisamment générique de R vers \mathbb{F}_q , et notons par M_p l'image réciproque de M . De plus, soit M_0 le \mathcal{D} -module holonome correspondant à la fibre complexe de M .

Selon les résultats mentionnés ci-dessus de [FFK23], le faisceau pervers M_p donne lieu à une famille de sommes exponentielles, dont l'équidistribution est régie par un groupe de symétrie $\text{Conv}(M_p)$. Dans le cas particulier où G est le groupe additif \mathbb{G}_a , le groupe de symétrie $\text{Conv}(M_p)$ peut être décrit comme un groupe de monodromie, comme on le voit dans la démonstration de l'équidistribution pour les sommes de Kloosterman.

Plutôt que de comparer le groupe de monodromie associé à M_p au groupe de Galois différentiel associé à M_0 , cette thèse se concentre sur l'étude d'un certain *groupe de symétrie de Rham*, noté $\text{Conv}(M_0)$, dont la définition est étroitement parallèle à celle de $\text{Conv}(M_p)$. De plus, lorsque G est le groupe additif, $\text{Conv}(M_0)$ peut être identifié à un groupe de Galois différentiel, et lorsque G est le groupe multiplicatif, $\text{Conv}(M_0)$ peut être décrit comme un groupe de Galois aux différences.

Tout comme la définition de $\text{Conv}(M_p)$ a nécessité plusieurs avancées conceptuelles, la définition de $\text{Conv}(M_0)$ en demande autant, et elle sera développée tout au long de

ce mémoire. Un théorème de comparaison précis entre $\text{Conv}(M_p)$ et $\text{Conv}(M_0)$ sous des hypothèses raisonnables fera l'objet de travaux futurs.



À partir de ce point, nous abandonnons la notation précédente et considérons que G est un groupe algébrique commutatif et connexe sur \mathbb{C} . Comme l'a noté Krämer dans [Krä14, §5], et utilisé par la suite dans [FFK23, §3.5], l'existence d'un tel groupe de symétrie est, en essence, une conséquence formelle d'un *théorème d'annulation générique*. Dans le contexte de la cohomologie de Rham, ce résultat devrait suivre le modèle ci-dessous.

PROTO-THÉORÈME. Soit G un groupe algébrique commutatif et connexe sur \mathbb{C} , et soit M un D-module holonome sur G . Le résultat suivant est vérifié :

$$\begin{aligned} H_c^i(G, M \otimes \mathcal{L}_\chi) &= H^i(G, M \otimes \mathcal{L}_\chi) = 0 \text{ pour } i \neq 0; \\ H_c^0(G, M \otimes \mathcal{L}_\chi) &\simeq H^0(G, M \otimes \mathcal{L}_\chi) \end{aligned}$$

pour presque tous les caractères χ .

Tel qu'il est formulé, le proto-théorème ci-dessus n'a pas beaucoup de sens. Pour le raffiner en un énoncé précis et significatif, nous devons procéder par les étapes suivantes :

- (1) Définir les *faisceaux caractères* : des fibrés en droites avec connexion intégrable correspondant aux systèmes locaux ℓ -adiques de rang un \mathcal{L}_χ en caractéristique positive.

- (2) Construire un espace de modules G^b de faisceaux caractères et analyser ses propriétés géométriques.
- (3) Étudier la géométrie des lieux dans G^b constitués des faisceaux caractères satisfaisant le proto-théorème ci-dessus.

La première étape est simple. Pour un groupe algébrique commutatif et connexe G sur \mathbb{F}_q , avec loi de groupe $m: G \times G \rightarrow G$, il existe une caractérisation claire des systèmes locaux ℓ -adiques de la forme \mathcal{L}_χ pour un certain caractère χ . Ce sont les systèmes locaux ℓ -adiques de rang un \mathcal{L} qui satisfont la condition $m^* \mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$ —un isomorphisme reflétant l'équation $\chi(xy) = \chi(x)\chi(y)$. Plus précisément, les morphismes de groupes

$$\left\{ \begin{array}{l} \text{Caractères} \\ \chi: G(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_\ell^\times \end{array} \right\} \begin{array}{c} \xrightarrow{\chi \mapsto \mathcal{L}_\chi} \\ \xleftarrow{\text{tr. } \mathcal{L} \leftrightarrow \mathcal{L}} \end{array} \left\{ \begin{array}{l} \text{Classes d'isomorphisme de systèmes} \\ \text{locaux } \ell\text{-adiques de rang un } \mathcal{L} \text{ sur } G \\ \text{satisfaisant } m^* \mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L} \end{array} \right\}$$

sont des inverses l'un de l'autre [ST21, Lem. 2.16]. Cette observation mène à la définition suivante.

DÉFINITION (Faisceau caractère). Un *faisceau caractère* est un fibré en droites muni d'une connexion intégrable (\mathcal{L}, ∇) sur G satisfaisant $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$.

Pour mieux comprendre ces objets, rappelons le théorème de Barsotti–Chevalley, qui affirme que le groupe algébrique G s'insère dans une suite exacte courte

$$0 \rightarrow L \rightarrow G \rightarrow A \rightarrow 0,$$

où L est un groupe linéaire et A est une variété abélienne. De plus, le groupe linéaire L se décompose en un produit d'un tore T et d'un groupe unipotent U . Pour des raisons qui deviendront bientôt apparentes¹, nous désignons par $H_m^1(G_{\text{dR}}, \mathbb{G}_m)$ le groupe des classes d'isomorphisme de faisceaux caractères sur G . Ces groupes ont des descriptions simples pour les groupes unipotents, les tores et les variétés abéliennes.

■ **EXEMPLE.** En caractéristique zéro, un groupe unipotent U est toujours de la forme \mathbb{G}_a^n pour un certain n . Le groupe $H_m^1(U_{\text{dR}}, \mathbb{G}_m)$ est alors naturellement isomorphe à \mathbb{C}^n via l'application

$$\begin{aligned} \mathbb{C}^n &\rightarrow H_m^1(U_{\text{dR}}, \mathbb{G}_m) \\ (\chi_1, \dots, \chi_n) &\mapsto (\mathcal{O}_U, d - \chi_1 dt_1 - \dots - \chi_n dt_n). \end{aligned}$$

¹La démonstration du Théorème A introduit l'espace de de Rham G_{dR} , et $H^1(G_{\text{dR}}, \mathbb{G}_m)$ est sa cohomologie fppf à coefficients dans \mathbb{G}_m . Le groupe $H_m^1(G_{\text{dR}}, \mathbb{G}_m)$ est le sous-groupe constitué des fibrés en droites satisfaisant la condition de caractère.

De même, sur \mathbb{C} , un tore T est isomorphe à une puissance du groupe multiplicatif \mathbb{G}_m^r . Ici, le groupe $H_m^1(T_{\text{dR}}, \mathbb{G}_m)$ est naturellement isomorphe à $(\mathbb{C}/\mathbb{Z})^r$ via l'application

$$(\mathbb{C}/\mathbb{Z})^r \rightarrow H_m^1(T_{\text{dR}}, \mathbb{G}_m)$$

$$(\chi_1, \dots, \chi_r) \mapsto \left(\mathcal{O}_T, d - \chi_1 \frac{dt_1}{t_1} - \dots - \chi_r \frac{dt_r}{t_r} \right).$$

Pour les variétés abéliennes A , tout fibré en droites muni d'une connexion intégrable est un faisceau caractère. En effet, un fibré en droites admettant une connexion intégrable a une première classe de Chern nulle, ce qui implique qu'il appartient à $\text{Pic}^0(A)$. ■

Dans le contexte de la théorie de Hodge non-abélienne, Simpson a construit un espace de modules de fibrés en droites munis d'une connexion plate sur une variété projective lisse [Sim94]. En particulier, il existe un espace de modules de faisceaux caractères sur les variétés abéliennes. Pour les groupes affines, cependant, le groupe des fibrés en droites avec connexion plate est trop grand², mais la condition de caractère résout efficacement ce problème.

THÉORÈME A (3.2.6). *Il existe un espace algébrique en groupes lisse et connexe G^b satisfaisant $\dim G \leq \dim G^b \leq 2 \dim G$, dont les \mathbb{C} -points paramètrent les faisceaux caractères.*

Les techniques employées dans la démonstration de ce théorème représentent la principale nouveauté de cette thèse. Ces méthodes sont distinctes des autres approches à l'annulation générique et conduisent à une nouvelle construction de l'espace de modules de Simpson pour les variétés abéliennes.

Esquisse de la démonstration du Théorème A. L'inspiration pour cette démonstration vient de la formule de Barsotti–Weil, que nous généralisons dans le Théorème F. Pour une variété abélienne A , cette formule affirme que le faisceau d'extension $\underline{\text{Ext}}^1(A, \mathbb{G}_m)$, calculé dans la catégorie des faisceaux abéliens sur le site $\text{fppf}(\text{Sch}/\mathbb{C})_{\text{fppf}}$, est représentable par la variété abélienne duale A' . Il s'agit d'un espace de modules de fibrés en droites \mathcal{L} sur A satisfaisant $m^* \mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$.

L'élément suivant est l'espace de de Rham. Pour une variété algébrique X , les fibrés en droites sur l'espace de de Rham X_{dR} correspondent aux fibrés en droites munis d'une connexion plate sur X . Par conséquent, on peut montrer que le faisceau abélien $A^{\natural} := \underline{\text{Ext}}^1(A_{\text{dR}}, \mathbb{G}_m)$ paramètre les faisceaux caractères sur A et est représentable par un groupe algébrique connexe. Ce groupe algébrique coïncide avec l'espace de modules de Simpson.

²Par exemple, « l'espace de modules » de fibrés en droites munis d'une connexion plate sur \mathbb{A}^1 est l'ind-schéma $\mathbb{A}^\infty := \text{colim}_n \mathbb{A}^n$, tandis que l'espace des modules des faisceaux caractères sur \mathbb{G}_a est \mathbb{G}_a lui-même.

Pour un groupe algébrique connexe commutatif général G , le Théorème 2.3.21 fournit un isomorphisme entre les \mathbb{C} -points du faisceau $G^\natural := \underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)$ et le groupe $H_m^1(G_{\text{dR}}, \mathbb{G}_m)$. Si G est semi-abélien (c'est-à-dire une extension d'une variété abélienne par un tore), alors ce faisceau abélien est représentable par un espace algébrique lisse et connexe, établissant ainsi le théorème.

Cependant, lorsque G contient un sous-groupe unipotent, le faisceau G^\natural n'est plus représentable par un espace algébrique. Dans le Chapitre 3, nous avons calculé G^\natural de la manière la plus explicite possible et identifié les constituants problématiques. En excluant ces parties problématiques, nous obtenons enfin l'espace de modules souhaité G^\flat . \square

À la suite des calculs explicites nécessaires pour la démonstration précédente, nous établissons une forme normale pour les faisceaux caractères. Soit $\psi: G \rightarrow A$ l'application issue de la décomposition de Barsotti–Chevalley, et soit Ω_G le groupe vectoriel des différentielles *invariantes* sur G . Chaque $\omega \in \Omega_G$ induit un faisceau caractère $\mathcal{L}_\omega := (\mathcal{O}_G, d + \omega)$ sur G . Enfin, soit \mathcal{L}_χ le faisceau caractère sur G correspondant à un point $\chi \in G^\flat$.

COROLLAIRE (3.1.10). *Tout faisceau caractère sur G peut être exprimé comme $\mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha$, pour certains $\omega \in \Omega_G$ et $\alpha \in A^\flat$.*

Bien que la décomposition ci-dessus ne soit pas unique, le Corollaire 3.1.10 fournit une caractérisation précise des différentes décompositions possibles d'un faisceau caractère. Après avoir examiné les Étapes (1) et (2), nous passons maintenant à l'Étape (3), en commençant par le cas des groupes unipotents.

THÉORÈME B (5.3.1). *Soit U un groupe unipotent commutatif sur \mathbb{C} , et soit M un \mathcal{D} -module holonome sur U . Il existe un sous-ensemble ouvert dense V de U^\flat tel que*

$$\begin{aligned} H_c^i(U, M \otimes \mathcal{L}_\chi) &= H^i(U, M \otimes \mathcal{L}_\chi) = 0 \text{ pour } i \neq 0; \\ H_c^0(U, M \otimes \mathcal{L}_\chi) &\simeq H^0(U, M \otimes \mathcal{L}_\chi) \end{aligned}$$

soit vérifié pour tout $\chi \in V$.

Un groupe unipotent U sur \mathbb{C} est nécessairement un groupe vectoriel, l'espace de modules U^\flat étant isomorphe au groupe vectoriel dual U^* . Pour un \mathcal{D} -module holonome donné sur U , nous suspectons que le lieu des caractères *bons* est bien plus structuré qu'un simple sous-ensemble ouvert dense de U^* . Nous conjecturons que son complément consiste en une union finie de sous-espaces affines de codimension positive. Cela conduit à la définition suivante.

DÉFINITION (3.3.1, 3.3.4). *Soit G un groupe algébrique commutatif et connexe sur \mathbb{C} . Pour un épimorphisme $\rho: G \twoheadrightarrow \tilde{G}$ avec noyau connexe, l'image de $\rho^\flat: \tilde{G}^\flat \hookrightarrow G^\flat$ est*

appelée un *sous-espace linéaire* de G^b . Un *sous-espace générique* de G^b est le complément d'une union finie de translatés de sous-espaces linéaires de G^b de codimension positive.

Ces sous-espaces sont simples à décrire pour les groupes algébriques linéaires et peuvent être reliés à des concepts établis dans la littérature sur les variétés abéliennes.

■ **EXEMPLE — 3.3.2.** Considérons un tore T avec groupe des caractères X . Un sous-espace linéaire de $T^b \simeq \Omega_T/X$ est de la forme V/Y , où Y est un sous-groupe de X et V est le sous-espace linéaire de Ω_T généré par Y , au sens de l'algèbre linéaire. Pour un groupe unipotent U , un sous-espace linéaire de $U^b = U^*$ correspond à un sous-espace linéaire de l'espace vectoriel sous-jacent de U^* , au sens de l'algèbre linéaire.

Pour une variété abélienne A , les sous-espaces linéaires de $A^b \simeq A^{\natural}$ ont été étudiés pour la première fois par Simpson, qui les a appelés *triple tori* [Sim93, p. 365]. Schnell désigne par *linear subvarieties* les translatés de sous-espaces linéaires de A^b [Sch15, Def. 2.3]. ■

Bien que nous n'ayons pas encore prouvé que le sous-espace des bons caractères dans U^b est générique³, nous sommes en mesure d'établir un résultat similaire pour les variétés semi-abéliennes. Ici, nous notons par $\varphi: L \rightarrow G$ l'application provenant de la décomposition de Barsotti–Chevalley.

THÉORÈME C (5.2.1, 5.4.3). *Soit G une variété semi-abélienne sur \mathbb{C} , et soit M un objet de $D_h^b(\mathcal{D}_G)$. Il existe alors un sous-espace générique V de L^b tel que l'application d'oubli de supports*

$$H_c^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha) \rightarrow H^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha)$$

soit un isomorphisme pour tout $\alpha \in A^b$ et tout $\omega \in \Omega_G$ tel que $\varphi^ \mathcal{L}_\omega$ appartienne à V . De plus, si M est concentré en degré zéro, pour chaque ω comme ci-dessus, il existe un sous-espace générique W de A^b tel que*

$$H_c^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha) \simeq H^*(G, M \otimes \mathcal{L}_\omega \otimes \psi^* \mathcal{L}_\alpha)$$

soit concentré en degré zéro pour tout $\alpha \in W$.

Le cas particulier des variétés abéliennes dans le théorème ci-dessus a été prouvé par Schnell [Sch15, Thms. 2.2 et 4.1]. De plus, Krämer prouve un résultat similaire pour M régulier dans [Krä14, Thm. 2.1]. Ce premier résultat sert d'ingrédient à notre démonstration, qui a été inspirée par le second.

Nous sommes maintenant en mesure d'introduire la construction du *groupe de symétrie de de Rham*, comme promis précédemment. L'idée clé est d'appliquer le formalisme des catégories tannakiennes, qui fournit des critères pour qu'une catégorie

³Cependant, la Remarque 5.3.5 présente une idée prometteuse pour une telle preuve.

soit équivalente à la catégorie des représentations de dimension finie d'un schéma en groupes affine.

L'exigence fondamentale pour les catégories tannakiennes est qu'elles soient monoïdales symétriques, ce qui signifie qu'elles possèdent une opération binaire correspondant au produit tensoriel des représentations. Dans notre contexte, cette opération est fournie par la convolution, qui se manifeste sous deux formes :

$$\begin{aligned} D_h^b(\mathcal{D}_G) \times D_h^b(\mathcal{D}_G) &\longrightarrow D_h^b(\mathcal{D}_G) & D_h^b(\mathcal{D}_G) \times D_h^b(\mathcal{D}_G) &\longrightarrow D_h^b(\mathcal{D}_G) \\ (M, N) &\longmapsto m_-(M \boxtimes N) & (M, N) &\longmapsto m_+(M \boxtimes N). \end{aligned}$$

Bien que ces opérations ne coïncident pas et ne préservent pas les \mathcal{D} -modules holonomes en degré zéro, elles le font à *objets négligeables près*.

DÉFINITION (6.2.1). Soit M un \mathcal{D} -module holonome sur G . On dit que M est *négligeable* s'il existe un sous-espace générique V de G^b tel que

$$H_c^*(G, M \otimes \mathcal{L}_\chi) = 0$$

pour tout $\chi \in V$. Nous désignons par $\text{Neg}(\mathcal{D}_G)$ la sous-catégorie pleine de $\text{Hol}(\mathcal{D}_G)$ constituée des \mathcal{D} -modules holonomes négligeables.

THÉORÈME D (6.2.12). Soit G le groupe additif, un tore ou une variété abélienne. La catégorie $\text{Neg}(\mathcal{D}_G)$ est une sous-catégorie épaisse de $\text{Hol}(\mathcal{D}_G)$, et le quotient $\text{Hol}(\mathcal{D}_G)/\text{Neg}(\mathcal{D}_G)$ est tannakienne sous convolution. En particulier, chaque \mathcal{D} -module holonome M sur G a un groupe de symétrie associé $\text{Conv}(M)$.

Ces groupes de symétrie peuvent être décrits de manière assez concrète lorsque G est soit le groupe additif \mathbb{G}_a , soit le groupe multiplicatif \mathbb{G}_m .

THÉORÈME E (6.3.11, 6.4.5). Soit M un \mathcal{D} -module holonome sur \mathbb{G}_a , et soit $\text{FT}(M)$ sa transformée de Fourier. La fibre générique $\text{FT}(M)_\eta$ est un module différentiel dont le groupe de Galois différentiel est isomorphe à $\text{Conv}(M)$.

De même, soit N un \mathcal{D} -module holonome sur \mathbb{G}_m , et soit $\text{MT}(N)$ sa transformée de Mellin. La fibre générique $\text{MT}(N)_\eta$ est un module aux différences dont le groupe de Galois aux différences est isomorphe à $\text{Conv}(N)$.

Dans [Krä22, Thm. 2.2], Krämer a établi un résultat similaire pour la transformée de Fourier–Mukai des \mathcal{D} -modules holonomes sur les variétés abéliennes. Cette caractérisation a ensuite été utilisée pour montrer que les groupes de symétrie associés sont presque connexes. La même conclusion s'applique pour \mathbb{G}_m dans notre contexte, malgré les méthodes différentes employées, et on montre que le résultat analogue pour \mathbb{G}_a est faux.

EXTENSIONS DE FAISCEAUX ABÉLIENS

La démonstration du Théorème A a nécessité une analyse détaillée et nuancée des faisceaux fppf d’extensions. En chemin, nous avons découvert plusieurs résultats qui méritent d’être signalés pour eux-mêmes. À notre surprise, nous avons constaté que la célèbre formule de Barsotti–Weil semble manquer d’une preuve complète publiée dans la littérature, en plus d’autres incohérences.

N’ayant pas trouvé de référence pour la formule de Barsotti–Weil dans le niveau de généralité dont nous avons besoin, nous avons établi la généralisation suivante.

THÉORÈME F (2.3.10). *Soit A un schéma abélien et B un schéma en groupes commutatif affine sur un schéma de base S . Pour un S -schéma T , les applications naturelles*

$$\underline{\mathrm{Ext}}^1(A, B)(T) \leftarrow \mathrm{Ext}_T^1(A, B) \rightarrow H_m^1(A_T, B_T)$$

sont des isomorphismes. (Voir 2.3.6 pour la définition de $H_m^1(A_T, B_T)$.) En particulier, le faisceau abélien $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ est représentable par le schéma abélien dual A' .

Pour un schéma abélien A sur une base régulière S , l’annulation de $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ a été affirmée dans [Bre75, Rem. 6], [Jos09, Thm. 1.2.5], et [Bro21, Cor. 11.5]. Malheureusement, dans chaque cas, ce résultat est attribué à [Bre69, §7], mais des arguments supplémentaires semblent manquer. Le théorème suivant, développé en collaboration avec Zev Rosengarten, vise à combler cette lacune, du moins en caractéristique zéro.

THÉORÈME G (2.4.4). *Soit A un schéma abélien sur un schéma de caractéristique zéro S . Alors le faisceau abélien $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ est nul.*

Un autre faisceau abélien qui joue un rôle important dans cette thèse est $\underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$, pour un groupe unipotent commutatif U sur un corps de caractéristique zéro. Dans [Ros23, Rem. 2.2.16], Rosengarten détaille la construction par Gabber d’une section non nulle de ce faisceau. Dans l’Exemple 2.3.17, nous fournissons ce qui pourrait être la section la plus simple de ce type.

À regret, l’annulation de ce faisceau a été affirmée dans la littérature. Cette affirmation se trouve dans [Pol11, Lem. 1.3.6], a été utilisée dans les démonstrations de [BB09, Lem. A.4.5] et [Ber14, Lem. 10], et peut être dérivée des calculs dans [Lau85, Ex. p. 25]. Notre dernier théorème démontre que, bien que ce faisceau ne soit pas nul, ses images s’annulent souvent.

THÉORÈME H (2.3.16). *Soit U un groupe unipotent commutatif sur un corps de caractéristique zéro k . Alors le groupe $\underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)(T)$ est nul pour tous les k -schémas seminormaux T .*

Nous notons qu’un résultat similaire a été annoncé par Gabber lors d’une conférence récente [Gab23]. Nos arguments sont indépendants, et nous n’étions pas au courant de la démonstration de Gabber lors de l’élaboration de cette thèse.

PART I.

PRELIMINARIES

1. QUASI-COHERENT SHEAVES AND \mathcal{D} -MODULES

Coherent and de Rham cohomology, like most cohomology theories in algebraic geometry, have natural coefficients: quasi-coherent sheaves and \mathcal{D} -modules. This chapter introduces the main concepts needed for their study in this thesis. Much of the material is classical, albeit sometimes lacking precise references.

SIX-FUNCTOR FORMALISMS

A guiding thread in our approach is the notion of a six-functor formalism. For each cohomology theory, the assignment of a (derived) category of coefficients $D(X)$ to a geometric object X should exhibit a rich functoriality. First, the categories $D(X)$ must be closed symmetric monoidal, meaning they are equipped with a tensor product $- \otimes -$ that is left adjoint to an internal hom functor $\underline{\mathrm{Hom}}(-, -)$.

A morphism $f: X \rightarrow S$ between geometric objects induces adjoint inverse and direct image functors $f^*: D(S) \rightarrow D(X)$ and $f_*: D(X) \rightarrow D(S)$, respectively. Additionally, for sufficiently nice morphisms f , there exist adjoint proper direct image and exceptional inverse image functors $f_!: D(X) \rightarrow D(S)$ and $f^!: D(S) \rightarrow D(X)$, respectively. Recently, Liu–Zheng and Mann succinctly encoded the myriad expected relations between these six functors in Definition 1.2.8.

Mann also introduced a method, detailed in Theorem 1.2.13, for constructing six-functor formalisms, which significantly simplifies classical constructions and provides more precise information. Furthermore, Scholze’s Theorem 1.2.18 demonstrates that six-functor formalisms naturally extend to stacks, with a broad class of morphisms having $!$ -functors [Sch22].

QUASI-COHERENT SHEAVES

We would like to apply this framework to construct a six-functor formalism for quasi-coherent sheaves on schemes. A fundamental property of six-functor formalisms is

1. Quasi-coherent sheaves and \mathcal{D} -modules

that a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

induces an isomorphism $g^*f_! \simeq f'_!g'^*$. Although there is no straightforward definition of the proper direct image functors, the functors g^*f_* and $f'_*g'^*$ are not far from being isomorphic. To illustrate this, suppose that the schemes $X = \text{Spec } B$, $S = \text{Spec } A$, and $S' = \text{Spec } A'$ are affine. Then, for a B -module M , we seek an isomorphism

$$M \otimes_B^{\mathbb{L}} (B \otimes_A A') \simeq M \otimes_A^{\mathbb{L}} A'.$$

It becomes evident that, for this isomorphism to hold, the tensor product $B \otimes_A A'$ should also be derived. Additionally, to consider the spectrum of $B \otimes_A^{\mathbb{L}} A'$, it must possess some sort of ring structure. This leads us into the world of higher algebra and derived algebraic geometry.

We assume that the reader believes in the existence of a theory of ∞ -categories, in which the ∞ -category \mathbf{An} of *anima* plays the role of the category of sets. The objects X of \mathbf{An} should be thought as derived sets, endowed with *homotopy groups* $\pi_i(X)$ for $i \geq 0$. There is a fully faithful functor $\text{Set} \hookrightarrow \mathbf{An}$, whose essential image consists of the anima X satisfying $\pi_i(X) = 0$ for $i > 0$. All additional necessary details about ∞ -categories will be explained in the main text.

In Section 1.3, we construct the ∞ -category $\mathbf{An}(\text{Ring})$ of *animated rings*, serving as the desired derived analogues of commutative rings. With the local theory established, the fundamental objects of algebraic geometry can be defined elegantly. A *derived stack* is a functor $\mathbf{An}(\text{Ring}) \rightarrow \mathbf{An}$ satisfying fppf descent. Then, a *derived scheme* is a derived stack that is locally, with respect to the Zariski topology, isomorphic to an affine derived scheme. The latter being the derived stack corepresented by an animated ring.

The functor mapping an animated ring R to its derived ∞ -category $D(R)$ forms a sheaf of ∞ -categories. This allows us to define the ∞ -category of quasi-coherent sheaves $D_{\text{qc}}(X)$ on a derived scheme X by descent. Mann's method yields in Theorem 1.3.20 a six-functor formalism for quasi-coherent sheaves on quasi-compact quasi-separated derived schemes. Remark 1.3.21 discusses the relationship between this formalism and other approaches to Grothendieck duality.

Finally, Scholze's Theorem 1.2.18 extends this six-functor formalism to derived stacks. For more detailed information, see Theorem 1.3.23. Interestingly, although the functor $f_!$ was *defined* as f_* for a morphism f between quasi-compact quasi-separated derived schemes, the forget-supports map $f_! \rightarrow f_*$ may not be an isomorphism when f is a morphism between derived stacks.

DE RHAM SPACES AND \mathcal{D} -MODULES

As first observed by Simpson [Sim96], given a cohomology theory H and a geometric object X , there often exists a stack X_H whose ∞ -category of quasi-coherent sheaves matches the ∞ -category of coefficients for H . Furthermore, the association $X \mapsto X_H$ preserves the natural functoriality of H .

In this chapter, we explore the de Rham side of this story. For a smooth algebraic variety X over a characteristic zero field k , the *de Rham space* X_{dR} is a derived stack obtained as a quotient of X by identifying infinitesimally close points. We note that the functor X_{dR} takes values in the full subcategory Set of An , making it essentially a sheaf of sets in the site $(\mathrm{Sch}/k)_{\mathrm{fppf}}$.

For a commutative algebraic group G , the difference of two infinitesimally close points lies in an infinitesimal neighborhood of the identity. Consequently, denoting its formal completion along the identity by \widehat{G} , the de Rham space G_{dR} is naturally isomorphic to the quotient G/\widehat{G} . This isomorphism will be of paramount importance in Part II.

Theorem 1.4.4 relates quasi-coherent sheaves on de Rham spaces to \mathcal{D} -modules. More precisely, for a smooth morphism $p: X \rightarrow S$ between k -schemes of finite type, there exists a natural morphism $\iota_p: X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ inducing a symmetric monoidal equivalence of ∞ -categories

$$D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S) \simeq D_{\mathrm{qc}}(\mathcal{D}_{X/S}).$$

Under this equivalence, the functor ι_p^* corresponds to the forgetful functor of left $\mathcal{D}_{X/S}$ -modules, $\iota_p^!$ corresponds to the forgetful functor of right $\mathcal{D}_{X/S}$ -modules, and $\iota_{p,!}$ corresponds to the induced $\mathcal{D}_{X/S}$ -module functor.

According to Corollaries 1.4.8 and 1.4.12, the natural functoriality of \mathcal{D} -modules can also be accessed from the de Rham spaces. Consider a commutative diagram of finite type k -schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & & \downarrow p \\ S' & \xrightarrow{g} & S, \end{array}$$

where p and q are smooth, and f is quasi-projective. The universal property of fiber products provides a natural map $(f, g)_{\mathrm{dR}}: X'_{\mathrm{dR}} \times_{S'_{\mathrm{dR}}} S' \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$. Then, the functor $(f, g)_{\mathrm{dR}}^*$ corresponds to the "naive" inverse image of \mathcal{D} -modules, while the functor $(f, g)_{\mathrm{dR},!}$ corresponds, up to a shift, to the direct image of \mathcal{D} -modules.

HOLONOMIC \mathcal{D} -MODULES

The six-functor formalism for quasi-coherent sheaves on de Rham spaces is quite general but possesses some unusual features. For instance, given an open immersion

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$j: U \rightarrow X$, one might expect the map $j_{\text{dR}}: U_{\text{dR}} \rightarrow X_{\text{dR}}$ to be cohomologically étale, meaning that $j_{\text{dR}}^* \simeq j_{\text{dR}}^!$. Yet, Corollary 1.4.3 shows that it is cohomologically proper, so $j_{\text{dR},!} \simeq j_{\text{dR},*}$.

Section 1.5 constructs a dual version of this six-functor formalism, after restricting to holonomic \mathcal{D} -modules. The four functors $f_!$, f^+ , f_+ , and $f^!$ associated to a morphism f are well-known in the \mathcal{D} -module literature. However, the requirement of having a six-functor formalism compels us to consider a tensor product \otimes_X and an internal hom functor $\underline{\text{Hom}}_X$ that are familiar to some experts but largely absent from the literature.

We also consider integrable connections, seen as \mathcal{O} -coherent \mathcal{D} -modules, systematically in degree $\dim X$. Remark 1.5.2 outlines many pleasing consequences of this choice. For now, the reader can take comfort in knowing that, under the Riemann–Hilbert correspondence, our tensor product and internal hom correspond to the conventional tensor product and internal hom of constructible sheaves. Moreover, the Riemann–Hilbert correspondence maps regular connections in degree $\dim X$ to local systems.

The reader can find a table comparing our notation with those of the most common references on page 83.

1.1. THE DE RHAM SPACE

Let k be a field and consider the category Aff/k of affine schemes over k . In order to simplify notation, we will often denote an object $\text{Spec } R$ of Aff/k as R .

DEFINITION 1.1.1 (de Rham space). Given a presheaf of sets X on Aff/k , its *de Rham space* X_{dR} is the presheaf defined by

$$X_{\text{dR}}(R) := \text{colim}_{I \subset R} X(R/I),$$

where the colimit runs over the filtered poset of nilpotent ideals of R . This presheaf comes equipped with a morphism $X \rightarrow X_{\text{dR}}$ induced by the trivial ideal $I = 0$.

This assignment is functorial: given a morphism of presheaves $f: X \rightarrow Y$, there is an induced map $f_{\text{dR}}: X_{\text{dR}} \rightarrow Y_{\text{dR}}$, making the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_{\text{dR}} \\ f \downarrow & & \downarrow f_{\text{dR}} \\ Y & \longrightarrow & Y_{\text{dR}} \end{array}$$

commute. As it will be formalized in Corollary 1.1.7, the geometric interpretation of X_{dR} , at least for smooth k -schemes X , is that it is a quotient of X where we identify infinitesimally close points.

We begin our study of the de Rham space with the following simple observation, which will prove useful later.

PROPOSITION 1.1.2. *The functor $(-)\text{dR}: \text{PSh}(\text{Aff}/k) \rightarrow \text{PSh}(\text{Aff}/k)$ preserves finite limits and arbitrary colimits.*

Proof. Since limits and colimits of presheaves are computed pointwise, this follows from the fact that filtered colimits in the category of sets commute with arbitrary colimits and finite limits. \square

For a finite type k -algebra R (or more generally, a noetherian k -algebra), the nilradical $\text{Nil}(R)$ is nilpotent as it is generated by finitely many nilpotent elements. Consequently, $X_{\text{dR}}(R) \simeq X(R/\text{Nil}(R)) = X(R_{\text{red}})$. This property holds for any k -algebra provided X is a finite type scheme.

PROPOSITION 1.1.3. *Let X be a locally of finite type scheme over k . Then $X_{\text{dR}}(R) \simeq X(R_{\text{red}})$ for every k -algebra R .*

Proof. Define $S := \text{colim}_{I \subset R} R/I$, where the colimit runs through the nilpotent ideals of R . As usual, elements of S are denoted as equivalence classes of the form $[I, x]$, where I is a nilpotent ideal in R and x is an element of R . Here, $[I, x] = [I', x']$ if there exists a nilpotent ideal J containing I and I' such that $x \equiv x' \pmod{J}$.

The natural map $R \rightarrow S$, corresponding to the ideal $I = 0$, sends every nilpotent in R to zero. In other words, it factors through the nilradical yielding a map $R_{\text{red}} \rightarrow S$. We affirm that this morphism is injective. Indeed, $[0, x] = 0$ means that there exists a nilpotent ideal J containing x . It follows that x is nilpotent and so vanishes on R_{red} . Since $[I, x] \in S$ is the image of $x \in R$, we have that $R_{\text{red}} \rightarrow S$ is an isomorphism.

As X is locally of finite type, [Stacks, Tag 01ZC] gives $X_{\text{dR}}(R) \simeq X(S) \simeq X(R_{\text{red}})$, concluding the proof. \square

Perhaps not surprisingly, given the aforementioned geometric interpretation of X_{dR} , formal completions of schemes can be written in terms of de Rham spaces.

PROPOSITION 1.1.4. *Let X be a k -scheme and let Z be a closed subscheme of X . The formal completion \widehat{X}_Z of X along Z is isomorphic to $X \times_{X_{\text{dR}}} Z_{\text{dR}}$.*

Proof. Let $\mathcal{J} \subset \mathcal{O}_X$ be the ideal sheaf defined by Z , and let R be a k -algebra. We aim to obtain a functorial isomorphism

$$\text{colim}_{I \subset R} X(R) \times_{X(R/I)} Z(R/I) \simeq \text{colim}_{n \geq 0} \text{Spec}_X(\mathcal{O}_X/\mathcal{J}^{n+1})(R),$$

where the colimit on the left runs through the nilpotent ideals of R . For an ideal $I \subset R$,

1. Quasi-coherent sheaves and \mathcal{D} -modules

let i_I denote the closed immersion $\mathrm{Spec} R/I \rightarrow \mathrm{Spec} R$. Then, we have that

$$\begin{aligned} \mathrm{colim}_{I \subset R} X(\mathbf{R}) \times_{X(\mathbf{R}/I)} Z(\mathbf{R}/I) &\simeq \mathrm{colim}_{I \subset R} \{x \in X(\mathbf{R}) \mid i_1^* x^* \mathcal{J} = 0\} \\ &\simeq \mathrm{colim}_{n \geq 0} \mathrm{colim}_{I^{n+1}=0} \{x \in X(\mathbf{R}) \mid i_1^* x^* \mathcal{J} = 0\} \\ &\simeq \mathrm{colim}_{n \geq 0} \{x \in X(\mathbf{R}) \mid x^* \mathcal{J}^{n+1} = 0\} \simeq \widehat{X}_Z(\mathbf{R}), \end{aligned}$$

where the last isomorphism is the universal property of the relative spectrum. \square

Note that the expression $X \times_{X_{\mathrm{dR}}} Z_{\mathrm{dR}}$ is meaningful even if $Z \rightarrow X$ is not a closed immersion. In such cases, it can be used to *define* formal completions. Moreover, this characterization makes it clear that the projection $\widehat{X}_Z \rightarrow X$ is a monomorphism whenever $Z \rightarrow X$ is.

COROLLARY 1.1.5. *If Z is a closed subscheme of a k -scheme X , the projection $\widehat{X}_Z \rightarrow X$ is a monomorphism of presheaves.*

Proof. According to [Stacks, Tag 01L7], the closed immersion $i: Z \rightarrow X$ is a monomorphism in the category of schemes. Now, the Yoneda embedding preserves limits and any functor that preserves limits preserves monomorphisms [Stacks, Tag 01L3]. In other words, i is a monomorphism of presheaves. Since the de Rham functor preserves finite limits, so is $Z_{\mathrm{dR}} \rightarrow X_{\mathrm{dR}}$. Finally, fibered products preserve monomorphisms and this finishes the proof. \square

For a morphism of k -schemes $p: X \rightarrow S$, the universal product of fibered products induces a map $\iota_p: X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$. As the following proposition shows, this map faithfully encodes the differential information contained in p .¹

PROPOSITION 1.1.6. *Let $p: X \rightarrow S$ be a morphism of k -schemes. Then p is formally smooth (resp. formally unramified) if and only if $\iota_p: X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ is an epimorphism (resp. monomorphism) of presheaves.*

Proof. Recall that p is formally smooth (resp. formally unramified) if, for every k -algebra R with a map $\mathrm{Spec} R \rightarrow S$ and for every nilpotent ideal $I \subset R$, the induced map

$$\mathrm{Hom}_S(\mathrm{Spec} R, X) \rightarrow \mathrm{Hom}_S(\mathrm{Spec} R/I, X)$$

is surjective (resp. injective). We will translate this condition into the surjectivity (resp. injectivity) of $X(\mathbf{R}) \rightarrow (X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)(\mathbf{R})$.

Suppose that p is formally smooth, let R be any k -algebra, and let $[x, s]$ be an element of

$$(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)(\mathbf{R}) \simeq \mathrm{colim}_{I \subset R} X(\mathbf{R}/I) \times_{S(\mathbf{R}/I)} S(\mathbf{R}).$$

¹This result is somewhat akin to the slogan "f is smooth (resp. unramified) if and only if df is surjective (resp. injective)".

That is, there exists a nilpotent ideal $I \subset R$ such that $\chi \in X(R/I)$ and $s \in S(R)$ agree on $S(R/I)$. By formal smoothness, we have a morphism $\bar{\chi}: \text{Spec } R \rightarrow X$ making the diagram

$$\begin{array}{ccc} \text{Spec } R/I & \xrightarrow{\chi} & X \\ \downarrow & \nearrow \bar{\chi} & \downarrow p \\ \text{Spec } R & \xrightarrow{s} & S \end{array}$$

commute. (The lower triangle commutes since $\bar{\chi}$ is a morphism over S , and the upper triangle commutes because $\bar{\chi}$ maps to χ .) This is an element of $X(R)$ mapping to $(X_{\text{dR}} \times_{S_{\text{dR}}} S)(R)$. Conversely, suppose that R is a k -algebra with a map $s: \text{Spec } R \rightarrow S$, $I \subset R$ is a nilpotent ideal, and χ is an element of $\text{Hom}_S(\text{Spec } R/I, X)$. This data defines an element of $(X_{\text{dR}} \times_{S_{\text{dR}}} S)(R)$ and so there exists $\bar{\chi} \in X(R)$ mapping to it. In other words, $\iota_p: X \rightarrow X_{\text{dR}} \times_{S_{\text{dR}}} S$ is an epimorphism if and only if p is formally smooth.

Now, if p is formally unramified, consider a k -algebra R and let $\chi, \psi \in X(R)$ be two elements whose images in $X_{\text{dR}} \times_{S_{\text{dR}}} S(R)$ coincide. That is, there exists a nilpotent ideal $I \subset R$ such that the diagram

$$\text{Spec } R/I \rightarrow \text{Spec } R \rightrightarrows X \xrightarrow{p} S$$

commutes. In particular, χ and ψ define elements of $\text{Hom}_S(\text{Spec } R, X)$ that coincide on $\text{Hom}_S(\text{Spec } R/I, X)$. Since p is formally unramified, we have that $\chi = \psi$.

Suppose that $\iota_p: X \rightarrow X_{\text{dR}} \times_{S_{\text{dR}}} S$ is a monomorphism and let $\chi, \psi \in \text{Hom}_S(\text{Spec } R, X)$ be two morphisms that coincide on $\text{Hom}_S(\text{Spec } R/I, X)$, for some ring k -algebra R with a map $\text{Spec } R \rightarrow S$ and a nilpotent ideal $I \subset R$. In particular, χ and ψ are elements of $X(R)$ that coincide on $(X_{\text{dR}} \times_{S_{\text{dR}}} S)(R)$. It follows that $\chi = \psi$ and so p is formally unramified. \square

Let $Y \rightarrow X$ be an immersion of k -schemes that factors as $Y \rightarrow U \rightarrow X$, where $Y \rightarrow U$ is a closed immersion with ideal \mathcal{I} and $U \rightarrow X$ is an open immersion. The formal completion of X along Y is usually defined as the colimit of $\text{Spec}_{\mathcal{U}}(\mathcal{O}_{\mathcal{U}}/\mathcal{I}^{n+1})$, for $n \geq 0$. The previous proposition shows that $U \simeq U_{\text{dR}} \times_{X_{\text{dR}}} X$ and so

$$U \times_{U_{\text{dR}}} Z_{\text{dR}} \simeq X \times_{X_{\text{dR}}} U_{\text{dR}} \times_{U_{\text{dR}}} Z_{\text{dR}} \simeq X \times_{X_{\text{dR}}} Z_{\text{dR}},$$

proving that the Proposition 1.1.4 also works for locally closed immersions.

COROLLARY 1.1.7. *Let $X \rightarrow S$ be a formally smooth morphism of k -schemes. Then $X_{\text{dR}} \times_{S_{\text{dR}}} S$ is the coequalizer of*

$$\widehat{(X \times_S X)}_{\Delta} \rightrightarrows X,$$

where $\widehat{(X \times_S X)}_{\Delta}$ is the formal completion of $X \times_S X$ along the diagonal.

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Proof. In order to simplify notation, let $Y = X_{\text{dR}} \times_{S_{\text{dR}}} S$. Since every epimorphism is effective in a topos, $X \rightarrow Y$ is the coequalizer of $X \times_Y X \rightrightarrows X$. Now, the result follows from general category theory: the pullback of

$$\begin{array}{ccc} & X \times_S X & \\ & \downarrow & \\ X_{\text{dR}} & \xrightarrow{\Delta_{\text{dR}}} & (X \times_S X)_{\text{dR}} \end{array}$$

is $X \times_Y X$. □

Recall that the functor of points of a scheme X is a sheaf for the étale and fppf topologies. We will now study the descent properties of X_{dR} , and we begin with a lemma.

LEMMA 1.1.8. *Let R be a k -algebra and let $\{R \rightarrow R_i\}_{i \in I}$ be an étale covering. Then the reduction $\{R_{\text{red}} \rightarrow R_{i,\text{red}}\}_{i \in I}$ is also an étale covering. Moreover, any étale covering of R_{red} arises in this way.*

Proof. Since $R \rightarrow R_i$ is étale, so is its base change $R_{\text{red}} \rightarrow R_{\text{red}} \otimes_R R_i$. By [Stacks, Tag 033B], we have that $R_{\text{red}} \otimes_R R_i$ is reduced and then [EGA I, Cor. 5.1.8] gives that

$$R_{\text{red}} \otimes_R R_i = (R_{\text{red}} \otimes_R R_i)_{\text{red}} \simeq (R_{\text{red}} \otimes_{R_{\text{red}}} R_{i,\text{red}})_{\text{red}} \simeq R_{i,\text{red}}.$$

It follows that $\{R_{\text{red}} \rightarrow R_{i,\text{red}}\}_{i \in I}$ is an étale covering of R_{red} .

Now, consider an étale covering $\{R_{\text{red}} \rightarrow S_i\}_{i \in I}$ of R_{red} . By the topological invariance of the étale site, there exists a covering $\{R \rightarrow R_i\}_{i \in I}$ along with isomorphisms $R_{\text{red}} \otimes_R R_i \simeq S_i$ for all $i \in I$ [Stacks, Tag 04DZ]. The same argument as above shows that S_i is reduced, and then $S_i \simeq R_{i,\text{red}}$. □

PROPOSITION 1.1.9. *Let X be a locally of finite type scheme over k . The de Rham space X_{dR} is an étale sheaf on Aff/k .*

Proof. Let R be a k -algebra and let $\{R \rightarrow R_i\}_{i \in I}$ be an étale covering of R . We want to prove that the diagram

$$X(R_{\text{red}}) \rightarrow \prod_i X(R_{i,\text{red}}) \rightrightarrows \prod_{i,j} X((R_i \otimes_R R_j)_{\text{red}})$$

is an equalizer. The lemma above says that $\{R_{\text{red}} \rightarrow R_{i,\text{red}}\}_{i \in I}$ is also an étale cover and then the fact that X is an étale sheaf implies that the diagram

$$X(R_{\text{red}}) \rightarrow \prod_i X(R_{i,\text{red}}) \rightrightarrows \prod_{i,j} X(R_{i,\text{red}} \otimes_{R_{\text{red}}} R_{j,\text{red}})$$

is an equalizer. The same argument as in the proof of the previous lemma shows that $R_{i,\text{red}} \otimes_{R_{\text{red}}} R_{j,\text{red}}$ is reduced. Then [EGA I, Cor. 5.1.8] gives isomorphisms $R_{i,\text{red}} \otimes_{R_{\text{red}}} R_{j,\text{red}} \simeq (R_i \otimes_R R_j)_{\text{red}}$, finishing the proof. \square

In particular, the preceding proposition implies that the de Rham space of a commutative algebraic group over k is an abelian étale sheaf on Aff/k .

PROPOSITION 1.1.10. *The functor $(-)_{\text{dR}}$ from commutative algebraic groups over k to abelian étale sheaves on Aff/k is exact.*

Proof. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of commutative algebraic groups over k . In particular, it is left-exact in the category of abelian presheaves on Aff/k . By Proposition 1.1.2, the induced exact sequence $0 \rightarrow A_{\text{dR}} \rightarrow B_{\text{dR}} \rightarrow C_{\text{dR}} \rightarrow 0$ is also left-exact in abelian presheaves. Since sheafification is exact, this sequence is left-exact in the category of abelian étale sheaves.

Let us verify that $B_{\text{dR}} \rightarrow C_{\text{dR}}$ is an epimorphism of abelian sheaves. Given a k -algebra R and an element $c \in C_{\text{dR}}(R) = C(R_{\text{red}})$, the fact that $B \rightarrow C$ is an epimorphism of étale sheaves implies that there exists a covering $\{R_{\text{red}} \rightarrow S_i\}_{i \in I}$ such that $c|_{S_i}$ is in the image of $B(S_i) \rightarrow C(S_i)$ for all $i \in I$ [Stacks, Tag 00WN]. Lemma 1.1.8 then gives a covering $\{R \rightarrow R_i\}_{i \in I}$ whose reduction is $\{R_{\text{red}} \rightarrow S_i\}_{i \in I}$. It follows that $c|_{R_i} = c|_{S_i}$ is in the image of $B_{\text{dR}}(R_i) \rightarrow C_{\text{dR}}(R_i)$ for all $i \in I$, concluding the proof. \square

We proved in Corollary 1.1.7 that X_{dR} is a quotient of X in which we identify infinitesimally close points. When X is a commutative algebraic group G , the difference of two such points has to live in an infinitesimal neighborhood of the identity. This heuristic leads to the result below.

PROPOSITION 1.1.11. *Let G be a smooth commutative algebraic group over k . Then G_{dR} is isomorphic to the presheaf quotient G/\widehat{G} , where \widehat{G} is the formal completion of G along the identity. In particular, G_{dR} is also isomorphic to the sheaf quotient G/\widehat{G} .*

Proof. In this proof, let us consider every (co)limit to be taken inside the category of abelian presheaves on Aff/k . As the cokernel of the identity section $e: \text{Spec } k \rightarrow G$ is G itself, a variant of Proposition 1.1.2 for abelian presheaves shows that the cokernel of $e_{\text{dR}}: \text{Spec } k \rightarrow G_{\text{dR}}$ is G_{dR} . The universal property of cokernels then induces the dashed map below.

$$\begin{array}{ccccc}
 \widehat{G} & \longrightarrow & G & \longrightarrow & G/\widehat{G} \\
 \downarrow & & \downarrow & & \downarrow \text{---} \\
 \text{Spec } k & \xrightarrow{e_{\text{dR}}} & G_{\text{dR}} & \xlongequal{\quad} & G_{\text{dR}}
 \end{array}$$

The square on the left is cartesian due to Proposition 1.1.4, and $G \rightarrow G_{\text{dR}}$ is an epimorphism since G is smooth. Then, [Stacks, Tag 08N4] implies that the square

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on the left is also cocartesian, and [Stacks, Tag 08N3] gives that $G/\widehat{G} \rightarrow G_{\text{dR}}$ is an isomorphism. Since G_{dR} is already an étale sheaf, the presheaf and the sheaf quotients G/\widehat{G} coincide. \square

We now study the descent properties of de Rham spaces with respect to the finer fppf topology. We remark that the following lemma, along with the subsequent proposition and its corollary, are the unique results in this section that need the base field k to have characteristic zero.

LEMMA 1.1.12 (Cartier). *Let G be a commutative algebraic group over a characteristic zero field k , and let \mathfrak{g} be its Lie algebra seen as a vector group. The formal completions of G and of \mathfrak{g} along the identity coincide.*

Proof. Since an algebraic group and its formal completion share the same Lie algebra, the composition

$$\left\{ \begin{array}{c} \text{Algebraic} \\ \text{groups over } k \end{array} \right\} \xrightarrow{\widehat{(-)}} \left\{ \begin{array}{c} \text{Infinitesimal formal} \\ \text{groups over } k \end{array} \right\} \xrightarrow{\text{Lie}(-)} \left\{ \begin{array}{c} \text{Lie algebras} \\ \text{over } k \end{array} \right\}$$

is the functor associating an algebraic group to its Lie algebra. In particular, G and \mathfrak{g} , seen as a vector group, have the same image by the composition above. Now, by [SGA 3_I, Exp. VII_B, Cor. 3.3.2], the functor on the right is an equivalence of categories. In particular, G and \mathfrak{g} have isomorphic formal completions. \square

PROPOSITION 1.1.13. *Let G be a commutative algebraic group over a characteristic zero field k . Then G_{dR} is an fppf sheaf isomorphic to G/\widehat{G} and the functor $(-)_{\text{dR}}$ from commutative algebraic groups over k to abelian fppf sheaves is exact.*

Proof. By the Lemma 1.1.12, the formal completion \widehat{G} is a direct sum of copies of \widehat{G}_a . Then, given a k -algebra R , [Bha22, Rem. 2.2.18] says that $H_{\text{fppf}}^1(R, \widehat{G}_a) = 0$ and so $(G/\widehat{G})(R) \simeq G(R)/\widehat{G}(R) \simeq G_{\text{dR}}(R)$, where the quotient on the left is taken on the fppf topology. In other words G_{dR} is an fppf sheaf isomorphic to G/\widehat{G} . The exactness of $(-)_{\text{dR}}$ here is a particular case of Proposition 1.1.10. \square

COROLLARY 1.1.14. *Let X be a locally of finite type scheme over a characteristic zero field k . Then X_{dR} is an fppf sheaf.*

Proof. Let R be a k -algebra and let $\{R \rightarrow R_i\}_{i \in I}$ be an fppf covering of R . By the previous proposition, the diagram

$$R_{\text{red}} \rightarrow \prod_i R_{i,\text{red}} \rightrightarrows \prod_{i,j} (R_i \otimes_R R_j)_{\text{red}}$$

is an equalizer in the category of k -algebras. The functor of points $X(-): (\text{Aff}/k)^{\text{op}} \rightarrow \text{Set}$ sends this diagram to an equalizer in the category of sets, finishing the proof. \square

■ **REMARK 1.1.15** — DE RHAM SPACES IN POSITIVE CHARACTERISTIC. Let k be a field of characteristic $p > 0$. Given a k -algebra R , the colimit R_{perf} of the tower

$$R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} \dots$$

is the so-called *colimit perfection* of R . It is always a perfect k -algebra and the natural map $R \rightarrow R_{\text{perf}}$ is universal among morphisms from R to a perfect algebra. We define a presheaf $\mathbb{G}_{a,\text{perf}}$ on Aff/k by $\mathbb{G}_{a,\text{perf}}(R) := R_{\text{perf}}$. As [Bha22, Rem. 2.2.18] shows, we have an exact sequence of abelian fppf sheaves

$$0 \rightarrow \widehat{\mathbb{G}}_a \rightarrow \mathbb{G}_a \rightarrow \mathbb{G}_{a,\text{perf}} \rightarrow 0.$$

It follows that the natural map of abelian étale sheaves $\mathbb{G}_{a,\text{dR}} \rightarrow \mathbb{G}_{a,\text{perf}}$ identifies $\mathbb{G}_{a,\text{perf}}$ with the fppf sheafification of $\mathbb{G}_{a,\text{dR}}$. ■

We end this section by extending the definition of de Rham spaces from functors on Aff/k to functors on Sch/k .

DEFINITION 1.1.16. Let X be a locally of finite type scheme over k , seen as its functor of points $(\text{Sch}/k)^{\text{op}} \rightarrow \text{Set}$. We define its *de Rham space* by taking the de Rham space of the restriction $(\text{Aff}/k)^{\text{op}} \rightarrow (\text{Sch}/k)^{\text{op}} \rightarrow \text{Set}$ and then right Kan extending to $(\text{Sch}/k)^{\text{op}}$.

This definition actually coincides with the naive one, but it will be more convenient. Indeed, given a k -scheme S , Proposition 1.1.3 implies that

$$X_{\text{dR}}(S) \simeq \lim_{\text{Spec } A \rightarrow S} X(\text{Spec } A_{\text{red}}),$$

where the limit runs through the affine k -schemes with a map to S . Since the functor $X(-) = \text{Hom}_k(-, X)$ commutes with limits, this is also

$$X\left(\text{colim}_{\text{Spec } A \rightarrow S} \text{Spec } A_{\text{red}}\right) \simeq X(S_{\text{red}}).$$

The usefulness of this definition comes from the fact that we have an equivalence of topoi $\text{Sh}((\text{Aff}/k)_{\text{ét}}) \simeq \text{Sh}((\text{Sch}/k)_{\text{ét}})$. Here, the functor $\text{Sh}((\text{Sch}/k)_{\text{ét}}) \rightarrow \text{Sh}((\text{Aff}/k)_{\text{ét}})$ is given by restriction and the functor $\text{Sh}((\text{Aff}/k)_{\text{ét}}) \rightarrow \text{Sh}((\text{Sch}/k)_{\text{ét}})$ is a right Kan extension [Stacks, Tag 021E].

Every result in this section that was true for sheaves generalizes to this setting. Take Proposition 1.1.11 as an example: it is no longer true that $G_{\text{dR}} \simeq G/\widehat{G}$ as presheaves on Sch/k , but G_{dR} is isomorphic to the quotient G/\widehat{G} taken in $\text{Ab}((\text{Sch}/k)_{\text{ét}})$. For the sake of completeness, we give precise statements below.

PROPOSITION 1.1.17. Let Z be a closed subscheme of a locally of finite type scheme X over k . Then the formal completion \widehat{X}_Z of X along Z is isomorphic to $X \times_{X_{\text{dR}}} Z_{\text{dR}}$ and the projection $\widehat{X}_Z \rightarrow X$ is a monomorphism of étale sheaves on Sch/k .

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PROPOSITION 1.1.18. *Let $f: X \rightarrow S$ be a morphism of locally of finite type k -schemes. Then f is formally unramified if and only if $X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ is a monomorphism of étale sheaves on Sch/k . Moreover, if f is formally smooth, then $X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ is an epimorphism of étale sheaves on Sch/k .*

PROPOSITION 1.1.19. *Let G be a commutative algebraic group over k . Then G_{dR} is isomorphic to the quotient G/\widehat{G} taken in $\mathrm{Ab}((\mathrm{Sch}/k)_{\mathrm{\acute{e}t}})$. Moreover, the functor $(-)_{\mathrm{dR}}$ from commutative algebraic groups over k to abelian étale sheaves on Sch/k is exact.*

Due to [Stacks, Tag 021V], all results in the last 3 propositions also hold for the fppf topology as long as k has characteristic zero.

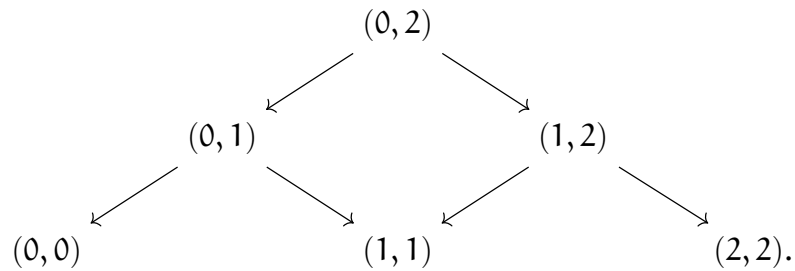
1.2. DIGRESSION ON SIX-FUNCTOR FORMALISMS

The objective of the upcoming sections is to extend the derived category of quasi-coherent sheaves on a scheme to a very general class of stacks. In particular, we will be able to consider quasi-coherent sheaves on de Rham spaces, along with all the associated functoriality. To achieve this, we explain the categorical formalization of a six-functor formalism developed in [Man22, App. A.5] and [Sch22].

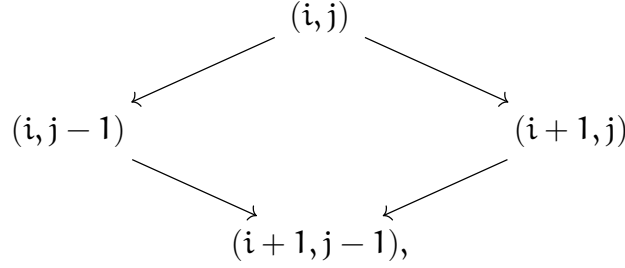
DEFINITION 1.2.1 (Geometric setup). *A geometric setup is a pair (\mathcal{C}, E) , where \mathcal{C} is an ∞ -category with finite limits, and E is a class of morphisms in \mathcal{C} that includes all isomorphisms and is stable under pullback and composition.*

One should consider \mathcal{C} as the category of *spaces* under consideration, while E represents the class of morphisms $f: X \rightarrow Y$ for which the functors $f_!$ and $f^!$ are defined. For instance, in the usual six-functor formalism of abelian sheaves on topological spaces, we could let \mathcal{C} be the category of topological spaces and E be the class of separated locally proper maps as described in [SS16].

For the next definition, we recall that the *twisted arrow category* $\mathrm{Tw}(\Delta^n)$ is the full subcategory of $(\Delta^n)^{\mathrm{op}} \times \Delta^n$ consisting of simplices $(i, j) \in \{0, 1, \dots, n\}^2$ with $i \geq j$ [Kerodon, Tag 00B2]. For example, the category $\mathrm{Tw}(\Delta^2)$ can be visually represented as



This representation generalizes for all $n \geq 0$. In this picture, we say that a *right arrow* is a morphism of the form $(i, j) \rightarrow (i + 1, j)$, for some i and j . Additionally, a diagram of the form

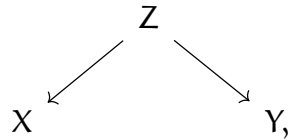


for some i and j , is going to be referred as a *square*.

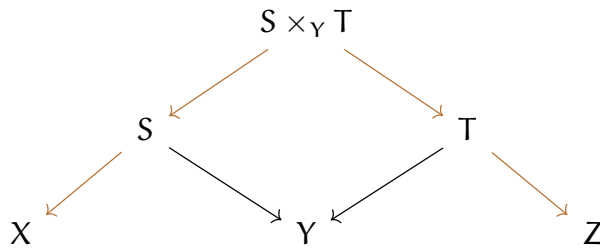
DEFINITION 1.2.2 (Category of correspondences). Given a geometric setup $(\mathcal{C}, \mathcal{E})$, the ∞ -category of correspondences $\text{Corr}(\mathcal{C}, \mathcal{E})$ is defined as the simplicial set whose n -simplices are maps $\text{Tw}(\Delta^n) \rightarrow \mathcal{C}$ sending right arrows to maps in \mathcal{E} and squares to pullbacks.

We remark that $\text{Corr}(\mathcal{C}, \mathcal{E})$ is indeed an ∞ -category [LZ17, Lem. 6.1.2]. This ∞ -category has a natural symmetric monoidal structure, in which the tensor product of two objects X and Y is their cartesian product $X \times Y$. We refer to [LZ17, Prop. 6.1.3] and [Sch22, Def. 3.11] for more details.

Unwrapping the definition, we see that the objects of $\text{Corr}(\mathcal{C}, \mathcal{E})$ are the same as the objects of \mathcal{C} . A morphism from X to Y is given by a *correspondence*



where the right arrow $Z \rightarrow Y$ lies in \mathcal{E} , and the composition of two correspondences $X \leftarrow S \rightarrow Y$ and $Y \leftarrow T \rightarrow Z$ is witnessed by the outer arrows in the diagram

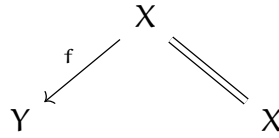


DEFINITION 1.2.3 (Three-functor formalism). A *three-functor formalism* on a geometric setup $(\mathcal{C}, \mathcal{E})$ is a lax monoidal functor $D: \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Cat}_\infty$.

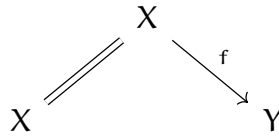
Let us explain why this definition deserves its name. An object X of \mathcal{C} is also an object of $\text{Corr}(\mathcal{C}, \mathcal{E})$, and so it gets mapped to an ∞ -category $D(X)$. This should be

1. *Quasi-coherent sheaves and \mathcal{D} -modules*

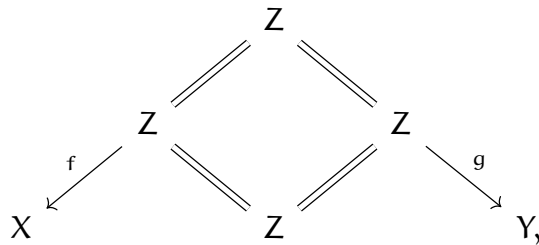
interpreted as the category of *coefficients* for the cohomology theory enhanced by D . For a morphism $f: X \rightarrow Y$ in \mathcal{C} , the correspondence



is mapped to a functor $f^*: D(Y) \rightarrow D(X)$. Similarly, for a morphism $f: X \rightarrow Y$ in \mathcal{E} , we define the functor $f_!: D(X) \rightarrow D(Y)$ as the image of the correspondence



by D . A general correspondence $X \xleftarrow{f} Z \xrightarrow{g} Y$ can be written as the composition



and so it gets mapped to $g_! f^*: D(X) \rightarrow D(Y)$.

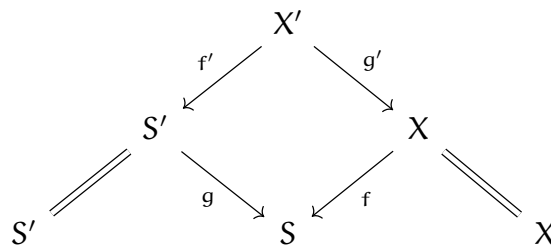
A three-functor formalism also contains the data of tensor products. Indeed, for every pair of objects X, Y in \mathcal{C} , the lax monoidality of D provides an *exterior tensor product* functor $\boxtimes: D(X) \times D(Y) \rightarrow D(X \times Y)$. If $\Delta_X: X \rightarrow X \times X$ is the diagonal map, the composition

$$D(X) \times D(X) \xrightarrow{\boxtimes} D(X \times X) \xrightarrow{\Delta_X^*} D(X)$$

sends a pair of objects M, N to the *tensor product* $M \otimes N$. The tensor unit of $D(X)$ is denoted by 1_X .

The following remarks explain some of the usual relations between the three functors $f_!$, f^* , and \otimes that are encoded in Definition 1.2.3.

■ **REMARK 1.2.4 — BASE CHANGE.** Let $f: X \rightarrow S$ be a morphism in \mathcal{C} , and $g: S' \rightarrow S$ be a morphism in \mathcal{E} . Denote by X' the fiber product $X \times_S S'$. Since the large diagram



gives the composition of the correspondences $S' = S' \xrightarrow{g} S$ and $S \xleftarrow{f} X = X$, the outer correspondence is mapped to the functor $f^*g_!$. On the other hand, the outer correspondence is $S' \xleftarrow{f'} X' \xrightarrow{g'} X$, and so it is mapped to the $g'_!f'^*$. Therefore, we have an isomorphism $f^*g_! \simeq g'_!f'^*$. ■

■ **REMARK 1.2.5 — INVERSE IMAGES ARE MONOIDAL.** The natural functor $\mathcal{C}^{\text{op}} \rightarrow \text{Corr}(\mathcal{C}, \mathcal{E})$, that sends a morphism $f: X \rightarrow Y$ to the correspondence

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow \\ Y & & X, \end{array}$$

is symmetric monoidal with respect to the coproduct monoidal structure on \mathcal{C}^{op} . Therefore, the composition $\mathcal{C}^{\text{op}} \rightarrow \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Cat}_{\infty}$ is lax symmetric monoidal, making $f^*: D(Y) \rightarrow D(X)$ symmetric monoidal.² ■

■ **REMARK 1.2.6 — EXTERIOR TENSOR PRODUCT.** Let X and Y be objects of \mathcal{C} , and denote by $\text{pr}_X: X \times Y \rightarrow X$ and $\text{pr}_Y: X \times Y \rightarrow Y$ the natural projections. The naturality of the lax monoidal structure implies that the diagram

$$\begin{array}{ccc} D(X) \times D(Y) & \xrightarrow{\boxtimes} & D(X \times Y) \\ \text{pr}_X^* \times \text{pr}_Y^* \downarrow & & \downarrow (\text{pr}_X \times \text{pr}_Y)^* \\ D(X \times Y) \times D(X \times Y) & \xrightarrow{\boxtimes} & D(X \times Y \times X \times Y) \\ & \searrow \Delta_{X \times Y}^* & \downarrow \\ & & D(X \times Y) \end{array}$$

\otimes

commutes. We deduce isomorphisms $M \boxtimes N \simeq \text{pr}_X^* M \otimes \text{pr}_Y^* N$ for all M in $D(X)$ and N in $D(Y)$. ■

■ **REMARK 1.2.7 — PROJECTION FORMULA.** Given an object X of \mathcal{C} , the diagonal $\Delta_X: X \rightarrow X \times X$ endows X with the structure of a commutative algebra in \mathcal{C}^{op} . Furthermore, a morphism $f: X \rightarrow Y$ in \mathcal{C} induces a morphism of commutative algebras, implying that X is a Y -module in \mathcal{C}^{op} .

Since the natural functor $\mathcal{C}^{\text{op}} \rightarrow \text{Corr}(\mathcal{C}, \mathcal{E})$ is symmetric monoidal, X inherits a structure of Y -module in $\text{Corr}(\mathcal{C}, \mathcal{E})$. If f belongs to \mathcal{E} , the fact that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ (\text{id}_X, f) \downarrow & & \downarrow \Delta_Y \\ X \times Y & \xrightarrow{f \times \text{id}_Y} & Y \times Y \end{array}$$

²According to [HA, Thm. 2.4.3.18], lax symmetric monoidal functors $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ are equivalent to functors $\mathcal{C}^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_{\infty})$.

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is cartesian implies that $X = X \xrightarrow{f} Y$ is a morphism of Y -modules in $\text{Corr}(\mathcal{C}, \mathcal{E})$. Consequently, $f_! : D(X) \rightarrow D(Y)$ is a morphism of $D(Y)$ -modules. In particular, we have isomorphisms $f_!(M \otimes f^*N) \simeq f_!M \otimes N$ that are natural in $M \in D(X)$ and $N \in D(Y)$.

For a concrete proof of the projection formula, based on the explicit description of the cocartesian fibration defining the monoidal structure of $\text{Corr}(\mathcal{C}, \mathcal{E})$, see [Sch22, End of Lect. III]. ■

We now define a six-functor formalism as a three-functor formalism in which the functors f_* , $f^!$, and \otimes have right adjoints.

DEFINITION 1.2.8 (Six-functor formalism). A *six-functor formalism* on a geometric setup $(\mathcal{C}, \mathcal{E})$ is a three-functor formalism

$$D : \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Cat}_\infty$$

such that

- (1) For each object X of \mathcal{C} , the symmetric monoidal ∞ -category $D(X)$ is closed;
- (2) For each map $f : X \rightarrow Y$ in \mathcal{C} , the functor $f^* : D(Y) \rightarrow D(X)$ has a right adjoint;
- (3) For each map $f : X \rightarrow Y$ in \mathcal{E} , the functor $f_! : D(X) \rightarrow D(Y)$ has a right adjoint.

The ∞ -categories $D(X)$ are often presentable, allowing the use of the adjoint functor theorem to infer the existence of the required right adjoints. (We refer to the Remark 1.2.14 for more on presentable ∞ -categories.) As is customary, we denote the inner hom in $D(X)$ by $\underline{\text{Hom}}$, the right adjoint of f^* by f_* , and the right adjoint of $f_!$ by $f^!$. The following remarks elucidates how the compatibilities among all six functors are encapsulated in Definition 1.2.8.

■ **REMARK 1.2.9 — LOCAL ADJUNCTIONS.** Consider a morphism $f : X \rightarrow Y$ in \mathcal{E} and objects $M \in D(Y)$, $N \in D(X)$, and $P \in D(Y)$. The projection formula and the adjunctions in the definition of a six-functor formalism provide the following isomorphisms of anima:

$$\begin{aligned} \text{Hom}(M, f_* \underline{\text{Hom}}(N, f^!P)) &\simeq \text{Hom}(f^*M, \underline{\text{Hom}}(N, f^!P)) \\ &\simeq \text{Hom}(f^*M \otimes N, f^!P) \\ &\simeq \text{Hom}(f_!(f^*M \otimes N), P) \\ &\simeq \text{Hom}(M \otimes f_!N, P) \\ &\simeq \text{Hom}(M, \underline{\text{Hom}}(f_!N, P)). \end{aligned}$$

The fully faithfulness of the Yoneda embedding then implies that we have isomorphisms

$$f_* \underline{\text{Hom}}(N, f^!P) \simeq \underline{\text{Hom}}(f_!N, P),$$

that are natural on $N \in D(X)$ and $P \in D(Y)$. Similar considerations show that we also have isomorphisms

$$\begin{aligned} f_* \underline{\mathrm{Hom}}(f^* M, N) &\simeq \underline{\mathrm{Hom}}(M, f_* N) \\ \underline{\mathrm{Hom}}(f^* M, f^! P) &\simeq f^! \underline{\mathrm{Hom}}(M, P) \\ \underline{\mathrm{Hom}}(M \otimes P, Q) &\simeq \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(P, Q)), \end{aligned}$$

that are natural in $M \in D(Y)$, $N \in D(X)$, $P \in D(Y)$, and $Q \in D(Y)$. ■

■ **REMARK 1.2.10** — POINCARÉ DUALITY. Let $f: X \rightarrow Y$ be a morphism in E and denote the dualizing object $f^! 1_Y$ as ω_f . The counit of the adjunction $f_! \dashv f^!$, combined with the projection formula, yields a natural transformation

$$f_!(f^! 1_Y \otimes f^* -) \simeq f_! f^! 1_Y \otimes - \rightarrow -.$$

By adjunction, we obtain a natural map $\rho_f: \omega_f \otimes f^* \rightarrow f^!$. Following [Sch22, Def. 5.1], we say that f is *cohomologically smooth* if

- (1) The natural transformation ρ_f is an isomorphism;
- (2) The dualizing object ω_f is invertible under the tensor product;
- (3) For any morphism $g: Y' \rightarrow Y$ in C with base changes $f': X' \rightarrow Y'$ and $g': X' \rightarrow X$, properties (1) and (2) also hold for f' , and the natural map $g'^* \omega_f \rightarrow \omega_{f'}$ is an isomorphism.

In reasonable six-functor formalisms, smooth morphisms (however they are defined in C) should be cohomologically smooth. While this is not a formal statement, [Zav23, Thm. 3.3.1] provides a straightforward criterion for its verification. There is also a robust strategy for computing the dualizing object ω_f based on deformation to the normal cone. For further details, see [CS22, Lect. XIII] or [Zav23, §4]. ■

■ **REMARK 1.2.11** — FORGET-SUPPORTS MAP. Consider a morphism $f: X \rightarrow Y$ in E . Scholze, in [Sch22, Def. 6.10], gave a definition of what it means for f to be *cohomologically proper*, a condition that ensures the functors $f_!$ and f_* are naturally isomorphic. As one might expect, proper morphisms (defined geometrically in C) are often cohomologically proper.

In most examples of six-functor formalisms, the morphisms within the class E are separated, implying that their associated diagonal maps are cohomologically proper. Under these conditions, [Sch22, Prop. 6.11] guarantees the existence of a natural transformation

$$f_! \rightarrow f_*.$$

Following Katz, we refer to it as the *forget-supports map*. This map is an isomorphism if and only if f is cohomologically proper. ■

1. Quasi-coherent sheaves and \mathcal{D} -modules

As demonstrated in this section, a lax monoidal functor $\text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Cat}_\infty$ contains a substantial amount of information. This complexity might suggest that constructing a three-functor formalism is a nearly impossible task. However, we will now outline a method for such constructions, due to Liu–Zheng and Mann.

For a geometric setup $(\mathcal{C}, \mathcal{E})$, it is often straightforward to concoct a functor $D_0: \mathcal{C}^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_\infty)$ that encodes the ∞ -categories of coefficients, the tensor products and the inverse image functors. According to [HA, Thm. 2.4.3.18], this is equivalent to a lax monoidal functor

$$\text{Corr}(\mathcal{C}, \text{isomorphisms}) \simeq \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty.$$

Our goal is to extend this functor to the larger ∞ -category $\text{Corr}(\mathcal{C}, \mathcal{E})$. This will be done with the help of the following gadget.

DEFINITION 1.2.12 (Compactification setup). Let $(\mathcal{C}, \mathcal{E})$ be a geometric setup, and let $D_0: \mathcal{C}^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_\infty)$ be a functor. A *compactification setup* with respect to this data is a pair (I, P) of classes of morphisms in \mathcal{C} that satisfy the following conditions:

- (1) The classes I and P include all isomorphisms, and are stable under pullback and composition;
- (2) Given two maps $f: X \rightarrow Y$ and $g: Y \rightarrow S$, if both g and $g \circ f$ belong to I (resp. to P), so does f ;
- (3) Every morphism in $I \cap P$ is n -truncated for some n ;
- (4) Every map in \mathcal{E} can be expressed as a composition $p \circ j$, with p in P and j in I ;
- (5) For each morphism j in I , the functor j^* admits a left adjoint $j_!$ satisfying base change and the projection formula;
- (6) For each morphism p in P , the functor p^* admits a right adjoint p_* satisfying base change and the projection formula;
- (7) For any cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{j'} & X \\ p' \downarrow & & \downarrow p \\ S' & \xrightarrow{j} & S, \end{array}$$

with j in I and p in P , the natural map $j_! p'_* \rightarrow p_* j'_!$ is an isomorphism.

Some remarks are in order. First, condition (3) is automatically satisfied if \mathcal{C} is a 1-category. The map $j_! p'_* \rightarrow p_* j'_!$ in condition (7) is defined as the adjoint to the counit

$$p^* j_! p'_* \simeq j'_! p'^* p'_* \rightarrow j'_!$$

In most instances, condition (7) is straightforward to verify, as noted in [Sch22, Rem. 4.2]. As the notation suggests, the classes I and P are to be imagined as consisting of open immersions and proper maps.

The most significant example is when \mathcal{C} is the category of quasi-compact and quasi-separated schemes, and \mathcal{E} is the class of separated morphisms of finite type. In this context, Nagata’s compactification theorem [Stacks, Tag 0F41] ensures that the classes I of open immersions and P of proper maps satisfy conditions (1) to (4) in Definition 1.2.12.

The following theorem, due to Mann and Liu–Zheng [Sch22, Thm. 4.6], provides a very efficient method for constructing three- and six-functor formalisms.

THEOREM 1.2.13. *Let $(\mathcal{C}, \mathcal{E})$ be a geometric setup, and let $D_0: \mathcal{C}^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_\infty)$ be a functor. Given a compactification setup (I, P) , there is an extension of D_0 to a lax symmetric monoidal functor*

$$D: \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Cat}_\infty$$

such that, for $j \in I$, the functor $j_!$ is left adjoint to j^ and, for $p \in P$, the functor $p_!$ is right adjoint to p^* .*

For the convenience of the reader, we remark that all truncated morphisms in I are cohomologically étale³, and all truncated morphisms in P are cohomologically proper.

We now delve into the final topic of this section: the extension of a six-functor formalism to stacks. Henceforth, assume that $D: \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Pr}^L$ is a *presentable* six-functor formalism. For the reader’s convenience, the following remark summarizes some essential concepts about presentable ∞ -categories.

■ **REMARK 1.2.14 — ON PRESENTABLE ∞ -CATEGORIES.** Most ∞ -categories encountered in practice are large. Yet, they are often quite manageable. Consider, for instance, the category Ab of abelian groups. This is a large category, but every abelian group can be expressed as a filtered colimit of objects from the small subcategory of finitely generated abelian groups. The theory of *presentable ∞ -categories*, a cornerstone of higher category theory, axiomatizes this situation.

An ∞ -category \mathcal{C} is *presentable* if there exists a small ∞ -category \mathcal{D} , a set of morphisms R in $\text{PSh}(\mathcal{D}, \text{An})$, and a fully faithful functor $\mathcal{C} \rightarrow \text{PSh}(\mathcal{D}, \text{An})$ whose essential image consists of presheaves $F: \mathcal{D}^{\text{op}} \rightarrow \text{An}$ such that, for every morphism $\varphi: G \rightarrow G'$ in R , the induced map $\text{Hom}(G', F) \rightarrow \text{Hom}(G, F)$ is an isomorphism of anima. This equivalence between \mathcal{C} and a full subcategory of $\text{PSh}(\mathcal{D}, \text{An})$ can be viewed as a presentation of \mathcal{C} with generators in \mathcal{D} and relations imposed by R .

The *raison d’être* of presentable ∞ -categories is the adjoint functor theorem. This theorem states that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between presentable ∞ -categories has a right

³See [Sch22, Def. 6.12] for the precise definition. This is a condition morally equivalent to being cohomologically smooth and having a trivial dualizing object.

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adjoint if and only if it preserves small colimits. Similarly, F has a left adjoint if and only if it preserves small limits and κ -filtered colimits, for some regular cardinal κ [HTT, Cor. 5.5.2.9].⁴

Following Lurie, we denote the ∞ -categories of presentable ∞ -categories and left or right adjoint functors by Pr^L and Pr^R , respectively. Since the forgetful functors $\mathrm{Pr}^L \rightarrow \mathrm{Cat}_\infty$ and $\mathrm{Pr}^R \rightarrow \mathrm{Cat}_\infty$ preserve small limits, limits in Pr^L and Pr^R are computed as in Cat_∞ [HTT, §5.5.3].

By mapping a functor to its right adjoint, we obtain an equivalence $(\mathrm{Pr}^L)^{\mathrm{op}} \rightarrow \mathrm{Pr}^R$. Consequently, colimits in Pr^L are computed by first taking right adjoints and then computing the result limit in Cat_∞ . A similar procedure also allows us to compute colimits in Pr^R . Chapter 5 in [HTT] describes many other ways of constructing presentable ∞ -categories. ■

Let $\widehat{\mathbf{C}}$ be the ∞ -category of presheaves of anima on \mathbf{C} , and let $\widehat{\mathbf{E}}$ be the class of morphisms in $\widehat{\mathbf{C}}$ whose pullback to any object of \mathbf{C} is in \mathbf{E} . We denote by $\widehat{\mathbf{C}}_{\widehat{\mathbf{E}}}$ the subcategory of $\widehat{\mathbf{C}}$ whose morphisms lie in $\widehat{\mathbf{E}}$. By [Man22, Prop. A.5.16], D can be extended to a six-functor formalism

$$\mathrm{Corr}(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}) \rightarrow \mathrm{Pr}^L,$$

which we still denote by D . This six-functor formalism is defined for a wide collection of spaces $\widehat{\mathbf{C}}$, though relatively few maps in $\widehat{\mathbf{C}}$ have $!$ -functors.

DEFINITION 1.2.15 (D-topology). Let $D: \mathrm{Corr}(\mathbf{C}, \mathbf{E}) \rightarrow \mathrm{Pr}^L$ be a presentable six-functor formalism. We denote by $D^*: \mathbf{C}^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ the functor that sends a map $f: X \rightarrow Y$ to $f^*: D(Y) \rightarrow D(X)$. Similarly, we denote by $D^!: \widehat{\mathbf{C}}_{\widehat{\mathbf{E}}}^{\mathrm{op}} \rightarrow \mathrm{Cat}_\infty$ the functor that sends a map $f: X \rightarrow Y$ to $f^!: D(Y) \rightarrow D(X)$. Now, consider a family of maps $f_i: X_i \rightarrow S$ in \mathbf{C} .

- (1) The maps f_i form a *cover of S in the canonical topology* if, for all objects Y and all maps $S' \rightarrow S$ in \mathbf{C} , the functor $\mathrm{Hom}(-, Y)$ satisfies descent along the maps $X_i \times_S S' \rightarrow S'$.
- (2) The maps f_i satisfy *universal $*$ -descent* if, for every map $S' \rightarrow S$ in \mathbf{C} , the functor D^* satisfies descent along the maps $X_i \times_S S' \rightarrow S'$.
- (3) The maps f_i satisfy *universal $!$ -descent* if, for every map $S' \rightarrow S$ in $\widehat{\mathbf{C}}_{\mathbf{E}}$, the functor $D^!$ satisfies descent along the maps $X_i \times_S S' \rightarrow S'$.

We say that the maps f_i form a *D-cover* if they lie in \mathbf{E} , form a cover of S in the canonical topology, and satisfy both universal $*$ -descent and universal $!$ -descent. The Grothendieck topology on \mathbf{C} generated by these covers is called the *D-topology*.

⁴In particular, if F preserves small limits and filtered colimits, then it has a left adjoint.

■ **REMARK 1.2.16.** There is a convenient criterion for verifying that a given morphism $f: X \rightarrow S$ in \mathcal{E} satisfies $*$ - and $!$ -descent. By [Sch22, Prop. 6.18], this holds if f is cohomologically smooth⁵ and the functor $f^*: D(S) \rightarrow D(X)$ is conservative. Moreover, if f is also a canonical cover, then it is a D -cover. ■

Let $\tilde{\mathcal{C}}$ be the ∞ -category of sheaves of anima on \mathcal{C} , with respect to the D -topology. We denote by $\tilde{\mathcal{E}}_0$ the class of morphisms in $\tilde{\mathcal{C}}$ that lie in $\hat{\mathcal{E}}$.

DEFINITION 1.2.17. Let $\tilde{\mathcal{E}}$ be a class of morphisms in $\tilde{\mathcal{C}}$, containing $\tilde{\mathcal{E}}_0$, that is stable under pullback and composition. Suppose the six-functor formalism $D: \text{Corr}(\tilde{\mathcal{C}}, \tilde{\mathcal{E}}_0) \rightarrow \text{Pr}^{\text{L}}$ extends uniquely to $\text{Corr}(\tilde{\mathcal{C}}, \tilde{\mathcal{E}})$.

- (1) The class $\tilde{\mathcal{E}}$ is *stable under disjoint unions* if, for all morphisms $X_i \rightarrow S$ in $\tilde{\mathcal{E}}$, their disjoint union $\coprod X_i \rightarrow S$ is also in $\tilde{\mathcal{E}}$.
- (2) The class $\tilde{\mathcal{E}}$ is *local on the target* if whenever $f: X \rightarrow S$ is a morphism in $\tilde{\mathcal{C}}$ such that, for every object S' in \mathcal{C} mapping to S , the pullback $X \times_S S' \rightarrow S'$ lies in $\tilde{\mathcal{E}}$, then $f \in \tilde{\mathcal{E}}$.
- (3) The class $\tilde{\mathcal{E}}$ is *local on the source* if whenever $f: X \rightarrow S$ is a morphism in $\tilde{\mathcal{C}}$ such that there is a map $g: X' \rightarrow X$ in $\tilde{\mathcal{E}}$ of universal $!$ -descent satisfying $f \circ g \in \tilde{\mathcal{E}}$, then $f \in \tilde{\mathcal{E}}$.
- (4) The class $\tilde{\mathcal{E}}$ is *tame* if, for any morphism $X \rightarrow S$ in $\tilde{\mathcal{E}}$ with S in \mathcal{C} , there exist morphisms $X_i \rightarrow S$ in \mathcal{E} and a morphism $\coprod X_i \rightarrow X$ over S that is in $\tilde{\mathcal{E}}$ and of universal $!$ -descent.

The following theorem is proven in [Sch22, Thm. 4.20].

THEOREM 1.2.18. *Let $D: \text{Corr}(\mathcal{C}, \mathcal{E}) \rightarrow \text{Pr}^{\text{L}}$ be a presentable six-functor formalism. There exists a minimal collection of morphisms $\tilde{\mathcal{E}}$ in $\tilde{\mathcal{C}}$, containing $\tilde{\mathcal{E}}_0$, such that D extends uniquely to $\text{Corr}(\tilde{\mathcal{C}}, \tilde{\mathcal{E}})$ and $\tilde{\mathcal{E}}$ satisfies all conditions in Definition 1.2.17.*

1.3. THE SIX-FUNCTOR FORMALISM OF QUASI-COHERENT SHEAVES

In this section, we construct a six-functor formalism for quasi-coherent sheaves on quasi-compact, quasi-separated schemes. Recall that in a six-functor formalism, each cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

⁵It suffices for 1_X to be f -smooth, in the sense of [Sch22, Def. 6.1].

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induces an isomorphism of functors $f^*g_! \simeq g'_!f'^*$. When dealing with (complexes of) quasi-coherent sheaves, unusual phenomena occur. Firstly, there is no clear definition of compactly supported direct images. However, an isomorphism $f^*g_* \simeq g'_*f'^*$ exists as long one of the maps f or g is flat [Stacks, Tag 08IB]. (Here, every functor is considered to be derived.)

The necessity of flatness becomes apparent when one recognizes the mismatch between derived and non-derived structures. On one hand, $X' = X \times_S S'$ is locally represented by a non-derived tensor product of rings. On the other hand, a relationship is expected between the derived functors f^*g_* and $g'_*f'^*$. When either f or g is flat, these tensor products become automatically derived, yielding the desired isomorphism.

To eliminate this hypothesis, we must consider a derived version of fiber products, locally represented by a derived tensor product of rings. This is provided by Quillen's non-abelian derived categories, which we now explain. Our exposition of this topic is greatly inspired by Khan's notes [Kha23].

DEFINITION 1.3.1 (Algebraic category). Let \mathcal{C} be a 1-category. We say that \mathcal{C} is *algebraic* if there exists an essentially small full subcategory \mathcal{P} of \mathcal{C} that is closed under finite coproducts, along with an equivalence

$$\mathrm{Fun}_{\Pi}(\mathcal{P}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\sim} \mathcal{C},$$

which restricts via the Yoneda embedding $\mathcal{P} \hookrightarrow \mathrm{Fun}_{\Pi}(\mathcal{P}^{\mathrm{op}}, \mathrm{Set})$ to the inclusion $\mathcal{P} \hookrightarrow \mathcal{C}$. Here, $\mathrm{Fun}_{\Pi}(\mathcal{P}^{\mathrm{op}}, \mathrm{Set})$ refers to the category of functors $\mathcal{P}^{\mathrm{op}} \rightarrow \mathrm{Set}$ that preserve finite products.

Readers may note that an algebraic category is the same as a category of models for a Lawvere theory. Furthermore, a category \mathcal{C} is algebraic if and only if the conditions in Definition 1.3.1 apply with \mathcal{P} as the full subcategory of compact projective objects.

■ **EXAMPLE 1.3.2.** Here are examples of algebraic categories:

- (1) The category Set of sets is algebraic, with its compact projective objects being finite sets.
- (2) The category Grp of groups is algebraic. Its compact projective objects are free groups on finite sets.
- (3) For a ring R , the category $\mathrm{Mod}(R)$ of R -modules is algebraic. Its compact projective objects are finite projective R -modules.
- (4) For a ring R , the category $\mathrm{Alg}(R)$ of R -algebras is algebraic. Its compact projective objects are retracts of finite type polynomial R -algebras.

For $\text{Mod}(R)$, \mathbf{P} can be the full subcategory of finite free R -modules. Likewise, for $\text{Alg}(R)$, \mathbf{P} can be taken as the full subcategory constituted of the R -algebras $R[x_1, \dots, x_n]$ for all $n \geq 0$. ■

By substituting the category of sets in the equivalence $\text{Fun}_{\Pi}(\mathbf{P}^{\text{op}}, \text{Set}) \simeq \mathbf{C}$ with the ∞ -category of anima, we derive a higher categorical analogue of \mathbf{C} .

DEFINITION 1.3.3 (Animation). Let $\mathbf{C} \simeq \text{Fun}_{\Pi}(\mathbf{P}^{\text{op}}, \text{Set})$ be an algebraic category. Its *animation*, denoted $\text{An}(\mathbf{C})$, is the ∞ -category $\text{Fun}_{\Pi}(\mathbf{P}^{\text{op}}, \text{An})$ of functors $\mathbf{P}^{\text{op}} \rightarrow \text{An}$ that preserve finite products.⁶

The animation of the category of sets is equivalent to the ∞ -category An . Indeed, since every finite set can be expressed as a finite coproduct of copies of the terminal set $\{*\}$, a functor $\{\text{finite sets}\}^{\text{op}} \rightarrow \text{An}$ preserving finite products is uniquely determined by the image of $\{*\}$. The next remark provides additional examples of animations.

■ **REMARK 1.3.4.** The animation $\text{An}(\mathbf{C})$ can be derived from the category of simplicial objects of \mathbf{C} by inverting weak equivalences according to a model structure induced by the Quillen model structure on simplicial sets [HTT, Cor. 5.5.9.3].

In particular, the ∞ -categories of animated groups, modules, and algebras can be obtained from the respective categories of simplicial objects by inverting weak equivalences. For a ring R , the Dold–Kan correspondence implies that $\text{An}(\text{Mod}(R))$ is equivalent to the full subcategory $D^{\leq 0}(R)$ of the derived ∞ -category $D(R)$, consisting of complexes whose cohomologies vanish in positive degrees. ■

The natural inclusion $\text{Set} \hookrightarrow \text{An}$ induces a fully faithful functor $\mathbf{C} \hookrightarrow \text{An}(\mathbf{C})$. Objects in its essential image are said to be *static*. This functor has a left adjoint $\pi_0: \text{An}(\mathbf{C}) \rightarrow \mathbf{C}$ that acts as

$$(\mathbf{P}^{\text{op}} \rightarrow \text{An}) \mapsto (\mathbf{P}^{\text{op}} \rightarrow \text{An} \xrightarrow{\pi_0} \text{Set}),$$

where $\pi_0: \text{An} \rightarrow \text{Set}$ maps an anima to its set of isomorphism classes of objects. More generally, we have functors $\pi_i: \text{An}(\mathbf{C}) \rightarrow \mathbf{C}$ for all $i \geq 0$.

The following result, available in [HTT, Prop. 5.5.8.15], provides a universal property of animation. Recall that reflexive coequalizers are the categorical generalization of quotients by equivalence relations. Geometric realizations serve as their ∞ -categorical counterparts, corresponding to stacky quotients.

PROPOSITION 1.3.5 (Universal property of animation). Let $\mathbf{C} \simeq \text{Fun}_{\Pi}(\mathbf{P}^{\text{op}}, \text{Set})$ be an algebraic category. For every ∞ -category \mathbf{D} that has filtered colimits and geometric realizations, the natural functor

$$\text{Fun}_s(\text{An}(\mathbf{C}), \mathbf{D}) \rightarrow \text{Fun}(\mathbf{P}, \mathbf{D})$$

⁶The animation $\text{An}(\mathbf{C})$ is independent of the choice of \mathbf{P} .

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is an equivalence. Here, $\text{Fun}_s(\text{An}(\mathcal{C}), \mathcal{D})$ denotes the ∞ -category of functors $\text{An}(\mathcal{C}) \rightarrow \mathcal{D}$ that preserve filtered colimits and geometric realizations.

An astute reader will recognize that a functor $\text{An}(\mathcal{C}) \rightarrow \mathcal{D}$ preserves filtered colimits and geometric realizations if and only if it preserves *sifted colimits*, thereby justifying the subscript s . This proposition enables the animation of functors as well.

DEFINITION 1.3.6 (Animated functors). Consider algebraic categories $\mathcal{C} \simeq \text{Fun}_{\Pi}(\mathcal{P}^{\text{op}}, \text{Set})$ and $\mathcal{D} \simeq \text{Fun}_{\Pi}(\mathcal{Q}^{\text{op}}, \text{Set})$, and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between them. The universal property of animation, applied to the composition $\mathcal{P} \hookrightarrow \mathcal{C} \rightarrow \mathcal{D} \hookrightarrow \text{An}(\mathcal{D})$, yields a unique functor

$$\text{An}(F): \text{An}(\mathcal{C}) \rightarrow \text{An}(\mathcal{D})$$

which preserves filtered colimits and geometric realizations, and restricts to F on \mathcal{P} . The functor $\text{An}(F)$ is referred to as the *animation* of F .

There is a natural isomorphism $\pi_0 \circ \text{An}(F) \simeq F \circ \pi_0$ of functors $\text{An}(\mathcal{C}) \rightarrow \mathcal{D}$. Similar to derived functors, for any two composable functors F and G , there exists a natural transformation $\text{An}(F) \circ \text{An}(G) \rightarrow \text{An}(F \circ G)$ that is not necessarily an isomorphism. This natural transformation becomes an isomorphism under certain additional conditions, which are detailed in [ČS24, Prop. 5.1.5]. Additionally, the functor $\text{An}(F)$ preserves all colimits if and only if F preserves finite coproducts.

Having established the necessary general theory, we will now examine some animated avatars of fundamental objects in commutative algebra. The category Ring of rings is algebraic, as noted in Remark 1.3.2, and we denote its animation by $\text{An}(\text{Ring})$. For an animated ring R , there exists a unit map $R \rightarrow \pi_0(R)$ that can be visualized similarly to the quotient map $R \rightarrow R_{\text{red}}$ in commutative algebra. Just as a ring R can be thought as an infinitesimally thickened version of R_{red} , an animated ring R can be imagined as a ring $\pi_0(R)$ enriched with additional higher-dimensional information.

Consider a diagram $R \leftarrow S \rightarrow T$ of animated rings. We define the animated ring $R \otimes_S T$ by animating the classical tensor product functor on rings. This tensor product satisfies the universal property of pushouts in the ∞ -category of animated rings. Furthermore, if S is static, the forgetful functor from animated S -algebras⁷ to animated S -modules identifies $R \otimes_S T$ with the standard derived tensor product in $D^{\leq 0}(S)$.

The next step is to define the ∞ -category $\text{AnMod}(R)$ of animated modules over an animated ring R . We approach this by first defining an algebraic category of modules over any ring.

DEFINITION 1.3.7. We denote by RingMod the category of pairs (R, M) , where R is a ring

⁷By [SAG, Cor. 25.1.4.3], the animation of the category of S -algebras is equivalent to the ∞ -category of animated rings equipped with a map from S .

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and M is an R -module. A morphism $(R, M) \rightarrow (R', M')$ consists of a morphism of rings $R \rightarrow R'$ and a morphism of R' -modules $M \otimes_R R' \rightarrow M'$.

The category RingMod is algebraic. We may take the full subcategory \mathcal{P} to be the category of pairs (R, M) , where R is a polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and M is a finite free R -module $R^{\oplus m}$, for some $n, m \geq 0$. Moreover, the natural functor $\text{RingMod} \rightarrow \text{Ring}$, which maps (R, M) to R , induces a bicartesian fibration $\text{An}(\text{RingMod}) \rightarrow \text{An}(\text{Ring})$. The following remark explains the necessary facts about such fibrations.

■ **REMARK 1.3.8 — ON (CO)CARTESIAN FIBRATIONS.** Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor of ∞ -categories. For each object X of \mathcal{D} , the fiber \mathcal{C}_X is an ∞ -category defined as the pullback

$$\begin{array}{ccc} \mathcal{C}_X & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow F \\ \{X\} & \longrightarrow & \mathcal{D}. \end{array}$$

In order to assemble these fibers into a functor $\mathcal{D}^{\text{op}} \rightarrow \text{Cat}_{\infty}$, a morphism $f: X_1 \rightarrow X_2$ in \mathcal{D} should induce to a functor $f^*: \mathcal{C}_{X_2} \rightarrow \mathcal{C}_{X_1}$. Let us describe how such a functor might intuitively be constructed.

Given an object M_2 in \mathcal{C}_{X_2} , one could expect the existence of an object M_1 in \mathcal{C}_{X_1} , together with a morphism $M_1 \rightarrow M_2$ mapping to f under the functor F , as indicated in the diagram

$$\begin{array}{ccc} \mathcal{C} & & M_1 \longrightarrow M_2 \\ F \downarrow & & \downarrow \\ \mathcal{D} & & X_1 \longrightarrow X_2. \end{array}$$

This object M_1 would be a natural candidate for f^*M_2 . Typically, there can be multiple objects M_1 that satisfy these conditions, so one might impose the map $M_1 \rightarrow M_2$ to satisfy some universal property. To determine what this universal property should be, let us examine the action of $f^*: \mathcal{C}_{X_2} \rightarrow \mathcal{C}_{X_1}$ on morphisms.

Let $\varphi: M_2 \rightarrow N_2$ be a morphism in \mathcal{C}_{X_2} , along with morphisms $M_1 \rightarrow M_2$ and $N_1 \rightarrow N_2$ that map to f under F .

$$\begin{array}{ccc} \mathcal{C} & & M_1 \longrightarrow M_2 \\ F \downarrow & & \downarrow \varphi \\ \mathcal{D} & & N_1 \longrightarrow N_2 \\ & & \downarrow \\ & & X_1 \longrightarrow X_2 \end{array}$$

By requiring $N_1 \rightarrow N_2$ to satisfy the universal property of being F -cartesian, we ensure the existence of a morphism $f^*\varphi: M_1 \rightarrow N_1$ making the square above commute. We

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now proceed to give precise definitions of F -cartesian morphisms and of cartesian fibrations; the functors $F: \mathbf{C} \rightarrow \mathbf{D}$ whose fibers assemble into a functor $\mathbf{D}^{\text{op}} \rightarrow \mathbf{Cat}_{\infty}$.

A morphism $\psi: N_1 \rightarrow N_2$ in \mathbf{C} is said to be F -cartesian if, for every object M in \mathbf{C} , the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(M, N_1) & \xrightarrow{\psi \circ -} & \text{Hom}_{\mathbf{C}}(M, N_2) \\ F \downarrow & & \downarrow F \\ \text{Hom}_{\mathbf{D}}(F(M), F(N_1)) & \xrightarrow{F(\psi) \circ -} & \text{Hom}_{\mathbf{D}}(F(M), F(N_2)) \end{array} \quad (*)$$

is a pullback in the ∞ -category \mathbf{An} . We say that $F: \mathbf{C} \rightarrow \mathbf{D}$ is a *cartesian fibration* if, for every morphism $f: X_1 \rightarrow X_2$ in \mathbf{D} and every object N_2 in \mathbf{C}_{X_2} , there exists an F -cartesian map $\psi: N_1 \rightarrow N_2$ that maps to f under F .⁸

Since a functor of ∞ -categories $\mathbf{D}^{\text{op}} \rightarrow \mathbf{Cat}_{\infty}$ contains more data than merely an action on objects and morphisms, the discussion above fails short of defining such a functor. A crowning achievement of [HTT] is the fact that this can indeed be accomplished. According to [HTT, Thm. 3.2.0.1], there exist equivalences of ∞ -categories

$$\text{St}: \text{Cart}(\mathbf{D}) \xrightleftharpoons{\sim} \text{Fun}(\mathbf{D}^{\text{op}}, \mathbf{Cat}_{\infty}) : \text{Un},$$

where $\text{Cart}(\mathbf{D})$ denotes the subcategory of $\mathbf{Cat}_{\infty}/\mathbf{D}$ consisting of cartesian fibrations and functors that preserve cartesian maps. The functor St is known as *straightening*, while Un is called *unstraightening*.

By inverting all the arrows, a functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is said to be a *cocartesian fibration* if the opposite functor $F^{\text{op}}: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ is a cartesian fibration. Furthermore, F is referred to as a *bicartesian fibration* if it is both a cartesian and a cocartesian fibration. Bicartesian fibrations are significant in our context due to their connection with adjoint functors.

Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a bicartesian fibration, and consider a map $f: X_1 \rightarrow X_2$ in \mathbf{D} . Viewed as a cartesian fibration, F induces a functor $f^*: \mathbf{C}_{X_2} \rightarrow \mathbf{C}_{X_1}$. Dually, as a cocartesian fibration, F induces a functor $f_*: \mathbf{C}_{X_1} \rightarrow \mathbf{C}_{X_2}$. We claim that f_* is left adjoint to f^* . To see this, consider the functor $\Delta^1 \rightarrow \mathbf{D}$ induced by f , and form the pullback diagram

$$\begin{array}{ccc} \mathbf{C}_f & \longrightarrow & \mathbf{C} \\ \downarrow & & \downarrow F \\ \Delta^1 & \longrightarrow & \mathbf{D}. \end{array}$$

The functor $\mathbf{C}_f \rightarrow \Delta^1$ remains a bicartesian fibration, with the functors f_* and f^* being induced by the unique morphism $0 \rightarrow 1$ in Δ^1 . Given objects M_1 and M_2 in \mathbf{C}_{X_1} and \mathbf{C}_{X_2} , respectively, we obtain equivalences

$$\text{Hom}_{\mathbf{C}_{X_1}}(M_1, f^*M_2) \simeq \text{Hom}_{\mathbf{C}_f}(M_1, M_2) \simeq \text{Hom}_{\mathbf{C}_{X_2}}(f_*M_1, M_2).$$

⁸We refer to [Maz19] for the equivalence between these definitions and their quasi-categorical counterparts in [HTT].

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The equivalence on the left arises from considering $\psi: f^*M_2 \rightarrow M_2$, a cartesian lift associated with the morphism $0 \rightarrow 1$, in the diagram (*). The equivalence on the right is dual. This proves that the functor f_* is indeed left adjoint to f^* . ■

For readers who found the preceding remark useful, we suggest the following exercise: prove that the functor $\text{RingMod} \rightarrow \text{Ring}$, which maps (R, M) to R , is a bicartesian fibration. Additionally, it would be valuable to explicitly describe its fibers and the corresponding pullback and pushforward functors. We now establish the animated counterpart of these statements.

PROPOSITION 1.3.9. *The animation of the functor $\text{RingMod} \rightarrow \text{Ring}$, that maps (R, M) to R , is a bicartesian fibration $\text{An}(\text{RingMod}) \rightarrow \text{An}(\text{Ring})$.*

Proof. Let Poly denote the full subcategory of Ring that consists of polynomial rings $\mathbb{Z}[x_1, \dots, x_n]$ for some $n \geq 0$. Similarly, let PolyMod be the full subcategory of RingMod consisting of pairs (R, M) , where R is a polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and M is a finite free R -module $R^{\oplus m}$, for some $n, m \geq 0$. Recall that the ∞ -categories $\text{An}(\text{RingMod})$ and $\text{An}(\text{Ring})$ are given by $\text{Fun}_{\Pi}(\text{PolyMod}^{\text{op}}, \text{An})$ and $\text{Fun}_{\Pi}(\text{Poly}^{\text{op}}, \text{An})$, respectively.

Let $F: \text{PolyMod} \rightarrow \text{Poly}$ be the restriction of the functor $\text{RingMod} \rightarrow \text{Ring}$ to PolyMod . One can verify that this functor admits a left adjoint $G: \text{Poly} \rightarrow \text{PolyMod}$, which sends a polynomial ring R to the pair $(R, 0)$. As a result, we obtain an adjunction between the corresponding functor ∞ -categories

$$\begin{array}{ccc}
 & \xrightarrow{- \circ G^{\text{op}}} & \\
 \text{Fun}(\text{PolyMod}^{\text{op}}, \text{An}) & \perp & \text{Fun}(\text{Poly}^{\text{op}}, \text{An}) \\
 & \xleftarrow{- \circ F^{\text{op}}} &
 \end{array}$$

In other words, the left Kan extension of a functor $\text{PolyMod}^{\text{op}} \rightarrow \text{An}$ along F^{op} is equivalent to precomposition with G^{op} . According to [HTT, Rem. 5.3.5.9], this implies that the functor $\text{An}(\text{RingMod}) \rightarrow \text{An}(\text{Ring})$ maps a presheaf $M: \text{PolyMod}^{\text{op}} \rightarrow \text{An}$ to the composition

$$\text{Poly}^{\text{op}} \xrightarrow{G^{\text{op}}} \text{PolyMod}^{\text{op}} \xrightarrow{M} \text{An}.$$

Now, let $\varphi: S \rightarrow R$ be a morphism of animated rings, and let $M: \text{PolyMod}^{\text{op}} \rightarrow \text{An}$ be an object of $\text{An}(\text{RingMod})$ such that $M \circ G^{\text{op}} \simeq R$. Denote by \bar{S} and \bar{R} the precompositions of S and R with F^{op} , respectively. The unit of the adjunction above induces a map $M \rightarrow M \circ G^{\text{op}} \circ F^{\text{op}} \simeq \bar{R}$.

We claim that the projection $\rho: M \times_{\bar{R}} \bar{S} \rightarrow M$ is a cartesian map over φ . Let N be an object of $\text{An}(\text{RingMod})$. Since limits in functor ∞ -categories are computed pointwise, the morphism ρ is mapped to φ under the functor $- \circ G^{\text{op}}: \text{An}(\text{RingMod}) \rightarrow \text{An}(\text{Ring})$.

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Consequently, the diagram (*) in Remark 1.3.8 becomes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{An}(\mathrm{RingMod})}(\mathbf{N}, M \times_{\overline{R}} \overline{S}) & \xrightarrow{\rho \circ -} & \mathrm{Hom}_{\mathrm{An}(\mathrm{RingMod})}(\mathbf{N}, M) \\ \downarrow - \circ G^{\mathrm{op}} & & \downarrow - \circ G^{\mathrm{op}} \\ \mathrm{Hom}_{\mathrm{An}(\mathrm{Ring})}(\mathbf{N} \circ G^{\mathrm{op}}, S) & \xrightarrow{\varphi \circ -} & \mathrm{Hom}_{\mathrm{An}(\mathrm{Ring})}(\mathbf{N} \circ G^{\mathrm{op}}, R). \end{array}$$

The universal property of the fiber product $M \times_{\overline{R}} \overline{S}$ implies that this diagram is cartesian, finishing the proof that $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$ is a cartesian fibration.

To prove that the functor $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$ is also a cocartesian fibration, we need to show that all the pullback functors admit left adjoints. Since the functors F and G preserve finite coproducts, the functor $- \circ F^{\mathrm{op}}: \mathrm{Fun}(\mathrm{Poly}^{\mathrm{op}}, \mathrm{An}) \rightarrow \mathrm{Fun}(\mathrm{PolyMod}^{\mathrm{op}}, \mathrm{An})$ restricts to provide a right adjoint to $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$. Consequently, the latter is a morphism in Pr^{R} .

Given that limits in Pr^{R} are computed in Cat_{∞} , we conclude that the fibers of $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$ are presentable ∞ -categories. As the pullback functors preserve limits and filtered colimits, the adjoint functor theorem (see Remark 1.2.14) implies that they admit left adjoints, proving that $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$ is indeed a cocartesian fibration. \square

DEFINITION 1.3.10 (Animated modules). Let R be an animated ring. We define its ∞ -category of animated modules $\mathrm{AnMod}(R)$ as the fiber of the bicartesian fibration $\mathrm{An}(\mathrm{RingMod}) \rightarrow \mathrm{An}(\mathrm{Ring})$ at R . For a morphism $S \rightarrow R$ of animated rings, the functor $\mathrm{AnMod}(S) \rightarrow \mathrm{AnMod}(R)$ is called the *base change functor* and is denoted as $- \otimes_S R$, while the functor $\mathrm{AnMod}(R) \rightarrow \mathrm{AnMod}(S)$ is called *restrictions of scalars* and is denoted $(-)_S$.

From the discussion in Remark 1.3.8, along with Proposition 1.3.9, we have that restriction of scalars is right adjoint to the base change functor. Furthermore, the proof of Proposition 1.3.9 implies that the ∞ -categories of animated modules are always presentable.

Given a ring R , considered as a static animated ring, the Dold–Kan correspondence gives an equivalence between its ∞ -category of animated modules and $D^{\leq 0}(R)$. By taking the limit of the ∞ -categories $D^{\leq n}(R)$ for all $n \geq 0$, we obtain the unbounded derived ∞ -category $D(R)$. Before extending this construction to an arbitrary animated ring, we provide a brief explanation of some ideas concerning stable ∞ -categories.

■ **REMARK 1.3.11 — ON STABLE ∞ -CATEGORIES.** Let \mathcal{C} be an ∞ -category that has finite limits and colimits. We say that \mathcal{C} is *stable* if every commutative square is cartesian if and only if it is cocartesian.⁹

⁹This is similar to the condition on abelian categories requiring the image of a morphism to be isomorphic to its coimage.

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Denote the zero object of a stable ∞ -category \mathbf{C} by 0 . From the definition, the functor $\Omega: \mathbf{C} \rightarrow \mathbf{C}$, mapping an object M to the fiber product $0 \times_M 0$, is an equivalence of categories. This induces a shift functor on the homotopy category \mathbf{hC} , which can be shown to be triangulated [HA, Thm. 1.1.2.14].

Now, suppose \mathbf{C} is any ∞ -category admitting finite limits. Particularly, \mathbf{C} has a terminal object $*$, so the coslice ∞ -category \mathbf{C}_* has a zero object. The *stabilization* $\mathrm{Sp}(\mathbf{C})$ of \mathbf{C} is the limit of the tower

$$\cdots \rightarrow \mathbf{C}_* \xrightarrow{\Omega} \mathbf{C}_* \xrightarrow{\Omega} \mathbf{C}_*.$$

This is a stable ∞ -category, equipped with a limit-preserving functor $\Omega^\infty: \mathrm{Sp}(\mathbf{C}) \rightarrow \mathbf{C}$ that is final among limit-preserving functors from stable ∞ -categories [HA, Cor. 1.4.2.23].

If \mathbf{C} is presentable, the functor $\Omega: \mathbf{C}_* \rightarrow \mathbf{C}_*$ has a left adjoint $\Sigma: \mathbf{C}_* \rightarrow \mathbf{C}_*$, and $\mathrm{Sp}(\mathbf{C})$ can be computed as the colimit of the tower

$$\mathbf{C}_* \xrightarrow{\Sigma} \mathbf{C}_* \xrightarrow{\Sigma} \mathbf{C}_* \rightarrow \cdots$$

in Pr^{L} . Indeed, since the forgetful functor $\mathrm{Pr}^{\mathrm{R}} \rightarrow \mathrm{Cat}_\infty$ preserves limits, the limit defining $\mathrm{Sp}(\mathbf{C})$ can also be computed in $\mathrm{Pr}^{\mathrm{R}} \simeq (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$. Thus, $\mathrm{Sp}(\mathbf{C})$ is presentable, and the functor $\Omega^\infty: \mathrm{Sp}(\mathbf{C}) \rightarrow \mathbf{C}$ has a left adjoint $\Sigma^\infty: \mathbf{C} \rightarrow \mathrm{Sp}(\mathbf{C})$, initial among colimit-preserving functors to presentable stable ∞ -categories. ■

DEFINITION 1.3.12 (Derived category of an animated ring). Let R be an animated ring. We define its derived ∞ -category $D(R)$ to be the stabilization of $\mathrm{AnMod}(R)$.

Considering that the ∞ -category $\mathrm{AnMod}(R)$ is presentable, we deduce the same for $D(R)$. The functor $\Sigma^\infty: \mathrm{AnMod}(R) \rightarrow D(R)$ is fully faithful, and its essential image $D^{\leq 0}(R)$ constitutes the connective part of a t-structure on $D(R)$. The heart of this t-structure is equivalent to the abelian category of modules over $\pi_0(R)$.

For a morphism of animated rings $S \rightarrow R$, the universal properties of stabilization of presentable ∞ -categories induce functors $D(S) \rightarrow D(R)$ and $D(R) \rightarrow D(S)$ making the diagrams

$$\begin{array}{ccc} \mathrm{AnMod}(S) & \xrightarrow{-\otimes_S R} & \mathrm{AnMod}(R) & & D(R) & \dashrightarrow & D(S) \\ \Sigma^\infty \downarrow & & \downarrow \Sigma^\infty & & \Omega^\infty \downarrow & & \downarrow \Omega^\infty \\ D(S) & \dashrightarrow & D(R) & & \mathrm{AnMod}(R) & \xrightarrow{(-)_S} & \mathrm{AnMod}(S) \end{array}$$

commute. We refer to and denote these functors as their restrictions to the ∞ -categories of animated modules. These universal properties also imply that the functor $-\otimes_S R: D(S) \rightarrow D(R)$ remains left adjoint to $(-)_S: D(R) \rightarrow D(S)$.

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PROPOSITION 1.3.13. *Let R be an animated ring. Then the tensor product of modules induces a symmetric monoidal structure on the ∞ -categories $\mathbf{AnMod}(R)$ and $\mathbf{D}(R)$. Furthermore, for a morphism of animated rings $S \rightarrow R$, the base change functors $\mathbf{AnMod}(S) \rightarrow \mathbf{AnMod}(R)$ and $\mathbf{D}(S) \rightarrow \mathbf{D}(R)$ are symmetric monoidal.*

Proof. According to [HA, Rem. 4.8.1.8], the functor $\mathrm{Fun}_{\Pi}((-)^{\mathrm{op}}, \mathbf{An})$, mapping ∞ -categories to sifted cocomplete ∞ -categories, is symmetric monoidal.¹⁰ Therefore, this functor preserves commutative algebra objects.

Let \mathbf{Poly} be the full subcategory of \mathbf{Ring} consisting of polynomial rings $\mathbb{Z}[x_1, \dots, x_n]$ for some $n \geq 0$. Similarly, let $\mathbf{PolyMod}$ denote the full subcategory of $\mathbf{RingMod}$ consisting of pairs (R, M) , where R is a polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and M is a finite free R -module $R^{\oplus m}$ for some $n, m \geq 0$. Both categories are symmetric monoidal, as is the natural functor $\mathbf{PolyMod} \rightarrow \mathbf{Poly}$.

All in all, we find that the induced functor $\mathbf{An}(\mathbf{RingMod}) \rightarrow \mathbf{An}(\mathbf{Ring})$ is symmetric monoidal. By unstraightening, we obtain a functor $\mathbf{An}(\mathbf{RingMod})^{\otimes} \rightarrow \mathbf{An}(\mathbf{Ring})^{\otimes}$ over the category \mathbf{Fin}_* of pointed finite sets. (See [HA, Def. 2.0.0.7].) An animated ring R then gives rise to a section $\mathbf{Fin}_* \rightarrow \mathbf{An}(\mathbf{Ring})^{\otimes}$ of $\mathbf{An}(\mathbf{Ring})^{\otimes} \rightarrow \mathbf{Fin}_*$, acting as

$$\langle n \rangle \mapsto \underbrace{(R, R, \dots, R)}_{n \text{ copies}}.$$

By Proposition 1.3.9, the functor $\mathbf{An}(\mathbf{RingMod})^{\otimes} \rightarrow \mathbf{An}(\mathbf{Ring})^{\otimes}$ is a cocartesian fibration. Consequently, we obtain a symmetric monoidal structure on $\mathbf{AnMod}(R)$ by pullback:

$$\begin{array}{ccc} \mathbf{AnMod}(R)^{\otimes} & \longrightarrow & \mathbf{An}(\mathbf{RingMod})^{\otimes} \\ \downarrow & & \downarrow \\ \mathbf{Fin}_* & \longrightarrow & \mathbf{An}(\mathbf{Ring})^{\otimes} \\ & \searrow & \downarrow \\ & & \mathbf{Fin}_* \end{array}$$

(Note: A curved arrow also points from $\mathbf{An}(\mathbf{RingMod})^{\otimes}$ to \mathbf{Fin}_* in the original diagram.)

It is clear that, for a morphism $S \rightarrow R$ of animated rings, the base change functor $\mathbf{AnMod}(S) \rightarrow \mathbf{AnMod}(R)$ is symmetric monoidal.

Let $\mathbf{Pr}_{\mathrm{st}}^{\mathrm{L}}$ be the full subcategory of \mathbf{Pr}^{L} consisting of presentable stable ∞ -categories. As in Remark 1.3.11, the stabilization functor $\mathrm{Sp}: \mathbf{Pr}^{\mathrm{L}} \rightarrow \mathbf{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is left adjoint to the inclusion $\mathbf{Pr}_{\mathrm{st}}^{\mathrm{L}} \hookrightarrow \mathbf{Pr}^{\mathrm{L}}$. Letting $\mathrm{Sp} := \mathrm{Sp}(\mathbf{An})$ denote the ∞ -category of *spectra*, Proposition 4.8.2.18 of [HA] identifies $\mathbf{Pr}_{\mathrm{st}}^{\mathrm{L}}$ with $\mathbf{Mod}_{\mathrm{Sp}}(\mathbf{Pr}^{\mathrm{L}})$, the ∞ -category of Sp -module objects in \mathbf{Pr}^{L} . Under this identification, the stabilization–inclusion adjunction corresponds to the free–forgetful adjunction between Sp -modules and their underlying objects.

¹⁰In the notation of that reference, we set $\mathcal{K} = \emptyset$ and \mathcal{K}' to be the collection of sifted ∞ -categories.

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As in classical algebra, the free Sp-module functor $\mathrm{Sp}: \mathrm{Pr}^{\mathrm{L}} \rightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ is symmetric monoidal with respect to Lurie's tensor product [HA, Thm. 4.5.3.1]. (See also [HA, Prop. 3.4.2.1 and Prop. 4.6.2.17].) Consequently, this functor preserves commutative algebra objects. In particular, it induces a symmetric monoidal structure on the derived ∞ -category of an animated ring, such that all pullback functors are symmetric monoidal. \square

DEFINITION 1.3.14 (Flat morphisms). Let $\varphi: S \rightarrow R$ be a morphism of animated rings. We define φ to be *flat* if the induced map $\pi_0(S) \rightarrow \pi_0(R)$ is flat and the natural maps

$$\pi_i(S) \otimes_{\pi_0(S)} \pi_0(R) \rightarrow \pi_i(R)$$

are isomorphisms for all $i \geq 0$. Furthermore, if $\mathrm{Spec} \pi_0(R) \rightarrow \mathrm{Spec} \pi_0(S)$ is surjective, then φ is said to be *faithfully flat*. We say that φ is an *fppf* cover if it is faithfully flat and $\pi_0(S) \rightarrow \pi_0(R)$ is of finite presentation.

Analogous to the classical setting, a morphism of animated rings $S \rightarrow R$ is flat if and only if the base change functor $- \otimes_S R: D(S) \rightarrow D(R)$ is t-exact. Smooth and étale morphisms of animated rings can be likewise defined. According to [SAG, Prop. A.3.2.1], fppf covers induce a Grothendieck topology on the ∞ -category $\mathrm{An}(\mathrm{Ring})$ of animated rings.

DEFINITION 1.3.15 (Derived stack). Let \mathcal{X} be a functor $\mathrm{An}(\mathrm{Ring}) \rightarrow \mathrm{An}$. We say that \mathcal{X} *satisfies fppf descent* if \mathcal{X} preserves finite products and, for every fppf cover $S \rightarrow R$, the diagram

$$\mathcal{X}(S) \rightarrow \mathcal{X}(R) \rightrightarrows \mathcal{X}(R \otimes_S R) \rightrightarrows \mathcal{X}(R \otimes_S R \otimes_S R) \rightrightarrows \dots$$

forms a limit diagram. A functor $\mathrm{An}(\mathrm{Ring}) \rightarrow \mathrm{An}$ satisfying fppf descent is called a *derived stack*. The category of derived stacks is denoted by dStk .

The simplest example of derived stack is the functor $\mathrm{An}(\mathrm{Ring}) \rightarrow \mathrm{An}$ corepresented by an animated ring R . This is the *affine derived scheme* $\mathrm{Spec} R$. A classical stack, viewed as a functor $\mathrm{Ring} \rightarrow \mathrm{Grpd}$ satisfying fppf descent, can also be considered as a derived stack. Indeed, it suffices to left Kan extend the composition $\mathrm{Ring} \rightarrow \mathrm{Grpd} \hookrightarrow \mathrm{An}$ along the inclusion $\mathrm{Ring} \hookrightarrow \mathrm{An}(\mathrm{Ring})$ and sheafify.

Before defining a derived scheme as a derived stack that is locally, with respect to the Zariski topology, isomorphic to an affine derived scheme, we define open immersions of derived stacks.

DEFINITION 1.3.16 (Open immersion). Let $j: \mathcal{U} \rightarrow \mathcal{X}$ be a morphism of derived stacks.

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- (1) If $\mathcal{U} = \mathrm{Spec} R$ and $\mathcal{X} = \mathrm{Spec} S$ are affine derived schemes, j is an *open immersion* if the morphism of animated rings $S \rightarrow R$ is flat and $\mathrm{Spec} \pi_0(R) \rightarrow \mathrm{Spec} \pi_0(S)$ is an open immersion.
- (2) If \mathcal{X} is an affine derived scheme, j is an *open immersion* if it is a monomorphism and there exists a collection of affine derived schemes $\{\mathcal{U}_i\}_{i \in I}$ together with an effective epimorphism $\mathcal{U}_i \rightarrow \mathcal{X}$ such that the compositions $\mathcal{U}_i \rightarrow \mathcal{X}$ are open immersions for all $i \in I$.
- (3) In general, j is an *open immersion* if for every affine derived scheme V endowed with a morphism $V \rightarrow \mathcal{X}$, the base change $\mathcal{U} \times_{\mathcal{X}} V \rightarrow V$ is an open immersion.

DEFINITION 1.3.17 (Derived scheme). A derived stack \mathcal{X} is a *derived scheme* if there exists a collection $\{\mathcal{U}_i \rightarrow \mathcal{X}\}_{i \in I}$ of open immersions, where each \mathcal{U}_i is an affine derived scheme, such that

$$\coprod_{i \in I} \mathcal{U}_i \rightarrow \mathcal{X}$$

is an effective epimorphism in the ∞ -topos dStk .

A classical scheme, when viewed as a derived stack, is also a derived scheme. Furthermore, for a derived scheme X , the precomposition of X with the natural inclusion $\mathrm{Ring} \hookrightarrow \mathrm{An}(\mathrm{Ring})$ yields the *underlying classical scheme* X_{cl} . This functor $\mathrm{Ring} \rightarrow \mathrm{An}$ indeed takes values in sets and is represented by a classical scheme.

A derived scheme X is defined to be *quasi-compact* or *quasi-separated* if its underlying classical scheme X_{cl} has the same property. Together, these conditions imply that X has a finite affine cover whose intersections are also quasi-compact. We denote the ∞ -category of quasi-compact quasi-separated derived schemes as dSch .

We can extend the definition of every property of morphisms of animated rings that is local on the source and target (such as flatness and being locally of finite presentation) to maps in dSch . We say a morphism $f: X \rightarrow S$ satisfies such a property P if there exist an affine cover V_i of S and affine covers \mathcal{U}_{ij} of $X \times_S V_i$ such that the compositions $\mathcal{U}_{ij} \rightarrow X \times_S V_i \rightarrow V_i$ satisfy P . Furthermore, f is said to be an *fppf cover* if it is flat, locally of finite presentation, and $f_{\mathrm{cl}}: X_{\mathrm{cl}} \rightarrow S_{\mathrm{cl}}$ is surjective.

DEFINITION 1.3.18 (Quasi-coherent sheaves). We define the functor $D_{\mathrm{qc}}: \mathrm{dStk}^{\mathrm{op}} \rightarrow \mathrm{Cat}_{\infty}$, which maps a derived stack \mathcal{X} to its *derived ∞ -category of quasi-coherent sheaves* $D_{\mathrm{qc}}(\mathcal{X})$, as the right Kan extension of the functor $\mathrm{An}(\mathrm{Ring}) \rightarrow \mathrm{Cat}_{\infty}$ along the natural inclusion $\mathrm{An}(\mathrm{Ring}) \hookrightarrow \mathrm{dStk}^{\mathrm{op}}$.

Let $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ be the full subcategory of Pr^{L} consisting of presentable stable ∞ -categories. Since the ∞ -category $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}})$ is complete and the natural functor $\mathrm{CAlg}(\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}) \rightarrow \mathrm{Cat}_{\infty}$

preserves limits [HA, Prop. 1.1.4.6 and Cor. 3.2.2.5], the ∞ -categories $D_{\text{qc}}(\mathcal{X})$ are presentable, stable, and endowed with symmetric monoidal structures. Furthermore, for a morphism of derived stacks $f: \mathcal{X} \rightarrow \mathcal{S}$, the pullback $f^*: D_{\text{qc}}(\mathcal{S}) \rightarrow D_{\text{qc}}(\mathcal{X})$ is symmetric monoidal.

The following animated analogue of faithfully flat descent, proved by Lurie and Töen [SAG, Cor. D.6.3.3], will be our main tool for understanding quasi-coherent sheaves on derived stacks.

THEOREM 1.3.19. *The functor $\text{An}(\text{Ring}) \rightarrow \text{Cat}_{\infty}$, which sends an animated ring R to its derived ∞ -category $D(R)$, satisfies faithfully flat descent. In other words, this functor preserves finite products and, for a faithfully flat morphism of animated rings $S \rightarrow R$, the diagram*

$$D(S) \rightarrow D(R) \rightrightarrows D(R \otimes_S R) \rightrightarrows D(R \otimes_S R \otimes_S R) \rightrightarrows \dots$$

forms a limit diagram.

Consequently, for an effective epimorphism $f: \mathcal{X} \rightarrow \mathcal{S}$ of derived stacks, the diagram

$$D_{\text{qc}}(\mathcal{S}) \rightarrow D_{\text{qc}}(\mathcal{X}) \rightrightarrows D_{\text{qc}}(\mathcal{X} \times_S \mathcal{X}) \rightrightarrows D_{\text{qc}}(\mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X}) \rightrightarrows \dots$$

forms a limit diagram.

THEOREM 1.3.20. *The functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{CAlg}(\text{Cat}_{\infty})$, which sends X to $D_{\text{qc}}(X)$, extends to a six-functor formalism on the geometric setup $(\text{dSch}, \text{all})$ valued in presentable stable ∞ -categories.*

Proof. Let I be the collection of all isomorphisms in dSch , and let P consist of every morphism in dSch . In order to apply Theorem 1.2.13, we need to verify that (I, P) is a compactification setup. The only non-vacuous condition is that, for every morphism $f: X \rightarrow S$ in dSch , the functor f^* has a right adjoint f_* satisfying base change and the projection formula. This can be verified as in [BFN10, §3.2]. \square

■ **REMARK 1.3.21 — THE EXCEPTIONAL INVERSE IMAGE FUNCTOR.** Let $f: X \rightarrow S$ be a morphism of quasi-compact quasi-separated schemes. The right adjoint to the direct image functor $f_*: D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(S)$ is sometimes denoted by f^{\times} in the Grothendieck duality literature. A functor that deserves to be called $f^!$ can then be constructed by factoring f as a composition $p \circ j$ of an open immersion j and a proper map p , and defining $f^!$ as $j^* \circ p^{\times}$.

Instead of imposing that $f_! = f_*$ as we did, one could also enlarge the ∞ -categories $D_{\text{qc}}(X)$ of quasi-coherent sheaves to the ∞ -categories $D_{\square}(X)$ of solid modules. There, one can define a compactly supported direct image functor $f_!: D_{\square}(X) \rightarrow D_{\square}(S)$ and its right adjoint can be restricted to $D_{\text{qc}}(S) \rightarrow D_{\text{qc}}(X)$, where it coincides with the functor $f^!$ defined in the previous paragraph.

As we have no need for the *correct* exceptional inverse image functor in this thesis, we will continue to denote the right adjoint of $f_*: D_{\text{qc}}(X) \rightarrow D_{\text{qc}}(S)$ as $f^!$. ■

1. Quasi-coherent sheaves and \mathcal{D} -modules

The final objective of this section is to extend this six-functor formalism from dSch to dStk . First, we note that a derived stack can also be regarded as an fppf sheaf on dSch . Just as in the classical setting, there is an equivalence of ∞ -categories between dStk and the category of fppf sheaves of anima on dSch [Hoy14, Lem. C.3].

Now, the theory exposed in the previous section gives an extension of this six-functor formalism to the ∞ -category $\widetilde{\mathrm{dSch}}$ of anima-valued sheaves on dSch with respect to the D_{qc} -topology. The next result gives a comparison between dStk and $\widetilde{\mathrm{dSch}}$.

PROPOSITION 1.3.22. *Let $f: X \rightarrow S$ be an fppf cover of quasi-compact quasi-separated derived schemes. Then f is a D_{qc} -cover.*

Proof. Given that every morphism of quasi-compact quasi-separated derived schemes is cohomologically proper, the criterion of [Sch22, Prop. 6.19] applies. Namely, it suffices to prove that $f_*\mathcal{O}_X$ is descendable in the sense of [Mat16, Def. 3.18]. Furthermore, up to taking affine covers, we can assume that both X and S are affine derived schemes. The result then follows from [Mat16, Cor. 3.33]. \square

As a consequence of the preceding proposition, there exists a fully faithful functor $\widetilde{\mathrm{dSch}} \hookrightarrow \mathrm{dStk}$. This functor has a left adjoint $L: \mathrm{dStk} \rightarrow \widetilde{\mathrm{dSch}}$, referred to as *sheafification* [HTT, Prop. 6.2.2.7]. The results of the previous section provide a collection of morphism \tilde{E} in $\widetilde{\mathrm{dSch}}$ such that the six-functor formalism of Theorem 1.3.20 extends to the geometric setup $(\widetilde{\mathrm{dSch}}, \tilde{E})$. The following corollary extends it further to dStk .

COROLLARY 1.3.23. *Let E be the collection of morphisms of derived stacks that lie in \tilde{E} after sheafification. Then, the six-functor formalism of quasi-coherent sheaves extends from quasi-compact quasi-separated derived schemes to derived stacks. In this extension, every morphism in E has $!$ -functors.*

Proof. Theorem 1.2.18 gives an extension of D_{qc} to anima-valued sheaves on dSch with respect to the D_{qc} -topology. Furthermore, [Cam24, Lem. 3.1.4] shows that the natural functor

$$\mathrm{Corr}(\mathrm{dStk}, E) \rightarrow \mathrm{Corr}(\widetilde{\mathrm{dSch}}, \tilde{E}),$$

given by sheafification, is symmetric monoidal. Consequently, the composition

$$\mathrm{Corr}(\mathrm{dStk}, E) \rightarrow \mathrm{Corr}(\widetilde{\mathrm{dSch}}, \tilde{E}) \rightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$$

is a six-functor formalism valued in presentable stable ∞ -categories. \square

By definition of the D_{qc} -topology, quasi-compact quasi-separated derived schemes lie in the full subcategory $\widetilde{\mathrm{dSch}}$ of dStk , and thus the six-functor formalism above is indeed an extension of the one in Theorem 1.3.20. Moreover, since D_{qc} -covers are of

universal $*$ -descent, the restriction

$$\mathrm{dStk}^{\mathrm{op}} \simeq \mathrm{Corr}(\mathrm{dStk}, \text{isomorphisms}) \hookrightarrow \mathrm{Corr}(\mathrm{dStk}, \mathbb{E}) \rightarrow \mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$$

agrees with the functor given in Definition 1.3.18. In particular, for a derived stack \mathcal{X} , pullback by the natural map $\mathcal{X} \rightarrow \mathrm{L}(\mathcal{X})$ induces an equivalence of ∞ -categories $\mathrm{D}_{\mathrm{qc}}(\mathcal{X}) \simeq \mathrm{D}_{\mathrm{qc}}(\mathrm{L}(\mathcal{X}))$.

Finally, we observe that, although we defined the functor $f_!$ as f_* for a morphism f of quasi-compact quasi-separated derived schemes, the functors $f_!$ and f_* may differ when f is a morphism of derived stacks. We first explain the simplest possible example, then we give a more interesting one, that is also going to be useful in the next section.

■ **EXAMPLE 1.3.24.** For a field k , denote by X a countable disjoint union of copies of $\mathrm{Spec} k$, and by $p: X \rightarrow \mathrm{Spec} k$ the structure map. By descent, an object M of $\mathrm{D}_{\mathrm{qc}}(X)$ amounts to a collection M_i of objects in $\mathrm{D}(k)$. Then, $p_!M$ is the direct sum of the M_i , while p_*M computes their cartesian product. Furthermore, the forget-supports map $p_!M \rightarrow p_*M$ can be identified to the natural inclusion. ■

PROPOSITION 1.3.25. *Let X be a quasi-compact quasi-separated scheme. Consider a closed immersion $i: Z \rightarrow X$ cut out by a locally finitely generated ideal sheaf. Then, the natural map $\hat{i}: \hat{X}_Z \rightarrow X$ is cohomologically étale.*

Proof. Let h be the natural morphism $Z \rightarrow \hat{X}_Z$. We claim that h is of universal $!$ -descent. The question is local on X , so we can suppose that X is affine and Z is the vanishing loci of n functions. Since being of universal $!$ -descent is stable under base change, we may assume that $n = 1$ and Z is the vanishing locus of t in $X = \mathrm{Spec} \mathbb{Z}[t]$.

The condition on [Sch22, Prop. 6.6] for verifying that \mathcal{O}_Z is h -smooth coincides with the one for verifying that \mathcal{O}_Z is i -smooth, in the sense of [Sch22, Def. 6.1]. The latter holds according to [Stacks, Tag 0AU3]. By the derived Nakayama lemma [Stacks, Tag 0G1U], pullback by h is conservative, thus Remark 1.2.16 implies that it is of universal $!$ -descent.

Given that having $!$ -functors is local on the source, we conclude that \hat{i} has $!$ -functors. We note that the pullback functor $\hat{i}^*: \mathrm{D}_{\mathrm{qc}}(X) \rightarrow \mathrm{D}_{\mathrm{qc}}(\hat{X}_Z)$ has a fully faithful left adjoint satisfying base change and the projection formula [HP23, Thms. 2.2.3 and 2.2.5]. Arguing as in [Sch22, Thm. 8.13], we conclude that $\mathcal{O}_{\hat{X}_Z}$ is \hat{i} -smooth. Since its diagonal is an isomorphism, \hat{i} is cohomologically étale. □

■ **EXAMPLE 1.3.26 — GREENLEES–MAY.** Let $i: Z \rightarrow X$ be a closed immersion defined by a locally finitely generated ideal sheaf \mathcal{J} . Denote by \hat{i} the natural map $\hat{X}_Z \rightarrow X$. Since \hat{i} is a cohomologically étale monomorphism, base change implies that both $\hat{i}^* \circ \hat{i}_!$ and $\hat{i}^* \circ \hat{i}_*$ are isomorphic to the identity functor. Consequently, $\hat{i}_!$ and \hat{i}_* are fully faithful.

The functor $\hat{i}_! \circ \hat{i}^*$, an avatar of *local cohomology*, maps an object M in $\mathrm{D}_{\mathrm{qc}}(X)$ to its \mathcal{J} -power torsion part. In contrast, the functor $\hat{i}_* \circ \hat{i}^*$ sends an object M of $\mathrm{D}_{\mathrm{qc}}(X)$ to

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its \mathcal{J} -completion. Consequently, the full subcategories of $D_{\text{qc}}(X)$ consisting of \mathcal{J} -power torsion sheaves and \mathcal{J} -complete sheaves are equivalent. This equivalence is known as *Greenlees–May duality*, a vast generalization of Grothendieck’s local duality theorem. For more details, we refer to [HP23, §2].

For example, consider the inclusion i of a prime $Z = \text{Spec } \mathbb{F}_p$ into $X = \text{Spec } \mathbb{Z}$. In this case, the functor $\hat{i}_! \circ \hat{i}^*$ maps the structure sheaf \mathbb{Z} to $\mathbb{Q}_p/\mathbb{Z}_p[-1]$. On the other hand, the functor $\hat{i}_* \circ \hat{i}^*$ maps \mathbb{Z} to the p -adic integers \mathbb{Z}_p . This gives another example where the forget-supports map fails to be an isomorphism. ■

1.4. QUASI-COHERENT SHEAVES ON DE RHAM SPACES

In this section, we fulfill a promise made at the beginning of Section 1.2 by studying the derived ∞ -categories of quasi-coherent sheaves on de Rham spaces and their relation to \mathcal{D} -modules. Additionally, many natural functors relating these ∞ -categories can be interpreted as natural operations on \mathcal{D} -modules.

The content of this section is similar to that in [Sch22, App. to Lec. VIII], but we focus on quasi-coherent sheaves instead of ind-coherent sheaves and we consider relative \mathcal{D} -modules. We begin by generalizing de Rham spaces to derived stacks.

DEFINITION 1.4.1 (de Rham space). Let \mathcal{X} be a derived stack over a field k . Its *de Rham space* \mathcal{X}_{dR} is the functor $\text{An}(\text{Ring}) \rightarrow \text{An}$ defined by

$$\mathcal{X}_{\text{dR}}(\mathbb{R}) := \text{colim}_{I \subset \pi_0(\mathbb{R})} \mathcal{X}(\pi_0(\mathbb{R})/I),$$

where the colimit runs over the filtered poset of nilpotent ideals of $\pi_0(\mathbb{R})$. This functor comes equipped with a natural transformation $\mathcal{X} \rightarrow \mathcal{X}_{\text{dR}}$ induced by the trivial ideal.

Some remarks are in order. First, the inclusion $\text{Set} \hookrightarrow \text{An}$ preserves filtered colimits, ensuring the consistency of the definition above with Definition 1.1.1. Moreover, for a derived scheme X , the natural map $X_{\text{cl}} \rightarrow X$ induces an isomorphism between their de Rham spaces. This enables the extension of most results in Section 1.1 to derived schemes.¹¹ That being said, for simplicity, this section will focus on classical schemes.

PROPOSITION 1.4.2. *Let $p: X \rightarrow S$ be a smooth morphism of relative dimension n between finite type k -schemes, and let $\iota_p: X \rightarrow X_{\text{dR}} \times_{S_{\text{dR}}} S$ be the natural map. Then, ι_p is cohomologically smooth.*

Proof. Cohomological smoothness is a property that can be verified locally on the source and target. Thus, by Proposition 1.1.6 and [Stacks, Tag 039Q], we may assume

¹¹However, the pullback functor $D_{\text{qc}}(X_{\text{dR}}) \rightarrow D_{\text{qc}}(X)$, corresponding to the forgetful functor from \mathcal{D}_X -modules to quasi-coherent sheaves, surely depends on the derived structure of X .

that p is the natural projection $\mathbb{A}_S^n \rightarrow S$ for some $n \geq 0$. In this case, the map ι_p becomes the product of n copies of $\iota: \mathbb{A}^1 \rightarrow \mathbb{A}_{\text{dR}}^1$ with the identity of S . Therefore, it suffices to prove that ι is cohomologically smooth.

Since ι is an effective epimorphism, we will check that the projection $\mathbb{A}^1 \times_{\mathbb{A}_{\text{dR}}^1} \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is cohomologically smooth. As in the proof of Corollary 1.1.7, this pullback can be identified with the projection $\mathbb{A}^1 \times \widehat{\mathbb{G}}_a \rightarrow \mathbb{A}^1$. Consequently, it is sufficient to verify that the structure map $\widehat{\mathbb{G}}_a \rightarrow \text{Spec } k$ is cohomologically smooth.

The formal group $\widehat{\mathbb{G}}_a$ can also be viewed as a formal completion of \mathbb{P}^1 . Hence, Proposition 1.3.25 implies that the natural map $\widehat{\mathbb{G}}_a \rightarrow \mathbb{P}^1$ is cohomologically étale. The structure map $\mathbb{P}^1 \rightarrow \text{Spec } k$ is cohomologically smooth by [Sch22, Prop. 8.14], concluding the proof that ι_p is cohomologically smooth. \square

To examine the natural functoriality of de Rham spaces in a relative setting, we consider the following commutative diagram of finite type k -schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & & \downarrow p \\ S' & \xrightarrow{g} & S, \end{array} \quad (*)$$

where p and q are smooth. We emphasize that this square is *not* necessarily supposed to be cartesian. The universal property of fiber products provides a natural map $(f, g)_{\text{dR}}: X'_{\text{dR}} \times_{S'_{\text{dR}}} S' \rightarrow X_{\text{dR}} \times_{S_{\text{dR}}} S$.

COROLLARY 1.4.3. *Consider the setting of the commutative square (*). Then, the morphism $(f, g)_{\text{dR}}: X'_{\text{dR}} \times_{S'_{\text{dR}}} S' \rightarrow X_{\text{dR}} \times_{S_{\text{dR}}} S$ admits $!$ -functors. Furthermore, $(f, g)_{\text{dR}}$ is cohomologically proper if the square (*) is cartesian, or if f is étale and g is a monomorphism. The morphism $(f, g)_{\text{dR}}$ is cohomologically smooth if f is smooth and proper. Additionally, if f is a closed immersion and g is the identity morphism, $(f, g)_{\text{dR}}$ is cohomologically étale.*

Proof. Given that the property of having $!$ -functors is local on the source, it suffices to verify this after precomposing with the cohomologically smooth map $\iota_{p'}$. The commutativity of the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \iota_{p'} \downarrow & & \downarrow \iota_p \\ X'_{\text{dR}} \times_{S'_{\text{dR}}} S' & \xrightarrow{(f, g)_{\text{dR}}} & X_{\text{dR}} \times_{S_{\text{dR}}} S, \end{array}$$

which can be verified after post composing with the projections $X_{\text{dR}} \times_{S_{\text{dR}}} S \rightarrow X_{\text{dR}}$ and $X_{\text{dR}} \times_{S_{\text{dR}}} S \rightarrow S$, implies that $(f, g)_{\text{dR}} \circ \iota_{p'}$ is the composition of ι_p and f . As both ι_p and f have $!$ -functors, the same holds for $(f, g)_{\text{dR}}$.

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If the diagram $(*)$ is cartesian, or if f is étale and g is monic, the square above becomes cartesian. Given that cohomological properness can be verified after pulling back via the cohomologically smooth map ι_p [Cam24, Prop. 3.1.14], the cohomological properness of f implies the same for $(f, g)_{\text{dR}}$.

Similarly, [Cam24, Prop. 3.1.24] implies that cohomological smoothness is local on the source. Thus, by the commutativity of the commutative diagram above, $(f, g)_{\text{dR}}$ is cohomologically smooth as long as f is. This holds if f is smooth and proper [Sch22, Prop. 8.14].

Suppose that g is the identity morphism and f is a closed immersion. Both the large rectangle and the bottom square in the diagram

$$\begin{array}{ccc}
 X'_{\text{dR}} \times_{X_{\text{dR}}} X & \longrightarrow & X \\
 \downarrow & & \downarrow \iota_p \\
 X'_{\text{dR}} \times_{S_{\text{dR}}} S & \xrightarrow{(f, g)_{\text{dR}}} & X_{\text{dR}} \times_{S_{\text{dR}}} S \\
 \downarrow & & \downarrow \\
 X'_{\text{dR}} & \xrightarrow{f_{\text{dR}}} & X_{\text{dR}}
 \end{array}$$

are cartesian. By the pasting law, so is the top square. Since being cohomologically étale is local on the target, it suffices to prove that $X'_{\text{dR}} \times_{X_{\text{dR}}} X \rightarrow X$ has this property. This is the content of Proposition 1.3.25. \square

For a smooth morphism of schemes $X \rightarrow S$ of relative dimension n , we denote by $\mathcal{D}_{X/S}$ its sheaf of differential operators as defined in [EGA IV₄, §16.8.7]. The ∞ -category $D_{\text{qc}}(\mathcal{D}_{X/S})$ is then defined as the full subcategory of the derived ∞ -category $D(\mathcal{D}_{X/S})$ of left $\mathcal{D}_{X/S}$ -modules, consisting of objects whose cohomology sheaves are quasi-coherent \mathcal{O}_X -modules.

This ∞ -category is endowed with a forgetful functor $D_{\text{qc}}(\mathcal{D}_{X/S}) \rightarrow D_{\text{qc}}(X)$, which can be viewed as a restriction of scalars. Since tensoring with the dualizing complex $\Omega_{X/S}^n[n]$ gives an equivalence of ∞ -categories from $D_{\text{qc}}(\mathcal{D}_{X/S})$ to the analogously defined ∞ -category of *right* $\mathcal{D}_{X/S}$ -modules [Ber00, §1.2.1], there exists another forgetful functor $D_{\text{qc}}(\mathcal{D}_{X/S}) \rightarrow D_{\text{qc}}(X)$.

As usual, the forgetful functor for right $\mathcal{D}_{X/S}$ -modules has a left adjoint, which sends M to $M \otimes \mathcal{D}_{X/S}$ with its natural structure of a right $\mathcal{D}_{X/S}$ -module. This is referred to as the *induced $\mathcal{D}_{X/S}$ -module functor*.¹²

THEOREM 1.4.4. *Let $p: X \rightarrow S$ be a smooth morphism between finite type k -schemes. Then, the ∞ -categories $D_{\text{qc}}(X_{\text{dR}} \times_{S_{\text{dR}}} S)$ and $D_{\text{qc}}(\mathcal{D}_{X/S})$ are monoidally equivalent. Under this*

¹²There is also an induced $\mathcal{D}_{X/S}$ -module functor for left $\mathcal{D}_{X/S}$ -modules. However, as noted by Saito [Sai89], the one for right $\mathcal{D}_{X/S}$ -modules has better compatibility with the direct image functors.

equivalence, the functor ι_p^* corresponds to the forgetful functor of left $\mathcal{D}_{X/S}$ -modules, the functor $\iota_p^!$ corresponds to the forgetful functor of right $\mathcal{D}_{X/S}$ -modules, and $\iota_{p,!}$ corresponds to the induced $\mathcal{D}_{X/S}$ -module functor.

We will first prove this result at the level of abelian categories. As in Definition 1.3.18, we can right Kan extend the functor $(\text{Sch}/k)^{\text{op}} \rightarrow \text{Cat}$, which sends a k -scheme to its abelian category of quasi-coherent sheaves, to the 2-category $\text{PSh}(\text{Sch}/k, \text{Grpd})$. We will still denote this extension by QCoh .

For a more concrete description of $\text{QCoh}(X_{\text{dR}} \times_{S_{\text{dR}}} S)$, consider the following diagram, where each map is a natural projection.

$$\widehat{(X \times_S X \times_S X)}_{\Delta} \rightrightarrows \widehat{(X \times_S X)}_{\Delta} \rightrightarrows X$$

We denote by pr_i (resp. pr_{ij}) the projection on the i -th factor (resp. on the i, j -th factors).

LEMMA 1.4.5. *Given a quasi-coherent sheaf M on X , the following data are equivalent:*

- (i) *An isomorphism of quasi-coherent sheaves $\varepsilon: \text{pr}_1^* M \rightarrow \text{pr}_2^* M$ satisfying the cocycle condition $\text{pr}_{12}^*(\varepsilon) \circ \text{pr}_{23}^*(\varepsilon) = \text{pr}_{13}^*(\varepsilon)$;*
- (ii) *A stratification on M as in [BO78, Def. 2.10];*
- (iii) *A morphism of \mathcal{O}_X -algebras $\mathcal{D}_{X/S} \rightarrow \underline{\text{End}}_{\mathcal{O}_S}(M)$.*

Proof. Let $(X \times_S X)_{\Delta}^{(n)}$ denote the n -th infinitesimal neighborhood of the diagonal in $X \times_S X$. Recall that the formal completion $\widehat{(X \times_S X)}_{\Delta}$ is the filtered colimit of the schemes $(X \times_S X)_{\Delta}^{(n)}$. In particular, $\text{QCoh}(\widehat{(X \times_S X)}_{\Delta})$ is the limit of the categories $\text{QCoh}((X \times_S X)_{\Delta}^{(n)})$. Thus, an isomorphism $\varepsilon: \text{pr}_1^* M \rightarrow \text{pr}_2^* M$ corresponds to a compatible system of isomorphisms $\varepsilon_n: (\text{pr}_1^{(n)})^* M \rightarrow (\text{pr}_2^{(n)})^* M$, where

$$(X \times_S X)_{\Delta}^{(n)} \begin{array}{c} \xrightarrow{\text{pr}_1^{(n)}} \\ \xrightarrow{\text{pr}_2^{(n)}} \end{array} X$$

are the natural projections. This establishes the equivalence between (i) and (ii). Finally, [BO78, Prop. 2.11] gives the equivalence between (ii) and (iii). The reader may want to see [BO78, Rem. 2.13] as well. \square

Note that, for a morphism of quasi-coherent sheaves $\varphi: M \rightarrow N$ on X along with isomorphisms $\varepsilon_M: \text{pr}_1^* M \rightarrow \text{pr}_2^* M$ and $\varepsilon_N: \text{pr}_1^* N \rightarrow \text{pr}_2^* N$ as above, the diagram

$$\begin{array}{ccc} \text{pr}_1^* M & \xrightarrow{\text{pr}_1^* \varphi} & \text{pr}_1^* N \\ \varepsilon_M \downarrow & & \downarrow \varepsilon_N \\ \text{pr}_2^* M & \xrightarrow{\text{pr}_2^* \varphi} & \text{pr}_2^* N \end{array}$$

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commutes if and only if φ is $\mathcal{D}_{X/S}$ -linear. Moreover, for quasi-coherent sheaves M and N on X endowed with isomorphisms $\varepsilon_M: \mathrm{pr}_1^* M \rightarrow \mathrm{pr}_2^* M$ and $\varepsilon_N: \mathrm{pr}_1^* N \rightarrow \mathrm{pr}_2^* N$, the tensor product $M \otimes N$ inherits the isomorphism

$$\mathrm{pr}_1^*(M \otimes N) \simeq \mathrm{pr}_1^* M \otimes \mathrm{pr}_1^* N \xrightarrow{\varepsilon_M \otimes \varepsilon_N} \mathrm{pr}_2^* M \otimes \mathrm{pr}_2^* N \simeq \mathrm{pr}_2^*(M \otimes N).$$

According to [Ber96, Cor. 2.3.3], this corresponds to the usual tensor product of $\mathcal{D}_{X/S}$ -modules constructed locally as in [HTT08, Prop. 1.2.9]. All in all, we obtain the following proposition.

PROPOSITION 1.4.6. *Let $X \rightarrow S$ be a smooth morphism of finite type k -schemes. Then $\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ is monoidally equivalent to the category $\mathrm{QCoh}(\mathcal{D}_{X/S})$ of quasi-coherent $\mathcal{D}_{X/S}$ -modules.*

Proof. Note that the functor $\mathrm{QCoh}: \mathrm{PSh}(\mathrm{Sch}/k, \mathrm{Grpd})^{\mathrm{op}} \rightarrow \mathrm{Cat}$ factors through the sheafification $\mathrm{PSh}(\mathrm{Sch}/k, \mathrm{Grpd}) \rightarrow \mathrm{Sh}((\mathrm{Sch}/k)_{\acute{\mathrm{e}}\mathrm{t}}, \mathrm{Grpd})$ to yield a limit-preserving functor $\mathrm{Sh}((\mathrm{Sch}/k)_{\acute{\mathrm{e}}\mathrm{t}}, \mathrm{Grpd})^{\mathrm{op}} \rightarrow \mathrm{Cat}$ [HTT, Prop. 5.5.4.20]. Proposition 1.1.18 then implies that $\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ is the limit of the diagram

$$\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(\widehat{(X \times_S X)}_{\Delta}) \rightrightarrows \mathrm{QCoh}(\widehat{(X \times_S X \times_S X)}_{\Delta}).$$

This is a category of descent data, whose objects are precisely quasi-coherent sheaves M on X endowed with the data of Lemma 1.4.5.(i). The result then follows from the aforementioned lemma. \square

To proceed from Proposition 1.4.6 to the proof of Theorem 1.4.4, we use the following lemma.

LEMMA 1.4.7. *We define two full subcategories $D_{\mathrm{qc}}^{\leq 0}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ and $D_{\mathrm{qc}}^{\geq 0}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ of $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$, consisting of objects whose inverse image by ι_p lies in $D_{\mathrm{qc}}^{\leq 0}(X)$ or $D_{\mathrm{qc}}^{\geq 0}(X)$, respectively.*

The pair $(D_{\mathrm{qc}}^{\leq 0}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S), D_{\mathrm{qc}}^{\geq 0}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S))$ defines a t-structure on $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$, whose heart is equivalent to $\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$. Furthermore, $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ is equivalent to the derived ∞ -category of its heart.

Proof. The map $\iota_p: X \rightarrow X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S$ being an effective epimorphism implies that the ∞ -category $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ is the limit of the diagram

$$D_{\mathrm{qc}}(X) \rightrightarrows D_{\mathrm{qc}}(\widehat{(X \times_S X)}_{\Delta}) \rightrightarrows D_{\mathrm{qc}}(\widehat{(X \times_S X \times_S X)}_{\Delta}) \rightrightarrows \cdots$$

Since all the functors in the diagram above are t-exact and $\iota_p^*: D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S) \rightarrow D_{\mathrm{qc}}(X)$ is conservative, [HP23, Lem. 2.1.12.(iv)] endows $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ with the desired

t-structure. The same result also states that the heart $D_{\text{qc}}^{\heartsuit}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$ is the limit of the diagram

$$\text{QCoh}(X) \rightrightarrows \text{QCoh}(\widehat{(\mathcal{X} \times_S \mathcal{X})_{\Delta}}) \rightrightarrows \text{QCoh}(\widehat{(\mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X})_{\Delta}}) \rightrightarrows \cdots$$

Here, Proposition 1.3.25 allows us to identify the abelian categories of quasi-coherent sheaves on the formal completions with the hearts of their corresponding stable ∞ -categories. According to [Car+17, Lem. 2.21], $D_{\text{qc}}^{\heartsuit}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$ is also the limit of the truncated diagram

$$\text{QCoh}(X) \rightrightarrows \text{QCoh}(\widehat{(\mathcal{X} \times_S \mathcal{X})_{\Delta}}) \rightrightarrows \text{QCoh}(\widehat{(\mathcal{X} \times_S \mathcal{X} \times_S \mathcal{X})_{\Delta}}),$$

and this limit is precisely $\text{QCoh}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$.

Let M and I be objects of $\text{QCoh}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S) \simeq D_{\text{qc}}^{\heartsuit}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$, and suppose that I is injective. We claim that $\pi_0 \text{Hom}_{\mathbf{C}}(M, I[n])$, where $\mathbf{C} = D_{\text{qc}}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$, vanishes for all $n > 0$. Given the description of \mathbf{C} as a limit, $\pi_0 \text{Hom}_{\mathbf{C}}(M, I[n])$ embeds into $\pi_0 \text{Hom}_{\mathbf{D}}(\iota_p^* M, \iota_p^* I[n])$, for $\mathbf{D} = D_{\text{qc}}(X)$. Therefore, it suffices to prove that the latter vanishes.

To establish that $\pi_0 \text{Hom}_{\mathbf{D}}(\iota_p^* M, \iota_p^* I[n]) = 0$, we will verify that $\iota_p^* I$ is an injective \mathcal{O}_X -module. Since X is noetherian, it suffices to prove that $\iota_p^* I$ is an injective object of $\text{QCoh}(X)$ [TT07, B.4]. Consider a monomorphism $\mathcal{F} \rightarrow \mathcal{G}$ of quasi-coherent sheaves on X , and a map $\mathcal{F} \rightarrow \iota_p^* I$. Since $\mathcal{D}_{X/S}$ is flat over \mathcal{O}_X , the induced map $\mathcal{D}_{X/S} \otimes \mathcal{F} \rightarrow \mathcal{D}_{X/S} \otimes \mathcal{G}$ remains a monomorphism. Furthermore, by the universal property of scalar extension, there exists a morphism $\mathcal{D}_{X/S} \otimes \mathcal{F} \rightarrow I$ making the diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \iota_p^*(\mathcal{D}_{X/S} \otimes \mathcal{F}) & \longrightarrow & \iota_p^*(\mathcal{D}_{X/S} \otimes \mathcal{G}) \\ \downarrow & & \downarrow \\ \iota_p^* I & & \end{array}$$

(A curved arrow also points from \mathcal{F} to $\iota_p^* I$)

commute. The injectivity of I then gives a morphism $\mathcal{D}_{X/S} \otimes \mathcal{G} \rightarrow I$ whose image under ι_p^* makes the triangle above commute. This confirms that $\iota_p^* I$ is injective, and so $\pi_0 \text{Hom}_{\mathbf{C}}(M, I[n])$ vanishes.

The conservativity and t-exactness of the functor $\iota_p^*: D_{\text{qc}}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S) \rightarrow D_{\text{qc}}(X)$, along with [HA, Prop. 1.2.1.19], implies that the t-structure of $D_{\text{qc}}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$ is both left and right complete. Then, [HA, Prop. 1.3.3.7] gives rise to a fully faithful functor $D^+(\text{QCoh}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)) \rightarrow D_{\text{qc}}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S)$ whose essential image is the full subcategory

$$D_{\text{qc}}^+(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S) := \bigcup_{n \leq 0} D_{\text{qc}}^{\geq n}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S) \subset D_{\text{qc}}(\mathcal{X}_{\text{dR}} \times_{S_{\text{dR}}} S).$$

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The desired equivalence $D(\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)) \xrightarrow{\sim} D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)$ then follows by taking left completions. Indeed, the category $D(\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S))$ is left complete since $\iota_p^*: D(\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)) \rightarrow D(\mathrm{QCoh}(X)) \simeq D_{\mathrm{qc}}(X)$ is conservative and t-exact [Stacks, Tag 09T4]. \square

Proof of Theorem 1.4.4. The same strategy as in the preceding lemma, based on [HA, Prop. 1.3.3.7], also shows that $D_{\mathrm{qc}}(\mathcal{D}_{X/S})$ is equivalent to $D(\mathrm{QCoh}(\mathcal{D}_{X/S}))$. All in all, we have equivalences

$$D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S) \simeq D(\mathrm{QCoh}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S)) \simeq D(\mathrm{QCoh}(\mathcal{D}_{X/S})) \simeq D_{\mathrm{qc}}(\mathcal{D}_{X/S}),$$

where the equivalence in the middle is given by Proposition 1.4.6. It is clear that, under these equivalences, the pullback ι_p^* corresponds to the forgetful functor. \square

Having related de Rham spaces to \mathcal{D} -modules, we now relate their natural functorialities. First, we note that the equivalence $D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S) \simeq D_{\mathrm{qc}}(\mathcal{D}_{X/S})$ preserves the tensor products on both sides. Next, we recall the setting of a not-necessarily-cartesian square of finite type k -schemes

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & & \downarrow p \\ S' & \xrightarrow{g} & S, \end{array} \quad (*)$$

where p and q are smooth. This square induces morphisms $(\widehat{X' \times_{S'} X'})_{\Delta} \rightarrow (\widehat{X \times_S X})_{\Delta}$ and $(\widehat{X' \times_{S'} X' \times_{S'} X'})_{\Delta} \rightarrow (\widehat{X \times_S X \times_S X})_{\Delta}$ making the diagram

$$\begin{array}{ccccc} (\widehat{X' \times_{S'} X' \times_{S'} X'})_{\Delta} & \rightrightarrows & (\widehat{X' \times_{S'} X'})_{\Delta} & \rightrightarrows & X' \\ \downarrow & & \downarrow & & \downarrow \\ (\widehat{X \times_S X \times_S X})_{\Delta} & \rightrightarrows & (\widehat{X \times_S X})_{\Delta} & \rightrightarrows & X \end{array}$$

commute, and allowing us to pullback the data of Lemma 1.4.5.(i). This defines an image inverse functor $(f, g)^*: \mathrm{QCoh}(\mathcal{D}_{X/S}) \rightarrow \mathrm{QCoh}(\mathcal{D}_{X'/S'})$.¹³ The following result follows directly from the proof of Theorem 1.4.4.

COROLLARY 1.4.8. *Let $X \rightarrow S$ and $X' \rightarrow S'$ be smooth morphisms of finite type k -schemes. Given morphisms $f: X' \rightarrow X$ and $g: S' \rightarrow S$ making the diagram $(*)$ commute, the diagram*

$$\begin{array}{ccc} D_{\mathrm{qc}}(X_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S) & \xrightarrow{\sim} & D_{\mathrm{qc}}(\mathcal{D}_{X/S}) \\ (f, g)_{\mathrm{dR}}^* \downarrow & & \downarrow L(f, g)^* \\ D_{\mathrm{qc}}(X'_{\mathrm{dR}} \times_{S'_{\mathrm{dR}}} S') & \xrightarrow{\sim} & D_{\mathrm{qc}}(\mathcal{D}_{X'/S'}) \end{array}$$

¹³See [Ber00, §2.1.1] for the analogous construction on arithmetic \mathcal{D} -modules.

commutes up to natural isomorphism.

■ **REMARK 1.4.9 — ON THE EXISTENCE OF LEFT DERIVED FUNCTORS.** Let $X \rightarrow S$ be a smooth morphism of schemes. A complex M of \mathcal{O}_X -modules is called \mathcal{O}_X -homotopically flat if, for every acyclic complex L of \mathcal{O}_X -modules, the tensor product $M \otimes_{\mathcal{O}_X} L$ is also acyclic. Similarly, a complex M of $\mathcal{D}_{X/S}$ -modules is said to be $\mathcal{D}_{X/S}$ -homotopically flat if $M \otimes_{\mathcal{D}_{X/S}} L$ is acyclic for every acyclic complex of $\mathcal{D}_{X/S}$ -modules L .

For a complex N of $\mathcal{D}_{X/S}$ -modules, [Stacks, Tag 06YF] provides an \mathcal{O}_X -homotopically flat complex of \mathcal{O}_X -modules M along with a quasi-isomorphism $M \rightarrow \widetilde{\mathrm{DR}}(N)$, where $\widetilde{\mathrm{DR}}(N)$ is defined as in [Sai89, Lem. 1.5]. Since $\mathcal{D}_{X/S}$ is a flat \mathcal{O}_X -module, applying the functor $\widetilde{\mathrm{DR}}^{-1} = - \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S}$ preserves the quasi-isomorphism. By the same result of Saito, the composition

$$M \otimes_{\mathcal{O}_X} \mathcal{D}_{X/S} = \widetilde{\mathrm{DR}}^{-1}(M) \rightarrow \widetilde{\mathrm{DR}}^{-1} \widetilde{\mathrm{DR}}(N) \rightarrow N$$

is also a quasi-isomorphism. Given that M is \mathcal{O}_X -homotopically flat, this provides a $\mathcal{D}_{X/S}$ -homotopically flat resolution of N .

Using these $\mathcal{D}_{X/S}$ -homotopically flat resolutions, one can compute the left derived functor of inverse images and tensor products of relative \mathcal{D} -modules on the unbounded derived category. As in [Stacks, Tags 08DW and 08DX], it is clear that both functors preserve complexes with quasi-coherent cohomologies. ■

We are now in position to define direct images of relative \mathcal{D} -modules in the setting of the commutative diagram (*). As in the absolute case, we define the *transfer module*

$$\mathcal{D}_{X/S \leftarrow X'/S'} := \omega_{X'/S'} \otimes_{\mathcal{O}_{X'}} (f, g)^* \mathcal{D}_{X/S} \otimes_{f^{-1}\mathcal{O}_X} f^{-1} \omega_{X/S}^{\otimes -1},$$

which is a $(f^{-1}\mathcal{D}_{X/S}, \mathcal{D}_{X'/S'})$ -bimodule. Observe that, whenever the square (*) is cartesian, the inverse image $(f, g)^* \mathcal{D}_{X/S}$ reduces to $\mathcal{D}_{X'/S'}$ and so $\mathcal{D}_{X/S \leftarrow X'/S'} \simeq \mathcal{D}_{X'/S'}$. In every case, we define the direct image functor $D_{\mathrm{qc}}(\mathcal{D}_{X'/S'}) \rightarrow D_{\mathrm{qc}}(\mathcal{D}_{X/S})$ as

$$(f, g)_+ := \mathrm{Rf}_* \left(\mathcal{D}_{X/S \leftarrow X'/S'} \otimes_{\mathcal{D}_{X'/S'}}^L - \right).$$

Akin to the absolute setting, given a commutative diagram of finite type k -schemes in which the vertical morphisms are smooth

$$\begin{array}{ccccc} X'' & \xrightarrow{f'} & X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \xrightarrow{g'} & S' & \xrightarrow{g} & S, \end{array}$$

we have a natural equivalence $(f, g)_+ \circ (f', g')_+ \simeq (f \circ f', g \circ g')_+$, and similarly for the inverse images [Ber00, §2.1.1].

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■ **REMARK 1.4.10 — ON THE DIRECT IMAGE FUNCTOR.** In the notation above, we begin by observing that $(f, g)_+$ is well-defined as a functor $D(\mathcal{D}_{X'/S'}) \rightarrow D(\mathcal{D}_{X/S})$. First, the derived tensor product in its definition can be computed using homotopically flat resolutions, as in the preceding remark. Next, recall that the category of modules over any sheaf of (not necessarily commutative) rings is Grothendieck [KS06, Thm. 18.1.6]. For complexes in such abelian categories, there exist homotopically injective resolutions [Stacks, Tags 070H and 079P], allowing us to compute the functor Rf_* .

It remains to prove that $(f, g)_+$ preserves the full subcategories of quasi-coherent sheaves. As in the proof of Proposition 1.4.11 below, we may assume that either the commutative square $(*)$ is cartesian or g is the identity map. In each case, the result follows from [Stacks, Tag 08D5]. For the second case, one needs to resolve a quasi-coherent complex by induced \mathcal{D} -modules as in [Ber02, Prop. 2.4.2]. ■

Even though the following result is interesting by itself, it will be particularly useful to us via its corollary that gives a compatibility between the direct image of relative \mathcal{D} -modules and the direct image on the respective de Rham spaces.

PROPOSITION 1.4.11. *Let $X \rightarrow S$ and $X' \rightarrow S'$ be smooth morphisms of finite type k -schemes, and let $d = \dim_S X - \dim_{S'} X'$. Suppose that k has characteristic zero. Consider morphisms $f: X' \rightarrow X$ and $g: S' \rightarrow S$ making the diagram $(*)$ commute and suppose that the square is cartesian or f is smooth. Then the functor $L(f, g)^*$ is left adjoint to $(f, g)_+[d]$.*

Proof. The case in which the square is cartesian can be proved using the same arguments in [Vig21, Cor. 2.1.19]. We now consider the case in which f is smooth. Remark that the universal property of pullbacks induces a smooth morphism $X' \rightarrow X \times_S S'$ making the diagram

$$\begin{array}{ccccc}
 & & f & & \\
 & & \curvearrowright & & \\
 X' & \longrightarrow & X \times_S S' & \longrightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 S' & \xrightarrow{\text{id}_{S'}} & S' & \xrightarrow{g} & S
 \end{array}$$

commute. In particular we may suppose that g is the identity map. This case can be proven as in [Kas03, Thm. 4.40]. □

The uniqueness of adjoints then gives the corollary below.

COROLLARY 1.4.12. *Let $X \rightarrow S$ and $X' \rightarrow S'$ be smooth morphisms of finite type k -schemes, and let $d = \dim_S X - \dim_{S'} X'$. Consider morphisms $f: X' \rightarrow X$ and $g: S' \rightarrow S$ making the diagram $(*)$ commute, and suppose that k has characteristic zero and f is quasi-projective. Then*

the diagram

$$\begin{array}{ccc} D_{\text{qc}}(X'_{\text{dR}} \times_{S'_{\text{dR}}} S') & \xrightarrow{\sim} & D_{\text{qc}}(\mathcal{D}_{X/S}) \\ (f, g)_{\text{dR},!} \downarrow & & \downarrow (f, g)_+[\text{d}] \\ D_{\text{qc}}(X_{\text{dR}} \times_{S_{\text{dR}}} S) & \xrightarrow{\sim} & D_{\text{qc}}(\mathcal{D}_{X/S}) \end{array}$$

commutes up to natural isomorphism.

Proof. As in the proof of the previous proposition, it suffices to prove this in the cases where g is the identity map or the square $(*)$ is cartesian. The case where the square $(*)$ is cartesian follows from Proposition 1.4.11, along with the fact that $(f, g)_{\text{dR}}$ is cohomologically proper in this case.

Now, suppose that g is the identity map. Since we are assuming that f is quasi-projective, we may factor it as a composition $q \circ i \circ j$, where j is an open immersion, i is a closed immersion, and q is smooth and proper. Corollary 1.4.3 says that $(j, \text{id})_{\text{dR}}$ is cohomologically proper, thus the result for j follows as in the previous paragraph.

Another application of Corollary 1.4.3 gives that $(i, \text{id})_{\text{dR}}$ is cohomologically étale. Consequently, $(i, \text{id})_{\text{dR},!}$ is left adjoint to $(i, \text{id})_{\text{dR}}^*$ and the result for i follows from [Car21, Cor. 4.2.3].

As in [Sch22, Thm. 8.35], we have that $(q, \text{id})_{\text{dR},!}$ is isomorphic to $(q, \text{id})_{\text{dR},*}[2n]$, where n is the relative dimension of q . In particular, $(q, \text{id})_{\text{dR},!}[-2n]$ is right adjoint to $(q, \text{id})_{\text{dR}}^*$. The result then follows once again from Proposition 1.4.11. \square

1.5. THE SIX-FUNCTOR FORMALISM OF HOLONOMIC \mathcal{D} -MODULES

The formalism developed in the previous section is quite general but exhibits some unusual behaviors. For instance, given an open immersion $j: U \rightarrow X$, the morphism $j_{\text{dR}}: U_{\text{dR}} \rightarrow X_{\text{dR}}$ is cohomologically proper. In this section, our goal is to construct a dual version of this formalism, one that more closely resembles familiar six-functor formalisms.

Let \mathbf{C} be the category of smooth quasi-projective varieties over a characteristic zero field k . From Corollary 1.4.3 and [Cam24, Lem. 3.1.4], we observe that the functor

$$\text{Corr}(\mathbf{C}, \text{all}) \xrightarrow{(-)_{\text{dR}}} \text{Corr}(\text{dStk}, E) \rightarrow \text{Cat}_{\infty}$$

defines a three-functor formalism on the geometric setup (\mathbf{C}, all) .¹⁴ Here, E denotes the class of morphisms given by Corollary 1.3.23.

We denote the external tensor product in this three-functor formalism as \boxtimes . Similarly, the tensor product on $D_{\text{qc}}(\mathcal{D}_X)$ is denoted as $\otimes_{\mathcal{O}_X}^L$. For a morphism f in \mathbf{C} , we denote

¹⁴This is even a six-functor formalism, but this section will make no use of the right adjoints.

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the inverse image functor as Lf^* . The insistence on the symbol L is due to the fact that most computations for \mathcal{D} -modules reduce to studying the underived analogues of these functors.

Moving forward, we make an initial modification to the three-functor formalism. We retain the action on objects and the lax monoidal structure but adjust the action on correspondences:

$$\left(\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array} \right) \mapsto g_{\mathrm{dR},!} f_{\mathrm{dR}}^* [2 \dim Z - \dim X - \dim Y].$$

This has the effect of replacing the tensor product $\otimes_{\mathcal{O}_X}^L$ with $\otimes_X^! := \otimes_{\mathcal{O}_X}^L[-\dim X]$, whose unit $1_X^!$ is isomorphic to $\mathcal{O}_X[\dim X]$. Similarly, the inverse image becomes $f^! := Lf^*[\dim Z - \dim X]$, and the direct image becomes $g_+ := g_{\mathrm{dR},!}[\dim Z - \dim Y]$.¹⁵

Denote by $D_h^b(\mathcal{D}_X)$ the full subcategory of $D_{\mathrm{qc}}(\mathcal{D}_X)$ consisting of objects M whose cohomologies $\mathcal{H}^i(M)$ are holonomic and vanish for sufficiently large $|i|$. We equip $D_h^b(\mathcal{D}_X)$ with its standard t -structure, and denote by $\mathrm{Hol}(\mathcal{D}_X)$ its heart. According to [HTT08, §3.2.1], the three functors described above preserve the stable ∞ -categories of holonomic \mathcal{D} -modules.

From [HTT08, Cor. 2.6.8], we know that the ∞ -category $D_h^b(\mathcal{D}_X)$ has a natural t -exact anti-equivalence $D_X: D_h^b(\mathcal{D}_X) \xrightarrow{\sim} D_h^b(\mathcal{D}_X)^{\mathrm{op}}$. This functor is its own inverse [HTT08, Prop. 2.6.5]. Now, we make the final modification to our three-functor formalism. Once again, we retain the action on objects and the lax monoidal structure but adjust the action on correspondences:

$$\left(\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array} \right) \mapsto D_Y g_+ f^! D_X.$$

This modification replaces the tensor product $\otimes_X^!$ with the functor \otimes_X , which acts as $(M, N) \mapsto D_X(D_X(M) \otimes_X^! D_X(N))$, and whose unit 1_X is isomorphic to $\mathcal{O}_X[-\dim X]$. Furthermore, the inverse image functor is transformed into $f^+ := D_Z f^! D_X$, and the direct image functor is transformed into $g_! := D_Y g_+ D_Z$.

Although the ∞ -categories $D_h^b(\mathcal{D}_X)$ are not presentable, the three functors just constructed admit right adjoints.

PROPOSITION 1.5.1. *The three-functor formalism D_h^b constructed above is a six-functor formalism. Moreover, for a morphism $f: X \rightarrow S$, the right adjoint of $f_!$ is $f^!$ and the right adjoint of f^+ is f_+ .*

¹⁵Corollary 1.4.12 ensures that this notation is not overloaded.

1.5. The six-functor formalism of holonomic \mathcal{D} -modules

For an object N of $D_h^b(\mathcal{D}_X)$, the right adjoint of $- \otimes_X N$ is the functor $\underline{\text{Hom}}_X(N, -)$ defined as $D_X(- \otimes_X D_X(N))$.

Proof. That $f^!$ and f_+ are right adjoint to $f_!$ and f^+ , respectively, is the content of [HTT08, Cor. 3.2.15]. The existence and expression for the right adjoint of the tensor product follow from [HTT08, Prop. 2.6.14]. \square

For the reader's convenience, we present a table comparing our notations with those of common references. In it, we consider a morphism $f: X \rightarrow S$ and let $d = \dim X - \dim S$.

Our notation	[Ber]	[Bor+87]	[HTT08]	[Meb88]
Lf^*	Lf^Δ	Lf°	Lf^*	$\mathbf{L}f^*$
$f^!$	$f^!$	$f^!$	f^\dagger	-
f_+	f_*	f_+	\int_f	\int_{f^*}
D_X	D	D_X	\mathbb{D}_X	$(-)^*$
f^+	f^*	f^+	f^*	$\mathbf{L}f^![-d]$
$f_!$	$f_!$	$f_!$	$\int_{f!}$	$\int_{f!}$
$\otimes_{\mathcal{O}_X}^L$	$\otimes_{\mathcal{O}_X}^L$	$\otimes_{\mathcal{O}_X}^L$	$\otimes_{\mathcal{O}_X}^L$	$\otimes_{\mathcal{O}_X}^L$
$\otimes_X^!$	Δ	-	-	-
\otimes_X	-	-	-	-
$\underline{\text{Hom}}_X$	Hom	-	-	-

As this table shows, except for the tensor product \otimes_X and its right adjoint $\underline{\text{Hom}}_X$, all the functors in this six-functor formalism are standard in the \mathcal{D} -module literature. Not only is the choice of the tensor product \otimes_X necessary in order to have a six-functor formalism, but this functor also agrees with the standard tensor product of constructible sheaves under the Riemann–Hilbert correspondence [HTT08, Prop. 4.7.8 and Thm. 7.1.1].

The following remarks highlight key aspects of this six-functor formalism, which will be essential in the subsequent chapters. We advise the reader to proceed to the next chapter and consult this section as necessary.

■ **REMARK 1.5.2 — ON CONNECTIONS.** As usual, we identify vector bundles (locally free sheaves) with integrable connections and \mathcal{O}_X -coherent \mathcal{D}_X -modules. Henceforth, we will refer to these objects simply as *connections*. We denote by $D_{\text{int}}^b(\mathcal{D}_X)$ the full subcategory of $D_h^b(\mathcal{D}_X)$ consisting of objects with connections as their cohomologies. As in the theory of perverse sheaves, we endow $D_{\text{int}}^b(\mathcal{D}_X)$ with the standard t-structure, shifted by $\dim X$. Thus, the heart of $D_{\text{int}}^b(\mathcal{D}_X)$ consists of connections in degree $\dim X$.

Consider a morphism $f: X \rightarrow S$, and let $d = \dim X - \dim S$. The inverse image $f^+: D_h^b(\mathcal{D}_S) \rightarrow D_h^b(\mathcal{D}_X)$ restricts to a t-exact functor $D_{\text{int}}^b(\mathcal{D}_S) \rightarrow D_{\text{int}}^b(\mathcal{D}_X)$. Furthermore, we have $\mathcal{H}^i(f^+M) \simeq f^* \mathcal{H}^{i-d}(M)$ for every object M in $D_h^b(\mathcal{D}_S)$ and all $i \in \mathbb{Z}$ [Bor+87,

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Rem. in §VI.4]. Here, the functor f^* coincides with the usual inverse image of connections [CF09, §2.3].

Since the objects in $D_{\text{int}}^b(\mathcal{D}_X)$ are precisely the dualizable objects in $D_h^b(\mathcal{D}_X)$, we have the isomorphisms

$$M \otimes_X N \simeq M \otimes_X^! N[2 \dim X] \simeq M \otimes_{\mathcal{O}_X} N[\dim X],$$

for M in $D_{\text{int}}^b(\mathcal{D}_X)$ and N in $D_h^b(\mathcal{D}_X)$. Consequently, \otimes_X restricts to t-exact functors $D_{\text{int}}^b(\mathcal{D}_X) \times D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_X)$ and $D_{\text{int}}^b(\mathcal{D}_X) \times D_{\text{int}}^b(\mathcal{D}_X) \rightarrow D_{\text{int}}^b(\mathcal{D}_X)$.

Finally, the duality functor $D_X: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_X)^{\text{op}}$ also restricts to a t-exact anti-equivalence $D_{\text{int}}^b(\mathcal{D}_X) \xrightarrow{\sim} D_{\text{int}}^b(\mathcal{D}_X)^{\text{op}}$. More precisely, we have an isomorphism $\mathcal{H}^i(D_X(M)) \simeq \underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{H}^{-i}(M), \mathcal{O}_X)$ for every object M in $D_h^b(\mathcal{D}_S)$ and all $i \in \mathbb{Z}$ [HTT08, Ex. 2.6.10]. Here, $\underline{\text{Hom}}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ denotes the usual dual of connections. ■

■ **REMARK 1.5.3 — ON COHOMOLOGY.** Let X be a smooth quasi-projective variety over k , and denote its structure map by $p: X \rightarrow \text{Spec } k$. As usual in six-functor formalisms, for an object M of $D_h^b(\mathcal{D}_X)$, we define its *de Rham cohomology* and *compactly supported de Rham cohomology groups* as

$$H^i(X, M) := \mathcal{H}^i(p_+ M) \quad \text{and} \quad H_c^i(X, M) := \mathcal{H}^i(p_! M).$$

According to [Dim+00, Prop. 1.4], the de Rham cohomology group $H^i(X, M)$ can be explicitly computed as the sheaf cohomology group $H^i(X, M \otimes \Omega_X^*)$, where the de Rham complex Ω_X^* is concentrated in degrees $[-\dim X, 0]$. This expression, along with the isomorphisms $H_c^i(X, M) \simeq H^{-i}(X, D_X(M))^\vee$, also enables the computation of compactly supported de Rham cohomology groups. ■

■ **REMARK 1.5.4 — ARTIN VANISHING.** Consider a morphism $f: X \rightarrow S$ between smooth quasi-projective k -varieties. A \mathcal{D} -module analogue of Artin vanishing states that, if f is affine, then $f_+: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_S)$ is right t-exact and $f_!: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_S)$ is left t-exact [Bor+87, Prop. VI.8.1].

While we will not use this, it is worth noting that, when f is quasi-finite, the functors $f^!: D_h^b(\mathcal{D}_S) \rightarrow D_h^b(\mathcal{D}_X)$ and $f_+: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_S)$ are left t-exact. Dually, $f_!: D_h^b(\mathcal{D}_X) \rightarrow D_h^b(\mathcal{D}_S)$ and $f^+: D_h^b(\mathcal{D}_S) \rightarrow D_h^b(\mathcal{D}_X)$ are right t-exact. This follows from [HTT08, Exs. 1.5.12 and 1.5.22] and [Bor+87, Prop. VI.8.1], along with Zariski's main theorem. ■

■ **REMARK 1.5.5 — RECOLLEMENT.** Consider a closed immersion $i: Z \hookrightarrow X$ and its complementary open immersion $j: U \hookrightarrow X$. Recall that i is proper (hence $i_! \simeq i_+$) and j is étale (hence $j^! \simeq j^+$). This data satisfies the *recollement conditions* [Bei+18, §1.4.3]. Specifically, the following hold:

- (1) $i^! j_+ = 0$, and by adjunction, $i^+ j_! = 0$ and $j^+ i_+ = 0$;

- (2) The adjunction maps $i_!i^! \rightarrow \text{id} \rightarrow j_+j^+$ and $j_!j^! \rightarrow \text{id} \rightarrow i_+i^+$ form fiber sequences;
- (3) The adjunction maps $i^+i_+ \rightarrow \text{id} \rightarrow i^!i_!$ and $j^+j_+ \rightarrow \text{id} \rightarrow j^!j_!$ are isomorphisms.

In the \mathcal{D} -module context, these axioms are essentially equivalent to Kashiwara's equivalence [HTT08, Cor. 1.6.2].

Since the counit $i^+i_+ \rightarrow \text{id}$ is an isomorphism, the functor i_+ is fully faithful, so the composition $i_+i^! \rightarrow \text{id} \rightarrow i_+i^+$ yields a map $i^! \rightarrow i^+$. Section 1.4 in [Bei+18] shows that $i^+j_+ \simeq i^!j_![1]$ and that

$$\begin{aligned} j_!M &\rightarrow j_+M \rightarrow i_+i^+j_+M \rightarrow j_!M[1], \\ i^!N &\rightarrow i^+N \rightarrow i^+j_+j^+N \rightarrow i^!N[1], \end{aligned}$$

for M in $D_h^b(\mathcal{D}_U)$ and N in $D_h^b(\mathcal{D}_X)$, are distinguished triangles. \blacksquare

■ **REMARK 1.5.6 — MAYER-VIETORIS.** Consider a smooth quasi-projective k -variety X and let $i_1: Z_1 \rightarrow X$ and $i_2: Z_2 \rightarrow X$ be closed immersions covering X , where Z_1 and Z_2 are also smooth.

$$\begin{array}{ccc} Z_1 \cap Z_2 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow i_2 \\ Z_1 & \xrightarrow{i_1} & X. \end{array}$$

Denoting by $i_{12}: Z_1 \cap Z_2 \rightarrow X$ the diagonal map above, we have a fiber sequence

$$M \rightarrow i_{1,!}i_1^+M \oplus i_{2,!}i_2^+M \rightarrow i_{12,!}i_{12}^+M$$

for every M in $D_h^b(\mathcal{D}_X)$. This is a consequence of the recollement conditions. \blacksquare

■ **REMARK 1.5.7 — MIDDLE DIRECT IMAGE.** Let $f: Y \hookrightarrow X$ be a locally closed immersion. Given a holonomic \mathcal{D} -module M over Y , we define the *middle direct image* $f_{i,+}M$ as the image of the forget-supports map $\mathcal{H}^0(f_!M) \rightarrow \mathcal{H}^0(f_+M)$.

For a holonomic \mathcal{D} -module N over X , we say that N is an *extension* of M if $\mathcal{H}^0(f^+N)$ is isomorphic to M . The \mathcal{D} -module $f_{i,+}M$ is characterized as the unique, up to isomorphism, extension of M supported on \bar{Y} with no non-trivial subobjects or quotients supported on $\bar{Y} \setminus Y$. In particular, if \mathcal{E} is a connection on X and Y is a dense open of X , then $f_{i,+}f^+\mathcal{E} \simeq \mathcal{E}$ [Kat90, Cor. 2.9.1.1]. \blacksquare

■ **REMARK 1.5.8 — NEARBY AND VANISHING CYCLES.** Let $i: Z \hookrightarrow X$ be a closed immersion of codimension one. Given this setup, a holonomic \mathcal{D} -module M on X can be endowed with the *V-filtration* as described in [Meb88, Chap. III.4]. This filtration is indexed by k , and its graded pieces $\text{gr}_\alpha^V(M)$ vanish unless $-\alpha$ is a root modulo \mathbb{Z} of the *Bernstein–Sato polynomial* associated to i [Meb88, Prop. 4.2.1].

The *unipotent nearby cycles functor* $\psi_1: \text{Hol}(\mathcal{D}_X) \rightarrow \text{Hol}(\mathcal{D}_Z)$ is defined as gr_{-1}^V , while the *unipotent vanishing cycles functor* $\phi_1: \text{Hol}(\mathcal{D}_X) \rightarrow \text{Hol}(\mathcal{D}_Z)$ is defined as gr_0^V . These

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functors come with natural transformations

$$\text{can}: \psi_1 \rightarrow \phi_1 \quad \text{and} \quad \text{var}: \phi_1 \rightarrow \psi_1$$

given locally by $-\partial_t$ and t , respectively, where t is a local equation for Z . Furthermore, the compositions $\text{can} \circ \text{var}$ and $\text{var} \circ \text{can}$ are nilpotent [Meb88, §4.3.3]. Finally, according to [Meb88, §§4.5.3 and 4.6.5], we have fiber sequences

$$\begin{aligned} i^+ \mathcal{M} &\rightarrow \psi_1(\mathcal{M}) \xrightarrow{\text{can}} \phi_1(\mathcal{M}), \\ i^! \mathcal{M} &\rightarrow \phi_1(\mathcal{M}) \xrightarrow{\text{var}} \psi_1(\mathcal{M}), \end{aligned}$$

for every holonomic \mathcal{D} -module on X . ■

PART II.

HARMONIC ANALYSIS ON STACKS

2. CARTIER DUALITY OF COMMUTATIVE GROUP STACKS

An underlying motif in this chapter is the concept of commutative group stacks. These entities generalize commutative group schemes and naturally emerge in their study. The *raison d'être* of commutative group stacks is that they provide the correct setting for a general form of Cartier duality, the algebro-geometric counterpart to Pontryagin duality.

COMMUTATIVE GROUP STACKS

Henceforth, we fix a base scheme S and consider classical fppf stacks over S . There is a standard method for constructing commutative group stacks. For a morphism $\mathcal{F} \rightarrow \mathcal{G}$ of abelian sheaves on $(\text{Sch}/S)_{\text{fppf}}$, the sheaf \mathcal{F} acts on \mathcal{G} via translation, enabling us to consider the stacky quotient $[\mathcal{G}/\mathcal{F}]$. Proposition 2.1.3, derived from the Dold–Kan correspondence, asserts that every commutative group stack arises in this manner.

The interplay between these perspectives is a fundamental characteristic of those objects. On one hand, commutative group stacks are geometric objects, prompting us to consider line bundles or even quasi-coherent sheaves on them. On the other hand, commutative group stacks can be studied using the tools of fppf cohomology, similarly to any abelian sheaf.

Consider a commutative group stack \mathcal{G} , with its group law denoted by $m: \mathcal{G} \times_S \mathcal{G} \rightarrow \mathcal{G}$. The *stacky Cartier dual* of \mathcal{G} , denoted \mathcal{G}^\vee , is the commutative group stack $\underline{\text{Hom}}(\mathcal{G}, \text{BG}_m)$. Since BG_m classifies line bundles, a morphism of group stacks $\mathcal{G} \rightarrow \text{BG}_m$ corresponds to a *multiplicative* line bundle on \mathcal{G} . More precisely, for an S -scheme T , the objects of $\mathcal{G}^\vee(T)$ are pairs (\mathcal{L}, α) , where \mathcal{L} is a line bundle on $\mathcal{G} \times_S T$ and α is an isomorphism $m^*\mathcal{L} \rightarrow \mathcal{L} \boxtimes \mathcal{L}$ making two diagrams encoding associativity and commutativity commute.

Denote by \mathcal{G}° the morphism of abelian sheaves, defined up to quasi-isomorphism, associated with a commutative group stack \mathcal{G} via the Dold–Kan correspondence. We view \mathcal{G}° as a two-term complex concentrated in degrees -1 and 0 . The stacky Cartier dual can be explicitly described as $(\mathcal{G}^\vee)^\circ = \tau_{\leq 0} \text{R}\underline{\text{Hom}}(\mathcal{G}^\circ, \mathbb{G}_m[1])$. This leads us to the study of classical Cartier duals $\underline{\text{Hom}}(\mathcal{G}, \mathbb{G}_m)$ and extension sheaves $\underline{\text{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$, for abelian sheaves \mathcal{G} .

ABELIAN SCHEMES AND TORI

Consider two fundamental examples of commutative group schemes over a general base: abelian schemes and tori. For a torus T over S , its Cartier dual $X := \underline{\mathrm{Hom}}(T, \mathbb{G}_m)$ is represented by a group scheme étale-locally isomorphic to the constant group scheme \mathbb{Z}^r , for some integer r . According to a theorem of Grothendieck, $\underline{\mathrm{Ext}}^1(T, \mathbb{G}_m)$ vanishes, implying that $T^\vee \simeq BX$. Similarly, the stacky Cartier dual of X is BT . For further details, see Propositions 2.2.1 and 2.3.1.

For an abelian S -scheme A , the Cartier dual $\underline{\mathrm{Hom}}(A, \mathbb{G}_m)$ vanishes. The stacky Cartier dual A^\vee , which then corresponds to the extension sheaf $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$, is represented by the dual abelian scheme of A . This result is known as the *Barsotti–Weil formula*. Surprisingly, a complete proof appears to be absent from the literature.¹ We present a generalization of this result in Theorem 2.3.10. In contrast to the approach of [Oor66], our method applies to general base schemes and does not require a reduction to the case of a base field.

Even though stacky Cartier duals contain no information about higher extension sheaves, those frequently appear in long exact sequences. For an abelian scheme A over a regular base S , the vanishing of $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ has been claimed in [Bre75, Rem. 6], [Jos09, Thm. 1.2.5], and [Bro21, Cor. 11.5]. Regrettably, in each instance, this result is attributed to [Bre69, §7], but additional arguments appear to be lacking.

Section 2.4, written in collaboration with Zev Rosengarten, aims to address this gap. Theorem 2.4.4 proves that $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ vanishes for a base scheme S of characteristic zero, without any regularity assumptions. Additionally, we show that $\underline{\mathrm{Ext}}^2(A, M) = 0$ for an abelian scheme A over a general base S and a quasi-coherent \mathcal{O}_S -module M . In particular, $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_a)$ vanishes.

THE CASE OF A BASE FIELD

We now consider a commutative connected algebraic group G over a field k of characteristic zero. According to the Barsotti–Chevalley theorem, G has a functorial decomposition as an extension of an abelian variety A by a product of a torus T and an unipotent group U . While the classical Cartier duals and extension sheaves of A and T were discussed in the previous subsection, we now also consider unipotent groups.

As a first observation, we note that a commutative unipotent group over a characteristic zero field is necessarily a vector group. In particular, U is isomorphic to some power of the additive group \mathbb{G}_a . Denoting by U^* the dual vector space, Proposition 2.2.3 computes the Cartier dual $\underline{\mathrm{Hom}}(U, \mathbb{G}_m)$, identifying it to the formal completion \widehat{U}^* of U^* along the identity.

¹For further details, see [Jos09, Footnote to Thm. 1.2.2] and [Jos10].

Due to its significance in the remainder of this thesis, we denote the abelian sheaf $\underline{\text{Ext}}^1(G, \mathbb{G}_m)$ by G' . This is the sheafification of the functor mapping a k -scheme S to the group $\text{Ext}_S^1(G, \mathbb{G}_m)$, which classifies extensions of G_S by \mathbb{G}_m over S . Consequently, there exists a natural morphism

$$\text{Ext}_S^1(G, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(G, \mathbb{G}_m)(S) = G'(S)$$

that is functorial on S . Furthermore, the group $\text{Ext}_S^1(G, \mathbb{G}_m)$ can be seen as the group of isomorphism classes of objects in $G^\vee(S)$. As such, there also exists a natural map²

$$\text{Ext}_S^1(G, \mathbb{G}_m) \rightarrow H^1(G_S, \mathbb{G}_m)$$

sending a pair (\mathcal{L}, α) to \mathcal{L} . Its image lies in the subgroup $H_m^1(G_S, \mathbb{G}_m)$, which consists of equivalence classes of line bundles \mathcal{L} on G_S satisfying $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$.

For a regular k -scheme S , Theorem 2.3.19 demonstrates that both morphisms constructed above

$$G'(S) = \underline{\text{Ext}}^1(G, \mathbb{G}_m)(S) \leftarrow \text{Ext}_S^1(G, \mathbb{G}_m) \rightarrow H_m^1(G_S, \mathbb{G}_m)$$

are isomorphisms. This substantially simplifies the geometric interpretation of elements in $G'(S)$.

A crucial element in proving Theorem 2.3.19 is a detailed study of the abelian sheaf $U' = \underline{\text{Ext}}^1(U, \mathbb{G}_m)$. The most important point being Proposition 2.3.16, which states that U' vanishes on seminormal k -schemes. In [Ros23, Rem. 2.2.16], Rosengarten details Gabber's construction of a non-zero section of this sheaf. In Example 2.3.17, we provide what is arguably the simplest non-zero section of this sheaf.

Unfortunately, the vanishing of U' has been asserted in the literature. This claim can be found in [Pol11, Lem. 1.3.6], was utilized in the proofs of [BB09, Lem. A.4.5] and [Ber14, Lem. 10], and can be derived from the computations in [Lau85, Ex. p. 25].³ We believe this led Laumon to consider a modified definition of the stacky Cartier dual in his more recent preprint [Lau96]. This modification plays a significant role in the following chapter.

THE STACKY CARTIER DUAL OF A DE RHAM SPACE

Using the de Rham spaces introduced in the previous chapter, we can also employ fppf cohomology techniques to the study of line bundles *with integrable connection* on G . Recall that the de Rham space G_{dR} , as in Definition 1.1.1, is isomorphic to the quotient of G by its formal completion \widehat{G} along the identity.

²Equivalently, this arrow sends an extension to its underlying \mathbb{G}_m -torsor.

³Indeed, Laumon claims that the morphism γ in Proposition 3.1.9 is an isomorphism.

2. Cartier duality of commutative group stacks

As a first observation, Proposition 2.2.6 explicitly verifies that the Cartier dual $\underline{\mathrm{Hom}}(\mathbb{G}_{\mathrm{dR}}, \mathbb{G}_m)$ vanishes. As a result, the stacky Cartier dual $\mathbb{G}_{\mathrm{dR}}^\vee$ is naturally isomorphic to $\underline{\mathrm{Ext}}^1(\mathbb{G}_{\mathrm{dR}}, \mathbb{G}_m)$. This abelian sheaf is of critical importance in this thesis, and we denote it by \mathbb{G}^\natural .

According to Theorem 1.4.4, the fppf cohomology group $H^1(\mathbb{G}_{\mathrm{dR}} \times S, \mathbb{G}_m)$ classifies isomorphism classes of line bundles \mathcal{L} on G_S with integrable connection ∇ relative to S . As before, we consider the subgroup $H_m^1(\mathbb{G}_{\mathrm{dR}} \times S, \mathbb{G}_m)$ consisting of pairs (\mathcal{L}, ∇) satisfying $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$. One of the main results in Section 2.3, Theorem 2.3.21, states that the natural maps

$$\mathbb{G}^\natural(S) = \underline{\mathrm{Ext}}^1(\mathbb{G}_{\mathrm{dR}}, \mathbb{G}_m)(S) \leftarrow \mathrm{Ext}_S^1(\mathbb{G}_{\mathrm{dR}}, \mathbb{G}_m) \rightarrow H_m^1(\mathbb{G}_{\mathrm{dR}} \times S, \mathbb{G}_m),$$

for a reduced k -scheme S , are isomorphisms.

For an abelian variety A , the abelian sheaf A^\natural is representable by an algebraic group known as the *universal vector extension* of the dual abelian variety A' . This object was extensively studied in [MM74], and our methods provide simpler proofs for many of their results. See Remark 2.3.24 for further details. The geometric properties of T^\natural and U^\natural are more subtle and will be discussed in the next chapter.

2.1. DEFINITIONS AND FIRST PROPERTIES

Let Lat be the full subcategory of Ab whose objects are free abelian groups of finite rank. Given an ∞ -category \mathcal{C} with finite products, we define its category of *abelian group objects* $\mathrm{Ab}(\mathcal{C})$ as the ∞ -category of functors $\mathrm{Lat}^{\mathrm{op}} \rightarrow \mathcal{C}$ that commute with finite products. (See [Lur17, §1.2] for more.)

When \mathcal{C} is the category of sets, $\mathrm{Ab}(\mathcal{C})$ coincides with the usual category of abelian groups. If \mathcal{C} is the ∞ -category of anima, $\mathrm{Ab}(\mathcal{C})$ is the ∞ -category of animated abelian groups as in Section 1.3. More generally, if X is an ∞ -topos, then $\mathrm{Ab}(X)$ is the ∞ -category of sheaves on X valued in animated abelian groups. This motivates the definition below.

DEFINITION 2.1.1 (Commutative group stack). Let \mathcal{C} be a 1-site. A *commutative group stack* on \mathcal{C} is a sheaf

$$\mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{An}_{\leq 1}(\mathrm{Ab}),$$

where $\mathrm{An}_{\leq 1}(\mathrm{Ab})$ is the full subcategory of $\mathrm{An}(\mathrm{Ab})$ consisting of animated abelian groups M with $\pi_i(M) = 0$ for $i \neq 0, 1$.

Equivalently, a commutative group stack on \mathcal{C} is an abelian group object in the 2-topos $\mathrm{Sh}(\mathcal{C}, \mathrm{Grpd})$. These objects were first suggested by Grothendieck and subsequently

studied by Deligne in [SGA 4_{III}, Exp. XVIII, §1.4] under the name *champs de Picard strictement commutatifs*.

■ **REMARK 2.1.2.** The forgetful functor $\text{An}(\text{Ab}) \rightarrow \text{An}$ restricts to $\text{An}_{\leq 1}(\text{Ab}) \rightarrow \text{Grpd}$. Concretely, an object of $\text{An}_{\leq 1}(\text{Ab})$ is a groupoid M endowed with a symmetric monoidal structure $+$ such that the set of isomorphism classes $\pi_0(M)$ is a group. Additionally, for all x, y in M , the symmetry constraints $x + x \rightarrow x + x$ and $x + y \rightarrow y + x \rightarrow x + y$ are supposed to be the identity maps. ■

The following proposition, a consequence of the Dold–Kan correspondence, provides a concrete characterization of commutative group stacks. This characterization will underpin virtually every computation involving these objects throughout this thesis.

PROPOSITION 2.1.3. *Let \mathbf{C} be a 1-site. Denote by $\text{Ch}^{[-1,0]}(\text{Ab}(\mathbf{C}))$ the category of complexes of abelian sheaves⁴ on \mathbf{C} concentrated in degrees -1 and 0 . Similarly, denote by $\text{D}^{[-1,0]}(\text{Ab}(\mathbf{C}))$ the full subcategory of $\text{D}(\text{Ab}(\mathbf{C}))$ consisting of the objects whose cohomologies are concentrated in degrees -1 and 0 . The functor*

$$\begin{aligned} \text{Ch}^{[-1,0]}(\text{Ab}(\mathbf{C})) &\rightarrow \text{Sh}(\mathbf{C}, \text{An}_{\leq 1}(\text{Ab})) \\ [\mathcal{F} \rightarrow \mathcal{G}] &\mapsto [\mathcal{G}/\mathcal{F}] \end{aligned}$$

factors through $\text{D}^{[-1,0]}(\text{Ab}(\mathbf{C}))$, and the induced functor $\text{D}^{[-1,0]}(\text{Ab}(\mathbf{C})) \rightarrow \text{Sh}(\mathbf{C}, \text{An}_{\leq 1}(\text{Ab}))$ is an equivalence of ∞ -categories.

We will use the proof of this result to explain some important concepts in higher topos theory. Readers who are already familiar with these concepts may skip to the conclusion of the proof.

Proof of Proposition 2.1.3. Let \mathbf{D} be an ∞ -category. Recall that a functor $\mathcal{F}: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ is a sheaf if it preserves finite products and if, for every covering $X \rightarrow S$ in \mathbf{C} , the natural morphism

$$\mathcal{F}(S) \rightarrow \lim \left[\mathcal{F}(X) \rightrightarrows \mathcal{F}(X \times_S X) \rightrightarrows \mathcal{F}(X \times_S X \times_S X) \rightrightarrows \cdots \right]$$

is invertible. Functors preserving finite products and satisfying the preceding descent condition with respect to all *hypercovers* in \mathbf{C} are called *hypersheaves*. We denote by $\text{HSh}(\mathbf{C}, \mathbf{D})$ the full subcategory of $\text{Sh}(\mathbf{C}, \mathbf{D})$ constituted of the hypersheaves.

Consider a sheaf of abelian groups \mathcal{G} on \mathbf{C} . Using the natural inclusion $\text{Ab} \hookrightarrow \text{D}(\text{Ab})$, \mathcal{G} can be viewed as a presheaf valued in the derived ∞ -category $\text{D}(\text{Ab})$. While this often fails to satisfy descent, one can sheafify to obtain a functor $\text{Ab}(\mathbf{C}) \rightarrow \text{Sh}(\mathbf{C}, \text{D}(\text{Ab}))$.

⁴Denoting the category of abelian sheaves on \mathbf{C} by $\text{Ab}(\mathbf{C})$ is an abuse of notation, since this is the category of abelian group objects on the associated 1-topos.

2. Cartier duality of commutative group stacks

This induces a t-exact fully faithful functor $D(\mathbf{Ab}(\mathbf{C})) \hookrightarrow \mathbf{Sh}(\mathbf{C}, D(\mathbf{Ab}))$, whose essential image is precisely $\mathbf{HSh}(\mathbf{C}, D(\mathbf{Ab}))$ [SAG, Cor. 2.1.2.3].

The distinction between hypersheaves and ordinary sheaves is a phenomenon unique to ∞ -categories. More precisely, the inclusion $\mathbf{HSh}(\mathbf{C}, D) \hookrightarrow \mathbf{Sh}(\mathbf{C}, D)$ is an equivalence as soon as D is a n -category for some *finite* n [HTT, Lem. 6.5.2.9]. Consequently,

$$D^{[-1,0]}(\mathbf{Ab}(\mathbf{C})) \rightarrow \mathbf{Sh}(\mathbf{C}, D^{[-1,0]}(\mathbf{Ab}))$$

is an equivalence of ∞ -categories.

Finally, recall that the Dold–Kan correspondence gives an equivalence of ∞ -categories $D^{\leq 0}(\mathbf{Ab}) \xrightarrow{\sim} \mathbf{An}(\mathbf{Ab})$. Here, the i -th homotopy group of an object on the right is isomorphic to the $-i$ -th cohomology group of the corresponding object on the left. In particular, this restricts to an equivalence $D^{[-1,0]}(\mathbf{Ab}) \xrightarrow{\sim} \mathbf{An}_{\leq 1}(\mathbf{Ab})$ leading to the desired equivalence $D^{[-1,0]}(\mathbf{Ab}(\mathbf{C})) \xrightarrow{\sim} \mathbf{Sh}(\mathbf{C}, \mathbf{An}_{\leq 1}(\mathbf{Ab}))$. \square

Let us momentarily denote by \mathcal{G}° the object in the derived category of abelian sheaves associated with a commutative group stack \mathcal{G} . (We will soon begin identifying these objects.) Here, we outline some consequences of the proof of Proposition 2.1.3.

■ REMARK 2.1.4.

1. Let \mathcal{G} be an abelian sheaf. Composing with the natural inclusion $\mathbf{Ab} \hookrightarrow \mathbf{An}_{\leq 1}(\mathbf{Ab})$, we obtain a commutative group stack denoted by the same symbol. Then, \mathcal{G}° is isomorphic to the complex $[0 \rightarrow \mathcal{G}]$. Similarly, $(B\mathcal{G})^\circ$ is isomorphic to $[\mathcal{G} \rightarrow 0]$.
2. Given a commutative group stack \mathcal{G} , the abelian sheaf $\mathcal{H}^0(\mathcal{G}^\circ)$ is isomorphic to the coarse moduli sheaf of \mathcal{G} . This is the sheafification of the presheaf sending an object X of \mathbf{C} to the group of isomorphism classes $\pi_0(\mathcal{G}(X))$ of $\mathcal{G}(X)$. Similarly, $\mathcal{H}^{-1}(\mathcal{G}^\circ)$ is isomorphic to the automorphism sheaf of a zero section of \mathcal{G} .
3. Let \mathcal{G} and \mathcal{H} be commutative group stacks. We denote by $\underline{\mathbf{Hom}}(\mathcal{G}, \mathcal{H})$ the internal Hom of commutative group stacks. This is another commutative group stack satisfying $\underline{\mathbf{Hom}}(\mathcal{G}, \mathcal{H})^\circ \simeq \tau_{\leq 0} \mathbf{R}\underline{\mathbf{Hom}}(\mathcal{G}^\circ, \mathcal{H}^\circ)$. \blacksquare

The following useful proposition, proven in [Bro21, Prop. 2.10], shows that fiber sequences provide a reasonable notion of *short exact sequences of commutative group stacks*. However, the reader should be aware that this notion sometimes behaves in unexpected ways. For example, given an abelian sheaf \mathcal{G} , the morphisms

$$\mathcal{G} \rightarrow 0 \rightarrow B\mathcal{G}$$

satisfy the equivalent conditions in the proposition below [Bro21, Rem. 2.12].

PROPOSITION 2.1.5. Let $\varphi: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ and $\psi: \mathcal{G}_2 \rightarrow \mathcal{G}_3$ be morphisms of commutative group stacks. Then

$$\mathcal{G}_1^\circ \rightarrow \mathcal{G}_2^\circ \rightarrow \mathcal{G}_3^\circ$$

is a fiber sequence if and only if ψ is an effective epimorphism and φ identifies \mathcal{G}_1 with the fiber of ψ over a zero section of \mathcal{G}_3 .

Henceforth, we will focus on the big fppf site $\mathbf{C} = (\mathrm{Sch}/S)_{\mathrm{fppf}}$ associated with a base scheme S . There is a distinguished abelian sheaf on this site: the multiplicative group \mathbb{G}_m , sending a S -scheme T to the group of units $\Gamma(T, \mathcal{O}_T)^\times$. Its classifying stack, $\mathbf{B}\mathbb{G}_m$, will play a role in this thesis similar to that of the circle group in the Pontryagin duality of locally compact abelian groups.

DEFINITION 2.1.6. Let \mathcal{G} be a commutative group stack on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. We define its Cartier dual \mathcal{G}^{D} as $\underline{\mathrm{Hom}}(\mathcal{G}, \mathbb{G}_m)$ and its stacky Cartier dual \mathcal{G}^\vee as $\underline{\mathrm{Hom}}(\mathcal{G}, \mathbf{B}\mathbb{G}_m)$.

From our perspective, the stacky Cartier dual \mathcal{G}^\vee is the *true* dual of a commutative group stack while \mathcal{G}^{D} is a good enough approximation in some situations. The following remark explains our motivation for considering \mathcal{G}^\vee .

■ REMARK 2.1.7. Let \mathcal{G} be a commutative group stack on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, and denote by $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ its group law. Given an S -scheme T , the groupoid $\mathcal{G}^\vee(T)$ can be described as follows: its objects are pairs (\mathcal{L}, α) , where \mathcal{L} is a line bundle on $\mathcal{G} \times_S T$ and α is an isomorphism $m^* \mathcal{L} \rightarrow \mathcal{L} \boxtimes \mathcal{L}$ making diagrams (A) and (B) just above [Bro21, Rem. 3.13] commute. A morphism from (\mathcal{L}, α) to (\mathcal{L}', α') is an isomorphism of line bundles $\varphi: \mathcal{L} \rightarrow \mathcal{L}'$ making the diagram

$$\begin{array}{ccc} m^* \mathcal{L} & \xrightarrow{\alpha} & \mathcal{L} \boxtimes \mathcal{L} \\ m^* \varphi \downarrow & & \downarrow \varphi \boxtimes \varphi \\ m^* \mathcal{L}' & \xrightarrow{\alpha'} & \mathcal{L}' \boxtimes \mathcal{L}' \end{array}$$

commute. The symmetric monoidal structure on $\mathcal{G}^\vee(T)$ sends (\mathcal{L}, α) and (\mathcal{L}', α') to the pair $(\mathcal{L} \otimes \mathcal{L}', \alpha \cdot \alpha')$, where $\alpha \cdot \alpha'$ is defined as the composition

$$m^*(\mathcal{L} \otimes \mathcal{L}') \simeq m^* \mathcal{L} \otimes m^* \mathcal{L}' \xrightarrow{\alpha \otimes \alpha'} (\mathcal{L} \boxtimes \mathcal{L}) \otimes (\mathcal{L}' \boxtimes \mathcal{L}') \simeq (\mathcal{L} \otimes \mathcal{L}') \boxtimes (\mathcal{L} \otimes \mathcal{L}').$$

The reader might also be interested in comparing this description to [SGA 7_I, Exp. VII, §1.1.6] or [Mor85, §I.2.3]. ■

In most cases of interest in this thesis, we will work with commutative group stacks \mathcal{G} defined by complexes $[\mathcal{F} \rightarrow \mathcal{G}]$, where the map $d: \mathcal{F} \rightarrow \mathcal{G}$ is a monomorphism. In this case, $\mathcal{G} \simeq \mathrm{coker} \, d$ takes values in the full subcategory Ab of $\mathrm{An}_{\leq 1}(\mathrm{Ab})$. The following proposition gives a convenient criterion for computing the stacky Cartier duals of these objects.

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PROPOSITION 2.1.8. *Let \mathcal{G} be an abelian sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. Then the stacky Cartier dual of the classifying stack $B\mathcal{G}$ is isomorphic to \mathcal{G}^{D} . If $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m) = 0$, then $\mathcal{G}^{\vee} \simeq B\mathcal{G}^{\mathrm{D}}$. Similarly, if $\mathcal{G}^{\mathrm{D}} = 0$, then $\mathcal{G}^{\vee} \simeq \underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$.*

Proof. The first isomorphism is simply the fact that $((B\mathcal{G})^{\vee})^{\circ}$ is given by

$$\tau_{\leq 0}\mathrm{RHom}(\mathcal{G}[1], \mathbb{G}_m[1]) \simeq \underline{\mathrm{Hom}}(\mathcal{G}, \mathbb{G}_m).$$

Similarly, using the explicit description of $(\mathcal{G}^{\vee})^{\circ}$ as $\tau_{\leq 0}\mathrm{RHom}(\mathcal{G}, \mathbb{G}_m[1])$, the other isomorphisms follow from the fact that an object of a derived category, whose cohomology is concentrated in a single degree, is isomorphic to that cohomology in the corresponding degree. \square

We refer the reader to [Bro21] for an in-depth study of the Cartier duality of commutative group stacks, whose focus is largely orthogonal to the content of the following sections.

2.2. COMPUTATIONS OF CARTIER DUALS

Consider a base scheme S , and let \mathcal{G} be an abelian sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$, regarded as a commutative group stack. Our approach to computing the stacky Cartier dual \mathcal{G}^{\vee} predominantly follows the method outlined in Proposition 2.1.8. We typically either prove the vanishing of $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$ and calculate \mathcal{G}^{D} , or we establish that \mathcal{G}^{D} vanishes and compute $\underline{\mathrm{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$. This section focuses on computing the Cartier duals \mathcal{G}^{D} for some abelian sheaves \mathcal{G} of interest.

Recall that an *abelian scheme* is a group scheme over S that is smooth, proper, and has connected fibers.⁵ Additionally, a *torus* is a finite type group scheme over S that is locally, with respect to the fpqc topology, isomorphic to a finite product of copies of the multiplicative group \mathbb{G}_m .⁶ As is customary, we will systematically denote the Cartier dual T^{D} of a torus T by X .

PROPOSITION 2.2.1. *Let A be an abelian scheme and T be a torus over a base scheme S . Then the Cartier dual A^{D} of A vanishes, and the Cartier dual X of T is representable by a group scheme that is étale locally isomorphic to a finite product of copies of the constant group scheme \mathbb{Z} . Moreover, the Cartier dual of X is isomorphic to T .*

Proof. Denote by $p: A \rightarrow S$ the structure map, and let T be an S -scheme. By the universal property of the global spectrum, a morphism of schemes $A_T \rightarrow \mathbb{G}_{m,T}$ over T

⁵An abelian scheme is automatically of finite presentation, has geometrically connected fibers, and has a commutative group law.

⁶A torus is automatically of finite presentation, affine, and faithfully flat over the base. Moreover, it is étale locally isomorphic to a finite product of copies of \mathbb{G}_m and has a commutative group law.

is equivalent to a morphism of \mathcal{O}_T -algebras

$$\mathcal{O}_T[t, t^{-1}] \rightarrow p_{T,*}\mathcal{O}_{A_T} \simeq \mathcal{O}_T,$$

where $p_T: A_T \rightarrow T$ is the base change of p [Stacks, Tag 0E0L]. It follows that every morphism of schemes $A_T \rightarrow \mathbb{G}_{m,T}$ over T must be constant. If it is a morphism of groups, it has to be trivial. This proves that A^D vanishes. The statements about tori are proven in [SGA 3II, Exp. X, Cor. 5.7]. \square

Moving forward, we focus on the case where the base scheme S is the spectrum of a characteristic zero field k . The following proposition recalls the standard setting for Cartier duality [SGA 3I, Exp. VII_B, Prop. 2.2.2]. (Proposition 2.2.2 is independent of the characteristic of k , but every other result below requires it to be zero.)

PROPOSITION 2.2.2. *Let $G = \text{Spec } R$ be an affine commutative group scheme over k . Its Cartier dual G^D is represented by the formal group $\text{Spf } R^*$, where R^* is the dual Hopf algebra. Moreover, the double dual $(G^D)^D$ is naturally isomorphic to G .*

Let U be a commutative unipotent algebraic group over k . Since the exponential map gives an isomorphism between U and its Lie algebra, U is necessarily a vector group [Mil17, Prop. 14.32] and we denote by U^* its vector space dual.

PROPOSITION 2.2.3. *Let U be a commutative unipotent algebraic group with dual U^* . Denote by \widehat{U} and \widehat{U}^* their formal completions along the zero sections. Then $U^D \simeq \widehat{U}^*$ and $\widehat{U}^D \simeq U^*$.*

Proof. The computation $U^D \simeq \widehat{U}^*$ follows from the fact that the dual of $\text{Sym}(U^*)$ is the completion of $\text{Sym}(U)$ at the ideal of degree one elements. (Upon a choice of basis, this is nothing but the isomorphism $k[x_1, \dots, x_n]^* \simeq k[[x_1, \dots, x_n]]$.) The other computation follows from this one by duality. \square

Given a k -algebra R , we have that $\widehat{\mathbb{G}}_a(R)$ is the group of nilpotent elements in R and $\widehat{\mathbb{G}}_m(R)$ is that of unipotent elements. We remark that the formal groups $\widehat{\mathbb{G}}_a$ and $\widehat{\mathbb{G}}_m$ are isomorphic via the map

$$\begin{aligned} \widehat{\mathbb{G}}_m(R) &\rightarrow \widehat{\mathbb{G}}_a(R) \\ 1 + x &\mapsto \log(1 + x). \end{aligned}$$

This phenomenon is a general property of formal groups in characteristic zero, and it simplifies their study. (In positive characteristic, divided powers come on the scene, and give rise to the group scheme⁷ \mathbb{G}_a^\sharp ; the crystalline counterpart of $\widehat{\mathbb{G}}_a$ [Dri22].) The following result already appeared in this thesis as Lemma 1.1.12.

⁷Not to be confused with the abelian sheaf \mathbb{G}_a^\natural of Definition 2.3.23.

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PROPOSITION 2.2.4 (Cartier). *Let G be a commutative algebraic group over k and let \mathfrak{g} be its Lie algebra, seen as a vector group. The formal completions of G and of \mathfrak{g} along their zero sections coincide.*

The preceding propositions enable us to compute the Cartier dual of the formal completions of commutative algebraic groups. This result, with a slightly different proof, also appears in [BB09, Lem. A.3.1].

COROLLARY 2.2.5. *Let G be a commutative algebraic group over k and denote by \mathfrak{g} its Lie algebra. The Cartier dual of the formal completion \widehat{G} is naturally isomorphic to \mathfrak{g}^* . This also coincides with the invariant differentials Ω_G of G .*

We are now in position to compute the Cartier dual of the de Rham space $G_{\mathrm{dR}} \simeq G/\widehat{G}$ of a commutative connected algebraic group G .

PROPOSITION 2.2.6. *For a commutative connected algebraic group G over k , the Cartier dual of G_{dR} vanishes.*

Proof. Recall from the Barsotti–Chevalley theorem [Mil17, Thm. 8.28 and Cor. 16.15] that G is an extension of an abelian variety A by a product of a torus T and a unipotent group U . Since the de Rham functor $(-)\mathrm{dR}$ is exact, the de Rham space G_{dR} is an extension of A_{dR} by $T_{\mathrm{dR}} \times U_{\mathrm{dR}}$. Thus, we have an induced exact sequence

$$0 \rightarrow \underline{\mathrm{Hom}}(A_{\mathrm{dR}}, \mathbb{G}_m) \rightarrow \underline{\mathrm{Hom}}(G_{\mathrm{dR}}, \mathbb{G}_m) \rightarrow \underline{\mathrm{Hom}}(T_{\mathrm{dR}}, \mathbb{G}_m) \times \underline{\mathrm{Hom}}(U_{\mathrm{dR}}, \mathbb{G}_m),$$

and so it suffices to prove the result when G is an abelian variety, a torus, or a unipotent group.

Because the functor $\underline{\mathrm{Hom}}(-, \mathbb{G}_m)$ is left-exact, the vanishing of $G_{\mathrm{dR}}^{\mathrm{D}}$ is equivalent to the morphism $\underline{\mathrm{Hom}}(G, \mathbb{G}_m) \rightarrow \underline{\mathrm{Hom}}(\widehat{G}, \mathbb{G}_m)$ being a monomorphism. We verify this in the relevant particular cases. For abelian varieties, their vanishing Cartier dual ensures this property. For a unipotent group U , the relevant morphism is isomorphic to $\widehat{U}^* \rightarrow U^*$, which is likewise monic per Corollary 1.1.5.

For a torus T , the question of whether $\underline{\mathrm{Hom}}(T, \mathbb{G}_m) \rightarrow \underline{\mathrm{Hom}}(\widehat{T}, \mathbb{G}_m)$ is a monomorphism is local on the base, allowing us to assume that $T = \mathbb{G}_m$. Then the statement boils down to the following: given a connected k -algebra R , if

$$\begin{aligned} \mathrm{Uni}(B) &\hookrightarrow B^\times \rightarrow B^\times \\ \chi &\mapsto \chi \mapsto \chi^n \end{aligned}$$

is the unit map for every R -algebra B , then $n = 0$. Here, $\mathrm{Uni}(B)$ denotes the group of unipotent elements in B . This can be proven by choosing $B = R[z]/(z-1)^r$ for some $r > n$, leading to the desired result. \square

The hypothesis that G is connected in the Proposition 2.2.6 is essential. In general, a commutative algebraic group G over k fits into a short exact sequence

$$0 \rightarrow G^0 \rightarrow G \rightarrow \pi_0(G) \rightarrow 0,$$

where G^0 is the connected component of G containing the identity, and $\pi_0(G)$ is the finite étale group of connected components. Proposition 1.1.6 states that the natural map $\pi_0(G) \rightarrow \pi_0(G)_{\text{dR}}$ is an isomorphism, and the preceding result then implies that the Cartier duals of G_{dR} and $\pi_0(G)$ coincide.

2.3. EXTENSIONS BY THE MULTIPLICATIVE GROUP

As in the previous section, consider a base scheme S and an abelian sheaf \mathcal{G} on $(\text{Sch}/S)_{\text{fppf}}$. To continue our strategy outlined in the beginning of Section 2.2, we now focus on computing the extension sheaves $\underline{\text{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$. We start by revisiting a result of Grothendieck.

PROPOSITION 2.3.1 (Grothendieck). *Let T be a torus over S with Cartier dual X . Then both $\underline{\text{Ext}}^1(T, \mathbb{G}_m)$ and $\underline{\text{Ext}}^1(X, \mathbb{G}_m)$ vanish.*

Proof. The vanishing of $\underline{\text{Ext}}^1(T, \mathbb{G}_m)$ was proved in [SGA 7_I, Exp. VIII, Prop. 3.3.1]. As for $\underline{\text{Ext}}^1(X, \mathbb{G}_m)$, its vanishing is a local property on S and so we may suppose that $X \simeq \mathbb{Z}$. Consequently, the result follows from the exactness of $\underline{\text{Hom}}(\mathbb{Z}, -)$, which is isomorphic to the identity functor. \square

For an abelian scheme A over S , the sheaf $\underline{\text{Ext}}^1(A, \mathbb{G}_m)$ is represented by the dual abelian scheme A' . This is the so-called *Barsotti–Weil formula*, yet surprisingly, a complete proof seems to be absent from the literature.⁸ A generalization of this result is going to be proven in Theorem 2.3.10.

We now describe a strategy for computing extension sheaves in a very general setting, in the hopes that it may also be useful elsewhere. Let \mathcal{G} and \mathcal{A} be abelian groups in a 1-topos \mathbf{X} . As $\underline{\text{Ext}}^i(\mathcal{G}, \mathcal{A})$ is the sheafification of the presheaf $S \mapsto \text{Ext}_S^i(\mathcal{G}, \mathcal{A})$, there is a natural morphism of groups

$$\text{Ext}_S^i(\mathcal{G}, \mathcal{A}) \rightarrow \underline{\text{Ext}}^i(\mathcal{G}, \mathcal{A})(S),$$

functorial on \mathcal{G} , \mathcal{A} , and S . The following simple result describes its kernel and cokernel for $i = 1$. We recall that the cohomology groups $H^i(S, -)$ are defined as the right derived functors of $\Gamma(S, -) := \text{Hom}_{\mathbf{X}}(S, -): \text{Ab}(\mathbf{X}) \rightarrow \text{Ab}$.

⁸For more, see [Jos09, Footnote to Thm. 1.2.2] and [Jos10].

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PROPOSITION 2.3.2. *Let S be an object and \mathcal{G}, \mathcal{A} be abelian groups in a 1-topos \mathbf{X} . Then the kernel of the sheafification map $\mathrm{Ext}_S^1(\mathcal{G}, \mathcal{A}) \rightarrow \underline{\mathrm{Ext}}^1(\mathcal{G}, \mathcal{A})(S)$ is $H^1(S, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{A}))$, and its cokernel is $H^2(S, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{A}))$.*

Proof. The group of sections of $\underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{A})$ over S is, by definition, $\mathrm{Hom}_S(\mathcal{G}, \mathcal{A})$. In particular, there is a Grothendieck spectral sequence, often called *local-to-global spectral sequence*, whose five-term exact sequence yields the desired result. \square

Now that we have established a connection between extension *sheaves* and extension *groups*, we will study a method for computing the latter. The following proposition, initially suggested by Grothendieck in [SGA 7_I, Exp. VII, Rem. 3.5.4] and partially developed by Breen in [Bre69], has been independently proven by Deligne (in a letter to Breen available in Appendix B) and by Clausen–Scholze [SC19, Thm. 4.10].

PROPOSITION 2.3.3 (Breen–Deligne resolution). *Let \mathcal{G} be an abelian group in a 1-topos \mathbf{X} . There exists a functorial resolution of the form*

$$\cdots \rightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[\mathcal{G}^{r_{i,j}}] \rightarrow \cdots \rightarrow \mathbb{Z}[\mathcal{G}^3] \oplus \mathbb{Z}[\mathcal{G}^2] \rightarrow \mathbb{Z}[\mathcal{G}^2] \rightarrow \mathbb{Z}[\mathcal{G}] \rightarrow \mathcal{G},$$

where the n_i and $r_{i,j}$ are all positive integers.

Clausen and Scholze’s proof shows that the first terms of the resolution can be chosen in the following way:

$$\mathbb{Z}[\mathcal{G}^4] \oplus \mathbb{Z}[\mathcal{G}^3] \oplus \mathbb{Z}[\mathcal{G}^3] \oplus \mathbb{Z}[\mathcal{G}^2] \oplus \mathbb{Z}[\mathcal{G}] \xrightarrow{d_3} \mathbb{Z}[\mathcal{G}^3] \oplus \mathbb{Z}[\mathcal{G}^2] \xrightarrow{d_2} \mathbb{Z}[\mathcal{G}^2] \xrightarrow{d_1} \mathbb{Z}[\mathcal{G}] \xrightarrow{d_0} \mathcal{G},$$

where the differentials are given by

$$\begin{aligned} d_3([x, y, z, t]) &= ([x + y, z, t] - [x, y + z, t] + [x, y, z + t] - [x, y, z] - [y, z, t], 0) \\ d_3([x, y, z]) &= (-[x, y, z] + [x, z, y] - [z, x, y], [x + y, z] - [x, z] - [y, z]) \\ d_3([x, y, z]) &= ([x, y, z] - [y, x, z] + [y, z, x], [x, y + z] - [x, y] - [x, z]) \\ d_3([x, y]) &= (0, [x, y] + [y, x]) \\ d_3([x]) &= (0, [x, x]) \\ d_2([x, y, z]) &= [x + y, z] - [x, y + z] + [x, y] - [y, z] \\ d_2([x, y]) &= [x, y] - [y, x] \\ d_1([x, y]) &= [x + y] - [x] - [y] \\ d_0([x]) &= x. \end{aligned}$$

Here, the top $d_3([x, y, z])$ acts on the first factor of $\mathbb{Z}[\mathcal{G}^3]$, while the bottom $d_3([x, y, z])$ acts on the second factor. In particular, this explicit description allows us to define two important invariants.

DEFINITION 2.3.4. Let \mathcal{G} and \mathcal{A} be abelian groups in a 1-topos \mathbf{X} . Applying the functor $\text{Hom}_{\text{Ab}(\mathbf{X})}(-, \mathcal{A})$ to the Breen–Deligne resolution of \mathcal{G} , we obtain the complex

$$\begin{array}{c} \Gamma(\mathcal{G}, \mathcal{A}) \rightarrow \Gamma(\mathcal{G}^2, \mathcal{A}) \rightarrow \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^2, \mathcal{A}) \\ \left. \vphantom{\Gamma(\mathcal{G}, \mathcal{A})} \right\} \\ \rightarrow \Gamma(\mathcal{G}^4, \mathcal{A}) \oplus \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^2, \mathcal{A}) \oplus \Gamma(\mathcal{G}, \mathcal{A}). \end{array}$$

We denote the first cohomology of this complex by $H_s^2(\mathcal{G}, \mathcal{A})$ and the second cohomology by $H_s^3(\mathcal{G}, \mathcal{A})$.

■ REMARK 2.3.5. The invariant $H_s^2(\mathcal{G}, \mathcal{A})$ is usually known as the symmetric subgroup of the second Hochschild cohomology group [Mil17, Chap. 15]. When \mathbf{X} is the topos of sets, it reduces to the subgroup of the group cohomology $H^2(\mathcal{G}, \mathcal{A})$ constituted of the symmetric cocycles. The (non-standard) notation $H_s^3(\mathcal{G}, \mathcal{A})$ indicates that this group is, in some sense, a variant of the third Hochschild cohomology which is better adapted to commutative groups. ■

As usual, given an object S and an abelian group \mathcal{A} in a 1-topos \mathbf{X} , the group $H^1(S, \mathcal{A})$ classifies \mathcal{A} -torsors over S , where the group operation corresponds to the contracted product [Gir71, §§III.2.4.2, III.2.4.5, III.3.5.4]. Moreover, a morphism $f: T \rightarrow S$ in \mathbf{X} induces a map of groups $f^*: H^1(S, \mathcal{A}) \rightarrow H^1(T, \mathcal{A})$. This map sends an \mathcal{A} -torsor $P \rightarrow S$ to the pullback $f^*P \rightarrow T$ [Gir71, §V.1.5.3]. When $S = \mathcal{G}$ is also an abelian group, we define a subgroup $H_m^1(\mathcal{G}, \mathcal{A})$ of $H^1(\mathcal{G}, \mathcal{A})$ constituted of the \mathcal{A} -torsors over \mathcal{G} compatible with the group structure on the latter.

DEFINITION 2.3.6. Let \mathcal{G} and \mathcal{A} be abelian groups in a 1-topos \mathbf{X} . Denote by $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ the group operation of \mathcal{G} , and by $\text{pr}_1, \text{pr}_2: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ the natural projections. We define $H_m^1(\mathcal{G}, \mathcal{A})$ as the kernel of the morphism $m^* - \text{pr}_1^* - \text{pr}_2^*$.

Put simply, $H_m^1(\mathcal{G}, \mathcal{A})$ is the group of isomorphism classes of \mathcal{A} -torsors P over \mathcal{G} satisfying $m^*P \simeq \text{pr}_1^*P \wedge \text{pr}_2^*P$. Often, we say that these \mathcal{A} -torsors are *multiplicative*. With this terminology established, we may now explain the computation of the extension groups.

PROPOSITION 2.3.7 (Breen). *Let \mathcal{G} and \mathcal{A} be abelian groups in a 1-topos \mathbf{X} . There exists an exact sequence*

$$0 \rightarrow H_s^2(\mathcal{G}, \mathcal{A}) \rightarrow \text{Ext}^1(\mathcal{G}, \mathcal{A}) \rightarrow H_m^1(\mathcal{G}, \mathcal{A}) \rightarrow H_s^3(\mathcal{G}, \mathcal{A}) \rightarrow \text{Ext}^2(\mathcal{G}, \mathcal{A}),$$

that is functorial in \mathcal{G} and \mathcal{A} .

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Before diving into the proof, let us explain the morphism $\text{Ext}^1(\mathcal{G}, \mathcal{A}) \rightarrow H_m^1(\mathcal{G}, \mathcal{A})$. Since $\text{Ext}^1(-, \mathcal{A})$ is an additive functor, an extension \mathcal{E} of \mathcal{G} by \mathcal{A} always satisfies

$$m^* \mathcal{E} = (\text{pr}_1 + \text{pr}_2)^* \mathcal{E} \simeq \text{pr}_1^* \mathcal{E} + \text{pr}_2^* \mathcal{E},$$

where the sum on the right is the Baer sum of extensions. Such an extension defines an \mathcal{A} -torsor P over \mathcal{G} , which satisfies $m^* P \simeq \text{pr}_1^* P \wedge \text{pr}_2^* P$.

Proof of Proposition 2.3.7. The universal property of free objects gives that $H^i(\mathcal{G}^n, \mathcal{A})$ is isomorphic to $\text{Ext}^i(\mathbb{Z}[\mathcal{G}^n], \mathcal{A})$ for all n and i . Then, the Breen–Deligne resolution yields an spectral sequence

$$E_1^{i,j}: \prod_{r=1}^{n_i} H^j(\mathcal{G}^{s_{i,r}}, \mathcal{A}) \implies \text{Ext}^{i+j}(\mathcal{G}, \mathcal{A}),$$

whose five-term exact sequence is precisely the one in the statement. □



The Breen–Deligne resolution is generally not acyclic with respect to extension groups. In the topos of sets, every torsor is trivial, and the proposition above recovers the classical isomorphism $H_s^2(\mathcal{G}, \mathcal{A}) \simeq \text{Ext}^1(\mathcal{G}, \mathcal{A})$, for any pair of abelian groups \mathcal{G} and \mathcal{A} . However, we will see numerous examples of fppf sheaves \mathcal{G} and \mathcal{A} for which $H_m^1(\mathcal{G}, \mathcal{A})$ is non-trivial.

■ **REMARK 2.3.8.** In [SGA 7_I, Exp. VII, §1.2], Grothendieck proved that the category of extensions of \mathcal{G} by \mathcal{A} is equivalent to a category of pairs (P, α) , where P is a \mathcal{A} -torsor over \mathcal{G} and $\alpha: m^* P \rightarrow \text{pr}_1^* P \wedge \text{pr}_2^* P$ is an isomorphism of \mathcal{A} -torsors over $\mathcal{G} \times \mathcal{G}$ making two diagrams (imposing that P admits an associative and commutative group law) commute. In particular, $\text{Ext}^1(\mathcal{G}, \mathcal{A})$ is isomorphic to the group of isomorphism classes of such pairs, and our invariants $H_s^2(\mathcal{G}, \mathcal{A})$ and $H_s^3(\mathcal{G}, \mathcal{A})$ govern how far the map

$$\begin{aligned} \text{Ext}^1(\mathcal{G}, \mathcal{A}) &\rightarrow H_m^1(\mathcal{G}, \mathcal{A}) \\ [P, \alpha] &\mapsto [P] \end{aligned}$$

is from being an isomorphism. ■

Even though the first terms of the Breen–Deligne resolution are explicit, the invariants $H_s^2(\mathcal{G}, \mathcal{A})$ and $H_s^3(\mathcal{G}, \mathcal{A})$ are usually quite hard to compute. The following observation will suffice for their computations in many interesting cases.

LEMMA 2.3.9. *Let \mathcal{G} and \mathcal{A} be abelian groups in a 1-topos \mathcal{X} . If every morphism $\mathcal{G}^n \rightarrow \mathcal{A}$ in \mathcal{X} (for $n = 2, 3$) can be expressed as a sum of maps $\mathcal{G} \rightarrow \mathcal{A}$, then both $H_s^2(\mathcal{G}, \mathcal{A})$ and $H_s^3(\mathcal{G}, \mathcal{A})$ vanish. In particular, this condition is satisfied if every morphism $\mathcal{G}^n \rightarrow \mathcal{A}$ is constant.*

Proof. First, we demonstrate that $H_s^2(\mathcal{G}, \mathcal{A})$ vanishes. The kernel of

$$\Gamma(\mathcal{G}^2, \mathcal{A}) \rightarrow \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^2, \mathcal{A})$$

is composed of the maps $f: \mathcal{G}^2 \rightarrow \mathcal{A}$ that satisfy $f(x+y, z) - f(y, z) = f(x, y+z) - f(x, y)$ and $f(x, y) = f(y, x)$. By applying the given hypothesis, we find morphisms $f_1, f_2: \mathcal{G} \rightarrow \mathcal{A}$ such that $f(x, y) = f_1(x) + f_2(y)$. This simplifies the first equation to

$$f_1(x+y) - f_1(y) = f_2(y+z) - f_2(y).$$

By setting $x = y = 0$ and $y = z = 0$, we observe that f must be constant. Next, the image of

$$\Gamma(\mathcal{G}, \mathcal{A}) \rightarrow \Gamma(\mathcal{G}^2, \mathcal{A})$$

consists of maps of the form $(x, y) \mapsto g(x+y) - g(x) - g(y)$, for some $g: \mathcal{G} \rightarrow \mathcal{A}$. Since every constant map is of this form, it follows that the cohomology vanishes.

We will use the same strategy to show that $H_s^3(\mathcal{G}, \mathcal{A})$ also vanishes. The kernel of the morphism

$$\Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^2, \mathcal{A}) \rightarrow \Gamma(\mathcal{G}^4, \mathcal{A}) \oplus \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^3, \mathcal{A}) \oplus \Gamma(\mathcal{G}^2, \mathcal{A}) \oplus \Gamma(\mathcal{G}, \mathcal{A})$$

is composed of the maps $p: \mathcal{G}^3 \rightarrow \mathcal{A}$ and $q: \mathcal{G}^2 \rightarrow \mathcal{A}$ satisfying

$$p(x, y, z) + p(x, y+z, t) + p(y, z, t) = p(x+y, z, t) + p(x, y, z+t) \quad (1.1)$$

$$p(x, y, z) + p(z, x, y) + q(x, z) + q(y, z) = p(x, z, y) + q(x+y, z) \quad (2.1)$$

$$p(x, y, z) + p(y, z, x) + q(x, y+z) = p(y, x, z) + q(x, y) + q(x, z) \quad (3.1)$$

$$q(x, y) + q(y, x) = 0 \quad (4.1)$$

$$q(x, x) = 0. \quad (5.1)$$

To organize the remainder of the proof, we will denote a simplified version of the Equation (n.i) as (n.i + 1). Once again, we write $p(x, y, z) = p_1(x) + p_2(y) + p_3(z)$ for some $p_i: \mathcal{G} \rightarrow \mathcal{A}$, and $q(x, y) = q_1(x) + q_2(y)$ for some $q_i: \mathcal{G} \rightarrow \mathcal{A}$. Using Equation (5.1), we replace every instance of q_2 by $-q_1$. This yields the following relations.

$$p_3(z) + p_1(x) + p_2(y+z) + p_1(y) + p_3(t) = p_1(x+y) + p_3(z+t) \quad (1.2)$$

$$p_2(y) + p_3(z) + p_1(z) + p_2(x) + q_1(x) + q_1(y) = p_2(z) + q_1(x+y) + q_1(z) \quad (2.2)$$

$$p_1(x) + p_2(y) + p_2(z) + p_3(x) + q_1(y) + q_1(z) = p_2(x) + q_1(y+z) + q_1(x) \quad (3.2)$$

By setting $x = y = 0$ in Equations (2.2) and (3.2), we obtain

$$p_2(0) + p_3(z) + p_1(z) + q_1(0) = q_1(z) \quad (2.3)$$

$$p_1(0) + p_2(z) + p_3(0) = 0. \quad (3.3)$$

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With this information, we can simplify Equation (1.2):

$$p_3(z) + p_1(x) + p_2(0) + p_1(y) + p_3(t) = p_1(x + y) + p_3(z + t). \quad (1.3)$$

Restricting this equation to $x = 0$ and $t = 0$ gives two new relations, implying that p_1 and p_3 are essentially morphisms of groups up to constants.

$$p_1(x) + p_1(y) - p_1(0) = p_1(x + y) \quad (1.4)$$

$$p_3(z) + p_3(t) - p_3(0) = p_3(z + t) \quad (1.5)$$

In summary, we have obtained the following equations:

$$p_1(x) + p_1(y) - p_1(0) = p_1(x + y) \quad (1.4)$$

$$p_3(z) + p_3(t) - p_3(0) = p_3(z + t) \quad (1.5)$$

$$p_1(z) - p_1(0) + p_3(z) - p_3(0) = q_1(z) - q_1(0) \quad (2.3)$$

$$p_1(0) + p_2(z) + p_3(0) = 0. \quad (3.3)$$

Finally, to prove that $H_s^3(\mathcal{G}, \mathcal{A})$ vanishes, we need to find a morphism $h: \mathcal{G}^2 \rightarrow \mathcal{A}$ such that

$$\begin{aligned} p(x, y, z) &= h(x + y, z) - h(x, y + z) + h(x, y) - h(y, z) \\ q(x, y) &= h(x, y) - h(y, x). \end{aligned}$$

A quick verification using the previously established equations shows that the morphism $h(x, y) := p_1(x) - p_3(y)$ satisfies this criterion. \square

In order to profit from this machinery, we return to the setting where X is the category of sheaves on $(\text{Sch}/S)_{\text{fppf}}$, for a given base scheme S . The promised generalization of the Barsotti–Weil formula now follows directly from the preceding results.

THEOREM 2.3.10 (Generalized Barsotti–Weil formula). *Let A be an abelian scheme and B an affine commutative group scheme over S . For an S -scheme T , the natural maps*

$$\underline{\text{Ext}}^1(A, B)(T) \leftarrow \text{Ext}_T^1(A, B) \rightarrow H_m^1(A_T, B_T)$$

are isomorphisms.

Proof. The argument used in the proof of Proposition 2.2.1 shows that $\underline{\text{Hom}}(A, B)$ vanishes. In particular, Proposition 2.3.2 provides the desired isomorphism on the left. Similarly, Lemma 2.3.9 yields the isomorphism on the right. \square

■ **REMARK 2.3.11.** Taking the Breen–Deligne resolution for granted, the proof of Theorem 2.3.10 is remarkably straightforward. While the application of Lemma 2.3.9 may seem intricate, the Generalized Barsotti–Weil formula actually follows from a simpler special case of the lemma, where every morphism $\mathcal{G}^n \rightarrow \mathcal{A}$ is constant—this case is significantly easier to establish. ■

Specifically, the Theorem 2.3.10 generalizes the results in the Section VII.3 of Serre's book [Ser88].

COROLLARY 2.3.12. *Let A be an abelian scheme over S and T an S -scheme. Then the natural maps*

$$\underline{\mathrm{Ext}}^1(A, \mathbb{G}_a)(T) \leftarrow \mathrm{Ext}_T^1(A, \mathbb{G}_a) \rightarrow H^1(A_T, \mathcal{O}_{A_T})$$

are isomorphisms.

Proof. The content of this corollary is that every element of $H^1(A_T, \mathcal{O}_{A_T})$ is multiplicative. Let $m: A_T \times_T A_T \rightarrow A_T$ denote the group law, and let $\mathrm{pr}_1, \mathrm{pr}_2: A_T \times_T A_T \rightarrow A_T$ be the natural projections. By the Künneth formula, the map

$$\begin{aligned} H^1(A_T, \mathcal{O}_{A_T}) \oplus H^1(A_T, \mathcal{O}_{A_T}) &\rightarrow H^1(A_T^2, \mathcal{O}_{A_T^2}) \\ (x, y) &\mapsto \mathrm{pr}_1^* x + \mathrm{pr}_2^* y \end{aligned}$$

is an isomorphism. In particular, given an element z of $H^1(A_T, \mathcal{O}_{A_T})$, there exist $x, y \in H^1(A_T, \mathcal{O}_{A_T})$ satisfying

$$\mathrm{pr}_1^* x + \mathrm{pr}_2^* y = m^* z.$$

Restricting this to $A_T \times_T T \hookrightarrow A_T \times_T A_T$, we obtain $x = z$. Similarly, restricting it to $T \times_T A_T \hookrightarrow A_T \times_T A_T$, we obtain $y = z$. \square

COROLLARY 2.3.13. *Let A be an abelian scheme over S . Then $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ is representable by the dual abelian scheme A' . More generally, if H is an extension of a finite locally free group scheme by a torus over S , we have an isomorphism $\underline{\mathrm{Ext}}^1(A, H) \simeq \underline{\mathrm{Hom}}(H^D, A')$.*

Proof. For an S -scheme T , the T -points of the dual abelian scheme A' are given by $H_m^1(A_T, \mathbb{G}_m)$.⁹ The first statement then follows as an application of Theorem 2.3.10. The second statement can be proven similarly to [Jos09, Prop. 1.2.3]. \square

Under additional hypotheses on the base scheme S , we can also apply Lemma 2.3.9 to a broader class of commutative group schemes over S . The following proposition extends a result by Colliot-Thélène and Gabber [Col08, Prop. 3.2 and Thm. 5.6]. Their method of proof can also be applied to our setting, but this result follows directly from our machinery.

THEOREM 2.3.14. *Let G be a commutative group scheme over a reduced base S . Assume that the morphism $G \rightarrow S$ is smooth, surjective, and has connected geometric fibers. Then the natural map*

$$\mathrm{Ext}_S^1(G, \mathbb{G}_m) \rightarrow H_m^1(G, \mathbb{G}_m)$$

is an isomorphism.

⁹See the discussion at the beginning of [GW23, §27.41].

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Proof. This result is a direct application of Lemma 2.3.9, using González-Avilés' generalization of Rosenlicht's lemma [Gon17, Thm. 1.1]. For the reader's convenience, we note that this result holds more generally in the following context: G is a commutative group scheme over a reduced base S such that $G \rightarrow S$ is syntomic, surjective, and has reduced, connected maximal geometric fibers. Here, a maximal geometric fiber of $G \rightarrow S$ is a scheme $G \times_S \overline{\text{Spec } \kappa(\eta)}$, for some generic point η of an irreducible component of S . \square

Henceforth, we concentrate on the case where the base scheme is the spectrum of a characteristic zero field. We start by using Proposition 2.3.2 to prove that the sheafification map is an isomorphism in certain cases.

PROPOSITION 2.3.15. *Let T be a torus and U be a unipotent group over a characteristic zero field k . The sheafification map $\text{Ext}_S^1(T, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(T, \mathbb{G}_m)(S)$ is an isomorphism for an irreducible geometrically unibranch k -scheme S . Similarly, the map $\text{Ext}_S^1(U, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(U, \mathbb{G}_m)(S)$ is an isomorphism for a k -scheme S that is either reduced or affine.*

Proof. Let S be an irreducible geometrically unibranch k -scheme. Proposition 2.3.1 states that the sheaf $\underline{\text{Ext}}^1(T, \mathbb{G}_m)$ vanishes, so the desired result is equivalent to the vanishing of the cohomology group $H^1(S, X) = 0$, where $X := T^\mathbb{D}$ is the Cartier dual of T . This holds due to [SGA 7_I, Exp. VIII, Prop. 5.1].

For the unipotent case, we may assume that $U = \mathbb{G}_a$. If S is an affine scheme, [Bha22, Rem. 2.2.18] implies that $H^i(S, \widehat{\mathbb{G}}_a)$ vanishes for all $i \neq 0$. We claim that, for a reduced k -scheme S , the cohomology groups $H^i(S, \widehat{\mathbb{G}}_a)$ vanish for all i . First, we prove that the étale cohomology groups $H_{\text{ét}}^i(S, \widehat{\mathbb{G}}_a)$ vanish, and then we identify these groups with their fppf analogues.

Recall that $\widehat{\mathbb{G}}_a(\text{Spec } R)$ is the nilradical of a k -algebra R . Thus, the sheaf condition implies that $\widehat{\mathbb{G}}_a(S)$ vanishes for reduced S . Next, the cohomology $H_{\text{ét}}^i(S, \widehat{\mathbb{G}}_a)$ can be computed on the small étale site of S and, according to [Stacks, Tag 03PC.(8)], the restriction of $\widehat{\mathbb{G}}_a$ to this site vanishes.

Since $\mathbb{G}_{a, \text{dR}}$ is the presheaf quotient of \mathbb{G}_a by $\widehat{\mathbb{G}}_a$, we obtain the following morphism of long exact sequences:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Gamma(S, \widehat{\mathbb{G}}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_{a, \text{dR}}) & \longrightarrow & H^1(S, \widehat{\mathbb{G}}_a) & \longrightarrow & H^1(S, \mathbb{G}_a) & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma(S, \widehat{\mathbb{G}}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_a) & \longrightarrow & \Gamma(S, \mathbb{G}_{a, \text{dR}}) & \longrightarrow & H_{\text{ét}}^1(S, \widehat{\mathbb{G}}_a) & \longrightarrow & H_{\text{ét}}^1(S, \mathbb{G}_a) & \longrightarrow & \cdots \end{array}$$

Given that S is reduced, we have shown that $H_{\text{ét}}^i(S, \widehat{\mathbb{G}}_a) = 0$ for all i . Moreover, since \mathbb{G}_a is smooth, the natural map $H^i(S, \mathbb{G}_a) \rightarrow H_{\text{ét}}^i(S, \mathbb{G}_a)$ is an isomorphism. Using these facts, a diagram chase gives that $H^i(S, \widehat{\mathbb{G}}_a)$ vanishes for all i . \square

To understand the sheafification map $\text{Ext}_S^1(\mathbb{G}, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(\mathbb{G}, \mathbb{G}_m)(S)$ for a connected commutative algebraic group \mathbb{G} over a characteristic zero field, we need the following vanishing result.

PROPOSITION 2.3.16. *Let U be a unipotent group over a characteristic zero field k . Then the group $\underline{\text{Ext}}^1(U, \mathbb{G}_m)(S)$ vanishes for all seminormal k -schemes S .*

Proof. Since seminormal schemes are reduced [Stacks, Tag 0EUQ], Theorem 2.3.14 and Proposition 2.3.15 imply that $\underline{\text{Ext}}^1(U, \mathbb{G}_m)(S) \simeq H_m^1(U_S, \mathbb{G}_{m,S})$. Denoting by $p: U_S \rightarrow S$ the structure map, Traverso's theorem [Sad21, Lem. 4.3] gives that

$$p^*: H^1(S, \mathbb{G}_m) \rightarrow H^1(U_S, \mathbb{G}_m)$$

is an isomorphism. In particular, $H_m^1(U_S, \mathbb{G}_m)$ is isomorphic to the subgroup of $H^1(S, \mathbb{G}_m)$ consisting of elements $x \in H^1(S, \mathbb{G}_m)$ that satisfy $p^*x \in H_m^1(U_S, \mathbb{G}_m)$. Next, denote by $m: U_S \times_S U_S \rightarrow U_S$ the group operation and by $\text{pr}_i: U_S \times_S U_S \rightarrow U_S$ the natural projections. Then, p^*x lies in $H_m^1(U_S, \mathbb{G}_m)$ if and only if

$$m^*p^*x = \text{pr}_1^*p^*x + \text{pr}_2^*p^*x.$$

However, the morphisms $p \circ m$, $p \circ \text{pr}_1$, and $p \circ \text{pr}_2$ are all equal to the structure map of U_S^2 , which has a section $S \rightarrow U_S^2$. Thus, $m^*p^*x = \text{pr}_1^*p^*x + \text{pr}_2^*p^*x$ holds if and only if $x = 0$, completing the proof. \square

■ **EXAMPLE 2.3.17 — SCHANUEL'S MODULE.** Let k be a characteristic zero field. Although the group $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S)$ vanishes for seminormal k -schemes S , the sheaf $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$ is non-trivial. In [Ros23, Rem. 2.2.16], Rosengarten constructs an example (due to Gabber) of an extension of \mathbb{G}_a by \mathbb{G}_m that does not split locally in the fppf topology. Here, we construct another non-zero section of $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$.

Let $R = k[x, y]/(y^2 - x^3)$ be the coordinate ring of a cusp. Since R is reduced, Theorem 2.3.14 and Proposition 2.3.15 provide isomorphisms

$$\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(R) \xleftarrow{\sim} \text{Ext}_R^1(\mathbb{G}_a, \mathbb{G}_m) \xrightarrow{\sim} H_m^1(\mathbb{G}_{a,R}, \mathbb{G}_m) \subset \text{Pic}(R[t]).$$

Now, consider the fractional ideals $I = (x, 1 + yt/x)$ and $J = (x, 1 - yt/x)$ of $R[t]$. As

$$1 = x^2t^4 + (1 + xt^2)(1 - xt^2) \in IJ = (x^2, x + yt, x - yt, 1 - xt^2),$$

the ideal I is invertible and thus defines an element of $\text{Pic}(R[t])$. The fractional ideal $m^*J \text{pr}_1^* I \text{pr}_2^* I$ of $R[t_1, t_2]$ is

$$\begin{aligned} & (x^3, x^2 + xyt_1, x^2 + xyt_2, x + y(t_1 + t_2) + x^2t_1t_2, x^2 - xy(t_1 + t_2), \\ & x - (x^2t_1 + y)(t_1 + t_2) + yt_1, x - (x^2t_2 + y)(t_1 + t_2) + yt_2, \\ & 1 + xt_1t_2 - x(t_1 + t_2)^2 - yt_1t_2(t_1 + t_2)). \end{aligned}$$

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As one can verify in their preferred programming language, this ideal also contains 1; proving that I lies in the subgroup $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(R) \simeq H_m^1(\mathbb{G}_{a,R}, \mathbb{G}_m)$ of $\text{Pic}(R[t])$. The following code provides an implementation in Macaulay2.

```

1  -- Define the ideal 'prod' with the given generators
2  Rt1t2 = ZZ[x, y, u, v] / (y^2 - x^3)
3  prod = ideal(
4      x^3,
5      x^2 + x*y*u,
6      x^2 + x*y*v,
7      x + y*(u + v) + x^2*u*v,
8      x^2 - x*y*(u + v),
9      x - (x^2*u + y)*(u + v) + y*u,
10     x - (x^2*v + y)*(u + v) + y*v,
11     1 + x*u*v - x*(u + v)^2 - y*u*v*(u + v)
12 )
13
14 -- Compute a Groebner basis and the change of basis matrix,
15 -- which provides the coefficients for writing 1 as a linear
16 -- combination of the generators of 'prod'
17 grob = gb(prod, ChangeMatrix => true)
18 mat = getChangeMatrix(grob)
19
20 -- Print the results
21 gens grob
22 mat

```

The reader is spared the pain of seeing 1 as a page-long linear combination of the generators of $m^*J_{\text{pr}_1^*}I_{\text{pr}_2^*}I$. ■

■ **REMARK 2.3.18** — **SCHANUEL'S MODULE IN POSITIVE CHARACTERISTIC.** Let k be a field of characteristic $p > 0$. The previous example gives a k -scheme S such that $\text{Ext}_S^1(\mathbb{G}_a, \mathbb{G}_m) \simeq H_m^1(\mathbb{G}_{a,S}, \mathbb{G}_m) \neq 0$. However, it is no longer true that $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S) \simeq \text{Ext}_S^1(\mathbb{G}_a, \mathbb{G}_m)$.

Given a k -algebra R , the colimit R_{perf} of the tower

$$R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} R \xrightarrow{x \mapsto x^p} \dots$$

is known as the *colimit perfection* of R . It is always a perfect k -algebra, and the natural map $R \rightarrow R_{\text{perf}}$ is universal among morphisms from R to perfect algebras. The kernel of $R \rightarrow R_{\text{perf}}$ is the nilradical of R , and $R \rightarrow R_{\text{perf}}$ is surjective if R is semiperfect.¹⁰

¹⁰An \mathbb{F}_p -algebra R is said to be *semiperfect* if the Frobenius endomorphism is surjective.

In [Bha22, Rem. 2.2.18], Bhatt proves that $R\Gamma(\mathbb{R}, \widehat{\mathbb{G}}_a) \simeq [\mathbb{R} \rightarrow \mathbb{R}_{\text{perf}}]$, where the complex is in degrees zero and one. In particular, $H^2(\mathbb{R}, \widehat{\mathbb{G}}_a)$ vanishes, but $H^1(\mathbb{R}, \widehat{\mathbb{G}}_a)$ may not. Therefore, for $S = \text{Spec } k[x, y]/(y^2 - x^3)$, the sheafification map

$$\text{Ext}_S^1(\mathbb{G}_a, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)(S)$$

is surjective but not injective. More precisely, Rosengarten has recently proven that the sheaf $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$ vanishes [Ros23, Prop. 2.2.14]. ■

We are now in position to prove the promised enhancement of Theorem 2.3.14, for commutative connected algebraic groups over a characteristic zero field.

THEOREM 2.3.19. *Let G be a commutative connected algebraic group over a characteristic zero field k . For a regular k -scheme S , the natural maps*

$$\underline{\text{Ext}}^1(G, \mathbb{G}_m)(S) \leftarrow \text{Ext}_S^1(G, \mathbb{G}_m) \rightarrow H_m^1(\mathbb{G}_S, \mathbb{G}_m)$$

are isomorphisms.

Proof. That the arrow on the right is an isomorphism was already proven in Theorem 2.3.14. This result holds even if S is merely assumed to be reduced. For the reader's convenience, we note that the hypotheses of Theorem 2.3.14 are satisfied due to [Mil17, Cor. 1.32 and 8.39].

We now turn our attention to the sheafification map on the left. As in the proof of Proposition 2.2.6, we know that G is an extension of an abelian variety A by a linear group L , that is a product of a torus T and a unipotent group U . This leads to the commutative diagram

$$\begin{array}{ccccccc} \text{Hom}_S(L, \mathbb{G}_m) & \longrightarrow & \text{Ext}_S^1(A, \mathbb{G}_m) & \longrightarrow & \text{Ext}_S^1(G, \mathbb{G}_m) & \longrightarrow & \text{Ext}_S^1(L, \mathbb{G}_m) \\ \parallel & & \downarrow \wr & & \downarrow & & \downarrow \wr \\ \underline{\text{Hom}}(L, \mathbb{G}_m)(S) & \longrightarrow & \underline{\text{Ext}}^1(A, \mathbb{G}_m)(S) & \longrightarrow & \underline{\text{Ext}}^1(G, \mathbb{G}_m)(S) & \longrightarrow & \underline{\text{Ext}}^1(L, \mathbb{G}_m)(S), \end{array}$$

whose rows are exact. Propositions 2.3.1 and 2.3.16 imply that $\underline{\text{Ext}}^1(L, \mathbb{G}_m)(S)$ vanishes, and Proposition 2.3.15 further ensures that $\text{Ext}_S^1(L, \mathbb{G}_m) = 0$. The desired result then follows from a diagram chase. □

The following example demonstrates that the map $\text{Ext}_S^1(G, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(G, \mathbb{G}_m)(S)$ may not be an isomorphism if S is not regular.

■ **EXAMPLE 2.3.20.** Let k be an algebraically closed field of characteristic zero, and consider the nodal curve $S = \text{Spec } k[x, y]/(y^2 - x^3 + x^2)$. The proof of Proposition 2.3.7 indicates that $\text{Ext}_S^1(\mathbb{G}_m, \mathbb{G}_m)$ can be computed in the étale topology. Since Grothendieck's

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proof of Proposition 2.3.1 also shows that $\underline{\text{Ext}}^1(\mathbb{G}_m, \mathbb{G}_m)$ vanishes on the étale site, Proposition 2.3.2 gives that

$$\text{Ext}_S^1(\mathbb{G}_m, \mathbb{G}_m) \simeq H_{\text{ét}}^1(S, \mathbb{Z}).$$

Denote by $f: \mathbb{P}_k^1 \rightarrow S$ the normalization map, and by $i: \text{Spec } k \rightarrow S$ the inclusion of the singular point. As one can verify, we have a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow f_*\mathbb{Z} \rightarrow i_*\mathbb{Z} \rightarrow 0,$$

implying that $H_{\text{ét}}^1(S, \mathbb{Z}) \simeq \mathbb{Z}$. In particular, the sheafification map $\text{Ext}_S^1(\mathbb{G}_m, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(\mathbb{G}_m, \mathbb{G}_m)(S)$ is not an isomorphism. ■

We also prove an analogue of Theorem 2.3.14 for de Rham spaces.

THEOREM 2.3.21. *Let G be a commutative connected algebraic group over a characteristic zero field k . For a reduced k -scheme S , the natural maps*

$$\underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)(S) \leftarrow \text{Ext}_S^1(G_{\text{dR}}, \mathbb{G}_m) \rightarrow H_m^1(G_{\text{dR}} \times S, \mathbb{G}_m)$$

are isomorphisms.

Proof. Proposition 2.2.6 asserts that the Cartier dual of G_{dR} vanishes. Consequently, Proposition 2.3.2 implies that the map on the left is an isomorphism for all k -schemes S . To show that the map on the right is also an isomorphism for reduced k -schemes S , we use Lemma 2.3.9.

Consider a morphism of S -schemes $f: G_{\text{dR}}^n \times S \rightarrow \mathbb{G}_{m,S}$, where n is either 2 or 3. As in the proof of Theorem 2.3.14, there exist morphisms $f_i: G_S \rightarrow \mathbb{G}_{m,S}$ for $i = 1, \dots, n$, such that their product equals the composition

$$G_S^n \rightarrow G_{\text{dR}}^n \times S \rightarrow \mathbb{G}_{m,S}.$$

Since epimorphisms in topoi are stable under base change, Proposition 1.1.19 implies that the map $G_S^n \rightarrow G_{\text{dR}}^n \times S$ is an epimorphism. Therefore, the morphisms f_i factor through the quotient, yielding maps

$$\bar{f}_i: G_{\text{dR}} \times S \rightarrow \mathbb{G}_{m,S},$$

whose product is equal to f . □

Our final proposition in this section is a vanishing result for extension sheaves of commutative formal groups, whose proof may be found in [Rus13, Lem. 1.14]. This result is primarily used for the formal completion \widehat{G} of a commutative connected algebraic group G over a field of characteristic zero, where it was first presented in [BB09, Lem. A.4.6].

PROPOSITION 2.3.22. *Let \mathcal{G} be a commutative formal group over a field whose Cartier dual is of finite type. Then $\underline{\text{Ext}}^1(\mathcal{G}, \mathbb{G}_m)$ vanishes.*

In order to have a bird's-eye view of this section, consider the following definition.

DEFINITION 2.3.23. Let G be a commutative connected algebraic group over a characteristic zero field k . We denote the abelian sheaf $\underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)$ by G^\natural , and the abelian sheaf $\underline{\text{Ext}}^1(G, \mathbb{G}_m)$ by G' .

Denote by $m: G \times G \rightarrow G$ the group law of G . For a reduced k -scheme S , Theorem 2.3.21 gives isomorphisms

$$G^\natural(S) := \underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)(S) \xleftarrow{\sim} \text{Ext}_S^1(G_{\text{dR}}, \mathbb{G}_m) \xrightarrow{\sim} H_m^1(G_{\text{dR}} \times S, \mathbb{G}_m)$$

that are functorial on G and S . In other words, $G^\natural(S)$ is isomorphic to the set of isomorphism classes of line bundles \mathcal{L} on G_S with integrable connection ∇ relative to S satisfying $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$. Moreover, the group structure of $G^\natural(S)$ corresponds to the tensor products of connections.

Similarly, for a regular k -scheme S , Theorem 2.3.14 yields isomorphisms

$$G'(S) := \underline{\text{Ext}}^1(G, \mathbb{G}_m)(S) \xleftarrow{\sim} \text{Ext}_S^1(G, \mathbb{G}_m) \xrightarrow{\sim} H_m^1(G_S, \mathbb{G}_m)$$

that are functorial on G and S . As above, this implies that $G'(S)$ can be identified to the group of isomorphism classes of line bundles \mathcal{L} on G_S satisfying $m^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$.

The following remark explains our choice of notation in Definition 2.3.23.

■ REMARK 2.3.24. Let A be an abelian variety over a characteristic zero field k . According to Corollary 2.3.13, the abelian sheaf A' in Definition 2.3.23 is represented by the dual abelian variety. Thus, our notation is not overloaded.

By considering the long exact sequence in cohomology associated with the Cartier duality functor $(-)^{\text{D}} := \underline{\text{Hom}}(-, \mathbb{G}_m)$ and the short exact sequence

$$0 \rightarrow \widehat{A} \rightarrow A \rightarrow A_{\text{dR}} \rightarrow 0,$$

we deduce that A^\natural is an extension of A' by Ω_A , the vector group of the invariant differentials of A . (Consequently, A^\natural is representable by an algebraic group.¹¹) We affirm that A^\natural is the universal vector extension of A' as defined in [MM74, §I.1].

The proof of Theorem 2.3.21 provides natural isomorphisms

$$A^\natural(S) := \underline{\text{Ext}}^1(A_{\text{dR}}, \mathbb{G}_m)(S) \xleftarrow{\sim} \text{Ext}_S^1(A_{\text{dR}}, \mathbb{G}_m) \xrightarrow{\sim} H_m^1(A_{\text{dR}} \times S, \mathbb{G}_m)$$

for *all* k -schemes S . This implies that the presheaf $S \mapsto H_m^1(A_{\text{dR}} \times S, \mathbb{G}_m)$ already satisfies fppf descent. Therefore, the sheafification in the definition of E^\natural [MM74, Def. I.4.1.6] is superfluous, leading to the identification $A^\natural \simeq E^\natural$.

¹¹We refer the reader to Remark 3.1.3 for more explanations.

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Mazur and Messing also define an abelian sheaf $\underline{\text{Ext}}^{\natural}(A, \mathbb{G}_m)$ which, by Remark 2.3.8, is isomorphic to $A^{\natural} = \underline{\text{Ext}}^1(A_{\text{dR}}, \mathbb{G}_m)$. In particular, our methods reprove their [MM74, Prop. I.4.2.1], which compares $\underline{\text{Ext}}^{\natural}(A, \mathbb{G}_m)$ and E^{\natural} . Finally, from [MM74, Props. I.2.6.7 and I.3.2.3], we conclude that A^{\natural} is indeed the universal vector extension of A' . ■

2.4. HIGHER EXTENSIONS OF ABELIAN SCHEMES

Even though the stacky Cartier dual \mathcal{G}^{\vee} of a commutative group stack \mathcal{G} contains no information about the higher extension sheaf $\underline{\text{Ext}}^2(\mathcal{G}^{\circ}, \mathbb{G}_m)$, some homological arguments may require its vanishing. For an abelian scheme A over a regular base S , the vanishing of $\underline{\text{Ext}}^2(A, \mathbb{G}_m)$ has been claimed in [Bre75, Rem. 6], [Jos09, Thm. 1.2.5], and [Bro21, Cor. 11.5].

However, in each instance, this result is deduced from [Bre69, §7], and additional arguments appear to be lacking. To address this gap, in this section, written in collaboration with Zev Rosengarten, we establish that both $\underline{\text{Ext}}^2(A, \mathbb{G}_a)$ and $\underline{\text{Ext}}^2(A, \mathbb{G}_m)$ vanish in characteristic zero without any regularity assumptions on the base scheme S .

We start by establishing a Künneth formula for the cohomology of an abelian scheme over an arbitrary ring.

LEMMA 2.4.1. *Let A be an abelian scheme over a ring R . Then the R -module $H^n(A^2, \mathcal{O}_{A^2})$ is the direct sum of $H^i(A, \mathcal{O}_A) \otimes_R H^j(A, \mathcal{O}_A)$ for $i + j = n$. In particular, $H^1(A^2, \mathcal{O}_{A^2})$ is isomorphic to $H^1(A, \mathcal{O}_A) \oplus H^1(A, \mathcal{O}_A)$.*

Proof. Since A is flat over R , we have $R\Gamma(A^2, \mathcal{O}_{A^2}) \simeq R\Gamma(A, \mathcal{O}_A) \otimes_R^{\mathbb{L}} R\Gamma(A, \mathcal{O}_A)$, as shown in [Stacks, Tag 0FLQ]. Consider a projective resolution P of $R\Gamma(A, \mathcal{O}_A)$. The Künneth spectral sequence, whose second page is

$$E_2^{p,q} = \bigoplus_{i+j=q} \text{Tor}_R^p(\mathcal{H}^i(P), \mathcal{H}^j(P)) \simeq \bigoplus_{i+j=q} \text{Tor}_R^p(H^i(A, \mathcal{O}_A), H^j(A, \mathcal{O}_A)),$$

converges to $\mathcal{H}^{p+q}(P \otimes_R P) \simeq H^{p+q}(A^2, \mathcal{O}_{A^2})$. Note that $H^i(A, \mathcal{O}_A)$ is a finite locally free R -module for all i due to [GW23, Thm. 27.203]. Consequently, these Tor_R^p all vanish for $p > 0$ and the spectral sequence degenerates. □

THEOREM 2.4.2. *Let A be an abelian scheme over a ring R , and let M be an R -module. Then $\text{Ext}_R^2(A, M) = 0$.*

Proof. Consider the structure map $p: A \rightarrow \text{Spec } R$. We claim that $H^j(A, p^*M)$ is isomorphic to $H^j(A, \mathcal{O}_A) \otimes_R M$ for all j . Using the projection formula, we obtain

$$R\Gamma(A, p^*M) \simeq R\Gamma(A, \mathcal{O}_A \otimes_{\mathcal{O}_A}^{\mathbb{L}} Lp^*M) \simeq R\Gamma(A, \mathcal{O}_A) \otimes_R^{\mathbb{L}} M.$$

The claim then follows from [GW23, Thm. 27.203], which states that $R\Gamma(A, \mathcal{O}_A)$ is a bounded complex with finite, locally free cohomology modules.

Next, we compute $\text{Ext}_R^2(A, M)$ using the Breen–Deligne spectral sequence, as in the proof of Proposition 2.3.7. To simplify notation, let $H^{i,j}$ denote the R -module $H^i(A^j, \mathcal{O}_{A^j}) \otimes_R M$. We explicitly write the beginning of the first page.

$$\begin{aligned} H^{2,1} &\longrightarrow H^{2,2} \longrightarrow H^{2,3} \oplus H^{2,2} \longrightarrow H^{2,4} \oplus H^{2,3} \oplus H^{2,3} \oplus H^{2,2} \oplus H^{2,1} \\ H^{1,1} &\longrightarrow H^{1,2} \longrightarrow H^{1,3} \oplus H^{1,2} \longrightarrow H^{1,4} \oplus H^{1,3} \oplus H^{1,3} \oplus H^{1,2} \oplus H^{1,1} \\ H^{0,1} &\longrightarrow H^{0,2} \longrightarrow H^{0,3} \oplus H^{0,2} \longrightarrow H^{0,4} \oplus H^{0,3} \oplus H^{0,3} \oplus H^{0,2} \oplus H^{0,1} \end{aligned}$$

Here, the maps on the left are given by $(m^* - \text{pr}_1^* - \text{pr}_2^*) \otimes \text{id}$, where $m: A \times A \rightarrow A$ is the group operation. The maps on the middle are given by

$$(\text{pr}_{1,2}^* - (\text{id} \times m)^* + (m \times \text{id})^* - \text{pr}_{2,3}^*, \text{id}^* - \tau^*) \otimes \text{id},$$

where $\tau: A \times A \rightarrow A \times A$ permutes the factors. Similarly, the maps on the right can be computed using the formulas described just after Proposition 2.3.3.

Although [GW23, Lem. 27.209] assumes that R is a field and $M = R$, the same proof shows that the map $H^{2,1} \rightarrow H^{2,2}$ is injective by replacing [GW23, Cor. 27.200] with [GW23, Cor. 27.204]. Furthermore, since $H^0(A^j, \mathcal{O}_{A^j}) = R$ for all j , the cohomology of the bottom row at $H^{0,3} \oplus H^{0,2}$ is the cohomology of the complex

$$\begin{aligned} M &\longrightarrow M^{\oplus 2} \longrightarrow M^{\oplus 5} \\ x &\longmapsto (0, 0) \\ (x, y) &\longmapsto (-x, -x - y, x - y, 2y, y), \end{aligned}$$

that is clearly exact.

To show that the cohomology of the middle row at $H^{1,2}$ vanishes, it suffices to prove that the map $H^{1,2} \rightarrow H^{1,3}$ is injective. According to Lemma 2.4.1, we have isomorphisms $H^{1,2} \simeq (H^{1,1})^{\oplus 2}$ and $H^{1,3} \simeq (H^{1,1})^{\oplus 3}$. Now, consider the composition

$$H^{1,1} \rightarrow (H^{1,1})^{\oplus 2} \simeq H^{1,2} \rightarrow H^{1,3} \simeq (H^{1,1})^{\oplus 3} \rightarrow H^{1,1},$$

where the middle map is one of $\{\text{pr}_{1,2}^*, (\text{id} \times m)^*, (m \times \text{id})^*, \text{pr}_{2,3}^*\}$, and the other two arrows are natural injections and projections. This composition can be geometrically described as the pullback by

$$A \rightarrow A \times_R A \times_R A \rightarrow A \times_R A \rightarrow A,$$

where the first map is the natural closed immersion of a factor, the middle map is one of $\{\text{pr}_{1,2}, \text{id} \times m, m \times \text{id}, \text{pr}_{2,3}\}$, and the last map is a projection.

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By considering all possible inclusions $H^{1,1} \rightarrow (H^{1,1})^{\oplus 2}$, middle maps, and projections $(H^{1,1})^{\oplus 3} \rightarrow H^{1,1}$, we see that the map $H^{1,2} \rightarrow H^{1,3}$ acts as

$$\begin{aligned} (H^{1,1})^{\oplus 2} &\longrightarrow (H^{1,1})^{\oplus 3} \\ (x, y) &\longmapsto (x, 0, -y), \end{aligned}$$

which is clearly injective. This finishes the proof. \square

COROLLARY 2.4.3. *Let A be an abelian scheme over a base scheme S , and let M be a quasi-coherent \mathcal{O}_S -module. Then $\underline{\mathrm{Ext}}^2(A, M)$ vanishes. In particular, $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_a) = 0$.*

We now turn to the main result of this section.

THEOREM 2.4.4. *Let A be an abelian scheme over a characteristic zero scheme S . Then $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m) = 0$.*

The proof of Theorem 2.4.4 requires several lemmas. Readers are encouraged to proceed directly to its proof and refer back to the lemmas as needed.

LEMMA 2.4.5. *Let A an abelian scheme over a base scheme S . For every non-zero integer n , the multiplication-by- n map on $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$ is injective.*

Proof. According to [GW23, Prop. 27.186], we have a short exact sequence of abelian sheaves

$$0 \rightarrow A[n] \rightarrow A \xrightarrow{n} A \rightarrow 0,$$

where $A[n]$ is a finite locally free group scheme over S . Passing to the long exact sequence in cohomology, we obtain the exact sequence

$$\underline{\mathrm{Ext}}^1(A[n], \mathbb{G}_m) \rightarrow \underline{\mathrm{Ext}}^2(A, \mathbb{G}_m) \xrightarrow{n} \underline{\mathrm{Ext}}^2(A, \mathbb{G}_m)$$

and $\underline{\mathrm{Ext}}^1(A[n], \mathbb{G}_m)$ vanishes due to [SGA 7_I, Exp. VIII, Prop. 3.3.1]. \square

LEMMA 2.4.6. *Let $S = \mathrm{Spec} R_0$ be an affine scheme, and let $T = \mathrm{Spec} R$ for an R_0 -algebra R . Suppose R is a filtered colimit $\mathrm{colim} R_\lambda$ of R_0 -algebras, and let $T_\lambda = \mathrm{Spec} R_\lambda$. Then, for all commutative group schemes G and H over S such that $H \rightarrow S$ is smooth, we have that $\mathrm{Ext}_T^i(G, H)$ is the filtered colimit of the $\mathrm{Ext}_{T_\lambda}^i(G, H)$ for all i .*

Proof. As filtered colimits of abelian groups are exact, this result follows from the Breen–Deligne spectral sequence in conjunction with [SGA 4_{II}, Exp. VII, Cor. 5.9]. \square

LEMMA 2.4.7. *Let A be an abelian scheme over a noetherian ring R . Then the restriction map $\mathrm{Ext}_R^2(A, \mathbb{G}_m) \rightarrow \mathrm{Ext}_{R_{\mathrm{red}}}^2(A, \mathbb{G}_m)$ is injective.*

Proof. Let $I \subset R$ be an arbitrary nilpotent ideal, and let n denote its index of nilpotency. We will show that the map $\mathrm{Ext}_R^2(A, \mathbb{G}_m) \rightarrow \mathrm{Ext}_{R/I}^2(A, \mathbb{G}_m)$ is injective, which suffices due to the noetherianity of R . Note that the quotient map $R \rightarrow R/I$ factors as

$$R = R/I^n \rightarrow R/I^{n-1} \rightarrow \dots \rightarrow R/I^2 \rightarrow R/I = R_{\mathrm{red}},$$

where every map is surjective and has a square-zero kernel. Consequently, we may assume that $n = 2$. Additionally, we remark that the Breen–Deligne spectral sequence allow us to compute the extension groups on the respective small étale sites.

Let $i: \mathrm{Spec} R/I \rightarrow \mathrm{Spec} R$ be the natural closed immersion. Since $I^2 = 0$, we have a short exact sequence of étale sheaves

$$0 \rightarrow I \rightarrow \mathbb{G}_{m,R} \rightarrow i_* \mathbb{G}_{m,R/I} \rightarrow 0.$$

The exactness of i_* then implies that

$$\mathrm{Ext}_R^2(A, I) \rightarrow \mathrm{Ext}_R^2(A, \mathbb{G}_m) \rightarrow \mathrm{Ext}_{R/I}^2(A, \mathbb{G}_m)$$

is exact, and the result follows from Theorem 2.4.2. \square

We extend our gratitude to Gabber for drawing our attention to the paper [KM23], which facilitated the proof of our next lemma.

LEMMA 2.4.8. *Let (R, \mathfrak{m}) be a complete noetherian local ring and let A be an abelian scheme over R . Assume that the residue field R/\mathfrak{m} has characteristic zero. Then the natural map $\mathrm{Ext}_R^2(A, \mathbb{G}_m) \rightarrow \lim_n \mathrm{Ext}_{R/\mathfrak{m}^n}^2(A, \mathbb{G}_m)$ is injective.*

Proof. Let n be a non-negative integer. For simplicity, we make the convention that the 0-th power of an ideal is the zero ideal, so $R/\mathfrak{m}^0 \simeq R$. As in Definition 2.3.6, we denote the kernel of the natural map $H^2(A_{R/\mathfrak{m}^n}, \mathbb{G}_m) \rightarrow H^2(A_{R/\mathfrak{m}^n}^2, \mathbb{G}_m)$ as $H_m^2(A_{R/\mathfrak{m}^n}, \mathbb{G}_m)$. By the Breen–Deligne spectral sequence, there is an exact sequence

$$0 \rightarrow Q_n \rightarrow \mathrm{Ext}_{R/\mathfrak{m}^n}^2(A, \mathbb{G}_m) \rightarrow H_m^2(A_{R/\mathfrak{m}^n}, \mathbb{G}_m),$$

where Q_n is the middle cohomology of the complex

$$M_n := \left[H^1(A_{R/\mathfrak{m}^n}, \mathbb{G}_m) \rightarrow H^1(A_{R/\mathfrak{m}^n}^2, \mathbb{G}_m) \rightarrow H^1(A_{R/\mathfrak{m}^n}^3, \mathbb{G}_m) \oplus H^1(A_{R/\mathfrak{m}^n}^2, \mathbb{G}_m) \right].$$

The left-exactness of the lim functor then provides the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_0 & \longrightarrow & \mathrm{Ext}_R^2(A, \mathbb{G}_m) & \longrightarrow & H_m^2(A, \mathbb{G}_m) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \lim Q_n & \longrightarrow & \lim \mathrm{Ext}_{R/\mathfrak{m}^n}^2(A, \mathbb{G}_m) & \longrightarrow & \lim H_m^2(A_{R/\mathfrak{m}^n}, \mathbb{G}_m), \end{array}$$

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where both rows are exact. Since [KM23, Thm. 11.11] implies that the right-most vertical arrow is injective, it suffices to prove the same for the left-most one.

By [KM23, Prop. 5.3], the natural morphism of complexes $M_0 \rightarrow \lim M_n$ is an isomorphism, making the map $Q_0 \rightarrow \lim Q_n$ a comparison between $\mathcal{H}^1(\lim M_n)$ and $\lim \mathcal{H}^1(M_n)$. Here is one way of understanding this comparison.

Given an additive functor $F: \mathbf{A} \rightarrow \mathbf{B}$ between Grothendieck abelian categories and a complex M in \mathbf{A} , there exist two "hypercohomology" spectral sequences

$$D_1^{i,j} = R^j F(M^i) \quad \text{and} \quad E_2^{i,j} = R^i F(\mathcal{H}^j(M)),$$

that converge to $\mathcal{H}^{i+j}(RF(M))$. If M is constituted of F -acyclic objects, the first spectral sequence degenerates at the second page, proving that the second spectral sequence converges to $\mathcal{H}^{i+j}(F(M))$. Furthermore, when $R^i F$ vanishes for $i > 1$, this spectral sequence yields a short exact sequence

$$0 \rightarrow R^1 F(\mathcal{H}^{i-1}(M)) \rightarrow \mathcal{H}^i(F(M)) \rightarrow F(\mathcal{H}^i(M)) \rightarrow 0.$$

An additional application of [KM23, Thm. 11.11] demonstrates that the terms in our complex M_n are acyclic with respect to the \lim functor. Therefore, our preceding discussion applies with $F = \lim$, yielding a short exact sequence

$$0 \rightarrow \lim^1 H_m^1(A_{R/m^n}, \mathbb{G}_m) \rightarrow Q_0 \rightarrow \lim Q_n \rightarrow 0.$$

The abelian group $H_m^1(A_{R/m^n}, \mathbb{G}_m)$ is isomorphic to the R/m^n -points of the dual abelian scheme A' . The formal smoothness of A' implies that the natural map

$$A'(R/m^{n+1}) \rightarrow A'(R/m^n)$$

is surjective for all n . Consequently, the pro-object $\{H_m^1(A_{R/m^n}, \mathbb{G}_m)\}$ satisfies the Mittag-Leffler property, concluding the proof. \square

We are now in position to prove the main result of this section.

Proof of Theorem 2.4.4. Let $T \rightarrow S$ be a morphism of schemes, and let E be an element of $\text{Ext}_T^2(A, \mathbb{G}_m)$. By the Lemma 2.4.5, we need to prove that E is fppf-locally torsion. Up to taking affine covers, we may suppose that $T = \text{Spec } R$ and $S = \text{Spec } R_0$. The Lemma 2.4.6 and [Stacks, Tag 00QN] imply that we can suppose R to be a finitely presented R_0 -algebra. By [GW23, Rem. 27.91] and [Stacks, Tag 05N9] we can further assume that R_0 is a finite type \mathbb{Z} -algebra. Lastly, Lemma 2.4.7 asserts that we may also assume T is reduced.

Let η be a generic point of T . Since T is reduced, we have that $\mathcal{O}_{T,\eta}$ is already the residue field $\kappa(\eta)$ [Stacks, Tag 00EU]. In particular, $\kappa(\eta)$ is the filtered colimit of coordinate rings of affine opens containing η . From Lemma 2.4.6 and [Bre69, §7], there

exists an open subset of T containing η where E is torsion. Taking the union of all such opens for each generic point yields a dense open subset U of T where E is torsion.

Consider an irreducible component Z of $T \setminus U$ with generic point ξ , and let $B := \mathcal{O}_{T,\xi}$ be its local ring. The completion \widehat{B} is a local ring with maximal ideal \mathfrak{m} and, for all $n \geq 1$, the composition

$$\widehat{B} \rightarrow \widehat{B}/\mathfrak{m}^n \rightarrow (\widehat{B}/\mathfrak{m}^n)_{\text{red}} \simeq \widehat{B}/\mathfrak{m}$$

is the quotient map. Thus, Lemmas 2.4.7 and 2.4.8 imply that both maps below

$$\text{Ext}_{\widehat{B}}^2(A, \mathbb{G}_m) \rightarrow \lim_n \text{Ext}_{\widehat{B}/\mathfrak{m}^n}^2(A, \mathbb{G}_m) \rightarrow \prod_{n=1}^{\infty} \text{Ext}_{\widehat{B}/\mathfrak{m}}^2(A, \mathbb{G}_m)$$

are injective. According to [Bre69, §7], $\text{Ext}_{\widehat{B}/\mathfrak{m}}^2(A, \mathbb{G}_m)$ is torsion, and then so is $\text{Ext}_{\widehat{B}}^2(A, \mathbb{G}_m)$.

Given that the local ring $B = \mathcal{O}_{T,\xi}$ is excellent, Popescu's theorem [Stacks, Tag 07GC] states that the completion \widehat{B} is a filtered colimit of smooth B -algebras B_λ . By Lemma 2.4.6, the pullback of E to $\text{Spec } B_\lambda$ is torsion for some λ , which we now fix.

Now, since B is the filtered colimit of the coordinate rings of affine open neighborhoods of η , we can spread out the map $\text{Spec } B_\lambda \rightarrow \text{Spec } B$ to a smooth morphism $Y \rightarrow V$, where V is an affine neighborhood of η . Furthermore, B_λ is the filtered colimit of the coordinate rings of all Y mapping smoothly to V with image containing ξ . According to Lemma 2.4.6, by further narrowing down Y , E becomes torsion on Y .

Let f be the composition $Y \rightarrow V \rightarrow T$. Since Z is irreducible, $Z \cap f(Y)$ is a dense open subset of it (and so its complement is lower-dimensional). By reiterating the same procedure, which finishes after a finite number of steps since T is noetherian, we can find an fppf cover of this complement where E is torsion. Gathering these maps for each irreducible component of $T \setminus U$, together with $U \rightarrow T$, we obtain an fppf cover of T where E is torsion. \square

COROLLARY 2.4.9. *Let A be an abelian variety over a characteristic zero field k . Then the abelian sheaf $\underline{\text{Ext}}^2(A_{\text{dR}}, \mathbb{G}_m)$ vanishes.*

Proof. The abelian sheaf $\underline{\text{Ext}}^2(A_{\text{dR}}, \mathbb{G}_m)$ fits into the exact sequence

$$\underline{\text{Ext}}^1(\widehat{A}, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^2(A_{\text{dR}}, \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^2(A, \mathbb{G}_m),$$

and the extremities vanish by the preceding theorem, along with Proposition 2.3.22. \square

3. MODULI OF CHARACTER SHEAVES

Let G be a commutative connected algebraic group over a field k of characteristic zero, with $m: G \times G \rightarrow G$ as its group operation. A primary goal of this thesis is to study line bundles with integrable connection (\mathcal{L}, ∇) on G satisfying the condition $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$. These objects are referred to as *character sheaves*, and this chapter is dedicated to studying the geometry of their moduli spaces.

LAUMON DUALITY OF GENERALIZED 1-MOTIVES

For a k -scheme S , Remark 2.3.8 and Theorem 2.3.21 show that the S -points of the sheaf $G^\natural := \underline{\text{Ext}}^1(G_{\text{dR}}, \mathbb{G}_m)$ correspond to isomorphism classes of triples $(\mathcal{L}, \nabla, \alpha)$. Here, \mathcal{L} is a line bundle on G_S with integrable connection ∇ relative to S , and α is an isomorphism $m^*(\mathcal{L}, \nabla) \rightarrow (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$ making two specific diagrams commute.

When S is reduced, the natural map $G^\natural(S) \rightarrow H_m^1(G_{\text{dR}} \times S, \mathbb{G}_m)$ is an isomorphism. Explicitly, the S -points of G^\natural also correspond to isomorphism classes of pairs (\mathcal{L}, ∇) , where \mathcal{L} is a line bundle on G_S with integrable connection ∇ relative to S satisfying $m^*(\mathcal{L}, \nabla) \simeq (\mathcal{L}, \nabla) \boxtimes (\mathcal{L}, \nabla)$. In particular, $G^\natural(k)$ is the group of isomorphism classes of character sheaves on G .

In Chapter 4, we will introduce a Fourier transform functor $D_{\text{qc}}(G_{\text{dR}}) \rightarrow D_{\text{qc}}(G^\natural)$. Given the existence of a widely studied Fourier transform $D_{\text{qc}}(\mathcal{D}_{\mathbb{A}^1}) \rightarrow D_{\text{qc}}(\mathcal{D}_{\mathbb{A}^1})$, one might expect G^\natural to be isomorphic to the de Rham space $\mathbb{G}_{\alpha, \text{dR}}$. However, this is not the case. Proposition 3.1.5 gives a monomorphism $\mathbb{G}_{\alpha, \text{dR}} \rightarrow G^\natural$, with the cokernel being the non-vanishing sheaf $\underline{\text{Ext}}^1(\mathbb{G}_\alpha, \mathbb{G}_m)$ studied in Remark 2.3.17.

This discrepancy led Laumon to define a class of commutative group stacks called *generalized 1-motives* in [Lau96], which includes all commutative connected algebraic groups and their de Rham spaces. Subsequently, Laumon extended Deligne's Cartier duality for 1-motives in [Del74] to this setting, ensuring the self-duality of $\mathbb{G}_{\alpha, \text{dR}}$.

Akin to the example of $\mathbb{G}_{\alpha, \text{dR}}$ above, the Laumon dual $\mathcal{G}^\triangleright$ of a generalized 1-motive \mathcal{G} naturally maps to the stacky Cartier dual \mathcal{G}^\vee . Proposition 3.1.5 shows that this map induces an isomorphism $\pi_1(\mathcal{G}^\triangleright) \xrightarrow{\sim} \pi_1(\mathcal{G}^\vee)$ and a short exact sequence

$$0 \rightarrow \pi_0(\mathcal{G}^\triangleright) \rightarrow \pi_0(\mathcal{G}^\vee) \rightarrow \underline{\text{Ext}}^1(L, \mathbb{G}_m) \rightarrow 0,$$

where L is *the linear part* of \mathcal{G} . (See the discussion below Definition 3.1.1 for a precise description of L .) In particular, this shows that Deligne's Cartier dual of a 1-motive

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coincides with its stacky Cartier dual.

A notable property of the Laumon dual is its computational ease. To set the stage for computing $G_{\text{dR}}^{\triangleright}$, express G as an extension

$$0 \rightarrow L \xrightarrow{\varphi} G \xrightarrow{\psi} A \rightarrow 0$$

of an abelian variety A by a linear group L . Let Ω_G denote the vector group of invariant differentials on G . There is a natural pullback map $\psi^*: \Omega_A \rightarrow \Omega_G$, and a morphism $\Omega_A \rightarrow A^{\natural}$ that maps an invariant differential η to the character sheaf $(\mathcal{O}_A, d + \eta)$. We denote by K the pushout $(\Omega_G \times A^{\natural})/\Omega_A$, which is representable by a commutative connected algebraic group.

There is also a natural morphism from the Cartier dual L^{D} of L to K . This morphism maps a character $\chi \in L^{\text{D}}$ to $[\omega, (\mathcal{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^*\omega = d\chi/\chi$, and (\mathcal{L}, ∇) is the unique element of A^{\natural} satisfying $\psi^*(\mathcal{L}, \nabla) \simeq (\mathcal{O}_G, d - \omega)$. This is a monomorphism, and Proposition 3.1.7 identifies the quotient K/L^{D} with the Laumon dual $G_{\text{dR}}^{\triangleright}$.

The morphism from the Laumon dual $G_{\text{dR}}^{\triangleright}$ to the stacky Cartier dual G^{\natural} can be described simply using this identification. Proposition 3.1.9 shows that this morphism maps $[\omega, (\mathcal{L}, \nabla)]$ to the character sheaf

$$(\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathcal{L}, \nabla).$$

Since Proposition 2.3.16 implies that $G_{\text{dR}}^{\triangleright}(S) \rightarrow G^{\natural}(S)$ is an isomorphism for seminormal k -schemes S , it follows that every character sheaf on G has the form above.

MODULI SPACE OF CHARACTER SHEAVES

By further decomposing the linear part L as a product of a torus T and a unipotent group U , the Cartier dual $L^{\text{D}} \simeq T^{\text{D}} \times U^{\text{D}}$ becomes isomorphic to $X \times \widehat{U}^*$. Here, X is the character group of T , and \widehat{U}^* is the formal completion along the identity of the dual vector space U^* . Since \widehat{U}^* is a formal group, the Laumon dual of G_{dR} cannot be represented by a scheme or even an algebraic space.

We define the moduli space of character sheaves G^b as the quotient K/X . As before, for seminormal k -schemes S , the natural morphism $G^b(S) \rightarrow H_m^1(G_{\text{dR}} \times S, \mathbb{G}_m)$ is an isomorphism. In particular, $G^b(k)$ is isomorphic to the group of isomorphism classes of character sheaves on G , thereby justifying its name.

Note that the group of isomorphism classes of character sheaves on \mathbb{G}_m is k/\mathbb{Z} . Accordingly, the moduli space \mathbb{G}_m^b is isomorphic to \mathbb{G}_a/\mathbb{Z} , a group algebraic space not representable by a scheme. The main result in this chapter, Theorem 3.2.5, asserts that the complex

$$0 \rightarrow A^b \rightarrow G^b \rightarrow T^b \times U^b \rightarrow 0$$

is exact. Consequently, descent implies that G^b is representable by a smooth commutative connected group algebraic space. The dimension of G^b satisfies

$$\dim G \leq \dim G^b \leq 2 \dim G,$$

with equality on the left if and only if G is affine, and equality on the right if and only if G is proper.

This theorem might lead the reader to think that $(-)^b$ is an exact functor, in the sense of exact categories. Proposition 3.2.9 demonstrates that this contravariant functor maps monomorphisms to epimorphisms and epimorphisms to monomorphisms. However, as illustrated by Example 3.2.10, the functor $(-)^b$ is not exact.

LINEAR AND GENERIC SUBSPACES OF THE MODULI SPACE

The guiding thread of this thesis is a vanishing theorem for de Rham cohomology on G , which asserts that the cohomology of twists by character sheaves on certain *large* subsets of the moduli space G^b vanish. These subsets exhibit a rigid structure derived from the algebraic and geometric properties of the group G under consideration.

For an epimorphism $\rho: G \rightarrow \tilde{G}$ with connected kernel, the image of $\rho^b: \tilde{G}^b \hookrightarrow G^b$ is said to be a *linear subspace* of G^b . When G is an abelian variety A , linear subspaces of $A^b \simeq A^{\natural}$ were initially studied by Simpson in the context of non-abelian Hodge theory, referring to them as *triple tori* [Sim93, p. 365]. These subsets also play a key role in Schnell's work [Sch15], where he terms translates of linear subspaces of A^b as *linear subvarieties*.

Given a torus T , a linear subspace of $T^b \simeq \Omega_T/X$ is of the form V/Y , where Y is a subgroup of X and V is the linear subspace of Ω_T generated by Y in the sense of linear algebra. For a unipotent group U , a linear subspace of $U^b = U^*$ corresponds to a linear subspace of the underlying vector space of U^* in the sense of linear algebra.

More generally, let L be a linear commutative connected algebraic group. By decomposing L as a product of a torus T and a unipotent group U , we obtain an isomorphism $L^b \simeq T^b \times U^b$. Then, a linear subspace of L^b is the product of linear subspaces of T^b and U^b .

We define a *generic subspace* of G^b as the complement of a finite union of translates of linear subspaces with positive codimension. These are the large subsets of G^b on which generic vanishing is supposed to hold. In most other approaches to generic vanishing, such as [Krä14; BSS18; LMW18; LMW19], the authors considered open dense subsets of some moduli space as their analogues of generic subspaces. Hereafter, we compare our notion with those in the literature.

According to Proposition 3.3.5, when G is an extension of an abelian variety by a unipotent group, a generic subspace of G^b is also open and dense. Since the Zariski topology of $\mathbb{G}_m^b \simeq \mathbb{G}_a/\mathbb{Z}$ is too coarse for such a result, we are led to suppose $k = \mathbb{C}$ and

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consider the analytification G_{an}^b . Although G^b is often not representable by a scheme, G_{an}^b is a complex Lie group.

Interestingly, for an abelian variety A , the complex manifold A_{an}^b is Stein, even though A^b itself is not affine. The moduli space A^b is even *anti-affine*, meaning that every morphism of algebraic varieties $A^b \rightarrow \mathbb{A}^1$ is constant. In general, Proposition 3.3.5 implies that a generic subspace of G^b defines an open subset of G_{an}^b whose complement has measure zero. In particular, it is an open dense subset of G_{an}^b .

Often, the moduli space considered for generic vanishing is the *character variety* $\text{Char}(G)$, defined as the spectrum of the group algebra $\mathbb{C}[\pi_1(G_{\text{an}})]$. This serves as a moduli space for rank-one local systems on G_{an} . The natural Hopf algebra structure on $\mathbb{C}[\pi_1(G_{\text{an}})]$ equips $\text{Char}(G)$ with a multiplication, encoding the tensor product of local systems.

The Riemann–Hilbert equivalence states that local systems on G_{an} correspond to *regular* connections on G . Since all non-trivial character sheaves on unipotent groups are irregular, $\text{Char}(G)$ cannot parametrize character sheaves in general. On the other hand, Proposition 3.3.9 states that the natural map $G_{\text{an}}^b \rightarrow \text{Char}(G)_{\text{an}}$ is an isomorphism for semiabelian varieties G .

3.1. GENERALIZED 1-MOTIVES AND THEIR LAUMON DUALS

Let k be a perfect field of any characteristic. In this chapter, we will systematically apply Proposition 2.1.3 to represent commutative group stacks over k as two-term complexes (in degrees -1 and 0) of abelian sheaves on the site $(\text{Sch}/k)_{\text{fppf}}$. For the reader’s convenience, we recall that the Cartier dual of a commutative formal group is represented by an affine commutative group scheme.

DEFINITION 3.1.1. A *generalized 1-motive* is a two-term complex of abelian fppf sheaves $[\mathcal{G} \rightarrow G]$, where G is a smooth commutative connected algebraic group over k , and \mathcal{G} is a commutative formal group over k whose Cartier dual is smooth and connected.

A usual 1-motive, as defined in [Del74, §10.1], is a special case of the definition above where k is algebraically closed, G is a semiabelian variety, and \mathcal{G} is a finitely generated free \mathbb{Z} -module. Our Definition 3.1.1 is inspired by the one in [Rus13] and extends Laumon’s [Lau96] to base fields that may have positive characteristic.

Let $[\mathcal{G} \rightarrow G]$ be a generalized 1-motive. The Barsotti–Chevalley theorem [Mil17, Thm. 8.27] states that G has a smallest subgroup L such that G/L is proper. This subgroup is affine, smooth and connected, and the quotient G/L is an abelian variety A . In other words, G can be functorially decomposed as an extension

$$0 \rightarrow L \rightarrow G \rightarrow A \rightarrow 0,$$

of an abelian variety A by an affine smooth commutative connected algebraic group L .



Every algebraic group over a characteristic zero field is smooth. In positive characteristic, a singular connected algebraic group is still an extension of an abelian variety by a linear group, but this decomposition may not be unique. See [Bri17a, Ex. 4.3.8].

The motivation for Definition 3.1.1 arises from the fact that the Cartier duality of Proposition 2.2.2 naturally restricts to an anti-equivalence of categories

$$\left\{ \begin{array}{c} \text{Affine smooth} \\ \text{commutative connected} \\ \text{algebraic groups over } k \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Commutative formal groups} \\ \text{over } k \text{ whose Cartier dual is} \\ \text{smooth and connected} \end{array} \right\}.$$

Using this, we will concoct a dual of $[\mathcal{G} \rightarrow G]$ of the form $[L^D \rightarrow K]$, for some smooth commutative connected algebraic group K fitting into the short exact sequence

$$0 \rightarrow \mathcal{G}^D \rightarrow K \rightarrow A' \rightarrow 0,$$

where A' is the dual abelian variety of A . A functorial definition of K is given by the following lemma.

LEMMA 3.1.2. *Let $[\mathcal{G} \rightarrow A]$ be the generalized 1-motive defined by the composition $\mathcal{G} \rightarrow G \rightarrow A$. The complex $\mathrm{RHom}([\mathcal{G} \rightarrow A], \mathbb{G}_m)$ has no cohomology in degrees 0 and 2. Moreover, $\mathrm{Ext}^1([\mathcal{G} \rightarrow A], \mathbb{G}_m)$ is representable by a smooth commutative connected algebraic group.*

Proof. Applying the functor $\mathrm{RHom}(-, \mathbb{G}_m)$ to the fiber sequence $\mathcal{G} \rightarrow A \rightarrow [\mathcal{G} \rightarrow A]$, we obtain the following long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}^0([\mathcal{G} \rightarrow A], \mathbb{G}_m) & \longrightarrow & 0 & \longrightarrow & \mathcal{G}^D \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Ext}^1([\mathcal{G} \rightarrow A], \mathbb{G}_m) \longrightarrow A' \longrightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & \mathrm{Ext}^2([\mathcal{G} \rightarrow A], \mathbb{G}_m) \longrightarrow 0. \end{array}$$

Here, we use Propositions 2.2.1 and 2.3.22, along with Theorem 2.4.4 to justify the non-trivial zeros above. The vanishing results follow directly, and the representability of $\mathrm{Ext}^1([\mathcal{G} \rightarrow A], \mathbb{G}_m)$ by a smooth commutative connected algebraic group follows by descent. (See the following remark for an explanation of how these descent arguments work.) \square

■ REMARK 3.1.3 — ON PROPERTIES INHERITED BY EXTENSIONS. Let S be a base scheme, and consider an extension of abelian sheaves on the site $(\mathrm{Sch}/S)_{\mathrm{fppf}}$:

$$0 \rightarrow \mathcal{H} \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{G} \rightarrow 0.$$

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The composition $\mathcal{H} \times \mathcal{E} \xrightarrow{\varphi \times \text{id}} \mathcal{E} \times \mathcal{E} \xrightarrow{m} \mathcal{E}$, where m is the group operation of \mathcal{E} , defines an action of \mathcal{H} on \mathcal{E} . This action fits into the cartesian diagram

$$\begin{array}{ccccc} \mathcal{H} \times \mathcal{E} & \xrightarrow{\varphi \times \text{id}} & \mathcal{E} \times \mathcal{E} & \xrightarrow{m} & \mathcal{E} \\ \text{pr}_2 \downarrow & & & & \downarrow \psi \\ \mathcal{E} & \xrightarrow{\quad \quad \quad \psi \quad \quad \quad} & & & \mathcal{G}, \end{array}$$

giving that \mathcal{E} is an \mathcal{H} -torsor over \mathcal{G} . In other words, we have an \mathcal{H} -equivariant isomorphism $\mathcal{H} \times \mathcal{E} \rightarrow \mathcal{E} \times_{\mathcal{G}} \mathcal{E}$.

Assume that \mathcal{G} and \mathcal{H} are represented by group algebraic spaces G and H over S , respectively. By [Stacks, Tag 00WN], there exists a scheme T with a map $T \rightarrow \mathcal{E}$ such that the composition

$$T \rightarrow \mathcal{E} \rightarrow G$$

is an fppf cover. This induces a trivialization $H \times T \simeq \mathcal{E} \times_G T$. Since the definition of an algebraic space is local in the fppf topology [Stacks, Tag 04SK], it follows that \mathcal{E} is representable by an algebraic space E over G . Therefore, E is also a group algebraic space over S .

Let \mathcal{P} be a property of morphisms of algebraic spaces that is stable under base change and fppf-local on the base. The same reasoning implies that if $H \rightarrow S$ satisfies \mathcal{P} , then so does $E \rightarrow G$. Furthermore, if \mathcal{P} is stable under composition and is satisfied by $G \rightarrow S$, then $E \rightarrow S$ also satisfies \mathcal{P} . (See [Stacks, Tags 03H8 and 03YE] for an extensive list of such properties \mathcal{P} .)

Now, suppose that S is the spectrum of a field k . If G and H are schemes, then the maps $G \rightarrow S$ and $H \rightarrow S$ are necessarily separated [Stacks, Tag 047L]. The preceding discussion implies that $E \rightarrow S$ is also separated, and so [Stacks, Tag 0B8G] shows that E is a scheme as well.

For the remainder of this discussion, assume that G and H are group algebraic spaces locally of finite type over k . Then, we note that E has dimension $\dim G + \dim H$. Indeed, [Stacks, Tag 0AFH] says that the relative dimension of $E \rightarrow G$ is $\dim E - \dim G$. However, this relative dimension is also the dimension of the fiber H , completing the proof.

Since $E \rightarrow G$ is faithfully flat of finite presentation, the induced morphism $|E| \rightarrow |G|$ between the underlying topological spaces is a quotient map [Stacks, Tag 0413]. Although the fibers of $|E| \rightarrow |G|$ may not be homeomorphic to $|H|$, there exists a continuous surjection from $|H|$ to them, as stated in [Stacks, Tag 03H4]. In particular, if G and H are connected, so is E . ■

Following Laumon, we will systematically denote by K the algebraic group representing $\underline{\text{Ext}}^1([\mathcal{G} \rightarrow \mathcal{A}], \mathbb{G}_m)$. The following variant of (stacky) Cartier duality was initially introduced in [Del74] and was subsequently generalized in [Lau96] and [Rus13].

DEFINITION 3.1.4. Let $[\mathcal{G} \rightarrow G]$ be a generalized 1-motive. We define the *Laumon dual* $[\mathcal{G} \rightarrow G]^\triangleright$ to be the generalized 1-motive $[L^D \rightarrow K]$, where $L^D \rightarrow K$ is the connecting morphism induced by the fiber sequence $L \rightarrow [\mathcal{G} \rightarrow G] \rightarrow [\mathcal{G} \rightarrow A]$ via the Cartier duality functor.

We note that the Laumon dual is functorial on the generalized 1-motive. Indeed, by the functoriality of the Barsotti–Chevalley theorem, a morphism of complexes

$$\begin{array}{ccc} \mathcal{G}_1 & \longrightarrow & G_1 \\ \downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & G_2 \end{array}$$

between two generalized 1-motives induces maps $L_1 \rightarrow L_2$ and $A_1 \rightarrow A_2$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & G_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & G_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

commute. Consequently, we obtain a commutative diagram in the derived category of abelian sheaves

$$\begin{array}{ccccc} L_1 & \longrightarrow & [\mathcal{G}_1 \rightarrow G_1] & \longrightarrow & [\mathcal{G}_1 \rightarrow A_1] \\ \downarrow & & \downarrow & & \downarrow \\ L_2 & \longrightarrow & [\mathcal{G}_2 \rightarrow G_2] & \longrightarrow & [\mathcal{G}_2 \rightarrow A_2]. \end{array}$$

The Cartier duality functor then induces a morphism $[\mathcal{G}_2 \rightarrow G_2]^\triangleright \rightarrow [\mathcal{G}_1 \rightarrow G_1]^\triangleright$.

Our next proposition gives a comparison between the Laumon dual defined above and the stacky Cartier dual of Definition 2.1.6.

PROPOSITION 3.1.5. *Let $[\mathcal{G} \rightarrow G]$ be a generalized 1-motive. There exists a natural map $[\mathcal{G} \rightarrow G]^\triangleright \rightarrow [\mathcal{G} \rightarrow G]^\vee$, whose cofiber is $\underline{\text{Ext}}^1(L, \mathbb{G}_m)$.*

Proof. The fiber sequence defining the Laumon dual induces the fiber sequence below.

$$\underline{\text{RHom}}([\mathcal{G} \rightarrow A], \mathbb{G}_m[1]) \rightarrow \underline{\text{RHom}}([\mathcal{G} \rightarrow G], \mathbb{G}_m[1]) \rightarrow \underline{\text{RHom}}(L, \mathbb{G}_m[1])$$

By Lemma 3.1.2, $\underline{\text{Ext}}^2([\mathcal{G} \rightarrow A], \mathbb{G}_m)$ vanishes and so $\underline{\text{Ext}}^1([\mathcal{G} \rightarrow G], \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1(L, \mathbb{G}_m)$ is an epimorphism. Then, [Bro21, Lem. 3.10] implies that

$$\tau_{\leq 0} \underline{\text{RHom}}([\mathcal{G} \rightarrow A], \mathbb{G}_m[1]) \rightarrow \tau_{\leq 0} \underline{\text{RHom}}([\mathcal{G} \rightarrow G], \mathbb{G}_m[1]) \rightarrow \tau_{\leq 0} \underline{\text{RHom}}(L, \mathbb{G}_m[1])$$

is also a fiber sequence. Yet another application of Lemma 3.1.2 gives that

$$\tau_{\leq 0} \underline{\text{RHom}}([\mathcal{G} \rightarrow A], \mathbb{G}_m[1]) \simeq K$$

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and so, up to a shift, the fiber sequence just obtained is $L^\vee[-1] \rightarrow K \rightarrow [\mathcal{G} \rightarrow G]^\vee$. Since $L^D \simeq \tau_{\leq 0}(L^\vee[-1])$, there is a natural map $L^D \rightarrow L^\vee[-1]$ making the square

$$\begin{array}{ccccc} L^D & \longrightarrow & K & \longrightarrow & [L^D \rightarrow K] \\ \downarrow & & \parallel & & \\ L^\vee[-1] & \longrightarrow & K & \longrightarrow & [\mathcal{G} \rightarrow G]^\vee \end{array}$$

commute and inducing a morphism of fiber sequences. In this way we obtain the desired comparison map.

Now, according to [Stacks, Tag 08J5], there exists a fiber sequence $L^D \rightarrow L^\vee[-1] \rightarrow \underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. Finally, the octahedral axiom [Stacks, Tag 05R0] gives that the cofiber of $[\mathcal{G} \rightarrow G]^\triangleright \rightarrow [\mathcal{G} \rightarrow G]^\vee$ is isomorphic to $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$. \square

This proposition implies that the comparison map $[\mathcal{G} \rightarrow G]^\triangleright \rightarrow [\mathcal{G} \rightarrow G]^\vee$ becomes an isomorphism precisely when $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) = 0$. This condition is satisfied when G is semiabelian, thereby demonstrating that the Cartier dual on 1-motives, as defined by Deligne, agrees with the stacky Cartier dual. Extending this result, we derive the following corollary.

COROLLARY 3.1.6. *The comparison morphism $[\mathcal{G} \rightarrow G]^\triangleright \rightarrow [\mathcal{G} \rightarrow G]^\vee$ is always an isomorphism when the base field k has positive characteristic. When k has characteristic zero, the comparison map is an isomorphism if and only if G is semiabelian.*

Proof. In positive characteristic, the fppf sheaf $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m)$ always vanishes due to [Ros23, Prop. 2.2.17]. If k has characteristic zero, L is a product of a torus and a vector group U [Mil17, Cor. 16.15]. In particular, Proposition 2.3.1 implies that $\underline{\mathrm{Ext}}^1(L, \mathbb{G}_m) \simeq \underline{\mathrm{Ext}}^1(U, \mathbb{G}_m)$. By Example 2.3.17, the latter sheaf vanishes precisely when U does. \square

For the remainder of this section, we assume the base field k has characteristic zero and $\mathcal{G} = \widehat{G}$, so that $[\widehat{G} \rightarrow G] \simeq G_{\mathrm{dR}}$. We now provide a more explicit description of $K = \underline{\mathrm{Ext}}^1([\widehat{G} \rightarrow A], \mathbb{G}_m)$. The long exact sequence derived from the extension

$$0 \rightarrow \widehat{A} \rightarrow A \rightarrow A_{\mathrm{dR}} \rightarrow 0,$$

via the Cartier duality functor, results in the short exact sequence

$$0 \rightarrow \Omega_A \rightarrow A^\natural \rightarrow A' \rightarrow 0,$$

as noted in Remark 2.3.24. Now, the quotient map $\psi: G \rightarrow A$ induces a pullback map

$\psi^*: \Omega_A \rightarrow \Omega_G$, and we consider the corresponding pushout extension.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^{\natural} & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow \psi^* & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & (\Omega_G \times A^{\natural})/\Omega_A & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

Here, and throughout this thesis, Ω_G represents the vector group consisting of the *invariant* differentials of G .

PROPOSITION 3.1.7. *The algebraic group $K = \underline{\text{Ext}}^1([\widehat{G} \rightarrow A], \mathbb{G}_m)$ is naturally isomorphic to the quotient $(\Omega_G \times A^{\natural})/\Omega_A$.*

Proof. Consider the following commutative diagram, whose rows are fiber sequences.

$$\begin{array}{ccccc} \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G} \rightarrow A] \\ \downarrow & & \parallel & & \downarrow \\ \widehat{A} & \longrightarrow & A & \longrightarrow & [\widehat{A} \rightarrow A] \end{array}$$

After applying $\text{RHom}(-, \mathbb{G}_m)$ and taking long exact sequences in cohomology, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_A & \longrightarrow & A^{\natural} & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow \psi^* & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \Omega_G & \longrightarrow & K & \longrightarrow & A' & \longrightarrow & 0. \end{array}$$

We assert that the square on the left is cocartesian. This means that the complex

$$0 \rightarrow \Omega_A \rightarrow \Omega_G \times A^{\natural} \rightarrow K \rightarrow 0,$$

with the action of Ω_A on $\Omega_G \times A^{\natural}$ as in the typical pushout construction, is exact. This complex is part of the following larger commutative diagram.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Omega_G & \xlongequal{\quad} & \Omega_G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_A & \longrightarrow & \Omega_G \times A^{\natural} & \longrightarrow & K & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_A & \longrightarrow & A^{\natural} & \longrightarrow & A' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

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In this diagram, every column is exact, and both the top and bottom rows are exact. Therefore, the middle row must also be exact. \square

Let us fix some notation. Given that the algebraic group K is a quotient of $\Omega_G \times A^\natural$, we will denote its points as (equivalence classes of) pairs $(\omega, (\mathcal{L}, \nabla))$, where $\omega \in \Omega_G$ and $(\mathcal{L}, \nabla) \in A^\natural$.¹ We will also systematically denote the natural maps $L \rightarrow G$ and $G \rightarrow A$ as φ and ψ , respectively.

■ **REMARK 3.1.8.** The morphism $L^D \rightarrow K$ in the Laumon dual of $[\widehat{G} \rightarrow G]$ can also be explicitly described: it maps a character $\chi \in L^D$ to $[\omega, (\mathcal{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^*\omega = d\chi/\chi$, and (\mathcal{L}, ∇) is the unique element of A^\natural satisfying $\psi^*(\mathcal{L}, \nabla) \simeq (\mathcal{O}_G, d - \omega)$. Note that, since the Cartier dual of G_{dR} vanishes, the map $L^D \rightarrow K$ is a monomorphism. In particular, the Laumon dual of G_{dR} is a usual abelian sheaf. ■

Finally, we describe the comparison map $[\widehat{G} \rightarrow G]^\triangleright \rightarrow [\widehat{G} \rightarrow G]^\vee$ from Proposition 3.1.5. The universal property of pushouts permits us to define the morphism of abelian sheaves

$$\begin{aligned} K &\rightarrow G^\natural \\ [\omega, (\mathcal{L}, \nabla)] &\mapsto (\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathcal{L}, \nabla). \end{aligned}$$

This map factors through the quotient, resulting in a morphism $\gamma: K/L^D \rightarrow G^\natural$.

PROPOSITION 3.1.9. *The morphism $\gamma: K/L^D \rightarrow G^\natural$, sending an equivalence class $[\omega, (\mathcal{L}, \nabla)]$ to the line bundle with integrable connection*

$$(\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathcal{L}, \nabla),$$

coincides with the comparison map $G_{\text{dR}}^\triangleright \rightarrow G_{\text{dR}}^\vee$. In particular, γ is a monomorphism and its cokernel is $\underline{\text{Ext}}^1(L, \mathbb{G}_m)$.

Proof. Consider the morphism $K \rightarrow G^\natural$, defined above using the universal property of pushouts. We affirm that the diagram

$$\begin{array}{ccc} K & \longrightarrow & G^\natural \\ \downarrow \wr & & \parallel \\ \underline{\text{Ext}}^1([\widehat{G} \rightarrow A], \mathbb{G}_m) & \longrightarrow & \underline{\text{Ext}}^1([\widehat{G} \rightarrow G], \mathbb{G}_m), \end{array}$$

¹This is a slight abuse of notation since the sheafification involved in defining the quotient sheaf may be non-trivial. That being said, by Serre vanishing, $K(S)$ indeed is the quotient of $\Omega_G(S) \times A^\natural(S)$ by $\Omega_A(S)$ for affine S .

on which the map $\underline{\text{Ext}}^1([\widehat{G} \rightarrow A], \mathbb{G}_m) \rightarrow \underline{\text{Ext}}^1([\widehat{G} \rightarrow G], \mathbb{G}_m)$ is induced by the natural morphism of complexes $[\widehat{G} \rightarrow G] \rightarrow [\widehat{G} \rightarrow A]$, commutes. This is the same as showing that the diagram

$$\begin{array}{ccc}
 & \underline{\text{Ext}}^1([\widehat{A} \rightarrow A], \mathbb{G}_m) & \\
 & \downarrow & \searrow \\
 \underline{\text{Hom}}(\widehat{G}, \mathbb{G}_m) & \longrightarrow \underline{\text{Ext}}^1([\widehat{G} \rightarrow A], \mathbb{G}_m) & \longrightarrow \underline{\text{Ext}}^1([\widehat{G} \rightarrow G], \mathbb{G}_m)
 \end{array}$$

commutes. The upper triangle clearly commutes by functoriality, and the lower triangle can be seen to commute by applying the functor $\text{R}\underline{\text{Hom}}(-, \mathbb{G}_m)$ to the morphism of fiber sequences

$$\begin{array}{ccccc}
 \widehat{G} & \longrightarrow & G & \longrightarrow & [\widehat{G} \rightarrow G] \\
 \parallel & & \downarrow \psi & & \downarrow \\
 \widehat{G} & \longrightarrow & A & \longrightarrow & [\widehat{G} \rightarrow A],
 \end{array}$$

and taking long exact sequences in cohomology. Now, as in the proof of Proposition 3.1.5, there are two dashed morphisms making the diagram

$$\begin{array}{ccccc}
 K & \longrightarrow & [X \times \widehat{U}^* \rightarrow K] & \longrightarrow & L^D[1] \\
 \parallel & & \downarrow \text{dashed} & & \downarrow \\
 K & \longrightarrow & G^\natural & \longrightarrow & L^\vee
 \end{array}$$

commute: the comparison map of Proposition 3.1.5 and γ . The fact that they coincide follows from [Stacks, Tag 0FWZ]. \square

Recall that the linear part L of G decomposes as a product of a torus T , whose Cartier dual is denoted as X , and a vector group U . Inasmuch as $\underline{\text{Ext}}^1(L, \mathbb{G}_m) \simeq \underline{\text{Ext}}^1(U, \mathbb{G}_m)$ has no k -points, due to Propositions 2.3.1 and 2.3.16, this computation is particularly useful for obtaining concrete information about character sheaves.

COROLLARY 3.1.10. *The group of isomorphism classes $H_m^1(G_{\text{dR}}, \mathbb{G}_m)$ of character sheaves on G fits into the short exact sequence*

$$0 \rightarrow X \rightarrow (\Omega_G \times A^\natural(k))/\Omega_A \rightarrow H_m^1(G_{\text{dR}}, \mathbb{G}_m) \rightarrow 0,$$

where the morphism on the left is described in Remark 3.1.8, and the morphism on the right is as in Proposition 3.1.9. In particular, every character sheaf on G is of the form

$$(\mathcal{O}_G, d + \omega) \otimes_{\mathcal{O}_G} \psi^*(\mathcal{L}, \nabla),$$

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for some $\omega \in \Omega_G$ and $(\mathcal{L}, \nabla) \in A^\natural(k)$.

Proof. As explained above, the preceding proposition provides an isomorphism $(K/L^D)(k) \rightarrow H_m^1(G_{\text{dR}}, \mathbb{G}_m)$. The group of k -points $(K/L^D)(k)$ fits into the long exact sequence

$$0 \rightarrow X \rightarrow K(k) \rightarrow (K/L^D)(k) \rightarrow H^1(k, X) \times H^1(k, \widehat{U}^*),$$

where the term on the right vanishes due to [SGA 7_I, Exp. VIII, Prop. 5.1] and [Bha22, Rem. 2.2.18]. Finally, as noted in Footnote 1, $K(k)$ is isomorphic to the quotient $(\Omega_G \times A^\natural(k))/\Omega_A$, completing the proof. \square

3.2. THE MODULI OF CHARACTER SHEAVES

As discussed in the previous section, Laumon defined a dual $[\widehat{G} \rightarrow G]^\triangleright$ that, in a sense, eliminates the enigmatic object $U' = \underline{\text{Ext}}^1(U, \mathbb{G}_m)$ from the stacky Cartier dual $G^\natural = G_{\text{dR}}^\vee$. Now, according to Remark 3.1.8, the Laumon dual of G_{dR} is isomorphic to the quotient sheaf $K/(X \times \widehat{U}^*)$. Since \widehat{U}^* is a formal group, it cannot be representable. This leads us to the following definition.

DEFINITION 3.2.1 (Moduli space of character sheaves). We denote by G^\flat the abelian sheaf K/X , where $X \hookrightarrow K$ is the morphism that maps $\chi \in X$ to $[\omega, (\mathcal{L}, \nabla)]$, where ω is any element of Ω_G satisfying $\varphi^* \omega = d\chi/\chi$ and (\mathcal{L}, ∇) is the unique element of A^\natural satisfying $\psi^*(\mathcal{L}, \nabla) \simeq (\mathcal{O}_G, d - \omega)$.

Note that although the map $X \rightarrow K$ is a monomorphism, it is not an immersion. Consequently, it is unclear whether the quotient $G^\flat = K/X$ is a scheme, and it may indeed fail to be one. A brief verification shows that the morphism

$$\begin{aligned} k &\rightarrow H_m^1(\mathbb{G}_{m, \text{dR}}, \mathbb{G}_m) \\ \alpha &\mapsto (\mathcal{O}_{\mathbb{G}_m}, d - \alpha dx/x) \end{aligned}$$

is surjective and induces an isomorphism $H_m^1(\mathbb{G}_{m, \text{dR}}, \mathbb{G}_m) \simeq k/\mathbb{Z}$. Accordingly, the moduli space \mathbb{G}_m^\flat is isomorphic to \mathbb{G}_a/\mathbb{Z} , an algebraic space² that is not a scheme. Nonetheless, the main result of this section, Theorem 3.2.5, asserts in Corollary 3.2.6 that G^\flat is as well-behaved as one could hope for.

■ **REMARK 3.2.2.** Let S be a k -scheme. According to Propositions 2.2.6, 2.3.2, and 3.1.5, there are natural morphisms

$$G^\flat(S) \xrightarrow{(1)} G_{\text{dR}}^\triangleright(S) \xrightarrow{(2)} G^\natural(S) \xrightarrow{(3)} H_m^1(G_{\text{dR}} \times S, \mathbb{G}_m).$$

²We emphasize that, as in [Stacks, Tag 025Y] and contrarily to [LM00, Déf. 1.1], we *do not* suppose that algebraic spaces are quasi-separated.

Proposition 2.3.15 and Theorem 2.3.21 imply that the maps (1) and (3) are isomorphisms if S is reduced. Furthermore, Proposition 2.3.16 states that (2) is an isomorphism for seminormal k -schemes S . In particular, $G^b(k)$ is isomorphic to the group of isomorphism classes of character sheaves on G , thereby justifying its name. ■

■ REMARK 3.2.3. The assignment $G \mapsto G^b$ is functorial. Specifically, consider a morphism $G_1 \rightarrow G_2$ between commutative connected algebraic groups over a characteristic zero field. Since the Barsotti–Chevalley decomposition is functorial, we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & G_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_2 & \longrightarrow & G_2 & \longrightarrow & A_2 & \longrightarrow & 0, \end{array}$$

where A_i are abelian varieties and L_i are affine. The linear parts L_i further decompose into a product of tori T_i and unipotent groups U_i . By [Mil17, Cor. 14.18], the morphism $L_1 \rightarrow L_2$ is a product of morphisms $T_1 \rightarrow T_2$ and $U_1 \rightarrow U_2$. In particular, this induces a map $X_2 \rightarrow X_1$ between the character groups of T_2 and T_1 , respectively. The functoriality of the Laumon dual ensures that the square on the right of the diagram

$$\begin{array}{ccccc} X_2 & \longrightarrow & X_2 \times \widehat{U}_2^* & \longrightarrow & K_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_1 \times \widehat{U}_1^* & \longrightarrow & K_1 \end{array}$$

commutes. Consequently, the universal property of cokernels gives the desired map $G_2^b \rightarrow G_1^b$. ■

■ EXAMPLE 3.2.4. As usual, let T be a torus, U a unipotent group, and A be an abelian variety. We have that

$$T^b \simeq T^\natural \simeq \Omega_T/X \simeq \mathfrak{t}^*/X, \quad U^b \simeq \Omega_U \simeq U^*, \quad A^b \simeq A^\natural.$$

The first one is a group algebraic space, while the other two are algebraic groups. ■

Before going further, we note that there exists a large diagram relying many of the objects appearing in this chapter. Since both formal completions and the de Rham

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functor are exact, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \widehat{T} \times \widehat{U} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{A} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T \times U & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T_{\text{dR}} \times U_{\text{dR}} & \longrightarrow & G_{\text{dR}} & \longrightarrow & A_{\text{dR}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0,
 \end{array}$$

in which every column and row is exact. By applying the Cartier duality functor, and passing to the long exact sequences in cohomology, we obtain

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & G^{\text{D}} & \xrightarrow{\varphi^*} & X \times \widehat{U}^* \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega_A & \xrightarrow{\psi^*} & \Omega_G & \xrightarrow{\varphi^*} & \Omega_T \times \Omega_U \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A^{\natural} & \xrightarrow{\psi^*} & G^{\natural} & \xrightarrow{\varphi^*} & T^{\natural} \times U^{\natural} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & A' & \xrightarrow{\psi^*} & G' & \xrightarrow{\varphi^*} & U' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Here, every row (including the zigzag path) and every column is exact. The necessary computations and vanishing results have already been discussed in the previous chapter. We note that the morphisms in the columns have natural geometric interpretations, as given by

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G^{\text{D}} & \longrightarrow & \Omega_G & \longrightarrow & G^{\natural} \longrightarrow G' \longrightarrow 0 \\
 & & \chi & \longmapsto & d\chi/\chi & & (\mathcal{L}, \nabla) \longmapsto \mathcal{L} \\
 & & & & \omega & \longmapsto & (\mathcal{O}_G, d + \omega).
 \end{array}$$

We are now in position to state the main theorem of this section.

THEOREM 3.2.5. *Let G be a commutative connected algebraic group over a characteristic zero field. Write G as an extension of an abelian variety A by a product $T \times U$ of a torus T and a unipotent group U . Then the complex $0 \rightarrow A^b \rightarrow G^b \rightarrow T^b \times U^b \rightarrow 0$ is exact.*

Proof. Consider the map $K \rightarrow \Omega_T \times \Omega_U$ induced by $\varphi^*: \Omega_G \rightarrow \Omega_T \times \Omega_U$ and $0: A^b \rightarrow \Omega_T \times \Omega_U$. We claim that the composition $K \rightarrow \Omega_T \times \Omega_U \rightarrow \Omega_T/X \times \Omega_U$ descends to the quotient K/X . According to the universal property of the quotient, we need to verify that the composition

$$X \rightarrow X \times \widehat{U}^* \rightarrow K \rightarrow \Omega_T \times \Omega_U \rightarrow \Omega_T/X \times \Omega_U$$

is zero. Applying the functor $\text{R}\underline{\text{H}}\text{om}(-, \mathbb{G}_m)$ to the morphism of fiber sequences

$$\begin{array}{ccccc} L & \longrightarrow & [\widehat{L} \rightarrow L] & \longrightarrow & \widehat{L}[1] \\ \parallel & & \downarrow & & \downarrow \\ L & \longrightarrow & [\widehat{G} \rightarrow G] & \longrightarrow & [\widehat{G} \rightarrow A], \end{array}$$

and taking long exact sequences in cohomology, we find that the composition $X \times \widehat{U}^* \rightarrow K \rightarrow \Omega_T \times \Omega_U$ the familiar map appearing on Page 132. In particular, this composition is the product of $X \rightarrow \Omega_T$ and $\widehat{U}^* \rightarrow \Omega_U$. It follows that our large composition vanishes, and we obtain a map $K/X \rightarrow \Omega_T/X \times \Omega_U$.

Now, we have every morphism needed to consider the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & X & \xlongequal{\quad} & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^b & \longrightarrow & K & \longrightarrow & \Omega_T \times \Omega_U & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A^b & \longrightarrow & K/X & \longrightarrow & \Omega_T/X \times \Omega_U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

whose columns are clearly exact. Since the top row is also exact, by the nine-lemma, it suffices to prove that the middle row is exact. This holds by an application of the snake lemma in the pushout extension defining K . \square

The theorem above implies that G^b is a "coarse moduli space" for G^b , in the sense

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that G^b is represented by an algebraic space with the same k -points as G^b .³

COROLLARY 3.2.6. *The abelian sheaf G^b is representable by a smooth commutative connected group algebraic space. Moreover, it satisfies $\dim G \leq \dim G^b \leq 2 \dim G$, with equality on the left if and only if G is affine, and equality on the right if and only if G is proper.*

Proof. The discussion in Remark 3.1.3 indicates that G^b is representable by a smooth commutative connected group algebraic space of dimension $\dim T^b + \dim U^b + \dim A^b$. Given that

$$\begin{aligned} \dim T^b &= \dim \Omega_T/X = \dim \Omega_T = \dim T \\ \dim U^b &= \dim U^* = \dim U \\ \dim A^b &= \dim A^{\natural} = \dim \Omega_A + \dim A' = 2 \dim A, \end{aligned}$$

we find that $\dim G^b = \dim T + \dim U + 2 \dim A = \dim G + \dim A$, thus completing the proof. \square

The full subcategory of $\mathbf{Ab}((\text{Sch}/k)_{\text{fppf}})$ consisting of commutative connected algebraic groups is closed under extensions. Consequently, it inherits a structure of exact category. One might believe that this makes the functor $G \mapsto G^b$ exact. The following results provide some support for this conjecture.

PROPOSITION 3.2.7. *Let $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ be a short exact sequence of abelian varieties over a characteristic zero field. Then the complex $0 \rightarrow A_3^b \rightarrow A_2^b \rightarrow A_1^b \rightarrow 0$ is exact.*

Proof. By Proposition 1.1.19, the complex $0 \rightarrow A_{1,\text{dR}} \rightarrow A_{2,\text{dR}} \rightarrow A_{3,\text{dR}} \rightarrow 0$ is exact. The Cartier duality functor then induces the exact sequence

$$\underline{\text{Hom}}(A_{1,\text{dR}}, \mathbb{G}_m) = 0 \rightarrow A_3^{\natural} \rightarrow A_2^{\natural} \rightarrow A_1^{\natural} \rightarrow 0 = \underline{\text{Ext}}^2(A_{3,\text{dR}}, \mathbb{G}_m),$$

where the vanishing results are due to Proposition 2.2.6 and Corollary 2.4.9. \square

PROPOSITION 3.2.8. *Let $0 \rightarrow L_1 \rightarrow L_2 \rightarrow L_3 \rightarrow 0$ be a short exact sequence of linear commutative connected algebraic groups over a characteristic zero field. Then the complex $0 \rightarrow L_3^b \rightarrow L_2^b \rightarrow L_1^b \rightarrow 0$ is exact.*

Proof. First, we decompose the linear groups L_i as a product of tori T_i and unipotent groups U_i . The exactness of the formal completion functor induces the short exact sequence

$$0 \rightarrow \widehat{T}_1 \times \widehat{U}_1 \rightarrow \widehat{T}_2 \times \widehat{U}_2 \rightarrow \widehat{T}_3 \times \widehat{U}_3 \rightarrow 0.$$

³However, it remains unclear whether there is a natural morphism $G^{\natural} \rightarrow G^b$, or if such a morphism would satisfy the universal property.

Then, the Cartier duality functor gives rise to the exact sequence

$$0 \rightarrow \Omega_{T_3} \times \Omega_{U_3} \rightarrow \Omega_{T_2} \times \Omega_{U_2} \rightarrow \Omega_{T_1} \times \Omega_{U_1} \rightarrow 0 = \underline{\text{Ext}}^1(\widehat{T}_3 \times \widehat{U}_3, \mathbb{G}_m).$$

Here, we use Corollary 2.2.5 and Proposition 2.3.22. As in Remark 3.2.3, we have induced maps $X_3 \rightarrow X_2 \rightarrow X_1$ between the character groups of the tori T_i , fitting into the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{T_3} \times \Omega_{U_3} & \longrightarrow & \Omega_{T_2} \times \Omega_{U_2} & \longrightarrow & \Omega_{T_1} \times \Omega_{U_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{T_3}/X_3 \times \Omega_{U_3} & \longrightarrow & \Omega_{T_2}/X_2 \times \Omega_{U_2} & \longrightarrow & \Omega_{T_1}/X_1 \times \Omega_{U_1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0. \end{array}$$

The first row above is exact due to Proposition 2.3.1, and the nine lemma implies that the bottom row is also exact. \square

PROPOSITION 3.2.9. *Let $0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$ be a short exact sequence of commutative connected algebraic groups over a characteristic zero field. Then $G_3^b \rightarrow G_2^b$ is a monomorphism and $G_2^b \rightarrow G_1^b$ is an epimorphism.*

Proof. Consider the following commutative diagram, composed of the short exact sequence of commutative connected algebraic groups G_i , along with the induced maps on their Barsotti–Chevalley decompositions.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_1 \times U_1 & \longrightarrow & T_2 \times U_2 & \longrightarrow & T_3 \times U_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

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The exactness of Barsotti–Chevalley decompositions has been thoroughly examined by Brion in [Bri17b]. He proved that the complex $0 \rightarrow \mathcal{U}_1 \rightarrow \mathcal{U}_2 \rightarrow \mathcal{U}_3 \rightarrow 0$ is exact, but

$$0 \rightarrow \mathcal{T}_1 \rightarrow \mathcal{T}_2 \rightarrow \mathcal{T}_3 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \rightarrow \mathcal{A}_3 \rightarrow 0$$

might only be exact up to isogeny [Bri17b, Thm. 2.9, Lems. 4.3 and 4.7]. That being said, the map $\mathcal{T}_1 \rightarrow \mathcal{T}_2$ is a monomorphism, while $\mathcal{T}_2 \rightarrow \mathcal{T}_3$ and $\mathcal{A}_2 \rightarrow \mathcal{A}_3$ are epimorphisms.⁴

Next, we apply the functor $(-)^b$ to the diagram above, obtaining the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{A}_3^b & \longrightarrow & \mathcal{A}_2^b & \longrightarrow & \mathcal{A}_1^b \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{G}_3^b & \longrightarrow & \mathcal{G}_2^b & \longrightarrow & \mathcal{G}_1^b \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{T}_3^b \times \mathcal{U}_3^b & \longrightarrow & \mathcal{T}_2^b \times \mathcal{U}_2^b & \longrightarrow & \mathcal{T}_1^b \times \mathcal{U}_1^b \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0,
 \end{array}$$

whose rows are complexes and columns are exact. According to the preceding propositions, the maps $\mathcal{A}_3^b \rightarrow \mathcal{A}_2^b$ and $\mathcal{T}_3^b \times \mathcal{U}_3^b \rightarrow \mathcal{T}_2^b \times \mathcal{U}_2^b$ are monomorphisms. The long exact sequence in cohomology associated with the short exact sequence of complexes above implies that $\mathcal{G}_3^b \rightarrow \mathcal{G}_2^b$ is also a monomorphism.

We claim that $\mathcal{A}_2^b \rightarrow \mathcal{A}_1^b$ is an epimorphism. Let F be the kernel of $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ and B be its cokernel. Note that F is a finite group, while B is an abelian variety. Consider the short exact sequences

$$0 \rightarrow F \rightarrow \mathcal{A}_1 \rightarrow \mathcal{A}_1/F \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{A}_1/F \rightarrow \mathcal{A}_2 \rightarrow B \rightarrow 0.$$

By passing to the de Rham spaces and considering the long exact sequences induced by the Cartier duality functor, we obtain that the maps $\mathcal{A}_2^b \rightarrow (\mathcal{A}_1/F)^b$ and $(\mathcal{A}_1/F)^b \rightarrow \mathcal{A}_1^b$ are epimorphisms. Hence, so is their composition. Since $\mathcal{T}_2^b \times \mathcal{U}_2^b \rightarrow \mathcal{T}_1^b \times \mathcal{U}_1^b$ is an epimorphism, the same argument as above proves the same for $\mathcal{G}_2^b \rightarrow \mathcal{G}_1^b$. \square

As it turns out, the lack of exactness in the Barsotti–Chevalley decompositions is more than just a nuisance in the proof above. Inspired by [Bri17b, Rem. 4.6], we provide below a counter-example to our conjecture.

⁴Brion’s results apply in arbitrary characteristic, while the characteristic zero case required here is significantly simpler.

■ EXAMPLE 3.2.10. Assume the base field k contains a non-trivial p -th root of unit, for some prime p . Given an elliptic curve E containing a rational point of order p , we let μ_p act diagonally on the product $E \times \mathbb{G}_m$ and consider the quotient G . This group fits into the short exact sequence

$$0 \rightarrow E \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 0.$$

Taking the Barsotti–Chevalley decomposition of each group, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & 0 & \longrightarrow & \mathbb{G}_m & \xlongequal{\quad} & \mathbb{G}_m & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \parallel & & \\
 & & 0 & \longrightarrow & E & \longrightarrow & G & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\
 & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mu_p & \longrightarrow & E & \longrightarrow & E/\mu_p & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0, & & & &
 \end{array}$$

whose rows and columns are exact. Passing to the de Rham spaces on the bottom row and taking the long exact sequence in cohomology induced by the Cartier dual, we obtain the exact sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow (E/\mu_p)^b \rightarrow E^b \rightarrow 0.$$

The functor $(-)^b$ can be applied to the portion of the preceding diagram consisting of commutative connected algebraic groups. This yields the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & 0 & \longrightarrow & (E/\mu_p)^b & \longrightarrow & E^b & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & \mathbb{G}_m^b & \longrightarrow & G^b & \longrightarrow & E^b & \longrightarrow & 0 & \\
 & & \parallel & & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & \mathbb{G}_m^b & \xlongequal{\quad} & \mathbb{G}_m^b & \longrightarrow & 0 & & & \\
 & & \downarrow & & \downarrow & & & & & \\
 & & 0 & & 0, & & & & &
 \end{array}$$

3. Moduli of character sheaves

where the columns are exact and the rows are complexes. The long exact sequence in cohomology induced from a short exact sequence of complexes then identifies the middle cohomology of

$$0 \rightarrow \mathbb{G}_m^b \rightarrow G^b \rightarrow E^b \rightarrow 0$$

with $\mathbb{Z}/p\mathbb{Z}$, thereby proving that the functor $(-)^b$ is not exact. ■

3.3. LINEAR AND GENERIC SUBSPACES OF THE MODULI SPACE

The overarching goal of this thesis is to study generic vanishing theorems, stating that the de Rham cohomology of twists by character sheaves vanish for character sheaves on a large subset of the moduli space G^b . The complements of these subsets consist of finite unions of subsets arising from quotients of G . This section focuses on their study.

DEFINITION 3.3.1 (Linear subspace). Let G be a commutative connected algebraic group over a characteristic zero field. For an epimorphism $\rho: G \twoheadrightarrow \tilde{G}$ with connected kernel, the image of $\rho^b: \tilde{G}^b \hookrightarrow G^b$ is said to be a *linear subspace* of G^b .

The following remark concretely characterizes linear subspaces of G^b when G is affine. Furthermore, when G is an abelian variety, this notion is related to existing concepts in the literature.

■ **REMARK 3.3.2.** Let $G \twoheadrightarrow \tilde{G}$ be an epimorphism between commutative connected algebraic groups with connected kernel N . If G is linear, unipotent, a torus, or an abelian variety, then \tilde{G} and N inherit the same properties. The following characterizations follow from this observation.

Consider a torus T with character group X . A linear subspace of $T^b \simeq \Omega_T/X$ is of the form V/Y , where Y is a subgroup of X and V is the linear subspace of Ω_T generated by Y , in the sense of linear algebra. For a unipotent group U , a linear subspace of $U^b = U^*$ corresponds to a linear subspace of the underlying vector space of U^* , in the sense of linear algebra.

Let L be a linear commutative connected algebraic group, and decompose it as a product of a torus T and a unipotent group U . If $\rho: L \twoheadrightarrow \tilde{L}$ is an epimorphism with connected kernel, then \tilde{L} is also linear and decomposes as $\tilde{L} \simeq \tilde{T} \times \tilde{U}$. Moreover, ρ is a product of epimorphisms $T \twoheadrightarrow \tilde{T}$ and $U \twoheadrightarrow \tilde{U}$. Thus, a linear subspace of L^b is the product of linear subspaces of T^b and U^b .

For an abelian variety A , linear subspaces of $A^b \simeq A^\natural$ were first studied by Simpson, who termed them *triple tori* [Sim93, p. 365]. Schnell refers to translates of linear subspaces of A^b as *linear subvarieties* [Sch15, Def. 2.3]. ■

3.3. Linear and generic subspaces of the moduli space

PROPOSITION 3.3.3. *Let G be a commutative connected algebraic group over a characteristic zero field, and express G as an extension of an abelian variety A by a linear group L . If $V \subset L^b$ is a (translate of a) linear subspace, so is its inverse image by $G^b \rightarrow L^b$.*

Proof. Let $L \twoheadrightarrow \tilde{L}$ be the epimorphism defining the linear subspace V , and let N be its kernel. Denote by \tilde{G} the quotient of G by N ; we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & \tilde{L} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \tilde{G} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & A & \xlongequal{\quad} & A \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0,
 \end{array}$$

whose rows and columns are exact. Applying the functor $(-)^b$, we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A^b & \xlongequal{\quad} & A^b & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \tilde{G}^b & \longrightarrow & G^b & \longrightarrow & N^b \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \tilde{L}^b & \longrightarrow & L^b & \longrightarrow & N^b \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

According to Theorem 3.2.5 and Proposition 3.2.8, its columns are exact, as well as the top and bottom rows. Consequently, the middle row is exact as well. A diagram chase shows that the square

$$\begin{array}{ccc}
 \tilde{G}^b & \longrightarrow & G^b \\
 \downarrow & & \downarrow \\
 \tilde{L}^b & \longrightarrow & L^b
 \end{array}$$

is cartesian, thereby concluding the proof. \square

3. Moduli of character sheaves

The following definition makes precise the *large* subsets of the moduli space on which generic vanishing is expected to hold.

DEFINITION 3.3.4 (Generic subspace). Let G be a commutative connected algebraic group over a characteristic zero field. A *generic subspace* of G^b is the complement of a finite union of translates of linear subspaces of G^b with positive codimension.

In algebraic geometry, a property is said to hold generically if it holds on an open dense subset. For extensions of abelian varieties by unipotent groups, this can be related with our definition above.

PROPOSITION 3.3.5. *Let G be a commutative algebraic group over a characteristic zero field that is an extension of an abelian variety by a unipotent group. If V is a generic subspace of G^b , then V is open and dense in G^b .*

Proof. Since the intersection of a finite number of open dense subsets is also open and dense, it suffices to prove that the complement of a linear subspace with positive codimension is open and dense. Let $G \twoheadrightarrow \tilde{G}$ be an epimorphism with connected kernel N defining the linear subspace \tilde{G}^b of G^b .

First, we claim that \tilde{G} is also an extension of an abelian variety by a unipotent group. Denote by U the maximal unipotent subgroup of G and by A the quotient G/U . The universal property of quotients yields a morphism $N/(U \cap N) \rightarrow A$ making the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & U \cap N & \longrightarrow & N & \longrightarrow & N/(U \cap N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & U & \longrightarrow & G & \longrightarrow & A \longrightarrow 0
 \end{array}$$

commute. A quick diagram chase shows that $N/(U \cap N) \rightarrow A$ is a monomorphism. The usual isomorphism theorems then imply that $(G/N)/(U/(U \cap N))$ is isomorphic to $(G/U)/(N/(U \cap N))$. Denoting by B this common group, we may complete the

diagram above to

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{U} \cap \mathbf{N} & \longrightarrow & \mathbf{N} & \longrightarrow & \mathbf{N}/(\mathbf{U} \cap \mathbf{N}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{G} & \longrightarrow & \mathbf{A} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{U}/(\mathbf{U} \cap \mathbf{N}) & \longrightarrow & \tilde{\mathbf{G}} & \longrightarrow & \mathbf{B} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Here every column and every row is exact. Since \mathbf{B} is a quotient of an abelian variety, it is also an abelian variety. Similarly, $\mathbf{U}/(\mathbf{U} \cap \mathbf{N})$ is a unipotent group.

Recall from Corollary 3.2.6 that \mathbf{G}^b and $\tilde{\mathbf{G}}^b$ are commutative connected algebraic groups. Since algebraic groups in characteristic zero are smooth, \mathbf{G}^b is irreducible [Stacks, Tag 047N].

According to [Mil17, Thm. 5.34], the map $\tilde{\mathbf{G}}^b \rightarrow \mathbf{G}^b$ is a closed immersion and so its complement is open. Since $\tilde{\mathbf{G}}^b$ is assumed to be of positive codimension, its complement is non-empty. The irreducibility of \mathbf{G}^b then implies that it is dense, thus completing the proof. \square

The preceding result does not hold for general commutative connected algebraic groups. The Zariski topology of $\mathbb{G}_m^b \simeq \mathbb{A}^1/\mathbb{Z}$ is too coarse. Indeed, the only open dense subset of \mathbb{G}_m^b is the entire space itself. This suggests the following definition.

DEFINITION 3.3.6 (Analytic moduli space). Let \mathbf{G} be a commutative connected algebraic group over \mathbb{C} . Taking the \mathbb{C} -points of the morphism $X \hookrightarrow K$ defining the moduli space \mathbf{G}^b , we obtain a monomorphism of abelian groups $X \hookrightarrow K(\mathbb{C})$. Equip X with the discrete topology and $K(\mathbb{C})$ with the analytic topology. The quotient $K(\mathbb{C})/X$ is said to be the *analytic moduli space* and is denoted by \mathbf{G}_{an}^b .

Interestingly, although \mathbf{G}^b is often not representable by a scheme, \mathbf{G}_{an}^b is a complex manifold. Furthermore, for an abelian variety \mathbf{A} , the complex manifold \mathbf{A}_{an}^b is Stein, even though \mathbf{A}^b is not affine. The moduli space \mathbf{A}^b is even *anti-affine*, meaning that every morphism of algebraic varieties $\mathbf{A}^b \rightarrow \mathbb{A}^1$ is constant [Bri17a, Prop. 5.5.8].

For a generic subspace V of \mathbf{G}^b , we denote by V_{an} the set $V(\mathbb{C})$ with the subspace topology inherited from \mathbf{G}_{an}^b . An analytic analogue of Proposition 3.3.5 holds in all generality.

3. Moduli of character sheaves

PROPOSITION 3.3.7. *Let G be a commutative connected algebraic group over \mathbb{C} . If V is a generic subspace of G^b , then V_{an} is an open subset of G_{an}^b whose complement has measure zero. In particular, it is an open dense subset.*

Proof. Let $G \twoheadrightarrow \tilde{G}$ be an epimorphism with connected kernel N defining a linear subspace \tilde{G}^b of G^b with positive codimension. As in the proof of Proposition 3.3.5, it suffices to prove that \tilde{G}_{an}^b is a closed subset of G_{an}^b with measure zero.

According to the functoriality of $(-)^b$, explained in Remark 3.2.3, the natural map $\tilde{G}_{\text{an}}^b \hookrightarrow G_{\text{an}}^b$ is induced by a morphism of schemes $\tilde{K} \rightarrow K$. Consequently, \tilde{G}_{an}^b is a complex Lie subgroup of G_{an}^b . It follows that $\tilde{G}_{\text{an}}^b \hookrightarrow G_{\text{an}}^b$ is a closed immersion [Bou06, Chap. III, §1.3.5]. Since we assumed that $\dim \tilde{G}^b < \dim G^b$, Sard's theorem implies that \tilde{G}_{an}^b has measure zero in G_{an}^b . \square

DEFINITION 3.3.8 (Character variety). Let G be a commutative connected algebraic group over \mathbb{C} . The *character variety* of G , denoted $\text{Char}(G)$, is the spectrum of the group algebra $\mathbb{C}[\pi_1(G_{\text{an}})]$.

Recall that the group algebra functor $\text{Grp} \rightarrow \text{Alg}(\mathbb{C})$ is left adjoint to the functor sending a \mathbb{C} -algebra R to its group of units R^\times . Thus, the \mathbb{C} -points of $\text{Char}(G)$ correspond to characters $\pi_1(G_{\text{an}}) \rightarrow \mathbb{C}^\times$ of the fundamental group $\pi_1(G_{\text{an}})$. In other words, $\text{Char}(G)$ is a moduli space for rank one local systems on G_{an} . The Hopf algebra structure in $\mathbb{C}[\pi_1(G_{\text{an}})]$ induces a structure of algebraic group in $\text{Char}(G)$, encoding the tensor product of local systems.

The Riemann–Hilbert equivalence states that local systems on G_{an} correspond to *regular* connections on G . Since all non-trivial character sheaves on unipotent groups are *irregular*, $\text{Char}(G)$ cannot parametrize character sheaves in general. On the other hand, character sheaves on semiabelian varieties are *regular*, allowing such a comparison.

PROPOSITION 3.3.9. *Let G be a semiabelian variety over \mathbb{C} . Then the complex Lie groups G_{an}^b and $\text{Char}(G)_{\text{an}}$ are isomorphic.*

Proof. Write G as an extension of an abelian variety A by a torus T . By analytification, we obtain a short exact sequence

$$0 \rightarrow T_{\text{an}} \rightarrow G_{\text{an}} \rightarrow A_{\text{an}} \rightarrow 0$$

of complex Lie groups. In particular, $G_{\text{an}} \rightarrow A_{\text{an}}$ is a fibration with fiber T_{an} . Considering the associated long exact sequence in homotopy, we have

$$0 = \pi_2(A_{\text{an}}) \rightarrow \pi_1(T_{\text{an}}) \rightarrow \pi_1(G_{\text{an}}) \rightarrow \pi_1(A_{\text{an}}) \rightarrow \pi_0(T_{\text{an}}) = 0.$$

Here, $\pi_0(T_{\text{an}})$ vanishes since T_{an} is connected, and $\pi_2(A_{\text{an}})$ vanishes since A_{an} is a Lie

group. This induces a short exact sequence of Hopf algebras

$$\mathbb{C} \rightarrow \mathbb{C}[\pi_1(\mathbf{T}_{\text{an}})] \rightarrow \mathbb{C}[\pi_1(\mathbf{G}_{\text{an}})] \rightarrow \mathbb{C}[\pi_1(\mathbf{A}_{\text{an}})] \rightarrow \mathbb{C},$$

followed by a short exact sequence of affine algebraic groups

$$0 \rightarrow \text{Char}(\mathbf{A}) \rightarrow \text{Char}(\mathbf{G}) \rightarrow \text{Char}(\mathbf{T}) \rightarrow 0.$$

There is a natural holomorphic morphism $\mathbf{G}_{\text{an}}^b \rightarrow \text{Char}(\mathbf{G})_{\text{an}}$ sending a character sheaf (\mathcal{L}, ∇) to the local system $\ker \nabla_{\text{an}}$. This is compatible with inverse images and tensor products, giving rise to the following commutative diagram of complex Lie groups.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A}_{\text{an}}^b & \longrightarrow & \mathbf{G}_{\text{an}}^b & \longrightarrow & \mathbf{T}_{\text{an}}^b & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Char}(\mathbf{A})_{\text{an}} & \longrightarrow & \text{Char}(\mathbf{G})_{\text{an}} & \longrightarrow & \text{Char}(\mathbf{T})_{\text{an}} & \longrightarrow & 0 \end{array}$$

Since character sheaves on \mathbf{A} and \mathbf{T} are regular, the Riemann–Hilbert correspondence implies that the morphisms $\mathbf{A}_{\text{an}}^b \rightarrow \text{Char}(\mathbf{A})_{\text{an}}$ and $\mathbf{T}_{\text{an}}^b \rightarrow \text{Char}(\mathbf{T})_{\text{an}}$ are injective. The fact that a line bundle with integrable connection on \mathbf{A} is automatically a character sheaf further says that $\mathbf{A}_{\text{an}}^b \rightarrow \text{Char}(\mathbf{A})_{\text{an}}$ is surjective.

To prove that $\mathbf{T}_{\text{an}}^b \rightarrow \text{Char}(\mathbf{T})_{\text{an}}$ is surjective, we may suppose that $\mathbf{T} = \mathbb{G}_m$. A line bundle with integrable connection on \mathbb{G}_m is of the form $(\mathcal{O}_{\mathbb{G}_m}, d + f(x) dx/x)$, for some $f \in k[x, x^{-1}]$. This connection is regular precisely if f is constant. In other words, regular line bundles with integrable connection on \mathbb{G}_m are the same as character sheaves. This proves that $(\mathbb{G}_m)_{\text{an}}^b \rightarrow \text{Char}(\mathbb{G}_m)_{\text{an}}$ is surjective. \square

COROLLARY 3.3.10. *Let \mathbf{G} be a semiabelian variety over \mathbb{C} . Then every character sheaf on \mathbf{G} is regular.*

4. FOURIER TRANSFORMS

In this chapter, we present a Fourier transform mapping quasi-coherent sheaves on a commutative group stack \mathcal{G} to quasi-coherent sheaves on its stacky Cartier dual \mathcal{G}^\vee . This construction mirrors the role of the classical Fourier transform in harmonic analysis, providing a powerful tool for translating problems about quasi-coherent sheaves on \mathcal{G} into equivalent problems on \mathcal{G}^\vee , where they may become more tractable.

FUNCTION-SHEAF DICTIONARY

Let ℓ be a prime number, and let X be an algebraic variety over a finite field k of characteristic different from ℓ . Denote by Frob_k the geometric Frobenius automorphism. For any object M in the derived category $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ of constructible ℓ -adic sheaves, we define a function $X(k) \rightarrow \overline{\mathbb{Q}}_\ell$ as

$$x \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(\text{Frob}_k, \mathcal{H}^i(M)_{\bar{x}}),$$

whose values are independent of a choice of geometric point \bar{x} above x . This is the so-called *trace function* tr_M of M .

The assignment $M \mapsto \text{tr}_M$ establishes a dictionary between objects of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and functions on $X(k)$. For a map $f: X \rightarrow Y$ of algebraic varieties over k , the table below illustrates how certain operations on sheaves translate into operations on trace functions.

Object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ or $D_c^b(Y, \overline{\mathbb{Q}}_\ell)$	Associated trace function
$M_1 \otimes M_2$	$\text{tr}_{M_1} \cdot \text{tr}_{M_2}$
f^*N	$\text{tr}_N \circ f$
$Rf_!M$	$y \mapsto \sum_{f(x)=y} \text{tr}_M(x)$

For each character χ of k , there exists an ℓ -adic local system \mathcal{L}_χ on \mathbb{A}_k^1 whose trace function is χ . This local system is commonly referred to as the *Artin–Schreier sheaf*. Recall that, once a character χ is fixed, any other character of k can be expressed as $x \mapsto \chi(tx)$ for some unique $t \in k$. As a result, the Fourier transform of a function f on k is defined by

$$\hat{f}: t \mapsto \sum_{x \in k} f(x)\chi(tx).$$

4. Fourier transforms

Every operation above has a corresponding sheaf-theoretic analogue. Indeed, when f is the trace function of a complex M on \mathbb{A}^1 , the Fourier transform \hat{f} is the trace function of the complex $R\mathrm{pr}_{2,!}(\mathrm{pr}_1^* M \otimes \sigma^* \mathcal{L}_\chi)$. Here, $\sigma: \mathbb{A}_k^1 \times \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ is the multiplication map, whereas $\mathrm{pr}_1, \mathrm{pr}_2: \mathbb{A}_k^1 \times \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ are the natural projections.

The functor $R\mathrm{pr}_{2,!}(\mathrm{pr}_1^* - \otimes \sigma^* \mathcal{L}_\chi)$ was originally introduced by Deligne in a letter to Kazhdan [FFK23, App. D], where he explored the fundamental results of harmonic analysis within this setting. This construction has since become a cornerstone in ℓ -adic cohomology, inspiring numerous adaptations across other cohomology theories [Muk81; Dai00; Lau03; Noo04].

FOURIER TRANSFORM ON COMMUTATIVE GROUP STACKS

Let \mathcal{G} be a commutative group stack. Recall that the points of \mathcal{G}^\vee correspond to multiplicative line bundles on \mathcal{G} . Just as $(t, x) \mapsto \chi(tx)$ is the universal character on k , the evaluation map $\mathcal{G} \times \mathcal{G}^\vee \rightarrow \mathrm{BG}_m$ gives rise to the universal multiplicative line bundle $\mathcal{P}_\mathcal{G}$. For each point $\alpha \in \mathcal{G}^\vee$, the restriction of $\mathcal{P}_\mathcal{G}$ to $\mathcal{G} \simeq \mathcal{G} \times \{\alpha\}$ is isomorphic to the multiplicative line bundle \mathcal{L}_α corresponding to α .

This leads to the definition of a Fourier transform functor $\mathrm{FT}_\mathcal{G}: D_{\mathrm{qc}}(\mathcal{G}) \rightarrow D_{\mathrm{qc}}(\mathcal{G}^\vee)$, sending M to $\mathrm{pr}_{2,!}(\mathrm{pr}_1^* M \otimes \mathcal{P}_\mathcal{G})$, where pr_1 and pr_2 are the natural projections from $\mathcal{G} \times \mathcal{G}^\vee$. The functors appearing in this definition come from the six-functor formalism for quasi-coherent sheaves on derived stacks developed in Section 1.3. To reassure the reader, we note that the functor $\mathrm{pr}_{2,!}$ coincides with $\mathrm{pr}_{2,*}$ in most situations of interest.

In the case of an abelian scheme, this operation is known in the literature under the name *Fourier–Mukai transform*. Having the aforementioned six-functor formalism at our disposal, the main properties of the Fourier transform on commutative group stacks can be established in a manner analogous to the classical proofs for the Fourier–Mukai transform. This is explored in detail in Sections 4.1 and 4.2.

One particularly significant property of the Fourier transform deserves special emphasis. For simplicity, consider the case where the commutative group stack \mathcal{G} is defined over a ring R . For an object M of $D_{\mathrm{qc}}(\mathcal{G})$, Proposition 4.1.8 provides isomorphisms

$$R\Gamma_c(\mathcal{G}, M \otimes \mathcal{L}_\alpha) \simeq \alpha^* \mathrm{FT}_\mathcal{G}(M),$$

where \mathcal{L}_α is the multiplicative line bundle determined by a point $\alpha \in \mathcal{G}^\vee(R)$. In other words, the complex $\mathrm{FT}_\mathcal{G}(M)$ encapsulates the cohomology groups of all twists of M by multiplicative line bundles.

THE FOURIER–LAUMON TRANSFORM

Let \mathcal{G} be a generalized 1-motive over a field k of characteristic zero. In Section 3.1, we constructed its Laumon dual $\mathcal{G}^\triangleright$ and established a comparison morphism $\mathcal{G}^\triangleright \rightarrow \mathcal{G}^\vee$.

For the reader's convenience, we recall that Corollary 3.1.6 states that this map is an isomorphism precisely when \mathcal{G} has no "unipotent part".

We define the *Fourier–Laumon transform* $LT_{\mathcal{G}}: D_{qc}(\mathcal{G}) \rightarrow D_{qc}(\mathcal{G}^{\triangleright})$ as the pullback of $FT_{\mathcal{G}}: D_{qc}(\mathcal{G}) \rightarrow D_{qc}(\mathcal{G}^{\vee})$ by the comparison map $\mathcal{G}^{\triangleright} \rightarrow \mathcal{G}^{\vee}$. We believe this approach offers a highly conceptual perspective on the Fourier transform originally defined by Laumon in [Lau96], significantly simplifying its definition. This viewpoint also demonstrates that the Fourier–Laumon transform inherits most of the key properties of the Fourier transform on commutative group stacks.

In this thesis, the most significant example of a generalized 1-motive is the de Rham space G_{dR} associated with a commutative connected algebraic group G . Its Fourier–Laumon transform $D_{qc}(\mathcal{D}_G) \rightarrow D_{qc}(G_{dR}^{\triangleright})$ extends several classical operations for \mathcal{D} -modules, including the Fourier transform on unipotent groups, the Mellin transform on tori, and the Fourier–Mukai transform on abelian varieties. For a more detailed discussion, see Section 4.3.

4.1. FOURIER TRANSFORM ON COMMUTATIVE GROUP STACKS

Throughout this chapter, we will systematically employ the six-functor formalism for quasi-coherent sheaves on derived stacks, as established in Corollary 1.3.23. A (classical) stack is viewed as a derived stack whose classical truncation, given by the composition $\text{Ring} \hookrightarrow \text{An}(\text{Ring}) \rightarrow \text{An}$, takes values in the 2-category Grpd of groupoids. Henceforth, we fix a base scheme S and assume, unless otherwise specified, that all stacks are over S .



In this chapter, every functor is implicitly derived, including fiber products, which are to be computed in the ∞ -category $d\text{Stk}$ of derived stacks. It is worth noting that all results also extend to classical fiber products under appropriate tor-independence conditions.

The following definition introduces a class of stacks to which the results of this chapter are applicable. According to Definition 1.2.17 and Proposition 1.3.22, any stack \mathcal{X} with an fppf cover by a scheme lies in this class. Notably, this includes algebraic stacks. Similarly, Proposition 1.4.2 implies that de Rham spaces also lie in this class.

DEFINITION 4.1.1. Let \mathcal{X} be a stack. We say that \mathcal{X} is *!-able* if its structure map $p: \mathcal{X} \rightarrow S$ has $!$ -functors.

The *raison d'être* of this condition, in the setting of this chapter, is that \mathcal{X} being $!$ -able allow us to consider the $!$ -direct image of the projection $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$. In many cases, such as for quasi-compact quasi-separated schemes, the $!$ -direct image coincides with the usual direct image. But it is the correct functor to consider for more general stacks,

4. Fourier transforms

such as de Rham spaces.

A myriad of interesting operators in analysis arise as *integral transforms*. For a function K on a product space $X \times Y$, called a *kernel*, those act as

$$f \mapsto \left(y \mapsto \int f(x)K(x, y) dx \right).$$

In words, the integral transform above takes a function f on X , treats it as a two-variable function on $X \times Y$, multiplies it by K , and then integrates over X . Each step has a sheaf-theoretic analogue, motivating the definition below.

DEFINITION 4.1.2 (Integral transform). Let \mathcal{X} and \mathcal{Y} be $!$ -able stacks together with an object K of $D_{qc}(\mathcal{X} \times \mathcal{Y})$. We say that the functor

$$\begin{aligned} D_{qc}(\mathcal{X}) &\rightarrow D_{qc}(\mathcal{Y}) \\ M &\mapsto \text{pr}_{2,!}(\text{pr}_1^* M \otimes K), \end{aligned}$$

where pr_1 and pr_2 are the natural projections from $\mathcal{X} \times \mathcal{Y}$, is the *integral transform with kernel* K .

The most important property of those operations, in the context of this chapter, is the fact that compositions of integral transforms are also integral transforms. This result, in the generality below, can be proven just as in [GW23, Prop. 27.241].

PROPOSITION 4.1.3. Let \mathcal{X} , \mathcal{Y} , and \mathcal{Z} be $!$ -able stacks together with objects K_1 of $D_{qc}(\mathcal{X} \times \mathcal{Y})$ and K_2 of $D_{qc}(\mathcal{Y} \times \mathcal{Z})$. Then the composition of integral transforms

$$D_{qc}(\mathcal{X}) \rightarrow D_{qc}(\mathcal{Y}) \rightarrow D_{qc}(\mathcal{Z})$$

is isomorphic to the integral transform with kernel

$$K_1 \circ K_2 := \text{pr}_{13,!}(\text{pr}_{12}^* K_1 \otimes \text{pr}_{23}^* K_2),$$

where pr_{ij} is the projection from $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ into the ij -th factors.

We will often use this result in the following form.

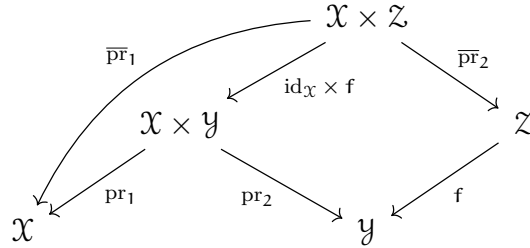
PROPOSITION 4.1.4. Let \mathcal{X} , \mathcal{Y} , and \mathcal{Z} be $!$ -able stacks together with an object K of $D_{qc}(\mathcal{X} \times \mathcal{Y})$ and a map $f: \mathcal{Z} \rightarrow \mathcal{Y}$. Then the composition

$$D_{qc}(\mathcal{X}) \rightarrow D_{qc}(\mathcal{Y}) \xrightarrow{f^*} D_{qc}(\mathcal{Z}),$$

where the functor on the left is the integral transform with kernel K , is isomorphic to the integral transform with kernel $(\text{id}_{\mathcal{X}} \times f)^* K$.

Since the integral transform with kernel $\Gamma_{f,!}\mathcal{O}_Z$ is isomorphic to the inverse image functor $f^*: D_{qc}(\mathcal{Y}) \rightarrow D_{qc}(\mathcal{Z})$, the following result is actually a corollary of the previous one. That being said, it is easier to just prove it directly.¹

Proof of Proposition 4.1.4. An application of base change in the diagram



yields the isomorphism $f^* pr_{2,!}(pr_1^* - \otimes K) \simeq \bar{pr}_{2,!}(\bar{pr}_1^* - \otimes (id_X \times f)^*K)$. □

Let \mathcal{G} be a commutative group stack. Its stacky Cartier dual $\mathcal{G}^\vee := \underline{\text{Hom}}(\mathcal{G}, \text{BG}_m)$, as studied in Chapter 2, parametrizes multiplicative line bundles on \mathcal{G} . Naturally, if \mathcal{G} is flat², there is a universal such line bundle on the product $\mathcal{G} \times \mathcal{G}^\vee$. This is the kernel for the most important kind of integral transform.

DEFINITION 4.1.5 (Fourier transform). Let \mathcal{G} be a !-able flat commutative group stack with stacky Cartier dual \mathcal{G}^\vee . The evaluation map $\mathcal{G} \times \mathcal{G}^\vee \rightarrow \text{BG}_m$ defines the *Poincaré line bundle* $\mathcal{P}_{\mathcal{G}}$ on $\mathcal{G} \times \mathcal{G}^\vee$. We define the *Fourier transform* functor $\text{FT}_{\mathcal{G}}: D_{qc}(\mathcal{G}) \rightarrow D_{qc}(\mathcal{G}^\vee)$ as the integral transform with kernel $\mathcal{P}_{\mathcal{G}}$.

We now present an interesting example of a Fourier transform over a general base scheme S . In the next sections, we will see more examples after specializing to a base field.

■ **EXAMPLE 4.1.6.** Let A be an abelian scheme over S . Since its Cartier dual $A^D = \underline{\text{Hom}}(A, \mathbb{G}_m)$ vanishes, the Barsotti–Weil formula (Corollary 2.3.13) implies that A^\vee is isomorphic to the dual abelian scheme A' . The associated Fourier transform $D_{qc}(A) \rightarrow D_{qc}(A')$ has been widely studied in the literature under the name *Fourier–Mukai transform*. See [GW23, §27.45] for its main properties.

The stacky Cartier dual of BA vanishes, making the associated Fourier transform rather trivial; it reduces to the functor $p_!: D_{qc}(BA) \rightarrow D_{qc}(S)$, where $p: BA \rightarrow S$ denotes the structure map. Furthermore, we note that p is cohomologically proper, as the pullback $A \rightarrow S$ by the effective epimorphism $S \rightarrow BA$ has this property. As a result, the forget-supports map $p_! \rightarrow p_*$ is an isomorphism. ■

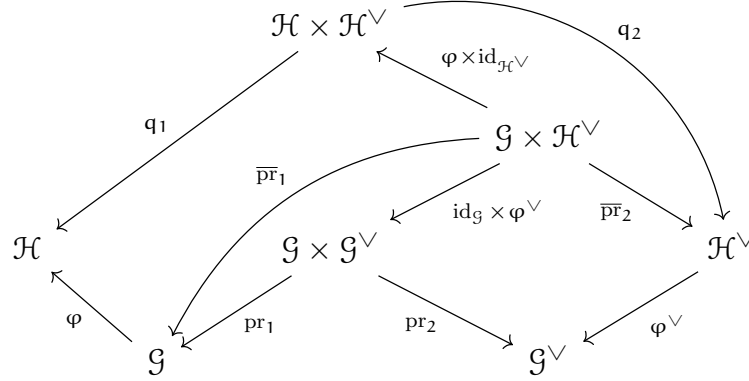
¹Furthermore, the direct proof does not require Γ_f to have !-functors and needs fewer tor-independence hypotheses when working with classical products.

²We say that \mathcal{G} is flat if the derived product $\mathcal{G} \times \mathcal{X}$ agrees with its classical truncation for all stacks \mathcal{X} .

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PROPOSITION 4.1.7. *Let $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of !-able flat commutative group stacks with dual $\varphi^\vee: \mathcal{H}^\vee \rightarrow \mathcal{G}^\vee$. Suppose that φ and φ^\vee have !-functors. Then the functors $\varphi^{\vee,*} \circ \mathrm{FT}_{\mathcal{G}}$ and $\mathrm{FT}_{\mathcal{H}} \circ \varphi_!$ are isomorphic. Similarly, the functors $\varphi_! \circ \mathrm{FT}_{\mathcal{H}}$ and $\mathrm{FT}_{\mathcal{G}} \circ \varphi^*$ are isomorphic.*

Proof. The Proposition 4.1.4 gives that $\varphi^{\vee,*} \circ \mathrm{FT}_{\mathcal{G}} \simeq \overline{\mathrm{pr}}_{2,!}(\overline{\mathrm{pr}}_1^* - \otimes (\mathrm{id}_{\mathcal{G}} \times \varphi^\vee)^* \mathcal{P}_{\mathcal{G}})$, where the maps are as in the diagram below.



By definition of the dual morphism, the two maps below coincide

$$\begin{aligned} \mathcal{G} \times \mathcal{H}^\vee &\xrightarrow{\mathrm{id}_{\mathcal{G}} \times \varphi^\vee} \mathcal{G} \times \mathcal{G}^\vee \rightarrow \mathrm{BG}_m \\ \mathcal{G} \times \mathcal{H}^\vee &\xrightarrow{\varphi \times \mathrm{id}_{\mathcal{H}^\vee}} \mathcal{H} \times \mathcal{H}^\vee \rightarrow \mathrm{BG}_m, \end{aligned}$$

and so $(\varphi \times \mathrm{id}_{\mathcal{H}^\vee})^* \mathcal{P}_{\mathcal{H}} \simeq (\mathrm{id}_{\mathcal{G}} \times \varphi^\vee)^* \mathcal{P}_{\mathcal{G}}$. The commutativity of the diagram above, together with the projection formula implies that

$$\begin{aligned} \varphi^{\vee,*} \circ \mathrm{FT}_{\mathcal{G}} &\simeq q_{2,!}(\varphi \times \mathrm{id}_{\mathcal{H}^\vee})_!(\overline{\mathrm{pr}}_1^* - \otimes (\varphi \times \mathrm{id}_{\mathcal{H}^\vee})^* \mathcal{P}_{\mathcal{H}}) \\ &\simeq q_{2,!}((\varphi \times \mathrm{id}_{\mathcal{H}^\vee})_! \overline{\mathrm{pr}}_1^* - \otimes \mathcal{P}_{\mathcal{H}}). \end{aligned}$$

The first natural isomorphism then follows by another application of base change. A dual version of the same arguments yields the desired isomorphism between $\varphi_! \circ \mathrm{FT}_{\mathcal{H}}$ and $\mathrm{FT}_{\mathcal{G}} \circ \varphi^*$. \square

The following result follows directly from Proposition 4.1.4. In the particular case where T and S are both the spectrum of a ring R , it says that the Fourier transform $\mathrm{FT}_{\mathcal{G}}(M)$ of an object M in $D_{\mathrm{qc}}(\mathcal{G})$ contains all the data of the compactly supported³ cohomology groups $H_c^*(\mathcal{G}, M \otimes \mathcal{L}_\alpha)$ for all $\alpha \in \mathcal{G}^\vee(R)$. Here, \mathcal{L}_α denotes the line bundle on \mathcal{G} associated with α .

³We say compactly supported cohomology because it is defined as a !-direct image. In most cases of interest this coincides with usual cohomology, though.

PROPOSITION 4.1.8. *Let \mathcal{G} be a !-able flat commutative group stack with stacky Cartier dual \mathcal{G}^\vee . Consider a S -scheme T and the natural projections $\mathrm{pr}_{\mathcal{G}}: \mathcal{G}_T \rightarrow \mathcal{G}$ and $\mathrm{pr}_T: \mathcal{G}_T \rightarrow T$. Then, for every $\alpha \in \mathcal{G}^\vee(T)$, we have an isomorphism $\mathrm{pr}_{T,!}(\mathrm{pr}_{\mathcal{G}}^* - \otimes \mathcal{L}_\alpha) \simeq \alpha^* \circ \mathrm{FT}_{\mathcal{G}}$, where \mathcal{L}_α is the multiplicative line bundle on \mathcal{G}_T associated with α .*

4.2. CONVOLUTION AND EQUIVALENCES OF CATEGORIES

Just like the Fourier transform in analysis sends a product of functions to the convolution of their Fourier transforms, our sheaf-theoretic Fourier transform also enjoys an analogous property. This convolution product provides a clear criterion for determining when the Fourier transform is an equivalence of categories.

DEFINITION 4.2.1 (Convolution). Let \mathcal{G} be a !-able flat commutative group stack with group operation $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$. We define the *convolution product* as the bifunctor $D_{\mathrm{qc}}(\mathcal{G}) \times D_{\mathrm{qc}}(\mathcal{G}) \rightarrow D_{\mathrm{qc}}(\mathcal{G})$ sending (M, N) to $M * N := m_!(M \boxtimes N)$.

PROPOSITION 4.2.2. *Let \mathcal{G} be a !-able flat commutative group stack with zero section $e: S \rightarrow \mathcal{G}$. The convolution product endows the stable ∞ -category $D_{\mathrm{qc}}(\mathcal{G})$ of a symmetric monoidal structure with unit $e_! \mathcal{O}_S$.*

Proof. Let \mathcal{C} be an ∞ -category admitting finite limits, and let E be a class of morphisms stable under pullback and composition, containing all isomorphisms. According to Definition 1.2.3, a three-functor formalism is a lax symmetric monoidal functor

$$D: \mathrm{Corr}(\mathcal{C}, E) \rightarrow \mathrm{Cat}_\infty,$$

where $\mathrm{Corr}(\mathcal{C}, E)$ is the symmetric monoidal ∞ -category of correspondences. Denote by \mathcal{C}_E the subcategory of \mathcal{C} composed of the morphisms lying in E . We have a natural functor $\mathcal{C}_E \rightarrow \mathrm{Corr}(\mathcal{C}, E)$ acting as the identity on objects and sending a morphism $f: X \rightarrow Y$ to the correspondence

$$\begin{array}{ccc} & X & \\ \mathrm{id}_X \swarrow & & \searrow f \\ X & & Y. \end{array}$$

This functor is symmetric monoidal, implying that the composition $\mathcal{C}_E \rightarrow \mathrm{Corr}(\mathcal{C}, E) \rightarrow \mathrm{Cat}_\infty$ is lax symmetric monoidal. As such, it preserves commutative algebra objects.

Now, consider the six-functor formalism on derived stacks D_{qc} defined in Section 1.3. Since a !-able flat commutative group stack is a commutative algebra in the category of derived stacks, the preceding discussion endows $D_{\mathrm{qc}}(\mathcal{G})$ with a symmetric monoidal structure given by convolution. \square

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Let $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of commutative group stacks. Suppose that φ admits $!$ -functors and that both \mathcal{G} and \mathcal{H} are $!$ -able and flat. As a byproduct of the preceding proof, the functor $\varphi_!$ is symmetric monoidal with respect to convolution. In particular, if $e: S \rightarrow \mathcal{G}$ is the identity section of \mathcal{G} , the functor $e_!$ maps tensor products to convolutions.

PROPOSITION 4.2.3. *Let \mathcal{G} be a $!$ -able flat commutative group stack. The Fourier transform $\mathrm{FT}_{\mathcal{G}}: D_{\mathrm{qc}}(\mathcal{G}) \rightarrow D_{\mathrm{qc}}(\mathcal{G}^{\vee})$ is symmetric monoidal with respect to the convolution product on the source and the tensor product on the target. Moreover, it is also symmetric monoidal with respect to the tensor product on the source and the convolution product on the target.*

Proof. The result follows directly from Proposition 4.1.7, noting that stacky Cartier duality interchanges diagonal maps with group operations. \square

DEFINITION 4.2.4. We say that a commutative group stack \mathcal{G} is *reflexive* if it is flat, $!$ -able, and the natural evaluation map $\mathrm{ev}_{\mathcal{G}}: \mathcal{G} \rightarrow (\mathcal{G}^{\vee})^{\vee}$ is an isomorphism. In this case, we define a functor $\mathrm{FT}_{\mathcal{G}}^{\vee}: D_{\mathrm{qc}}(\mathcal{G}^{\vee}) \rightarrow D_{\mathrm{qc}}(\mathcal{G})$ as the composition of $\mathrm{FT}_{\mathcal{G}^{\vee}}$ with the inverse image by $\mathrm{ev}_{\mathcal{G}}$.

We remark that this Fourier transform $\mathrm{FT}_{\mathcal{G}}^{\vee}: D_{\mathrm{qc}}(\mathcal{G}^{\vee}) \rightarrow D_{\mathrm{qc}}(\mathcal{G})$ really is what one expects it to be. Indeed, if $\tau: \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathcal{G}^{\vee} \times \mathcal{G}$ switches the factors, the maps

$$\mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathrm{BG}_m \quad \text{and} \quad \mathcal{G} \times \mathcal{G}^{\vee} \xrightarrow{\tau} \mathcal{G}^{\vee} \times \mathcal{G} \xrightarrow{\mathrm{id}_{\mathcal{G}^{\vee}} \times \mathrm{ev}_{\mathcal{G}}} \mathcal{G}^{\vee} \times (\mathcal{G}^{\vee})^{\vee} \rightarrow \mathrm{BG}_m$$

coincide, and so $\mathcal{P}_{\mathcal{G}} \simeq \tau^*(\mathrm{id}_{\mathcal{G}^{\vee}} \times \mathrm{ev}_{\mathcal{G}})^* \mathcal{P}_{\mathcal{G}^{\vee}}$. Then Proposition 4.1.4 implies that $\mathrm{FT}_{\mathcal{G}}^{\vee}$ is nothing but the functor

$$M \mapsto \mathrm{pr}_{1,!}(\mathrm{pr}_2^* M \otimes \mathcal{P}_{\mathcal{G}}),$$

where pr_1 and pr_2 are the natural projections from $\mathcal{G} \times \mathcal{G}^{\vee}$.

In the author's experience, Fourier transforms of reflexive commutative group stacks are often equivalences of categories. However, this does not seem to be a formal consequence of the theory developed in this chapter. The proposition below describes one method for proving such equivalences.⁴

PROPOSITION 4.2.5. *Let \mathcal{G} be a reflexive commutative group stack with Cartier dual \mathcal{G}^{\vee} and Poincaré bundle $\mathcal{P}_{\mathcal{G}}$. Denote by $\mathrm{pr}_1: \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathcal{G}$ and $\mathrm{pr}_2: \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathcal{G}^{\vee}$ the natural projections. Then the Fourier transforms $\mathrm{FT}_{\mathcal{G}}: D_{\mathrm{qc}}(\mathcal{G}) \rightarrow D_{\mathrm{qc}}(\mathcal{G}^{\vee})$ and $\mathrm{FT}_{\mathcal{G}^{\vee}}^{\vee}: D_{\mathrm{qc}}(\mathcal{G}^{\vee}) \rightarrow D_{\mathrm{qc}}(\mathcal{G})$ are equivalences of categories if and only if $\mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}$ and $\mathrm{pr}_{2,!} \mathcal{P}_{\mathcal{G}}$ are invertible with respect to the convolution product.*

⁴See [Bei+18, App. B] for a slightly different approach.

Proof. If both Fourier transforms above are equivalences of categories, so are the compositions $\mathrm{FT}_{\mathcal{G}} \circ \mathrm{FT}_{\mathcal{G}}^{\vee}$ and $\mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}}$. Conversely, if these compositions are equivalences of categories, then $(\mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}})^{-1} \circ \mathrm{FT}_{\mathcal{G}}^{\vee}$ is a left-inverse and $\mathrm{FT}_{\mathcal{G}}^{\vee} \circ (\mathrm{FT}_{\mathcal{G}} \circ \mathrm{FT}_{\mathcal{G}}^{\vee})^{-1}$ is a right-inverse to $\mathrm{FT}_{\mathcal{G}}$, proving that it is an equivalence and similarly for $\mathrm{FT}_{\mathcal{G}}^{\vee}$. By symmetry, it suffices to prove that $\mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}}$ is an equivalence of categories if and only if $\mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}$ is invertible with respect to the convolution product.

Proposition 4.1.3, along with the discussion following Definition 4.2.4, shows that $\mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}}$ is the integral transform with kernel $\mathcal{P}_{\mathcal{G}} \circ \sigma^* \mathcal{P}_{\mathcal{G}}$, where $\sigma: \mathcal{G}^{\vee} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}^{\vee}$ switches the factors. Now, consider the morphism $\mu: \mathcal{G} \times \mathcal{G}^{\vee} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}^{\vee}$ given by $(x, \varphi, y) \mapsto (x + y, \varphi)$. By comparing the maps

$$\begin{aligned} \mathcal{G} \times \mathcal{G}^{\vee} \times \mathcal{G} &\xrightarrow{\mu} \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathrm{BG}_m \\ \mathcal{G} \times \mathcal{G}^{\vee} \times \mathcal{G} &\xrightarrow{\mathrm{pr}_{12}} \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathrm{BG}_m \\ \mathcal{G} \times \mathcal{G}^{\vee} \times \mathcal{G} &\xrightarrow{\mathrm{pr}_{23}} \mathcal{G}^{\vee} \times \mathcal{G} \xrightarrow{\sigma} \mathcal{G} \times \mathcal{G}^{\vee} \rightarrow \mathrm{BG}_m, \end{aligned}$$

we see that $\mathcal{P}_{\mathcal{G}} \circ \sigma^* \mathcal{P}_{\mathcal{G}} \simeq \mathrm{pr}_{13,!} \mu^* \mathcal{P}_{\mathcal{G}}$. Then, an application of base change on the diagram

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{G}^{\vee} \times \mathcal{G} & \xrightarrow{\mu} & \mathcal{G} \times \mathcal{G}^{\vee} \\ \mathrm{pr}_{13} \downarrow & & \downarrow \mathrm{pr}_1 \\ \mathcal{G} \times \mathcal{G} & \xrightarrow{m} & \mathcal{G} \end{array}$$

gives that $\mathrm{pr}_{13,!} \mu^* \mathcal{P}_{\mathcal{G}} \simeq m^* \mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}$.

Denote by p_1 and p_2 the natural projections from $\mathcal{G} \times \mathcal{G}$, and by $\mathrm{inv}_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{G}$ the map sending an element of \mathcal{G} to its inverse. Remark that the morphism $\delta: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ defined by $(x, y) \mapsto (-x, x + y)$ is an isomorphism whose inverse is itself. It follows that

$$\begin{aligned} \mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}} &\simeq p_{2,!} (p_1^* - \otimes m^* \mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}) \\ &\simeq p_{2,!} \delta_! \delta^* (p_1^* - \otimes m^* \mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}) \\ &\simeq m_! (p_1^* \mathrm{inv}_{\mathcal{G}}^* - \otimes p_2^* \mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}) \simeq (\mathrm{inv}_{\mathcal{G}}^* -) * (\mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}). \end{aligned}$$

Since $\mathrm{inv}_{\mathcal{G}}$ is an isomorphism, the functor $\mathrm{FT}_{\mathcal{G}}^{\vee} \circ \mathrm{FT}_{\mathcal{G}}$ is an equivalence of categories precisely when $\mathrm{pr}_{1,!} \mathcal{P}_{\mathcal{G}}$ is invertible with respect to the convolution product. \square

■ **EXAMPLE 4.2.6.** Let $A \rightarrow S$ be an abelian scheme of relative dimension g with dual A' , and let \mathcal{P}_A be its Poincaré bundle. As above, we denote by pr_1 and pr_2 the natural projections from $A \times_S A'$. Moreover, let $e: S \rightarrow A$ and $e': S \rightarrow A'$ be the zero sections. By [GW23, Prop. 27.229], we have that

$$\begin{aligned} \mathrm{pr}_{1,*} \mathcal{P}_A &\simeq e_* \det(\mathcal{C}_e)^{-1}[-g] \\ \mathrm{pr}_{2,*} \mathcal{P}_A &\simeq e'_* \det(\mathcal{C}_e)^{-1}[-g], \end{aligned}$$

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where $\mathcal{C}_e = e^* \Omega_{A/S}^1$ is the conormal bundle of the zero section. Since the convolution inverse of $e_* \det(\mathcal{C}_e)^{-1}[-g]$ is $e_* \det(\mathcal{C}_e)[g]$, and similarly for $e'_* \det(\mathcal{C}_e)^{-1}[-g]$, this recovers Mukai's original result [Muk81, Thm. 2.2]. ■

4.3. THE FOURIER–LAUMON TRANSFORM

Throughout this section, the base scheme S is the spectrum of a characteristic zero field k .⁵ There is an equivalence of categories $D_{\text{qc}}(\mathbb{G}_a) \simeq D_{\text{qc}}(\widehat{B\mathbb{G}}_a)$, where $\widehat{B\mathbb{G}}_a$ denotes the formal completion of \mathbb{G}_a at the identity [Bha22, Ex. 2.2.12]. Thus, one might expect a Fourier transform witnessing this equivalence. While $\widehat{B\mathbb{G}}_a$ is the Cartier dual of \mathbb{G}_a , the coarse moduli sheaf of \mathbb{G}_a^\vee is the non-vanishing $\underline{\text{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m)$ studied in Remark 2.3.17.

In this subsection, we treat $\widehat{B\mathbb{G}}_a$ as the *correct* dual. This is the *Laumon dual*, denoted $\mathbb{G}_a^{\triangleright}$ and defined in Section 3.1. More generally, let \mathcal{G} be a generalized 1-motive, regarded as a commutative group stack. In the aforementioned section, we constructed the Laumon dual $\mathcal{G}^{\triangleright}$ along with a natural map $\mathcal{G}^{\triangleright} \rightarrow \mathcal{G}^\vee$. It may be reassuring to note that Corollary 3.1.6 shows that the comparison map $\mathcal{G}^{\triangleright} \rightarrow \mathcal{G}^\vee$ is frequently an isomorphism.

PROPOSITION 4.3.1. *Let \mathcal{G} be a generalized 1-motive over k . Then \mathcal{G} is !-able.*

Proof. Write \mathcal{G} as a quotient of smooth commutative connected algebraic group G by a commutative formal group \mathcal{G} . By Cartier duality, \mathcal{G} is the Cartier dual L^D of an affine smooth connected algebraic group L . Because L decomposes as a product of a torus T and a unipotent group U , the formal group \mathcal{G} is isomorphic to $X \times \widehat{U}^*$, where X is the character group of T and U^* is the dual vector group.

It is clear that the map $X \rightarrow \text{Spec } k$ is cohomologically smooth, and Proposition 1.4.2 established that the map $\widehat{U}^* \rightarrow \text{Spec } k$ is cohomologically smooth as well. Since the diagram

$$\begin{array}{ccc} X \times \widehat{U}^* & \longrightarrow & \text{Spec } k \\ \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & B\mathcal{G} \end{array}$$

is cartesian, it follows that the unit section $\text{Spec } k \rightarrow B\mathcal{G}$ is cohomologically smooth. Finally, the diagram

$$\begin{array}{ccc} G & \longrightarrow & \text{Spec } k \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & B\mathcal{G} \end{array}$$

⁵As noted in Corollary 3.1.6, the Fourier transform discussed in this section is required only in characteristic zero.

being cartesian gives that the quotient map $G \rightarrow \mathcal{G}$ is cohomologically smooth. Remark 1.2.16 yields that $G \rightarrow \mathcal{G}$ satisfies universal $!$ -descent. Given that having $!$ -functors is local on the source, we conclude that \mathcal{G} is $!$ -able. \square

As a result of the preceding proposition, all the results from the previous two sections apply to generalized 1-motives. In particular, we can define a variant of the Fourier transform that maps quasi-coherent sheaves on \mathcal{G} to quasi-coherent sheaves on the Laumon dual $\mathcal{G}^\triangleright$.

DEFINITION 4.3.2. Let \mathcal{G} be a generalized 1-motive. The *Fourier–Laumon transform* is the functor $LT_{\mathcal{G}}: D_{qc}(\mathcal{G}) \rightarrow D_{qc}(\mathcal{G}^\triangleright)$ defined as the composition of $FT_{\mathcal{G}}$ with the inverse image by $\mathcal{G}^\triangleright \rightarrow \mathcal{G}^\vee$.



This definition is inspired by the one presented in [Lau96, §6.3]. Although we have not verified if they coincide (up to shifts and tensoring with invertible modules), we believe that Definition 4.3.2 is highly conceptual and meets all essential properties expected of a Fourier transform.

The following examples illustrate how the Fourier–Laumon transform generalizes some well-known operations for \mathcal{D} -modules. The comparison with the classical functors follows directly from the computation of the Laumon duals via the Proposition 3.1.7, along with the Corollaries 1.4.8 and 1.4.12.

■ EXAMPLE 4.3.3 — FOURIER TRANSFORM FOR \mathcal{D} -MODULES ON UNIPOTENT GROUPS. Let U be a unipotent commutative algebraic group, and let U^* be its dual vector space seen as a vector group. The Laumon dual of U_{dR} is U^*_{dR} , inducing the Fourier–Laumon transform $D_{qc}(\mathcal{D}_U) \rightarrow D_{qc}(\mathcal{D}_{U^*})$.⁶ Using the notation from [Dai00, §2.2.2], this functor coincides with $\mathcal{F}_*[-\dim U]$. In particular, since \mathcal{F}_* is t -exact [Dai00, Cor. 2.2.2.1], this Fourier–Laumon transform maps a \mathcal{D}_U -module concentrated in degree zero to a \mathcal{D}_{U^*} -module concentrated in degree $\dim U$. (See also [Lau85, §1].)

After a choice of bases $U \simeq \text{Spec } k[x_1, \dots, x_n]$ and $U^* \simeq \text{Spec } k[\xi_1, \dots, \xi_n]$, this Fourier–Laumon transform becomes rather explicit. Indeed, consider the Weyl algebras⁷ $A_n = k[x_1, \dots, x_n]\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle$ and $A_n^* = k[\xi_1, \dots, \xi_n]\langle \partial_{\xi_1}, \dots, \partial_{\xi_n} \rangle$. The isomorphism

$$\begin{aligned} k[x_1, \dots, x_n]\langle \partial_{x_1}, \dots, \partial_{x_n} \rangle &\rightarrow k[\xi_1, \dots, \xi_n]\langle \partial_{\xi_1}, \dots, \partial_{\xi_n} \rangle \\ x_i &\mapsto \partial_{\xi_i} \\ \partial_{x_i} &\mapsto -\xi_i \end{aligned}$$

⁶This provides another example where the stacky Cartier dual and the Laumon dual do not coincide.

⁷The free noncommutative algebra generated by the variables x_i and the differential operators ∂_{x_i} , satisfying the commutation relations $\partial_{x_i} x_j - x_j \partial_{x_i} = \delta_{ij}$ for $i, j = 1, \dots, n$.

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allows us to define a functor $\text{Mod}(A_n) \rightarrow \text{Mod}(A_n^*)$ given by $M \mapsto A_n^* \otimes_{A_n} M$. This induces an equivalence of ∞ -categories $D(\text{Mod}(A_n)) \rightarrow D(\text{Mod}(A_n^*))$ which, under the identifications $D_{\text{qc}}(\mathcal{D}_U) \simeq D(\text{Mod}(A_n))$ and $D_{\text{qc}}(\mathcal{D}_{U^*}) \simeq D(\text{Mod}(A_n^*))$, coincides with the functor \mathcal{F}_* mentioned above [KL85, Lem. 7.1.4]. ■

■ **EXAMPLE 4.3.4 — MELLIN TRANSFORM FOR \mathcal{D} -MODULES ON TORI.** Let T be a k -torus. Denote by Ω_T the vector space of invariant differential forms over T , seen as a vector group, and by X its character group. Both the stacky Cartier and the Laumon dual of T_{dR} are given by Ω_T/X , where X acts on Ω_T as

$$\begin{aligned} X \times \Omega_T &\rightarrow \Omega_T \\ (\chi, \omega) &\mapsto \omega + \chi^*(dt/t). \end{aligned}$$

Here, t denotes the canonical coordinate of $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$. We obtain a Fourier transform $D_{\text{qc}}(\mathcal{D}_T) \rightarrow D_{\text{qc}}^X(\Omega_T)$, where $D_{\text{qc}}^X(\Omega_T)$ is the derived ∞ -category of quasi-coherent X -equivariant sheaves on Ω_T .

This functor coincides with $F[-\dim T]$, with F defined as in [Lau85, §2]. For a split torus T , this operation can be made explicit. After choosing coordinates, we have isomorphisms $T \simeq \text{Spec } k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, $\Omega_T \simeq \text{Spec } k[s_1, \dots, s_n]$ and $X \simeq \mathbb{Z}^n$. Under these identifications, an X -equivariant quasi-coherent sheaf on Ω_T corresponds to a left $k[s_1, \dots, s_n]\langle \tau_1, \tau_1^{-1}, \dots, \tau_n, \tau_n^{-1} \rangle$ -module.⁸ Then, F coincides with the functor induced by the isomorphism

$$\begin{aligned} k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]\langle \theta_{x_1}, \dots, \theta_{x_n} \rangle &\rightarrow k[s_1, \dots, s_n]\langle \tau_1, \tau_1^{-1}, \dots, \tau_n, \tau_n^{-1} \rangle \\ x_i &\mapsto \tau_i \\ \theta_i &\mapsto -s_i, \end{aligned}$$

where $\theta_i := x_i \partial_{x_i}$, as in the previous example [LS91b, Lem. 1.2.2]. In particular, F is t -exact with respect to the standard t -structures. ■

■ **EXAMPLE 4.3.5 — FOURIER–MUKAI TRANSFORM FOR \mathcal{D} -MODULES ON ABELIAN VARIETIES.** Let A be an abelian variety, and denote by A^\natural the universal vector extension of the dual abelian variety A' . The stacky Cartier and the Laumon dual of A_{dR} coincide and are isomorphic to A^\natural . As a result, there exists a Fourier transform $D_{\text{qc}}(\mathcal{D}_A) \rightarrow D_{\text{qc}}(A^\natural)$, which is naturally isomorphic to $\tilde{\mathcal{F}}[-\dim A]$, in the notation of [Lau96, §3].

According to [Lau96, Thm. 3.2.1], the functor $\tilde{\mathcal{F}}$ is an equivalence of categories. In particular, it restricts to compact objects, giving $D_{\text{coh}}^b(\mathcal{D}_A) \rightarrow D_{\text{coh}}^b(A^\natural)$. This functor is not t -exact with respect to any shifts of the standard t -structures. However, it becomes t -exact with respect to a perverse t -structure on the target after restricting to holonomic complexes [Sch15, Thm. 4.2]. ■

⁸Similar to the Weyl algebra, $k[s_1, \dots, s_n]\langle \tau_1, \tau_1^{-1}, \dots, \tau_n, \tau_n^{-1} \rangle$ is the free noncommutative algebra generated by the variables s_i and the invertible difference operators τ_i , with commutation relations $\tau_i s_j - s_j \tau_i = \tau_i \delta_{ij}$ for $i, j = 1, \dots, n$.

We now explain the main properties of the Fourier–Laumon transform, beginning by those inherited from properties of the Fourier transform studied in the previous sections.

PROPOSITION 4.3.6. *Let $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ be a morphism of generalized 1-motives with Laumon dual $\varphi^\triangleright: \mathcal{H}^\triangleright \rightarrow \mathcal{G}^\triangleright$. Then the functors $\varphi^{\triangleright,*} \circ \mathrm{LT}_{\mathcal{G}}$ and $\mathrm{LT}_{\mathcal{H}} \circ \varphi_!$ are isomorphic.*

Proof. The comparison map between the Laumon and the stacky Cartier dual being natural on the generalized 1-motive implies that the diagram

$$\begin{array}{ccc} \mathcal{H}^\triangleright & \longrightarrow & \mathcal{H}^\vee \\ \varphi^{\triangleright} \downarrow & & \downarrow \varphi^\vee \\ \mathcal{G}^\triangleright & \longrightarrow & \mathcal{G}^\vee \end{array}$$

commutes. The result then follows from Proposition 4.1.7. □

PROPOSITION 4.3.7. *Let \mathcal{G} be a generalized 1-motive. Then the Fourier–Laumon transform $\mathrm{LT}_{\mathcal{G}}: D_{\mathrm{qc}}(\mathcal{G}) \rightarrow D_{\mathrm{qc}}(\mathcal{G}^\triangleright)$ is symmetric monoidal with respect to the convolution product on the source and the tensor product on the target.*

Proof. This result can either be seen as a consequence of Proposition 4.2.3, or one could use the preceding proposition to argue as in the proof of Proposition 4.2.3. □

Note that not every commutative group stack is reflexive. For instance, the stacky Cartier dual of the classifying stack of an abelian scheme vanishes. But every generalized 1-motive is reflexive with respect to the Laumon dual.

PROPOSITION 4.3.8. *Let \mathcal{G} be a generalized 1-motive. Then there is a natural isomorphism $\mathcal{G} \xrightarrow{\sim} (\mathcal{G}^\triangleright)^\triangleright$.*

Proof. We will freely use the notations of Section 3.1 in this proof. Writing the generalized 1-motive \mathcal{G} as a two-term complex $[\mathcal{G} \rightarrow G]$, its Laumon dual becomes $[L^\triangleright \rightarrow K]$, for $K = \underline{\mathrm{Ext}}^1([\mathcal{G} \rightarrow A], \mathbb{G}_m)$. As in the proof of Lemma 3.1.2, we have a short exact sequence

$$0 \rightarrow (L^\triangleright)^\triangleright \rightarrow \underline{\mathrm{Ext}}^1([L^\triangleright \rightarrow A'], \mathbb{G}_m) \rightarrow \underline{\mathrm{Ext}}^1(A', \mathbb{G}_m) \rightarrow 0.$$

Cartier duality gives that the natural map $L \rightarrow (L^\triangleright)^\triangleright$ is an isomorphism, while the Barsotti–Weil formula along with the duality of abelian varieties implies that the natural map $A \rightarrow \underline{\mathrm{Ext}}^1(A', \mathbb{G}_m)$ is an isomorphism. The uniqueness in the Barsotti–Chevalley decomposition then yields that $\underline{\mathrm{Ext}}^1([L^\triangleright \rightarrow A'], \mathbb{G}_m) \simeq G$. □

PART III.

TANNAKIAN GROUPS OF \mathcal{D} -MODULES

5. THE GENERIC VANISHING THEOREM

Let G be a commutative connected algebraic group over \mathbb{C} . In the previous part, we constructed the moduli space of character sheaves G^b and studied its geometry. Specifically, we identified a class of large subspaces within G^b , referred to as generic subspaces, which reflect the geometric structure of G . We now turn our attention to vanishing theorems for the de Rham cohomology of holonomic \mathcal{D} -modules on G , twisted by character sheaves on these generic subspaces of G^b .

GENERAL OUTLINE

In this chapter, all algebraic varieties are assumed to be defined over \mathbb{C} , and every point is considered to be closed. We adopt the \mathcal{D} -module conventions introduced in Section 1.5, which includes a table aligning our notation with that of common references. Following Remark 1.5.2, we denote by \mathcal{L}_χ the character sheaf associated with a point $\chi \in G^b$, taken *in degree* $\dim G$. For an invariant form $\omega \in \Omega_G$, we denote the character sheaf $(\mathcal{O}_G, d + \omega)[- \dim G]$ by \mathcal{L}_ω .

Let M be a holonomic \mathcal{D} -module on G . The desired generic vanishing theorem should assert the existence of a generic subspace $V \subset G^b$ such that, for all $\chi \in V$:

- (1) the forget-supports map $H_c^*(G, M \otimes_G \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\chi)$ is an isomorphism;
- (2) the complex $H^*(G, M \otimes_G \mathcal{L}_\chi)$ is concentrated in degree zero.

If each holonomic \mathcal{D} -module of G admits a generic subspace of G^b with these properties, we say that G *satisfies generic vanishing*.

A fundamental observation is that property (1) is closely related to linear groups, while property (2) is naturally associated with abelian varieties. For an abelian variety A , the forget-supports map $H_c^*(A, M \otimes_A \mathcal{L}_\chi) \rightarrow H^*(A, M \otimes_A \mathcal{L}_\chi)$ is always an isomorphism, making property (2) the primary focus. On the other hand, for a linear group L , whenever the forget-supports map $H_c^*(L, M \otimes_L \mathcal{L}_\chi) \rightarrow H^*(L, M \otimes_L \mathcal{L}_\chi)$ is an isomorphism, Artin vanishing ensures that both sides are concentrated in degree zero (see Proposition 5.1.2 for more details).

For a general commutative connected algebraic group G , the Barsotti–Chevalley theorem provides a decomposition of G as an extension of an abelian variety A by a linear group L :

$$0 \rightarrow L \xrightarrow{\varphi} G \xrightarrow{\psi} A \rightarrow 0.$$

5. The generic vanishing theorem

Moreover, from Corollary 3.1.10, every character sheaf on G can be expressed as $\mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha$, for some $\omega \in G$ and $\alpha \in A^b$. The projection formula then gives isomorphisms

$$\begin{aligned} H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) &\simeq H^*(A, \psi_!(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha) \\ H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) &\simeq H^*(A, \psi_+(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha). \end{aligned}$$

Consequently, properties (1) and (2), for $\mathcal{L}_\chi = \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha$, can be derived from the following conditions:

- (1') the forget-supports map $\psi_!(M \otimes_G \mathcal{L}_\omega) \rightarrow \psi_+(M \otimes_G \mathcal{L}_\omega)$ is an isomorphism;
- (2') the complex $H^*(A, \psi_+(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha)$ is concentrated in degree zero.

Building on Schnell's work, we can focus primarily on condition (1'). Indeed, a relative version of Artin vanishing guarantees that, for each $\omega \in \Omega_G$ such that the forget-supports map $\psi_!(M \otimes_G \mathcal{L}_\omega) \rightarrow \psi_+(M \otimes_G \mathcal{L}_\omega)$ is an isomorphism, both sides are concentrated in degree zero. Thus, condition (2') follows from the fact that abelian varieties satisfy generic vanishing; a result of Schnell [Sch15, Thms. 2.2 and 4.1].

RELATIVE GENERIC VANISHING

The Barsotti–Chevalley decomposition implies that G is an L -torsor over A . In particular, there exists an étale covering $S \rightarrow A$ that trivializes this torsor and identifies the map $\psi: G \rightarrow A$ with the projection $\text{pr}_S: L \times S \rightarrow S$:

$$\begin{array}{ccc} L \times S & \xrightarrow{\text{pr}_S} & S \\ \downarrow & & \downarrow \\ G & \xrightarrow{\psi} & A. \end{array}$$

As in Proposition 5.4.1, the verification of condition (1') reduces to the following assertion: for each smooth variety S and each object M of $D_h^b(\mathcal{D}_{L \times S})$, there exists a generic subspace $V \subset L^b$ such that, for all $\chi \in V$,

- (1'') the forget-supports map $\text{pr}_{S,!}(M \otimes_L \text{pr}_L^+ \mathcal{L}_\chi) \rightarrow \text{pr}_{S,+}(M \otimes_L \text{pr}_L^+ \mathcal{L}_\chi)$ is an isomorphism.

Here, $\text{pr}_L: L \times S \rightarrow L$ denotes the natural projection. A linear group L with this property is said to satisfy *relative generic vanishing*.

The following result is the culmination of all the *déviassages* discussed above. When combined with the verification that tori satisfy relative generic vanishing, it stands as one of the most significant theorems in this thesis.

THEOREM (5.4.3). *Let M be an object of $D_h^b(\mathcal{D}_G)$. Suppose that L satisfies relative generic vanishing. Then there exists a generic subspace V of L^b such that the forget-supports map*

$$H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha),$$

is an isomorphism for every $\alpha \in A^b$ and every $\omega \in \Omega_G$ such that $\varphi^+ \mathcal{L}_\omega$ lies in V . Furthermore, if M is concentrated in degree zero, for each ω as above, there exists a generic subspace W of A^b such that

$$H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \simeq H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha)$$

is concentrated in degree zero for every $\alpha \in W$.

As explained in Corollary 5.4.2, for an object M of $D_h^b(\mathcal{D}_G)$, this theorem guarantees the existence of a generic subspace W of G^b such that the forget-supports map

$$H_c^*(G, M \otimes_G \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\chi)$$

is an isomorphism for every $\chi \in W$. If M is concentrated in degree zero, it provides a "large" subset W' of G^b such that

$$H_c^*(G, M \otimes_G \mathcal{L}_\chi) \simeq H^*(G, M \otimes_G \mathcal{L}_\chi)$$

is concentrated in degree zero for every $\chi \in W'$. However, it remains unclear whether W' contains a generic subspace of G^b .

THE PROOF OF RELATIVE GENERIC VANISHING

In Section 5.1, we present a general strategy for proving that a forget-supports map is an isomorphism based on nearby and vanishing cycles. (Which, in the de Rham setting, are modeled by the V -filtration of Kashiwara, Malgrange, and Sabbah.) This strategy was successfully applied in Theorem 5.2.1 to demonstrate that tori satisfy relative generic vanishing, thereby reproving and generalizing a theorem of Sabbah [Sab92, Thm. 2.2].

The unipotent case, however, presents significantly more challenges. In Example 5.3.4, we apply the previously outlined strategy to establish a specific instance of relative generic vanishing for unipotent groups. We believe this example is highly indicative of the general case, potentially even serving as a model for proving relative generic vanishing for arbitrary linear groups. Although we are not yet able to verify every detail, Remark 5.3.5 outlines a possible approach to such a proof.

Nonetheless, in Proposition 5.3.1, we prove that a unipotent group U satisfies a weak form of generic vanishing. Specifically, for each object M of $D_h^b(\mathcal{D}_U)$, there exists an open dense subset V of U^b such that the forget-supports map

$$H_c^*(G, M \otimes_U \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_U \mathcal{L}_\chi)$$

5. The generic vanishing theorem

is an isomorphism for all $\chi \in V$. Furthermore, if M is concentrated in degree zero, those cohomology groups vanish in all non-zero degrees.

5.1. RELATIVE GENERIC VANISHING

In this section, we explain our strategy for proving a significant part of the generic vanishing theorem. Given that a general commutative connected algebraic group G is an extension of an abelian variety A by a linear group L , there exists an étale covering $S \rightarrow A$ that trivializes the L -torsor $G \rightarrow A$ and identifies it with the projection $L \times S \rightarrow S$. This brings us to the following definition.

DEFINITION 5.1.1 (Relative generic vanishing). Let L be a linear commutative connected algebraic group. We say that L *satisfies relative generic vanishing* if, for each smooth algebraic variety S and each object M of $D_h^b(\mathcal{D}_{L \times S})$, there exists a generic subspace V of L^b such that the forget-supports map

$$\mathrm{pr}_{S,!}(M \otimes_{L \times S} \mathrm{pr}_L^+ \mathcal{L}_\chi) \rightarrow \mathrm{pr}_{S,+}(M \otimes_{L \times S} \mathrm{pr}_L^+ \mathcal{L}_\chi),$$

where $\mathrm{pr}_S : L \times S \rightarrow S$ and $\mathrm{pr}_L : L \times S \rightarrow L$ are the projections, is an isomorphism for every $\chi \in V$.

As one could expect, a linear commutative connected algebraic group that satisfies relative generic vanishing also satisfies a generic vanishing theorem.

PROPOSITION 5.1.2. *Let L be a linear commutative connected algebraic group, and suppose that L satisfies relative generic vanishing. Then, for each holonomic \mathcal{D} -module M over L , there exists a generic subspace V of L^b such that*

$$\begin{aligned} H_c^i(L, M \otimes_L \mathcal{L}_\chi) &= H^i(L, M \otimes_L \mathcal{L}_\chi) = 0 \text{ for } i \neq 0; \\ H_c^0(L, M \otimes_L \mathcal{L}_\chi) &\simeq H^0(L, M \otimes_L \mathcal{L}_\chi) \end{aligned}$$

for every $\chi \in V$.

Proof. Applying relative generic vanishing to $S = \mathrm{Spec} \mathbb{C}$ provides a generic subspace V such that the forget-supports map

$$H_c^*(L, M \otimes_L \mathcal{L}_\chi) \rightarrow H^*(L, M \otimes_L \mathcal{L}_\chi)$$

is an isomorphism for every $\chi \in V$. Since L is affine, a \mathcal{D} -module analogue of Artin vanishing implies that the left-hand side is concentrated in positive degrees, whereas the right-hand side is concentrated in negative degrees. (See Remark 1.5.4 for more details on Artin vanishing.) Consequently, whenever both sides are isomorphic, they must be concentrated in degree zero. \square

In the remainder of this section, we outline a strategy for the more general problem of verifying that a forget-supports map is an isomorphism. Here, we employ the six-functor formalism of holonomic \mathcal{D} -modules; however, the reader will note that these ideas are also applicable in most six-functor formalisms equipped with notions of nearby and vanishing cycles. Indeed, we were inspired by [Lau81] and [KL85], where similar techniques were used in the setting of ℓ -adic cohomology.

Let $f: X \rightarrow S$ be a morphism between smooth algebraic varieties, and let N be a holonomic \mathcal{D} -module over X . Our goal is to determine conditions on N implying that the forget-supports map $f_!N \rightarrow f_+N$ is an isomorphism. As a first observation, the theorem on Nagata compactifications [Stacks, Tag 0F41] provides an open immersion $X \hookrightarrow \bar{X}$ and a proper map $\bar{X} \rightarrow S$ whose composition is equal to f .

According to the theorem of resolution of singularities [Kol07, Thm. 36], there exists a smooth projective variety \tilde{X} and a projective morphism $\tilde{X} \rightarrow \bar{X}$ that is an isomorphism over the smooth locus of \bar{X} . Consequently, $X \hookrightarrow \bar{X}$ factors through \tilde{X} , and we obtain a factorization

$$\begin{array}{ccccc}
 & & \tilde{X} & & \\
 & \nearrow j & \downarrow & \searrow p & \\
 X & \hookrightarrow & \bar{X} & \longrightarrow & S, \\
 & \searrow f & & &
 \end{array}$$

where j is an open immersion, p is proper, and \tilde{X} is a smooth algebraic variety. Since $f_!N \rightarrow f_+N$ is the image of $j_!N \rightarrow j_+N$ by p_+ , it suffices to prove that the latter is an isomorphism.



Currently, resolutions of singularities are only known to exist in characteristic zero. However, most six-functor formalisms that one might consider in positive characteristic work without any regularity assumptions. In such cases, the factorization $X \hookrightarrow \bar{X} \rightarrow S$ suffices for these arguments.

The following lemma provides a general criterion this problem using the V-filtration of Kashiwara, Malgrange and Sabbah.

LEMMA 5.1.3. *Let $j: X \hookrightarrow \tilde{X}$ be an open immersion between smooth algebraic varieties, with complementary closed immersion $i: Z \hookrightarrow \tilde{X}$. Suppose that Z is smooth and of codimension 1. Given a holonomic \mathcal{D} -module N on X , the forget-supports map*

$$j_!N \rightarrow j_+N$$

is an isomorphism if and only if $\text{gr}_0^V(j_+N)$ vanishes.

Since the V-filtration is the \mathcal{D} -module analogue of nearby and vanishing cycles, we will give a proof of this result focusing on standard properties these functors. We refer the reader to Remark 1.5.8 for more on the V-filtration.

5. The generic vanishing theorem

Proof of Lemma 5.1.3. Recall that $\mathrm{gr}_0^V(j_+N)$ is the \mathcal{D} -module analogue of the unipotent vanishing cycle functor $\phi_1(j_+N)$. Firstly, note that we have a *recollement* fiber sequence

$$j_!N \rightarrow j_+N \rightarrow i_+i^+j_+N.$$

Hence, the forget-supports map is an isomorphism precisely if $i_+i^+j_+N = 0$. Moreover, since i^+i_+ is isomorphic to the identity functor, this occurs if and only if $i^+j_+N = 0$.

Next, the standard theory of nearby and vanishing cycles provides two additional fiber sequences:

$$\begin{aligned} i^+j_+N &\rightarrow \psi_1(j_+N) \xrightarrow{\mathrm{can}} \phi_1(j_+N) \\ i^!j_+N &\rightarrow \phi_1(j_+N) \xrightarrow{\mathrm{var}} \psi_1(j_+N). \end{aligned}$$

The first sequence demonstrates that the forget supports map is an isomorphism if and only if can is. Since $i^!j_+ = 0$, the second sequence implies that $\psi_1(j_+N)$ and $\phi_1(j_+N)$ are isomorphic. In particular, if $\phi_1(j_+N)$ vanishes, then so does $\psi_1(j_+N)$ and can is an isomorphism.

Conversely, if the forget supports map is an isomorphism, our reasoning shows that can is an isomorphism as well. Furthermore, var is also an isomorphism (since $i^!j_+ = 0$) and thus so is $\mathrm{can} \circ \mathrm{var}$. However, this morphism is nilpotent, implying that it is zero. It follows that $\phi_1(j_+N) = 0$. \square

Although we can always choose a factorization $X \hookrightarrow \tilde{X} \rightarrow S$ such that the complement Z of X in \tilde{X} is a divisor, we often cannot ensure that Z is smooth. In most other cohomology theories, all the smoothness hypotheses in the preceding lemma are unnecessary; however, this is not the case for de Rham cohomology. A standard way of circumventing this issue is the so-called *graph trick* of Kashiwara.

Assume that the divisor Z is not-necessarily smooth but is the zero set of a regular function f on \tilde{X} . Denoting by $\Gamma_f: \tilde{X} \hookrightarrow \tilde{X} \times \mathbb{A}^1$ its graph, we note that $\varphi: j_!N \rightarrow j_+N$ is an isomorphism if and only if $\Gamma_{f,+}\varphi$ is. Indeed, Γ_f has a retraction. The graph Γ_f is a closed immersion, and thus $\Gamma_{f,+}\varphi$ is the forget-supports map $(\Gamma_f \circ j)_!N \rightarrow (\Gamma_f \circ j)_+N$. Next, we have the following commutative diagram

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \tilde{X} \\ \tilde{\Gamma}_f \downarrow & & \downarrow \Gamma_f \\ \tilde{X} \times \mathbb{G}_m & \xhookrightarrow{\tilde{j}} & \tilde{X} \times \mathbb{A}^1, \end{array}$$

where $\tilde{\Gamma}_f: X \hookrightarrow \tilde{X} \times \mathbb{G}_m$ is the restriction of Γ_f to X and \tilde{j} is the natural open immersion. Consequently, the morphism $(\Gamma_f \circ j)_!N \rightarrow (\Gamma_f \circ j)_+N$ agrees with $\tilde{j}_!\tilde{\Gamma}_{f,+}N \rightarrow \tilde{j}_+\tilde{\Gamma}_{f,+}N$. Since the complement of $\tilde{X} \times \mathbb{G}_m$ in $\tilde{X} \times \mathbb{A}^1$ is $\tilde{X} \times \{0\} \simeq \tilde{X}$, which is supposed to be smooth, we can apply Lemma 5.1.3 to $\tilde{\Gamma}_{f,+}N$.

Consider the general setting where $j: X \hookrightarrow \tilde{X}$ is an open immersion between smooth algebraic varieties, and the complement $i: Z \hookrightarrow \tilde{X}$ has codimension one. Verifying whether $\varphi: j_!N \rightarrow j_+N$ is an isomorphism can be done on an open (even étale) cover of \tilde{X} . We may then choose an open cover $\{V_\lambda\}_{\lambda \in \Lambda}$ of \tilde{X} such that $Z_\lambda := Z \cap V_\lambda$ is the zero set of some regular function for all λ . Using proper and smooth base change on the diagram

$$\begin{array}{ccccc} U_\lambda & \xrightarrow{j_\lambda} & V_\lambda & \xleftarrow{i_\lambda} & Z_\lambda \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{j} & \tilde{X} & \xleftarrow{i} & Z, \end{array}$$

where $U_\lambda := X \cap V_\lambda$, we see that the restriction of φ to V_λ is the forget-supports map $j_{\lambda,!}N|_{U_\lambda} \rightarrow j_{\lambda,+}N|_{U_\lambda}$. This can be checked to be an isomorphism using the graph trick and Lemma 5.1.3.

In the context of proving relative generic vanishing, where X is the product of a linear commutative connected algebraic group and a smooth algebraic variety, one can always ensure that the complement Z of X in \tilde{X} is a simple normal crossings divisor. This further simplifies the geometry of the problem. Up to working locally, as in the previous paragraph, we are led to consider the forget-supports map $j_!N \rightarrow j_+N$, where j is the product of $\mathbb{A}^r \setminus \{x_1 \dots x_r = 0\} \hookrightarrow \mathbb{A}^r$ and the identity morphism of some smooth algebraic variety.

For notational simplicity, we will ignore the product with a smooth algebraic variety and assume j is the natural open immersion $\mathbb{A}^r \setminus \{x_1 \dots x_r = 0\} \hookrightarrow \mathbb{A}^r$. This morphism can be factored as

$$\begin{array}{ccccccc} \mathbb{A}^r \setminus \{x_1 \dots x_r = 0\} & \xrightarrow{j_1} & \mathbb{A}^r \setminus \{x_2 \dots x_r = 0\} & \xleftarrow{j_2} & \mathbb{A}^n \setminus \{x_3 \dots x_r = 0\} & \xleftarrow{\quad} & \dots \\ & & & & & & \xleftarrow{\quad} \\ & & & & & & \mathbb{A}^r \setminus \{x_{r-1}x_r = 0\} \\ & & & & & & \xleftarrow{j_{r-1}} \mathbb{A}^n \setminus \{x_r = 0\} \xleftarrow{j_r} \mathbb{A}^r. \end{array}$$

Although the complement of j is a singular simple normal crossings divisor, the complements of each of the open immersions j_i are smooth:

$$\begin{array}{ccccc} \mathbb{A}^r \setminus \{x_1 \dots x_r = 0\} & \xrightarrow{j_1} & \mathbb{A}^r \setminus \{x_2 \dots x_r = 0\} & \xleftarrow{i_1} & \{0\} \times \mathbb{G}_m^{r-1} \\ \mathbb{A}^r \setminus \{x_2 \dots x_r = 0\} & \xrightarrow{j_2} & \mathbb{A}^r \setminus \{x_3 \dots x_r = 0\} & \xleftarrow{i_2} & \mathbb{A}^1 \times \{0\} \times \mathbb{G}_m^{r-2} \\ \vdots & & \vdots & & \vdots \\ \mathbb{A}^r \setminus \{x_{r-1}x_r = 0\} & \xrightarrow{j_{r-1}} & \mathbb{A}^r \setminus \{x_r = 0\} & \xleftarrow{i_{r-1}} & \mathbb{A}^{r-2} \times \{0\} \times \mathbb{G}_m \\ \mathbb{A}^r \setminus \{x_r = 0\} & \xrightarrow{j_r} & \mathbb{A}^r & \xleftarrow{i_r} & \mathbb{A}^{r-1} \times \{0\}. \end{array}$$

5. The generic vanishing theorem

Since the forget-supports map $j_!N \rightarrow j_+N$ also factors accordingly, an application of Lemma 5.1.3 implies that $j_!N \rightarrow j_+N$ is an isomorphism as long as

$$\mathrm{gr}_0^V(j_{1,+}N), \mathrm{gr}_0^V(j_{2,+}j_{1,+}N), \dots, \mathrm{gr}_0^V(j_{r,+} \dots j_{1,+}N)$$

all vanish.

5.2. THE CASE OF TORI

This section is dedicated to proving that tori satisfy relative generic vanishing, as defined in Definition 5.1.1. We will employ the strategy outlined in the previous section.

THEOREM 5.2.1. *Let T be a torus over \mathbb{C} . Then T satisfies relative generic vanishing.*

Before delving into the proof of Theorem 5.2.1, let us set the stage. Since \mathbb{C} is algebraically closed, T must be isomorphic to \mathbb{G}_m^r for some r . The moduli space of character sheaves T^b is isomorphic to $(\mathbb{G}_a/\mathbb{Z})^r$, and we denote by \mathcal{L}_χ the character sheaf

$$(\mathcal{O}_{\mathbb{G}_m}, d - \chi_1 dt_1/t_1)[-1] \boxtimes \dots \boxtimes (\mathcal{O}_{\mathbb{G}_m}, d - \chi_r dt_r/t_r)[-1]$$

associated with $\chi = (\chi_1, \dots, \chi_r) \in T^b \simeq (\mathbb{G}_a/\mathbb{Z})^r$. It is often convenient to view the connection $(\mathcal{O}_{\mathbb{G}_m}, d - \chi_i dt_i/t_i)$ as the \mathcal{D} -module $\mathcal{D}_{\mathbb{G}_m}/\mathcal{D}_{\mathbb{G}_m}(t_i \partial_{t_i} + \chi_i)$.

Let S be a smooth algebraic variety, and let M be an object of $D_h^b(\mathcal{D}_{T \times S})$. Furthermore, consider the natural embedding $j: T \hookrightarrow \mathbb{P}^n$. According to the general strategy outlined in Section 5.1, to prove Theorem 5.2.1, it suffices to obtain a generic subspace V such that the forget-supports map

$$j_!(M \otimes_{T \times S} \mathrm{pr}_T^+ \mathcal{L}_\chi) \rightarrow j_+(M \otimes_{T \times S} \mathrm{pr}_T^+ \mathcal{L}_\chi), \quad (*)$$

where $\mathrm{pr}_T: T \times S \rightarrow T$ is the natural projection, is an isomorphism for every $\chi \in V$.

Since both tensoring with a connection and taking the (compactly supported) direct image by j are t -exact functors, we can suppose that M is concentrated in degree zero. For the remainder of this section, we also assume that $r = 2$ and $S = \mathrm{Spec} \mathbb{C}$. The general case does not seem to follow from this particular case, but the same arguments prove the general result, albeit with heavier notation.

Next, we examine the geometry of the open immersion $j: \mathbb{G}_m^2 \hookrightarrow \mathbb{P}^2$ and its complementary closed immersion. The projective space \mathbb{P}^2 has three natural affine charts:

$$\begin{aligned} V_x &= \{[x : y : z] \in \mathbb{P}^2 \mid x \neq 0\} \simeq \mathbb{A}_{y,z}^2 \\ V_y &= \{[x : y : z] \in \mathbb{P}^2 \mid y \neq 0\} \simeq \mathbb{A}_{x,z}^2 \\ V_z &= \{[x : y : z] \in \mathbb{P}^2 \mid z \neq 0\} \simeq \mathbb{A}_{x,y}^2 \end{aligned}$$

where subscripts denote the natural coordinates of these spaces. The image of \mathbb{G}_m^2 in \mathbb{P}^2 is contained in all three charts. However, since the charts V_x and V_z have different coordinates, we will give different names to the copies of \mathbb{G}_m^2 in each chart.

$$\begin{aligned} \mathbb{U}_x &= \mathbb{G}_{m,y,z}^2 \xrightarrow{j_x} V_x = \mathbb{A}_{y,z}^2 \\ \mathbb{U}_y &= \mathbb{G}_{m,x,z}^2 \xrightarrow{j_y} V_y = \mathbb{A}_{x,z}^2 \\ \mathbb{U}_z &= \mathbb{G}_{m,x,y}^2 \xrightarrow{j_z} V_z = \mathbb{A}_{x,y}^2 \end{aligned}$$

Here, the natural isomorphism $\mathbb{U}_x = \mathbb{G}_{m,y,z}^2 \rightarrow \mathbb{U}_z = \mathbb{G}_{m,x,y}^2$ acts as $(y, z) \mapsto (1/z, y/z)$, and the natural isomorphism $\mathbb{U}_y = \mathbb{G}_{m,x,z}^2 \rightarrow \mathbb{U}_z = \mathbb{G}_{m,x,y}^2$ acts as $(x, z) \mapsto (x/z, 1/z)$.

As explained in Section 5.1, the map $(*)$ is an isomorphism if and only if each of the following morphisms

$$\begin{aligned} j_{x,!}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)|_{\mathbb{U}_x} &\rightarrow j_{x,+}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)|_{\mathbb{U}_x} \\ j_{y,!}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)|_{\mathbb{U}_y} &\rightarrow j_{y,+}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)|_{\mathbb{U}_y} \\ j_{z,!}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) &\rightarrow j_{z,+}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) \end{aligned}$$

are. Let us outline how to verify that the forget-supports map associated with j_z is an isomorphism, since the others are similar. The open immersion $j_z: \mathbb{U}_z \hookrightarrow V_z$ factors as

$$\mathbb{A}_{x,y}^2 \setminus \{xy = 0\} \xrightarrow{j_z^x} \mathbb{A}_{x,y}^2 \setminus \{y = 0\} \xrightarrow{j_z^y} \mathbb{A}_{x,y}^2,$$

inducing a factorization of the forget-supports map $j_{z,!}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) \rightarrow j_{z,+}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)$ as

$$j_{z,!}^y j_{z,!}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) \rightarrow j_{z,!}^y j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) \rightarrow j_{z,+}^y j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X).$$

Consequently, $j_{z,!}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) \rightarrow j_{z,+}(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)$ is an isomorphism as long as both

$$\mathrm{gr}_0^V(j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)) \quad \text{and} \quad \mathrm{gr}_0^V(j_{z,+}^y(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X))$$

vanish. We now prove Theorem 5.2.1 by computing these associated graded parts.

Proof of Theorem 5.2.1. Let us start by studying the \mathcal{D} -module $j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X)$. According to Remark 1.5.2, the following isomorphisms hold¹

$$\begin{aligned} j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}} \mathcal{L}_X) &\simeq j_{z,+}^x(\mathcal{M} \otimes_{\mathcal{T}}^! \mathcal{L}_X)[4] \\ &\simeq j_{z,+}^x \mathcal{M} \otimes^! j_{z,+}^x \mathcal{L}_X[4] \\ &\simeq j_{z,+}^x \mathcal{M} \otimes_0^! j_{z,+}^x \mathcal{L}_X[2]. \end{aligned}$$

¹Let $j: \mathbb{U} \rightarrow X$ be an open immersion, and let $M, N \in D_h^b(\mathcal{D}_{\mathbb{U}})$. By recollement and the projection formula, we have $j_! M \otimes_X j_! N \simeq j_!(M \otimes_{\mathbb{U}} j^+ j_! N) \simeq j_!(M \otimes_{\mathbb{U}} N)$. Dually, we have $j_+ M \otimes_X^! j_+ N \simeq j_+(M \otimes_{\mathbb{U}}^! N)$. This formula is used in the second isomorphism below.

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Since $j_{z,+}^x \mathcal{L}_X[2] \simeq j_{z,*}^x \mathcal{L}_X[2]$ is isomorphic to $\mathcal{O}(*\mathbb{G}_{m,x})$ as an \mathcal{O} -module, and tensoring over \mathcal{O} is the same as tensoring over $\mathcal{O}(*\mathbb{G}_{m,x})$, we conclude that

$$j_{z,+}^x(M \otimes_{\mathbb{T}} \mathcal{L}_X) \simeq j_{z,+}^x M \otimes_{\mathcal{O}(*\mathbb{G}_{m,x})} j_{z,+}^x \frac{\mathcal{D}}{\mathcal{D}(x\partial_x + \chi_1, y\partial_y + \chi_2)}.$$

This isomorphism implies that $j_{z,+}^x(M \otimes_{\mathbb{T}} \mathcal{L}_X)$ has the same underlying \mathcal{O} -module as $j_{z,+}^x M$, but the element $x\partial_x$ in the ring of differential operators of $\mathbb{A}_{x,y}^2 \setminus \{y=0\}$ acts on it as

$$m \otimes 1 \mapsto x\partial_x(m) \otimes 1 + m \otimes x\partial_x(1) = x\partial_x(m) \otimes 1 - m \otimes \chi_1 1 = (x\partial_x - \chi_1)(m \otimes 1).$$

It follows that $\mathrm{gr}_0^V(j_{z,+}^x(M \otimes_{\mathbb{T}} \mathcal{L}_X)) \simeq \mathrm{gr}_{\chi_1}^V(j_{z,+}^x M)$. The same arguments show that $\mathrm{gr}_0^V(j_{z,+}^x(M \otimes_{\mathbb{T}} \mathcal{L}_X))$ is isomorphic to $\mathrm{gr}_{\chi_2}^V(j_{z,+}^x M)$.

The change of coordinates from U_z to U_x turns the characters (χ_1, χ_2) into the pair $(\chi_2, -\chi_1 - \chi_2)$. Similarly, the change of coordinates from U_z to U_y maps the characters (χ_1, χ_2) to $(\chi_1, -\chi_1 - \chi_2)$. As above, we have that $j_{x,!}(M \otimes_{\mathbb{T}} \mathcal{L}_X)|_{U_x} \rightarrow j_{x,+}(M \otimes_{\mathbb{T}} \mathcal{L}_X)|_{U_x}$ is an isomorphism if and only if

$$\mathrm{gr}_{\chi_2}^V(j_{y,+}^x M|_{U_x}) \quad \text{and} \quad \mathrm{gr}_{-\chi_1 - \chi_2}^V(j_{x,+} M|_{U_x})$$

vanish. Furthermore, the forget-supports map $j_{y,!}(M \otimes_{\mathbb{T}} \mathcal{L}_X)|_{U_y} \rightarrow j_{y,+}(M \otimes_{\mathbb{T}} \mathcal{L}_X)|_{U_y}$ is an isomorphism precisely when both

$$\mathrm{gr}_{\chi_1}^V(j_{y,+}^x M|_{U_y}) \quad \text{and} \quad \mathrm{gr}_{-\chi_1 - \chi_2}^V(j_{y,+} M|_{U_y})$$

vanish.

Since the V -filtration is discrete, the six aforementioned graded pieces all vanish unless $-\chi_1, -\chi_2, -\chi_2, \chi_1 + \chi_2, -\chi_1$, or $\chi_1 + \chi_2$ are roots modulo \mathbb{Z} of the respective Bernstein–Sato polynomials. The subsets of T^b cut out by the (χ_1, χ_2) satisfying these conditions form the desired translates of linear subspaces. \square

5.3. THE UNIPOTENT CASE

Let U be an n -dimensional unipotent commutative algebraic group over \mathbb{C} . While we believe that U satisfies relative generic vanishing, a formal proof remains elusive. Therefore, this section is devoted to establishing an absolute generic vanishing theorem for U . Additionally, in Remark 5.3.5, we outline a potential proof that U satisfies relative generic vanishing.

Recall that the moduli space U^b is isomorphic to the vector group associated with the dual vector space U^* . Under a choice of coordinates, U becomes isomorphic to \mathbb{G}_a^n

and similarly for U^* . A point $\chi = (\chi_1, \dots, \chi_n) \in U^*$ then corresponds to the character sheaf

$$\mathcal{L}_\chi = (\mathcal{O}_{G_a}, d - \chi_1 dt_1)[-1] \boxtimes \cdots \boxtimes (\mathcal{O}_{G_a}, d - \chi_n dt_n)[-1].$$

With this notation, we state the main result of this section.

PROPOSITION 5.3.1. *Let M be an object of $D_h^b(\mathcal{D}_U)$. There exists an open dense subset V of U^\flat such that the forget-supports map*

$$H_c^*(G, M \otimes_U \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_U \mathcal{L}_\chi)$$

is an isomorphism for all $\chi \in V$. Furthermore, if M is concentrated in degree zero, those cohomology groups vanish in non-zero degrees.

The key tool for the proof of Proposition 5.3.1 is the Fourier transform described in Example 4.3.3. For the reader's convenience, we recall its main properties. Consider the evaluation map $\sigma: U \times U^* \rightarrow \mathbb{A}^1$ and the exponential character sheaf $\mathcal{E} := (\mathcal{O}_{\mathbb{A}^1}, d - dx)[-1]$ on \mathbb{A}^1 . We define the Fourier transform functor $FT_U: D_h^b(\mathcal{D}_U) \rightarrow D_h^b(\mathcal{D}_{U^*})$ as

$$FT_U := pr_{2,+} (pr_1^+(-) \otimes_{U \times U^*} \sigma^+ \mathcal{E}[n]),$$

where $pr_1: U \times U^* \rightarrow U$ and $pr_2: U \times U^* \rightarrow U^*$ are the canonical projections. This functor is a shift of the Fourier–Laumon transform associated with U_{dR} , and coincides with the functor \mathcal{F}_* from [Dai00, §2.2.2].

This operation has a plethora of wonderful properties, but we will content ourselves with explaining those strictly needed for our purposes:

- The functor FT_U is t-exact with respect to the canonical t-structures on $D_h^b(\mathcal{D}_U)$ and $D_h^b(\mathcal{D}_{U^*})$;
- Denoting by $inv_{U^*}: U^* \rightarrow U^*$ the inverse map, we have a natural isomorphism $D_{U^*} \circ FT_U \simeq inv_{U^*}^+ \circ FT_U \circ D_U$.

The importance of the Fourier transform on the generic vanishing theorem lies in the fact that $FT_U(M)$ simultaneously contains the data of all cohomology groups for every character twist of M . The second half of this result was already proven in Proposition 4.1.8; however, we provide a direct proof here using the six-functor formalism of holonomic \mathcal{D} -modules.

PROPOSITION 5.3.2. *Let M be an object of $D_h^b(\mathcal{D}_U)$ and denote by $p: U \rightarrow \text{Spec } \mathbb{C}$ the structure map. Then,*

$$p_!(M \otimes_U \mathcal{L}_\chi) \simeq \chi^+ FT_U(M)[-n] \quad \text{and} \quad p_+(M \otimes_U \mathcal{L}_\chi) \simeq \chi^! FT_U(M)[n]$$

hold for every $\chi \in U^$.*

5. The generic vanishing theorem

Proof. An application of the proper base change theorem on the cartesian diagram

$$\begin{array}{ccc}
 \mathbf{U} \times \text{Spec } \mathbb{C} & \xrightarrow{\text{id} \times \chi} & \mathbf{U} \times \mathbf{U}^* \\
 \bar{p}_1 \downarrow & & \downarrow \text{pr}_2 \\
 \mathbf{U} & & \mathbf{U}^* \\
 p \downarrow & & \downarrow \chi \\
 \text{Spec } \mathbb{C} & \xrightarrow{\chi} & \mathbf{U}^*
 \end{array}$$

yields $\chi^+ \circ \text{pr}_{2,!} \simeq p_! \circ \bar{p}_{1,!} \circ (\text{id} \times \chi)^+$. By applying $\text{pr}_1^+ M \otimes_{\mathbf{U} \times \mathbf{U}^*} \sigma^+ \mathcal{E}$ to both sides, we get

$$\chi^+ \text{FT}_{\mathbf{U}}(M)[-n] \simeq p_! \bar{p}_{1,!} N,$$

where $N = (\text{id} \times \chi)^+ \text{pr}_1^+ M \otimes_{\mathbf{U} \times \text{Spec } \mathbb{C}} (\text{id} \times \chi)^+ \sigma^+ \mathcal{E}$. Since (id, p) is the inverse of \bar{p}_1 , the functors $\bar{p}_{1,!}$ and $(\text{id}, p)^+$ are isomorphic. In particular,

$$\bar{p}_{1,!} N \simeq (\text{id}, p)^+ (\text{id} \times \chi)^+ \text{pr}_1^+ M \otimes_{\mathbf{U}} (\text{id}, p)^+ (\text{id} \times \chi)^+ \sigma^+ \mathcal{E} \simeq M \otimes_{\mathbf{U}} \sigma(-, \chi)^+ \mathcal{E},$$

for $\text{pr}_1 \circ (\text{id} \times \chi) \circ (\text{id}, p) = \text{id}$ and $\sigma \circ (\text{id} \times \chi) \circ (\text{id}, p) = \sigma(-, \chi)$.

Upon a choice of coordinates, both \mathbf{U} and \mathbf{U}^* become isomorphic to \mathbb{G}_a^n , and thus the map $\sigma(-, \chi)$ acts as

$$(t_1, \dots, t_n) \mapsto \chi_1 t_1 + \dots + \chi_n t_n.$$

Therefore, $\sigma(-, \chi)^+ \mathcal{E} \simeq \mathcal{L}_\chi$, giving us the first desired isomorphism.

By applying the result just proved to $D_{\mathbf{U}}(M)$ and taking the dual of both sides, we obtain

$$D_{\mathbb{C}} p_! (D_{\mathbf{U}}(M) \otimes_{\mathbf{U}} \mathcal{L}_\chi) \simeq D_{\mathbb{C}} \chi^+ \text{FT}_{\mathbf{U}}(D_{\mathbf{U}}(M))[n].$$

The left-hand side is

$$\begin{aligned}
 D_{\mathbb{C}} p_! (D_{\mathbf{U}}(M) \otimes_{\mathbf{U}} \mathcal{L}_\chi) &\simeq p_+ D_{\mathbf{U}}(D_{\mathbf{U}}(M) \otimes_{\mathbf{U}} \mathcal{L}_\chi) \\
 &\simeq p_+(M \otimes_{\mathbf{U}}^! D_{\mathbf{U}}(\mathcal{L}_\chi)) \\
 &\simeq p_+(M \otimes_{\mathbf{U}}^! \mathcal{L}_\chi^\vee[2n]) \\
 &\simeq p_+(M \otimes_{\mathbf{U}}^! \mathcal{L}_{-\chi}[2n]) \\
 &\simeq p_+(M \otimes_{\mathbf{U}} \mathcal{L}_{-\chi}),
 \end{aligned}$$

while the right-hand side is

$$\begin{aligned}
 D_{\mathbb{C}} \chi^+ \text{FT}_{\mathbf{U}}(D_{\mathbf{U}}(M))[n] &\simeq \chi^! D_{\mathbf{U}^*} \text{FT}_{\mathbf{U}}(D_{\mathbf{U}}(M))[n] \\
 &\simeq \chi^! \text{inv}_{\mathbf{U}^*}^+ \text{FT}_{\mathbf{U}}(M)[n] \\
 &\simeq \chi^! \text{inv}_{\mathbf{U}^*}^! \text{FT}_{\mathbf{U}}(M)[n] \\
 &\simeq (-\chi)^! \text{FT}_{\mathbf{U}}(M)[n].
 \end{aligned}$$

By switching χ and $-\chi$, we obtain the desired result. \square

Proposition 5.3.1 now follows directly from the main properties of the Fourier transform.

Proof of Proposition 5.3.1. As $\mathrm{FT}_U(M)$ is a bounded complex of holonomic \mathcal{D} -modules, there exists an open dense subset $V \subset U^*$ such that $\mathcal{H}^i(\mathrm{FT}_U(M)|_V)$ is locally free for all $i \in \mathbb{Z}$. Then, whenever χ lies in V , we have that

$$H^i(\mathbb{G}, M \otimes_{\mathbb{G}} \mathcal{L}_\chi) \simeq \mathcal{H}^{i+n}(\chi^! \mathrm{FT}_U(M)|_V) \simeq \chi^* \mathcal{H}^i(\mathrm{FT}_U(M)|_V)$$

and

$$H_c^i(\mathbb{G}, M \otimes_{\mathbb{G}} \mathcal{L}_\chi) \simeq \mathcal{H}^{i-n}(\chi^+ \mathrm{FT}_U(M)|_V) \simeq \chi^* \mathcal{H}^i(\mathrm{FT}_U(M)|_V).$$

The isomorphisms on the right hold due to Remark 1.5.2. \square

Although Proposition 5.3.1 does not establish that the set of characters where generic vanishing holds forms a generic subspace, its proof provides more precise information than its statement. This is clarified in the following remark. Notably, this result is even more precise than its positive characteristic analogue [FFK23, Thm. 2.3].

■ **REMARK 5.3.3 — STRATIFIED VANISHING.** Let M be an object of $D_h^b(\mathcal{D}_U)$. According to [HTT08, Thm. 3.3.1], there exists a decreasing sequence of closed subschemes

$$U^* = Z_0 \supset Z_1 \supset \cdots \supset Z_m \supset Z_{m+1} = \emptyset$$

such that the complements $V_j := Z_j \setminus Z_{j+1}$ are smooth and the cohomology sheaves $\mathcal{H}^*(\mathrm{FT}_U(M)|_{V_j})$ are connections for all j . Denote by d_j the integer $n - \dim V_j$. Then, for any χ in V_j , we have that

$$\begin{aligned} H^i(U, M \otimes_U \mathcal{L}_\chi) &\simeq \chi^* \mathcal{H}^{i+d_j}(\mathrm{FT}_U(M)|_{V_j}), \\ H_c^i(U, M \otimes_U \mathcal{L}_\chi) &\simeq \chi^* \mathcal{H}^{i-d_j}(\mathrm{FT}_U(M)|_{V_j}). \end{aligned}$$

Consequently, $H_c^{i+d_j}(U, M \otimes_U \mathcal{L}_\chi)$ is isomorphic to $H^{i-d_j}(U, M \otimes_U \mathcal{L}_\chi)$ for all i . If M is concentrated in degree zero, these groups vanish for $i \neq 0$. ■

We now outline a potential proof of relative generic vanishing for unipotent groups. The following example illustrates some of the main ideas, which will be formalized in the subsequent discussion.

■ **EXAMPLE 5.3.4.** Consider the unipotent group $U = \mathbb{G}_{a,x}$, the smooth algebraic variety $S = \mathbb{A}_t^1$, and the exponential connection $M = (\mathcal{O}_{U \times S}, d - x dt - t dx)$. Denoting by $j: \mathbb{G}_{a,x} \times \mathbb{A}_t^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{A}_t^1$ the natural open immersion, we claim that the forget-supports map

$$j_!(M \otimes_{U \times S} \mathrm{pr}_U^+ \mathcal{L}_\chi) \rightarrow j_+(M \otimes_{U \times S} \mathrm{pr}_U^+ \mathcal{L}_\chi), \quad (*)$$

where $\mathrm{pr}_U: U \times S \rightarrow U$ is the natural projection, is an isomorphism for every $\chi \in \mathbb{C}^\times$.

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As explained in Section 5.1, it suffices to prove this after restricting to the affine chart

$$\{([x : y], t) \in \mathbb{P}^1 \times \mathbb{A}_x^1 \mid x \neq 0\} \simeq \mathbb{A}_{y,t}^2.$$

Denoting by $j': \mathbb{G}_{m,y} \times \mathbb{A}_t^1 \hookrightarrow \mathbb{A}_{y,t}^2$ the natural inclusion, the problem reduces to proving that the forget-supports map

$$j'_! \left(\mathcal{O}_{\mathbb{G}_{m,y} \times \mathbb{A}_t^1}, d - \frac{dt}{y} + (t + \chi) \frac{dy}{y^2} \right) \rightarrow j'_+ \left(\mathcal{O}_{\mathbb{G}_{m,y} \times \mathbb{A}_t^1}, d - \frac{dt}{y} + (t + \chi) \frac{dy}{y^2} \right)$$

is an isomorphism. Consider the blow-up $f: \text{Bl}_0 \mathbb{A}_{y,t}^2 \rightarrow \mathbb{A}_{y,t}^2$ of the origin in $\mathbb{A}_{y,t}^2$, viewed as the closed subscheme of $\mathbb{A}_{y,t}^2 \times \mathbb{P}^1$ cut out by the points $((y, t), [u : v])$ satisfying $yu = tv$. Since $j': \mathbb{G}_{m,y} \times \mathbb{A}_t^1 \hookrightarrow \mathbb{A}_{y,t}^2$ factors as

$$\begin{array}{ccccc} & & \overset{j'}{\curvearrowright} & & \\ \mathbb{G}_{m,y} \times \mathbb{A}_t^1 & \xrightarrow{\tilde{j}} & \text{Bl}_0 \mathbb{A}_{y,t}^2 & \xrightarrow{f} & \mathbb{A}_{y,t}^2 \\ & \longmapsto & ((y, t), [t : y]) & & \\ & & ((y, t), [u : v]) & \longmapsto & (y, t), \end{array}$$

and $f: \text{Bl}_0 \mathbb{A}_{y,t}^2 \rightarrow \mathbb{A}_{y,t}^2$ is proper, it suffices to prove that the forget-supports map

$$\tilde{j}'_! \left(\mathcal{O}_{\mathbb{G}_{m,y} \times \mathbb{A}_t^1}, d + \frac{dt}{y} - (t + \chi) \frac{dy}{y^2} \right) \rightarrow \tilde{j}'_+ \left(\mathcal{O}_{\mathbb{G}_{m,y} \times \mathbb{A}_t^1}, d + \frac{dt}{y} - (t + \chi) \frac{dy}{y^2} \right)$$

is an isomorphism.

As before, this can be checked locally. Therefore, we consider the affine charts of $\text{Bl}_0 \mathbb{A}_{y,t}^2$:

$$\begin{aligned} V_u &:= \{((y, t), [u : v]) \in \mathbb{A}_{y,t}^2 \times \mathbb{P}^1 \mid yu = tv, u \neq 0\} \simeq \mathbb{A}_{t,v}^2 \\ V_v &:= \{((y, t), [u : v]) \in \mathbb{A}_{y,t}^2 \times \mathbb{P}^1 \mid yu = tv, v \neq 0\} \simeq \mathbb{A}_{y,u}^2. \end{aligned}$$

The intersections of $\mathbb{G}_{m,y} \times \mathbb{A}_t^1$ with V_u and V_v are given, respectively, by

$$\begin{array}{ccc} (t, v) & \mathbb{A}_{t,v}^2 \setminus \{tv = 0\} & \xleftarrow{j_u} \mathbb{A}_{t,v}^2 \\ \downarrow & \downarrow & \downarrow \\ (tv, t) & \mathbb{G}_{m,y} \times \mathbb{A}_t^1 & \hookrightarrow \text{Bl}_0 \mathbb{A}_{y,t}^2 \end{array} \quad \text{and} \quad \begin{array}{ccc} (y, u) & \mathbb{G}_{m,y} \times \mathbb{A}_u^1 & \xleftarrow{j_v} \mathbb{A}_{y,u}^2 \\ \downarrow & \downarrow & \downarrow \\ (y, yu) & \mathbb{G}_{m,y} \times \mathbb{A}_t^1 & \hookrightarrow \text{Bl}_0 \mathbb{A}_{y,t}^2. \end{array}$$

The inverse images of our connection from $\mathbb{G}_{m,y} \times \mathbb{A}_u^1$ to $\mathbb{A}_{t,v}^2 \setminus \{tv = 0\}$ and $\mathbb{G}_{m,y} \times \mathbb{A}_t^1$, respectively, are

$$\left(\mathcal{O}, d + \frac{\chi}{vt^2} dt + \frac{(t + \chi)}{v^2 t} dv \right) \quad \text{and} \quad \left(\mathcal{O}, d - du + \chi \frac{dy}{y^2} \right).$$

For $\chi = 0$, these connections simplify to

$$\left(\mathcal{O}, d + \frac{dv}{v^2}\right) \quad \text{and} \quad (\mathcal{O}, d - du),$$

whereas, for $\chi \neq 0$, the variable changes $t \mapsto \chi t$ and $y \mapsto \chi y$ result in

$$\left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t}dv\right) \quad \text{and} \quad \left(\mathcal{O}, d - du + \frac{dy}{y^2}\right).$$

All in all, the morphism (*) is an isomorphism for all $\chi \in \mathbb{C}^\times$ as long as the forget-supports maps

$$j_{u,!} \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t}dv\right) \rightarrow j_{u,+} \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t}dv\right) \quad (5.1)$$

$$j_{v,!} \left(\mathcal{O}, d - du + \frac{dy}{y^2}\right) \rightarrow j_{v,+} \left(\mathcal{O}, d - du + \frac{dy}{y^2}\right) \quad (5.2)$$

are isomorphisms.

Since the divisor $y = 0$ associated with j_v is smooth, Lemma 5.1.3 can be applied directly. This lemma asserts that the (5.2) is an isomorphism if and only if

$$\text{gr}_0^V \left(j_{v,+} \left(\mathcal{O}, d - du + \frac{dy}{y^2} \right) \right)$$

vanishes. The underlying $\mathbb{C}[u, y]$ -module of $j_{v,+}(\mathcal{O}, d - du + dy/y^2)$ is $\mathbb{C}[u, y^\pm]$, and the differential operators ∂_u and ∂_y act on $p \in \mathbb{C}[u, y^\pm]$ as

$$\partial_u \cdot p = \frac{\partial p}{\partial u} - p \quad \text{and} \quad \partial_y \cdot p = \frac{\partial p}{\partial y} + \frac{p}{y^2}.$$

We claim that the V -filtration on this \mathcal{D} -module is constant. The only potentially non-trivial axiom of the V -filtration is that $\mathbb{C}[u, y^\pm]$ must be a finitely generated $\mathbb{C}[u, y, \partial_u, y\partial_y]$ -module. For this, note that $\{1\}$ serves as a generating set. Indeed, the relation

$$(y\partial_y + n)(y\partial_y + n - 1) \dots (y\partial_y + 1)(y\partial_y) \cdot 1 = 1/y^{n+1}$$

holds for all $n \geq 0$.

The divisor $tv = 0$ associated with j_u is singular, but has simple normal crossings. As a result, the approach outlined in Section 5.1 is applicable. We factor j_u as

$$\mathbb{A}_{t,v}^2 \setminus \{tv = 0\} \xrightarrow{j_u^t} \mathbb{A}_{t,v}^2 \setminus \{v = 0\} \xrightarrow{j_u^v} \mathbb{A}_{t,v}^2,$$

after which it suffices to verify that

$$\text{gr}_0^V \left(j_{u,+}^t \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t}dv \right) \right) \quad \text{and} \quad \text{gr}_0^V \left(j_{u,+} \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t}dv \right) \right)$$

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vanish. The underlying $\mathbb{C}[t, v^\pm]$ - and $\mathbb{C}[t, v]$ -modules of the \mathcal{D} -modules

$$j_{u,+}^t \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t} dv \right) \quad \text{and} \quad j_{u,+} \left(\mathcal{O}, d + \frac{dt}{vt^2} + \frac{(t+1)}{v^2t} dv \right)$$

are both $\mathbb{C}[t^\pm, v^\pm]$. The differential operators ∂_t and ∂_v act on $p \in \mathbb{C}[t^\pm, v^\pm]$ as follows:

$$\partial_t \cdot p = \frac{\partial p}{\partial t} + \frac{p}{vt^2} \quad \text{and} \quad \partial_v \cdot p = \frac{\partial p}{\partial v} + \frac{(t+1)p}{vt^2}.$$

Again, we claim that the associated V -filtrations are constant. To demonstrate this, it suffices to verify that $\mathbb{C}[t^\pm, v^\pm]$ is a finitely generated $\mathbb{C}[t, v^\pm, t\partial_t, \partial_v]$ -module as well as a finitely generated $\mathbb{C}[t, v, \partial_t, v\partial_v]$ -module. In both cases, $\{1\}$ is a generating set, as indicated by the following formulas

$$\begin{aligned} v^{n+1} (t\partial_t + n)(t\partial_t + n - 1) \dots (t\partial_t + 1)(t\partial_t) \cdot 1 &= 1/t^{n+1} \\ (t^2\partial_t)^{n+1} \cdot 1 &= 1/v^{n+1}, \end{aligned}$$

which hold for all $n \geq 0$. ■

We finish this section with some ideas on how to generalize the previous example and prove relative generic vanishing for unipotent groups. It should be taken with a large grain of salt, as we are currently unable to verify some of its details.

■ **REMARK 5.3.5 — ON RELATIVE GENERIC VANISHING.** Let S be a smooth algebraic variety, and M be an object of $D_h^b(\mathcal{D}_{U \times S})$. Additionally, consider the natural embedding $j: U \times S \hookrightarrow \mathbb{P}^n \times S$. We believe that there exists a generic subspace V of U^b such that the forget-supports map

$$j_!(M \otimes_{U \times S} \text{pr}_U^+ \mathcal{L}_\chi) \rightarrow j_+(M \otimes_{U \times S} \text{pr}_U^+ \mathcal{L}_\chi), \quad (*)$$

where $\text{pr}_U: U \times S \rightarrow U$ is the natural projection, is an isomorphism for every χ in V .

Since holonomic complexes are generically connections, one should be able to assume that the cohomology sheaves of M are connections. As every functor above is t -exact, we can further suppose that M is concentrated in degree zero. In what follows, we will also assume that $S = \text{Spec } \mathbb{C}$ for simplicity, even though one should probably keep track of S .

According to the Kedlaya–Mochizuki theorem [Ked11; Moc11], there exists a projective birational morphism $f: X \rightarrow \mathbb{P}^n$, which is an isomorphism above U , such that $f^!j_+M$ has a *good formal structure*. Specifically, there exists a commutative diagram

$$\begin{array}{ccccc} U & \xleftarrow{\tilde{j}} & X & \xleftarrow{\tilde{i}} & Z \\ \parallel & & \downarrow f & & \downarrow \tilde{f} \\ U & \xleftarrow{j} & \mathbb{P}^n & \xleftarrow{i} & \mathbb{P}^{n-1}, \end{array}$$

where both squares are cartesian and Z is a simple normal crossings divisor, such that, for each $z \in Z$, the formal module $\widehat{\mathcal{O}_{X,z}}(*Z) \otimes_{\mathcal{O}_{X,z}} \tilde{j}_+ M$ can, up to ramified coverings, be decomposed as a direct sum of tensor products of exponential modules with regular ones.

From the properness of f , we see that the forget-supports map $(*)$ is an isomorphism as long as

$$\tilde{j}_!(M \otimes_{\mathbb{U}} \mathcal{L}_\chi) \rightarrow \tilde{j}_+(M \otimes_{\mathbb{U}} \mathcal{L}_\chi)$$

is. Following the approach outlined in Section 5.1, we may suppose that Z is smooth. Then, Lemma 5.1.3 states that the forget-supports map above is an isomorphism precisely if the \mathcal{D} -module $\mathrm{gr}_0^V(\tilde{j}_+(M \otimes_{\mathbb{U}} \mathcal{L}_\chi))$ on Z vanishes. The V -filtration being compatible with formal completions and ramification, it suffices to prove that

$$\mathrm{gr}_*^V(\widehat{\mathcal{O}_{X,z}}(*Z) \otimes_{\mathcal{O}_{X,z}} \tilde{j}_+(M \otimes_{\mathbb{U}} \mathcal{L}_\chi)) \simeq \mathrm{gr}_*^V(\widehat{\mathcal{O}_{X,z}}(*Z) \otimes_{\mathcal{O}_{X,z}} \tilde{j}_+ M \otimes_{\mathcal{O}_X(*Z)} \tilde{j}_+ \mathcal{L}_\chi[n])$$

vanishes for χ in a generic subspace of \mathbb{U}^\flat . (Recall that $\tilde{j}_+ \mathcal{L}_\chi[n]$ is concentrated in degree zero.) Finally, the normal form provided by the Kedlaya–Mochizuki theorem should enable us to use the character χ to force the V -filtration to be constant, as illustrated in the example above. ■

5.4. THE GENERAL CASE

We now return to the general case where G is a commutative connected algebraic group. Recall that such a group necessarily fits into a short exact sequence

$$0 \rightarrow L \xrightarrow{\varphi} G \xrightarrow{\psi} A \rightarrow 0,$$

in which L is a linear group and A is an abelian variety. We denote by \mathcal{L}_ω the character sheaf $(\mathcal{O}_G, d + \omega)[- \dim G]$ defined by an invariant differential $\omega \in \Omega_G$. Using this notation, the main observation needed for this section comes from Corollary 3.1.10: every character sheaf on G is of the form

$$\mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha,$$

for some $\omega \in \Omega_G$ and $\alpha \in A^\flat$.

The following proposition is the *raison d'être* of the notion of relative generic vanishing.

PROPOSITION 5.4.1. *Let M be an object of $D_h^b(\mathcal{D}_G)$. Suppose that L satisfies relative generic vanishing. Then there exists a generic subspace V of L^\flat such that the forget-supports map*

$$\psi_!(M \otimes_G \mathcal{L}_\omega) \rightarrow \psi_+(M \otimes_G \mathcal{L}_\omega),$$

is an isomorphism for every $\omega \in \Omega_G$ such that $\varphi^+ \mathcal{L}_\omega$ lies in V .

5. The generic vanishing theorem

Proof. Since $\psi: G \rightarrow A$ is an L-torsor, there exists an étale covering $S \rightarrow A$ and an L-equivariant isomorphism $\mu: L \times S \rightarrow G \times_A S$ making the square

$$\begin{array}{ccc} L \times S & \xrightarrow{\text{pr}_S} & S \\ \mu \downarrow & & \downarrow \\ G \times_A S & & \\ \text{pr}_G \downarrow & & \downarrow \\ G & \xrightarrow{\psi} & A \end{array}$$

cartesian. It suffices to check that $\psi_!(M \otimes_G \mathcal{L}_\omega) \rightarrow \psi_+(M \otimes_G \mathcal{L}_\omega)$ is an isomorphism after pulling back via $S \rightarrow A$. By proper and smooth base change, this map becomes

$$\text{pr}_{S,!}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \mu^+ \text{pr}_G^+ \mathcal{L}_\omega) \rightarrow \text{pr}_{S,+}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \mu^+ \text{pr}_G^+ \mathcal{L}_\omega).$$

Next, consider the following diagram, whose commutativity is ensured by the L-equivariance of μ .

$$\begin{array}{ccccc} L \times S & & & & L \times S \\ \text{id}_L \times e \times \text{id}_S \searrow & & \xrightarrow{m_L \times \text{id}_S} & & \downarrow \mu \\ L \times L \times S & & & & L \times S \\ \text{id}_L \times \mu \downarrow & & & & \downarrow \mu \\ L \times G \times_A S & \xrightarrow{\varphi \times \text{id}} & G \times G \times_A S & \xrightarrow{m_G \times \text{id}_S} & G \times_A S \\ & & \text{pr}_{G \times G} \searrow & & \downarrow \text{pr}_G \\ & & G \times G & \xrightarrow{m_G} & G \end{array}$$

Pulling back \mathcal{L}_ω through the two colored paths results in the isomorphism

$$\mu^+ \text{pr}_G^+ \mathcal{L}_\omega \simeq \text{pr}_L^+ \varphi^+ \mathcal{L}_\omega \otimes \text{pr}_S^+(e \times \text{id}_S)^+ \mu^+ \text{pr}_G^+ \mathcal{L}_\omega.$$

Consequently, $\text{pr}_{S,!}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \mu^+ \text{pr}_G^+ \mathcal{L}_\omega)$ is isomorphic to

$$\begin{aligned} & \text{pr}_{S,!}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \text{pr}_L^+ \varphi^+ \mathcal{L}_\omega \otimes \text{pr}_S^+(e \times \text{id}_S)^+ \mu^+ \text{pr}_G^+ \mathcal{L}_\omega) \\ & \simeq \text{pr}_{S,!}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \text{pr}_L^+ \varphi^+ \mathcal{L}_\omega) \otimes (e \times \text{id}_S)^+ \mu^+ \text{pr}_G^+ \mathcal{L}_\omega, \end{aligned}$$

and similarly for $\text{pr}_{S,+}(\mu^+ \text{pr}_G^+ M \otimes_{L \times S} \mu^+ \text{pr}_G^+ \mathcal{L}_\omega)$. The result then follows from the assumption that L satisfies relative generic vanishing. \square

COROLLARY 5.4.2. *Let M be an object of $D_h^b(\mathcal{D}_G)$. Suppose that L satisfies relative generic vanishing. Then there exists a generic subspace V of L^b such that the forget-supports map*

$$H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha),$$

is an isomorphism for every $\alpha \in A^b$ and every $\omega \in \Omega_G$ such that $\varphi^+ \mathcal{L}_\omega$ lies in V . In particular, there exists a generic subspace W of G^b such that the forget-supports map

$$H_c^*(G, M \otimes_G \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\chi)$$

is an isomorphism for every $\chi \in W$.

Proof. Let $p: G \rightarrow \text{Spec } \mathbb{C}$ and $q: A \rightarrow \text{Spec } \mathbb{C}$ be the structure maps. If $\omega \in \Omega_G$ is such that $\varphi^+ \mathcal{L}_\omega$ lies in the generic subspace V given by the preceding proposition, the projection formula implies that

$$\begin{aligned} p_!(M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) &\simeq q_! \psi_!(M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \\ &\simeq q_+(\psi_!(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha) \\ &\simeq q_+(\psi_+(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha) \\ &\simeq q_+ \psi_+(M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \\ &\simeq p_+(M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha). \end{aligned}$$

This proves the first half of the statement. The second half follows by taking W to be the inverse image of V by the morphism $\varphi^b: G^b \rightarrow L^b$, as in Proposition 3.3.3. \square

As the preceding corollary shows, relative generic vanishing for L implies half of the generic vanishing theorem for G . Specifically, it implies that the forget-supports map

$$H_c^*(G, M \otimes_G \mathcal{L}_\chi) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\chi)$$

is an isomorphism for χ in a generic subspace of G^b . Furthermore, the generic vanishing theorem for abelian varieties proven by Schnell implies that both sides vanish in non-zero degrees for most choices of character sheaves.

THEOREM 5.4.3. *Let M be an object of $D_h^b(\mathcal{D}_G)$. Suppose that L satisfies relative generic vanishing. Then there exists a generic subspace V of L^b such that the forget-supports map*

$$H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \rightarrow H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha),$$

is an isomorphism for every $\alpha \in A^b$ and every $\omega \in \Omega_G$ such that $\varphi^+ \mathcal{L}_\omega$ lies in V . Furthermore, if M is concentrated in degree zero, for each ω as above, there exists a generic subspace W of A^b such that

$$H_c^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \simeq H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha)$$

is concentrated in degree zero for every $\alpha \in W$.

5. The generic vanishing theorem

Proof. The first half of the statement was already proven in Corollary 5.4.2. As in its proof, the projection formula implies that

$$H^*(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) \simeq H^*(A, \psi_+(M \otimes_G \mathcal{L}_\omega) \otimes_A \mathcal{L}_\alpha).$$

If $\varphi^+ \mathcal{L}_\omega$ lies in V , the forget-supports map $\psi_!(M \otimes_G \mathcal{L}_\omega) \rightarrow \psi_+(M \otimes_G \mathcal{L}_\omega)$ is an isomorphism. Consequently, similar to the proof of Proposition 5.1.2, Artin vanishing implies that both sides are concentrated in degree zero.

For each $\omega \in \Omega_G$ as above, [Sch15, Thms. 2.2 and 4.1] provide a generic subspace W of A^b such that the right hand side is concentrated in degree zero for every $\alpha \in W$. \square

6. COMPARISON OF TANNAKIAN GROUPS

Let G be a commutative connected algebraic group over \mathbb{C} that satisfies generic vanishing. In this chapter, we explore a particular tannakian group associated with a holonomic \mathcal{D} -module on G , known as the *convolution group*. These groups not only serve as direct analogues to those governing the equidistribution of exponential sums, as discussed in the introduction of this thesis, but also generalize both differential and difference Galois groups.

CONVOLUTION AND ASSOCIATED TANNAKIAN GROUPS

Let $m: G \times G \rightarrow G$ be the group operation of G . The convolution of \mathcal{D} -modules on G comes in two variants: $!$ -convolution and $+$ -convolution:

$$M *_! N := m_!(M \boxtimes N) \quad \text{and} \quad M *_{+} N := m_+(M \boxtimes N).$$

Each of these operations defines a symmetric monoidal structure on the derived ∞ -category $D_h^b(\mathcal{D}_G)$ of holonomic \mathcal{D} -modules. A central aim of this thesis is to define a tannakian group associated with holonomic \mathcal{D} -modules under these convolution operations.

Two significant challenges arise immediately. Tannakian categories are, by definition, abelian categories equipped with a rigid—meaning that every object is dualizable—symmetric monoidal structure. Yet, neither variant of convolution necessarily preserves holonomic \mathcal{D} -modules in degree zero, and often there are objects that are not dualizable with respect to either convolution.

To better understand the failure of rigidity, note that, according to Proposition 6.1.7, an object M of $D_h^b(\mathcal{D}_G)$ is dualizable with respect to $!$ -convolution if and only if the forget-supports map

$$M *_! N \rightarrow M *_{+} N$$

is an isomorphism for every N in $D_h^b(\mathcal{D}_G)$.

We define an object M of $D_h^b(\mathcal{D}_G)$ to be *negligible* if there exists a generic subspace $V \subset G^b$ such that $H_c^*(G, M \otimes_G \mathcal{L}_\chi)$ vanishes for all $\chi \in V$. This concept is motivated by the result established in Proposition 6.2.9, which shows that although the two forms of convolution may not coincide and do not preserve holonomic \mathcal{D} -modules in degree zero, they do so *up to negligible objects*.

6. Comparison of tannakian groups

By modding out negligible objects, we define the quotient category $\text{Hol}(\mathcal{D}_G)/\text{Neg}(\mathcal{D}_G)$. This quotient is precisely constructed so that the convolution products descend to it, thereby endowing the quotient category with a rigid symmetric monoidal structure. Moreover, our Theorem 6.2.12 demonstrates that this quotient is a neutral tannakian category. Consequently, every holonomic \mathcal{D} -module M on G admits a tannakian group $\text{Conv}(M)$, which we refer to as the *convolution group*.

DIFFERENTIAL GALOIS THEORY

To our knowledge, there are two primary approaches to differential Galois theory. The first approach, pioneered by Kolchin and Picard–Vessiot, is distinctly algebraic in nature. In this framework, the Galois group of a differential operator is defined as the group of automorphisms of its Picard–Vessiot extension, which serves as an analogue to the splitting field of a polynomial.

The second approach is geometric and was popularized by Katz in his foundational works [Kat82a; Kat87]. This approach focuses on the category of vector bundles with integrable connection on a smooth connected algebraic variety. As this category is tannakian, each vector bundle with an integrable connection is associated with a tannakian group, which is its differential Galois group.

Although experts are likely aware of it, we were unable to find any references that directly compare these two approaches. As a result, the first main result in Section 6.3 is Theorem 6.3.9, which establishes an isomorphism between the Galois groups arising from each theory. Our focus was on the one-dimensional setting, as is often the case on the algebraic theory. However, we believe that the same arguments should naturally extend to higher dimensions.

Let M be a holonomic \mathcal{D} -module on the additive group \mathbb{G}_a , and let $\text{FT}(M)$ denote its Fourier transform. As with any holonomic \mathcal{D} -module, $\text{FT}(M)$ restricts to a vector bundle with integrable connection on a dense open subset $U \subset \mathbb{G}_a$. According to the discussion above, $\text{FT}(M)|_U$ has an associated differential Galois group, which coincides with the differential Galois group of the generic fiber $\text{FT}(M)_\eta$ in the algebraic approach.

Notably, Theorem 6.3.11 establishes an isomorphism between this differential Galois group and the convolution group $\text{Conv}(M)$. This isomorphism not only provides an effective method for computing $\text{Conv}(M)$, but also offers valuable theoretical insights into these objects. For instance, it implies that every linear algebraic group can be realized as the convolution group of some holonomic \mathcal{D} -module on \mathbb{G}_a , thereby solving the "inverse problem" for convolution groups.

DIFFERENCE GALOIS THEORY

The Fourier transform for holonomic \mathcal{D} -modules on \mathbb{G}_a has a multiplicative counterpart: the Mellin transform. This functor maps a holonomic \mathcal{D} -module on the multiplicative group \mathbb{G}_m to a \mathbb{Z} -equivariant quasi-coherent sheaf on \mathbb{G}_a . Given that the generator of \mathbb{Z} acts on the ring of functions $\mathbb{C}[s]$ of \mathbb{G}_a by $p(s) \mapsto p(s + 1)$, the generic fiber of the Mellin transform corresponds to a *difference equation*.

A classic example from analysis illustrates this operation: the Gamma function Γ is the Mellin transform of the exponential function, and it satisfies the difference equation

$$\Gamma(s + 1) = s\Gamma(s).$$

Just as differential operators have associated Galois groups, so too do difference operators. This is the essence of difference Galois theory, as presented in [VS97].

Analogous to the additive setting, Theorem 6.4.5 establishes an isomorphism between the convolution group $\text{Conv}(M)$ of a holonomic \mathcal{D} -module M on \mathbb{G}_m and the difference Galois group of $\text{MT}(M)_\eta$. Here as well, this isomorphism gives an effective method for computing $\text{Conv}(M)$ and yields some theoretical insights about those objects.

However, unlike the differential case, not every linear algebraic group can be realized as the convolution group of a holonomic \mathcal{D} -module on \mathbb{G}_m . We demonstrate that the group of connected components of $\text{Conv}(M)$ is always cyclic. Additionally, we offer a partial answer to the inverse problem: every connected linear algebraic group is isomorphic to the convolution group of some holonomic \mathcal{D} -module on \mathbb{G}_m .

6.1. CONVOLUTION OF HOLONOMIC \mathcal{D} -MODULES

Let G be a commutative connected algebraic group over a field k of characteristic zero, with $m: G \times G \rightarrow G$ as its group operation. The convolution of \mathcal{D} -modules can be viewed as a particular case of the convolution of quasi-coherent sheaves on commutative group stacks, as introduced in Section 4.2. However, in this section, we undertake a more detailed analysis. While many of the results discussed here are familiar to experts, we believe it could be useful to compile them comprehensively in one location.

DEFINITION 6.1.1 (Convolution). We define the $!$ -convolution and $+$ -convolution products as bifunctors $D_h^b(\mathcal{D}_G) \times D_h^b(\mathcal{D}_G) \rightarrow D_h^b(\mathcal{D}_G)$ that map a pair (M, N) to $M *_! N := m_!(M \boxtimes N)$ and $M *_+ N := m_+(M \boxtimes N)$, respectively.

An elementary but frequently useful observation is that both convolution products reduce to the tensor product when $G = \text{Spec } k$. As we will see, most properties of the tensor product in a six-functor formalism have their counterparts in convolution.

6. Comparison of tannakian groups

We start by observing that convolution defines a symmetric monoidal structure, accompanied by a tensor-hom adjunction.

PROPOSITION 6.1.2. *Both convolutions equip $D_h^b(\mathcal{D}_G)$ with a symmetric monoidal structure, with $\delta_e := e_+k$ as identity. Furthermore, the symmetric monoidal ∞ -category $(D_h^b(\mathcal{D}_G), *)$ is closed, with its inner hom given by*

$$\underline{\mathrm{Hom}}_G^*(M, N) := \mathrm{pr}_{1,+} \underline{\mathrm{Hom}}_{G \times G}(\mathrm{pr}_2^+ M, m^! N),$$

where $\mathrm{pr}_1, \mathrm{pr}_2: G \times G \rightarrow G$ denote the natural projections.

Before proceeding with the proof, we will first clarify some concepts regarding closed symmetric monoidal categories.

■ REMARK 6.1.3 — ON CLOSED SYMMETRIC MONOIDAL CATEGORIES. Let (\mathcal{C}, \otimes) be a symmetric monoidal ∞ -category, with 1 denoting its identity object. The symmetric monoidal ∞ -category (\mathcal{C}, \otimes) is said to be *closed* if, for every object N in \mathcal{C} , the functor $- \otimes N$ admits a right adjoint $\underline{\mathrm{Hom}}(N, -)$. In this case, the adjunction can be enhanced to an isomorphism

$$\underline{\mathrm{Hom}}(M \otimes N, P) \simeq \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(N, P)),$$

which is natural in M, N , and P . The dual of an object N , denoted N^\vee , is defined as $\underline{\mathrm{Hom}}(N, 1)$. By adjunction, the identity map $N^\vee \rightarrow N^\vee$ induces the *evaluation map*

$$\mathrm{ev}: N \otimes N^\vee \rightarrow 1.$$

The object N is said to be *reflexive* if the adjunct $N \rightarrow (N^\vee)^\vee$ of the evaluation map is an isomorphism. In this case, the enhanced adjunction gives an isomorphism

$$\underline{\mathrm{Hom}}(M, N) \simeq (M \otimes N^\vee)^\vee.$$

For objects M and N in \mathcal{C} , consider the morphism $\mu_{M,N}: M^\vee \otimes N \rightarrow \underline{\mathrm{Hom}}(M, N)$, defined as the composition

$$M^\vee \otimes N \simeq M^\vee \otimes \underline{\mathrm{Hom}}(1, N) \xrightarrow{\circ} \underline{\mathrm{Hom}}(M, N).$$

An object M is said to be *dualizable* if $\mu_{M,M}$ is an isomorphism, in which case M is also reflexive and $\mu_{M,N}$ is an isomorphism for every N . If M is dualizable, there exists a *coevaluation map* coev , defined as the composition

$$1 \rightarrow \underline{\mathrm{Hom}}(M, M) \xrightarrow{\mu_{M,M}^{-1}} M^\vee \otimes M,$$

with the property that both compositions

$$\begin{aligned} M &\simeq M \otimes 1 \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} M \otimes M^\vee \otimes M \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} 1 \otimes M \simeq M \\ M^\vee &\simeq 1 \otimes M^\vee \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} M^\vee \otimes M \otimes M^\vee \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} M^\vee \otimes 1 \simeq M^\vee \end{aligned}$$

are homotopic to the identity morphisms. Conversely, if there exists an object N , together with morphisms $\varepsilon: M \otimes N \rightarrow 1$ and $\eta: 1 \rightarrow N \otimes M$, such that both compositions

$$\begin{aligned} M &\simeq M \otimes 1 \xrightarrow{\text{id} \otimes \eta} M \otimes N \otimes M \xrightarrow{\varepsilon \otimes \text{id}} 1 \otimes M \simeq M \\ N &\simeq 1 \otimes N \xrightarrow{\eta \otimes \text{id}} N \otimes M \otimes N \xrightarrow{\text{id} \otimes \varepsilon} N \otimes 1 \simeq N \end{aligned}$$

are homotopic to the identities, then M is dualizable and N is isomorphic to M^\vee . Moreover, under this identification, ε corresponds to ev and η corresponds to coev [DP80, Thm. 1.3]. If every object is dualizable, the closed symmetric monoidal ∞ -category (\mathcal{C}, \otimes) is said to be *rigid*. ■

Proof of Proposition 6.1.2. Following the same reasoning as in the proof of Proposition 4.2.2, and applying it to the six-functor formalism of holonomic \mathcal{D} -modules established in Proposition 1.5.1, along with the variant that maps a correspondence

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array}$$

to $g_+ f^!$, it follows that both convolutions equip $D_h^b(\mathcal{D}_G)$ with a symmetric monoidal structure, with $\delta_e := e_+ k$ serving as identity object. The explicit description for the inner hom in $(D_h^b(\mathcal{D}_G), *_!)$ is a consequence of the standard adjunctions within a six-functor formalism. □

Similarly to how inverse images are symmetric monoidal with respect to the tensor product, direct images are symmetric monoidal with respect to convolution. The corollary below follows directly from the proof of the previous proposition.

COROLLARY 6.1.4. *Let $\varphi: G \rightarrow H$ be a morphism of commutative connected algebraic groups over k . The direct image functor φ_+ (resp. the proper direct image functor $\varphi_!$) is symmetric monoidal with respect to $+$ -convolution (resp. $!$ -convolution). Furthermore, the duality functor D_G is also symmetric monoidal but switches both versions of convolution.*

Lastly, in our analogies with tensor products, the following proposition provides an analogue of the projection formula for convolution.

PROPOSITION 6.1.5. *Let $\varphi: G \rightarrow H$ be a morphism of commutative connected algebraic groups over k . Then, there are isomorphisms*

$$\varphi^+(\varphi_! M *_! N) \simeq M *_! \varphi^+ N \quad \text{and} \quad \varphi^!(\varphi_+ M *_+ N) \simeq M *_+ \varphi^! N,$$

for every M in $D_h^b(\mathcal{D}_G)$ and N in $D_h^b(\mathcal{D}_H)$.

6. Comparison of tannakian groups

Proof. An application of proper base change in the cartesian diagram

$$\begin{array}{ccc}
 G \times G & \xrightarrow{\text{id} \times \varphi} & G \times H \\
 \downarrow m & & \downarrow \varphi \times \text{id} \\
 & & H \times H \\
 & & \downarrow m \\
 G & \xrightarrow{\varphi} & H
 \end{array}$$

yields that

$$\varphi^+ \circ m_! \circ (\varphi \times \text{id})_! \simeq m_! \circ (\text{id} \times \varphi)^+ \quad \text{and} \quad \varphi^! \circ m_+ \circ (\varphi \times \text{id})_+ \simeq m_+ \circ (\text{id} \times \varphi)^!.$$

The desired isomorphisms are then obtained by evaluating these natural isomorphisms on $M \boxtimes N$. \square

While the inner hom for $!$ -convolution is seldom discussed in the literature, a duality functor is frequently encountered (e.g., in [Kat12] and [Krä14]). The following lemma demonstrates that these two functors are naturally compatible. As in the rest of this thesis, we denote by $\text{inv}: G \rightarrow G$ the inverse map.

LEMMA 6.1.6. *For an object M of $D_h^b(\mathcal{D}_G)$, denote by M^\heartsuit the convolution dual $\underline{\text{Hom}}_G^*(M, \delta_e)$. Then, there is a natural isomorphism $(-)^{\heartsuit} \simeq \text{inv}^+ \circ D_G$. In particular, every object of $D_h^b(\mathcal{D}_G)$ is reflexive with respect to $!$ -convolution.*

Proof. By unraveling the definitions and utilizing that δ_e is auto-dual, we find that

$$\begin{aligned}
 \underline{\text{Hom}}_G^*(M, \delta_e) &= \text{pr}_{1,+} \underline{\text{Hom}}_{G \times G}(\text{pr}_2^+ M, m^! \delta_e) \\
 &= \text{pr}_{1,+} D_{G \times G}(\text{pr}_2^+ M \otimes_{G \times G} D_{G \times G}(m^! \delta_e)) \\
 &\simeq \text{pr}_{1,+} D_{G \times G}(\text{pr}_2^+ M \otimes_{G \times G} m^+ D_G(\delta_e)) \\
 &\simeq \text{pr}_{1,+} D_{G \times G}(\text{pr}_2^+ M \otimes_{G \times G} m^+ \delta_e) \\
 &\simeq D_G \text{pr}_{1,!}(\text{pr}_2^+ M \otimes_{G \times G} m^+ \delta_e).
 \end{aligned}$$

Since the rectangle in the diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{\text{(inv, id)}} & & \\
 G & \xrightarrow{\Delta_G} & G \times G & \xrightarrow{\text{inv} \times \text{id}} & G \times G \\
 \downarrow p & & & & \downarrow m \\
 \text{Spec } k & \xrightarrow{e} & & & G
 \end{array}$$

is cartesian, it follows that

$$m^+ \delta_e = m^+ e_! k \simeq (\text{inv} \times \text{id})_! \Delta_{G,!} p^+ k \simeq (\text{inv} \times \text{id})_! \Delta_{G,!} 1_G.$$

All in all, we obtain $\underline{\mathrm{Hom}}_G^*(M, \delta_e) \simeq D_G \mathrm{pr}_{1,!}(\mathrm{pr}_2^+ M \otimes_{G \times G} (\mathrm{inv} \times \mathrm{id})_! \Delta_G, !1_G)$. Applying the projection formula, this simplifies to

$$\begin{aligned} D_G \mathrm{pr}_{1,!}(\mathrm{inv} \times \mathrm{id})_!((\mathrm{inv} \times \mathrm{id})^+ \mathrm{pr}_2^+ M \otimes_{G \times G} \Delta_G, !1_G) \\ \simeq D_G \mathrm{inv}_! \mathrm{pr}_{1,!}(\mathrm{pr}_2^+ M \otimes_{G \times G} \Delta_G, !1_G), \end{aligned}$$

since $\mathrm{pr}_2 \circ (\mathrm{inv} \times \mathrm{id}) = \mathrm{pr}_2$ and $\mathrm{pr}_1 \circ (\mathrm{inv} \times \mathrm{id}) = \mathrm{inv} \circ \mathrm{pr}_1$. By applying the projection formula one more time, we find

$$D_G \mathrm{inv}_! \mathrm{pr}_{1,!} \Delta_G, !(\Delta_G^+ \mathrm{pr}_2^+ M \otimes_{G \times G} 1_G) \simeq D_G \mathrm{inv}_! M,$$

as $\mathrm{pr}_1 \circ \Delta_G = \mathrm{pr}_2 \circ \Delta_G = \mathrm{id}_G$. Finally, since $\mathrm{inv}: G \rightarrow G$ is an isomorphism, this is nothing but $\mathrm{inv}^+ D_G(M)$. \square

Our next result, that gives an interesting criterion for dualizable objects with respect to $!$ -convolution, is a consequence of [BD13, Cor. 4.6]. That being said, in our setting we can also give a direct proof.

PROPOSITION 6.1.7. *An object M in $D_h^b(\mathcal{D}_G)$ is dualizable with respect to $!$ -convolution if and only if the forget-supports map $M *_! N \rightarrow M *_+ N$ is an isomorphism for every N in $D_h^b(\mathcal{D}_G)$.*

Proof. Let $M^\heartsuit \simeq \mathrm{inv}^+ D_G(M)$ denote the dual of M in the closed symmetric monoidal ∞ -category $(D_h^b(\mathcal{D}_G), *_!)$. By definition, M is dualizable if and only if the natural map $\mu: M^\heartsuit *_! N \rightarrow \underline{\mathrm{Hom}}_G^*(M, N)$ is an isomorphism for every N in $D_h^b(\mathcal{D}_G)$. According to Lemma 6.1.6, one can replace N with N^\heartsuit without loss of generality. By unraveling the definitions, we find that μ is the image of the morphism

$$\mathrm{pr}_{1,!}(\mathrm{pr}_2^+ M \otimes_{G \times G} m^+ \mathrm{inv}^+ N) \rightarrow m_+(\mathrm{inv} \times \mathrm{inv})^+(M \boxtimes N)$$

through the duality functor D_G .

Since $\mathrm{pr}_1 = m \circ \tau$, where $\tau: G \times G \rightarrow G \times G$ is the *shear map* $(g, h) \mapsto (gh, h^{-1})$, we have that $\mathrm{pr}_{1,!} \simeq m_! \tau_!$. Given that τ is an involution, it follows that $\tau_! \simeq \tau^+$. Consequently, $D_G(\mu)$ corresponds to the forget-supports map

$$m_!(\mathrm{inv} \times \mathrm{inv})^+(M \boxtimes N) \rightarrow m_+(\mathrm{inv} \times \mathrm{inv})^+(M \boxtimes N).$$

Finally, applying proper base change to the cartesian square

$$\begin{array}{ccc} G \times G & \xrightarrow{\mathrm{inv} \times \mathrm{inv}} & G \times G \\ m \downarrow & & \downarrow m \\ G & \xrightarrow{\mathrm{inv}} & G, \end{array}$$

we conclude that μ is the image of the forget-supports map $M *_! N \rightarrow M *_+ N$ through the anti-equivalence of categories $(-)^{\heartsuit}$. This completes the proof. \square

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COROLLARY 6.1.8. *The symmetric monoidal ∞ -category $(D_h^b(\mathcal{D}_G), *_{!})$ is rigid if and only if G is an abelian variety.*

Proof. If G is proper, so is the multiplication map $m: G \times G \rightarrow G$ and then the forget-supports map $M *_{!} N \rightarrow M *_{+} N$ is always an isomorphism. Conversely, suppose that $M *_{!} N \xrightarrow{\sim} M *_{+} N$ for all $M, N \in D_h^b(\mathcal{D}_G)$. Let $\varphi: L \rightarrow G$ be the kernel of the Albanese morphism, as in the Barsotti–Chevalley decomposition, and take $M = N$ to be $\varphi_{!}1_L$.

The given isomorphism says that $m_{!}(\varphi \times \varphi)_{!}1_{L \times L} \simeq m_{+}(\varphi \times \varphi)_{+}1_{L \times L}$. Since φ is a group morphism, this is the same as $\varphi_{!}m_{L,!}1_{L \times L} \simeq \varphi_{+}m_{L,+}1_{L \times L}$, where $m_L: L \times L \rightarrow L$ is the group law of L . Applying φ^+ to both sides, we simplify this expression to $m_{L,!}1_{L \times L} \simeq m_{L,+}1_{L \times L}$.

Let $p: L \rightarrow \text{Spec } k$ be the structure map. Since the multiplication map m_L is isomorphic to a projection $L \times L \rightarrow L$, the map $m_{L,!}1_{L \times L} \rightarrow m_{L,+}1_{L \times L}$ is the image by p^+ of the forget-supports map $p_{!}1_L \rightarrow p_{+}1_L$. The latter is an isomorphism since p has a section.

Without loss of generality, we may suppose that k is algebraically closed. Then, the linear group L decomposes as $\mathbb{G}_a^r \times \mathbb{G}_m^s$, for some r, s . As a result, the Künneth formula gives isomorphisms

$$H_c^0(\mathbb{A}^1)^{\otimes r} \otimes_k H_c^0(\mathbb{G}_m)^{\otimes s} \simeq \mathcal{H}^0(p_{!}1_L) \xrightarrow{\sim} \mathcal{H}^0(p_{+}1_L) \simeq H^0(\mathbb{A}^1)^{\otimes r} \otimes_k H^0(\mathbb{G}_m)^{\otimes s}.$$

This implies that $r = s = 0$, since otherwise the left-hand side vanishes whereas the right-hand side does not. \square

The final proposition of this section gives a compatibility between convolution and character sheaves. This compatibility is a counterpart of the equality

$$\begin{aligned} ((f \cdot \chi) * (g \cdot \chi))(t) &= \int f(x)\chi(x)g(tx^{-1})\chi(tx^{-1}) dx \\ &= \left(\int f(x)g(tx^{-1}) dx \right) \chi(t) = (f * g)(t) \cdot \chi(t), \end{aligned}$$

under the analogy between functions and sheaves.

PROPOSITION 6.1.9. *Let $\chi \in G^b(k)$, and let $t_\chi: D_h^b(\mathcal{D}_G) \rightarrow D_h^b(\mathcal{D}_G)$ be the functor $- \otimes_G \mathcal{L}_\chi$. Then t_χ is t -exact and symmetric monoidal with respect to both convolutions.*

Proof. For complexes M and N in $D_h^b(\mathcal{D}_X)$, the projection formula implies that

$$\begin{aligned} t_\chi(M) *_{!} t_\chi(N) &= m_{!}((M \otimes_G \mathcal{L}_\chi) \boxtimes (N \otimes_G \mathcal{L}_\chi)) \\ &\simeq m_{!}((M \boxtimes N) \otimes_{G \times G} (\mathcal{L}_\chi \boxtimes \mathcal{L}_\chi)) \\ &\simeq m_{!}((M \boxtimes N) \otimes_{G \times G} m^+ \mathcal{L}_\chi) \\ &\simeq m_{!}(M \boxtimes N) \otimes_G \mathcal{L}_\chi \\ &= t_\chi(M *_{!} N). \end{aligned}$$

If we replace $*_!$ by $*_+$, all of these isomorphisms still hold, with the possible exception of the projection formula. Nevertheless, the isomorphism

$$m_+((M \boxtimes N) \otimes_{G \times G} m^+ \mathcal{L}_\chi) \simeq m_+(M \boxtimes N) \otimes_G \mathcal{L}_\chi$$

holds because \mathcal{L}_χ is dualizable with respect to the tensor product. Moreover, the functor t_χ sends the convolution identity δ_e to itself. Indeed, another application of the projection formula gives

$$\delta_e \otimes_G \mathcal{L}_\chi = e_! k \otimes_G \mathcal{L}_\chi \simeq e_!(k \otimes_k \underbrace{e^+ \mathcal{L}_\chi}_{\simeq k}) \simeq e_! k = \delta_e,$$

finishing the proof. □

6.2. THE CONVOLUTION GROUP

The goal of this section is to construct a tannakian category of holonomic \mathcal{D} -modules using the convolution products. As first observed by Gabber and Loeser [GL96] in a related setting, there exists a quotient ∞ -category of $D_h^b(\mathcal{D}_G)$ where both convolutions coincide and become t-exact. The heart of this stable ∞ -category is going to be the desired tannakian category. Most of the contents of this section appear in some way or another in [GL96], [Krä14], or [FFK23]. From this point onward, we assume that $k = \mathbb{C}$.

DEFINITION 6.2.1 (Negligible objects). Let M be a holonomic \mathcal{D} -module on G . We say that M is *negligible* if there exists a generic subspace V of G^b such that $H_c^*(G, M \otimes_G \mathcal{L}_\chi) = 0$ for all $\chi \in V$. An object of $D_h^b(\mathcal{D}_G)$ is said to be *negligible* if its cohomology sheaves are. The full subcategories of $\text{Hol}(\mathcal{D}_G)$ and $D_h^b(\mathcal{D}_G)$ consisting of negligible objects are denoted by $\text{Neg}(\mathcal{D}_G)$ and $D_{\text{neg}}^b(\mathcal{D}_G)$, respectively.

The study of negligible is highly dependent on a generic vanishing theorem. Since we have yet to prove such a result in absolute generality, we make it an axiom.

DEFINITION 6.2.2 (Generic vanishing). We say that G *satisfies generic vanishing* if, for every holonomic \mathcal{D} -module M on G , there exists a generic subspace V of G^b such that

$$\begin{aligned} H_c^i(G, M \otimes_G \mathcal{L}_\chi) &= H^i(G, M \otimes_G \mathcal{L}_\chi) = 0 \text{ for } i \neq 0; \\ H_c^0(G, M \otimes_G \mathcal{L}_\chi) &\simeq H^0(G, M \otimes_G \mathcal{L}_\chi) \end{aligned}$$

for every $\chi \in V$.

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The results in the previous chapter establish that the additive group¹ \mathbb{G}_a , tori, and abelian varieties satisfy generic vanishing. A simple spectral sequence argument also implies the following seemingly stronger result.

LEMMA 6.2.3. *Assume that G satisfies generic vanishing and let M be an object of $D_h^b(\mathcal{D}_G)$. Then, there exists a generic subspace V of G^b such that*

$$H_c^i(G, M \otimes_G \mathcal{L}_\chi) \simeq H^i(G, M \otimes_G \mathcal{L}_\chi) \simeq H_c^0(G, \mathcal{H}^i(M) \otimes_G \mathcal{L}_\chi)$$

holds for all $i \in \mathbb{Z}$ and all $\chi \in V$.

Proof. Let χ be a character in G^b . Since the functor $- \otimes_G \mathcal{L}_\chi$ is t-exact, it follows that $\mathcal{H}^j(M \otimes_G \mathcal{L}_\chi) \simeq \mathcal{H}^j(M) \otimes_G \mathcal{L}_\chi$ for all j . As a result, there are spectral sequences

$$\begin{aligned} E_2^{i,j} &= H_c^i(G, \mathcal{H}^j(M) \otimes_G \mathcal{L}_\chi) \implies H_c^{i+j}(G, M \otimes_G \mathcal{L}_\chi) \\ F_2^{i,j} &= H^i(G, \mathcal{H}^j(M) \otimes_G \mathcal{L}_\chi) \implies H^{i+j}(G, M \otimes_G \mathcal{L}_\chi). \end{aligned}$$

For each j , there exists a generic subspace V_j of G^b such that $H_c^*(G, \mathcal{H}^j(M) \otimes_G \mathcal{L}_\chi)$ coincides with $H^*(G, \mathcal{H}^j(M) \otimes_G \mathcal{L}_\chi)$ and is concentrated in degree zero for $\chi \in V_j$. These subspaces may only differ from G^b for a finite number of indices. Thus, their intersection V is also a generic subspace where the desired conclusion holds. \square

The following proposition connects our definition of negligible objects with those found in [GL96; Kat12; Krä14]. We refer the reader to Example 4.3.4 for further details on the Mellin transform of \mathcal{D} -modules on tori.

PROPOSITION 6.2.4. *Let G be an abelian variety or a torus, and let M be a holonomic \mathcal{D} -module on G . Then M is negligible if and only if its Euler characteristic $\chi(G, M)$ vanishes. Furthermore, when G is a torus, M is negligible precisely when the generic fiber of its Mellin transform vanishes.*

Proof. Before starting the proof, we note that Theorem A.1.1 asserts that the Euler characteristic $\chi(G, M)$ coincides with its compactly supported counterpart $\chi_c(G, M)$. In particular, it suffices to prove that M is negligible if and only if its compactly supported Euler characteristic $\chi_c(G, M)$ vanishes.

Assume that M is negligible. Since generic subspaces are non-empty, there exists an element $\chi \in G^b$ such that

$$H_c^*(G, M \otimes_G \mathcal{L}_\chi) = 0.$$

As a result, Lemma A.2.5 and Proposition A.2.6 imply that $\chi_c(G, M)$ coincides with $\chi_c(G, M \otimes_G \mathcal{L}_\chi) = 0$. Conversely, if the compactly supported Euler characteristic

¹Proposition 5.3.1 implies that unipotent groups satisfy a weaker version of generic vanishing. Both versions agree in dimension one.

$\chi_c(G, M)$ vanishes, the generic vanishing theorems ensure the existence of a generic subspace V of G^b such that $H_c^i(G, M \otimes_G \mathcal{L}_\chi) = 0$ for $i \neq 0$ and

$$\dim H_c^0(G, M \otimes_G \mathcal{L}_\chi) = \chi_c(G, M \otimes_G \mathcal{L}_\chi) = \chi_c(G, M) = 0$$

for all χ in V . This proves that M is negligible. The statement regarding the Mellin transform follows from [LS92, Thm. 2]. \square

The following proposition provides a recipe for constructing negligible holonomic \mathcal{D} -modules. In particular, it asserts that character sheaves are always negligible.

PROPOSITION 6.2.5. *Let $\rho: G \rightarrow H$ be an epimorphism with a non-trivial connected kernel N and let $d = \dim N$. Then, for any holonomic \mathcal{D} -module M on H and any $\chi \in G^b$, the object $\rho^+ M \otimes_G \mathcal{L}_\chi[d]$ is negligible. If G is a torus or an abelian variety, every simple negligible holonomic \mathcal{D} -module is of this form.*

The proof of this proposition requires a lemma akin to the orthogonality of characters in classical harmonic analysis.

LEMMA 6.2.6. *Let χ be a non-zero element of G^b . Then, the de Rham cohomology groups $H^*(G, \mathcal{L}_\chi)$ and their compactly supported counterparts $H_c^*(G, \mathcal{L}_\chi)$ all vanish.*

Proof. Recall that the category of vector bundles with integrable connection on G is tannakian. Given a fiber functor ω , with associated tannakian group Γ , we affirm that $H^0(G, \mathcal{E}) \simeq \omega(\mathcal{E})^\Gamma$ holds for every integrable connection \mathcal{E} . Indeed, the left-hand side is $\text{Hom}(\mathcal{O}_G, \mathcal{E})$, whereas the right-hand side is $\text{Hom}(\omega(\mathcal{O}_G), \omega(\mathcal{E}))$.² If \mathcal{E} is simple, so is the representation $\omega(\mathcal{E})$. In particular, it follows that $H^0(G, \mathcal{L}_\chi) = \omega(\mathcal{L}_\chi)^\Gamma = 0$.

We will prove by induction that $H^i(G, \mathcal{L}_\chi)$ vanishes for higher i . Assume it holds for all $0 \leq i < r$. Factor the identity map $\text{id}: G \rightarrow G$ as

$$G \xrightarrow{(\text{id}, e)} G \times G \xrightarrow{m} G,$$

to obtain a factorization $H^r(G, \mathcal{L}_\chi) \rightarrow H^r(G \times G, m^+ \mathcal{L}_\chi) \rightarrow H^r(G, \mathcal{L}_\chi)$ of the identity. Now,

$$H^r(G \times G, m^+ \mathcal{L}_\chi) \simeq H^r(G \times G, \mathcal{L}_\chi \boxtimes \mathcal{L}_\chi) \simeq \bigoplus_{i+j=r} H^i(G, \mathcal{L}_\chi) \otimes_k H^j(G, \mathcal{L}_\chi),$$

and every term on the right vanishes. Thus, $H^r(G, \mathcal{L}_\chi)$ vanishes as well. The statement regarding compactly supported cohomology follows by Poincaré duality. \square

²Recall from Remark 1.5.2 our convention that when connections are regarded as \mathcal{D} -modules, they should be shifted to be in degree equal to the dimension of the underlying space.

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Proof of Proposition 6.2.5. Let χ' be any point of G^b distinct from χ . By the projection formula, we have

$$H_c^*(G, \rho^+ M \otimes_G \mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'}[d]) \simeq H_c^{*+d}(H, M \otimes_H \rho_!(\mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'})).$$

We claim that $\rho_!(\mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'})$ vanishes, implying the desired result. To prove this, it suffices to verify that $h^+ \rho_!(\mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'}) = 0$ for all $h \in H(k)$. Since character sheaves are translation-invariant, proper base change identifies this object with $\bar{\rho}_!(\mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'})$, where $\bar{\rho}: N \rightarrow \text{Spec } k$ denotes the structure map. Given that χ' is distinct from χ , the character sheaf $\mathcal{L}_\chi \otimes_G \mathcal{L}_{\chi'}$ is non-trivial, and the result follows from Lemma 6.2.6.

The characterizations of simple negligible holonomic \mathcal{D} -modules follow from Proposition 6.2.4, as well as [Sch15, Cor. 5.2] and [GL96, Thm. 9.2]. \square

Let us recall some fundamental facts about quotients of categories. For an abelian category A , a non-empty full subcategory B is said to be *thick* if it is stable under subquotients and extensions. In this case, B is also an abelian category, and there exists an abelian category A/B equipped with an exact functor $A \rightarrow A/B$ universal among exact functors to abelian categories mapping the objects of B to zero [Stacks, Tag 02MS].

For a stable ∞ -category C , a *stable subcategory* of C is a non-empty full subcategory D that is stable under the formation of fibers and cofibers. In this case, there exists a stable ∞ -category C/D equipped with an exact functor $C \rightarrow C/D$, which is universal among exact functors to stable ∞ -categories mapping the objects of D to zero [NS18, Thm. I.3.3].³

Assume that C has a symmetric monoidal structure \otimes where the tensor product is exact in each variable. We say that D is a \otimes -*ideal* if $M \otimes N$ lies in D for all M in C and N in D . Then, the quotient C/D inherits a unique symmetric monoidal structure such that the functor $C \rightarrow C/D$ is symmetric monoidal [NS18, Thm. I.3.6].

PROPOSITION 6.2.7. *Assuming that G satisfies generic vanishing, the full subcategory $\text{Neg}(\mathcal{D}_G)$ of $\text{Hol}(\mathcal{D}_G)$ is thick. Furthermore, $D_{\text{neg}}^b(\mathcal{D}_G)$ is a stable subcategory of $D_h^b(\mathcal{D}_G)$ and is a \otimes -ideal with respect to $!$ -convolution.*

Proof. Consider a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ of holonomic \mathcal{D} -modules on G , and let χ belong to the intersection of the generic subspaces associated with M , N , and P . Note that this intersection remains a generic subspace. The long exact sequence in cohomology gives a short exact sequence

$$0 \rightarrow H_c^0(G, M \otimes_G \mathcal{L}_\chi) \rightarrow H_c^0(G, N \otimes_G \mathcal{L}_\chi) \rightarrow H_c^0(G, P \otimes_G \mathcal{L}_\chi) \rightarrow 0.$$

Consequently, N is negligible if and only if both M and P are. It follows that $\text{Neg}(\mathcal{D}_G)$ is a thick subcategory of $\text{Hol}(\mathcal{D}_G)$.

³The homotopy category $h(C/D)$ is equivalent to the Verdier quotient of triangulated categories hC/hD , as constructed in classical references.

Let $M \rightarrow N \rightarrow P$ be a fiber sequence in $D_h^b(\mathcal{D}_G)$, and assume that M and N are negligible. Taking the associated long exact sequence in cohomology, we obtain an exact sequence

$$\mathcal{H}^i(N) \xrightarrow{\varphi_i} \mathcal{H}^i(P) \xrightarrow{\psi_i} \mathcal{H}^{i+1}(M)$$

for all i . This induces a short exact sequence

$$0 \rightarrow \mathcal{H}^i(N)/\ker \varphi_i \rightarrow \mathcal{H}^i(P) \rightarrow \operatorname{im} \psi_i \rightarrow 0,$$

where both extremities are negligible. Since $\operatorname{Neg}(\mathcal{D}_G)$ is a thick subcategory of $\operatorname{Hol}(\mathcal{D}_G)$, it follows that P is also negligible. The same argument also shows that negligible objects are stable under the formation of fibers.

Finally, let M be an object of $D_h^b(\mathcal{D}_G)$ and N an object of $D_{\operatorname{neg}}^b(\mathcal{D}_G)$. Let χ be in the generic subspace associated with N , as in Lemma 6.2.3. According to Corollary 6.1.4 and Proposition 6.1.9, we have an isomorphism

$$H_c^*(G, (M *_! N) \otimes_G \mathcal{L}_\chi) \simeq H_c^*(G, M \otimes_G \mathcal{L}_\chi) \otimes_k H_c^*(G, N \otimes_G \mathcal{L}_\chi),$$

where $H_c^*(G, N \otimes_G \mathcal{L}_\chi)$ vanishes due to Lemma 6.2.3. \square

DEFINITION 6.2.8. For G satisfying generic vanishing, the *abelian quotient category* $\overline{\operatorname{Hol}}(\mathcal{D}_G)$ is defined as the quotient of $\operatorname{Hol}(\mathcal{D}_G)$ by $\operatorname{Neg}(\mathcal{D}_G)$, whereas the *stable quotient ∞ -category* $\overline{D}_h^b(\mathcal{D}_G)$ is defined as the quotient of $D_h^b(\mathcal{D}_G)$ by $D_{\operatorname{neg}}^b(\mathcal{D}_G)$.

Proposition 6.2.7 implies that $\overline{\operatorname{Hol}}(\mathcal{D}_G)$ is an abelian category, and $\overline{D}_h^b(\mathcal{D}_G)$ is a stable ∞ -category equipped with a symmetric monoidal structure given by $!$ -convolution. Furthermore, $\overline{D}_h^b(\mathcal{D}_G)$ inherits a t -structure whose heart is equivalent to $\overline{\operatorname{Hol}}(\mathcal{D}_G)$ [GL96, Prop. 3.6.1]. Consequently, $\overline{\operatorname{Hol}}(\mathcal{D}_G)$ can be naturally viewed as a full subcategory of $\overline{D}_h^b(\mathcal{D}_G)$.

Our next result is the *raison d'être* for considering negligible objects. Informally, it asserts that while the two variants of convolution on $D_h^b(\mathcal{D}_G)$ may not coincide and do not preserve holonomic \mathcal{D} -modules in degree zero, they do so *up to negligible objects*.

PROPOSITION 6.2.9. *Assume that G satisfies generic vanishing and let M and N be objects of $D_h^b(\mathcal{D}_G)$. The cofiber of the forget-supports map $M *_! N \rightarrow M *_+ N$ is negligible. Moreover, if M and N are holonomic \mathcal{D} -modules in degree zero, then the cohomology sheaves $\mathcal{H}^i(M *_! N)$ and $\mathcal{H}^i(M *_+ N)$ are negligible for $i \neq 0$.*

Proof. Let C be the cofiber of the forget-supports map $M *_! N \rightarrow M *_+ N$. Since G satisfies generic vanishing, there exists a generic subspace V of G^b such that

$$\begin{aligned} H_c^j(G, \mathcal{H}^i(C) \otimes_G \mathcal{L}_\chi) &= 0 \text{ for } j \neq 0; \\ H_c^0(G, \mathcal{H}^i(C) \otimes_G \mathcal{L}_\chi) &\simeq H_c^i(G, C \otimes_G \mathcal{L}_\chi) \end{aligned}$$

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holds for all $i \in \mathbb{Z}$ and $\chi \in V$. Thus, it suffices to prove that $H_c^*(G, C \otimes_G \mathcal{L}_\chi)$ vanishes for generic χ . This is equivalent to proving that the induced map

$$\varphi: H_c^*(G, (M *! N) \otimes_G \mathcal{L}_\chi) \rightarrow H_c^*(G, (M *+ N) \otimes_G \mathcal{L}_\chi)$$

is an isomorphism. According to Proposition 6.1.9 and Corollary 6.1.4, the map φ fits into the following commutative diagram:

$$\begin{array}{ccc} H_c^*(G, (M *! N) \otimes_G \mathcal{L}_\chi) & \xrightarrow{\sim} & H_c^*(G, M \otimes_G \mathcal{L}_\chi) \otimes_k H_c^*(G, N \otimes_G \mathcal{L}_\chi) \\ \varphi \downarrow & & \downarrow \\ H_c^*(G, (M *+ N) \otimes_G \mathcal{L}_\chi) & & \\ \downarrow & & \downarrow \\ H^*(G, (M *+ N) \otimes_G \mathcal{L}_\chi) & \xrightarrow{\sim} & H^*(G, M \otimes_G \mathcal{L}_\chi) \otimes_k H^*(G, N \otimes_G \mathcal{L}_\chi). \end{array}$$

After possibly shrinking V , we may further assume that the forget-supports maps

$$\begin{aligned} H_c^*(G, M \otimes_G \mathcal{L}_\chi) &\rightarrow H^*(G, M \otimes_G \mathcal{L}_\chi) \\ H_c^*(G, N \otimes_G \mathcal{L}_\chi) &\rightarrow H^*(G, N \otimes_G \mathcal{L}_\chi) \\ H_c^*(G, (M *+ N) \otimes_G \mathcal{L}_\chi) &\rightarrow H^*(G, (M *+ N) \otimes_G \mathcal{L}_\chi) \end{aligned}$$

are isomorphisms, proving that φ is an isomorphism for generic χ .

Next, assume that M and N are holonomic \mathcal{D} -modules in degree zero. By generic vanishing, there exists a generic subspace W of G^b such that

$$H_c^*(G, (M *! N) \otimes_G \mathcal{L}_\chi) \simeq H_c^*(G, M \otimes_G \mathcal{L}_\chi) \otimes_k H_c^*(G, N \otimes_G \mathcal{L}_\chi)$$

is concentrated in degree zero for all $\chi \in W$. Additionally, the preceding lemma provides a generic subspace W' of G^b such that

$$H_c^i(G, (M *! N) \otimes_G \mathcal{L}_\chi) \simeq H_c^0(G, \mathcal{H}^i(M *! N) \otimes_G \mathcal{L}_\chi)$$

holds for all $i \in \mathbb{Z}$ and all $\chi \in W'$. As a result, $H_c^0(G, \mathcal{H}^i(M *! N) \otimes_G \mathcal{L}_\chi)$ vanishes whenever $i \neq 0$ and $\chi \in W \cap W'$. Thus, the cohomology sheaves $\mathcal{H}^i(M *! N)$ are negligible for $i \neq 0$. The proof for $M *+ N$ is analogous. \square

PROPOSITION 6.2.10. *Assume that G satisfies generic vanishing. Then, the symmetric monoidal ∞ -categories $\overline{\text{Hol}}(\mathcal{D}_G)$ and $\overline{\text{D}}_h^b(\mathcal{D}_G)$ are rigid.*

We start by proving that the inner hom for $!$ -convolution descends to the quotient category $\overline{\text{D}}_h^b(\mathcal{D}_G)$.

LEMMA 6.2.11. Assume that G satisfies generic vanishing, and let M and N be objects of $D_h^b(\mathcal{D}_G)$. The inner hom for $!$ -convolution $\underline{\text{Hom}}_G^*(M, N)$, defined in Proposition 4.2.2, becomes negligible whenever either M or N is negligible. As a consequence, $\underline{\text{Hom}}_G^*$ descends to a bifunctor $\overline{D}_h^b(\mathcal{D}_G) \times \overline{D}_h^b(\mathcal{D}_G) \rightarrow \overline{D}_h^b(\mathcal{D}_G)$.

Proof. Let n be the dimension of G , and let $\sigma: G \times G \rightarrow G \times G$ be the isomorphism that maps (x, y) to (xy, y) . The commutativity of the diagram

$$\begin{array}{ccc}
 & & G \\
 & \overset{m}{\curvearrowright} & \nearrow \\
 G \times G & \xrightarrow{\sigma} & G \times G \\
 & \underset{pr_2}{\curvearrowleft} & \searrow \\
 & & G
 \end{array}$$

pr_1 (arrow from $G \times G$ to G)
 pr_2 (arrow from $G \times G$ to G)

implies that

$$\begin{aligned}
 \underline{\text{Hom}}_G^*(M, N) &= pr_{1,+} \underline{\text{Hom}}_{G \times G}(pr_2^+ M, m^! N) \\
 &\simeq pr_{1,+} \underline{\text{Hom}}_{G \times G}(\sigma^+ pr_2^+ M, \sigma^! pr_1^! N) \\
 &\simeq pr_{1,+} \sigma^! \underline{\text{Hom}}_{G \times G}(pr_2^+ M, pr_1^! N) \\
 &\simeq pr_{1,+} \sigma_+ \underline{\text{Hom}}_{G \times G}(pr_2^+ M, pr_1^! N) \simeq m_+ \underline{\text{Hom}}_{G \times G}(pr_2^+ M, pr_1^! N).
 \end{aligned}$$

Consequently, for a character $\chi \in G^b$, we find that

$$\begin{aligned}
 \underline{\text{Hom}}_G^*(M, N) \otimes_G \mathcal{L}_\chi &\simeq m_+(\underline{\text{Hom}}_{G \times G}(pr_2^+ M, pr_1^! N) \otimes_{G \times G} m^+ \mathcal{L}_\chi) \\
 &\simeq m_+(\underline{\text{Hom}}_{G \times G}(pr_2^+ M, pr_1^! N) \otimes_{G \times G}^! pr_1^! \mathcal{L}_\chi \otimes_{G \times G}^! pr_2^! \mathcal{L}_\chi)[4n] \\
 &\simeq m_+(pr_2^! D_G(M) \otimes_{G \times G}^! pr_1^! N \otimes_{G \times G}^! pr_1^! \mathcal{L}_\chi \otimes_{G \times G}^! pr_2^! \mathcal{L}_\chi)[4n] \\
 &\simeq m_+(pr_2^!(D_G(M) \otimes_G^! \mathcal{L}_\chi) \otimes_{G \times G}^! pr_1^!(N \otimes_G^! \mathcal{L}_\chi))[4n] \\
 &\simeq m_+(pr_2^! D_G(M \otimes_G \mathcal{L}_{-\chi}) \otimes_{G \times G}^! pr_1^!(N \otimes_G \mathcal{L}_\chi)) \\
 &\simeq m_+ \underline{\text{Hom}}_{G \times G}(pr_2^+(M \otimes_G \mathcal{L}_{-\chi}), pr_1^!(N \otimes_G \mathcal{L}_\chi)) \\
 &\simeq \underline{\text{Hom}}_G^*(M \otimes_G \mathcal{L}_{-\chi}, N \otimes_G \mathcal{L}_\chi).
 \end{aligned}$$

Let $p: G \rightarrow \text{Spec } k$ denote the structure map. As in the proof of Proposition 6.2.9, it suffices to show that $p_+(\underline{\text{Hom}}_G^*(M, N) \otimes_G \mathcal{L}_\chi)$ vanishes for generic χ . Using the usual adjunctions, along with base change and [HTT08, Cor. 2.6.15], we have

$$\begin{aligned}
 p_+(\underline{\text{Hom}}_G^*(M, N) \otimes_G \mathcal{L}_\chi) &\simeq p_+ m_+ \underline{\text{Hom}}_{G \times G}(pr_2^+(M \otimes_G \mathcal{L}_{-\chi}), pr_1^!(N \otimes_G \mathcal{L}_\chi)) \\
 &\simeq (p \times p)_+ \underline{\text{Hom}}_{G \times G}(pr_2^+(M \otimes_G \mathcal{L}_{-\chi}), pr_1^!(N \otimes_G \mathcal{L}_\chi)) \\
 &\simeq R\text{Hom}_{\mathcal{D}_{G \times G}}(pr_2^+(M \otimes_G \mathcal{L}_{-\chi}), pr_1^!(N \otimes_G \mathcal{L}_\chi)) \\
 &\simeq R\text{Hom}_{\mathcal{D}_G}(pr_{1,!} pr_2^+(M \otimes_G \mathcal{L}_{-\chi}), N \otimes_G \mathcal{L}_\chi) \\
 &\simeq R\text{Hom}_{\mathcal{D}_G}(p^+ p!(M \otimes_G \mathcal{L}_{-\chi}), N \otimes_G \mathcal{L}_\chi).
 \end{aligned}$$

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If M is negligible, then the complex $p_!(M \otimes_{\mathcal{G}} \mathcal{L}_{-\chi})$ vanishes for generic χ , and consequently so does $p_+(\underline{\mathrm{Hom}}_{\mathcal{G}}^*(M, N) \otimes_{\mathcal{G}} \mathcal{L}_{\chi})$. The proof for the case when N is negligible follows similarly. \square

Proof of Proposition 6.2.10. According to Proposition 6.2.7 and Lemma 6.2.11, the bi-functors

$$- *_{!} - \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathcal{G}}^*(-, -)$$

both descend to the quotient ∞ -category $\overline{\mathcal{D}}_{\mathcal{G}}^b$. It follows from the counit-unit equations⁴ that these functors stay adjoint in the quotient, establishing that $\overline{\mathcal{D}}_{\mathcal{G}}^b$ is a closed symmetric monoidal ∞ -category.

The dualizability of every object in $\overline{\mathcal{D}}_{\mathcal{G}}^b$ follows directly from the proof of Proposition 6.1.7, along with Proposition 6.2.9. As in Lemma 6.1.6, denote by $(-)^{\vee}$ the functor $\underline{\mathrm{Hom}}_{\mathcal{G}}^*(-, \delta_e)$. Since both functors

$$\overline{\mathcal{D}}_{\mathcal{G}}^b \times \overline{\mathcal{D}}_{\mathcal{G}}^b \xrightarrow{- *_{!} -} \overline{\mathcal{D}}_{\mathcal{G}}^b \quad \text{and} \quad \overline{\mathcal{D}}_{\mathcal{G}}^b \text{op} \xrightarrow{(-)^{\vee}} \overline{\mathcal{D}}_{\mathcal{G}}^b$$

are t-exact, the heart $\overline{\mathrm{Hol}}(\mathcal{D}_{\mathcal{G}})$ inherits a rigid symmetric monoidal structure from the stable ∞ -category $\overline{\mathcal{D}}_{\mathcal{G}}^b$. \square

Before presenting the main theorem of this section, we first review some fundamental concepts related to tannakian categories. Let k be a field, and \mathcal{A} be a non-trivial rigid abelian k -linear symmetric monoidal category. We call \mathcal{A} *tannakian* if there exists a k -algebra R and an exact k -linear symmetric monoidal functor $\omega: \mathcal{A} \rightarrow \mathrm{Mod}(R)$. Such a functor is referred to as an *R -valued fiber functor* for \mathcal{A} .

A tannakian category that admits a fiber functor with values in k is said to be *neutral*. The main theorem concerning neutral tannakian categories states that, for a neutral tannakian category \mathcal{A} over k , there exists an affine group scheme H over k and a symmetric monoidal equivalence $\mathcal{A} \simeq \mathrm{Rep}(H)$ [DM82, Thm. 2.11]. Here, $\mathrm{Rep}(H)$ denotes the category of finite-dimensional representations of H .

There are specific criteria for determining when a non-trivial rigid abelian k -linear symmetric monoidal category \mathcal{A} is tannakian. Recall that for every object M of \mathcal{A} with dual M^{\vee} , there exists evaluation and coevaluation maps

$$\mathrm{ev}: M \otimes M^{\vee} \rightarrow 1 \quad \text{and} \quad \mathrm{coev}: 1 \rightarrow M \otimes M^{\vee}.$$

The element $\dim M$ of $k \simeq \mathrm{End}(1)$ corresponding to the composition $\mathrm{ev} \circ \mathrm{coev}$ is said to be the *dimension* of M . If k has characteristic zero, the category \mathcal{A} is tannakian if the dimension of every object M of \mathcal{A} is a non-negative integer [Del07, Thm. 7.1]. Furthermore, when k is algebraically closed, every tannakian category is neutral [Cou20, Thm. 6.4.1].

⁴See [RV22, Prop. F.5.6] for the counit-unit equations in the quasi-categorical setting.

THEOREM 6.2.12. *Let G be a commutative connected algebraic group over \mathbb{C} satisfying generic vanishing. Then, $\overline{\text{Hol}}(\mathcal{D}_G)$ is a neutral tannakian category.*

To prove this theorem, we need a more concrete description of the quotient category $\overline{\text{Hol}}(\mathcal{D}_G)$. We begin by outlining a general result, due to Gabber and Loeser, that describes quotients of abelian categories as full subcategories [GL96, Prop. 3.7.2].

Let \mathbf{B} be a thick subcategory of an abelian category \mathbf{A} . For an object M of \mathbf{A} , let $M_{\mathbf{B}}$ denote the largest subobject of M that belongs to \mathbf{B} . Similarly, let $M^{\mathbf{B}}$ denote the smallest subobject N of M such that M/N lies in \mathbf{B} . We write $[\mathbf{A} : \mathbf{B}]$ for the full subcategory of \mathbf{A} consisting of objects M for which $M_{\mathbf{B}} = 0$ and $M^{\mathbf{B}} = M$.

LEMMA 6.2.13 (Gabber–Loeser). *Let \mathbf{B} be a thick subcategory of an abelian category \mathbf{A} . Then the functor*

$$\begin{aligned} T: \mathbf{A} &\rightarrow [\mathbf{A} : \mathbf{B}] \\ M &\mapsto (M^{\mathbf{B}} + M_{\mathbf{B}})/M_{\mathbf{B}} \end{aligned}$$

factors through the quotient category \mathbf{A}/\mathbf{B} , inducing a quasi-inverse $S: \mathbf{A}/\mathbf{B} \rightarrow [\mathbf{A} : \mathbf{B}]$ for the restriction of the quotient functor $Q: \mathbf{A} \rightarrow \mathbf{A}/\mathbf{B}$ to $[\mathbf{A} : \mathbf{B}]$. In particular, S defines an equivalence of categories between \mathbf{A}/\mathbf{B} and $[\mathbf{A} : \mathbf{B}]$.



The category $[\mathbf{A} : \mathbf{B}]$ is abelian, but the inclusion functor $[\mathbf{A} : \mathbf{B}] \hookrightarrow \mathbf{A}$ is not necessarily exact. As a simple example, let \mathbf{A} be the category of abelian presheaves on a topological space X , and let \mathbf{B} be the full subcategory consisting of presheaves with trivial stalks. In this case, $[\mathbf{A} : \mathbf{B}]$ is the category of abelian sheaves on X , and its inclusion into \mathbf{A} fails to be exact.

The preceding lemma, applied to the thick subcategory $\text{Neg}(\mathcal{D}_G)$ of $\text{Hol}(\mathcal{D}_G)$, implies that the quotient $\overline{\text{Hol}}(\mathcal{D}_G)$ is equivalent to the full subcategory $\text{Hol}_{\text{int}}(\mathcal{D}_G)$ of $\text{Hol}(\mathcal{D}_G)$ consisting of holonomic \mathcal{D} -modules with no negligible subquotients. Assuming that G satisfies generic vanishing, the category $\text{Hol}_{\text{int}}(\mathcal{D}_G)$ inherits a rigid symmetric monoidal structure $*_{\text{mid}}$ from $\overline{\text{Hol}}(\mathcal{D}_G)$.

Proof of Theorem 6.2.12. According to all the discussion above, it suffices to prove that the dimension of every object M of $\text{Hol}_{\text{int}}(\mathcal{D}_G)$ is a non-negative integer. By definition, the dimension of M is the image of the composition

$$\delta_e \xrightarrow{\text{coev}} M *_{\text{mid}} M^{\vee} \xrightarrow{\text{ev}} \delta_e$$

under the isomorphism $\text{End}(\delta_e) \simeq \mathbb{C}$. Twisting by a character χ and taking compactly supported cohomology, we obtain morphisms

$$\mathbb{C} \rightarrow H_c^*(G, (M *_{\text{mid}} M^{\vee}) \otimes_G \mathcal{L}_\chi) \rightarrow \mathbb{C}.$$

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For a sufficiently generic character χ , $(M *_{\text{mid}} M^{\vee}) \otimes_G \mathcal{L}_\chi$ coincides with $(M *! M^{\vee}) \otimes_G \mathcal{L}_\chi$. As a result,

$$\begin{aligned} H_c^*(G, (M *_{\text{mid}} M^{\vee}) \otimes_G \mathcal{L}_\chi) &\simeq H_c^*(G, (M *! M^{\vee}) \otimes_G \mathcal{L}_\chi) \\ &\simeq H_c^*(G, M \otimes_G \mathcal{L}_\chi) \otimes_{\mathbb{C}} H_c^*(G, M \otimes_G \mathcal{L}_\chi)^*. \end{aligned}$$

From generic vanishing, there exists a character χ such that the above holds and $H_c^*(G, M \otimes_G \mathcal{L}_\chi)$ is concentrated in degree zero. As in the proof of [Krä14, Thm. 5.2], the morphisms

$$\mathbb{C} \rightarrow \text{End}(H_c^0(G, M \otimes_G \mathcal{L}_\chi)) \rightarrow \mathbb{C}$$

coincide with the evaluation and coevaluation maps of finite-dimensional vector spaces. Thus, the dimension of M is the same as the dimension of $H_c^0(G, M \otimes_G \mathcal{L}_\chi)$ as a \mathbb{C} -vector space. \square

Let M be a holonomic \mathcal{D} -module on G with image \overline{M} in $\overline{\text{Hol}}(\mathcal{D}_G)$. We denote by $\langle M \rangle$ the full subcategory of $\overline{\text{Hol}}(\mathcal{D}_G)$ whose objects are subquotients of finite direct sums of

$$\underbrace{M *! \cdots *! M *!}_{n \text{ factors}} \underbrace{M^{\vee} *! \cdots *! M^{\vee}}_{m \text{ factors}},$$

for some $n, m \geq 0$. By construction, the categories $\langle M \rangle$ are also tannakian.

COROLLARY 6.2.14. *Assume that G satisfies generic vanishing, and let M be a holonomic \mathcal{D} -module on G . Then, there exists an affine algebraic group $\text{Conv}(M)$ such that the full subcategory $\langle M \rangle$ of $\overline{\text{Hol}}(\mathcal{D}_G)$ is equivalent to the category $\text{Rep}(\text{Conv}(M))$ of finite-dimensional representations of $\text{Conv}(M)$. Furthermore, if M is semisimple, then $\text{Conv}(M)$ is reductive.*

Proof. The existence of $\text{Conv}(M)$ follows from the main theorem on tannakian categories [DM82, Thm. 2.11], and it is algebraic due to [DM82, Prop. 2.20]. The last statement is a consequence of [Mil17, Thm. 22.42], along with [DM82, Cor. 2.22]. \square

For the reader's convenience, we recall that the tannakian group of $\overline{\text{Hol}}(\mathcal{D}_G)$ can be recovered as the cofiltered limit of the algebraic groups $\text{Conv}(M)$, as $\langle M \rangle$ runs through the full subcategories of $\overline{\text{Hol}}(\mathcal{D}_G)$ generated by a single object. For more details, see the proof of [DM82, Prop. 2.8].⁵ Notably, this tannakian group is pro-algebraic.

6.3. DIFFERENTIAL GALOIS THEORY

In this section, we explain two approaches to differential Galois theory and examine their interrelation. Their equivalence, that we present as Theorem 6.3.9, appears to be

⁵Note that in [DM82], the notation \mathcal{C}_M is used for what we denote by $\langle M \rangle$. (They use $\langle M \rangle$ for a different subcategory.) However, our notation has since become standard.

well-known, although we were unable to locate any references describing it. Finally, we connect the convolution group discussed in the previous section to a differential Galois group.

We begin with a geometric approach to differential Galois theory, popularized by Katz in his influential papers [Kat82a; Kat87], which have significantly shaped our understanding of the subject. Let X be a geometrically connected algebraic variety that is smooth over a field k of characteristic zero, and assume that X has at least one rational point. Following Katz, we denote by $\mathrm{DE}(X/k)$ the category of vector bundles with integrable connection on X , which we refer to simply as *connections*.

Since every *coherent* \mathcal{O}_X -module equipped with a (not necessarily integrable) connection is locally free [And01, Cor. 2.5.2.2], $\mathrm{DE}(X/k)$ is a k -linear abelian category. Furthermore, it is endowed with a rigid symmetric monoidal structure, given by the tensor product of connections. The following proposition explains how connections behave under restriction to open subsets.

PROPOSITION 6.3.1. *Let U be an open dense subset of X . Then the restriction functor $\mathrm{DE}(X/k) \rightarrow \mathrm{DE}(U/k)$ is symmetric monoidal, exact, and fully faithful. Moreover, if (\mathcal{E}, ∇) is a connection on X and (\mathcal{E}', ∇') is a subquotient of $(\mathcal{E}, \nabla)|_U$, there exists a unique subquotient $(\tilde{\mathcal{E}}, \tilde{\nabla})$ of (\mathcal{E}, ∇) such that $(\mathcal{E}', \nabla') \simeq (\tilde{\mathcal{E}}, \tilde{\nabla})|_U$.*

Proof. Let $j: U \hookrightarrow X$ be the natural open immersion. It is clear that the restriction functor $j^*: \mathrm{DE}(X/k) \rightarrow \mathrm{DE}(U/k)$ is exact and symmetric monoidal, as the restriction of coherent sheaves possesses these properties. To prove that this functor is fully faithful, consider a morphism $\varphi: (\mathcal{E}, \nabla)|_U \rightarrow (\mathcal{E}', \nabla')|_U$ of connections on U . We need to show that φ has a unique extension to a morphism $\bar{\varphi}: (\mathcal{E}, \nabla) \rightarrow (\mathcal{E}', \nabla')$.

The existence of such an extension follows from Remark 1.5.7, as $\bar{\varphi} = j_{!+} \varphi$ is a morphism in $\mathrm{DE}(X/k)$ that restricts to φ on U . To establish uniqueness, suppose that φ is the zero-morphism, and let $\bar{\varphi}$ be any extension of φ . The image of $\bar{\varphi}$ would then be a locally free sheaf supported on $X \setminus U$. This implies that $\bar{\varphi}$ must vanish, thereby completing the proof of uniqueness.

To demonstrate the second part of the statement, consider a connection (\mathcal{E}, ∇) on X , and let (\mathcal{E}', ∇') be a subobject of $(\mathcal{E}, \nabla)|_U$. Since right-adjoints are left-exact, $j_* \mathcal{E}'$ is a subsheaf of $j_* j^* \mathcal{E} = j_* \mathcal{E}|_U$. Furthermore, the unit map $\eta: \mathcal{E} \rightarrow j_* j^* \mathcal{E}$ is a monomorphism as $\ker \eta$ is a coherent subsheaf of \mathcal{E} supported on $X \setminus U$. Thus, we may take the intersection of $j_* \mathcal{E}'$ and \mathcal{E} in $j_* j^* \mathcal{E}$.

We assert that the connection ∇ on \mathcal{E} restricts to a connection $\tilde{\nabla}$ on $\tilde{\mathcal{E}} := j_* \mathcal{E}' \cap \mathcal{E}$. This will imply that $(\tilde{\mathcal{E}}, \tilde{\nabla})$ is an object of $\mathrm{DE}(X/k)$ whose restriction to U is isomorphic to (\mathcal{E}', ∇') . Such an object must be unique, as the restriction functor $\mathrm{DE}(X/k) \rightarrow \mathrm{DE}(U/k)$ is fully faithful.

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Now, let V be an open subset of X , and consider a section s in

$$\Gamma(V, j_* \mathcal{E}' \cap \mathcal{E}) \simeq \Gamma(U \cap V, \mathcal{E}') \cap \Gamma(V, \mathcal{E}).$$

Since (\mathcal{E}', ∇') is a subobject of $(\mathcal{E}, \nabla)|_U$, the connection ∇' is the restriction of ∇ to U . Consequently, $\nabla(s)$ lies in

$$\Gamma(U \cap V, \mathcal{E}' \otimes_{\mathcal{O}_U} \Omega_{U/k}^1) = \Gamma(V, j_*(\mathcal{E}' \otimes_{\mathcal{O}_U} \Omega_{U/k}^1)) \simeq \Gamma(V, j_* \mathcal{E}' \otimes_{\mathcal{O}_X} \Omega_{X/k}^1).$$

Thus, we have

$$\nabla(s) \in \Gamma(V, j_* \mathcal{E}' \otimes_{\mathcal{O}_X} \Omega_{X/k}^1) \cap \Gamma(V, \mathcal{E} \otimes_{\mathcal{O}_X} \Omega_{X/k}^1) \simeq \Gamma(V, (j_* \mathcal{E}' \cap \mathcal{E}) \otimes_{\mathcal{O}_X} \Omega_{X/k}^1),$$

proving that ∇ indeed restricts to $\tilde{\mathcal{E}} = j_* \mathcal{E}' \cap \mathcal{E}$. Given that the functor j^* is exact, the corresponding statement about quotients follows directly from this result. \square

Let x be a k -point of X . By mapping a connection (\mathcal{E}, ∇) to its fiber $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_X} k$, we obtain an exact k -linear symmetric monoidal functor $\omega_x : \text{DE}(X/k) \rightarrow \text{Vect}(k)$. This functor serves as a fiber functor, establishing that $\text{DE}(X/k)$ is a tannakian category.

DEFINITION 6.3.2. Let x be a k -point of X . The tannakian group of $\text{DE}(X/k)$ associated with the fiber functor ω_x is called the *differential fundamental group* of X and is denoted by $\pi_1^{\text{diff}}(X, x)$. For a connection (\mathcal{E}, ∇) , the tannakian group of $\langle\langle \mathcal{E}, \nabla \rangle\rangle$ is referred to as *Katz' differential Galois group* and is denoted by $\text{Gal}_K((\mathcal{E}, \nabla), x)$.

The differential fundamental group $\pi_1^{\text{diff}}(X, x)$ is significantly larger than its Betti and étale counterparts. This observation led Deligne to consider the tannakian group of *unipotent* connections as his *de Rham fundamental group* in [Del89, §12.4]. Katz' differential Galois group, on the other hand, is a more manageable object, as we will demonstrate in this section.

Our next result follows directly from Proposition 6.3.1. We first conjectured that this statement might hold after examining [Kin15, Lem. 2.5], where the author considers a positive-characteristic analogue.

COROLLARY 6.3.3. *Let U be an open dense subset of X , and let (\mathcal{E}, ∇) be a connection on X . Then the restriction functor*

$$\langle\langle \mathcal{E}, \nabla \rangle\rangle \rightarrow \langle\langle \mathcal{E}, \nabla \rangle\rangle|_U$$

is an equivalence of symmetric monoidal categories. In particular, we have an isomorphism

$$\text{Gal}_K((\mathcal{E}, \nabla), x) \simeq \text{Gal}_K((\mathcal{E}, \nabla)|_U, x)$$

for every k -point x of U .

The following remark connects Katz' differential Galois groups with the classical monodromy groups.

■ **REMARK 6.3.4 — MONODROMY GROUPS.** Assume that k is the field of complex numbers \mathbb{C} . As above, the category $\text{DE}(X_{\text{an}})$ of holomorphic connections on the complex manifold X_{an} forms a neutral tannakian category. For an algebraic connection (\mathcal{E}, ∇) and a point $x \in X(\mathbb{C})$, the *monodromy group* $\text{Mon}((\mathcal{E}, \nabla), x)$ is defined as the tannakian group of the category $\langle (\mathcal{E}, \nabla)_{\text{an}} \rangle \subset \text{DE}(X_{\text{an}})$, with respect to the fiber functor given by x .

Let $\rho: \pi_1(X_{\text{an}}, x) \rightarrow \text{GL}(\mathcal{E}(x))$ be the representation associated with $(\mathcal{E}, \nabla)_{\text{an}}$ under the equivalence of categories $\text{DE}(X_{\text{an}}) \simeq \text{Rep}(\pi_1(X_{\text{an}}, x))$. The monodromy group $\text{Mon}((\mathcal{E}, \nabla), x)$ can be described as the Zariski closure in $\text{GL}(\mathcal{E}(x))$ of the image $\rho(\pi_1(X_{\text{an}}, x))$. There is a natural closed immersion

$$\text{Mon}((\mathcal{E}, \nabla), x) \hookrightarrow \text{Gal}_k((\mathcal{E}, \nabla), x)$$

induced by the analytification functor $\text{DE}(X/\mathbb{C}) \rightarrow \text{DE}(X^{\text{an}})$. From the Riemann–Hilbert correspondence, we find that this morphism is an isomorphism when (\mathcal{E}, ∇) is a regular connection. ■

Our next objective is to relate the geometric theory presented above to the differential Galois theory of Kolchin and Picard–Vessiot, as presented in [VS03]. For the reader's convenience, we briefly review some of its main concepts. To keep the discussion accessible, we focus on the one-dimensional case, referring to [Del07, Rmq. 9.9] for the more general setting.

DEFINITION 6.3.5. A field K equipped with a derivation $\partial: K \rightarrow K$ is called a *differential field*. A *differential module* over a differential field (K, ∂) is a finite-dimensional K -vector space M equipped with an additive map $\nabla: M \rightarrow M$ satisfying the Leibniz rule

$$\nabla(fm) = \partial(f)m + f\nabla(m),$$

for $f \in K$ and $m \in M$. A morphism of differential modules $(M, \nabla) \rightarrow (M', \nabla')$ is a K -linear map $\varphi: M \rightarrow M'$ such that $\nabla' \circ \varphi = \varphi \circ \nabla$. The category of differential modules over (K, ∂) is denoted by $\text{Diff}(K, \partial)$.

The category $\text{Diff}(K, \partial)$ naturally inherits a symmetric monoidal structure: the tensor product of two differential modules (M, ∇) and (M', ∇') is defined as the K -vector space $M \otimes_K M'$, with the connection acting on pure tensors as

$$m \otimes m' \mapsto \nabla(m) \otimes m' + m \otimes \nabla'(m'),$$

and extended by linearity. With this structure, $\text{Diff}(K, \partial)$ becomes a rigid abelian k -linear symmetric monoidal category, where k is the field of constants, i.e., the kernel of ∂ .

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THEOREM 6.3.6. *Let (K, ∂) be a differential field with field of constants k . Then the category $\text{Diff}(K, \partial)$ is tannakian. Moreover, if k is algebraically closed, $\text{Diff}(K, \partial)$ is a neutral tannakian category.*

Proof. The forgetful functor $\text{Diff}(K, \partial) \rightarrow \text{Vect}(K)$ serves as a K -valued fiber functor for $\text{Diff}(K, \partial)$, thereby establishing that $\text{Diff}(K, \partial)$ is a tannakian category. Furthermore, a theorem of Deligne asserts that every k -linear tannakian category is neutral when k is algebraically closed [Cou20, Thm. 6.4.1]. \square

Henceforth, we assume that the field of constants k is algebraically closed. The preceding theorem implies that every differential module possesses a certain symmetry group: its *differential Galois group*.

DEFINITION 6.3.7. Let (K, ∂) be a differential field with an algebraically closed field of constants, and let (M, ∇) be a differential module over (K, ∂) . The tannakian group associated with a fiber functor ω on $\langle (M, \nabla) \rangle$ is called the *differential Galois group* of (M, ∇) and is denoted by $\text{Gal}((M, \nabla), \omega)$.

To relate this algebraic theory with its geometric counterpart, let X be a smooth connected algebraic curve over an algebraically closed field k of characteristic zero. Let K be a function field of X . The K -module of Kähler differentials $\Omega_{K/k}^1$ is one-dimensional [Eis13, Thm. 16.14], and we let $\partial \in \text{Der}_k(K) \simeq (\Omega_{K/k}^1)^\vee$ be a non-zero derivation. This endows K with a structure of differential field.

LEMMA 6.3.8. *Let K be the function field of a smooth connected curve X over an algebraically closed field k of characteristic zero, and let $\partial \in \text{Der}_k(K)$ be a non-zero derivation. Then $\ker \partial = k$.*

Proof. We begin by observing that $L := \ker \partial$ is a subfield of K that contains k . Moreover, L is algebraically closed in K . Indeed, suppose $f \in K$ satisfies $P(f) = 0$ for some irreducible polynomial $P \in L[t]$. Then $P'(f)\partial(f) = \partial(P(f)) = 0$, and so $f \in L$.

We claim that K has exactly two subfields containing k that are algebraically closed in K : the algebraic closure of k in K , and K itself. To see this, let $x \in K$ be such that K is a finite extension of $k(x)$ and consider any subfield F of K containing k that is algebraically closed in K . If F/k is an algebraic extension, then F must be the algebraic closure of k in K . Otherwise, F contains a transcendental element $y \in K$.

Note that the extension $K/k(x, y)$ is finite (and thus algebraic), as $k(x) \subset k(x, y) \subset K$ and $K/k(x)$ is finite. This implies that $F = K$. Indeed, any element of K is a root of a polynomial in $k(x, y)[t]$ since $K/k(x, y)$ is algebraic. However, any element of K that is a root of a polynomial in $F[t] \supset k(x, y)[t]$ must belong to F , for F is algebraically closed in K .

Given that ∂ is a non-zero derivation, we conclude that L is the algebraic closure of k in K . According to [Stacks, Tag 0BY3], there exists a smooth projective curve Y with function field K . We affirm that Y is also connected. Indeed, any idempotent of $\Gamma(Y, \mathcal{O}_Y)$ is algebraic over k . Due to the normality of X , such an element must lie in $\Gamma(X, \mathcal{O}_X)$, proving that Y is connected.

Let $U = \text{Spec } R$ be an affine open of Y . Since Y is normal, any element of $K = \text{Frac } R$ that is algebraic over k must belong to R . Therefore, we have

$$L \subset \bigcap_{U \text{ affine}} \mathcal{O}_Y(U) = \bigcap_{U \text{ affine}} \bigcap_{y \in Y} \mathcal{O}_{Y,y} = \bigcap_{y \in Y} \mathcal{O}_{Y,y} = \mathcal{O}_Y(Y).$$

The result then follows from the fact that $\mathcal{O}_Y(Y) = k$, as in [Stacks, Tag 04L2]. \square

As a consequence of the preceding lemma, the category $\text{Diff}(K, \partial)$ is k -linear. For a connection (\mathcal{E}, ∇) on X , its generic fiber \mathcal{E}_η is a finite-dimensional K -vector space equipped with an additive map

$$\mathcal{E}_\eta \rightarrow \mathcal{E}_\eta \otimes_K \Omega_{K/k}^1.$$

By composing this morphism with the evaluation map $\text{ev}_\partial: \Omega_{K/k}^1 \rightarrow K$ defined by ∂ , we obtain an additive map $\nabla_\partial: \mathcal{E}_\eta \rightarrow \mathcal{E}_\eta$ satisfying the Leibniz rule. This endows \mathcal{E}_η with a structure of differential module.

THEOREM 6.3.9. *Let X be a smooth connected curve over an algebraically closed field k of characteristic zero. Let η be the generic point of X and K be its function field. For an object (\mathcal{E}, ∇) of $\text{DE}(X/k)$ and a non-zero derivation $\partial \in \text{Der}_k(K)$, the construction above yields a symmetric monoidal equivalence of categories*

$$\langle\langle \mathcal{E}, \nabla \rangle\rangle \xrightarrow{\sim} \langle\langle \mathcal{E}_\eta, \nabla_\partial \rangle\rangle,$$

where $\langle\langle \mathcal{E}, \nabla \rangle\rangle$ is a full subcategory of $\text{DE}(X/k)$ and $\langle\langle \mathcal{E}_\eta, \nabla_\partial \rangle\rangle$ is a full subcategory of $\text{Diff}(K, \partial)$.

Proof. By taking the colimit of the restriction functors in Corollary 6.3.3, we obtain a symmetric monoidal equivalence of categories

$$\langle\langle \mathcal{E}, \nabla \rangle\rangle \xrightarrow{\sim} \langle\langle \mathcal{E}, \nabla \rangle_\eta \rangle \subset \text{DE}(\text{Spec } K/k).$$

It then suffices to prove that our construction provides an equivalence of categories between $\text{DE}(\text{Spec } K/k)$ and $\text{Diff}(K, \partial)$.

The construction above defines a functor $\text{DE}(\text{Spec } K/k) \rightarrow \text{Diff}(K, \partial)$ that acts as $(M, \nabla) \mapsto (M, \nabla_\partial)$ on objects and is the identity on morphisms. To verify the image

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of a morphism $\varphi: (M, \nabla) \rightarrow (M', \nabla')$ under this functor is indeed a morphism on $\text{Diff}(K, \partial)$, we consider the following diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\nabla} & M \otimes_K \Omega_{K/k}^1 & \xrightarrow{\text{id} \otimes \text{ev}_\partial} & M \\ \varphi \downarrow & & \downarrow \varphi \otimes \text{id} & & \downarrow \varphi \\ M' & \xrightarrow{\nabla'} & M' \otimes_K \Omega_{K/k}^1 & \xrightarrow{\text{id} \otimes \text{ev}_\partial} & M'. \end{array}$$

The left square commutes since φ is a morphism in $\text{DE}(\text{Spec } K/k)$, and the right square commutes trivially.

This functor is clearly fully faithful. To show that it is essentially surjective, consider an object $(M, \widetilde{\nabla})$ of $\text{Diff}(K, \partial)$. We construct a connection $\nabla: M \rightarrow M \otimes_K \Omega_{K/k}^1$ by setting

$$\nabla(m) := \widetilde{\nabla}(m) \otimes \omega,$$

where $\omega \in \Omega_{K/k}^1$ is the dual of $\partial \in \text{Der}_k(K)$. The image of (M, ∇) under our functor is $(M, \widetilde{\nabla})$, since $\nabla_\partial: M \rightarrow M$ sends m to $\partial(\omega) \widetilde{\nabla}(m) = \widetilde{\nabla}(m)$. Therefore, the functor is an equivalence of categories. \square

In particular, the proof above implies that the category $\text{Diff}(K, \partial)$ is independent, up to equivalence, of the choice of ∂ . This result also allows us to compare the different notions of differential Galois groups.

It is worth noting that, over an algebraically closed field k , any two fiber functors on a neutral tannakian category \mathcal{A} generated by a single object are isomorphic. Specifically, [DM82, Thm. 3.2] asserts that the category of fiber functors for \mathcal{A} with values in k is equivalent to the category of G -torsors over k , where G is the tannakian group associated with any of the fiber functors. Since G is an algebraic group and k is algebraically closed, it follows that any G -torsor has a k -point. Consequently, we can omit explicit reference to the fiber functors in most of our notations.

COROLLARY 6.3.10. *With the notation of Theorem 6.3.9, the groups $\text{Gal}_K(\mathcal{E}, \nabla)$ and $\text{Gal}(\mathcal{E}_\eta, \nabla_\partial)$ are isomorphic.*

The remainder of this section will be dedicated to a comparison between the convolution group studied in the previous section and a differential Galois group. To facilitate this comparison, it is useful to introduce an alternative description of differential modules.

Consider the (non-commutative) rings $\mathbb{C}[t]\langle \partial_t \rangle$ and $\mathbb{C}(t)\langle \partial_t \rangle$, whose elements are finite sums

$$\sum_{n=0}^{\infty} p_n(t) \partial_t^n,$$

where the coefficients p_n belong to $\mathbb{C}[t]$ and $\mathbb{C}(t)$, respectively. The multiplication in these rings is governed by the relation $\partial_t t = t\partial_t + 1$. In this context, a differential module over the differential field $\mathbb{C}(t)$ with its standard derivation corresponds to a $\mathbb{C}(t)\langle\partial_t\rangle$ -module that is finite-dimensional over $\mathbb{C}(t)$.

Using this notation, the Fourier transform discussed in Example 4.3.3 can be interpreted as the equivalence of categories $\text{FT}: \text{Mod}(\mathbb{C}[x]\langle\partial_x\rangle) \rightarrow \text{Mod}(\mathbb{C}[t]\langle\partial_t\rangle)$ induced by the isomorphism

$$\mathbb{C}[x]\langle\partial_x\rangle \rightarrow \mathbb{C}[t]\langle\partial_t\rangle$$

that maps $x \mapsto \partial_t$ and $\partial_x \mapsto -t$. Given that a holonomic \mathcal{D} -module on \mathbb{A}^1 is a locally free \mathcal{O} -module of finite rank on a dense open subset, we obtain a functor

$$\text{Hol}(\mathbb{C}[x]\langle\partial_x\rangle) \xrightarrow{\text{FT}} \text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle) \xrightarrow{-\otimes_{\mathbb{C}[t]}\mathbb{C}(t)} \text{Diff}(\mathbb{C}(t)).$$

As established in the proof of Proposition 5.3.1, a holonomic $\mathbb{C}[x]\langle\partial_x\rangle$ -module is negligible if and only if its image in $\text{Diff}(\mathbb{C}(t))$ is zero. Consequently, the universal property of quotients induces a functor

$$\text{Hol}(\mathbb{C}[x]\langle\partial_x\rangle)/\text{Neg}(\mathbb{C}[x]\langle\partial_x\rangle) \rightarrow \text{Diff}(\mathbb{C}(t)).$$

Note that the category on the left carries a symmetric monoidal structure defined by convolution, while the category on the right carries a symmetric monoidal structure defined by tensor products. By Proposition 4.3.7, this functor is symmetric monoidal.⁶

THEOREM 6.3.11. *The functor $\text{Hol}(\mathbb{C}[x]\langle\partial_x\rangle)/\text{Neg}(\mathbb{C}[x]\langle\partial_x\rangle) \rightarrow \text{Diff}(\mathbb{C}(t))$ obtained by taking the generic fiber of the Fourier transform is a symmetric monoidal equivalence of categories.*

Proof. For a holonomic $\mathbb{C}[t]\langle\partial_t\rangle$ -module M , let us say that M is *dual-negligible* if it is the Fourier transform of a negligible holonomic $\mathbb{C}[x]\langle\partial_x\rangle$ -module. Equivalently, M is dual-negligible if its generic fiber $M \otimes_{\mathbb{C}[t]}\mathbb{C}(t)$ vanishes. We denote by $\text{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle)$ the full subcategory of $\text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)$ consisting of dual-negligible objects.

Note that the universal property of quotients induces functors making the diagram

$$\begin{array}{ccccc} \text{Hol}(\mathbb{C}[x]\langle\partial_x\rangle) & \xrightarrow{\text{FT}} & \text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle) & \xrightarrow{-\otimes_{\mathbb{C}[t]}\mathbb{C}(t)} & \text{Diff}(\mathbb{C}(t)) \\ & \searrow & \searrow & & \nearrow \\ & \frac{\text{Hol}(\mathbb{C}[x]\langle\partial_x\rangle)}{\text{Neg}(\mathbb{C}[x]\langle\partial_x\rangle)} & \xrightarrow{\quad} & \frac{\text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)}{\text{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle)} & \end{array}$$

commute. Since both horizontal functors on the left are equivalences of categories, it suffices to prove that the functor

$$-\otimes_{\mathbb{C}[t]}\mathbb{C}(t): \text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)/\text{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \text{Diff}(\mathbb{C}(t))$$

⁶The abstract machinery of Section 4.1 should not obscure the simplicity of this result, which can also be proven directly.

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is also an equivalence of categories. We will demonstrate this in several steps.

Part 1: The adjunction $F_0: \mathbf{Mod}(\mathbb{C}[t]) \rightleftarrows \mathbf{Vect}(\mathbb{C}(t)): G_0$ between extension and restriction of scalars induces an adjunction $F_1: \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle) \rightleftarrows \mathbf{Mod}(\mathbb{C}(t)\langle\partial_t\rangle): G_1$.

Let M be a $\mathbb{C}[t]$ -module and N be a $\mathbb{C}(t)$ -vector space. We recall that the adjunction maps are given explicitly by

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}(t)}(M \otimes_{\mathbb{C}[t]} \mathbb{C}(t), N) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}[t]}(M, N) \\ \varphi &\longmapsto (m \mapsto \varphi(m \otimes 1)) \\ (m \otimes p \mapsto p\psi(m)) &\longleftarrow \psi. \end{aligned}$$

Now suppose that M and N are endowed with structures of $\mathbb{C}[t]\langle\partial_t\rangle$ - and $\mathbb{C}(t)\langle\partial_t\rangle$ -modules, respectively. The adjunction maps above send morphisms of $\mathbb{C}(t)\langle\partial_t\rangle$ -modules to morphism of $\mathbb{C}[t]\langle\partial_t\rangle$ -modules, and inversely. Indeed we have that

$$\begin{aligned} \varphi(\partial(m) \otimes 1) &= \varphi(\partial(m) \otimes 1 + m \otimes \partial(1)) = \varphi(\partial(m \otimes 1)) = \partial(\varphi(m \otimes 1)) \\ p\psi(\partial(m)) + \partial(p)\psi(m) + p\partial(\psi(m)) + \partial(p)\psi(m) &= \partial(p\psi(m)), \end{aligned}$$

for all $m \in M$ and $p \in \mathbb{C}(t)$.

Part 2: The functor $G_1: \mathbf{Mod}(\mathbb{C}(t)\langle\partial_t\rangle) \rightarrow \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle)$ is fully faithful.

It is clear that $G_0: \mathbf{Vect}(\mathbb{C}(t)) \rightarrow \mathbf{Mod}(\mathbb{C}[t])$ is fully faithful, since an additive morphism φ between $\mathbb{C}(t)$ -vector spaces is $\mathbb{C}(t)$ -linear if and only if it is $\mathbb{C}[t]$ -linear. As a consequence, the counit of adjunction $F_0 \circ G_0 \rightarrow \mathrm{id}$ is an isomorphism. This implies the same for the counit $F_1 \circ G_1 \rightarrow \mathrm{id}$.

Part 3: The functor $F_1: \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \mathbf{Mod}(\mathbb{C}(t)\langle\partial_t\rangle)$ induces an equivalence of categories $F_2: \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle)/\ker F_1 \rightarrow \mathbf{Mod}(\mathbb{C}(t)\langle\partial_t\rangle)$.

This is a classical result about reflective localizations, which can be found in [Bor94, Prop. 5.3.1].

Part 4: The natural inclusion $\mathrm{Hol}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle)$ induces a fully faithful functor $\mathrm{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)/\mathrm{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \mathbf{Mod}(\mathbb{C}[t]\langle\partial_t\rangle)/\ker F_1$.

This is another categorical fact: if \mathbf{B} and \mathbf{C} are thick subcategories of an abelian category \mathbf{A} , the natural functor $\mathbf{B}/(\mathbf{B} \cap \mathbf{C}) \rightarrow \mathbf{A}/\mathbf{C}$ is fully faithful. Its proof is a simple exercise using calculus of fractions.

Part 5: Conclusion.

Composing the functor F_2 with the fully faithful functor obtained in Part 4, we obtain a fully faithful functor

$$\mathrm{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)/\mathrm{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \mathbf{Mod}(\mathbb{C}(t)\langle\partial_t\rangle).$$

As previously remarked, this functor takes values in the full subcategory $\text{Diff}(\mathbb{C}(t))$ of differential modules, leading to a fully faithful functor

$$\text{Hol}(\mathbb{C}[t]\langle\partial_t\rangle)/\text{DNeg}(\mathbb{C}[t]\langle\partial_t\rangle) \rightarrow \text{Diff}(\mathbb{C}(t)),$$

which is precisely the one obtained via the universal property of quotients.

To establish the essential surjectivity of this functor, let M be a difference module over $\mathbb{C}(t)$. According to the cyclic vector theorem [VS03, Prop. 2.9], there exists an element L of $\mathbb{C}(t)\langle\partial_t\rangle$ and an isomorphism

$$M \simeq \mathbb{C}(t)\langle\partial_t\rangle/\mathbb{C}(t)\langle\partial_t\rangle L.$$

By clearing the denominators in the coefficients of L , we may assume that L lies in $\mathbb{C}[t]\langle\partial_t\rangle$. Then M is isomorphic to the image of $\mathbb{C}[t]\langle\partial_t\rangle/\mathbb{C}[t]\langle\partial_t\rangle L$ under the functor above. \square

COROLLARY 6.3.12. *Let M be a holonomic \mathcal{D} -module on \mathbb{G}_a . Then, the groups $\text{Conv}(M)$ and $\text{Gal}(\text{FT}(M)_\eta)$ are isomorphic.*

This theorem also provides a solution to the inverse Galois problem for the convolution group on \mathbb{G}_a . The following is a direct consequence of the preceding corollary, along with the main theorem in [TT79].

COROLLARY 6.3.13. *Let G be a linear algebraic group over \mathbb{C} . Then, there exists a holonomic \mathcal{D} -module M on \mathbb{G}_a such that $\text{Conv}(M) \simeq G$.*

6.4. DIFFERENCE GALOIS THEORY

In this section, we present an analogue of Theorem 6.3.11 for the multiplicative group. We begin by briefly outlining the core concepts of difference Galois theory from a tannakian perspective. Although difference Galois theory has certain nuances that set it apart from differential Galois theory, these differences are not relevant to our discussion. Therefore, we will focus on the parallels with differential Galois theory. For a more comprehensive introduction to these ideas, we refer to [VS97].

DEFINITION 6.4.1. A field K equipped with an automorphism $\tau: K \rightarrow K$ is called a *difference field*. A *difference module* over a difference field (K, τ) is a finite-dimensional K -vector space M equipped with a bijective additive map $T: M \rightarrow M$ satisfying

$$T(fm) = \tau(f)T(m),$$

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for $f \in K$ and $m \in M$. A morphism of difference modules $(M, T) \rightarrow (M', T')$ is a K -linear map $\varphi: M \rightarrow M'$ such that $T' \circ \varphi = \varphi \circ T$. The category of difference modules over (K, τ) is denoted by $\text{Diff}(K, \tau)$.⁷

The category $\text{Diff}(K, \tau)$ naturally inherits a symmetric monoidal structure: the tensor product of two difference modules (M, T) and (M', T') is defined as the K -vector space $M \otimes_K M'$, with the difference operator acting on pure tensors as

$$m \otimes m' \mapsto T(m) \otimes T'(m'),$$

and extended by linearity. With this structure, $\text{Diff}(K, \tau)$ becomes a rigid abelian k -linear symmetric monoidal category, where k is the field of constants, i.e., the kernel of $\tau - \text{id}$.

THEOREM 6.4.2. *Let (K, τ) be a difference field with field of constants k . Then the category $\text{Diff}(K, \tau)$ is tannakian. Moreover, if k is algebraically closed, $\text{Diff}(K, \tau)$ is a neutral tannakian category.*

Proof. The same arguments as in Theorem 6.3.6 apply here. □

Henceforth, we assume that the field of constants k is algebraically closed. The preceding theorem implies that every difference module possesses a certain symmetry group: its *difference Galois group*.

DEFINITION 6.4.3. Let (K, τ) be a difference field with an algebraically closed field of constants, and let (M, T) be a difference module over (K, τ) . The tannakian group of $\langle (M, T) \rangle$ is called the *difference Galois group* of (M, T) and is denoted by $\text{Gal}(M, T)$.

Our primary example of difference field is the function field $\mathbb{C}(s)$ in one variable, equipped with the difference operator $\mathbb{C}(s) \rightarrow \mathbb{C}(s)$ that maps a rational function $f(s)$ to $f(s+1)$. Consider the (non-commutative) rings $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ and $\mathbb{C}(s)\langle \tau, \tau^{-1} \rangle$, whose elements are finite sums

$$\sum_{n \in \mathbb{Z}} f_n(s) \tau^n,$$

where the coefficients f_n lie in $\mathbb{C}[s]$ and $\mathbb{C}(s)$, respectively. The multiplication in these rings is defined by the relation $\tau s = (s+1)\tau$. With this notation, a difference module over $\mathbb{C}(s)$ corresponds to a $\mathbb{C}(s)\langle \tau, \tau^{-1} \rangle$ -module that is finite-dimensional over $\mathbb{C}(s)$.

Let us revisit the Mellin transform, as discussed in Example 4.3.4. The ring of differential operators on the multiplicative group \mathbb{G}_m is the non-commutative algebra

⁷The context will make it clear whether we are referring to a category of differential or difference modules.

$\mathbb{C}[x, x^{-1}]\langle \partial_x \rangle$, whose elements are finite sums of the form

$$\sum_{n=0}^{\infty} p_n(x) \partial_x^n,$$

where the coefficients p_n are Laurent polynomials in $\mathbb{C}[x, x^{-1}]$. The multiplication in this ring is defined by the relation $\partial_x x = x \partial_x + 1$. It will be convenient to consider the equivalent presentation $\mathbb{C}[x, x^{-1}]\langle \theta \rangle$, where $\theta := x \partial_x$.

From this point of view, the Mellin transform can be described as the equivalence of categories $\text{MT}: \text{Mod}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) \rightarrow \text{Mod}(\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle)$ induced by the isomorphism

$$\mathbb{C}[x, x^{-1}]\langle \theta \rangle \rightarrow \mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$$

that maps $x \mapsto \tau$ and $\theta \mapsto -s$. This equivalence of categories, along with Proposition 6.2.4, motivates the following definitions.

DEFINITION 6.4.4. Let M be a $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ -module. We say that M is *holonomic* if it is the Mellin transform of a holonomic \mathcal{D} -module on \mathbb{G}_m . A holonomic $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ -module M is said to be *dual-negligible* if its generic fiber $M \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ vanishes. We denote by $\text{Hol}(\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle)$ the category of holonomic $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ -modules, and by $\text{DNeg}(\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle)$ its full subcategory consisting of dual-negligible objects.

According to a theorem of Loeser–Sabbah [LS92, Thm. 2], the Euler characteristic of a holonomic \mathcal{D} -module M on \mathbb{G}_m coincides with the dimension of $\text{MT}(M)_\eta$ over $\mathbb{C}(s)$. As a result, the functor $- \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ maps holonomic $\mathbb{C}[s]\langle \tau, \tau^{-1} \rangle$ -modules to difference modules over $\mathbb{C}(s)$. In other words, the generic fiber of the Mellin transform defines a functor

$$\text{Hol}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) \rightarrow \text{Diff}(\mathbb{C}(s)).$$

The universal property of quotients and Proposition 6.2.4 imply that this functor factors through the quotient

$$\text{Hol}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) / \text{Neg}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) \rightarrow \text{Diff}(\mathbb{C}(s)).$$

Recall that the category on the left has a symmetric monoidal structure given by convolution, while the category on the right has a symmetric monoidal structure given by tensor products. The functor above is symmetric monoidal due to Proposition 4.2.3.

THEOREM 6.4.5. *The functor $\text{Hol}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) / \text{Neg}(\mathbb{C}[x, x^{-1}]\langle \theta \rangle) \rightarrow \text{Diff}(\mathbb{C}(s))$ obtained by taking the generic fiber of the Mellin transform is a symmetric monoidal equivalence of categories.*

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Proof. Exactly as in the proof of Theorem 6.3.11, it suffices to prove that the functor

$$- \otimes_{\mathbb{C}[s]} \mathbb{C}(s): \text{Hol}(\mathbb{C}[s]\langle\tau, \tau^{-1}\rangle) / \text{DNeg}(\mathbb{C}[s]\langle\tau, \tau^{-1}\rangle) \rightarrow \text{Diff}(\mathbb{C}(s))$$

is essentially surjective. In the difference setting, this result is a consequence of [LS91b, Thm. 1.2.1.2]. \square

COROLLARY 6.4.6. *Let M be a holonomic \mathcal{D} -module on \mathbb{G}_m . Then, the groups $\text{Conv}(M)$ and $\text{Gal}(\text{MT}(M)_\eta)$ are isomorphic.*

Unlike in the differential case, not every linear algebraic group can serve as a difference Galois group. According to [VS97, Prop. 1.20], if G is a difference Galois group over $\mathbb{C}(s)$, then G must be *almost connected*, meaning that the group of connected components $\pi_0(G) \simeq G/G^0$ is cyclic. From Corollary 6.4.6, we deduce the same property for convolution groups. This result can be seen as the de Rham counterpart to [Kat12, Thm. 6.5].

COROLLARY 6.4.7. *Let M be a holonomic \mathcal{D} -module on \mathbb{G}_m . Then the group of connected components $\pi_0(\text{Conv}(M))$ of $\text{Conv}(M)$ is cyclic.*

The inverse problem for difference equations conjectures that any linear algebraic group over \mathbb{C} with a cyclic group of connected components must be the Galois group of a difference module over $\mathbb{C}(s)$. To our knowledge, this problem remains open. However, it is known that every *connected* linear algebraic group is a difference Galois group [VS97, Thm. 3.1]. We can therefore deduce an analogous result for convolution groups.

COROLLARY 6.4.8. *Let G be a connected linear algebraic group over \mathbb{C} . Then, there exists a holonomic \mathcal{D} -module M on \mathbb{G}_m such that $\text{Conv}(M) \simeq G$.*

PART IV.
APPENDICES

A. EULER CHARACTERISTICS OF HOLONOMIC \mathcal{D} -MODULES

A.1. COMPARISON OF EULER CHARACTERISTICS

Let X be a smooth quasi-projective variety over a characteristic zero field k . Denote by $p: X \rightarrow \text{Spec } k$ the structure map. For an object M of $D_h^b(\mathcal{D}_X)$, we define its *Euler characteristic* as

$$\chi(X, M) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k H^i(X, M),$$

where the de Rham cohomology groups $H^i(X, M)$ are defined as $\mathcal{H}^i(p_+ M)$. Similarly, by replacing p_+ with $p_!$, we obtain the compactly supported cohomology groups $H_c^i(X, M)$ and the corresponding *compactly supported Euler characteristic* $\chi_c(X, M)$.

For Betti cohomology, due to the absence of ramification phenomena, it is rather straightforward to prove that both variants of Euler characteristic coincide for local systems. This result can be extended to complexes of constructible sheaves through a formal *dévissage* argument [Dim04, Cor. 4.1.23]. An analogous result also holds in the ℓ -adic setting, as demonstrated by Laumon [Lau81].

Laumon's argument is highly versatile, applying to most "motivic" cohomology theories that admit enhancements as six-functor formalisms with nearby and vanishing cycles. However, it does not directly apply to de Rham cohomology. A critical part of Laumon's proof involves considering a compactification \bar{X} of X . The difficulty with de Rham cohomology arises because the complement $\bar{X} \setminus X$ is often not smooth.

Although the six-functor formalism for holonomic \mathcal{D} -modules, as established in Proposition 1.5.1, can be extended to singular varieties,¹ some care must be taken regarding nearby and vanishing cycles. In this section, we prove that both variants of Euler characteristic agree for de Rham cohomology.

THEOREM A.1.1. *Let X be a smooth quasi-projective variety over a characteristic zero field k , and let M be an object of $D_h^b(\mathcal{D}_X)$. Then, the Euler characteristic $\chi_c(X, M)$ and $\chi(X, M)$ coincide.*

Before delving into the proof of this theorem, we encapsulate Laumon's argument in the following lemma. We refer the reader to Remark 1.5.8 for our conventions

¹See [Ayo17, §3.8] for one possible approach.

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regarding nearby and vanishing cycles functors for holonomic \mathcal{D} -modules.

LEMMA A.1.2. *Let $j: X \hookrightarrow \bar{X}$ be an open immersion from X to a smooth projective algebraic variety \bar{X} . Denote by $i: Z \hookrightarrow \bar{X}$ the complementary closed immersion, and suppose that Z is a smooth divisor. Then, for an object M of $D_h^b(\mathcal{D}_X)$, we have that $\chi(Z, i^+j_+M) = 0$.*

Proof. The same reduction as in the third part of the proof of Theorem A.1.1 below allows us to assume that M is a holonomic \mathcal{D} -module concentrated in degree zero. According to Remark 1.5.8, i^+j_+M is the fiber of the canonical map $\text{can}: \psi_1(j_+M) \rightarrow \phi_1(j_+M)$. In other words, i^+j_+M is isomorphic to the complex

$$\psi_1(j_+M) \xrightarrow{\text{can}} \phi_1(j_+M),$$

where $\psi_1(j_+M)$ is in degree -2 . The long exact sequence in cohomology arising from the short exact sequence of complexes

$$0 \rightarrow \ker \text{can}[2] \rightarrow [\psi_1(j_+M) \xrightarrow{\text{can}} \phi_1(j_+M)] \rightarrow \text{coker can}[1] \rightarrow 0$$

then implies that $\chi(Z, i^+j_+M) = \chi(Z, \ker \text{can}) - \chi(Z, \text{coker can})$.

We can also consider the following short exact sequences of holonomic \mathcal{D} -modules:

$$\begin{aligned} 0 &\rightarrow \ker \text{can} \rightarrow \psi_1(j_+M) \rightarrow \text{im can} \rightarrow 0 \\ 0 &\rightarrow \text{im can} \rightarrow \phi_1(j_+M) \rightarrow \text{coker can} \rightarrow 0. \end{aligned}$$

As before, these give rise to the relations $\chi(Z, \psi_1(j_+M)) = \chi(Z, \ker \text{can}) + \chi(Z, \text{im can})$ and $\chi(Z, \phi_1(j_+M)) = \chi(Z, \text{im can}) + \chi(Z, \text{coker can})$. Now, another application of Remark 1.5.8 shows that the variation map $\text{var}: \phi_1(j_+M) \rightarrow \psi_1(j_+M)$ is an isomorphism, and so

$$\begin{aligned} \chi(Z, \ker \text{can}) + \chi(Z, \text{im can}) &= \chi(Z, \psi_1(j_+M)) \\ &= \chi(Z, \phi_1(j_+M)) = \chi(Z, \text{im can}) + \chi(Z, \text{coker can}). \end{aligned}$$

We conclude that $\chi(Z, \ker \text{can}) = \chi(Z, \text{coker can})$, and so $\chi(Z, i^+j_+M) = 0$. \square

Proof of Theorem A.1.1. The proof of this result is based on a series of *déviassages*.

Part 1: Suppose that X is a curve and M is a connection in degree zero.

Let \bar{X} be the smooth compactification of X , and let Z be the complement of X in \bar{X} . Deligne's formula for Euler characteristics says that

$$\chi(X, M) = (\text{rank } M)\chi(X) - \sum_{z \in Z} \text{ir}_z(M),$$

where $\text{ir}_z(M)$ is the *irregularity* of M at the point $z \in Z$ [Del70, Eq. 6.21.1]. Now, by [ABC20, Lem. 17.2.2], we have that $\text{ir}_z(M) = \text{ir}_z(M^\vee)$ for all $z \in Z$. Poincaré duality then implies that

$$\begin{aligned}\chi_c(X, M) &= \chi(X, M^\vee) \\ &= (\text{rank } M^\vee)\chi(X) - \sum_{z \in Z} \text{ir}_z(M^\vee) \\ &= (\text{rank } M)\chi(X) - \sum_{z \in Z} \text{ir}_z(M) = \chi(X, M).\end{aligned}$$

Part 2: Suppose that X is a curve and M is a holonomic \mathcal{D} -module in degree zero.

Let $j: U \hookrightarrow X$ be the locus where M is a connection, and let $i: Z \hookrightarrow X$ be its complement. As in Remark 1.5.5, the adjunction maps form fiber sequences

$$j_!j^!M \rightarrow M \rightarrow i_+i^+M \quad \text{and} \quad i_!i^!M \rightarrow M \rightarrow j_+j^+M.$$

Consequently, there are long exact sequences

$$\begin{aligned}\cdots &\rightarrow H_c^i(U, j^!M) \rightarrow H_c^i(X, M) \rightarrow H_c^i(Z, i^+M) \rightarrow \cdots \\ \cdots &\rightarrow H^i(Z, i^!M) \rightarrow H^i(X, M) \rightarrow H^i(U, j^+M) \rightarrow \cdots.\end{aligned}$$

In particular, $\chi_c(X, M) = \chi_c(U, j^!M) + \chi_c(Z, i^+M)$ and $\chi(X, M) = \chi(U, j^+M) + \chi(Z, i^!M)$. The restriction $j^+M \simeq j^!M$ is a connection on U , thus the preceding part yields that $\chi_c(U, j^!M) = \chi(U, j^+M)$. Another application of Remark 1.5.5 provides a fiber sequence

$$i^!M \rightarrow i^+M \rightarrow i^+j_+j^+M.$$

Since Z is proper, Lemma A.1.2 implies that

$$\begin{aligned}\chi_c(Z, i^+M) &= \chi_c(Z, i^!M) + \chi_c(Z, i^+j_+j^+M) \\ &= \chi(Z, i^!M) + \chi(Z, i^+j_+j^+M) \\ &= \chi(Z, i^!M).\end{aligned}$$

All in all, we find that $\chi_c(X, M) = \chi(X, M)$.

Part 3: Suppose that X is a curve and M is an object of $D_h^b(\mathcal{D}_X)$.

We argue by induction on the amplitude of M . If M only has cohomology in a single degree, the result follows from the preceding part. Now, suppose that M has at most n non-zero cohomologies, and let $j \in \mathbb{Z}$ be the smallest index i such that $\mathcal{H}^i(M) \neq 0$. The truncation fiber sequence

$$\mathcal{H}^j(M)[-j] \rightarrow M \rightarrow \tau_{\geq j+1}M,$$

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as in [Stacks, Tag 08J5], implies that

$$\begin{aligned}\chi_c(X, M) &= (-1)^j \chi_c(X, \mathcal{H}^j(M)) + \chi_c(X, \tau_{\geq j+1} M) \\ \chi(X, M) &= (-1)^j \chi(X, \mathcal{H}^j(M)) + \chi(X, \tau_{\geq j+1} M).\end{aligned}$$

Since $\tau_{\geq j+1} M$ has at most $n - 1$ non-zero cohomologies, the induction hypothesis implies that $\chi_c(X, \mathcal{H}^j(M)) = \chi(X, \mathcal{H}^j(M))$ and $\chi_c(X, \tau_{\geq j+1} M) = \chi(X, \tau_{\geq j+1} M)$.

Part 4: Suppose that X is the affine space \mathbb{A}^n , for some n , and M is an object of $D_h^b(\mathcal{D}_X)$.

We argue by induction. The case $n = 1$ having been proven in the previous part, we suppose the result true from \mathbb{A}^n and prove it for \mathbb{A}^{n+1} . Let $q: \mathbb{A}^{n+1} \rightarrow \mathbb{A}^n$ be the projection onto the last n factors, and consider the following compactification

$$\begin{array}{ccccc}\mathbb{A}^{n+1} & \xrightarrow{j} & \mathbb{P}^1 \times \mathbb{A}^n & \xleftarrow{i} & \mathbb{A}^n \\ & \searrow q & \downarrow \bar{q} & \swarrow & \\ & & \mathbb{A}^n & & \end{array}$$

Remark 1.5.5 provides a fiber sequence $q_! M \rightarrow q_+ M \rightarrow i^+ j_+ M$. Thus, Lemma A.1.2 implies that $\chi(\mathbb{A}^n, q_! M) = \chi(\mathbb{A}^n, q_+ M)$. As a result, we find that

$$\chi_c(\mathbb{A}^{n+1}, M) = \chi_c(\mathbb{A}^n, q_! M) = \chi(\mathbb{A}^n, q_! M) = \chi(\mathbb{A}^n, q_+ M) = \chi(\mathbb{A}^{n+1}, M).$$

Part 5: Suppose that X is an affine algebraic variety and M is an object of $D_h^b(\mathcal{D}_X)$.

Noether normalization provides a finite map $f: X \rightarrow \mathbb{A}^n$, for some n . As a result, $\chi(X, M) = \chi(\mathbb{A}^n, f_+ M)$ and $\chi_c(X, M) = \chi_c(\mathbb{A}^n, f_! M) = \chi_c(\mathbb{A}^n, f_+ M)$. The result then follows from the preceding part.

Part 6: The general case.

For notational simplicity, assume that X is the union of two affine opens $j_1: U_1 \hookrightarrow X$ and $j_2: U_2 \hookrightarrow X$. Denote by $j_{12}: U_{12} \hookrightarrow X$ their intersection, which is also affine. As in Remark 1.5.6, there are fiber sequences

$$j_{12, !} j_{12}^+ M \rightarrow j_{1, !} j_1^+ M \oplus j_{2, !} j_2^+ M \rightarrow M \quad \text{and} \quad M \rightarrow j_{1, +} j_1^+ M \oplus j_{2, +} j_2^+ M \rightarrow j_{12, +} j_{12}^+ M.$$

In particular, we obtain the following long exact sequences in cohomology:

$$\begin{aligned}\cdots &\rightarrow H_c^i(U_{12}, j_{12}^+ M) \rightarrow H_c^i(U_1, j_1^+ M) \oplus H_c^i(U_2, j_2^+ M) \rightarrow H_c^i(X, M) \rightarrow \cdots \\ \cdots &\rightarrow H^i(X, M) \rightarrow H^i(U_1, j_1^+ M) \oplus H^i(U_2, j_2^+ M) \rightarrow H^i(U_{12}, j_{12}^+ M) \cdots\end{aligned}$$

These imply that $\chi_c(X, M) = \chi_c(U_1, j_1^+ M) + \chi_c(U_2, j_2^+ M) - \chi_c(U_{12}, j_{12}^+ M)$, and that $\chi(X, M) = \chi(U_1, j_1^+ M) + \chi(U_2, j_2^+ M) - \chi(U_{12}, j_{12}^+ M)$. Consequently, the result follows from the previous part. \square

A.2. EULER CHARACTERISTICS OF REGULAR TWISTS

In this section, we explore a de Rham analogue of another result concerning Euler characteristics that is known in the ℓ -adic setting. The following theorem stems directly from a result of Deligne, presented by Illusie in [Ill81].

THEOREM A.2.1 (Deligne). *Let k be an algebraically closed field of characteristic p , and let ℓ be a prime number distinct from p . Let X be a normal quasi-projective variety over k , and let M be an object of $D_c^b(X, \overline{\mathbb{Q}}_\ell)$. Then, for every tame rank-one ℓ -adic local system \mathcal{L} on X , we have that $\chi(X, M \otimes \mathcal{L}) = \chi(X, M)$.*

Proof. From a Mayer–Vietoris argument, as in the proof of Theorem A.1.1, we may assume that X is connected. Taking the closure of X inside of projective space, we obtain an open immersion from X to a projective variety \overline{X} [Stacks, Tag 08J5]. Since X is normal, this map factors through the normalization \tilde{X} of \overline{X} , giving rise to an open immersion $j: X \rightarrow \tilde{X}$.

The variety \tilde{X} is connected, normal, and proper over k . Consequently, a result of Deligne [Ill81, Thm. 2.1] applies, implying that $j_!(M \otimes \mathcal{L})$ and $j_!M$ have the same compactly supported Euler characteristics. The result then follows from [Lau81]. \square

Unfortunately, the proof of [Ill81, Thm. 2.1] appears to be deeply rooted in ℓ -adic cohomology and does not readily adapt to de Rham cohomology. Nevertheless, we believe that Theorem A.2.1 should hold true in the de Rham setting as well.

CONJECTURE A.2.2. *Let X be a smooth quasi-projective variety over \mathbb{C} , and let M be an object of $D_{\text{R}}^b(\mathcal{D}_X)$. Then, for every regular line bundle with integrable connection² \mathcal{L} , we have that $\chi(X, M \otimes_X \mathcal{L}) = \chi(X, M)$.*

The remainder of this appendix will be dedicated to presenting evidence for this conjecture. We begin by proving it in the one-dimensional case.

PROPOSITION A.2.3. *Conjecture A.2.2 holds when X is a curve.*

Proof. If M is an integrable connection concentrated in degree zero, this result is a direct consequence of Deligne’s formula for Euler characteristics, as in the first part of the proof of Theorem A.1.1. The same *déviissage* used in Parts 2 and 3 of that proof similarly implies the desired result. \square

PROPOSITION A.2.4. *Conjecture A.2.2 holds when X is projective.*

²Recall from Remark 1.5.2 our convention that when connections are regarded as \mathcal{D} -modules, they should be shifted to be in degree equal to the dimension of the underlying space.

A. Euler characteristics of holonomic \mathcal{D} -modules

We start the proof of this proposition with the particular case of abelian varieties. The following is an adaptation of the beautiful proof presented in [KW15, Cor. 4.2].

LEMMA A.2.5 (Krämer–Weissauer). *Conjecture A.2.2 holds when X is an abelian variety A .*

Proof. Let $p: A \rightarrow \text{Spec } \mathbb{C}$ be the structure map. The Euler characteristic $\chi(A, M)$ is the same as the dimension of p_+M , as an object of the rigid symmetric monoidal ∞ -category $D_{\text{coh}}^b(\mathbb{C})$.³ From Corollaries 6.1.4 and 6.1.8, we know that $D_h^b(\mathcal{D}_A)$ is also a rigid symmetric monoidal ∞ -category under $!$ -convolution, and the functor p_+ is symmetric monoidal. As a result, $\chi(A, M)$ is also the dimension of M in $D_h^b(\mathcal{D}_A)$.

Since any line bundle with integrable connection \mathcal{L} on A is a character sheaf, Proposition 6.1.9 asserts that the functor $- \otimes_A \mathcal{L}$ is an equivalence of ∞ -categories $D_h^b(\mathcal{D}_A) \rightarrow D_h^b(\mathcal{D}_A)$ that is symmetric monoidal with respect to $!$ -convolution. In particular, tensoring with \mathcal{L} preserves the dimension of objects in $D_h^b(\mathcal{D}_A)$. \square

Proof of Proposition A.2.4. Fix a base point on X and let $\alpha: X \rightarrow A$ be the Albanese map. Consider a line bundle with integrable connection (\mathcal{L}, ∇) on X . Since \mathcal{L} admits a connection, its first Chern class vanishes, implying that \mathcal{L} belongs to $\text{Pic}^0(X)$. Given that the Albanese map induces an isomorphism

$$\alpha^*: \text{Pic}^0(A) \xrightarrow{\sim} \text{Pic}^0(X),$$

there exists a line bundle \mathcal{L}' on A such that $\mathcal{L} \simeq \alpha^* \mathcal{L}'$. Furthermore, since the first Chern class of \mathcal{L}' vanishes, it admits an integrable connection ∇' .

The underlying line bundle of $\alpha^*(\mathcal{L}', \nabla') \otimes (\mathcal{L}, \nabla)^\vee$ being trivial, its connection is of the form $d + \omega$ for some $\omega \in H^0(X, \Omega_X)$. By Hodge theory, ω is of the form $\alpha^* \omega'$, for some translation-invariant differential $\omega' \in H^0(A, \Omega_A)$. Consequently, (\mathcal{L}, ∇) is isomorphic to $\alpha^*(\mathcal{L}', \nabla' - \omega')$.

Henceforth, we return to \mathcal{D} -module notation, where we denote $(\mathcal{L}, \nabla)[- \dim X]$ and $(\mathcal{L}', \nabla' - \omega')[- \dim A]$ simply as \mathcal{L} and \mathcal{L}' , respectively. Finally, we have that

$$\chi(X, M \otimes_X \mathcal{L}) = \chi(A, \alpha_+(M \otimes_X \alpha^+ \mathcal{L}')) = \chi(A, \alpha_+ M \otimes_A \mathcal{L}'),$$

thus the result follows from Lemma A.2.5. \square

PROPOSITION A.2.6. *Conjecture A.2.2 holds when X is a commutative connected linear algebraic group L .*

Proof. A commutative connected linear algebraic group L over \mathbb{C} is necessarily a product $\mathbb{G}_a^r \times \mathbb{G}_m^s$, for some r, s . Additionally, a regular line bundle with integrable connection \mathcal{L} on L is of the form

$$\text{pr}_1^+ \mathcal{L}_1 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s,$$

³Refer to the discussion preceding Theorem 6.2.12 for more details.

where $\text{pr}_i: L \rightarrow \mathbb{G}_m$ is the projection onto the i -th copy of \mathbb{G}_m , for some character sheaves \mathcal{L}_i on \mathbb{G}_m . The structure map $p: L \rightarrow \text{Spec } \mathbb{C}$ factors as

$$L \xrightarrow{\text{pr}_1} \mathbb{G}_m \xrightarrow{q} \text{Spec } \mathbb{C},$$

where q is the structure map of \mathbb{G}_m . As a result, Proposition A.2.3 implies that

$$\begin{aligned} \chi(L, M \otimes_L \mathcal{L}) &= \chi(L, M \otimes_L \text{pr}_1^+ \mathcal{L}_1 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s) \\ &= \chi(\mathbb{G}_m, \text{pr}_{1,+}(M \otimes_L \text{pr}_1^+ \mathcal{L}_1 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s)) \\ &= \chi(\mathbb{G}_m, \text{pr}_{1,+}(M \otimes_L \text{pr}_2^+ \mathcal{L}_2 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s) \otimes_{\mathbb{G}_m} \mathcal{L}_1) \\ &= \chi(\mathbb{G}_m, \text{pr}_{1,+}(M \otimes_L \text{pr}_2^+ \mathcal{L}_2 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s)) \\ &= \chi(L, M \otimes_L \text{pr}_2^+ \mathcal{L}_2 \otimes_L \cdots \otimes_L \text{pr}_s^+ \mathcal{L}_s). \end{aligned}$$

Continuing in this fashion, we find that $\chi(L, M \otimes_L \mathcal{L}) = \chi(L, M)$. \square

We end this section with some remarks on a possible proof of Conjecture A.2.2 based on the results above.

■ **REMARK A.2.7.** Let G be a semiabelian variety over \mathbb{C} . In other words, let G be an extension of an abelian variety A by a torus T :

$$0 \rightarrow T \xrightarrow{\varphi} G \xrightarrow{\psi} A \rightarrow 0.$$

In Proposition 3.3.9, we demonstrated that the regular line bundles with integrable connection on G are precisely the character sheaves. In particular, using notation from Section 5.4, Corollary 3.1.10 asserts that any line bundle with integrable connection on G must be of the form $\mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha$.

Building on the argument used in the proof of Proposition A.2.6, combined with Proposition A.2.4, we conclude that

$$\chi(G, M \otimes_G \mathcal{L}_\omega \otimes_G \psi^+ \mathcal{L}_\alpha) = \chi(G, M \otimes_G \mathcal{L}_\omega).$$

We believe that this is also equal to $\chi(G, M)$. If this equality holds, it would provide a proof of the conjecture for semiabelian varieties. We assert that this would suffice to prove the conjecture in its entirety.

For a smooth quasi-projective variety X , there exists a generalized Albanese map $\alpha: X \rightarrow G$, where G is a semiabelian variety. Just as in the proof of Proposition A.2.4, every regular line bundle with flat connection on X comes from G .⁴ This would prove the conjecture using the same arguments as in Proposition A.2.4. ■

⁴See [LMW20, §2.2] for further details. The paper [Fuj24] is another useful reference.

B. DELIGNE'S 1970 LETTER TO BREEN

Bures-sur-Yvette, le 4 août 1970.

Dear Breen,

For me, a "Moore complex" means data of the following kind:

— a functorial left resolution of G (G abelian group) by a complex M such that

a) M_n is a sum of functors $\mathbb{Z}^{(G^k)}$

b) $d_n : M_n \rightarrow M_{n-1}$ is a matrix of morphisms $f : \mathbb{Z}^{(G^k)} \rightarrow \mathbb{Z}^{(G^l)}$ of the form $f = \sum n_i f_i$, where each f_i is deduced from

$$f'_i : G^k \rightarrow G^l \quad : \quad f'_i((g^\alpha)) = \left(\sum_{\alpha=1}^k a_{\beta\alpha} g^\alpha \right) \quad (a_{\beta\alpha} \in \mathbb{Z}).$$

c) $M_0 = \mathbb{Z}^{(G)}$ and $\varepsilon : M_0 \rightarrow G$ is the evident map.

I hope that I have succeeded in proving the existence of a Moore complex, by using the idea of the "acyclic model theorem" + the "bar construction". I am sending you notes about it by the same mail.

Yours sincerely,
Pierre Deligne

1. DÉFINITIONS

1.0. J'utiliserai les définitions suivantes

- catégorie sous-additive : où les $\text{Hom}(X, Y)$ sont munis de lois de groupes abéliens, la composition étant distributive
- catégorie additive : sous-additive + existence de sommes (= produits) finies [en particulier, existence d'un 0]
- catégorie karoubienne : additive, et telle que tout projecteur ait un noyau:
 $\forall X \forall p : X \rightarrow X$, si $p^2 = p$, alors $X = Y \oplus Z$, avec $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Pour toute catégorie \mathcal{C} , il existe à équivalence près une unique catégorie karoubienne \mathcal{C}^k , munie d'un foncteur "universel" $[] : \mathcal{C} \rightarrow \mathcal{C}^k$, telle que pour toute catégorie karoubienne \mathcal{D} , le foncteur $\text{Hom}_{\text{add}}(\mathcal{C}^k, \mathcal{D}) \rightarrow \text{Hom}(\mathcal{C}, \mathcal{D})$ soit une équivalence de catégories. On l'obtient ainsi

a) Soit \mathcal{C}''

$$\begin{cases} \text{Ob}(\mathcal{C}'') = \text{Ob}(\mathcal{C}) \\ \text{Hom}_{\mathcal{C}''}(X, Y) = \mathbb{Z}^{\text{Hom}(X, Y)} \\ []'' : \mathcal{C} \rightarrow \mathcal{C}'' : X \mapsto X, f \mapsto f \end{cases}$$

la catégorie sous-additive engendrée par \mathcal{C} .

b) Soit \mathcal{C}' la catégorie additive engendrée par \mathcal{C}'' : un objet de \mathcal{C}' est une famille finie $(X_i)_{i \in I}$, notée $\sum_{i \in I} X_i$, et

$$\text{Hom}_{\mathcal{C}'} \left(\sum_{i \in I} X_i, \sum_{j \in J} Y_j \right) = \prod_{i, j} \text{Hom}(X_i, Y_j).$$

c) Enfin, \mathcal{C}^k est la catégorie karoubienne engendrée par \mathcal{C}' : un objet est un couple (X, p) ($p : X \rightarrow X$ tel que $p^2 = p$), noté $\text{Im}(p)$, et

$$\text{Hom}_{\mathcal{C}^k}(\text{Im}(p), \text{Im}(q)) = q \text{ Hom}_{\mathcal{C}'}(X, Y) p \subset \text{Hom}_{\mathcal{C}'}(X, Y).$$

On a par construction

$$(1.0.1) \quad \text{Hom}_{\mathcal{C}^k}([X], [Y]) = \mathbb{Z}^{\text{Hom}_{\mathcal{C}}(X, Y)}.$$

Pour simplifier les notations, on désignera par la suite par Al le foncteur $X \mapsto \mathbb{Z}^{(X)} : (\text{Ens}) \rightarrow (\text{Ab})$.

DÉFINITION 1.0.2. $Al(E) = \mathbb{Z}^{(E)}$.

■ REMARQUE 1.0.3. Le théorème de Dold–Puppe sur l'équivalence des objets semi-simpliciaux avec les complexes différentiels nuls en degré > 0 est valable dans les catégories karoubiennes. ■

■ REMARQUE 1.0.4. Soit $F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ un foncteur, si \mathcal{C} est karoubienne, il se prolonge en F^k , additif en la première variable $\mathcal{A}^k \times \mathcal{B} \rightarrow \mathcal{C}$. C'est clair sur la formule

$$\text{Hom}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) = \text{Hom}(\mathcal{B}, \text{Hom}(\mathcal{A}, \mathcal{C})) \approx \text{Hom}(\mathcal{B}, \text{Hom}_{\text{add}}(\mathcal{A}^k, \mathcal{C})). \quad \blacksquare$$

1.1. On désignera par \mathcal{C}_0 la catégorie des groupes abéliens libres de type fini, par \mathcal{C}_1 la catégorie karoubienne \mathcal{C}_0^k engendrée par \mathcal{C}_0 et par $[] : \mathcal{C}_0 \rightarrow \mathcal{C}_1$ le foncteur (non additif) canonique.

Soit $T : \mathcal{A} \rightarrow \mathcal{B}$ un foncteur (non nécessairement additif) entre catégories karoubiennes. \mathcal{A} étant additive, on dispose de

$$\otimes : \mathcal{C}_0 \times \mathcal{A} \rightarrow \mathcal{A} \quad : \quad (X, A) \mapsto X \otimes A.$$

En effet, si \mathcal{C}_{-1} est la catégorie réduite à un objet 1 et à sa flèche identique, alors $\mathcal{C}_0 \approx \mathcal{C}_{-1}^k$ (avec $[1] = \mathbb{Z}$) ; le foncteur $X \otimes A$ est caractérisé par son additivité en X et par $\mathbb{Z} \otimes A = A$.

$$(1.1.1) \quad \mathbb{Z}^n \otimes A = A^n$$

Composant avec T , on trouve

$$T' : \mathcal{C}_0 \times \mathcal{A} \rightarrow \mathcal{B} \quad : \quad (X, A) \mapsto T(X \otimes A)$$

qui se prolonge en un foncteur additif en la première variable (1.0.4)

$$T_1 : \mathcal{C}_1 \times \mathcal{A} \rightarrow \mathcal{B}.$$

On posera $T_X(A) = T_1(X, A)$.

$$(1.1.2) \quad T_{\sum [X_i]}(A) = \sum T(X_i \otimes A)$$

Plus généralement, si K est un complexe $K \in C(\mathcal{C}_1)$, alors T_K est un foncteur

$$T_K : \mathcal{A} \rightarrow C(\mathcal{B}).$$

On notera que les débuts des complexes de Moore connus sont de la forme $Al_K [Al : 1.0.2]$. Par exemple

$$K = \dots \rightarrow [\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}] \oplus [\mathbb{Z} \oplus \mathbb{Z}] \xrightarrow{\begin{matrix} \sigma \\ \partial_0 - \partial_1 + \partial_2 - \partial_3 \end{matrix}} [\mathbb{Z} \oplus \mathbb{Z}] \xrightarrow{p_1 - \mu + p_2} [\mathbb{Z}]$$

donne

$$\mathrm{Al}_K = \dots \rightarrow \mathbb{Z}^{(G \times G \times G)} \oplus \mathbb{Z}^{(G \times G)} \rightarrow \mathbb{Z}^{(G \times G)} \rightarrow \mathbb{Z}^{(G)}.$$

On a

$$(1.1.3) \quad T_{[\mathbb{Z}]}(A) = T(A)$$

$$(1.1.4) \quad [\]_K(\mathbb{Z}) = K \quad \text{pour } [\] : \mathcal{C}_0 \rightarrow \mathcal{C}_1.$$

La formule essentielle suivante est cas particulier de (1.0.1)

$$(1.1.5) \quad \mathrm{Hom}_{\mathcal{C}_1}([\mathbb{Z}], K) = \mathrm{Al}_K(\mathbb{Z}) \quad \text{ou plus généralement}$$

$$(1.1.6) \quad \mathrm{Hom}_{\mathcal{C}_1}([X], K) = \mathrm{Al}_K(\check{X}).$$

1.2. Pour T un foncteur entre catégories karoubiennes, on définit le "cross effect" (effets croisés) par la formule

$$(1.2.1) \quad T\left(\sum_{i \in I} X_i\right) = \sum_{P \subset I} T_{|P|}((X_i)_{i \in P}).$$

Par construction, si π_Q est le projecteur sur $\sum_{i \in Q} X_i$, alors $T_{|P|}((X_i)_{i \in I}) \subset T(\sum_{i \in I} X_i)$ est défini par le projecteur

$$(1.2.2) \quad P_P = \sum_{Q \subset P} (-1)^{\#(P-Q)} T(\pi_Q)$$

On pose aussi,

$$(1.2.3) \quad T^n(X) = T_n(X, \dots, X).$$

Pour I à un élément dans (1.2.1), on pose $T^+ = T_1$

$$(1.2.4) \quad T(X) = T_0 + T^+(X), \quad \begin{cases} T_0 \in \mathrm{Ob}(\mathcal{B}) \\ T^+(0) = 0 \end{cases}.$$

Pour tout complexe différentiel L avec $L^i = 0$ pour $i > 0$, on désignera par L_{DP} l'objet semi-simplicial (CSS) dont L est normalisé. Ainsi, $\mathbb{Z}[1]_{\mathrm{DP}}$ est le CSS

$$\dots \quad \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} \mathbb{Z} \oplus \mathbb{Z} \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} \mathbb{Z} \rightrightarrows 0.$$

Pour L objet semi-simplicial en groupes abéliens libres de type fini, on désignera par [L] (resp. [L]⁺) son image par [] (rep par []⁺) : $\mathcal{C}_0 \rightarrow \mathcal{C}_1$. Un lemme évident et fondamental de "bar-construction" dit que le CSS

$$(1.2.5) \quad T_{[\mathbb{Z}[1]_{\mathrm{DP}}]}(X) : \dots T(X^3) \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} T(X^2) \begin{array}{c} \rightrightarrows \\ \rightleftarrows \end{array} T(X) \rightrightarrows T(0)$$

a pour normalisé le complexe différentiel des $T^i(X)$

$$(1.2.6) \quad \dots \quad T^3(X) \xrightarrow{d_3} T^2(X) \xrightarrow{d_2} T^1(X) \xrightarrow{0} T^0,$$

où le morphisme $d_n : T^n(X) \rightarrow T^{n-1}(X)$ est la somme alternée des morphismes induits par les $X^n \rightarrow X^{n-1} : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_i + x_{i+1}, \dots, x_n)$. Pour d_3 :

$$[\text{induit par } (x, y, z) \mapsto (x + y, z)] \cdot [\text{induit par } (x, y, z) \mapsto (x, y + z)].$$

■ REMARQUE 1.2.7. Les foncteurs $T^i(X)$ sont de la forme $T_A(X)$ ($A \in \mathcal{C}_1$) et ce qui précède est une identité dans \mathcal{C}_1 . ■

1.3. J'appelle "théorème des modèles acycliques" l'énoncé suivant

– Soit \mathcal{A} une catégorie additive. Les foncteurs contravariants représentables $\text{Hom}(*, X)$ sont des projectifs de la catégorie abélienne $\text{Hom}_{\text{add}}(\mathcal{A}^{\text{opp}}, (\text{Ab}))$ des foncteurs additifs de \mathcal{A}^{opp} dans la catégorie des groupes abéliens.

Il a comme corollaires

COROLLAIRE 1.3.1. Soit $f : K \rightarrow L$ un morphisme de complexes bornés supérieurement dans une catégorie additive \mathcal{A} ; pour que f soit une équivalence d'homotopie, il faut et il suffit que pour $A \in \text{Ob}(\mathcal{A})$,

$$f : \text{Hom}^\bullet(A, K) \rightarrow \text{Hom}^\bullet(A, L)$$

soit un quasi-isomorphisme (i.e. induise un isomorphisme sur la (co)homologie).

COROLLAIRE 1.3.2. Soient $K, L \in \text{Ob}(\mathcal{C}^-(\mathcal{A}))$ (\mathcal{A} additive). Si, pour tout $A \in \text{Ob}(\mathcal{A})$,

$$H^i(\text{Hom}^\bullet(A, K)) = H^i(\text{Hom}^\bullet(A, L)) = 0 \quad \text{pour } i \neq 0,$$

alors tout morphisme de foncteurs

$$f : H^0(\text{Hom}^\bullet(*, K)) \rightarrow H^0(\text{Hom}^\bullet(*, L))$$

se prolonge de façon unique à homotopie près en $\tilde{f} : K \rightarrow L$.

1.4. Bar-construction

Soit $K \in \text{Ob}(\mathcal{C}^-(\mathcal{C}_1))$ et dérivons, au sens de Dold–Puppe, le foncteur $[]_K : \mathcal{C}_0 \rightarrow \mathcal{C}^-(\mathcal{C}_1)$. On posera

DÉFINITION 1.4.1. $SK = ([]_K(\mathbb{Z}[1]_{\text{DP}}))[-1]$.

DÉFINITION 1.4.2. $'S^+K = ([]_K^+(\mathbb{Z}[1]_{\text{DP}}))[-1]$.

Concrètement, les

$$\cdots []_K(\mathbb{Z}^3) \overset{\rightrightarrows}{\underset{\rightrightarrows}{\rightrightarrows}} []_K(\mathbb{Z}^2) \overset{\rightrightarrows}{\underset{\rightrightarrows}{\rightrightarrows}} []_K(\mathbb{Z}) \overset{\rightrightarrows}{\underset{\rightrightarrows}{\rightrightarrows}} []_K(0)$$

\parallel
 K

forment un objet semi-simplicial en complexes différentiels, et SK est le complexe différentiel simple associé au complexe double correspondant, poussé de 1 vers la droite. De même pour $'S^+K$, qui est homotope au complexe S^+K , défini par le double complexe

$$\cdots []_K(\mathbb{Z}^3) \xrightarrow{\partial_0 - \partial_1 + \partial_2} []_K(\mathbb{Z}^2) \xrightarrow{\partial_0 - \partial_1} []_K(\mathbb{Z}).$$

\parallel
 K

On dispose de morphismes naturels

$$K \xrightarrow{s^+} S^+K \rightarrow SK = S^+k \oplus []_K(0)[-1].$$

Le complexe $[]_{S^+K}(0)$ est homotope à zéro.

DÉFINITION 1.4.3. On dit que $K \in \text{Ob}(\mathcal{C}^-(\mathcal{C}_1))$ est stable si $s^+ : K \rightarrow S^+K$ est une équivalence d'homotopie.¹

■ REMARQUE 1.4.4. A ce moment, $[]_K(0) \sim []_{S^+K}(0) \sim 0$, de sorte que $s : K \rightarrow SK$ est une équivalence d'homotopie.² ■

PROPOSITION 1.4.5. Soit $K \in \text{Ob}(\mathcal{C}^-(\mathcal{C}_1))$. Les conditions suivantes sont équivalentes :

- (a) Le foncteur $X \mapsto H_i \text{Al}_K(X) : \mathcal{C}_0 \rightarrow (\text{Ab})$, est additif pour $i \leq q$.
- (b) Le morphisme de foncteurs $H_i \text{Al}_K(X) \rightarrow H_i \text{Al}_{S^+K}(X)$ est un isomorphisme pour $i \leq q$ ($X \in \text{Ob}(\mathcal{C}_0)$), un épimorphisme pour $i = q + 1$.
- (c) Il existe un diagramme

$$\begin{array}{ccc} K & \xrightarrow{s} & S^+K \\ & \searrow a & \swarrow v \\ & & K_1 \end{array}$$

tel que v soit un équivalence d'homotopie et que a^i soit un isomorphisme pour $i \leq q + 1$.

Si ces conditions sont vérifiés, alors :

- (d) Les foncteurs $X \mapsto H_i \text{Al}_{S^+K}(X)$ sont additifs pour $i \leq q + 1$.

¹Editor's note: In the letter, Deligne labels this definition as 1.4.1, which coincides with an earlier definition. Consequently, the numbering in this subsection differs from that in the original letter.

²Editor's note: Here Deligne writes "Réciproque", but does not continue.

COROLLAIRE 1.4.6. Pour que K soit stable, il faut et il suffit que les foncteurs $X \mapsto H_i \text{Al}_K(X)$ de \mathcal{C}_0 dans (Ab) soient additifs.

L'équivalence (b) \iff (c) résulte du théorème des modèles acycliques (1.3) et de (1.1.6) ; de même, (b) pour tout q équivaut à la stabilité, et (1.4.5) \implies (1.4.6).

Procédons par récurrence sur q pour prouver que

$$(a)_q \implies (b)_q \text{ et } (d)_q$$

Ceci suffit à prouver (1.4.5), car pour $r < q$, on aura alors

$$(b)_q + (a)_r \implies (b)_q + (d)_r \implies (b)_q + (a)_{r+1},$$

et que $(a)_N$ est clair pour $N \ll 0$.

Une des suites spectrales du double complexe $\text{Al}_{S+K}(X)$ s'écrit

$$E_{p,q}^1 = H_q(\text{Al}_K(X^{p+1})) \implies H_{p+q}(\text{Al}_{S+K}(X)).$$

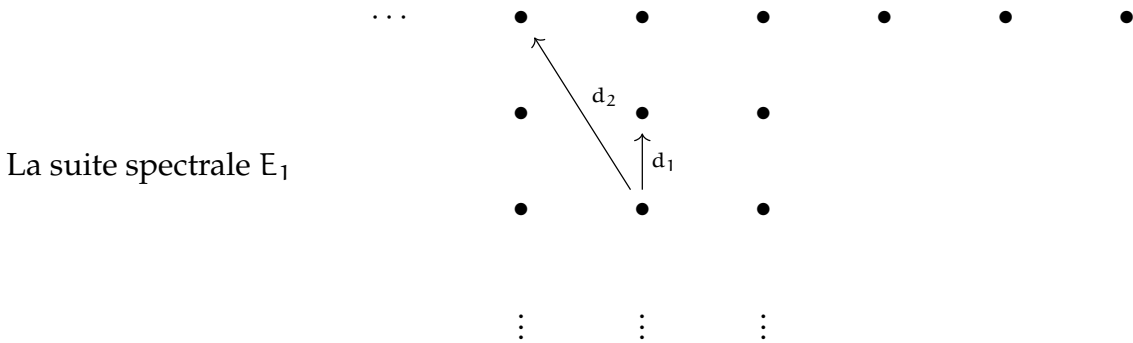
Les différentielles d_1 sont $\sum (-1)^i H_q(\text{Al}_K(\partial_i)) : \partial_i : X^{p+1} \rightarrow X^p$. Si T est le foncteur $H_q(\text{Al}_K(*))$, alors, par (1.2.5) (1.2.6), E_{*q}^2 est la cohomologie du complexe des $T^i(X)$ ($i > 0$)³ (cross effects), tandis que

$$s : H_q(\text{Al}_K(X)) \rightarrow H_q(\text{Al}_{S+K}(X))$$

est la flèche bord

$$T(X) \rightarrow T^1(X) = E_{0q}^2 \rightarrow H_q(\text{Al}_{S+K}(X)).$$

Si T est additif, alors $T = T^1$ et $T^i = 0$ pour $i \neq 0$, $E_{pq}^2 = 0$ pour $p \neq 0$ (et la valeur donnée de q), d'où $(a)_q \implies (b)_q$.



La suite spectrale donne de plus

$$H_{q+1}(\text{Al}_{S+K}(X)) = \text{coker}(T^2(X) \rightarrow T^1(X))$$

pour $T = H_{q+1}(\text{Al}_K(X))$.⁴ On achève la démonstration par le lemme

³Editor's note: This part is cut from my copy of the letter, so I'm unsure if it is correct.

⁴Editor's note: I'm unsure if this is what Deligne chose as T .

B. Deligne's 1970 letter to Breen

LEMME 1.4.7. Pour tout foncteur T d'une catégorie additive dans une catégorie abélienne, le foncteur

$$T' = \text{coker}(T^2(X) \rightarrow T^1(X))$$

est additif.

(a) On a $(T')' = T'$: on peut supposer que $T = T^+ (= T^1)$, i.e. $T(0) = 0$. Alors, on regarde le diagramme :

$$\begin{array}{ccccccc}
 T^2(X+X) & \longrightarrow & T(X+X) & \longrightarrow & T'(X+X) & \longrightarrow & 0 \\
 \downarrow p_1-(+)+p_2 & & \downarrow p_1-(+)+p_2 & \searrow & \downarrow & \searrow & \\
 & & & T^2(X) & & T'^2(X) & \\
 & & & \swarrow & \searrow 0 & \swarrow & \\
 T^2(X) = T_2(X, X) & \longrightarrow & T(X) & \longrightarrow & T'(X) & \longrightarrow & 0.
 \end{array}$$

(b) Soit T tel que $T = T'$, le $+$ induit $0 : T^2(X) \rightarrow T(X)$. Soit $f : X \oplus Y \rightarrow Z$. On factorise f en

$$X \oplus Y \xrightarrow{(f_1, 0) + (0, f_2)} Z \oplus Z \xrightarrow{+} Z, \quad \text{d'où}$$

$$\begin{array}{ccccc}
 T(X \oplus Y) & \longrightarrow & T(Z \oplus Z) & \longrightarrow & T(Z) \\
 \uparrow & & \uparrow & & \parallel \\
 T_2(X, Y) & \longrightarrow & T_2(Z, Z) & \xrightarrow[\text{car } T=T']{0} & T(Z),
 \end{array}$$

et f induit $0 : T_2(X, Y) \rightarrow T(Z)$. Ceci, pour f l'application 1 dans $Z = X \oplus Y$ entraîne que $T_2(X, Y) = 0$, d'où le lemme.

COROLLAIRE 1.4.8. Soit $f : K \rightarrow L$ un homomorphisme de complexes stables dans $C^-(\mathcal{C}_0)$. Alors f est une équivalence d'homotopie si et seulement si

$$Al_f(\mathbb{Z}) : H^* Al_K(\mathbb{Z}) \xrightarrow{\sim} H^* Al_L(\mathbb{Z}).$$

Résulte de (1.3.1) + (1.4.5). De (1.3.2) + (1.4.5), on tire de même

COROLLAIRE 1.4.9. Un complexe stable tel que $H^i(Al_K(\mathbb{Z})) = 0$ pour $i \neq 0$ est déterminé à homotopie près par $H^0(Al_K(\mathbb{Z}))$.

DÉFINITION 1.4.10. Un complexe de Moore est un complexe stable $K \in \text{Ob}(C^-(\mathcal{C}_1))$ tel que $H^i(Al_K(\mathbb{Z})) = 0$ pour $i \neq 0$ et que $H^0(Al_K(\mathbb{Z})) = \mathbb{Z}$.

D'après (1.3.6)⁵, s'il existe, un tel complexe est unique à homotopie près ; il vérifie

$$\begin{cases} H^0(\text{Al}_K(X)) = X \\ H^i(\text{Al}_K(X)) = 0, \quad \text{pour } i \neq 0 \end{cases}$$

pour X libre de type fini (done, par passage à la limite, pour X sans torsion).⁶

La Proposition (1.4.5) (c) + (d) permet d'itérer infiniment la construction S^+ (à homotopie près) en ne changeant à chaque pas que les composants de degré (homologique) de plus en plus élevé du complexe. Ceci permet de passer à la limite pour définir un itéré infini S^t , muni de $s^t : K \rightarrow S^t K$. Il me semble clair que

PROPOSITION 1.4.11. *Le foncteur $S^t : K^-(\mathcal{C}_1) \rightarrow K^-(\mathcal{C}_1)$ est un adjoint à gauche de l'inclusion dans $K^-(\mathcal{C}_1)$ de la sous-catégorie triangulée pleine des complexes stables, la flèche d'adjonction est s^t .*

■ EXEMPLE 1.4.12. Il doit être clair que $S^t([\mathbb{Z}])$ est le complexe de Mac-Lane A . Il doit être clair aussi que $S^t(K)$ peut se calculer "degré par degré". En formules, ce doit s'exprimer comme suit : on dispose dans \mathcal{C}_1 d'une unique operation $[\otimes]$, rendant commmutatif

$$\begin{array}{ccc} \mathcal{C}_0 \times \mathcal{C}_0 & \xrightarrow{[\otimes]} & \mathcal{C}_0 \\ \downarrow [\] \times [\] & & \downarrow [\] \\ \mathcal{C}_1 \times \mathcal{C}_1 & \xrightarrow{[\otimes]} & \mathcal{C}_1 \end{array}$$

et biadditive. On a

$$\begin{aligned} (1.4.13) \quad SK &= S([\mathbb{Z}]) [\otimes] K \\ S^+K &= S^+([\mathbb{Z}]) [\otimes] K \\ S^tK &= S^t([\mathbb{Z}]) [\otimes] K. \end{aligned}$$

■

Il doit être clair que $H_* \text{Hom}^\bullet([\mathbb{Z}], S^t([\mathbb{Z}]))$ est l'homologie stable des $K(\mathbb{Z}, n)$ ($n \rightarrow \infty$) [cf. (1.1.5)]. Par (1.4.11), c'est aussi

$$\text{Hom}^\bullet(S^t([\mathbb{Z}], S^t([\mathbb{Z}])),$$

donc une algèbre \mathcal{A} . Traduction en terme classiques ? \mathcal{A} est graduée : \mathcal{A}_* , et $\mathcal{A}_0 = \mathbb{Z}$.

Plus généralement, si K est un complexe stable, la dernière identité (1.4.13) définit une action, dans $K^-(\mathcal{C}_1)$, de $\mathcal{A} = \text{End}^\bullet(S^t([\mathbb{Z}]))$ sur K . Pour tout foncteur $T : \mathcal{A} \rightarrow \mathcal{B}$, $H_*(T_K(A))$ apparait ainsi comme un \mathcal{A} -module dans \mathcal{B} . En particulier, $H_* \text{Al}_K(\mathbb{Z})$ est un \mathcal{A} -module (aussi parce que $H_*(\text{Al}_K(\mathbb{Z})) \approx H_* \text{Hom}^\bullet([\mathbb{Z}], K) \approx H_* \text{Hom}^\bullet(S^t([\mathbb{Z}], K))$.

⁵Editor's note: There is no (1.3.6) in the letter. I believe that the reference should be to (1.4.9).

⁶Editor's note: Here Deligne writes "Réciproque", but does not continue.

2. COMPLEXES DE MOORE

THÉORÈME 2.1. *Il existe des complexes de Moore (1.3.7)⁷.*

C'est devenu plutôt clair. On part de $S^t(\mathbb{Z})$, stable, et pour lequel $H^0(\text{Al}_K(\mathbb{Z})) = \mathbb{Z}$. Pour K stable et $x \in H_q(\text{Al}_K(\mathbb{Z})) = H_q(\text{Hom}_{\mathbb{Z}}, K)$, il existe $\underline{x} : \mathbb{Z}[q] \rightarrow K$ qui le définit, et, par (1.4.11), on peut prolonger \underline{x} en $S^t(\mathbb{Z}[-q]) = S^t(\mathbb{Z})[-q] \rightarrow K$. Remplacer K par le cône de \underline{x} ne change par les $H_i(\text{Al}_K[\mathbb{Z}])$ pour $i < q$, tue $x \in H_q$ et modifie les H_i pour $i > q$. Procédant par récurrence, et notant que les complexes stables forment une catégorie triangulée, on fabrique le complexe de Moore.

Cette construction montre une relation entre complexe de Moore et résolution projective du \mathcal{A} -module d'augmentation \mathbb{Z} .

PROPOSITION 2.2. *Soit K un complexe stable. Si*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

est une suite exacte de groupes abéliens (libres ou non), alors la suite

$$\Sigma : \text{Al}_K(X) \xrightarrow{\alpha} \text{Al}_K(Y) \xrightarrow{\beta} \text{Al}_K(Z)$$

est "aussi bonne qu'une suite exacte courte":

- (a) *Remplaçant K par un complexe homotope, on se ramène au cas où $\text{Al}_K(0) = 0 : \text{Al}_K(X) = \text{Al}_K(0) \oplus \text{Al}_K^+(X)$ et $\text{Al}_K(0) \sim 0, \text{Al}_K^+(X) \sim \text{Al}'_{K^+}(X)$.*
- (b) *Ceci fait, α est injectif, β surjectif et*

$$\text{Al}_K(X) \xrightarrow{\alpha} \ker(\beta) \quad \text{et} \quad \text{coker}(\alpha) \rightarrow \text{Al}_K(Z)$$

soit des quasi-isomorphismes.

Pour tout K , $S^t(K)$ peut se calculer "degré par degré". Grâce à ce fait, on se ramène au cas de $S^t(\mathbb{Z}^n)$, ou au cas de $S^t(\mathbb{Z})$ (car $S^t(\mathbb{Z}^n) \approx S^t(\mathbb{Z})^n$). Pour le complexe de Moore, cela est aussi évident sur la construction donnée. Enfin, pour $K = S^t(\mathbb{Z})$ (1.4.12), j'espère que c'est classique.

COROLLAIRE 2.3. *Si K est un complexe de Moore, alors, pour tout groupe abélien X , on a*

$$H_i(\text{Al}_K(X)) = \begin{cases} 0 & \text{si } i \neq 0 \\ X & \text{si } i = 0 \end{cases} .$$

Pour X de type fini, on prend $0 \rightarrow \mathbb{Z}^n \rightarrow \mathbb{Z}^m \rightarrow X$ et on applique la suite exacte de cohomologie plus la proposition précédente et la remarque après (1.4.10).

Le cas général se obtient par \varinjlim .

⁷Editor's note: There is no (1.3.7) in the letter. I believe that the reference should be to (1.4.10).

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Titre : Annulation Générique pour les \mathcal{D} -modules Holonomes : Une Étude via la Dualité de Cartier

Mots clés : Géométrie Algébrique, \mathcal{D} -modules, Dualité de Cartier

Résumé : Motivée par des questions issues de la théorie analytique des nombres et de la géométrie complexe, cette thèse explore des théorèmes d'annulation générique pour la cohomologie de de Rham des \mathcal{D} -modules holonomes sur certains groupes algébriques commutatifs connexes G .

Nous commençons par construire un espace algébrique G^b qui paramétrise les fibrés en droites multiplicatifs munis d'une connexion plate sur G , appelés faisceaux caractères. Pour les variétés abéliennes, cela fournit une nouvelle construction de l'espace de modules de Simpson des fibrés en droites munis d'une connexion intégrable. Cette construction a l'avantage de s'appliquer également aux groupes G non propres. La nouveauté de notre approche réside dans l'utilisation systématique d'une incarnation « champêtre » de la dualité de Cartier, conduisant à l'étude de diverses extensions de faisceaux abéliens, pouvant être d'un intérêt indépendant.

En examinant la géométrie de l'espace de modules G^b , nous identifions une classe de sous-espaces appelés sous-espaces linéaires. Ensuite, le théorème d'annulation générique affirme que, pour chaque \mathcal{D} -module holonome, il existe un nombre fini de translates de ces sous-espaces linéaires telle que la cohomologie de de Rham des twists par des faisceaux caractères dans leur complément soit concentrée en degré zéro.

En utilisant le théorème d'annulation générique, nous déduisons qu'une certaine catégorie quotient des \mathcal{D} -modules holonomes est tannakienne. En d'autres termes, chaque \mathcal{D} -module de cette catégorie équivaut à une représentation d'un groupe algébrique, appelé groupe tannakien. Notamment, pour le groupe additif et le groupe multiplicatif, nous établissons un résultat de comparaison identifiant ces groupes tannakiens aux groupes de Galois différentiels ou aux groupes de Galois aux différences, respectivement.

Title : Generic Vanishing for Holonomic \mathcal{D} -modules: A Study via Cartier Duality

Keywords : Algebraic Geometry, \mathcal{D} -modules, Cartier Duality

Abstract : Motivated by questions in analytic number theory and complex geometry, this thesis studies generic vanishing theorems for the de Rham cohomology of holonomic \mathcal{D} -modules on certain commutative connected algebraic groups G .

We begin by constructing an algebraic space G^b that parametrizes multiplicative line bundles with flat connection on G , referred to as character sheaves. For abelian varieties, this provides a new construction of Simpson's moduli space of line bundles with integrable connection. Importantly, our construction also applies to non-proper groups G . The novelty of our approach lies in the systematic use of a stacky incarnation of Cartier duality, leading to the study of various extensions of abelian sheaves, which may be of independent interest.

By examining the geometry of the moduli space G^b , we identify a class of subspaces known as linear subspaces. The generic vanishing theorem then asserts that, for each holonomic \mathcal{D} -module, there exists a finite number of translates of these linear subspaces such that the de Rham cohomology of twists by character sheaves in their complement is concentrated in degree zero.

From the generic vanishing theorem, we deduce that a specific quotient category of holonomic \mathcal{D} -modules is tannakian. In other words, each such \mathcal{D} -module corresponds to a representation of an algebraic group, referred to as a tannakian group. Notably, for the additive and multiplicative groups, we establish a comparison result, identifying these tannakian groups with differential and difference Galois groups, respectively.