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Adrien Beguinet

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Numerical analysis of a poroelastic problem with random coefficients: Application to geothermal energy

*Analyse numérique d'un problème poroélastique à coefficients aléatoires :
Application à la géothermie*

Thèse de doctorat de l'Université Paris-Saclay

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sous la direction de Ludovic GOUDENÉGE (Directeur de Recherche CNRS, Université Paris-Saclay (LAMME)), le
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Thèse présentée et soutenue à Gif-sur-Yvette, le 19 décembre 2024, par

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Chapter 1

Introduction (français)

1.1 Introduction (français)

L'objectif de ce chapitre est de présenter les différents sujets abordés dans cette thèse, tout en les replaçant dans le cadre plus large du projet de recherche ANR (Agence Nationale de la Recherche) UPGEO, sur le stockage de l'énergie géothermique. Le sous-sol est généralement composé de roches ou de sédiments plus ou moins perméables, et si la perméabilité est suffisamment élevée, les fluides peuvent circuler librement. Dans sa forme la plus élémentaire, l'écoulement monophasique des eaux souterraines est modélisé par une seule loi de conservation, basée sur la loi de Darcy.

La géomécanique se concentre sur la mécanique des sols, une discipline d'ingénierie dans laquelle une représentation détaillée des sous-sols est cruciale. Elle s'appuie sur des données et des modèles géologiques et géophysiques et joue un rôle essentiel dans des domaines tels que l'exploration des hydrocarbures, l'exploitation de l'énergie géothermique et le stockage du dioxyde de carbone.

L'objectif du projet UPGEO est de mieux comprendre les systèmes géothermiques, en proposant des solutions qui optimisent le développement de nouveaux outils. Pour ce faire, une connaissance précise de l'écoulement des réservoirs grâce à des simulations numériques fiables est nécessaire. Les échelles de temps et d'espace variant considérablement selon le contexte, une méthode de résolution robuste est nécessaire. Par exemple, l'étude de l'évolution d'un bassin sédimentaire nécessite l'analyse de centaines de kilomètres carrés sur des millions d'années. En revanche, l'évaluation de l'impact d'une infrastructure humaine telle qu'un puits se concentre sur quelques kilomètres carrés et une période de quelques décennies.

La poroélasticité concerne l'interaction entre la déformation mécanique élastique et l'écoulement des fluides dans un matériau poreux. Elle peut donc être considérée comme une combinaison de l'écoulement dans les milieux poreux et de petites déformations mécaniques, modélisées au moyen de l'élasticité linéaire classique. La géomécanique des milieux poreux est un domaine dans lequel les principes de la mécanique des milieux continus sont imbriqués dans les caractéristiques géologiques et le comportement des fluides. Ce domaine d'étude vise à modéliser et à comprendre les interactions entre les déformations mécaniques des solides poreux et les flux de fluides qui les traversent.

Un défi majeur est de développer des modèles mathématiques qui capturent avec précision les interactions entre l'écoulement et la déformation du sol. La nécessité d'améliorer la compréhension des phénomènes géomécaniques a conduit Terzaghi [41] et Biot [8] à proposer des modèles qui tiennent compte de la nature couplée des effets de la poroélasticité.

Biot [9] a initialement introduit un couplage elliptique-parabolique. Plus précisément, il a introduit deux versions, couplant la déformation mécanique avec la pression ou la déformation mécanique avec la température sur un domaine D et un intervalle de temps $[0, T]$. Nous sommes conduits au système poro-élastique suivant, avec \bar{u} la déformation mécanique et \bar{p} la pression du fluide,

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}) = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}} = 0, \bar{p} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot) & \text{in } D, \end{cases} \quad (1.1)$$

où les coefficients et opérateurs seront introduits plus tard.

Biot a également considéré le problème du couplage de la température et du déplacement, avec $\bar{\mathbf{u}}$ la déformation mécanique et $\bar{\theta}$ la température,

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \beta \bar{\theta} \operatorname{Id}) = \tilde{\mathbf{f}}, & \text{in } D \times [0, T), \\ \partial_t (\chi \bar{\theta} + \bar{\theta}_0 \beta \operatorname{div}(\bar{\mathbf{u}})) - \operatorname{div}(\kappa_0 \nabla \bar{\theta}) = \tilde{h}, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}} = 0, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot), & \text{in } D. \end{cases}$$

Dans l'étude de la poroélasticité, les coefficients de Lamé, notés λ et μ , sont des paramètres fondamentaux qui décrivent le comportement élastique des matériaux. Ces coefficients sont intrinsèquement liés au module d'Young E et au coefficient de Poisson ν , qui sont plus couramment utilisés dans les applications d'ingénierie.

Le module d'Young E mesure la rigidité d'un matériau, indiquant dans quelle mesure il se déforme sous une charge donnée. Le coefficient de Poisson ν décrit le rapport entre la déformation transversale et la déformation axiale dans un matériau soumis à une contrainte uniaxiale, ce qui donne une idée de la réponse volumétrique du matériau à la contrainte.

Les relations entre les coefficients de Lamé et ces paramètres sont données par :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Ici, λ est le premier paramètre de Lamé, lié au module de masse, qui affecte la rigidité volumétrique du matériau, et μ est le module de cisaillement, qui indique la résistance du matériau à la déformation par cisaillement.

La perméabilité intrinsèque

$$\kappa := \frac{k}{\eta}$$

mesure la capacité du milieu à transmettre les fluides, en influençant la diffusion de la pression des fluides, avec k la perméabilité et η la viscosité du fluide. Il est mesuré en Darcy(mD). Le coefficient de Biot α représente le couplage entre la pression du fluide et la déformation mécanique. Il est défini par

$$\alpha = 1 - \frac{K}{K_S}$$

avec K le module d'inertie du milieu drainé et K_S le module d'inertie du composant solide. Le module de Biot M est défini comme suit

$$M = \frac{K_u - K}{\alpha^2}$$

avec K_u le module apparent non drainé du matériau.

Dans ce contexte, l'étude du système de Biot avec des coefficients déterministes est une étape fondamentale pour comprendre les interactions entre la déformation élastique de la matrice solide et l'écoulement des fluides dans un milieu poreux. Cependant, les formations géologiques sont intrinsèquement hétérogènes, avec des incertitudes inhérentes à leurs propriétés physiques telles que la perméabilité, la porosité et les modules de déformation. L'utilisation de coefficients déterministes, bien qu'instructive, n'est pas suffisante pour capturer la variabilité naturelle et les incertitudes associées aux propriétés géologiques des sous-sols.

C'est pourquoi il est pertinent d'introduire des coefficients aléatoires dans la modélisation du système de Biot. Cette approche permet de prendre en compte les variations spatiales des paramètres géophysiques et géométriques, offrant ainsi une représentation plus réaliste et plus robuste des processus du sous-sol. Les coefficients

aléatoires permettent de modéliser les incertitudes naturelles et les variations locales des propriétés environnementales, qui sont souvent imprévisibles et non uniformes.

L'intégration de ces coefficients aléatoires permet d'évaluer l'impact des incertitudes sur les prédictions du comportement des réservoirs et sur la fiabilité des simulations. Cela permet non seulement de mieux comprendre les mécanismes sous-jacents, mais aussi d'améliorer la prise de décision dans les applications pratiques. Par exemple, dans le domaine du stockage de l'énergie géothermique, de la gestion des réservoirs d'hydrocarbures ou du stockage géologique du CO₂, la précision et la fiabilité des modèles sont essentielles pour optimiser l'exploitation des ressources et minimiser les risques.

Le passage à des coefficients aléatoires se justifie également par la nécessité de développer des méthodes de simulation plus robustes, capables de gérer la complexité et l'incertitude des systèmes géomécaniques. En effectuant des simulations stochastiques, nous pouvons quantifier les marges d'erreur et la probabilité d'occurrence de différents scénarios, fournissant ainsi une base plus solide pour l'évaluation des risques et la planification stratégique. En résumé, l'introduction de coefficients aléatoires dans la modélisation du système de Biot améliore considérablement la précision, la robustesse et la fiabilité des simulations géomécaniques, répondant ainsi aux exigences des applications industrielles modernes.

La modélisation des réservoirs ou des aquifères comporte une incertitude quant à la représentation réelle des paramètres du sous-sol. C'est pourquoi nous considérons certains paramètres liés aux phénomènes géomécaniques (perméabilité κ , coefficients de Lamé μ et λ) comme des champs aléatoires. Les problèmes liés à l'hétérogénéité du sous-sol et à l'imprécision des mesures justifient une description stochastique des paramètres concernés.

Les méthodes de quantification de l'incertitude (UQ) et la modélisation du système de Biot introduites par [15] montrent la nécessité de considérer des équations différentielles partielles avec des coefficients aléatoires.

Pour modéliser les coefficients aléatoires du système de Biot, nous utilisons le théorème de Karhunen-Loève, un outil puissant pour la décomposition et la représentation des processus stochastiques. Cette méthode transforme un processus aléatoire en une série de composantes déterministes pondérées par des variables aléatoires indépendantes. Grâce à cette décomposition, les coefficients aléatoires peuvent être exprimés comme une combinaison linéaire de fonctions propres, chacune multipliée par une loi gaussienne correspondante.

L'expansion de Karhunen-Loève est particulièrement bien adaptée pour traiter les incertitudes des paramètres géophysiques et géomécaniques, en capturant efficacement les caractéristiques statistiques essentielles des coefficients aléatoires tout en réduisant la complexité du problème [7], [31], [18]. Dans la pratique, cette approche permet d'approximer des champs aléatoires complexes avec un nombre réduit de termes, tout en préservant une grande partie de la variabilité et de la structure spatiale des coefficients originaux.

Cette décomposition présente plusieurs avantages, notamment une meilleure gestion des incertitudes et une simplification des simulations numériques. En réduisant le nombre de variables aléatoires nécessaires, elle améliore l'efficacité des calculs tout en maintenant la précision du modèle. La méthode Karhunen-Loève devient ainsi un outil essentiel pour la modélisation de systèmes complexes où les paramètres incertains jouent un rôle crucial.

Cependant, ce modèle est trop simpliste pour les applications géothermiques, car le transfert de chaleur doit également être pris en compte. Plus précisément, le transfert de chaleur dans le sous-sol se fait principalement par diffusion et convection. Pour tenir compte de ce phénomène, une équation de conservation de l'énergie est ajoutée et couplée à l'équation de conservation de la masse par un terme de transport convectif, c'est-à-dire un transfert de chaleur avec mouvement des particules de fluide. Le modèle de Biot peut également être étendu aux effets thermiques, ce qui conduit à un système thermo-poroélastique, c'est-à-dire au couplage de l'écoulement, de la déformation élastique et du transfert de chaleur dans un milieu poreux. Même avec une densité de fluide constante, ce système présente une structure entièrement couplée entre les trois processus impliqués. Le dernier chapitre de cette thèse concerne différents aspects de la thermo-poroélasticité, en particulier la validation et l'analyse mathématique d'un système thermo-poroélastique, et enfin son implémentation numérique.

$$\left\{ \begin{array}{ll} -\operatorname{div}\left(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\right) + \alpha \nabla \bar{p} + (1 - \alpha)K_S \nabla \bar{\theta} = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}})\right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M}\right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S(1 - \alpha) \operatorname{div}(\bar{\mathbf{u}})\right) - \eta \cdot \nabla \bar{\theta} - \operatorname{div}(\beta \nabla \bar{\theta}) = h, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}}, \bar{p}, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot), \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot), & \text{in } D. \end{array} \right. \quad (1.2)$$

Van Duijn, Mikelic et Wick ont étudié une version non linéaire du système (1.2) dans [43]. Cependant, ils ont seulement prouvé que le problème était bien posé dans une version semi-discrétisée du problème. De plus, Van Duijn, Mikelic, Wheeler et Wick ont dérivé dans [42] un tel système par la méthode d'homogénéisation. On peut remarquer que ce système possède une fonction de Lyapunov. Dans [3, 14], les auteurs ont proposé une méthode Galerkin discontinue pour obtenir une solution approximative du problème non linéaire entièrement couplé.

La diffusivité thermique (notée β) mesure la vitesse de diffusion de la chaleur à travers un matériau. K_S représente le coefficient d'expansion solide, tandis que K_T représente le paramètre de couplage thermoélastique. Le stockage souterrain d'énergie thermomécanique consiste à injecter un fluide froid à haute pression dans un réservoir souterrain et à extraire de l'eau chaude pour récupérer l'énergie thermique.

1.2 Contenu de la thèse

1.2.1 Écoulement dans les milieux poreux et petites déformations du sol

Dans le chapitre 3, nous introduisons tout d'abord l'équation d'élasticité linéaire pour le déplacement dans le milieu poreux. Ce modèle de base de la mécanique déformable sera utilisé tout au long de cette thèse. Il est donné par le modèle de Biot (1.1), couplant la pression d'une seule phase fluide dans un milieu poreux saturé, où les déformations sont considérées comme quasi-statiques. Le fait que le problème de Biot soit bien posé a été prouvé par [36], ainsi que d'autres mathématiciens.

Nous montrons des résultats de régularité supplémentaires en espace et en temps pour la solution, ainsi que des estimations a priori, qui seront utilisées plus tard dans la thèse.

Enfin, nous introduisons un couplage thermo-poroélastique, modélisant l'évolution de la chaleur dans un milieu poroélastique saturé (1.2). Nous rappelons quelques résultats d'existence et d'unicité relatifs à ce système.

1.2.2 Méthode de discrétisation de gradient pour la solution numérique du problème de Biot

Dans le chapitre 4, nous rappelons la méthode de discrétisation de gradient (GDM), décrite pour la première fois par Droniou & al [24].

Les méthodes de discrétisation de gradient offrent un cadre d'analyse de convergence unifié pour diverses méthodes numériques appliquées aux équations aux dérivées partielles elliptiques et paraboliques, y compris les méthodes d'éléments finis conformes et non conformes, les éléments finis mixtes et les méthodes de volumes finis généralisés. Un résultat de convergence uniforme dans le temps et forte dans l'espace pour les approximations du schéma de gradient de ces équations est établi grâce au développement d'outils discrets.

L'étude des systèmes poroélastiques est essentielle pour comprendre l'interaction entre les déformations mécaniques et l'écoulement des fluides dans les milieux poreux. Leur discrétisation nécessite des méthodes numériques robustes. Ce chapitre se concentre sur l'application de la méthode de discrétisation du gradient (GDM) aux modèles poroélastiques, illustrant sa polyvalence et son efficacité dans le traitement de ces systèmes couplés.

La méthode de discrétisation de gradient (GDM) est un cadre complet conçu pour créer et analyser des schémas numériques pour les modèles de diffusion. La méthode GDM utilise un contexte d'analyse fonctionnelle discrète pour établir la forte convergence L^2 des gradients approximatifs et la convergence uniforme en temps pour les solutions approximatives. Afin de définir la méthode de discrétisation de gradient pour le déplacement mécanique $\bar{\mathbf{u}}$, nous définissons

- $X_u^{\mathcal{D}}$ un espace vectoriel de dimension finie,
- $\Pi_u^{\mathcal{D}} : X_u^{\mathcal{D}} \mapsto L^2(D)$ une fonction linéaire de reconstruction,
- $e_u^{\mathcal{D}} : X_u^{\mathcal{D}} \mapsto L^2(D)^d$ une reconstruction linéaire du gradient, de sorte que $\|e_u^{\mathcal{D}} \cdot\|_{L^2(D)}$ définit une norme sur $X_u^{\mathcal{D}}$,

et pour les flux de pression, la méthode de discrétisation de gradient est donnée par

- $X_p^{\mathcal{D}}$ un espace vectoriel de dimension finie,,
- $\Pi_p^{\mathcal{D}} : X_p^{\mathcal{D}} \mapsto L^2(D)$ une fonction linéaire de reconstruction,
- $\nabla_p^{\mathcal{D}} : X_p^{\mathcal{D}} \mapsto L^2(D)^d$ une reconstruction linéaire du gradient, de sorte que $\|\nabla_p^{\mathcal{D}} \cdot\|_{L^2(D)}$ définit une norme sur $X_p^{\mathcal{D}}$.

Nous définissons un pas de temps Δt et un maillage $h_{\mathcal{D}}$, et dénotons par E_u^{ini} et E_p^{ini} l'erreur d'approximation initiale. Nous prouvons que l'estimation d'erreur suivante est valable entre la solution du problème de Biot et son approximation numérique,

$$\begin{aligned} & \|e(\bar{\mathbf{u}}) - e_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T),L^2(D))}^2 + \|\operatorname{div}(\bar{\mathbf{u}}) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T),L^2(D))}^2 \\ & + \|\bar{p} - \Pi_p^{\mathcal{D}} p\|_{L^\infty(0,T;L^2(D))}^2 + \|\nabla \bar{p} - \nabla_p^{\mathcal{D}} p\|_{L^2(0,T;L^2(D))}^2 \\ & \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_p^{ini})^2 + (E_u^{ini})^2) \end{aligned} \quad (1.3)$$

où $(\bar{\mathbf{u}}, \bar{p})$ est la paire de solutions du problème de Biot et (\mathbf{u}, p) sa solution approximative. Des tests numériques utilisant la méthode des éléments finis $P1$ pour l'écoulement et la méthode des éléments finis $P2$ pour les déformations mécaniques ont été effectués. La méthode des éléments finis $P2$ pour les déformations mécaniques ([26]) mettent en évidence le comportement du modèle et la précision de la solution.

Notons qu'un problème plus complexe d'écoulement diphasique impliquant également des fractures a été étudié par [11], alors que [11] prouve la convergence de leur méthode numérique, nous pouvons également prouver l'estimation de l'erreur (1.3) en raison de la linéarité du problème. Les simulations ont été réalisées en Python, en utilisant la bibliothèque FeNiCs, disponible à sur <https://fenicsproject.org/>.

1.2.3 Coefficients aléatoires et décomposition de Karhunen-Loeve

Dans le chapitre 5, nous discutons de la modélisation du système de Biot avec des coefficients aléatoires, en nous concentrant sur la nature stochastique de la perméabilité et des coefficients de Lamé. Les équations différentielles partielles à coefficients aléatoires ont été étudiées à plusieurs reprises dans la littérature, afin de modéliser les incertitudes sur les modèles. La modélisation et la discrétisation ont été introduites dans [7] ou [25]. En introduisant un caractère aléatoire dans ces paramètres, nous visons à capturer les incertitudes inhérentes présentes dans les milieux poreux du monde réel. Cette approche améliore non seulement le réalisme de nos simulations, mais fournit également un cadre robuste pour analyser la sensibilité et la variabilité de la réponse du système à ces incertitudes.

La méthode de Karhunen-Loève nous permet de décomposer les champs aléatoires en une série de fonctions orthogonales, pondérées par des variables aléatoires non corrélées, fournissant ainsi un moyen systématique d'approximer le caractère aléatoire avec un nombre fini de termes.

Nous traitons des variables aléatoires de la forme $a := \varphi(W)$, avec W un champ aléatoire gaussien. Soit Q l'opérateur de covariance du champ gaussien W , et soit $(\eta_i, e_i)_{i \in \mathbb{N}}$ les valeurs propres et vecteurs propres de l'opérateur

$$T_Q : L^2(D) \rightarrow L^2(D), \quad f \mapsto T_Q(f) = \int_D Q(x, \cdot) f(x) dx.$$

Le champ aléatoire gaussien W admet la représentation

$$W(\mathbf{x}, \omega) = \sum_{i \in \mathbb{N}} \eta_i e_i(\mathbf{x}) Y_i(\omega).$$

La série converge dans $L^2(\Omega, L^2(D))$ et presque sûrement (voir [4], [30]).

Le système de Biot décrit l'interaction entre l'écoulement des fluides et la déformation mécanique dans un milieu poreux. Traditionnellement, ce système est modélisé avec des coefficients déterministes, qui peuvent ne pas refléter avec précision la variabilité naturelle des formations géologiques. Pour résoudre ce problème, nous modélisons la perméabilité et les coefficients de Lamé comme des champs aléatoires, en utilisant l'expansion de Karhunen-Loève pour représenter ces processus stochastiques. Après avoir défini les champs aléatoires, nous tronquons l'expansion de Karhunen-Loève à un nombre raisonnable de termes, ce qui donne un système tronqué avec une approximation à dimension finie des coefficients aléatoires.

L'expansion de Karhunen-Loève tronquée du champ aléatoire gaussien W est donnée par

$$W_N(\mathbf{x}, \omega) = \sum_{i=0}^N \eta_i e_i(\mathbf{x}) Y_i(\omega),$$

avec $N \in \mathbb{N}$ un indice de cut-off pour W .

Ce modèle tronqué simplifie les calculs tout en conservant les caractéristiques essentielles de la nature stochastique des paramètres.

Un aspect clé de cette étude est de quantifier l'erreur introduite par la troncature de l'expansion de Karhunen-Loève. Nous comparons les solutions du système de Biot original avec des coefficients entièrement aléatoires à celles du système tronqué. Notre analyse révèle que l'erreur entre les solutions du système initial et du système tronqué est du même ordre que l'erreur de troncature des coefficients. Ce résultat est crucial car il valide l'efficacité de la méthode de troncature pour préserver la précision du modèle tout en réduisant de manière significative les exigences de calcul. Le principal résultat du Chapitre 5 est l'estimation suivante de l'erreur entre la solution tronquée et la solution analytique, à savoir

$$\mathbb{E} \left(\sup_{t \in (0, T)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left(\sup_{t \in (0, T)} \|(\bar{p} - \bar{p}_N)(t)\|_{L^2(D)}^2 \right) \leq C \Theta_N^2,$$

où C est une constante positive, et Θ_N décrit l'erreur entre le coefficient

$$\mu, \lambda, \kappa$$

and their truncated versions

$$\mu_N, \lambda_N, \kappa_N.$$

1.2.4 Modèle thermo-poroélastique à coefficients aléatoires

Dans le chapitre 6, nous couplons le problème de Biot avec l'équation de la température, afin que notre modèle devienne plus directement applicable à l'application géothermique impliquée dans le projet ANR UPGEO.

L'étude de la thermo-poroélasticité linéarisée avec des coefficients aléatoires est essentielle pour comprendre et prédire le comportement des matériaux et des structures dans divers domaines scientifiques et techniques. Ce travail se concentre sur un modèle thermo-poroélastique où la perméabilité, les coefficients de Lamé et la diffusivité thermique sont considérés comme des variables aléatoires. Cette approche tient compte des incertitudes inhérentes aux propriétés des matériaux et aux conditions externes, qui sont souvent rencontrées dans des applications réelles telles que la modélisation de l'écoulement souterrain et la géomécanique.

Ces dernières années, des progrès significatifs ont été réalisés dans l'approximation numérique des équations différentielles partielles à coefficients aléatoires, en particulier dans le contexte de la quantification de l'incertitude (UQ). Par exemple, l'approximation par éléments finis des équations aux dérivées partielles elliptiques à coefficients aléatoires, couramment utilisées dans la modélisation de l'écoulement souterrain, a démontré les défis posés par la régularité limitée du coefficient de diffusion [7], [19]. En traitant soigneusement le problème du modèle presque sûrement dans le paramètre aléatoire, les chercheurs ont été en mesure d'établir des limites uniformes sur l'erreur des éléments finis, facilitant une analyse rigoureuse des méthodes Monte Carlo à plusieurs niveaux pour ces problèmes elliptiques.

Sur cette base, le présent travail étend l'étude au problème thermo-poroélastique avec des coefficients aléatoires. Ce problème est pertinent pour les scénarios où les mesures sont insuffisantes ou incertaines. Grâce à un schéma de discrétisation basé sur des méthodes d'éléments finis, cette recherche garantit la mesurabilité de la solution, en dérivant des limites de moment sur l'erreur d'approximation quadratique moyenne. Cette approche, associée à une approximation appropriée des coefficients et à un pas de temps stable, conduit à un algorithme entièrement discret pour résoudre les équations aux dérivées partielles paraboliques aléatoires, fournissant des limites d'erreur globales et une validation numérique.

Dans le chapitre 6, nous présentons la bonne formulation du modèle thermo-poroélastique linéarisé avec des coefficients aléatoires, ainsi que les estimations a priori et l'estimation de l'erreur à l'aide d'une méthode d'éléments finis avec des éléments P2 pour la mécanique, des éléments P1 pour la pression et des éléments P1 pour la température. Un schéma d'Euler implicite est utilisé pour la discrétisation temporelle. En outre, nous explorons l'application des méthodes de Monte Carlo à ce cadre et fournissons une estimation de l'erreur, combinant des contributions numériques et statistiques. Ce travail contribue aux efforts en cours pour développer des méthodes numériques robustes et fiables pour les systèmes thermo-poroélastiques complexes avec des paramètres incertains, améliorant ainsi notre capacité à prédire et à gérer leur comportement dans des applications pratiques.

Chapter 2

Introduction

2.1 Introduction

The aim of this chapter is to provide an introduction to the various topics covered in this thesis, while placing them within the broader framework of the ANR (Agence Nationale de la Recherche) research project UPGEO, on geothermal energy storage. The subsoil is generally composed of rock types or sediments of varying permeability, and if the permeability is sufficiently high, fluids can flow freely. In its most elementary form, single-phase groundwater flow is modeled by a single conservation law, based on Darcy's law.

Geomechanics focuses on soil mechanics, an engineering discipline where a detailed representation of subsoils is crucial. It relies on geological and geophysical data and models, playing an essential role in fields such as hydrocarbon exploration, geothermal energy exploitation and carbon dioxide storage.

The aim of the UPGEO project is to gain a better understanding of geothermal systems, by proposing solutions that optimize the development of new tools. To achieve this, accurate knowledge of reservoir flow through reliable numerical simulations is required. As time and space scales vary considerably depending on the context, a robust method of solution is required. For example, studying the evolution of a sedimentary basin requires the analysis of hundreds of square kilometers over millions of years. On the other hand, assessing the impact of a human infrastructure such as a well concentrates on a few square kilometers and a period of a few decades.

Poroelasticity concerns the interaction between elastic mechanical deformation and fluid flow in a porous material. It can therefore be seen as a combination of flow in porous media and small mechanical deformations, modeled by means of classical linear elasticity. Geomechanics in porous media is a field in which the principles of continuum mechanics are interwoven with geological characteristics and fluid behavior. This field of study aims to model and understand the interactions between the mechanical deformations of porous solids and the fluid flows that move through them.

A major challenge is to develop mathematical models that accurately capture interactions between flow and deformation of the ground. The need to improve the understanding of geomechanical phenomena led Terzaghi [41] and Biot [8] to propose models that take into account the coupled nature of poroelasticity effects.

Biot [9] originally introduced an elliptic-parabolic coupling. More precisely, he introduced two versions, coupling mechanical deformation with pressure or mechanical deformation with temperature over a domain $D \subset \mathbb{R}^d$ and a time interval $[0, T]$. We are led to the following poro-elastic system, with $\bar{\mathbf{u}}$ the mechanical deformation and \bar{p} the fluid pressure,

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}) = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}} = 0, \bar{p} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot) & \text{in } D, \end{cases} \quad (2.1)$$

where all coefficients, unknowns and operators will be introduced later.

Biot also considered the problem coupling with the temperature $\bar{\theta}$ and the mechanical deformation $\bar{\mathbf{u}}$

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \beta \bar{\theta} \operatorname{Id}) = \tilde{\mathbf{f}}, & \text{in } D \times [0, T), \\ \partial_t (\chi \bar{\theta} + \bar{\theta}_0 \beta \operatorname{div}(\bar{\mathbf{u}})) - \operatorname{div}(\kappa_0 \nabla \bar{\theta}) = \tilde{h}, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}} = 0, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot), & \text{in } D. \end{cases}$$

In the study of poroelasticity, the Lamé coefficients, denoted as λ and μ , are fundamental parameters that describe the elastic behavior of materials. These coefficients are intrinsically related to the Young's modulus E and the Poisson's ratio ν , which are more commonly used in engineering applications.

The Young's modulus E measures the stiffness of a material, indicating how much it will deform under a given load. The Poisson's ratio ν describes the ratio of transverse strain to axial strain in a material subjected to uniaxial stress, providing insight into the material's volumetric response to stress.

The relationships between the Lamé coefficients and these parameters are given by:

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu := \frac{E}{2(1+\nu)}.$$

Here, λ is the first Lamé parameter, related to the bulk modulus, which affects the material's volumetric stiffness, and μ is the shear modulus, indicating the material's resistance to shear deformation.

The intrinsic permeability

$$\kappa := \frac{k}{\eta}$$

measures the medium's ability to transmit fluids, influencing fluid pressure diffusion, with k the permeability and η the viscosity of the fluid. It is measured in Darcy(mD). The Biot coefficient α represents the coupling between fluid pressure and mechanical deformation. It is defined by

$$\alpha := 1 - \frac{K}{K_S}$$

with K the bulk modulus of the drained medium and K_S the bulk modulus of the solid component. The Biot modulus M is defined as

$$M := \frac{K_u - K}{\alpha^2}$$

with K_u the undrained bulk modulus of the material.

In this context the study of the Biot system with deterministic coefficients is a fundamental step towards understanding the interactions between elastic deformation of the solid matrix and fluid flow in a porous medium. However, geological formations are intrinsically heterogeneous, with inherent uncertainties in their physical properties such as permeability, porosity and deformation moduli. The use of deterministic coefficients, while instructive, is not sufficient to capture the natural variability and uncertainties associated with the geological properties of subsoils.

This is why it is relevant to introduce random coefficients into the modeling of the Biot system. This approach enables spatial variations in geophysical and geomechanical parameters to be taken into account, providing a more realistic and robust representation of subsurface processes. Random coefficients make it possible to model natural uncertainties and local variations in environmental properties, which are often unpredictable and non-uniform.

By incorporating these random coefficients, we can assess the impact of uncertainties on predictions of reservoir behavior and on the reliability of simulations. This not only provides a better understanding of the underlying mechanisms, but also improves decision-making in practical applications. For example, in geothermal energy storage, hydrocarbon reservoir management or geological CO_2 storage, the accuracy and reliability of models are essential to optimize resource exploitation and minimize risks.

The transition to random coefficients is also justified by the need to develop more robust simulation methods capable of handling the complexity and uncertainty of geomechanical systems. By performing stochastic simulations, we can quantify the margins of error and probability of occurrence of different scenarios, thus providing a more solid basis for risk assessment and strategic planning. In summary, the introduction of random coefficients

into the modeling of the Biot system significantly improves the accuracy, robustness and reliability of geomechanical simulations, meeting the requirements of modern industrial applications.

Modelling reservoirs or aquifers entails uncertainty as to the actual representation of subsoil parameters. Therefore we consider certain parameters linked to geomechanical phenomena (permeability κ , Lamé coefficients μ and λ) as random fields. The problems associated with subsoil heterogeneity and measurement inaccuracies justify a stochastic description of the parameters involved.

The uncertainty quantification (UQ) methods and Biot system modeling introduced by [15] show the need to consider partial differential equations with random coefficients.

To model the random coefficients of the Biot system, we use the Karhunen-Loève theorem, a powerful tool for the decomposition and representation of stochastic processes. This method transforms a random process into a series of deterministic components weighted by uncorrelated random variables. Thanks to this decomposition, the random coefficients can be expressed as a linear combination of eigenfunctions, each multiplied by a corresponding gaussian law.

The Karhunen-Loève expansion is particularly well suited to deal with uncertainties in geophysical and geomechanical parameters, effectively capturing the essential statistical characteristics of random coefficients while reducing the complexity of the problem (see [7], [18], [31]). In practice, this approach enables complex random fields to be approximated with a reduced number of terms, while preserving much of the variability and spatial structure of the original coefficients.

This decomposition offers several advantages, including better management of uncertainties and simplification of numerical simulations. By reducing the number of random variables required, it improves computational efficiency while maintaining model accuracy. The Karhunen-Loève method thus becomes an essential tool for modeling complex systems where uncertain parameters play a crucial role.

However, this model is too simplistic for geothermal applications, as heat transfer also needs to be included. More specifically, underground heat transfer occurs mainly by diffusion and convection. To account for this phenomenon, an energy conservation equation is added and coupled to the mass conservation equation by a convective transport term, i.e. heat transfer with fluid particle motion. Biot's model can also be extended to include thermal effects, leading to a thermo-poroelastic system, i.e. the coupling of flow, elastic deformation and heat transfer in a porous medium. Even with constant fluid density, this system exhibits a fully coupled structure between the three processes involved. The last chapter of this thesis concerns different aspects of thermo-poroelasticity, in particular the validation and mathematical analysis of a thermo-poroelastic system, and finally its numerical implementation.

$$\left\{ \begin{array}{ll} -\operatorname{div}\left(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\right) + \alpha \nabla \bar{p} + (1 - \alpha)K_S \nabla \bar{\theta} = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}})\right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M}\right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S(1 - \alpha) \operatorname{div}(\bar{\mathbf{u}})\right) - \eta \cdot \nabla \bar{\theta} - \operatorname{div}(\beta \nabla \bar{\theta}) = h, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}}, \bar{p}, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot), \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot), & \text{in } D. \end{array} \right. \quad (2.2)$$

Van Duijn, Mikelić, and Wick studied a nonlinear version of the system (2.2) in [43]. However, they have only proved the well-posedness of a time semi-discretized version of the problem. Moreover, Van Duijn, Mikelić, Wheeler and Wick have derived in [42] such a system by homogenization method. As a remark, we can see that such system possesses a Lyapunov function. In [3, 14], the authors have proposed a discontinuous Galerkin method to approximate solution of the fully coupled non-linear problem.

The thermal diffusivity (denoted β) measures how quickly heat diffuses through a material. K_S represents the solid expansion coefficient, while K_T represent the thermoelastic coupling parameter. Underground thermo-mechanical energy storage involves injecting a cold fluid at high pressure into an underground reservoir and extracting warm water to recover thermal energy.

2.2 Content of the thesis

2.2.1 Flow in porous media and small deformations of the ground

In Chapter 3, we first introduce the linear elasticity equation for the displacement in the porous medium. This basic model of deformable mechanics will be used throughout this thesis. The Biot model (2.1) couples the pressure of a single fluid phase in a saturated porous medium, where deformations are considered quasi-static. The well-posedness of the Biot problem has been proved by [36] as well as other mathematicians.

We show extra regularity results in space and time for the solution, as well as a priori estimates, which will be used later in the thesis.

Finally, we introduce a thermo-poroelastic coupling, modeling the evolution of heat in a saturated poroelastic medium (2.2). We recall some existence and uniqueness results related to this system.

2.2.2 Gradient schemes for the numerical solution of the Biot problem

In Chapter 4, we introduce the Gradient Discretization Method (GDM), first described by Droniou & al [24].

Gradient schemes offer a unified convergence analysis framework for various numerical methods applied to elliptic and parabolic partial differential equations, including conforming and non-conforming Finite Element, Mixed Finite Element, and Generalized Finite Volume methods. A uniform in time, strong in space convergence result for gradient scheme approximations of these equations is established through the development of discrete tools.

The study of poroelastic systems is essential for understanding the interaction between mechanical deformations and fluid flow in porous media. Their discretization requires robust numerical methods. This chapter focuses on the application of the Gradient Discretisation Method (GDM) to poroelastic models, illustrating its versatility and effectiveness in handling such coupled systems.

The Gradient Discretisation Method (GDM) is a comprehensive framework designed for creating and analysing numerical schemes for diffusion models. The GDM methods employs a discrete functional analysis background to establish the strong L^2 -convergence of the approximate gradients and the uniform in time convergence for the approximate solutions. In order to define the gradient discretization method for the mechanical displacement $\bar{\mathbf{u}}$, we define

- $X_u^{\mathcal{D}}$ a finite-dimensional vector space,
- $\Pi_u^{\mathcal{D}} : X_u^{\mathcal{D}} \mapsto L^2(D)$ a linear function reconstruction,
- $e_u^{\mathcal{D}} : X_u^{\mathcal{D}} \mapsto L^2(D)^d$ a linear gradient reconstruction, so that $\|e_u^{\mathcal{D}} \cdot \cdot\|_{L^2(D)}$ defines a norm on $X_u^{\mathcal{D}}$,

and for the pressure flows the gradient discretization method is given by

- $X_p^{\mathcal{D}}$ a finite-dimensional vector space,
- $\Pi_p^{\mathcal{D}} : X_p^{\mathcal{D}} \mapsto L^2(D)$ a linear function reconstruction,
- $\nabla_p^{\mathcal{D}} : X_p^{\mathcal{D}} \mapsto L^2(D)^d$ a linear gradient reconstruction, so that $\|\nabla_p^{\mathcal{D}} \cdot \cdot\|_{L^2(D)}$ defines a norm on $X_p^{\mathcal{D}}$.

We define a time step Δt and a meshsize $h_{\mathcal{D}}$, and denote by E_u^{ini} and E_p^{ini} the initial approximation error. We prove that the following error estimate holds between the solution of the Biot problem and its numerical approximation,

$$\begin{aligned} & \|e(\bar{\mathbf{u}}) - e_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T),L^2(D))}^2 + \|\operatorname{div}(\bar{\mathbf{u}}) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T),L^2(D))}^2 \\ & + \|\bar{p} - \Pi_p^{\mathcal{D}} p\|_{L^\infty(0,T;L^2(D))}^2 + \|\nabla \bar{p} - \nabla_p^{\mathcal{D}} p\|_{L^2(0,T;L^2(D))}^2 \\ & \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_p^{ini})^2 + (E_{\mathbf{u}}^{ini})^2) \end{aligned} \quad (2.3)$$

where $(\bar{\mathbf{u}}, \bar{p})$ is the solution pair of the Biot problem and (\mathbf{u}, p) its approximate solution. Numerical tests using the scalar P_1 finite element method for the flow and the vectorial \mathbf{P}_2 finite element for the mechanical deformations (see [26] for basic definitions of P_1 and \mathbf{P}_2 methods) highlight the model's behaviour and the solution accuracy.

Note that a more complex two-phase flow problem also involving fractures has been studied by [11]. While the authors in [11] prove the convergence of their numerical method, we can also prove the error estimate (2.3) because of the linearity of the problem. The simulations have been made in Python, using the FeNiCs library, available at <https://fenicsproject.org/>.

2.2.3 Random coefficients and the Karhunen-Loève decomposition

In Chapter 5, we discuss the modeling of the Biot system with random coefficients, focusing on the stochastic nature of permeability and Lamé coefficients. Partial differential equations with random coefficients have been studied several times in the literature, to modelize the uncertainties on the models. Modelization and discretization has been introduced in [7] or [25]. By introducing randomness in these parameters, we aim to capture the inherent uncertainties present in real-world porous media. This approach not only enhances the realism of our simulations but also provides a robust framework for analyzing the sensitivity and variability of the system's response to these uncertainties.

The Karhunen-Loève method allows us to decompose the random fields into a series of orthogonal functions e_i , weighted by uncorrelated random variables Y_i , thus providing a systematic way to approximate the randomness with a finite number of terms.

We deal with random fields of the form $a := \varphi(W)$, with W a Gaussian random field with mean zero. Let Q be the covariance function of the Gaussian field W , and let $(\eta_i, e_i)_{i \in \mathbb{N}}$ be the eigensequence of the operator

$$T_Q : L^2(D) \rightarrow L^2(D), \quad f \mapsto T_Q(f) = \int_D Q(x, \cdot) f(x) dx.$$

The Gaussian random field W admits the representation

$$W(\mathbf{x}, \omega) = \sum_{i \in \mathbb{N}} \eta_i e_i(\mathbf{x}) Y_i(\omega).$$

Under assumptions on the random variables, this series converges in $L^2(\Omega, L^2(D))$ and almost surely (see [4], [30]).

The Biot system describes the interaction between fluid flow and mechanical deformation in a porous medium. Traditionally, this system is modeled with deterministic coefficients, which may not accurately reflect the natural variability of geological formations. To address this, we model the permeability and Lamé coefficients as random fields, using the Karhunen-Loève expansion to represent these stochastic processes. After defining the random fields, we truncate the Karhunen-Loève expansion to a manageable number of terms, resulting in a truncated system with a finite-dimensional approximation of the random coefficients.

The truncated Karhunen-Loève expansion of the Gaussian random field W is given by

$$W_N(\mathbf{x}, \omega) = \sum_{i=0}^N \eta_i e_i(\mathbf{x}) Y_i(\omega),$$

with $N \in \mathbb{N}$ is a cut-off index for W .

This truncated model simplifies the computational complexity while retaining the essential characteristics of the stochastic nature of the parameters.

A key aspect of this study is to quantify the error introduced by truncating the Karhunen-Loève expansion. We compare the solutions of the original Biot system with fully random coefficients to those of the truncated system. Our analysis reveals that the error between the solutions of the initial system and the truncated system is of the same order as the truncation error of the coefficients. This result is crucial as it validates the efficiency of the truncation method in preserving the accuracy of the model while significantly reducing computational demands. The main result of Chapter 5 is the Theorem 5.2.3 where we describe the error estimate between the truncated solution and the analytical solution, namely

$$\mathbb{E} \left(\sup_{t \in (0, T)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left(\sup_{t \in (0, T)} \|(\bar{p} - \bar{p}_N)(t)\|_{L^2(D)}^2 \right) \leq C \Theta_N^2,$$

where C is a positive constant, and Θ_N describes an error between the random fields μ , λ , κ and their truncated versions μ_N , λ_N , κ_N with Karhunen-Loève approximation.

2.2.4 Thermo-poroelastic model with random coefficients

In Chapter 6, we couple the Biot problem with the temperature equation, so that our model becomes more directly applicable to the geothermic application involved in the ANR project UPGEO.

The study of the linearized thermo-poroelasticity with random coefficients is essential for understanding and predicting the behavior of materials and structures in various scientific and engineering fields. This work focuses on a thermo-poroelastic model where the permeability, Lamé coefficients, and thermal diffusivity are considered as random variables. This approach addresses the inherent uncertainties in material properties and external conditions, which are often encountered in real-world applications such as subsurface flow modeling and geomechanics.

In recent years, significant progress has been made in the numerical approximation of partial differential equations with random coefficients, particularly in the context of uncertainty quantification (UQ). For instance, the finite element approximation of elliptic partial differential equations with random coefficients, commonly used in subsurface flow modeling, has demonstrated the challenges posed by limited regularity of the diffusion coefficient regularity [7], [19]. By carefully treating the model problem almost surely in the random parameter, researchers have been able to establish uniform bounds on the finite element error, facilitating a rigorous analysis of multilevel Monte Carlo methods for these elliptic problems.

Building upon this foundation, the current work extends the study to the thermo-poroelastic problem with random coefficients. This problem is relevant for scenarios with insufficient measurements or uncertain material properties. Through a discretization scheme based on finite element methods, this research ensures the measurability of the solution, deriving moment bounds on the mean-squared approximation error. This approach, coupled with suitable coefficient approximation and stable time stepping, leads to a fully discrete algorithm for solving random parabolic partial differential equations, providing overall error bounds and numerical validation.

In Chapter 6, we present the well-posedness of the linearized thermo-poroelastic model (2.2) with random coefficients, along with a priori estimates and error estimation using a finite element method with \mathbf{P}_2 elements for mechanics, P_1 elements for pressure, and P_1 elements for temperature. An implicit Euler scheme is employed for time discretization. Furthermore, we explore the application of Monte Carlo methods to this framework, and provide an error estimate, combining numerical and statistical contributions. This work contributes to the ongoing efforts to develop robust and reliable numerical methods for complex thermo-poroelastic systems with uncertain parameters, enhancing our ability to predict and manage their behavior in practical applications.

Chapter 3

The Biot Problem

3.1 Introduction

Geomechanics and poroelasticity have various applications in a large range of engineering fields. A model coupling geomechanics with flow has been introduced by Biot in [8], [9]. As a matter of fact, Biot has developed two versions of this model, one coupling displacement and flow, and the other one coupling displacement and temperature. The problem (2.1) that we recall below couples pressure \bar{p} and displacement $\bar{\mathbf{u}}$,

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}) = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}}, \bar{p} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot) & \text{in } D, \end{cases}$$

whereas Biot also considered the problem coupling temperature $\bar{\theta}$ and displacement $\bar{\mathbf{u}}$,

$$\begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \beta \bar{\theta} \operatorname{Id}) = \tilde{\mathbf{f}}, & \text{in } D \times [0, T), \\ \partial_t (\chi \bar{\theta} + \bar{\theta}_0 \beta \operatorname{div}(\bar{\mathbf{u}})) - \operatorname{div}(\kappa_0 \nabla \bar{\theta}) = \tilde{h}, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}}, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot) & \text{in } D, \end{cases}$$

where \mathbf{f} , g , $\tilde{\mathbf{f}}$ and \tilde{h} are external terms, \bar{p}_0 and $\bar{\theta}_0$ are initial data, α , β , M and χ are constants, and λ , μ , κ and κ_0 are parameters of the Biot model.

The project UPGEO which has been subsidizing the preparation of this PhD Thesis deals with geothermy. Geothermal energy, the capture of heat from the subsurface at various temperature levels, is a key method for achieving the energy transition. However, France's new renewable energy plan has lowered targets for the deployment of deep geothermal energy due to high geological risks. These risks include the possibility that new operations may not yield geothermal resources with sufficient flow rates and temperatures to ensure project profitability over its lifespan. This geological risk is a significant barrier to the future development of geothermal energy in France and the Île-de-France region. The project aims to develop innovative modeling methods to optimize and explore new zones in sedimentary areas. This optimization requires (1) a precise understanding of reservoir heterogeneity in terms of sedimentary geometries, porosity/permeability, and reservoir connectivity, and (2) reliable numerical simulations of flow and heat transfer over 30 to 100 years after production begins. In view of the UPGEO project, we are led to study the following three-component problem involving mechanical displacement, pressure, and temperature (which is a linear version inspired from [16, 17])

$$\left\{ \begin{array}{ll} -\operatorname{div}\left(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\right) + \alpha \nabla \bar{p} + (1 - \alpha)K_S \nabla \bar{\theta} = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}})\right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M}\right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S(1 - \alpha) \operatorname{div}(\bar{\mathbf{u}})\right) - \eta \cdot \nabla \bar{\theta} - \operatorname{div}(\beta \nabla \bar{\theta}) = h, & \text{in } D \times (0, T), \\ \bar{\mathbf{u}}, \bar{p}, \bar{\theta} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(0, \cdot) = \bar{p}_0(\cdot), \bar{\theta}(0, \cdot) = \bar{\theta}_0(\cdot), & \text{in } D, \end{array} \right.$$

where \mathbf{f} , g and h are external terms, \bar{p}_0 and $\bar{\theta}_0$ initial data, α , β , η , M , K_T , K_S are constants, and λ , μ and κ are parameters of the Biot model.

The study and resolution of coupled poro-elastic models are well-developed research fields. Well-posedness of such system has been proved by Showalter [39], and regularity results have been introduced by Mikelić et al. in [36], where they introduced algorithm based on the iterative coupling of the mechanics and the fluid flow.

The numerical resolution of the coupled poro-elastic system relies either on a specific geometry which is compatible with two resolutions, or to introduce some interpolation operator for the coupled terms in each equation, see [29]. Thus several approaches are frequently employed to solve the coupled poro-elastic system. We can cite Coulet et al. [22] where the problem is solved by using Virtual Elements for the discretisation of the mechanics, and a TPFA finite volume method to solve the flow equation. Otherwise, iterative schemes applied to the coupled poro-elastic system (see, e.g. [1]) allow to solve geomechanics and flow in decoupled equation. They are known as fixed-stress split, fixed-strain split, undrained-split, and drained split schemes. Either we can obtain a contraction of the iterative schemes, or at least conditionally stable.

Thus, specific numerical discretization are used to solve numerically the poro-elastic system. We propose to introduce a particular framework, adapted to the problem which we can extract some numerical convergence results. This framework is the Gradient Discretization Methods (GDM) introduced first by Droniou, Eymard, Gallouët and Herbin in [24]. This framework covered several numerical methods such as

- Multi-point flux approximation (see e.g. Nordbotten [37] for the application to a Biot system),
- Crouzeix-Raviart elements (see [1] with an iterative schemes),
- Galerkin and Discontinuous Galerkin methods (see e.g. [20] for a DG schemes for the Biot's consolidation problem).

There are several approaches to solve a coupled flow and geomechanics problem, referred as implicit scheme, explicit scheme and iterative scheme. Our work focuses on the numerical approximation of a poroelastic problem, describes by the quasi-static Biot system, introduced in [8]-[9] and on the gradient discretization of the Biot system with homogeneous Dirichlet boundary conditions. We refer in particular to the original Biot article [9] and to Mikelić and Wheeler [36] for a well-posedness result. We prove regularity properties of the solutions of the continuous problem to ensure the validity of the error estimates.

In this section, we study a linear system for flow in porous media coupled to small mechanical deformations of the ground proposed by Biot and by Mikelić and Wheeler [9] and [36]. For the space discretization, we apply Gradient Discretization Methods which include numerical schemes such as finite element methods and generalized finite volume methods. Both flow and mechanical deformation are discretized by means of a gradient method and our numerical schemes couples both equations. We prove the existence and uniqueness of the discrete solution by means of an iteration scheme and then present error estimates and speed of convergence for this time-space approximation.

3.2 The linear elasticity equation

Elasticity is a fundamental concept in the study of materials and structures, describing how solid objects deform and return to their original shape when subjected to external forces. In engineering and applied physics, the equations of linear elasticity form the backbone of many practical applications, from the design of buildings and bridges to

the development of advanced materials and biomedical devices. Here we consider the quasi-static regime of linear elasticity, a crucial approximation that simplifies the analysis by neglecting inertial effects.

In many practical scenarios, the rate of loading is sufficiently slow such that the dynamic effects, like inertia and damping, can be neglected. This assumption leads to the quasi-static approximation, where the time-dependence of the problem is simplified, making it possible to solve the equilibrium equations of elasticity without considering the time derivatives of displacement.

The governing equations of linear elasticity are derived from fundamental physical principles: equilibrium of forces, compatibility of strains, and constitutive relations. Thus, given D a bounded domain, with ∂D its smooth boundary, the linear elasticity problem consists in finding a vector-valued displacement fields $\bar{\mathbf{u}} : D \mapsto \mathbb{R}^d$ in the classical Sobolev space $H_0^1(D)$ with vanishing trace at the boundary ∂D such that

$$\begin{cases} -\nabla \cdot \sigma(\bar{\mathbf{u}}) = \mathbf{f}, & \text{in } D, \\ \bar{\mathbf{u}} = 0, & \text{on } \partial D. \end{cases}$$

The vector valued $\mathbf{f} : D \mapsto \mathbb{R}^d$ is the body force per unit volume and it is assumed to be in the classical Lebesgue space $L^2(D)$. The mechanical behavior of the soil is described by $\sigma(\bar{\mathbf{u}})$ as the Cauchy stress-tensor, which reads in the case of linear isotropic elasticity as

$$\sigma(\bar{\mathbf{u}}) = 2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id}, \quad e(\bar{\mathbf{u}}) = \frac{\nabla \bar{\mathbf{u}} + {}^t \nabla \bar{\mathbf{u}}}{2}, \quad (3.1)$$

with $e(\bar{\mathbf{u}})$ represents the infinitesimal stress tensor which is matrix valued.

Here, μ and λ are the Lamé coefficients. They can be written in terms of the Young's modulus E (Pa) which measure the relation between tensile or compressive stress and the deformation, and the Poisson ratio ν (\emptyset) which measure the deformation of the soil. Thus we have

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \quad (3.2)$$

There could be mathematical issue when λ is unbounded, which occurs when ν increases to $1/2$. It has been studied in several work including [32]. In the following we will assume that μ and λ are bounded and strictly positive, which could be infer from boundedness of E and ν , and assuming ν is strictly positive and less than $1/2$.

We recall some classical results, which will be used concerning the elasticity equation. To clarify the notation, we define a scalar product for real matrices.

Definition 3.2.1. For all square matrices A and B of size $d \times d$, we define

$$A : B = \operatorname{Tr}(A^t B) = \sum_{i=1}^d \sum_{j=1}^d (A)_{ij} (B)_{ij}. \quad (3.3)$$

Moreover, we also define the associated norm $\|A\|^2 = A : A$.

Lemma 3.2.1 (Korn inequality). For $\mathbf{v} \in H_0^1(D)^d$, there exists a positive constant $C_K := C_K(D)$ such that

$$\|\mathbf{v}\|_{H_0^1(D)}^2 \leq C_K \|e(\mathbf{v})\|_{L^2(D)}^2. \quad (3.4)$$

Next, we prove the following result

Lemma 3.2.2. For all $\mathbf{v} \in H_0^1(D)^d$, we have

$$\operatorname{Id} : e(\mathbf{v}) = \operatorname{div}(\mathbf{v}) \in L^2(D). \quad (3.5)$$

Proof. Indeed, we have

$$\operatorname{Id} : e(\mathbf{v}) = \sum_{i=1}^d \sum_{j=1}^d (\operatorname{Id})_{ij} (e(\mathbf{v}))_{ij}$$

Since $(\text{Id})_{ij} = 0$ if $i \neq j$ and $(\text{Id})_{ii} = 1$, it leads to

$$\sum_{i=1}^d \sum_{j=1}^d (\text{Id})_{ij} (e(\mathbf{v}))_{ij} = \sum_{i=1}^d (e(\mathbf{v}))_{ii} = \sum_i \partial_{x_i} v_i = \text{div}(\mathbf{v}) \text{ in } L^2(D).$$

□

Lemma 3.2.3. *Let $\mathbf{v} \in H_0^1(D)^d$, there holds*

$$\|\text{div}(\mathbf{v})\|_{L^2(D)} \leq \sqrt{d} \|e(\mathbf{v})\|_{L^2(D)}. \quad (3.6)$$

Proof. Given $\mathbf{v} \in H_0^1(D)^d$ there holds that

$$\text{div}(\mathbf{v}) = \text{Tr}(e(\mathbf{v})) = \text{Tr}(e(\mathbf{v})\text{Id}).$$

Let's also notice that

$$\|e(\mathbf{v})\text{Id}\| \leq \|e(\mathbf{v})\| \|\text{Id}\|,$$

and

$$\|\text{Tr}(e(\mathbf{v}))\| = \|\text{Tr}(e(\mathbf{v})^t e(\mathbf{v}))^{\frac{1}{2}}\|.$$

Thus we have

$$\begin{aligned} \|\text{div}(\mathbf{v})\|_{L^2(D)} &= \|\text{Tr}(e(\mathbf{v})\text{Id})\|_{L^2(D)} \\ &\leq \|\text{Tr}(e(\mathbf{v})^t e(\mathbf{v}))^{\frac{1}{2}} \text{Tr}(\text{Id})^{\frac{1}{2}}\|_{L^2(D)} \\ &\leq \sqrt{d} \|e(\mathbf{v})\|_{L^2(D)}. \end{aligned}$$

where by (3.3) $\text{Tr}(e(\mathbf{v})^t e(\mathbf{v})) = \|e(\mathbf{v})\|_{L^2(D)}^2$.

□

Lemma 3.2.4. *Given $\mathbf{u}, \mathbf{v} \in H_0^1(D)$, there holds*

$$\begin{aligned} \int_D -\text{div}(e(\mathbf{u}))\mathbf{v} dx &= \int_D e(\mathbf{u}) : e(\mathbf{v}) dx, \\ \int_D -\text{div}(\text{div}(\mathbf{u})\text{Id})\mathbf{v} dx &= \int_D \text{div}(\mathbf{u})\text{div}(\mathbf{v}) dx. \end{aligned} \quad (3.7)$$

We impose the following hypothesis on the parameters.

Hypothesis 1. *There holds that*

$$\mathbf{f} \in L^2(D)^d.$$

Moreover, there exists some constants $\mu_0, \lambda_0, \mathcal{M}_\mu$ and \mathcal{M}_λ such that for all $x \in D$

$$0 < \mu_0 \leq \mu(x) \leq \mathcal{M}_\mu, \quad \text{and} \quad 0 < \lambda_0 \leq \lambda(x) \leq \mathcal{M}_\lambda.$$

Thus the following result holds

Theorem 3.2.1. *Under Hypothesis 1, there exists a unique solution $\bar{\mathbf{u}} \in H_0^1(D)^d$, in the sense of distribution, to the system*

$$\begin{aligned} -\text{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \text{div}(\bar{\mathbf{u}})\text{Id}) &= \mathbf{f}, & \text{in } D, \\ \bar{\mathbf{u}} &= 0, & \text{on } \partial D. \end{aligned} \quad (3.8)$$

Proof. Since the parameters λ and μ are bounded functions, the bilinear form

$$a(\mathbf{u}, \mathbf{v}) := \int_D 2\mu e(\mathbf{u}) : e(\mathbf{v}) + \lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v})$$

is continuous with respect to the $H_0^1(D)$ norm, and it satisfies for $\mathbf{u} \in H_0^1(D)^d$

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}) &= \int_D 2\mu |e(\mathbf{u})|^2 + \lambda |\operatorname{div}(\mathbf{u})|^2 dx \\ &\geq 2\mu_0 \|e(\mathbf{u})\|_{L^2(D)}^2 + \lambda_0 \|\operatorname{div}(\mathbf{u})\|_{L^2(D)}^2. \end{aligned}$$

Applying Korn's inequality 3.4 we deduce that this bilinear form is coercive. Since the form

$$l(\mathbf{v}) := \int_D \mathbf{f} \cdot \mathbf{v} dx$$

is continuous, Lax-Milgram theorem ensures the existence and uniqueness of the solution $\bar{\mathbf{u}} \in H_0^1(D)$ to the problem $a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v})$ for all $\mathbf{v} \in H_0^1(D)$ which coincides with a solution to the problem (3.8) in the sense of distributions since the smooth test functions with compact support are in $H_0^1(D)$. \square

3.3 Biot system and poroelasticity

The domain D represents an elastic porous medium saturated with slightly compressible and viscous monophasic fluid, such that the domain is considered as quasi-static (with no inertia) [8], [9]. Let $\bar{\mathbf{u}}$ be the mechanical deformation vector and \bar{p} the fluid pressure.

Thus the total stress-tensor induced on the soil is modelled by the Cauchy stress-tensor introduced in Equation (3.1), to which we sum an additional term representing the pore pressure,

$$\sigma(\bar{\mathbf{u}}, \bar{p}) = 2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id},$$

to obtain the displacement equation

$$-\operatorname{div}(\sigma(\bar{\mathbf{u}}, \bar{p})) = \mathbf{f}. \quad (3.9)$$

The Biot coefficient α is the variation of the total stress induced by the pressure in the porous medium. Darcy's law for a compressible fluid represents the relationship describing how the fluid flows through a porous medium, where the flow rate depends on the pressure gradient, the permeability of the medium, and the fluid viscosity. It models how pressure differences drive fluid movement in porous materials. Mathematically, it is represented by

$$\mathbf{v}_f = -\frac{k}{\eta} (\nabla p - \rho_f \mathbf{g}) \quad (3.10)$$

where \mathbf{g} is the gravity vector field. It will not be considered in the thesis.

The pore-pressure equation in the Biot model is derived from the conservation of mass for the fluid phase, coupled with Darcy's law for fluid flow through the porous medium. The equation can be expressed as:

$$\partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) + \operatorname{div}(\mathbf{v}_f) = g. \quad (3.11)$$

In this context, several parameters play a crucial role :

- **Permeability** k : The permeability is a measure of the ability of the porous medium to transfer fluids. It is a property of the material that depends on the pore structure and connectivity.
- **Fluid Viscosity** η : The dynamic viscosity of the fluid affects the resistance to flow through the porous medium. It plays a significant role in determining the rate at which the fluid can move in response to pressure gradients.

- **Fluid Storage Coefficient** M : This parameter characterizes the compressibility of the fluid-saturated porous medium. It represents the ability of the medium to store fluid when subjected to changes in pore pressure.
- **Biot's coefficient** α : This dimensionless parameter quantifies the coupling between the pore pressure and the volumetric strain of the solid matrix. It reflects how changes in pore pressure contribute to the overall deformation of the material.
- **Source term** g : This term accounts for any external addition or removal of fluid within the porous medium. It is used to model scenarios such as injection or extraction of fluids.
- **Gravity Term** $\rho_f \mathbf{g}$: This term represents the gravitational force acting on the fluid within the pores, where ρ_f is the fluid density and \mathbf{g} is the gravitational acceleration vector. Gravity impacts fluid flow and pressure distribution, especially in large-scale or vertical systems.

The Gravity term $\rho_f \mathbf{g}$ will be added to the right-hand-side of (3.11), and we denote by

$$\kappa := \frac{k}{\eta}$$

the diffusion coefficient in the pressure equation. The Biot system with homogeneous Dirichlet boundary conditions is given by

$$(\mathcal{P}) \begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}) = \mathbf{f}, & \text{in } D \times [0, T], \\ \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \bar{p} = 0, \quad \bar{\mathbf{u}} = 0, & \text{on } \partial D \times [0, T], \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot) & \text{in } D. \end{cases}$$

Since $\bar{p}(\cdot, 0)$ is known, we can solve the first equation of the system at time $t = 0$, which is an elliptic equation for $\bar{\mathbf{u}}(\cdot, 0)$ in D . Setting $\bar{\mathbf{u}}_0(x) := \bar{\mathbf{u}}(x, 0)$ for all $x \in D$, we deduce that $\bar{\mathbf{u}}_0 \in H_0^1(D)^d$ is the unique weak solution of the elliptic problem

$$\begin{aligned} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}_0) + \lambda \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id} - \alpha \bar{p}_0 \operatorname{Id}) &= \mathbf{f}(0), & \text{in } D, \\ \bar{\mathbf{u}}_0 &= 0, & \text{on } \partial D. \end{aligned} \tag{3.12}$$

Hypothesis 2. *There holds that*

$$\mathbf{f} \in C^1([0, T]; L^2(D)), \quad \text{and} \quad g \in C([0, T]; L^2(D)).$$

The initial data satisfies

$$\bar{p}_0 \in H_0^1(D).$$

The functions μ , λ , and κ satisfy

$$\mu, \lambda, \kappa \in C^\infty(\bar{D}),$$

and there exist constants μ_0 , λ_0 , κ_0 and \mathcal{M}_μ , \mathcal{M}_λ , \mathcal{M}_κ such that for all $x \in D$

$$0 < \mu_0 \leq \mu(x) \leq \mathcal{M}_\mu, \quad 0 < \lambda_0 \leq \lambda(x) \leq \mathcal{M}_\lambda, \quad \text{and} \quad 0 < \kappa_0 \leq \kappa(x) \leq \mathcal{M}_\kappa.$$

Moreover, M and α are positive constants.

First, under Hypothesis 2, we deduce from Theorem 3.2.1 that the problem (3.12) possesses a unique weak solution $\bar{\mathbf{u}}_0$ on which we have the following estimate.

Lemma 3.3.1. *There holds that*

$$\begin{aligned} \frac{\mu_0}{C_K} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 &\leq \mu_0 \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\ &\leq \frac{C_K}{4\mu_0} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{\alpha^2}{2\lambda_0} \|\bar{p}_0\|_{L^2(D)}^2. \end{aligned} \tag{3.13}$$

Proof. We multiply Equation (3.12) by $\bar{\mathbf{u}}_0$, and integrate over the domain D to obtain

$$\int_D 2\mu |e(\bar{\mathbf{u}}_0)|^2 dx + \int_D \lambda |\operatorname{div}(\bar{\mathbf{u}}_0)|^2 dx = \int_D \mathbf{f}(0) \cdot \bar{\mathbf{u}}_0 dx + \alpha \int_D \bar{p}_0 \operatorname{div}(\bar{\mathbf{u}}_0) dx, \quad (3.14)$$

so that

$$\begin{aligned} & 2\mu_0 \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \lambda_0 \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\ & \leq \frac{C_K}{4\mu_0} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{\mu_0}{C_K} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 \\ & + \frac{\alpha^2}{2\lambda_0} \|\bar{p}_0\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2. \end{aligned} \quad (3.15)$$

Applying the Korn inequality (3.4) to $\bar{\mathbf{u}}_0$, we deduce that

$$\frac{\mu_0}{C_K} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 \leq \frac{\mu_0 C_K}{C_K} \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2.$$

Then

$$\begin{aligned} & \mu_0 \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\ & \leq \frac{C_K}{4\mu_0} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{\alpha^2}{2\lambda_0} \|\bar{p}_0\|_{L^2(D)}^2, \end{aligned}$$

which coincide with (3.13). □

3.3.1 Function spaces

We define the spaces of functions

$$\begin{aligned} \mathbb{V} &= \{\mathbf{v} \in C([0, T]; H_0^1(D)^d) : \partial_t e(\mathbf{v}) \in L^2(D \times (0, T))^{d \times d}\}, \\ W &= \{q \in H^1(D \times (0, T)) : q \in C([0, T]; H_0^1(D))\}. \end{aligned}$$

We define a weak solution of problem (\mathcal{P}) as follows:

Definition 3.3.1. *The function pair $(\bar{\mathbf{u}}, \bar{p})$ is a weak solution of problem (\mathcal{P}) if, for all $t \in [0, T]$, we have for all $\mathbf{v} \in H_0^1(D)^d$ and $q \in H_0^1(D)$*

$$\int_D \{2\mu e(\bar{\mathbf{u}}) : e(\mathbf{v}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \cdot \operatorname{div}(\mathbf{v}) - \alpha \bar{p} \operatorname{div}(\mathbf{v})\} dx = \int_D \mathbf{f} \mathbf{v} dx, \quad (3.16)$$

$$\int_D \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) q dx + \int_D \kappa \nabla \bar{p} \cdot \nabla q dx = \int_D g q dx, \quad (3.17)$$

$$\bar{p}(x, 0) = \bar{p}_0(x), \quad \text{for all } x \in D.$$

For the sake of completeness, we recall the following well-posedness result and its proof which have already been given by Mikelić and Wheeler in [36].

Theorem 3.3.1 (Mikelić & Wheeler). *There exists a unique weak solution $(\bar{\mathbf{u}}, \bar{p}) \in \mathbb{V} \times W$ to the problem (\mathcal{P}) .*

The proof is based on the following section where we prove that there exists a contraction with a unique fixed point, which coincides with the unique solution of problem (\mathcal{P}) . We cite also [33] where they develop inf-sup theory to prove existence and uniqueness of solution.

3.3.2 Existence and uniqueness of the solution

We recall the undrained split iterative method, from [36], which consists in imposing constant fluid mass during the structure deformation.

We set $\bar{p}^0 = \bar{p}_0$ and $\bar{\mathbf{u}}_0 = 0$. We want to iterate the relation and ensure that

$$\bar{p}^{n+1} = \bar{p}^n - \alpha M \operatorname{div}(\bar{\mathbf{u}}^{n+1} - \bar{\mathbf{u}}^n). \quad (3.18)$$

Thus, for all $n \in \mathbb{N}$, our iterative process is to solve the following system

$$\begin{aligned} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}^{n+1}) + \lambda \operatorname{div}(\bar{\mathbf{u}}^{n+1}) \operatorname{Id} + M\alpha^2 \operatorname{div}(\bar{\mathbf{u}}^{n+1}) \operatorname{Id}) = \\ -\operatorname{div}(\alpha \bar{p}^n + M\alpha^2 \operatorname{div}(\bar{\mathbf{u}}^n)) + \mathbf{f}, \quad \text{in } D \times [0, T), \\ \bar{\mathbf{u}}^{n+1} = 0, \quad \text{on } \partial D \times [0, T), \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} \partial_t \left(\frac{1}{M} \bar{p}^{n+1} + \alpha \operatorname{div}(\bar{\mathbf{u}}^{n+1}) \right) - \operatorname{div}(\kappa \nabla \bar{p}^{n+1}) = g, \quad \text{in } D \times (0, T), \\ \bar{p}^{n+1} = 0, \quad \text{on } \partial D \times [0, T), \\ \bar{p}^{n+1}(\cdot, 0) = \bar{p}_0(\cdot), \quad \text{in } D. \end{aligned} \quad (3.20)$$

We define a distance d_{us} on the space $\mathbb{V} \times W$ as follows :

$$\begin{aligned} d_{us}((\mathbf{z}, \gamma), (\mathbf{w}, \zeta)) := & \frac{\kappa_0}{M} \max_{0 \leq t \leq T} \|\nabla(\gamma - \zeta)(t)\|_{L^2(D)}^2 \\ & + \frac{2\mu_0\alpha^2}{\lambda_0 + M\alpha^2} \|\partial_t e(\mathbf{z} - \mathbf{w})\|_{L^2(D \times (0, T))}^2 \\ & + \alpha^2 \|\partial_t \operatorname{div}(\mathbf{z} - \mathbf{w})\|_{L^2(D \times (0, T))}^2 \\ & + \left\| \frac{1}{M} \partial_t(\gamma - \zeta) + \alpha \partial_t \operatorname{div}(\mathbf{z} - \mathbf{w}) \right\|_{L^2(D \times (0, T))}^2 \end{aligned} \quad (3.21)$$

Up to the boundary condition that are slightly different, we obtain the same result as in [36].

Theorem 3.3.2. *Under the Hypothesis 2, the mapping $\mathcal{S} : \mathbb{V} \times W \rightarrow \mathbb{V} \times W$ defines a strict contraction so that it admits a unique fixed point in $\mathbb{V} \times W$.*

Proof. For $(\mathbf{w}_1, \zeta_1) \in \mathbb{V} \times W$ and $(\mathbf{w}_2, \zeta_2) \in \mathbb{V} \times W$, we define (\mathbf{z}_1, γ_1) and (\mathbf{z}_2, γ_2) as

$$\mathcal{S}(\mathbf{w}_1, \zeta_1) = (\mathbf{z}_1, \gamma_1), \quad \mathcal{S}(\mathbf{w}_2, \zeta_2) = (\mathbf{z}_2, \gamma_2) \quad (3.22)$$

so that

$$\begin{aligned} -\operatorname{div}(2\mu e(\mathbf{z}_i) + \lambda \operatorname{div}(\mathbf{z}_i) \operatorname{Id} + M\alpha^2 \operatorname{div}(\mathbf{z}_i) \operatorname{Id}) \\ = -\operatorname{div}(\alpha \zeta_i + M\alpha^2 \operatorname{div}(\mathbf{w}_i)) + \mathbf{f} \text{ in } D \times (0, T), \\ \mathbf{z}_i = 0 \text{ on } \partial D \times (0, T), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \partial_t \left(\frac{1}{M} \gamma_i + \alpha \operatorname{div}(\mathbf{z}_i) \right) - \operatorname{div}(\kappa \nabla \gamma_i) = f, \quad \text{in } D \times (0, T), \\ \gamma_i = 0, \text{ on } \partial D \times (0, T), \\ \gamma_i(t = 0) = p_0, \quad \text{in } D. \end{aligned} \quad (3.24)$$

Taking the difference of equation (3.23) for $i = 2$ and $i = 1$ and differentiating in time yields

$$\begin{aligned} & -\operatorname{div}(2\mu\partial_t e(\mathbf{z}_2 - \mathbf{z}_1) + \lambda\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)\operatorname{Id} + M\alpha^2\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)\operatorname{Id}) \\ & = -\partial_t \nabla \left((\zeta_2 - \zeta_1) + M\alpha^2 \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right) \text{ in } D \times (0, T), \\ & \partial_t \mathbf{z}_2 - \partial_t \mathbf{z}_1 = 0 \text{ on } \partial D \times (0, T). \end{aligned} \quad (3.25)$$

We take the matrix scalar product " : " of (3.25) with $\partial_t(\mathbf{z}_2 - \mathbf{z}_1)$, and integrate in D which gives in view of (3.5)

$$\begin{aligned} & \int_D 2\mu\partial_t e(\mathbf{z}_2 - \mathbf{z}_1) : \partial_t e(\mathbf{z}_2 - \mathbf{z}_1) \, dx \\ & + \int_D (\lambda + M\alpha^2)\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \, dx \\ & = \int_D \partial_t (\alpha(\zeta_2 - \zeta_1) + M\alpha^2 \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1))\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \, dx \\ & = M\alpha \int_D \partial_t \left(\frac{1}{M}(\zeta_2 - \zeta_1) + \alpha \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right) \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \, dx \\ & = M \int_D \partial_t \left(\frac{1}{M}(\zeta_2 - \zeta_1) + \alpha \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right) (\alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)) \, dx \end{aligned} \quad (3.26)$$

and thus by applying Young's inequality for all $\varepsilon > 0$

$$\begin{aligned} & 2\mu_0 \int_D |\partial_t e(\mathbf{z}_2 - \mathbf{z}_1)|^2 \, dx + (\lambda_0 + M\alpha^2) \int_D |\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)|^2 \, dx \\ & \leq \frac{M}{2\varepsilon} \int_D \left| \frac{1}{M} \partial_t (\zeta_2 - \zeta_1) + \alpha \partial_t \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right|^2 \, dx \\ & \quad + \frac{M\alpha^2\varepsilon}{2} \int_D |\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)|^2 \, dx \end{aligned} \quad (3.27)$$

We multiply (3.27) by $\frac{\alpha^2}{\lambda_0 + M\alpha^2(1 - \frac{\varepsilon}{2})}$ which yields

$$\begin{aligned} & \frac{2\mu_0\alpha^2}{\lambda_0 + M\alpha^2(1 - \frac{\varepsilon}{2})} \int_D |\partial_t e(\mathbf{z}_2 - \mathbf{z}_1)|^2 \, dx + \alpha^2 \int_D |\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)|^2 \, dx \\ & \leq \frac{M\alpha^2}{2\varepsilon(\lambda_0 + M\alpha^2(1 - \frac{\varepsilon}{2}))} \int_D \left| \frac{1}{M} \partial_t (\zeta_2 - \zeta_1) + \alpha \partial_t \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right|^2 \, dx. \end{aligned} \quad (3.28)$$

Taking the differences of equation (3.24) for $i = 2$ and $i = 1$ yields

$$\begin{aligned} & \partial_t \left(\frac{1}{M}(\gamma_2 - \gamma_1) + \alpha \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) - \operatorname{div}(\kappa \nabla(\gamma_2 - \gamma_1)) = 0 \text{ in } D \times (0, T), \\ & \gamma_1 - \gamma_2 = 0 \text{ on } \partial D \times (0, T), \\ & (\gamma_2 - \gamma_1)(t = 0) = 0 \text{ in } \Omega. \end{aligned} \quad (3.29)$$

We take the scalar product of the partial differential equation in (3.29) with $\partial_t(\gamma_2 - \gamma_1)$ and integrate in $D \times (0, T)$ which gives

$$\begin{aligned} & \int_0^t \int_D \left(\frac{1}{M} \partial_t (\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) \partial_t (\gamma_2 - \gamma_1) \, dx \, d\tau \\ & \quad + \int_0^t \int_D \kappa \frac{\partial_t}{2} |\nabla(\gamma_2 - \gamma_1)|^2 \, dx \, d\tau = 0. \end{aligned} \quad (3.30)$$

Since $\gamma_2 - \gamma_1 = 0$ at $t = 0$, there holds that

$$\begin{aligned} & \int_0^t \int_D \left(\frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) \partial_t(\gamma_2 - \gamma_1) dx d\tau \\ & \quad + \frac{\kappa_0}{2} \int_D |\nabla(\gamma_2 - \gamma_1)(t)|^2 dx dt \leq 0. \end{aligned} \quad (3.31)$$

Thus we have

$$\begin{aligned} & \int_0^t \int_D \left| \frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right|^2 dx dt \\ & = \int_0^t \int_D \left(\frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) dx dt \\ & \quad + \int_0^t \int_D \left(\frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) \frac{1}{M} \partial_t(\gamma_2 - \gamma_1) dx dt \\ & \leq \int_0^t \int_D \left(\frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right) \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) dx dt \\ & \quad - \frac{\kappa_0}{M} \int_D |\nabla(\gamma_2 - \gamma_1)(t)|^2 dx dt. \end{aligned} \quad (3.32)$$

Finally, applying Holder's inequality on the right hand side of (3.32), we have that

$$\begin{aligned} & \int_0^t \int_D \left| \frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right|^2 dx dt \\ & \quad + \frac{\kappa_0}{M} \int_D |\nabla(\gamma_2 - \gamma_1)(t)|^2 dx \leq \int_0^t \int_D \alpha^2 |\partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1)|^2 dx dt. \end{aligned} \quad (3.33)$$

Integrating (3.28) in time and adding (3.33) yields

$$\begin{aligned} & \frac{2\mu_0\alpha^2}{\lambda_0 + M\alpha^2(1 - \frac{\varepsilon}{2})} \int_0^t \int_D |\partial_t e(\mathbf{z}_2 - \mathbf{z}_1)|^2 dx dt \\ & \quad + \int_0^t \int_D \left| \frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right|^2 dx dt \\ & \quad \quad + \frac{\kappa_0}{M} \int_D |\nabla(\gamma_2 - \gamma_1)(t)|^2 dx \\ & \leq \frac{M\alpha^2}{2\varepsilon(\lambda_0 + M\alpha^2(1 - \frac{\varepsilon}{2}))} \int_0^t \int_D \left| \frac{1}{M} \partial_t(\zeta_2 - \zeta_1) + \alpha \partial_t \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right|^2 dx dt. \end{aligned} \quad (3.34)$$

We set $\varepsilon = \frac{\lambda_0}{M\alpha^2} + 1$ in order to minimize the coefficient on the right hand side of (3.34), which yields

$$\begin{aligned} & \frac{4\mu_0\alpha^2}{\lambda_0 + M\alpha^2} \int_0^t \int_D |\partial_t e(\mathbf{z}_2 - \mathbf{z}_1)|^2 dx \\ & \quad + \int_0^t \int_D \left| \frac{1}{M} \partial_t(\gamma_2 - \gamma_1) + \alpha \partial_t \operatorname{div}(\mathbf{z}_2 - \mathbf{z}_1) \right|^2 dx dt \\ & \quad \quad + \frac{\kappa_0}{2} \int_D |\nabla(\gamma_2 - \gamma_1)(t)|^2 dx \\ & \leq \left(\frac{M\alpha^2}{\lambda_0 + M\alpha^2} \right)^2 \int_0^t \int_D \left| \frac{1}{M} \partial_t(\zeta_2 - \zeta_1) + \alpha \partial_t \operatorname{div}(\mathbf{w}_2 - \mathbf{w}_1) \right|^2 dx dt. \end{aligned} \quad (3.35)$$

Since $0 < \frac{M\alpha^2}{\lambda_0 + M\alpha^2} < 1$, we have that

$$d_{us}((\mathbf{z}_2, \gamma_2) - (\mathbf{z}_1, \gamma_1)) \leq \left(\frac{M\alpha^2}{\lambda_0 + M\alpha^2} \right)^2 d_{us}((\mathbf{w}_2, \zeta_2) - (\mathbf{w}_1, \zeta_1)),$$

so that the operator \mathcal{S} is a contraction. □

Corollary 3.3.1. *The unique fixed point $(\bar{\mathbf{u}}, \bar{p}) \in \mathbf{V} \times W$ of \mathcal{S} is the unique solution of problem (\mathcal{P}) .*

3.4 Extra space regularities of the solution

We propose to detail some regularity results on the solution $(\bar{\mathbf{u}}, \bar{p})$ that will be needed later.

Theorem 3.4.1. *There exists a unique weak solution $(\bar{\mathbf{u}}, \bar{p})$ of the problem (\mathcal{P}) such that $\bar{\mathbf{u}} \in \mathbb{V}$ and $\bar{p} \in W$. Moreover, $\bar{\mathbf{u}}$ and \bar{p} satisfy*

$$2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} \in L^2(0, T; H_{\operatorname{div}}(D)), \quad (3.36)$$

$$\bar{p} \in L^2(0, T; H^2(D)), \quad (3.37)$$

so that the partial differential equations in problem (\mathcal{P}) are satisfied in $L^2(D \times (0, T))$.

Proof. We recall the parabolic equation

$$\partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g,$$

and the following regularity properties :

- $\mathbf{f} \in C^1(0, T; L^2(D)^d)$
- $g \in C(0, T; L^2(D))$,
- $\partial_t \operatorname{div}(\bar{\mathbf{u}}) = \partial_t \operatorname{Tr}(e(\bar{\mathbf{u}})) \in L^2(D \times (0, T))$,
- $\partial_t \bar{p} \in L^2(D \times (0, T))$.

It follows from [27, Theorem 7.5] we deduce (3.37).

Inspecting the terms of the elasticity tensor $2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id}$, we have that $\alpha \operatorname{div}(\bar{p} \operatorname{Id}) \in L^2(D \times (0, T))$, and since

$$-\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id}) = -\alpha \operatorname{div}(\bar{p} \operatorname{Id}) + \mathbf{f},$$

we deduce (3.36). □

3.5 Extra time regularity of the solution

In this section we suppose that the initial data are smooth enough to prove smoothness in time properties for the solution pair $(\bar{\mathbf{u}}, \bar{p})$.

We consider the following problem, which corresponds to Problem (\mathcal{P}) which has been differentiated in time.

$$(\mathcal{P}') \begin{cases} -\operatorname{div}\{2\mu e(\tilde{\mathbf{u}}) + \lambda \operatorname{div}(\tilde{\mathbf{u}}) \operatorname{Id} - \alpha \tilde{p} \operatorname{Id}\} = \partial_t \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\tilde{p}}{M} + \alpha \operatorname{div}(\tilde{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \tilde{p}) = \partial_t g, & \text{in } D \times (0, T), \\ \tilde{p} = 0, \tilde{\mathbf{u}} = 0, & \text{on } \partial D \times [0, T), \\ \frac{1}{M} \tilde{p}(\cdot, 0) + \alpha \operatorname{div}(\tilde{\mathbf{u}}(\cdot, 0)) = g(0) + \operatorname{div}(\kappa \nabla \bar{p}_0), & \text{in } D. \end{cases}$$

Hypothesis 3. We make the following assumptions on the data :

$$\begin{cases} \partial_t \mathbf{f} \in C([0, T], L^2(D)), \\ \partial_t g \in L^2(D \times (0, T)), \\ \bar{p}_0 \in H_0^1(D) \cap H^2(D). \end{cases}$$

From [36], we have the following

Theorem 3.5.1 (Mikelic-Wheeler). *Suppose that the initial data and the right-hand side satisfy Hypothesis 3. Then there exists a unique weak solution $(\tilde{\mathbf{u}}, \tilde{p})$ of the problem (\mathcal{P}') , such that $\tilde{\mathbf{u}} \in \mathbb{V}$ and $\tilde{p} \in W$.*

Proof. The differentiate problem (\mathcal{P}') coincides with the problem (\mathcal{P}) , except for the source terms and initial functions. Thus the proof is exactly the same than Theorem 3.3.1 and in [36]. \square

As far as we know, the following properties about the time regularity of the weak solution of problem (\mathcal{P}) are new. Indeed, we prove that the solution pair is in fact continuously differentiable in time. This property is necessary to prove the error estimates that we will study later.

Theorem 3.5.2. *Suppose that the initial data and the right-hand side satisfy Hypothesis 3. Then, the weak solution pair $(\tilde{\mathbf{u}}, \tilde{p})$ of problem (\mathcal{P}') is the time derivative of the weak solution pair $(\bar{\mathbf{u}}, \bar{p})$ of problem (\mathcal{P}) , namely $\tilde{\mathbf{u}} = \partial_t \bar{\mathbf{u}}$ and $\tilde{p} = \partial_t \bar{p}$. Consequently, $\tilde{\mathbf{u}} \in \mathbb{V}$ and $\tilde{p} \in W$, and in particular, we have that*

$$\bar{\mathbf{u}} \in C^1([0, T]; H_0^1(D)), \quad (3.38)$$

and

$$\bar{p} \in C^1([0, T]; H_0^1(D)). \quad (3.39)$$

Moreover, there holds that

$$2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} \in H^1(0, T; H_{\operatorname{div}}(D)) \subset C^{1/2}([0, T]; H_{\operatorname{div}}(D)), \quad (3.40)$$

$$\bar{p} \in H^1(0, T; H^2(D)) \subset C^{1/2}([0, T]; H^2(D)). \quad (3.41)$$

We recall that the spaces $C^{1/2}([0, T]; H)$ are the spaces of Hölder continuous functions of exponent 1/2 with values in the space H .

Proof of Theorem 3.5.2.

We set

$$\hat{p} = \int_0^t \tilde{p}(\tau) + \bar{p}_0, \quad \text{and} \quad \hat{\mathbf{u}} = \int_0^t \tilde{\mathbf{u}}(\tau) + \bar{\mathbf{u}}_0.$$

We will show that the pair $(\hat{\mathbf{u}}, \hat{p})$ coincides with the unique solution of the problem (\mathcal{P}) .

We integrate in time the pressure equation in problem (\mathcal{P}') :

$$\int_0^t \left(\partial_t \left(\frac{\tilde{p}}{M} + \alpha \operatorname{div}(\tilde{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \tilde{p}) \right) d\tau = \int_0^t \partial_t g \, d\tau$$

which implies

$$\frac{1}{M} \tilde{p}(t) + \alpha \operatorname{div}(\tilde{\mathbf{u}}(t)) - \operatorname{div} \left(\kappa \nabla \left(\int_0^t \tilde{p}(\tau) \, d\tau \right) \right) = \mathbf{f}(t) - \mathbf{f}(0) + \operatorname{div}(\kappa \nabla \bar{p}_0) + \mathbf{f}(0),$$

where we have used the initial condition for $(\tilde{p}, \tilde{\mathbf{u}})$, namely

$$\frac{1}{M} \tilde{p}(0) + \alpha \operatorname{div}(\tilde{\mathbf{u}}(0)) = \operatorname{div}(\kappa \nabla \bar{p}_0) + \mathbf{f}(0).$$

Then we have for almost all $t \in (0, T)$:

$$\frac{1}{M} \partial_t \widehat{p}(t) + \alpha \partial_t \operatorname{div}(\widehat{\mathbf{u}}(t)) - \operatorname{div} \left(\kappa \left(\int_0^t \widehat{p}(\tau) \, d\tau + \bar{p}_0 \right) \right) = \mathbf{f}(t).$$

Finally we conclude that the pair $(\widehat{\mathbf{u}}, \widehat{p})$ satisfies the following equation in $D \times (0, T)$:

$$\frac{1}{M} \partial_t \widehat{p} + \alpha \partial_t \operatorname{div}(\widehat{\mathbf{u}}) - \operatorname{div}(\kappa \nabla \widehat{p}) = g \quad (3.42)$$

We recall that in $D \times (0, T)$, the pair $(\widetilde{\mathbf{u}}, \widetilde{p})$ satisfies:

$$-\operatorname{div}(2\mu e(\widetilde{\mathbf{u}}) + \lambda \operatorname{div}(\widetilde{\mathbf{u}}) \operatorname{Id} - \alpha \widetilde{p} \operatorname{Id}) = \partial_t \mathbf{f}. \quad (3.43)$$

We integrate (3.43) in time to obtain :

$$-\operatorname{div} \left(2\mu e \left(\int_0^t \widetilde{\mathbf{u}} \, d\tau \right) + \lambda \operatorname{div} \left(\int_0^t \widetilde{\mathbf{u}} \, d\tau \right) \operatorname{Id} - \alpha \int_0^t \widetilde{p} \, d\tau \operatorname{Id} \right) = \int_0^t \partial_t \mathbf{f} \, d\tau. \quad (3.44)$$

And since $\widehat{p} = \int_0^t \widetilde{p} \, d\tau + \bar{p}_0$, and $\widehat{\mathbf{u}} = \int_0^t \widetilde{\mathbf{u}} \, d\tau + \bar{\mathbf{u}}_0$, the equation (3.44) implies that

$$-\operatorname{div}(2\mu e(\widehat{\mathbf{u}} - \bar{\mathbf{u}}_0) + \lambda \operatorname{div}(\widehat{\mathbf{u}} - \bar{\mathbf{u}}_0) \operatorname{Id} + \alpha (\widehat{p} - \bar{p}_0) \operatorname{Id}) = \mathbf{f} - \mathbf{f}(0), \text{ in } D \times [0, T]$$

or else

$$-\operatorname{div}(2\mu e(\widehat{\mathbf{u}}) + \lambda \operatorname{div}(\widehat{\mathbf{u}}) \operatorname{Id} + \alpha \widehat{p} \operatorname{Id}) = \mathbf{f} - \mathbf{f}(0) - \operatorname{div}(2\mu e(\bar{\mathbf{u}}_0) + \lambda \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id} + \alpha \bar{p}_0 \operatorname{Id}).$$

Since the pair $(\bar{\mathbf{u}}_0, \bar{p}_0)$ satisfies (3.12), we deduce that

$$-\operatorname{div}(2\mu e(\widehat{\mathbf{u}}) + \lambda \operatorname{div}(\widehat{\mathbf{u}}) \operatorname{Id} + \alpha \widehat{p} \operatorname{Id}) = \mathbf{f}. \quad (3.45)$$

Thus, we have shown that $(\widehat{\mathbf{u}}, \widehat{p})$ satisfies the equations (3.42) and (3.45), together with the initial conditions

- $\widehat{p}(\cdot, 0) = \bar{p}_0$ in D ,
- $\widehat{\mathbf{u}}(\cdot, 0) = \bar{\mathbf{u}}_0$ in D ,

and the boundary condition

- $\widehat{\mathbf{u}}(\cdot, t) = 0$, $\widehat{p} = (\cdot, t)$ on ∂D , for all $t \in (0, T)$.

We conclude that the pair $(\widehat{\mathbf{u}}, \widehat{p})$ coincides with the unique solution $(\bar{\mathbf{u}}, \bar{p})$ of problem (\mathcal{P}) .

Moreover, from (3.37) and (3.36), by applying Theorem 3.4.1 to $(\widetilde{\mathbf{u}}, \widetilde{p}) := (\partial_t \bar{\mathbf{u}}, \partial_t \bar{p})$ there holds that

$$2\mu e(\partial_t \bar{\mathbf{u}}) + \lambda \operatorname{div}(\partial_t \bar{\mathbf{u}}) \operatorname{Id} \in L^2(0, T; H_{\operatorname{div}}(D)),$$

$$\partial_t \bar{p} \in L^2(0, T; H^2(D)).$$

Thus we can ensure that $2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} \in H^1(0, T; H_{\operatorname{div}}(D))$ and $\bar{p} \in H^1(0, T; H^2(D))$. □

Finally, one can prove that, given more regularity on the data, the solution $(\bar{\mathbf{u}}, \bar{p})$ is smooth in time.

Corollary 3.5.1. *Assume that $\mathbf{f}^{(k)} \in C(0, T; L^2(D))$, $g^{(k)} \in L^2(D \times (0, T))$, and $\Delta \bar{p}_0^{(k-1)} \in L^2(D)$; then the associated k -times differentiated problem $(\mathcal{P}^{(k)})$ has a unique solution $(\bar{\mathbf{u}}^{(k)}, \bar{p}^{(k)}) \in \mathbb{V} \times W$.*

3.6 A priori estimates

We introduce the following Grönwall's lemma,

Lemma 3.6.1. *Given z a continuous function of $[0, T]$ in \mathbb{R}^+ , and $a, b \in \mathbb{R}^+$ such that*

$$z(t) \leq b + a \int_0^t z(s) ds.$$

Then,

$$z(t) \leq b e^{at}. \quad (3.46)$$

Theorem 3.6.1. *Let $(\bar{\mathbf{u}}, \bar{p})$ be the weak solution of the problem (\mathcal{P}) ; there exists a positive constant C which depends on T , the initial conditions and the source terms such that*

$$\begin{aligned} & \sup_{t \in [0, T]} \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \sup_{t \in [0, T]} \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\ & + \sup_{t \in [0, T]} \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 \leq C. \end{aligned} \quad (3.47)$$

Proof. Let $(\bar{\mathbf{u}}, \bar{p})$ be the solution of problem \mathcal{P} . We multiply the equation for $\bar{\mathbf{u}}$ by $\partial_t \bar{\mathbf{u}}$ and the equation for \bar{p} by \bar{p} , and integrate by parts in D to obtain

$$\begin{aligned} & \int_D 2\mu e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}}) - \alpha p \partial_t \operatorname{div}(\bar{\mathbf{u}}) dx = \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx, \\ & \int_D \frac{1}{M} \partial_t \bar{p} \bar{p} + \alpha \partial_t \operatorname{div}(\bar{\mathbf{u}}) \bar{p} dx + \int_D \kappa \nabla \bar{p} \cdot \nabla \bar{p} dx = \int_D g \bar{p} dx. \end{aligned} \quad (3.48)$$

There holds

$$\begin{aligned} & \int_D \mu \partial_t |e(\bar{\mathbf{u}})|^2 + \frac{\lambda}{2} \partial_t |\operatorname{div}(\bar{\mathbf{u}})|^2 dx - \int_D \alpha \bar{p} \partial_t \operatorname{div}(\bar{\mathbf{u}}) dx = \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx, \\ & \int_D \frac{1}{2M} \partial_t |\bar{p}|^2 + \alpha \partial_t \operatorname{div}(\bar{\mathbf{u}}) \bar{p} dx + \int_D \kappa |\nabla \bar{p}|^2 dx = \int_D g \bar{p} dx. \end{aligned} \quad (3.49)$$

We sum the two equations of (3.49) and integrate them over $(0, t)$ for $t \in (0, T)$ to deduce that

$$\begin{aligned} & \mu_0 \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 \\ & \leq \int_0^t \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx d\tau + \int_0^t \int_D g \bar{p} dx d\tau + \mathcal{M}_\mu \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\ & \quad + \frac{\mathcal{M}_\lambda}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \frac{1}{2M} \|\bar{p}_0\|_{L^2(D)}^2. \end{aligned} \quad (3.50)$$

Integration by parts gives

$$\int_0^t \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx d\tau = \int_D \mathbf{f}(t) \cdot \bar{\mathbf{u}}(t) dx - \int_0^t \int_D \partial_t \mathbf{f} \cdot \bar{\mathbf{u}} dx d\tau - \int_D \mathbf{f} \cdot \bar{\mathbf{u}}_0 dx,$$

so that,

$$\begin{aligned} & \int_0^t \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx d\tau \\ & \leq \frac{C_K}{2\mu_0} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{\mu_0}{2C_K} \|\bar{\mathbf{u}}(t)\|_{L^2(D)}^2 \\ & + \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\mu_0}{2C_K} \|\bar{\mathbf{u}}\|_{L^2(D \times (0, T))}^2 \\ & \quad + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2, \end{aligned} \quad (3.51)$$

where C_K is the constant from Korn's inequality.

Finally, from Korn inequality, since $\|\bar{\mathbf{u}}(t)\|_{L^2(D)}^2 \leq C_K \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2$, there holds

$$\begin{aligned} \int_0^t \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx d\tau &\leq \frac{C_K}{2\mu_0} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\ &+ \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ &+ \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2. \end{aligned} \quad (3.52)$$

Moreover, there holds

$$\int_0^t \int_D g \bar{p} dx d\tau \leq \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}\|_{L^2(D \times (0, T))}^2. \quad (3.53)$$

We deduce from (3.50), (3.52) and (3.53) that

$$\begin{aligned} &\frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla \bar{p}\|_{L^2(D)}^2 \\ &\leq \frac{C_K}{2\mu_0} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ &\quad + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 \\ &\quad + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}_0\|_{L^2(D)}^2. \end{aligned} \quad (3.54)$$

Thus, we have

$$\begin{aligned} &\frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 \\ &\leq \frac{C_K}{2\mu_0} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ &\quad + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 \\ &\quad + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}_0\|_{L^2(D)}^2. \end{aligned} \quad (3.55)$$

Under Hypothesis 2, and with (3.3.1), we obtain that the following quantity is bounded

$$\frac{C_K}{2\mu_0} \sup_{t \in [0, T]} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2.$$

So that by applying Gronwall's Lemma on (3.55) we obtain

$$\frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 \leq C e^t. \quad (3.56)$$

Thus there holds that

$$\frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\bar{p}\|_{L^2(D \times (0, T))}^2 \leq TC e^T. \quad (3.57)$$

Reinject this in (3.54) to conclude. \square

Theorem 3.6.2. We set $\gamma_0 = \min(\mu_0, \lambda_0, \frac{\kappa_0}{2}, \frac{1}{2M})$ and note C_K the positive constant from Korn inequality (3.4). Under Hypotheses 2 and 3, there holds

$$\begin{aligned} &\|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ &\quad + \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\nabla \bar{p}(t)\|_{L^2(D)}^2 \\ &\leq \frac{1}{\gamma_0} \left(\frac{C_K}{4\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{\mathcal{M}_\kappa}{2} \|\nabla \bar{p}_0\|_{L^2(D)}^2 \right). \end{aligned} \quad (3.58)$$

In other words, there exists a positive constant $C = C(C_K, \mu_0, \gamma_0, \mathbf{f}, g)$ depending on the Korn constant C_K , μ_0 , γ_0 and the sources terms \mathbf{f} and g such that

$$\begin{aligned} & \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ & + \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\nabla \bar{p}(t)\|_{L^2(D)}^2 \leq C(C_K, \mu_0, \gamma_0, \mathbf{f}, g). \end{aligned} \quad (3.59)$$

Proof. Therefore since $(\bar{\mathbf{u}}, \bar{p})$ is the solution of problem (\mathcal{P}) there holds

$$\partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, \quad \text{in } D \times (0, T). \quad (3.60)$$

Moreover $(\partial_t \bar{\mathbf{u}}, \partial_t \bar{p})$ is the solution of (\mathcal{P}') so that

$$-\operatorname{div}(2\mu \partial_t e(\bar{\mathbf{u}}) + \lambda \partial_t \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \partial_t \bar{p} \operatorname{Id}) = \partial_t \mathbf{f}, \quad \text{in } D \times (0, T). \quad (3.61)$$

Thus multiplying (3.60) by $q \in W$ and taking the integral over D , multiplying (3.61) by $\mathbf{v} \in \mathbb{V}$ and taking the integral over D leads to

$$\begin{aligned} & \int_D 2\mu \partial_t e(\bar{\mathbf{u}}) : e(\mathbf{v}) + \lambda \partial_t \operatorname{div}(\bar{\mathbf{u}}) \operatorname{div}(\mathbf{v}) - \alpha \partial_t \bar{p} \operatorname{div}(\mathbf{v}) \, dx = \int_D \partial_t \mathbf{f} \mathbf{v} \, dx, \\ & \int_D \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) q \, dx + \kappa \int_D \nabla \bar{p} \cdot \nabla q \, dx = \int_D g q \, dx. \end{aligned}$$

We set $\mathbf{v} = \partial_t \bar{\mathbf{u}}$, $q = \partial_t \bar{p}$ and integrate in time to obtain

$$\begin{aligned} & \int_0^t \int_D (2\mu |\partial_t e(\bar{\mathbf{u}})|^2 + \lambda |\partial_t \operatorname{div}(\bar{\mathbf{u}})|^2) \, dx dt \\ & - \alpha \int_0^t \int_D \partial_t \bar{p} \partial_t \operatorname{div}(\bar{\mathbf{u}}) \, dx dt = \int_0^t \int_D \partial_t \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} \, dx dt, \end{aligned} \quad (3.62)$$

$$\begin{aligned} & \int_0^t \int_D \frac{1}{M} |\partial_t \bar{p}|^2 + \alpha \partial_t \operatorname{div}(\bar{\mathbf{u}}) \partial_t \bar{p} \, dx dt \\ & + \int_0^t \int_D \kappa \nabla \bar{p} \cdot \partial_t \nabla \bar{p} \, dx dt = \int_0^t \int_D g \partial_t \bar{p} \, dx dt. \end{aligned} \quad (3.63)$$

We take the sum of (3.62) and (3.63) to obtain

$$\begin{aligned} & \int_0^t \int_D 2\mu |\partial_t e(\bar{\mathbf{u}})|^2 + \lambda |\partial_t \operatorname{div}(\bar{\mathbf{u}})|^2 \, dx dt \\ & + \int_0^t \int_D \frac{1}{M} |\partial_t \bar{p}|^2 \, dx dt + \frac{1}{2} \int_D \kappa (|\nabla \bar{p}(t)|^2 - |\nabla \bar{p}_0|^2) \, dx \\ & = \int_0^t \int_D \partial_t \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} + g \partial_t \bar{p} \, dx dt. \end{aligned} \quad (3.64)$$

Applying Young's inequality to the equation (3.64) and Hypothesis 2 and 3, we have for any $\varepsilon_1 > 0$

$$\begin{aligned} & 2\mu_0 \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \lambda_0 \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ & + \frac{1}{M} \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \frac{\kappa_0}{2} \|\nabla \bar{p}(t)\|_{L^2(D)}^2 \\ & \leq \frac{1}{2\varepsilon_1} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\varepsilon_1}{2} \|\partial_t \bar{\mathbf{u}}\|_{L^2(D \times (0, T))}^2 \\ & + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{2M} \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 \\ & \quad + \frac{\mathcal{M}_\kappa}{2} \|\nabla \bar{p}_0\|_{L^2(D)}^2 \end{aligned} \quad (3.65)$$

By Korn inequality, there exists C_K , depending on D , such that

$$\|\partial_t \bar{\mathbf{u}}\|_{L^2(D \times (0, T))}^2 \leq C_K \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2, \quad (3.66)$$

We set $\varepsilon_1 = \frac{2\mu_0}{C_K}$, so that (3.65) reads as

$$\begin{aligned} & \mu_0 \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \lambda_0 \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ & + \frac{1}{2M} \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \frac{\kappa_0}{2} \|\nabla \bar{p}(t)\|_{L^2(D)}^2 \\ & \leq \frac{C_K}{4\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{\mathcal{M}_\kappa}{2} \|\nabla \bar{p}_0\|_{L^2(D)}^2. \end{aligned} \quad (3.67)$$

Then we set $\gamma_0 := \min(\mu_0, \lambda_0, \frac{\kappa_0}{2}, \frac{1}{2M})$ to obtain

$$\begin{aligned} & \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ & + \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\nabla \bar{p}(t)\|_{L^2(D)}^2 \\ & \leq \frac{1}{\gamma_0} \left(\frac{C_K}{4\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 + \frac{\mathcal{M}_\kappa}{2} \|\nabla \bar{p}_0\|_{L^2(D)}^2 \right). \end{aligned} \quad (3.68)$$

□

3.7 Thermo-poroelastic problem

In geothermy, the temperature play a role alongside the pressure and the displacement. Therefore, we consider a thermo-poroelastic system, which can be seen as a generalization of the Biot system. There has been several study of a thermo-poroelastic system including non-linear convective therms, coupling the Darcy flow \mathbf{v}_D and the gradient of the temperature $\nabla \theta$. In Chapter 6, we propose to focus on a thermo-poroelastic problem with a linear convective terms, which we introduce below

$$\begin{aligned} & -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})) + \alpha \nabla \bar{p} + (1 - \alpha) K_S \nabla \bar{\theta} = \mathbf{f}, & \text{in } D \times [0, T], \\ & \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times [0, T], \\ & \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M} \right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S (1 - \alpha) \operatorname{div}(\bar{\mathbf{u}}) \right) - \eta \cdot \nabla \bar{\theta} - \operatorname{div}(\beta \nabla \bar{\theta}) = h, & \text{in } D \times [0, T] \\ & \bar{\mathbf{u}}, \bar{p}, \bar{\theta} = 0 & \text{in } \partial D \times [0, T], \\ & \bar{p}(\cdot, 0) = \bar{p}_0, \bar{\theta}(\cdot, 0) = \bar{\theta}_0, & \text{in } D. \end{aligned} \quad (3.69)$$

We introduce the following parameters, which comes fom the temperature coupling in the thermo-poroelastic system. Here, the coefficient η is a constant to replace a non-linear term in the problem studied in [17].

- **Solid Expansion Coefficient** K_S : This coefficient represents the degree to which the solid matrix of the porous material expands or contracts in response to temperature changes. It is a measure of the thermal expansion of the solid framework.
- **Thermoelastic Coupling Parameter** K_T : This parameter quantifies the interaction between thermal effects and elastic deformation in the material. It describes how temperature variations can induce mechanical stresses and strains in the solid matrix.
- **Thermal diffusivity** β : Thermal diffusivity characterizes the rate at which heat spreads through the porous medium. It is a key parameter in determining the transient thermal response of the material to temperature changes.

The initial condition on \bar{p}_0 and $\bar{\theta}_0$ permits to compute an initial solution $\bar{\mathbf{u}}_0 := \bar{\mathbf{u}}(t = 0)$, as done in the Biot system.

We look for $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$, with homogeneous Dirichlet boundary condition, a weak solution of the following variational formulation such that there holds

$$\begin{aligned}
(2\mu e(\bar{\mathbf{u}}), e(\mathbf{v})) + (\lambda \operatorname{div}(\bar{\mathbf{u}}) - \alpha \bar{p} - K_S(1 - \alpha)\bar{\theta}, \operatorname{div}(\mathbf{v})) &= (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in H^1(D), \\
(\partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \lambda \operatorname{div}(\bar{\mathbf{u}}) \right), q) + (\kappa \nabla \bar{p}, \nabla q) &= (g, q), & \forall q \in H^1(D), \\
(\partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M} \right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S(1 - \alpha) \operatorname{div}(\bar{\mathbf{u}}) \right), S) + (\eta \cdot \nabla \bar{\theta}, S) + (\beta \nabla \bar{\theta}, \nabla S) &= (h, S), & \forall S \in H^1(D).
\end{aligned} \tag{3.70}$$

An existence and uniqueness result has been proved in [17, Theorem 3.1], where they assume suitable regularity for the initial data and source terms.

Chapter 4

The Gradient Discretization Method for the Biot problem

4.1 Gradient schemes for the Biot problem

A Gradient Discretization Method (GDM) is based on finite-dimensional vector spaces and operators, first introduced in [24] by Droniou et al. Given the variational formulation of problem (\mathbb{P}) , the continuous spaces and operators are replaced by corresponding discrete spaces and operators. This method has been used to study linear elliptic or parabolic problems (in [23]). The system which we study is essentially composed of two parts, a flow computation and a mechanical deformation computation. For both of these parts, we present a Gradient Discretization Method. Such a coupling between flow and mechanical deformation has already been studied in [10, 11, 12, 13] in a nonlinear context and they prove the convergence of the algorithm. Here we deal with a linear case which permits us to present an error estimate for the discrete solution obtained by means of this Gradient Discretization Method.

4.1.1 Known results for the Biot problem

The Biot system with homogeneous Dirichlet boundary conditions is given by

$$(\mathcal{P}) \begin{cases} -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}) = \mathbf{f}, & \text{in } D \times [0, T), \\ \partial_t \left(\frac{\bar{p}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) = g, & \text{in } D \times (0, T), \\ \bar{p} = 0, \quad \bar{\mathbf{u}} = 0, & \text{on } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot) & \text{in } D. \end{cases}$$

The following hypothesis is a version of Hypothesis 1 with three variables and with time dependent second member.

Hypothesis 4. *We assume that*

$$\mathbf{f} \in C^1([0, T]; L^2(D)), \quad g \in C([0, T]; L^2(D)).$$

The initial data satisfies

$$\bar{p}_0 \in H_0^1(D) \cap H^2(D).$$

The functions μ , λ , and κ satisfy

$$\mu, \lambda, \kappa \in C^\infty(\bar{D}),$$

and there exist constants $\mu_0, \lambda_0, \kappa_0, \mathcal{M}_\mu, \mathcal{M}_\lambda$ and \mathcal{M}_κ such that for all $x \in D$

$$0 < \mu_0 \leq \mu(x) \leq \mathcal{M}_\mu, \quad 0 < \lambda_0 \leq \lambda(x) \leq \mathcal{M}_\lambda, \quad 0 < \kappa_0 \leq \kappa(x) \leq \mathcal{M}_\kappa.$$

Moreover, M and α are positive constants.

We recall that under Hypothesis 4 we have the following regularity which is enough for the numerical approximation. This regularity has been proved in Chapter 3 (see Theorem 3.5.2) and it is the only novelty compared to known literature.

Theorem 4.1.1 (from Theorem 3.5.2). *There exists a unique weak solution $(\bar{\mathbf{u}}, \bar{p})$ of problem (\mathbb{P}) which satisfies*

$$\bar{\mathbf{u}} \in C^1([0, T]; H_0^1(D)), \quad \bar{p} \in C^1([0, T]; H_0^1(D)).$$

and

$$\begin{aligned} 2\mu\epsilon(\bar{\mathbf{u}}) + \lambda\operatorname{div}(\bar{\mathbf{u}})\operatorname{Id} &\in H^1(0, T; H_{\operatorname{div}}(D)) \subset C^{1/2}(0, T; H_{\operatorname{div}}(D)), \\ \bar{p} &\in H^1(0, T; H^2(D)) \subset C^{1/2}(0, T; H^2(D)). \end{aligned}$$

In view of the regularity of $\bar{\mathbf{u}}$ and \bar{p} , the system of equations is satisfied almost everywhere.

4.1.2 Space-time discretization for the pressure and the mechanical deformations

The Gradient Discretization Method for the Darcy flow with homogeneous Dirichlet boundary conditions is defined by a vector of unknowns as well as suitable operators defined as follows :

Definition 4.1.1. Gradient Discretization Method for the pressure

The GDM for the pressure p is defined by $\mathcal{D}_p = (X_p, \Pi_p^{\mathcal{D}}, \nabla_p^{\mathcal{D}})$ where:

- X_p is a finite dimensional real vector space of discrete unknowns,
- the linear function reconstruction $\Pi_p^{\mathcal{D}} : X_p \rightarrow L^2(D)$,
- the linear gradient reconstruction $\nabla_p^{\mathcal{D}} : X_p \rightarrow L^2(D)$,
- an interpolator $\mathcal{I}_p^{\mathcal{D}} : H_0^1(D) \rightarrow X_p$ such that

$$\mathcal{I}_p^{\mathcal{D}} \psi = \arg \min_{q \in X_p} (\|\Pi_p^{\mathcal{D}} q - \psi\|_{L^2(D)}^2 + \|\nabla_p^{\mathcal{D}} q - \nabla \psi\|_{L^2(D)}^2). \quad (4.1)$$

We also assume that $\|\psi\|_p^{\mathcal{D}} := \|\nabla_p^{\mathcal{D}} \psi\|_{L^2(D)}$ defines a norm on X_p .

The Gradient Discretization Method for the displacement with homogeneous Dirichlet boundary conditions follow almost the same definition, with a slightly different definition of the differential operator.

Definition 4.1.2. Gradient Discretization Method for the displacement

The GDM for the displacement \mathbf{u} is defined by $\mathcal{D}_u = (X_u, \Pi_u^{\mathcal{D}}, e_u^{\mathcal{D}})$ where:

- X_u is a finite dimensional real vector space of discrete unknowns,
- a displacement reconstruction linear operator $\Pi_u^{\mathcal{D}} : X_u \rightarrow L^2(D)$,
- a symmetric gradient linear operator $e_u^{\mathcal{D}} : X_u \rightarrow L^2(D, \mathcal{S}_d(\mathbb{R}))$, where $\mathcal{S}_d(\mathbb{R})$ is the space of symmetric matrices of size $d \times d$,
- an interpolator $\mathcal{I}_u^{\mathcal{D}} : H_0^1(D) \rightarrow X_u$ such that

$$\mathcal{I}_u^{\mathcal{D}} \boldsymbol{\varphi} = \arg \min_{\mathbf{v} \in X_u} (\|\Pi_u^{\mathcal{D}} \mathbf{v} - \boldsymbol{\varphi}\|_{L^2(D)}^2 + \|e_u^{\mathcal{D}}(\mathbf{v}) - e(\boldsymbol{\varphi})\|_{L^2(D)}^2). \quad (4.2)$$

We define a norm on X_u such that

$$\|\mathbf{v}\|_u := \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \quad (4.3)$$

We define, for $\mathbf{v} \in X_u$, the discrete divergence operator

$$\operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) = \operatorname{Tr}(e_u^{\mathcal{D}}(\mathbf{v})), \quad (4.4)$$

so that $\|\operatorname{div}_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \leq \sqrt{d} \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}$.

For the time discretization of problem (\mathbb{P}) , we will apply a backward Euler scheme with a constant time step which involves :

- a discretization of the interval $[0, T] 0 = t^0 < t^1 < \dots < t^N = T$,
- We define $\Delta t = t^{k+1} - t^k$, for all $0 \leq k \leq N - 1$, as the constant time step, so that $T = N\Delta t$,
- Given a function g continuous in time, we note

$$g^{k+1} = \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} g(s) \, ds. \quad (4.5)$$

As soon as we have a sequence of iterates given by the backward Euler scheme in spaces $(\mathcal{D}_u, \mathcal{D}_p)$, we can associate a piecewise constant function in the time variable using the following

Definition 4.1.3. We define for all $0 \leq k \leq N - 1$, for all $t \in (t^k, t^{k+1}]$, for all $\mathbf{v} \in X_u$ and $q \in X_p$

$$\begin{aligned} \mathbf{v}(t) &:= \mathbf{v}^{k+1}, \\ q(t) &:= q^{k+1}, \end{aligned} \quad (4.6)$$

where \mathbf{v}^{k+1} and q^{k+1} are given by the backward Euler scheme that will be defined later.

4.1.3 Properties of the Gradient Discretization Methods

Under the abstract formulation of a Gradient Discretization Method, we will define a “space size” $h_{\mathcal{D}}$ of \mathcal{D} . But, in the study of convergence of a GDM, we will have a sequence of “space sizes” that converges to 0 corresponding to a sequence of GDM. We denote this sequence $(\mathcal{D}^n)_{n \in \mathbb{N}}$.

Since we have two variables, we denote $(\mathcal{D}_p^n)_{n \in \mathbb{N}}$ and $(\mathcal{D}_u^n)_{n \in \mathbb{N}}$ respectively for the pressure and the displacement.

They will have corresponding “space sizes” $(h_p^{\mathcal{D}^n})_{n \in \mathbb{N}}$ and $(h_u^{\mathcal{D}^n})_{n \in \mathbb{N}}$, and we will assume that $h_{\mathcal{D}^n} := \max\{h_p^{\mathcal{D}^n}, h_u^{\mathcal{D}^n}\}$ will converge to 0 as n grows to infinity.

Next, we recall the properties of the pressure discretization as presented by [24].

Definition 4.1.4 (Properties for the pressure discretization).

- **Coercivity of \mathcal{D}_p .**

For a Gradient Discretization Method \mathcal{D}_p , let $C_p^{\mathcal{D}}$ be defined by

$$C_p^{\mathcal{D}} := \max_{q \in X_p \setminus \{0\}} \left\{ \frac{\|\Pi_p^{\mathcal{D}} q\|_{L^2(D)}}{\|q\|_p} \right\}. \quad (4.7)$$

A sequence $(\mathcal{D}_p^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is coercive if there exists $C_p > 0$ such that $C_p \geq C_p^{\mathcal{D}^n}$, for all $n \in \mathbb{N}$. This coercivity property is analogous to the Poincaré inequality in the continuous case.

- **Consistency of \mathcal{D}_p**

For a Gradient Discretization Method \mathcal{D}_p , let $S_p^{\mathcal{D}} : H_0^1(D) \rightarrow [0, +\infty)$ be defined by

$$\begin{aligned} &\text{for all } \psi \in H_0^1(D), \\ S_p^{\mathcal{D}}(\psi) &:= \min_{q \in X_p \setminus \{0\}} \left\{ \|\psi - \Pi_p^{\mathcal{D}} q\|_{L^2(D)} + \|\nabla \psi - \nabla_p^{\mathcal{D}} q\|_{L^2(D)} \right\}. \end{aligned} \quad (4.8)$$

A sequence $(\mathcal{D}_p^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is said GD-consistent if

$$\text{for all } \psi \in H_0^1(D), \quad \lim_{n \rightarrow \infty} S_p^{\mathcal{D}^n}(\psi) = 0. \quad (4.9)$$

- **Limit-conforming of \mathcal{D}_p**

We note $H_{\text{div}}(D) := \{\boldsymbol{\psi} \in L^2(D) : \text{div}(\boldsymbol{\psi}) \in L^2(D)\}$.

For a Gradient Discretization Method \mathcal{D}_p , let $W_p^{\mathcal{D}} : H_{\text{div}}(D) \rightarrow [0, +\infty)$ be defined, for all $\boldsymbol{\psi} \in H_{\text{div}}(D)$, by

$$W_p^{\mathcal{D}}(\boldsymbol{\psi}) := \max_{q \in X_p \setminus \{0\}} \frac{1}{\|q\|_p} \left| \int_D (\text{div}(\boldsymbol{\psi}) \Pi_p^{\mathcal{D}} q + \boldsymbol{\psi} \cdot \nabla_p^{\mathcal{D}} q) \, dx \right|. \quad (4.10)$$

A sequence $(\mathcal{D}_p^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is said limit-conforming if

$$\text{for all } \boldsymbol{\psi} \in H_{\text{div}}(D), \quad \lim_{n \rightarrow \infty} W_p^{\mathcal{D}_p^n}(\boldsymbol{\psi}) = 0. \quad (4.11)$$

Next we present the definition of the space size for the pressure discretization.

Definition 4.1.5. Let \mathcal{D}_p be a Gradient Discretization Method for the pressure in the sense above. We introduce the following quantities:

$$\begin{aligned} s_p^{\mathcal{D}}(H_0^1(D)) &:= \sup \left\{ \frac{S_p^{\mathcal{D}}(\varphi)}{\|\varphi\|_{H_0^1(D)}}, \varphi \in H_0^1(D) \setminus \{0\} \right\}, \\ w_p^{\mathcal{D}}(H_{\text{div}}(D)) &:= \sup \left\{ \frac{W_p^{\mathcal{D}}(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{H_{\text{div}}(D)}}, \boldsymbol{\psi} \in H_{\text{div}}(D) \setminus \{0\} \right\}. \end{aligned} \quad (4.12)$$

We define the space size of \mathcal{D}_p as

$$h_p^{\mathcal{D}} := \max\{s_p^{\mathcal{D}}(H_0^1(D)), w_p^{\mathcal{D}}(H_0^1(D))\}. \quad (4.13)$$

Next, we recall the properties of the displacement discretization as presented by [24].

Definition 4.1.6 (Properties for the displacement discretization).

- **Coercivity of \mathcal{D}_u**

For a GDM \mathcal{D}_u , let $C_u^{\mathcal{D}}$ be defined by

$$C_u^{\mathcal{D}} := \max_{\mathbf{v} \in X_u \setminus \{0\}} \left\{ \frac{\|\Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)}}{\|\mathbf{v}\|_u} \right\}. \quad (4.14)$$

A sequence $(\mathcal{D}_u^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is coercive if there exists $C_u > 0$ such that $C_u \geq C_u^{\mathcal{D}_u^n}$, for all $n \in \mathbb{N}$.

This coercivity property is analogous to the Korn inequality in the continuous case.

- **Consistency of \mathcal{D}_u**

Let $S_u^{\mathcal{D}} : H_0^1(D) \rightarrow [0, +\infty)$ define by

$$\begin{aligned} &\text{for all } \varphi \in H_0^1(D), \\ S_u^{\mathcal{D}}(\varphi) &:= \min_{\mathbf{v} \in X_u \setminus \{0\}} \left\{ \|\varphi - \Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)} + \|e(\varphi) - e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \right\}. \end{aligned} \quad (4.15)$$

A sequence $(\mathcal{D}_u^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is said to be GD-consistent if

$$\text{for all } \varphi \in H^1(D), \quad \lim_{n \rightarrow \infty} S_u^{\mathcal{D}_u^n}(\varphi) = 0. \quad (4.16)$$

- **Limit-conformity of \mathcal{D}_u**

We note $H_{\text{div}}(D) := \{\boldsymbol{\psi} \in L^2(D) : \text{div}(\boldsymbol{\psi}) \in L^2(D)\}$.

Let $W_u^{\mathcal{D}} : H_{\text{div}}(D) \rightarrow [0, +\infty)$ be defined, for all $\boldsymbol{\psi} \in H_{\text{div}}(D)$ by

$$W_u^{\mathcal{D}}(\boldsymbol{\psi}) := \max_{\mathbf{v} \in X_u \setminus \{0\}} \frac{1}{\|\mathbf{v}\|_u} \left| \int_D (\boldsymbol{\psi} : e_u^{\mathcal{D}}(\mathbf{v}) + \text{div} \boldsymbol{\psi} \cdot \Pi_u^{\mathcal{D}} \mathbf{v}) \, dx \right|. \quad (4.17)$$

A sequence $(\mathcal{D}_u^n)_{n \in \mathbb{N}}$ of Gradient Discretization Method is limit-conforming if

$$\text{for all } \boldsymbol{\psi} \in H_{\text{div}}(D), \lim_{n \rightarrow \infty} W_u^{\mathcal{D}_u^n}(\boldsymbol{\psi}) = 0. \quad (4.18)$$

We now introduce a space size depending on the regularity of the approximation:

Definition 4.1.7. Let \mathcal{D}_u be a Gradient Discretization Method for the displacement in the sense above. We introduce the following quantities:

$$\begin{aligned} s_u^{\mathcal{D}}(H_0^1(D)) &= \sup \left\{ \frac{S_u^{\mathcal{D}}(\boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_{H_0^1(D)}}, \boldsymbol{\varphi} \in H_0^1(D) \setminus \{0\} \right\}, \\ w_u^{\mathcal{D}}(H_{\text{div}}(D)) &= \sup \left\{ \frac{W_u^{\mathcal{D}}(\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{H_{\text{div}}}}, \boldsymbol{\psi} \in H_{\text{div}} \setminus \{0\} \right\}. \end{aligned} \quad (4.19)$$

We define the space size of \mathcal{D}_u as

$$h_u^{\mathcal{D}} = \max\{s_u^{\mathcal{D}}(H_0^1(D)), w_u^{\mathcal{D}}(H_{\text{div}}(D))\}. \quad (4.20)$$

Now that the space sizes $h_p^{\mathcal{D}}$ and $h_u^{\mathcal{D}}$ of the Gradient Discretization Method (respectively for pressure and displacement) have been defined, we introduce the total space size as being the maximum size of the mesh for both discretizations.

Definition 4.1.8 (Space size). We note

$$h_{\mathcal{D}} := \max(h_u^{\mathcal{D}}, h_p^{\mathcal{D}}) \quad (4.21)$$

Proposition 4.1.1. Let $\boldsymbol{\varphi} \in H_{\text{div}}(D)$; then

$$W_u^{\mathcal{D}}(\boldsymbol{\varphi}) \leq h_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{H_{\text{div}}(D)}. \quad (4.22)$$

Proof. Given $\boldsymbol{\varphi} \in H_{\text{div}}(D)$ we have that

$$W_u^{\mathcal{D}}(\boldsymbol{\varphi}) = \frac{W_u^{\mathcal{D}}(\boldsymbol{\varphi})}{\|\boldsymbol{\varphi}\|_{H_{\text{div}}(D)}} \|\boldsymbol{\varphi}\|_{H_{\text{div}}(D)} \leq w_u^{\mathcal{D}} \|\boldsymbol{\varphi}\|_{H_{\text{div}}(D)} \leq h_{\mathcal{D}} \|\boldsymbol{\varphi}\|_{H_{\text{div}}(D)}.$$

□

Before introducing a variational formulation for the gradient problem, we show the following Lemma which is an analogous result to that of Lemma 3.2.2:

Lemma 4.1.1. For $\mathbf{v} \in X_u$,

$$\text{Id} : e_u^{\mathcal{D}}(\mathbf{v}) = \text{div}_u^{\mathcal{D}}(\mathbf{v}) \text{ in } L^2(D). \quad (4.23)$$

Proof. We consider $\mathbf{v} \in X_u$. We have that $\{\text{Id}\}_{ij} = 0$ if $i \neq j$, and $\{\text{Id}\}_{ii} = 1$. In view of (4.4), it follows that

$$\begin{aligned} \text{Id} : e_u^{\mathcal{D}}(\mathbf{v}) &= \sum_i \sum_j \{\text{Id}\}_{ij} \{e_u^{\mathcal{D}}(\mathbf{v})\}_{ij} = \sum_i \{e_u^{\mathcal{D}}(\mathbf{v})\}_{ii} \\ &= \text{Tr}(e_u^{\mathcal{D}}(\mathbf{v})) = \text{div}_u^{\mathcal{D}}(\mathbf{v}). \end{aligned}$$

□

By application of (4.23), we define the gradient problem as a variational formulation with the discrete operator defined above.

Definition 4.1.9. A solution $(\mathbf{U}, P) \in X_u^{N+1} \times X_p^{N+1}$ of the gradient scheme associated to the problem (\mathbb{P}) is a pair $\mathbf{U} := (\mathbf{u}^0, \dots, \mathbf{u}^N)$ and $P := (p^0, \dots, p^N)$ such that for all $k \in [0, N]$

$$\begin{aligned} & 2 \int_D \mu e_u^{\mathcal{D}}(\mathbf{u}^k) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + \int_D \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & - \alpha \int_D \Pi_p^{\mathcal{D}} p^k \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx = \int_D \mathbf{f}^k \Pi_u^{\mathcal{D}} \mathbf{v}, \text{ for all } \mathbf{v} \in X_u, \end{aligned} \quad (4.24)$$

and for all $k \in [1, N]$

$$\begin{aligned} & \frac{1}{M} \int_D \frac{\Pi_p^{\mathcal{D}}(p^k - p^{k-1})}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx + \alpha \int_D \frac{\operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k - \mathbf{u}^{k-1})}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx \\ & + \int_D \kappa \nabla_p^{\mathcal{D}} p^k \cdot \nabla_p^{\mathcal{D}} q \, dx = \int_D g^k \Pi_p^{\mathcal{D}} q \, dx, \text{ for all } q \in X_p, \end{aligned} \quad (4.25)$$

with

$$p^0 = \mathcal{I}_p^{\mathcal{D}} \bar{p}_0. \quad (4.26)$$

We remark that, given (4.23), we deduce the following equality from (4.24), for all $k \in [0, N]$:

$$\begin{aligned} & \int_D \left(2\mu e_u^{\mathcal{D}}(\mathbf{u}^k) + \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} p^k \operatorname{Id} \right) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & = 2 \int_D \mu e_u^{\mathcal{D}}(\mathbf{u}^k) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + \int_D \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & - \alpha \int_D \Pi_p^{\mathcal{D}} p^k \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx = \int_D \mathbf{f}^k \Pi_u^{\mathcal{D}} \mathbf{v}, \text{ for all } \mathbf{v} \in X_u. \end{aligned}$$

The existence and uniqueness of a solution of problem (4.24)-(4.26) will be proved in the next section.

4.2 Existence and uniqueness of the solution of the discrete problem (4.24)-(4.26)

In order to study the discrete problem (4.24)-(4.26) we will apply the undrained iterative split method, which has been introduced by Mikelić and Wheeler in [36]; the proof of the existence and uniqueness of the solution to the gradient scheme (4.24)-(4.26) relies on the Banach-Picard fixed point theorem.

At each time step k , given (\mathbf{u}^k, p^k) , we compute a sequence $(\mathbf{u}^{n,k}, p^{n,k})_{n \in \mathbb{N}}$ such that the limit fixed point of this sequence is set to be $(\mathbf{u}^{k+1}, p^{k+1})$. In what follows we define

$$L := \frac{M\alpha^2}{2}. \quad (4.27)$$

Definition 4.2.1. Let $k \geq 1$ be fixed. First we set $(\mathbf{u}^{0,k}, p^{0,k}) := (\mathbf{u}^k, p^k)$. Then, given $\mathbf{u}^{n,k}$ and $p^{n,k}$, we define the solution of the discrete problem at the iteration $n+1$ as the pair $(\mathbf{u}^{n+1,k}, p^{n+1,k}) \in X_u \times X_p$ solving the following system. First, we define $\mathbf{u}^{n+1,k}$ such that, for all $\mathbf{v} \in X_u$, we have

$$\begin{aligned} & \int_D 2\mu e_u^{\mathcal{D}}(\mathbf{u}^{n+1,k}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + (\lambda + L) \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{n+1,k}) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx = \\ & \alpha \int_D \left(\Pi_p^{\mathcal{D}} p^{n,k} + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{n,k}) \right) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) + \mathbf{f}^k \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx. \end{aligned} \quad (4.28)$$

Then, given $\mathbf{u}^{n+1,k}$, we define $p^{n+1,k}$ such that, for all $q \in X_p$, we have

$$\begin{aligned} \frac{1}{M} \int_D \frac{\Pi_p^{\mathcal{D}}(p^{n+1,k} - p^k)}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx + \alpha \int_D \frac{\operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{n+1,k} - \mathbf{u}^k)}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx \\ + \int_D \kappa \nabla_p^{\mathcal{D}} p^{n+1,k} \cdot \nabla_p^{\mathcal{D}} q \, dx = \int_D g^k \Pi_p^{\mathcal{D}} q \, dx. \end{aligned} \quad (4.29)$$

4.2.1 Well-posedness for the discretized decoupled equations

We first prove that there exists a unique solution of (4.28) and then there exists a unique solution of (4.29).

The displacement equation

We remark that the application $\mathbf{u}, \mathbf{v} \in X_u \mapsto \int_D e_u^{\mathcal{D}}(\mathbf{u}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx$ defines a scalar product on the space X_u .

In Equation (4.28), we remark that the right-hand-side will be a linear map and the left-hand-side is a bilinear form. Thus, it justifies the following definition with $\mathcal{F}^{n,k} \in L^2(D)$ being given as $\alpha \left(\Pi_p^{\mathcal{D}} p^{n,k} + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{n,k}) \right)$.

- $a(\mathbf{u}, \mathbf{v}) = 2 \int_D \mu e_u^{\mathcal{D}}(\mathbf{u}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + (\lambda + L) \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx,$
- $\ell(\mathbf{v}) = \int_D \mathcal{F}^{n,k} \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) + \mathbf{f}^k \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx.$

The problem (4.28) has the form

$$\text{Find } \mathbf{u}^{n+1,k} \in X_u, \text{ such that, for all } \mathbf{v} \in X_u, a(\mathbf{u}^{n+1,k}, \mathbf{v}) = \ell(\mathbf{v}). \quad (4.30)$$

In order to show the continuity of the linear form ℓ , we deduce from the Hölder inequality that

$$\left| \int_D \mathcal{G} \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \right| \leq \|\mathcal{G}\|_{L^2(D)} \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}.$$

Moreover, from Definition 4.1.2, we have $\|\operatorname{div}_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \leq \sqrt{d} \|e_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)}$, and from the coercivity assumptions (4.14) there holds $\|\Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)} \leq C_u \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}$, so that

$$|\ell(v)| \leq (C_u + \sqrt{d} \|\mathcal{G}\|_{L^2(D)}) \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}.$$

Next we show that the bilinear form $a(\mathbf{u}, \mathbf{v})$ is coercive. Indeed, given $\mathbf{v} \in X_u$, we have

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= 2\mu \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}^2 + (\lambda + L) \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}^2 \\ &\geq 2\mu \|\mathbf{v}\|_{L^2(D)}^2. \end{aligned}$$

Finally we prove that the bilinear form $a(\mathbf{u}, \mathbf{v})$ is continuous. Indeed, given $(\mathbf{u}, \mathbf{v}) \in X_u \times X_u$, we have

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &= \left| 2 \int_D \mu e_u^{\mathcal{D}}(\mathbf{u}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + (\lambda + L) \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \right| \\ &\leq 2\mu \|e_u^{\mathcal{D}}(\mathbf{u})\|_{L^2(D)} \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} + (\lambda + L) \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{u})\|_{L^2(D)} \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \\ &\leq (2\mu + d(\lambda + L)) \|e_u^{\mathcal{D}}(\mathbf{u})\|_{L^2(D)} \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \end{aligned}$$

It then follows from Lax-Milgram theorem that there exists a unique solution $\mathbf{u}^{n+1,k} \in X_u$ of the variational problem (4.30); this also implies the existence and uniqueness of the solution $\mathbf{u}^{n+1,k} \in X_u$ of problem (4.28).

The pressure equation

The pressure equation reads as

$$\frac{1}{M} \int_D \Pi_p^{\mathcal{D}} p^{n+1,k} \Pi_p^{\mathcal{D}} q \, dx + \kappa \Delta t \int_D \nabla_p^{\mathcal{D}} p^{n+1,k} \cdot \nabla_p^{\mathcal{D}} q \, dx = \Delta t \int_D \mathcal{F} \Pi_p^{\mathcal{D}} q \, dx,$$

where the source term \mathcal{F} depends on the source term of (4.29) and thus on p^{k-1} and $\mathbf{u}^{n+1,k}$.

We remark that the bilinear application $p, q \in X_p \mapsto \int_D \nabla_p^{\mathcal{D}} p \cdot \nabla_p^{\mathcal{D}} q \, dx$ defines a scalar product on the space X_p . We can then define the following bilinear and linear forms

- $m(p, q) = \frac{1}{M} \int_D \Pi_p^{\mathcal{D}} p \Pi_p^{\mathcal{D}} q \, dx + \kappa \Delta t \int_D \nabla_p^{\mathcal{D}} p \cdot \nabla_p^{\mathcal{D}} q \, dx,$
- $b(q) = \int_D \mathcal{F} \Pi_p^{\mathcal{D}} q \, dx.$

Next, we show that the linear form b is continuous. Indeed, we deduce from the coercivity assumptions (4.7) that

$$|b(q)| \leq \|\mathcal{F}\|_{L^2(D)} \|\Pi_p^{\mathcal{D}} q\|_{L^2(D)} \leq C_p \|\mathcal{F}\|_{L^2(D)} \|\Pi_u^{\mathcal{D}} q\|_{L^2(D)}.$$

Moreover, since

$$\|\Pi_p^{\mathcal{D}} q\|_{L^2(D)}^2 + \Delta t \kappa \|\nabla_p^{\mathcal{D}} q\|_{L^2(D)}^2 \geq C \|\nabla_p^{\mathcal{D}} q\|_{L^2(D)}^2,$$

then the coercivity of the bilinear form $m(p, q)$ follows.

Finally, we deduce from Lax-Milgram theorem that there exists a unique solution $p^{n+1,k}$ of problem (4.29).

4.2.2 Existence and uniqueness of the solution of the discrete problem (4.24)-(4.26)

In this section, we prove that a suitable map defines a strict contraction so that it possesses a unique fixed point which we identify as the solution of (4.24)-(4.26). Thus, given $(\mathbf{u}^{k-1}, p^{k-1})$ at the previous time step $k-1$, we consider the operator

$$\mathcal{S} : (\mathbf{w}, \zeta) \rightarrow (\mathbf{z}, \gamma)$$

such that (\mathbf{z}, γ) is the unique solution of the problem for all $\mathbf{v} \in X_u$ and $q \in X_p$

$$\begin{aligned} & \int_D 2\mu e_u^{\mathcal{D}}(\mathbf{z}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + (\lambda + L) \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & = \alpha \int_D (\Pi_p^{\mathcal{D}} \zeta + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w})) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) + \mathbf{g}^k \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx, \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} & \frac{1}{M} \int_D \frac{(\Pi_p^{\mathcal{D}} \gamma - p^{k-1})}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx + \alpha \int_D \frac{\operatorname{div}_u^{\mathcal{D}}(\mathbf{z} - \mathbf{u}^{k-1})}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx \\ & + \int_D \kappa \nabla_p^{\mathcal{D}} \gamma \cdot \nabla_p^{\mathcal{D}} q \, dx = \int_D f^k \Pi_p^{\mathcal{D}} q \, dx, \end{aligned} \quad (4.32)$$

which is given by solving the iterative variational problem (4.28)-(4.29).

We define the distance on the space $X_u \times X_p$ by

$$\begin{aligned} d^2((\mathbf{u}, p), (\mathbf{v}, q)) & = 4\mu L \|e_u^{\mathcal{D}}(\mathbf{u} - \mathbf{v})\|_{L^2(D)}^2 + \lambda L \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{u} - \mathbf{v})\|_{L^2(D)}^2 \\ & + 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(p - q)\|_{L^2(D)}^2 + \|\alpha \Pi_p^{\mathcal{D}}(p - q) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{u} - \mathbf{v})\|_{L^2(D)}^2. \end{aligned} \quad (4.33)$$

Theorem 4.2.1. *The mapping operator $\mathcal{S} : X_u \times X_p \rightarrow X_u \times X_p$ defines a strict contraction so that it admits a unique fixed point in $X_u \times X_p$, which coincides with the unique solution of the problem (4.24)-(4.25).*

Proof. For $(\mathbf{w}_1, \zeta_1) \in X_u \times X_p$ and $(\mathbf{w}_2, \zeta_2) \in X_u \times X_p$, we define (\mathbf{z}_1, γ_1) and (\mathbf{z}_2, γ_2) as

$$\mathcal{S}(\mathbf{w}_1, \zeta_1) = (\mathbf{z}_1, \gamma_1), \quad \mathcal{S}(\mathbf{w}_2, \zeta_2) = (\mathbf{z}_2, \gamma_2). \quad (4.34)$$

Since (\mathbf{z}_1, γ_1) and (\mathbf{z}_2, γ_2) satisfy the equations (4.31)-(4.32), we have that

$$\begin{aligned} & \int_D 2\mu e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx + (\lambda + L) \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ &= \int_D (\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)) \operatorname{div}_u^{\mathcal{D}}(\mathbf{v}) \, dx, \end{aligned} \quad (4.35)$$

and

$$\begin{aligned} & \frac{1}{M} \int_D \frac{\Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1)}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx + \alpha \int_D \frac{\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)}{\Delta t} \Pi_p^{\mathcal{D}} q \, dx \\ & \quad + \int_D \kappa \nabla(\gamma_2 - \gamma_1) \cdot \nabla q \, dx = 0. \end{aligned} \quad (4.36)$$

We set $\mathbf{v} = \mathbf{z}_2 - \mathbf{z}_1$, and deduce from (4.35) and Young's inequality that for any $\varepsilon > 0$

$$\begin{aligned} & 2\mu \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + (\lambda + L) \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\ &= \int_D \left(\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1) \right) \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \, dx \\ &\leq \frac{1}{2\varepsilon} \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2 + \frac{\varepsilon}{2} \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2. \end{aligned} \quad (4.37)$$

Thus

$$\begin{aligned} & 2\mu \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + \left(\lambda + L - \frac{\varepsilon}{2} \right) \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\ & \leq \frac{1}{2\varepsilon} \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2. \end{aligned} \quad (4.38)$$

We take $\varepsilon = \lambda + L$ in (4.38) and multiply this inequality by $2L$ which gives

$$\begin{aligned} & 4\mu L \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + (\lambda L + L^2) \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\ & \leq \frac{L}{\lambda + L} \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2. \end{aligned} \quad (4.39)$$

Setting $q = 2L\Delta t(\gamma_2 - \gamma_1)$ in (4.36) yields

$$\begin{aligned} & \frac{2L}{M} \|\Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + 2L\alpha \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) \, dx \\ & \quad + 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 = 0. \end{aligned} \quad (4.40)$$

Adding (4.39) and (4.40), we obtain

$$\begin{aligned} & 4\mu L \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + \lambda L \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\ & \quad + 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + L^2 \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\ & + \frac{2L}{M} \|\Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + 2L\alpha \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) \, dx \\ & \leq \frac{L}{\lambda + L} \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2. \end{aligned} \quad (4.41)$$

We remark that

$$\begin{aligned}
& \|\alpha \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&= \alpha^2 \|\Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + L^2 \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&\quad + 2L\alpha \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) \, dx \\
&= \frac{2L}{M} \|\Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + L^2 \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&\quad + 2L\alpha \int_D \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1) \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) \, dx.
\end{aligned} \tag{4.42}$$

where we have use (4.27).

Substituting (4.42) into (4.41) yields

$$\begin{aligned}
& 4\mu L \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + \lambda L \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&+ 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + \|\alpha \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&\leq \frac{L}{\lambda + L} \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2.
\end{aligned} \tag{4.43}$$

Thus, adding positive terms on the right-hand-side, we obtain

$$\begin{aligned}
& 4\mu L \|e_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 + \lambda L \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&+ 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(\gamma_2 - \gamma_1)\|_{L^2(D)}^2 + \|\alpha \Pi_p^{\mathcal{D}}(\gamma_2 - \gamma_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{z}_2 - \mathbf{z}_1)\|_{L^2(D)}^2 \\
&\leq \frac{L}{\lambda + L} \left(4\mu L \|e_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2 + \lambda L \|\operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2 \right. \\
&\quad \left. + 2\Delta t \kappa L \|\nabla_p^{\mathcal{D}}(\zeta_2 - \zeta_1)\|_{L^2(D)}^2 + \|\alpha \Pi_p^{\mathcal{D}}(\zeta_2 - \zeta_1) + L \operatorname{div}_u^{\mathcal{D}}(\mathbf{w}_2 - \mathbf{w}_1)\|_{L^2(D)}^2 \right)
\end{aligned} \tag{4.44}$$

which in view of (4.33) can be rewritten as

$$d^2(\mathcal{S}(\mathbf{w}_2, \zeta_2), \mathcal{S}(\mathbf{w}_1, \zeta_1)) \leq \frac{L}{\lambda + L} d^2((\mathbf{w}_2, \zeta_2), (\mathbf{w}_1, \zeta_1)). \tag{4.45}$$

Thus \mathcal{S} define a strict contraction on $X_u \times X_p$, so that it possesses a unique fixed point.

We concludes that given $(\mathbf{u}^{k-1}, p^{k-1})$ we can define (\mathbf{u}^k, p^k) as the unique fixed point of \mathcal{S} , which completes the proof. \square

4.3 Error estimates

4.3.1 Error estimate for the GDM

In this section, we obtain an error estimate between the exact solution $(\bar{\mathbf{u}}, \bar{p})$ of problem (\mathbb{P}) and the approximate solution (\mathbf{u}, p) obtained by means of a Gradient Discretization Method. As far as we know, these results are new since these error estimates have been obtained for special numerical methods (Coulet Giraud et Almani et al.), whereas we develop an error estimate for a Gradient Discretization Method that includes all these Finite Element and Generalized Finite Volume methods. Moreover, we obtained previously enough regularity on the continuous solution, so that we don't have to make the usual assumption about existence of a smooth solution.

We define the following initial interpolation errors

$$E_u^{ini} := \|e(\bar{\mathbf{u}}_0) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}_0\|_{L^2(D)} + \|\operatorname{div}(\bar{\mathbf{u}}_0) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}_0\|_{L^2(D)}; \quad (4.46)$$

$$E_p^{ini} := \|\bar{p}_0 - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}_0\|_{L^2(D)}. \quad (4.47)$$

and we suppose that $p_0 \in H^2(D)$ and $f \in H^1(0, T; L^2(D))$.

In the sequel, we will prove the following error estimate.

Theorem 4.3.1. *Let assume Hypothesis 4. Let $(\bar{\mathbf{u}}, \bar{p})$ be the weak solution of problem (\mathbb{P}) , and (\mathbf{u}, p) obtained by means of a Gradient Discretization Method defined in (4.24)-(4.26). Then it holds:*

$$\begin{aligned} & \|e(\bar{\mathbf{u}}) - e_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T);L^2(D))}^2 + \|\operatorname{div}(\bar{\mathbf{u}}) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u})\|_{L^\infty((0,T);L^2(D))}^2 \\ & + \|\bar{p} - \Pi_p^{\mathcal{D}} p\|_{L^\infty((0,T);L^2(D))}^2 + \|\nabla \bar{p} - \nabla_p^{\mathcal{D}} p\|_{L^2((0,T);L^2(D))}^2 \\ & \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_p^{ini})^2 + (E_u^{ini})^2) \end{aligned} \quad (4.48)$$

where E_p^{ini} , E_u^{ini} are defined in (4.46)-(4.47), and the constant C does not depend on $h_{\mathcal{D}}$ and Δt .

In the following, Theorem 4.3.1 will be proved by means of several successive lemmas, that we will described below. But first, we recall some known results about numerical error estimates for such problems. In [1], [22] and [34], the authors have obtained error estimates for other numerical methods.

Concerning the GDM, remark that the following result about control of consistency of the approximations holds.

Lemma 4.3.1. *Let $\varphi \in H_0^1(D)^d$, there holds*

$$\left(\|\varphi - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 + \|e(\varphi) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \leq S_u^{\mathcal{D}}(\varphi), \quad (4.49)$$

and

$$\left(\|\varphi - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 + \|\operatorname{div}(\varphi) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \leq \sqrt{d} S_u^{\mathcal{D}}(\varphi). \quad (4.50)$$

Let $\psi \in H_0^1(D)$, there holds

$$\left(\|\psi - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \psi\|_{L^2(D)}^2 + \|\nabla \psi - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \psi\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \leq S_p^{\mathcal{D}}(\psi). \quad (4.51)$$

Proof. To prove the inequality (4.49), we recall the definition of $S_u^{\mathcal{D}}(\varphi)$:

$$S_u^{\mathcal{D}}(\varphi) = \min_{\mathbf{v} \in X_u} \left\{ \|\varphi - \Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)} + \|e(\varphi) - e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} \right\}.$$

By definition of $\mathcal{I}_u^{\mathcal{D}} \varphi$ in (4.2), for all $\mathbf{v} \in X_u$, we have that

$$\begin{aligned} & \left(\|\Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi - \varphi\|_{L^2(D)}^2 + \|e(\varphi) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \\ & \leq \left(\|\Pi_u^{\mathcal{D}} \mathbf{v} - \varphi\|_{L^2(D)}^2 + \|e_u^{\mathcal{D}}(\mathbf{v}) - e(\varphi)\|_{L^2(D)}^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and in particular for $\mathbf{v} \in X_u$ minimizing the quantity $S_u^{\mathcal{D}}$.

$$\begin{aligned} & (\|\varphi - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2 + \|e(\varphi) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi\|_{L^2(D)}^2)^{\frac{1}{2}} \\ & \leq (\|\varphi - \Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)}^2 + \|e(\varphi) - e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}^2)^{\frac{1}{2}}. \end{aligned} \quad (4.52)$$

Since all the quantities in (4.52) are positive we deduce

$$\begin{aligned} & (\|\varphi - \Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)}^2 + \|e(\varphi) - e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}^2)^{\frac{1}{2}} \\ & \leq (\|\varphi - \Pi_u^{\mathcal{D}} \mathbf{v}\|_{L^2(D)} + \|e(\varphi) - e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}) \leq S_u^{\mathcal{D}}(\varphi) \end{aligned} \quad (4.53)$$

and (4.49) immediately follows.

From (4.49) we apply trace operator and recall that $\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi = \operatorname{Tr}(e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \varphi)$ and $\operatorname{div}(\varphi) = \operatorname{Tr}(e(\varphi))$ and from the same computations performed in (3.6), we have the result (4.50). Similarly, the inequality (4.51) follows from the definitions in (4.1) and (4.8). \square

4.3.2 Interpolation inequalities for the exact solution $(\bar{\mathbf{u}}, \bar{p})$.

First, we present the time-space approximation of the true displacement.

Lemma 4.3.2. *There exists two constants C_1, C_2 such that*

$$\begin{aligned} & \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \\ & + \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \leq C_1(\Delta t + h_{\mathcal{D}}), \end{aligned} \quad (4.54a)$$

$$\left\| \frac{e(\bar{\mathbf{u}}(t^{k+1})) - e(\bar{\mathbf{u}}(t^k))}{\Delta t} - \frac{e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^k))}{\Delta t} \right\|_{L^2(D)} \leq C_2 h_{\mathcal{D}}, \quad (4.54b)$$

and

$$\left\| \frac{\operatorname{div}(\bar{\mathbf{u}}(t^{k+1})) - \operatorname{div}(\bar{\mathbf{u}}(t^k))}{\Delta t} - \frac{\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)}{\Delta t} \right\|_{L^2(D)} \leq C_2 h_{\mathcal{D}}. \quad (4.54c)$$

Proof. We recall from Theorem 3.5.2 that the solution $\bar{\mathbf{u}}$ belongs to $C^1([0, T]; H_0^1(D))$. We remark that:

$$\begin{aligned} & \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \\ & \leq \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} + \|\bar{\mathbf{u}}(t^{k+1}) - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1})\|_{L^2(D)} \end{aligned} \quad (4.55)$$

and

$$\begin{aligned} & \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \\ & \leq \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e(\bar{\mathbf{u}}(t^{k+1})) \right\|_{L^2(D)} + \|e(\bar{\mathbf{u}}(t^{k+1})) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1})\|_{L^2(D)}. \end{aligned} \quad (4.56)$$

It follows from (4.49) that

$$\begin{aligned}
& \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} + \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \\
& \leq \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} + \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e(\bar{\mathbf{u}}(t^{k+1})) \right\|_{L^2(D)} \\
& \quad + \|\bar{\mathbf{u}}(t^{k+1}) - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1})\|_{L^2(D)} + \|e(\bar{\mathbf{u}}(t^{k+1})) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1})\|_{L^2(D)} \\
& \leq \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} + \left\| \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e(\bar{\mathbf{u}}(t^{k+1})) \right\|_{L^2(D)} + 2 S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})).
\end{aligned} \tag{4.57}$$

Two possibilities occurs :

- if $\bar{\mathbf{u}}(t^{k+1}) = 0$, then we have that $S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})) = 0$ so that in particular $S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})) \leq h_u^{\mathcal{D}}$,
- else, in view of (4.20) we have that

$$\begin{aligned}
S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})) &= \frac{S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1}))}{\|\bar{\mathbf{u}}(t^{k+1})\|_{H_0^1(D)}} \|\bar{\mathbf{u}}(t^{k+1})\|_{H_0^1(D)} \\
&\leq h_u^{\mathcal{D}} \|\bar{\mathbf{u}}(t^{k+1})\|_{H_0^1(D)}.
\end{aligned}$$

Then using Definition 4.1.8 and Theorem 3.5.2, there exists a constant C_1 such that

$$S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1})) \leq C_1 h_u^{\mathcal{D}}/2. \tag{4.58}$$

The regularity properties of $\bar{\mathbf{u}}$ and $e(\bar{\mathbf{u}})$ in time of Theorem 3.5.2 imply that :

$$\begin{aligned}
& \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \leq C_1 \Delta t/2, \\
& \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e(\bar{\mathbf{u}}(t^{k+1})) \right\|_{L^2(D)} \leq C_1 \Delta t/2.
\end{aligned}$$

Finally from (4.57) we have

$$\begin{aligned}
& \left\| \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt - \Pi_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} + \\
& \left\| e \left(\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \bar{\mathbf{u}}(t) dt \right) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) \right\|_{L^2(D)} \leq C_1(\Delta t + h_u^{\mathcal{D}}) \leq C_1(\Delta t + h_{\mathcal{D}}),
\end{aligned}$$

which completes the proof of (4.54a).

Since $\partial_t \bar{\mathbf{u}} \in C(0, T; L^2(D))$ we have

$$\frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} \partial_t \bar{\mathbf{u}}(t) dt = \frac{1}{\Delta t} \left(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k) \right).$$

and so by linearity of the operator $\mathcal{I}_u^{\mathcal{D}}$ defined in (4.2), we have in view of (4.15) and Lemma 4.3.1, inequality (4.50)

$$\begin{aligned} & \left\| \frac{\operatorname{div} \bar{\mathbf{u}}(t^{k+1}) - \operatorname{div} \bar{\mathbf{u}}(t^k)}{\Delta t} - \frac{\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)}{\Delta t} \right\|_{L^2(D)} \\ &= \left\| \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \right\|_{L^2(D)} \\ &\leq \sqrt{d} S_u^{\mathcal{D}} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right). \end{aligned}$$

which in view of (4.58) completes the proof of (4.54c). The proof of (4.54b) follows in the same way. \square

Next we present our result for the flow interpolation.

Lemma 4.3.3. *There exists two constants C_1, C_2 such that*

$$\begin{aligned} & \left\| \int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \\ &+ \left\| \nabla \left(\int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt \right) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \leq C_1 (\Delta t + h_{\mathcal{D}}). \end{aligned} \quad (4.59a)$$

and

$$\left\| \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right\| \leq C_2 h_{\mathcal{D}}. \quad (4.59b)$$

Proof. We follow the proof of Lemma 4.3.2. Theorem 3.5.2 ensures that \bar{p} belongs to $C^1([0, T]; H_0^1(D))$. We also have

$$\begin{aligned} \left\| \int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} &\leq \left\| \int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt - \bar{p}(t^{k+1}) \right\|_{L^2(D)} \\ &+ \left\| \bar{p}(t^{k+1}) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \end{aligned}$$

and

$$\begin{aligned} \left\| \nabla \left(\int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt \right) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} &\leq \left\| \nabla \left(\int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt \right) - \nabla \bar{p}(t^{k+1}) \right\|_{L^2(D)} \\ &+ \left\| \nabla \bar{p}(t^{k+1}) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)}. \end{aligned}$$

We remark by (4.51) that

$$\left\| \bar{p}(t^{k+1}) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} + \left\| \nabla \bar{p}(t^{k+1}) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \leq 2S_p^{\mathcal{D}}(\bar{p}(t^{k+1}))$$

Two possibilities occurs :

- if $\bar{p}(t^{k+1}) = 0$, then we have that $S_p^{\mathcal{D}}(\bar{p}(t^{k+1})) = 0$ so that in particular $S_p^{\mathcal{D}}(\bar{p}(t^{k+1})) \leq h_p^{\mathcal{D}}$,
- else, in view of (4.13) we have that

$$\begin{aligned} S_p^{\mathcal{D}}(\bar{p}(t^{k+1})) &= \frac{S_p^{\mathcal{D}}(\bar{p}(t^{k+1}))}{\|\bar{p}(t^{k+1})\|_{H_0^1(D)}} \|\bar{p}(t^{k+1})\|_{H_0^1(D)} \\ &\leq h_p^{\mathcal{D}} \|\bar{p}(t^{k+1})\|_{H_0^1(D)}. \end{aligned}$$

In view of Definition 4.1.8 and Theorem 3.5.2, there exists a constant C_1 such that

$$S_p^{\mathcal{D}}(\bar{p}(t^{k+1})) \leq C_1 h_p^{\mathcal{D}}/2. \quad (4.60)$$

We conclude, applying again Theorem 3.5.2, that

$$\begin{aligned} & \left\| \int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \\ + & \left\| \nabla \left(\int_{t^k}^{t^{k+1}} \frac{\bar{p}(t)}{\Delta t} dt \right) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right\|_{L^2(D)} \leq C_1 (\Delta t + h_p^{\mathcal{D}}), \end{aligned}$$

which completes the proof of (4.59a). Moreover, by (4.51) and linearity of the operator $\mathcal{I}_p^{\mathcal{D}}$, there holds

$$\begin{aligned} & \left\| \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right\|_{L^2(D)} \\ & \leq S_p^{\mathcal{D}} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right). \end{aligned}$$

In view of Theorem 3.5.2, $\|\partial_t \nabla \bar{p}\|_{H_0^1(D)}$ is bounded. Thus there exists a positive constant C_2 such that

$$\left\| \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right\|_{H_0^1(D)} \leq C_2,$$

and by the same arguments as in (4.60) we conclude that

$$S_p^{\mathcal{D}} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right) \leq C_2 h_p^{\mathcal{D}},$$

which completes the proof (4.59b). \square

4.3.3 Error estimates

We note

$$\varepsilon_u^k := \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - \mathbf{u}^k, \quad (4.61)$$

$$\varepsilon_p^k := \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) - p^k. \quad (4.62)$$

the interpolation error between the continuous solution and the numerical solution defined in Definition 4.2.1.

We will apply the Abel's formula given by

$$\sum_{k=1}^N a_k (b_k - b_{k-1}) = - \sum_{k=1}^{N-1} (a_{k+1} - a_k) b_k + a_N b_N - a_1 b_0. \quad (4.63)$$

Moreover, to simplify the notations we define

$$\sigma(\bar{\mathbf{u}}, \bar{p}) := 2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}, \quad (4.64)$$

and

$$\sigma_{u,p}^{\mathcal{D}}(\mathbf{u}, p) := 2\mu e_u^{\mathcal{D}}(\mathbf{u}) + \lambda \left(\operatorname{div}_u^{\mathcal{D}}(\mathbf{u}) \right) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} p \operatorname{Id}. \quad (4.65)$$

We have by (3.5) and (4.23):

$$\begin{aligned}
& \int_D \left(2\mu e(\bar{\mathbf{u}}(t^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right) dx \\
& - \int_D \left(2\mu e_u^{\mathcal{D}}(\mathbf{u}^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) + \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right) dx \\
& - \left\{ \alpha \int_D \bar{p}(t^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx - \alpha \int_D \Pi_p^{\mathcal{D}} p^k \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \right\} \\
= & 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \\
& + \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \\
& - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx
\end{aligned} \tag{4.66}$$

We sum on k the right-hand-side of (4.66) and apply the Abel formula (4.63) with

$$\begin{aligned}
a_k &= 2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} \\
&= \sigma(\bar{\mathbf{u}}(t^k), \bar{p}(t^k)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^k, p^k),
\end{aligned}$$

where σ and $\sigma_{u,p}^{\mathcal{D}}$ have been defined in (4.64)-(4.65), and

$$b_k = e_u^{\mathcal{D}}(\varepsilon_u^k)$$

to obtain

$$\begin{aligned}
& \sum_{k=1}^N 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \\
& + \sum_{k=1}^N \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \\
& - \sum_{k=1}^N \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \\
= & - \sum_{k=1}^{N-1} 2 \int_D \mu (e(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k) dx \\
& - \sum_{k=1}^{N-1} \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) dx \\
& + \sum_{k=1}^N \alpha \int_D (\bar{p}(t^{k+1}) - \bar{p}(t^k) - \Pi_p^{\mathcal{D}}(p^{k+1} - p^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) dx \\
& + \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) dx \\
& - \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) dx.
\end{aligned} \tag{4.67}$$

In what follows, we will obtain a bound for the quantity

$$\begin{aligned} \mathcal{E} &= \sum_{k=1}^{N-1} 2 \int_D \mu (e(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &+ \sum_{k=1}^{N-1} \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &\quad - \sum_{k=1}^{N-1} \alpha \int_D (\bar{p}(t^{k+1}) - \bar{p}(t^k) - \Pi_p^{\mathcal{D}}(p^{k+1} - p^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx. \end{aligned}$$

We define

$$\begin{aligned} \mathcal{E}_k &= 2 \int_D \mu (e(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &+ \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &\quad - \alpha \int_D (\bar{p}(t^{k+1}) - \bar{p}(t^k) - \Pi_p^{\mathcal{D}}(p^{k+1} - p^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \end{aligned} \quad (4.68)$$

so that by (4.67)

$$\begin{aligned} &\sum_{k=1}^N 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ &\quad + \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ &\quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ &= \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ &\quad - \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx - \sum_{k=1}^{N-1} \mathcal{E}_k. \end{aligned}$$

Lemma 4.3.4. *For all $1 \leq k \leq N$, the general term of the sequence \mathcal{E} satisfies the following inequalities*

$$|\mathcal{E}_k| \leq C \Delta t h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}, \quad (4.69)$$

and

$$|\mathcal{E}| \leq C \sum_{k=1}^{N-1} \Delta t h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}, \quad (4.70)$$

with a constant C which only depends on T .

Proof. We remark that (4.68) can be written as

$$\begin{aligned} \mathcal{E}_k &= \Delta t \int_D \left(2\mu e \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) + \lambda \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &\quad - \Delta t \int_D \alpha \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &\quad - \Delta t \int_D \left(2\mu e_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) + \lambda \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &\quad + \Delta t \int_D \alpha \Pi_p^{\mathcal{D}} \frac{p^{k+1} - p^k}{\Delta t} \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx. \end{aligned}$$

In order to prove Lemma 4.3.4, we will search a suitable estimate for the terms

$$\begin{aligned} & \int_D \left(2\mu e \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) + \lambda \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \operatorname{Id} \right. \\ & \quad \left. - \alpha \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ & - \int_D \left(2\mu e_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) + \lambda \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \operatorname{Id} \right. \\ & \quad \left. - \alpha \Pi_p^{\mathcal{D}} \frac{p^{k+1} - p^k}{\Delta t} \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \end{aligned} \quad (4.71)$$

We recall the definition of $W_u^{\mathcal{D}}$, used in the limit conformity properties (4.17). For $\varphi \in H_{\operatorname{div}}(D)$, we have

$$W_u^{\mathcal{D}}(\varphi) = \max_{\mathbf{v} \in X_u} \frac{1}{\|\mathbf{v}\|_u} \left| \int_D (e_u^{\mathcal{D}}(\mathbf{v})(x) : \varphi(x) + \Pi_u^{\mathcal{D}} \mathbf{v}(x) \cdot \operatorname{div} \varphi(x)) \, dx \right|.$$

We set

$$\begin{aligned} \bar{\varphi}_k &= 2\mu e \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) + \lambda \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \operatorname{Id} \\ & - \alpha \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \operatorname{Id}. \end{aligned} \quad (4.72)$$

It follows from the partial differential equation for $\bar{\mathbf{u}}$ that almost everywhere in D

$$-\operatorname{div}(\bar{\varphi}_k) = \frac{\mathbf{f}(t^{k+1}) - \mathbf{f}(t^k)}{\Delta t}. \quad (4.73)$$

Since (\mathbf{U}, P) satisfies the displacement equation (4.24), for all k there holds that, for all \mathbf{v} in X_u ,

$$\begin{aligned} - \int_D \operatorname{div}(\bar{\varphi}_k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx &= \int_D \frac{\mathbf{f}(t^{k+1}) - \mathbf{f}(t^k)}{\Delta t} \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx \\ &= \int_D \left(2\mu e_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) + \lambda \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & - \alpha \int_D \left(\Pi_p^{\mathcal{D}} \frac{p^{k+1} - p^k}{\Delta t} \operatorname{Id} \right) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \end{aligned} \quad (4.74)$$

Moreover, since $\bar{\varphi}_k$ verify the equality (4.73), for all \mathbf{v} in X_u , we have

$$\int_D \bar{\varphi}_k : e_u^{\mathcal{D}}(\mathbf{v}) - \frac{\mathbf{f}(t^{k+1}) - \mathbf{f}(t^k)}{\Delta t} \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx = \int_D (\bar{\varphi}_k : e_u^{\mathcal{D}}(\mathbf{v}) + \operatorname{div}(\bar{\varphi}_k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v}) \, dx \quad (4.75)$$

and that the quantity $W_u^{\mathcal{D}}(\bar{\varphi}_k)$ is well defined. From the definition of $W_u^{\mathcal{D}}$ there holds

$$\int_D \operatorname{div}(\bar{\varphi}_k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} + \bar{\varphi}_k : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \leq W_u^{\mathcal{D}}(\bar{\varphi}_k) \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \quad (4.76)$$

Next we look for a suitable bound for $W_u^{\mathcal{D}}(\bar{\varphi}_k)$. We have that

$$W_u^{\mathcal{D}}(\bar{\varphi}_k) = \frac{W_u^{\mathcal{D}}(\bar{\varphi}_k)}{\|\bar{\varphi}_k\|_{H_{\operatorname{div}}(D)}} \|\bar{\varphi}_k\|_{H_{\operatorname{div}}(D)},$$

which in view of (4.19), (4.20), (4.21) implies that

$$W_u^{\mathcal{D}}(\bar{\varphi}_k) \leq h_{\mathcal{D}} \|\bar{\varphi}_k\|_{H_{\operatorname{div}}(D)}. \quad (4.77)$$

Substituting $\mathbf{v} = \varepsilon_u^k$ in (4.76) leads, for all k , to

$$\int_D (\operatorname{div}(\bar{\varphi}_k) \cdot \Pi_u^{\mathcal{D}} \varepsilon_u^k + \bar{\varphi}_k : e_u^{\mathcal{D}}(\varepsilon_u^k)) \, dx \leq h_{\mathcal{D}} \|\bar{\varphi}_k\|_{H_{\operatorname{div}}(D)} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}. \quad (4.78)$$

We remark that by (3.36), $\bar{\varphi}_k$ is uniformly bounded in $H_{\operatorname{div}}(D)$. Thus there exists a constant C which only depends on T , such that

$$\int_D (\operatorname{div} \bar{\varphi}_k \cdot \Pi_u^{\mathcal{D}} \varepsilon_u^k + \bar{\varphi}_k : e_u^{\mathcal{D}}(\varepsilon_u^k)) \, dx \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}, \text{ for all } k. \quad (4.79)$$

Thus by substituting (4.74) and (4.75) in (4.79) leads to

$$\begin{aligned} & - \int_D \left(2\mu e_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k) + \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^{k+1} - \mathbf{u}^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}}(p^{k+1} - p^k) \operatorname{Id} \right) \\ & \quad : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx + \Delta t \int_D \bar{\varphi}_k : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \leq C \Delta t h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}. \end{aligned} \quad (4.80)$$

From (3.5), (4.23) we recognize the term \mathcal{E}_k on the left-hand-side of (4.80) which completes the proof of (4.69) and (4.70). \square

To obtain an estimate of the right-hand-side of (4.67), we also have to estimate the term

$$\begin{aligned} & \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ & - \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx, \end{aligned} \quad (4.81)$$

which in view of (4.64)-(4.65) can be rewritten as

$$\begin{aligned} & \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ & - \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \\ & = \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}}(\mathbf{u}^N)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^N)) \operatorname{Id} \right) e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ & \quad - \alpha \int_D (\bar{p}(t^N) - \Pi_p^{\mathcal{D}} p^N) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ & - \left\{ \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^1)) - e_u^{\mathcal{D}}(\mathbf{u}^1)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^1)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^1)) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \right. \\ & \quad \left. - \alpha \int_D (\bar{p}(t^1) - \Pi_p^{\mathcal{D}} p^1) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \right\}. \end{aligned}$$

More generally, we will prove the following result.

Lemma 4.3.5. *For all $k \in \{1, \dots, N\}$, for all $\mathbf{v} \in X_w$ there exists a constant C such that*

$$\begin{aligned} & \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\mathbf{v}) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \right) e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \end{aligned} \quad (4.82)$$

Proof. We recall that by (3.36), (3.37), (3.38) and (3.39),

- $2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id} \in H_{\operatorname{div}}(D)$ for all k ,
- $2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div} \bar{\mathbf{u}} \operatorname{Id} - \alpha \bar{p} \operatorname{Id} \in C^1([0, T]; L^2(D))$.

We follow the same steps as in the proof of Lemma 4.3.4. We recall that almost everywhere in D

$$-\operatorname{div} (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) = \mathbf{f}(t^k). \quad (4.83)$$

Since $(\bar{\mathbf{u}}, \bar{p})$ is the solution to the discrete displacement equation (4.24), it follows from (4.83) that for all $\mathbf{v} \in X_u$

$$\begin{aligned} & \int_D \operatorname{div}(2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx = - \int_D \mathbf{f}(t^k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx \\ & = - \int_D (2\mu e_u^{\mathcal{D}}(\mathbf{u}^k) + \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} p^k \operatorname{Id}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx. \end{aligned} \quad (4.84)$$

Thus we have that for all $\mathbf{v} \in X_u$

$$\begin{aligned} & \int_D (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & = \int_D (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx - \int_D \mathbf{f}(t^k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx \\ & \quad + \int_D \operatorname{div} (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) \cdot \Pi_u^{\mathcal{D}} \mathbf{v} \, dx \\ & \leq W_u^{\mathcal{D}} \left(2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id} \right) \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)} - \int_D \mathbf{f}(t^k) \cdot \Pi_u^{\mathcal{D}} \mathbf{v}. \end{aligned} \quad (4.85)$$

Substituting (4.84) into (4.85) leads to the following equation

$$\begin{aligned} & \int_D (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & - \int_D (2\mu e_u^{\mathcal{D}}(\mathbf{u}^k) + \lambda \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} p^k \operatorname{Id}) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \leq W_u^{\mathcal{D}} (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div} \bar{\mathbf{u}}(t^k) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \end{aligned} \quad (4.86)$$

By rearranging the terms in (4.86), we have that

$$\begin{aligned} & \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div} \bar{\mathbf{u}}(t^k) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \leq W_u^{\mathcal{D}} (2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div} \bar{\mathbf{u}}(t^k) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id}) \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \end{aligned}$$

In view of (4.22) and (4.23), and where we have used the regularity results in Theorem 4.1.1, we have

$$\begin{aligned} & \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} : e_u^{\mathcal{D}}(\mathbf{v}) \, dx \\ & \leq h_{\mathcal{D}} \|2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \operatorname{Id} - \alpha \bar{p} \operatorname{Id}\|_{L^\infty(0, T; H_{\operatorname{div}}(D))} \|e_u^{\mathcal{D}}(\mathbf{v})\|_{L^2(D)}. \end{aligned}$$

□

We set $k = 1$ and $\mathbf{v} = \varepsilon_u^0$ in (4.82) which gives:

$$\begin{aligned} & \int_D \left(2\mu(e(\bar{\mathbf{u}}(t^1)) - e_u^{\mathcal{D}}(\mathbf{u}^1)) + \lambda(\operatorname{div}(\bar{\mathbf{u}}(t^1)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^1)) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \\ & - \alpha \int_D (\bar{p}(t^1) - \Pi_p^{\mathcal{D}} p^1) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}. \end{aligned} \quad (4.87)$$

Then, with $k = N$ and $\mathbf{v} = \varepsilon_u^N$, from (4.82) we have:

$$\begin{aligned} & 2 \int_D \mu(e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}}(\mathbf{u}^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) + \lambda(\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^N)) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^N) \\ & - \alpha (\bar{p}(t^N) - \Pi_p^{\mathcal{D}} p^N) \operatorname{Id} : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}. \end{aligned} \quad (4.88)$$

We can now conclude by the following Lemma,

Lemma 4.3.6. *There exists a positive constant C such that the following inequality holds*

$$\begin{aligned} & \sum_{k=1}^N 2 \int_D \mu(e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & + \int_D \lambda(\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & \leq Ch_{\mathcal{D}} \left(\sum_{k=1}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \right). \end{aligned} \quad (4.89)$$

Proof. From Lemma 4.3.4, we have the following result

$$|\mathcal{E}| \leq C \sum_{k=1}^{N-1} \Delta t h_{\mathcal{D}} \|\varepsilon_u^k\|.$$

From Lemma 4.3.5, we have the following result

$$\begin{aligned} & \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}, \\ & \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \leq Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}. \end{aligned}$$

We recall from (3.5), (4.23) and (4.67)

$$\begin{aligned} & \sum_{k=1}^N 2 \int_D \mu(e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & + \int_D \lambda(\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & = -\mathcal{E} + \int_D (\sigma(\bar{\mathbf{u}}(t^N), \bar{p}(t^N)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^N, p^N)) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ & - \int_D (\sigma(\bar{\mathbf{u}}(t^1), \bar{p}(t^1)) - \sigma_{u,p}^{\mathcal{D}}(\mathbf{u}^1, p^1)) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx, \end{aligned}$$

so that it complete the proof of Lemma 4.3.6. \square

Thus the left-hand-side of (4.67) is bounded. let us remark that, given (4.46) and (4.47), it follows that

$$\begin{aligned} e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k) &= \left(e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) \right) \\ &\quad + (e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - e_u^{\mathcal{D}}(\mathbf{u}^k)), \end{aligned} \quad (4.90a)$$

$$\begin{aligned} \operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) &= \left(\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) \right) \\ &\quad + (\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)), \end{aligned} \quad (4.90b)$$

$$\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k = \left(\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) \right) + (\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k). \quad (4.90c)$$

To obtain an error estimate, we will get a suitable estimate of the second terms on the right-hand-side of (4.90a),(4.90b) and (4.90c) and bound the first terms of (4.90a),(4.90b) and (4.90c) which amounts to prove interpolations inequalities.

Thus, we substitute (4.90a),(4.90b) and (4.90c) in the left hand side of (4.67) to obtain

$$\begin{aligned} &\sum_{k=1}^N \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \right. \\ &\quad \left. - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx = \\ &\sum_{k=1}^N \int_D (2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} \\ &\quad - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id}) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ &\quad + \\ &\sum_{k=1}^N \int_D (2\mu (e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \\ &\quad - \alpha (\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id}) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \end{aligned} \quad (4.91)$$

and substitution (4.61), (4.62) in (4.91) we have

$$\begin{aligned} &\sum_{k=1}^N \int_D (2\mu e_u^{\mathcal{D}}(\varepsilon_u^k) + \lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \operatorname{Id}) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) = \\ &\sum_{k=1}^N \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)) \operatorname{Id} \right. \\ &\quad \left. - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k) \operatorname{Id} \right) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ &- \sum_{k=1}^N \int_D (2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) - (\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} \\ &\quad - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id}) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx. \end{aligned} \quad (4.92)$$

Since the first term on the right-hand-side of (4.92) is bounded by Lemma 4.3.6, it remains to bound the second

term, namely

$$\begin{aligned}
& \sum_{k=1}^N \int_D \left(2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} \right. \\
& \quad \left. - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id} \right) : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
&= \sum_{k=1}^N 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad + \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx.
\end{aligned} \tag{4.93}$$

We prove the following interpolation inequality.

Lemma 4.3.7.

$$\begin{aligned}
& \sum_{k=1}^N 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad + \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \\
& \leq Ch_{\mathcal{D}} \sum_{k=0}^N \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}
\end{aligned} \tag{4.94}$$

Proof. We will use again Abel's formula, so we define

$$\begin{aligned}
a_k &= 2\mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) + \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} \\
& \quad - \alpha (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id}
\end{aligned}$$

and

$$b_k = e_u^{\mathcal{D}}(\varepsilon_u^k).$$

We set

$$\bar{\varphi}^k = 2\mu e(\bar{\mathbf{u}}(t^k)) + \lambda \operatorname{div}(\bar{\mathbf{u}}(t^k)) \operatorname{Id} - \alpha \bar{p}(t^k) \operatorname{Id} \tag{4.95}$$

and

$$\varphi_{\Pi}^k = 2\mu e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) + \lambda \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) \operatorname{Id} - \alpha \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) \operatorname{Id} \tag{4.96}$$

such that the left-hand-side of (4.94) satisfies

$$\begin{aligned}
& \sum_{k=1}^N 2 \int_D \mu (e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad + \int_D \lambda (\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad - \alpha \int_D (\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)) \operatorname{Id} : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \\
&= \sum_{k=1}^N \int_D (\bar{\varphi}^k - \varphi_{\Pi}^k) : e_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx.
\end{aligned} \tag{4.97}$$

Applying the Abel formula to the right-hand-side of (4.97) gives

$$\begin{aligned} \sum_{k=1}^N \int_D (\bar{\varphi}^k - \varphi_{\Pi}^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx &= - \sum_{k=1}^{N-1} \int_D ((\bar{\varphi}^{k+1} - \bar{\varphi}^k) - (\varphi_{\Pi}^{k+1} - \varphi_{\Pi}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \\ &+ \int_D (\bar{\varphi}^N - \varphi_{\Pi}^N) : e_u^{\mathcal{D}}(\varepsilon_u^N) - (\bar{\varphi}^1 - \varphi_{\Pi}^1) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx. \end{aligned} \quad (4.98)$$

It remains to obtain a suitable estimate for the right-hand-side of (4.98).

We have from Lemma 4.3.2 and Lemma 4.3.3, for a positive constant C ,

- $\left\| \frac{e(\bar{\mathbf{u}}(t^{k+1})) - e(\bar{\mathbf{u}}(t^k))}{\Delta t} - \frac{e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k))}{\Delta t} \right\|_{L^2(D)} \leq Ch_{\mathcal{D}},$
- $\left\| \frac{\operatorname{div}(\bar{\mathbf{u}}(t^{k+1})) - \operatorname{div}(\bar{\mathbf{u}}(t^k))}{\Delta t} - \frac{\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k))}{\Delta t} \right\|_{L^2(D)} \leq Ch_{\mathcal{D}},$
- $\left\| \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} - \frac{\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}}(\bar{p}(t^{k+1}) - \bar{p}(t^k))}{\Delta t} \right\|_{L^2(D)} \leq Ch_{\mathcal{D}}.$

Moreover, we have immediately from (4.95) and (4.96)

$$\begin{aligned} &\frac{\bar{\varphi}^{k+1} - \bar{\varphi}^k}{\Delta t} - \frac{\varphi_{\Pi}^{k+1} - \varphi_{\Pi}^k}{\Delta t} = \\ &2\mu \left(\frac{e(\bar{\mathbf{u}}(t^{k+1})) - e(\bar{\mathbf{u}}(t^k))}{\Delta t} - \frac{e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k))}{\Delta t} \right) \\ &+ \lambda \left(\frac{\operatorname{div}(\bar{\mathbf{u}}(t^{k+1})) - \operatorname{div}(\bar{\mathbf{u}}(t^k))}{\Delta t} \operatorname{Id} - \frac{\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k))}{\Delta t} \operatorname{Id} \right) \\ &- \alpha \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \operatorname{Id} - \frac{\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}}(\bar{p}(t^{k+1}) - \bar{p}(t^k))}{\Delta t} \operatorname{Id} \right). \end{aligned} \quad (4.99)$$

We deduce from (4.95), (4.96) and the Lemmas 4.3.2 and 4.3.3 that

$$\left\| \frac{\bar{\varphi}^{k+1} - \bar{\varphi}^k}{\Delta t} - \frac{\varphi_{\Pi}^{k+1} - \varphi_{\Pi}^k}{\Delta t} \right\| \leq Ch_{\mathcal{D}}. \quad (4.100)$$

Finally in the right-hand-side of (4.98), we deduce from (4.3)

$$\sum_{k=1}^{N-1} \int_D ((\bar{\varphi}^{k+1} - \bar{\varphi}^k) - (\varphi_{\Pi}^{k+1} - \varphi_{\Pi}^k)) : e_u^{\mathcal{D}}(\varepsilon_u^k) \, dx \leq Ch_{\mathcal{D}} \sum_{k=1}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}. \quad (4.101)$$

The remaining terms on the right-hand-side of (4.98) can be bounded in a similar way. Indeed

$$\begin{aligned} &\int_D (\bar{\varphi}^N - \varphi_{\Pi}^N) : e_u^{\mathcal{D}}(\varepsilon_u^N) \, dx \\ &\leq \|\bar{\varphi}^N - \varphi_{\Pi}^N\|_{L^2(D)} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)} \\ &\leq Ch_{\mathcal{D}} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}, \end{aligned} \quad (4.102)$$

and

$$\begin{aligned} &\int_D (\bar{\varphi}^1 - \varphi_{\Pi}^1) : e_u^{\mathcal{D}}(\varepsilon_u^0) \, dx \\ &\leq \|\bar{\varphi}^1 - \varphi_{\Pi}^1\|_{L^2(D)} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \\ &\leq Ch_{\mathcal{D}} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}. \end{aligned} \quad (4.103)$$

We sum (4.101) with (4.102) and (4.103), and for some constant C , in view of (4.95) and (4.96) it concludes the proof of Lemma 4.3.7. \square

We can now prove the following result

Lemma 4.3.8.

$$\begin{aligned}
& \sum_{k=1}^N \int_D 2\mu e_u^{\mathcal{D}}(\varepsilon_u^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) + \lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \\
& \quad - \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \leq Ch_{\mathcal{D}} \left(\sum_{k=1}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \right)
\end{aligned} \tag{4.104}$$

Proof. It follows from (4.92) that the left-hand-side of (4.104) coincides with the sum of left-hand-side of (4.89) and (4.94). Summing the inequalities (4.89) and (4.94) gives (4.104) \square

The pressure equation

Next we present a similar estimate for the pressure equation, which will permit to conclude the proof Theorem 4.3.1. It follows from the definition of $W_p^{\mathcal{D}}$ in (4.10) that

$$\begin{aligned}
& \int_D \int_{t^k}^{t^{k+1}} \frac{\Delta \bar{p}(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx + \int_D \int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \cdot \nabla_p^{\mathcal{D}} w \, ds \, dx \\
& \leq W_p \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds \right) \|w\|_p^{\mathcal{D}}, \text{ for all } w \in X_p,
\end{aligned}$$

and thus by multiplying by κ we have

$$\begin{aligned}
& \int_D \kappa \int_{t^k}^{t^{k+1}} \frac{\Delta \bar{p}(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx + \int_D \kappa \int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \cdot \nabla_p^{\mathcal{D}} w \, ds \, dx \\
& \leq \kappa W_p \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds \right) \|w\|_p^{\mathcal{D}}, \text{ for all } w \in X_p.
\end{aligned} \tag{4.105}$$

Moreover since is the pair $(\bar{\mathbf{u}}, \bar{p})$ is the solution of problem (\mathbb{P}) , it follows from integration of the second equation in Problem (\mathbb{P}) between t^k and t^{k+1} that

$$\begin{aligned}
& \frac{1}{M} \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} + \alpha \operatorname{div} \frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \\
& \quad - \kappa \int_{t^k}^{t^{k+1}} \frac{\Delta \bar{p}(s)}{\Delta t} \, ds = \int_{t^k}^{t^{k+1}} \frac{g(s)}{\Delta t} \, ds
\end{aligned} \tag{4.106}$$

and thus for all $w \in X_p$

$$\begin{aligned}
& \int_D \kappa \int_{t^k}^{t^{k+1}} \frac{\Delta \bar{p}(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx = \frac{1}{M} \int_D \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \alpha \int_D \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad - \int_D \int_{t^k}^{t^{k+1}} \frac{g(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx.
\end{aligned} \tag{4.107}$$

Next we plug (4.107) into (4.105) to deduce that

$$\begin{aligned}
& \int_D \left(\frac{1}{M} \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} + \alpha \operatorname{div} \frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \Pi_p^{\mathcal{D}} w \, dx \\
& - \int_D \int_{t^k}^{t^{k+1}} \frac{g(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx + \int_D \kappa \int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \cdot \nabla_p^{\mathcal{D}} w \, ds \, dx \\
& \leq \kappa W_p^{\mathcal{D}} \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds \right) \|w\|_p^{\mathcal{D}}.
\end{aligned} \tag{4.108}$$

Next we recall that (\mathbf{U}, P) is defined as the unique solution of the gradient scheme (4.24)-(4.25). In particular, and in view of (4.5), (\mathbf{u}^k, p^k) satisfies for all k and for all $w \in X_p$

$$\begin{aligned}
& \int_D \int_{t^k}^{t^{k+1}} \frac{f(s)}{\Delta t} \Pi_p^{\mathcal{D}} w \, ds \, dx \\
& = \int_D \left(\frac{1}{M} \Pi_p^{\mathcal{D}} \frac{p^{k+1} - p^k}{\Delta t} + \alpha \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \int_D \kappa \nabla_p^{\mathcal{D}} p^{k+1} \cdot \nabla_p^{\mathcal{D}} w \, dx.
\end{aligned}$$

Then substituting the source term in (4.108) leads to the following, for all w in X_p ,

$$\begin{aligned}
& \int_D \left(\frac{1}{M} \frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} + \alpha \operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \int_D \kappa \int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \cdot \nabla_p^{\mathcal{D}} w \, ds \, dx \\
& - \int_D \left(\frac{1}{M} \Pi_p^{\mathcal{D}} \frac{p^{k+1} - p^k}{\Delta t} + \alpha \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad - \int_D \kappa \nabla_p^{\mathcal{D}} p^{k+1} \cdot \nabla_p^{\mathcal{D}} w \, dx \\
& \leq \kappa W_p^{\mathcal{D}} \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds \right) \|w\|_p^{\mathcal{D}}.
\end{aligned} \tag{4.109}$$

In what follows we perform the substitutions

- $\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k) = \operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) + \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k),$
- $\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k = \bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) + \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k,$
- $\nabla \bar{p}^k - \nabla_p^{\mathcal{D}} p^k = \nabla \bar{p}^k - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) + \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k) - \nabla_p^{\mathcal{D}} p^k.$

The equation (4.109) then reads as

$$\begin{aligned}
& \int_D \frac{1}{M} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& + \alpha \int_D \left(\operatorname{div} \frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \int_D \kappa \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right) \cdot \nabla_p^{\mathcal{D}} w \, dx \\
& + \int_D \frac{1}{M} \left(\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right) - \Pi_p^{\mathcal{D}} \left(\frac{p^{k+1} - p^k}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& + \alpha \int_D \left(\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) - \operatorname{div}_u^{\mathcal{D}} \left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \int_D \kappa (\nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) - \nabla_p^{\mathcal{D}} p^{k+1}) \cdot \nabla_p^{\mathcal{D}} w \, dx \\
& \leq \kappa W_p^{\mathcal{D}} \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds \right) \|w\|_p^{\mathcal{D}}.
\end{aligned} \tag{4.110}$$

Next we present an error bound for the first three terms of (4.110).

Lemma 4.3.9. *There exists a positive constant $C = C(T)$ such that, for all w in X_{p^r} there holds*

$$\begin{aligned}
& \frac{1}{M} \left| \int_D \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \left(\frac{\bar{p}(t^{k+1}) - \bar{p}(t^k)}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \right| \\
& + \alpha \left| \int_D \left(\operatorname{div} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \left(\frac{\bar{\mathbf{u}}(t^{k+1}) - \bar{\mathbf{u}}(t^k)}{\Delta t} \right) \right) \Pi_p^{\mathcal{D}} w \, dx \right| \\
& \quad + \kappa \left| \int_D \left(\int_{t^k}^{t^{k+1}} \frac{\nabla \bar{p}(s)}{\Delta t} \, ds - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) \right) \cdot \nabla_p^{\mathcal{D}} w \, dx \right| \\
& \leq Ch_{\mathcal{D}} \|\Pi_p^{\mathcal{D}} w\|_{L^2(D)} + C(\Delta t + h_{\mathcal{D}}) \|\nabla_p^{\mathcal{D}} w\|_{L^2(D)}.
\end{aligned} \tag{4.111}$$

Proof. We apply Cauchy-Schwarz, inequality, and use (4.54c) from Lemma 4.3.2 and (4.59a)-(4.59b) from Lemma 4.3.3. \square

We recall the notations $\varepsilon_u^{k+1} = \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^{k+1}) - \mathbf{u}^{k+1}$ and $\varepsilon_p^{k+1} = \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^{k+1}) - p^{k+1}$.

Lemma 4.3.10. *There exists a positive constant $C = C(T)$ such that*

$$\begin{aligned}
& \sum_{k=1}^N \int_D \frac{1}{M} \Pi_p^{\mathcal{D}} (\varepsilon_p^k - \varepsilon_p^{k-1}) \Pi_p^{\mathcal{D}} \varepsilon_p^k + \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \cdot \operatorname{div}_u^{\mathcal{D}} (\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\
& \quad + \frac{\Delta t}{2} \int_D \kappa |\nabla_p^{\mathcal{D}} \varepsilon_p^k|^2 \, dx \leq C(h_{\mathcal{D}} + \Delta t)^2.
\end{aligned} \tag{4.112}$$

Proof. We deduce from (4.110) and (4.111), and the notation recall above, we have

$$\begin{aligned}
& \frac{1}{M} \int_D \Pi_p^{\mathcal{D}} \left(\frac{\varepsilon_p^{k+1} - \varepsilon_p^k}{\Delta t} \right) \Pi_p^{\mathcal{D}} w \, dx + \alpha \int_D \operatorname{div}_u^{\mathcal{D}} \left(\frac{\varepsilon_u^{k+1} - \varepsilon_u^k}{\Delta t} \right) \Pi_p^{\mathcal{D}} w \, dx \\
& \quad + \int_D \kappa \nabla_p^{\mathcal{D}} \varepsilon_p^{k+1} \cdot \nabla_p^{\mathcal{D}} w \, dx \leq C(\Delta t + h_{\mathcal{D}}) \|\nabla_p^{\mathcal{D}} w\|_{L^2(D)}.
\end{aligned} \tag{4.113}$$

Then we chose $w = \Delta t \varepsilon_p^k$ and sum over k (4.113) which gives

$$\begin{aligned} & \sum_{k=1}^N \int_D \frac{1}{M} \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1}) \Pi_p^{\mathcal{D}} \varepsilon_p^k + \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \cdot \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & + \Delta t \int_D \kappa |\nabla_p^{\mathcal{D}} \varepsilon_p^k|^2 \, dx \leq \sum_{k=1}^N C \Delta t (h_{\mathcal{D}} + \Delta t) \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)} \end{aligned}$$

We apply Hölder's inequality as follows :

$$C \Delta t (h_{\mathcal{D}} + \Delta t) \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \leq \frac{\Delta t \kappa}{2} \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 + \frac{\Delta t}{2\kappa} (C(h_{\mathcal{D}} + \Delta t))^2$$

gives the result

$$\begin{aligned} & \sum_{k=1}^N \int_D \frac{1}{M} \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1}) \Pi_p^{\mathcal{D}} \varepsilon_p^k + \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \cdot \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \, dx \\ & + \Delta t \frac{\kappa}{2} \int_D |\nabla_p^{\mathcal{D}} \varepsilon_p^k|^2 \, dx \leq \sum_{k=1}^N C \Delta t (h_{\mathcal{D}} + \Delta t)^2. \end{aligned}$$

□

4.3.4 Proof of theorem 4.3.1

Given the previous results, by summing the two equations (4.104) and (4.112), we avoid to estimates the coupled term in the system (\mathbb{P}) .

We can ensure the following Lemmas,

Lemma 4.3.11. *There exists a positive constant $C = C(T)$, such that*

$$\begin{aligned} & \sum_{k=1}^N \int_D \frac{1}{M} \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1}) \Pi_p^{\mathcal{D}} \varepsilon_p^k + \Delta t \frac{\kappa}{2} \int_D |\nabla_p^{\mathcal{D}} \varepsilon_p^k|^2 \, dx \\ & + \sum_{k=1}^N \int_D 2 \left(\mu e_u^{\mathcal{D}}(\varepsilon_u^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) + \lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right) \, dx \\ & \leq C \Delta t \sum_{k=0}^N \left((h_{\mathcal{D}} + \Delta t)^2 + h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)} \right) \\ & \quad + C h_{\mathcal{D}} (\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}). \end{aligned} \tag{4.114}$$

Proof. We deduce from (4.104) in Lemma 4.3.8 and (4.23) that

$$\begin{aligned} & \sum_{k=1}^N \int_D \left(2\mu e_u^{\mathcal{D}}(\varepsilon_u^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) + \lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right. \\ & \quad \left. - \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right) \, dx \\ & \leq C h_{\mathcal{D}} \left(\sum_{k=1}^N \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \right). \end{aligned}$$

From (4.112) in Lemma 4.3.10 we have

$$\begin{aligned} \sum_{k=1}^N \int_D \left(\frac{1}{M} \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1}) \Pi_p^{\mathcal{D}} \varepsilon_p^k + \alpha \Pi_p^{\mathcal{D}} \varepsilon_p^k \cdot \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \right) dx \\ + \Delta t \frac{\kappa}{2} \int_D |\nabla_p^{\mathcal{D}} \varepsilon_p^k|^2 dx \leq \sum_{k=1}^N C \Delta t (h_{\mathcal{D}} + \Delta t)^2. \end{aligned}$$

Summing these two inequalities yields (4.114). \square

Lemma 4.3.12. *There exists a positive constant C such that*

$$\begin{aligned} & \mu \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \\ & + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\ & \leq C \Delta t \sum_{k=0}^N ((h_{\mathcal{D}} + \Delta t)^2 + h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}) \\ & + C (\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2) \\ & + C h_{\mathcal{D}} (\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}). \end{aligned} \quad (4.115)$$

Proof. We recall that for all $a, b \in \mathbb{R}$, the following equality holds :

$$a(a - b) = \frac{a^2 - b^2}{2} + \frac{1}{2}(a - b)^2. \quad (4.116)$$

In view of (4.116), we have for all k

$$\frac{\Pi_p^{\mathcal{D}} \varepsilon_p^k \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1})}{M} \geq \frac{(\Pi_p^{\mathcal{D}} \varepsilon_p^k)^2 - (\Pi_p^{\mathcal{D}} \varepsilon_p^{k-1})^2}{2M}, \quad (4.117a)$$

$$2\mu e_u^{\mathcal{D}}(\varepsilon_u^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \geq \mu (e_u^{\mathcal{D}}(\varepsilon_u^k))^2 - \mu (e_u^{\mathcal{D}}(\varepsilon_u^{k-1}))^2, \quad (4.117b)$$

$$\lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) \geq \lambda \frac{(\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k))^2 - (\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^{k-1}))^2}{2}. \quad (4.117c)$$

Summing (4.117a), (4.117b) and (4.117c) leads to :

$$\sum_{k=1}^N \int_D \frac{\Pi_p^{\mathcal{D}} \varepsilon_p^k \Pi_p^{\mathcal{D}}(\varepsilon_p^k - \varepsilon_p^{k-1})}{M} dx \geq \frac{1}{2M} (\|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 - \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2), \quad (4.118a)$$

$$\sum_{k=1}^N 2\mu e_u^{\mathcal{D}}(\varepsilon_u^k) : e_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \geq \mu (\|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 - \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2), \quad (4.118b)$$

and

$$\sum_{k=1}^N \lambda \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k) \operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^k - \varepsilon_u^{k-1}) dx \geq \frac{\lambda}{2} (\|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 - \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2). \quad (4.118c)$$

Substituting (4.118) in the left hand side of (4.114) permits to conclude the proof of Lemma 4.3.12. \square

Lemma 4.3.13. For Δt such that $(1 - \Delta t) \geq \frac{1}{2}$,

$$\begin{aligned} & \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\ & \leq CT(h_{\mathcal{D}} + \Delta t)^2 + \frac{\mu}{2} \sum_{k=0}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2 \\ & \quad + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2), \end{aligned} \quad (4.119)$$

where ε_u^k and ε_p^k are defined by (4.61) and (4.62) respectively, for $0 \leq k \leq N$.

Proof. We have from Hölder's inequality

$$Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)} \leq \frac{C^2 h_{\mathcal{D}}^2}{2\mu} + \frac{\mu}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2. \quad (4.120)$$

Substituting (4.120) in (4.115) gives

$$\begin{aligned} & \frac{\mu}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\ & \leq C\Delta t \sum_{k=0}^N ((h_{\mathcal{D}} + \Delta t)^2 + h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}) + \frac{C^2 h_{\mathcal{D}}^2}{2\mu} \\ & \quad + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2) + Ch_{\mathcal{D}} (\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}). \end{aligned} \quad (4.121)$$

On the right-hand-side of (4.121), we have by Hölder's inequality

$$\begin{aligned} C\Delta t \sum_{k=0}^N h_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)} & \leq \sum_{k=0}^N \Delta t \left(\frac{C^2}{\mu} h_{\mathcal{D}}^2 + \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2 \right) \\ & = \Delta t \sum_{k=0}^N \frac{C^2}{\mu} h_{\mathcal{D}}^2 + \Delta t \sum_{k=0}^{N-1} \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2 + \frac{\Delta t \mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2. \end{aligned}$$

We remark that

$$\begin{aligned} & \sum_{k=0}^N \Delta t \frac{C^2}{2\mu} h_{\mathcal{D}}^2 \leq \frac{C^2}{2\mu} (T+1) h_{\mathcal{D}}^2, \\ & \sum_{k=1}^N \Delta t (h_{\mathcal{D}} + \Delta t)^2 \leq T(h_{\mathcal{D}} + \Delta t)^2, \\ & Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \leq \frac{C}{2} h_{\mathcal{D}}^2 + \frac{C}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2, \end{aligned}$$

then (4.121) leads to

$$\begin{aligned} & (1 - \Delta t) \frac{\mu}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \\ & \quad + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \Delta t \sum_{k=1}^N \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\ & \leq \frac{\mu}{2} \sum_{k=0}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2 + CT(h_{\mathcal{D}} + \Delta t)^2 \\ & \quad + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2) + Ch_{\mathcal{D}} \|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}. \end{aligned} \quad (4.122)$$

So that (4.122) becomes

$$\begin{aligned}
& (1 - \Delta t) \frac{\mu}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\
& \leq \frac{\mu}{2} \sum_{k=0}^{N-1} \Delta t \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2 + CT(h_{\mathcal{D}} + \Delta t)^2 \\
& + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2).
\end{aligned}$$

This concludes the proof of Lemma 4.3.13. \square

We recall the discrete Gronwall Lemma which coincides with Lemma 6 in [22].

Lemma 4.3.14 (Discrete Gronwall Lemma). *Let $\{a_k\}_{k=0,1,2,\dots,N}$, $\{b_k\}_{k=0,1,2,\dots,N}$ be two positive sequences. We assume that*

$$a_0 + b_0 \leq \mathcal{C}$$

and that there exist a positive constant K such that

$$a_N + b_N \leq K \sum_{k=0}^{N-1} a_k + \mathcal{C}, \text{ for all } N \geq 1.$$

Then there holds

$$a_N + b_N \leq \mathcal{C} \exp(KN), \text{ for all } N \geq 0.$$

It will permit to obtain the following Lemma.

Lemma 4.3.15. *There exists a constant $C = C(T)$ such that*

$$\begin{aligned}
& \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\
& \leq \left(\sum_{k=1}^N C \Delta t (h_{\mathcal{D}}^2 + \Delta t^2) + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2) \right) \exp(CT).
\end{aligned} \tag{4.123}$$

Proof. We set :

- $a_k = \frac{\mu}{2} \|e_u^{\mathcal{D}}(\varepsilon_u^k)\|_{L^2(D)}^2$ and $a_N = \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2$;
- $b_N = \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2$;
- $K = \Delta t$;
- $\mathcal{C} = CT(h_{\mathcal{D}} + \Delta t)^2 + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2)$.

We apply the discrete Gronwall Lemma 4.3.14 to the inequality (4.119), which gives (4.123). \square

Lemma 4.3.16. *Let $(\bar{\mathbf{u}}, \bar{p})$ be the solution of problem (\mathbb{P}) . There exists a constant $C = C(T)$ such that for all $1 \leq k \leq N$ there holds*

- $\|e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)\|_{L^2(D)} \leq h_{\mathcal{D}} \|\bar{\mathbf{u}}\|_{L^\infty(0,T;H_0^1(D))} \leq Ch_{\mathcal{D}}$;
- $\|\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)\|_{L^2(D)} \leq h_{\mathcal{D}} \|\bar{\mathbf{u}}\|_{L^\infty(0,T;H_0^1(D))} \leq Ch_{\mathcal{D}}$,

- $\|\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)\|_{L^2(D)} \leq h_{\mathcal{D}} \|\bar{p}\|_{L^\infty(0,T;H_0^1(D))} \leq Ch_{\mathcal{D}}$;
- $\|\nabla \bar{p}(t^k) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)\|_{L^2(D)} \leq h_{\mathcal{D}} \|\bar{p}\|_{L^\infty(0,T;H_0^1(D))} \leq Ch_{\mathcal{D}}$.

Proof. We deduce from Lemma 4.3.1 that

- $\|e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)\|_{L^2(D)} \leq S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^k))$,
- $\|\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^k)\|_{L^2(D)} \leq \sqrt{d} S_u^{\mathcal{D}}(\bar{\mathbf{u}}(t^k))$,
- $\|\bar{p}(t^k) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)\|_{L^2(D)} \leq S_p^{\mathcal{D}}(\bar{p}(t^k))$.

The result of Lemma 4.3.16 follows from the definition of $S_u^{\mathcal{D}}$ in (4.15) and $S_p^{\mathcal{D}}$ in (4.8), as well as from (4.19), (4.20), (4.12), (4.13) and (4.21). \square

Lemma 4.3.17. *At the initial time, in view of the definitions of E_u^{ini} and E_p^{ini} in (4.46), (4.47) and of $\varepsilon_u^0, \varepsilon_p^0$ in (4.61), (4.62) and (4.26) we have*

- $\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)} \leq Ch_{\mathcal{D}} + E_u^{ini}$,
- $\|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)} \leq Ch_{\mathcal{D}} + E_p^{ini}$.

Proof. We recall that, also using (4.26), and Lemma 4.3.1

- $\|e_u^{\mathcal{D}}(\varepsilon_u^0)\| \leq \|e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}_0 - e(\bar{\mathbf{u}}_0)\|_{L^2(D)} + \|e(\bar{\mathbf{u}}_0) - e_u^{\mathcal{D}} \mathcal{S}_u^{\mathcal{D}} \bar{\mathbf{u}}_0\|_{L^2(D)} \leq Ch_{\mathcal{D}} + E_u^{ini}$
- $\|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^0)\| \leq \|\operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}_0 - \operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)} + \|\operatorname{div}(\bar{\mathbf{u}}_0) - \operatorname{div}_u^{\mathcal{D}} \mathcal{S}_u^{\mathcal{D}} \bar{\mathbf{u}}_0\|_{L^2(D)} \leq Ch_{\mathcal{D}} + E_u^{ini}$
- $\|\Pi_p^{\mathcal{D}} \varepsilon_p^0\| \leq \|\Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}_0 - \bar{p}_0\|_{L^2(D)} + \|\bar{p}_0 - \Pi_p^{\mathcal{D}} \mathcal{S}_p^{\mathcal{D}} \bar{p}_0\|_{L^2(D)} \leq Ch_{\mathcal{D}} + E_p^{ini}$

\square

Next we prove the following error estimate.

Lemma 4.3.18. *There exists a constant $C = C(T)$ such that*

$$\begin{aligned} & \|e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 + \|\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 \\ & + \|\bar{p}(t^N) - \Pi_p^{\mathcal{D}} p^N\|_{L^2(D)}^2 + \sum_{k=1}^N \Delta t \|\nabla \bar{p}(t^k) - \nabla_p^{\mathcal{D}} p^k\|_{L^2(D)}^2 \\ & \leq C \left(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2 \right). \end{aligned} \quad (4.124)$$

Proof. From (4.123), we have

$$\begin{aligned} & \frac{\mu}{4} \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 + \frac{\lambda}{2} \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \\ & + \frac{1}{2M} \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 + \frac{\kappa}{2} \sum_{k=1}^N \Delta t \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \\ & \leq \left(\sum_{k=1}^N C \Delta t (h_{\mathcal{D}} + \Delta t)^2 + C (\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2) \right) \exp(CT). \end{aligned}$$

In view of (4.61), (4.62), there holds

$$\|e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 \leq 2 \left(\|e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^N)\|_{L^2(D)}^2 + \|e_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \right),$$

$$\begin{aligned} \|\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 &\leq 2 \left(\|\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}} \mathcal{I}_u^{\mathcal{D}} \bar{\mathbf{u}}(t^N)\|_{L^2(D)}^2 + \|\operatorname{div}_u^{\mathcal{D}}(\varepsilon_u^N)\|_{L^2(D)}^2 \right), \\ \|\bar{p}(t^N) - \Pi_p^{\mathcal{D}} p^N\|_{L^2(D)}^2 &\leq 2 \left(\|\bar{p}(t^N) - \Pi_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^N)\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^N\|_{L^2(D)}^2 \right), \end{aligned}$$

and for all $0 \leq k \leq N$

$$\|\nabla(\bar{p}(t^k)) - \nabla_p^{\mathcal{D}} p^k\|_{L^2(D)}^2 \leq 2 \left(\|\nabla(\bar{p}(t^k)) - \nabla_p^{\mathcal{D}} \mathcal{I}_p^{\mathcal{D}} \bar{p}(t^k)\|_{L^2(D)}^2 + \|\nabla_p^{\mathcal{D}} \varepsilon_p^k\|_{L^2(D)}^2 \right).$$

Thus in view of (4.123) and Lemma 4.3.16, it follows

$$\begin{aligned} &\|e(\bar{\mathbf{u}}(t^N)) - e_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 + \|\operatorname{div}(\bar{\mathbf{u}}(t^N)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^N)\|_{L^2(D)}^2 \\ &\quad + \|\bar{p}(t^N) - \Pi_p^{\mathcal{D}} p^N\|_{L^2(D)}^2 + \sum_{k=1}^N \Delta t \|\nabla \bar{p}(t^k) - \nabla_p^{\mathcal{D}} p^k\|_{L^2(D)}^2 \\ &\leq \left(\sum_{k=1}^N C \Delta t (h_{\mathcal{D}}^2 + \Delta t^2) + C(\|e_u^{\mathcal{D}}(\varepsilon_u^0)\|_{L^2(D)}^2 \right. \\ &\quad \left. + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{L^2(D)}^2) \right) \exp(CT) + C(1+T)h_{\mathcal{D}}^2 \\ &\leq C(h_{\mathcal{D}}^2 + \Delta t^2) + \exp(CT)(\|e_u^{\mathcal{D}} \varepsilon_u^0\|_{L^2(D)}^2 + \|\Pi_p^{\mathcal{D}} \varepsilon_p^0\|_{M^2(D)}^2) \end{aligned} \quad (4.125)$$

In view of Lemma 4.3.17 this completes the proof of 4.3.18. \square

Proof of Theorem 4.3.1. Using the same arguments than in Lemma 4.3.18, we have the same estimation for all indexes $k \in \{1, \dots, N\}$

$$\|e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2 \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2) \quad (4.126a)$$

$$\|\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2 \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2), \quad (4.126b)$$

$$\|\bar{p}(t^k) - p^k\|_{L^2(D)}^2 \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2). \quad (4.126c)$$

$$\sum_{k'=1}^k \Delta t \|\nabla \bar{p}(t^{k'}) - \nabla_p^{\mathcal{D}} p^{k'}\|_{L^2(D)}^2 \leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2). \quad (4.126d)$$

Moreover, we have from Theorem 3.5.2, for all t such that $t \in (t^{k-1}, t^k]$.

$$\begin{aligned} \|e(\bar{\mathbf{u}}(t)) - e_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2 &\leq C(\|e(\bar{\mathbf{u}}(t)) - e(\bar{\mathbf{u}}(t^k))\|_{L^2(D)}^2 \\ &\quad + \|e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2) \\ &\leq C(\Delta t^2 + \|e(\bar{\mathbf{u}}(t^k)) - e_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2), \end{aligned} \quad (4.127a)$$

$$\begin{aligned} \|\operatorname{div}(\bar{\mathbf{u}}(t)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2 &\leq C(\|\operatorname{div}(\bar{\mathbf{u}}(t)) - \operatorname{div}(\bar{\mathbf{u}}(t^k))\|_{L^2(D)}^2 + \|\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2) \\ &\leq C(\Delta t^2 + \|\operatorname{div}(\bar{\mathbf{u}}(t^k)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}^k)\|_{L^2(D)}^2), \end{aligned} \quad (4.127b)$$

and

$$\begin{aligned} \|\bar{p}(t) - \Pi_p^{\mathcal{D}} p^k\|_{L^2(D)}^2 &\leq C(\|\bar{p}(t) - \bar{p}(t^k)\|_{L^2(D)}^2 + \|\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k\|_{L^2(D)}^2) \\ &\leq C(\Delta t^2 + \|\bar{p}(t^k) - \Pi_p^{\mathcal{D}} p^k\|_{L^2(D)}^2). \end{aligned} \quad (4.127c)$$

From Theorem 3.5.2 , Lemma 4.3.18 and (4.6)

$$\begin{aligned}
\int_0^T \|\nabla \bar{p}(t) - \nabla_p^{\mathcal{D}} p(t)\|_{L^2(D)}^2 &= \sum_{k=0}^{N-1} \int_{t^k}^{t^{k+1}} \|\nabla \bar{p}(\tau) - \nabla_p^{\mathcal{D}} p(\tau)\|_{L^2(D)}^2 d\tau \\
&\leq \sum_{k=0}^{N-1} \int_{t^k}^{t^{k+1}} \|\nabla \bar{p}(\tau) - \nabla \bar{p}(t^{k+1})\|_{L^2(D)}^2 d\tau \\
&\quad + \sum_{k=0}^{N-1} \Delta t \|\nabla \bar{p}(t^{k+1}) - \nabla_p^{\mathcal{D}} p^{k+1}\|_{L^2(D)}^2 \\
&\leq \sum C \Delta t^2 + \sum_{k=0}^{N-1} \Delta t \|\nabla \bar{p}(t^{k+1}) - \nabla_p^{\mathcal{D}} p^{k+1}\|_{L^2(D)}^2 \\
&\leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2).
\end{aligned} \tag{4.128}$$

Thus, by combining (4.126) with (4.127) and (4.128),

$$\begin{aligned}
&\sup_{t \in (0, T)} \|e(\bar{\mathbf{u}})(t) - e_u^{\mathcal{D}} \mathbf{u}(t)\|_{L^2(D)}^2 \\
&\quad + \sup_{t \in (0, T)} \|\operatorname{div}(\bar{\mathbf{u}}(t)) - \operatorname{div}_u^{\mathcal{D}}(\mathbf{u}(t))\|_{L^2(D)}^2 \\
&\quad + \sup_{t \in (0, T)} \|\bar{p}(t) - \Pi_p^{\mathcal{D}} p(t)\|_{L^2(D)}^2 + \|\nabla \bar{p} - \nabla_p^{\mathcal{D}} p\|_{L^2(D \times (0, T))}^2 \\
&\leq C(h_{\mathcal{D}}^2 + \Delta t^2 + (E_u^{ini})^2 + (E_p^{ini})^2).
\end{aligned} \tag{4.129}$$

□

4.4 Numerical method

4.4.1 The Finite Element Method

The aim of this section is to propose a conforming method which provides a specific Gradient Discretization Method satisfying the assumptions of coercivity, consistency, and limit-conformity. The numerical results obtained in Theorem 4.3.1 include the special case of the following $(\mathbf{P}_2) - (P_1)$ method. Since in the system, we need to combine the function p with a first order space derivative of u , the choice $(\mathbf{P}_2) - (P_1)$ ensures that we can sum these two components.

Given an integer l , we introduce the vector space

$$P_{l,d} = \operatorname{span} \left\{ x_1^{\alpha_1} \dots x_d^{\alpha_d}, 0 \leq \alpha_1, \dots, \alpha_d \leq l, \sum_{i=1}^d \alpha_i \leq l \right\}. \tag{4.130}$$

The vector space $P_{l,d}$ has the following dimension

$$\dim(P_{l,d}) = \binom{l+d}{d}. \tag{4.131}$$

From now on we omit the subscript d so that we write P_l when the context is unambiguous. We define the space \mathbf{P}_l as the vector polynomial space,

$$\mathbf{P}_l := (P_l)^d.$$

We chose to apply the \mathbf{P}_2 finite element method for the computation of the displacement which are piecewise

quadratic, and the P_1 finite element method which are piecewise linear for the computation of the pressure. We set a triangulation \mathcal{T}_h of \bar{D} made of triangle ($d = 2$) or tetrahedra ($d = 3$) denotes T . The diameter of each element T is denote h_T , such that we define the meshsize of \mathcal{T}_h as

$$h := \max_{T \in \mathcal{T}_h} \{h_T\}.$$

For each $T \in \mathcal{T}_h$ we let $\mathcal{E}(T)$ the set of edges of T , and \mathcal{E}_h the set of edges of \mathcal{T}_h , such that $\mathcal{E}_h := \mathcal{E}_h^I + \mathcal{E}_h^\partial$ with

- \mathcal{E}_h^I the collection of all interior edges,
- \mathcal{E}_h^∂ the set of boundary edges.

The $(\mathbf{P}_2) - (P_1)$ Lagrangian element is given by the pair (V_h, Q_h) such that

$$\begin{aligned} V_h &= \{\mathbf{v}_h \in C(\bar{D}), \mathbf{v}_h|_T \in \mathbf{P}^2(T), \forall T \in \mathcal{E}_h^\partial, \mathbf{v}_h = 0\}; \\ Q_h &= \{q_h \in C(\bar{D}), q_h|_T \in P^1(T), \forall T \in \mathcal{E}_h^\partial, q_h = 0\}. \end{aligned} \quad (4.132)$$

4.4.2 FEM as a Gradient Discretization Method

We recall some notations of the Finite Element Method. Given the triangulation \mathcal{T}_h of D , we let \mathcal{V}_h^I be the set of vertices of each triangle T of \mathcal{T}_h . Moreover, we let \mathcal{V}_h^M be the set of midpoints of each edge of \mathcal{E} . Given (4.131), there holds that

$$\dim(P_{2,2}) = 6, \dim(P_{1,2}) = 3.$$

Thus, it is necessary to evaluate the functions of V_h at 6 points, and to evaluate the functions of Q_h at 3 points. There holds that

Lemma 4.4.1. *Each function $\mathbf{v} \in V_h$ is uniquely determined by its values on all vertices of \mathcal{V}_h^I and the midpoints \mathcal{V}_h^M . Each function $q \in Q_h$ is uniquely determined by its values on all the vertices of \mathcal{V}_h^I .*

We define the gradients spaces X_u and X_p by

- $X_u = \{\mathbf{v} = (\mathbf{v}_i)_{i \in \mathcal{V}_h^I \cup \mathcal{V}_h^M}, \mathbf{v}_i \in \mathbb{R}, \text{ for all } i \in \mathcal{V}_h^I \cup \mathcal{V}_h^M\},$
- $X_p = \{q = (q_i)_{i \in \mathcal{V}_h^I}, q_i \in \mathbb{R}, \text{ for all } i \in \mathcal{V}_h^I\},$

which maps one-to-one to finite dimensional real vector spaces.

We set $(\psi_i)_{i \in \mathcal{V}_h^I \cup \mathcal{V}_h^M}$, the basis of the (\mathbf{P}_2) space, and $(\phi_i)_{i \in \mathcal{V}_h^I}$ the basis of the (P_1) space .

The Gradients operators are defined as follows, for $\mathbf{v} \in X_u$ and $q \in X_p$,

- $\Pi_u^D \mathbf{v} = \sum_{i \in \mathcal{V}_h^I \cup \mathcal{V}_h^M} \mathbf{v}_i \psi_i,$
- $e_u^D(\mathbf{v}) = e(\Pi_u^D \mathbf{v}),$
- $\Pi_p^D q = \sum_{i \in \mathcal{V}_h^I} q_i \phi_i,$
- $\nabla_p^D(q) = \nabla(\Pi_p^D q).$

The following properties holds

Lemma 4.4.2.

- X_u is a vector-space on \mathbb{R} , and

$$\|\cdot\cdot\cdot\|_u : \mathbf{v} \in X_u \mapsto \|e_u^D(\mathbf{v})\|_{L^2(D)}$$

defines a norm on X_u ,

- X_p is a vector-space on \mathbb{R} , and

$$\|\cdots\|_p : q \in X_p \mapsto \|\nabla_p^{\mathcal{D}} q\|_{L^2(D)}$$

defines a norm on X_p .

Finally we recall Theorem 8.10 from [24], which proves the needed properties of $(\mathbf{P}_2) - (P_1)$ method as a Gradient Discretization Method, and the consistency, coercivity and limit-conforming properties of Definition 4.1.6 and 4.1.4 for a sequence of GDM $(\mathcal{D}_u^m, \mathcal{D}_p^m)_{m \in \mathbb{N}}$.

Theorem 4.4.1 ([24, Theorem 8.10]). *We note $\mathcal{D}_u^m = (X_u^{\mathcal{D}_u^m}, \Pi_u^{\mathcal{D}_u^m}, e_u^{\mathcal{D}_u^m})$ and $\mathcal{D}_p^m = (X_p^{\mathcal{D}_p^m}, \Pi_p^{\mathcal{D}_p^m}, \nabla_u^{\mathcal{D}_p^m})$. Then \mathcal{D}_u^m and \mathcal{D}_p^m are Gradient Discretisation Methods. They are consistent, coercive and limit-conforming.*

4.4.3 Test cases

We will use the FENICS framework [2] using Python langage. This framework permits to describe equations in weak form and to specify a large variety of numerical finite elements method. Solving the linear systems is devoted to the FENICS framework that usually call UMFPAK library.

Parameter values

For the displacement equation, we recall the definition of the Lamé parameters as a function of the Poisson coefficient ν and the Young modulus E

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

We set the Poisson coefficient as $\nu = 0.25$ and the Young modulus as $E = 1$, so that the Lamé parameters satisfy

$$\mu = 0.4, \quad \lambda = 0.4.$$

For the pressure equation we set the permeability $\kappa = 1$.

The coupled Biot coefficients M and α are given by

$$\frac{1}{M} = 1, \quad \alpha = 1.$$

We compute the solution on the time interval $[0, T]$ with $T = 1$.

The domain $D := (0, 1) \times (0, 1)$ is discretized with a uniform triangulation, so that in each direction we have $2^{\mathcal{M}}$ cells, which gives $2^{2\mathcal{M}+1}$ triangles and $(2^{\mathcal{M}} + 1)^2$ vertices. We note h the diameter of the triangulation which is given by $h = \frac{1}{2^{\mathcal{M}}}$, and set $\Delta t = h$ for each triangulation of \mathcal{M} so that the time-step and the diameter of the mesh are of the same order. In the context of a Gradient Discretization Method, we can check that the physical space size h is a bound for the "space size" $h_{\mathcal{D}}$ defined in Definition 4.1.8 in the sense that there exists a constant C depending on the domain \mathcal{D} such that $h_{\mathcal{D}} \leq Ch$.

Analytical test cases

We consider two different cases. First we consider the solutions of \mathbb{P} which satisfy

$$-\operatorname{div}(0.8 e(\bar{\mathbf{u}}) + 0.4 \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id} - \bar{p}\operatorname{Id}) = \mathbf{f},$$

and

$$\partial_t(\bar{p} + \operatorname{div}(\bar{\mathbf{u}})) - \Delta \bar{p} = g.$$

We set the analytical solution $(\bar{\mathbf{u}}, \bar{p})$ which satisfy the previous problem (\mathbb{P}) as

$$\bar{\mathbf{u}}(t, x, y) = \begin{pmatrix} tx(1-x)y(1-y) \\ tx(1-x)y(1-y) \end{pmatrix}, \quad \bar{p}(t, x, y) = tx(1-x)y(1-y),$$

so that

$$\bar{p}_0 = 0, \bar{p} = 0 \text{ on } \partial D, \bar{\mathbf{u}} = 0 \text{ on } \partial D.$$

and we are able to compute numerically \mathbf{f} and g . We distinguish the error for the mechanical displacement $\bar{\mathbf{u}}$ and for the pressure \bar{p} with norms in the spaces $L^2((0, T); L^2(D))$, $L^\infty((0, T); L^2(D))$ and $L^2(0, T; H_0^1(D))$, $L^\infty(0, T; H_0^1(D))$. The following errors are plot in a loglog scale.

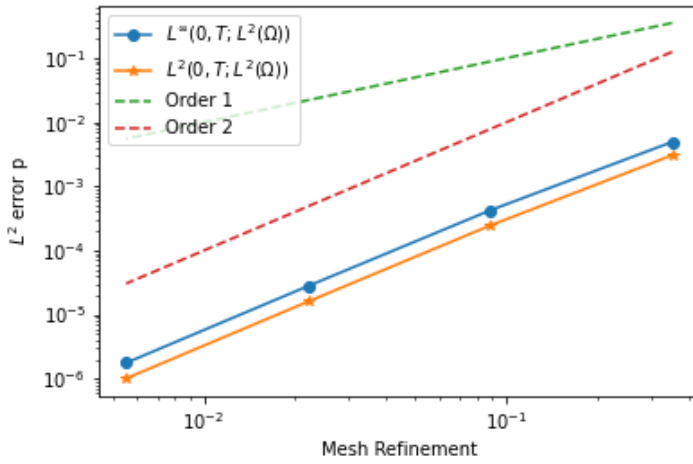


Figure 4.1: Errors $L^2((0, T); L^2(D))$ and $L^\infty((0, T); L^2(D))$ for the pressure

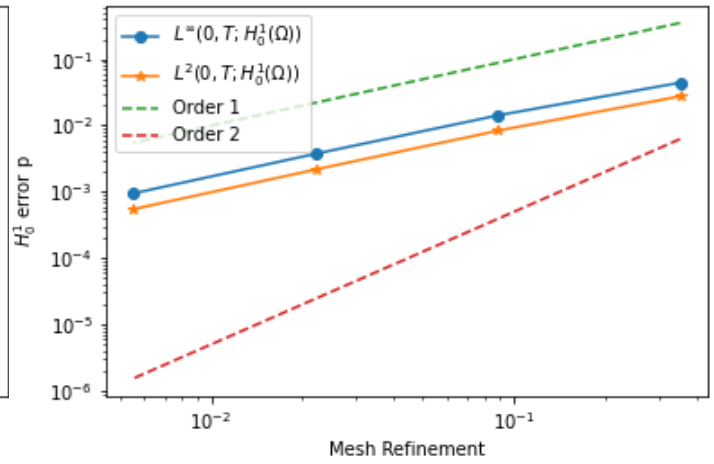


Figure 4.2: Error $L^2(0, T; H_0^1(D))$ and $L^\infty(0, T; H_0^1(D))$ for the pressure

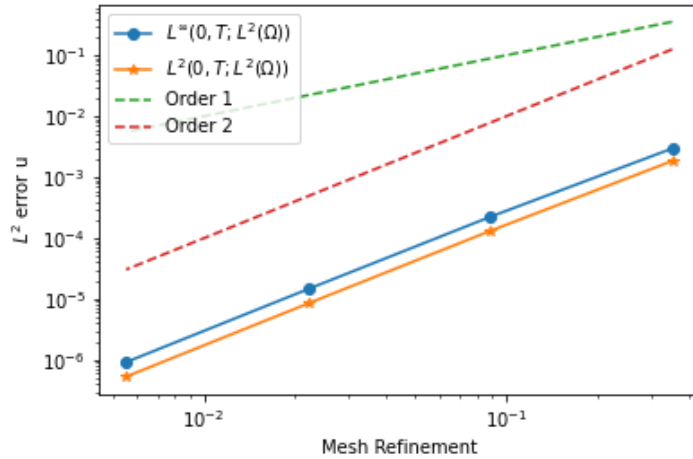


Figure 4.3: Error $L^2((0, T); L^2(D))$ and $L^\infty((0, T); L^2(D))$ for the displacement

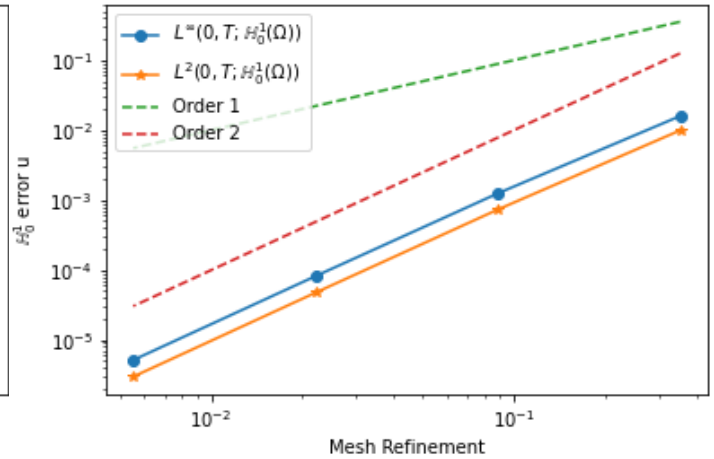


Figure 4.4: Error $L^2(0, T; H_0^1(D))$ and $L^\infty(0, T; H_0^1(D))$ for the displacement

We estimate the rates of convergence on the numerical tests by fitting the slope by linear regression. For the pressure, these slopes are listed in the following Table 4.1.

Thus, the convergence of p to \bar{p} is almost order 1 in the spaces $L^2((0, T); H_0^1(D))$ and $L^\infty((0, T); H_0^1(D))$, and almost order 2 in the spaces $L^2((0, T); L^2(D))$ and $L^\infty((0, T); L^2(D))$.

Mesh size h	8.8e-02	2.2e-02	5.5-03
	Rates of convergence		
$\ \bar{p} - p\ _{L^2((0,T);L^2(D))}$	1.82	1.97	1.99
$\ \bar{p} - p\ _{L^2((0,T);H_0^1(D))}$	0.86	0.97	0.99
$\ \bar{\mathbf{u}} - \mathbf{u}\ _{L^2((0,T);L^2(D))}$	1.90	1.98	1.99
$\ \bar{\mathbf{u}} - \mathbf{u}\ _{L^2((0,T);H_0^1(D))}$	1.99	1.97	1.99

Table 4.1: Convergence of (\mathbf{u}, p) to the solution $(\bar{\mathbf{u}}, \bar{p})$

Thus, the convergence of u to $\bar{\mathbf{u}}$ is almost order 2 (but at least of order 1) in the spaces $L^2((0, T); H_0^1(D))$ and $L^\infty((0, T); H_0^1(D))$, and almost order 2 in the spaces $L^2((0, T); L^2(D))$ and $L^\infty((0, T); L^2(D))$. These results illustrate the estimates of Theorem 4.3.1.

The second analytical test cases which we consider follows from the resolution of (\mathbb{P}) with \mathbf{f} given by

$$\mathbf{f}(\mathbf{x}, t) = \begin{pmatrix} \frac{e^{-t}}{\sqrt{2}} \cos(x/\sqrt{2}) \sin(y/\sqrt{2}) - 2.10^{-2} e^{-t} y \\ \sqrt{2} \sin(x/\sqrt{2}) \cos(y/\sqrt{2}) + 2.10^{-2} e^{-t} x \end{pmatrix}$$

and g given by

$$g(\mathbf{x}, t) = 0.5 e^{-t} \sin(x/\sqrt{2}) \cos(y/\sqrt{2}),$$

with

$$\bar{p}_0 = \sin\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{y}{\sqrt{2}}\right) \text{ as initial condition}$$

and

$$\bar{\mathbf{u}}(x, y) = 10^{-2} e^{-t} \begin{pmatrix} x^2 y \\ -x y^2 \end{pmatrix}, \quad \bar{p}(t, (x, y)) = e^{-t} \sin\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{y}{\sqrt{2}}\right) \text{ on } \partial D.$$

Thus the exact solution are given by

$$\bar{\mathbf{u}}(x, y) = 10^{-2} e^{-t} \begin{pmatrix} x^2 y \\ -x y^2 \end{pmatrix}, \quad \bar{p}(t, (x, y)) = e^{-t} \sin\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{y}{\sqrt{2}}\right) \text{ on } \partial D.$$

which has already been studied by [35] and [21]. The order of convergence follows that of the previous case, The time-step is given as $\Delta t := h$. The rates of convergence are given in the following Table 4.2.

Mesh size h	8.8e-02	2.2e-02	5.5-03
	Rates of convergence		
$\ \bar{p} - p\ _{L^2((0,T);L^2(D))}$	1.49	1.96	2.05
$\ \bar{p} - p\ _{L^2((0,T);H_0^1(D))}$	0.80	0.97	0.99
$\ \bar{\mathbf{u}} - \mathbf{u}\ _{L^2((0,T);L^2(D))}$	1.90	1.98	1.99
$\ \bar{\mathbf{u}} - \mathbf{u}\ _{L^2((0,T);H_0^1(D))}$	1.99	1.97	1.99

Table 4.2: Convergence of (\mathbf{u}, p) to the solution $(\bar{\mathbf{u}}, \bar{p})$

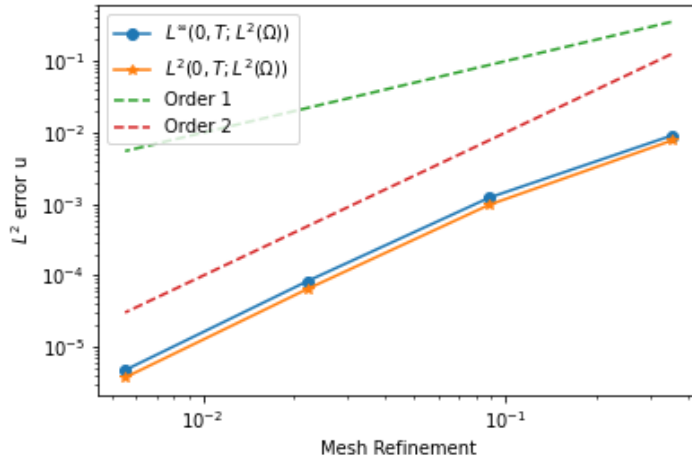


Figure 4.5: Error $L^2((0, T); L^2(D))$ and $L^\infty((0, T); L^2(D))$ for the displacement

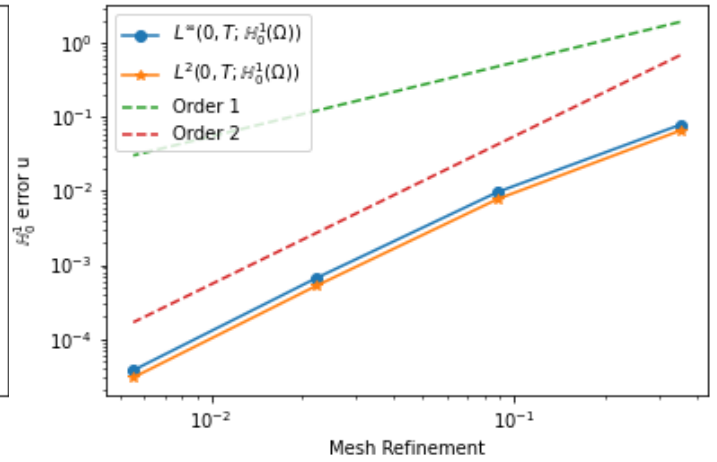


Figure 4.6: Error $L^2(0, T; H_0^1(D))$ and $L^\infty(0, T; H_0^1(D))$ for the displacement

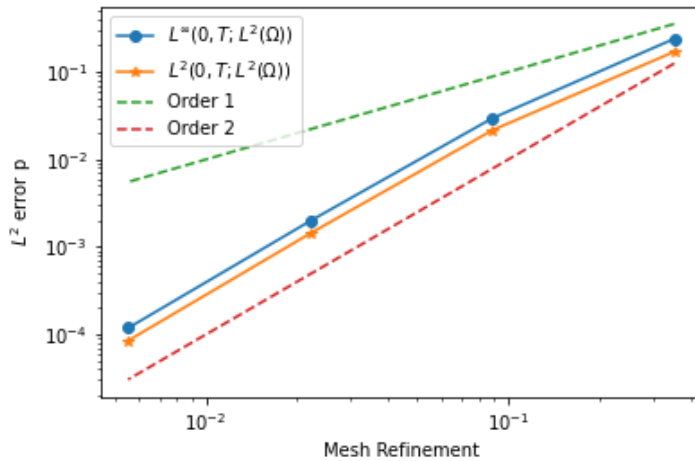


Figure 4.7: Error L^2 in space for the pressure

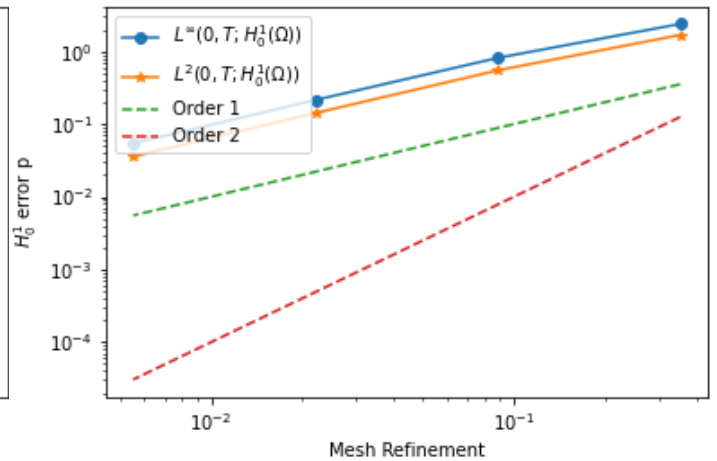


Figure 4.8: Error H_0^1 in space for the pressure

Test case with extraction and injection wells

In this subsection, we consider source terms as an injection and extraction wells. Thus, we set $\mathbb{B}_1 := \{(x, y) \in D, |x - 0.2| \leq 0.1, |y - 0.6| \leq 0.1\}$, $\mathbb{B}_2 := \{(x, y) \in D, |x - 0.8| \leq 0.1, |y - 0.3| \leq 0.1\}$ and $\mathbb{B}_0 := \{(x, y) \in D, \|(x, y) - (0.5, 0.5)\| \leq 0.25\}$.

$$\mathbf{f} = (0, 0), \quad g(t, (x, y)) := -5t \mathbf{1}_{(x, y) \in \mathbb{B}_1} + 5t \mathbf{1}_{(x, y) \in \mathbb{B}_2},$$

with the initial condition

$$\bar{p}_0 = \begin{cases} 5, & (x, y) \in \mathbb{B}_0 \\ 2, & (x, y) \notin \mathbb{B}_0 \end{cases}$$

The computation error is given as follows. We compute (\mathbf{u}, \mathbf{p}) the solution on a fine mesh of 524288 triangles, and compare the solution to this *precise* solution. We remark that the orders in $H_0^1(D)$ are respected, but that the orders

2 have been lost in $L^2(D)$, both in displacement and pressure variables.

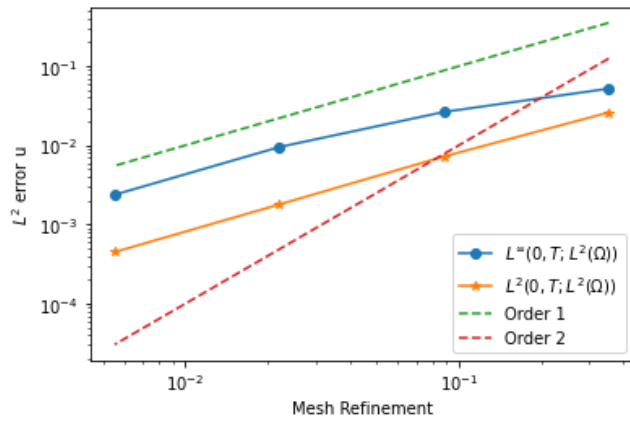


Figure 4.9: Error L^2 in space for the displacement

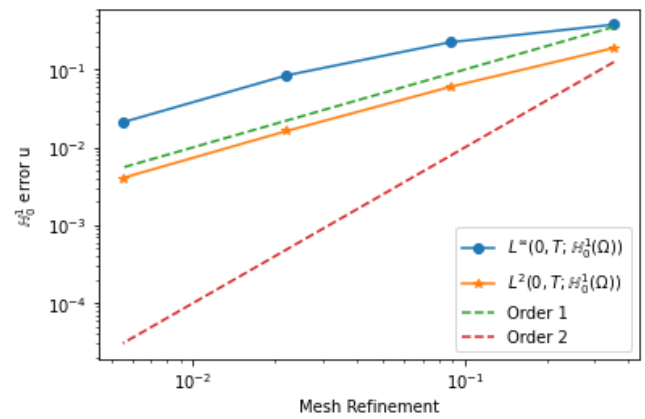


Figure 4.10: Error H_0^1 in space for the displacement

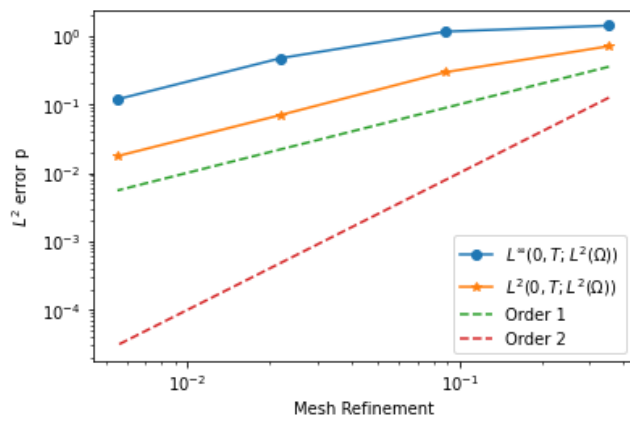


Figure 4.11: Error L^2 in space for the pressure

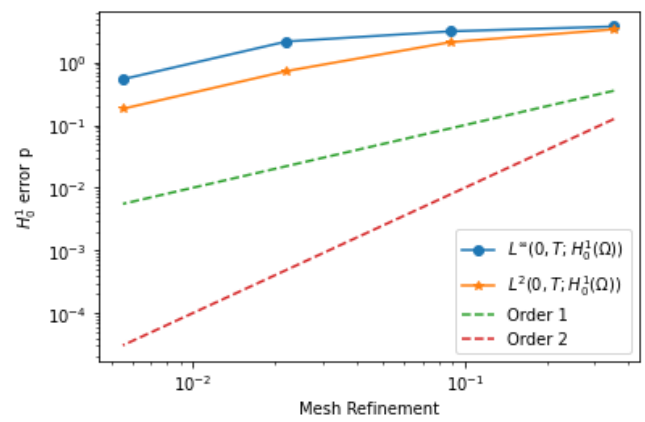


Figure 4.12: Error H_0^1 in space for the pressure

Chapter 5

Error estimate for a Biot problem with truncated random coefficients

5.1 Poroelastic problems and their solutions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T a positive constant and let $D \subset \mathbb{R}^d$ a bounded domain with a smooth boundary ∂D , where d is the space dimension.

We consider again the linear poro-elastic system, which we have studied in Chapter 3, but some of the coefficients are now random. This is motivated by the fact that the coefficients are poorly known by the geologists

$$\begin{aligned}
 -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}) + \alpha \nabla \bar{p} &= \mathbf{f} && \mathbb{P}\text{-a.s. in } D \times [0, T], \\
 \partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) &= g && \mathbb{P}\text{-a.s. in } D \times (0, T), \\
 \bar{p} = 0, \quad \bar{\mathbf{u}} = 0 &&& \mathbb{P}\text{-a.s. on } \partial D \times (0, T), \\
 \bar{p}(0, \cdot) &= \bar{p}_0(x) && \mathbb{P}\text{-a.s. on } D.
 \end{aligned} \tag{5.1}$$

where $\mathbf{f} \in C^1([0, T]; L^2(D))$ and $g \in C^1([0, T]; L^2(D))$ are deterministic source terms, and $\bar{p}_0 \in H^2(D)$. Through out this chapter, we will assume that

Hypothesis 5. *The functions μ, λ, κ are measurable on $\Omega \times D$ such that*

1. *There exist constants $0 < \kappa_0 \leq \mathcal{M}_\kappa < +\infty$ such that*

$$\kappa_0 \leq \kappa \leq \mathcal{M}_\kappa \text{ in } L^\infty(\Omega \times D);$$

2. *There exist constants $0 < \mu_0 \leq \mathcal{M}_\mu < +\infty$ such that*

$$\mu_0 \leq \mu \leq \mathcal{M}_\mu \text{ in } L^\infty(\Omega \times D);$$

3. *There exist constants $0 < \lambda_0 \leq \mathcal{M}_\lambda < +\infty$ such that*

$$\lambda_0 \leq \lambda \leq \mathcal{M}_\lambda \text{ in } L^\infty(\Omega \times D).$$

We recall that, for $\mathbf{f}_0 := \mathbf{f}(t = 0)$ and \bar{p}_0 given, the initial data $\bar{\mathbf{u}}_0$ is defined as the solution of the following problem which is the same as problem (3.12) where we substitute \mathbf{f} by $\mathbf{f}_0 - \alpha \nabla \bar{p}_0$.

$$\begin{aligned}
 -\operatorname{div}(2\mu e(\bar{\mathbf{u}}_0) + \lambda \operatorname{div}(\bar{\mathbf{u}}_0)\operatorname{Id}) &= \mathbf{f}_0 - \alpha \nabla \bar{p}_0 && \mathbb{P}\text{-a.s. in } D \\
 \bar{\mathbf{u}}_0 &= 0 && \mathbb{P}\text{-a.s. on } \partial D.
 \end{aligned} \tag{5.2}$$

Lemma 5.1.1. *There holds*

$$\begin{aligned} \mu_0 \|e(\bar{\mathbf{u}}_0)\|_{L^\infty(\Omega; L^2(D))}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^\infty(\Omega; L^2(D))}^2 \\ \leq \frac{C_K}{4\mu_0} \|\mathbf{f}_0\|_{L^2(D)}^2 + \frac{\alpha^2}{2\lambda_0} \|\bar{p}_0\|_{L^2(D)}^2, \end{aligned} \quad (5.3)$$

and thus also

$$\begin{aligned} \mu_0 \|e(\bar{\mathbf{u}}_0)\|_{L^2(\Omega \times D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(\Omega \times D)}^2 \\ \leq \frac{C_K}{4\mu_0} \|\mathbf{f}_0\|_{L^2(D)}^2 + \frac{\alpha^2}{2\lambda_0} \|\bar{p}_0\|_{L^2(D)}^2. \end{aligned} \quad (5.4)$$

Proof. Both (5.3) and (5.4) follow from (3.13) in Lemma 3.3.1. \square

Next we define the spaces

$$\mathcal{H} := L^2(\Omega, C^1([0, T], H_0^1(D))), \quad \text{and} \quad \mathcal{T} := L^2(\Omega, C^0([0, T], H_0^1(D))) \cap L^2(\Omega, H^1(0, T; L^2(D))). \quad (5.5)$$

Theorem 5.1.1. *There exists a unique solution $(\bar{\mathbf{u}}, \bar{p})$ of Problem (5.1), which is such that*

$$\bar{\mathbf{u}} \in \mathcal{H}, \quad \bar{p} \in \mathcal{T}.$$

More precisely, there exists a positive constant C such that

$$\begin{aligned} \|e(\bar{\mathbf{u}})\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\operatorname{div}(\bar{\mathbf{u}})\|_{L^2(\Omega \times D \times (0, T))}^2 \\ + \|\bar{p}\|_{L^2(\Omega \times D \times (0, T))}^2 \leq C, \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(\Omega \times D \times (0, T))}^2 \\ + \|\partial_t \bar{p}\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\nabla \bar{p}\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))}^2 \\ \leq \frac{16}{\gamma_0} \left(\frac{C_K}{4\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{M}{2} \|f\|_{L^2(D \times (0, T))}^2 \right). \end{aligned} \quad (5.7)$$

Proof. By Hypothesis 5, for \mathbb{P} -almost every $\omega \in \Omega$, the random coefficients satisfy the deterministic hypothesis. Thus Chapter 3 ensures that there exists a unique solution $(\bar{\mathbf{u}}(\omega), \bar{p}(\omega))$ such that

$$\bar{\mathbf{u}}(\omega) \in C^1([0, T], H_0^1(D)),$$

$$\bar{p}(\omega) \in C^0([0, T], L^2(D)) \cap H^1(0, T; L^2(D)).$$

By (3.58), there holds \mathbb{P} -almost surely

$$\begin{aligned} \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\ + \|\partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 + \|\nabla \bar{p}(t)\|_{L^2(D \times (0, T))}^2 \\ \leq \frac{16}{\gamma_0} \left(\frac{C_K}{4\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{M}{2} \|g\|_{L^2(D \times (0, T))}^2 \right), \end{aligned} \quad (5.8)$$

where $\gamma_0 = \min(\mu_0, \lambda_0, \frac{\kappa_0}{2}, \frac{1}{2M})$. Moreover, the mapping

$$\omega \in \Omega \mapsto (\bar{\mathbf{u}}(\omega), \bar{p}(\omega)) \quad (5.9)$$

is measurable as a composition between measurable random coefficients and continuous function of the input coefficients μ , λ and κ . Note that (5.8) implies (5.7).

Similarly, we deduce from (3.47) that there exists a non-random constant $C = C(T)$ such that \mathbb{P} -almost surely

$$\begin{aligned} & \sup_{t \in [0, T]} \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \sup_{t \in [0, T]} \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\ & + \sup_{t \in [0, T]} \frac{1}{2M} \|\bar{p}(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 \leq C, \end{aligned} \quad (5.10)$$

Taking the expectation in (5.10) under Hypothesis 5 yields (5.6). \square

5.2 The Karhunen-Loève expansions

5.2.1 Description

Given an open bounded domain $D \subset \mathbb{R}^d$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Karhunen-Loève expansion of a stochastic process

$$Z : \Omega \times \bar{D} \mapsto \mathbb{R}$$

is a spectral decomposition, that decomposes the process in terms of components that are both orthogonal on \bar{D} and Ω . More precisely we consider a process $Z \in L^2(\Omega; L^2(\bar{D}))$ which verifies the following hypothesis.

Hypothesis 6. • For all $x \in \bar{D}$, $\mathbb{E}[Z(x)^2] < \infty$,

- Z is a zero-mean process, $\mathbb{E}[Z(x)] = 0$ for all $x \in \bar{D}$,
- For all $x, y \in \bar{D}$, the covariance function $\operatorname{cov}[Z](x, y) = \mathbb{E}(Z(x)Z(y))$ is well-defined, and continuous on $\bar{D} \times \bar{D}$.

We define the self-adjoint operator T_Z as

$$T_Z : f \in L^2(\bar{D}) \mapsto \int_{\bar{D}} \operatorname{cov}[Z](x, \cdot) f(x) dx \in L^2(\bar{D}) \quad (5.11)$$

and note $\{\lambda_n, \beta_n\}$ the sequence of eigenpairs associated with the operator T_Z .

Theorem 5.2.1. *The stochastic process Z admits the following representation*

$$Z(\omega, x) = \sum_{n=1}^{\infty} \sqrt{\eta_n} \beta_n(x) Y_n(\omega), \quad (5.12)$$

where the real random variables $Y_n(\omega)$ are uniquely defined by

$$Y_n(\omega) = \frac{1}{\sqrt{\eta_n}} \int_{\bar{D}} Z(\omega, x) \beta_n(x) dx. \quad (5.13)$$

They are uncorrelated random variables with zero mean and unit variance.

Moreover, we define

$$Z_N(\omega, x) := \sum_{n=1}^N \sqrt{\eta_n} \beta_n(x) Y_n(\omega) \quad (5.14)$$

so that there holds

$$\sup_{x \in \bar{D}} \mathbb{E}[(Z - Z_N)^2(x)] \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

Proof. The results of Theorem 5.2.1 are presented in [5], [40], [28]. \square

5.2.2 Convergence Results

We follow the articles of Andrea Barth and Andreas Stein ([6] and [7]), to construct well-defined coefficients. We suppose that the following hypothesis holds. It will ensure that Hypothesis 5 is verified.

Hypothesis 7. *There exist nonnegative, continuous and bounded function $\bar{\kappa}, \bar{\mu}, \bar{\lambda} \in C^1(\bar{D})$, continuously differentiable positive functions $\varphi_\kappa, \varphi_\mu, \varphi_\lambda \in C^1(\mathbb{R})$, and a Gaussian field $W \in L^2(\Omega, \bar{D})$, such that*

- $\kappa(\omega, x) = \bar{\kappa}(x) + \varphi_\kappa(W(\omega, x))$,
- $\mu(\omega, x) = \bar{\mu}(x) + \varphi_\mu(W(\omega, x))$,
- $\lambda(\omega, x) = \bar{\lambda}(x) + \varphi_\lambda(W(\omega, x))$,

where W is a zero-mean Gaussian random field associated to a covariance operator Q . Moreover, we assume for all $x \in \bar{D}$ that

$$\kappa_0/2 \leq \bar{\kappa}(x) \leq \mathcal{M}_\kappa/2, \quad \mu_0/2 \leq \bar{\mu}(x) \leq \mathcal{M}_\mu/2, \quad \lambda_0/2 \leq \bar{\lambda}(x) \leq \mathcal{M}_\lambda/2,$$

and $\varphi_\kappa, \varphi_\mu, \varphi_\lambda$ verifies that there exists ϕ_0 and $\phi_1 > 0$ such that for all $x \in \mathbb{R}$ $|\varphi'_\kappa(x)| + |\varphi'_\mu(x)| + |\varphi'_\lambda(x)| \leq \phi_0 \exp(\phi_1 x)$ and that

$$\kappa_0/2 \leq \varphi_\kappa(x) \leq \mathcal{M}_\kappa/2, \quad \mu_0/2 \leq \varphi_\mu(x) \leq \mathcal{M}_\mu/2, \quad \lambda_0/2 \leq \varphi_\lambda(x) \leq \mathcal{M}_\lambda/2,$$

so that the Hypothesis 5 still holds, i.e. \mathbb{P} -almost surely we have for all $x \in \bar{D}$

$$\kappa_0 \leq \kappa(\omega, x) \leq \mathcal{M}_\kappa, \quad \mu_0 \leq \mu(\omega, x) \leq \mathcal{M}_\mu, \quad \lambda_0 \leq \lambda(\omega, x) \leq \mathcal{M}_\lambda. \quad (5.15)$$

We write the random field W in the form of its Karhunen-Loève expansion

$$W(\omega, x) = \sum_{n=1}^{\infty} \sqrt{\eta_n} \beta_n(x) Y_n(\omega). \quad (5.16)$$

The corresponding truncated expansion W_N is given by

$$W_N(\omega, x) = \sum_{n=1}^N \sqrt{\eta_n} \beta_n(x) Y_n(\omega). \quad (5.17)$$

We recall below some results from literature ([6], [7], [18]) which ensure the convergence of the Karhunen-Loève truncature. To that purpose, we make the following

Hypothesis 8. *The eigenfunctions β_n are continuously differentiable on D , there exists positive constants C_1, C_2, ζ, γ such that $2\zeta \leq \gamma$, there holds for all $n \in \mathbb{N}$*

$$\|\beta_n\|_{L^\infty(D)} \leq C_1, \quad \max_{l=1, \dots, d} \|\partial_{x_l} \beta_n\|_{L^\infty(D)} \leq C_1 n^\zeta, \quad \sum_{n=1}^{\infty} \eta_n n^\gamma \leq C_2$$

Define for all $N \in \mathbb{N}$, $\Theta_N := \sum_{n=N}^{\infty} \eta_n$, then under Hypothesis 8 we have

$$\lim_{N \rightarrow +\infty} \Theta_N = 0. \quad (5.18)$$

Theorem 5.2.2. *Let Hypothesis 8 holds. Then for any $N \in \mathbb{N}^*$ we have that*

$$\|W - W_N\|_{L^2(\Omega, L^\infty(D))} \leq C \Theta_N^{1/2},$$

where the positive constant C is independent of N .

Proof. We apply Theorem 3.8 in [6]. □

The approximate coefficients are defined by

$$\kappa_N(\omega) = \bar{\kappa}(x) + \varphi_\kappa(W_N(\omega, x)) \quad (5.19a)$$

$$\mu_N(\omega) = \bar{\mu}(x) + \varphi_\mu(W_N(\omega, x)) \quad (5.19b)$$

$$\lambda_N(\omega) = \bar{\lambda}(x) + \varphi_\lambda(W_N(\omega, x)), \quad (5.19c)$$

so that

$$\kappa_0 \leq \kappa_N \leq \mathcal{M}_\kappa, \quad \mu_0 \leq \mu_N \leq \mathcal{M}_\mu, \quad \lambda_0 \leq \lambda_N \leq \mathcal{M}_\lambda. \quad (5.20)$$

We can also deduce the following results from [6] Theorem 3.12,

Theorem 5.2.3. *Let Hypothesis 8 hold. Then there exists $C > 0$ such that for any $N \in \mathbb{N}$,*

$$\begin{aligned} \|\mu_N - \mu\|_{L^2(\Omega, L^\infty(D))} &\leq C\Theta_N^{1/2}, \\ \|\lambda_N - \lambda\|_{L^2(\Omega, L^\infty(D))} &\leq C\Theta_N^{1/2}, \\ \|\kappa_N - \kappa\|_{L^2(\Omega, L^\infty(D))} &\leq C\Theta_N^{1/2}. \end{aligned}$$

Hence, we deduce from (5.18) that $(\mu_N, \lambda_N, \kappa_N)$ strongly converge to (μ, λ, κ) in $L^2(\Omega, L^\infty(D))$ as $N \rightarrow +\infty$.

5.3 The Biot problem with truncated parameters

Next, we introduce the Biot problem with truncated coefficients.

$$\begin{aligned} -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_N) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_N) \operatorname{Id}) + \alpha \nabla \bar{p}_N &= \mathbf{f} && \mathbb{P} - a.s. \text{ in } D \times [0, T) \\ \partial_t \left(\frac{1}{M} p_N + \alpha \operatorname{div}(\bar{\mathbf{u}}_N) \right) - \operatorname{div}(\kappa_N \nabla \bar{p}_N) &= g && \mathbb{P} - a.s. \text{ in } D \times (0, T) \\ p_N = 0, \quad \bar{\mathbf{u}}_N = 0 &&& \mathbb{P} - a.s. \text{ on } \partial D \times [0, T) \\ p_N(\omega, x, 0) &= \bar{p}_0(x) && \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega \text{ and for all } x \in D. \end{aligned} \quad (5.21)$$

The following result holds

Theorem 5.3.1. *We suppose that Hypothesis 7 holds and fix $N \in \mathbb{N}$. For \mathbb{P} -almost every $\omega \in \Omega$, there exists a unique solution $(\bar{\mathbf{u}}_N, p_N) \in \mathcal{H} \times \mathcal{T}$ of Problem (5.21). It is such that there exists a deterministic constant $C = C(T)$ such that*

$$\|e(\bar{\mathbf{u}}_N)\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\operatorname{div}(\bar{\mathbf{u}}_N)\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\bar{p}_N\|_{L^2(\Omega \times D \times (0, T))}^2 \leq C, \quad (5.22)$$

and

$$\begin{aligned} &\|\partial_t e(\bar{\mathbf{u}}_N)\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\partial_t \operatorname{div}(\bar{\mathbf{u}}_N)\|_{L^2(\Omega \times D \times (0, T))}^2 \\ &+ \|\partial_t p_N\|_{L^2(\Omega \times D \times (0, T))}^2 + \|\nabla \bar{p}_N\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))}^2 \\ &\leq C \left(\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \|g\|_{L^2(D \times (0, T))}^2 \right), \end{aligned} \quad (5.23)$$

where the constant C does not depend on N .

Proof. The proof is similar to that of Theorem 5.1.1. □

5.4 The main error estimate

The main result of this chapter are the following error estimates.

Theorem 5.4.1. *There exists a deterministic constant $C = C(T)$ such that*

$$\begin{aligned} & \mathbb{E} \left(\sup_{t \in [0, T]} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left(\sup_{t \in [0, T]} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) \\ & + \mathbb{E} \left(\sup_{t \in (0, T)} \|(\bar{p} - \bar{p}_N)(t)\|_{L^2(D)}^2 \right) + \mathbb{E} \left(\int_0^T \int_D \|\nabla(\bar{p} - \bar{p}_N)(t)\|^2 dx dt \right) \\ & \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right) \end{aligned} \quad (5.24)$$

which tends to 0 as $N \rightarrow +\infty$ by Theorem 5.2.3.

Before proving Theorem 5.4.1 we show a convergence result for the initial function $\bar{\mathbf{u}}_{N,0}$.

We recall that given $\bar{p}_N(t=0) := \bar{p}_0$, $\bar{\mathbf{u}}_{N,0}$ is defined as the solution of

$$\begin{aligned} -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_{N,0}) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_{N,0})) &= \mathbf{f}(0) - \alpha \nabla \bar{p}_0, & \mathbb{P} - a.s. \text{ in } D, \\ \bar{\mathbf{u}}_{N,0} &= 0, & \mathbb{P} - a.s. \text{ on } \partial D. \end{aligned} \quad (5.25)$$

In view of the regularity of the data, the equation (5.25) is satisfied almost everywhere in D . Thus, given $\bar{\mathbf{u}}_0$ and $\bar{\mathbf{u}}_{N,0}$ the following results holds

Lemma 5.4.1. *Given $\bar{\mathbf{u}}_0$ the solution of (5.2) and $\bar{\mathbf{u}}_{N,0}$ the solution of (5.25), then there exists a deterministic constant $C = C(T)$ which does not depend on N such that*

$$\begin{aligned} & \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right) \\ & \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right). \end{aligned} \quad (5.26)$$

Proof. Let $\bar{\mathbf{u}}_0$ be the solution of (5.2) and let $\bar{\mathbf{u}}_{N,0}$ be the solution of (5.25). There holds that

$$\begin{aligned} & -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) \operatorname{Id}) \\ &= -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_0) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id}) \\ & \quad - \left(-\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_{N,0}) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_{N,0}) \operatorname{Id}) \right) \\ &= -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_0) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id}) \\ & \quad - \left(-\operatorname{div}(2\mu e(\bar{\mathbf{u}}_0) + \lambda \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id}) \right) \\ &= -\operatorname{div}(2(\mu_N - \mu)e(\bar{\mathbf{u}}_0) + (\lambda_N - \lambda)\operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{Id}). \end{aligned} \quad (5.27)$$

We multiply (5.27) by $\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}$ and integrate in $\Omega \times D$ to obtain in view of (3.7)

$$\begin{aligned} & \mathbb{E} \left(\int_D 2\mu_N |e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 + \lambda_N |\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 dx \right) \\ &= \mathbb{E} \left(\int_D 2(\mu_N - \mu)e(\bar{\mathbf{u}}_0) : e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) + (\lambda_N - \lambda)\operatorname{div}(\bar{\mathbf{u}}_0)\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) dx \right) \\ & \leq \mathbb{E} \left(2\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\ & \quad + \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right). \end{aligned} \quad (5.28)$$

From (5.3) there exists a positive constant C such that

$$\begin{aligned} & \mathbb{E} \left(\int_D 2\mu_N |e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 + \lambda_N |\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 dx \right) \\ & \leq C \left(\mathbb{E} (\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}) \right. \\ & \quad \left. + \mathbb{E} (\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}) \right). \end{aligned} \quad (5.29)$$

Applying Young's inequality in (5.29) we obtain,

$$\begin{aligned} & 2\mu_0 \mathbb{E} \left(\|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right) + \lambda_0 \mathbb{E} \left(\|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 dx \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 + \mu_0 \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right) \\ & + C^2 \mathbb{E} \left(\frac{1}{2\lambda_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 + \frac{1}{2\lambda_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right) \\ & + \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right), \end{aligned}$$

which in turn implies (5.26). □

Next, we prove Theorem 5.4.1.

Proof of Theorem 5.4.1. We remark that by (5.1) and (5.21), there holds

$$\begin{aligned} & -\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}) + \alpha \nabla \bar{p} \\ & = -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_N) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_N)\operatorname{Id}) + \alpha \nabla \bar{p}_N \end{aligned} \quad (5.30)$$

in $L^2(\Omega \times D \times (0, T))$, and that

$$\begin{aligned} & \partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) \\ & = \partial_t \left(\frac{1}{M} \bar{p}_N + \alpha \operatorname{div}(\bar{\mathbf{u}}_N) \right) - \operatorname{div}(\kappa_N \nabla \bar{p}_N) \end{aligned} \quad (5.31)$$

$L^2(\Omega \times D \times (0, T))$. We deduce from (5.30) that

$$\begin{aligned} & -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) + \lambda_N \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\operatorname{Id}) + \alpha \nabla (\bar{p} - \bar{p}_N) \\ & = -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}) + \lambda_N \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}) + \alpha \nabla \bar{p} \\ & \quad - \left(-\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}_N) + \lambda_N \operatorname{div}(\bar{\mathbf{u}}_N)\operatorname{Id}) + \alpha \nabla \bar{p}_N \right) \\ & = -\operatorname{div}(2\mu_N e(\bar{\mathbf{u}}) + \lambda_N \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}) + \alpha \nabla \bar{p} \\ & \quad - \left(-\operatorname{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}) + \alpha \nabla \bar{p} \right) \\ & = -\operatorname{div}(2(\mu_N - \mu)e(\bar{\mathbf{u}}) + (\lambda_N - \lambda)\operatorname{div}(\bar{\mathbf{u}})\operatorname{Id}). \end{aligned} \quad (5.32)$$

We deduce from (5.31) that

$$\begin{aligned}
& \partial_t \left(\frac{1}{M} (\bar{p} - \bar{p}_N) + \alpha \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) \right) - \operatorname{div}(\kappa_N \nabla (\bar{p} - \bar{p}_N)) \\
&= \partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa_N \nabla \bar{p}) \\
&\quad - \left(\partial_t \left(\frac{1}{M} \bar{p}_N + \alpha \operatorname{div}(\bar{\mathbf{u}}_N) \right) - \operatorname{div}(\kappa_N \nabla \bar{p}_N) \right) \\
&= \partial_t \left(\frac{1}{M} \bar{p} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) - \operatorname{div}(\kappa \nabla \bar{p}) + \operatorname{div}(\kappa \nabla \bar{p}) - \operatorname{div}(\kappa_N \nabla \bar{p}) \\
&\quad - \left(\partial_t \left(\frac{1}{M} \bar{p}_N + \alpha \operatorname{div}(\bar{\mathbf{u}}_N) \right) - \operatorname{div}(\kappa_N \nabla \bar{p}_N) \right) \\
&= -\operatorname{div}((\kappa_N - \kappa) \nabla \bar{p}).
\end{aligned} \tag{5.33}$$

We multiply (5.32) by $\partial_t(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)$ and integrate in $\Omega \times D \times (0, T)$ to obtain in view of (3.7)

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D \mu_N \partial_t |e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)|^2 + \frac{\lambda_N}{2} \partial_t |\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)|^2 dx d\tau \right) \\
&\quad - \alpha \mathbb{E} \left(\int_0^t \int_D (\bar{p} - \bar{p}_N) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
&= \mathbb{E} \left(\int_0^t \int_D 2(\mu - \mu_N) e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) + (\lambda - \lambda_N) \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right).
\end{aligned} \tag{5.34}$$

We multiply (5.33) by $\bar{p} - \bar{p}_N$ and integrate in $\Omega \times D \times (0, T)$ to obtain

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D \frac{1}{2M} \partial_t |\bar{p} - \bar{p}_N|^2 + \kappa_N |\nabla(\bar{p} - \bar{p}_N)|^2 dx d\tau \right) \\
&\quad + \alpha \mathbb{E} \left(\int_0^t \int_D \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) (\bar{p} - \bar{p}_N) dx d\tau \right) \\
&= \mathbb{E} \left(\int_0^t \int_D (\kappa_N - \kappa) \nabla \bar{p} \cdot \nabla (\bar{p} - \bar{p}_N) dx d\tau \right).
\end{aligned} \tag{5.35}$$

We deduce from (5.34)

$$\begin{aligned}
& \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D)}^2 \right) \\
&\quad - \mathbb{E} \left(\int_0^t \int_D \alpha (\bar{p} - \bar{p}_N) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
&\leq \mathbb{E} \left(\int_0^t \int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) + (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
&\quad + \mathbb{E} \left(\int_D \mu_N |e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 + \frac{\lambda_N}{2} |\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 dx \right)
\end{aligned} \tag{5.36}$$

and since $p_{N,0} = \bar{p}_0$

$$\begin{aligned}
& \mathbb{E} \left(\int_D \frac{1}{2M} |p(t) - p_N(t)|^2 dx \right) \\
& + \mathbb{E} \left(\int_0^t \int_D \alpha \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(p - p_N) dx d\tau \right) \\
& \quad + \mathbb{E} \left(\int_0^t \int_D \kappa_0 |\nabla(\bar{p} - \bar{p}_N)|^2 dx d\tau \right) \\
& \leq \mathbb{E} \left(\int_0^t \int_D (\kappa_N - \kappa) \nabla \bar{p} \cdot \nabla(\bar{p} - \bar{p}_N) dx d\tau \right).
\end{aligned} \tag{5.37}$$

We sum (5.36) with (5.37) to obtain

$$\begin{aligned}
& \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) \\
& + \mathbb{E} \left(\frac{1}{2M} \|p(t) - p_N(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla(p - p_N)\|_{L^2(D \times (0,t))}^2 \right) \\
& \leq \mathbb{E} \left(\int_0^t \int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& \quad + \mathbb{E} \left(\int_0^t \int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& \quad + \mathbb{E} \left(\int_0^t \int_D (\kappa_N - \kappa) \nabla \bar{p} \cdot \nabla(\bar{p} - \bar{p}_N) dx d\tau \right) \\
& + \mathbb{E} \left(\int_D \mu_N |e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 + \frac{\lambda_N}{2} |\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})|^2 dx \right).
\end{aligned} \tag{5.38}$$

Next we integrate by parts the first term on the right-hand-side of (5.38). There holds

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& = \mathbb{E} \left(\int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}(t)) : e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t) dx \right) \\
& - \mathbb{E} \left(\int_0^t \int_D 2(\mu_N - \mu) \partial_t e(\bar{\mathbf{u}}) : e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& - \mathbb{E} \left(\int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}_0) : e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) dx \right).
\end{aligned} \tag{5.39}$$

We estimate each term on the right-hand-side of (5.39) by Hölder's inequality. By (5.10), there exists a positive constant C such that

$$\|e(\bar{\mathbf{u}}(t))\|_{L^2(D)} \leq C \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}(t)) : e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t) dx \right) \\
& \leq \mathbb{E} \left(2 \|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right).
\end{aligned} \tag{5.40}$$

By (5.8) there exists a positive constant C such that

$$\|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0,T))} \leq C, \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D 2(\mu_N - \mu) \partial_t e(\bar{\mathbf{u}}) : e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& \leq 2 \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|\partial_t e(\bar{\mathbf{u}})\|_{L^2(D \times (0,t))} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))} \right) \\
& \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))} \right).
\end{aligned} \tag{5.41}$$

By (3.13) there exists a positive constant C such that

$$\|e(\bar{\mathbf{u}}_0)\|_{L^2(D)} \leq C \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_D 2(\mu_N - \mu) e(\bar{\mathbf{u}}_0) : e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) \right) \\
& \leq 2 \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)} \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\
& \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right).
\end{aligned} \tag{5.42}$$

Then we integrate by parts the second term on the right-hand-side of (5.38). We obtain

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& = \mathbb{E} \left(\int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}(t)) \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t) dx \right) \\
& - \mathbb{E} \left(\int_0^t \int_D (\lambda_N - \lambda) \partial_t \operatorname{div}(\bar{\mathbf{u}}) \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& - \mathbb{E} \left(\int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) dx \right).
\end{aligned} \tag{5.43}$$

Next we estimate each term on the right-hand-side of (5.43) by applying Hölder's inequality. By (5.10), there exists a positive constant C such that

$$\|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)} \leq C \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}(t)) \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t) dx \right) \\
& \leq \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& \leq C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right).
\end{aligned} \tag{5.44}$$

By (5.8) there exists a positive constant C such that

$$\|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0,T))} \leq C \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D (\lambda_N - \lambda) \partial_t \operatorname{div}(\bar{\mathbf{u}}) \operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N) dx d\tau \right) \\
& \leq \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\partial_t \operatorname{div}(\bar{\mathbf{u}})\|_{L^2(D \times (0,t))} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))} \right) \\
& \leq C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))} \right).
\end{aligned} \tag{5.45}$$

By (5.3) there exists a positive constant C such that

$$\|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)} \leq C \mathbb{P}\text{-almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_D (\lambda_N - \lambda) \operatorname{div}(\bar{\mathbf{u}}_0) \operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0}) \right) \\
& \leq \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\
& \leq C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right).
\end{aligned} \tag{5.46}$$

By (5.10) there exists a positive constant C such that

$$\|\nabla \bar{p}\|_{L^2(D \times (0, T))} \leq C \text{ almost surely,}$$

so that

$$\begin{aligned}
& \mathbb{E} \left(\int_0^t \int_D (\kappa_N - \kappa) \nabla \bar{p} \cdot \nabla (\bar{p} - \bar{p}_N) dx d\tau \right) \\
& \leq \mathbb{E} \left(\|\kappa_N - \kappa\|_{L^\infty(D)} \|\nabla \bar{p}\|_{L^2(D \times (0, t))} \|\nabla (\bar{p} - \bar{p}_N)\|_{L^2(D \times (0, t))} \right) \\
& \leq C \mathbb{E} \left(\|\kappa_N - \kappa\|_{L^\infty(D)} \|\nabla (\bar{p} - \bar{p}_N)\|_{L^2(D \times (0, t))} \right).
\end{aligned} \tag{5.47}$$

Substituting (5.39), (5.40), (5.41), (5.42) and (5.43), (5.44), (5.45), (5.46) and (5.47) in the right-hand-side of (5.38), we deduce that there exists a positive constant C such that

$$\begin{aligned}
& \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) \\
& + \mathbb{E} \left(\frac{1}{2M} \|p(t) - p_N(t)\|_{L^2(D)}^2 + \kappa_0 \|\nabla(p - p_N)\|_{L^2(D \times (0, t))}^2 \right) \\
& \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& + C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0, t))} \right) \\
& + C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& + C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0, t))} \right) \\
& + C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\
& + C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\
& + C \mathbb{E} \left(\|\kappa_N - \kappa\|_{L^\infty(D)} \|\nabla(\bar{p} - \bar{p}_N)\|_{L^2(D \times (0, t))} \right).
\end{aligned} \tag{5.48}$$

We apply Young's inequality to the right-hand-side of (5.48). There holds that

$$\begin{aligned}
& C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& \leq C^2 \mathbb{E} \left(\frac{1}{2\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right),
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
& C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0, t))} \right) \\
& \leq C^2 \mathbb{E} \left(\frac{1}{2\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0, t))}^2 \right),
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
& C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)} \right) \\
& \leq C^2 \mathbb{E} \left(\frac{1}{\lambda_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right),
\end{aligned} \tag{5.51}$$

and

$$\begin{aligned} & C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))} \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{\lambda_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))}^2 \right). \end{aligned} \quad (5.52)$$

There holds that

$$\begin{aligned} & C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{4\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\mu_0 \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right), \end{aligned} \quad (5.53)$$

and

$$\begin{aligned} & C \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{2\lambda_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)}^2 \right). \end{aligned} \quad (5.54)$$

It follows from (5.53), (5.54) and (5.26) that there exist a positive constant C such that

$$\begin{aligned} & \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)} \|e(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\ & + \mathbb{E} \left(\|\lambda_N - \lambda\|_{L^\infty(D)} \|\operatorname{div}(\bar{\mathbf{u}}_0 - \bar{\mathbf{u}}_{N,0})\|_{L^2(D)} \right) \\ & \leq C \mathbb{E} \left(\frac{1}{4\mu_0} \|\mu_N - \mu\|_{L^\infty(D)}^2 + \frac{1}{2\mu_0} \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right) \\ & + C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 \right). \end{aligned} \quad (5.55)$$

Finally there holds that

$$\begin{aligned} & C \mathbb{E} \left(\|\kappa_N - \kappa\|_{L^\infty(D)} \|\nabla(\bar{p} - \bar{p}_N)\|_{L^2(D \times (0,t))} \right) \\ & \leq C^2 \mathbb{E} \left(\frac{1}{2\kappa_0} \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right) + \mathbb{E} \left(\frac{\kappa_0}{2} \|\nabla(\bar{p} - \bar{p}_N)\|_{L^2(D \times (0,t))}^2 \right). \end{aligned} \quad (5.56)$$

Thus from (5.48), there exists a positive constant C such that

$$\begin{aligned} & \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) \\ & + \mathbb{E} \left(\frac{1}{2M} \|\bar{p}(t) - \bar{p}_N(t)\|_{L^2(D)}^2 + \frac{\kappa_0}{2} \|\nabla(\bar{p} - \bar{p}_N)\|_{L^2(D \times (0,t))}^2 \right) \\ & \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right) \\ & + \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^2(D \times (0,t))}^2 \right). \end{aligned} \quad (5.57)$$

We set

$$z(t) = \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right),$$

and

$$b = C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right),$$

so that there holds $z(t) \leq b + \int_0^t z(\tau) d\tau$. By Gronwall's Lemma 3.6.1 there holds that

$$\begin{aligned} & \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)(t)\|_{L^2(D)}^2 \right) \\ & \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right) e^t. \end{aligned}$$

We substitute that into the right-hand-side of Equation (5.57), thus we obtain a uniform bound in time of the left-hand-side. There exists a deterministic constant $C = C(T)$ which does not depend on N such that

$$\begin{aligned} & \mathbb{E} \left(\frac{\mu_0}{2} \|e(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^\infty((0,T);L^2(D))}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\bar{\mathbf{u}} - \bar{\mathbf{u}}_N)\|_{L^\infty((0,T);L^2(D))}^2 \right) \\ & + \mathbb{E} \left(\frac{1}{2M} \|p - p_N\|_{L^\infty((0,T);L^2(D))}^2 + \frac{\kappa_0}{2} \|\nabla(p - p_N)\|_{L^2(D \times (0,T))}^2 \right) \\ & \leq C \mathbb{E} \left(\|\mu_N - \mu\|_{L^\infty(D)}^2 + \|\lambda_N - \lambda\|_{L^\infty(D)}^2 + \|\kappa_N - \kappa\|_{L^\infty(D)}^2 \right), \end{aligned}$$

which completes the proof of (5.24). □

Chapter 6

Quantification of uncertainties for a thermoporoelastic problem

6.1 Introduction

Thermo-poroelasticity is a multidisciplinary field that explores the coupled interactions between thermal, hydraulic, and mechanical processes within porous materials. These interactions are critical for understanding and predicting the behavior of geological formations, engineered systems, and biological tissues under various environmental conditions. Traditional models of thermo-poroelasticity typically assume deterministic coefficients (see [44]); however, in real-world applications, many of these coefficients exhibit inherent uncertainties due to spatial variability and measurement limitations.

In this chapter, we extend the classical thermo-poroelasticity framework to incorporate random coefficients, specifically focusing on the permeability, Lamé coefficients, and thermal diffusivity. By considering these coefficients as random fields, we aim to capture the uncertainty and variability present in natural and engineered porous media.

To address the computational challenges posed by the stochastic nature of the problem, we apply a Monte Carlo method combined with a finite element discretization. Several methods can be used to discretize thermo-poroelastic problem as finite element method or finite volume method, see [38]. We use P2 elements for the mechanical displacements, P1 elements for the pressure, and P1 elements for the temperature. This discretization scheme ensures a high level of accuracy and stability in the numerical solutions.

We derive an error estimate for the numerical method, which quantifies the accuracy of the finite element approximation in the presence of random coefficients.

This chapter is structured as follows. We present the mathematical formulation of the thermo-poroelasticity equations with random coefficients. Then we describe the numerical method and the finite element discretization. We prove an error estimate combining the finite element approximation and a statistical error from the Monte-Carlo method. Numerical results are presented at the end of this chapter.

The problem which we consider is the following linear thermo-poroelastic problem, modeling the pressure flow and the temperature in the ground.

$$(\mathbb{P}) \begin{cases} -\operatorname{div}\left(2\mu e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}})\right) + \alpha \nabla \bar{p} + (1 - \alpha)K_S \nabla \bar{\theta} = \mathbf{f} & \text{in } D \times [0, T) \\ \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}})\right) - \operatorname{div}(\kappa \nabla \bar{p}) = g & \text{in } D \times [0, T) \\ \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M}\right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S(1 - \alpha) \operatorname{div}(\bar{\mathbf{u}})\right) - \eta \cdot \nabla \bar{\theta} - \operatorname{div}(\beta \nabla \bar{\theta}) = h & \text{in } D \times [0, T) \\ \bar{\mathbf{u}}, \bar{p}, \bar{\theta} = 0 & \text{in } \partial D \times [0, T), \\ \bar{p}(\cdot, 0) = \bar{p}_0(\cdot), \bar{\theta}(\cdot, 0) = \bar{\theta}_0(\cdot) & \text{in } D. \end{cases} \quad (6.1)$$

Problem (\mathbb{P}) is similar to the linear problem introduced in [17] (2.7a) – (2.7f), where they study a corresponding mixed-formulation. We will use their results about existence and uniqueness of the solution. We introduce η to

replace the convective transport term $\nabla \bar{p} \cdot \nabla \bar{\theta}$, so that the convective terms in the heat equation in (6.1) is replaced by a linear terms. In the following, η is a constant, but it could be a space function, as soon as $\eta \in L^\infty(D)$ is small enough.

We remark that given $\bar{p}(0, \cdot)$ and $\bar{\theta}(0, \cdot) \in L^2(D)$ being defined, we can solve the mechanical equation at time $t = 0$ by applying Lax-Milgram theorem on a suitable variational formulation, together with Korn inequality.

Introducing random parameters into the system of thermo-poroelasticity equations is crucial due to the inherent uncertainties in real-world geothermal phenomena. The material properties, such as Young's modulus (E), permeability (κ), Lamé coefficients (ν) and thermal diffusivity (β), can vary significantly based on environmental factors and the specific conditions of the study site. These variations and uncertainties may stem from measurement errors, material imperfections, or spatial variations that are difficult to precisely characterize. Therefore, modeling these parameters as random variables is essential for achieving more realistic and robust simulations. The stochastic approach helps capture uncertainties more effectively and provides more reliable estimates of geothermal behaviors, facilitating uncertainty quantification (UQ) and statistical analysis of the results.

Later, in Section 6.3, we introduce the **random** linear thermo-poroelastic problem, which contains randomness on the parameters of the problem. The parameters μ , λ , κ and β are considered to be random, so that they depend on a parameter $\omega \in \Omega$. We will discuss of their regularity later.

6.2 Existence, uniqueness and a priori estimates

We first consider the problem with deterministic coefficients. To ensure the existence and uniqueness of the solution, and to ensure bounds on the solution, we assume the following hypotheses on the parameters and functions involved in the system.

Hypothesis 9. *The coefficient μ , λ , κ and β satisfy*

$$\mu \in C^1(D), \lambda \in C^1(D), \kappa \in C^1(D), \beta \in C^1(D).$$

Moreover, there exists positive constants

$$\mu_0, \lambda_0, \kappa_0, \beta_0$$

and

$$\mathcal{M}_\mu, \mathcal{M}_\lambda, \mathcal{M}_\kappa, \mathcal{M}_\beta$$

such that

$$\begin{aligned} 0 < \mu_0 \leq \mu(x) \leq \mathcal{M}_\mu < \infty, \quad 0 < \lambda_0 \leq \lambda(x) \leq \mathcal{M}_\lambda < \infty, \\ 0 < \kappa_0 \leq \kappa(x) \leq \mathcal{M}_\kappa < \infty, \quad 0 < \beta_0 \leq \beta(x) \leq \mathcal{M}_\beta < \infty, \end{aligned} \tag{6.2}$$

and $\beta_0 - \frac{K_S \eta^2}{K_T} > 0$. Moreover, there holds that

$$\mathbf{f} \in C^2([0, T], L^2(D));$$

$$g \in C^1([0, T], L^2(D));$$

$$h \in C^1([0, T], L^2(D));$$

and

$$\bar{p}_0 \in H_0^1(D);$$

$$\bar{\theta}_0 \in H_0^1(D).$$

In [17], the authors proved the existence and uniqueness of a weak solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ under the Hypothesis 9. Gathering the regularity of [17, Theorem 3.1], we can state the following result.

Theorem 6.2.1 ([17, Theorem 3.1]). *There exists a unique solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta}) \in H_0^1(D)$ of the thermo-poroelastic problem such that*

$$\begin{aligned}\bar{\mathbf{u}} &\in H^1(0, T; L^2(D)), \\ e(\bar{\mathbf{u}}) &\in H^1(0, T; L^2(D)), \\ \bar{p} &\in H^1(0, T; H_0^1(D)), \\ \bar{\theta} &\in H^1(0, T; H_0^1(D)).\end{aligned}$$

Remark 6.2.1. *The proof of this theorem closely follows the analysis presented in [17], where the well-posedness of the thermo-poroelastic problem is established in a mixed variational framework. The existence and uniqueness of a weak solution are obtained using a Galerkin method and a priori estimates.*

This regularity can be improved which we do below. This regularity will be used later.

Corollary 6.2.1. *The solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ of (P) verifies*

$$\begin{aligned}e(\bar{\mathbf{u}}) &\in W^{1,\infty}(0, T; H_{\text{div}}(D)), \\ \bar{p} &\in L^2(0, T; H^2(D)), \\ \bar{\theta} &\in L^2(0, T; H^2(D)).\end{aligned}$$

Proof. Theorem 3.1 from [17] states that

$$\partial_t (\alpha \nabla \bar{p} + (1 - \alpha) K_S \nabla \bar{\theta}) \in L^\infty(0, T; L^2(D)).$$

The mechanical equation from Problem (P) leads to

$$\begin{cases} -\text{div}(2\mu e(\bar{\mathbf{u}}) + \lambda \text{div}(\bar{\mathbf{u}})\text{Id}) = -\alpha \nabla \bar{p} - (1 - \alpha) K_S \nabla \bar{\theta} + \mathbf{f}, & \text{in } D \times [0, T], \\ \bar{\mathbf{u}} = 0, & \text{on } \partial D \times [0, T]. \end{cases}$$

Thus, there holds

$$2\mu e(\bar{\mathbf{u}}) + \lambda \text{div}(\bar{\mathbf{u}})\text{Id} \in W^{1,\infty}(0, T; H_{\text{div}}(D)).$$

We recall that $\text{div}(\bar{\mathbf{u}}) = \text{Tr}(e(\bar{\mathbf{u}}))$, and from (3.2.3) almost everywhere in $(0, T)$

$$\|\text{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)} \leq \sqrt{d} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}.$$

Thus we have that

$$e(\bar{\mathbf{u}}) \in W^{1,\infty}(0, T; H_{\text{div}}(D)).$$

From the pressure equation, there holds that

$$\begin{cases} -\text{div}(\kappa \nabla \bar{p}) = g - \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \text{div}(\bar{\mathbf{u}}) \right), & \text{in } D \times [0, T], \\ \bar{p} = 0, & \text{on } \partial D \times [0, T]. \end{cases}$$

From the regularity of g and Theorem 3.1 of [17] there holds that

$$\bar{p} \in L^2(0, T; H^2(D)).$$

Finally, there holds that

$$\begin{cases} -\text{div}(\beta \nabla \bar{\theta}) = h - \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M} \right) \bar{\theta} - \frac{K_S}{M} \bar{p} + K_S (1 - \alpha) \text{div}(\bar{\mathbf{u}}) \right) + \eta \cdot \nabla \bar{\theta}, & \text{in } D \times [0, T], \\ \bar{\theta} = 0, & \text{on } \partial D \times [0, T]. \end{cases}$$

We deduce from the regularity of h and β , and from Theorem 3.1 of [17] that the results holds. \square

6.2.1 A priori estimates

We recall the following inequalities, used in the proof of the next theorem.

Lemma 6.2.1 (Korn inequality). *For $\mathbf{v} \in H_0^1(D)^d$, there exists a positive constant $C_K := C_K(D)$ such that*

$$\|\mathbf{v}\|_{H_0^1(D)}^2 \leq C_K \|e(\mathbf{v})\|_{L^2(D)}^2. \quad (6.3)$$

Lemma 6.2.2 (Poincaré inequality). *For $q \in H_0^1(D)$, there exists a positive constant $C_P := C_P(D)$ such that*

$$\|q\|_{L^2(D)} \leq C_P \|\nabla q\|_{L^2(D)}. \quad (6.4)$$

Theorem 6.2.2. *There exists a positive constant C (depending on constants of the problem) such that*

$$\begin{aligned} & \sup_{t \in (0, T)} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \sup_{t \in (0, T)} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\ & + \sup_{t \in (0, T)} \|\bar{p}(t)\|_{L^2(D)}^2 + \sup_{t \in (0, T)} \|\bar{\theta}(t)\|_{L^2(D)}^2 \\ & + \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 + \|\nabla \bar{\theta}\|_{L^2(D \times (0, T))}^2 \leq C(\mu_0, \lambda_0, \kappa_0, \beta_0, \mathcal{M}_\mu, \mathcal{M}_\lambda, \mathcal{M}_\kappa, \mathcal{M}_\beta) \\ & \left(\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \|g\|_{L^2(D \times (0, T))}^2 + \|h\|_{L^2(D)}^2 \right. \\ & \left. + \|\bar{p}_0\|_{L^2(D)}^2 + \|\bar{\theta}_0\|_{L^2(D)}^2 \right). \end{aligned} \quad (6.5)$$

Proof. From Corollary 6.2.1, we multiply respectively the mechanical equation in (\mathbb{P}) by $\partial_t \bar{\mathbf{u}}$, the pressure equation by \bar{p} and the temperature equation by $\bar{\theta}$, and integrate over D . For a fixed $t \in [0, T]$, there holds that

$$\begin{aligned} & \int_D 2\mu e(\bar{\mathbf{u}}) : \partial_t e(\bar{\mathbf{u}}) + \lambda \operatorname{div}(\bar{\mathbf{u}}) \partial_t \operatorname{div}(\bar{\mathbf{u}}) \\ & - \int_D \alpha \bar{p} \partial_t \operatorname{div}(\bar{\mathbf{u}}) + (1 - \alpha) K_S \bar{\theta} \partial_t \operatorname{div}(\bar{\mathbf{u}}) dx = \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx \end{aligned} \quad (6.6a)$$

$$\int_D \partial_t \left(\frac{\bar{p} - K_S \bar{\theta}}{M} + \alpha \operatorname{div}(\bar{\mathbf{u}}) \right) \bar{p} dx + \int_D \kappa |\nabla \bar{p}|^2 dx = \int_D g \bar{p} dx \quad (6.6b)$$

$$\begin{aligned} & \int_D \partial_t \left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M} \right) \bar{\theta} - \frac{K_S}{M} \bar{p} + (1 - \alpha) K_S \operatorname{div}(\bar{\mathbf{u}}) \right) \bar{\theta} dx \\ & - \int_D \eta \cdot \nabla \bar{\theta} \bar{\theta} dx + \int_D \beta |\nabla \bar{\theta}|^2 dx = \int_D h \bar{\theta} dx. \end{aligned} \quad (6.6c)$$

First we consider the right-hand-side of (6.6a). By taking the integral over $(0, T)$ and integrate by parts in time, we have

$$\int_0^t \int_D \mathbf{f} \cdot \partial_t \bar{\mathbf{u}} dx dt = \int_D (\mathbf{f}(t) \cdot \bar{\mathbf{u}}(t) - \mathbf{f}(0) \cdot \bar{\mathbf{u}}(0)) dx - \int_0^t \int_D \partial_t \mathbf{f} \cdot \bar{\mathbf{u}} dx dt. \quad (6.7)$$

Moreover, we remark that

$$\frac{1}{2M} \partial_t |\bar{p} - K_S \bar{\theta}|^2 = \left(\partial_t \frac{\bar{p} - K_S \bar{\theta}}{M} \right) \bar{p} - \left(\partial_t \frac{K_S \bar{p} - K_S^2 \bar{\theta}}{M} \right) \bar{\theta}.$$

Thus, by taking the integral over $(0, T)$ of the sum of (6.6a), (6.6b) and (6.6c), we obtain in view of (6.7)

$$\begin{aligned}
& \int_0^t \int_D \mu \partial_t |e(\bar{\mathbf{u}})|^2 + \frac{\lambda}{2} \partial_t |\operatorname{div}(\bar{\mathbf{u}})|^2 dx dt \\
& + \int_0^t \int_D \frac{\partial_t}{2M} |\bar{p} - K_S \bar{\theta}|^2 dx dt + \int_0^t \int_D \frac{K_S}{2K_T} \partial_t |\bar{\theta}|^2 dx dt \\
& + \int_0^t \int_D \kappa |\nabla \bar{p}|^2 + \beta |\nabla \bar{\theta}|^2 - \eta \cdot \nabla \bar{\theta} \bar{\theta} dx dt \\
& = \int_D \mathbf{f}(t) \cdot \bar{\mathbf{u}}(t) - \mathbf{f}(0) \cdot \bar{\mathbf{u}}(0) dx - \int_0^t \int_D \partial_t \mathbf{f} \cdot \bar{\mathbf{u}} dx dt \\
& \quad \int_0^t \int_D g \bar{p} dx dt + \int_0^t \int_D h \bar{\theta} dx dt.
\end{aligned} \tag{6.8}$$

The following inequality holds,

$$\begin{aligned}
& \mu_0 \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\bar{p}(t) - K_S \bar{\theta}(t)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\bar{\theta}(t)\|_{L^2(D)}^2 \\
& + \kappa_0 \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 + \beta_0 \|\nabla \bar{\theta}\|_{L^2(D \times (0, T))}^2 \\
& \leq \int_D \mathbf{f}(t) \cdot \bar{\mathbf{u}}(t) dx - \int_0^t \int_D \partial_t \mathbf{f} \cdot \bar{\mathbf{u}} dx dt + \int_0^t \int_D g \bar{p} + h \bar{\theta} dx dt \\
& - \int_D \mathbf{f}(0) \cdot \bar{\mathbf{u}}(0) dx + \mathcal{M}_\mu \|e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \frac{\mathcal{M}_\lambda}{2} \|\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\bar{p}_0 - K_S \bar{\theta}_0\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\bar{\theta}_0\|_{L^2(D)}^2 + \int_0^t \int_D \eta \cdot \bar{\theta} \bar{\theta} dx dt.
\end{aligned} \tag{6.9}$$

We apply Hölder's inequality on the right hand side of (6.9), and with Poincaré and Korn inequality,

$$\begin{aligned}
& \int_D \mathbf{f}(t) \cdot \bar{\mathbf{u}}(t) dx \leq \frac{C_K}{2\mu_0} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\
& \int_0^t \int_D \partial_t \mathbf{f} \cdot \bar{\mathbf{u}} dx dt \leq \frac{C_K}{2\mu_0} \|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 \\
& \int_0^t \int_D g \bar{p} + h \bar{\theta} dx dt \leq \frac{C_P}{2\kappa_0} \|g\|_{L^2(D \times (0, T))}^2 + \frac{C_P}{\beta_0} \|h\|_{L^2(D \times (0, T))}^2 \\
& \quad + \frac{\kappa_0}{2} \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 + \frac{\beta_0}{4} \|\nabla \bar{\theta}\|_{L^2(D \times (0, T))}^2 \\
& + \int_0^t \int_D \eta \cdot \nabla \bar{\theta} \bar{\theta} dx dt \leq \frac{K_T \eta^2}{2K_S} \|\nabla \bar{\theta}\|_{L^2(D \times (0, T))}^2 + \frac{K_S}{2K_T} \|\bar{\theta}\|_{L^2(D \times (0, T))}^2.
\end{aligned} \tag{6.10}$$

Thus from (6.9)

$$\begin{aligned}
& \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\bar{p}(t) - K_S \bar{\theta}(t)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\bar{\theta}(t)\|_{L^2(D)}^2 \\
& + \frac{\kappa_0}{2} \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 + \frac{1}{2} \left(\beta_0 - \frac{K_S \eta^2}{K_T} \right) \|\nabla \bar{\theta}(t)\|_{L^2(D)}^2 \\
& \leq \frac{C_K}{2\mu_0} (\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \|\mathbf{f}\|_{L^2(D \times (0, T))}^2) \\
& \quad + \frac{C_P}{2\kappa_0} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{\beta_0 C_P} \|h\|_{L^2(D)}^2 \\
& \quad + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D)}^2 \\
& + \frac{\mu_0}{2} \|e(\bar{\mathbf{u}})\|_{L^2(D \times (0, T))}^2 + \frac{K_S}{2K_T} \|\bar{\theta}\|_{L^2(D \times (0, T))}^2.
\end{aligned} \tag{6.11}$$

We set

$$\begin{aligned}
A & := \frac{C_K}{2\mu_0} (\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\mathbf{f}(t)\|_{L^2(D)}^2) \\
& + \frac{C_P}{2\kappa_0} \|g\|_{L^2(D \times (0, T))}^2 + \frac{1}{\beta_0 C_P} \|h\|_{L^2(D)}^2 \\
& + \frac{1}{2} \|\mathbf{f}(0)\|_{L^2(D)}^2 + \frac{1}{2} \|\bar{\mathbf{u}}_0\|_{L^2(D \times (0, T))}^2.
\end{aligned} \tag{6.12}$$

Thus by applying Gronwall Lemma's, there holds that (for a small $\varepsilon > 0$)

$$\begin{aligned}
& \frac{\mu_0}{2} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\bar{p}(t) - K_S \bar{\theta}(t)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\bar{\theta}(t)\|_{L^2(D)}^2 \\
& + \frac{\kappa_0}{2} \|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2 + \varepsilon \|\nabla \bar{\theta}(t)\|_{L^2(D)}^2 \\
& \leq \exp(T) A + T \exp(T) A + A.
\end{aligned} \tag{6.13}$$

We set

$$\chi := \min \left\{ \frac{\mu_0}{2}, \frac{\lambda_0}{2}, \frac{1}{2M}, \frac{K_S}{2K_T}, \frac{\kappa_0}{2}, \frac{1}{\varepsilon} \right\},$$

and dividing equation (6.13) by χ leads to the results. \square

6.3 Random Parametrization

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T a positive constant and let $D \subset \mathbb{R}^d$ a bounded domain with a smooth boundary ∂D , where d is the space dimension. We suppose that the Lamé parameters λ and μ are random, namely we will write $\lambda(\omega, x)$ and $\mu(\omega, x)$ and that the diffusion coefficients κ and β are also random, namely we will write $\kappa(\omega, x)$ and $\beta(\omega, x)$ for $\omega \in \Omega$.

Then $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ satisfy the system

$$\begin{aligned}
& -\operatorname{div}\left(2\mu(\omega, x)e(\bar{\mathbf{u}}(\omega, x, t)) + \lambda(\omega, x)\operatorname{div}(\bar{\mathbf{u}}(\omega, x, t))\right) + \alpha\nabla\bar{p}(\omega, x, t) + (1 - \alpha)K_S\nabla\bar{\theta}(\omega, x, t) = \mathbf{f}(x, t) \\
& \partial_t\left(\frac{1}{M}\bar{p}(\omega, x, t) - \frac{K_S}{M}\bar{\theta}(\omega, x, t) + \alpha\operatorname{div}(\bar{\mathbf{u}}(\omega, x, t))\right) - \operatorname{div}(\kappa(\omega, x)\nabla\bar{p}(\omega, x, t)) = g(x, t) \\
& \partial_t\left(\left(\frac{K_S}{K_T} + \frac{K_S^2}{M}\right)\bar{\theta}(\omega, x, t) - \frac{K_S}{M}\bar{p}(\omega, x, t) + K_S(1 - \alpha)\operatorname{div}(\bar{\mathbf{u}}(\omega, x, t))\right) \\
& - \eta \cdot \nabla\bar{\theta}(\omega, x, t) - \operatorname{div}(\beta(\omega, x)\nabla\bar{\theta}(\omega, x, t)) = h(x, t),
\end{aligned} \tag{6.14}$$

together with the boundary conditions, \mathbb{P} -almost surely

$$\bar{\mathbf{u}} = 0, \bar{p} = 0, \bar{\theta} = 0, \text{ on } \partial D \times [0, T), \tag{6.15}$$

and the deterministic initial conditions \mathbb{P} -almost surely

$$\bar{p}(\omega, \cdot, 0) = \bar{p}_0(\cdot), \bar{\theta}(\omega, \cdot, 0) = \bar{\theta}_0(\cdot), \text{ in } D. \tag{6.16}$$

We generalize Hypothesis 9 as follows.

Hypothesis 10. *We suppose that almost surely*

$$E \in C^1(D),$$

and there exists positive constant E_0, E_{\max} such that, for \mathbb{P} -almost all $\omega \in \Omega$, we have

$$0 < E_0 \leq E(\omega, x) \leq E_{\max} < \infty \text{ in } D.$$

We define

$$\mu(\omega, x) = \frac{E(\omega, x)}{2(1 + \nu)}, \quad \lambda(\omega, x) = \frac{E(\omega, x)}{(1 + \nu)(1 - 2\nu)}.$$

We suppose that

$$\mu \in L^\infty(\Omega \times D), \lambda \in L^\infty(\Omega \times D), \kappa \in L^\infty(\Omega \times D), \beta \in L^\infty(\Omega \times D).$$

so that there exists positive constants

$$\mu_0, \lambda_0, \kappa_0, \beta_0$$

and

$$\mathcal{M}_\mu, \mathcal{M}_\lambda, \mathcal{M}_\kappa, \mathcal{M}_\beta$$

such that for all $x \in D$, \mathbb{P} -almost all $\omega \in \Omega$, we have

$$\begin{aligned}
0 < \mu_0 &\leq \mu(\omega, x) \leq \mathcal{M}_\mu < \infty, \quad 0 < \lambda_0 \leq \lambda(\omega, x) \leq \mathcal{M}_\lambda < \infty, \\
0 < \kappa_0 &\leq \kappa(\omega, x) \leq \mathcal{M}_\kappa < \infty, \quad 0 < \beta_0 \leq \beta(\omega, x) \leq \mathcal{M}_\beta < \infty,
\end{aligned} \tag{6.17}$$

The following result holds :

Theorem 6.3.1. *There exists a unique solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ such that*

$$\begin{aligned}
\bar{\mathbf{u}} &\in L^2(\Omega, L^\infty(0, T; H^1(D))), \\
\bar{p} &\in L^2(\Omega, L^\infty(0, T; H^1(D))) \\
\bar{\theta} &\in L^2(\Omega, L^\infty(0, T; H^1(D))).
\end{aligned}$$

Moreover, there exists a deterministic constant C such that $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ satisfies

$$\begin{aligned}
& \mathbb{E}[\sup_{t \in (0, T)} \|e(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2] + \mathbb{E}[\sup_{t \in (0, T)} \|\operatorname{div}(\bar{\mathbf{u}}(t))\|_{L^2(D)}^2] \\
& + \mathbb{E}[\sup_{t \in (0, T)} \|\bar{p}(t)\|_{L^2(D)}^2] + \mathbb{E}[\sup_{t \in (0, T)} \|\bar{\theta}(t)\|_{L^2(D)}^2] \\
& + \mathbb{E}[\|\nabla \bar{p}\|_{L^2(D \times (0, T))}^2] + \mathbb{E}[\|\nabla \bar{\theta}\|_{L^2(D \times (0, T))}^2] \\
& \leq C \left(\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\mathbf{f}(t)\|_{L^2(D)}^2 \right. \\
& \quad + \|g\|_{L^2(D \times (0, T))}^2 + \|h\|_{L^2(D)}^2 \\
& \quad \left. + \|\bar{p}_0\|_{L^2(D)}^2 + \|\bar{\theta}_0\|_{L^2(D)}^2 \right). \tag{6.18}
\end{aligned}$$

Proof. We choose $\omega \in \Omega$, such that Hypothesis 10 is fulfilled. Thus, there exists a unique solution

$$\bar{\mathbf{u}}(\omega), \bar{p}(\omega), \bar{\theta}(\omega).$$

Consider the measurable mapping

$$J : \omega \in \Omega \mapsto (\mu(\omega), \lambda(\omega), \kappa(\omega), \beta(\omega)),$$

since the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ is continuous in its coefficients $(\mu, \lambda, \kappa, \beta)$ (see [7] p.1550), then the mapping

$$S : \omega \in \Omega \mapsto (\bar{\mathbf{u}}(\omega), \bar{p}(\omega), \bar{\theta}(\omega))$$

is measurable.

From Equation (6.5), we have for \mathbb{P} -almost every $\omega \in \Omega$

$$\begin{aligned}
& \sup_{t \in (0, T)} \|e(\bar{\mathbf{u}}(\omega, t))\|_{L^2(D)}^2 + \sup_{t \in (0, T)} \|\operatorname{div}(\bar{\mathbf{u}}(\omega, t))\|_{L^2(D)}^2 \\
& + \sup_{t \in (0, T)} \|\bar{p}(\omega, t)\|_{L^2(D)}^2 + \sup_{t \in (0, T)} \|\bar{\theta}(\omega, t)\|_{L^2(D)}^2 \\
& + \|\nabla \bar{p}(\omega)\|_{L^2(D \times (0, T))}^2 + \|\nabla \bar{\theta}(\omega)\|_{L^2(D \times (0, T))}^2 \leq C(\mu_0, \lambda_0, \kappa_0, \beta_0, \mathcal{M}_\mu, \mathcal{M}_\lambda, \mathcal{M}_\kappa, \mathcal{M}_\beta) \\
& \left(\|\partial_t \mathbf{f}\|_{L^2(D \times (0, T))}^2 + \sup_{t \in [0, T]} \|\mathbf{f}(t)\|_{L^2(D)}^2 + \|g\|_{L^2(D \times (0, T))}^2 + \|h\|_{L^2(D)}^2 \right. \\
& \quad \left. + \|\bar{p}_0\|_{L^2(D)}^2 + \|\bar{\theta}_0\|_{L^2(D)}^2 \right),
\end{aligned}$$

and since the left-hand-side is independent of ω , by taking the means the result holds. \square

6.4 Pathwise Discretization Schemes

We consider a conforming triangulation $\mathcal{K}_l = \{K_1, \dots, K_{n_l}\}$ of D , and we define the diameter of the partition of D as

$$h_l := \max_{K \in \mathcal{K}_l} \operatorname{diam}(K). \tag{6.19}$$

We introduce the finite element spaces for the displacement \mathbf{u} , the pressure p and the temperature θ :

$$\mathcal{V}_l := \{\mathbf{v}_l \in C^0(\bar{D})^d \mid \mathbf{v}_l|_{\partial D} = 0 \text{ and } \mathbf{v}_l|_K \in \mathcal{P}_2(K)^d, K \in \mathcal{K}_l\} \subset \mathcal{V} \tag{6.20a}$$

$$\mathcal{Q}_l := \{q_l \in C^0(\bar{D})^d \mid q_l|_{\partial D} = 0 \text{ and } q_l|_K \in \mathcal{P}_1(K), K \in \mathcal{K}_l\} \subset \mathcal{Q} \tag{6.20b}$$

$$\Theta_l := \{\vartheta_l \in C^0(\bar{D})^d \mid \vartheta_l|_{\partial D} = 0 \text{ and } \vartheta_l|_K \in \mathcal{P}_1(K), K \in \mathcal{K}_l\} \subset \Theta. \tag{6.20c}$$

The set $\mathcal{P}_1(K)$ denotes the space of polynomials function of degree 1 on the triangle K . The set $\mathcal{P}_2(K)$ denotes the space of polynomials function of degree 2 on the triangle K . (\mathbf{v}_i) correspond to the basis functions of \mathcal{V}_l , which are computed at the vertices and the middle points of the edges of \mathcal{K}_l . (q_i) and (ϑ_i) are the basis functions of \mathcal{Q}_l and Θ_l , which are computed at the vertices of the triangle \mathcal{K}_l .

Given an element $K \in \mathcal{K}_l$, let $x_1, \dots, x_j \in \overline{D}$ denote the vertices of K . We define three families of linear nodal basis of $\mathcal{P}_1(K)$, and a local interpolation operator on K for each set of functions,

$$\mathbf{v}_1^K, \dots, \mathbf{v}_j^K, \mathcal{I}^{\mathbf{u}}(K) : C^0(\overline{K})^d \rightarrow \mathcal{P}_1(K)^d, \mathbf{v} \mapsto \sum_{i=1}^j \mathbf{v}(x_i^K) \mathbf{v}_i^K \quad (6.21a)$$

$$q_1^K, \dots, q_j^K, \mathcal{I}^p(K) : C^0(\overline{K}) \rightarrow \mathcal{P}_1(K), q \mapsto \sum_{i=1}^j q(x_i^K) q_i^K \quad (6.21b)$$

$$\vartheta_1^K, \dots, \vartheta_j^K, \mathcal{I}^\theta(K) : C^0(\overline{K}) \rightarrow \mathcal{P}_1(K), \vartheta \mapsto \sum_{i=1}^j \vartheta(x_i^K) \vartheta_i^K. \quad (6.21c)$$

Thus, from each operators in (6.21), we can define a global interpolation operator

$$[\mathcal{I}_l^{\mathbf{u}} \mathbf{v}](x) := [\mathcal{I}^{\mathbf{u}}(K) \mathbf{v}](x), \text{ for } x \in D, \text{ where } K \in \mathcal{K} \text{ is such that } x \in \overline{K} \quad (6.22a)$$

$$[\mathcal{I}_l^p q](x) := [\mathcal{I}^p(K) q](x), \text{ for } x \in D, \text{ where } K \in \mathcal{K} \text{ is such that } x \in \overline{K} \quad (6.22b)$$

$$[\mathcal{I}_l^\theta \vartheta](x) := [\mathcal{I}^\theta(K) \vartheta](x), \text{ for } x \in D, \text{ where } K \in \mathcal{K} \text{ is such that } x \in \overline{K}. \quad (6.22c)$$

The two following lemmas are classical results from approximation by P_1 and \mathbf{P}_2 elements.

Lemma 6.4.1. *Given h the diameter of the triangulation, for $\mathbf{v} \in H^2(D)$, $q \in H^1(D)$, there exists a constant C such that*

$$\|(1 - \mathcal{I}_l^{\mathbf{u}}) \mathbf{v}\|_{H^2(D)} \leq Ch, \|(1 - \mathcal{I}_l^p) q\|_{H^1(D)} \leq Ch, \|(1 - \mathcal{I}_l^\theta) \vartheta\|_{H^1(D)} \leq Ch.$$

Lemma 6.4.2. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $T > 0$.*

- Assume $\overline{p} \in L^2(0, T; H^2(D))$ and $\overline{\theta} \in L^2(0, T; H^2(D))$, and they are approximated by piecewise linear finite elements P_1 on a triangulation \mathcal{T}_h with maximum element diameter h .
- Let $\overline{\mathbf{u}} \in L^2(0, T; H^1(D)^d)$, where its symmetric gradient $e(\overline{\mathbf{u}}) = \frac{1}{2}(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T)$ belongs to $L^2(0, T; H_{\text{div}}(D))$, meaning that $\nabla \cdot \overline{\mathbf{u}} \in L^2(D)$. Assume that $\overline{\mathbf{u}}$ is approximated by quadratic finite elements \mathbf{P}_2 .

Then, for all $t \in [0, T]$, there exists a constant C , independent of h , such that the following error estimates hold:

1. Error estimate for \overline{p} and $\overline{\theta}$ (approximated by P_1):

$$\|\overline{p}(t) - \overline{p}_h(t)\|_{H^1(D)} + \|\overline{\theta}(t) - \overline{\theta}_h(t)\|_{H^1(D)} \leq Ch (\|\overline{p}(t)\|_{H^2(D)} + \|\overline{\theta}(t)\|_{H^2(D)}),$$

where $\overline{p}_h(t)$ and $\overline{\theta}_h(t)$ are the P_1 finite element approximations of $\overline{p}(t)$ and $\overline{\theta}(t)$, respectively.

2. Error estimate for $\overline{\mathbf{u}}$ (approximated by \mathbf{P}_2):

$$\|\overline{\mathbf{u}}(t) - \overline{\mathbf{u}}_h(t)\|_{H^1(D)} \leq Ch \|e(\overline{\mathbf{u}})(t)\|_{H_{\text{div}}(D)}.$$

Here, $\overline{\mathbf{u}}_h(t)$ is the \mathbf{P}_2 finite element approximation of $\overline{\mathbf{u}}(t)$, and $e(\overline{\mathbf{u}})(t)$ denotes the symmetric gradient of $\overline{\mathbf{u}}(t)$.

We introduce the semi-discrete version of the weak formulation of (6.14).

Suppose that the source terms \mathbf{f} , g and h satisfy the hypothesis above; the semi-discrete formulation can be written as : find $(\mathbf{u}_l, p_l, \theta_l) \in H^1(0, T; \mathcal{V}_l) \times H^1(0, T; \mathcal{Q}_l) \times H^1(0, T; \Theta_l)$, so that for $t \in [0, T]$, for all $(\mathbf{v}, q, \vartheta) \in \mathcal{V}_l \times \mathcal{Q}_l \times \Theta_l$, for almost all $\omega \in \Omega$, there holds that

$$\begin{aligned} & \int_D 2\mu(\omega, x)e(\mathbf{u}_l(\omega, x, t)) : e(\mathbf{v}_l(x)) + \lambda(\omega, x)\operatorname{div}(\mathbf{u}_l(\omega, x, t))\operatorname{div}(\mathbf{v}(x))dx \\ & - \alpha \int_D p_l(\omega, x, t)\operatorname{div}(\mathbf{v}(x))dx - K_S(1 - \alpha) \int_D \theta_l(\omega, x, t)\operatorname{div}(\mathbf{v}(x))dx \\ & = \int_D \mathbf{f}(x, t) \cdot \mathbf{v}(x)dx \end{aligned} \quad (6.23a)$$

$$\begin{aligned} & \int_D \partial_t \left(\frac{p_l - K_S \theta_l}{M} + \lambda \operatorname{div}(\mathbf{u}_l) \right) (\omega, x, t) q(x) dx \\ & + \int_D \kappa(x) \nabla p_l(\omega, x, t) \cdot \nabla q(x) dx = \int_D g(x) q(x) dx \end{aligned} \quad (6.23b)$$

$$\begin{aligned} & \int_D \partial_t \left(a_0 \theta_l - \frac{K_S}{M} p_l + K_S(1 - \alpha) \operatorname{div}(\mathbf{u}_l) \right) (\omega, x, t) \vartheta(x) dx \\ & + \int_D \beta(x) \nabla \theta_l(\omega, x, t) \cdot \nabla \vartheta(x) dx + \int_D \eta \cdot \nabla \theta_l(\omega, x, t) \vartheta(x) dx = \int_D h(x) \vartheta(x) dx \end{aligned} \quad (6.23c)$$

$$\mathbf{u}_l(\omega, \cdot, 0) = \mathcal{I}_l^{\mathbf{u}} \mathbf{u}_0, \quad p_l(\omega, \cdot, 0) = \mathcal{I}_l^p p_0, \quad \theta_l(\omega, \cdot, 0) = \mathcal{I}_l^\theta \theta_0. \quad (6.23d)$$

The interpolation of the initial data are well-defined since \mathbf{u}_0 , p_0 and θ_0 are continuous in \bar{D} in the space variable.

Theorem 6.4.1. *Under previous assumptions described in Lemmas 6.4.1 and 6.4.1, let $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ be the solution to problem (6.14) and $(\mathbf{u}_l, p_l, \theta_l)$ be the solution to the system (6.23). Then, there is a positive constant $C > 0$ such that*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \left(\|e(\bar{\mathbf{u}} - \mathbf{u}_l)(t)\|_{L^2(D)}^2 + \|\operatorname{div}(\bar{\mathbf{u}} - \mathbf{u}_l)(t)\|_{L^2(D)}^2 + \|(\bar{p} - K_S \bar{\theta} - p_l + K_S \theta_l)(t)\|_{L^2(D)}^2 + \|(\bar{\theta} - \theta_l)(t)\|_{L^2(D)}^2 \right) \right] \leq C \\ \left(\|(1 - \mathcal{I}_l^{\mathbf{u}})e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \|(1 - \mathcal{I}_l^p)\operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 + \|(1 - \mathcal{I}_l^p)\bar{p}_0\|_{L^2(D)}^2 + \|(1 - \mathcal{I}_l^\theta)\bar{\theta}_0\|_{L^2(D)}^2 \right). \end{aligned}$$

Proof. We define the following error

$$\varepsilon_u^l := \bar{\mathbf{u}} - \mathbf{u}_l, \quad \varepsilon_p^l := \bar{p} - p_l, \quad \varepsilon_\theta^l := \bar{\theta} - \theta_l. \quad (6.24)$$

We recall that $a_0 := \frac{K_S}{K_T} + \frac{K_S^2}{M}$. Thus, for a fixed $\omega \in \Omega$, for a fixed $t \in [0, T]$, there holds

$$\int_D 2\mu e(\varepsilon_u^l) : e(\mathbf{v}) + \lambda \operatorname{div}(\varepsilon_u^l) \operatorname{div}(\mathbf{v}) - \alpha \varepsilon_p^l \operatorname{div}(\mathbf{v}) - K_S(1 - \alpha) \varepsilon_\theta^l \operatorname{div}(\mathbf{v}) dx = 0, \quad (6.25a)$$

$$\int_D \partial_t \left(\frac{\varepsilon_p^l - K_S \varepsilon_\theta^l}{M} + \alpha \operatorname{div}(\varepsilon_u^l) \right) q dx + \int_D \kappa \nabla \varepsilon_p^l \cdot \nabla q dx = 0, \quad (6.25b)$$

$$\begin{aligned} & \int_D \partial_t \left(a_0 \varepsilon_\theta^l - \frac{K_S}{M} \varepsilon_p^l + K_S(1 - \alpha) \operatorname{div}(\varepsilon_u^l) \right) \vartheta dx \\ & + \int_D \beta \nabla \varepsilon_\theta^l \cdot \nabla \vartheta dx + \int_D \eta \cdot \nabla \varepsilon_\theta^l \vartheta dx = 0, \end{aligned} \quad (6.25c)$$

$$\varepsilon_p^l(\omega, \cdot, 0) = (\bar{p}_0 - \mathcal{I}_l^p \bar{p}_0)(\omega, \cdot), \quad \varepsilon_\theta^l(\omega, \cdot, 0) = (\bar{\theta}_0 - \mathcal{I}_l^\theta \bar{\theta}_0)(\omega, \cdot), \quad (6.25d)$$

for all $(\mathbf{v}, q, \vartheta) \in \mathcal{V}_l \times \mathcal{Q}_l \times \Theta_l$. Next we set

$$\mathbf{v} := \partial_t (\mathcal{I}_l^{\mathbf{u}} \bar{\mathbf{u}}(\omega, \cdot, t) - \mathbf{u}_l(\omega, \cdot, t)), \quad q := \mathcal{I}_l^p \bar{p}(\omega, \cdot, t) - p_l(\omega, \cdot, t), \quad \vartheta := \mathcal{I}_l^\theta \bar{\theta}(\omega, \cdot, t) - \theta_l(\omega, \cdot, t)$$

and integrate over $[0, t]$. Since

$$\mathcal{I}_l^u \bar{\mathbf{u}} - \mathbf{u}_l = (\mathcal{I}_l^u \bar{\mathbf{u}} - \bar{\mathbf{u}}) + (\bar{\mathbf{u}} - \mathbf{u}_l), \quad \mathcal{I}_l^p \bar{p} - p_l = (\mathcal{I}_l^p \bar{p} - \bar{p}) + (\bar{p} - p_l), \quad \mathcal{I}_l^\theta \bar{\theta} - \theta_l = (\mathcal{I}_l^\theta \bar{\theta} - \bar{\theta}) + (\bar{\theta} - \theta_l),$$

from (6.25a), we obtain

$$\begin{aligned} & \int_0^t \int_D \mu \partial_t |e(\varepsilon_u^l)|^2 + \frac{\lambda}{2} \partial_t |\operatorname{div}(\varepsilon_u^l)|^2 dx dt \\ & - \int_0^t \int_D \alpha \varepsilon_p^l \partial_t \operatorname{div}(\varepsilon_u^l) - K_S (1 - \alpha) \varepsilon_\theta^l \partial_t \operatorname{div}(\varepsilon_u^l) dx dt \\ & = \int_0^t \int_D 2\mu e(\varepsilon_u^l) : (1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}} + \lambda \operatorname{div}(\varepsilon_u^l) \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) dx d\tau \\ & - \int_0^t \int_D \alpha \varepsilon_p^l \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) + K_S (1 - \alpha) \varepsilon_\theta^l \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) dx d\tau. \end{aligned} \quad (6.26a)$$

From (6.25b) there holds

$$\begin{aligned} & \int_0^t \int_D \partial_t \left(\frac{\varepsilon_p^l - K_S \varepsilon_\theta^l}{M} + \alpha \operatorname{div}(\varepsilon_u^l) \right) \varepsilon_p^l + \kappa |\nabla \varepsilon_p^l|^2 dx d\tau \\ & = \int_0^t \int_D \partial_t \left(\frac{\varepsilon_p^l - K_S \varepsilon_\theta^l}{M} + \alpha \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^p) \bar{p} dx d\tau + \int_0^t \int_D \kappa \nabla \varepsilon_p^l \nabla (1 - \mathcal{I}_l^p) \bar{p} dx d\tau. \end{aligned} \quad (6.26b)$$

As well from (6.25c) we have

$$\begin{aligned} & \int_0^t \int_D \partial_t \left(a_0 \varepsilon_\theta^l - \frac{K_S}{M} \varepsilon_p^l + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l) \right) \varepsilon_\theta^l + \beta |\nabla \varepsilon_\theta^l|^2 dx dt \\ & = \int_0^t \int_D \partial_t \left(a_0 \varepsilon_\theta^l - \frac{K_S}{M} \varepsilon_p^l + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^\theta) \bar{\theta} dx d\tau \\ & + \int_0^t \int_D \beta \nabla \varepsilon_\theta^l \cdot \nabla (1 - \mathcal{I}_l^\theta) \bar{\theta} + \eta \cdot \nabla \varepsilon_\theta^l (1 - \mathcal{I}_l^\theta) \bar{\theta} + \eta \cdot \nabla \varepsilon_\theta^l \varepsilon_\theta^l dx dt. \end{aligned} \quad (6.26c)$$

Since

$$\varepsilon_p^l(0) = (1 - \mathcal{I}_l^p) \bar{p}(0)$$

and by integrating by parts the first terms on the right-hand-side of (6.26b), we have

$$\begin{aligned}
& \int_0^t \int_D \partial_t \left(\frac{\varepsilon_p^l - K_S \varepsilon_\theta^l}{M} + \alpha \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^p) \bar{p} dx d\tau \\
& \leq \int_D \left(\frac{\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)}{M} + \alpha \operatorname{div}(\varepsilon_u^l(t)) \right) (1 - \mathcal{I}_l^p) \bar{p}(t) dx \\
& \quad - \int_D \int_0^t \left(\frac{\varepsilon_p^l - K_S \varepsilon_\theta^l}{M} + \alpha \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^p) \partial_t \bar{p} dx d\tau \\
& \quad - \int_D \left(\frac{\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)}{M} + \alpha \operatorname{div}(\varepsilon_u^l(0)) \right) (1 - \mathcal{I}_l^p) \bar{p}(0) dx \\
& \leq \frac{\varepsilon_1}{2M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{1}{2\varepsilon_1 M} \|(1 - \mathcal{I}_l^p) \bar{p}(t)\|_{L^2(D)}^2 \\
& \quad + \frac{\varepsilon_2}{2} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2 + \frac{1}{2\varepsilon_4} \|(1 - \mathcal{I}_l^p) \bar{p}(t)\|_{L^2(D)}^2 \\
& + \frac{\varepsilon_3}{2M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{1}{2\varepsilon_3 M} \|(1 - \mathcal{I}_l^p) \partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 \\
& \quad + \frac{\varepsilon_4}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 + \frac{1}{2\varepsilon_4} \|(1 - \mathcal{I}_l^p) \partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 \\
& \quad - \int_D \left(\frac{\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)}{M} + \alpha \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_p^l(0) dx.
\end{aligned} \tag{6.27}$$

Since $\varepsilon_\theta^l(0) = (1 - \mathcal{I}_l^\theta) \bar{p}(0)$ and by integrating by parts the first terms on the right-hand-side of (6.26c) and since $a_0 = \frac{K_S}{K_T} + \frac{K_S^2}{M}$, we have

$$\begin{aligned}
& \int_0^t \int_D \partial_t \left(\frac{K_S}{K_T} \varepsilon_\theta^l - \frac{K_S}{M} (\varepsilon_p^l - K_S \varepsilon_\theta^l) + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^\theta) \bar{\theta} dx d\tau \\
& \leq \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l(t) - \frac{K_S}{M} (\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)) + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l(t)) \right) (1 - \mathcal{I}_l^\theta) \bar{\theta}(t) dx d\tau \\
& \quad - \int_0^t \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l - \frac{K_S}{M} (\varepsilon_p^l - K_S \varepsilon_\theta^l) + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l) \right) (1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta} dx d\tau \\
& \quad - \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l(0) - \frac{K_S}{M} (\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)) + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l(0)) \right) (1 - \mathcal{I}_l^\theta) \bar{\theta}(0) dx d\tau \\
& \leq \frac{K_S \varepsilon_5}{2K_T} \|\varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{K_S}{2K_T \varepsilon_5} \|(1 - \mathcal{I}_l^\theta) \bar{\theta}(t)\|_{L^2(D)}^2 \\
& \quad + \frac{\varepsilon_6}{M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{K_S^2}{2\varepsilon_6 M} \|(1 - \mathcal{I}_l^\theta) \bar{\theta}(t)\|_{L^2(D)}^2 \\
& \quad + \frac{\varepsilon_7}{2} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2 + \frac{(K_S(1 - \alpha))^2}{2\varepsilon_7} \|(1 - \mathcal{I}_l^\theta) \bar{\theta}(t)\|_{L^2(D)}^2 \\
& \quad + \frac{K_S \varepsilon_8}{2K_T} \|\varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{K_S}{2K_T \varepsilon_8} \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta}\|_{L^2(D)}^2 \\
& \quad + \frac{\varepsilon_9}{M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{K_S^2}{2\varepsilon_9 M} \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta}\|_{L^2(D \times (0, T))}^2 \\
& \quad + \frac{\varepsilon_{10}}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 + \frac{(K_S(1 - \alpha))^2}{2\varepsilon_{10}} \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta}\|_{L^2(D \times (0, T))}^2 \\
& \quad - \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l(0) - \frac{K_S}{M} (\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)) + K_S (1 - \alpha) \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_\theta^l(0) dx d\tau.
\end{aligned} \tag{6.28}$$

Moreover, by applying Young's inequality to (6.26a) we have that

$$\begin{aligned}
& \mu_0 \int_0^t \int_D \partial_t |e(\varepsilon_u^l)|^2 dx d\tau + \frac{\lambda_0}{2} \int_0^t \int_D \partial_t |\operatorname{div}(\varepsilon_u^l)|^2 dx d\tau \\
& - \int_0^t \int_D \alpha \varepsilon_p^l \partial_t \operatorname{div}(\varepsilon_u^l) - K_S(1-\alpha) \varepsilon_\theta^l \partial_t \operatorname{div}(\varepsilon_u^l) dx d\tau \\
& \leq \int_0^t \int_D 2\mu e(\varepsilon_u^l) : (1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}} + \lambda \operatorname{div}(\varepsilon_u^l) \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) dx d\tau \\
& - \int_0^t \int_D \alpha \varepsilon_p^l \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) + K_S(1-\alpha) \varepsilon_\theta^l \operatorname{div}((1 - \mathcal{I}_l^u) \partial_t \bar{\mathbf{u}}) dx d\tau.
\end{aligned} \tag{6.29}$$

Thus, we sum (6.26a) with (6.26b) and (6.26c), and we inject (6.27) and (6.28). Thus for I_1, I_2, I_3, I_4 which will be specified later, there holds

$$\begin{aligned}
& \mu_0 \|e(\varepsilon_u^l(t))\|_{L^2(D)}^2 + \frac{\lambda_0}{2} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\varepsilon_\theta^l(t)\|_{L^2(D)}^2 \\
& + \kappa_0 \|\nabla \varepsilon_p^l\|_{L^2(D \times (0,T))}^2 + (\beta_0 - \frac{\eta^2}{2\varepsilon_{13}}) \|\nabla \varepsilon_\theta^l\|_{L^2(D \times (0,T))}^2 \\
& \leq I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{6.30}$$

with

$$\begin{aligned}
I_1 &= \frac{\varepsilon_1}{2M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{\varepsilon_2}{2} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2 \\
& + \frac{\varepsilon_5 K_S}{2K_T} \|\varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{\varepsilon_6}{M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{\varepsilon_7}{2} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2,
\end{aligned}$$

$$\begin{aligned}
I_2 &= M_\mu \|e(\varepsilon_u^l(0))\|_{L^2(D)}^2 + \frac{M_\lambda}{2} \|\operatorname{div}(\varepsilon_u^l(0))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\varepsilon_\theta^l(0)\|_{L^2(D)}^2 - \int_D \left(\frac{\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)}{M} + \alpha \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_p^l(0) dx \\
& - \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l(0) - \frac{K_S}{M} (\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)) + K_S(1-\alpha) \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_\theta^l(0) dx d\tau,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \frac{\varepsilon_3}{2M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0,T))}^2 + \frac{\varepsilon_4}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0,T))}^2 \\
& + \frac{K_S \varepsilon_8}{2K_T} \|\varepsilon_\theta^l\|_{L^2(D \times (0,T))}^2 + \frac{\varepsilon_9}{M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0,T))}^2 + \frac{\varepsilon_{10}}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D)}^2 \\
& + M_\mu \varepsilon_{11} \|e(\varepsilon_u^l)\|_{L^2(D \times (0,T))}^2 + M_\lambda \varepsilon_{12} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0,T))}^2 + \frac{\varepsilon_{13}}{2} \|\varepsilon_\theta^l\|_{L^2(D \times (0,T))}^2,
\end{aligned}$$

$$\begin{aligned}
I_4 &= \left(\frac{1}{2\varepsilon_1 M} + \frac{1}{2\varepsilon_2} \right) \|(1 - \mathcal{I}_l^p) \bar{p}(t)\|_{L^2(D)}^2 + \left(\frac{1}{2\varepsilon_3 M} + \frac{1}{2\varepsilon_4} \right) \|(1 - \mathcal{I}_l^p) \partial_t \bar{p}\|_{L^2(D \times (0,T))}^2 \\
& + \left(\frac{K_S}{2K_T \varepsilon_5} + \frac{K_S^2}{2\varepsilon_6 M} + \frac{(K_S(1-\alpha))^2}{2\varepsilon_7} \right) \|(1 - \mathcal{I}_l^\theta) \bar{\theta}(t)\|_{L^2(D)}^2 \\
& + \left(\frac{K_S}{2K_T \varepsilon_8} + \frac{K_S^2}{2\varepsilon_9 M} + \frac{(K_S(1-\alpha))^2}{2\varepsilon_{10}} \right) \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta}\|_{L^2(D \times (0,T))}^2.
\end{aligned}$$

The error on the initial data are defined in

$$\begin{aligned}
I_2 = & M_\mu \|e(\varepsilon_u^l(0))\|_{L^2(D)}^2 + \frac{M_\lambda}{2} \|\operatorname{div}(\varepsilon_u^l(0))\|_{L^2(D)}^2 \\
& + \frac{1}{2M} \|\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)\|_{L^2(D)}^2 + \frac{K_S}{2K_T} \|\varepsilon_\theta^l(0)\|_{L^2(D)}^2 - \int_D \left(\frac{\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)}{M} + \alpha \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_p^l(0) dx \\
& - \int_D \left(\frac{K_S}{K_T} \varepsilon_\theta^l(0) - \frac{K_S}{M} (\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)) + K_S(1 - \alpha) \operatorname{div}(\varepsilon_u^l(0)) \right) \varepsilon_\theta^l(0) dx d\tau.
\end{aligned} \tag{6.31}$$

We have I_3 defined by

$$\begin{aligned}
I_3 = & \frac{\varepsilon_3}{2M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{\varepsilon_4}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 \\
& + \frac{K_S \varepsilon_8}{2K_T} \|\varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{\varepsilon_9}{M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{\varepsilon_{10}}{2} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D)}^2 \\
& + M_\mu \varepsilon_{11} \|e(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 + M_\lambda \varepsilon_{12} \|\operatorname{div}(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 + \frac{\varepsilon_{13}}{2} \|\varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2.
\end{aligned} \tag{6.32}$$

The interpolation error I_4 is defined by

$$\begin{aligned}
I_4 = & \left(\frac{1}{2\varepsilon_1 M} + \frac{1}{2\varepsilon_2} \right) \|(1 - \mathcal{I}_l^p) \bar{p}(t)\|_{L^2(D)}^2 + \left(\frac{1}{2\varepsilon_3 M} + \frac{1}{2\varepsilon_4} \right) \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{p}\|_{L^2(D \times (0, T))}^2 \\
& + \left(\frac{K_S}{2K_T \varepsilon_5} + \frac{K_S^2}{2\varepsilon_6 M} + \frac{(K_S(1 - \alpha))^2}{2\varepsilon_7} \right) \|(1 - \mathcal{I}_l^\theta) \bar{\theta}(t)\|_{L^2(D)}^2 \\
& + \left(\frac{K_S}{2K_T \varepsilon_8} + \frac{K_S^2}{2\varepsilon_9 M} + \frac{(K_S(1 - \alpha))^2}{2\varepsilon_{10}} \right) \|(1 - \mathcal{I}_l^\theta) \partial_t \bar{\theta}\|_{L^2(D \times (0, T))}^2.
\end{aligned} \tag{6.33}$$

Inside I_1 we set $\varepsilon_1 = 1/4$, $\varepsilon_6 = 1/8$ inside I_1 , so that

$$\frac{\varepsilon_1}{2M} + \frac{\varepsilon_6}{M} = \frac{1}{4M},$$

we set $\varepsilon_5 = 1/2$, so that

$$\frac{\varepsilon_5 K_S}{2K_T}$$

and we set $\varepsilon_2 = \varepsilon_7 = \frac{\lambda_0}{4}$ so that

$$\frac{\varepsilon_2}{2} + \frac{\varepsilon_7}{2} = \frac{\lambda_0}{4}.$$

Inside I_3 , we set $\varepsilon_3 = \frac{1}{4}$ and $\varepsilon_9 = \frac{1}{8}$ so that

$$\frac{\varepsilon_3}{2M} + \frac{\varepsilon_9}{M} = \frac{1}{4M},$$

and we set $\varepsilon_4 = \frac{\lambda_0}{8}$, $\varepsilon_{10} = \frac{\lambda_0}{8}$, $\varepsilon_{12} = \frac{\lambda_0}{8M_\lambda}$ so that

$$\frac{\varepsilon_4 + \varepsilon_{10}}{2} + M_\lambda \varepsilon_{12} = \frac{\lambda_0}{4},$$

and we set $\varepsilon_8 = \frac{1}{4}$, $\varepsilon_{13} = \frac{K_S}{4K_T}$ so that

$$\frac{K_S \varepsilon_8}{2K_T} + \frac{\varepsilon_{13}}{2} = \frac{K_S}{4K_T},$$

and we set $\varepsilon_{11} = \frac{\mu_0}{M_\mu}$ so that

$$M_\mu \varepsilon_{11} = \mu_0.$$

Then we have from (6.30)

$$\begin{aligned}
& \mu_0 \|e(\varepsilon_u^l)(t)\|_{L^2(D)}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\varepsilon_u^l(t))\|_{L^2(D)}^2 \\
& + \frac{1}{4M} \|\varepsilon_p^l(t) - K_S \varepsilon_\theta^l(t)\|_{L^2(D)}^2 + \frac{K_S}{4K_T} \|\varepsilon_\theta^l(t)\|_{L^2(D)}^2 \\
& + \kappa_0 \|\nabla \varepsilon_p^l\|_{L^2(D \times (0, T))}^2 + (\beta_0 - \frac{\eta^2}{2}) \|\nabla \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 \\
& \leq \mu_0 \|e(\varepsilon_u^l)\|_{L^2(D \times (0, T))}^2 + \frac{\lambda_0}{4} \|\operatorname{div}(\varepsilon_\theta^l)\|_{L^2(D \times (0, T))}^2 \\
& + \frac{1}{4M} \|\varepsilon_p^l - K_S \varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 + \frac{K_S}{4K_T} \|\varepsilon_\theta^l\|_{L^2(D \times (0, T))}^2 \\
& + I_2 + I_4.
\end{aligned} \tag{6.34}$$

We recall that

$$\begin{aligned}
\|e(\varepsilon_u^l(0))\|_{L^2(D)}^2 &= \|(1 - \mathcal{I}_l^u) e(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\
\|\operatorname{div}(\varepsilon_u^l(0))\|_{L^2(D)}^2 &= \|(1 - \mathcal{I}_l^u) \operatorname{div}(\bar{\mathbf{u}}_0)\|_{L^2(D)}^2 \\
\|\varepsilon_p^l(0) - K_S \varepsilon_\theta^l(0)\|_{L^2(D)}^2 &= \|(1 - \mathcal{I}_l^p) \bar{p}_0 - K_S (1 - \mathcal{I}_l^\theta) \bar{\theta}_0\|_{L^2(D)}^2 \leq 2\|(1 - \mathcal{I}_l^p) \bar{p}_0\|_{L^2(D)}^2 + 2K_S \|(1 - \mathcal{I}_l^\theta) \bar{\theta}_0\|_{L^2(D)}^2 \\
\|\varepsilon_\theta^l(0)\|_{L^2(D)}^2 &= \|(1 - \mathcal{I}_l^\theta) \bar{\theta}_0\|_{L^2(D)}^2.
\end{aligned}$$

If we apply the Gronwall Lemma, we obtain the bound for a fixed ω , and since this bound does not depend on ω we obtain the conclusion on the expectation. \square

6.4.1 Time discretization

We introduce the temporal discretization for the semi-discrete problem (6.23), and thus we denote

$$\mathbf{u}_l(\omega, \cdot, \cdot), p_l(\omega, \cdot, \cdot), \theta_l(\omega, \cdot, \cdot)$$

the solution to (6.23). We consider a fully discretisation of this problem, so that we consider the following time discretization with an homogeneous step time

$$0 = t_0 < t_1 < \dots < t_N = T,$$

for some $N \in \mathbb{N}$, such that $t_{i+1} - t_i = \Delta t$.

The temporal derivative is approximated by an Euler backward schemes

$$\partial_t \zeta(\omega, \cdot, t_i) \simeq \frac{\zeta(\omega, \cdot, t_i) - \zeta(\omega, \cdot, t_{i-1})}{\Delta t}. \tag{6.35}$$

We recall the following result,

Thus, following the regularity of 6.2.1, we have

Theorem 6.4.2. *Under Hypothesis 10, let*

$$\mathbf{u}_l^i, p_l^i, \theta_l^i, \quad i = 0, \dots, N,$$

the solution to the fully-discrete problem. We consider its linear interpolation in $[0, T]$ as follows,

$$\begin{aligned}
\mathbf{u}_l^N(t) &= \mathbf{u}_l^i, \quad t \in [t_i, t_{i+1}) \\
p_l^N(t) &= p_l^i, \quad t \in [t_i, t_{i+1}) \\
\theta_l^N(t) &= \theta_l^i, \quad t \in [t_i, t_{i+1}).
\end{aligned}$$

Then there holds that

$$\mathbb{E} \left[\sup_{t \in (0, T)} \|e(\mathbf{u}_l - \mathbf{u}_l^N)\|_{L^2(D)} \right]^{1/2} \leq C \Delta t \quad (6.36a)$$

$$\mathbb{E} \left[\sup_{t \in (0, T)} \|p_l - p_l^N\|_{L^2(D)}^2 + \|\nabla p_l - \nabla p_l^N\|_{L^2(D \times (0, T))}^2 \right]^{1/2} \leq C \Delta t \quad (6.36b)$$

$$\mathbb{E} \left[\sup_{t \in (0, T)} \|\theta_l - \theta_l^N\|_{L^2(D)}^2 + \|\nabla \theta_l - \nabla \theta_l^N\|_{L^2(D \times (0, T))}^2 \right]^{1/2} \leq C \Delta t. \quad (6.36c)$$

Proof. Under the Lipschitz regularity of the solutions, there holds that these following estimates holds. \square

6.5 Monte Carlo estimators

The Monte Carlo method is a powerful statistical technique used to approximate solutions to complex problems by sampling random variables. In the context of our thermo-poroelastic problem, where material properties such as permeability, thermal conductivity, and elastic moduli exhibit randomness, the Monte Carlo method provides a means to quantify the uncertainty in the system's response. By performing numerous simulations with randomly sampled inputs, we can estimate the distribution of the solution and derive statistical quantities of interest, such as the mean or variance.

For assessing the accuracy of the Monte Carlo simulation in this context, we consider the Root Mean Square Error (RMSE). RMSE measures the average magnitude of the errors between the numerical predictions and the true or reference solution, accounting for the variability introduced by the random inputs. It provides insight into the convergence of the simulation as the number of samples increases and allows us to validate the robustness and reliability of the stochastic model in capturing the essential physics of the thermo-poroelastic system.

As we have generated samples from the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$, and we are able to control the discretization error, we may estimate its expected value.

Thus, considering a sufficiently fine spatial tessellation \mathcal{T}_l with a time discretization Δt_l , we denote the time-discrete finite element solution $(\mathbf{u}_l, p_l, \theta_l)$. In this section we note $(\mathbf{u}_0, p_0, \theta_0)$ the solution for an initial discretization \mathcal{T}_0 , for the following time-step Δt_0 .

Thus to approximates

$$\mathbb{E}[\bar{\mathbf{u}}], \mathbb{E}[\bar{p}] \text{ and } \mathbb{E}[\bar{\theta}],$$

we will consider the means of the finite element solutions

$$\mathbb{E}[\mathbf{u}_l], \mathbb{E}[p_l] \text{ and } \mathbb{E}[\theta_l].$$

Thus, we will compute estimators to this previous quantities, so that for $M \in \mathbb{N}^*$, we draw the samples $\omega^{(i)}$, for $1 \leq i \leq M$ and compute

$$\begin{aligned} \mathbb{E}_M[\mathbf{u}_l] &:= \frac{1}{M} \sum_{i=1}^M \mathbf{u}_l(\omega^{(i)}), \\ \mathbb{E}_M[p_l] &:= \frac{1}{M} \sum_{i=1}^M p_l(\omega^{(i)}), \\ \mathbb{E}_M[\theta_l] &:= \frac{1}{M} \sum_{i=1}^M \theta_l(\omega^{(i)}). \end{aligned} \quad (6.37)$$

The accuracy of our approximations is quantify with the RMSE, so that we aim to control

$$\begin{aligned} \epsilon(\mathbf{u}) &:= \left(\mathbb{E} \left(\mathbb{E}_M[\mathbf{u}_l] - \mathbb{E}[\bar{\mathbf{u}}] \right)^2 \right)^{1/2}, \\ \epsilon(p) &:= \left(\mathbb{E} \left(\mathbb{E}_M[p_l] - \mathbb{E}[\bar{p}] \right)^2 \right)^{1/2}, \\ \epsilon(\theta) &:= \left(\mathbb{E} \left(\mathbb{E}_M[\theta_l] - \mathbb{E}[\bar{\theta}] \right)^2 \right)^{1/2}. \end{aligned} \quad (6.38)$$

The approximation by Monte Carlo methods is bounded by the following Lemmas,

Lemma 6.5.1. *Let $M \in \mathbb{N}^*$ and $\zeta \in L^2(\Omega; V)$. Then*

$$\|\mathbb{E}[\zeta] - \mathbb{E}_M[\zeta]\|_{L^2(\Omega; V)} \leq \frac{\|\zeta\|_{L^2(\Omega; V)}}{\sqrt{M}}, \text{ and } \|\mathbb{E}_M[\zeta]\|_{L^2(\Omega; V)} \leq \|\zeta\|_{L^2(\Omega; V)}.$$

Proof. We consider $M \in \mathbb{N}^*$.

Thus there holds that

$$(\mathbb{E}[X] - \mathbb{E}_M[X])^2 = \left(\frac{1}{M} \sum_{i=1}^M \mathbb{E}[X] - X_i \right)^2.$$

Thus by taking the expectations,

$$\begin{aligned} \mathbb{E} \left(\left(\frac{1}{M} \sum_{i=1}^M \mathbb{E}[X] - X_i \right)^2 \right) \\ = \mathbb{V} \left(\frac{1}{M} \sum_{i=1}^M X \right), \end{aligned}$$

and by the variance operator and since X are iid variables, there holds

$$\mathbb{V} \left(\frac{1}{M} \sum_{i=1}^M X \right) = \frac{1}{\sqrt{M}} \mathbb{V}(X).$$

□

In addition to this Lemma, we recall some notations and classical results, Thus, by considering the computed solution and the analytic solution, we can ensure that

Theorem 6.5.1. *Let Hypothesis 10 holds. For $M \in \mathbb{N}^*$, the root mean-square error (RMSE) is bounded by*

$$\begin{aligned} \|\mathbb{E}(\bar{\mathbf{u}}) - \mathbb{E}_M(\mathbf{u}_l)\|_{L^2(\Omega, L^\infty(0, T; H_0^1(D)))} + \|\mathbb{E}(\bar{p}) - \mathbb{E}_M(p_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \\ + \|\mathbb{E}(\bar{\theta}) - \mathbb{E}_M(\theta_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \leq C(h_l + \Delta t_l + 1/\sqrt{M}) \end{aligned} \quad (6.39)$$

where h_l is the size of the spatial discretization, Δt_l the size of the time discretization and where C is a positive constant which depends on the initial data.

Proof. We set

$$V := L^\infty(0, T; L^2(D)).$$

Thus, we consider each elements on the left hand side of Equation (6.39).

For the mechanical displacements, there holds that

$$\|\mathbb{E}[\bar{\mathbf{u}}] - \mathbb{E}_M(\mathbf{u}_l)\|_{L^2(\Omega; V)} \leq \|\mathbb{E}[\bar{\mathbf{u}}] - \mathbb{E}_M[\bar{\mathbf{u}}]\|_{L^2(\Omega; V)} + \|\mathbb{E}_M[\bar{\mathbf{u}} - \mathbf{u}_l]\|_{L^2(\Omega; V)}. \quad (6.40)$$

From Lemma 6.5.1, we bound the right hand side of Equation (6.40) by

$$\begin{aligned} \|\mathbb{E}[\bar{\mathbf{u}}] - \mathbb{E}_M[\bar{\mathbf{u}}]\|_{L^2(\Omega; V)} &\leq \frac{1}{\sqrt{M}} \|\bar{\mathbf{u}}\|_{L^2(\Omega; V)}, \\ \|\mathbb{E}_M[\bar{\mathbf{u}} - \mathbf{u}_l]\|_{L^2(\Omega; V)} &\leq \|\bar{\mathbf{u}} - \mathbf{u}_l\|_{L^2(\Omega; V)} \end{aligned} \quad (6.41)$$

By Theorem 6.2.2, there exists a positive constant C such that $\|\bar{\mathbf{u}}\|_{L^2(\Omega; V)} \leq C$, and by Theorem 6.4.2 there exists a positive constant C such that $\|\bar{\mathbf{u}} - \mathbf{u}_l\|_{L^2(\Omega; V)} \leq C(h_l + \Delta t_l)$.

We can extend this results to the solutions $(\bar{p}, \bar{\theta})$ and (p_l, θ_l) , so that Theorem 6.5.1 is proved. □

The control of the root mean-square error and the uncertainty by an $\epsilon > 0$, thus follows by

- $h_l, \Delta t_l = \mathcal{O}(\epsilon)$,
- $M = \mathcal{O}(\epsilon^{-2})$.

Standard Monte-Carlo simulation is often computationally expensive. The complexity of the thermo-poroelastic system, the need for high-dimensional meshes or the large number of samples required to reduce statistical error arises a high computational cost.

Thus, we can introduce the Multi-Level Monte Carlo method designed to reduce the computational cost of estimating expectations in problems where standard Monte Carlo simulations are expensive. The main idea behind MLMC is to exploit a hierarchy of discretizations of the PDEs, combining coarse meshes providing cheaper solutions and fine meshes giving more accurate solutions. Combining these solution across different levels reduce the overall cost, without sacrificing accuracy.

6.5.1 Multi-level Monte Carlo

We are interested in finding the expected values of some quantity of interest Q , which is expressed as a functional of the solution $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$,

$$Q := \mathcal{G}(\bar{\mathbf{u}}, \bar{p}, \bar{\theta}).$$

From the numerical schemes introduced previously, we approximate Q by

$$Q_h := \mathcal{G}(\mathbf{u}_h, p_h, \theta_h),$$

defined on a spatial grid \mathcal{T}_h .

Thus for $L \in \mathbb{N}$ let define a sequence of subspaces

$$V_{h_0} \subset \dots \subset V_{h_L},$$

with refinement size defined as $h_0 > \dots > h_L > 0$. For sake of simplicity, we assume that the time-step Δt depends directly on the spatial discretization, so that there exists a linear function φ such that

$$\Delta t_l = \varphi(h_l). \quad (6.42)$$

Thus for each $0 \leq l \leq L$, we are able to define Q_{h_l} , and by linearity of the expectation, $\mathbb{E}[Q_L]$ is written as a telescoping sum:

$$\mathbb{E}[Q_{h_L}] = \mathbb{E}[Q_{h_0}] + \sum_{l=1}^L \mathbb{E}[Q_{h_l} - Q_{h_{l-1}}] \quad (6.43)$$

so that :

- Q_{h_l} is the approximation of the quantity of interest as the l -th discretization level in space and time,
- The term $Q_{h_l} - Q_{h_{l-1}}$ represents the difference in the quantities of interest between successive levels, which typically becomes smaller as the mesh is refined,
- Q_{h_0} correspond to the coarsest level (least accurate, but cheap),
- L is the finest level of discretization.

The multi level Monte Carlo approximates these expectations using independent Monte Carlo averages at each level. The number of samples N_l at each level is optimized based on the variance of $Q_l - Q_{l-1}$ and the computational cost at that level. The MLMC estimator is given by

$$\mathbb{E}_{MLMC}[Q_L] := \frac{1}{N_0} \sum_{i=1}^{N_0} Q_0(\omega_i) + \sum_{l=1}^L \left(\frac{1}{N_l} \sum_{i=1}^{N_l} (Q_l(\omega_i) - Q_{l-1}(\omega_i)) \right) \quad (6.44)$$

where

- N_l is the number of samples at level l ,
- ω_i represents an independent realization of the random parameter ω . The same sample is used on both discretization.

Theorem 6.5.2. *Let Hypothesis 10 holds. There exists a positive constant C such that the RMSE is bounded as follows :*

$$\begin{aligned} & \|\mathbb{E}(\bar{\mathbf{u}}) - \mathbb{E}_M(\mathbf{u}_l)\|_{L^2(\Omega, L^\infty(0, T; H_0^1(D)))} + \|\mathbb{E}(\bar{p}) - \mathbb{E}_M(p_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \\ & + \|\mathbb{E}(\bar{\theta}) - \mathbb{E}_M(\theta_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \leq C(h_L + \Delta t_L + \sum_{l=0}^{L-1} \frac{h_l + \Delta t_l}{\sqrt{M_{l+1}}} + \frac{h_{l_0} + \Delta t_{l_0}}{\sqrt{M_0}} + \frac{1}{\sqrt{M_0}}) \end{aligned} \quad (6.45)$$

Proof. Following the proof of Theorem 6.5.1, we have that

$$\begin{aligned} \|\mathbb{E}[\bar{\mathbf{u}}] - \mathbb{E}_{MLMC}[\mathbf{u}_L]\| & \leq \|\mathbb{E}[\bar{\mathbf{u}}] - \mathbb{E}[\mathbf{u}_L]\|_{L^2(\Omega; V)} + \|\mathbb{E}[\mathbf{u}_L] - \mathbb{E}_{MLMC}[\mathbf{u}_L]\|_{L^2(\Omega; V)}^2 \\ & \leq \|\bar{\mathbf{u}} - \mathbf{u}_L\|_{L^2(\Omega; V)} + \left\| \sum_{l=1}^L \left(\mathbb{E}[\mathbf{u}_l - \mathbf{u}_{l-1}] - \frac{1}{M_l} \sum_{i=1}^{M_l} (\mathbf{u}_l(\omega_i) - \mathbf{u}_{l-1}(\omega_i)) \right) \right\|_{L^2(\Omega; V)} \\ & \quad + \left\| \mathbb{E}[\mathbf{u}_{h_0}] - \frac{1}{M_0} \sum_{i=1}^{M_0} (\mathbf{u}_{h_0}(\omega_i)) \right\|_{L^2(\Omega; V)} \\ & \leq C(h_L + \Delta t_L) + \sum_{l=0}^L \frac{1}{\sqrt{M_l}} \|\mathbf{u}_l - \mathbf{u}_{l-1}\|_{L^2(\Omega; V)} + \frac{1}{\sqrt{M_0}} \|\mathbf{u}_{l_0}\|_{L^2(\Omega; V)}. \end{aligned}$$

For all $1 \leq l \leq L$, there holds that for some positive constant C

$$\begin{aligned} \|\mathbf{u}_l - \mathbf{u}_{l-1}\|_{L^2(\Omega; V)} & \leq \|\mathbf{u}_l - \bar{\mathbf{u}}\|_{L^2(\Omega; V)} + \|\mathbf{u}_{l-1} - \bar{\mathbf{u}}\|_{L^2(\Omega; V)} \\ & \leq C(h_l + \Delta t_l + h_{l-1} + \Delta t_{l-1}). \end{aligned}$$

Moreover,

$$\|\mathbf{u}_{l_0}\|_{L^2(\Omega; V)} \leq C(h_{l_0} + \Delta t_{l_0}) + \|\bar{\mathbf{u}}\|_{L^2(\Omega; V)}.$$

From Theorem 6.2.2, $\|\bar{\mathbf{u}}\|_{L^2(\Omega; V)}$ is bounded, and thus we have the following estimate

$$\|\mathbb{E}(\bar{\mathbf{u}}) - \mathbb{E}_M(\mathbf{u}_l)\|_{L^2(\Omega, L^\infty(0, T; H_0^1(D)))} \leq C(h_L + \Delta t_L + \sum_{l=0}^{L-1} \frac{h_l + \Delta t_l}{\sqrt{M_{l+1}}} + \frac{h_{l_0} + \Delta t_{l_0}}{\sqrt{M_0}} + \frac{1}{\sqrt{M_0}}). \quad (6.46)$$

We can conclude by applying the same reasoning to \bar{p} and $\bar{\theta}$, and their numerical approximations. \square

Corollary 6.5.1. *Let the hypothesis of Theorem 6.5.2 holds. For $L \in \mathbb{N}$, considering a decreasing sequences $h_0 > \dots > h_L$, $\Delta t_0 > \dots > \Delta t_L$ and $M_l \in \mathbb{N}$ such that*

$$\frac{1}{M_0} \simeq (h_L + \Delta t_L)^2 \quad \text{and} \quad \frac{1}{M_l} \simeq \frac{(h_L + \Delta t_L)^2}{(h_{l-1} + \Delta t_{l-1})^2} \gamma_l^2,$$

where $\gamma_l \geq 0$ for all $l \in \mathbb{N}$, such that

$$\sum_{l=1}^L \gamma_l = 1.$$

Then, there exists a positive constant C such that

$$\begin{aligned} & \|\mathbb{E}(\bar{\mathbf{u}}) - \mathbb{E}_M(\mathbf{u}_l)\|_{L^2(\Omega, L^\infty(0, T; H_0^1(D)))} + \|\mathbb{E}(\bar{p}) - \mathbb{E}_M(p_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \\ & + \|\mathbb{E}(\bar{\theta}) - \mathbb{E}_M(\theta_l)\|_{L^2(\Omega, L^\infty(0, T; L^2(D)))} \leq C(h_L + \Delta t_L). \end{aligned} \quad (6.47)$$

Proof. We apply the hypothesis on the coefficients M_l to Theorem 6.5.2. The result follows. \square

6.6 Applications to geothermal systems

This thesis was part of the UPGEO project from the French National Agency for Research (ANR) about the UPscaling and the simulation of heat flows to improve the efficiency of deep GEOthermal systems.

This section will explain the main objectives of the UPGEO project, and how this thesis has partially answered to practical questions of geologists.

6.6.1 The UPGEO project

In France, heat networks are largely dependent on fossil fuels (42%), and the share of deep geothermal energy accounts for less than 5%. Its deployment in metropolitan areas is limited by a geological risk that is difficult to predict. UPGEO has worked on developing a predictive digital tool that will enable thermo-hydro-mechanical simulations and assess the performance of a specific location in Île-de-France regarding its geothermal potential.

Geothermal energy, which refers to the mobilization of heat contained in the subsurface at very low, low, or high temperatures, is one of the methods for achieving the energy transition. The new multiannual programming for renewable energy has recently revised downward the targets for deploying deep geothermal energy in France, particularly due to the high geological risks that a new operation may not yield a geothermal resource with sufficient flow and temperature characteristics to ensure the project's profitability over its lifespan. This geological risk poses an obstacle to the future development of geothermal energy in France and the Île-de-France region. The UPGEO project has focused on innovative modeling methods, proposing solutions that optimize and explore the development of new areas in sedimentary domains. This optimization requires a precise understanding of the reservoir's heterogeneity in terms of sedimentary geometries, porosity/permeability, and reservoir connectivity, and reliable numerical simulations of flows and heat transfer over 30 years and even up to 100 years after production begins.

The main objective of the UPGEO project was to successfully scale the permeability measured in the laboratory to the sedimentary connectivity of reservoir bodies at a kilometer scale. The way to homogenize effective coefficients such as porosity, permeability, mechanical deformation (Gassmann tensor and Biot coefficient), or effective thermal dispersion through equations valid at all points for fluid and solid constituents has constituted the major challenge of this project, which has required coupling geological and mathematical data and concepts.

Among all the objectives of UPGEO project, there were questions of modelling and simulation codes:

- Proposing new equations for upscaling, i.e., Thermo-Hydro-Mechanical (THM) equations, including Biot's quasi-static equations, for simulating flow in unfractured sands and in fractured siliciclastic and carbonate reservoirs.
- Implementing simulation codes that integrate upscaling into specific plugins on platforms such as Eclipse, PumaFlow, OpenGeoSys, or DuMuX.

One of the challenges was to bring together two scientific communities that rarely collaborate in France: geologists and mathematicians.

6.6.2 Orsay Valley and the Saclay Plateau

In the context of the UPGEO project, we aim to apply the models and their numerical solutions to the Orsay Valley and the Saclay Plateau, where geothermal activities are conducted daily.

Our primary objective is to demonstrate that our models align with the collected data and accurately reproduce long-term behavior based on documented real-world data.

We use the data collected by the geology team at Orsay. The works focuses on the following domain (see Figure 6.1), which is the geothermal complex at the Moulon District in Paris-Saclay.

CentraleSupélec and Orsay's laboratory are closed to a geothermal production well, see Figure 6.2 for today's satellite view.

The water flow is in the direction of Paris (south-west to north-east), with a pressure differential between incoming and outgoing flow of 4 Bars. We consider the following set of parameter,

$$E = 5.10^9 \text{Pa}, \nu = 0.3,$$

with

$$\mu = \frac{E}{2(1+\nu)}, \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

We set the coupling parameter α as $\alpha = 1$. The water compressibility modulus verifies

$$c_f = 4.5E - 5\text{Pa}^{-1}$$

and the global porosity is set to 0.2. The permeability κ is measured globally in the domain as

$$\kappa = 917.45 \text{ in mD}.$$

Finally, the viscosity is set at $0.8 \times 10^{-3} \text{ kg}\cdot\text{m}^{-3}$.

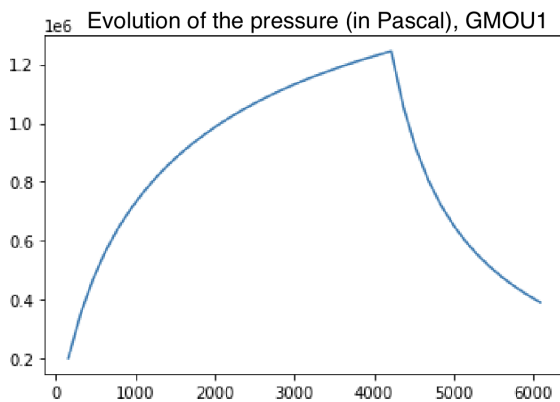


Figure 6.3: Pressure over time in GMOU1

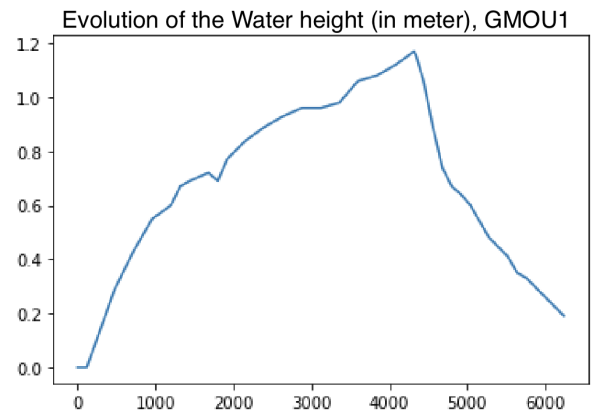


Figure 6.4: Water height over time in GMOU1

Figure 6.5: Matching between simulated data and real-world data.

Figure 6.4 represents the evolution of the water height, measured at GMOU1, while water is pumped at 200 Pa, until 4320 min, inside GMOU2.

Since the water height is proportional to the pressure, the computational results representing the pressure evolution is in accordance with the measured data on the well GMOU1 for water height.

6.6.3 Uncertainties Quantification

The soil is randomly filled to reproduce a complete soil profile. This approach accounts for the natural heterogeneity of soils, which is often difficult to model deterministically.

To quantify the uncertainties associated with soil modeling, we use the Monte Carlo method. This technique involves conducting a large number of simulations by randomly sampling input parameters such as porosity, permeability, and mineral composition. Each simulation generates a set of results that reflect the variability of soil conditions.

By studying the results to these uncertainties, we can identify the most influential parameters on the behavior of the system. This analysis helps us understand how variations in soil properties can affect critical aspects such as fluid flow, heat transport, and the mechanical stability of geothermal structures.

Furthermore, by integrating the results of the Monte Carlo simulations into a decision-making framework, we can better assess the risks and opportunities associated with geothermal projects. This also allows us to formulate evidence-based recommendations while considering the inherent uncertainties in our modeling.

In conclusion, quantifying uncertainties through the Monte Carlo method is essential for a reliable assessment of the performance of geothermal systems, thereby ensuring informed decision-making and effective planning.

6.7 Numerical results

6.7.1 Random parameters

Given that Y is a random variable following a normal distribution $\mathcal{N}(\mu, \sigma)$, we consider the following law $X = X(\zeta)$ which read as

$$X = \exp\left(\zeta Y - \frac{\zeta^2}{2}\right),$$

where ζ is a positive parameter.

Thus, given $\mu(x)$, $\lambda(x)$, $\kappa(x)$ and $\beta(x)$, we define the random parameter in the system (6.14) as

$$\begin{aligned}\mu(\omega, x) &= \mu(x)X(\zeta_\mu), \quad \lambda(\omega, x) = \lambda(x)X(\zeta_\lambda), \\ \kappa(\omega, x) &= \kappa(x)X(\zeta_\kappa), \quad \beta(\omega, x) = \beta(x)X(\zeta_\beta).\end{aligned}$$

The expectation of X is given by:

$$E(X) = E\left(\exp\left(\zeta Y - \frac{\zeta^2}{2}\right)\right)$$

We can rewrite the expectation as:

$$E(X) = \exp\left(-\frac{\zeta^2}{2}\right) \cdot E(\exp(\zeta Y)),$$

where $\exp\left(-\frac{\zeta^2}{2}\right)$ is a constant factor that can be taken out of the expectation.

Since Y follows a normal distribution $\mathcal{N}(\sigma, \mu)$, its probability density function (PDF) is:

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

The expectation of $\exp(\zeta Y)$ is given by:

$$E(\exp(\zeta Y)) = \int_{-\infty}^{+\infty} \exp(\zeta y) f_Y(y) dy$$

Substituting the probability density function of Y :

$$E(\exp(\zeta Y)) = \int_{-\infty}^{+\infty} \exp(\zeta y) \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

We combine the exponential terms:

$$E(\exp(\zeta Y)) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(\zeta y - \frac{(y-\mu)^2}{2\sigma^2}\right) dy.$$

Expanding the quadratic term in the exponent:

$$-\frac{(y-\mu)^2}{2\sigma^2} = -\frac{y^2 - 2y\mu + \mu^2}{2\sigma^2} = -\frac{y^2}{2\sigma^2} + \frac{y\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}.$$

Combining with ζy :

$$\zeta y - \frac{(y-\mu)^2}{2\sigma^2} = \frac{1}{2\sigma^2} (2(\zeta\sigma^2 + \mu)y - y^2 - \mu^2)$$

To evaluate the integral, we complete the square in the exponent, which results in a Gaussian integral for the first term:

$$\zeta y - \frac{(y-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2} (y - (\zeta\sigma^2 + \mu))^2 + \frac{(\zeta\sigma^2 + \mu)^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}.$$

The integral resolves to:

$$E(\exp(\zeta Y)) = \exp\left(\zeta\mu + \frac{\zeta^2\sigma^2}{2}\right)$$

Recalling the constant factor $\exp\left(-\frac{\zeta^2}{2}\right)$, we have:

$$E(X) = \exp\left(-\frac{\zeta^2}{2}\right) \cdot \exp\left(\zeta\mu + \frac{\zeta^2\sigma^2}{2}\right).$$

Simplifying the expression, we get:

$$E(X) = \exp\left(\zeta\mu + \frac{\zeta^2\sigma^2}{2} - \frac{\zeta^2}{2}\right).$$

Thus we conclude that the expectation of X is:

$$E(X) = \exp\left(\zeta\mu + \frac{\zeta^2\sigma^2}{2} - \frac{\zeta^2}{2}\right)$$

This result is due to the exponential transformation applied to the normal variable, which results in a log-normal-like distribution, adjusted by the $-\frac{\zeta^2}{2}$ term.

6.7.2 Numerical solutions

We consider the following analytical solutions given

$$\eta(t, x, y) := tx(1-x)y(1-y).$$

We set the functions $(\bar{\mathbf{u}}, \bar{p}, \bar{\theta})$ as $\bar{\mathbf{u}}(t, x, y) = (\zeta(t, x, y), \zeta(t, x, y))$, $\bar{p} = \zeta(t, x, y)$ and $\bar{\theta} = \zeta(t, x, y)$.

We consider the following parameters for the problem. The Young's Modulus E is set to 1, and the Poisson ratio ν_P is set to 0.25. Thus, since

$$\bar{\mu} = \frac{E}{2(1+\nu_P)}, \quad \bar{\lambda} = \frac{E\nu_P}{(1+\nu_P)(1-2\nu_P)}$$

we set the Lamé parameters as $\bar{\mu} = 0.8$ and $\bar{\lambda} = 0.4$. The diffusion coefficient are set as $\kappa = 1$ and $\beta = 1$, while the Biot coefficient is set as $\alpha = 0.5$, the Biot modulus verify $M = 1$. The solid expansion coefficient and the thermoelastic coupling parameter verifies $K_S = 1$, $K_T = 1$.

The randomness is introduced as following. Given a positive parameter ζ , we set

$$\begin{aligned}\mu(\omega) &= \bar{\mu}X(\zeta_\mu), \lambda(\omega) = \bar{\lambda}X(\zeta_\lambda), \\ \kappa(\omega) &= \bar{\kappa}X(\zeta_\kappa), \beta(\omega) = \bar{\beta}X(\zeta_\beta).\end{aligned}$$

We propose several realisations, by randomizing one of the parameter μ , λ , κ or β .

First, setting the parameters $\zeta_\mu = \zeta_\lambda = \zeta_\beta = 0$, and $\zeta_\kappa = 1$, i.e. introducing randomness only on the parameter κ , we illustrate in Figure 6.6 the error for the pressure between the exact solution and the Monte-carlo computed solutions over 200 samples of the permeability $\{\kappa(\omega_i)\}_{i=1\dots 200}$. This error is given by

$$\left\| \mathbb{E}[\bar{p}] - \frac{1}{200} \sum_{i=1}^{200} p_h(\omega_i) \right\|_{L^2(D \times (0,T))},$$

for multiple sizes of the meshes $h \in \{2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}\}$ and corresponding time step $\Delta t = h$.

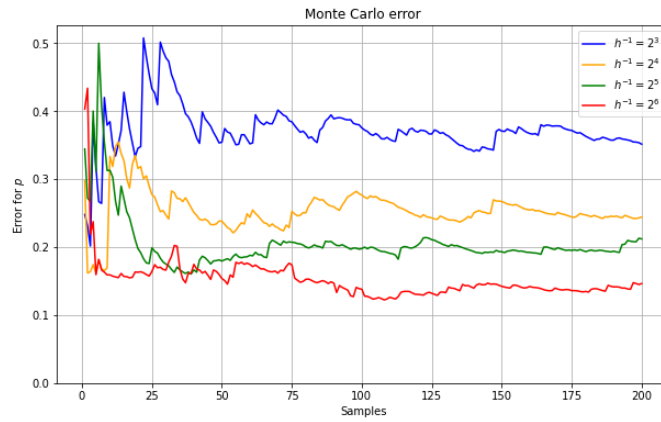


Figure 6.6: Error for \bar{p} at several mesh-size

By setting $\zeta_\mu = \zeta_\lambda = \zeta_\kappa = 0$, $\zeta_\beta = 1$, i.e. introducing randomness only on the parameter β , we illustrate in Figure 6.7 the error for the temperature between the exact solution and the Monte-carlo computed solutions over 200 samples of the thermal diffusion $\{\beta(\omega_i)\}_{i=1\dots 200}$. This error is given by

$$\left\| \mathbb{E}[\bar{\theta}] - \frac{1}{200} \sum_{i=1}^{200} \bar{\theta}_h(\omega_i) \right\|_{L^2(D \times (0,T))}$$

for multiple sizes of the meshes $h \in \{2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}\}$ and corresponding time step $\Delta t = h$.

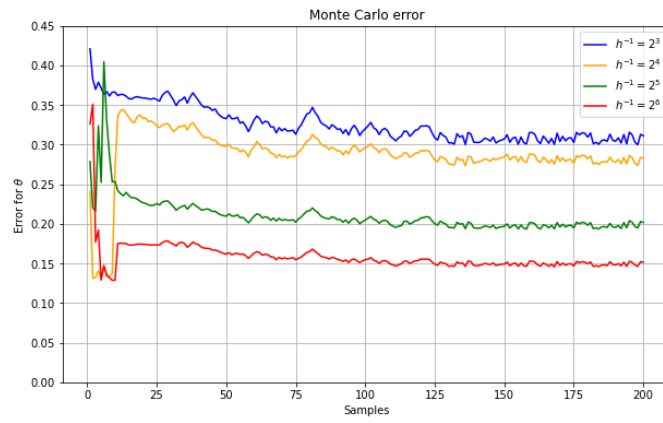
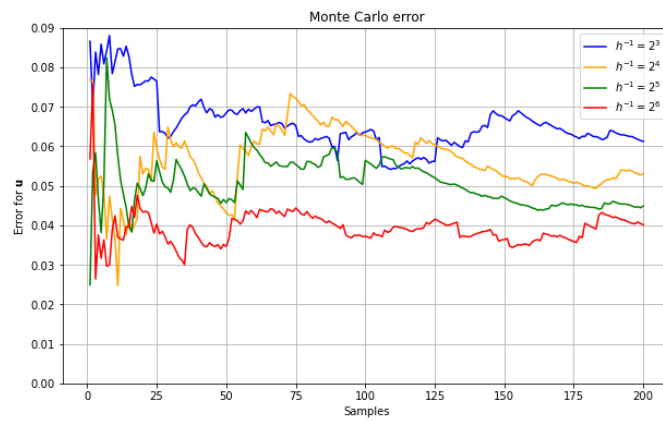
By setting $\zeta_\mu = \zeta_\lambda = 1$, and $\zeta_\kappa = \zeta_\beta = 0$, i.e. introducing randomness only on the Lamé parameters μ , λ , we illustrate in Figure 6.8 the error for the mechanical displacement between the exact solution and the Monte-carlo computed solutions over 200 samples of the Lamé parameters $\{\mu(\omega_i), \lambda(\omega_i)\}_{i=1\dots 200}$.

This error for 200 samples is given by

$$\left\| \mathbb{E}[\text{div}(\bar{\mathbf{u}})] - \sum_{i=1}^{200} \text{div}(\bar{\mathbf{u}}_h(\omega_i)) \right\|_{L^2(D \times (0,T))}$$

for multiple sizes of the meshes $h \in \{2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}\}$ and corresponding time step $\Delta t = h$.

Due to the approximation in space and time, there exists a remaining error of approximation, related to the size of the meshes and time step, which explains why the Monte Carlo error does not converge to 0 on Figures 6.6, 6.7 and 6.8.

Figure 6.7: Error for $\bar{\theta}$ at several mesh-sizeFigure 6.8: Error for \bar{u} at several mesh-size

Chapter 7

Conclusion and Perspectives

This thesis focuses on the modeling of geological phenomena in a poroelastic medium and the quantification of uncertainties applied to a (thermo-)poroelastic model. It addresses aspects such as finite element convergence, time approximation, and the errors associated with Monte Carlo methods and Karhunen-Loève expansions. The thermo-poroelastic model studied captures the interactions between the elastic deformation of a porous material, fluid diffusion within the pores, and thermal variations. This type of model is crucial in various industrial fields, where it is essential to assess the impact of uncertainties inherent in material properties, measurements, boundary conditions, and applied loads.

As a first step, we examined the regularity of the weak solution of a system based on the Biot model, excluding thermal diffusion. In this case, we proved in Theorem 3.4.1 and Theorem 3.5.2 that, under natural assumptions—such as sufficient spatial regularity of the initial pressure, adequate time and space regularity of the forcing terms, and smooth parameters (see Hypothesis 2 and Hypothesis 3)—the solution achieves additional regularity in both space and time. In a second step, we introduced a numerical approximation based on the Gradient Discretization Method. We demonstrated the existence and uniqueness of a solution to the discrete problem and estimated the numerical error in Theorem 4.3.1. This method was instantiated using $(\mathbf{P}_2) - (P_1)$ Lagrangian elements. In a third step, we studied a Biot problem with random coefficients. Using uniform estimates with respect to the randomness, we controlled the error between the true random solution and the solution obtained from a truncated Karhunen-Loève expansion. This error, described in Theorem 5.4.1, is determined by the convergence rate of the truncated Karhunen-Loève expansion for the random parameters μ , λ and κ .

Finally, we investigated the robustness and convergence of numerical methods, particularly the Finite Element Method applied to both poroelastic and thermo-poroelastic models. Our analyses demonstrated that space-time discretization via finite elements ensures the convergence of solutions towards the exact solution of the system under certain conditions. Convergence studies were conducted to determine the rate of convergence and to minimize the numerical error associated with this method.

To quantify uncertainties associated with model parameters, we applied the Monte Carlo method. This probabilistic approach evaluates the effects of these uncertainties on the results of the thermo-poroelastic model, providing estimates of the system's outputs. This error quantification is described in Theorem 6.5.1. The Monte Carlo error, which depends on the number of simulations and the variance of the input parameters, was illustrated through numerical simulations performed on a personal computer.

The combined use of finite element methods and Monte Carlo simulations enables accurate modeling while rigorously quantifying uncertainties. Furthermore, applying the Multilevel Monte Carlo method offers more refined quantification and resolution, with a reduced computational workload. Future research could focus on optimizing simulation algorithms to achieve a better balance between accuracy and computational cost, as well as integrating advanced uncertainty reduction techniques for more complex, large-scale applications.

Several aspects of this work could be further improved or expanded. For example, the Gradient Discretization Method could be tested on different meshes adapted to the studied domain. Additionally, employing polynomial chaos expansion on the poroelastic problem to account for parameter randomness while coupling it with the Gradient Discretization Method, could provide a more generalized framework to address these problems.

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Titre : Analyse numérique d'un problème poroélastique à coefficients aléatoires :
Application à la géothermie

Mots clés : poroélastique, coefficients aléatoires, discrétisation de gradient, GDM, géothermie.

Résumé : Ce document présente une analyse numérique d'un système poroélastique avec des coefficients aléatoires, en se concentrant sur les applications dans le stockage de l'énergie géothermique. Il vise à améliorer la compréhension des systèmes géothermiques grâce à une modélisation mathématique avancée de l'écoulement des fluides et des déformations mécaniques dans les milieux poreux, en particulier à l'aide du modèle de Biot. L'étude incorpore des coefficients aléatoires pour tenir compte de la variabilité naturelle et des incertitudes des formations géologiques, ce qui rend les modèles plus réalistes et plus robustes pour des applications pratiques telles que l'énergie géothermique, l'extraction d'hydrocarbures et le stockage du dioxyde de carbone.

Le travail met en évidence la méthode de discrétisation de gradient (GDM) en tant qu'approche

numérique générique pour résoudre les systèmes couplés dans les modèles poroélastiques. On utilise également la décomposition de Karhunen-Loève pour représenter les données aléatoires, réduisant ainsi la complexité de calcul tout en capturant les caractéristiques statistiques essentielles du système.

En outre, un modèle thermo-poroélastique est développé pour inclure le transfert de chaleur, qui est crucial pour les applications géothermiques. Ce modèle intègre l'écoulement, la déformation élastique et les effets thermiques afin d'améliorer la prévision, la gestion du stockage et l'extraction de l'énergie souterraine. La recherche met l'accent sur la quantification de l'incertitude et présente des méthodes numériques robustes pour tenir compte de la nature stochastique des paramètres du sol, afin d'améliorer la prise de décision et réduire les risques pour des applications en géothermie.

Title: Numerical analysis of a poroelastic problem with random coefficients:
Application to geothermal energy

Keywords: poroelastic, random coefficients, Gradient Discretization Method, GDM, geothermal energy

Abstract: The document presents a numerical analysis of a poroelastic system with random coefficients, focusing on applications in geothermal energy storage. It aims to improve understanding of geothermal systems through advanced mathematical modeling of fluid flow and mechanical deformations in porous media, specifically using the Biot model. The study incorporates random coefficients to account for the natural variability and uncertainties in geological formations, which makes the models more realistic and robust for practical applications like geothermal energy.

The work highlights the Gradient Discretization Method (GDM) as a generic numerical approach for solving coupled systems in poroelastic models. We

also employ the Karhunen-Loève decomposition to represent random processes, reducing the computational complexity while capturing essential statistical characteristics of the system.

Furthermore, a thermo-poroelastic model is developed to include heat transfer, which is crucial for geothermal applications. This model integrates flow, elastic deformation, and thermal effects to improve prediction and management of subsurface energy storage and extraction. The research emphasizes uncertainty quantification and presents robust numerical methods to address the stochastic modelisation of the subsurface parameters.

