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# Tracking Correlations through Autonomous Collisional Models - Impact on Dynamics and Energetics

Samyak Pratyush Prasad

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THÈSE

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**Suivi des corrélations via des modèles de collision autonomes -  
Impact sur la dynamique et l'énergie**

**Tracking Correlations through Autonomous Collisional Models -  
Impact on Dynamics and Energetics**

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# Abstract (English)

Waveguide QED refers to the physics of quantum emitters coupled to reservoirs of light modes confined in one dimension, such as superconducting and photonic circuits. Owing to constant technological improvements it is now possible to measure the state of the light with high efficiency. These new experimental capacities mandate to update theoretical tools, to close formerly open quantum systems. This is the purpose of Autonomous Collisional Models (ACM). They model the emitter-field interaction as repeated interactions in a closed manner, with no external influence. By construction, an ACM captures the unitary dynamics of the quantum emitter and the light modes, unlocking access to their correlations. Moreover, they are energy conserving and give rise to a thermodynamics framework with symmetric and accurate energy balances between quantum systems.

We first build an ACM to describe a qubit coupled to a displaced thermal field, which is the regime of the Optical Bloch Equations (OBE). Through this study, we track fundamental correlations created within each collision and in doing so find that each sub-system is driven by an effective Hamiltonian while a remnant term captures the effect of the correlations. They respectively impact the field amplitude and fluctuations, resulting in a physically observable splitting. We then explore the thermodynamic consequences of our framework by developing a general paradigm valid for any isolated bipartite system, i.e., bipartite quantum energetics (BQE). Exploiting the global energy conservation, we define work-like (heat-like) flows as the energy changes stemming from the effective Hamiltonian dynamics (the dynamics induced by the correlations). We show that these quantities are accessible through measurements performed on the field, such as dyne or spectroscopic measurements. Our approach differs from former analyses by the emitter self  $b$ -work, which yields a tighter expression of the second law. We quantitatively relate this tightening to the extra-knowledge about the field, compared with usual treatment of the atom as an open system.

Lastly, to extract intuitions about the energetic impact of correlations, we apply BQE to the interaction of light fields within a BS. It reveals that the energies exchanged

are directly measurable in the fields' phase-space distributions. We find that the field is deformed whenever heat-like energy is exchanged, while displacements are caused through transfer of work-like energy. We again find a tightening of the second law and show that in this framework squeezing can transfer heat from a colder to a hotter field, while maintaining a positive entropy production. The concepts and effects we introduce deepen our understanding of thermodynamics in the quantum regime and its potential for energy management at quantum scales.

# Résumé (Français)

Waveguide QED fait référence à la physique des émetteurs quantiques couplés à des réservoirs de modes lumineux confinés dans une dimension, tels que les circuits supraconducteurs et photoniques. Grâce aux améliorations technologiques constantes, il est désormais possible de mesurer l'état de la lumière avec une grande efficacité. Ces nouvelles capacités expérimentales imposent de mettre à jour les outils théoriques, de fermer des systèmes quantiques jusqu'alors ouverts. C'est le but des modèles collisionnels autonomes (ACM). Ils modélisent l'interaction émetteur-champ comme des interactions répétées de manière fermée, sans influence extérieure. Par construction, un ACM capture la dynamique unitaire de l'émetteur quantique et des modes lumineux, ouvrant l'accès à leurs corrélations. De plus, ils sont économes en énergie et donnent lieu à un cadre thermodynamique avec des bilans énergétiques symétriques et précis entre les systèmes quantiques.

Nous construisons d'abord un ACM pour décrire un qubit couplé à un champ thermique déplacé, qui est le régime des équations de Bloch optiques (OBE). Dans cette étude, nous suivons les corrélations fondamentales créées au sein de chaque collision et ce faisant, nous constatons que chaque sous-système est piloté par un hamiltonien effectif tandis qu'un terme résiduel capture l'effet des corrélations. Ils impactent respectivement l'amplitude et les fluctuations du champ, ce qui entraîne une séparation physiquement observable. Nous explorons ensuite les conséquences thermodynamiques de notre cadre en développant un paradigme général valable pour tout système bipartite isolé, c'est-à-dire l'énergétique quantique bipartite (BQE). En exploitant la conservation globale de l'énergie, nous définissons les flux de type travail (de type chaleur) comme les changements d'énergie découlant de la dynamique hamiltonienne effective (la dynamique induite par les corrélations). Nous montrons que ces quantités sont accessibles via des mesures effectuées sur le champ, telles que des mesures dynes ou spectroscopiques. Notre approche diffère des analyses précédentes par self *b*-work de l'émetteur, qui donne une expression plus précise de la deuxième loi. Nous relierons quantitativement ce resserrement à la connaissance

supplémentaire sur le champ, par rapport au traitement habituel de l'atome comme un système ouvert.

Enfin, pour extraire des intuitions sur l'impact énergétique des corrélations, nous appliquons la BQE à l'interaction des champs lumineux au sein d'un BS. Elle révèle que les énergies échangées sont directement mesurables dans les distributions de l'espace des phases des champs. Nous constatons que le champ est déformé chaque fois qu'une énergie de type chaleur est échangée, tandis que les déplacements sont causés par le transfert d'énergie de type travail. Nous trouvons à nouveau un resserrement de la deuxième loi et montrons que dans ce cadre, la compression peut transférer la chaleur d'un champ plus froid vers un champ plus chaud, tout en maintenant une production d'entropie positive. Les concepts et les effets que nous introduisons approfondissent notre compréhension de la thermodynamique dans le régime quantique et son potentiel pour la gestion de l'énergie aux échelles quantiques.

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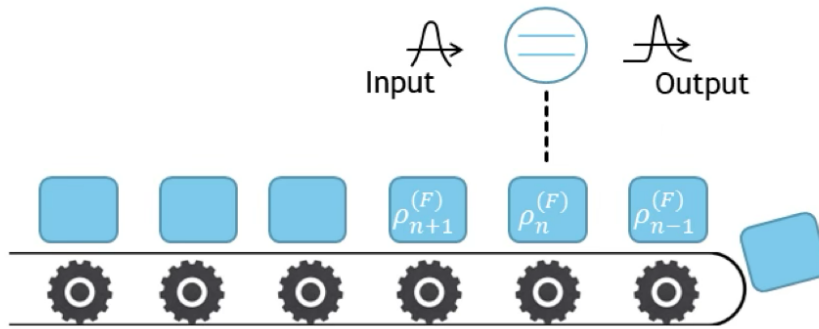
One of the most important and widely used approaches to analyze quantum interactions is the theory of open quantum systems. The field emerged owing to the fact that the Schrödinger equation or unitary dynamics could not describe experiments as attaining a single wavefunction for all of its components is impossible. The theory instead focuses on the study of the dynamics of small quantum systems interacting with a large immeasurable environment. This field advanced in 1976 with the discovery of the Gorini–Kossakowski–Sudarshan–Lindblad equation [Lin76] (GKSL or Lindblad equation). This equation is used to describe the dynamics of the quantum systems under consideration. It splits the description of the dynamics of the quantum system into two parts, one that describes coherent (quantum) processes and the other that describes losses due to interaction with the environment. This fundamental equation is, till date, used to model processes in multiple areas of research, from errors in quantum computing [NC10] to emission processes of atoms [BP+02; TGD92].

As the theory developed, it was used to model fundamental processes such as measurements and decoherence, the process by which quantum superpositions lose coherence due to environmental interactions. Works by W. H. Zurek [Zur81] popularized the idea that decoherence is a key mechanism explaining the classical behavior of macroscopic systems. This work laid the groundwork for understanding quantum-to-classical transitions.

Following a similar line of thought, works such as [Ali79] parallelly prescribed a thermodynamic viewpoint to the study of open quantum systems. They considered the environment as a thermal bath from which energy cannot be extracted and all energy going into it is lost. This setup was then used to describe quantum engines [GK92; BS95] and hence, to study irreversibility of quantum processes and the second law of thermodynamics for quantum systems. The primary assumption of all these works was to place a strong “thermodynamic cut” on the thermal bath or the environment, i.e., the environment cannot be accessed. However, this led to apparent violations of the second law at the quantum level, if the fluctuations and the initial quantum resources of the environment were not taken into account [Wan+02; Nie+16].

## 1.1 Motivations

Experimental advancements were made parallelly with the development of the theory of open quantum systems. Waveguide quantum electrodynamics (WQED) emerged to explore



**Fig. 1.1:** Depiction of interaction between environment units  $\rho_n^F$  and a two-level atom through a collisional model. The dashed-line depicts interaction. The input units (on the left of the atom) interact with the atom sequentially, as if on a conveyor belt. The output units are deformed due to the interaction with the atom.

the interaction between light and matter at the quantum level within structured environments, using waveguides. Waveguides form narrow channels to confine photons. This confinement is then used to enhance light-matter interactions, leading to novel phenomena and potential applications in quantum information processing, quantum optics, and solid-state physics. This study first emerged as a way to describe collective quantum effects such as superradiance of coupled atoms through the Dicke model [Dic54]. Experimental observation of these effects using real atoms proved to be difficult as coherences are easily lost to the environment of the atoms which cannot be probed. Advancements made in quantum technologies allowed for enhanced control of the coupling using artificial atom such as superconducting qubits and quantum dots, in waveguides [Ste+22; Mai+23] and even enabled the manipulation of single photons and quantum states of light with high precision, making it a promising platform for the development of quantum networks and photonic quantum computers [She+23].

With these advancements made in the experimental WQED, the environment can no longer be considered as a solely lossy channel. Now structured environments can be engineered and the previously considered lossy processes can be probed with high detection efficiencies such that no information is lost. Even though the waveguide acts as an environment, it forms a quantum system, and needs to be described. Hence, these experimental advances bring the need to update the theoretical tools to close open quantum systems.

A route towards taking into account the state of the environment is the collisional model [Cic+22]. It models a small part of the environment as a set of quantum systems which interact with the system under consideration (see Fig. 1.1). These models are able to re-derive the dynamical equations of the open approach when the degrees of freedom of

the environment are again removed [Sca+02; GP12a]. It can furthermore be used to track the state of the environment bringing new information and insights which could not be accessed earlier [Cic+22]. In WQED, the collisional model arises in the 1D atom model description, where a point-like emitter interacts with “collisional units” of the field in the waveguide. From a thermodynamic viewpoint, however, the concepts used have remained the same so far: the waveguide is considered a bath, and the energy it receives is heat (energy “lost”). These concepts require an update as the joint system-environment state has an autonomous description in the collisional model, i.e., it closes the open systems and energy is globally conserved. The thermodynamic cut needs to be redefined as now some of the energy of the environment can be accessed. This would help in better resource accounting, gives access to the coherent processes in the environment and another way to resolve the apparent violations of the second law [Rod+19; LP21].

In this thesis, we close open systems and go a step further. Rather than only analyzing the sub-systems, we take into account the effects of the correlations built between the system and the environment. These correlations are rarely considered while modelling the dynamics of optical interactions. Works such as [Ali+20] have shown how dynamical equations can be derived while taking correlation into account. Tracking the correlations build up in optical interactions can be useful to understand fundamental processes occurring during the interaction and can help to “close” open systems. On the thermodynamics side, this changes the thermodynamic cut to be placed on the correlations and leads to a symmetric consideration of the sub-system of the environment and the small quantum system. Doing so, sheds light on the energy exchanges occurring during the dynamics, the irreversibility of quantum processes, the second law of thermodynamics and the quantum-classical boundaries.

## 1.2 Thesis Structure

Motivated by the need to track correlations and studying its consequences, throughout the thesis we make comparisons with results existing in literature on the dynamics and thermodynamics of such interactions. In order to make a fair comparison with our framework, we make a few restrictions on the approach to be compared with:

- The dynamical equation:
  - Must be widely used to model optical interactions;
  - Notion of work and heat defined in literature;

Thermodynamics of the optical Bloch equations (OBE) [Elo+20] and energy exchanges between Gaussian fields are used for comparison.

- The Platform:
  - Must be isolated, bipartite and solvable;
  - The reduced dynamics of subsystems must be the same as the open approach;One-dimensional atom model and beam splitter interactions are studied.

We have split the study into two parts. In the first two chapters we analyze the dynamics of emitters coupled to a bosonic field. In the next two chapters, we study the thermodynamics of optical interactions, where we introduce a new framework which we have dubbed bipartite quantum energetics. We place the detailed calculations used and interesting connections towards the end of each part to focus on the physics and the main results in the chapters.

### 1.2.1 Analysis of Dynamics

In chapter 2, we begin by analyzing the interaction between a quantum emitter and a propagating field inside a waveguide. Treating this as a completely isolated bipartite system, we introduce the 1D atom and we derive a collisional model to solve the dynamics. Through this we provide the regime of validity for the analysis conducted in this thesis.

In chapter 3, we then address the question of closing open systems. We do so by analyzing the regime of the OBE and computing the dynamics of the correlations as they are created per collision. We pinpoint the coherent and the incoherent processes and their magnitude of effect on the dynamics of the sub-systems and the correlations. Finally we find measurable ways of probing these processes.

### 1.2.2 Thermodynamic Analysis

In chapter 4, we conduct the thermodynamic analysis of the OBE. We first introduce the framework of bipartite quantum energetics. This allows us to analyze the effect that the correlations have on the energy that is exchanged during the interaction. We redefine work and heat in our framework and compare it to those in literature. To provide intuitions, we moreover, analyze the case of spontaneous emission processes. Finally, we present the second law and how it changes when the thermodynamic cut is placed only on the correlations. We show this can be understood as the energy lost in ignoring the correlations developed.

Finally in chapter 5, we go beyond emitter-field interactions and analyze the energetics of coupled baths. We model this via beam splitter interactions occurring between Gaussian fields. Through this, we build simple intuitions and provide an operational viewpoint on the definitions of work and heat used in our framework. We moreover, find useful bounds on the extraction of energy into quantum batteries. To end the analysis, we show how the insertion or extraction of resources such as squeezing and coherences can modify the second law and even make it tighter, strengthening the intuition that loss of correlations leads to irreversibility.

We end the thesis by summarizing the main results and providing future directions in the form of open questions that can possibly be resolved using the framework we develop in this thesis.



L'une des approches les plus importantes et les plus largement utilisées pour analyser les interactions quantiques est la théorie des systèmes quantiques ouverts. Ce domaine est né du fait que l'équation de Schrödinger ou la dynamique unitaire ne pouvait pas décrire les expériences car il était impossible d'obtenir une fonction d'onde unique pour toutes ses composantes. La théorie se concentre plutôt sur l'étude de la dynamique de petits systèmes quantiques en interaction avec un grand environnement incommensurable. Ce domaine a progressé en 1976 avec la découverte de l'équation de Gorini-Kossakowski-Sudarshan-Lindblad [Lin76] (GKSL ou équation de Lindblad). Cette équation est utilisée pour décrire la dynamique des systèmes quantiques considérés. Elle divise la description de la dynamique du système quantique en deux parties, l'une décrivant les processus cohérents (quantiques) et l'autre décrivant les pertes dues à l'interaction avec l'environnement. Cette équation fondamentale est, jusqu'à ce jour, utilisée pour modéliser des processus dans de nombreux domaines de recherche, depuis les erreurs dans l'informatique quantique [NC10] jusqu'aux processus d'émission d'atomes [BP+02; TGD92].

Au fur et à mesure que la théorie se développait, elle a été utilisée pour modéliser des processus fondamentaux tels que les mesures et la décohérence, le processus par lequel les superpositions quantiques perdent leur cohérence en raison des interactions environnementales. Les travaux de W. H. Zurek [Zur81] ont popularisé l'idée que la décohérence est un mécanisme clé expliquant le comportement classique des systèmes macroscopiques. Ces travaux ont jeté les bases de la compréhension des transitions du quantique au classique.

Suivant une ligne de pensée similaire, des travaux tels que [Ali79] ont parallèlement prescrit un point de vue thermodynamique à l'étude des systèmes quantiques ouverts. Ils ont considéré l'environnement comme un bain thermal d'où l'énergie ne peut pas être extraite et toute l'énergie qui y pénètre est perdue. Cette configuration a ensuite été utilisée pour décrire les moteurs quantiques [GK92; BS95] et donc, pour étudier l'irréversibilité des processus quantiques et la deuxième loi de la thermodynamique pour les systèmes quantiques. L'hypothèse principale de tous ces travaux était de placer une forte « coupure thermodynamique » sur le bain thermal ou l'environnement, c'est-à-dire que l'environnement n'est pas accessible. Cependant, cela a conduit à des violations apparentes de la deuxième loi au niveau quantique, si les fluctuations et les ressources quantiques initiales de l'environnement n'étaient pas prises en compte [Wan+02; Nie+16].

## 2.1 Motivations

Des avancées expérimentales ont été réalisées parallèlement au développement de la théorie des systèmes quantiques ouverts. L'électrodynamique quantique des guides d'ondes (WQED) a émergé pour explorer l'interaction entre la lumière et la matière au niveau quantique dans des environnements structurés, en utilisant des guides d'ondes. Les guides d'ondes forment des canaux étroits pour confiner les photons. Ce confinement est ensuite utilisé pour améliorer les interactions lumière-matière, conduisant à de nouveaux phénomènes et à des applications potentielles dans le traitement de l'information quantique, l'optique quantique et la physique du solide. Cette étude est apparue pour la première fois comme un moyen de décrire les effets quantiques collectifs tels que la superradiance des atomes couplés à travers le modèle de Dicke [Dic54]. L'observation expérimentale de ces effets à l'aide d'atomes réels s'est avérée difficile car les cohérences sont facilement perdues dans l'environnement des atomes qui ne peuvent pas être sondés. Les progrès réalisés dans les technologies quantiques ont permis un meilleur contrôle du couplage à l'aide d'atomes artificiels tels que des qubits supraconducteurs et des points quantiques, dans des guides d'ondes [Ste+22; Mai+23] et ont même permis la manipulation de photons uniques et d'états quantiques de lumière avec une grande précision, ce qui en fait une plate-forme prometteuse pour le développement de réseaux quantiques et d'ordinateurs quantiques photoniques [She+23].

Grâce à ces avancées réalisées dans le domaine expérimental WQED, l'environnement ne peut plus être considéré comme un canal uniquement avec pertes. Il est désormais possible de concevoir des environnements structurés et de sonder les processus avec pertes précédemment considérés avec une efficacité de détection élevée, de sorte qu'aucune information ne soit perdue. Même si le guide d'ondes agit comme un environnement, il forme un système quantique et doit être décrit. Par conséquent, ces avancées expérimentales entraînent la nécessité de mettre à jour les outils théoriques pour fermer les systèmes quantiques ouverts.

Une voie vers la prise en compte de l'état de l'environnement est le modèle collisionnel [Cic+22]. Il modélise une petite partie de l'environnement comme un ensemble de systèmes quantiques qui interagissent avec le système considéré (voir Fig. 1.1). Ces modèles sont capables de redériver les équations dynamiques de l'approche ouverte lorsque les degrés de liberté de l'environnement sont à nouveau supprimés [Sca+02; GP12a]. Il peut en outre être utilisé pour suivre l'état de l'environnement en apportant de nouvelles informations et perspectives auxquelles on n'avait pas accès auparavant [Cic+22]. Dans WQED, le modèle collisionnel apparaît dans la description du modèle atomique 1D, où

un émetteur ponctuel interagit avec des « unités collisionnelles » du champ dans le guide d'ondes. D'un point de vue thermodynamique, les concepts utilisés sont toutefois restés les mêmes jusqu'à présent : le guide d'ondes est considéré comme un bain, et l'énergie qu'il reçoit est de la chaleur (énergie « perdue »). Ces concepts nécessitent une mise à jour car l'état conjoint système-environnement a une description autonome dans le modèle collisionnel, c'est-à-dire qu'il ferme les systèmes ouverts et que l'énergie est globalement conservée. La coupure thermodynamique doit être redéfinie car une partie de l'énergie de l'environnement est désormais accessible. Cela aiderait à une meilleure comptabilité des ressources, donnerait accès aux processus cohérents dans l'environnement et constituerait un autre moyen de résoudre les violations apparentes de la deuxième loi [Rod+19; LP21].

Dans cette thèse, nous fermons les systèmes ouverts et allons plus loin. Plutôt que d'analyser uniquement les sous-systèmes, nous prenons en compte les effets des corrélations construites entre le système et l'environnement. Ces corrélations sont rarement prises en compte lors de la modélisation de la dynamique des interactions optiques. Des travaux tels que [Ali+20] ont montré comment des équations dynamiques peuvent être dérivées en prenant en compte la corrélation. Suivre les corrélations construites dans les interactions optiques peut être utile pour comprendre les processus fondamentaux se produisant pendant l'interaction et peut aider à « fermer » des systèmes ouverts. Du côté de la thermodynamique, cela change la coupure thermodynamique à placer sur les corrélations et conduit à une considération symétrique du sous-système de l'environnement et du petit système quantique. Ce faisant, on met en lumière les échanges d'énergie se produisant pendant la dynamique, l'irréversibilité des processus quantiques, la deuxième loi de la thermodynamique et les frontières quantiques-classiques.

## 2.2 Structure de la thèse

Motivés par la nécessité de suivre les corrélations et d'étudier leurs conséquences, nous faisons tout au long de la thèse des comparaisons avec des résultats existant dans la littérature sur la dynamique et la thermodynamique de telles interactions. Afin de faire une comparaison équitable avec notre cadre, nous faisons quelques restrictions sur l'approche à comparer :

- L'équation dynamique:
  - Doit être largement utilisé pour modéliser les interactions optiques;
  - Notion de travail et de chaleur définie dans la littérature;

La thermodynamique des équations optiques de Bloch (OBE) [Elo+20] et les échanges d'énergie entre champs gaussiens sont utilisés à des fins de comparaison.

- La Plateforme:
  - Doit être isolé, bipartite et soluble;
  - La dynamique réduite des sous-systèmes doit être la même que l'approche ouverte;

Le modèle atomique unidimensionnel et les interactions entre séparateur de faisceau sont étudiés.

Nous avons divisé l'étude en deux parties. Dans les deux premiers chapitres, nous analysons la dynamique des émetteurs couplés à un champ bosonique. Dans les deux chapitres suivants, nous étudions la thermodynamique des interactions optiques, où nous introduisons un nouveau cadre que nous avons baptisé énergétique quantique bipartite. Nous plaçons les calculs détaillés utilisés et les connexions intéressantes vers la fin de chaque partie pour nous concentrer sur la physique et les principaux résultats des chapitres.

### 2.2.1 Analyse de la dynamique

Dans le chapitre 2, nous commençons par analyser l'interaction entre un émetteur quantique et un champ de propagation à l'intérieur d'un guide d'ondes. En traitant cela comme un système bipartite complètement isolé, nous introduisons l'atome 1D et nous dérivons un modèle collisionnel pour résoudre la dynamique. Grâce à cela, nous fournissons le régime de validité pour l'analyse menée dans cette thèse.

Dans le chapitre 3, nous abordons ensuite la question de la fermeture des systèmes ouverts. Nous le faisons en analysant le régime de l'OBE et en calculant la dynamique des corrélations telles qu'elles sont créées par collision. Nous identifions les processus cohérents et incohérents et leur ampleur d'effet sur la dynamique des sous-systèmes et des corrélations. Enfin, nous trouvons des moyens mesurables de sonder ces processus.

### 2.2.2 Analyse thermodynamique

Dans le chapitre 4, nous effectuons l'analyse thermodynamique de l'OBE. Nous introduisons d'abord le cadre de l'énergétique quantique bipartite. Cela nous permet d'analyser l'effet des corrélations sur l'énergie échangée lors de l'interaction. Nous redéfinissons le travail et la chaleur dans notre cadre et le comparons à ceux de la littérature. Pour fournir des intuitions, nous analysons en outre le cas des processus d'émission spontanée. Enfin, nous présentons la deuxième loi et comment elle change lorsque la coupure thermodynamique

est placée uniquement sur les corrélations. Nous montrons que cela peut être compris comme l'énergie perdue en ignorant les corrélations développées.

Enfin, dans le chapitre 5, nous allons au-delà des interactions émetteur-champ et analysons l'énergétique des bains couplés. Nous modélisons cela via des interactions de type séparateur de faisceaux se produisant entre des champs gaussiens. Grâce à cela, nous construisons des intuitions simples et fournissons un point de vue opérationnel sur les définitions du travail et de la chaleur utilisées dans notre cadre. Nous trouvons en outre des bornes utiles sur l'extraction d'énergie dans les batteries quantiques. Pour terminer l'analyse, nous montrons comment l'insertion ou l'extraction de ressources telles que la compression et les cohérences peuvent modifier la deuxième loi et même la rendre plus serrée, renforçant l'intuition selon laquelle la perte de corrélations conduit à l'irréversibilité.

Nous terminons la thèse en résumant les principaux résultats et en fournissant des orientations futures sous forme de questions ouvertes qui peuvent éventuellement être résolues en utilisant le cadre que nous développons dans cette thèse.



# Part I

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Dynamics

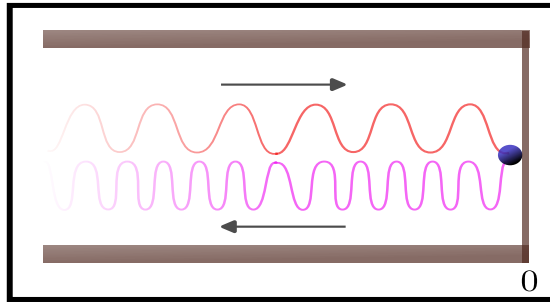


# Collisional Model in One-Dimensional Atoms

Collisional models have been demonstrated to model open system dynamics for various interactions [Cic+22]. They were originally considered to be toy models for interactions between point-like emitters and large environments [Sca+02; GP12b]. Here the “environment units” collide and interact once with the emitter and are lost, never to interact again. This has been useful in the studies including that of irreversibility, entanglement and energetic costs of processes. Works such as [Rod+19; Cic17] have shown that emitter-field interaction can be fundamentally modelled as collisional models in an autonomous manner.

We begin this chapter by introducing the one-dimensional atom which allows us to analyse emitter-field interactions in an isolated bipartite manner. We then depict how autonomous collisional model (ACM) arises in this setting, with the assuming the flat-coupling approximation. We relax this approximations in section 3.3, where we introduce a coarse grained procedure to derive ACM with new coupling dependent bosonic operators. Finally, we find corrections and provide the orders of errors in using ACM for this setting.

## 3.1 One-Dimensional Atom



**Fig. 3.1:** A schematic of the 1D atom. The incoming field (in red) interacts with the stationary emitter (in blue) at position 0 and is reflected (in violet) after being transformed by the emitter. The emitter and the field form an isolated bipartite system.

Let us consider a general emitter with multiple energy levels, each of which is characterized by a transition frequency  $\omega$  and ladder operator  $L(\omega)$ . The bare Hamiltonian of the emitter  $H_S$  obeys the relation  $[H_S, L(\omega)] = -\hbar\omega L(\omega)$ . The emitter is coupled to a one-dimensional (1D) reservoir of electromagnetic modes of wavevector  $\vec{k}$ , frequency  $\omega_k$  and annihilation operators  $a_k$ , of bare Hamiltonian  $H_f = \sum_k \hbar\omega_k a_k^\dagger a_k$  (see Fig.3.1). The discretization procedure of the continuous frequencies of the field is given in section 5.1.

In this setup, the emitter is considered to be stationary and the size of the emitter is much smaller than the wavelength of light such that its position degrees of freedom has no effect on the dynamics. Moreover, the absorption and emission processes of the emitter occur only in one direction, i.e., the wavevector becomes a scalar wavenumber  $k$  and is always positive  $k \geq 0$ . The light modes follow a linear dispersion relation,  $\omega_k = vk$ ,  $v$  being the group velocity. This situation being typical of waveguide QED, we dub this reservoir a “waveguide” – however, we should keep in mind that waveguide QED covers a wide range of physical situations, e.g. emitters coupled to adiabatically eliminated directional cavities [Koj+04]. All along the paper, we dub the joint system formed by the emitter and the 1D field, a 1D atom<sup>1</sup>. Importantly, note that the terminology “1D atom” refers to the joint system of the light field and the quantum emitter which together are considered to be isolated. An interesting aspect of this setup is that the emitted light, which in general is “lost”, can now be collected and measured [Mai+23]. The framework of the 1D atom is convenient to model light-matter functionalities (which include quantum interfaces, gates, networks and repeaters) in a variety of physical platforms, from circuit quantum electrodynamics [Cot+17], semiconducting quantum dots [Mai+23] to atomic physics [Dis+21], to name a few.

Throughout this thesis, we will consider exchange interactions in bipartite systems assuming the rotating-wave approximation. As we will be only interested in the dynamics of the interaction, we place ourselves in the interaction picture with respect to  $H_S + H_f$  and the interaction is given by

$$V_{Sf}(t) = i\hbar \sum_{\omega} \sum_{k \geq 0} g_k \left( e^{-i(\omega_k - \omega)t} L^\dagger(\omega) a_k - e^{i(\omega_k - \omega)t} L(\omega) a_k^\dagger \right) \quad (3.1)$$

where,  $g_k$  is the coupling strength to the field mode with momentum  $\hbar k$ . Eventually, the 1D atom dynamics is governed by the following equation in the interaction picture:

$$\dot{\rho}_{Sf} = -\frac{i}{\hbar} [V_{Sf}(t), \rho_{Sf}(t)], \quad (3.2)$$

where  $\rho_{Sf}(t)$  is the quantum state of the atom-field system. Note that, Eq.(3.2) is valid at all timescales. We will assume the coupling strength to be weak. With this, we can already model driven-dissipative settings. In chapter 4, we will see that when the emitter

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<sup>1</sup>One can call this a 0.5 dimensional system as the setup is such that the input and output “ports” are the same and emitter absorbs and emits on the same side. The light interacts with the emitter and goes back the same way it came. For example, this is the situation for interactions with quantum dots in micropillar cavities. In a fully one dimension setup, the emitter will be able to absorb and emit on both sides.

is a two-level atom and the field is a displaced-thermal state, this setup gives rise to the optical Bloch equations (OBE) [TGD92].

## 3.2 Collisional Model from Flat Coupling

Let us now briefly show how a collisional model emerges from the 1D atom setting – providing a microscopic model for the interaction. Here, we will follow [Cic+22]. For simplicity, let us consider only one of the interacting frequencies  $\omega_0$  of the emitter such that  $L \equiv L(\omega_0)$ . We first restrict the coupling strength of the interaction in Eq. (3.1) to be flat around  $\omega_0$ , i.e.,  $g_k = g_{k_0} = \sqrt{\gamma/\varrho}$ , for all  $k$ . Here,  $\gamma$  is the line-width of the emitter for transition with frequency  $\omega_0$  and  $\varrho$  is the density of states of the field (see section 5.1). This interaction Hamiltonian gives rise to a unitary evolution  $\mathcal{U}(T, t_0)$  between times  $T$  and  $t_0$  which evolves the bipartite state  $\rho_{Sf}$ . We now discretize the unitary into a product of  $N$  discrete-time dynamical maps of the form  $\mathcal{U}_n \equiv \mathcal{U}(t_n, t_{n+1}) = \mathcal{T} \exp \left\{ -i \int_{t_n}^{t_{n+1}} du V_{Sf}(u)/\hbar \right\}$  by discretizing time into  $N$  steps of  $\Delta t$  such that  $t_n = n(T - t_0)/N\Delta t$  with  $n \in \mathbb{Z}^+$ . Here,  $\mathcal{T}$  is the time-ordering operator. Defining discrete temporal bin field operators  $b_n = \sqrt{\Delta t/\varrho} \sum_k a_k e^{-i\omega_k t_n}$  which follow the bosonic commutation relations  $[b_n, b_m^\dagger] = \delta_{n,m}$  and  $[b_n, b_m] = 0$ , we find that the interaction takes the form  $V_n \equiv \int_{t_n}^{t_{n+1}} du V_{Sf}(u)/\hbar = i\sqrt{\gamma\Delta t}(b_n L^\dagger - b_n^\dagger L)$ . Using this and the Magnus expansion [Mag54; Bla+10], the dynamical maps can be expanded to second order in  $V_n$ :

$$\mathcal{U}_n = \mathbb{I} - iV_n - \frac{1}{2}V_n^2. \quad (3.3)$$

We can stop at second order as we have assumed the coupling to be weak and  $\Delta t \rightarrow 0$  such,  $\gamma\Delta t$  is a small parameter. Moreover, in the same limit, it is possible to show for certain field states that the contribution of the second term in the Magnus expansion, i.e.,  $\int_{t_n}^{t_{n+1}} du \int_{t_n}^u dv [V(u), V(v)]$  is negligible – we show this explicitly in the next section. Eq. (3.3) shows that the interaction only occurs with input units of the field at each time  $t_n$ .

Throughout this thesis, we will consider the initial state of the field to be a product state of identical temporal units, i.e., the density matrix of the initial field is given by  $\rho_f(t_0) \equiv \text{Tr}_S \{ \rho_{Sf}(t_0) \} = \bigotimes_{n=0}^N \rho_n$ . The label  $n$  on the state denotes the time  $t_n$  at which it interacts with the emitter. We will show in chapter 4 that this can be done when the field is quasi-monochromatic, where each  $\rho_n$  depends on the discretization  $\Delta t$ . At each time  $t_n$ , a unit  $\rho_n$  interacts with the emitter and leaves, never to interact again. This is the standard collisional model arising from the microscopic modelling of the emitter-field interaction. As

stated, such maps are considered in the limit  $\Delta t$  approaches zero giving rise to continuous time dynamics where, one retrieves the master equations of the emitter  $\rho_S(t) \equiv \text{Tr}_f\{\rho_{Sf}(t)\}$ , that of the field  $\rho_f(t)$  and the input-output relations of the field [GC85].

### 3.3 Coarse-Grained Autonomous Collisional Model

In the previous section, we discussed how the collisional model emerges from the microscopic modelling of the interaction between the emitter and the field. We imposed the restrictions on the form of the interaction and took  $\Delta t \rightarrow 0$  stating that these give rise to master equations for the emitter. However, the open dynamics of the emitter is generally derived at a coarse grained level, where the evolution over a small time scale  $\tau_c$  known as the correlation time is ignored. Ignorance of this dynamics gives rise to Markovian master equations such as the OBEs [TGD92; BP+02]. This procedure, provides a lower bound to the discretization interval, i.e.,  $\tau_c \ll \Delta t$  and hence,  $\Delta t \rightarrow 0$  can't be taken in principle. Here, we bridge the gap between open dynamics and the collision model in a more realistic setting. In the following, we will report our original results, starting from Eq. (3.1), where we do not impose the flat coupling assumption or  $\Delta t \rightarrow 0$ . We will coarse grain the dynamics in a similar manner, but provide correction terms arising from the Magnus expansion when  $\Delta t$  has a lower bound  $\tau_c$  and the coupling is not flat.

We begin with Eq. (3.1), again considering only one of the interacting frequencies of the emitter  $\omega_0$ . Without making any approximations on the form of the interaction Hamiltonian, we find again the discrete dynamical maps  $\mathcal{U}_n$  using the Suzuki-Trotter decomposition when discretizing time in steps of  $\Delta t$ . As before, the time ordering in  $\mathcal{U}_n$  is removed using the Magnus expansion. The procedure assumes the existence of a unique solution to each unitary map to be  $\mathcal{U}_n = \exp\{\Omega(t_{n+1}, t_n)\}$  where,  $\Omega(t_{n+1}, t_n)$  is an anti-Hermitian operator and has a power series expansion in terms of a parameter  $\lambda$ ,  $\Omega(t_{n+1}, t_n) = \sum_k \lambda^k \Omega_k(t_{n+1}, t_n)$ . Each of the operators  $\Omega_k(t_{n+1}, t_n)$  gives the  $k^{\text{th}}$  order of the Magnus expansion. Using the evolution equations, one can find the solutions to the first two orders in terms of  $V_{Sf}(t)$  [Bla+10; Mag54]:

$$\begin{aligned}\Omega_1(t_{n+1}, t_n) &= -\frac{i}{\hbar} \int_{t_n}^{t_{n+1}} du V_{Sf}(u) \\ \Omega_2(t_{n+1}, t_n) &= -\frac{1}{2\hbar^2} \int_{t_n}^{t_{n+1}} du \int_{t_n}^u dv [V_{Sf}(u), V_{Sf}(v)].\end{aligned}\tag{3.4}$$

Setting the arbitrary value of  $\lambda = 1$ , we find  $\mathcal{U}_n = \exp \{ \Omega_1(t_{n+1}, t_n) + \Omega_2(t_{n+1}, t_n) + \dots \}$ . To simplify the form of the interaction in Eq. (3.4), we define new coupling dependent temporal operators

$$B(x, t) = \sum_k g_k e^{-i\Delta\omega_k(t-x/v)} a_k,$$

with  $\Delta\omega_k \equiv \omega_k - \omega_0$ . Here, unlike the collisional models studied in present literature, we keep the dependency on the coupling strength. In the limit, where the coupling is flat, these operators are proportional to the usual definitions of the temporal operators such as in [Gro+18]. Using these, the interaction takes the form

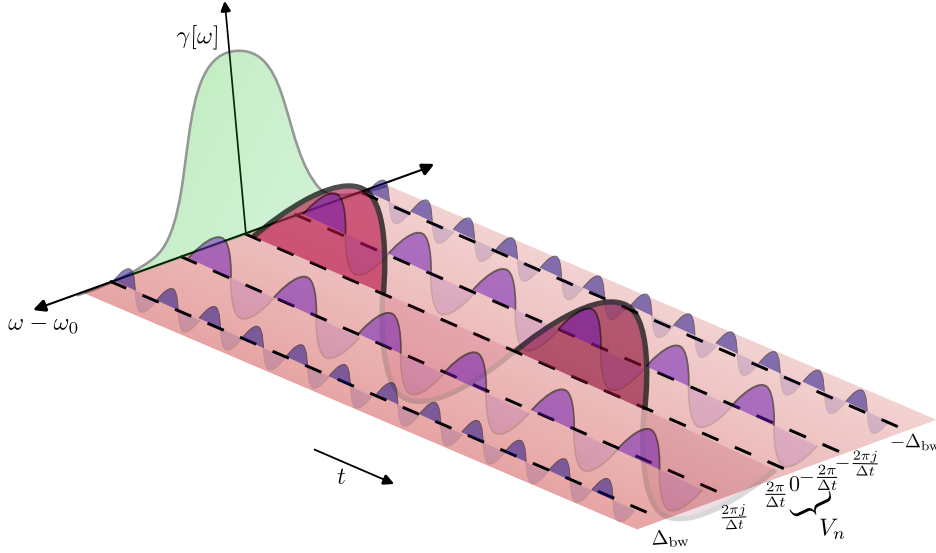
$$V_{Sf}(t) = i\hbar (L^\dagger B(0, t) - B^\dagger(0, t) L). \quad (3.5)$$

Let us now show that by using the bound on the discretization interval  $\tau_c \ll \Delta t$ , only the first two orders of the Magnus expansion (Eq. (3.4)) contribute.

To evaluate the integrals in Eq. (3.4), we coarse grain the operators  $B(0, t)$ . We define,

$$b_{n,j} = \frac{1}{\sqrt{\gamma_j \Delta t}} \int_{t_n}^{t_n + \Delta t} dt e^{i\frac{2\pi}{\Delta t} j t} B(0, t). \quad (3.6)$$

We have introduced the parameter  $\gamma_j = 2\pi \sum_k |g_k|^2 \delta_D(\Delta\omega_k - \frac{2\pi j}{\Delta t})$  with  $\delta_D$  being the Dirac distribution and noting that  $\gamma_0 = \gamma$ . As shown in section 5.2, within the coarse graining interval  $\Delta t$ , these act as bosonic operators satisfying the commutation relations  $[b_{n,j}, b_{m,k}^\dagger] = \delta_{n,m} \delta_{j,k}$  and  $[b_{n,j}, b_{m,k}] = 0$ .



**Fig. 3.2:** Schematic of different field modes oscillating in time with its height depicting the amplitude. The coupling  $\gamma[\omega]$  over the different detunings  $\omega - \omega_0$  is depicted in light green. The Gaussian distribution is taken as an example. A low frequency mode, i.e., a mode with  $j = 0$  is depicted in red while a mode corresponding to  $b_{n,j \neq 0}$  which is detuned from the central frequency by  $2\pi j/\Delta t$  is depicted in blue.  $\Delta_{\text{bw}} \propto \tau_c^{-1}$  is the band-width, where the coupling becomes negligible. Notice that fast oscillating  $b_{n,j}$  modes can couple with non-negligible strengths as it is within the band-width. Nevertheless they appear with small amplitudes. We indicate by the braces the interaction range of  $V_n$  having detunings in the range  $(-\frac{2\pi}{\Delta t}, \frac{2\pi}{\Delta t})$ , i.e., with the  $b_{n,j=0} = 0$  modes.

### Complete Temporal Bin Decomposition

The temporal bin operators  $b_{n,j}$ , similarly introduced in [Gro+18; Cil+20] for flat coupling, form a complete set of normal operators that describe the coarse grained field modes. Analogous to the discretization of time in intervals of  $\Delta t$ , the frequency modes are also discretized into intervals of  $2\pi\Delta t^{-1}$ .  $b_{n,j=0}$  describe a set of quasi-resonant modes while the other describe modes detuned by  $2\pi j\Delta t^{-1}$  with  $j \neq 0$ . This is depicted in Fig. 3.2, where, we see that the modes detuned by  $2\pi\Delta t^{-1}$  oscillate much faster than the ones with  $j = 0$ . Usually interactions with modes  $j \neq 0$  are not considered in collisional models as the average frequency of these modes  $2\pi j\Delta t^{-1}$  become divergent as  $\Delta t \rightarrow 0$ . These interaction can then be treated as interactions with the vacuum state of these modes. However, here we consider a non-flat coupling giving a lower bound on  $\Delta t$  that is determined by the correlation time:  $\tau_c \ll \Delta t$ . As a consequence, we see in Fig. 3.2 that some the  $j \neq 0$  modes exist within the bandwidth  $\Delta_{\text{bw}} \propto \tau_c^{-1}$ . We show in section 5.3 that modes with  $j \neq 0$  do contribute, but only to negligible frequency shifts (light shifts) as their frequency is much greater than  $\omega_0$  and do not contribute significantly to the coarse grained dynamics.

We now evaluate  $\Omega_1(t_{n+1}, t_n)$  by substituting Eq. (3.5) and Eq. (3.6) in Eq. (3.4),

$$\begin{aligned}\Omega_1(t_{n+1}, t_n) &= \int_{t_n}^{t_n+\Delta t} dt L^\dagger B(0, t) - \text{h.c.} \\ &= \sqrt{\gamma\Delta t} (L^\dagger b_n - b_n^\dagger L),\end{aligned}\quad (3.7)$$

where, h.c. is the Hermitian conjugate and  $b_n \equiv b_{n,0}$ . From the above equation, one can already notice the emergence of the collisional model interaction in Eq. (3.3).

In section 5.3 we evaluate  $\Omega_2(t_{n+1}, t_n)$  by similarly substituting Eq. (3.6) into it. We show that  $\Omega_2(t_{n+1}, t_n)$  is written in terms of  $b_{n,j}$  as

$$\Omega_2(t_{n+1}, t_n) = \frac{i\Delta t}{2\pi} \sum_{j' \neq 0} \left( \frac{\gamma_{j'}}{j'} (b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L] + L^\dagger L) - \frac{\sqrt{\gamma\gamma_{j'}}}{j'} (b_{n,0}^\dagger b_{n,j'} + b_{n,j'}^\dagger b_{n,0}) [L^\dagger, L] \right) \quad (3.8)$$

We see that  $\Omega_2(t_{n+1}, t_n)$  contains the term:  $\Omega_{2,1} = -\hbar\delta_{\text{LS}}L^\dagger L + \frac{i\Delta t}{2\pi} \sum_{j' \neq 0} \frac{\gamma_{j'}}{j'} b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L]$ , with  $\delta_{\text{LS}} = \frac{1}{2\pi} \sum_{j' > 0} \frac{\gamma_{j'}}{j'}$ .  $\delta_{\text{LS}}$  induces a correction to the emitter frequency (known as Lamb shift in atomic physics) while there is also a dispersive coupling with the higher frequency modes ( $j \neq 0$ ) which causes a shift of the emitter's frequency (light shift) that we can evaluate by taking the average over a thermal state of the field in each time bin. We find that, it's magnitude is

$$\begin{aligned}\Delta t \delta_{ls} &= \left| \frac{i\Delta t}{2\pi} \sum_{j \neq 0} \frac{\gamma_j \langle b_{n,j}^\dagger b_{n,j} \rangle}{j} \right| \lesssim \Delta t \gamma \left| \int_{\frac{\pi}{\Delta t}}^{\Delta_{bw}/2} d\omega \frac{n[\omega_0 + \omega] - n[\omega_0 - \omega]}{\omega} \right| \\ &\simeq \Delta t \gamma \left| \int_{\frac{\pi}{\Delta t}}^{\Delta_{bw}/2} d\omega \frac{2\bar{n}_{\text{th}}(\bar{n}_{\text{th}} + 1)\omega}{\omega} \right| \\ &\simeq \Delta t \bar{n}_{\text{th}}(\bar{n}_{\text{th}} + 1) \gamma \beta \Delta_{bw}.\end{aligned}\quad (3.9)$$

Here,  $\Delta_{bw} \propto \frac{1}{\tau_c}$  is the band-width of the emitter-field coupling and we have approximated the sum over  $j$  with an integral over frequencies  $\omega \equiv \omega_j = \frac{2\pi j}{\Delta t}$ . In the first line, we have introduced  $n[\omega] = (e^{-\beta\hbar\omega} - 1)^{-1}$  which is the Bose-Einstein distribution of the field, and to go to the second line we have assumed that the bandwidth  $\Delta_{bw}$  is much smaller than  $\omega_0$  so that we can expand the distribution around  $\omega_0$ . One can notice that  $\delta_{ls} \ll \omega_0$  from the fact that  $\tau_c \geq \hbar\beta$ ,  $\Delta_{bw}^{-1}$  [BP+02] and  $\gamma\tau_c \ll 1$  (and similarly for  $\delta_{LS}$ ).

One can recover the usual expression to the Lamb shift (appearing e.g. in the context of the derivation of the OBE) by taking into account the limit  $\Delta t \gg \tau_c \geq \Delta_{\text{bw}}^{-1}, \hbar\beta$ :

$$\begin{aligned}
\delta_{\text{LS}} &= \frac{1}{2\pi} \sum_{j' > 0} \frac{1}{j'} \sum_k |g_k|^2 \delta_D(\Delta\omega_k - \frac{2\pi j'}{\Delta t}) \\
&\simeq \frac{1}{2\pi} \text{P} \int d\omega \sum_k |g_k|^2 \frac{\delta_D(\Delta\omega_k - \omega)}{\omega} \\
&= \frac{1}{2\pi} \sum_{k \neq 0} \frac{|g_k|^2}{\Delta\omega_k},
\end{aligned} \tag{3.10}$$

where P stands for Cauchy's principal value and again we have approximated the sum over  $j'$  with an integral over frequencies  $\omega \equiv \omega_j$ . The other terms of  $\Omega_2$ , induces an effective coupling, mediated by the emitter, between the modes  $b_{n,j}$  for  $j = 0$  (the closer to the emitter's frequency) to modes further detuned from the emitter  $j \neq 0$ . This coupling term is already of first order in  $\gamma\Delta t$ . Assuming a factorized initial thermal state of the modes  $b_{n,j}$  (and no initial correlation with the emitter) ensures that  $\langle b_{n,j}(0) \rangle = 0$ , such that this coupling term yield no contribution at first order to the reduced dynamics of the emitter or of the  $b_{n,j}$  modes.

Reabsorbing the Lamb shift in the definition of  $\omega_0$ , we can approximate  $\Omega(t_n, t_{n+1}) = \sqrt{\gamma\Delta t}(b_{n,0}L^\dagger + b_{n,0}^\dagger L) + o(\gamma\Delta t)$ . To first order in  $\gamma\Delta t$ , the emitter therefore, effectively interacts only with the modes  $b_{n,j=0} \equiv b_n$ . To be consistent with our first order truncation of  $\Omega(t_n, t_{n+1})$ , we finally expand the exponential in  $\mathcal{U}_n$  to obtain

$$\mathcal{U}_n = \mathbb{I} + \sqrt{\gamma\Delta t} (b_n L^\dagger - b_n^\dagger L) + \frac{\gamma\Delta t}{2} (b_n L^\dagger - b_n^\dagger L)^2 + o(\gamma\Delta t), \tag{3.11}$$

which has the same form as Eq. (3.3) with corrections from coupling to higher frequency modes. Hence, we have shown that, from the coarse graining of the dynamics between the emitter and the field with non-flat coupling between them, we retrieve the collisional model unitary up to order of  $\gamma\Delta t$  by simplifying the form of the interaction. Moreover, we find correction terms to the frequency through this procedure.

In general, we notice that, following this procedure, without taking  $\Delta t \rightarrow 0$ , we can find the evolution equations of the emitter, the field and even the correlation terms using Eq. (3.11) up to order  $\gamma\Delta t$  [MCA22]. We do so in the next chapter to track the correlations created during a collision in the regime of the Optical Bloch Equations.

# Closed Optical Bloch Equations: Dynamics

Optical Bloch Equations (OBE) model the dynamics of classically-driven two-level atoms coupled to thermal baths [TGD92; BP+02]. They are hardware-universal, they accurately model various setups including neutral atoms, superconducting qubits, quantum dots, and trapped ions. They provide a solid basis to explore a broad range of questions of fundamental and practical interest in quantum thermodynamics, quantum optics and quantum technologies, from the performances and efficiencies of quantum devices like quantum interfaces and quantum gates [Ste+22; Maf+23], to quantum heat engines and their quantum advantages [ULK15; Kla+19; Kos19]. OBE are derived by treating such driven-dissipative quantum emitters according to the theory of open quantum systems: their dynamics are solved by coarse-graining dynamical equations written at the fundamental level, and by tracing out the environmental degrees of freedom to which they are coupled, provided the conditions of weak coupling and Markovianity are fulfilled [BP+02; Ali79; DLK18].

At the fundamental level, the OBE derive from the joint dynamics of the atom and the field, where the field contains both the drive and the bath. This was demonstrated in the case of the one-dimensional atom that we considered in the previous chapter. The 1D field is treated as a sequence of so-called “collision units” interacting with the atom once [Sca+02; GP12b; GP12a; Cic+22], their dynamics being captured by ACMs. When the 1D field is initially prepared in a displaced-thermal state, the OBE naturally emerge from the ACM [Cic17; Gro+18]. In these models, atom-field correlations building up during one collision never explicitly appear: they are absorbed in the dissipative part of OBE, while the Hamiltonian is set by the external drive.

Here we begin by reviewing how the OBE emerge in the 1D atom setting. Next we propose a new way of analyzing the same emitter-field dynamics, where fundamental correlations become instrumental. We first build a new kind of ACM keeping track of these fundamental correlations. To do so, in section 4.3, we propose how to “close” the dynamics of the OBE and analyze the evolution of the emitter, the field and the correlations that develop. Namely within each collision, each system is driven by an effective Hamiltonian while a remnant term captures the effect of the correlations. On the atom side, this results in splitting the standard dissipator appearing in the OBE in two terms: a self-driving term proportional to the atom coherences in the energy basis, and a correlation term. On the

field side, the effective Hamiltonian and the correlation term respectively impact the field amplitude and fluctuations, resulting in a physically observable splitting. Finally in section 4.3.4, we show how the components of the output field spectra are affected by the dynamics within the collision.

## 4.1 Open Optical Bloch Equations

Let us consider the 1D atom setup introduced in chapter 3, but with the emitter being a two-level atom (qubit). The ladder operators of the atom are given by  $L = \sigma_- \equiv |g\rangle\langle e|$  and  $L^\dagger = \sigma_+ \equiv \sigma_-^\dagger$ , where  $|e(g)\rangle$  is the excited (ground) state of the qubit. We take  $\gamma$  to be the line width of this atom. Starting from Eq. (3.5), this gives the discrete time evolution operator,

$$V_n = i\sqrt{\gamma\Delta t}(b_n\sigma_+ - b_n^\dagger\sigma_-). \quad (4.1)$$

To derive the dynamics of the two-level atom interacting with the field, the standard approach is to exploit the inequality  $\Delta t \ll \gamma^{-1}$  (naturally fulfilled in the case of a vanishing coarse-graining time) to develop  $\mathcal{U}_n$  up to first order in  $\gamma\Delta t$ , i.e. at second order in  $V_n$  (c.f. Eq. (3.11)). Introducing  $\Delta\rho(t_n) \equiv \rho(t_{n+1}) - \rho(t_n)$ , we obtain the two first terms of the Dyson series,

$$\begin{aligned} \Delta\rho &= \Delta\rho^{(1)} + \Delta\rho^{(2)} + o(\gamma\Delta t) \\ \Delta\rho^{(1)} &= -i[V_n, \rho(t_n)] \\ \Delta\rho^{(2)} &= -\frac{1}{2}\{\rho(t_n), V_n^2\} + V_n\rho(t_n)V_n. \end{aligned} \quad (4.2)$$

Let us focus on the reduced atom dynamical equation obtained by tracing over the field degrees of freedom in Eq.(4.2). As already noticed by [Cic17; Cil+20; Gro+18; Rod+19] for flat atom-field couplings, the OBE emerge from Eq.(4.2) for very specific initial field states, i.e. displaced thermal states. Following these tracks, we take as our initial state  $\rho_{Sf}(t_0) = \rho_S(t_0) \otimes \rho_f(t_0)$ , with  $\rho_f(t_0) = \bigotimes_{n=0}^{\infty} \mu_n$ . Considering the  $n^{th}$  collision mode, we have introduced  $\mu_n = \mathcal{D}_n(\alpha_n)\eta_n^\beta\mathcal{D}_n^\dagger(\alpha_n)$ , where  $\mathcal{D}_n(\alpha_n)$  features a displacement by an amplitude  $\alpha_n = |\alpha|\sqrt{\Delta t}e^{-i(\omega_L - \omega_0)t_n}$  and  $\eta_n^\beta \equiv e^{-\beta\hbar\omega_0 b_n^\dagger b_n}/(1 - e^{-\beta\hbar\omega_0})$  the thermal state of inverse temperature  $\beta$  (see section 4.3.1). The atom reduced state change deduced by tracing Eq.(4.2) over the field degrees of freedom allows us to express the coarse-grained

time derivative  $\Delta\rho_S(t_n)/\Delta t$ . In the continuous time limit  $\gamma\Delta t \rightarrow 0$  and  $t_n \rightarrow t$ , it obeys the equation known as the OBE [TGD92; BP+02]:

$$\frac{\Delta\rho_S(t)}{\Delta t} = -\frac{i}{\hbar} [H_D(t), \rho_S(t)] + \mathcal{L}_{\text{OBE}}[\rho_S(t)]. \quad (4.3)$$

We have kept the notation  $\Delta t$  in Eq.(4.3) to make explicit that it results from a coarse-graining. The Hamiltonian part of the equation stems from the first term of the Dyson series. It captures a constant external drive at the frequency  $\omega_L$ , and is unambiguously set by the initial state of the field, namely, the coherent components of the input units [Rod+19],

$$H_D(t) = \frac{i\hbar\Omega}{2} (e^{i(\omega_0-\omega_L)t}\sigma_+ - e^{-i(\omega_0-\omega_L)t}\sigma_-) \quad (4.4)$$

with  $\Omega/2 = \lim_{\gamma\Delta t \rightarrow 0} \sqrt{\gamma}\alpha/\sqrt{\Delta t}$ . The second term of the Dyson series gives rise to the so-called dissipator, a Completely Positive and Trace-Preserving map (CPTP) capturing the effect of the thermal bath:

$$\begin{aligned} \mathcal{L}_{\text{OBE}}[\rho_S] = & \gamma\bar{n}_{\text{th}} \left( \sigma_+\rho_S\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho_S\} \right) \\ & + \gamma(\bar{n}_{\text{th}} + 1) \left( \sigma_-\rho_S\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_S\} \right). \end{aligned} \quad (4.5)$$

$\bar{n}_{\text{th}}$  is the mean number of thermal photons in the mode resonant with the atom. We dub Eq.(4.3) the open OBE, as they have primarily been established to model the dynamics of the atom, treated as a quantum open system.

## 4.2 Bipartite Dynamics

In order to close the open dynamics, i.e., take into account the degrees of freedom of the field and all the correlations, we will take a step back and analyze the dynamics of arbitrary bipartite systems. We will be concerned with dynamics within  $\Delta t$  and hence, we will now formulate dynamical equations valid at all time-scales. Here, we follow the treatment of the dynamics done by [HOO15; Wei+08; MCA21; Mai+23] which will also be applicable to the atom-field interactions.

Consider two isolated systems  $A$  and  $B$  with bare Hamiltonians  $H_A$  and  $H_B$ . They interact via a time-independent Hamiltonian  $V_{AB}^{(\text{lab})}$  where, the superscript  $(\text{lab})$  denotes

operators in the Schrödinger (laboratory) frame. At each time  $t$  during the evolution, the bipartite state  $\rho_{AB}^{(\text{lab})}(t)$  can be separated into the correlated part and uncorrelated parts

$$\rho_{AB}^{(\text{lab})}(t) = \rho_A^{(\text{lab})}(t) \otimes \rho_B^{(\text{lab})}(t) + \chi^{(\text{lab})}(t), \quad (4.6)$$

where,  $\rho_{A(B)}^{(\text{lab})}(t) \equiv \text{Tr}_{B(A)}\{\rho_{AB}^{(\text{lab})}(t)\}$  and  $\chi^{(\text{lab})}(t)$  is the correlation matrix and is defined through Eq. (4.6). Note that  $\text{Tr}\{\chi^{(\text{lab})}(t)\} = 0$ . The dynamics of each sub-system is affected by the state of the other system and the correlations created at each time. This is made clear by taking the trace of one of the sub-systems in the von-Neumann equation of  $\rho_{AB}^{(\text{lab})}(t)$  and using Eq. (4.6) in the interaction picture<sup>1</sup> with respect to  $H_A + H_B$ ,

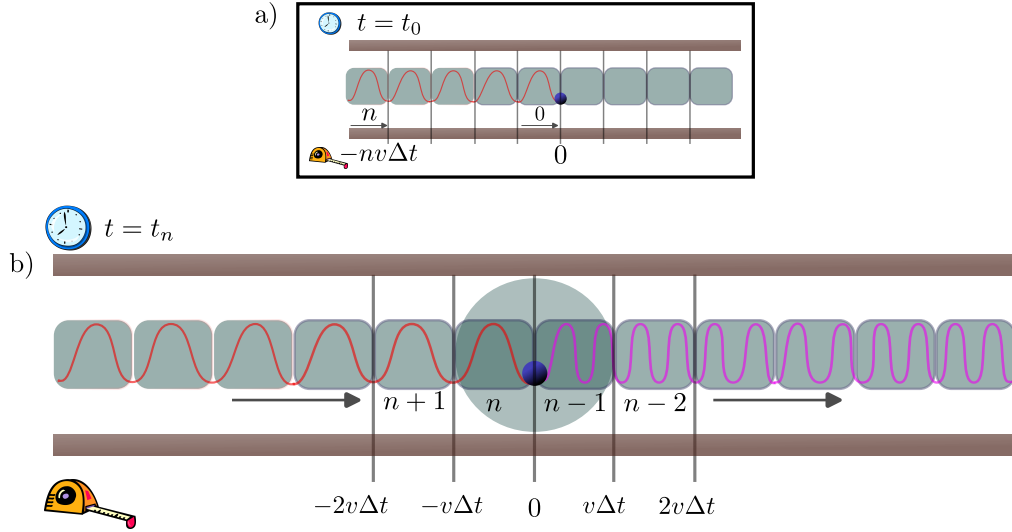
$$i\hbar\dot{\rho}_{A(B)}(t) = [\mathcal{H}_{A(B)}(t), \rho_{A(B)}(t)] + \text{Tr}_{B(A)}\{[V_{AB}(t), \chi(t)]\}. \quad (4.7)$$

This equations is valid at all time-scales and is symmetric for the systems  $A$  and  $B$ . The above equation contains  $\mathcal{H}_{A(B)}(t) = \text{Tr}_{B(A)}\{V_{AB}(t)\rho_{B(A)}(t)\}$  which is the drive operator the system  $B(A)$  performs on  $A(B)$  and  $\mathcal{L}_{A(B),\chi}[\rho_{A(B)}(t)] \equiv (-i/\hbar) \text{Tr}_{B(A)}\{[V_{AB}(t), \chi(t)]\}$  is the dynamical contribution coming from the correlations. Equation (4.7) highlights the two distinct effects of the coupling on the reduced dynamics of  $A$  and  $B$ : the instantaneous state of  $B(A)$  exerts a unitary drive on  $A(B)$ ; the correlations between  $A$  and  $B$  induce a non-unitary change of the reduced state via  $\mathcal{L}_{A(B),\chi}[\rho_{A(B)}(t)]$ . In other words, the drive operator  $\mathcal{H}_{A(B)}(t)$  changes the state of the system  $A(B)$  in an entropy-preserving fashion, while the correlations change the state of  $A(B)$  in a way that may or may not lead to an entropy change. Henceforth, we refer to the first (second) term in the right-hand side of Eq. (4.7) as the effective drive (correlation contribution) of the reduced equation of motion of system. This decomposition of the equations of motion introduced implies that the rate of change of any average value including those of observables can, in turn, also be separated into an uncorrelated and a correlated contribution as we will see in the next section and the chapters that follow.

## 4.3 Closed Optical Bloch Equations

In this section, we will again study the dynamics of the 1D atom, but now keeping track of the evolution of the field while separating the correlations. We will begin by showing how a displaced-thermal field can be written as a product of displaced-thermal fields in

<sup>1</sup>Note that this separation of the state into correlated parts does not remain the same in arbitrary reference frames where the correlations in the state can change, such as in the Heisenberg frame. Nevertheless, in chapter 6, we will show that the interaction frame with respect to the bare Hamiltonians of the sub-systems does not change the equations of the observables used in this thesis.



**Fig. 4.1:** a) Snapshot of the system at the initial time,  $t = t_0$ . The  $n^{\text{th}}$  unit of the input field is at position  $x_n = -nv\Delta t$ . The gray boxes represent the field's collisional units of size  $v\Delta t$ . b) Snapshot of the system at time  $t_n$ . The circular area denotes the interaction region. The atom starts to interact with the input unit ( $n^{\text{th}}$  unit), and stops to interact with the output one ( $(n-1)^{\text{th}}$  unit). Unlike the output units, the input units are prepared in a product of states that only differ by a relative phase.

temporal bins. Following this, we will analyze the evolution of the atom and the field units while tracking the creation of correlations during the collision interval  $\Delta t$ . Through this we will extract the effect of the uncorrelated part and correlations on the dynamics of the two sub-systems at different orders of the interaction strength.

### 4.3.1 Structure of field in temporal bins

A thermal field is described by the density matrix  $\rho_f^\beta = \exp\{-\beta H_f\}/Z$ . Using the equivalence relation Eq. (5.20) in section 5, we find

$$\sum_{n,j} \left( \omega_0 + \frac{2\pi j}{\Delta t} \right) b_{n,j}^\dagger b_{n,j} \simeq \sum_k \omega_k a_k^\dagger a_k. \quad (4.8)$$

From this we see that such state is well approximated by a factorized of the modes  $b_{n,j}$ :  $\rho_f^\beta \simeq \bigotimes_{n,j} e^{-\beta(\omega_0 + \frac{2\pi j}{\Delta t}) b_{n,j}^\dagger b_{n,j}} / Z$ . In particular, the modes  $b_n \equiv b_{n,j=0}$  involved in the collision model description of the dynamics are described by the state:

$$\rho_f^{\beta, (j=0)} = \bigotimes_n \frac{e^{-\beta \omega_0 b_n^\dagger b_n}}{Z_0} = \bigotimes_n \eta_n^\beta, \quad (4.9)$$

with  $Z_0 = \text{Tr}\{e^{-\beta\omega_0 \sum_n b_n^\dagger b_n}\}$ .

Let the field be displaced by an amplitude  $\alpha_L$  at frequency  $\omega_L$ . The displacement operator has the form  $\mathcal{D}(\alpha_L) = \exp\{\alpha_L a_L^\dagger - \alpha_L^* a_L\}$  in terms of the operator  $a_L$  destroying excitations of the field's mode of frequency  $\omega_L$ . Eq. (5.18) in section 5, implies that all the modes  $b_n$  are displaced by an amplitude  $\alpha_n$  which verifies:

$$\alpha_n = \sqrt{\frac{\Delta t}{\gamma}} g_L \alpha_L \text{sinc}\left((\omega_L - \omega_0) \frac{\Delta t}{2}\right) e^{-i(\omega_L - \omega_0)(t_n + \frac{\Delta t}{2})}, \quad (4.10)$$

$$|\alpha_L| = \lim_{\gamma \Delta t \rightarrow 0} \sqrt{\frac{\gamma}{\Delta t}} \frac{|\alpha_n|}{g_L}. \quad (4.11)$$

The displaced-thermal field can hence be written as,

$$\rho_f^{(j=0)}(t_0) = \bigotimes_n \mathcal{D}(\alpha_n) \eta_n^\beta \mathcal{D}^\dagger(\alpha_n), \quad (4.12)$$

where, we find displaced-thermal fields in the temporal bins.

### 4.3.2 Field evolution

We can now compute the evolution of these coarse-grained modes using Eq.(4.2). To do so, it is convenient to define  $b_{x_n}(t_m) \equiv (v\sqrt{\gamma\Delta t})^{-1} \int_{x_{n-1}}^{x_n} dx B(x, t_m)$ , the mode localized between  $x_n = nv\Delta t$  and  $x_{n-1}$  at time  $t_m$ . These modes also obey bosonic relations  $[b_{x_n}(t_m), b_{x_k}(t_l)] = 0$  and  $[b_{x_n}(t_m), b_{x_k}^\dagger(t_l)] = \delta_{n-m, k-l}$ . The linear dispersion relation in the waveguide yields  $B(x, t) = B(0, t - x/v)$ , such that the free dynamics gives rise to the intuitive map  $b_{x_n}(t_{m+1}) = b_{x_{n-1}}(t_m)$  [see Fig.4.1 b)] and hence  $b_n = b_{x_{-n}}(t_0)$ : the mode  $b_n$  can equivalently be understood as the mode coupled to the atom between  $t_n$  and  $t_{n+1}$ , or as the mode positioned between  $x_{-n}$  and  $x_{-(n+1)}$  at time  $t_0$  [see Fig.4.1 a)]. In particular, the mode  $b_n$  [resp.  $b_{n-1}$ ] starts (resp. stops) interacting with the atom at time  $t_n$ . Taking the trace over the atom degrees of freedom in Eq. (4.2) yields the evolution of the field up to second order in  $V_n$ :

$$\Delta_n \rho_f^{(1)} = -i\Delta t [\mathcal{H}_f(t_n), \rho_f(t_n)] \quad (4.13)$$

$$\begin{aligned} \Delta_n \rho_f^{(2)} = & -\frac{\gamma\Delta t}{2} \{p_e(t_n) b_n b_n^\dagger + p_g(t_n) b_n^\dagger b_n\}, \rho_f(t_n) \} \\ & + \gamma\Delta t (p_g(t_n) b_n \rho_f(t_n) b_n^\dagger + p_e(t_n) b_n^\dagger \rho_f(t_n) b_n). \end{aligned} \quad (4.14)$$

We have introduced  $p_e(t_n) = \langle \sigma_+ \sigma_- \rangle_{t_n}$  and  $p_g(t_n) = \langle \sigma_- \sigma_+ \rangle_{t_n}$  the atom populations of the excited and the ground levels at time  $t_n$ . The first term of the Dyson series gives rise to the

Hamiltonian evolution, where  $\mathcal{H}_f(t_n) = \frac{1}{\Delta t} \text{Tr}_S \{V_n \rho_S(t_n)\} = i\sqrt{\frac{\gamma}{\Delta t}} (b_n \langle \sigma_+ \rangle_{t_n} - b_n^\dagger \langle \sigma_- \rangle_{t_n})$  stands for the effective drive induced by the atom on the field, proportional to the atomic coherences. Conversely, the second term takes the form of a dissipator for the field, which captures incoherent emissions and absorptions by the atom.

To complete the analysis of the field evolution, it is convenient to consider the changes of the first (second) moment of the field induced by the atom, namely  $\text{Tr}_f \{b_n \Delta_n \rho_f\}$  ( $\text{Tr}_f \{b_n^\dagger b_n \Delta_n \rho_f\}$ ). Eqs. (4.13) and (4.14) yield

$$\begin{aligned} \text{Tr}_f \{b_n \Delta_n \rho_f\} &= -\sqrt{\gamma \Delta t} \langle \sigma_- \rangle_{t_n} \\ &\quad - \frac{\gamma \Delta t}{2} (p_g(t_n) \langle b_n \rangle_{t_n} + p_e(t_n) \langle b_n^\dagger \rangle_{t_n}) + o(\gamma \Delta t), \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \text{Tr}_f \{b_n^\dagger b_n \Delta_n \rho_f^{(2)}\} &= \gamma \Delta t (\langle \sigma_+ \sigma_- \rangle_{t_n} + \langle b_n^\dagger b_n \rangle_{t_n} \langle \sigma_z \rangle_{t_n}) \\ &\quad - \sqrt{\gamma \Delta t} [\langle b_n \rangle_{t_n} \langle \sigma_+ \rangle_{t_n} + \langle b_n^\dagger \rangle_{t_n} \langle \sigma_- \rangle_{t_n}] + o(\gamma \Delta t). \end{aligned} \quad (4.16)$$

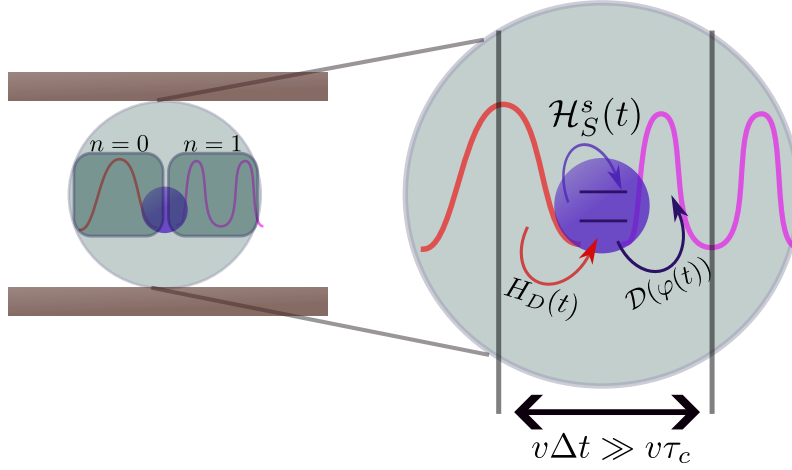
### Relation to Input-Output formalism

We can now relate our framework established in the interaction picture to the input-output formalism introduced in [GC85] in the Heisenberg picture. To do so, we define the input (output) operators in the continuous-time limit  $\gamma \Delta t \rightarrow 0$  and  $t_n \rightarrow t$ :

$$b_{\text{out(in)}}(t) \equiv \lim_{\gamma \Delta t \rightarrow 0} \frac{b_{x_1(x_0)}(t_n)}{\sqrt{\Delta t}}. \quad (4.17)$$

Unlike the collision modes  $b_n$  which have no dimension, the field operators  $b_{\text{out(in)}}(t)$  have the dimension of  $[t^{-1/2}]$ . In the interaction representation used in our ACM, the first moment evolution captures the difference  $\text{Tr} \{b_n \Delta_n \rho_f(t_n)\} = \langle b_{x_1}(t_{n+1}) \rangle_{t_{n+1}} - \langle b_{x_0}(t_n) \rangle_{t_n}$ , hence Eq. (4.15) gives rise to the mean input-output equation:

$$\langle b_{\text{out}}(t) \rangle_t = \langle b_{\text{in}}(t) \rangle_t - \sqrt{\gamma} \langle \sigma_- \rangle_t, \quad (4.18)$$



**Fig. 4.2:** Unitary processes between the atom and the field during a collision, i.e., within a time interval  $\Delta t \gg \tau_c$ . The input field (red) imparts an external drive ( $H_D(t)$ ) on the atom, which in turn displaces the field ( $\mathcal{D}(\varphi(t))$ ) with the magnitude proportional to the atom's coherence  $\varphi(t) = \sqrt{\gamma \Delta t} \langle \sigma_- \rangle_t$ . The change of the field state into the outgoing field (violet) induces a self-drive ( $\mathcal{H}_S^s(t)$ ) on the atom with strength proportional to its own coherence.

evidencing that only the first term of the Dyson series contributes to the change in the field amplitude. This is not the case for the second moment evolution [Eq. (4.16)] that becomes

$$\begin{aligned} \langle b_{\text{out}}^\dagger(t) b_{\text{out}}(t) \rangle_t &= \langle b_{\text{in}}^\dagger(t) b_{\text{in}}(t) \rangle_t \\ &\quad - \sqrt{\gamma} (\langle \sigma_+ \rangle_t \langle b_{\text{in}}(t) \rangle_t + \langle \sigma_- \rangle_t \langle b_{\text{in}}^\dagger(t) \rangle_t) \\ &\quad + \gamma \left( \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle \sigma_z \rangle_t + \frac{1}{2} \right), \end{aligned} \quad (4.19)$$

revealing that the change in the photon rate due to the interaction stems from stimulated (spontaneous) emission captured by the first (second) term of the Dyson series.

Finally, it is interesting to notice that we can take into account the coarse graining by simply writing the average of the coupling dependent temporal operator  $\langle B(0, t) \rangle$  in terms of  $\langle b_{\text{out(in)}}(t) \rangle$ . This can be done by writing  $\langle b_{\text{out(in)}}(t) \rangle$  in terms of the coarse grained operators using Eq. (4.17). Adding the two and using Eq. (3.6), in the limit, leads to the useful relation

$$\langle B(0, t) \rangle = \frac{\langle b_{\text{in}}(t) \rangle + \langle b_{\text{in}}^\dagger(t) \rangle}{2}. \quad (4.20)$$

### 4.3.3 Anatomy of a collision

In the resolution presented above, the correlations building up between the atom and the field during one collision are never explicitly written. In this section, we track these

fundamental correlations, resuming from the expansion Eq. (4.2). We rewrite the first order term as

$$\Delta_n \rho^{(1)} = \Delta_n \rho_S^{(1)} \otimes \rho_f(t_n) + \rho_S(t_n) \otimes \Delta_n \rho_f^{(1)} + \Delta_n \chi^{(1)}, \quad (4.21)$$

where

$$\begin{aligned} \Delta_n \rho_S^{(1)} &= \text{Tr}_f \{ \Delta_n \rho^{(1)} \} = -i [ \langle V_n \rangle_f, \rho_S(t_n) ] \\ \Delta_n \rho_f^{(1)} &= \text{Tr}_S \{ \Delta_n \rho^{(1)} \} = -i [ \langle V_n \rangle_S, \rho_f(t_n) ] \end{aligned} \quad (4.22)$$

respectively refer to the change at first order in  $V_n$  of the atom and field reduced state. We have introduced  $\langle V_n \rangle_{S(f)} = \text{Tr} \{ V_n \rho_{S(f)}(t_n) \}$ . In addition,

$$\Delta_n \chi^{(1)} = -i [ V_n - \langle V_n \rangle_S - \langle V_n \rangle_f, \rho(t_n) ] \quad (4.23)$$

captures the change in the atom-field correlation matrix during the  $n^{\text{th}}$  collision. We now inject Eq. (4.21) into the second term of the Dyson series  $\Delta \rho^{(2)} = -\frac{i}{2} [ V_n, \Delta \rho^{(1)} ]$ , leading to three second-order contributions to the atom-field state variation

$$\begin{aligned} \Delta_n \rho^{(2,S)} &= -\frac{1}{2} [ V_n, [ \langle V_n \rangle_f, \rho(t_n) ] ] \\ \Delta_n \rho^{(2,f)} &= -\frac{1}{2} [ V_n, [ \langle V_n \rangle_S, \rho(t_n) ] ] \\ \Delta_n \rho^{(2,\chi)} &= -\frac{1}{2} [ V_n, [ V_n - \langle V_n \rangle_f - \langle V_n \rangle_S, \rho(t_n) ] ]. \end{aligned} \quad (4.24)$$

These three terms respectively stem from the first-order atom and field evolution, and from the atom-field correlations.

From Eqs.(4.24), it appears that the second order evolution of each subsystem now splits into three terms. We first identify terms  $\Delta_n \rho_{S(f)}^{(2,S(f))}$  of the form

$$\Delta_n \rho_{S(f)}^{(2,S(f))} = -\frac{1}{2} [ \langle V_n \rangle_{f(S)}, [ \langle V_n \rangle_{f(S)}, \rho_{S(f)}(t_n) ] ]. \quad (4.25)$$

These terms correspond to the second-order term in the expansion of the unitary evolution generated by the effective Hamiltonian  $\langle V_n \rangle_{f(S)}$ , whose first-order is given in Eq. (4.22). Second, we define  $\Delta_n \rho_{S(f)}^{(2,f(S))}$  as

$$\begin{aligned} \Delta_n \rho_{S(f)}^{(2,f(S))} &= -\frac{1}{2} \text{Tr}_{f(S)} \{ V_n [ \langle V_n \rangle_{S(f)}, \rho_{f(S)}(t_n) ] \} \rho_{S(f)}(t_n) \\ &\quad - \rho_{S(f)}(t_n) \frac{1}{2} \text{Tr}_{f(S)} \{ [ \rho_{f(S)}(t_n), \langle V_n \rangle_{S(f)} ] V_n \} \end{aligned} \quad (4.26)$$

These terms capture the self-driving of system  $S(f)$ , via the perturbation of the other system (see below). Finally,  $\Delta_n \rho^{(2,\chi)}$  corresponds to the non-unitary evolution of the reduced field and atom states induced by the atom-field correlations.

Remarkably, this framework yields a new splitting in the state variation during a single collision, where we distinguish the contribution of the reduced evolution stemming from correlations (able to generate non-unitary evolution) from unitary contributions. Namely, we now write:

$$\Delta_n \rho^{(1)} + \Delta_n \rho^{(2)} = \Delta_n \rho^\chi + \Delta_n \rho^\otimes, \quad (4.27)$$

where

$$\begin{aligned} \Delta_n \rho^\otimes &= \Delta_n \rho_S^{(1)} \otimes \rho_f(t_n) + \rho_S(t_n) \otimes \Delta_n \rho_f^{(1)} \\ &\quad + \Delta_n \rho^{(2,S)} + \Delta_n \rho^{(2,f)} \\ \Delta_n \rho^\chi &= \Delta_n \chi^{(1)} + \Delta_n \rho^{(2,\chi)}. \end{aligned} \quad (4.28)$$

So far, this section's analysis is general for any bipartite system  $S - f$  complying with the Magnus expansion treatment. We now explore how the tracking of correlations sheds new light on the atom-field dynamics, in the regime of emergence of the OBE.

### Atom evolution

To begin with, we recall that  $\langle V_n \rangle_f = H_D(t_n) \Delta t$ , where  $H_D(t)$  is defined in Eq.(4.4). Thus, the first order term  $\Delta_n \rho_S^{(1)}$  evolves under the external drive term appearing in the OBE (see Eq.(4.3) and red arrow in Fig. 4.2). As this term is of first order in  $\Delta t$ , the term  $\Delta_n \rho_S^{(2,S)}$  is of order  $\mathcal{O}(\Omega^2 \Delta t^2)$  and can therefore be neglected in the single-collision atom dynamics. Second, Eq. (4.26) reads:

$$\Delta_n \rho_S^{(2,f)} = -\frac{i\Delta t}{\hbar} [\mathcal{H}_S^s(t_n), \rho_S(t_n)], \quad (4.29)$$

with

$$\mathcal{H}_S^s(t_n) = -i\hbar \frac{\gamma}{2} (\langle \sigma_- \rangle_{t_n} \sigma_+ - \langle \sigma_+ \rangle_{t_n} \sigma_-) \quad (4.30)$$

Remarkably, this term describes a process in which the atom generates a drive on itself, mediated by the field. This contribution, that we dub "self-drive", is a first important outcome of our approach. We observe that it can appear even in the absence of external drive,

just by taking into account the evolution of the local field state, i.e. during spontaneous emission, provided that the atom carries coherences in the energy basis (see blue arrow in Fig. 4.2). Eventually, the coarse-grained atom evolution equation reads in the continuous limit  $t_n \rightarrow t$ :

$$\dot{\rho}_S(t) = -\frac{i}{\hbar} [H_D(t) + \mathcal{H}_S^s(t), \rho_S(t)] + \mathcal{L}_{S,\chi}[\rho_S(t)]. \quad (4.31)$$

We have introduced the term  $\mathcal{L}_{S,\chi}(t_n) = \lim_{\Delta t \rightarrow 0} \frac{\Delta \rho_S^{(2,\chi)}}{\Delta t}$  which quantifies the effect of the correlations during the collision on the reduced state of the atom.

Equations (4.3) and (4.31) feature two equivalent evolution equations governing the atom reduced dynamics, which give rise to the same CPTP map. The former [Eq. (4.3)] corresponds to the textbook OBE, for which the CPTP map splits into two terms: the standard thermal dissipator and a Hermitian part solely involving the external driving term  $H_D(t)$ . Unlike the textbook OBE, the Hermitian part of the evolution equation captured by [Eq. (4.31)] also entails the second order self-drive  $\mathcal{H}_S^s(t)$ . The sum of the correlation term and the self-drive give back the dissipator appearing in the standard OBE. The self-drive is a major new concept brought by our approach. It has fundamental thermodynamical consequences as we show in Chapter 6.

### Field evolution

In the same way, the evolution equation of the field is obtained from Eq. (4.22). It is identical to Eq. (4.13) and corresponds to the coarse-grained effective drive exerted by the atom on the field,  $\langle V_n \rangle_S = \Delta t \mathcal{H}_f(t_n) = i\hbar\sqrt{\gamma\Delta t}(\langle \sigma_+ \rangle_{t_n} b_n - \langle \sigma_- \rangle_{t_n} b_n^\dagger)$ . It entails the reduced state of the atom at time  $t_n$ , and gives rise to displacement of the field with amplitude  $\varphi_n = -\sqrt{\gamma\Delta t}\langle \sigma_- \rangle_{t_n}$  consistently with the mean input-output relation derived in Eq. (4.18) (see violet arrow in Fig. 4.2).

At second order, we find that, unlike the case of the atom, the term  $\Delta \rho_f^{(2,S)}$  yields a negligible contribution to the field dynamics (there is therefore no “self-drive” for the field), while the second-order drive  $\Delta \rho_f^{(2,f)}$  must be kept (see section 5.4). This is noticeable in the change in field photon number across the collision, which contains only two non-negligible contributions:

$$\text{Tr} \left\{ b_n^\dagger b_n \Delta_n \rho_f^{(2,f)} \right\} = \gamma \Delta t |\langle \sigma_- \rangle_{t_n}|^2 \quad (4.32)$$

$$\begin{aligned} \text{Tr} \left\{ b_n^\dagger b_n \Delta_n \rho_f^{(2,\chi)} \right\} &= \gamma \Delta t \left( \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle \sigma_z \rangle_{t_n} + \frac{1}{2} \right) \\ &\quad - \gamma \Delta t |\langle \sigma_- \rangle_{t_n}|^2 \end{aligned} \quad (4.33)$$

where we have used that  $\Delta t \langle b_n^\dagger b_n \rangle_{t_n} = \Delta t \bar{n}_{\text{th}} + \Delta t |\alpha_n|^2 = \Delta t \bar{n}_{\text{th}} + |\alpha|^2 \mathcal{O}(\Delta t^2)$ . Eq.(4.32) are consistent with the evolution of the second moment captured in the standard approach (Eq.(4.16)). Owing to our new splitting, we observe that the unitary  $\Delta\rho^{(2,f)}$  part of the field evolution (resp. the correlation part  $\Delta\rho^{(2,x)}$ ) solely gives rise to a change of the field amplitude (resp. fluctuations). As a consequence, the splitting we propose has a clear physical imprint on the field observables and can be traced back in measurable quantities. In the next Section, we exploit this splitting for thermodynamical purposes.

### 4.3.4 Spectral signature of the correlations

The correlations generated between the field and atom can also be tracked in the field spectrum. We define the input (output) spectrum as  $S_{\text{in}}(\omega) = S(\omega, t_0)$  ( $S_{\text{out}}(\omega) = S(\omega, t_\infty)$ ) where we have introduced  $S(\omega, t) \equiv \sum_k \delta_D(\omega_k - \omega) \text{Tr}\{a_k^\dagger a_k \rho_f(t)\}$ .  $t_\infty$  is long enough to reach the steady regime of the OBE. We show in section 8.3 that the input spectrum reads

$$\begin{aligned} S_{\text{in}}(\omega) &= \frac{\gamma}{2\pi\gamma[\omega]} \iint dt d\tau e^{-i\Delta\omega\tau} \langle b_{\text{in}}^\dagger(t+\tau) b_{\text{in}}(t) \rangle \\ &= \int dt \left( \delta(\omega - \omega_L) \frac{\Omega^2}{4\gamma[\omega]} + \frac{\gamma \bar{n}_{\text{th}}}{2\pi\gamma[\omega]} \right). \end{aligned} \quad (4.34)$$

As expected, the input spectrum consists of a single spectral component at the drive frequency and a constant thermal noise. On the other hand, the output spectrum depends on two-collision processes:

$$S_{\text{out}}(\omega) = S_{\text{in}}(\omega) + \lim_{\gamma\Delta t \rightarrow 0} \frac{\gamma\Delta t}{2\pi\gamma[\omega]} \sum_{m,n} e^{-i\Delta\omega(t_m - t_n)} \text{Tr}\{b_m^\dagger b_n \Delta_{m,n}\rho_f\}. \quad (4.35)$$

We have introduced  $\Delta_{m,n}\rho_f$ , which stands for the change of the joint state of the modes  $n$  and  $m$  induced by their collisions with the atom:

$$\Delta_{m,n}\rho_f = \text{Tr}_S \left\{ U_m \eta_m \mathcal{E}_S^{t_m - t_n} [U_n \eta_n \rho_S(t_n) U_n^\dagger] U_m^\dagger \right\} - \eta_m \otimes \eta_n. \quad (4.36)$$

We have denoted  $\mathcal{E}_S^\tau\{\cdot\}$  the super-operator capturing the atom reduced evolution over a time interval  $\tau$  (obtained by solving Eq. (4.3)).  $\Delta_{m,n}\rho_f$  captures the correlations between field temporal modes  $m$  and  $n$  mediated by the atom, whose memory time typically spans a large number of consecutive collisions of order  $1/\gamma\Delta t$ . Remarkably, the fundamental correlations generated between the atom and the field at time  $t_n$  generate these field-field

correlations, leaving a measurable imprint in the output spectrum. To show this, we inject the expression (4.27) in Eq. (4.38), leading to the splitting:

$$\Delta_{m,n}\rho_f = \Delta_{m,n}\rho_f^\otimes + \Delta_{m,n}\rho_f^\chi. \quad (4.37)$$

with

$$\begin{aligned} \Delta_{m,n}\rho_f^\otimes &= \text{Tr}_S\{U_m\eta_m\mathcal{E}_S^{t_m-t_n}[\rho_S(t_n)\otimes\eta_m+\Delta_n\rho^\otimes]\} \\ &\quad -\eta_m\otimes\eta_n \\ \Delta_{m,n}\rho_f^\chi &= \text{Tr}_S\{U_m\eta_m\mathcal{E}_S^{t_m-t_n}[\Delta_n\rho^\chi]\}. \end{aligned} \quad (4.38)$$

By inserting those two terms in Eq. (4.35), we identify the contributions to the spectrum resulting from correlated and uncorrelated evolution of the field and atom, namely  $S_{\text{out}}(\omega) - S_{\text{in}}(\omega) = \int dt (\dot{S}^\otimes(\omega) + \dot{S}^\chi(\omega))$ .  $\dot{S}^\otimes(\omega)$  reads (see section 8.3.3):

$$\begin{aligned} \dot{S}^\otimes(\omega) &= \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i\Delta\omega\tau} \langle b_{\text{out}}^\dagger(t+\tau) \rangle \langle b_{\text{out}}(t) \rangle - \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i\Delta\omega\tau} \langle b_{\text{in}}^\dagger(t+\tau) \rangle \langle b_{\text{in}}(t) \rangle \\ &= -\frac{\gamma\Omega}{2\gamma[\omega]} \delta_D(\omega - \omega_L) (e^{i(\omega_L - \omega_0)t_\infty} \langle \sigma_- \rangle_{t_\infty} + \text{c.c.}) + \frac{\gamma^2}{\gamma[\omega]} \delta_D(\omega - \omega_L) |\langle \sigma_- \rangle_{t_\infty}|^2. \end{aligned} \quad (4.39)$$

$\dot{S}^\otimes(\omega)$  consists of a single spectral component centered at  $\omega_L$ . The weight of this component involves two contributions, the first one (the second one) scaling like  $\gamma\Omega$  ( $\gamma^2$ ) and being due to stimulated emission (spontaneous emission). On the other hand, the correlation-induced spectrum takes the form (see section 8.3.3):

$$\dot{S}^\chi(\omega) = \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i\Delta\omega\tau} \langle \delta b_{\text{out}}^\dagger(t+\tau) \delta b_{\text{out}}(t) \rangle, \quad (4.40)$$

where we have introduced  $\delta b_{\text{out}}(t) = b_{\text{out}}(t) - \langle b_{\text{out}}(t) \rangle$ .  $\dot{S}^\chi(\omega)$  corresponds to the so-called incoherent emission spectrum. It is composed of three Lorentzian functions peaked at  $\omega_L, \omega_L \pm \sqrt{\delta^2 + \Omega^2}$  (a.k.a. Mollow's fluorescence triplet). Hence, we have found that the correlation induced evolution causes fluctuations in the field observables and leads to loss of purity.

Notice that the correlations have consequences that are difficult to reverse. They hence, have a clear thermodynamic consequence. These signatures of correlations are revealed even in the energy exchanged. In the next chapter, we will study the thermodynamic impact of the correlation created during the evolution.



# Detailed Calculations:

## Coarse-Grained Field Modes

Here, we will detail the calculations to discretize continuous modes to discrete modes and find the discrete bosonic operators from the coupling dependent operators  $B(0, t)$  within the coarse graining time  $\Delta t$ . We will use this to find the error in the decomposition of the evolution into a collisional model and finally the coarse-grained evolution of the field units.

### 5.1 Discretization of frequency and time

Here, we will illustrate the discretization of operators in time and frequency and derive useful expressions. For simplicity, here we will work with the usual temporal bin operators which do not take into account the coupling strength. We start with the interaction picture operators [MCA22] but in continuous frequency and time. We begin with the operator

$$b(t) = \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} a(\omega) \quad (5.1)$$

We discretize time in intervals of  $\Delta t$  and frequency in intervals of  $\Delta\omega$ . We use,  $\int_{t_n}^{t_n+\Delta t} dt b(t) = \Delta t b(t_n)$  and  $\int_{\omega_k}^{\omega_k+\Delta\omega} d\omega a(\omega) = \Delta\omega a(\omega_k)$  and define

$$b_n = \sqrt{\Delta t} b(t_n) = \frac{1}{\sqrt{\Delta t}} \int_{t_n}^{t_n+\Delta t} dt b(t) \quad (5.2)$$

$$a_k = \sqrt{\Delta\omega} a(\omega_k) = \frac{1}{\sqrt{\Delta\omega}} \int_{\omega_k}^{\omega_k+\Delta\omega} d\omega a(\omega) \quad (5.3)$$

where,  $2\pi\Delta\omega^{-1} = \varrho$  which is the density of states. Using stationary wave boundary conditions, it typically takes the value of  $L/v$  where,  $v$  is the speed of light in the medium and  $L$  is the length of the waveguide. These give the correct commutation relations for the discrete operators. The commutator gives

$$[a_k, a_j] = \Delta\omega [a(\omega_k), a(\omega_j)] = \frac{2\pi}{\varrho} [a(\omega_k), a(\omega_j)] = \frac{2\pi}{\varrho} \delta_D(\omega_k - \omega_j) \quad (5.4)$$

Discretizing the Dirac delta gives and using  $\varrho = L/v$  gives the Kronecker delta:

$$[a_k, a_j] = \frac{2\pi v}{L} \delta_D(\omega_k - \omega_j) = \frac{2\pi v}{L} \frac{\Delta t}{2\pi} \sum_n e^{i\frac{v\Delta t}{L} 2\pi(j-k)n} = \delta_{k,j}. \quad (5.5)$$

The discrete time operator instead is

$$\begin{aligned}
b_n &= \sqrt{\frac{\Delta t}{2\pi}} \int d\omega e^{-i\omega t_n} a(\omega) \\
&= \sqrt{\frac{\Delta t}{2\pi}} \sum_j \Delta\omega e^{-i\omega_j t_n} a(\omega_j) \\
&= \sqrt{\frac{\Delta t}{2\pi}} \sqrt{\frac{2\pi v}{L}} \sum_j e^{-i\omega_j t_n} a_j.
\end{aligned} \tag{5.6}$$

which is related by Fourier transform

$$\begin{aligned}
\sum_n e^{i\omega_k t_n} b_n &= \sqrt{\frac{\Delta t}{2\pi}} \sqrt{\frac{2\pi v}{L}} \frac{L}{v\Delta t} \sum_j \frac{v\Delta t}{L} \sum_n e^{i\frac{2\pi(k-j)n}{v\Delta t}} a_j \\
&= \sqrt{\frac{L}{v\Delta t}} \sum_j \delta_{k,j} a_j \\
&= \sqrt{\frac{L}{v\Delta t}} a_k \\
a_k &= \sqrt{\frac{v\Delta t}{L}} \sum_n e^{i\omega_k t_n} b_n
\end{aligned} \tag{5.7}$$

Hence,

$$a_k = \sqrt{\frac{v\Delta t}{L}} \sum_n e^{i\omega_k t_n} b_n, \tag{5.8}$$

and

$$b_n = \sqrt{\frac{v\Delta t}{L}} \sum_j e^{-i\omega_j t_n} a_j. \tag{5.9}$$

These temporal operators are relevant in the case of flat coupling, where the frequency shifts don't depend on the distribution of the coupling strength. In this case, the relation between the frequency and temporal displacement and coherent amplitude is

$$D(\beta) = \exp \left\{ \sum_n \beta \sqrt{\frac{\Delta t}{\varrho}} e^{-i\omega_k t_n} a_n^\dagger \right\} = \bigotimes D(\alpha_n), \tag{5.10}$$

where,

$$\alpha_n = \sqrt{\frac{\Delta t}{\varrho}} \beta e^{-i\omega_k t_n}. \tag{5.11}$$

The relation between the line width at emitter frequency  $\omega_0$  and the discrete coupling strengths is ( $\omega_j = \frac{2\pi v}{L}j$ ),

$$\begin{aligned}\gamma(\omega_0) &= 2\pi \sum_j g_j^2 \delta_D(\omega_j - \omega_0) = \sum_j g_j^2 \int dt e^{-i(\omega_j - \omega_0)t} \\ &= \sum_j g_j^2 \sum_n \Delta t e^{-i2\pi \frac{v}{L}(j-0)n\Delta t} = \sum_j g_j^2 \frac{L}{v} \delta_{j,0} = g_0^2 \frac{L}{v} = g_0^2 \varrho,\end{aligned}\quad (5.12)$$

and hence,  $g_0 = \sqrt{\frac{\gamma(\omega_0)}{\varrho}}$ . And finally the relation to the continuous coupling strength is  $g_j = g(\omega_j) \sqrt{\frac{1}{\varrho}}$  and hence,

$$\gamma(\omega_0) = 2\pi \sum_j g_j^2 \delta_D(\omega_j - \omega_0) = 2\pi \int d\omega g^2(\omega) \delta_D(\omega - \omega_0) = 2\pi g^2(\omega_0). \quad (5.13)$$

## 5.2 Coarse grained operator relations

Here, we take into account the distribution of the coupling strength over a range of frequencies and find bosonic operators by decomposing the field operators in discrete time. To obtain a discrete decomposition of the functions of time, we introduce the basis of functions  $w_{n,j}(t) = \frac{1}{\sqrt{\Delta t}} \Theta_n(t) e^{-i\frac{2\pi}{\Delta t}jt}$ , for  $n, j \in \mathbb{Z}$ , where  $\Theta_n(t) = 1$  for  $t \in [t_n, t_{n+1}]$  and 0 everywhere else. The function  $w_{n,j}$  has therefore only support on the time interval  $[t_n, t_{n+1}]$  and has an average frequency  $2\pi j/\Delta t$ . The basis is orthonormal in the sense that  $\langle w_{n,j}, w_{n',j'} \rangle = \int_{-\infty}^{\infty} dt w_{n,j}^*(t) w_{n',j'}(t) = \delta_{j,j'} \delta_{n,n'}$ . We then have:

$$B(0, t) = \sum_{n,j} \sqrt{\gamma_j} w_{n,j}(t) b_{n,j}, \quad (5.14)$$

with  $b_{n,j}$  the discrete field mode localized within time interval  $[t_n, t_{n+1}]$  and with average frequency  $2\pi j/\Delta t$ :

$$b_{n,j} = \frac{1}{\sqrt{\gamma_j \Delta t}} \int_{t_n}^{t_n + \Delta t} dt e^{i\frac{2\pi}{\Delta t}jt} B(0, t). \quad (5.15)$$

We have introduced the parameter  $\gamma_j = 2\pi \sum_k |g_k|^2 \delta_D(\Delta\omega_k - \frac{2\pi j}{\Delta t})$  with  $\Delta\omega_k = \omega_k - \omega_0$ . The  $b_{n,j}$  are normalized to verify:

$$\begin{aligned} [b_{n,j}^\dagger, b_{n',j'}] &= \left[ \int dt \sum_k g_k e^{-i\Delta\omega_k t} w_{n,j}(t) a_k, \int dt' \sum_{k'} g_{k'}^* e^{-i\Delta\omega_{k'} t'} w_{n',j'}^*(t') a_{k'}^\dagger \right] \\ &= \int dt \int dt' \sum_k |g_k|^2 e^{i\Delta\omega_k(t-t')} w_{n,j}(t) w_{n',j'}^*(t'). \end{aligned} \quad (5.16)$$

The correlation function of the field  $C(t, t') = C_{B^\dagger}(t, t') - C_B(t, t') = \sum_k |g_k|^2 e^{i\Delta\omega_k(t-t')}$  has a correlation time  $\tau_c$ . Given the choice of coarse-graining time  $\Delta t \gg \tau_c$ , terms with  $n \neq n'$  in Eq (5.16) will only involve values of  $|t - t'|$  much larger than  $\tau_c$  and can then be neglected.

$$\begin{aligned} [b_{n,j}^\dagger(t), b_{n',j'}(t)] &\simeq \frac{1}{\gamma_j} \frac{\delta_{n,n'}}{\Delta t} \int dt \Theta_n(t) \int dt' \Theta_n(t') C(t-t') e^{-\frac{2i\pi}{\Delta t}(jt-j't')} \\ &= \frac{1}{\gamma_j} \frac{\delta_{nn'}}{\Delta t} \int dt e^{-\frac{2i\pi}{\Delta t}(j-j')t} \Theta_n(t) \int d\tau \Theta_n(t-\tau) C(\tau) e^{-\frac{2i\pi}{\Delta t}j'\tau} \\ &\simeq \delta_{n,n'} \frac{\int dt e^{-\frac{2i\pi}{\Delta t}(j-j')t} \Theta_n(t) \int_{-\infty}^{\infty} d\tau C(\tau) e^{-\frac{2i\pi}{\Delta t}j'\tau}}{\Delta t \gamma_j} \\ &= \delta_{n,n'} \delta_{j,j'}. \end{aligned} \quad (5.17)$$

To go to the second line we have changed integration variable from  $t'$  to  $\tau = t - t'$ . To go to the third line, we have used that  $C(\tau)$  is non zero for  $|\tau| \lesssim \tau_c$  such that  $\Theta(t - \tau)$  can be approximated by  $\Theta_n(t)$  in the integral over  $\tau$ .

Finally, it is useful to have in mind the correspondence relation that we derive from the explicit expression of  $B(0, t)$ :

$$b_{n,j} = \sqrt{\frac{\Delta t}{\gamma_j}} \sum_k g_k e^{-i(\Delta\omega_k - \frac{2\pi j}{\Delta t})(t_n + \frac{\Delta t}{2})} \text{sinc}\left(\frac{1}{2}(\Delta\omega_k \Delta t - 2\pi j)\right) a_k. \quad (5.18)$$

This relation imply the equivalence of the two description in term of number of excitations in the field:

$$\begin{aligned}
\sum_{n,j} b_{n,j}^\dagger b_{n,j} &= \sum_{n,j} \frac{\Delta t}{\gamma_j} \sum_{k,l} g_k^* g_l a_k^\dagger a_l e^{i(\omega_l - \omega_k)(t_n + \Delta t)} \text{sinc} \left( \frac{\Delta \omega_k \Delta t}{2} - \pi j \right) \text{sinc} \left( \frac{\Delta \omega_l \Delta t}{2} - \pi j \right) \\
&\simeq \sum_j \frac{\Delta t}{\gamma_j} \sum_{k,l} g_k^* g_l a_k^\dagger a_l \frac{\delta_D(\omega_l - \omega_k)}{\Delta t} \text{sinc} \left( \frac{\Delta \omega_k \Delta t}{2} - \pi j \right) \text{sinc} \left( \frac{\Delta \omega_l \Delta t}{2} - \pi j \right) \\
&= \sum_j \frac{1}{\gamma_j} \sum_k |g_k|^2 a_k^\dagger a_k \text{sinc}^2 \left( \frac{\Delta \omega_k \Delta t}{2} - \pi j \right) \sum_l \delta_D(\omega_l - \omega_k) \\
&\simeq \sum_j \sum_k a_k^\dagger a_k \text{sinc}^2 \left( \frac{\Delta \omega_k \Delta t}{2} - \pi j \right) \\
&= \sum_k a_k^\dagger a_k.
\end{aligned} \tag{5.19}$$

To go to the fourth line, we have used that the sinc functions are peaked around  $\Delta \omega_k = 2\pi j / \Delta t$ , such that  $|g_k|^2 \sum_l \delta_D(\omega_k - \omega_l) \simeq \gamma_j$  provided the coupling coefficients can be considered constant over the frequency interval  $2\pi / \Delta t$ , which holds true in the limit  $\Delta t \gg \tau_c$  considered in this article. To go to the last line, we have used the identity  $\sum_{j=-\infty}^{\infty} \text{sinc}^2(x - \pi j) = 1$ .

Using that  $2\pi / \Delta t \ll \omega_k$  over the frequency range in which the atom and field interact, we can also deduce the equivalence relation in terms of the field Hamiltonian:

$$\begin{aligned}
\sum_{n,j} \left( \omega_0 + \frac{2\pi j}{\Delta t} \right) b_{n,j}^\dagger b_{n,j} &\simeq \sum_j \left( \omega_0 + \frac{2\pi j}{\Delta t} \right) \sum_k a_k^\dagger a_k \text{sinc}^2 \left( \omega_0 + \frac{2\pi j}{\Delta t} \right) \\
&\simeq \sum_j \sum_k \omega_k a_k^\dagger a_k \text{sinc}^2 \left( \frac{\Delta \omega_k \Delta t}{2} - \pi j \right) \\
&= \sum_k \omega_k a_k^\dagger a_k.
\end{aligned} \tag{5.20}$$

## 5.3 Interaction with far detuned field modes

Now we will show that the  $\Omega_2(t_{n+1}, t_n)$  term in Eq. (3.4) can be written as Eq. (3.8). Substituting Eq. (3.5) and Eq. (5.14) in Eq. (3.4) we find:

$$\begin{aligned}
\Omega_2 &= -\frac{1}{2\hbar^2} \int_{t_n}^{t_{n+1}} du \int_{t_n}^u dv [V_{Sf}(u), V_{Sf}(v)] \\
&= -\frac{1}{2} \int_{t_n}^{t_{n+1}} du \int_{t_n}^u dv (B(0, u) B^\dagger(0, v) L^\dagger L + B^\dagger(0, u) B(0, v) LL^\dagger - \text{H.c.}) \\
&= -\frac{1}{2} \int_{t_n}^{t_{n+1}} du \int_{t_n}^u dv \sum_{j'} \sqrt{\frac{\gamma_{j'}}{\Delta t}} \left( e^{i\frac{2\pi}{\Delta t} j' v} B(0, u) b_{n,j'}^\dagger L^\dagger L + e^{-i\frac{2\pi}{\Delta t} j' v} B^\dagger(0, u) b_{n,j'} LL^\dagger - \text{H.c.} \right) \\
&= -\frac{1}{2} \sum_{j' \neq 0} \sqrt{\frac{\gamma_{j'}}{\Delta t}} \int_{t_n}^{t_n + \Delta t} du \left[ \frac{\Delta t}{i2\pi j'} \left[ \left( e^{i\frac{2\pi}{\Delta t} j' u} - 1 \right) B(0, u) b_{n,j'}^\dagger L^\dagger L \right. \right. \\
&\quad \left. \left. - \left( e^{-i\frac{2\pi}{\Delta t} j' u} - 1 \right) B^\dagger(0, u) b_{n,j'} LL^\dagger \right] - \text{H.c.} \right] \\
&\quad - \frac{1}{2} \sqrt{\frac{\gamma}{\Delta t}} \int_{t_n}^{t_{n+1}} du \left( (u - t_n) B(0, u) b_{n,0}^\dagger L^\dagger L + (u - t_n) B^\dagger(0, u) b_{n,0} LL^\dagger - \text{H.c.} \right)
\end{aligned} \tag{5.21}$$

where in the last line we have split the sum over  $j'$  into  $j' = 0$  and  $j' \neq 0$ . Taking the last term with  $j' = 0$  as  $\Omega_{2,2}/2$ <sup>1</sup>, we can re-write it as:

$$\begin{aligned}
\frac{\Omega_{2,2}}{2} &= -\frac{1}{2} \sqrt{\frac{\gamma}{\Delta t}} \int_{t_n}^{t_{n+1}} du \left( (u - t_n) B(0, u) b_{n,0}^\dagger L^\dagger L + (u - t_n) B^\dagger(0, u) b_{n,0} LL^\dagger - \text{H.c.} \right) \\
&= -\frac{1}{2} \sqrt{\frac{\gamma}{\Delta t}} \int_{t_n}^{t_{n+1}} du \left( (u - t_n) B(0, u) b_{n,0}^\dagger L^\dagger L + (u - t_n) B^\dagger(0, u) b_{n,0} LL^\dagger \right) \\
&\quad - \frac{1}{2} \sqrt{\frac{\gamma}{\Delta t}} \int_{t_n}^{t_{n+1}} du \left( -(u - t_n) b_{n,0} B^\dagger(0, u) L^\dagger L - (u - t_n) b_{n,0}^\dagger B(0, u) LL^\dagger \right) \\
&= -\frac{1}{2} \sum_{j \neq 0} \frac{\sqrt{\gamma \gamma_j}}{\Delta t} \int_{t_n}^{t_{n+1}} du (u - t_n) \left( e^{-i\frac{2\pi}{\Delta t} j u} b_{n,0}^\dagger b_{n,j} - e^{i\frac{2\pi}{\Delta t} j u} b_{n,j}^\dagger b_{n,0} \right) [L^\dagger, L] \\
&= -\sum_{j \neq 0} \frac{i\sqrt{\gamma \gamma_j} \Delta t}{4\pi j} \left( b_{n,0}^\dagger b_{n,j} + b_{n,j}^\dagger b_{n,0} \right) [L^\dagger, L]
\end{aligned} \tag{5.22}$$

This features an effective coupling, mediated by the emitter, between the modes  $b_{n,0}$  which are the modes closer to the emitter frequency (see Fig. 3.2), to modes further detuned

<sup>1</sup>Reason for division by 2 is that some terms with  $j' \neq 0$  have the same form as  $\Omega_{2,2}/2$  which will together contribute to the exchange type coupling.

from the emitter with  $j \neq 0$ . Now again substituting Eq. (5.14) and rewriting  $\Omega_{2,2}$  in  $\Omega_{2,2}$ , we find:

$$\begin{aligned}
\Omega_2 &= -\frac{1}{2} \sum_{j' \neq 0} \sqrt{\frac{\gamma_{j'}}{\Delta t}} \int_{t_n}^{t_n + \Delta t} du \left[ \frac{\Delta t}{i2\pi j'} \left[ \left( e^{i\frac{2\pi}{\Delta t} j' u} - 1 \right) B(0, u) b_{n,j'}^\dagger L^\dagger L \right. \right. \\
&\quad \left. \left. - \left( e^{-i\frac{2\pi}{\Delta t} j' u} - 1 \right) B^\dagger(0, u) b_{n,j'} L L^\dagger \right] - \text{H.c.} \right] + \frac{\Omega_{2,2}}{2} \\
&= -\frac{\Delta t}{4\pi} \sum_{j' \neq 0} \left[ \frac{\sqrt{\gamma_{j'}}}{i j'} \left[ \sqrt{\gamma_{j'}} \left( b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L] + L^\dagger L \right) \right. \right. \\
&\quad \left. \left. + \sqrt{\gamma} \left( b_{n,0}^\dagger b_{n,j'} L L^\dagger - b_{n,0} b_{n,j'}^\dagger L^\dagger L \right) \right] - \text{H.c.} \right] + \frac{\Omega_{2,2}}{2} \\
&= -\frac{\Delta t}{4\pi} \sum_{j' \neq 0} \frac{\sqrt{\gamma_{j'}}}{i j'} \left[ \sqrt{\gamma_{j'}} \left( b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L] + L^\dagger L \right) - \text{H.c.} \right] + \Omega_{2,2} \\
&= \frac{i\Delta t}{2\pi} \sum_{j' \neq 0} \left( \frac{\gamma_{j'}}{j'} \left( b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L] + L^\dagger L \right) - \frac{\sqrt{\gamma_{j'}}}{j'} \left( b_{n,0}^\dagger b_{n,j'} + b_{n,j'}^\dagger b_{n,0} \right) [L^\dagger, L] \right). \quad (5.23)
\end{aligned}$$

We hence, find

$$\Omega_{2,1} = \frac{i\Delta t}{2\pi} \sum_{j' \neq 0} \frac{\gamma_{j'}}{j'} \left( b_{n,j'}^\dagger b_{n,j'} [L^\dagger, L] + L^\dagger L \right)$$

which induces frequency shifts (Lamb and light shifts). In section 3.3 we find the regime in which  $\Omega_{2,1}$  and  $\Omega_{2,2}$  can be considered negligible.

## 5.4 Field's reduced dynamics

We start from Eqs. (4.25)-(4.26) in the case of the field  $f$ , and inject that  $\langle V_n \rangle_S = \Delta t \mathcal{H}_f(t_n) = i\hbar \sqrt{\gamma \Delta t} (\langle \sigma_+ \rangle_{t_n} b_n - \langle \sigma_- \rangle_{t_n} b_n^\dagger)$ . We first focus on the term  $\Delta \rho_f^{(2,f)}$ . Unlike the case of the atom, this term is of order  $\mathcal{O}(\sqrt{\gamma \Delta t})$ , such that

$$\Delta \rho_f^{(2,f)} = -\frac{\gamma \Delta t}{2} [b_n \langle \sigma_+ \rangle_{t_n} - b_n^\dagger \langle \sigma_- \rangle_{t_n}, [b_n \langle \sigma_+ \rangle_{t_n} - b_n^\dagger \langle \sigma_- \rangle_{t_n}, \rho_f(t_n)]] \quad (5.24)$$

is of first order in  $\Delta t$ . This term is in particular important to compute the change of field photon number across a collision. The self-drive Hamiltonian for the field takes the explicit form:

$$\mathcal{H}_f^s(t_n) = -\frac{i\gamma}{2} \langle \sigma_z \rangle_{t_n} (\langle b_n^\dagger \rangle b_n - \langle b_n \rangle b_n^\dagger) \quad (5.25)$$

As  $\langle b_n \rangle = \mathcal{O}(\Omega \sqrt{\frac{\Delta t}{\gamma}})$  (see section 4.3.1), we see that the term  $\Delta \rho_f^{(2,S)}$  is of order  $\mathcal{O}(\Delta t^{3/2})$  and hence the field's self-drive is negligible in the continuous limit.



# Part II

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Thermodynamics



# Closed Optical Bloch Equations: Thermodynamics

Thermodynamics provides relations between energy and entropy exchanges between different systems, leading to fundamental results, such as the first and second laws, as well as practical results, such as bounds on engine efficiency [Cal98]. Generally this study begins by establishing quantities or systems that are useful or measurable and those that form sources of loss or error. For interactions between macroscopic quantities, this is clear, the large thermal environment cannot be measured while an engine (system of interest) is useful and measurable.

Quantum thermodynamics extends this theory to the quantum regime. Following the same methodology, two main strategies prevail, depending on if the systems of interest are open or closed. The “open” approach explores energy and entropy flows received by quantum open systems coupled to environments whose degrees of freedom are traced out. For example, this situation is encapsulated by the OBE discussed in section 4.1 of the previous chapter. Conversely, the “closed” approach focuses on interaction between quantum systems without tracing any degrees of freedom. For example, the 1D atom model in the previous chapter “closed” the “open” OBE dynamics. When the global dynamics is solvable, adopting this viewpoint allows keeping all information about quantum states as the systems jointly evolve. Moreover, the global evolution is autonomous, meaning that the total energy of the joint systems is a constant of motion. The closed strategy provides a solid basis to establish a consensual framework for energetic analyses. It first brought out new interpretations of the second law, where irreversibility could ultimately be traced back to the erasure of environment information and correlations between systems [LP21; ELV10].

Here, we analyze energy exchanges between the field and the atom considered in the previous chapter in the regime of the OBE. Our starting point to study the thermodynamics is the assumption that the loss in our model is only the correlations developed between the two systems, i.e., the field and the atom are individually completely measurable but joint measurements are disallowed. To do so, we develop upon a recent framework extending the concepts of work and heat within a closed, isolated bipartite system [Wei+08; HOO15; MCA22] which we dub bipartite quantum energetics (BQE). The work-like (resp. heat-like) flow is defined as the energy flow stemming from the effective unitary drive exerted by one system onto the other (resp. from the build-up of correlations between the two systems).

To make a clear distinction from the concepts of work and heat investigated in the context of open quantum systems, we dub these quantities bipartite work (*b*-work) and bipartite heat (*b*-heat). In the case of the 1D atom under study, we show that the *b*-work and *b*-heat received by the field are equal to the change of the coherent (resp. incoherent) part of the field energy. Building on this property, we propose protocols to measure directly *b*-work and *b*-heat flows through dyne or spectroscopic experiments. Remarkably, our capacity to track correlations allows us to single out an atom self-drive and its subsequent self *b*-work, which constitutes the main outcome of our model. The latter is equal to the coherent energy spontaneously radiated in the field - a quantity usually considered a contribution to the heat in the standard framework. As a consequence, we show that the *b*-heat satisfies a second law that is tighter than the one obtained with the heat and work flows defined in the open approach [Elo+20; Rod+19]. We quantitatively relate the smaller entropy production to the increase of knowledge about the field state allowed by the closure of the OBE – a knowledge that could be used to recover work and reduce energy dissipation.

We begin by reviewing the thermodynamics of the OBE. Then we introduce the BQE framework to separate the contributions of the correlations to the energetics and provide conservation laws. Next, we analyze the coarse grained dynamics of the OBE using BQE which provides useful expressions for the measurement of the *b*-work and *b*-heat. Finally, we contrast between the open and the closed thermodynamics and find a tighter bound of the second law which is attributed to the extraction of the self *b*-work.

## 6.1 Thermodynamics of Open Optical Bloch Equations

Here, we briefly recall the thermodynamic results established within the context of open OBE [Elo+20; Ali79; Bin+19; AF13]. In this textbook case, both the drive and the bath are taken as external classical entities, the atom being the open quantum system under study. Thus, the atom and the field play fundamentally different roles. The atom internal energy entails the external drive and is defined as [Ali79]  $U(t) \equiv \text{Tr}_S \left\{ \rho_S^{(\text{lab})}(t) \left( H_S + H_D^{(\text{lab})}(t) \right) \right\}$ . Its time derivative  $\Delta U(t)/\Delta t \equiv \dot{U}(t)$  splits into two components:

$$\dot{U}(t) = \dot{W}(t) + \dot{Q}(t), \quad (6.1)$$

where

$$\dot{W} = \text{Tr}_S \left\{ \rho_S^{(\text{lab})}(t) \frac{\Delta H_D^{(\text{lab})}(t)}{\Delta t} \right\}, \quad (6.2)$$

is the work flow received by the atom and  $H_D^{(\text{lab})}(t) = (i\hbar\Omega/2)(e^{-i\omega_L t}\sigma_+ - e^{i\omega_L t}\sigma_-)$  stands for the external drive. The heat flow  $\dot{Q}(t)$  is defined as equal and opposite to the time derivative of the bath energy [ELV10]. Within the approximations leading to the OBE, it reads [Elo+20]

$$\dot{Q}(t) = \text{Tr}_S \left\{ \left( H_S + H_D^{(\text{lab})}(t) \right) \mathcal{L}_{\text{OBE}} \left[ \rho_S^{(\text{lab})}(t) \right] \right\}. \quad (6.3)$$

The rationale for such a splitting is that it brings out consistent expressions of the second law of thermodynamics [Elo+20]. Indeed, one recovers an equivalent of Clausius inequality by defining the quantity  $\Sigma_{\text{open}}(t) = \Delta S_S(t) - \beta Q(t)$ .  $\Delta S_S = S(\rho_S(t)) - S(\rho_S(t_0))$  is the change of the atom von Neumann entropy and  $Q(t)$  the total heat received by the atom between times  $t_0$  and  $t$ . Introducing the relative entropy  $D(\rho||\sigma) = \text{Tr}\{\rho(\log\rho - \log\sigma)\}$  between  $\rho$  and  $\sigma$ , one can rewrite  $\Sigma_{\text{open}}$  as

$$\Sigma_{\text{open}}(t) = D\left(\rho_{Sf}(t) || \rho_S(t) \otimes \rho_f^\beta\right) \geq 0. \quad (6.4)$$

Owing to the positivity of the relative entropy,  $\Sigma_{\text{open}}(t)$  has been interpreted as a thermodynamic entropy production and Eq. (6.4) as a manifestation of the second law in the quantum regime [Elo+20; LP21; ELV10].

## 6.2 Bipartite Quantum Energetics

Let us consider the general scenario described in section 4.2 of chapter 4, where quantum systems  $\rho_A$  and  $\rho_B$  interact. Importantly, the two quantum systems form an isolated bipartite system. The interaction Hamiltonian is then time independent in the Schrödinger picture, i.e., the laboratory frame and is given by  $V_{AB}^{(\text{lab})}$ . The Hamiltonian for the evolution of the bipartite system is

$$H^{(\text{lab})} = H_A + H_B + V_{AB}^{(\text{lab})}. \quad (6.5)$$

The evolution brought about by this Hamiltonian is energy conserving, i.e.,  $\mathcal{U}_{\text{tot}} \equiv \langle H^{(\text{lab})} \rangle$ , which is the initial amount of energy injected in the bipartite system, is constant in time. Without loss of generality, we will consider these systems to be initially uncorrelated, i.e., the initial state of the bipartite system  $\rho_{AB}(t_0) = \rho_A(t_0) \otimes \rho_B(t_0)$  with  $\rho_{A(B)}(t) \equiv \text{Tr}_{B(A)}\{\rho_{AB}(t)\}$ . This state evolves under a unitary  $\exp\{-iH^{(\text{lab})}t/\hbar\}$  to the state  $\rho_{AB}(t) = \exp\{-iH^{(\text{lab})}t/\hbar\}(\rho_A(t_0) \otimes \rho_B(t_0))\exp\{iH^{(\text{lab})}t/\hbar\}$ . Our task is to describe the energy exchanges taking place during this unitary interaction between the sub-system.

## 6.2.1 Energy conservation and notation

In the open approach recalled in section 6.1, energetic quantities are defined for the atom only. However, since the onset of quantum thermodynamics, the community has searched for protocols to access work and heat flows at their sources, i.e. within driving fields, batteries or baths. This quest brought out fine thermometry experiments [PK22; Kar+20], energy measurements of microwave pulses in superconducting circuits [Cot+17], of light pulses emitted by semiconducting quantum dots [Mai+23], or of mechanical vibrations in carbon nanotubes [Vig+22] to cite a few of them. These protocols naturally point towards “closing” open thermodynamic systems [MCA22; Mon+20; EL23]. From a theoretical point of view, this calls for new frameworks to characterize energy and entropy flows between parties of globally closed, isolated systems [HOO15; MCA21; Mai+23; Wei+08; CB22]. In this mindset, we now introduce and analyse energy flows in the case where the the atom on the one hand, the drive and the bath on the other, are treated as the quantum systems  $A$  and  $B$  that form an isolated bipartite system. As we will see, the bath and the drive roles are both played by the waveguide field, in line with the dynamics presented in section 6.3. To make a clear distinction from the open approach, we shall use curly symbols to denote all energetic quantities defined within this closed approach.

The internal energies of the systems  $A$  and  $B$  read  $\mathcal{U}_j(t) \equiv \text{Tr}_j \left\{ H_j \rho_j^{(\text{lab})}(t) \right\}$ , where  $j \in \{A, B\}$ , while the coupling energy term is defined as  $\mathcal{V}_{AB}(t) = \text{Tr}[V_{AB}^{(\text{lab})} \rho_{AB}^{(\text{lab})}(t)]$ . Thus Eq. (6.5) results in

$$\Delta \mathcal{U}_{\text{tot}} = \Delta \mathcal{U}_A(t) + \Delta \mathcal{U}_B(t) + \Delta \mathcal{V}_{AB}(t) = 0 \quad (6.6)$$

which is a statement of energy conservation, where we have defined  $\Delta \mathcal{O}(t) \equiv \mathcal{O}(t) - \mathcal{O}(t_0)$  for a quantity  $\mathcal{O}$ . The coupling energy  $\mathcal{V}_{AB}(t)$  arises exactly when the internal energies don't compensate each other [Bre+21; Lin+24]. Global energy conservation thus allows us to analyze individual energy changes as resulting from energy *ex*-changes between the atom, the field, and the coupling term.

## 6.2.2 First law

To treat the energetic exchanges during the unitary dynamics symmetrically, we again make the separation into uncorrelated and correlated parts as made in section 4.2 in chapter 4 following [HOO15; Wei+08; MCA21; Mai+23]. Using Eq. (4.7), the energies gained/lost by a sub-system can now be separated into *path-dependent* quantities, one that comes from

this effective unitary driving and the rest from the formation of correlations. We start with the rate of change of internal energy, or the internal energy flux for each sub-system

$$\frac{d}{dt}\mathcal{U}_{A(B)}(t) = \underbrace{-\frac{i}{\hbar}\text{Tr}_{A(B)}\left\{\left[H_{A(B)}, \mathcal{H}_{A(B)}^{(\text{lab})}(t)\right]\rho_{A(B)}(t)\right\}}_{\text{Effective Driving Contribution}} \underbrace{-\frac{i}{\hbar}\text{Tr}\left\{\left[H_{A(B)}, V_{AB}^{(\text{lab})}\right]\chi(t)\right\}}_{\text{Correlation Contribution}} \quad (6.7)$$

The effective driving component of the internal energy is entropy preserving and hence, we define bipartite-work flux or *b*-work flux as this component of the gain/loss in internal energy, i.e.,

$$\frac{d}{dt}b\mathcal{W}_{A(B)}(t) \equiv -\frac{i}{\hbar}\text{Tr}_{A(B)}\left\{\left[H_{A(B)}, \mathcal{H}_{A(B)}^{(\text{lab})}(t)\right]\rho_{A(B)}(t)\right\}. \quad (6.8)$$

*b*-Work arises by a driving exerted on one sub-system by the instantaneous state of the other. Unlike open approaches, it can be done on large environments or thermal baths too. On the other hand, the correlation component that leads to mixedness and fluctuations of *each* sub-system, we dub as the bipartite heat flux or *b*-heat flux, i.e.,

$$\frac{d}{dt}b\mathcal{Q}_{A(B)}(t) \equiv -\frac{i}{\hbar}\text{Tr}\left\{\left[H_{A(B)}, V_{AB}^{(\text{lab})}\right]\chi(t)\right\}, \quad (6.9)$$

Unlike the open approaches, here *b*-heat is defined for both sub-systems. In terms of the general idea of heat as energy “lost”, this can be thought of as the energy lost in the creation of correlations. Interestingly, we can do the same kind of separation in the dynamical contributions to the coupling energy. The coupling energy flux has an uncorrelated component  $\mathcal{V}^{\otimes}(t)$  (uncorrelated coupling energy) and a correlated component  $\mathcal{V}^{\times}(t)$  (correlated coupling energy). With these, the *b*-work and *b*-heat adhere to global conservation rules,

$$\frac{d}{dt}(b\mathcal{W}_S(t) + b\mathcal{W}_f(t) + \mathcal{V}_{Sf}^{\otimes}(t)) = 0 \text{ and} \quad (6.10)$$

$$\frac{d}{dt}(b\mathcal{Q}_S(t) + b\mathcal{Q}_f(t) + \mathcal{V}_{Sf}^{\times}(t)) = 0. \quad (6.11)$$

The explicit expressions for the (un)correlated coupling energy can be derived from the equations above. These equations are derived for and can be applied to bipartite settings such as that of chapter 3, and interactions in a beam splitter as we will see in chapter 7 amongst many others. Nevertheless, they are easily extendable to concatenated bipartite dynamics hence, opening a route to the study the energetics of multiple connected emitters where the systems can become highly correlated.

It is important to note that these definitions of *b*-work, *b*-heat, correlated and uncorrelated components of coupling energy are not only path-dependent, but also frame

dependent. For example, a correlation matrix does not exist in the Heisenberg frame. Nevertheless, we can show that, any frame which does not change the type of correlations (such as the standard frame with respect to the bare Hamiltonians), does not alter the definition of the correlated and uncorrelated components of internal energies or the coupling energy.

This approach of defining work and heat has several advantages. In quantum optical setups, they have natural consequences and hence, are directly measurable [Mai+23] (see section 6.5). They can be related to the concepts such as ergotropy used in charging of batteries [MCA21]. These are mainly true because, remarkably the one-dimensional nature of the light field provides full access to its state, a capacity that we now exploit to measure its energy change as well as the  $b$ -work and  $b$ -heat exchanges in the driven-dissipative setting of section 6.1.

### 6.2.3 BQE for coarse-grained dynamics

Before the end of this section, we go a step further and introduce the energetics at the coarse-grained level. Building on the analysis of a collision performed in section 4.3 of the previous chapter, let us now recast the energy changes locally experienced by the atom and the field during the  $n^{\text{th}}$  collision into  $b$ -work and  $b$ -heat. Discretization of (6.8) and (6.9) leads to:

$$\Delta[b\mathcal{W}_j](t_n) = \text{Tr}_j \{ H_j \Delta_n \rho_j^{\otimes} \} \quad (6.12)$$

and

$$\Delta[b\mathcal{Q}_j](t_n) = \text{Tr} \{ H_j \Delta_n \rho^{(2,x)} \}. \quad (6.13)$$

We have introduced  $\Delta_n \rho_{S(f)}^{\otimes} = \text{Tr}_{f(S)} \{ \Delta_n \rho^{\otimes} \}$  with  $\Delta \rho^{\otimes}$  defined in Eq. (4.28). This makes it clear that the amount of energy  $\Delta[b\mathcal{W}]_j(t_n)$  received by the atom (resp. the field) stems from the effective unitary exerted by the field (resp. the atom), and is therefore entropy-preserving. To form a comparison with the concept of work established for open systems, we write the energy flow in the continuous time limit as  $b\dot{\mathcal{W}}_j(t) \equiv \Delta[b\mathcal{W}]_j(t)/\Delta t$  and  $b\dot{\mathcal{Q}}_j(t) \equiv \Delta[b\mathcal{Q}]_j(t)/\Delta t$ . From here we will analyze the energy exchanged at the coarse grained level.

## Changing frames in BQE

As we have already noted, the BQE framework is path-dependent. The splitting of energy into  $b$ -work and  $b$ -heat are respectively entropy preserving and entropy changing. The simplest way to understand the dependence on the frame is to look at the  $b$ -work exchanged in different frames.

Let us consider an arbitrary unitary transformation  $U_{AB}$  that is used to change the frame from the laboratory frame. We will denote the operators in this frame with a superscript  $I$ . Written in this frame, the  $b$ -work done of the system  $A$  reads,

$$\begin{aligned}
 b\dot{W}_A(t) &= -\frac{i}{\hbar} \text{Tr}_A \left\{ \left[ H_A, \mathcal{H}_A^{(\text{lab})}(t) \right] \rho_A(t) \right\} \\
 &= -\frac{i}{\hbar} \text{Tr} \left\{ \left[ U_{AB} H_A U_{AB}^\dagger, U_{AB} V_{AB}^{(\text{lab})} U_{AB}^\dagger \right] U_{AB} (\rho_A(t) \otimes \rho_B(t)) U_{AB}^\dagger \right\} \\
 &= -\frac{i}{\hbar} \text{Tr} \left\{ \left[ H_A^I, V_{AB}^I \right] U_{AB} (\rho_A(t) \otimes \rho_B(t)) U_{AB}^\dagger \right\} \\
 &= -\frac{i}{\hbar} \text{Tr} \left\{ \left[ H_A^I, V_{AB}^I \right] \rho_{AB}^I(t) \right\}
 \end{aligned}$$

Notice from the last line that the  $b$ -work can no longer be written in terms of just a effective driving component, i.e.,  $b\dot{W}_A(t) \neq -\frac{i}{\hbar} \text{Tr} \left\{ \left[ H_A^I, \mathcal{H}_A^I \right] \rho_A^I(t) \right\}$ . This is because an arbitrary frame transformation can change the correlations present in the state. In general we have  $\rho_{AB}^I(t) = U_{AB} (\rho_A(t) \otimes \rho_B(t)) U_{AB}^\dagger = \rho_A^I(t) \otimes \rho_B^I(t) + \chi^I(t)$  and so the  $b$ -work written in this frame has an entropy changing component

$$\dot{W}_A(t) = -\frac{i}{\hbar} \text{Tr} \left\{ \left[ H_A^I, V_{AB}^I \right] \rho_A^I(t) \otimes \rho_B^I(t) \right\} - \underbrace{\frac{i}{\hbar} \text{Tr} \left\{ \left[ H_A^I, V_{AB}^I \right] \chi^I(t) \right\}}_{\text{Entropy changing component}}.$$

This becomes striking when transforming to the Heisenberg frame, where all the contribution of the correlations are in the definition of  $b$ -work and  $b$ -heat does not appear.

To remedy this, we consider only frame transformations of the form  $U_{AB} = U_A \otimes U_B$  such that the frame transformation does not create correlations, i.e.,  $\chi^I(t) = \mathbf{0}$ . In this case, one can use the expressions of  $b$ -work and  $b$ -heat while also maintaining the property that they are entropy preserving and changing respectively. For frame transformations which change correlations, one must also consider the correlations created between observables in that frame. In section 8.4 of chapter 8 we relate the  $b$ -work and  $b$ -heat to averages and fluctuations of observables in the Heisenberg frame.

## 6.3 Energetics of Closed OBE

As argued above,  $b$ -work and  $b$ -heat represent two different ways of exchanging energy between coupled quantum systems. In this section, we show that these concepts are operationally relevant. We first specify the conditions of validity of our analysis. We then propose two experimental protocols to measure these energy flows, respectively based on the statistical and spectral properties of the 1D field. To keep the text light, technical calculations are presented in Chapter 8.

### 6.3.1 Scope of the analysis

All proceeding equalities provided in this chapter are exact at steady state, or when the field is resonant with the atom (i.e.  $\omega_L = \omega_0$ ). In all other cases, those equalities hold up to an error of order  $\mathcal{O}(\gamma\Omega, \gamma\delta)$  on the field energy flows (while leading terms of those flows typically scale with  $\gamma\omega_0$ ) unless stated otherwise. We first relate the internal energy flows to the input and output field moments:

$$\dot{U}_S(t) = -\hbar\omega_0(\langle b_{\text{out}}^\dagger b_{\text{out}}(t) \rangle - \langle b_{\text{in}}^\dagger b_{\text{in}}(t) \rangle) \quad (6.14)$$

and

$$\dot{U}_f(t) = \hbar\omega_L(\langle b_{\text{out}}^\dagger b_{\text{out}}(t) \rangle - \langle b_{\text{in}}^\dagger b_{\text{in}}(t) \rangle). \quad (6.15)$$

In using Eq. (6.15) to measure the energy of the field, we have assumed that only frequencies around  $\omega_L$  matter. This gives a relative error of  $\mathcal{O}(\Omega/\omega_0 + (\omega_L - \omega_0)/\omega_0)$  which is negligible in the regime of the OBE. Nevertheless when the drive is resonant with the atom, one gets  $\dot{U}_S(t) = -\dot{U}_f(t)$ : the energy received by the atom equals the energy change of the field, and the whole situation can be interpreted as a simple energy exchange between light and matter. In the general case where the drive is not resonant with the atom, however, the unbalance between  $\dot{U}_f$  and  $\dot{U}_S$  is compensated by a non-zero coupling term  $\dot{\mathcal{V}} = -\dot{U}_f - \dot{U}_S \neq 0$ . This term leads to the simultaneous fulfilment of the conservation of the total energy and of the number of photons; it has been interpreted as a binding energy between the atom and the field [Elo+20]. Remarkably, the one-dimensional nature of the light field provides full access to its state, a capacity that we now exploit to measure its energy change as well as the  $b$ -work and  $b$ -heat exchanges.

### 6.3.2 Field statistics

As a first important result, we show in section 8.2 that the splitting between  $b$ -work and  $b$ -heat flows exactly maps the splitting between mean values and fluctuations of the field energy flow. Namely, the  $b$ -work flows read

$$b\dot{\mathcal{W}}_S(t) = -\hbar\omega_0 (|\langle b_{\text{out}}(t) \rangle|^2 - |\langle b_{\text{in}}(t) \rangle|^2) \quad (6.16)$$

$$b\dot{\mathcal{W}}_f(t) = \hbar\omega_L (|\langle b_{\text{out}}(t) \rangle|^2 - |\langle b_{\text{in}}(t) \rangle|^2). \quad (6.17)$$

Whether received by the atom or the field, the  $b$ -work flow can be related to the change of the field coherent energy, i.e. the energy stored in the field coherent component. This quantity is routinely extracted from dyne measurements and is therefore accessible experimentally. Notice that, at resonance, the  $b$ -work received by the atom exactly compensates the  $b$ -work received by the field. This property can be seen as a fully quantum version of the action-reaction principle.

The link between work and coherent component of the field was postulated and measured in [Cot+17; Mon+20], was demonstrated within a consistent bipartite energetic framework in [MCA21] in the case of a zero-temperature field and recently measured in [Mai+23]. Remarkably, as predicted in [Mon+20] and reported in [Mai+23], work exchanges can take place in the absence of a drive, along with spontaneous emission. Then the extracted work is equal to the coherent energy of the spontaneously emitted field, which is non-zero as soon as the atom initially carries coherences in its energy basis. As elaborated in Section 6.4.1, this kind of work directly stems from the atom self-drive - below we dub it self  $b$ -work, an instrumental quantity to capture the difference between the open and the closed approach. The framework introduced here provides a general consistency to this body of work, by extending its validity to the general case of an arbitrary drive and a finite bath temperature.

We now consider the  $b$ -heat flows. Introducing  $\delta b_{\text{out(in)}} = b_{\text{out(in)}}(t) - \langle b_{\text{out(in)}}(t) \rangle$  the average fluctuation of the input and the output fields respectively, they read

$$b\dot{\mathcal{Q}}_S(t) = \hbar\omega_0 (\langle \delta b_{\text{in}}^\dagger(t) \delta b_{\text{in}}(t) \rangle - \langle \delta b_{\text{out}}^\dagger(t) \delta b_{\text{out}}(t) \rangle) \quad (6.18)$$

$$b\dot{\mathcal{Q}}_f(t) = -\hbar\omega_L (\langle \delta b_{\text{in}}^\dagger(t) \delta b_{\text{in}}(t) \rangle - \langle \delta b_{\text{out}}^\dagger(t) \delta b_{\text{out}}(t) \rangle). \quad (6.19)$$

The  $b$ -heat flows can thus be traced back in the change of the field incoherent component. These results shed new light on phase-space deformations of the field resulting from its interaction with the atom, such as squeezing. From our energetic framework, these

deformations appear as the direct consequence of the fact that field and atom have exchanged energy through correlations - even if these correlations do not exist anymore. In this view, the  $b$ -heat has been interpreted as a witness of past correlations in [MCA21; Mai+23].

In the remainder of this section, it is instructive to consider the steady state of the OBE, which is reached after driving the atom longer than the spontaneous emission time,  $t_\infty \gg \gamma^{-1}$ . Internal energies are state functions. Hence, they verify  $\dot{U}_S(t_\infty) = \dot{U}_f(t_\infty) = 0$  such that the  $b$ -work and the  $b$ -heat flows exactly compensate each other within each subsystem,  $b\dot{W}_{S(f)}(t_\infty) + b\dot{Q}_{S(f)}(t_\infty) = 0$ . Eventually, all stationary flows can be trivially recovered from  $b\dot{W}_S(t_\infty)$ . It is convenient to introduce the saturation parameter  $s = 2\Omega^2 / (4(\omega_0 - \omega_L)^2 + \gamma^2 (2\bar{n}_{\text{th}} + 1)^2)$  [TGD92]. In terms of this parameter, the stationary work flow received by the atom reads

$$b\dot{W}_S(t_\infty) = \frac{\gamma\hbar\omega_0}{2} \left( \frac{s}{(1+s)^2} \right) \left[ 1 + s - \frac{1}{(2\bar{n}_{\text{th}} + 1)^2} \right]. \quad (6.20)$$

$b\dot{W}_S(t_\infty)$  is always positive: the atom continuously receives  $b$ -work from the field, and turns it into  $b$ -heat, i.e. energy incoherently radiated in the field. The  $b$ -work flow increases with the intensity of the drive  $\Omega$ , until it saturates at  $\gamma\hbar\omega_0/2$  where the atom reaches its maximal stationary excited state population. While increasing the bath temperature and  $\bar{n}_{\text{th}}$ , the emission of the atom becomes more and more incoherent, and the stationary  $b$ -work flow decreases. While this behavior is qualitatively similar to that of the stationary open work flow  $\dot{W}$  [Elo+20], significant differences occur when the saturation parameter is close to one. We explore these differences in depth in section 6.4.1.

### 6.3.3 Field energy spectrum

In this section, we propose an alternative way to measure the  $b$ -work and  $b$ -heat flows. Our proposal is based on the field spectral density of energy, a.k.a the field energy spectrum. It is defined at any time as  $u_f(\omega, t) \equiv \sum_k \delta_D(\omega_k - \omega) \hbar\omega_k \text{Tr}[a_k^\dagger a_k \rho(t)]$ . The change of energy spectrum between the initial time  $t_0$  and the final time  $t_\infty$  reads  $\Delta u_f(\omega) = u_f(\omega, t_\infty) - u_f(\omega, t_0)$ . Throughout this section we consider  $t_\infty$  sufficiently large such that the stationary regime of the OBE is reached. As shown in section 8.3, it is possible to rewrite this change of spectrum as:

$$\Delta u_f(\omega) = \frac{\gamma\hbar\omega}{2\pi\gamma[\omega]} \int_{t_0}^{t_\infty} du \int d\tau e^{-i(\omega-\omega_0)\tau} \left[ \langle b_{\text{out}}^\dagger(u+\tau) b_{\text{out}}(u) \rangle - \langle b_{\text{in}}^\dagger(u+\tau) b_{\text{in}}(u) \rangle \right]. \quad (6.21)$$

Following Eqs. (6.17) and (6.19), we split the correlation functions appearing in Eq. (6.21) into their mean values and fluctuations, giving rise to the spectra corresponding to the  $b$ -work and  $b$ -heat flows received by the field:

$$b\dot{w}_f(\omega) \equiv \hbar\omega\dot{S}^\otimes(\omega) = \frac{\gamma\hbar\omega}{2\pi\gamma[\omega]} \int d\tau e^{-i(\omega-\omega_0)\tau} [\langle b_{\text{out}}^\dagger(t+\tau) \rangle \langle b_{\text{out}}(t) \rangle - \langle b_{\text{in}}^\dagger(t+\tau) \rangle \langle b_{\text{in}}(t) \rangle], \quad (6.22)$$

$$b\dot{q}_f(\omega) \equiv \hbar\omega\dot{S}^x(\omega) = \frac{\gamma\hbar\omega}{2\pi\gamma[\omega]} \int d\tau e^{-i(\omega-\omega_0)\tau} [\langle \delta b_{\text{out}}^\dagger(t+\tau) \delta b_{\text{out}}(t) \rangle - \langle \delta b_{\text{in}}^\dagger(t+\tau) \delta b_{\text{in}}(t) \rangle]. \quad (6.23)$$

Integrating Eq. (6.21) (resp. Eqs (6.22) and (6.23)) over  $\omega$  naturally yields  $\int d\omega \Delta u_f(\omega) = \mathcal{U}_f(t_\infty) - \mathcal{U}_f(t_0)$ , (resp.  $\int d\omega b\dot{q}_f(\omega) = b\dot{Q}_f(t_\infty)$  and  $\int d\omega b\dot{w}_f(\omega) = b\dot{W}_f(t_\infty)$ ).

Let us rewrite these spectra as a function of atomic steady state observables. The spectrum of the work flow reads (see section 8.4)

$$b\dot{w}_f(\omega) = -\delta_D(\omega_L - \omega) \frac{\gamma\hbar\omega}{\gamma[\omega]} \left( \frac{\Omega}{2} \langle \sigma_x \rangle_{t_\infty} - \gamma |\langle \sigma_- \rangle_{t_\infty}|^2 \right). \quad (6.24)$$

Equation (6.24) reveals that  $b\dot{w}_f(\omega)$  features only one spectral component at the driving frequency  $\omega_L$ . This is expected since the  $b$ -work captures the coherent component of the field radiated by the atom, which has the drive frequency. This component involves two terms respectively scaling like the Rabi frequency  $\Omega$  and the spontaneous emission rate  $\gamma$ . The former term accounts for the work exchanges with the driving field through stimulated emission. Remarkably, the latter captures work exchanges triggered by spontaneous emission. As we show in Section 6.4.1, it corresponds to a self  $b$ -work induced by the atom self-drive. Conversely, the spectrum of the  $b$ -heat flow received by the field takes the following form:

$$b\dot{q}_f(\omega) = \frac{\gamma^2\hbar\omega}{2\pi\gamma[\omega]} \int d\tau e^{-i(\omega-\omega_0)\tau} (\text{Tr} \{ \sigma_+ \mathcal{E}_S^\tau \{ \sigma_- \rho_S(t_\infty) \} \} - \bar{n}_{\text{th}} \text{Tr} \{ \sigma_+ \mathcal{E}_S^\tau \{ [\rho_S(t_\infty), \sigma_-] \} \}) - \frac{\gamma^2\hbar\omega}{\gamma[\omega]} \delta_D(\omega - \omega_L) |\langle \sigma_- \rangle_{t_\infty}|^2. \quad (6.25)$$

The first term of Eq. (6.25) corresponds to the fluctuations of the atomic dipole and gives rise to the Mollow triplet [TGD92]. The last term is equal and opposite to the self  $b$ -work flow. Remarkably this term exactly compensates for the elastic component of the Mollow triplet, i.e., the monochromatic light at frequency  $\omega_L$  of the laser emitted by the atom through stimulated or elastic scattering processes. The second term deforms the Mollow triplet due to the presence of thermal fluctuations in the field. Hence, the spectrum of

the  $b$ -heat flow solely entails the inelastic components of the Mollow triplet, i.e. three peaks of typical width  $\gamma$  respectively located at  $\omega_L$  and  $\omega_L \pm \Omega$ . By realizing a spectroscopic analysis of the input and output fields, one can experimentally distinguish the spectra of the  $b$ -work and  $b$ -heat flows at steady state. These results comprise another evidence for the operational value of our framework and enrich the set of experimental tools to characterize energy exchanges between light and matter.

## 6.4 Open vs. Closed Approach

In this section we wrap up our results to extract the main differences between the open and the closed frameworks. Let us first focus on internal energies. We recall that the atom internal energy in the open framework incorporates the coupling to the external drive,  $U(t) = \mathcal{U}_S(t) + \langle H_D^{(\text{lab})}(t) \rangle$ . Conversely, the closed framework preserves a complete symmetry between light and matter, both internal energies being defined. We show in section 8.5 that for an initially displaced thermal field, the identity

$$\dot{U}(t) = \dot{\mathcal{U}}_S + \dot{\mathcal{V}}_{Sf}(t) = -\dot{\mathcal{U}}_f(t) \quad (6.26)$$

holds exactly at any time. Hence in the open approach, the energy flow  $\dot{U}(t)$  received by the atom was postulated to be provided by the field, while this simply results from energy conservation in the closed approach. In the remainder of this section, we focus on the first and second laws of thermodynamics. We shall dub “open work” and “open heat” flows the work and heat flows defined in the open approach.

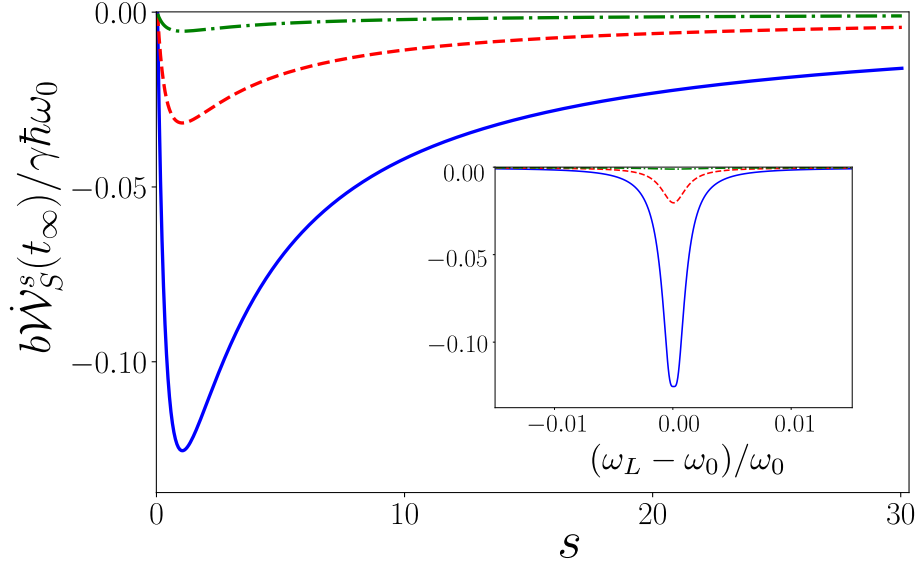
### 6.4.1 Self $b$ -work

We first consider the resonant case and rewrite the  $b$ -work flow received by the atom as

$$\begin{aligned} b\dot{\mathcal{W}}_S(t) &= -\frac{i}{\hbar} \text{Tr}_S \{ [H_S, H_D(t) + \mathcal{H}_S^s(t)] \rho_S(t) \} \\ &= \dot{W} + b\dot{\mathcal{W}}_S^s(t). \end{aligned} \quad (6.27)$$

We have recalled the expression of the open work flow

$$\dot{W} = -\frac{i}{\hbar} \text{Tr}_S \{ [H_S, H_D(t)] \rho_S(t) \} = \text{Tr}_S \left\{ \rho_S(t) \frac{dH_D(t)}{dt} \right\}.$$



**Fig. 6.1:** Resonant self  $b$ -work flow and at steady state in units of  $\gamma \hbar \omega_0$  against the saturation parameter  $s$  with  $\bar{n}_{\text{th}} = 0$  (solid blue line), 0.5 (dashed red line), 2 (dash-dotted green line). Inset: Self  $b$ -work flow at steady state in units of  $\gamma \hbar \omega_0$  against the detuning, with  $\Omega/\gamma = 1/\sqrt{2}$ .

It differs from the  $b$ -work flow by

$$b\dot{\mathcal{W}}_S^s(t) \equiv -\frac{i}{\hbar} \text{Tr}_S \{ [H_S, \mathcal{H}_S^s(t)] \rho_S(t) \} = -\gamma \hbar \omega_0 |\langle \sigma_- \rangle|^2. \quad (6.28)$$

This quantity corresponds to the work flow induced by the self-drive, that we have introduced above as the self  $b$ -work flow. It appears that  $b\dot{\mathcal{W}}_S^s(t)$  is always negative: this is expected as the atom cannot provide work to itself. The same kind of relation can be derived between the open heat and the  $b$ -heat flows, yielding

$$-b\dot{\mathcal{Q}}_S(t) - (-\dot{\mathcal{Q}}(t)) = b\dot{\mathcal{W}}_S^s(t) = b\dot{\mathcal{Q}}_f(t) - (-\dot{\mathcal{Q}}(t)). \quad (6.29)$$

The interpretation of the difference between the open and the closed approach is transparent. In the former, all the energy received by the field is counted as heat, while in the latter, part of the energy received by the field corresponds to the self  $b$ -work. It is the energetic consequence of the fact that the self-drive, despite being unitary, is included in the dissipator of the open OBE. Note that this difference is also visible in Eq. (6.25): integrating the last term gives rise to the steady state flow of self  $b$ -work, while integrating the first term, i.e. the Mollow triplet, gives rise to the negative of the open heat flow  $-\dot{\mathcal{Q}}^\infty$ .

We now focus on the steady state regime. In this regime Eqs. (6.27) is modified for non-resonant fields (see section 8.6) such that

$$b\dot{\mathcal{W}}_f(t) = -\dot{W} - \frac{\omega_L}{\omega_0} b\dot{\mathcal{W}}_S^s(t), \quad (6.30)$$

where, work and self-work are now done at the frequency of the field  $\omega_L$  (see also Eq. (4.39)). Hence the steady-state self  $b$ -work flow captures the difference between open- and closed-system energetic analysis, even for non-resonant driving fields. It takes a simple analytical expression as a function of the saturation parameter  $s$ :

$$b\dot{\mathcal{W}}_S^s(t_\infty) = -\gamma\hbar\omega_0|\langle\sigma_-\rangle_\infty|^2 = \frac{-\gamma\hbar\omega_0}{2(2\bar{n}_{\text{th}} + 1)^2} \left( \frac{s}{(1+s)^2} \right). \quad (6.31)$$

Figure 6.1(a) displays the flow of self  $b$ -work as a function of  $s$ . It vanishes for  $s \rightarrow 0$  where the drive is too small to stabilize any atomic coherence against the thermal component of the field. The self  $b$ -work also vanishes in the opposite case  $s \rightarrow \infty$ . Here stimulated emission takes over and coherent emission in the driving mode dominates over empty modes. In these two limits, there is no quantitative difference between the open and the closed approaches. Conversely, the self  $b$ -work reaches the maximal value of  $-\gamma\hbar\omega_0/8(2\bar{n}_{\text{th}} + 1)^2$  for  $s = 1$ , i.e.  $\Omega = \gamma(2\bar{n}_{\text{th}} + 1)/\sqrt{2}$ . This is where the drive is large enough to induce atomic coherence, but not enough to switch off spontaneous emission. This situation is ideal to explore the deep quantum regime of OBE as already noticed in [Elo+20]. Finally Fig. 6.1(b) shows the self  $b$ -work flow as a function of the detuning for  $\Omega/\gamma = 1/\sqrt{2}$  and different values of  $\bar{n}_{\text{th}}$ .

## 6.5 Measurement of Bipartite-work during Spontaneous Emission

Let us now, briefly summarize how  $b$ -work was measured in the case of a spontaneously emitting qubit in [Mai+23]. The  $b$ -work is experimentally accessed during spontaneous emission through self-homodyne measurements with the emitted light field.

The qubit is a quantum dot (QD) coupled to a micropillar cavity. This system operates in the weak coupling regime where the qubit is effectively coupled to a one-dimensional continuum of optical modes. This represents the 1D atom system that we have already discussed. In order to observe spontaneous emission, we first, the resonantly excite a qubit by a laser drive and bring it into the superposition state  $\cos(\theta/2)|g\rangle + e^{i\alpha}\sin(\theta/2)|e\rangle$  where,  $\theta$  is the pulse area and  $\alpha$  is the classical phase of the driving laser. The driving

laser is then switched off and after time of a few  $\gamma^{-1}$ , all the energy is transmitted by spontaneous emission to the vacuum field around the qubit. This emitted field is given by  $|\psi_f\rangle = \cos(\theta/2)|0\rangle + e^{i\alpha}\sin(\theta/2)|1\rangle$  with energy  $\mathcal{U}_f$ . By integrating Eq. (6.17), it is easy to see that the  $b$ -work transferred during the spontaneous emission process is given by  $b\mathcal{W}_{\text{spn}} = \hbar\omega_0 \cos^2(\theta/2) \sin^2(\theta/2)$  as the input field is vacuum. We show in chapter 8 that,  $b$ -work corresponds to a change in the coherent part of the field. Moreover, if spontaneous emission is seen as a charging process of the vacuum field, this is the maximum amount of coherence that can be discharged into another field via a beam splitter (see section 7.3). Nevertheless, in order to measure this  $b$ -work, we instead use the emitted field to do a self-homodyne measurement. This constitutes interfering two copies of the emitted field on a balanced beam splitter. The two input states are identical except for a relative phase  $\varphi$  which is due to the difference in path lengths. The visibility of the output fields is proportional to the  $b$ -work. Taking the input and output operators of the beam splitter as  $c_{1(2)}$  and  $c_{3(4)}$ , we find that the visibility is:

$$v = \frac{2\text{Re} \left\{ \int dt e^{i\varphi} \langle c_1^\dagger(t) c_2(t) \rangle \right\}}{\int dt \left( \langle c_3^\dagger c_3(t) \rangle + \langle c_4^\dagger c_4(t) \rangle \right)} = \frac{2\hbar\omega_0 \cos(\varphi) \int dt |\langle c_1(t) \rangle|^2}{2\mathcal{U}_f} = \frac{\hbar\omega_0 \cos(\varphi) b\mathcal{W}_{\text{spn}}}{\mathcal{U}_f}, \quad (6.32)$$

where, in the second equality we have used that the two interfering fields are the same apart from the phase. The visibility of the interference varies with the relative phase between the two inputs,  $\varphi$ , which reaches maximum constructive (destructive) interference when  $\varphi = 0(\pi)$ . Hence, the  $b$ -work done during a spontaneous emission process can be measured from the visibility attained by doing a self-homodyne with the emitted field.

## 6.6 Tightening the Second Law

A thermodynamic framework is expected to connect energetic and entropic quantities, thus providing useful quantitative relations, e.g. between irreversibility (as quantified by entropy production) and energy dissipation. In this last section, we show that the bipartite energetic framework leads to a tighter expression of the second law. We relate the reduction of entropy production to an increased knowledge about the output field state, that could be exploited to recycle energy.

Being more specific, the entropy production in the open approach,  $\Sigma_{\text{open}}(t) = D(\rho_{Sf}(t) || \rho_S(t) \otimes \rho_f^\beta)$  [Eq. (6.4)] can be decomposed as

$$\Sigma_{\text{open}}(t) = I_{Sf}(t) + D\left(\rho_f(t) || \rho_f^\beta\right), \quad (6.33)$$

where  $I_{Sf}(t) = S(\rho_S(t)) + S(\rho_f(t)) - S(\rho_{Sf}(t))$  is the mutual information between the atom and the field at time  $t$ . The splitting in Eq. (6.33) suggests an information-theoretic interpretation of the second law, in which irreversibility is attributed to the lack of information about the joint atom-field state at time  $t$ , as quantified by the right-hand side terms [ELV10; LP21]. The term  $I_{Sf}$  accounts for the ignorance on the atom-field correlations. Conversely, the relative entropy  $D(\rho_f(t) \parallel \rho_f^\beta)$  measures the informational distance between the actual output field at time  $t$  and the estimate of its state, i.e. the thermal equilibrium state.

In contrast, the closed approach contains more information about the output field, as it keeps track of the imprint left by the atom through the driving term  $\mathcal{H}_f(t)$ . This additional information modifies the field reference state appearing in Eq.(6.33), thus the entropy production. We have computed the field displacement induced by the atom between the initial time  $t_0$  and the final time  $T$  (see section 8.7):

$$\mathcal{D}(T) = \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^T ds \mathcal{H}_f(s) \right\} = \bigotimes_{n=0}^N \mathcal{D}(\varphi_n), \quad (6.34)$$

with  $\mathcal{D}(\varphi_n) = \exp\{(\varphi_n b_n^\dagger - \varphi_n^* b_n)\}$  and  $\varphi_n = -\sqrt{\gamma \Delta t} \langle \sigma_- \rangle_{t_n}$ . This leads us to consider the state  $\mathcal{D}(T) \rho_f^\beta \mathcal{D}^\dagger(T)$  as our new field reference state to describe our estimation about the field. Therefore, the entropy production is modified in the following way

$$\begin{aligned} \Sigma_{\text{closed}} &= D \left( \rho_{Sf}(T) \parallel \rho_S(T) \otimes \mathcal{D}(T) \rho_f^\beta \mathcal{D}^\dagger(T) \right) \\ &= D \left( \rho_{Sf}(T) \parallel \rho_S(T) \bigotimes_n \mathcal{D}(\varphi_n) \eta_n^\beta \mathcal{D}^\dagger(\varphi_n) \right) \\ &= I_{Sf}(T) + D \left( \rho_f(T) \parallel \bigotimes_n \mathcal{D}(\varphi_n) \eta_n^\beta \mathcal{D}^\dagger(\varphi_n) \right). \end{aligned} \quad (6.35)$$

Crucially, this entropy production gives rise to an inequality featuring the  $b$ -heat received by the field (see section 8.7 for the derivation):

$$\Sigma_{\text{closed}} = \Delta S_S + \beta b Q_f \geq 0, \quad (6.36)$$

which we interpret as the formulation of the second law consistent with our closed approach. At resonance, we retrieve a Clausius-like inequality, where  $\Sigma_{\text{closed}} = \Delta S_S - \beta b Q_S \geq 0$ . In this case, using Eq. (6.29), it is straightforward to notice that  $-Q(T) \geq -b Q_S(T)$  as  $b \mathcal{W}_S^g \leq 0$  and hence,  $\Sigma_{\text{closed}} \leq \Sigma_{\text{open}}$ . In general,  $-Q(T) \geq b Q_f(T)$  (see section 8.6.1), and hence,  $\Sigma_{\text{closed}} \leq \Sigma_{\text{open}}$ , leading to a smaller entropy production and also a tighter expression of the second law.

This smaller entropy production is directly related to a more accurate accounting of the non-equilibrium resources present in the field state to include the work exchanges via displacements of the temporal modes. This is operationally justified as such displacements can be detected via dyne or spectral measurements (see Section 6.3). As they give rise to coherent fields, they can be used in principle to coherently drive other emitters [Mai+23]. When even more knowledge (or control) about the local state of the bath can be assumed, a similar approach was used by some of us to derive an expression of the second law in general autonomous quantum setups, accounting for all non-equilibrium resources [EL23].

In order to take the thermodynamic analysis of optical interaction a step forward, in the next chapter we will look at the energy exchanges occurring between thermal baths. This will help us develop further intuitions about the BQE framework developed in this chapter.



# Thermodynamics of Beam Splitter Operations

The interference of two electromagnetic fields in a beam splitter (BS) has been ubiquitous in quantum optics since its early history. It plays a crucial role in the implementation of photon-photon gates and in the generation of cluster states for quantum computation, in quantum simulators, and boson samplers, hence representing an essential building block for all optical-based quantum technologies [Wan+20]. Owing to its simplicity, the energetic analysis of a BS interference has already been the subject of fundamental inquiry in quantum thermodynamics [AMD23].

Here we present a consistent energetic analysis of the BS interaction based on BQE introduced in Chapter 6. As we saw in the previous chapters, it allowed us to analyze the impact of the correlations' built up between interacting systems. Here we apply this framework to extract simple intuitions for linear interactions.

Applying this paradigm to the BS interference, the definitions of  $b$ -work and  $b$ -heat assume a clear operational meaning: they are respectively produced by the change of the first, and the second moments of the fields' phase-space distributions, e.g. their Husimi functions. Furthermore, we apply the earlier formulation of entropy production presented for 1D atoms and show that even in this setting, extracting  $b$ -work leads to a smaller entropy production and a tighter Clausius-like inequality than the one obtained without extracting it. Merging the energetic analysis of the two settings has already proven useful to study the role of coherence in the charge and discharge of an optical quantum battery fueled by a quantum dot [Mai+23].

We begin by applying BQE on the interference of light fields. Next we consider the interference of Gaussian fields. We show that these are fully described by their Husimi functions, the proposed analysis produces results that match classical intuitions: coherent fields exchange only  $b$ -work, thermal fields exchange only  $b$ -heat. In the first case no entropy is produced in the process, in the second a positive entropy production is given by the  $b$ -heat flowing from the hotter to the colder field. We then present the charge and discharge of an optical quantum battery and find useful bounds on  $b$ -work that can be extracted. Finally, we compute the entropy production for the interference. We show that in this framework squeezing can be interpreted as a kind of Maxwell's demon, as it allows the transfer of  $b$ -heat from a colder to a hotter thermal field [Nie+16; Nie+18], without violating the second law.

## 7.1 BQE of Beam Splitter Interference

The framework described in the previous chapter can be applied to an integral part of an optical network, i.e., the beam splitter (BS). It can be thought of as a highly leaky cavity providing an effective interaction between two waveguides attached to it, where each waveguide itself forms an input and an output port of the BS. Let us consider the interaction of two resonant monochromatic optical modes (or plane waves)  $A$  and  $B$  via a BS. The interaction Hamiltonian is the exchange interaction between these two modes with strength  $g$ :

$$H_I = i\hbar g (h^\dagger v - v^\dagger h), \quad (7.1)$$

where,  $h$  and  $v$  represent the annihilation operator of each of the optical modes (or plane waves), and  $H_I$  is in the interaction picture with respect to the fields. The action of the evolution operator at time  $t$ ,  $U_{\text{BS}}(t) \equiv \exp\{- (i/\hbar) H_I t\}$  hence, gives

$$U_{\text{BS}}(t) \begin{pmatrix} h \\ v \end{pmatrix} U_{\text{BS}}^\dagger(t) = \begin{pmatrix} \cos(\frac{\varphi}{2}) & \sin(\frac{\varphi}{2}) \\ -\sin(\frac{\varphi}{2}) & \cos(\frac{\varphi}{2}) \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}, \quad (7.2)$$

which is equivalent to the standard transfer matrix [GK04] relating the input operators  $c_{1(2)}$ , with the output operators  $c_{3(4)}$ , where 1 and 2 (3 and 4) label the input (resp. the output) ports of a beam splitter with angle  $\varphi = 2gt$ . In terms of average values we have:  $\langle h(v) \rangle_t = \langle c_{3(4)} \rangle$  and  $\langle h(v) \rangle_0 = \langle c_{1(2)} \rangle$ . As the fields have same frequency, the average value of the interaction Hamiltonian in Eq. (7.1) is equal to zero at any time. This implies that the energy change is equal and opposite and it reads

$$\begin{aligned} \Delta\mathcal{U}_A &\equiv \hbar\omega_0 \langle h^\dagger h \rangle_\tau - \hbar\omega_0 \langle h^\dagger h \rangle_0 \\ &= \hbar\omega_0 [\sin^2(\varphi/2) (\langle c_2^\dagger c_2 \rangle - \langle c_1^\dagger c_1 \rangle) \\ &\quad + \sin(\varphi) \text{Re} \langle c_2 c_1^\dagger \rangle] = \mathcal{U}_3 - \mathcal{U}_1 = -\Delta\mathcal{U}_B, \end{aligned} \quad (7.3)$$

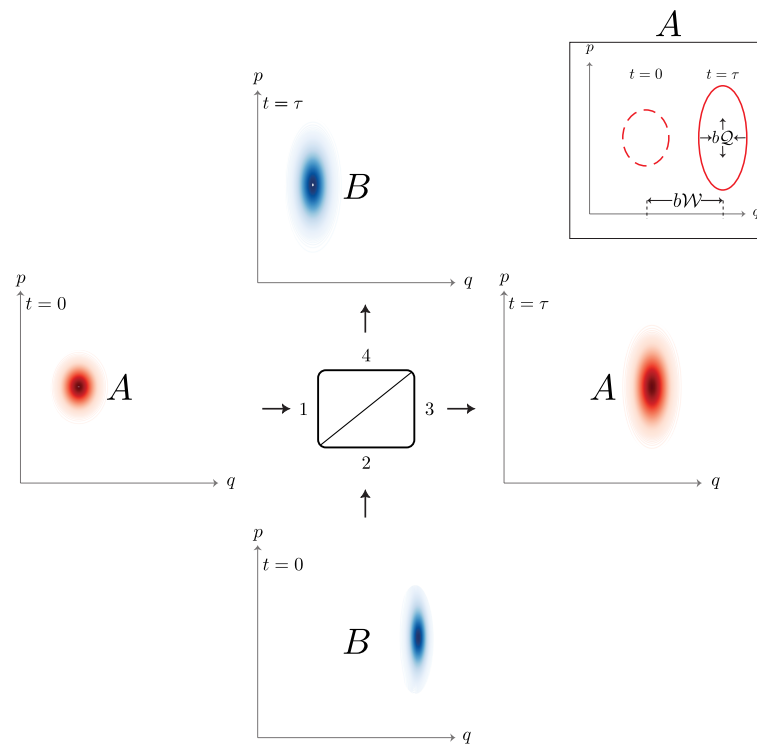
while the bipartite-work and bipartite-heat can as expected be written in terms of the input and output operators of the beam splitter

$$b\mathcal{W}_A = \hbar\omega_0 (|\langle c_3 \rangle|^2 - |\langle c_1 \rangle|^2) = -b\mathcal{W}_B, \quad (7.4)$$

$$b\mathcal{Q}_A = \hbar\omega_0 \left\{ \left( \langle c_3^\dagger c_3 \rangle - |\langle c_3 \rangle|^2 \right) - \left( \langle c_1^\dagger c_1 \rangle - |\langle c_1 \rangle|^2 \right) \right\} = -b\mathcal{Q}_B. \quad (7.5)$$

These quantities are measurable when we look at the Husimi representation of the fields in the phase space [GK04; Lou00], i.e.  $\mathbb{H}_\rho(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle$ , where  $\rho$  is an arbitrary field's state. The first moment of this distribution,  $\langle c \rangle$ , characterizes the coherent component of the field, while one of its second moments,  $\langle c^\dagger c \rangle - |\langle c \rangle|^2$ , is related to the incoherent component. The bipartite work  $b\mathcal{W}_A$  and the bipartite heat  $b\mathcal{Q}_A$  respectively, depend on how much these two moments change from the input to the output. Then, they are observable quantities, measurable experimentally through the reconstruction of the fields' Husimi distributions.

## 7.2 Gaussian Field Interference



**Fig. 7.1:** The optical modes  $A$  (red) and  $B$  (blue) are coupled by the linear mode-mixing Hamiltonian of Eq. (7.1) for a time  $\tau$ . Husimi functions of modes  $A$  and  $B$  before ( $t = 0$ ) and after ( $t = \tau$ ) the interaction,  $p$  and  $q$  are the phase-space coordinates. Inset: contour of the Husimi functions of mode  $A$  before (dotted) and after (solid) the interaction:  $b\mathcal{W}$  is associated with the displacement of the Husimi function, while  $b\mathcal{Q}$  is associated with its deformation.

To better understand the operational meaning of the BS energetics, we focus on input states fully characterized by their first and second moments, i.e., uncorrelated Gaussian fields. The most general Gaussian field can be written as a displaced squeezed thermal state

$$\rho = \mathcal{D}(\alpha) \hat{S}(z) \left( \sum_n \frac{e^{-\beta \hbar \omega n}}{Z} |n\rangle \langle n| \right) \hat{S}^\dagger(z) \mathcal{D}^\dagger(\alpha), \quad (7.6)$$

where,  $\mathcal{D}(\alpha) = \exp\{\alpha c^\dagger - \alpha^* c\}$  is the displacement operator,  $\hat{S}(z) = \exp\{\frac{1}{2}(z^* c^2 - z c^{\dagger 2})\}$  is the squeezing operator,  $Z$  is the partition function,  $\bar{n}_{\text{th}}$  is the average number of thermal excitation such that  $\beta = \ln\left(\frac{1+\bar{n}_{\text{th}}}{\bar{n}_{\text{th}}}\right) = \frac{1}{k_B T}$ , with  $T$  being the temperature. As the BS transformation is a Gaussian transformation, Gaussian input fields are transformed into Gaussian output fields, i.e. the interference only changes the values of the three parameters  $\alpha$ ,  $z$  and  $T$  of each input. The center of the Husimi function of a Gaussian field is given by the first moment of the distribution  $\langle c \rangle$ , while its shape is characterized by the covariance,  $\langle c^\dagger c \rangle - |\langle c \rangle|^2$ . Hence, in this case, the bipartite-work is the energy change due to the displacement of the fields, while the bipartite-heat is the one due to the change of their shapes in the phase-space (change of squeezing and temperature), see Fig. 7.1 inset.

It is worthy to notice that the bipartite-work exchanged among two Gaussian fields stems solely from the displacement operators, or, in other words, two thermal-squeezed fields can only exchange bipartite-heat. This becomes clear by noticing that the action of the BS transformation on the displacement operator of one of the two modes reads:

$$U_{\text{BS}}^\dagger(\tau) \mathcal{D}_A(\alpha) U_{\text{BS}}(\tau) = \mathcal{D}_A\left(-\sin\left(\frac{\varphi}{2}\right)\alpha\right) \otimes \mathcal{D}_B\left(\cos\left(\frac{\varphi}{2}\right)\alpha\right) \quad (7.7)$$

where,  $\mathcal{D}_A$  and  $\mathcal{D}_B$  are the displacement operators acting on the fields  $A$  and  $B$ . Hence, the displaced parts of Gaussian fields give rise to local unitary transformations, associated to bipartite-work exchange. When the two input fields are thermal, the 50:50 BS interaction results in an effective thermalization, as the outputs attain the same temperature being the average of the two input temperatures. In this case the energy is exchanged under the form of bipartite-heat flowing from the hotter to the colder field in agreement with the classical intuitions.

## 7.3 Charge and Discharge of an Optical Quantum Battery

Using the BS as a paradigmatic example, we have analyzed the consequences of BQE on the first law of thermodynamics. Now let us apply this framework to discuss the consequences on energy storage (extraction) into (from) batteries. We will first extract work from an arbitrary emitter into a field  $B$  which will form our battery. Then we will transfer this extracted work using a BS to another arbitrary field  $A$  which will form our receiver.

For this protocol we ask, what is the maximum work that we can transfer to the receiver? To find this, let us take the emitter such that it emits a field in vacuum  $B$ . Of course, in this case, the battery is initially empty. It is charged by the work done on it by

the emitter which is given by  $b\mathcal{W}_B = \hbar\omega_0 \int dt |\langle b_{out}(t) \rangle|^2$  (see Eq. (6.17)) where  $b_{out}(t)$  is the output temporal mode annihilation operator of  $B$ . Now we input this field into port 2 of the BS and interfere it with  $A$  which forms the input at the other port 1. We show in the section 8.9 that the work done on the field  $A$  is given by

$$b\mathcal{W}_A = \hbar\omega_0 \left[ \sin(\varphi) \operatorname{Re} \left\{ \int dt \langle c_1(t) \rangle \langle c_2(t) \rangle^* \right\} + \sin^2\left(\frac{\varphi}{2}\right) \left( \int dt |\langle c_2(t) \rangle|^2 - \int dt |\langle c_1(t) \rangle|^2 \right) \right], \quad (7.8)$$

where,  $c_1(t)$  and  $c_2(t)$  form the temporal mode annihilation operators at the input ports 1 and 2 respectively. The input at port 2 is the emitted field  $B$  where we identify  $\langle c_2(t) \rangle = \langle b_{out}(t) \rangle$ . Hence, we re-write  $b\mathcal{W}_A$  as

$$b\mathcal{W}_A = \hbar\omega_0 \sin(\varphi) \operatorname{Re} \left\{ \int dt \langle c_1(t) \rangle \langle b_{out}(t) \rangle^* \right\} + \left( b\mathcal{W}_B - \hbar\omega_0 \int dt |\langle c_1(t) \rangle|^2 \right) \sin^2\left(\frac{\varphi}{2}\right). \quad (7.9)$$

For clarity and without loss of generality we use a 50:50 BS to get that the work done on  $A$  is

$$b\mathcal{W}_A = \hbar\omega_0 \int dt \operatorname{Re} \left\{ \langle c_1(t) \rangle \langle b_{out}(t) \rangle^* \right\} - \frac{\hbar\omega_0}{2} \int dt |\langle c_1(t) \rangle|^2 + \frac{1}{2} b\mathcal{W}_B. \quad (7.10)$$

We can show that  $b\mathcal{W}_A$  is bounded above by  $b\mathcal{W}_B$ , i.e., the work extracted from the battery can't be more than the work transferred to it. Notice that  $\operatorname{Re} \left\{ \langle c_1(t) \rangle \langle b_{out}(t) \rangle^* \right\} \leq |\langle c_1(t) \rangle| |\langle b_{out}(t) \rangle|$ . Using this we maximize the first two terms of  $b\mathcal{W}_A$

$$\hbar\omega_0 \int dt \operatorname{Re} \left\{ \langle c_1(t) \rangle \langle b_{out}(t) \rangle^* \right\} - \frac{\hbar\omega_0}{2} \int dt |\langle c_1(t) \rangle|^2 \leq \hbar\omega_0 \int dt |\langle c_1(t) \rangle| \left( |\langle b_{out}(t) \rangle| - \frac{1}{2} |\langle c_1(t) \rangle| \right). \quad (7.11)$$

When the right-hand term of the inequality is maximized over  $|\langle c_1(t) \rangle|$ , we find that it attains the maximum value for  $|\langle c_1(t) \rangle| = |\langle b_{out}(t) \rangle|$  with the maximum value being  $\frac{\hbar\omega_0}{2} \int dt |\langle b_{out}(t) \rangle|^2 = \frac{1}{2} b\mathcal{W}_B$ . Hence, substituting this result in Eq. (7.10), we find that

$$b\mathcal{W}_A \leq b\mathcal{W}_B. \quad (7.12)$$

Hence, the work done by an emitter on an empty battery ( $b\mathcal{W}_B$ ) is the maximum work that can be extracted from it into a work receiver using a BS.

## 7.4 Entropy Production and Squeezing as a Resource

### 7.4.1 Extracting coherent energy

So far we have analyzed the energy exchanges in a BS interference in terms of  $b$ -work and  $b$ -heat formulating the first law of thermodynamics in the context of quantum optics. As the  $b$ -work is associated with an entropy-preserving transformation, we expect the fields entropy change to be linked to a  $b$ -heat exchange by some formulation of the second law of thermodynamics. To show that this is indeed the case, let us once again consider the fields being monochromatic Gaussian fields (see Eq. (7.6)) as in section 7.1, but now let us prepare field  $B$  in purely thermal state with inverse temperature  $\beta$ ,  $\rho_B^{\text{th}}$ .

Like in standard thermodynamic cycles, let us consider that, after the interaction, the system  $B$  undergoes thermalization with a reservoir at inverse temperature  $\beta$ . As  $B$  starts in a thermal state and undergoes thermalization after the interaction, the energy  $\Delta\mathcal{U}_B = -\Delta\mathcal{U}_A$  can be interpreted as an heat-like quantity that the system  $A$  loses into the thermal reservoir [Bin+19], hence we can set  $\Delta\mathcal{U}_B = -Q_A$ , where we used a different notation in order to distinguish this heat-like quantity from the  $b$ -heat. Thermalization leads to irreversibility of the process and hence, entropy production  $\Sigma$  which satisfies a generalized Clausius inequality:

$$\Sigma = I_{AB} + D(\rho_B(t) \parallel \rho_B^{\text{th}}) = \Delta S_A - \beta Q_A \geq 0, \quad (7.13)$$

where  $I_{AB}$  is the mutual information created during the interaction,  $D(\rho_B(t) \parallel \rho_B^{\text{th}})$  is the divergence,  $\Delta S_A$  is the change in von-Neumann entropy of  $A$ .

Now let us consider that, after the BS interaction, but before the thermalization, one applies a displacement operation,  $\mathcal{D}_B(\alpha)$ , to the output of  $B$  (port 4 of the BS) in order to unitarily extract some of its energy. In this case, the entropy production becomes

$$\Sigma_{\text{ext}}(\alpha) = I_{AB} + D\left(\mathcal{D}_B^\dagger(\alpha) \rho_B(t) \mathcal{D}_B(\alpha) \parallel \rho_B^{\text{th}}\right), \quad (7.14)$$

where  $\mathcal{D}_B^\dagger(\alpha) \rho_B(t) \mathcal{D}_B(\alpha)$  is the state of system  $B$  after the displacement operation. It is clear that, when  $\alpha = 0$ ,  $\Sigma_{\text{ext}}(0) = \Sigma$ . In section 8.10 we show that the displacement  $\mathcal{D}_B(\alpha_{BQE})$  with  $\alpha_{BQE} = -\langle c_4 \rangle$  can be used to extract all the bipartite work received by system  $B$  during the BS interaction. Expanding  $\Sigma_{\text{ext}}(\alpha_{BQE})$  we obtain another generalized

Clausius inequality that now explicitly contains  $bQ_A$  instead of  $Q_A$  (see section 8.10 for the derivation):

$$\Sigma_{\text{ext}}(\alpha_{BQE}) = \Delta S_A - \beta bQ_A \geq 0. \quad (7.15)$$

We will now show that  $\Sigma \geq \Sigma_{\text{ext}}(\alpha_{BQE})$ , i.e., Eq. (7.15) is a tighter inequality than Eq. (7.13). Indeed we have:  $-Q_A = \Delta U_B = b\mathcal{W}_B + bQ_B = b\mathcal{W}_B - bQ_A$ . Using the  $b$ -work definition in Eq. (7.4), one finds  $b\mathcal{W}_B = \hbar\omega_0 (|\langle c_4 \rangle|^2 - |\langle c_2 \rangle|^2)$ . Furthermore, as system  $B$  is initially thermal,  $|\langle c_2 \rangle|^2 = 0$  and hence  $b\mathcal{W}_B = \hbar\omega_0 |\langle c_4 \rangle|^2 \geq 0$ . Thus we have:

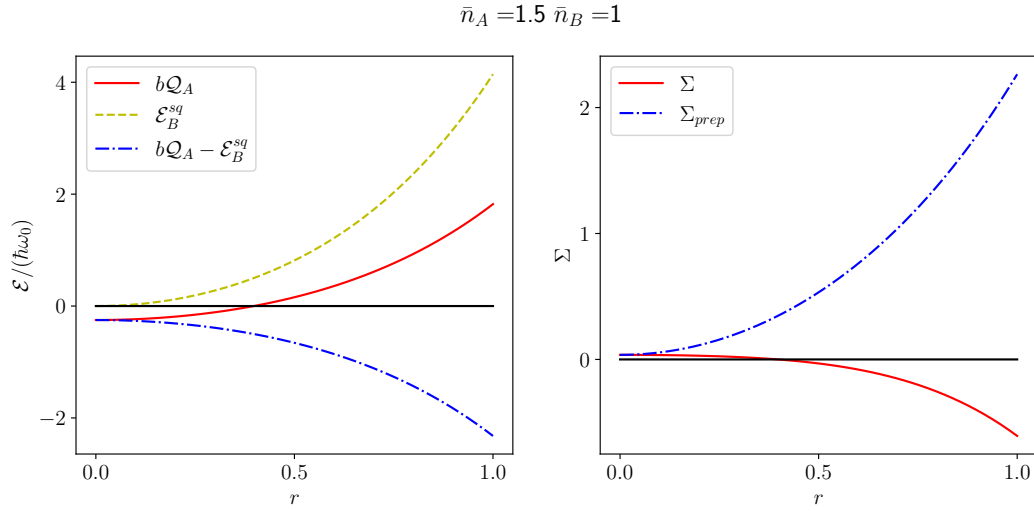
$$\begin{aligned} \Sigma &= \Delta S_A - \beta Q_A = \Delta S_A - \beta bQ_A + \beta b\mathcal{W}_B \\ &= \Sigma_{\text{ext}}(\alpha_{BQE}) + \beta b\mathcal{W}_B \\ &\geq \Sigma_{\text{ext}}(\alpha_{BQE}), \end{aligned} \quad (7.16)$$

with the equality being reached when the first moment (coherent part) of both the initial states is zero, i.e.,  $b\mathcal{W}_B = 0$ . Thus extracting from the field  $B$  the energy that it received in the form of  $b$ -work, using a local (displacement) operation, reduces the amount of entropy produced. The inequality in Eq. (7.16) can be tested experimentally by interfering a thermal field ( $B$ ) with a coherent or a displaced-thermal field ( $A$ ) by measuring: the change of energy of  $B$ , its deformation in the phase-space ( $b$ -heat) and the entropy change of  $A$ .

## 7.4.2 Squeezing as a resource

As we have seen, entropy production can be lowered by applications of specific protocols through which we gain information and (or) energy, yet it remains positive for any process. The positivity of this quantity is a way to formulate the second law of thermodynamics [Bin+19]. One could expect that positive entropy production is always equivalent to the fact that heat flows from the hotter to the colder system. However specific protocols can lead to surprising situations where this is not the case. These protocols, which apparently violate the second law, only require a more refined formulation of Clausius inequality. Indeed it has been shown that initial state preparation can make engines operate above the Carnot bound [GK14]. In the following we analyze an example of such a protocol obtained by suitably squeezing a colder thermal field before interacting it with an hotter one and explicate how violations of the second law can be resolved.

Let us first analyze a ‘‘classical’’ situation where the direction of the heat flow from the hotter to the colder system is equivalent to positive entropy production. This is two thermal fields interacting via a 50:50 BS. Let’s take their inverse temperatures to



**Fig. 7.2:** Squeezing a colder thermal field to revert the direction of the heat flow. The fields interacting in the BS are a thermal state with  $\bar{n}_A = 1.5$ , and a squeezed-thermal state with  $\bar{n}_B = 1$ , such that their inverse temperatures respect  $\beta_B < \beta_A$ . (a) Plot of  $bQ_A = Q_A$  and  $\mathcal{E}_B^{sq}$  against the (real) squeezing strength  $r$  in units of  $\hbar\omega_0$ . Importantly  $bQ_A - \mathcal{E}_B^{sq} < 0$  for all  $r$  even when  $bQ_A > 0$ . (b) Entropy production against  $r$ .  $\Sigma_{\text{prep}} > 0$  for all  $r$ , even when  $bQ_A > 0$  and  $\Sigma < 0$ .

be  $\beta_A$  and  $\beta_B$ , and their average excitation numbers to be  $N_A \equiv \langle c_1^\dagger c_1 \rangle = \bar{n}_A$ , and  $N_B \equiv \langle c_2^\dagger c_2 \rangle = \bar{n}_B$  respectively. Since the BS is 50:50, the fields attain same final excitation number  $N_f \equiv \langle c_3^\dagger c_3 \rangle = \langle c_4^\dagger c_4 \rangle = \frac{N_A + N_B}{2} = \frac{\bar{n}_A + \bar{n}_B}{2} = \bar{n}_f$ , and same final temperature  $\beta_f = \ln \left\{ \frac{1 + N_f}{N_f} \right\}$  (see section 8.8). Let us notice that, in this case the  $b$ -work is null and hence  $-\Delta\mathcal{U}_B = Q_A = bQ_A$ . Using Eq. (7.5), we find that: if  $\beta_B > \beta_A$  then  $bQ_A < 0$ , i.e. heat flows from the hotter to the colder field. Considering thermalization of system  $B$  back to temperature  $\beta_B$  (there is no  $b$ -work to be extracted after interaction), the entropy produced is  $\Sigma = \Delta S_A - \beta_B bQ_A$ . Hence in this case, the fact that heat goes from the hotter to the colder field, i.e.  $bQ_A < 0$ , is equivalent to the positivity of the entropy production.

Now, let us squeeze the field  $B$  (which we have assumed to have a lower temperature) before the BS interaction, with strength  $z = r \in \mathbb{R}$  getting  $\rho_B(t) = \hat{S}(r)\rho_B^{\text{th}}\hat{S}^\dagger(r)$  where,  $\hat{S}(r) = \exp\{(r/2)(c_2^2 - c_2^{\dagger 2})\}$  is a single mode squeezing unitary. Squeezing changes the initial average excitation number of  $B$  to  $N_B = [(\bar{n}_B + \frac{1}{2}) \cosh(2r) - \frac{1}{2}]$  by changing the second moment of the field while leaving the first one unaffected. Hence, the initial thermal field  $B$  receives energy in the form of squeezing, i.e.,  $\mathcal{E}_B^{sq} = \hbar\omega_0(N_B - \bar{n}_B)$ . Next, we make the fields interact. At the end of the 50:50 BS transformation, not only the temperatures, but also the squeezing parameters become the same for the two fields, so that they attain the same excitation number,  $N_f = \frac{N_A + N_B}{2} = \frac{\bar{n}_A + N_B}{2}$  (see section 8.8). At the end of the

interaction, the field  $B$  is thermalized back to  $\rho_B^{\text{th}}$ . Let us stress that, like before, the  $b$ -work is null ( $-\Delta\mathcal{U}_B = Q_A = b\mathcal{Q}_A$ ). Figure 7.2(a) shows the  $b$ -heat received by  $A$  for different squeezing strengths (red line). We notice that  $b\mathcal{Q}_A$  is not always negative as one would expect considering that  $\beta_B > \beta_A$ , indeed the amount of squeezing can change the direction of heat flow. In particular, as we show in section 8.10, whenever the (real) squeezing parameter satisfies

$$\frac{\bar{n}_A - \bar{n}_B}{(1 + 2\bar{n}_B)} \leq \sinh^2(r), \quad (7.17)$$

$b$ -heat flows from the colder to the hotter field, i.e.  $b\mathcal{Q}_A > 0$ . Nevertheless, the blue dashed-dotted line shows that  $b\mathcal{Q}_A < \mathcal{E}_B^{sq}$  for any real value of  $r$ .

Figure 7.2(b) shows that when  $b\mathcal{Q}_A > 0$ ,  $\Sigma = \Delta S_A - \beta_B b\mathcal{Q}_A < 0$  (red line) with an apparent violation of the second law of thermodynamics. However if we consider the entropy production of the whole protocol including the preparation stage, i.e. the initial squeezing operation, (see section 8.10):

$$\begin{aligned} \Sigma_{\text{prep}} &= I_{AB} + D(\rho_B(\tau) \parallel \rho_B^{\text{th}}), \\ &= \Delta S_A - \beta_B(b\mathcal{Q}_A - \mathcal{E}_B^{sq}) \geq 0, \end{aligned} \quad (7.18)$$

the fact that  $\mathcal{E}_B^{sq} - b\mathcal{Q}_A > 0$  ensures that the entropy production is always positive, see Fig. 7.2(b). Hence, we see that, the initial resource provided can reverse the direction of the heat flow, without violating the second law if the preparation stage of the protocol is included into the entropy production.



# Detailed Calculations: Thermodynamics of the 1D Atom and Beam Splitter

Here we will begin with the computation for the BQE quantities in terms of the atom and field operators for the 1D atom setup. We then go on to define the spectral density functions for distributed coupling strengths and the energy densities. Following this we show how we can lower the entropy produced using this setup and the collisional model. Finally we detail the calculations for the BQE of BS interference and relate entropy produced to the resources extracted or consumed.

## 8.1 Expressions of Energetic Quantities in terms of Atomic Averages for 1D Atom

Here, we compute the rate of change (fluxes) of the BQE quantities in terms of the averages of atomic operators and list them for convenience. To do so we use the definitions introduced in section 6.2 along with the balance equations and substitute the OBE (Eq. (4.3)). This gives the exact expressions. We also introduce the approximation to relate the atom and field energetic quantities. The BQE quantities are

- Atom internal energy flux:

$$\begin{aligned}
 \dot{U}_S(t) &= \text{Tr} \{ H_S \dot{\rho}_S \} \\
 &= -i \frac{\omega_0}{2} \text{Tr} \{ [\sigma_z, H_D(t)] \rho_S(t) \} + \frac{\hbar \omega_0 \gamma \bar{n}_{\text{th}}}{2} \text{Tr} \left( \sigma_z \sigma_+ \rho_S(t) \sigma_- - \frac{1}{2} \sigma_z \{ \sigma_- \sigma_+, \rho_S(t) \} \right) \\
 &\quad + \frac{\gamma \hbar \omega_0 (\bar{n}_{\text{th}} + 1)}{2} \text{Tr} \left( \sigma_z \sigma_- \rho_S(t) \sigma_+ - \frac{1}{2} \sigma_z \{ \sigma_+ \sigma_-, \rho_S(t) \} \right) \\
 &= \frac{\hbar \omega_0 \Omega}{2} \left( \langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle + \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle \right) - \gamma \hbar \omega_0 \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle \sigma_z \rangle - \frac{\gamma \hbar \omega_0}{2}.
 \end{aligned} \tag{8.1}$$

- Coupling energy flux (see section 8.5):

$$\begin{aligned}
\dot{\mathcal{V}}_{Sf}(t) &= \frac{d}{dt} \text{Tr} \{ H_D(t) \rho_S(t) \} \\
&= \frac{i\hbar\Omega}{2} \frac{d}{dt} \text{Tr} \{ (e^{-i(\omega_L - \omega_0)t} \sigma_+ - e^{i(\omega_L - \omega_0)t} \sigma_-) \rho_S(t) \} \\
&= \frac{i\hbar\Omega}{2} \text{Tr} \left\{ \frac{d}{dt} (e^{-i(\omega_L - \omega_0)t} \sigma_+ - e^{i(\omega_L - \omega_0)t} \sigma_-) \rho_S(t) \right\} + \frac{i\hbar\Omega}{2} \text{Tr} \{ (e^{-i(\omega_L - \omega_0)t} \sigma_+ - e^{i(\omega_L - \omega_0)t} \sigma_-) \dot{\rho}_S(t) \} \\
&= \frac{\hbar\Omega(\omega_L - \omega_0)}{2} (e^{-i(\omega_L - \omega_0)t} \langle \sigma_+ \rangle + e^{i(\omega_L - \omega_0)t} \langle \sigma_- \rangle) - \frac{i\hbar\Omega\gamma\bar{n}_{\text{th}}}{4} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle - \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) \\
&\quad - \frac{i\hbar\Omega\gamma(\bar{n}_{\text{th}} + 1)}{4} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle - \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) \\
&= \frac{\hbar\Omega(\omega_L - \omega_0)}{2} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle + \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) - \gamma \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle H_D(t) \rangle. \quad (8.2)
\end{aligned}$$

- Field internal energy flux:

$$\begin{aligned}
\dot{\mathcal{U}}_f(t) &= -\dot{\mathcal{U}}_S(t) - \dot{\mathcal{V}}_{Sf}(t) \\
&= -\frac{\hbar\omega_0\Omega}{2} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle + \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) + \hbar\omega_0\gamma \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle \sigma_z \rangle + \frac{\gamma\hbar\omega_0}{2} \\
&\quad - \frac{\hbar\Omega(\omega_L - \omega_0)}{2} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle + \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) + \gamma \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) \langle H_D(t) \rangle \\
&= -\frac{\hbar\omega_L\Omega}{2} (\langle \sigma_+ e^{-i(\omega_L - \omega_0)t} \rangle + \langle \sigma_- e^{i(\omega_L - \omega_0)t} \rangle) + \gamma \left( \bar{n}_{\text{th}} + \frac{1}{2} \right) (\hbar\omega_0 \langle \sigma_z \rangle + \langle H_D(t) \rangle) + \frac{\gamma\hbar\omega_0}{2}. \quad (8.3)
\end{aligned}$$

From the last lines of Eq. (8.1) and Eq. (8.3), we notice that for  $\omega_{0(L)} \gg \Omega > \gamma\bar{n}_{\text{th}}$ ,  $\dot{\mathcal{U}}_{S(f)}(t)$  largely depends on the variation of the first term. As this is in the regime of the OBE, it is straightforward to see that  $\dot{\mathcal{U}}_S = -(\hbar\omega_0/\hbar\omega_L)\dot{\mathcal{U}}_f$  is a good approximation and hence, the rate of change of the internal energy of the atom can be inferred by measuring it for the field.

- Atom bipartite work flux:

$$\begin{aligned}
b\dot{\mathcal{W}}_S(t) &= -\frac{i}{\hbar}\text{Tr}\{[H_S, \mathcal{H}_S(t)]\rho_S(t)\} \\
&= \text{Tr}\left\{\left[\frac{\hbar\omega_0}{2}\sigma_z, (\sigma_+\langle B(0,t)\rangle - \langle B^\dagger(0,t)\rangle\sigma_-)\right]\rho_S(t)\right\} \\
&= 2\hbar\omega_0\text{Re}\{\langle\sigma_+\rangle\langle B(0,t)\rangle\} \\
&= 2\hbar\omega_0\sqrt{\gamma}\text{Re}\{\langle\sigma_+\rangle\langle b_{\text{in}}(t)\rangle\} - \gamma\hbar\omega_0|\langle\sigma_-\rangle|^2 \\
&= \hbar\omega_0\Omega\text{Re}\{\langle\sigma_+e^{-i(\omega_L-\omega_0)t}\rangle\} - \gamma\hbar\omega_0|\langle\sigma_-\rangle|^2 \\
&= \frac{\hbar\omega_0\Omega}{2}(\langle\sigma_+e^{-i(\omega_L-\omega_0)t}\rangle + \langle\sigma_-e^{i(\omega_L-\omega_0)t}\rangle) - \gamma\hbar\omega_0|\langle\sigma_-\rangle|^2. \tag{8.4}
\end{aligned}$$

- Field bipartite work flux:

$$\begin{aligned}
b\dot{\mathcal{W}}_f(t) &= -\dot{W} - \dot{W}_{\text{self}} \\
&= -\text{Tr}\left\{\rho_S(t)\frac{d}{dt}H_D(t)\right\} - \text{Tr}\left\{\rho_S(t)\frac{d}{dt}\mathcal{H}_S^s(t)\right\} \\
&\quad + \frac{i}{\hbar}\text{Tr}\{\rho_S(t)[H_S, H_D(t)]\} + \frac{i}{\hbar}\text{Tr}\{\rho_S(t)[H_S, \mathcal{H}_S^s(t)]\} \\
&= -\frac{\hbar\omega_L\Omega}{2}(\langle\sigma_+e^{-i(\omega_L-\omega_0)t}\rangle + \langle\sigma_-e^{i(\omega_L-\omega_0)t}\rangle) - \frac{\gamma}{2}\langle\sigma_z\rangle\langle H_D(t)\rangle + \gamma\hbar\omega_0|\langle\sigma_-\rangle|^2. \tag{8.5}
\end{aligned}$$

- Atom bipartite heat flux:

$$\begin{aligned}
b\dot{\mathcal{Q}}_S(t) &= \dot{\mathcal{U}}_S(t) - b\dot{\mathcal{W}}_S(t) \\
&= \gamma\hbar\omega_0|\langle\sigma_-\rangle|^2 - \frac{\gamma\hbar\omega_0}{2} - \gamma\hbar\omega_0\left(\bar{n}_{\text{th}} + \frac{1}{2}\right)\langle\sigma_z\rangle. \tag{8.6}
\end{aligned}$$

- Field bipartite heat flux:

$$\begin{aligned}
b\dot{\mathcal{Q}}_f(t) &= \dot{\mathcal{U}}_f(t) - b\dot{\mathcal{W}}_f(t) \\
&= -\gamma\hbar\omega_0|\langle\sigma_-\rangle|^2 + \frac{\gamma\hbar\omega_0}{2} + \gamma\left(\bar{n}_{\text{th}} + \frac{1}{2}\right)(\hbar\omega_0\langle\sigma_z\rangle + \langle H_D(t)\rangle) + \frac{\gamma}{2}\langle\sigma_z\rangle\langle H_D(t)\rangle. \tag{8.7}
\end{aligned}$$

## 8.2 Expressions of Energetic Quantities in terms of Field Averages for 1D Atom

Here, we apply the input-output formalism presented in section 4.3.2 to derive measurable expressions for the BQE quantities for the atom and the field. From Eq. (4.17) and Eq. (5.20), taking  $\lim_{\gamma\Delta t \rightarrow 0} t_m = t$  we can easily see that for a quasi-monochromatic field when  $\omega_L \gg \Omega$ , i.e. within the regime of the OBE,  $\dot{\mathcal{U}}_f(t) = \hbar\omega_L \lim_{\gamma\Delta t \rightarrow 0} \langle b_{x_1}^\dagger(t_m)b_{x_1}(t_m) - b_{x_0}^\dagger(t_m)b_{x_0}(t_m) \rangle / (\Delta t) = \hbar\omega_L \langle b_{\text{out}}^\dagger(t)b_{\text{out}}(t) - b_{\text{in}}^\dagger(t)b_{\text{in}}(t) \rangle = \hbar\omega_L(I_{\text{out}} - I_{\text{in}})$  is a good approximation for the internal energy flux of the field. As we can relate the internal energy flux of the atom with that of the field (see the previous section), we will now compute the  $b$ -work fluxes in terms of the field operators.

### 8.2.1 Bipartite work done on the atom

Using Eq. (4.20), we rewrite the effective drive exerted by the propagating field on the atom (see Eq. (4.7)) as

$$\begin{aligned} \mathcal{H}_S(t) &= i\hbar (\langle B(0,t) \rangle \sigma_+ - \langle B^\dagger(0,t) \rangle \sigma_-) \\ &= i\hbar\sqrt{\gamma} \left( \left( \frac{\langle b_{\text{out}}(t) \rangle + \langle b_{\text{in}}(t) \rangle}{2} \right) \sigma_+ - \left( \frac{\langle b_{\text{out}}^\dagger(t) \rangle + \langle b_{\text{in}}^\dagger(t) \rangle}{2} \right) \sigma_- \right). \end{aligned} \quad (8.8)$$

Now using the commutation condition for the emitter  $[H_S, \sigma_-] = -\hbar\omega_0\sigma_-$  in Eq. (6.8) we get:

$$\begin{aligned} b\dot{\mathcal{W}}_S(t) &= -\frac{i}{\hbar} \text{Tr} \{ [H_S, \mathcal{H}_S(t)] \rho_S(t) \} \\ &= \hbar\omega_0 \text{Re} \left\{ \left( \langle b_{\text{out}}^\dagger(t) \rangle + \langle b_{\text{in}}^\dagger(t) \rangle \right) \sqrt{\gamma} \langle \sigma_- \rangle \right\}. \end{aligned} \quad (8.9)$$

Now using the input-output relation Eq. (4.18), we arrive at

$$-\sqrt{\gamma} \langle \sigma_- \rangle = \langle b_{\text{out}}(t) \rangle - \langle b_{\text{in}}(t) \rangle, \quad (8.10)$$

and hence,

$$\begin{aligned}
b\dot{\mathcal{W}}_S(t) &= -\hbar\omega_0 \text{Re} \left\{ \left( \langle b_{\text{out}}^\dagger(t) \rangle + \langle b_{\text{in}}^\dagger(t) \rangle \right) \left( \langle b_{\text{out}}(t) \rangle - \langle b_{\text{in}}(t) \rangle \right) \right\} \\
&= -\hbar\omega_0 \text{Re} \left\{ |\langle b_{\text{out}}(t) \rangle|^2 - |\langle b_{\text{in}}(t) \rangle|^2 + \langle b_{\text{in}}^\dagger(t) \rangle \langle b_{\text{out}}(t) \rangle - \langle b_{\text{out}}^\dagger(t) \rangle \langle b_{\text{in}}(t) \rangle \right\} \\
&= \hbar\omega_0 \left( |\langle b_{\text{in}}(t) \rangle|^2 - |\langle b_{\text{out}}(t) \rangle|^2 \right). \tag{8.11}
\end{aligned}$$

The above equations show that the work done on the atom by a propagating field (Eq. (6.16)) can be attributed to the change of the first moment of the latter and is measurable.

## 8.2.2 Bipartite work done on the propagating field

The effective drive exerted by the atom on the propagating field (Eq. (4.7)) reads:

$$\mathcal{H}_f(t) = i\hbar \left( B(0, t) \langle \sigma_+ \rangle - B^\dagger(0, t) \langle \sigma_- \rangle \right) \tag{8.12}$$

So the bipartite work flux (Eq. (6.8)) provided to the field is,

$$b\dot{\mathcal{W}}_f(t) = -\frac{i}{\hbar} \text{Tr} \{ [H_f, \mathcal{H}_f(t)] \rho_f(t) \} \tag{8.13}$$

First, we evaluate the commutator by using the expression of the field operators  $B(0, t) = \sum_k g_k e^{-i\Delta\omega_k t} a_k$ ,

$$\begin{aligned}
-\frac{i}{\hbar} [H_f, \mathcal{H}_f(t)] &= \left[ \sum_k \hbar\omega_k a_k^\dagger a_k, \sum_l \left( g_l e^{-i\Delta\omega_l t} a_l \langle \sigma_+ \rangle - g_l e^{i\Delta\omega_l t} a_l^\dagger \langle \sigma_- \rangle \right) \right] \\
&= -\sum_k \hbar\omega_k g_k \left[ \langle \sigma_+ \rangle e^{-i\Delta\omega_k t} a_k + \langle \sigma_- \rangle e^{i\Delta\omega_k t} a_k^\dagger \right] \\
&= -\sum_k \hbar (\Delta\omega_k + \omega_0) g_k \left[ \langle \sigma_+ \rangle e^{-i\Delta\omega_k t} a_k + \langle \sigma_- \rangle e^{i\Delta\omega_k t} a_k^\dagger \right] \\
&= -\hbar \left[ \langle \sigma_+ \rangle \sum_k \Delta\omega_k g_k e^{-i\Delta\omega_k t} a_k + \langle \sigma_- \rangle \sum_k \Delta\omega_k g_k e^{i\Delta\omega_k t} a_k^\dagger \right] \\
&\quad -\hbar\omega_0 \left[ \langle \sigma_+ \rangle B(0, t) + \langle \sigma_- \rangle B^\dagger(0, t) \right]. \tag{8.14}
\end{aligned}$$

Now we use the rate of change of the field operator  $i\dot{B}(0, t) = \sum_k \Delta\omega_k g_k e^{-i\Delta\omega_k t} a_k$  and Eq. (4.20), giving the expression of the work flux to be

$$\begin{aligned} b\dot{W}_f(t) &= -i\hbar \left[ \langle \sigma_+ \rangle \langle \dot{B}(0, t) \rangle - \langle \sigma_- \rangle \langle \dot{B}^\dagger(0, t) \rangle \right] - \hbar\omega_0 \left[ \langle \sigma_+ \rangle \langle B(0, t) \rangle + \langle \sigma_- \rangle \langle B^\dagger(0, t) \rangle \right] \\ &= 2\hbar \text{Im} \left[ \langle \sigma_+ \rangle \langle \dot{B}(0, t) \rangle \right] - 2\hbar\omega_0 \text{Re} \left[ \langle \sigma_+ \rangle \langle B(0, t) \rangle \right] \\ &= 2\hbar \text{Im} \left[ \langle \sigma_+ \rangle \langle \dot{B}(0, t) \rangle \right] - \hbar\omega_0 \sqrt{\gamma} \text{Re} \left[ \langle \sigma_+ \rangle (\langle b_{\text{in}}(t) \rangle + \langle b_{\text{out}}(t) \rangle) \right]. \end{aligned} \quad (8.15)$$

We rewrite this by taking

$$\begin{aligned} 2\hbar \text{Im} \left[ \langle \sigma_+ \rangle \langle \dot{B}(0, t) \rangle \right] &= 2\hbar \text{Im} \left\{ \langle \sigma_+ \rangle \left[ \frac{d}{dt} \langle B(0, t) \rangle - \text{Tr} \{ B(0, t) \dot{\rho}_f(t) \} \right] \right\} \\ &= 2\hbar \text{Im} \left\{ \langle \sigma_+ \rangle \left[ \frac{d}{dt} \langle B(0, t) \rangle - \text{Tr}_{S_f} \{ B(0, t) \dot{\rho}_{S_f}(t) \} \right] \right\}, \end{aligned} \quad (8.16)$$

where using Eq. (3.5) and the fact that  $[B(0, t), B^\dagger(0, t)] = \sum_k |g_k|^2$ , we get

$$\begin{aligned} 2\hbar \text{Im} \left\{ \langle \sigma_+ \rangle \text{Tr}_{S_f} \{ B(0, t) \dot{\rho}_{S_f}(t) \} \right\} &= -2\hbar \text{Im} \left\{ \frac{i}{\hbar} \langle \sigma_+ \rangle \text{Tr}_{S_f} \{ B(0, t) [V_{S_f}(t), \rho_{S_f}(t)] \} \right\} \\ &= -2\hbar |\langle \sigma_- \rangle|^2 \text{Im} \left\{ \text{Tr} \{ [B(0, t), B^\dagger(0, t)] \rho_f(t) \} \right\} \\ &= -2\hbar |\langle \sigma_- \rangle|^2 \text{Im} \left\{ \sum_k |g_k|^2 \right\} \\ &= 0. \end{aligned} \quad (8.17)$$

Hence, using Eq. (4.20), the bipartite work flux is rewritten as

$$b\dot{W}_f(t) = 2\hbar\sqrt{\gamma} \text{Im} \left\{ \langle \sigma_+ \rangle \frac{d}{dt} (\langle b_{\text{in}}(t) \rangle + \langle b_{\text{out}}(t) \rangle) \right\} - \hbar\omega_0 \sqrt{\gamma} \text{Re} \left[ \langle \sigma_+ \rangle (\langle b_{\text{in}}(t) \rangle + \langle b_{\text{out}}(t) \rangle) \right]. \quad (8.18)$$

First we notice that  $\sqrt{\gamma} \text{Im} \left\{ \langle \sigma_+ \rangle_t \frac{d}{dt} \langle b_{\text{in}}(t) \rangle \right\} = -\sqrt{\gamma}(\omega_L - \omega_0) \text{Im} \left\{ i \langle \sigma_+ \rangle_t \langle b_{\text{in}}(t) \rangle \right\}$ . We use

$$\sqrt{\gamma} \text{Im} \left\{ \langle \sigma_+ \rangle_t \frac{d}{dt} \langle b_{\text{out}}(t) \rangle \right\} \approx -\sqrt{\gamma}(\omega_L - \omega_0) \text{Im} \left\{ i \langle \sigma_+ \rangle_t \langle b_{\text{out}}(t) \rangle \right\}. \quad (8.19)$$

which is a good approximation whenever  $\Omega_R, \delta \ll \omega_0, \omega_L$  and becomes exact as we approach steady state. Using this and Eq.(8.10), we find

$$\begin{aligned} b\dot{\mathcal{W}}_f(t) &= \text{Im} \left[ i\hbar(\omega_L - \omega_0) \left( \langle b_{\text{out}}^\dagger(t) \rangle - \langle b_{\text{in}}^\dagger(t) \rangle \right) (\langle b_{\text{in}}(t) \rangle + \langle b_{\text{out}}(t) \rangle) \right] \\ &\quad + \hbar\omega_0 \text{Re} \left[ \left( \langle b_{\text{out}}^\dagger(t) \rangle - \langle b_{\text{in}}^\dagger(t) \rangle \right) (\langle b_{\text{in}}(t) \rangle + \langle b_{\text{out}}(t) \rangle) \right] \\ &= \hbar\omega_L (|\langle b_{\text{out}}(t) \rangle|^2 - |\langle b_{\text{in}}(t) \rangle|^2), \end{aligned} \quad (8.20)$$

which is the measurable expression of the rate of bipartite work done on the field by the atom, i.e. Eq. (6.17). Comparing to Eq. (8.11), we find the simple relation  $b\dot{\mathcal{W}}_S = -(\hbar\omega_0/\hbar\omega_L)b\dot{\mathcal{W}}_f$  which is exact at steady state.

### 8.2.3 Expressions of energetic quantities at steady state

By definition, at steady state  $\langle \dot{\sigma}_x \rangle_{t_\infty} = \langle \dot{\sigma}_y \rangle_{t_\infty} = \langle \dot{\sigma}_z \rangle_{t_\infty} = 0$ , implying  $\dot{\mathcal{U}}_S(t_\infty) = \dot{\mathcal{U}}_f(t_\infty) = \dot{\mathcal{V}}_{Sf}(t_\infty) = 0$ . This results in the atom and the field exchanging  $b$ -work and  $b$ -heat such that  $b\dot{\mathcal{W}}_{S(f)}(t_\infty) = -b\dot{\mathcal{Q}}_{S(f)}(t_\infty)$ . As shown in section 8.2.2, at steady state,  $b\dot{\mathcal{W}}_f(t_\infty) = -\frac{\omega_L}{\omega_0}b\dot{\mathcal{W}}_S(t_\infty)$ . Hence, the all energetic quantities at steady state can be derived from  $b\dot{\mathcal{W}}_S(t_\infty)$  and  $b\dot{\mathcal{W}}_S^s(t_\infty)$  which can be computed using Eq. (8.4). The steady state averages are expressed in terms of the saturation parameter  $s = 2\Omega^2 / (4\delta^2 + \gamma^2(2\bar{n}_{\text{th}} + 1)^2)$ :

$$b\dot{\mathcal{W}}_S(t_\infty) = -\frac{\gamma\hbar\omega_0}{2(2\bar{n}_{\text{th}} + 1)^2} \left( \frac{s}{(1+s)^2} \right) + \frac{\gamma\hbar\omega_0}{2} \left( \frac{s}{1+s} \right), \quad (8.21)$$

$$\dot{\mathcal{W}}_S^s(\infty) = -\frac{\gamma\hbar\omega_0}{2(2\bar{n}_{\text{th}} + 1)^2} \left( \frac{s}{(1+s)^2} \right). \quad (8.22)$$

## 8.3 Spectral signatures of correlations

### 8.3.1 Spectrum in terms of collision modes

In this appendix, we relate the Fourier transform of the correlation function of field at position  $x_1$ , i.e., after the interaction with the atom, to the output photon-number spectrum  $\text{Tr}\{a_k^\dagger a_k \rho_f(t_\infty)\}$ . The state  $\rho_f(t_\infty)$  is the state of the field at long times, it includes both, a large number of units that have already interacted and the ones that will interact even after the atom has already reached its limit cycle (or steady state ( $\rho_{ss}$ ) if the atom is in the frame rotating with the laser frequency  $\omega_L$ ). This field state is given by

$$\rho_f(t_\infty) = \text{Tr}_S \left\{ \Pi_{k=0}^\infty U_k(\rho_S(0) \otimes \rho_{in}) \Pi_{k=0}^\infty U_k^\dagger \right\}. \quad (8.23)$$

We use this state because the measurement of a single frequency requires access to a large number of temporal correlations created between all the interacted units. We hence, start with the Fourier transform of the steady-state field correlation function at position  $x_1$  at a frequency  $\omega_D$ . Using Eq. (5.18) we find:

$$\begin{aligned}
S_{\text{out}}(\omega_D) &= \frac{\gamma \Delta t}{2\pi\gamma[\omega_D]} \sum_{m,n} e^{-i\Delta\omega_D t_m} e^{i\Delta\omega_D t_n} \text{Tr} \{ b_{x_1}^\dagger(t_m) b_{x_1}(t_n) \rho_f(t_\infty) \} \\
&= \frac{\Delta t^2}{2\pi\gamma[\omega_D]} \sum_{m,n} \sum_{k,l} g_k g_l^* e^{-i(\omega_D - \omega_l)t_m} e^{i(\omega_D - \omega_k)t_n} \text{Tr} \{ a_l^\dagger a_k \rho_f(t_\infty) \} \\
&\quad \times e^{-i(\omega_l - \omega_k) \frac{\Delta t}{2}} \text{sinc} \left( \frac{1}{2} \Delta\omega_k \Delta t \right) \text{sinc} \left( \frac{1}{2} \Delta\omega_D \Delta t \right) \\
&= \frac{L\Delta t}{2\pi v \gamma[\omega_D]} \sum_n \sum_k g_k g_D^* e^{-i(\omega_D - \omega_k)t_n} \text{Tr} \{ a_D^\dagger a_k \rho_f(t_\infty) \} \\
&\quad \times e^{-i(\omega_D - \omega_k) \frac{\Delta t}{2}} \text{sinc} \left( \frac{1}{2} \Delta\omega_k \Delta t \right) \text{sinc} \left( \frac{1}{2} \Delta\omega_D \Delta t \right) \\
&= \text{Tr} \left\{ \sum_k \delta_D(\omega_k - \omega_D) a_k^\dagger a_k \rho_f(t_\infty) \right\} \text{sinc}^2 \left( \frac{1}{2} \Delta\omega_D \Delta t \right). \tag{8.24}
\end{aligned}$$

where  $\Delta\omega_D = \omega_D - \omega_0$ ,  $L$  is the typical length of the waveguide and  $v$  is the group velocity of the light.  $L/v$  is required to get the density of modes used in defining a Kronecker delta. This is a photon-number spectrum up to  $\text{sinc}^2(\Delta\omega_D \Delta t/2)$ . The sinc function restricts the frequencies being measured to be between  $\omega_0 - \pi/2\Delta t$  and  $\omega_0 + \pi/2\Delta t$ . As  $\Omega, (\omega_L - \omega_0) \ll \Delta t^{-1}$  in the regime of the OBEs, we find the Mollow triplet in this range of frequencies. Hence,  $S_{\text{out}}(\omega_D)$  gives the photon-number spectrum within the regime of interest. It can be re-written in terms of a delay time ( $t_p = t_m - t_n$ ):

$$S_{\text{out}}(\omega_D) = \frac{\gamma \Delta t}{2\pi\gamma[\omega_D]} \text{Re} \sum_{m,p} e^{-i\Delta\omega_D t_p} \text{Tr}_f \{ b_{m+p}^\dagger b_m \rho_f(t_\infty) \}, \tag{8.25}$$

which in the continuous limit gives:

$$S_{\text{out}}(\omega) = \frac{\gamma}{2\pi\gamma[\omega]} \iint dt d\tau e^{-i(\omega - \omega_0)\tau} \langle b_{\text{out}}^\dagger(t + \tau) b_{\text{out}}(t) \rangle_\infty. \tag{8.26}$$

For simplicity of notation, from now we don't write Re. This transformation can be made by splitting the sum in the first line of Eq. (8.24) into positive and negative time delays (using Heaviside functions) and noticing that they are complex conjugates of each other.

### 8.3.2 Separating signatures of correlations

Here, we will separate the contribution to the spectrum that appear due to the generation correlation during the interaction. We are interested in its effect on the Mollow triplet and hence, we will only consider collisions after the atom has reached its limit cycle.

To do so, let us consider that after  $n_{ss}$  collisions, the atom reaches its limit cycle. We now, split the spectrum into contributions from the transients and that from the collisions occurring after the atom has just reached the limit cycle. To do so we simply split the sum and hence, the spectrum can be split as:

$$S_{\text{out}}(\omega_D) = \frac{\gamma \Delta t}{2\pi\gamma[\omega_D]} \sum_{m,p} e^{-i(\omega_D - \omega_0)t_p} \text{Tr}_f \{ b_{m+p}^\dagger b_m \rho_f(t_\infty) \} = S_{\text{trs}}(\omega_D) + S_{\text{ss}}(\omega_D), \quad (8.27)$$

where,

$$S_{\text{trs}}(\omega_D) = \frac{\gamma}{2\pi\gamma[\omega_D]} \sum_{m=0}^{n_{ss}-1} \Delta t \sum_p e^{-i(\omega_D - \omega_0)t_p} \text{Tr}_f \left\{ b_{m+p}^\dagger b_m \text{Tr}_S \left\{ \prod_{k=0}^{n_{ss}-1} U_k \rho_S(0) \otimes \rho_{in} U_k^\dagger \right\} \right\}, \quad (8.28)$$

$$S_{\text{ss}}(\omega_D) = \frac{\gamma}{2\pi\gamma[\omega_D]} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D - \omega_0)t_p} \text{Tr}_f \left\{ b_{m+p}^\dagger b_m \text{Tr}_S \left\{ \prod_{k=n_{ss}}^{\infty} U_k \rho_S(t_m) \otimes \rho_{in} U_k^\dagger \right\} \right\}, \quad (8.29)$$

where, now we use  $t_m \geq n_{ss}\Delta t$ . We will ignore the contributions of  $S_{\text{trs}}(\omega_D)$  and only analyze  $S_{\text{ss}}(\omega_D)$ , i.e., the contributions to the spectrum after the atom is in its limit cycle. Hence,  $S_{\text{out}}(\omega_D) = S_{\text{ss}}(\omega_D)$ . In terms of the input spectrum, we can write

$$S_{\text{out}}(\omega_D) = S_{\text{in}}(\omega_D) + \frac{\gamma}{2\pi\gamma[\omega_D]} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D - \omega_0)t_p} \text{Tr}_f \{ b_{m+p}^\dagger b_m \Delta_{m+p,m} \rho_f \}, \quad (8.30)$$

where,

$$\Delta_{m+p,m} \rho_f = \text{Tr}_S \left\{ U_{m+p} \eta_{m+p} \mathcal{E}_S^{t_{m+p}-t_m} [U_m \eta_m \rho_S(t_m) U_m^\dagger] U_{m+p}^\dagger \right\}, \quad (8.31)$$

and  $\mathcal{E}_S^{t_{m+p}-t_m}$  is the evolution super-operator for the atom state over time  $t_p$  and is given by

$$\mathcal{E}_S^{t_{m+p}-t_m} [\rho_S] = \prod_{k=m}^{m+p} \text{Tr}_f \{ U_m \rho_S \eta_m U_m^\dagger \}. \quad (8.32)$$

For simplicity, we define for  $m \geq n_{ss}$ ,

$$G_{mp} = \text{Tr}_f \{ b_{m+p}^\dagger b_m \Delta_{m+p,m} \rho_f \}, \quad (8.33)$$

such that  $S_{\text{out}}(\omega_D) = S_{\text{in}}(\omega_D) + (\gamma/2\pi\gamma[\omega_D]) \sum_{m=n_{\text{ss}}}^{\infty} \Delta t \sum_p e^{-i(\omega_D - \omega_0)t_p} G_{mp}$ .  $G_{mp}$  contains the change of two collisional units of the field. To separate the correlation induced evolution during these two collisions, we first separate that for the first collision and hence find  $\Delta_{m+p,m}\rho_f = \Delta_{m+p,m}\rho_f^{\otimes} + \Delta_{m+p,m}\rho_f^{\chi}$ , where

$$\Delta_{m+p,m}\rho_f^{\otimes} = \text{Tr}_S \left\{ U_{m+p}\eta_{m+p} \mathcal{E}_S^{t_{m+p}-t_m} [\Delta_m\rho^{\otimes} + \rho_S(t_m) \otimes \eta_m] U_{m+p}^{\dagger} \right\} - \eta_{m+p} \otimes \eta_m, \quad (8.34)$$

$$\Delta_{m+p,m}\rho_f^{\chi} = \text{Tr}_S \left\{ U_{m+p}\eta_{m+p} \mathcal{E}_S^{t_{m+p}-t_m} [\Delta_m\rho^{\chi}] U_{m+p}^{\dagger} \right\}. \quad (8.35)$$

Their contribution to the spectrum is separated as  $G_{mp} = G_{mp}^{\otimes} + G_{mp}^{\chi}$  with

$$G_{mp}^{\otimes(\chi)} = \text{Tr}_f \left\{ b_{m+p}^{\dagger} b_m \Delta_{m+p,m}\rho_f^{\otimes(\chi)} \right\}. \quad (8.36)$$

We will show that this separation is enough to separate the effect of the correlation induced evolution, as the evolution induced by the correlation in the second collision has negligible contribution to  $G_{mp}^{\otimes}$ . We will now compute the two contributions to  $G_{mp}$ . Substituting Eq.(4.28) in the above we find:

$$\begin{aligned} G_{mp}^{\otimes} = & \text{Tr}_{S,m+p,m} \left\{ b_{m+p}^{\dagger} b_m U_{m+p}\eta_{m+p} \left[ \rho_S(t_{m+p}) \otimes \Delta_m\rho_f^{(1)} + \rho_S(t_{m+p}) \otimes \eta_m \right] U_{m+p}^{\dagger} \right\} \\ & + \text{Tr}_{S,m+p,m} \left\{ b_{m+p}^{\dagger} b_m U_{m+p}\eta_{m+p} \mathcal{E}_S^{t_{m+p}-t_m} [\Delta_m\rho^{(2,S)} + \Delta_m\rho^{(2,f)}] U_{m+p}^{\dagger} \right\} - \langle b_{m+p}^{\dagger} \rangle_{m+p} \langle b_m \rangle_m, \end{aligned} \quad (8.37)$$

and

$$G_{mp}^{\chi} = \text{Tr}_{S,m+p,m} \left\{ b_{m+p}^{\dagger} b_m U_{m+p}\eta_{m+p} \mathcal{E}_S^{t_{m+p}-t_m} [\Delta_m\chi^{(1)} + \Delta_m\rho^{(2,\chi)}] U_{m+p}^{\dagger} \right\}, \quad (8.38)$$

where, we used that the contribution of  $\Delta_m\rho_S^{(1)}$  is of order  $\alpha o(\delta\Delta t)$  as  $\rho_S(t_m)$  is in its limit cycle and  $\Delta_m\rho_S^{(2,S)} = 0$ .

Let us first compute  $G_{mp}^{\otimes}$ . Substituting Eq.(4.24) in  $G_{mp}^{\otimes}$ , we find that the contribution of  $\Delta_m\rho^{(2,S)}$  is of order  $\alpha o(\gamma\Delta t)$  while that of  $\Delta_m\rho^{(2,f)}$  is of order  $(1 + \bar{n}_{\text{th}}) o(\gamma\Delta t)$ . Using Eq. (4.13), we find  $\text{Tr} \left\{ b_m \Delta_m\rho_f^{(1)} \right\} = -\sqrt{\gamma\Delta t} \langle \sigma_- \rangle_{t_m}$  and hence,

$$\begin{aligned} G_{mp}^{\otimes} = & e^{-i(\omega_L - \omega_0)t_p} \text{Tr}_{S,m+p} \left\{ b_{m+p}^{\dagger} \left[ -\sqrt{\gamma\Delta t} \langle \sigma_- \rangle_{t_m} + \langle b_m \rangle_{t_m} \right] U_{m+p} (\rho_S(t_{m+p}) \otimes \eta_{m+p}) U_{m+p}^{\dagger} \right\} \\ & - \langle b_{m+p}^{\dagger} \rangle_{m+p} \langle b_m \rangle_m. \end{aligned} \quad (8.39)$$

The action of the second collision, similarly gives

$$G_{mp}^{\otimes} = -\sqrt{\gamma\Delta t} \left( \langle \sigma_- \rangle_{t_m} \langle b_{m+p}^\dagger \rangle_{t_{m+p}} + \langle b_m \rangle_{t_m} \langle \sigma_+ \rangle_{t_{m+p}} \right) + \gamma\Delta t \langle \sigma_+ \rangle_{t_{m+p}} \langle \sigma_- \rangle_{t_m}. \quad (8.40)$$

where, we use that  $\text{Tr}_{S,m+p} \left\{ b_{m+p}^\dagger \Delta_{m+p} \chi^{(1)} \right\} = 0$ ,  $\text{Tr}_{S,m+p} \left\{ \Delta_{m+p} \rho_S^{(1)} \otimes b_{m+p}^\dagger \rho_f(t_{m+p}) \right\} = 0$ ,  $\text{Tr}_{S,m+p} \left\{ \rho_S(t_{m+p}) \otimes b_{m+p}^\dagger \Delta_{m+p} \rho_f^{(1)} \right\} = -\sqrt{\gamma\Delta t} \langle \sigma_+ \rangle_{t_{m+p}}$ , and  $\text{Tr}_{S,m+p} \left\{ \rho_S(t_{m+p}) \otimes b_{m+p}^\dagger \eta_{m+p} \right\} = \langle b_{m+p}^\dagger \rangle_{t_{m+p}}$ . As one can notice, there is no contribution from the correlation induced evolution during the second collision to  $G_{mp}^{\otimes}$ . Hence, Eq.(8.38) is enough to separate the contributions of the correlations on the spectrum.

Now let us compute  $G_{mp}^x$ . Using Eq. (4.24), we find that the contribution of the second order correlation evolution is  $\text{Tr}_m \left\{ b_m \Delta_m \rho^{(2,x)} \right\} = o(\gamma\Delta t)$ . Therefore, the only contribution from the first collision is from the creation of correlation during dynamics at first order. Using Eq. (4.23) in  $G_{mp}^x$ , and taking the trace over the first collisional unit (mode  $m$ ) we find

$$\begin{aligned} G_{mp}^x &= -\sqrt{\gamma\Delta t} \text{Tr}_{S,m+p} \left\{ b_{m+p}^\dagger U_{m+p} \left( \mathcal{E}_S^{t_{m+p}-t_m} \{ \sigma_- \rho_S(t_m) \} \otimes \eta_{m+p} \right) U_{m+p}^\dagger \right\} \\ &\quad + \bar{n}_{\text{th}} \sqrt{\gamma\Delta t} \text{Tr}_{S,m+p} \left\{ b_{m+p}^\dagger U_{m+p} \left( \mathcal{E}_S^{t_{m+p}-t_m} \{ [\rho_S(t_m), \sigma_-] \} \otimes \eta_{m+p} \right) U_{m+p}^\dagger \right\} \\ &\quad + \sqrt{\gamma\Delta t} \langle \sigma_- \rangle_{t_m} \text{Tr}_{S,m+p} \left\{ b_{m+p}^\dagger U_{m+p} \left( \rho_S(t_{m+p}) \otimes \eta_{m+p} \right) U_{m+p}^\dagger \right\}. \end{aligned} \quad (8.41)$$

Only the first order evolution terms contribute during the second collision as the second order evolution contribution to  $G_{mp}^x$  are of order  $o(\gamma\Delta t)$ . By expanding  $G_{mp}^x$  in the form of Eq. (4.2) and Eq. (4.28), we find that the uncorrelated evolution terms of the second collision do not contribute. Therefore, only the first order correlating part of the evolution during the second collision contributes to the correlated part of the spectrum and by taking the trace over the second collisional unit (mode  $m+p$ ) we find

$$\begin{aligned} G_{mp}^x &= \gamma\Delta t \left( \text{Tr} \left\{ \sigma_+ \mathcal{E}_S^{t_{m+p}-t_m} \{ \sigma_- \rho_S(t_m) \} \right\} - \bar{n}_{\text{th}} \text{Tr} \left\{ \sigma_+ \mathcal{E}_S^{t_{m+p}-t_m} \{ [\rho_S(t_m), \sigma_-] \} \right\} \right) \\ &\quad - \gamma\Delta t \langle \sigma_+ \rangle_{t_{m+p}} \langle \sigma_- \rangle_{t_m}. \end{aligned} \quad (8.42)$$

Hence, the contribution to the correlated and uncorrelated part of the spectrum is

$$\begin{aligned}
S^{\otimes}(\omega_D) &= \frac{\gamma}{2\pi\gamma(\omega_D)} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D-\omega_0)t_p} G_{mp}^{\otimes} \\
&= -\frac{\gamma\sqrt{\gamma\Delta t}}{2\pi\gamma(\omega_D)} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D-\omega_0)t_p} \left( \langle \sigma_- \rangle_{t_m} \langle b_{m+p}^\dagger \rangle_{t_{m+p}} + \langle b_m \rangle_{t_m} \langle \sigma_+ \rangle_{t_{m+p}} \right) \\
&\quad + \frac{\gamma^2 \Delta t}{2\pi\gamma(\omega_D)} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D-\omega_0)t_p} \langle \sigma_+ \rangle_{t_{m+p}} \langle \sigma_- \rangle_{t_m}, \tag{8.43}
\end{aligned}$$

$$\begin{aligned}
S^X(\omega_D) &= \frac{\gamma}{2\pi\gamma(\omega_D)} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D-\omega_0)t_p} G_{mp}^X \\
&= \frac{\gamma^2 \Delta t}{2\pi\gamma(\omega_D)} \sum_{m=n_{ss}}^{\infty} \Delta t \sum_p e^{-i(\omega_D-\omega_0)t_p} \left( \text{Tr} \left\{ \sigma_+ \mathcal{E}_S^{t_{m+p}-t_m} \{ \sigma_- \rho_S(t_m) \} \right\} \right. \\
&\quad \left. - \bar{n}_{\text{th}} \text{Tr} \left\{ \sigma_+ \mathcal{E}_S^{t_{m+p}-t_m} \{ [\rho_S(t_m), \sigma_-] \} \right\} - \langle \sigma_+ \rangle_{t_{m+p}} \langle \sigma_- \rangle_{t_m} \right) \tag{8.44}
\end{aligned}$$

### 8.3.3 Spectral rates

By taking the sum over the delay time  $t_p$ , we can find the contributions to the spectral density rate  $\dot{S}(\omega_D)$ . We rewrite the expressions using only atomic operators, using  $\langle b_{\text{in}}(t) \rangle = \frac{\Omega}{2\sqrt{\gamma}} e^{-i(\omega_L-\omega_0)t}$  and  $\langle \sigma_+ \rangle_{t_{m+p}} = e^{i(\omega_L-\omega_0)t_p} \langle \sigma_+ \rangle_{t_m}$  as the atom is in its limit cycle. We get the input field contribution as

$$\begin{aligned}
\dot{S}_{\text{in}}(\omega_D) &= \frac{\gamma}{2\pi\gamma[\omega_D]} \sum_p e^{-i(\omega-\omega_0)t_p} \langle b_{m+p}^\dagger b_m \rangle_{t_0} \\
&= \sum_p e^{-i\Delta\omega t_p} \text{Tr} \left\{ \bigotimes_n \mathcal{D}^\dagger(\alpha_n) b_{x_0}^\dagger(t_{m+p}) b_{x_0}(t_m) \mathcal{D}(\alpha_n) \eta_n^\beta \right\} \\
&= \sum_p e^{-i\Delta\omega t_p} \alpha_{m+p}^* \alpha_m + \sum_p e^{-i\Delta\omega t_p} \text{Tr} \left\{ \bigotimes_n b_{x_0}^\dagger(t_{m+p}) b_{x_0}(t_m) \eta_n^\beta \right\} \\
&= \sum_p e^{-i\Delta\omega t_p} \langle b_{x_0}^\dagger(t_{m+p}) \rangle \langle b_{x_0}(t_m) \rangle + \langle b_{x_0}^\dagger(t_m) b_{x_0}(t_m) \rangle \\
&= \sum_p e^{-i\Delta\omega t_p} \langle b_{x_0}^\dagger(t_{m+p}) \rangle \langle b_{x_0}(t_m) \rangle + \bar{n}_{\text{th}}, \tag{8.45}
\end{aligned}$$

which in the continuous limit gives,

$$\begin{aligned}
\dot{S}_{\text{in}}(\omega) &= \frac{\gamma}{2\pi\gamma[\omega]} \left( \int d\tau e^{-i\omega\tau} e^{i\omega_0\tau} \langle b_{\text{in}}^\dagger(t+\tau) \rangle_{t+\tau} \langle b_{\text{in}}(t) \rangle_t + \bar{n}_{\text{th}} \right) \\
&= \frac{\gamma}{2\pi\gamma[\omega_D]} \left( \frac{\Omega^2}{4\gamma} \int d\tau e^{i(\omega_L - \omega)\tau} + \bar{n}_{\text{th}} \right) \\
&= \delta_D(\omega_L - \omega) \frac{\Omega^2}{4\gamma[\omega]} + \frac{\gamma}{2\pi\gamma[\omega]} \bar{n}_{\text{th}}.
\end{aligned} \tag{8.46}$$

Similarly, the rate of the uncorrelated contribution to the spectrum in the continuous limit gives

$$\begin{aligned}
\dot{S}^{\otimes}(\omega) &= -\delta_D(\omega_L - \omega) \frac{\gamma}{\gamma[\omega]} \frac{\Omega}{2} \left( e^{i(\omega_L - \omega_0)t} \langle \sigma_- \rangle_{t_\infty} + e^{-i(\omega_L - \omega_0)t} \langle \sigma_+ \rangle_{t_\infty} \right) \\
&\quad + \delta_D(\omega_L - \omega) \frac{\gamma^2}{\gamma[\omega]} |\langle \sigma_- \rangle_{t_\infty}|^2.
\end{aligned} \tag{8.47}$$

while the correlated contribution in the continuous limit is

$$\begin{aligned}
\dot{S}^{\times}(\omega) &= \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i(\omega - \omega_0)\tau} \left( \text{Tr} \{ \sigma_+ \mathcal{E}_S^\tau \{ \sigma_- \rho_S(t) \} \} - \bar{n}_{\text{th}} \text{Tr} \{ \sigma_+ \mathcal{E}_S^\tau \{ [\rho_S(t), \sigma_-] \} \} \right) \\
&\quad - \frac{\gamma^2}{2\pi\gamma[\omega]} \int d\tau e^{-i(\omega - \omega_0)\tau} \langle \sigma_+ \rangle_{t_\infty + \tau} \langle \sigma_- \rangle_{t_\infty}.
\end{aligned} \tag{8.48}$$

### Relation with Input-Output operators

We can rewrite  $\dot{S}^{\otimes(x)}(\omega)$  in terms of input and output mode operators. First notice that in Eq. (8.39), we can substitute Eq. (4.18) to get average of the output mode after the first collision. Then applying the unitary on the mode operator for the second collision and taking the trace gives the average of the output mode after the second collision. Using  $\dot{S}^{\otimes}(\omega_D) = (\gamma/2\pi\gamma(\omega_D)) \sum_p e^{-i(\omega_D - \omega_0)t_p} G_{mp}^{\otimes}$ , we find in the continuous limit

$$\dot{S}^{\otimes}(\omega) = \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i\Delta\omega\tau} \left( \langle b_{\text{out}}^\dagger(t+\tau) \rangle \langle b_{\text{out}}(t) \rangle - \langle b_{\text{in}}^\dagger(t+\tau) \rangle \langle b_{\text{in}}(t) \rangle \right). \tag{8.49}$$

Substituting Eq. (8.31) in  $G_{mp}$ , and using  $\dot{S}^{\times}(\omega_D) = (\gamma/2\pi\gamma(\omega_D)) \sum_p e^{-i(\omega_D - \omega_0)t_p} (G_{mp} - G_{mp}^{\otimes})$  as  $G_{mp}^{\times} = G_{mp} - G_{mp}^{\otimes}$ , we find in the continuous limit

$$\dot{S}^{\times}(\omega) = \frac{\gamma}{2\pi\gamma[\omega]} \int d\tau e^{-i\Delta\omega\tau} \langle \delta b_{\text{out}}^\dagger(t+\tau) \delta b_{\text{out}}(t) \rangle. \tag{8.50}$$

Hence, we see that the correlated contribution is due to the fluctuations of the output operator which the uncorrelated contribution is due to the change of its mean value.

## 8.4 Relation between Heisenberg Input-Output Operators, BQE and Spectral density functions

In this appendix, we derive input-output operators for coupling that is not flat in the Heisenberg picture from those in the interaction picture defined in chapter 3. We will then use these operators to compute two-time correlation functions and hence, the spectrum. We will show how BQE quantities arise from the Heisenberg operators and relate them to the spectrum and the mean and the fluctuations of the change in internal energy.

### 8.4.1 Input-Output Relations

To define the input-operator we assume that the interaction with the atom occurs at the  $t$ . Using Eq. (4.17), we have

$$\begin{aligned}
 b_{\text{in}}(t) &= \lim_{\gamma\Delta t \rightarrow 0} \frac{b_{x_0}(t_m)}{\sqrt{\Delta t}} = \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \sum_k e^{-i\Delta\omega_k u} g_k a_k(0) \\
 &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \sum_k e^{-i\Delta\omega_k u} g_k a_k^H(0) \\
 &= b_{\text{in}}^H(t).
 \end{aligned} \tag{8.51}$$

where, we take  $\lim_{\gamma\Delta t \rightarrow 0} t_m = t$ . This is true as the initial time operators in Heisenberg or the interaction picture are the same. The output operator in the Heisenberg picture at time  $t$ , is  $b_{x_0}(t_m)$  evolved through the unitary  $\mathcal{U}_{m-1}$ , i.e.

$$\begin{aligned}
 b_{\text{out}}^H(t) &= \lim_{\gamma\Delta t \rightarrow 0} \frac{\mathcal{U}_{m-1}^\dagger b_{x_0}(t_m) \mathcal{U}_{m-1}}{\sqrt{\Delta t}} \\
 &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \int_0^{\Delta x} dx \sum_k e^{-i\Delta\omega_k(t_{m+1}-x/v)} g_k \mathcal{U}_{m-1}^\dagger a_k(0) \mathcal{U}_{m-1} \\
 &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \sum_k e^{-i\Delta\omega_k u} g_k \mathcal{U}_{m-1}^\dagger a_k(0) \mathcal{U}_{m-1}.
 \end{aligned} \tag{8.52}$$

To make the connection with Heisenberg operators  $a_k^H$ , at this point, let us imagine that we place a detector at a position  $x = \mathcal{L}$  such that light interacts with the atom and is detected after time  $T = t + \mathcal{L}/v$ . We could either choose to fix the position of the detector  $\mathcal{L}$  or the time of detection  $T$ . Using the evolution of the interaction picture operator,

we have  $a_k(0) = e^{i\omega_k(t+\mathcal{L}/v)}a_k(t+\mathcal{L}/v) = e^{i\omega_k T}a_k(T)$ . Taking,  $\lim_{\gamma\Delta t \rightarrow 0} x_l = \mathcal{L}$  such that  $\lim_{\gamma\Delta t \rightarrow 0} (t_m + x_l/v) = T$ , we get

$$b_{\text{out}}^H(t) = \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma\Delta t}} \int_{t_m}^{t_m+\Delta t} du e^{i\omega_0(t_m+x_l/v)} \sum_k e^{-i\Delta\omega_k(u-t_m-x_l/v)} g_k \mathcal{U}_{m-1}^\dagger a_k(t_m+x_l/v) \mathcal{U}_{m-1}. \quad (8.53)$$

This is the output operator for the interaction at time  $t$  with photodetection at time  $T$  in the interaction picture. To define the Heisenberg output operator from this, we use the fact that  $[b_m, b_{n \neq m}] = 0$  and  $[H_S + H_f, V_{Sf}] = 0$  and hence,

$$\begin{aligned} b_{\text{out}}^H(t) &= \lim_{\gamma\Delta t \rightarrow 0} \frac{(\prod_n^{m-1} \mathcal{U}_n^\dagger) \mathcal{U}_m^\dagger b_{x_0}(t_m) \mathcal{U}_m (\prod_n^{m-1} \mathcal{U}_n)}{\sqrt{\Delta t}} \\ &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma\Delta t}} \int_{t_m}^{t_m+\Delta t} du e^{i\omega_0(t_m+x_l/v)} \sum_k e^{-i\Delta\omega_k(u-t_m-x_l/v)} g_k \left( \prod_n^{m-1} \mathcal{U}_n^\dagger \right) a_k(t_m+x_l/v) \left( \prod_n^{m-1} \mathcal{U}_n \right) \\ &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma\Delta t}} \int_{t_m}^{t_m+\Delta t} du e^{i\omega_0(t_m+x_l/v)} \sum_k e^{-i\Delta\omega_k(u-t_m-x_l/v)} g_k a_k^H(t_m+x_l/v), \end{aligned} \quad (8.54)$$

where,  $a_k^H(t_m) = (\prod_n^{m-1} \mathcal{U}_n^\dagger) a_k(t_m) (\prod_n^{m-1} \mathcal{U}_n)$  as  $a_k(t_m)$  is already in the interaction picture wrt.  $H_S + H_f$ . These input-output operators in the Heisenberg picture, are not the same as the standards ones used in [GC85]. They are coarse grained operators for each collision defined with frequency mode operators represented in the Heisenberg picture weighted with a coupling that is not flat. To verify that these definitions form the input-output operators starting from the Heisenberg picture, we solve the Heisenberg equations of motion between times  $[0, t]$ , and  $[t, T]$  to get

$$g_k a_k^H(t) = e^{-i\omega_k t} g_k a_k^H(0) - g_k^2 \int_0^t e^{-i\omega_k(t-t')} \sigma_-^H(t') dt', \quad (8.55)$$

$$g_k a_k^H(t) = e^{-i\omega_k(t-T)} g_k a_k^H(T) + g_k^2 \int_t^T e^{-i\omega_k(t-t')} \sigma_-^H(t') dt'. \quad (8.56)$$

Integrating over the region where the interaction with the atom takes place, and normalizing we get

$$\begin{aligned}
\lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \sum_k \int_{t_m}^{t_{m+1}} du g_k e^{i\omega_0 u} a_k^H(u) &= \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \sum_k e^{-i\Delta\omega_k u} g_k a_k^H(0) \\
&- \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \sum_k g_k^2 \int_{t_m}^{t_{m+1}} du \int_0^u dt' e^{-i\omega_k(u-t')} e^{i\omega_0 u} \sigma_-^H(t') \\
&= b_{\text{in}}^H(t) - \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \sum_k g_k^2 \int_{t_m}^{t_{m+1}} du \int_0^u dt' e^{-i\omega_k(u-t')} e^{i\omega_0 u} \sigma_-^H(t'),
\end{aligned} \tag{8.57}$$

and similarly,

$$\begin{aligned}
\lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \sum_k \int_{t_m}^{t_{m+1}} du g_k e^{i\omega_0 u} a_k^H(u) &= \lim_{\gamma\Delta t \rightarrow 0} \int_{t_m}^{t_{m+1}} du \frac{e^{i\omega_0(t_{m+1}+x_l/v)}}{\sqrt{\gamma}\Delta t} \sum_k e^{-i\Delta\omega_k(u-t_m-x_l/v)} g_k a_k^H(t_m + x_l/v) \\
&+ \lim_{\gamma\Delta t \rightarrow 0} \sum_k \frac{g_k^2}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \int_u^{t_m+x_l/v} dt' e^{-i\omega_k(u-t')} e^{i\omega_0 u} \sigma_-^H(t') \\
&= b_{\text{out}}^H(t) \\
&+ \lim_{\gamma\Delta t \rightarrow 0} \sum_k \frac{g_k^2}{\sqrt{\gamma}\Delta t} \int_{t_m}^{t_{m+1}} du \int_u^{t_m+x_l/v} dt' e^{-i\omega_k(u-t')} e^{i\omega_0 u} \sigma_-^H(t').
\end{aligned} \tag{8.58}$$

Finally, the general input-output relation is

$$b_{\text{out}}^H(t) = b_{\text{in}}^H(t) - \lim_{\gamma\Delta t \rightarrow 0} \frac{1}{\sqrt{\gamma}\Delta t} \sum_k g_k^2 \left( \int_{t_m}^{t_{m+1}} du \int_0^{t_m+x_l/v} dt' e^{-i\omega_k(u-t')} e^{i\omega_0 u} \sigma_-^H(t') \right). \tag{8.59}$$

On average, up to  $o(\gamma\Delta t)$  (integrating over  $u$  then changing variable to  $\tau = t' - t_m$ ), we get

$$\lim_{\gamma\Delta t \rightarrow 0} \frac{e^{i\omega_0 t_m}}{\sqrt{\gamma}} \left( \int_0^{x_l/v} d\tau \sum_k g_k^2 e^{i\omega_k \tau} \sigma_-^H(t_m + \tau) \right) = \sqrt{\gamma} \sigma_-^H(t) + o(\gamma\Delta t), \tag{8.60}$$

where, we use the fact that  $x_l/v \gg \Delta t \gg \tau_c$  such that in the relevant integration interval  $\sigma_-^H(t_m + \tau) \approx \sigma_-^H(t_m)$ . Hence,

$$b_{\text{out}}^H(t) = b_{\text{in}}^H(t) + \sqrt{\gamma} \sigma_-^H(t) + o(\gamma\Delta t), \tag{8.61}$$

which verifies the input-output relation.

## 8.4.2 Spectrum and BQE in the Heisenberg picture

Now we use the input-output operators in the Heisenberg picture to construct the spectrum and hence, the energetic quantities. For this, it is useful to define the operators for continuous frequency modes. We have

$$b_{\text{in}}^H(t) = \frac{1}{\sqrt{2\pi\gamma\Delta t}} \lim_{\gamma\Delta t \rightarrow 0} \int_{t_m}^{t_{m+1}} du \int d\nu e^{-i\Delta\nu u} \sqrt{\gamma[\nu]} a^H(\nu, 0), \quad (8.62)$$

$$b_{\text{out}}^H(t) = \frac{1}{\sqrt{2\pi\gamma\Delta t}} \lim_{\gamma\Delta t \rightarrow 0} \int_{t_m}^{t_{m+1}} du e^{i\omega_0(t_m+x_l/v)} \int d\nu e^{-i\Delta\nu(u-t_m-x_l/v)} \sqrt{\gamma[\nu]} a^H(\nu, t_m+x_l/v). \quad (8.63)$$

where, we defined  $\gamma[\nu] = \sum_k 2\pi g_k^2 \delta_D(\nu - \omega_k)$ ,  $\Delta\nu = \nu - \omega_0$  and  $\gamma[\omega_0] \equiv \gamma$ . These continuous mode operators  $a^H(\nu, t)$  have the dimension of  $[t^{\frac{1}{2}}]$ . To construct the spectrum, we define a weighted Fourier transform of the two-point correlation function of the Heisenberg output (input) operators

$$\mathcal{G}_{\text{out(in)}}(\omega, t) = f(\omega) \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \langle b_{\text{out(in)}}^{H\dagger}(t+\tau) b_{\text{out(in)}}^H(t) \rangle. \quad (8.64)$$

This is defined, for the time of interaction being  $t$  with a fixed photo-detection time  $T > t_\infty$  and  $f(\omega)$  is an arbitrary function which assigns a weight at each frequency  $\omega$ . We will consider interactions up to steady state, i.e.  $T \rightarrow \infty$ . Taking  $\lim_{\gamma\Delta t \rightarrow 0} t_n = t + \tau$  while  $\lim_{\gamma\Delta t \rightarrow 0} t_m = t$  where  $\tau = \lim_{\gamma\Delta t \rightarrow 0} (t_n - t_m)$ , we compute the ‘‘output weighted spectrum’’

$$\begin{aligned} \int_{t_0}^{t_\infty} dt \mathcal{G}_{\text{out}}(\omega, t) &= \frac{f(\omega)}{2\pi} \iint_{t_0}^{t_\infty} dt d\tau e^{-i(\omega-\omega_0)\tau} \langle b_{\text{out}}^{H\dagger}(t+\tau) b_{\text{out}}^H(t) \rangle \\ &= \frac{f(\omega) \gamma[\omega]}{\gamma} \langle a^{\dagger H}(\omega, T) a^H(\omega, T) \rangle, \end{aligned} \quad (8.65)$$

which we have shown to be true in section 8.3 upto a sinc function. Taking the weight as  $f(\omega) = f_\gamma(\omega) \equiv \gamma \hbar \omega / \gamma[\omega]$ , we find that  $u_{\text{out}}(\omega) = \int_{t_0}^{t_\infty} dt \mathcal{G}_{\text{out}}(\omega, t)$ , i.e. the energy of the photons at frequency  $\omega$  that have already interacted during the steady state times, i.e., after the atom has reach the steady state/its limit cycle. Similarly, the ‘‘weighted input spectrum’’  $\int_{t_0}^{t_\infty} dt \mathcal{G}_{\text{in}}(\omega, t)$  with the same weight gives the input energy of the field  $u_{\text{in}}(\omega) = \hbar \omega \langle a^{\dagger H}(\omega, 0) a^H(\omega, 0) \rangle$ . Hence, the spectral density of change in energy of the field is  $\Delta u_f(\omega) = u_{\text{out}}(\omega) - u_{\text{in}}(\omega) = \int_{t_0}^{t_\infty} dt (\mathcal{G}_{\text{out}}(\omega, t) - \mathcal{G}_{\text{in}}(\omega, t))$ . Hence, we find that the rate of change of internal energy of the field per frequency must be  $\dot{u}_f(\omega) = \mathcal{G}_{\text{out}}(\omega, t) - \mathcal{G}_{\text{in}}(\omega, t)$ .

At this point, we can define the  $b$ -work and  $b$ -heat rates per frequency. To do so, we define the fluctuations in the field operators as  $\delta b_{\text{out(in)}}^H(t) = b_{\text{out(in)}}^H(t) - \langle b_{\text{out(in)}}^H(t) \rangle$ , which implies  $\langle \delta b_{\text{out(in)}}^H(t) \rangle = 0$ . The rate of change of energy per frequency can hence, be split into a part that comes from the coherences and a part which comes from the fluctuations

$$\begin{aligned} \dot{u}_f(\omega) &= \mathcal{G}_{\text{out}}(\omega, t) - \mathcal{G}_{\text{in}}(\omega, t) \\ &= \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \left( \langle b_{\text{out}}^{H\dagger}(t+\tau) \rangle \langle b_{\text{out}}^H(t) \rangle - \langle b_{\text{in}}^\dagger(t+\tau) \rangle \langle b_{\text{in}}^H(t) \rangle \right) \\ &\quad + \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \left( \langle \delta b_{\text{out}}^{H\dagger}(t+\tau) \delta b_{\text{out}}^H(t) \rangle - \langle \delta b_{\text{in}}^{H\dagger}(t+\tau) \delta b_{\text{in}}^H(t) \rangle \right). \end{aligned} \quad (8.66)$$

At steady state, we can identify that the  $b$ -work rate done per frequency stems from the coherent part of the input and output operators

$$\begin{aligned} b\dot{w}_f(\omega) &= \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \left( \langle b_{\text{out}}^{H\dagger}(t+\tau) \rangle \langle b_{\text{out}}^H(t) \rangle - \langle b_{\text{in}}^{H\dagger}(t+\tau) \rangle \langle b_{\text{in}}^H(t) \rangle \right) \\ &= \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \left( -\sqrt{\frac{\gamma[\omega_L]}{2\pi}} \alpha_L e^{i\Delta\omega_L(t+\tau)} \langle \sigma_-^H(t) \rangle - \sqrt{\frac{\gamma[\omega_L]}{2\pi}} \langle \sigma_+^H(t+\tau) \rangle \alpha_L e^{-i\Delta\omega_L t} \right) \\ &\quad + \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \gamma \langle \sigma_+^H(t+\tau) \rangle \langle \sigma_-^H(t) \rangle \\ &= \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau \left( -\sqrt{\frac{\gamma[\omega_L]}{2\pi}} \alpha_L e^{i(\omega_L-\omega)\tau} \langle \sigma_-^I \rangle - \sqrt{\frac{\gamma[\omega_L]}{2\pi}} \alpha_L e^{i(\omega_L-\omega)\tau} \langle \sigma_+^I \rangle + \gamma e^{i(\omega_L-\omega)\tau} \langle \sigma_+^I \rangle \langle \sigma_-^I \rangle \right) \\ &= -\delta(\omega_L - \omega) \frac{\gamma \hbar \omega}{\gamma[\omega]} \left( \frac{\Omega}{2} \langle \sigma_x^I \rangle_{t_\infty} - \gamma \left| \langle \sigma_-^I \rangle_{t_\infty} \right|^2 \right). \end{aligned} \quad (8.67)$$

Given the short correlation time  $\tau_c \sim \Delta_{\text{bw}}^{-1}$ , we must have a very smooth variation of the coupling strength with  $\omega$ , such that we can neglect the variation of  $\gamma[\omega]$  over the range  $\omega_0 - \omega_L \ll \Delta_{\text{bw}}$ . This implies that in the regime of the OBE  $\gamma/\gamma[\omega_L] = 1$  such that  $\int d\omega b\dot{w}_f(\omega) = b\dot{W}_f(t_\infty)$ . The averages  $\langle \cdot \rangle_{t_\infty}$  are computed at steady state and the superscript  $I$  denotes the frame of the atom rotating only with the drive frequency  $\omega_L$ .  $b$ -Heat rate done per frequency stems from the fluctuations of the input and output operators

$$\begin{aligned} b\dot{q}_f(\omega) &= \frac{f_\gamma(\omega)}{2\pi} \int_{t_0}^{t_\infty} d\tau e^{-i\Delta\omega\tau} \left( \langle \delta b_{\text{out}}^{H\dagger}(t+\tau) \delta b_{\text{out}}^H(t) \rangle - \langle \delta b_{\text{in}}^{H\dagger}(t+\tau) \delta b_{\text{in}}^H(t) \rangle \right) \\ &= \frac{\gamma^2 \hbar \omega}{2\pi \gamma[\omega]} \int_{t_0}^{t_\infty} d\tau e^{-i\omega\tau} \left( \langle \sigma_+^H(t+\tau) \sigma_-^H(t) \rangle \right) - \delta(\omega_L - \omega) \frac{\gamma^2 \hbar \omega}{\gamma[\omega]} \left| \langle \sigma_-^I \rangle_{t_\infty} \right|^2, \end{aligned} \quad (8.68)$$

where, for simplicity we have taken zero temperature in the last line (for non-zero temperature see  $\dot{S}^\chi(\omega)$  in section 8.3.3). In this case we have  $\int d\omega b\dot{q}_f(\omega) = b\dot{Q}_f(t_\infty)$ . Eq. (8.67) and Eq. (8.68) are the energetic spectral densities which lead to Eq. (6.27)

and Eq. (6.29) for energy exchanges per frequency at steady state as one can identify  $b\dot{w}_S^s(\omega) = \delta_D(\omega_L - \omega) \gamma \hbar \omega |\langle \sigma_- \rangle_{t_\infty}|^2$  as the spectral density of self  $b$ -work rate which is delta peaked at  $\omega_L$  such that  $\int_{-\infty}^{\infty} d\omega b\dot{w}_S^s(\omega) = b\dot{W}_S^s(t_\infty)$ .

## 8.5 Coupling Energy for 1D atom using the Collisional Model

Here, we derive the average value of the interaction picture coupling Hamiltonian  $V_{Sf}(t)$ . We start with its discrete-time version given in Eq. (3.3)

$$V_{Sf}(t_n) = V_n = i\hbar \sqrt{\frac{\gamma}{\Delta t}} (\sigma_+ b_n - b_n^\dagger \sigma_-). \quad (8.69)$$

Throughout we will use the fact that  $\text{Tr}_f\{V_n \eta_n^\beta\} = 0$ . The change of the coupling energy after the  $n^{\text{th}}$  collision is

$$\begin{aligned} \Delta \mathcal{V}_{Sf} &= \mathcal{V}_{Sf}(t_{n+1}) - \mathcal{V}_{Sf}(t_n) \\ &= \text{Tr} \{V_{n+1} \rho_{Sf}(t_{n+1})\} - \text{Tr} \left\{ V_n \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right\}. \end{aligned} \quad (8.70)$$

The initial state of the field may contain all the field units that have yet to interact. The evolution of the bipartite state reads

$$\rho_{Sf}(t_{n+1}) = \mathcal{U}_n \left( \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right) \mathcal{U}_n^\dagger. \quad (8.71)$$

Expanding the unitary to the first order in  $\gamma \Delta t$ ,

$$\begin{aligned} \rho_{Sf}(t_{n+1}) &= \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) - \frac{i}{\hbar} \left[ V_n, \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right] \Delta t \\ &\quad - \frac{1}{2} \left[ V_n, \left[ V_n, \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right] \right] \Delta t^2. \end{aligned} \quad (8.72)$$

Substituting this in the coupling energy, we get

$$\begin{aligned} \Delta \mathcal{V}_{Sf} = & \text{Tr}_{Sf} \left\{ (V_{n+1} - V_n) \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right\} \\ & - \frac{i}{\hbar} \Delta t \text{Tr}_{Sf} \left\{ [V_{n+1}, V_n] \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right\} \\ & - \frac{1}{2} \Delta t^2 \text{Tr}_{Sf} \left\{ V_{n+1} \left[ V_n, \left[ V_n, \rho_S(t_n) \bigotimes_m \mathcal{D}(\alpha_m) \eta_m^\beta \mathcal{D}^\dagger(\alpha_m) \right] \right] \right\}. \end{aligned} \quad (8.73)$$

We can trace over all the units that are not involved in the evolution and use the cyclic property of the trace to apply the displacement on the coupling Hamiltonian to get

$$\begin{aligned} \Delta \mathcal{V}_{Sf} = & \text{Tr}_{Sf} \left\{ (H_D(t_{n+1}) + V_{n+1}) \left( \rho_S(t_n) \otimes \eta_{m+1}^\beta \right) - (H_D(t_n) + V_n) \rho_S(t_n) \otimes \eta_n^\beta \right\} \\ & - \frac{i}{\hbar} \Delta t \text{Tr}_{Sf} \left\{ [H_D(t_{n+1}) + V_{n+1}, H_D(t_n) + V_n] \left( \rho_S(t_n) \otimes \eta_m^\beta \right) \right\} \\ & - \frac{1}{2} \Delta t^2 \text{Tr}_{Sf} \left\{ [[H_D(t_{n+1}) + V_{n+1}, H_D(t_n) + V_n], H_D(t_n) + V_n] \rho_S(t_n) \otimes \eta_m^\beta \right\}. \end{aligned} \quad (8.74)$$

Now we use that  $\text{Tr}_{Sf} \left\{ V_{n+1} \rho_S(t_n) \eta_{m+1}^\beta \right\} = 0$  and  $\Delta H_D(t_n) = H_D(t_{n+1}) - H_D(t_n)$ ,

$$\begin{aligned} \Delta \mathcal{V}_{Sf} = & \text{Tr}_S \left\{ \Delta H_D(t_n) \rho_S(t_n) \right\} \\ & - \frac{i}{\hbar} \Delta t \text{Tr}_{Sf} \left\{ [H_D(t_{n+1}) + V_{n+1}, H_D(t_n) + V_n] \left( \rho_S(t_n) \otimes \eta_n^\beta \right) \right\} \\ & - \frac{1}{2} \Delta t^2 \text{Tr}_{Sf} \left\{ [[H_D(t_{n+1}) + V_{n+1}, H_D(t_n) + V_n], H_D(t_n) + V_n] \rho_S(t_n) \otimes \eta_m^\beta \right\}. \end{aligned} \quad (8.75)$$

Re-writing the second and third lines we get

$$\begin{aligned} \text{Tr}_{Sf} \left\{ [H_D(t_{n+1}) + V_{n+1}, H_D(t_n) + V_n] \left( \rho_S(t_n) \otimes \eta_m^\beta \right) \right\} \\ = \text{Tr}_{Sf} \left\{ H_D(t_{n+1}) [H_D(t_n) + V_n, (\rho_S(t_n) \otimes \eta_n^\beta)] \right\} \\ + \text{Tr}_{Sf} \left\{ [V_{n+1}, H_D(t_n) + V_n] \left( \rho_S(t_n) \otimes \eta_{m+1}^\beta \right) \right\} \end{aligned}$$

where, we see that  $\text{Tr}_{Sf} \{ [V_{n+1}, H_D(t_n) + V_n] (\rho_S(t_n) \otimes \eta_m^\beta) \} = 0$  as  $V_{n+1}$  gets traced only with  $\eta_{n+1}^\beta$  and hence,  $\text{Tr} \{ V_{n+1} V_n \eta_{n+1}^\beta \} = 0$ . Defining  $\tilde{V}_n = H_D(t_n) + V_n$

$$\begin{aligned} \text{Tr}_{Sf} \left\{ \left[ [H_D(t_{n+1}) + V_{n+1}, \tilde{V}_n], \tilde{V}_n \right] \rho_S(t_n) \otimes_m \eta_m^\beta \right\} &= \text{Tr}_{Sf} \left\{ \left[ [H_D(t_{n+1}), \tilde{V}_n], \tilde{V}_n \right] \rho_S(t_n) \otimes_m \eta_m^\beta \right\} \\ &+ \text{Tr}_{Sf} \left\{ \left[ [V_{n+1}, \tilde{V}_n], \tilde{V}_n \right] \rho_S(t_n) \otimes_m \eta_m^\beta \right\}. \end{aligned}$$

Again, the second term is zero as  $V_{n+1}$  gets traced only with  $\eta_{n+1}^\beta$  and hence,  $\text{Tr} \{ V_{n+1} \tilde{V}_n \eta_{n+1}^\beta \} = 0$ . Finally, we get

$$\begin{aligned} \Delta \mathcal{V}_{Sf} &= \text{Tr}_S \{ \Delta H_D(t_n) \rho_S(t_n) \} + \text{Tr}_{Sf} \{ H_D(t_{n+1}) [H_D(t_n) + V_n, (\rho_S(t_n) \otimes \eta_m^\beta)] \} \\ &+ \text{Tr}_{Sf} \left\{ H_D(t_{n+1}) \left[ (H_D(t_n) + V_n), \left[ (H_D(t_n) + V_n), \rho_S(t_n) \otimes_m \eta_m^\beta \right] \right] \right\} \\ &= \text{Tr}_S \{ \Delta H_D(t_n) \rho_S(t_n) \} + \text{Tr}_S \left\{ H_D(t_{n+1}) \text{Tr}_f \left\{ \tilde{\mathcal{U}}_n \left( \rho_S(t_n) \otimes_m \eta_m^\beta \right) \tilde{\mathcal{U}}_n^\dagger - \rho_S(t_n) \otimes_m \eta_m^\beta \right\} \right\}. \end{aligned} \quad (8.76)$$

Noticing that  $\text{Tr}_f \{ \tilde{\mathcal{U}}_n \rho_S(t_n) \otimes_m \eta_m^\beta \tilde{\mathcal{U}}_n^\dagger - \rho_S(t_n) \otimes_m \eta_m^\beta \} = \rho_S(t_{n+1}) - \rho_S(t_n) = \Delta \rho_S$  is the change of the atom's state after the n-th collision, we get  $\Delta \mathcal{V}_{Sf} = \Delta \langle H_D(t_n) \rangle$ , which in the continuous limit results in,

$$\dot{\mathcal{V}}_{Sf}(t) = \frac{d}{dt} \langle H_D(t) \rangle \implies \Delta \mathcal{V}_{Sf}(t) = \Delta \langle H_D(t) \rangle. \quad (8.77)$$

## 8.6 Derivation and Analysis of the Energetic Relations

We begin by showing that out of resonance, at any time  $t$ ,  $b\dot{\mathcal{W}}_f + \dot{W} = -\text{Tr}_S \{ \rho_S^{(\text{lab})}(t) d\mathcal{H}_S^{(\text{lab})}(t)/dt \}$ , where for simplicity we define  $\dot{W}_{\text{self}}(t) = \text{Tr}_S \{ \rho_S^{(\text{lab})}(t) d\mathcal{H}_S^{(\text{lab})}(t)/dt \}$  to make a parallel with the definition of open work. We can then use this result to get Eq. (6.27) and (6.29) of the main text by taking  $\mathcal{V}_{Sf} = \mathcal{V}_{Sf}^\otimes = \mathcal{V}_{Sf}^X = 0$  which is true at resonance [MCA21] and that at resonance or steady state we have

$$\dot{W}_{\text{self}}(t) = b\dot{\mathcal{W}}_S^s(t).$$

Here, we work in the lab frame with respect to the atom (still rotating with  $H_f$ ) and remove the notation <sup>lab</sup> in this section. Taking the derivative of the self-drive on the system ( $\mathcal{H}_S^s(t)$ , see Eq. (4.30)) we get

$$\dot{W}_{\text{self}}(t) = -\text{Tr} \left\{ \rho_S(t) \frac{d}{dt} H_D(t) \right\} + \left\langle \dot{V}_{Sf}(t) \right\rangle_{\otimes} + \text{Tr}_{Sf} \left[ V_{Sf}(t) \rho_S(t) \otimes \frac{d}{dt} \rho_f(t) \right], \quad (8.78)$$

where,  $\left\langle \dot{V}_{Sf}(t) \right\rangle_{\otimes} \equiv \text{Tr}_{Sf} \left[ \dot{V}_{Sf}(t) \rho_S(t) \otimes \rho_f(t) \right]$ . To simplify the second term, we use Eq. (4.7) for the field to get

$$\text{Tr}_{Sf} \left[ V_{Sf}(t) \rho_S(t) \otimes \frac{d}{dt} \rho_f(t) \right] = \text{Tr}_{Sf} \{ [\mathcal{H}_f(t), V_{Sf}(t)] \chi_{Sf}(t) \}. \quad (8.79)$$

Now substituting Eq. (3.5) in the second term of  $\dot{W}_{\text{self}}$  gives

$$\begin{aligned} \text{Tr}_{Sf} \{ [\mathcal{H}_f(t), V_{Sf}(t)] \chi_{Sf}(t) \} &= -\hbar^2 \text{Tr}_{Sf} \{ [B(0, t), B^\dagger(0, t)] (\langle \sigma_+ \rangle \sigma_- - \langle \sigma_- \rangle \sigma_+) \chi_{Sf}(t) \} \\ &= -\hbar^2 \text{Tr} \left\{ \sum_k |g_k|^2 (\langle \sigma_+ \rangle \sigma_- - \langle \sigma_- \rangle \sigma_+) \chi_{Sf}(t) \right\} = 0 \end{aligned} \quad (8.80)$$

where, we used the fact that  $[B(0, t), B^\dagger(0, t)] = \sum_k |g_k|^2$  and that  $\text{Tr}_f [\chi_{Sf}(t)] = 0$ . Therefore, we have

$$\dot{W}_{\text{self}}(t) = -\text{Tr} \left\{ \rho_S(t) \frac{d}{dt} H_D(t) \right\} + \left\langle \dot{V}_{Sf}(t) \right\rangle_{\otimes}. \quad (8.81)$$

After using  $\frac{d}{dt} (V_{Sf}(t)) = \frac{i}{\hbar} [H_f, V_{Sf}(t)]$ , which is the equation of motion for the coupling Hamiltonian, we get that the bipartite work done on the field is

$$b\dot{\mathcal{W}}_f(t) = -\left\langle \frac{d}{dt} V_{Sf}(t) \right\rangle_{\otimes} = -\dot{W}(t) - \dot{W}_{\text{self}}(t), \quad (8.82)$$

where the first term is the work rate (Eq. (6.2) computed in the lab frame). At resonance, we use the fact that  $\mathcal{V}(t) = \mathcal{V}^\times(t) = 0$  to get Eq.(6.29).

## 8.6.1 Sign of self $b$ -work

We begin with  $\dot{W}_{\text{self}} = -b\dot{W}_f - \dot{W} = \text{Tr}\{\rho_S^{(\text{lab})}(t)d\mathcal{H}_S^{s(\text{lab})}(t)/dt\}$  (see Eq. (4.30)). This can be expressed as,  $\dot{Q}_S + b\dot{Q}_f = \dot{W}_{\text{self}} - \dot{V}_{Sf}$ . We first show that  $\dot{W}_{\text{self}} \leq 0$ . We compute

$$\dot{W}_{\text{self}} = -i\hbar\frac{\gamma}{2}(\langle\dot{\sigma}_-\rangle\langle\sigma_+\rangle - \langle\dot{\sigma}_+\rangle\langle\sigma_-\rangle). \quad (8.83)$$

Using the atom's master equation, we find

$$\langle\dot{\sigma}_-\rangle = -i\omega_0\langle\sigma_-\rangle - \frac{\Omega}{2}e^{-i\omega_L t}\langle\sigma_z\rangle - \gamma\left(\frac{2\bar{n}_{\text{th}} + 1}{2}\right)\langle\sigma_-\rangle. \quad (8.84)$$

Introducing for readability the Bloch coordinates  $q(t) = \langle\sigma_q\rangle$  for  $q = x, y, z$ , we obtain

$$\dot{W}_{\text{self}} = -\frac{\gamma\hbar\omega_0}{4}\left((x^2(t) + y^2(t)) + \frac{\Omega}{\omega_0}y(t)z(t)\right). \quad (8.85)$$

When sweeping the admissible atom states (i.e. verifying  $x^2(t) + y^2(t) + z^2(t) \leq 1$  and  $x(t), y(t), z(t) \in [-1, 1]$ ), it is easy to show that  $\dot{W}_{\text{self}} \leq 0$  except when we have both

$$x(t) \in [-\sqrt{-y^2(t) - \epsilon y(t)z(t)}, \sqrt{-y^2(t) - \epsilon y(t)z(t)}],$$

$$y(t)(y(t) + \epsilon z(t)) < 0,$$

with  $\epsilon = \Omega/\omega_0 \ll 1$ . In this range of parameter, it is easy to check that the maximum value taken by  $\dot{W}_{\text{self}}(t)$  is  $\frac{\gamma\hbar\omega_0}{8}(\sqrt{1 + \epsilon^2} - 1) \simeq \gamma\hbar\omega_0\frac{\Omega^2}{16\omega_0^2} \ll \gamma\hbar\omega_0$ . Both the maximum positive value of  $\dot{W}_{\text{self}}$  and the range of parameters allowing to have  $\dot{W}_{\text{self}} \geq 0$  vanish as  $\mathcal{O}(\Omega^2/\omega_0^2)$ . Therefore, except in that narrow interval, where the work-like quantities are sensibly equivalent,  $\dot{W}_{\text{self}} < 0$ . To show that,  $\Sigma_{\text{closed}} \leq \Sigma_{\text{open}}$ , we need that  $\dot{W}_{\text{self}} - \dot{V}_{Sf} \leq 0$  where,

$$\dot{W}_{\text{self}} - \dot{V}_{Sf} = -\frac{\gamma\hbar\omega_0}{4}\left((x^2(t) + y^2(t)) + \epsilon y(t)z(t) + 2\epsilon\left(\bar{n}_{\text{th}} + \frac{1}{2}\right)y(t) + \frac{2\epsilon(\omega_L - \omega_0)}{\gamma}x(t)\right). \quad (8.86)$$

Similarly, we can notice, for a quasi-resonant field, where  $\epsilon\sqrt{(\bar{n}_{\text{th}} + \frac{1}{2})^2 + (\frac{\omega_L - \omega_0}{\gamma})^2} \ll 1$ , for the range of values considered, this quantity is negative, and hence, the field's bipartite heat is smaller than the heat received by the bath according to the OBE. This finally implies a smaller entropy production for the closed analysis, i.e.,  $\Sigma_{\text{closed}} \leq \Sigma_{\text{open}}$ .

## 8.7 Derivation of Clausius relation from the Collisional Model

In order to derive the closed Clausius relation, we employ the collision model description of the interaction between the atom and the waveguide field as described in chapter 3. Here we work in the displaced frame with respect to the initial displacement of the field and denote by  $\rho_n$  the state of the  $n^{\text{th}}$  unit of the field in the input cell before the collision and  $\rho'_n$  as the state of the same unit in the output cell after the collision. We begin the derivation with Eq. (6.35), where we rewrite the displacement operator (see Eq. (6.34)) in terms of displacements on the collisional units using Eq. (5.18)

$$\begin{aligned}\mathcal{D}(t) &= \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^T d\tau \mathcal{H}_f(\tau) \right\} \\ &= \bigotimes_n^N \exp \left\{ \sqrt{\gamma \Delta t} (\langle \sigma_+ \rangle_{t_n} b_n - \langle \sigma_- \rangle_{t_n} b_n^\dagger) \right\} \\ &= \bigotimes_n^N \mathcal{D}(\varphi_n)\end{aligned}\quad (8.87)$$

where,  $\varphi_n = -\sqrt{\gamma \Delta t} \langle \sigma_- \rangle_{t_n}$  and  $N = (T - t_0)/\Delta t$ . Substituting in Eq. (6.35) we get

$$\begin{aligned}\Sigma_{\text{closed}} &= -\text{Tr} \{ \rho_S(T) \ln \rho_S(T) \} + \text{Tr} \{ \rho_{Sf}(T) \ln \rho_{Sf}(T) \} \\ &\quad - \sum_n \text{Tr} \{ \rho'_n \ln (\mathcal{D}(\varphi_n) \rho_n \mathcal{D}^\dagger(\varphi_n)) \}.\end{aligned}\quad (8.88)$$

Now we use the fact that the state of the collisional unit is a thermal white noise where  $\rho_n = \eta_n^\beta$  to get

$$\begin{aligned}\Sigma_{\text{closed}} &= -\text{Tr} \{ \rho_S(T) \ln \rho_S(T) \} + \text{Tr} \{ \rho_{Sf}(T) \ln \rho_{Sf}(T) \} \\ &\quad + \beta \sum_n \text{Tr} \{ \mathcal{D}^\dagger(\varphi_n) \rho'_n \mathcal{D}(\varphi_n) H_n \} + \ln(Z_n) \\ &= \beta \sum_n \text{Tr} \{ (\mathcal{D}^\dagger(\varphi_n) \rho'_n \mathcal{D}(\varphi_n) - \rho_n) H_n \} + \Delta S_S,\end{aligned}\quad (8.89)$$

where  $\Delta S_S$  is the change of the entropy of the atom and we used that fact that  $\text{Tr} \{ \rho_{Sf}(t) \ln \rho_{Sf}(t) \} = \text{Tr} \{ \rho_{Sf}(0) \ln \rho_{Sf}(0) \}$  as it is a closed system. Up to an order of  $\gamma \Delta t$ , the action of the displacement on  $\rho'_n$  is

$$\mathcal{D}^\dagger(\varphi_n) \rho'_n \mathcal{D}(\varphi_n) = \rho'_n + \frac{i}{\hbar} \Delta t [\mathcal{H}_f(t_n), \rho'_n]. \quad (8.90)$$

Furthermore, at the order of  $\gamma\Delta t$ , the evolution of each of the field units reduces to

$$\rho'_n = \rho_n - \frac{i}{\hbar}\Delta t [\mathcal{H}_f(t_n), \rho_n] - \frac{i}{\hbar}\Delta t \text{Tr}_S \{ [V(t_n), \chi(t_n)] \}. \quad (8.91)$$

Substituting Eq. (8.90) and Eq. (8.91) in Eq.(8.89), we get

$$\begin{aligned} \Sigma_{\text{closed}} &= \Delta S_S + \beta \sum_n \text{Tr} \{ (\rho'_n - \rho_n) H_n \} \\ &\quad + \frac{i}{\hbar} \beta \sum_n \text{Tr} \{ \Delta t [\mathcal{H}_f(t_n), \rho_n] H_n \}. \end{aligned} \quad (8.92)$$

Here, we find the total change of internal energy of all the bath units

$$\begin{aligned} \Delta \mathcal{U}_f &= \sum_n \text{Tr} \{ (\rho'_n - \rho_n) H_n \} \\ &= -\frac{i}{\hbar} \sum_n \Delta t \text{Tr} \{ [H_n, \mathcal{H}_f(t_n)] \rho_n \} \\ &\quad - \frac{i}{\hbar} \sum_n \Delta t \text{Tr} \{ [H_n, V(t_n)] \chi(t_n) \}, \end{aligned} \quad (8.93)$$

where, the second equality is the discrete time version of  $\int dt \dot{\mathcal{U}}_f$ . We identify  $b\dot{\mathcal{W}}_n = -\frac{i}{\hbar} \text{Tr} \{ [H_n, \mathcal{H}_f(t_n)] \rho_n \}$  as the rate of bipartite work done on each bath unit and  $b\dot{\mathcal{Q}}_n = -\frac{i}{\hbar} \text{Tr} \{ [H_n, V(t_n)] \chi(t_n) \}$  as the rate of bipartite heat given to each bath unit. Hence, substituting  $\Delta \mathcal{U}_f$  and  $b\dot{\mathcal{W}}_n$  in Eq. (8.92) and taking the continuous limit we get

$$\begin{aligned} \Sigma_{\text{closed}} &= \Delta S_S + \beta \Delta \mathcal{U}_f - \beta \sum_n \Delta t b \dot{\mathcal{W}}_n \\ &\rightarrow \Delta S_S + \beta \Delta \mathcal{U}_f - \beta b \mathcal{W}_f \\ &= \Delta S_S + \beta b \mathcal{Q}_f. \end{aligned} \quad (8.94)$$

At resonance,  $b\mathcal{Q}_f = -b\mathcal{Q}_S$ , using which we find the Clausius relation in Eq. (6.36). The positivity of  $\Sigma_{\text{closed}}$  arises from the positivity of the relative entropy.

## 8.8 Gaussian State Transformations and Energetics in BS

A Gaussian state is fully described by its covariance matrix:

$$\mathcal{C} = \begin{pmatrix} \langle c^2 \rangle - \langle c \rangle^2 & \langle c^\dagger c \rangle - |\langle c \rangle|^2 \\ \langle c^\dagger c \rangle - |\langle c \rangle|^2 & \langle c^{\dagger 2} \rangle - \langle c^\dagger \rangle^2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (8.95)$$

$$= \frac{1}{2\hbar\omega_0} \begin{pmatrix} \omega_0^2 V(x) - V(p) + i\omega_0 (\langle [x, p]_+ \rangle - 2\langle x \rangle \langle p \rangle) & \omega_0^2 V(x) + V(p) \\ \omega_0^2 V(x) + V(p) & \omega_0^2 V(x) - V(p) - i\omega_0 (\langle [x, p]_+ \rangle - 2\langle x \rangle \langle p \rangle) \end{pmatrix}, \quad (8.96)$$

where,  $V(x) = \langle x^2 \rangle - \langle x \rangle^2$  and  $V(p) = \langle p^2 \rangle - \langle p \rangle^2$  and  $\omega_0$  is the frequency of the mode. The covariance matrix can also be defined as

$$\mathcal{C}' = \begin{pmatrix} V(x) & \frac{1}{2} \langle [x, p]_+ \rangle - \langle x \rangle \langle p \rangle \\ \frac{1}{2} \langle [x, p]_+ \rangle - \langle x \rangle \langle p \rangle & V(p) \end{pmatrix}. \quad (8.97)$$

The most general Gaussian state is determined by the three parameters,  $\alpha$ ,  $z = re^{i\theta}$  and  $\bar{n}$  which are the displacement, squeezing strength and the average thermal excitation number [FOP05]. For a general Gaussian state with triple  $(\alpha, (z = re^{i\theta}), \bar{n})$ , substituting Eq. (7.6) in Eq. (8.95) we can rewrite the covariance matrix as

$$\mathcal{C} = \begin{pmatrix} -(\bar{n} + \frac{1}{2}) e^{i\theta} \sinh(2r) & (\bar{n} + \frac{1}{2}) \cosh(2r) \\ (\bar{n} + \frac{1}{2}) \cosh(2r) & -(\bar{n} + \frac{1}{2}) e^{-i\theta} \sinh(2r) \end{pmatrix}. \quad (8.98)$$

The determinant of the covariance matrix gives the average thermal excitation number of the state,

$$|\mathcal{C}| = -\left(\bar{n} + \frac{1}{2}\right)^2 \implies \bar{n} = -\frac{1}{2} + \sqrt{-|\mathcal{C}|}. \quad (8.99)$$

The  $b$ -heat exchanged during the BS interaction between field modes  $A$  and  $B$  in section 7.1 is computed using Eq. (7.5), i.e., the change of the off-diagonal elements of  $\mathcal{C}$  after the BS transformation as

$$\begin{aligned} b\mathcal{Q}_A &= \hbar\omega_0 \left( \langle c_3^\dagger c_3 \rangle - |\langle c_3 \rangle|^2 \right) - \hbar\omega_0 \left( \langle c_1^\dagger c_1 \rangle - |\langle c_1 \rangle|^2 \right) \\ &= \hbar\omega_0 \left( \left( \bar{n}_3 + \frac{1}{2} \right) \cosh(2r_3) \right) - \left( \left( \bar{n}_1 + \frac{1}{2} \right) \cosh(2r_1) \right), \end{aligned} \quad (8.100)$$

which relates the average thermal excitation numbers and squeezing parameters of input states  $(\bar{n}_{1(2)}, z_{1(2)})$  and output states  $(\bar{n}_{3(4)}, z_{3(4)})$  of  $A(B)$  with the  $b$ -heat. As the BS transformation is a linear Gaussian transformation, the covariance matrix of the input states just transforms by the transformation of the operators inside with the BS transformations Eq. (7.2). This means that the triple  $(\alpha, (z = re^{i\theta}), \bar{n})$  of the outputs are related to those of the two inputs. For a BS with angle  $\varphi$  and initial input correlations  $\chi_{12}$ , the diagonal element of the covariance matrix of the output of  $A$  ( $\mathcal{C}_3$ ), in terms of the input parameters are

$$\begin{aligned} \langle c_3^2 \rangle - \langle c_3 \rangle^2 &= \cos^2 \left( \frac{\varphi}{2} \right) (\langle c_1^2 \rangle - \langle c_1 \rangle^2) + \sin^2 \left( \frac{\varphi}{2} \right) (\langle c_2^2 \rangle - \langle c_2 \rangle^2) + \sin(\varphi) (\langle c_1 c_2 \rangle - \langle c_1 \rangle \langle c_2 \rangle) \\ &\implies - \left( \bar{n}_3 + \frac{1}{2} \right) e^{i\theta_3} \sinh(2r_3) = -\cos^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_1 + \frac{1}{2} \right) e^{i\theta_1} \sinh(2r_1) \\ &\quad - \sin^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_2 + \frac{1}{2} \right) e^{i\theta_2} \sinh(2r_2) + \sin(\varphi) (\text{Tr} \{c_1 c_2 \chi_{12}\}) = (\mathcal{C}_3)_d. \end{aligned} \quad (8.101)$$

The off-diagonal element of  $\mathcal{C}_3$  are given by

$$\begin{aligned} \langle c_3^\dagger c_3 \rangle - |\langle c_3 \rangle|^2 &= \cos^2 \left( \frac{\varphi}{2} \right) (\langle c_1^\dagger c_1 \rangle - |\langle c_1 \rangle|^2) + \sin^2 \left( \frac{\varphi}{2} \right) (\langle c_2^\dagger c_2 \rangle - |\langle c_2 \rangle|^2) + \sin(\varphi) \text{ReTr} \{c_1 c_2^\dagger \chi_{12}\} \\ &\implies \left( \bar{n}_3 + \frac{1}{2} \right) \cosh(2r_3) = \cos^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_1 + \frac{1}{2} \right) \cosh(2r_1) + \sin^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_2 + \frac{1}{2} \right) \cosh(2r_2) \\ &\quad + \sin(\varphi) \text{ReTr} \{c_1 c_2^\dagger \chi_{12}\} = (\mathcal{C}_3)_{od}. \end{aligned} \quad (8.102)$$

Similarly, the diagonal element of the covariance matrix of the output of  $B$  ( $\mathcal{C}_4$ )

$$\begin{aligned} \langle c_4^2 \rangle - \langle c_4 \rangle^2 &= \sin^2 \left( \frac{\varphi}{2} \right) (\langle c_1^2 \rangle - \langle c_1 \rangle^2) + \cos^2 \left( \frac{\varphi}{2} \right) (\langle c_2^2 \rangle - \langle c_2 \rangle^2) + \sin(\varphi) (\langle c_1 c_2 \rangle - \langle c_1 \rangle \langle c_2 \rangle) \\ &\implies - \left( \bar{n}_4 + \frac{1}{2} \right) e^{i\theta_4} \sinh(2r_4) = -\sin^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_1 + \frac{1}{2} \right) e^{i\theta_1} \sinh(2r_1) \\ &\quad - \cos^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_2 + \frac{1}{2} \right) e^{i\theta_2} \sinh(2r_2) - \sin(\varphi) (\text{Tr} \{c_1 c_2 \chi_{12}\}) = (\mathcal{C}_4)_d, \end{aligned} \quad (8.103)$$

and the off-diagonal element of  $\mathcal{C}_4$

$$\begin{aligned}
\langle c_4^\dagger c_4 \rangle - |\langle c_4 \rangle|^2 &= \sin^2 \left( \frac{\varphi}{2} \right) \left( \langle c_1^\dagger c_1 \rangle - |\langle c_1 \rangle|^2 \right) + \cos^2 \left( \frac{\varphi}{2} \right) \left( \langle c_2^\dagger c_2 \rangle - |\langle c_2 \rangle|^2 \right) \\
&\quad - \sin(\varphi) \operatorname{ReTr} \left\{ c_1 c_2^\dagger \chi_{12} \right\} \\
\implies \left( \bar{n}_4 + \frac{1}{2} \right) \cosh(2r_4) &= \sin^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_1 + \frac{1}{2} \right) \cosh(2r_1) + \cos^2 \left( \frac{\varphi}{2} \right) \left( \bar{n}_2 + \frac{1}{2} \right) \cosh(2r_2) \\
&\quad - \sin(\varphi) \operatorname{ReTr} \left\{ c_1 c_2^\dagger \chi_{12} \right\} = (\mathcal{C}_4)_{od}. \tag{8.104}
\end{aligned}$$

Notice that when there are no initial correlations ( $\chi_{12} = 0$ ) and the BS is 50:50 ( $\varphi = \frac{\pi}{2}$ ),  $(\mathcal{C}_3)_d = (\mathcal{C}_4)_d$  and  $(\mathcal{C}_3)_{od} = (\mathcal{C}_4)_{od}$  and moreover, when there is no initial displacement, the average excitation number only depends on the thermal excitation and the squeezing parameter,  $N = (\bar{n} + \frac{1}{2}) \cosh(2r) - \frac{1}{2}$ . This implies that, under these conditions, the states attain the same excitation number  $N_3 = N_4 = \frac{N_1 + N_2}{2}$  and squeezing parameter  $z_3 = z_4$  after exchanging  $b\mathcal{Q}_A = \frac{\hbar\omega_0}{2} (N_2 - N_1) = -b\mathcal{Q}_B$  amount of  $b$ -heat.

The  $b$ -work exchanged is instead computed by taking the rate of  $b$ -work done,

$$\begin{aligned}
b\dot{\mathcal{W}}_A(u) &= -\frac{i}{\hbar} \operatorname{Tr} \{ [H_A, \mathcal{H}_A(u)] \rho_A(u) \} \\
&= \hbar\omega_0 \operatorname{Tr} \{ [h^\dagger h, h^\dagger v - v^\dagger h] \rho_A(u) \} \\
&= 2\hbar\omega_0 g \operatorname{Re} \left( \langle h \rangle_u \langle v^\dagger \rangle_u \right) \\
&= 2\hbar\omega_0 g \operatorname{Re} \left( (\cos(gu) \langle h \rangle_0 + \sin(gu) \langle v \rangle_0) (\cos(gu) \langle h^\dagger \rangle_0 - \sin(gu) \langle v^\dagger \rangle_0) \right) \\
&= 2\hbar\omega_0 g \operatorname{Re} \left( (\cos^2(gu) \langle h \rangle_0 \langle v^\dagger \rangle_0 + \sin(gu) \cos(gu) |\langle v \rangle_0|^2) \right) \\
&\quad - 2\hbar\omega_0 g \operatorname{Re} \left( (\cos(gu) \sin(gu) |\langle h \rangle_0|^2 + \sin^2(gu) \langle h^\dagger \rangle_0 \langle v \rangle_0) \right) \\
&= 2\hbar\omega_0 g \sin(gu) \cos(gu) (|\langle v \rangle_0|^2 - |\langle h \rangle_0|^2) \\
&\quad + 2\hbar\omega_0 g (\cos^2(gu) - \sin^2(gu)) \operatorname{Re} \{ \langle h \rangle_0 \langle v \rangle_0^* \}, \tag{8.105}
\end{aligned}$$

where, we have used Eq. (7.2). Integrating till time  $t$

$$\begin{aligned}
b\mathcal{W}_A(t) &= \int_0^t b\dot{\mathcal{W}}_A(u) du = \hbar\omega_0 g \int_0^t \sin(2gu) (|\langle v \rangle_0|^2 - |\langle h \rangle_0|^2) du \\
&\quad + \hbar\omega_0 g \int_0^t \cos(2gu) \operatorname{Re} \{ \langle h \rangle_0 \langle v \rangle_0^* \} du \\
&= \frac{\hbar\omega_0}{2} (\cos(\varphi) - 1) (|\langle h \rangle_0|^2 - |\langle v \rangle_0|^2) + \hbar\omega_0 \sin(\varphi) \operatorname{Re} \{ \langle h \rangle_0 \langle v \rangle_0^* \} \\
&= -\hbar\omega_0 \sin^2 \left( \frac{\varphi}{2} \right) (|\langle h \rangle_0|^2 - |\langle v \rangle_0|^2) + \hbar\omega_0 \sin(\varphi) \operatorname{Re} \{ \langle h \rangle_0 \langle v \rangle_0^* \}, \tag{8.106}
\end{aligned}$$

where, we have used that  $2gt = \varphi$ .

## 8.9 Generalization to the Pulsed Case

Here, we consider the BS interaction but between the orthogonal field modes that are not monochromatic, but are wave-packets with a certain temporal shape and spectrum centred around the same frequency  $\omega_0$ , i.e. each field is a reservoir of modes of frequency  $\omega_k$  and wave-number  $k = \omega_k/v$  with  $v$  the group velocity in the medium. The interaction is dictated by a multi-mode mixing Hamiltonian:

$$H = \sum_{k,l} \left\{ \hbar\omega_k a_k^\dagger a_k + \hbar\omega_k b_k^\dagger b_k + i\hbar g \left( a_k^\dagger b_l - b_l^\dagger a_k \right) \right\}. \quad (8.107)$$

It is convenient to do the Fourier transform of the operators  $a_k$  and  $b_k$  and define bosonic operators which create and destroy photons at a given time and position (cf. section 5.1):

$$\begin{aligned} \tilde{a}(x, t) &= \frac{1}{\sqrt{\varrho}} \sum_k e^{-i(\omega_k t - kx)} a_k, \\ \tilde{b}(y, t) &= \frac{1}{\sqrt{\varrho}} \sum_k e^{-i(\omega_k t - ky)} b_k, \end{aligned} \quad (8.108)$$

where,  $\varrho$  is the density of modes, having the dimension of a time. Let us notice that, since we are considering linear dispersion relations for both fields,  $\tilde{a}(x, t) = \tilde{a}(0, t - x/v)$ , (resp.  $\tilde{b}(y, t) = \tilde{b}(0, t - y/v)$ ).

Injecting Eqs. (8.108) in the interaction picture Hamiltonian Eq. (7.1) we get:

$$H_I(t) = \frac{i\hbar\varphi}{2} \left( \tilde{a}(0, t)^\dagger \tilde{b}(0, t) - \tilde{b}^\dagger(0, t) \tilde{a}(0, t) \right), \quad (8.109)$$

where  $\varphi = 2gt$  as defined in the main text. The interaction Hamiltonian in Eq. (8.109) is still a two-mode mixing Hamiltonian as Eq. (7.1), but now it is mixing temporal modes instead of frequency modes. Physically this can be pictured as a local interaction coupling the temporal modes of the two fields propagating along the BS axes, in the moment where they cross in the BS position ( $x = 0 = y$ ).

The annihilation operators in Eq. (8.108) can be discretized in time (with a spacing  $\Delta t$ ) as

$$\tilde{a}_n = \frac{1}{\sqrt{\Delta t}} \int_{n\Delta t}^{(n+1)\Delta t} dt \tilde{a}(0, t) \approx \frac{1}{\sqrt{\Delta t}} \tilde{a}(0, t_n) \Delta t = \tilde{a}(0, t_n) \sqrt{\Delta t}. \quad (8.110)$$

These quantities are similarly defined for the field  $B$  by replacing  $\tilde{a}$  by  $\tilde{b}$ . The unitary evolution operator over a time  $\Delta t$  can be written as:

$$U_n \equiv U_{\text{BS}}((n+1)\Delta t, n\Delta t) = e^{-\frac{i}{\hbar} \int_{n\Delta t}^{(n+1)\Delta t} dt H_I(t)} \approx e^{-\varphi(\tilde{a}^\dagger(0,t_n)\tilde{b}(0,t_n) - \tilde{b}^\dagger(0,t_n)\tilde{a}(0,t_n))\Delta t} = e^{-\varphi(\tilde{a}_n^\dagger\tilde{b}_n - \tilde{b}_n^\dagger\tilde{a}_n)}. \quad (8.111)$$

The unitary evolution operator over a time  $t$  is

$$U_{\text{BS}}(t, 0) = \otimes_{n=0}^{t/\Delta t} U_n. \quad (8.112)$$

Using the above relation and the commutation relation  $[\tilde{a}_n, \tilde{a}_m^\dagger] = \delta_{n,m}$  (resp.  $[\tilde{b}_n, \tilde{b}_m^\dagger] = \delta_{n,m}$ ), we obtain:

$$U_{\text{BS}}^\dagger(t_f, 0) \begin{pmatrix} \tilde{a}_n \\ \tilde{b}_n \end{pmatrix} U_{\text{BS}}(t_f, 0) = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \\ -\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right) \end{pmatrix} \begin{pmatrix} \tilde{a}_n \\ \tilde{b}_n \end{pmatrix}, \quad (8.113)$$

where  $t_f > t_n$  is the final time of the interaction.

The  $b$ -work  $b\mathcal{W}_A$  gained by the  $n$ -th temporal mode of the field  $A$  during the interaction in terms of  $\tilde{a}_n$  and  $\tilde{b}_n$  respectively is:

$$b\mathcal{W}_A = \hbar\omega_0 \left[ \sin(\varphi) \operatorname{Re} \left\{ \langle \tilde{a}_n \rangle_0 \langle \tilde{b}_n \rangle_0^* \right\} - \sin^2\left(\frac{\varphi}{2}\right) \left( |\langle \tilde{a}_n \rangle_0|^2 - |\langle \tilde{b}_n \rangle_0|^2 \right) \right]. \quad (8.114)$$

The total  $b$ -work gained by  $A$  at the final time  $t_f$  is then the sum of the  $b$ -work received by all the temporal modes:

$$\begin{aligned} b\mathcal{W}_A(t_f) &= \sum_{n=1}^{t_f/\Delta t} b\mathcal{W}_A \\ &= \sum_{n=0}^{t_f/\Delta t} \Delta t \hbar\omega_0 \left( \sin(\varphi) \operatorname{Re} \left\{ \langle \tilde{a}(0, t_n) \rangle_0 \langle \tilde{b}(0, t_n) \rangle_0^* \right\} - \sin^2\left(\frac{\varphi}{2}\right) \left( |\langle \tilde{a}(0, t_n) \rangle_0|^2 - |\langle \tilde{b}(0, t_n) \rangle_0|^2 \right) \right) \\ &\rightarrow \hbar\omega_0 \left( \sin(\varphi) \operatorname{Re} \left\{ \int_0^{t_f} dt \langle c_1(t) \rangle \langle c_2(t) \rangle^* \right\} - \sin^2\left(\frac{\varphi}{2}\right) \int_0^{t_f} dt \left( |\langle c_1(t) \rangle|^2 - |\langle c_2(t) \rangle|^2 \right) \right) \\ &= \int_0^{t_f} dt \hbar\omega_0 \left( |\langle c_3(t) \rangle|^2 - |\langle c_1(t) \rangle|^2 \right) \end{aligned} \quad (8.115)$$

where we defined  $\langle c_{1(3)}(t) \rangle = \langle \tilde{a}(0, t) \rangle_{0(t_f)}$  and  $\langle c_{2(4)}(t) \rangle = \langle \tilde{b}(0, t) \rangle_{0(t_f)}$  and we used the limit  $\sum \Delta t \rightarrow \int dt$ .

The  $b$ -heat can be obtained subtracting the  $b$ -work from the total energy change of  $A$ :

$$\begin{aligned} b\mathcal{Q}_A(t_f) &= \Delta\mathcal{U}_A(t_f) - b\mathcal{W}_A(t_f) \\ &= \int_0^{t_f} dt \hbar\omega_0 \left( \langle c_3^\dagger c_3(t) \rangle - |\langle c_3(t) \rangle|^2 \right) - \left( \langle c_1^\dagger c_1(t) \rangle - |\langle c_1(t) \rangle|^2 \right). \end{aligned} \quad (8.116)$$

## 8.10 Entropy Production

### Entropy production for coherent energy extraction

Let us begin by showing that, in the protocol to extract energy as discussed in section 7.4.1, all the  $b$ -work can be extracted by taking  $\alpha_{BQE} = -\langle c_4 \rangle$ , where one can infer the value of  $\langle c_4 \rangle$  from Eq. (7.2). Notice that, by doing an analysis similar to that in section 6.3, the  $b$ -work given to the field  $B$  is

$$b\mathcal{W}_B = \hbar\omega_0 (|\langle c_4 \rangle|^2 - |\langle c_2 \rangle|^2). \quad (8.117)$$

However,  $\langle c_2 \rangle = 0$  as the initial state of the field is purely thermal. Hence,  $b\mathcal{W}_B = \hbar\omega_0 |\langle c_4 \rangle|^2$ . The energy extracted when the displacement is applied on the state of field  $B$  after the BS interaction is

$$\begin{aligned} \mathcal{E}_B^{disp} &= \left\langle \mathcal{D}_B^\dagger(\alpha_{BQE}) H_B \mathcal{D}_B(\alpha_{BQE}) \right\rangle - \langle H_B \rangle \\ &= (b^\dagger + \alpha_{BQE}^*) (b + \alpha_{BQE}) \\ &= \hbar\omega_0 \left( \alpha_{BQE} \langle c_4^\dagger \rangle + \alpha_{BQE}^* \langle c_4 \rangle + |\alpha_{BQE}|^2 \right), \end{aligned} \quad (8.118)$$

where,  $H_B = \hbar\omega_0 b^\dagger b$ . Taking  $\alpha_{BQE} = -\langle c_4 \rangle$ , we get

$$\mathcal{E}_B^{disp} = -\hbar\omega_0 |\langle c_4 \rangle|^2 = -b\mathcal{W}_B. \quad (8.119)$$

Now we show that, by substituting  $\alpha_{BQE}$  in Eq. (7.14), we get a Clausius type relation with the  $b$ -heat. We start with Eq. (7.14)

$$\begin{aligned}
\Sigma_{\text{ext}}(\alpha_{BQE}) &= \mathcal{I}_{AB} + D\left(\mathcal{D}_B^\dagger(\alpha_{BQE})\rho_B(\tau)\mathcal{D}_B(\alpha_{BQE})\|\rho_B^{\text{th}}\right) \\
&= \Delta S_A + \beta \text{Tr}\left\{\left(\mathcal{D}_B^\dagger(\alpha_{BQE})\rho_B(\tau)\mathcal{D}_B(\alpha_{BQE}) - \rho_B^{\text{th}}\right)H_B\right\} \\
&= \Delta S_A + \beta \text{Tr}\left\{\left(\mathcal{D}_B^\dagger(\alpha_{BQE})\rho_B(\tau)\mathcal{D}_B(\alpha_{BQE}) - \rho_B(\tau) + \rho_B(\tau) - \rho_B^{\text{th}}\right)H_B\right\} \\
&= \Delta S_A + \beta \mathcal{E}_B^{\text{disp}} + \beta \Delta \mathcal{U}_B.
\end{aligned} \tag{8.120}$$

Using Eq. (8.118), we finally get

$$\Sigma_{\text{ext}}(\alpha_{BQE}) = \Delta S_A - \beta b \mathcal{W}_B + \beta \Delta \mathcal{U}_B = \Delta S_A - \beta b \mathcal{Q}_A \geq 0, \tag{8.121}$$

where, we have used the fact that the field's are resonant and hence,  $b\mathcal{Q}_B = -b\mathcal{Q}_A$ .

## Entropy production of squeezing protocol

Here we consider a BS interaction (of arbitrary angle  $\varphi$ ) between a thermal field ( $A$ ) and a squeezed thermal field  $B$  such that the squeezed field is at a lower temperature. Field  $B$  is squeezed with strength  $z = r \in \mathbb{R}$  to give  $\rho_B(t)$  while  $A$  is a thermal state, each having initial average excitation numbers  $N_B = (\bar{n}_B + \frac{1}{2}) \cosh(2r) - \frac{1}{2}$  and  $N_A = \bar{n}_A$  as described in section 7.4. We assume that the fields are initially uncorrelated. They exchange  $b$ -heat through the BS interaction. Substituting Eq. (8.102) in Eq. (7.5), we find that the  $b$ -heat received by the 'hotter' field  $A$  is

$$b\mathcal{Q}_A = \sin^2\left(\frac{\varphi}{2}\right) \left[ \left(\bar{n}_B + \frac{1}{2}\right) \cosh(2r) - \left(\bar{n}_A + \frac{1}{2}\right) \right]. \tag{8.122}$$

Hence, for  $b\mathcal{Q}_A \geq 0$  we need that the squeezing must satisfy

$$\frac{\bar{n}_A - \bar{n}_B}{(1 + 2\bar{n}_B)} \leq \sinh^2(r). \tag{8.123}$$

The entropy production of the squeezing protocol  $\Sigma_{\text{prep}}$  can be written in terms of  $b$ -heat provided to  $B$  as

$$\begin{aligned}
\Sigma_{\text{prep}} &= \mathcal{I}_{AB}(\tau) + D(\rho_B(\tau) \parallel \rho_B^{\text{th}}) \\
&= \beta_B \text{Tr}(\rho_B(\tau) H) + \ln Z + \Delta S_A - S(\rho_B^{\text{th}}) \\
&= \Delta S_A + \beta_B (\text{Tr}(\rho_B(\tau) H) - \text{Tr}(\rho_B(t) H) + \text{Tr}(\rho_B(t) H) - \text{Tr}(\rho_B^{\text{th}} H)) \\
&= \Delta S_A + \beta_B (b\mathcal{Q}_B + \mathcal{E}_B^{\text{sq}}) \\
&= \Delta S_A - \beta_B (b\mathcal{Q}_A - \mathcal{E}_B^{\text{sq}}), \tag{8.124}
\end{aligned}$$

where,  $\mathcal{E}_B^{\text{sq}} = \text{Tr}\{H(\rho(t) - \rho_B^{\text{th}})\}$  is the energy provided by the initial squeezing. In the second line, we have used the fact that the total entropy change of the bipartite system is zero and hence, the mutual information is the sum of the change of von-Neumann entropy of each subsystem.



# Conclusion and Future Directions (English)

Creation of correlations between quantum systems gives rise to a myriad of interesting phenomenon. On the one hand, knowledge about them leads to desirable protocols such as quantum teleportation and on the other, disregarding them can lead to various losses. This knowledge has moreover a clear thermodynamic impact affecting the loss or gain of energy. In this thesis, we have proposed a route to track these correlations as they are created. Through this we have determined the impact that they have on interacting quantum systems, furthermore paving a way to closing open system dynamics and tightening the second law of thermodynamics.

## 9.1 Summary and Inferences

One of the aims of this thesis was to study and track correlations that are usually considered to be lost. To fulfill this aim, we began by analyzing the dynamics between an emitter and a field in an autonomous manner. We showed that an autonomous collisional model (ACM) arises in the weak-coupling regime of Jaynes-Cummings interaction. We upgraded the existing collisional models in order to take into account arbitrary distributions of the coupling and found how the well known Lamb shifts and temperature induced frequency shifts may arise during the interaction and the order of correction induced by these shifts. We provided useful errors bounds in using the collisional model which at the very least, indicates the regime of validity of our work.

To make a fair comparison between the effects of closing open systems and already existing literature in the theory of open quantum systems, we analyzed the regime of the Optical Bloch Equations (OBE). First, we treated the emitter-field interaction in an isolated bipartite manner in this regime and systematically separated the contributions of correlations created between them. In order to do so, we bridged the gap between the ACM that we developed and the Dyson series. Utilizing this connection, we tracked the change of state of the emitter, field and even the creation of the correlations occurring during each collision at different orders of interaction strength. This led to us singling out a fundamental driving term that is mediated by the field on the emitter, that was previously hidden in the standard dissipator of the OBE. Next, we analyzed the spectrum of the output field. Again applying our framework and separating the correlations, we found

observable signatures of correlations on the spectrum of the field providing physicality to our findings. This analysis deepens our understanding of fundamental processes at play during emitter-field interactions.

These processes also have an energetic imprint. To analyze the energy exchanges we developed upon a symmetric energetic framework to separate the contributions of correlations to the energy. This bipartite quantum energetics (BQE) framework puts the thermodynamic cut at the correlations, i.e., assumes complete knowledge of only the sub-systems. We compared this starkly different approach to the existing thermodynamics of the OBE and found additional work contributions, even during spontaneous emission processes. Remarkably we showed that this additional work can be measured and even extracted. Consequently this extraction led to a tightening of the second law, where the entropy is bound by the energy brought by correlations that are considered lost, revealing that more knowledge leads to lesser production of entropy.

To build intuitions on the energetics and consequences on the second law, we considered energy transfers between baths. We did this by studying Gaussian fields interfering via a beam splitter and found measurable indicators of correlations and coherences being exchanged. We again found tighter bounds on the entropy produced provided all the initial resources are taken into account. The analysis of the thermodynamics in this framework, clearly indicates the connections between the experimental access to systems and the entropy produced. It reveals that with more information and the ability to use this information, smaller entropy is produced and hinting towards better reversibility of processes.

## 9.2 Future Directions

The framework presented in this thesis is relatively new and presents answers to multiple questions, both in the dynamics and thermodynamics of interactions with light. It provides a new perspective on heat and work exchanged between quantum systems. It paves the way to address fundamental questions and brings new questions to the field of quantum optics. Let us ask some of these questions here:

- We began this thesis with the assumption that the thermodynamic cut was placed on the correlations. But what if experiments could access elements in a correlated basis too? It would be interesting to apply this framework to the same regime of the OBE where elements of the dressed basis could be measured. All the  $b$ -heat would no longer be lost and it could be possible to extract it too.

- In chapters 6 and 7, we saw that displacement operations could be used to extract work. What if more single body unitaries were allowed without an energy expense? One could in principle extract all the ergotropy. Indeed  $b$ -work has already been related to the ergotropy in [MCA21]. This study will have clear consequences on the energetics of quantum batteries.
- We have shown that  $b$ -heat is an indicator of past correlations and is measurable on the fluctuations of the field output modes. This could, in principle, be used to detect errors in protocols and quantum networks involving light fields as correlations lead to loss of purity of the information carrier.
- In chapter 7, we saw that the initial squeezing of the bath had to be taken into account to find a tighter second law. It would be interesting to connect this energy provided by the squeezing to the correlations created during this squeezing process. In this case, only fundamentally lost correlations could be related to the second law.
- Throughout the thesis, we have separated the correlations created. However, we make no distinction between classical and quantum correlations. To do so, capturing the relation between  $b$ -heat and quantum correlation measures could be revealing.
- This thesis has been a fundamental analysis of dynamics and energetics of quantum systems. However, tracking correlations in setups such as the 1D atom and the beam splitter could have practical applications as they form the basic building blocks for superconducting and optical devices. We have shown throughout the thesis that formation of correlations is required to deform a field. Hence, Wigner negativity from Wigner non-negative states is also created through creation then loss of correlations. It would be interesting to study the dynamics of creation of the negativity through this framework. This would allow for optimization in the way negativity is generated. These Wigner negative states would then be useful for photonic computation.

From the above questions, it is clear that there are multiple areas to apply the framework developed in this thesis both in fundamental and applied research. This work is hence, a step towards building a clear connection between fundamental and applied research which is becoming a growing need in the scientific community and is important for further development of today's society.



La création de corrélations entre systèmes quantiques donne lieu à une myriade de phénomènes intéressants. D'un côté, la connaissance de ces corrélations conduit à des protocoles souhaitables tels que la téléportation quantique et de l'autre, leur méconnaissance peut entraîner diverses pertes. Cette connaissance a en outre un impact thermodynamique clair affectant la perte ou le gain d'énergie. Dans cette thèse, nous avons proposé une voie pour suivre ces corrélations au fur et à mesure de leur création. Grâce à cela, nous avons déterminé l'impact qu'elles ont sur les systèmes quantiques en interaction, ouvrant en outre la voie à la fermeture de la dynamique des systèmes ouverts et au renforcement de la deuxième loi de la thermodynamique.

## 10.1 Résumé et conclusions

L'un des objectifs de cette thèse était d'étudier et de suivre les corrélations qui sont généralement considérées comme perdues. Pour atteindre cet objectif, nous avons commencé par analyser la dynamique entre un émetteur et un champ de manière autonome. Nous avons montré qu'un modèle collisionnel autonome (ACM) apparaît dans le régime de couplage faible de l'interaction Jaynes-Cummings. Nous avons mis à niveau les modèles collisionnels existants afin de prendre en compte les distributions arbitraires du couplage et avons trouvé comment les décalages de Lamb et les décalages de fréquence induits par la température bien connus peuvent apparaître pendant l'interaction et l'ordre de correction induit par ces décalages. Nous avons fourni des bornes d'erreur utiles en utilisant le modèle collisionnel qui, à tout le moins, indiquent le régime de validité de notre travail.

Pour faire une comparaison équitable entre les effets de la fermeture des systèmes ouverts et la littérature déjà existante dans la théorie des systèmes quantiques ouverts, nous avons analysé le régime des équations de Bloch optiques (OBE). Tout d'abord, nous avons traité l'interaction émetteur-champ de manière bipartite isolée dans ce régime et avons systématiquement séparé les contributions des corrélations créées entre elles. Pour ce faire, nous avons comblé l'écart entre l'ACM que nous avons développé et la série de Dyson. En utilisant cette connexion, nous avons suivi le changement d'état de l'émetteur, du champ et même la création des corrélations se produisant lors de chaque collision à différents ordres de force d'interaction. Cela nous a conduit à distinguer un terme moteur

fondamental qui est médiatisé par le champ sur l'émetteur, qui était auparavant caché dans le dissipateur standard de l'OBE. Ensuite, nous avons analysé le spectre du champ de sortie. En appliquant à nouveau notre cadre et en séparant les corrélations, nous avons trouvé des signatures observables de corrélations sur le spectre du champ donnant une matérialité à nos résultats. Cette analyse approfondit notre compréhension des processus fondamentaux en jeu lors des interactions émetteur-champ.

Ces processus ont également une empreinte énergétique. Pour analyser les échanges d'énergie, nous avons développé un cadre énergétique symétrique pour séparer les contributions des corrélations à l'énergie. Ce cadre d'énergétique quantique bipartite (BQE) place la coupure thermodynamique au niveau des corrélations, c'est-à-dire qu'il suppose une connaissance complète des seuls sous-systèmes. Nous avons comparé cette approche radicalement différente à la thermodynamique existante de l'OBE et avons trouvé des contributions de travail supplémentaires, même pendant les processus d'émission spontanée. De manière remarquable, nous avons montré que ce travail supplémentaire peut être mesuré et même extrait. Par conséquent, cette extraction a conduit à un resserrement de la deuxième loi, où l'entropie est liée à l'énergie apportée par les corrélations qui sont considérées comme perdues, révélant qu'une plus grande connaissance conduit à une moindre production d'entropie.

Pour construire des intuitions sur l'énergétique et les conséquences sur la deuxième loi, nous avons considéré les transferts d'énergie entre les bains. Nous l'avons fait en étudiant les champs gaussiens interférant via un séparateur de faisceau et avons trouvé des indicateurs mesurables des corrélations et des cohérences échangées. Nous avons de nouveau trouvé des limites plus strictes sur l'entropie produite à condition que toutes les ressources initiales soient prises en compte. L'analyse de la thermodynamique dans ce cadre indique clairement les liens entre l'accès expérimental aux systèmes et l'entropie produite. Elle révèle qu'avec plus d'informations et la capacité d'utiliser ces informations, une entropie plus faible est produite et laisse entrevoir une meilleure réversibilité des processus.

## 10.2 Orientations futures

Le cadre présenté dans cette thèse est relativement nouveau et apporte des réponses à de multiples questions, tant dans la dynamique que dans la thermodynamique des interactions avec la lumière. Il offre une nouvelle perspective sur la chaleur et le travail échangés entre les systèmes quantiques. Il ouvre la voie à des questions fondamentales et apporte de

nouvelles questions au domaine de l'optique quantique. Posons-nous ici certaines de ces questions :

- Nous avons commencé cette thèse avec l'hypothèse que la coupure thermodynamique était placée sur les corrélations. Mais que se passerait-il si les expériences pouvaient aussi accéder à des éléments dans une base corrélée ? Il serait intéressant d'appliquer ce cadre au même régime de l'OBE où des éléments de la base vestimentaire pourraient être mesurés. Toute la  $b$ -heat ne serait plus perdue et il pourrait être possible de l'extraire également.
- Dans les chapitres 6 et 7, nous avons vu que les opérations de déplacement pouvaient être utilisées pour extraire du travail. Et si davantage d'unités à corps unique étaient autorisées sans dépense d'énergie ? On pourrait en principe extraire toute l'ergotropie. En effet, le  $b$ -work a déjà été lié à l'ergotropie dans [MCA21]. Cette étude aura des conséquences claires sur l'énergétique des batteries quantiques.
- Nous avons montré que la chaleur  $b$  est un indicateur des corrélations passées et qu'elle est mesurable sur les fluctuations des modes de sortie du champ. Cela pourrait en principe être utilisé pour détecter des erreurs dans les protocoles impliquant des champs lumineux car les corrélations conduisent à une perte de pureté du support d'information.
- Dans le chapitre 7, nous avons vu que la compression initiale du bain devait être prise en compte pour trouver une seconde loi plus stricte. Il serait intéressant de relier cette énergie apportée par la compression aux corrélations créées lors de ce processus de compression. Dans ce cas, seules les corrélations fondamentalement perdues pourraient être reliées à la seconde loi.
- Tout au long de la thèse, nous avons séparé les corrélations créées. Cependant, nous ne faisons aucune distinction entre les corrélations classiques et quantiques. Pour ce faire, capturer la relation entre la  $b$ -heat et les mesures de corrélation quantique pourrait être révélateur.
- Cette thèse a été une analyse fondamentale de la dynamique et de l'énergétique des systèmes quantiques. Cependant, le suivi des corrélations dans des configurations telles que l'atome 1D et le séparateur de faisceau pourrait avoir des applications pratiques car elles constituent les éléments de base des dispositifs supraconducteurs et optiques. Nous avons montré tout au long de la thèse que la formation de corrélations est nécessaire pour déformer un champ. Par conséquent, la négativité de Wigner est

également créée par des corrélations. Il serait intéressant d'étudier la dynamique de création de la négativité à travers ce cadre. Cela permettrait d'optimiser la manière dont la négativité est générée. Ces états négatifs de Wigner seraient alors utiles pour le calcul photonique.

Les questions ci-dessus montrent clairement qu'il existe de nombreux domaines d'application du cadre développé dans cette thèse, tant en recherche fondamentale qu'en recherche appliquée. Ce travail constitue donc une étape vers l'établissement d'un lien clair entre la recherche fondamentale et la recherche appliquée, qui devient un besoin croissant dans la communauté scientifique et qui est important pour le développement futur de la société d'aujourd'hui.

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