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Emmanuel Kammerer

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INSTITUT
POLYTECHNIQUE
DE PARIS

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Thèse de doctorat



Cartes aléatoires $3/2$ -stables

Thèse de doctorat de l'Institut Polytechnique de Paris
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Thèse présentée et soutenue à Palaiseau, le 30/06/2025, par

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Résumé

Cette thèse se concentre sur l'étude du comportement asymptotique de modèles de graphes aléatoires issus de deux grandes familles : les cartes planaires aléatoires et les arbres aléatoires. Dans une première partie, nous étudions la géométrie de grandes cartes aléatoires avec des sommets de grands degrés, qui correspondent au dual des cartes à grandes faces et sont appelées des cartes stables, avec un intérêt tout particulier pour le cas critique : celui des cartes $3/2$ -stables. Nous démontrons que ces cartes apparaissent naturellement dans l'étude des cartes aléatoires couplées au modèle de boucles $O(n)$, modèle classique de mécanique statistique, dans le cas $n = 2$. Nous obtenons le comportement asymptotique des distances entre deux sommets uniformes et démontrons que le diamètre est du même ordre de grandeur. Une conséquence est que les cartes $3/2$ -stables ne satisfont pas de limite d'échelle aux sens habituels de Gromov-Hausdorff ou de Gromov-Prokhorov. Cependant, en nous concentrant sur les sommets de grands degrés, nous obtenons la limite d'échelle des distances entre les sommets de grands degrés et la racine. Cette limite d'échelle correspond à une distance entre les boucles d'un ensemble conforme de boucles (CLE) et le bord du disque, mesurée à l'aide d'une gravité quantique de Liouville indépendante, qui est la transformée de Lamperti de la distance quantique au bord introduite par Aru, Holden, Powell et Sun. Nous explorons aussi les liens entre cette distance et la distance des boucles du CLE au bord introduite par Werner et Wu. Au passage, nous obtenons une propriété de Markov pour des exploration du CLE sur un 2-disque quantique indépendant en combinant un résultat de Miller, Sheffield et Werner avec un résultat de convergence obtenu par Lehmkuehler. Nous étudions ensuite les géodésiques à la racine pour la distance de percolation de premier passage dans les cartes stables et obtenons notamment que le diamètre des cartes stables dans le cas dense est logarithmique. Nous construisons la limite d'échelle des géodésiques au moyen d'un flot coalescent de diffusions à sauts. Ce dernier nous permet aussi de construire des espaces métriques aléatoires que l'on conjecture être les limites d'échelle des cartes aléatoires à grands degrés.

Dans une seconde partie, nous introduisons une généralisation du modèle des arbres récursifs uniformes en ajoutant un mécanisme de gel. À certaines étapes, un sommet choisi au hasard est gelé et plus aucun nouveau sommet ne peut s'attacher à celui-ci. Les arbres plans uniformes et l'arbre des infections dans un modèle SIR sur le graphe complet rentrent dans le cadre de ce modèle. Selon la répartition des instants de gel, la géométrie de l'arbre est radicalement différente. Nous obtenons les limites locales de ces arbres et les limites d'échelle des distances entre la racine et un sommet typique, de la distance entre deux sommets typiques et de la hauteur totale. Lorsque le nombre de sommets non gelés évolue linéairement, la hauteur totale de l'arbre est logarithmique et il n'y a pas de limite d'échelle aux sens de Gromov-Hausdorff ou de Gromov-Prokhorov. Dans le cas sous-linéaire où le nombre de sommets non gelés évolue comme une puissance donnée du nombre de sommets, nous identifions une transition de phase. Dans le régime critique, nous démontrons que l'arbre satisfait une limite d'échelle vers un arbre réel compact aléatoire relié étroitement à un coalescent de Kingman inhomogène en temps identifié par Aldous. En utilisant des techniques similaires de processus coalescents, nous obtenons enfin la limite d'échelle d'un autre modèle d'arbres aléatoires : des arbres aléatoires uniformes dont on fixe le degré et la hauteur de chaque sommet. Nous montrons, sous des conditions naturelles de convergence du profil, que ces arbres, une fois renormalisés, convergent. En guise d'application, nous construisons des limites d'échelle d'arbres de Bienaymé-Galton-Watson en environnement variable.

Abstract

This thesis focuses on the asymptotic behaviour of models of random graphs from two large families: random planar maps and random trees. In a first part, we study the geometry of large random planar maps with high degree vertices, which correspond to the dual of maps with large faces and are called stable maps, with a particular interest in the critical case: the case of $3/2$ -stable maps. We prove that these maps appear naturally in the study of random planar maps coupled with a loop $O(n)$ model, a classical model of statistical mechanics, in the case $n = 2$. We obtain the asymptotic behaviour of the distances between two uniform random vertices and we prove that the diameter is of the same order. A consequence of these results is that $3/2$ -stable maps do not satisfy scaling limits in the usual sense of Gromov-Hausdorff or Gromov-Prokhorov. However, focusing on high degree vertices, we get the scaling limit of the distances between high degree vertices and the root. This scaling limit corresponds to a distance between the loops of a conformal loop ensemble (CLE) and the boundary of the disc, measured using an independent Liouville quantum gravity, which is the Lamperti transform of the “quantum distance” to the boundary introduced by Aru, Holden, Powell and Sun. We also explore the relations between this distance and the distance to the boundary introduced by Werner and Wu. En route, we obtain a Markov property for CLE explorations on an independent 2-quantum disc by combining a result of Miller, Sheffield and Werner with a convergence result of Lehmkuehler. We then study the geodesics to the root for the first passage percolation distance on stable maps and obtain for instance that the diameter of stable maps is logarithmic in the dense case. We construct the scaling limit of the geodesics by means of a coalescing flow of pure jump diffusions. This flow also enables us to construct random metric spaces which we conjecture to be the scaling limit of random maps with high degrees.

In a second part, we introduce a generalisation of the model of random recursive trees by adding a freezing mechanism. At some steps, a uniform random vertex is frozen and new vertices can no longer be attached to this vertex. Random uniform plane trees and the infection tree in an SIR model on the complete graph fall within this framework. Depending on the distribution of the freezing steps, the geometry of the tree is drastically different. We obtain local limits of these trees and scaling limits of the distances between the root and a typical vertex, of the distance between two uniform typical vertices and of the total height of the tree obtained after n steps. When the number of non-frozen vertices grows linearly in n , the total height of the tree is logarithmic and there is no scaling limit in the sense of Gromov-Hausdorff or Gromov-Prokhorov. In the sub-linear case where the number of non-frozen vertices evolves as a given power of n , we identify a phase transition. In the critical regime, we prove that the tree satisfies a scaling limit towards a random compact real tree, closely related to a time non-homogenous Kingman coalescent process identified by Aldous. Lastly, using similar techniques of coalescing processes, we get the scaling limit of another model of random trees. More precisely, we consider large uniform random trees where we fix for each vertex its degree and height. We prove, under natural conditions of convergence for the profile, that those trees properly renormalised converge. As an application, we obtain scaling limits of Bienaymé-Galton-Watson trees in varying environment.

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Chapitre 1

Introduction

Dans cette introduction, nous présentons d’abord les cartes planaires aléatoires qui sont l’objet d’étude de la première partie du manuscrit, consacrée à l’étude des cartes planaires aléatoires à grands degrés et constituée des articles [Kam23b, Kam23a, Kam24a, Kam24b]. Ensuite, nous présentons les modèles des arbres récursifs gelés et des arbres aléatoires à degrés et hauteurs fixés qui sont introduits et étudiés dans la deuxième partie de cette thèse qui regroupe les articles [BBRKK23a, BBRKK23b, BRK24].

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1.1 Cartes planaires aléatoires

L’étude de la géométrie de cartes planaires aléatoires à grands degrés est l’objet de la première partie de cette thèse. Nous donnons dans cette première section les définitions des cartes planaires et des modèles de cartes planaires aléatoires qui seront étudiés. Nous décrivons aussi comment les cartes planaires aléatoires à grands degrés, appelées cartes stables, apparaissent naturellement dans l’étude de modèles de mécanique statistique couplés à des cartes aléatoires.

1.1.1 Cartes planaires

Les cartes planaires peuvent être considérées comme des graphes dessinés sur la sphère vus à déformation près.

Définition 1.1.1. *Une carte planaire est un (multi-)graphe (boucles et arêtes multiples autorisées) connexe non orienté, localement fini, plongé proprement dans la sphère (euclidienne, dans l’espace à trois dimensions) et vu à homéomorphisme de la sphère préservant l’orientation près.*

Plus précisément, chaque sommet du graphe est envoyé sur un point de la sphère de manière injective et chaque arête est envoyée sur un chemin joignant les deux points correspondants ne croisant pas les autres chemins et sommets. Les composantes connexes de la sphère privée des chemins sont appelées faces et sont délimitées par des arêtes. De manière équivalente, une carte planaire peut aussi être vue comme un recollement connexe de polygones homéomorphe à la sphère.

Une carte planaire \mathbf{m} est dite finie si elle a un nombre fini d'arêtes. Dans ce cas, les nombres de faces, de sommets et d'arêtes de \mathbf{m} sont finis et sont notoirement reliés par la formule d'Euler :

$$\#\text{Sommets}(\mathbf{m}) + \#\text{Faces}(\mathbf{m}) - \#\text{Arêtes}(\mathbf{m}) = 2.$$

Pour faciliter leur énumération, on munit les cartes d'une arête orientée distinguée appelée arête racine. La face à droite de l'arête racine est appelée la face racine ou face externe et sera notée f_r . Le nombre d'arêtes bordant une face f d'une carte est appelé le degré de cette face et est noté $\text{deg}(f)$. Le degré de la face racine $\text{deg}(f_r)$ est aussi appelé le périmètre de la carte.

À toute carte \mathbf{m} peut être associée sa carte duale \mathbf{m}^\dagger obtenue en plaçant un sommet sur chaque face de \mathbf{m} et en reliant deux sommets correspondant à deux faces adjacentes. Comme dans [Cur23], nous définissons l'arête racine de la carte duale comme l'arête duale correspondant à l'arête à droite de l'arête racine en tournant autour de la face racine et on l'oriente vers l'intérieur de la face racine. Voir la figure 1.1.

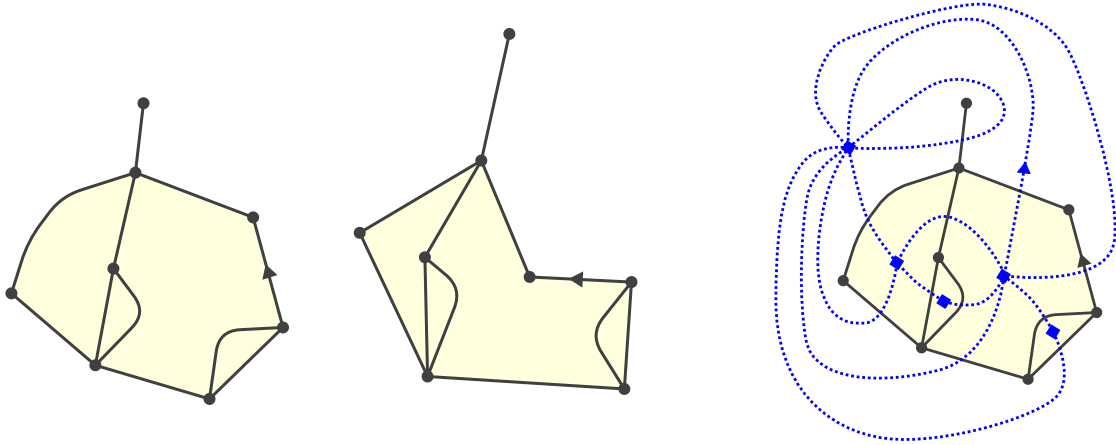


FIGURE 1.1 – Gauche et centre : la même carte planaire bipartie enracinée \mathbf{m} de périmètre 8. La face racine est en blanc. Droite : sa carte duale \mathbf{m}^\dagger est dessinée en pointillés bleus.

Encore pour des raisons combinatoires, on se restreint aux cartes biparties, c'est-à-dire aux cartes dont les faces ont toutes un degré pair. Leur énumération est bien plus simple que celle des cartes non biparties. Tutte, pionnier dans l'énumération des cartes, a montré dans [Tut62] que le nombre de cartes planaires biparties non enracinées avec k faces étiquetées de 1 à k de degrés respectifs $2\ell_1, \dots, 2\ell_k$, où $\ell_1, \dots, \ell_k \geq 1$ sont des entiers, ayant chacune un coin (secteur angulaire autour d'un sommet délimité par deux arêtes) distingué est égal à

$$\frac{(e-1)!}{v!} \prod_{i=1}^k \frac{(2\ell_i)!}{\ell_i!(\ell_i-1)!},$$

où $e = \sum_i \ell_i$ et $v = 2 + e - k$ sont respectivement le nombre d'arêtes et le nombre de sommets d'une telle carte. La formule ci-dessus est souvent appelée formule des tranches, du fait que Tutte considérait des objets combinatoires équivalents aux cartes obtenus en saucissonnant une sphère privée d'un nombre fini de composantes simplement connexes délimitées par des lacets simples disjoints en reliant des sommets placés sur ces lacets.

On note \mathcal{M} l'ensemble des cartes (planaires, biparties, enracinées) finies et, pour tout entier $\ell \geq 1$, on note $\mathcal{M}^{(\ell)}$ l'ensemble des cartes finies de périmètre 2ℓ . Une carte \mathbf{m} peut être munie de différentes distances, notamment :

- La distance de graphe (primal) d_{gr} sur les sommets de \mathbf{m} , qui est obtenue en donnant une longueur 1 à chaque arête ;
- La distance de graphe dual d_{gr}^\dagger sur les faces de \mathbf{m} , obtenue en donnant une longueur 1 à chaque arête duale. Elle correspond à la distance de graphe sur la carte duale \mathbf{m}^\dagger ;
- La distance de percolation de premier passage d_{fpp}^\dagger sur les faces de \mathbf{m} , obtenue en donnant des longueurs aléatoires exponentielles indépendantes de paramètre 1 à chaque arête duale.

1.1.2 Cartes aléatoires de Boltzmann

Pour tirer une carte aléatoire, on peut par exemple se restreindre aux cartes à n arêtes et en tirer une uniformément au hasard. Une façon générale de tirer une carte aléatoire consiste à associer un poids de Boltzmann à chaque face selon son degré et de tirer chaque carte proportionnellement au produit de ces poids. Ce modèle de cartes aléatoires dites de Boltzmann a été introduit dans [MM07]. Étant donnée une suite non nulle de poids positifs $\mathbf{q} = (q_k)_{k \geq 1}$, on définit le poids d'une carte $\mathbf{m} \in \mathcal{M}$ par

$$w_{\mathbf{q}}(\mathbf{m}) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r\}} q_{\deg(f)/2}.$$

Pour tout entier $\ell \geq 1$, la fonction de partition des cartes de périmètre 2ℓ est définie par

$$W^{(\ell)} = \sum_{\mathbf{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}).$$

Lorsque $W^{(\ell)} < \infty$ pour tout $\ell \geq 1$, on dit que la suite de poids \mathbf{q} est admissible. En fait, on sait que \mathbf{q} est admissible dès que $W^{(\ell)} < \infty$ pour un certain $\ell \geq 1$, voir le corollaire 3.15 dans [Cur23]. Par le lemme 3.13 de [Cur23], on sait aussi que si \mathbf{q} est admissible, alors il existe une constante $c_{\mathbf{q}} > 0$ telle que

$$\frac{W^{(\ell+1)}}{W^{(\ell)}} \xrightarrow{\ell \rightarrow +\infty} c_{\mathbf{q}}. \quad (1.1.1)$$

Lorsque \mathbf{q} est admissible, pour tout entier $\ell \geq 1$, on note $\mathbb{P}^{(\ell)}$ la mesure de probabilité sur $\mathcal{M}^{(\ell)}$, appelée mesure de Boltzmann, définie par

$$\forall \mathbf{m} \in \mathcal{M}^{(\ell)}, \quad \mathbb{P}^{(\ell)}(\{\mathbf{m}\}) = \frac{w_{\mathbf{q}}(\mathbf{m})}{W^{(\ell)}}.$$

On note $\mathbb{E}^{(\ell)}$ l'espérance associée. Une carte aléatoire de loi $\mathbb{P}^{(\ell)}$ est appelée une carte aléatoire de \mathbf{q} -Boltzmann de périmètre 2ℓ .

La géométrie des cartes de Boltzmann est intimement liée au comportement asymptotique de la fonction de partition $W^{(\ell)}$ quand $\ell \rightarrow +\infty$. En calculant la fonction de partition et en utilisant la méthode de Laplace, on peut montrer que

$$W^{(\ell)} = O(c_{\mathbf{q}}^{\ell} \ell^{-3/2}) \quad \text{et} \quad W^{(\ell)} = \Omega(c_{\mathbf{q}}^{\ell} \ell^{-5/2})$$

quand $\ell \rightarrow +\infty$. Voir la section 5.3 de [Cur23]. Comme dans le cours [Cur23], on dit que la suite de poids \mathbf{q} est sous-critique (ou de type $a = 3/2$) s'il existe une constante $p_{\mathbf{q}} > 0$ telle que

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-3/2}.$$

L'exposant $-3/2$ évoque les asymptotiques obtenus dans l'énumération des arbres.

La suite \mathbf{q} est dite critique générique (de type $a = 5/2$) lorsqu'il existe $p_{\mathbf{q}} > 0$ tel que

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-5/2}.$$

Lorsque $q_k = 0$ pour k assez grand, i.e. quand le degré des faces est borné, et \mathbf{q} est admissible alors soit \mathbf{q} est sous-critique, soit elle est critique générique. Pour obtenir des exposants intermédiaires, il faut favoriser l'apparition de grands degrés en choisissant \mathbf{q} de sorte que les poids décroissent "assez lentement".

On dit que \mathbf{q} est critique non-générique de type a avec $3/2 < a < 5/2$ lorsqu'il existe $p_{\mathbf{q}} > 0$ tel que

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-a}.$$

Dans des définitions plus générales, dans [Ric18, CR20], on autorise en facteur une fonction à variation lente dans l'équivalent ci-dessus. Une carte de Boltzmann associée à une suite de poids \mathbf{q} de type a sera aussi appelée une carte stable de paramètre $a - 1/2$.

1.1.3 Cartes décorées par un modèle de mécanique statistique

Une motivation importante dans l'étude des cartes à grands degrés vient de la gravité quantique et de la physique statistique. En effet, les cartes stables apparaissent naturellement en considérant les interfaces de modèles de mécanique statistique couplés à des cartes aléatoires. Nous présentons dans cette sous-section le modèle classique des cartes aléatoires décorées de boucles $O(n)$, modèle considérablement étudié par les physiciens (voir par exemple [EK95, EK96] et les références à l'intérieur) puis par les mathématiciens, et expliquons comment les cartes stables permettent de les caractériser, grâce à la décomposition en *gasket* introduite par Bouttier, Borot et Guitter dans [BBG12b].

Une carte décorée de boucles (\mathbf{m}, \mathbf{L}) est une carte (finie, enracinée, bipartie, planaire) \mathbf{m} munie d'un ensemble $\mathbf{L} = \{\mathcal{L}_1, \dots, \mathcal{L}_k\}$ de chemins fermés disjoints non-orientés sur la carte duale \mathbf{m}^\dagger . De plus, les boucles \mathcal{L}_i ne traversent pas la face racine de \mathbf{m} et sont rigides au sens où elles ne visitent que des quadrangles et entrent et sortent des quadrangles en passant par des côtés opposés. Voir la figure 1.2. On note \mathcal{LM} l'ensemble des cartes décorées de boucles et pour tout entier $\ell \geq 1$, on note $\mathcal{LM}^{(\ell)}$ l'ensemble des cartes décorées de boucles de périmètre 2ℓ .

Soient $\tilde{\mathbf{q}} = (\tilde{q}_k)_{k \geq 1}$ une suite (possiblement nulle) de réels positifs et deux réels $h, n \geq 0$. Le poids d'une carte décorée de boucles $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}$ est défini par

$$w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}) = \left(\prod_{\mathcal{L} \in \mathbf{L}} nh^{\#\mathcal{L}} \right) \left(\prod_{f \in \text{Faces}(\mathbf{m}) \setminus (\{f_r\} \cup \bigcup_{\mathcal{L} \in \mathbf{L}} \mathcal{L})} \tilde{q}_{\deg(f)/2} \right),$$

où $\#\mathcal{L}$ est la longueur de la boucle \mathcal{L} , c'est-à-dire le nombre de quadrangles traversés. Pour tout entier $\ell \geq 1$, définissons la fonction de partition des cartes décorées de boucles $O(n)$ de triplet $(\tilde{\mathbf{q}}, h, n)$ de périmètre 2ℓ par

$$F^{(\ell)}(\tilde{\mathbf{q}}, h, n) = \sum_{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}).$$

Le triplet $(\tilde{\mathbf{q}}, h, n)$ est dit admissible si $F^{(\ell)}(\tilde{\mathbf{q}}, h, n) < \infty$ pour tout $\ell \geq 1$, ce qui permet de définir la mesure de Boltzmann sur les cartes décorées de boucles $O(n)$ de la même façon que pour les cartes aléatoires de Boltzmann.

Rappelons désormais la décomposition en *gasket* d'une carte décorée de boucles introduite par [BBG12b]. Le *gasket* d'une carte décorée de boucles $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}$ est une carte planaire $\mathbf{g} \in \mathcal{M}$ obtenue en enlevant l'intérieur des boucles les plus à l'extérieur de (\mathbf{m}, \mathbf{L}) . Voir la figure 1.2. Remarquons que si l'on définit la suite de poids $\mathbf{q} = (q_k)_{k \geq 1}$ en posant pour tout $k \geq 1$,

$$q_k = \tilde{q}_k + nh^{2k} F^{(k)}(\tilde{\mathbf{q}}, h, n),$$

alors pour tout $\mathbf{g} \in \mathcal{M}$,

$$w_{\mathbf{q}}(\mathbf{g}) = \sum_{\substack{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM} \\ \mathbf{g} \text{ est le } gasket \text{ de } (\mathbf{m}, \mathbf{L})}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}).$$

En particulier, en sommant sur $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}$, on en déduit que $W^{(\ell)} = F^{(\ell)}(\tilde{\mathbf{q}}, h, n)$ pour tout $\ell \geq 1$ et que \mathbf{q} est admissible si et seulement si le triplet $(\tilde{\mathbf{q}}, h, n)$ est admissible.

Comme dans [Bud18], on dit que le triplet $(\tilde{\mathbf{q}}, h, n)$ est critique non générique s'il est admissible, $n > 0$ et $h = 1/c_{\tilde{\mathbf{q}}}$. Par conséquent, une suite \mathbf{q} est la suite de poids du *gasket* d'une carte décorée de boucles $O(n)$ critique non générique si et seulement si \mathbf{q} est admissible et

$$\forall k \geq 1, \quad q_k - nc_{\mathbf{q}}^{-2k} W^{(k)} \geq 0. \quad (1.1.2)$$

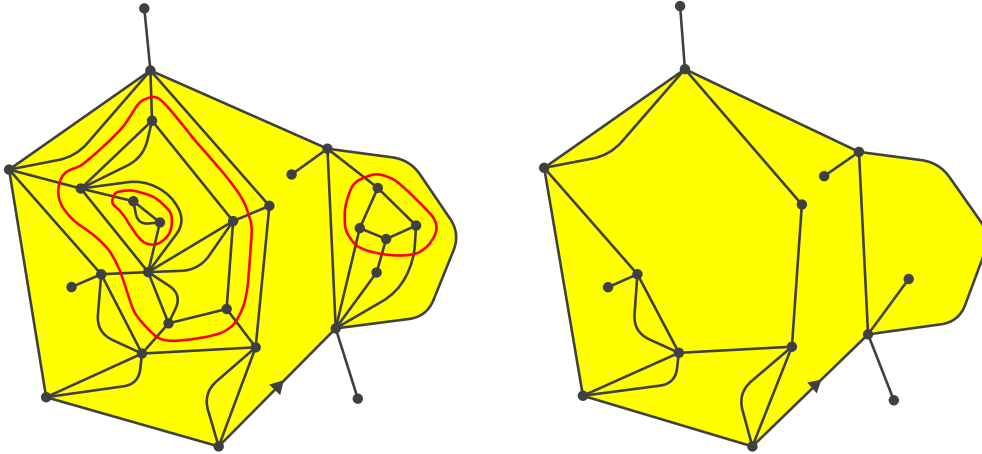


FIGURE 1.2 – Gauche : une carte décorée de boucles. Droite : son *gasket*.

Il est connu que pour $n \in (0, 2)$, lorsque le triplet $(\tilde{\mathbf{q}}, h, n)$ est critique non générique, on peut observer les comportements

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-a}$$

avec $p_{\mathbf{q}} > 0$, $c_{\mathbf{q}} = 1/h$ et $a = 2 \pm \arccos(n/2)/\pi$. Voir par exemple l'équation (2.9) dans [BBD23]. Cela signifie que la suite \mathbf{q} est critique non-générique de type a . Autrement dit, le *gasket* d'une carte décorée de boucles $O(n)$ critique non-générique est une carte stable. Cependant, le cas $n = 2$ restait ouvert. Dans le cas $n = 2$ nous montrons dans le chapitre 4 que le *gasket* d'une carte décorée de boucles $O(2)$ critique est bien $3/2$ -stable, avec l'apparition d'une fonction à variation lente. Rappelons qu'une fonction $L : \mathbb{R}_+ \rightarrow \mathbb{R}$ est dite à variation lente si pour tout $\lambda > 0$, on a $L(\lambda x) \sim L(x)$ quand $x \rightarrow +\infty$.

Théorème 1.1.1. (théorème 4.1.1 du chapitre 4) Soit $\mathbf{q} = (q_k)_{k \geq 1}$ une suite de poids admissible satisfaisant (1.1.2) avec $n = 2$. Alors il existe une fonction à variation lente $L_{\mathbf{q}}$ telle que $1 = O(L_{\mathbf{q}}(\ell))$, $L_{\mathbf{q}}(\ell) = O(\log \ell)$ et

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} c_{\mathbf{q}}^{\ell+1} \frac{L_{\mathbf{q}}(\ell)}{2\ell^2}.$$

Simultanément, Aïdékon, Da Silva et Hu ont montré ce résultat dans le cas particulier des quadrangulations dans une nouvelle version de [ASH24] et l'ont appliqué à l'étude des boucles et du volume de ces cartes décorées. Leur approche consiste à approximer le modèle $O(2)$ grâce au modèle $O(n)$ en faisant tendre n vers 2.

Nous démontrons le théorème 1.1.1 de façon analytique et élémentaire, en utilisant la factorisation de Wiener–Hopf pour la marche aléatoire dont les incréments suivent la loi de probabilité ν sur \mathbb{Z} reliée à la suite de poids \mathbf{q} par

$$\nu(k) = q_{k+1} c_{\mathbf{q}}^k \text{ pour } k \geq 0 \quad \text{et} \quad \nu(-k) = 2W^{(-1+k)} c_{\mathbf{q}}^{-k} \text{ pour } k \geq 1. \quad (1.1.3)$$

La factorisation de Wiener–Hopf, outil fondamental et classique dans l'étude des marches aléatoires, a été introduite dans l'étude des cartes aléatoires par Timothy Budd dans [Bud17, Bud18].

1.1.4 Exploration de cartes aléatoires

Un outil particulièrement utile dans l'étude des cartes aléatoires est l'exploration d'épluchage, une exploration markovienne découverte en physique dans [Wat95] et dont la version s'appliquant au cadre général des cartes biparties a été introduite par Budd dans [Bud16]. Nous renvoyons aux sections 2.3 et 3.6.1 ou aux notes de cours [Cur23, Bud17] pour plus de détails mais donnons dès à présent un avant-goût de cet outil.

Cartes aléatoires avec une cible et cartes infinies. Une carte avec une face cible $\mathbf{m}_\square = (\mathbf{m}, \square)$ est une carte (finie) $\mathbf{m} \in \mathcal{M}$ avec une face distinguée \square différente de la face racine f_r . Pour tous $\ell, p \geq 1$, notons $\mathcal{M}_p^{(\ell)}$ l'ensemble des cartes finies de périmètre 2ℓ avec une face cible de périmètre $2p$. Nous posons pour tout $\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}$,

$$w_{\mathbf{q}}(\mathbf{m}_\square) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r, \square\}} q_{\deg(f)/2}.$$

Nous définissons aussi la fonction de partition $W_p^{(\ell)} = \sum_{\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}_\square)$. De même, une carte avec un sommet cible $\mathbf{m}_\bullet = (\mathbf{m}, \bullet)$ est une carte (finie) $\mathbf{m} \in \mathcal{M}$ avec un sommet distingué \bullet . Nous notons $\mathcal{M}_0^{(\ell)}$ l'ensemble des cartes finies de périmètre 2ℓ avec un sommet cible. On pose $w_{\mathbf{q}}(\mathbf{m}_\bullet) = w_{\mathbf{q}}(\mathbf{m})$ pour tout $\mathbf{m}_\bullet \in \mathcal{M}_0^{(\ell)}$, et $W_0^{(\ell)} = \sum_{\mathbf{m}_\bullet \in \mathcal{M}_0^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}_\bullet)$.

On peut munir les cartes (avec ou sans cible) d'une distance locale qui peut être définie en posant pour tous \mathbf{m}, \mathbf{m}' ,

$$d_{\text{loc}}(\mathbf{m}, \mathbf{m}') = \left(1 + \sup\{r \geq 0, \text{Ball}_r^\dagger(\mathbf{m}) = \text{Ball}_r^\dagger(\mathbf{m}')\}\right)^{-1},$$

où $\text{Ball}_r^\dagger(\mathbf{m})$ est la carte obtenue à partir de \mathbf{m} en ne gardant que les faces à distance de graphe dual au plus r de la face racine f_r et les arêtes duales entre celles-ci. On peut compléter \mathcal{M} muni de cette distance en un espace métrique complet. Par le théorème 2 de [Bud17], on peut définir une carte aléatoire infinie $\mathfrak{M}_\infty^{(\ell)}$ comme la limite en loi quand $p \rightarrow \infty$ de cartes aléatoires de loi $\mathbb{P}_p^{(\ell)}$ dans le complété de \mathcal{M} muni de la distance locale. La carte $\mathfrak{M}_\infty^{(\ell)}$ est presque sûrement localement finie, n'a qu'un seul bout (i.e. si on enlève un nombre fini d'arêtes, il ne restera toujours qu'une seule composante connexe infinie) et est appelée une carte aléatoire infinie de \mathbf{q} -Boltzmann du plan. Sa loi est notée $\mathbb{P}_\infty^{(\ell)}$.

Dans le paragraphe suivant, on notera \mathbf{m}_* une carte avec une cible $* \in \{\square, \bullet, \infty\}$. On écrit $* = \square$ pour dire que \mathbf{m}_* est une carte avec une face cible, $* = \bullet$ pour dire que \mathbf{m}_* est une carte avec un sommet cible et $* = \infty$ pour dire que \mathbf{m}_* est une carte infinie avec un seul bout.

Exploration d'épluchage. L'exploration d'épluchage découvre peu à peu les faces de la carte \mathbf{m} , en partant de la face racine f_r . À chaque étape, on regarde ce qu'il y a derrière une arête au bord de la région déjà explorée qui est choisie en suivant un algorithme d'épluchage.

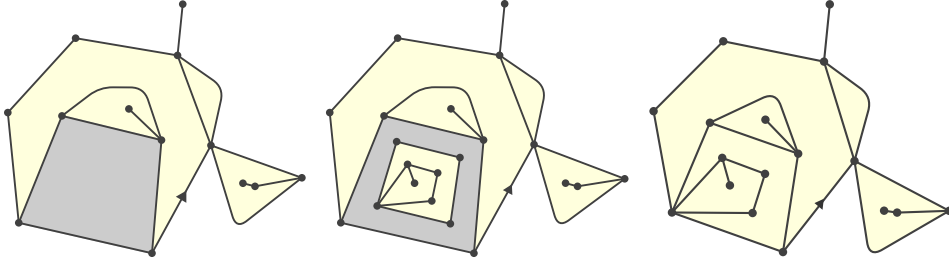


FIGURE 1.3 – Gauche : une carte ϵ avec un trou qui est une sous-carte de la carte \mathbf{m} à droite. La carte u dessinée dans le trou de ϵ au centre permet de récupérer \mathbf{m} en collant le bord de u au bord du trou.

On dit qu'une carte ϵ est une carte avec un trou si ϵ a une face distinguée avec un bord simple qui est différente de f_r . De plus, on dit que ϵ est une sous-carte de \mathbf{m} si l'on peut obtenir \mathbf{m} en collant une carte dans le trou de ϵ (voir la figure 3.5). Le bord $\partial\epsilon$ de ϵ est l'ensemble des arêtes au bord de son trou. Un algorithme d'épluchage \mathcal{A} est une fonction qui prend une carte avec un trou ϵ et renvoie une arête de $\partial\epsilon$. L'algorithme \mathcal{A} peut être aléatoire mais ne peut dépendre que du passé de l'exploration.

Soit $\mathbf{m} \in \mathcal{M}$ ou \mathbf{m}_* une carte avec une cible $* \in \{\square, \bullet, \infty\}$. Une exploration d'épluchage de \mathbf{m} (resp. \mathbf{m}_*) suivant l'algorithme \mathcal{A} est une suite croissante de sous-cartes $\bar{\epsilon}_0 \subset \bar{\epsilon}_1 \subset \dots \subset \bar{\epsilon}_n \subset \dots$ de \mathbf{m} (resp. \mathbf{m}_*) telle que $\bar{\epsilon}_0$ est la carte obtenue en collant f_r à un trou de même degré, et telle que pour tout $n \geq 0$, la sous-carte $\bar{\epsilon}_{n+1}$ est obtenue à partir de $\bar{\epsilon}_n$ de la façon suivante :

- Si la face f derrière l'arête $\mathcal{A}(\bar{\epsilon}_n)$ dans \mathbf{m} (resp. \mathbf{m}_*) n'est pas dans $\bar{\epsilon}_n$ (et n'est pas la face \square dans le cas $* = \square$), alors on "découvre" cette face en collant f à $\bar{\epsilon}_n$ sur l'arête $\mathcal{A}(\bar{\epsilon}_n)$. Voir le côté gauche de la figure 1.4 ;

- Si la face derrière $\mathcal{A}(\bar{\epsilon}_n)$ dans \mathbf{m} (resp. \mathbf{m}_*) est déjà dans $\bar{\epsilon}_n$, alors elle est nécessairement adjacente au trou. On colle l'arête $\mathcal{A}(\bar{\epsilon}_n)$ à l'arête dans $\partial\bar{\epsilon}_n$ en suivant les relations d'adjacence dans \mathbf{m} . Le trou se divise alors en deux. Pour \mathbf{m}_* , si $*$ = \square ou $*$ = \bullet , on remplit le trou qui ne contient pas la cible, et si $*$ = ∞ , on remplit le trou qui est de taille finie. Pour \mathbf{m} , en l'absence de cible, on choisit de remplir le trou qui a le plus petit périmètre (et en cas d'égalité on remplit l'un des deux en faisant un choix déterministe et arbitraire). Voir le côté droit de la figure 1.4.
- Dans le cas de l'exploration de \mathbf{m}_* avec $*$ = \square , si la face derrière l'arête $\mathcal{A}(\bar{\epsilon}_n)$ dans \mathbf{m}_* est \square , alors on pose $\bar{\epsilon}_{n+1} = \mathbf{m}_*$ et l'exploration s'arrête.

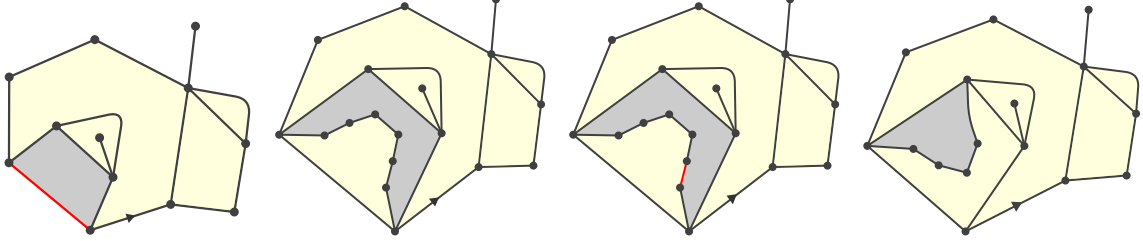


FIGURE 1.4 – De gauche à droite : deux étapes de l'exploration d'épluchage de la carte \mathbf{m} de la figure 1.3 partant de la sous-carte ϵ à gauche. L'arête rouge est l'arête choisie par l'algorithme d'épluchage. La première étape correspond à la découverte d'une face. La deuxième étape correspond au recollement de deux arêtes du bord et au remplissage du trou de plus petit périmètre.

Loi de l'exploration. On note $(P(n))_{n \geq 0}$ le processus du périmètre défini comme la moitié du nombre d'arêtes de $\partial\bar{\epsilon}_n$. L'exploration de $\mathbf{m} \in \mathcal{M}$ ou de \mathbf{m}_\bullet s'arrête quand P atteint zéro. Sous $\mathbb{P}^{(\ell)}$, $\mathbb{P}_p^{(\ell)}$ avec $p \geq 0$ ou sous $\mathbb{P}_\infty^{(\ell)}$, l'exploration d'épluchage $(\bar{\epsilon}_n)_{n \geq 0}$ et le processus $(P(n))_{n \geq 0}$ sont des chaînes de Markov avec des probabilités de transition explicites reliées à la marche aléatoire $(S_n)_{n \geq 0}$ de pas suivant la loi ν définie en (1.1.3). Notamment :

- Sous $\mathbb{P}_\infty^{(\ell)}$, le processus de périmètre a la loi d'une marche aléatoire $(S_n)_{n \geq 0}$ partant de ℓ conditionnée à rester toujours strictement positive.
- Sous $\mathbb{P}_0^{(\ell)}$, le processus de périmètre a la loi d'une marche aléatoire $(S_n)_{n \geq 0}$ partant de ℓ conditionnée à entrer dans $\mathbb{Z}_{<0}$ en touchant zéro et qui est absorbée en zéro.
- Sous $\mathbb{P}_p^{(\ell)}$ avec $p \geq 1$, en ajoutant la convention que le processus du périmètre atteint $-p$ lors de la découverte de la face cible \square , le processus de périmètre a la loi d'une marche aléatoire $(S_n)_{n \geq 0}$ partant de ℓ conditionnée à entrer dans $\mathbb{Z}_{<0}$ en touchant $-p$ et qui est absorbée en $-p$.

L'exploration $(\bar{\epsilon}_n)_{n \geq 0}$ satisfait aussi une propriété cruciale appelée propriété de Markov spatiale : sous $\mathbb{P}^{(\ell)}$ (resp. $\mathbb{P}_\infty^{(\ell)}$ ou $\mathbb{P}_p^{(\ell)}$ pour $p \geq 0$), conditionnellement à la région explorée $\bar{\epsilon}_n$, la carte qui remplit le trou de $\bar{\epsilon}_n$ dans \mathbf{m} (resp. \mathbf{m}_*) est une carte de Boltzmann de loi $\mathbb{P}^{(k)}$ (resp. $\mathbb{P}_\infty^{(k)}$ ou $\mathbb{P}_p^{(k)}$ pour $p \geq 0$), où k est la moitié du degré du trou. De plus, lorsqu'on remplit un trou après avoir recollé deux arêtes, la carte à l'intérieur est indépendante du reste et a aussi une loi $\mathbb{P}^{(j)}$ où j est la moitié du degré du trou.

Algorithmes d'épluchage. La loi du processus du périmètre ne dépend pas de l'algorithme d'épluchage \mathcal{A} choisi. La diversité des choix d'algorithmes possible explique la polyvalence et les nombreuses applications de l'exploration d'épluchage : on peut par exemple suivre une marche aléatoire comme dans [BC13, CM21] ou longer les interfaces de percolation ou du modèle d'Ising comme dans [CT20, AMS21, BC17, BCM18]. Nous présentons ici deux algorithmes qui permettent de suivre les distances à la face racine pour les distances d_{gr}^\dagger et d_{fpp}^\dagger qui sont utilisés notamment dans [BC17, BCM18, BBCK18].

Un algorithme d'épluchage particulièrement adapté à l'étude de d_{fpp}^\dagger est l'algorithme d'exploration uniforme. Il choisit uniformément au hasard une arête au bord de la sous-carte. En utilisant cet algorithme avec un couplage adéquat avec les longueurs aléatoires exponentielles définissant d_{fpp}^\dagger , on peut voir que pour tout $n \geq 0$, en supposant que l'exploration de s'est pas arrêtée avant, la distance d_{fpp}^\dagger de f_r au trou

\mathbf{u}_n de $\bar{\mathbf{e}}_n$ peut être écrite sous la forme

$$T(n+1) := d_{\text{fpp}}^\dagger(f_r, \mathbf{u}_n) = \sum_{i=0}^n \frac{\mathcal{E}_i}{2P(i)}, \quad (1.1.4)$$

où les \mathcal{E}_i sont des v.a.i.i.d. exponentielles de paramètre 1. Plus précisément, $\bar{\mathbf{e}}_n$ correspond à la boule fermée de centre f_r et de rayon $T(n)$ où on a rempli les trous qui sont remplis lors de l’exploration. En particulier, une face f découverte au temps n (i.e. $f \in \text{Faces}(\bar{\mathbf{e}}_n) \setminus \text{Faces}(\bar{\mathbf{e}}_{n-1})$) est à distance $T(n)$ de la face racine f_r . Sous $\mathbb{P}^{(\ell)}$, les \mathcal{E}_i sont indépendantes du processus du périmètre. Voir la section 13.1 of [Cur23] pour plus de détails.

Pour la distance de graphe d_{gr}^\dagger , un algorithme utile est l’algorithme d’épluchage par couche. Étant donnée une sous-carte \mathbf{e} de \mathbf{m} , on définit la hauteur de $e \in \partial\mathbf{e}$ comme la distance de graphe dual entre f_r et la face à côté de e qui n’est pas dans le trou. Supposons qu’il existe un entier $h \geq 0$ tel que toutes les arêtes de $\partial\mathbf{e}$ soient de hauteur h ou $h+1$ et que les arêtes de hauteur h forment un intervalle de $\partial\mathbf{e}$. Alors on choisit d’éplucher l’arête la plus à gauche de hauteur h (voir la figure 3.7) et quand toutes les arêtes de $\partial\mathbf{e}$ ont la même hauteur on choisit l’une d’entre elles arbitrairement. Pour suivre la distance à la racine, il suffit donc de savoir combien de temps on met pour compléter chaque couche, c’est-à-dire combien de temps il faut pour passer de la boule de rayon h autour de f_r à la boule de rayon $h+1$. Voir la section 13.2 de [Cur23] pour plus de détails.

1.2 Limites d’échelle de cartes aléatoires

Une question naturelle pour les cartes aléatoire est l’étude de leurs limites d’échelle. Autrement dit, vue de loin, à quoi ressemble une grande carte aléatoire? Nous commençons par expliquer comment décrire la convergence d’une suite d’espace métriques, puis nous décrivons les résultats connus et les conjectures pour les limites d’échelle de cartes aléatoires. Enfin, nous évoquons la théorie de la gravité quantique de Liouville qui permet de décrire ces objets limites à l’aide du champ libre gaussien.

1.2.1 Convergence d’espaces métriques

Il existe plusieurs manières de décrire une convergence d’espaces métriques, dont certaines peuvent s’interpréter en quelque sorte comme la convergence de la “matrice” des distances entre des paires de points. Nous décrivons brièvement ici les convergences au sens de Gromov–Hausdorff, Gromov–Prokhorov et Gromov–Hausdorff–Prokhorov qui sont les plus couramment étudiées dans le cadre des limites d’échelle. Nous renvoyons à la section 7.6 du chapitre 7 pour plus de détails sur ces notions de convergence.

Gromov–Prokhorov. Une première manière de voir les limites d’espaces métriques munis d’une mesure de probabilité revient à considérer la matrice des distances entre des points tirés indépendamment. Si (X, d, μ) et (X^n, d^n, μ^n) pour $n \geq 0$ sont des espaces métriques munis de mesures de probabilité μ et μ^n boréliennes, soient $(Y_i^n)_{i \in \mathbb{N}}$ pour tout $n \geq 0$ et $(Y_i)_{i \in \mathbb{N}}$ des familles de variables aléatoires i.i.d. de lois respectives μ^n pour $n \geq 0$ et μ . On dit que (X^n, d^n, μ^n) converge vers (X, d, μ) pour au sens de Gromov–Prokhorov (GP) si on a la convergence en loi

$$(d^n(Y_i^n, Y_j^n))_{i,j \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} (d(Y_i, Y_j))_{i,j \in \mathbb{N}}$$

pour la topologie produit. La définition ci-dessus n’est pas la définition historique de la section 3 $\frac{1}{2}$.3 de [Gro07], mais elle est équivalente grâce au théorème 5 de [GPW09]. Voir aussi la synthèse [Jan20]. La topologie associée à cette notion de convergence sur l’ensemble de certaines classes d’équivalence d’espaces métriques munis d’une mesure de probabilité est celle d’un espace polonais.

Gromov–Hausdorff. Une deuxième approche consiste à plonger de manière isométrique les espaces métriques dans un même espace et à regarder la distance de Hausdorff entre ces deux plongements. Rappelons que la distance de Hausdorff entre deux parties A, B d’un espace métrique X est définie par

$$d_H(A, B) = \inf\{\varepsilon > 0, A \subset B^\varepsilon, B \subset A^\varepsilon\},$$

où A^ε et B^ε sont les ε -voisinages de A et B et où $\inf \emptyset = \infty$ par convention. La (pseudo-)distance de Gromov–Hausdorff (GH) entre deux espaces métriques (X, d) et (X', d') est définie par

$$d_{\text{GH}}((X, d), (X', d')) = \inf_{E, \varphi, \varphi'} d_{\text{H}}(\varphi(X), \varphi'(X')),$$

où l'infimum est pris sur tous les espaces métriques E et les plongements isométriques $\varphi : X \rightarrow E$, $\varphi' : X' \rightarrow E$. La topologie associée à d_{GH} sur l'ensemble des espaces métriques compacts vus à isométrie près est celle d'un espace polonais.

Gromov–Hausdorff–Prokhorov. Finalement, on dit qu'une suite $(X^n, d^n, \mu^n)_{n \geq 0}$ d'espaces métriques munis d'une mesure de probabilité converge vers (X, d, μ) au sens de Gromov–Hausdorff–Prokhorov (GHP) si (X^n, d^n) converge vers (X, d) pour la distance de GH et si (X^n, d^n, μ^n) converge vers (X, d, μ) au sens de GP. Encore une fois, ce n'est pas la définition historique de la convergence GHP donnée dans [ADH13], mais les deux définitions sont équivalentes grâce au lemme 4 de [BR21]. La topologie associée à cette notion de convergence sur l'ensemble de certaines classes d'équivalence d'espaces métriques compacts munis d'une mesure de probabilité est celle d'un espace polonais.

1.2.2 Limites d'échelle de cartes aléatoires

L'étude des limites d'échelle de cartes aléatoires a connu une grande avancée avec la définition de la sphère brownienne dans [MM06], et la preuve de limites d'échelle des quadrangulations uniformes à n sommets quand $n \rightarrow +\infty$ vers la sphère brownienne obtenue indépendamment par [Mie13] et [LG13]. Depuis, on sait même grâce à [Mar18b] que les cartes de \mathbf{q} -Boltzmann sous $\mathbb{P}^{(1)}$ conditionnées à avoir n sommets satisfont la même limite d'échelle au sens de Gromov–Hausdorff lorsque \mathbf{q} est critique générique. Lorsque \mathbf{q} est sous-critique, la carte ressemble à un arbre. En effet, les points (3) et (4) du théorème 6.3 de [Mar22] donnent la limite d'échelle vers un arbre brownien pour des cartes à n arêtes ou au moins n arêtes et un périmètre prescrit dépendant de n . Combinés avec la limite d'échelle du volume donnée par la proposition 10.4 de [Cur23], il y a peu de doutes que ces résultats donnent facilement la limite d'échelle sous $\mathbb{P}^{(\ell)}$ vers un arbre continu brownien. Des résultats très proches de limite d'échelle vers un arbre continu brownien lorsqu'on considère des cartes sous $\mathbb{P}^{(1)}$ conditionnées à avoir n arêtes par exemple ont été démontrés dans [JS15]. Nous nous concentrons donc dans ce qui suit sur les cas critiques. De plus, par souci de cohérence, nous n'énoncerons que des résultats de convergence sous $\mathbb{P}^{(\ell)}$ quand $\ell \rightarrow +\infty$. Pour tout $\ell \geq 1$, soit $\mathfrak{M}^{(\ell)}$ une carte aléatoire de loi $\mathbb{P}^{(\ell)}$.

Cas générique. Lorsque \mathbf{q} est de type $a = 5/2$, la limite d'échelle est connue et il s'agit du disque brownien, une surface aléatoire dans la classe d'universalité de la sphère brownienne.

Théorème 1.2.1. (théorème 8 de [BM17]) *Si \mathbf{q} est de type $a = 5/2$, alors on a la limite d'échelle*

$$\left(\mathfrak{M}^{(\ell)}, \left(\frac{2\ell}{3} \right)^{-1/2} d_{\text{gr}} \right)_{\ell \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \text{FBD}_1,$$

pour la topologie de Gromov–Hausdorff, où FBD_1 est une surface aléatoire de dimension de Hausdorff 4 homéomorphe à un disque qui est appelée le disque brownien (libre) de périmètre 1.

En fait, on peut aussi vérifier que la convergence ci-dessus a aussi lieu au sens de GHP en munissant la carte $\mathfrak{M}^{(\ell)}$ de la mesure uniforme sur les sommets, en s'appuyant sur le point (2) du théorème 6.3 de [Mar22] qui donne la limite d'échelle des cartes à volume et périmètre prescrits et en utilisant la limite d'échelle du volume sous $\mathbb{P}^{(\ell)}$ énoncée dans la proposition 10.4 de [Cur23]. Le résultat ci-dessus est énoncé pour la distance d_{gr} définie sur les sommets. Si l'on considère les distances d_{gr}^\dagger ou d_{fpp}^\dagger définies sur les faces, il est conjecturé que le résultat tient toujours en identifiant les points du bord du disque brownien à un seul point. Le fait que les distances primales d_{gr} et duales d_{gr}^\dagger et d_{fpp}^\dagger se comportent de la même façon dans le cas générique vient du fait que les faces et les sommets ont des petits degrés.

Cas stable. Quand \mathbf{q} est de type $a \in]3/2, 5/2[$, la géométrie de la carte est fortement influencée par les grandes faces. L'étude de la géométrie des cartes stables munies de la distance d_{gr} a été initiée par Le Gall et Miermont dans [LGM11], qui ont montré que les cartes aléatoires sous $\mathbb{P}^{(1)}$ conditionnées à avoir n sommets satisfont une limite d'échelle le long de sous-suites vers un espace métrique aléatoire de dimension de Hausdorff $2a - 1$. Leur résultat a ensuite été étendu par [Mar18a] et la limite d'échelle a été construite tout récemment dans [CMR25]. L'outil principal dans l'étude de la distance d_{gr} est l'utilisation de bijections avec des arbres, en particulier la bijection de Bouttier-Di Francesco-Guitter [BDFG04]. Dans cette thèse, on se concentre sur les distances duales d_{gr}^\dagger et d_{fpp}^\dagger qui se comportent de façon radicalement différente dans la mesure où les faces de grand degré jouent un rôle de "hubs" et raccourcissent considérablement les distances et dont la géométrie est beaucoup moins connue. L'étude des distances d_{gr}^\dagger et d_{fpp}^\dagger se fait au moyen de l'exploration d'épluchage.

Distances duales sur les cartes stables. Les cartes stables munies des distances d_{gr}^\dagger et d_{fpp}^\dagger ont été d'abord étudiées sous l'angle de leur limite locale, une version infinie de la carte, par [BC17] pour $a \neq 2$ et par [BCM18] pour $a = 2$. Une transition de phase apparaît au niveau de la croissance du volume des boules avec le rayon : polynomiale quand $a > 2$, exponentielle quand $a < 2$ et intermédiaire dans le cas critique $a = 2$. Dans [BBCK18], Bertoin, Budd, Curien et Kortchemski ont établi pour $a > 2$ la limite d'échelle des distances à la racine sur $\mathfrak{M}^{(\ell)}$ munie de d_{gr}^\dagger ou d_{fpp}^\dagger , qui est décrite à l'aide d'un processus de branchement appelé croissance-fragmentation. Il est conjecturé que ces cartes satisfont une limite d'échelle dans le cas $a > 2$ au sens de GHP. Dans le chapitre 3, nous montrons l'analogie de leur résultat dans le cas critique $a = 2$ dans la proposition 3.6.6 où un facteur $\log \ell$ apparaît pour la distance de graphe dual. Cependant, la limite d'échelle de $\mathfrak{M}^{(\ell)}$ munie de d_{gr}^\dagger ou d_{fpp}^\dagger reste inconnue.

Dans le chapitre 2 nous montrons que dans le cas $a = 2$, les cartes $\mathfrak{M}^{(\ell)}$ ne satisfont pas de limite d'échelle au sens de GH ou de GP. Plus précisément nous montrons que la distance entre la racine et un point typique satisfait une limite d'échelle en probabilité vers une constante.

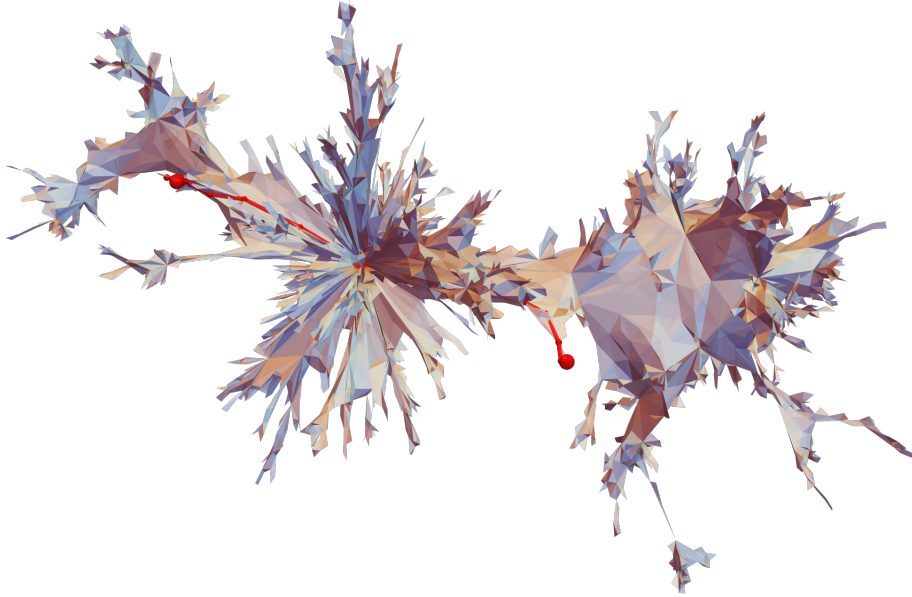


FIGURE 1.5 – Simulation d'une carte 3/2-stable $\mathfrak{M}^{(500, \dagger)}$ de périmètre 1000. Les deux grandes sphères rouges sont deux sommets uniformes de $\mathfrak{M}^{(500, \dagger)}$ et le chemin rouge est le plus court chemin entre eux.

Théorème 1.2.2. (théorème 2.1.1 du chapitre 2) Supposons \mathbf{q} de type $a = 2$. Pour tout $\ell \geq 1$, soit $F^{(\ell)}$ une face uniforme de $\mathfrak{M}^{(\ell)}$ (prise conditionnellement à $\mathfrak{M}^{(\ell)}$). Alors

$$\frac{d_{\text{fpp}}^\dagger(f_r, F^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}} \quad \text{et} \quad \frac{d_{\text{gr}}^\dagger(f_r, F^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2}.$$

La première étape dans la preuve du résultat ci-dessus est d'échanger en quelque sorte les rôles de la face racine f_r et de la face uniforme $F^{(\ell)}$. On peut alors démarrer une "exploration uniforme de la carte

partant de la face uniforme vers la face racine". De plus, dans le cas $a = 2$, la limite d'échelle du processus du périmètre, obtenue dans [BCM18] est reliée à un processus de Cauchy de sorte que le périmètre $P(n)$ est de l'ordre de n . En partant de (1.1.4), on obtient que $d_{\text{fpp}}^\dagger(f_r, F^{(\ell)})$ est de l'ordre de $\log \ell$. La limite d'échelle de la distance de graphe dual est obtenue à l'aide de la même idée de ré-enracinement et en utilisant l'algorithme d'épluchage par couche, l'analyse de ce dernier étant plus délicate : le facteur $\log \ell$ supplémentaire vient du fait que le temps pour compléter une couche de périmètre p pendant l'exploration de la carte est de l'ordre de $p/\log p$ comme démontré dans [BCM18] dans le cadre de la limite locale.

Dans le même chapitre, on montre que les géodésiques entre la racine et deux points typiques se "séparent rapidement" (voir la figure 1.5), ce qui permet d'avoir la limite d'échelle de la matrice des distances entre des points typiques. Pour tout $\ell \geq 1$, soit $(F_i^{(\ell)})_{i \geq 1}$ une suite de faces uniformes indépendantes dans $\mathfrak{M}^{(\ell)}$ prises conditionnellement à $\mathfrak{M}^{(\ell)}$. Alors

$$\left(\frac{d_{\text{fpp}}^\dagger(F_i^{(\ell)}, F_j^{(\ell)})}{\log \ell} \right)_{i \neq j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \left(\frac{2}{\pi^2 p_{\mathbf{q}}} \right)_{i \neq j \geq 1} \quad \text{et} \quad \left(\frac{d_{\text{gr}}^\dagger(F_i^{(\ell)}, F_j^{(\ell)})}{(\log \ell)^2} \right)_{i \neq j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \left(\frac{1}{\pi^2} \right)_{i \neq j \geq 1}. \quad (1.2.1)$$

En introduisant des martingales associées à l'exploration d'épluchage, nous montrons aussi que le diamètre de $\mathfrak{M}^{(\ell)}$ pour la distance d_{fpp}^\dagger (resp. d_{gr}^\dagger) est majoré avec probabilité $1 - o(1)$ par une constante fois $\log \ell$ (resp. $(\log \ell)^2$) quand $\ell \rightarrow +\infty$. Cela implique bien que de telles cartes ne satisfont pas de limite d'échelle au sens de GH ou GP. En effet, une limite au sens de GH est forcément compacte, or un espace métrique ayant une infinité de points à distance constante les uns des autres ne peut pas être compact. Pour GP, comme expliqué par exemple dans la remarque 4.5 de [Jan20], on peut remarquer que si (X, d, μ) était une limite d'échelle le long d'une sous-suite au sens de GP, alors deux points Y_1, Y_2 indépendants de loi μ seraient à distance constante. En particulier, si $\varepsilon > 0$ est assez petit, le deuxième point Y_2 de loi μ n'est presque sûrement pas dans la boule de rayon ε centrée en Y_1 , ce qui signifie que cette boule est presque sûrement de μ -mesure nulle, donc p.s. Y_1 n'est pas dans le support $\text{supp } \mu$, i.e. $\mu(\text{supp } \mu) = 0$, absurde.

Cependant, en regardant à une échelle plus petite autour de la racine, dans le chapitre 3, nous montrons que dans le cas $a = 2$ les distances des faces de grand degré à la face racine satisfont une limite d'échelle, qui est obtenue comme dans [BBCK18] à l'aide d'un processus de croissance-fragmentation.

Pour aller plus loin que les distances à la face racine dans l'étude des limites d'échelle des cartes stables, on étudie les géodésiques vers la faces racine. Remarquons que si \mathfrak{m} est une carte munie de la distance de percolation de premier passage d_{fpp}^\dagger , il existe un unique plus court chemin entre deux faces de \mathfrak{m} puisque la loi exponentielle n'a pas d'atome. Ainsi, les plus courts chemins des faces de \mathfrak{m} à f_r forment un arbre noté $\mathcal{T}(\mathfrak{m})$. On note $d_{\mathcal{T}(\mathfrak{m})}$ la distance induite par d_{fpp}^\dagger sur $\mathcal{T}(\mathfrak{m})$. Dans le chapitre 5 nous comptons le nombre de faces le long des géodésiques vers f_r et obtenons notamment comme corollaire une majoration du diamètre pour la distance de graphe dual dans le cas $a \in]3/2, 2[$: avec probabilité $1 - o(1)$ quand $\ell \rightarrow +\infty$, le diamètre de $\mathfrak{M}^{(\ell)}$ est majoré par une constante fois $\log \ell$ (voir le théorème 5.1.4). Nous montrons aussi une limite d'échelle de l'arbre $\mathcal{T}(\mathfrak{M}^{(\ell)})$ au sens de la convergence de la matrice des distances entre les faces énumérées dans l'ordre décroissant des degrés. Notons $(f_i^{(\ell)})_{i \geq 1}$ les faces de $\mathfrak{M}^{(\ell)}$ énumérées dans l'ordre décroissant des degrés.

Théorème 1.2.3. (théorème 5.1.5 du chapitre 5) *Si \mathbf{q} est de type $a \in]3/2, 5/2[$, alors on a la convergence*

$$\left(\ell^{2-a} d_{\mathcal{T}(\mathfrak{M}^{(\ell)})}(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i, j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} d_{\mathcal{T}_a}(w_i, w_j) \right)_{i, j \geq 1},$$

pour la topologie produit, où $c_a := \pi/\Gamma(a)$ et $\mathcal{T}_a = (\{w_i; i \geq 1\}, d_{\mathcal{T}_a})$ est un espace métrique aléatoire dénombrable.

La principale technique dans la preuve du théorème ci-dessus, et que nous introduisons et utilisons systématiquement dans le chapitre 5 est un renversement du temps dans l'exploration uniforme que nous décrivons brièvement ci-dessous.

Soit $\mathfrak{m} \in \mathcal{M}$ une carte. Notons $P = (P(n))_{n \geq 0}$ le processus du périmètre associé à l'exploration uniforme de \mathfrak{m} . Conditionnellement à P on reconstruit une carte de même loi dans le sens inverse :

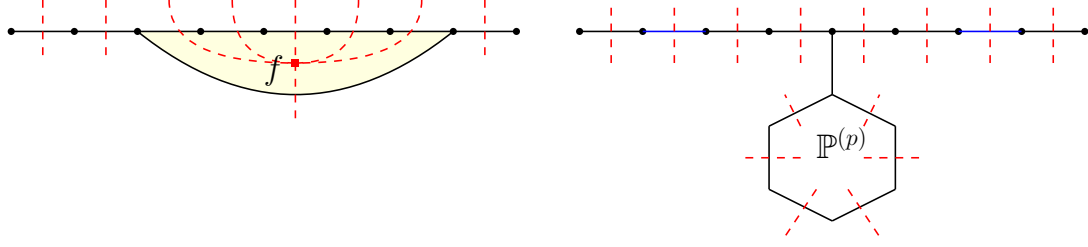


FIGURE 1.6 – Illustration du renversement du temps dans l’exploration uniforme. Gauche : recollement d’une nouvelle face. Droite : attachement d’une arête reliée à une carte de loi $\mathbb{P}^{(p)}$. Les traits en pointillés rouge décrivent des portions de géodésiques vers la face racine pour la distance $d_{\text{fpp}}^{\uparrow}$.

- Si $\Delta P(k) \geq 0$, alors l’exploration uniforme a découvert une face. On colle donc une nouvelle face f de degré $2\Delta P(k) + 2$ à $2\Delta P(k) + 1$ arêtes consécutives où la position de la première arête est choisie uniformément au hasard sur le bord. Voir la figure 1.6, côté gauche.
- Si $\Delta P(k) < 0$, alors l’exploration uniforme a recollé deux arêtes et rempli le trou de périmètre $2p$ avec $p := -\Delta P(k) - 1$. On attache donc une arête en une position choisie uniformément au hasard sur le bord reliée à une carte de Boltzmann de loi $\mathbb{P}^{(p)}$. Voir la figure 1.6, côté droit.

Le renversement du temps permet de suivre les géodésiques pour la distance $d_{\text{fpp}}^{\uparrow}$. Comme illustré dans la figure 1.6, les géodésiques coalescent lorsque l’on colle une nouvelle face.

Ce renversement du temps de l’exploration uniforme peut être encodé par des trajectoires coalescentes à valeurs dans le tore \mathbb{R}/\mathbb{Z} . Les coalescences de géodésiques se transcrivent alors comme des coalescences de trajectoires. L’espace \mathcal{T}_a est construit grâce à un flot coalescent de diffusions à sauts qui apparaît comme la limite d’échelle de ces trajectoires coalescentes. Soit $N(dt, dz, du)$ un processus ponctuel de Poisson (PPP) sur $] -\infty, 0] \times (\mathbb{R}_+^* \setminus \{1\}) \times [0, 1]$ d’intensité $dt\lambda(dz)du$ où la mesure λ est donnée par

$$\lambda(dx) := \frac{\Gamma(a)}{\pi} \left(\frac{1}{(x(1-x))^a} \mathbf{1}_{1/2 < x < 1} + \cos(a\pi) \frac{1}{(x(x-1))^a} \mathbf{1}_{x > 1} \right) dx.$$

Pour tous $x \in \mathbb{R}$, $z \in \mathbb{R}_+^*$ et $u \in [0, 1]$, notons

$$g(x, z, u) = \left(\{x - u\} - \left(\frac{z - 1}{z} \right)_+ \right)_+ z - \{x - u\} + \frac{1}{2} \left(1 - \left(\frac{1}{z} \wedge z \right) \right),$$

où $\{x\}$ est la partie fractionnaire de x et x_+ est sa partie positive. Le flot $(X_{s,t}(v))_{s \leq t \leq 0, v \in \mathbb{R}}$ décrivant la limite d’échelle des géodésiques à la racine est défini par l’équation différentielle stochastique

$$X_{s,t}(v) = v + \int_s^t \int_0^\infty \int_0^1 g(X_{s'-}(v), z, u) \tilde{N}(ds', dz, du),$$

où \tilde{N} est le processus ponctuel de Poisson compensé associé à N . Voir la figure 1.7 pour une simulation de deux trajectoires du flot. La variable z du PPP correspond à la limite d’échelle des sauts du processus du périmètre tandis que la variable u correspond à la position uniforme qui est choisie à chaque étape pour coller une face ou attacher l’arête dans le renversement du temps de l’exploration uniforme. L’existence et l’unicité de solutions fortes à cette EDS ne découle pas des résultats classiques reposant sur la méthode des itérations successives de Picard en supposant que les coefficients soient lipschitziens en un certain sens (voir par exemple le théorème 9.1 du chapitre IV de [IW89] ou le chapitre 6 de [App09]), mais est une application d’un théorème de [LP12] qui requiert un peu moins de régularité pour le coefficient sous une hypothèse de monotonie de $x \mapsto x + g(x, z, u)$.

Plus précisément, puisque l’exploration de la carte a différentes branches, l’espace \mathcal{T}_a est défini à l’aide d’une famille dénombrable de tels flots qui sont “recollés” les uns aux autres. Les grandes faces de la carte correspondent aux z strictement plus grands que 1 apparaissant dans le PPP qui sont chacun associés à un point de \mathcal{T}_a . La distance entre deux points de \mathcal{T}_a est définie à l’aide du temps de coalescence des trajectoires associées. Pour démontrer le théorème 1.2.3, nous montrons d’abord que le flot ci-dessus est bien la limite d’échelle des trajectoires coalescentes qui codent le renversement en temps de l’exploration

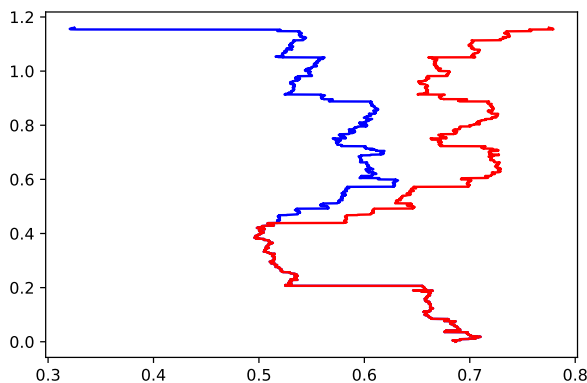


FIGURE 1.7 – Simulation de deux trajectoires du flot coalescent de diffusions à sauts qui décrit la limite d'échelle des géodésiques fpp à la racine dans les cartes à grands degrés. En abscisse : les coordonnées dans le tore \mathbb{R}/\mathbb{Z} . En ordonnée : le temps (en valeur absolue).

uniforme. Ensuite, pour obtenir la limite d'échelle des distances, nous montrons la limite d'échelle des temps de coalescence. Cette dernière est obtenue de façon différente selon si $a \in]3/2, 2[$ ou $a \in [2, 5/2[$. Dans le cas $a \in]3/2, 2[$ nous montrons en quelque sorte que le temps de coalescence correspond forcément à un temps où l'attachement d'une très grande face contraint les deux trajectoires à coalescer. Dans le cas $a \in [2, 5/2[$, ce n'est pas le cas, mais après la coalescence, la trajectoire rencontre très vite des grandes faces, ce qui va tout de même permettre d'obtenir la limite d'échelle du temps de coalescence.

1.2.3 Gravité quantique de Liouville

Les cartes aléatoires, en tant que modèle de surfaces aléatoires, peuvent être vues comme des modèles discrets pour la gravité quantique. Mais une autre approche est de construire des modèles directement dans le continu en s'appuyant sur l'invariance conforme. Cette approche a été initiée par Polyakov dans [Pol81] et est appelée la gravité quantique de Liouville. L'équivalence entre les limites d'échelle des modèles de cartes aléatoires avec des modèles issus de la gravité quantique de Liouville n'a à ce jour été établie que pour les cartes de Boltzmann dans le cas générique $a = 5/2$, i.e. pour la classe d'universalité de la sphère brownienne.

Champ libre gaussien et gravité quantique. La gravité quantique de Liouville est construite à partir du champ libre gaussien qui peut être vu comme une généralisation du mouvement brownien. Il joue par ailleurs un rôle important en mécanique statistique, voir par exemple [Ken01], et dans plusieurs constructions de la théorie quantique des champs, voir par exemple [GK83]. Définissons ici le champ libre gaussien avec condition au bord libre. Soit $D \subset \mathbb{C}$ un domaine simplement connexe différent de \mathbb{C} . Notons $H(D)$ l'espace de Hilbert obtenu comme l'adhérence de l'ensemble des fonctions f de classe \mathcal{C}^∞ sur D à valeurs réelles modulo une constante qui ont une énergie de Dirichlet finie (c'est-à-dire $\int_D |\nabla f|^2 < \infty$) muni du produit scalaire défini pour tout couple (f, g) de telles fonctions par

$$(f|g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2z,$$

où d^2z est la mesure de Lebesgue sur \mathbb{C} . Soit $(f_n)_{n \geq 0}$ une base de Hilbert de $H(D)$. Soit $(X_n)_{n \geq 0}$ une famille de v.a.i.i.d. de loi $\mathcal{N}(0, 1)$. Le champ libre gaussien avec condition au bord libre sur D est défini comme la somme

$$h = \sum_{n \geq 0} X_n f_n,$$

qui converge dans l'espace des distributions modulo une constante. On peut fixer cette constante en assignant une valeur particulière de h en une certaine fonction test f . Le champ libre gaussien est

invariant en loi par transformation conforme au sens où $h \circ \varphi$ a la même loi que h à une constante additive près pour toute transformation conforme $\varphi : D \rightarrow D$.

Nous ne détaillons pas ici la définition de la gravité quantique de Liouville, mais donnons simplement une intuition sur ces objets, en particulier les disques quantiques de paramètre $\gamma \in]0, 2]$, et renvoyons au chapitre 3 ou aux cours [BP24] et [WP21]. Dans la suite, h est obtenu à partir du champ libre gaussien défini ci-dessus en prenant une projection sur un sous-espace bien choisi et en ajoutant une fonction continue aléatoire bien choisie dépendant de γ de façon à obtenir un couple (D, h) appelé γ -disque quantique. Cette surface quantique a été introduite dans [DMS21].

Ce disque quantique est muni d'une mesure positive aléatoire appelée mesure d'aire quantique μ_h^γ définie formellement comme l'exponentielle de γh , qui est aussi connue sous le nom de chaos multiplicatif gaussien. Elle peut être définie via différentes régularisations, le cas $\gamma = 2$, dit critique, étant plus délicat. On peut aussi définir une mesure de longueur de bord quantique ν_h^γ sur ∂D . Plus précisément, comme expliqué dans la section 4 de [AHPS23], on a les convergences suivantes en probabilité : lorsque $\gamma \in]0, 2[$, pour tout ouvert borné $A \subset D$, on a

$$\mu_h^\gamma(A) = \lim_{\delta \rightarrow 0} (8(2 - \gamma))^{-1} \int_A \exp(\gamma h_\delta(z)) \delta^{\gamma^2/2} d^2z$$

et pour tout ouvert borné $B \subset \partial D$

$$\nu_h^\gamma(B) = \lim_{\delta \rightarrow 0} (8(2 - \gamma))^{-1} \int_B \exp\left(\frac{\gamma}{2} h_\delta(z)\right) \delta^{\gamma^2/4} dz,$$

tandis que dans le cas critique $\gamma = 2$, pour tout ouvert borné $A \subset D$ on a la limite en probabilité

$$\mu_h^2(A) = \lim_{\delta \rightarrow 0} \int_A \left(-h_\delta + \log\left(\frac{1}{\delta}\right)\right) e^{2h_\delta(z)} \delta d^2z$$

et pour tout ouvert borné $B \subset \partial D$

$$\nu_h^2(B) = \lim_{\delta \rightarrow 0} \int_B \left(-\frac{h_\delta}{2} + \log\left(\frac{1}{\delta}\right)\right) e^{h_\delta(z)} \delta dz,$$

où d^2z est la mesure de Lebesgue sur \mathbb{C} , dz est la mesure de Lebesgue sur \mathbb{R} quand D est le demi-plan de Poincaré \mathbb{H} et $h_\delta(z)$ est la moyenne circulaire de h sur le cercle de centre z et de rayon δ .

Un γ -disque quantique de longueur de bord quantique $\nu_h^\gamma(\partial D)$ égale à 1 a une aire quantique finie, i.e. $\mu_h^\gamma(D) < \infty$. On peut obtenir un γ -disque quantique de longueur b en remplaçant h par $h + (2/\gamma) \log b$. Deux γ -disques quantiques $(D, h), (\tilde{D}, \tilde{h})$ sont identifiés s'il existe une transformation conforme (autrement dit un biholomorphisme) $\varphi : D \rightarrow \tilde{D}$ tel que

$$h = \tilde{h} \circ \varphi + Q_\gamma \log |\varphi'| \quad \text{où} \quad Q_\gamma = \frac{2}{\gamma} + \frac{\gamma}{2}.$$

Les mesures d'aire et de longueur de bord quantiques sont alors reliées par $\mu_h^\gamma = \varphi_* \mu_{\tilde{h}}^\gamma$ et $\nu_h^\gamma = \varphi_* \nu_{\tilde{h}}^\gamma$.

La mesure d'aire quantique μ_h^γ permet aussi de définir une distance sur le disque quantique. Voir par exemple [GM21, DDG23]. L'espace métrique mesuré aléatoire ainsi obtenu est la limite d'échelle conjecturale des cartes décorées de boucles $O(n)$ critiques pour la convergence GHP avec $\gamma = 2\sqrt{1 - \arccos(n/2)}/\pi$ ou $\gamma = 2/\sqrt{1 + \arccos(n/2)}/\pi$. Voir la conjecture 2.1 de [HL24] pour une conjecture précise qui décrit aussi la limite d'échelle des boucles, qui correspond aux ensembles de boucles conformes évoqués ci-dessus. Le résultat de [ASH24] conforte cette conjecture en identifiant la limite d'échelle du nombre de sommets. Dans le cas de "gravité pure" $\gamma = \sqrt{8/3}$, cette distance avait déjà été construite par Miller et Sheffield et l'espace métrique aléatoire obtenu a la même loi que le disque brownien, voir [MS20, MS21a, MS21b]. Dans ce travail, nous n'utiliserons pas cette distance.

Évolutions de Schramm–Loewner et ensembles de boucles conformes. Une seconde famille d'objets omniprésents en géométrie conforme aléatoire est formée par les évolutions de Schramm–Loewner de paramètre $\kappa > 0$ (SLE_κ), des courbes aléatoires introduites par Schramm dans [Sch00], et leurs homologues à boucles : les ensembles conformes de boucles (CLE_κ pour *conformal loop ensemble*), introduits

dans [She09, SW12]. Donnons la définition d'un SLE "cordal" sur le demi-plan de Poincaré \mathbb{H} allant de zéro à l'infini. Soit $(B_t)_{t \geq 0}$ un mouvement brownien standard issu de zéro. Définissons pour tout $z \in \mathbb{H}$ la solution de l'équation différentielle ordinaire

$$g_0(z) = z \quad \text{et} \quad \partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}.$$

Notons τ_z le temps d'explosion de la solution. Pour tout $t \geq 0$, notons $K_t = \{z \in \mathbb{H}, \tau_z \leq t\}$, autrement dit, K_t est le complémentaire dans \mathbb{H} du domaine de définition de $z \mapsto g_t(z)$. Alors il est connu que $(K_t)_{t \geq 0}$ peut s'écrire sous la forme $K_t = \gamma([0, t])$ pour tout $t \geq 0$, où $\gamma : \mathbb{R}_+ \rightarrow \mathbb{H} \cup \{0\}$ est un chemin continu tel que $\gamma(t) \rightarrow \infty$ quand $t \rightarrow +\infty$. Lorsque $\kappa \leq 4$, le chemin γ est simple (i.e. γ est injective) ce qui n'est plus le cas quand $\kappa > 4$.

Un CLE sur un domaine $D \subset \mathbb{C}$ est un ensemble dénombrable aléatoire de courbes fermées dont chacune ressemble à un SLE. Nous ne définissons pas précisément les CLE dans ce paragraphe, qui peuvent être par exemple construits à l'aide de SLE branchants ou à l'aide des soupes de boucles browniennes et nous renvoyons aux cours [Law05, Wer04, BP24] pour plus de détails sur les SLE_κ , au chapitre 3 et aux articles [She09, SW12] pour les CLE_κ . Les SLE_κ et CLE_κ satisfont une transition de phase en $\kappa = 4$: la courbe et les boucles sont simples pour $\kappa \leq 4$, ce qui n'est plus le cas lorsque $\kappa > 4$. Ils entretiennent un lien étroit avec la gravité quantique de Liouville via la fameuse fermeture éclair quantique¹ introduite dans [She16]. Il est possible de définir la mesure de bord quantique ν_h^γ sur les courbes de type SLE_κ avec $\kappa = \gamma^2$ ou $\kappa = 4/\gamma^2$ et donc sur les boucles d'un CLE_κ . La fermeture éclair quantique établit informellement que si l'on trace un SLE sur une surface quantique, alors conditionnellement à la longueur de bord quantique, les surfaces quantiques découpées sont indépendantes.

Distances sur le CLE_4 . Dans le chapitre 3, nous nous intéressons particulièrement au CLE_4 . Ce dernier peut être notamment construit via une exploration branchante partant du bord du disque définie par Werner et Wu dans [WW13] à l'aide d'un processus de Poisson de "bulles" SLE_4 qui est appelée l'exploration uniforme du CLE_4 . Cette exploration est définie comme suit.

Notons E l'ensemble des excursions positives

$$E = \{e \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+); e(0) = 0 \text{ and } \tau(e) := \sup\{s > 0; e(s) > 0\} \in (0, \infty)\}.$$

Soit λ la mesure d'excursions d'Itô sur E , caractérisée par le fait que pour tout $x > 0$, la mesure du sous-ensemble E_x des excursions de hauteur au moins x est finie et si on renormalise λ pour avoir une mesure de probabilité sur E_x , alors après avoir atteint x l'excursion a la loi d'un brownien arrêté quand il atteint zéro.

Si $e \in E$ est une excursion de durée $\tau(e)$, on définit la fonction pilote w sur $[0, \tau(e)]$ par

$$\forall t \in [0, \tau(e)], \quad w_t = 2e(t) - \int_0^t \frac{ds}{e(s)}.$$

On sait par [SW12] que pour λ -presque tout $e \in E$, les transformations conformes g_t caractérisées par l'équation différentielle ordinaire

$$g_0(z) = z \quad \text{et} \quad \forall t \geq 0, \quad \partial_t g_t(z) = \frac{2}{g_t(z) - w_t}$$

définissent une boucle simple $\gamma = \gamma(e)$ dans le demi-plan partant de zéro, dans le sens où le domaine de définition de g_t peut s'écrire $\mathbb{H} \setminus \gamma((0, t])$. On définit la mesure μ^0 comme la mesure image $\gamma_* \lambda$. Pour tout $x \in \mathbb{R}$, notons μ^x la mesure sur les boucles enracinées en x obtenue en translatant μ^0 .

Soit $z \in \mathbb{H}$. Définissons d'abord l'exploration vers z . Soit $(\hat{\gamma}_u)_{u \geq 0}$ un PPP d'intensité $M = \int_{\mathbb{R}} dx \mu^x$. Pour toute boucle $\hat{\gamma}_u$, nous notons \hat{f}_u la transformation conforme de la composante connexe de $\mathbb{H} \setminus \hat{\gamma}_u$ contenant z vers \mathbb{H} qui fixe z et telle que $\hat{f}'_u(z) \in \mathbb{R}_+$. Posons $\hat{F}_u^z = \hat{F}_u := \circ_{0 \leq v < u} \hat{f}_v$ où la composition se fait dans l'ordre d'apparition des \hat{f}_u . Notons aussi τ^z le premier temps u pour lequel la boucle $\hat{\gamma}_u$ entoure z . Posons $(\tilde{\gamma}_u^z)_{u \leq \tau^z} := (\hat{F}_u^{-1}(\hat{\gamma}_u))_{u \leq \tau^z}$. Par le lemme 8 de [WW13], on peut coupler les processus $(\tilde{\gamma}_u^z)_{u \leq \tau^z}$ et $(\tilde{\gamma}_u^{z'})_{u \leq \tau^{z'}}$ de façon à ce qu'ils coïncident jusqu'au moment où ils séparent z de z' . En réalisant ce

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couplage avec un ensemble dénombrable dense de points z dans \mathbb{H} , l'ensemble des boucles entourant les points de cet ensemble dense est le CLE_4 sur le domaine \mathbb{H} . Les processus $(\tilde{\gamma}_u^z)_{u \leq \tau^z}$ pour $z \in \mathbb{H}$ peuvent alors être vus comme des branches de l'exploration uniforme.

On se ramène au disque unité \mathbb{D} en appliquant une transformation conforme $\Phi : \mathbb{H} \rightarrow \mathbb{D}$. Le choix de Φ ne change pas la loi de l'exploration par la proposition 9 de [WW13]. Le temps de découverte de chaque boucle \mathcal{L} du CLE_4 peut être vu comme une distance au bord et est noté $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L})$. Plus récemment, dans [AHPS23], Aru, Holden, Powell et Sun montrent un couplage entre l'exploration uniforme d'un CLE_4 sur un 2-disque quantique indépendant avec une excursion brownienne dans le demi-plan. Grâce à [AdS22], ce couplage permet de décrire l'évolution de la longueur quantique des boucles en utilisant un processus de branchement qui est un processus de croissance-fragmentation. Ils appellent la paramétrisation en temps du processus de croissance fragmentation la "distance quantique" au bord du disque $\partial\mathbb{D}$. Le temps de découverte de chaque boucle \mathcal{L} du CLE_4 est vu comme la "distance quantique" entre $\partial\mathbb{D}$ et \mathcal{L} , que l'on notera $d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})$.

Notons $(Y_{\mathcal{L}}(t))_{0 \leq t \leq d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})}$ la longueur de bord quantique de la région inexplorée dans la branche de l'exploration uniforme qui découvre la boucle \mathcal{L} , paramétrée par la distance quantique. On définit une troisième distance au bord en posant

$$d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}) = \int_0^{d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})} \frac{1}{Y_{\mathcal{L}}(t)} dt.$$

Le théorème ci-dessous montre que la distance d_{Lamperti} décrit la limite d'échelle des distances de graphe dual entre la racine et les faces de grands degrés vers les distances entre le bord et les boucles du CLE_4 .

Théorème 1.2.4. *(extrait du théorème 3.1.4 du chapitre 3) Supposons \mathbf{q} de type $a = 2$. Soit $(f_i^{(\ell)})_{i \geq 1}$ la famille des faces de $\mathfrak{M}^{(\ell)}$ rangées dans l'ordre décroissant des degrés. Soit $(\mathcal{L}_i)_{i \geq 1}$ la famille des boucles du CLE_4 rangées dans l'ordre décroissant de la longueur de bord quantique (en voyant $\partial\mathbb{D}$ comme une boucle). Notons $\nu_h^2(\mathcal{L})$ la longueur quantique d'une boucle \mathcal{L} . Alors, pour la topologie produit,*

$$\left(\frac{\deg(f_i^{(\ell)})}{2\ell}, \frac{2d_{\text{gr}}^{\dagger}(f_r, f_i^{(\ell)})}{\log \ell} \right)_{i \geq 1} \xrightarrow[\ell \rightarrow +\infty]{(d)} (\nu_h^2(\mathcal{L}_i), d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}_i))_{i \geq 1}.$$

Un résultat analogue a lieu en munissant la carte de la distance d_{fpp}^{\dagger} , où dans ce cas le facteur $\log \ell$ est remplacé par une constante. La preuve du théorème ci-dessus ne repose que sur des outils d'épluchage de cartes aléatoires en utilisant les techniques de [BCM18] et les outils développés dans [BCK18, BBCK18]. Il suffit alors d'identifier la loi de la limite grâce aux travaux [AdS22, AHPS23].

Dans le chapitre 3 nous relierons aussi la distance au bord d_{Lamperti} à la distance d_{WW} le long d'une branche de l'exploration uniforme appelée la composante localement la plus grande : à chaque fois que la région inexplorée se découpe en deux, on continue l'exploration de la région qui a la plus grande longueur de bord quantique. Le long de cette branche, d_{Lamperti} et d_{WW} sont alors reliées par un processus continu croissant à accroissements stationnaires (voir le théorème 3.1.3). Si les accroissements étaient indépendants, alors ce processus serait une fonction linéaire déterministe, donc les distances d_{Lamperti} et d_{WW} seraient égales à une constante multiplicative près. Nous donnons à la fin du chapitre 3 une heuristique qui semble aller à l'encontre de cette égalité. Pour montrer ce résultat, nous démontrons une propriété de Markov quantique pour l'exploration uniforme du CLE_4 : pour tout $t \geq 0$, conditionnellement aux longueurs de bord quantique des boucles déjà explorées et à celle des régions inexplorées jusqu'à la distance quantique t , les régions inexplorées sont des disques quantiques indépendants (voir le théorème 3.1.2). Ce résultat étend ceux de [MSW22, MSW21] au cas critique $\gamma = 2$. Nous l'obtenons essentiellement grâce aux travaux de Lehmkuehler [Leh23] qui approxime des explorations du CLE_4 avec des explorations de CLE_{κ} pour $\kappa < 4$ et de [AHPS23] qui montrent la convergence du γ -disque quantique vers le 2-disque quantique lorsque $\gamma \uparrow 2$.

1.2.4 Perspectives

Nous rassemblons ici quelques questions et éléments de réponses qui émergent des résultats présentés jusqu'ici. Concernant l'étude des limites d'échelle des cartes à grands degrés, plusieurs directions apparaissent naturellement.

Distances entre faces uniformes dans le cas $a \in]3/2, 2[$. Les techniques introduites dans le chapitre 2 sont générales et peuvent tout à fait s'appliquer en dehors du cas critique $a = 2$, comme le montre l'usage des martingales du chapitre 2 dans le chapitre 5 pour démontrer la majoration du diamètre de la carte $\mathfrak{M}^{(\ell)}$ de loi $\mathbb{P}^{(\ell)}$ par une constante fois $\log \ell$ dans le cas $a \in]3/2, 2[$ donnée par le théorème 5.1.4. Nous pensons qu'il est tout à fait faisable de montrer l'analogie du théorème 1.2.2 et de l'équation (1.2.1) dans le cas $a \in]3/2, 2[$ pour la distance de graphe dual, autrement dit, si $(F_i^{(\ell)})_{i \geq 1}$ est une suite de faces uniformes indépendantes dans $\mathfrak{M}^{(\ell)}$ prises conditionnellement à $\mathfrak{M}^{(\ell)}$, qu'il existe une constante $C_a > 0$ telle que

$$\left(\frac{d_{\text{gr}}^{\dagger}(f_r, F_i^{(\ell)})}{\log \ell} \right)_{i \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} (C_a)_{i \geq 1} \quad \text{et} \quad \left(\frac{d_{\text{gr}}^{\dagger}(F_i^{(\ell)}, F_j^{(\ell)})}{\log \ell} \right)_{i \neq j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} (2C_a)_{i \neq j \geq 1}.$$

Ces convergences devraient être obtenues en combinant les outils développés dans [Kam23b] avec les résultats de [BC17]. En prenant en compte la majoration du diamètre par une constante fois $\log \ell$ du théorème 5.1.4, cela entraînerait l'absence de limite d'échelle au sens de Gromov-Hausdorff ou Gromov-Prokhorov dans le cas $a \in]3/2, 2[$ avec la distance de graphe duale. Pour la distance de percolation de premier passage, comme observé dans [BC17] dans le cadre de la limite locale, en échangeant la face racine et la face uniforme comme dans le chapitre 2, on devrait pouvoir voir que si $F^{(\ell)}$ est une face aléatoire uniforme de $\mathfrak{M}^{(\ell)}$, la distance $d_{\text{fpp}}^{\dagger}(f_r, F^{(\ell)})$ est tendue dans le cas $a \in]3/2, 2[$.

Limite d'échelle des géodésiques pour la distance de graphe dual. Au vu de la limite d'échelle de l'arbre des géodésiques pour la distance de percolation de premier passage énoncée dans le théorème 1.2.3, on peut conjecturer une limite d'échelle analogue vers \mathcal{T}_a pour les géodésiques pour la distance de graphe dual, en divisant les distances par ℓ^{a-2} dans le cas $a \in]2, 5/2[$ et par $\log \ell$ dans le cas critique $a = 2$. Plus précisément, si $\mathfrak{m} \in \mathcal{M}$, notons $\mathcal{G}(\mathfrak{m})$ le graphe constitué de la réunion des plus courts chemins à la face racine pour la distance d_{gr}^{\dagger} et notons $d_{\mathcal{G}(\mathfrak{m})}$ la distance de graphe sur $\mathcal{G}(\mathfrak{m})$. Pour tout $\ell \geq 1$, notons $(f_i^{(\ell)})_{i \geq 1}$ la suite des faces de $\mathfrak{M}^{(\ell)}$ rangées dans l'ordre décroissant des degrés. Rappelons que $c_a = \pi/\Gamma(a)$ et que $\mathcal{T}_a = (\{w_i; i \geq 1\}, d_{\mathcal{T}_a})$ est l'espace métrique aléatoire que dans le théorème 1.2.3. Nous conjecturons la limite d'échelle suivante.

Conjecture 1.2.1. *Les convergences suivantes ont lieu en loi pour la topologie produit.*

— Si \mathfrak{q} est de type $a \in]2, 5/2[$, alors

$$\left(\ell^{2-a} d_{\mathcal{G}(\mathfrak{m})}(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i, j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1 + g_{\mathfrak{q}}}{2c_a p_{\mathfrak{q}}} d_{\mathcal{T}_a}(w_i, w_j) \right)_{i, j \geq 1},$$

où $g_{\mathfrak{q}} = (1/2) \sum_{k \geq 1} \nu(-k)(2k-1)$.

— Si \mathfrak{q} est de type $a = 2$, alors

$$\left(\frac{1}{\log \ell} d_{\mathcal{G}(\mathfrak{m})}(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i, j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a} d_{\mathcal{T}_a}(w_i, w_j) \right)_{i, j \geq 1}.$$

Pour montrer les limites d'échelle ci-dessus, une approche similaire au renversement du temps de l'exploration uniforme peut être de renverser le temps dans l'exploration d'épluchage par couche. Une autre approche serait de construire une décomposition analogue à celle introduite par Krikun [Kri05] dans le cas des triangulations. C'est en partie ce qui motive l'étude des arbres aléatoires à catastrophes locales par [CCC24]. Un argument d'ergodicité permettrait probablement de comparer les distances d_{gr}^{\dagger} et d_{fpp}^{\dagger} dans le cas $a \in]2, 5/2[$.

Limite d'échelle des cartes stables. Par ailleurs, le flot coalescent de diffusions à sauts nous permet de construire un espace métrique aléatoire dénombrable que l'on conjecture être la limite d'échelle des cartes stables munies de d_{fpp}^{\dagger} ou de d_{gr}^{\dagger} , voir la dernière partie du chapitre 5 pour la construction précise. L'idée principale est que les seules informations manquantes à l'arbre $\mathcal{T}(\mathfrak{m})$ des géodésiques fpp pour retrouver la carte \mathfrak{m} sont les recollements d'arêtes lors de l'exploration uniforme. Ces recollements créent des raccourcis, modifiant ainsi les distances. La même idée est mise en œuvre dans la dernière partie du chapitre 5 qui définit un espace métrique aléatoire $\mathcal{D}_a = (\{w_i; i \geq 1\}, d_{\mathcal{D}_a})$ en partant, non pas de $\mathcal{T}(\mathfrak{m})$,

mais de sa limite d'échelle donnée par le flot coalescent de diffusions. Rappelons que $\mathfrak{M}^{(\ell)}$ est une carte de loi $\mathbb{P}^{(\ell)}$ pour tout $\ell \geq 1$ et que $(f_i^{(\ell)})_{i \geq 1}$ est la suite des faces de $\mathfrak{M}^{(\ell)}$ rangées dans l'ordre décroissant des degrés.

Conjecture 1.2.2. *Nous conjecturons les limites d'échelle suivantes :*

— (conjecture 5.6.2) *Supposons $a \in]3/2, 5/2[$. Alors, pour la topologie produit,*

$$\left(\ell^{2-a} d_{\text{fpp}}^\dagger(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1}.$$

— (conjecture 5.6.3) *Supposons $a \in]2, 5/2[$. Alors, pour la topologie produit,*

$$\left(\ell^{2-a} d_{\text{gr}}^\dagger(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1 + g_{\mathbf{q}}}{2c_a p_{\mathbf{q}}} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1}.$$

De plus, \mathcal{D}_a peut être complété en un espace compact et la convergence peut aussi se réécrire comme une convergence au sens de Gromov-Hausdorff.

— (conjecture 5.6.4) *Supposons $a = 2$. Alors, pour la topologie produit,*

$$\left(\frac{1}{\log \ell} d_{\text{gr}}^\dagger(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1}.$$

Pour obtenir la limite d'échelle des cartes à grands degrés, l'obstacle principal est d'obtenir la convergence des distances modifiées par ces raccourcis.

Lien entre les distances d_{WW} et d_{Lamperti} . Du côté de la gravité quantique et des CLE, de nombreuses questions naturelles apparaissent. Tout d'abord, il paraîtrait naturel de penser que les distances au bord d_{WW} et d_{Lamperti} seraient les mêmes à une constante multiplicative près. La question est d'ailleurs soulevée dans la section 5.2.3 de [AHPS23]. Nous donnons une heuristique dans la section 3.7.2 du chapitre 3 qui semble aller dans l'autre direction. Soit Z un point tiré au hasard dans \mathbb{D} de loi $\mu_h/\mu_h(\mathbb{D})$ conditionnellement au 2-disque quantique et au CLE_4 . Considérons la boucle \mathcal{L} du CLE_4 qui entoure le point Z . La distance d_{WW} du bord $\partial\mathbb{D}$ aux boucles du CLE_4 est indépendante du disque quantique, de sorte que, par invariance conforme, $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L})$ a la même loi que la distance d_{WW} du bord à la boucle entourant le point 0. La loi de cette dernière est connue depuis [WW13] et est une loi exponentielle. Pour montrer que les distances d_{WW} et d_{Lamperti} ne sont pas les mêmes, il suffirait de montrer que $d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L})$ ne suit pas une loi exponentielle. Considérons l'ensemble des points entourés par des boucles à distance d_{Lamperti} au moins $h \geq 0$ du bord. Par la propriété de Markov pour les disques quantiques, les composantes connexes de cet ensemble sont des 2-disques quantiques indépendants conditionnellement à leur longueur de bord quantique. Le fait de prendre Z de loi $\mu_h/\mu_h(\mathbb{D})$ va biaiser l'aire quantique de la composante connexe qui contient Z , ce qui empêcherait intuitivement l'absence de mémoire et donc le fait que $d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L})$ suive une loi exponentielle.

Distances entre les boucles. Une autre question importante est la définition des distances entre les boucles du CLE_4 . La construction de la distance d_{WW} entre les boucles du CLE_4 a été abordée par Watson dans la toute fin de sa thèse [Wat15], sans tous les détails. L'idée est d'utiliser l'exploration uniforme du CLE_4 de [WW13] en la démarrant non pas au bord du disque $\partial\mathbb{D}$ mais en une boucle du CLE_4 , conditionnellement au CLE_4 . La distance d'une boucle \mathcal{L}_1 à une boucle \mathcal{L}_2 est alors construite comme le temps de découverte de \mathcal{L}_2 en partant de \mathcal{L}_1 . L'extension de d_{Lamperti} en une distance entre les boucles du CLE_4 est une autre question qui émerge du théorème 1.2.4 et du fait que d_{Lamperti} est différente de d_{WW} . Nous conjecturons que l'extension de d_{Lamperti} aux distances entre boucles du CLE_4 , une fois construite, décrit la limite d'échelle des cartes 3/2-stables. Plus précisément, en supposant construite la distance d_{Lamperti} entre les boucles du CLE_4 , si $(\mathcal{L}_i)_{i \geq 1}$ est la suite des boucles du CLE_4 rangées dans l'ordre décroissant de longueur quantique de bord, on peut conjecturer que, pour la topologie produit,

$$\left(\frac{1}{\log \ell} d_{\text{gr}}^\dagger(f_i^{(\ell)}, f_j^{(\ell)}) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2} d_{\text{Lamperti}}(\mathcal{L}_i, \mathcal{L}_j) \right)_{i,j \geq 1}.$$

L’exploration uniforme et les géodésiques dans le CLE_4 . Par ailleurs, les théorèmes 1.2.4, 1.2.3 et la construction de \mathcal{T}_a dans le cas $a = 2$ motivent l’idée que l’exploration uniforme du CLE_4 devrait être caractérisée par la propriété de Markov quantique et par l’invariance par des “rotations quantiques” : informellement, si l’on “décolle” une région inexplorée et qu’on la recolle selon la mesure de bord quantique mais avec un décalage, l’exploration a toujours la même loi. Voir la section 5.2.1 de [AHPS23] pour plus de détails sur un problème très proche. Cette propriété devrait impliquer en particulier que les géodésiques des boucles du CLE_4 vers le bord $\partial\mathbb{D}$, i.e. la généalogie des boucles découvertes par l’exploration uniforme, sont données par l’espace métrique \mathcal{T}_a dans le cas $a = 2$ et peuvent donc être décrites par le même flot coalescent de diffusions à sauts qui apparaît dans la limite d’échelle des géodésiques à la racine dans les cartes 3/2-stables.

Géométrie des cartes décorées de boucles $O(2)$. Enfin, dans la mesure où les cartes 3/2-stables peuvent être vues comme les *gaskets* des cartes décorées de boucles $O(2)$ critiques, tous les résultats sur les distances d_{fpp}^\dagger et d_{gr}^\dagger dans les cartes 3/2-stables peuvent être vus comme des résultats sur les distances sur les cartes décorées de boucles $O(2)$ critiques où par convention on se déplace “instantanément” le long des boucles, comme expliqué dans [Bud18], puisque dans le *gasket* les boucles deviennent des bords de faces. Voir par exemple la section 3.7.1 du chapitre 3. Cependant, si l’on enlève cette identification, qui peut paraître artificielle, la compréhension des distances sur les cartes décorées de boucles $O(n)$ critiques reste un défi majeur.

1.3 Arbres aléatoires et coalescence

Dans une deuxième partie, indépendante de la première, nous introduisons le modèle des arbres récursifs gelés, qui généralise les arbres récursifs uniformes et les arbres de Bienaymé(-Galton-Watson) de loi de reproduction géométrique et permet de décrire la diffusion d’une épidémie pour le modèle classique SIR (sain, infecté, guéri) sur un graphe complet. Nous étudions notamment ses limites locales et limites d’échelle, avec pour outil principal une construction par coalescence avec naissance généralisant le coalescent de Kingman. Ces travaux ont été réalisés en collaboration avec Étienne Bellin, Arthur Blanc-Renaudie et Igor Kortchemski. Dans le dernier chapitre, dans un travail avec Arthur Blanc-Renaudie, nous utilisons aussi des processus de coalescence avec naissance pour déterminer les limites d’échelle d’arbres aléatoires uniformes dont les degrés des sommets sont prescrits à chaque hauteur.

1.3.1 Arbres récursifs gelés

Arbres récursifs uniformes. Les arbres récursifs uniformes sont construits en partant d’un seul sommet, la racine, et en attachant l’un après l’autre de nouveaux sommets à un sommet existant choisi uniformément au hasard. Ces arbres, aussi appelés arbres à attachement uniforme, ont été introduits dans [NR70]. Ils ont été proposés comme modèle pour des phénomènes allant de la propagation d’épidémies [Moo74] aux liens les plus empruntés sur internet [JKZVM02], en passant par les systèmes de vente pyramidale [Gas77] ou les copies de manuscrits au Moyen Âge [NH82].

Sur le plan mathématique, ce modèle a été considérablement étudié, voir par exemple [Pit94] ou la synthèse [SM95]. On sait notamment que la hauteur d’un sommet typique dans un arbre récursif uniforme à n sommets est équivalente en probabilité à $\log n$ (voir par exemple [Dev88]) et la hauteur totale de l’arbre est équivalente p.s. à $e \log n$ quand $n \rightarrow +\infty$ (voir par exemple [Pit94]). Le modèle a été généralisé dans de nombreuses directions, notamment en ajoutant des poids déterministes comme dans [KR01] ou aléatoires comme dans [BV06].

Arbres récursifs gelés. Nous introduisons une nouvelle généralisation des arbres récursifs uniformes en autorisant des étapes où au lieu d’attacher un nouveau sommet, on “gèle” un sommet choisi uniformément au hasard parmi les sommets pas encore gelés, sur lequel les sommets ajoutés ultérieurement ne pourront plus s’attacher.

Plus précisément, soit $\mathbf{x} = (x_i)_{i \geq 1}$ une suite d’éléments de $\{\pm 1\}$. On part d’un seul sommet actif, c’est-à-dire non gelé, qui est la racine de l’arbre. À chaque étape $i \geq 1$, tant qu’il reste des sommets actifs,

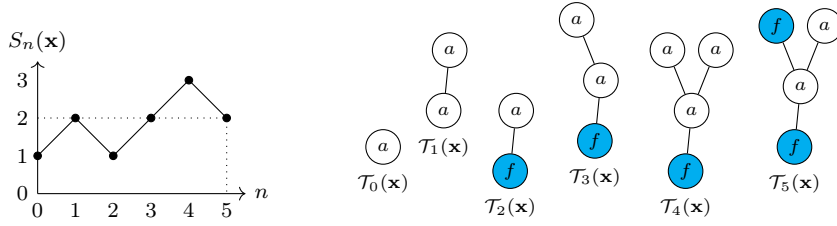


FIGURE 1.8 – À gauche, la marche $(S_n(\mathbf{x}))_{n \geq 0}$ jusqu’au temps $n = 5$ associée à la suite $\mathbf{x} = +1, -1, +1, +1, -1, \dots$. À droite, une réalisation possible des arbres $\mathcal{T}_0(\mathbf{x})$ jusqu’à $\mathcal{T}_5(\mathbf{x})$. Les sommets gelés sont coloriés en bleu et étiquetés f tandis que les sommets actifs sont étiquetés a .

- Si $x_i = 1$, alors on attache un nouveau sommet actif sur un sommet actif choisi uniformément au hasard ;
- Si $x_i = -1$, alors on gèle un sommet qui était jusqu’à maintenant actif choisi uniformément au hasard.

Pour tout $n \geq 0$, notons $\mathcal{T}_n(\mathbf{x})$ l’arbre obtenu après n étapes à partir de la suite \mathbf{x} . Par ailleurs, posons $S_0(\mathbf{x}) := 1$ et pour tout $n \geq 1$,

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i \quad \text{et} \quad \tau(\mathbf{x}) = \inf\{n \geq 0, S_n(\mathbf{x}) = 0\}.$$

Voir la figure 1.8 pour un exemple. Remarquons que si $\tau(\mathbf{x})$ est fini, alors il correspond au premier temps auquel tous les sommets sont gelés et pour tout $n \in \llbracket 0, \tau(\mathbf{x}) \rrbracket$, il y a $S_n(\mathbf{x})$ sommets actifs dans $\mathcal{T}_n(\mathbf{x})$. Remarquons que lorsque $x_i = 1$ pour tout $i \geq 1$, on retrouve les arbres récursifs uniformes. De plus, si $\mathbf{X} = (X_i)_{i \geq 1}$ est une suite de v.a.i.i.d. uniformes dans $\{\pm 1\}$, alors conditionnellement à $\tau(\mathbf{X}) = n$, l’arbre $\mathcal{T}_n(\mathbf{X})$ est un arbre plan uniforme quand l’ordre plan des sommets de $\mathcal{T}_n(\mathbf{X})$ est choisi uniformément au hasard conditionnellement à $\mathcal{T}_n(\mathbf{X})$, ce dernier point est une conséquence du théorème énoncé dans le paragraphe suivant.

Limites locales. Si τ est un arbre enraciné localement fini, pour tout $h \geq 0$, notons $\llbracket \tau \rrbracket_h$ le sous-arbre obtenu en ne gardant que les sommets de hauteur au plus h et les arêtes entre ces sommets. La distance locale entre deux arbres τ, τ' peut être par exemple définie en posant

$$d_{\text{loc}}(\tau, \tau') = (1 + \sup\{h \geq 0, \llbracket \tau \rrbracket_h = \llbracket \tau' \rrbracket_h\})^{-1}.$$

Il est bien connu que l’espace métrique des arbres enracinés localement finis muni de d_{loc} est un espace métrique complet. Une suite d’arbres τ_n converge pour d_{loc} vers τ si et seulement si pour tout $h \geq 0$ il existe un rang n_h à partir duquel $\llbracket \tau_n \rrbracket_h = \llbracket \tau \rrbracket_h$. On dit que la suite converge localement. Nous montrons d’abord que la suite $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$ converge localement en loi (et presque sûrement dans ce cas) si et seulement si la série $\sum_{i \geq 1} (1/S_i(\mathbf{x})) \mathbb{1}_{x_i = -1}$ diverge. C’est un cas particulier du théorème 6.1.1 du chapitre 6 qui traite aussi le cas plus général où on fait varier la suite \mathbf{x} avec n . Nous notons alors $\mathcal{T}_\infty(\mathbf{x})$ la limite locale de $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$. Lorsqu’on prend une suite aléatoire $\mathbf{X} = (X_i)_{i \geq 1}$, nous montrons le résultat suivant :

Théorème 1.3.1. (Théorème 6.1.2 du chapitre 6) Soit $p \in [0, 1[$. Soit $\mathbf{X} = (X_i)_{i \geq 1}$ une suite de v.a.i.i.d. dans $\{\pm 1\}$ telle que $\mathbb{P}(X_1 = 1) = p$. Alors $\mathcal{T}_\infty(\mathbf{X})$ est un arbre de Bienaymé(-Galton-Watson) de loi de reproduction géométrique μ telle que $\mu(k) = (1-p)p^k$ pour tout entier $k \geq 0$.

La preuve de ce théorème repose sur une caractérisation des arbres de Bienaymé à loi de reproduction géométrique : un arbre aléatoire \mathcal{T} est un arbre de Bienaymé de loi de reproduction géométrique de paramètre p si et seulement si la probabilité que la racine n’ait pas d’enfant est $1-p$ et, conditionnellement à l’évènement où la racine a au moins un enfant, le sous-arbre formé par le premier enfant de la racine et ses descendants et le sous-arbre formé par la racine et ses autres descendants sont indépendants et ont la même loi que \mathcal{T} . Pour vérifier que $\mathcal{T}_\infty(\mathbf{X})$ satisfait cette caractérisation, on remarque que la racine est gelée dès le début si et seulement si $X_1 = -1$, donc avec probabilité $1-p$. Dans le cas où la racine n’est pas gelée au début, on lui attache un premier enfant et on peut considérer le sous-arbre constitué du premier

enfant et de ses descendants ainsi que le sous-arbre formé par la racine et de ses autres descendants. On peut alors se convaincre que ces deux sous-arbres, à chaque fois qu'ils sont modifiés, évoluent de manière indépendante comme deux suites d'arbres récursifs gelés et en particulier leurs limites locales ont la même loi que $\mathcal{T}_\infty(\mathbf{X})$. La difficulté principale de la preuve est essentiellement technique et consiste à formuler et démontrer cette dernière assertion.

Limites d'échelle. Dans tout ce paragraphe, on se donne des suites $\mathbf{x}^n = (x_i^n)_{i \geq 1}$ pour tout $n \geq 0$ à valeurs dans $\{\pm 1\}$ telles que $\tau(\mathbf{x}^n) > n$ et on considère l'arbre $\mathcal{T}_n(\mathbf{x}^n)$. Pour tout $n \geq 0$, notons

$$h_n^+ = \sum_{i=1}^n \frac{1}{S_i(\mathbf{x}^n)} \mathbb{1}_{x_i^n=1}. \quad (1.3.1)$$

Dans le théorème 6.1.3 du chapitre 6 nous montrons que la hauteur $H(U^n)$ d'un sommet actif U^n de $\mathcal{T}_n(\mathbf{x}^n)$ tiré uniformément au hasard satisfait pour tout $p \geq 1$,

$$\frac{H(U^n)}{h_n^+} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} 1.$$

Lorsque h_n^+ augmente assez vite, i.e. lorsque $\log(n)/h_n^+ \rightarrow 0$ quand $n \rightarrow +\infty$, la hauteur totale $\text{Height}(\mathcal{T}_n(\mathbf{x}^n))$ de $\mathcal{T}_n(\mathbf{x}^n)$ satisfait la même convergence que $H(U^n)$. À l'inverse, dans le régime "linéaire" où informellement $S_i(\mathbf{x}^n) \simeq ci$ pour une certaine constante $c \in]0, 1]$, nous montrons dans le théorème 6.1.4 que

$$\frac{h_n^+}{\log n} \xrightarrow[n \rightarrow \infty]{} \frac{c+1}{2c} \quad \text{alors que} \quad \frac{\text{Height}(\mathcal{T}_n(\mathbf{x}^n))}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \frac{1+c}{2c} f(c), \quad (1.3.2)$$

où $f(c)$ est l'unique solution telle que $f(c) > 1$ à $f(c)(\log f(c) - 1) = (c-1)/(c+1)$. On montre dans le même théorème que la hauteur d'un sommet typique (pas nécessairement actif) est aussi équivalente en probabilité à h_n^+ tandis que la distance entre deux sommets typiques est équivalente à $2h_n^+$ en probabilité, ce qui exclut toute limite d'échelle au sens GP ou GH comme expliqué dans la discussion suivant le théorème 1.2.2. Dans le cas $c = 1$, on retrouve les résultats évoqués plus haut pour les arbres récursifs uniformes.

Dans le chapitre 7, nous examinons le cas sous-linéaire où informellement $S_i(\mathbf{x}^n) \simeq f(i/n)i^\alpha$, où $f : [0, 1] \rightarrow \mathbb{R}_+$ est une fonction mesurable et $\alpha \in]0, 1[$, de sorte que la hauteur satisfait pour tout $p \geq 1$,

$$\frac{\text{Height}(\mathcal{T}_n(\mathbf{x}^n))}{n^{1-\alpha}} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} \int_0^1 \frac{dt}{2f(t)}.$$

Pour tout $n \geq 1$, notons $\mathcal{T}_n(\mathbf{x}^n)/n^{1-\alpha}$ l'espace métrique obtenu en divisant les distances de graphe sur l'arbre par $n^{1-\alpha}$ et muni de la mesure uniforme sur les sommets. Nous identifions une transition de phase en $\alpha = 1/2$ (voir la figure 1.3.1) :

- Quand $\alpha \in]0, 1/2[$, l'espace métrique mesuré $\mathcal{T}_n(\mathbf{x}^n)/n^{1-\alpha}$ converge en loi au sens de GHP vers le segment $[0, \int_0^1 dt/(2f(t))]$ muni de la mesure image de la mesure uniforme sur $[0, 1]$ par $x \mapsto \int_0^x dt/(2f(t))$.
- Quand $\alpha \in]1/2, 1[$, l'arbre $\mathcal{T}_n(\mathbf{x}^n)$ ressemble à une étoile avec des branches de taille de l'ordre de $n^{1-\alpha}$. Plus précisément, si V_0^n est la racine de $\mathcal{T}_n(\mathbf{x}^n)$ et si V_1^n, \dots, V_k^n sont des sommets tirés uniformément au hasard dans $\mathcal{T}_n(\mathbf{x}^n)$, alors en notant d^n la distance de graphe sur l'arbre,

$$\left(\frac{1}{n^{1-\alpha}} d^n(V_i^n, V_j^n) \right)_{0 \leq i \neq j \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_0^{U_i} \frac{dt}{2f(t)} + \int_0^{U_j} \frac{dt}{2f(t)} \right)_{0 \leq i \neq j \leq k},$$

où U_1, \dots, U_k sont des v.a.i.i.d. uniformes dans $[0, 1]$ et $U_0 = 0$. En particulier, par un raisonnement semblable à celui suivant le théorème 1.2.2, on peut voir qu'il n'y a pas de limite d'échelle au sens GP ou GH.

Dans le cas critique $\alpha = 1/2$, le théorème 7.2.3 donne la limite d'échelle au sens GHP vers un arbre réel compact mesuré aléatoire $\mathcal{T}(f)$:

$$\frac{\mathcal{T}_n(\mathbf{x}^n)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}(f).$$

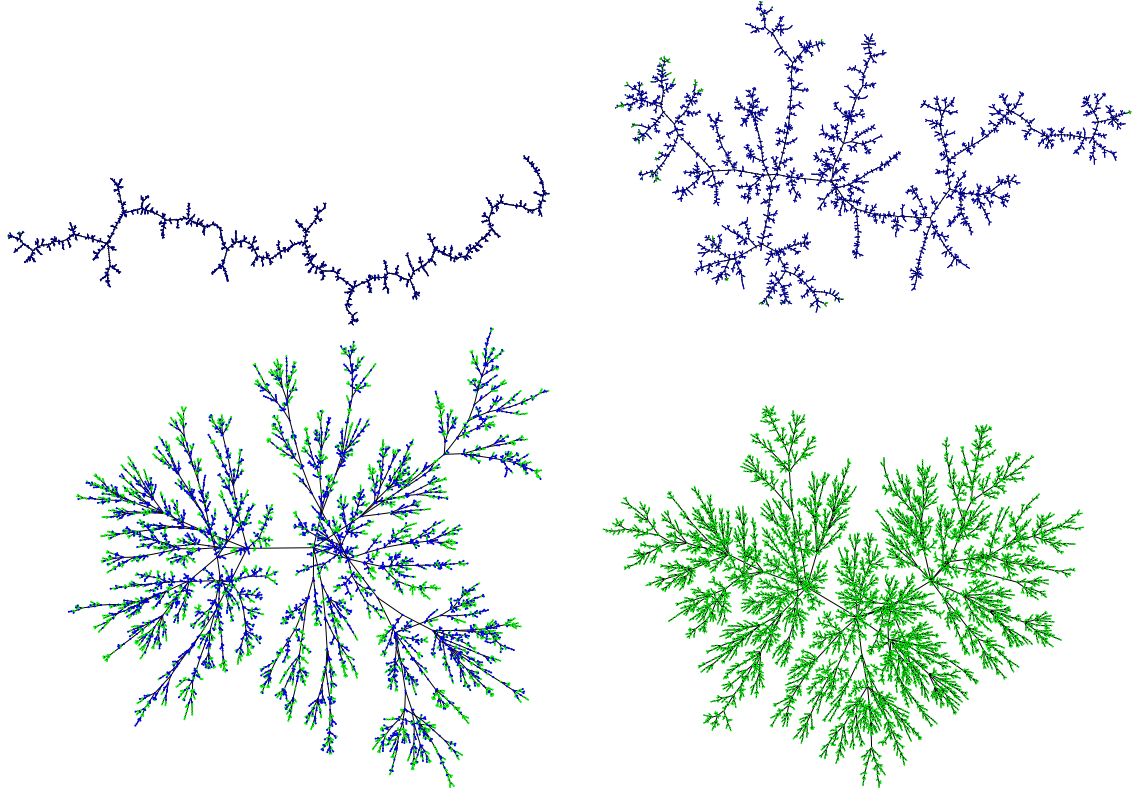


FIGURE 1.9 – Simulations d’arbres récursifs gelés, où le nombre de sommets actifs évolue comme le nombre d’étapes à la puissance α ; haut : $\alpha = 0.2$ et 0.5 , bas : $\alpha = 0.8$ et 1 . Les sommets gelés sont bleus ; les sommets actifs sont verts.

L’arbre réel $\mathcal{T}(f)$ n’est pas construit de manière classique via des fonctions de contour mais grâce à un changement de temps d’une variante inhomogène en temps d’un coalescent de Kingman introduite par Aldous dans [Ald98] : des particules naissent à des temps uniformes dans $[0, 1]$ et sur chaque intervalle $[t - dt, t]$ elles coalescent à taux $1/f(t)^2$. L’arbre $\mathcal{T}(f)$ n’a que des points de branchement de degré 2, sauf éventuellement la racine qui a un degré infini lorsque $\int_0 1/f^2 < \infty$. Une conséquence du lien avec les arbres plans uniformes est que si e est une excursion brownienne normalisée, alors $\mathcal{T}(e)$ est un arbre brownien.

La plupart des limites d’échelle ci-dessus sont démontrées en utilisant une construction des arbres récursifs gelés via un processus de coalescence avec naissance en renversant le temps qui généralise le coalescent de Kingman : les étapes de gel d’un sommet correspondent à la naissance d’un sommet tandis que les étapes d’attachement deviennent des étapes de coalescence. Pour connaître la hauteur d’un sommet étant donné son temps de naissance, il suffit de savoir le nombre d’évènements de coalescence qu’il a traversés, et puisque les coalescences se font aléatoirement de manière uniforme parmi les $S_n(\mathbf{x})$ particules, la hauteur d’un sommet actif de $\mathcal{T}_n(\mathbf{x})$ peut s’écrire sous la forme $\sum_{i=1}^n X_i$ où les X_i sont des v.a. indépendantes de Bernoulli de paramètre $(1/S_i(\mathbf{x})) \mathbb{1}_{x_i=1}$ et la hauteur d’un sommet gelé à un temps b s’écrit $\sum_{i=1}^b X_i$. Ceci explique le rôle de la quantité (1.3.1). De manière analogue, pour étudier la généalogie de l’arbre $\mathcal{T}_n(\mathbf{x})$, on peut remarquer que deux particules coalescent au temps i avec probabilité $1/\binom{S_i(\mathbf{x})}{2}$ lorsque $x_i = 1$.

La limite d’échelle GHP dans le cas critique $\alpha = 1/2$ est obtenue en montrant la convergence de la généalogie vers celle de $\mathcal{T}(f)$ qui est décrite par le coalescent évoqué précédemment, puis en appliquant la technique du chaînage. Le chaînage est un outil important en théorie de la concentration présenté dans le livre [Tal05] qui permet de majorer une fonction sur un espace en utilisant une suite croissante de sous-espaces.

1.3.2 Arbre des infections dans le modèle SIR

Modèle SIR. Le modèle SIR (susceptible, infecté, rétabli) est un modèle compartimental classique pour l'évolution d'épidémies. Historiquement, les modèles épidémiologiques ont été utilisés dès le début du XX^{ème} siècle et se sont largement développés depuis, devenant un outil essentiel à la prise de décision face aux pandémies. Voir par exemple [AB12, BPB⁺19] pour une synthèse sur les modèles stochastiques d'épidémies et leur étude statistique. Le modèle SIR sépare les individus en trois catégories :

- Les individus infectés, qui peuvent infecter des individus susceptibles ou guérir ;
- Les individus susceptibles, qui peuvent être infectés par d'autres individus infectés ;
- Les individus rétablis, qui ne peuvent plus être infectés ni infecter les autres.

Le modèle SIR à temps continu a été introduit par Bartlett dans [Bar49] et a été étudié et généralisé dans de nombreuses directions, notamment dans un cadre multi-types, ou sur différents graphes. Voir par exemple [AB12].

L'arbre des infections. On se concentre ici sur l'étude de l'arbre des infections obtenu en suivant l'épidémie où une arête est présente entre deux individus si l'un d'eux a infecté l'autre. Cet arbre apparaît naturellement [CSBV16, KFH06], mais n'a à notre connaissance jamais été étudié mathématiquement.

Plus précisément, on commence avec 1 infecté et $n \geq 1$ susceptibles. Chaque infecté guérit à taux 1 et pour toute paire d'individus infecté et susceptible, l'infecté infecte le susceptible à taux $\lambda_n > 0$. On appelle étape de l'infection un événement où un susceptible devient infecté ou lorsqu'un infecté guérit. Notons τ_n le nombre d'étapes quand l'épidémie disparaît. Notons pour tout $k \geq 0$, \mathcal{T}_k^n l'arbre des infections après k étapes. On pose aussi $\mathcal{T}^n = \mathcal{T}_{\tau_n}^n$. Notons par ailleurs, $(U_k^n, I_k^n)_{0 \leq k \leq \tau_n}$ les nombres de susceptibles et infectés à chaque étape de l'épidémie. Remarquons que $(U_k^n, I_k^n)_{0 \leq k \leq \tau_n}$ est une chaîne de Markov avec pour état initial $(U_0^n, I_0^n) = (n, 1)$ qui est absorbée lorsque I_k^n atteint zéro.

Définissons la suite $\mathbf{X}^n = (X_i^n)_{1 \leq i \leq \tau_n}$ de variables aléatoires dans $\{\pm 1\}$ comme suit : pour tout $i \in [1, \tau_n]$, on pose $X_i^n = I_i^n - I_{i-1}^n$. Alors, par construction, on peut voir que $(\mathcal{T}_k^n)_{0 \leq k \leq \tau_n}$ a la même loi que la suite d'arbres récursifs gelés $(\mathcal{T}_k(\mathbf{X}^n))_{0 \leq k \leq \tau_n}$, où les infectés correspondent aux sommets actifs et où les guéris correspondent aux sommets gelés.

Dans la dernière partie du chapitre 6 nous appliquons les résultats obtenus sur les arbres récursifs gelés à l'arbre épidémique. En utilisant notamment le théorème 1.3.1, nous obtenons facilement la limite locale :

Théorème 1.3.2. (Théorème 6.6.1 du chapitre 6) *Nous avons les résultats suivants.*

- Supposons $\lambda_n \sim \lambda/n$ pour un $\lambda > 0$. Alors \mathcal{T}^n converge localement en loi vers un arbre de Bienaymé-(Galton-Watson) de loi de reproduction géométrique de paramètre $1/(1 + \lambda)$.
- Supposons $\lambda_n \gg 1/n$. Alors la suite $(\mathcal{T}^n)_{n \geq 0}$ n'est pas tendue pour la topologie de la convergence locale.

Pour la limite d'échelle de la hauteur et des distances entres sommets uniformes, on montre dans le théorème 6.6.2 que le nombre d'infectés a une limite fluide, au sens où $t \mapsto I_{[nt]}^n/n$ converge vers une fonction solution d'une équation différentielle ordinaire. Lorsque $\lambda_n \sim \lambda/n$ avec $\lambda > 1$, sur l'évènement de "survie" de l'épidémie, qui correspond à la survie du processus de Bienaymé dans le théorème ci-dessus, il est alors possible de voir que le nombre d'infectés augmente linéairement, de sorte que la hauteur est logarithmique. Dans le cas $\lambda_n \gg 1/n$, on montre que \mathcal{T}^n ressemble à un arbre récursif uniforme. En utilisant (1.3.2), nous montrons le théorème suivant.

Théorème 1.3.3. (Extrait du théorème 6.6.2 du chapitre 6) *On a les résultats suivants.*

- Supposons $\lambda_n \sim \lambda/n$ avec $\lambda > 1$. Soit A l'évènement de survie d'un processus de Bienaymé de loi de reproduction géométrique de paramètre $1/(1 + \lambda)$. Pour tout $n \geq 1$, conditionnellement à \mathcal{T}^n , soient V_1^n et V_2^n deux sommets indépendants uniformes dans \mathcal{T}^n . Alors

$$\frac{H(V_1^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(d)} \frac{\lambda}{\lambda - 1} \mathbb{1}_A \quad \text{et} \quad \frac{d^n(V_1^n, V_2^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(d)} \frac{2\lambda}{\lambda - 1} \mathbb{1}_A,$$

où $H(V_1^n)$ est la hauteur de V_1^n dans \mathcal{T}^n et d^n est la distance de graphe dans \mathcal{T}^n .

- Supposons $\lambda_n \gg 1/n$. Alors

$$\frac{H(V_1^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1, \quad \frac{d^n(V_1^n, V_2^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 2 \quad \text{et} \quad \frac{\text{Height}(\mathcal{T}^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} e.$$

La hauteur totale de l'arbre dans le cas $\lambda_n \sim \lambda/n$ est l'objet d'un travail en cours avec Igor Kortchemski et Delphin Sénizergues.

1.3.3 Arbres uniformes à degrés et hauteurs fixés

Dans le chapitre 8 nous introduisons un modèle simple d'arbre aléatoire pour lequel les degrés des sommets sont fixés à chaque hauteur. Formellement, considérons un ensemble de sommets $\{(i, j)\}_{i \geq 0, j \geq 1}$ et pour tout $n \in \mathbb{N}$ une famille d'entiers positifs $(d_{i,j}^n)_{i \geq 0, j \geq 1}$. Chaque sommet (i, j) a pour hauteur i et son degré est $d_{i,j}^n$. Sans perdre de généralité, on suppose que sur toute ligne $i \geq 0$, la suite $(d_{i,j}^n)_{j \geq 1}$ est décroissante. De plus, pour faire en sorte que tout sommet (i, j) de degré non nul soit relié à la racine $(0, 1)$, on suppose que $d_{0,2}^n = 0$ et

$$\forall i \geq 0, \quad \#\{j \geq 1, d_{i+1,j}^n > 0\} \leq D_i^n := \sum_{j \geq 1} d_{i,j}^n < \infty.$$

L'arbre \mathcal{T}^n est construit de la façon suivante. Le sommet $(0, 1)$ est la racine et, indépendamment, à chaque hauteur $i \geq 0$, on attache les sommets dans $\{(i+1, j)\}_{1 \leq j \leq D_i^n}$ uniformément au hasard à des sommets dans $\{(i, j), j \geq 1, d_{i,j}^n \geq 1\}$ de sorte que tout sommet (i, j) a exactement $d_{i,j}^n$ enfants. Remarquons que la hauteur de \mathcal{T}^n est $h^n := \inf\{i \geq 0; d_{i,1}^n = 0\}$. Un exemple est donné par la figure 1.10.

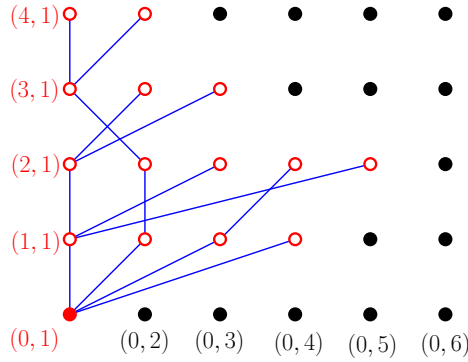


FIGURE 1.10 – Un exemple d'arbre \mathcal{T} à degrés et hauteurs fixés avec $d_{0,1} = 4, d_{1,1} = 3, d_{1,2} = 1, d_{1,3} = 1, d_{2,1} = 2, d_{2,2} = 1, d_{3,1} = 2$ et tous les autres degrés sont nuls. Les sommets de \mathcal{T} sont représentés par des cercles rouge, la racine est un point rouge, tandis que les autres sommets sont des points noirs. Ici, $D_0 = 4, D_1 = 5, D_2 = 3, D_3 = 2, D_4 = 0$.

Nous donnons des conditions suffisantes sur $(d_{i,j}^n)_{i \geq 0, j \geq 1}$ pour que l'arbre \mathcal{T}^n admette une limite d'échelle au sens de GHP, en autorisant les sommets de grands degrés. Dans la suite, sans perdre de généralité, quitte à prendre une sous suite, la hauteur de \mathcal{T}^n sera d'ordre n . Ces conditions peuvent être résumées de la manière suivante. Nous supposons :

- La convergence du profil : il existe une mesure sans atome de support $[0, 1]$ telle qu'on ait la convergence faible

$$\sum_{i \geq 0} \delta_{i/n} \frac{D_i^n}{\sum_{k \geq 0} D_k^n} \xrightarrow{n \rightarrow \infty} \nu;$$

- La convergence des grands degrés : il existe une famille $\Theta = ((\theta_j(t))_{j \geq 1})_{t \in]0, 1[}$ de réels positifs telle qu'on ait la convergence faible de mesures sur $]0, 1[\times \mathbb{R}_+^*$

$$\sum_{1 \leq i < n; d_{i,j}^n > 0} \delta_{(i/n, d_{i,j}^n / D_i^n)} \xrightarrow{n \rightarrow \infty} \sum_{t, j; \theta_j(t) > 0} \delta_{(t, \theta_j(t))};$$

- La convergence du "taux de coalescence" : il existe une mesure ρ localement finie telle qu'on ait la convergence faible de mesures

$$\sum_{1 \leq i < n; d_{i,j}^n > 0} \delta_{i/n} \frac{\binom{d_{i,j}^n}{2}}{\binom{D_i^n}{2}} \xrightarrow{n \rightarrow \infty} \rho + \sum_{j, t; \theta_j(t) > 0} \theta_j(t)^2 \delta_t.$$

Nous faisons quelques autres hypothèses techniques : que deux grands degrés à hauteurs proches finissent par être à la même hauteur quand n est assez grand, ou encore que pour tout $0 < x < y < 1$, la somme $\sum_{x \leq t \leq y} \sum_{j \geq 1} \theta_j(t)^2$ est finie. Enfin pour avoir un espace métrique à la limite, on suppose que pour tout $0 < a < b < 1$, on a $\rho([a, b]) > 0$ ou $\sum_{j \geq 1} \sum_{t \in [a, b]; \theta_j(t) > 0} \theta_j(t) = \infty$. Remarquons que la convergence des grands degrés donne par ailleurs l'inégalité $\sum_{j \geq 1} \theta_j(t) \leq 1$ pour tout $t \geq 0$.

Sous les hypothèses ci-dessus, on montre que \mathcal{T}^n/n converge en loi au sens de GP vers un espace métrique mesuré $\mathcal{T}(\nu, \rho, \Theta)$ (qui est un arbre réel compact mesuré). Sous des hypothèses supplémentaires, la convergence a aussi lieu au sens de GHP.

La convergence et la construction de $\mathcal{T}(\nu, \rho, \Theta)$ sont obtenues en observant la généalogie de sommets typiques, et cette généalogie est construite à l'aide de processus coalescents avec naissance. L'arbre réel $\mathcal{T}(\nu, \rho, \Theta)$ est construit à l'aide du coalescent suivant : des particules naissent indépendamment à des instants de $[0, 1]$ de loi ν , puis sur chaque intervalle $[t, t - dt]$, deux particules coalescent à taux $\rho(dt)$ et en chaque instant $t \geq 0$ tel que $\theta_1(t) > 0$, les particules sont associées "au j -ème atome" avec probabilité $\theta_j(t)$ et les particules associées au même atome coalescent. La convergence GP découle de la convergence du coalescent discret associé à la généalogie dans \mathcal{T}^n vers le coalescent décrit ci-dessus.

Le fait que $\mathcal{T}(\nu, \rho, \Theta)$ est bien défini et la convergence GHP sont établis à l'aide de la technique du chaînage.

Applications. Nous conjecturons que nos résultats s'appliquent à de nombreux modèles d'arbres aléatoires, tels que les arbres uniformes à degrés fixés étudiés dans [BR21, Mar18b, AU20], les arbres de Bienaymé-(Galtson-Watson) en environnement aléatoire étudiés notamment dans [CKKM22].

Nous décrivons ici l'application de nos résultats aux arbres de Bienaymé-(Galtson-Watson) en environnement variable dont la limite d'échelle du profil a notamment été étudiée par [Kur78] et plus récemment avec des conditions très générales par [BS15] et [FLL22]. Les processus de Bienaymé en environnement variable visent à modéliser l'évolution d'une population au cours du temps lorsque le milieu devient plus ou moins favorable à l'expansion de la population, par exemple à cause des variations du climat.

Pour tout $n \geq 1$, indépendamment pour tout $i \in \llbracket 1, n \rrbracket$, soit $(\xi_{i,n}(j))_{j \geq 1}$ une famille de v.a.i.i.d. à valeurs dans les entiers positifs. Le processus de Bienaymé en environnement variable $(Z_{i,n})_{0 \leq i \leq n}$ est défini en posant $Z_{0,n} = 1$ et pour tout $i \geq 1$,

$$Z_{i,n} = \sum_{j=1}^{Z_{i-1,n}} \xi_{i,n}(j).$$

La loi des $\xi_{i,n}(j)$ dépend de i (et de n) et est souvent appelée l'environnement. L'arbre de Bienaymé en environnement variable \mathcal{T}_n de hauteur n associé peut être construit (en loi) conditionnellement aux $(\xi_{i,n}(j))_{j \geq 1, 0 \leq i \leq n}$ comme l'arbre uniforme à degrés et hauteurs fixés dont les degrés à chaque hauteur $i \geq 0$ sont obtenus en réordonnant dans l'ordre décroissant les $\xi_{i+1,n}(j)$ pour $1 \leq j \leq Z_{i,n}$ et en complétant la suite avec des zéros, obtenant ainsi une suite $(d_{i,j}^n)_{j \geq 1}$.

Pour appliquer notre résultat, il suffit donc de vérifier que les hypothèses précédentes sont satisfaites par les degrés. C'est ce que nous faisons dans la section 8.5 du chapitre 8 en nous appuyant sur la limite d'échelle du profil obtenue par Bansaye et Simatos dans [BS15].

1.3.4 Perspectives

De nombreuses questions restent ouvertes pour les arbres récursifs gelés. Certaines sont évoquées dans la section 6.7.

Nombre de bouts. On peut notamment se demander combien la limite locale $\mathcal{T}_\infty(\mathbf{x})$ a de bouts, en fonction du choix de \mathbf{x} . Évidemment, si $\tau(\mathbf{x}) < \infty$, la limite locale n'a aucun bout. Dans le cas où $\tau(\mathbf{x}) = \infty$, en regardant les résultats de limites d'échelle décrits précédemment, on peut conjecturer une transition de phase : lorsque $S_n \gg \sqrt{n}$, il y aurait une infinité de bouts et lorsque $S_n \ll \sqrt{n}$, il y aurait un seul bout. Dans le cas critique $S_n \simeq \sqrt{n}$, par analogie avec l'arbre de Kesten, on peut imaginer qu'il n'y a qu'un seul bout.

Taille des sous-arbres et degrés. On peut aussi étudier l'évolution des tailles des sous-arbres et des degrés dans la construction des arbres récursifs gelés. Par exemple, si l'on considère l'évolution de la taille des sous-arbres le long de la suite $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$, on peut se rendre compte que celle-ci est décrite par une urne de Pólya "avec suppressions". Plus précisément, supposons $x_1 = 1$. Pour tout $n \geq 1$, notons $\mathcal{T}_n^{(1)}(\mathbf{x})$ le sous arbre de $\mathcal{T}_n(\mathbf{x})$ constitué du premier enfant de la racine de $\mathcal{T}_n(\mathbf{x})$ et de ses descendants et notons $\mathcal{T}_n^{(2)}(\mathbf{x})$ le sous-arbre constitué de la racine de $\mathcal{T}_n(\mathbf{x})$ et de ses descendants qui ne sont pas dans $\mathcal{T}_n^{(1)}(\mathbf{x})$. Alors les tailles des sous-arbres $\mathcal{T}_n^{(1)}(\mathbf{x})$ et $\mathcal{T}_n^{(2)}(\mathbf{x})$ évoluent comme le nombre de boules rouges et bleues dans l'urne suivante : on part d'une urne qui contient une boule rouge et une boule bleue. À chaque étape $n \geq 2$, on tire une boule uniformément au hasard dans l'urne. Si $x_n = 1$, alors on remet la boule dans l'urne avec une nouvelle boule de la même couleur. Si $x_n = -1$, alors on retire la boule.

Hauteur minimale. Par ailleurs, compte tenu des résultats obtenus sur le comportement de la hauteur, on peut conjecturer que la hauteur de $\mathcal{T}_n(\mathbf{x}^n)$ est d'ordre au moins $e \log n$. Plus précisément, on conjecture que pour toute famille $(\mathbf{x}^n)_{n \geq 1}$ de suites de ± 1 telle que $\tau(\mathbf{x}^n) \geq n$ pour tout $n \geq 1$, pour tout $\varepsilon > 0$, on a

$$\mathbb{P}(\text{Height}(\mathcal{T}_n(\mathbf{x}^n)) \geq (e - \varepsilon) \log n) \xrightarrow[n \rightarrow \infty]{} 1.$$

Dimension de Hausdorff. La dimension de Hausdorff de $\mathcal{T}(f)$ reste inconnue. On pense que les dimensions de Hausdorff et de Minkowski sont au moins 2. Mais ce qui se passe autour de la racine dépend fortement du comportement de f autour de zéro.

Trouver le patient zéro. Une question très étudiée dans l'inférence sur les graphes aléatoires est comment retrouver la racine d'un arbre comme dans [BDL17]. La motivation peut être la recherche du patient zéro dans une épidémie, du responsable d'une attaque informatique ou de l'origine d'une rumeur se propageant dans un réseau social. Ce type de question est parfois appelé l'archéologie sur les réseaux [NK11] et a été abordé pour de nombreux modèles d'arbres aléatoires : par exemple pour les arbres récursifs uniformes dans [Hai70, BDL17, LP19, CX21, ABBB⁺24], pour les arbres de Bienaymé conditionnés dans [BDG22] ou pour les arbres à attachement préférentiel dans [BDL17, CX21, BH23, CCL⁺24]. Retrouver la racine de $\mathcal{T}_n(\mathbf{x}^n)$ sous certaines conditions sur \mathbf{x}^n est un problème ouvert qui peut être motivé par le fait que l'arbre des infections du modèle SIR est un arbre récursif gelé.

Attachement préférentiel avec gel. Le modèle des arbres récursifs gelés peut être généralisé dans plusieurs directions, par exemple en remplaçant l'attachement uniforme des sommets par un attachement préférentiel où les nouveaux sommets s'attachent à un sommet existant avec une probabilité proportionnelle à son degré.

SIR sur d'autres graphes. Concernant l'arbre des infections pour le modèle SIR, une question naturelle est s'il est possible d'étendre notre approche à un modèle SIR sur d'autres graphes que le graphe complet.

Arbres multi-types. Enfin, les limites d'échelle obtenues pour le modèle des arbres uniformes à degrés et hauteurs fixés pourraient s'appliquer à de nombreux modèles d'arbres aléatoires où l'approche par les fonctions de hauteur ou de contour échoue. Cependant, le modèle des arbres uniformes à degrés et hauteurs fixés pourrait être davantage généralisé pour pouvoir traiter d'autres modèles d'arbres aléatoires, par exemple les modèles d'arbres multi-types.

Première partie

Cartes aléatoires à grands degrés

Chapitre 2

On large $3/2$ -stable maps

Ce chapitre est issu de l'article [Kam23b] à paraître dans les Annales de l'Institut Henri Poincaré Probabilités et Statistiques.

Abstract We discuss asymptotics of large Boltzmann random planar maps such that every vertex of degree k has weight of order k^{-2} . Infinite maps of that kind were studied by Budd, Curien and Marzouk. These maps can be seen as the dual of the discrete α -stable maps studied by Le Gall and Miermont for $\alpha = 3/2$ or as the gaskets of critical $O(2)$ -decorated random planar maps. We compute the asymptotics of the graph distance and of the first passage percolation distance between two uniform vertices, which are respectively equivalent in probability to $(\log \ell)^2/\pi^2$ and $2(\log \ell)/(\pi^2 p_{\mathbf{q}})$ when the perimeter of the map ℓ goes to ∞ , where $p_{\mathbf{q}}$ is a constant which depends on the model. We also show that the diameter is of the same order as those distances for both metrics and obtain in particular that these maps do not satisfy scaling limits in the sense of Gromov-Prokhorov or Gromov-Hausdorff for lack of tightness. To study the peeling exploration of these maps, we prove local limit and scaling limit theorems for a class of random walks with heavy tails conditioned to remain positive until they die at $-\ell$ towards processes that we call stable Lévy processes conditioned to stay positive until they jump and die at -1 .

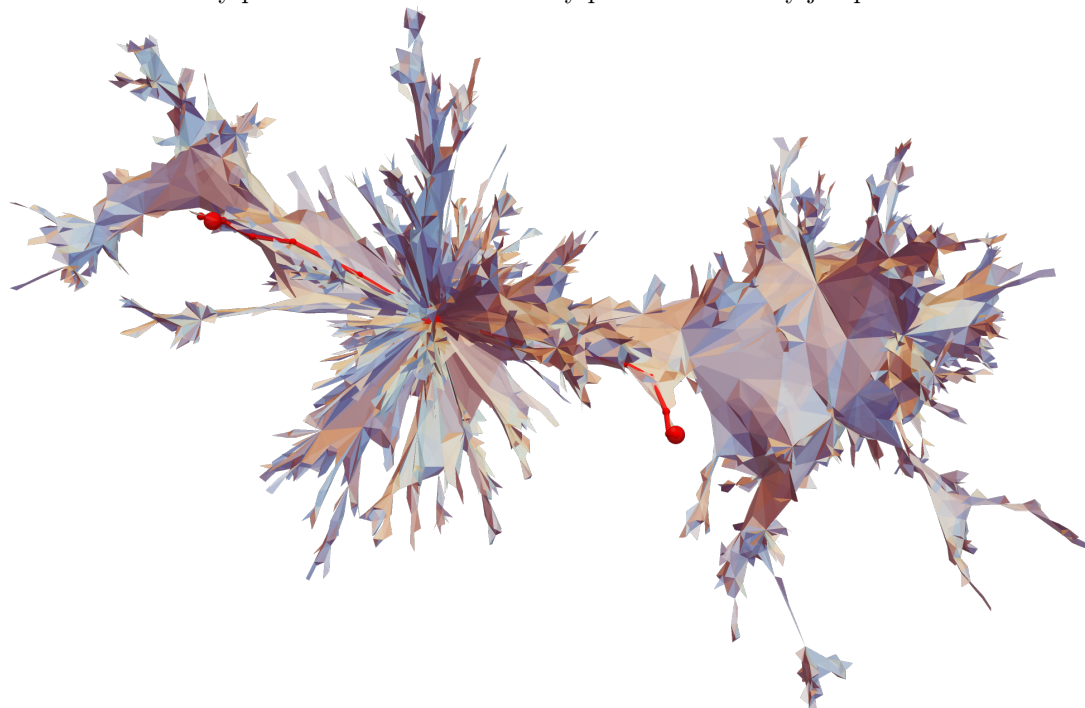


Figure 2.1 – Simulation of a $3/2$ -stable map $\mathfrak{M}^{(500, \dagger)}$ of perimeter 1000. The two large red spheres are two uniform random vertices of $\mathfrak{M}^{(500, \dagger)}$ and the red path is the shortest path between them.

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2.1 Introduction

2.1.1 Boltzmann 3/2-stable planar maps

Random planar maps have been studied very actively, motivated partly by their connections with two dimensional Liouville quantum gravity. In this chapter, we study the geometry of large random 3/2-stable Boltzmann planar maps. A planar map \mathfrak{m} is a connected planar graph embedded in the sphere seen up to orientation-preserving homeomorphism and equipped with a distinguished oriented edge \vec{e}_r called the root edge. The face f_r adjacent to the right of the root edge is called the root face. The degree of f_r is also called the perimeter of \mathfrak{m} . We also require the map \mathfrak{m} to be bipartite, i.e. that all its faces have even degree, inasmuch as the enumeration of bipartite maps is particularly simple. We write \mathcal{M} for the set of finite bipartite rooted planar maps.

Random planar maps can be picked uniformly at random among maps of a given size. A more general way to pick a random map $\mathfrak{m} \in \mathcal{M}$ is to assign Boltzmann weights to them, following [MM07]. Let $\mathbf{q} = (q_k)_{k \geq 1}$ be a non-zero sequence of non-negative numbers called the sequence of weights. The weight of a map $\mathfrak{m} \in \mathcal{M}$ is defined as

$$w_{\mathbf{q}}(\mathfrak{m}) = \prod_{f \in \text{Faces}(\mathfrak{m}) \setminus \{f_r\}} q_{\deg(f)/2}.$$

For all $\ell \geq 1$, let $\mathcal{M}^{(\ell)} \subset \mathcal{M}$ be the set of all (finite) maps with perimeter 2ℓ . We introduce the partition function

$$W^{(\ell)} = \sum_{\mathfrak{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathfrak{m}).$$

When $W^{(1)} < \infty$, we say that \mathbf{q} is admissible. In what follows, we also require the weight sequence to be critical of type $a \in (3/2, 5/2]$, which is equivalent by Proposition 5.10 of [Cur23] to having

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-a}, \quad (2.1.1)$$

for some constant $p_{\mathbf{q}} > 0$. The cases $a \in (3/2, 5/2)$ are called non-generic. Such sequences \mathbf{q} do exist, see Lemma 6.1 of [BC17] for explicit examples. If \mathbf{q} is admissible, we write $\mathbb{P}^{(\ell)}$ the probability measure on $\mathcal{M}^{(\ell)}$ associated with the weight sequence \mathbf{q} , called the Boltzmann measure, defined by

$$\mathbb{P}^{(\ell)}(\mathbf{m}) = \frac{w_{\mathbf{q}}(\mathbf{m})}{W^{(\ell)}}.$$

We also write $\mathbb{E}^{(\ell)}$ for the associated mathematical expectation. A random map $\mathfrak{M}^{(\ell)}$ following the law $\mathbb{P}^{(\ell)}$ is called a \mathbf{q} -Boltzmann random map of perimeter 2ℓ . The \mathbf{q} -Boltzmann random maps for \mathbf{q} of type a are also called $(a - 1/2)$ -stable maps. The object of study in this work is not the map $\mathfrak{M}^{(\ell)}$ in itself but rather its dual map $\mathfrak{M}^{(\ell, \dagger)}$ obtained by exchanging the roles of the faces and of the vertices.

For $a = 2$, the local limit, the infinite $3/2$ -stable map, was studied in [BCM18] by Budd, Curien and Marzouk. In particular, they established the scaling limit of the perimeter associated with the exploration of the infinite map, and studied its basic metric properties and their results will be especially helpful in Section 2.4.

The peeling exploration of the map makes appear a random walk S on \mathbb{Z} with step distribution ν which will play a crucial part in this work. It is related to \mathbf{q} by

$$\forall k \geq 0, \nu(k) = q_{k+1} c_{\mathbf{q}}^k \quad \text{and} \quad \nu(-k-1) = 2W^{(k)} c_{\mathbf{q}}^{-k-1}. \quad (2.1.2)$$

When \mathbf{q} is critical of type $a = 2$, we know in particular by Proposition 5.10 of [Cur23] that

$$\nu(-k) \sim p_{\mathbf{q}} k^{-2} \quad \text{and} \quad \nu([k, \infty)) \sim p_{\mathbf{q}} k^{-1} \quad \text{as } k \rightarrow \infty, \quad (2.1.3)$$

so that ν is in the domain of attraction of the symmetric Cauchy process (with no drift).

2.1.2 Main results

Throughout this work, unless explicitly stated otherwise, \mathbf{q} will be a critical non-generic weight sequence of type $a = 2$ and, for all $\ell \geq 1$, $\mathfrak{M}^{(\ell)}$ will be a \mathbf{q} -Boltzmann random map of perimeter 2ℓ .

Before stating the main results, let us introduce the distance functions on a map \mathbf{m} . For every faces f_1, f_2 of \mathbf{m} , we denote by $d_{\text{gr}}^{\dagger}(f_1, f_2)$ the graph distance between f_1 and f_2 in the dual map \mathbf{m}^{\dagger} . Another distance, easier to study in our context, is the first-passage percolation distance d_{fpp}^{\dagger} defined as follows: to each edge of \mathbf{m}^{\dagger} we assign an independent random variable following the exponential law of parameter 1 (that we will write $\mathcal{E}(1)$), interpreted as its random length. For every faces f_1, f_2 of \mathbf{m} , we then denote by $d_{\text{fpp}}^{\dagger}(f_1, f_2)$ the distance between f_1 and f_2 in the dual map \mathbf{m}^{\dagger} whose edges are equipped with those random lengths.

The first theorem describes the distance from the root face to a uniform random face (we also prove that the same results hold for a uniform random edge or a uniform random vertex).

Theorem 2.1.1. *For all $\ell \geq 1$, let $F^{(\ell)}$ be a random uniform face in $\mathfrak{M}^{(\ell)}$ (conditionally on $\mathfrak{M}^{(\ell)}$) and recall that f_r is the root face. Then*

$$\frac{d_{\text{fpp}}^{\dagger}(f_r, F^{(\ell)})}{\log \ell} \underset{\ell \rightarrow \infty}{\xrightarrow{(\mathbb{P})}} \frac{1}{\pi^2 p_{\mathbf{q}}} \quad \text{and} \quad \frac{d_{\text{gr}}^{\dagger}(f_r, F^{(\ell)})}{(\log \ell)^2} \underset{\ell \rightarrow \infty}{\xrightarrow{(\mathbb{P})}} \frac{1}{2\pi^2}.$$

The scales for the distances in $\log \ell$ and $(\log \ell)^2$ are specific to the $3/2$ -stable case on account of its criticality and of the singular behavior of Cauchy processes. Deeply related behaviors have already been observed for the volume growth of balls in the infinite map in Theorems A and B of [BCM18].

We then prove that the geodesics from the root to two uniform random faces “split very quickly”, so that we get the asymptotic of the distance between two uniform random faces, see Corollary 2.5.1. This

corollary implies that if $(F_i^{(\ell)})_{i \geq 1}$ are i.i.d. uniform random faces of $\mathfrak{M}^{(\ell)}$ for all $\ell \geq 1$, then, for the product topology,

$$\left(\frac{d_{\text{fpp}}^\dagger(F_i, F_j)}{\log \ell} \right)_{i \neq j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \left(\frac{2}{\pi^2 p_{\mathbf{q}}} \right)_{i \neq j \geq 1} \quad \text{and} \quad \left(\frac{d_{\text{gr}}^\dagger(F_i, F_j)}{(\log \ell)^2} \right)_{i \neq j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \left(\frac{1}{\pi^2} \right)_{i \neq j \geq 1}. \quad (2.1.4)$$

So as to study the diameter in Section 2.6, we control the distances uniformly on all the faces. Aside from the first and second moment methods, a convenient tool is a martingale associated with the perimeter process, hence enabling us to show that the diameter is of the same order as the distance between two faces picked uniformly at random.

Theorem 2.1.2. *Let $\text{Diam}_{\text{gr}}^\dagger(\mathfrak{M}^{(\ell)})$ (resp. $\text{Diam}_{\text{fpp}}^\dagger(\mathfrak{M}^{(\ell)})$) be the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the graph distance (resp. fpp distance). Let $\delta > 0$. There exists a constant $K(\mathbf{q})$ depending on \mathbf{q} such that, with probability $1 - o(1)$ when $\ell \rightarrow \infty$,*

$$\left(\frac{3}{4(1 - q_1^2)} \vee \frac{2}{\pi^2 p_{\mathbf{q}}} \right) - \delta \leq \frac{\text{Diam}_{\text{fpp}}^\dagger(\mathfrak{M}^{(\ell)})}{\log \ell} \leq K(\mathbf{q}) \quad \text{and} \quad \frac{1}{\pi^2} - \delta \leq \frac{\text{Diam}_{\text{gr}}^\dagger(\mathfrak{M}^{(\ell)})}{(\log \ell)^2} \leq 18 + \delta.$$

These bounds on the diameter indicate in particular the scale at which one should look for studying Gromov-Hausdorff convergence. Actually, (2.1.4) shows that at the macroscopic scale, $\mathfrak{M}^{(\ell, \dagger)}$ looks like a star with many branches when $\ell \rightarrow \infty$, see the simulation in Figure 2.1. As a consequence, the sequence $(\mathfrak{M}^{(\ell, \dagger)})_{\ell \geq 1}$ equipped with the rescaled distances is not tight, neither for the Gromov-Hausdorff topology, nor for the Gromov-Prokhorov topology. Indeed, on the one hand, the limit of any subsequence can not be compact due to (2.1.4), while on the other hand any limiting random separable metric measured space (X, d, μ) of some subsequence for Gromov-Prokhorov would have an empty support since two uniform points of law μ would be at constant distance by (2.1.4), which contradicts the separability (see e.g. Remark 4.5 of [Jan20] for details on this reasoning).

More precisely, for all $\ell \geq 1$, let μ_ℓ be the uniform measure on the faces of $\mathfrak{M}^{(\ell)}$. We have thus shown the following corollary.

Corollary 2.1.3. *The sequences $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{(\log \ell)^2} d_{\text{gr}}^\dagger, \mu_\ell)_{\ell \geq 1}$ and $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{\log \ell} d_{\text{fpp}}^\dagger, \mu_\ell)_{\ell \geq 1}$ are not tight for the Gromov-Prokhorov topology. Besides, the sequences $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{(\log \ell)^2} d_{\text{gr}}^\dagger)_{\ell \geq 1}$ and $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{\log \ell} d_{\text{fpp}}^\dagger)_{\ell \geq 1}$ are not tight for the Gromov-Hausdorff topology. In particular, there is no Gromov-Hausdorff scaling limit of any subsequence of $(\mathfrak{M}^{(\ell, \dagger)}, d_{\text{gr}}^\dagger)_{\ell \geq 1}$ or $(\mathfrak{M}^{(\ell, \dagger)}, d_{\text{fpp}}^\dagger)_{\ell \geq 1}$*

Still, following [ET22] and [Jan21], the convergence (2.1.4) can also be interpreted as the convergence in probability of $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{(\log \ell)^2} d_{\text{gr}}^\dagger, \mu_\ell)_{\ell \geq 1}$ and $(\mathfrak{M}^{(\ell, \dagger)}, \frac{1}{\log \ell} d_{\text{fpp}}^\dagger, \mu_\ell)_{\ell \geq 1}$ towards some long dendrons defined in Example 7.2 of [Jan21].

2.1.3 Conditioned random walks

The main technique to study the metric properties of $\mathfrak{M}^{(\ell, \dagger)}$ is the filled-in peeling exploration. It was first introduced in [Wat95] and [Ang03] for triangulations. A general version is defined in [Bud16] and is presented in Section 2.3. See [Bud17] and [Cur23] for a more detailed introduction. The peeling process consists in a step-by-step Markovian exploration of the map starting from the root face according some peeling algorithm. At each step, the peeling process reveals the face behind the peeled edge chosen by the peeling algorithm. Various choices of peeling algorithms enable to understand different properties of the map. Our use of the peeling algorithm is similar to that of [BC17] for the study of infinite $(a - 1/2)$ -stable Boltzmann maps for \mathbf{q} critical non-generic of type $a \in (3/2, 5/2) \setminus \{2\}$ and [BCM18] for the study of infinite $3/2$ -stable Boltzmann maps. A difference is that most of the time we will work with the peeling algorithm on the map which has been rerooted (see Subsection 2.3.4).

In our case, the evolution of the perimeter of the explored part has the same law as the ν -random walk conditioned to stay positive until it jumps and dies at $-\ell$. This is, to the best of our knowledge, a new type of conditioning which we study in details in Section 2.2. In that section we introduce a key coupling (Lemma 2.2.5) between this conditioned random walk and the ν -random walk conditioned to stay positive forever, using that such conditionings can be performed via Doob h -transforms. This coupling enables us in particular to use some results of [BCM18]. This lemma states that for all $n \geq 1$,

there exists a coupling between the two conditioned random walks such that if the second one has not been killed yet at time n , then the two conditioned random walks coincide.

The rest of Section 2.2 can be seen as an asymptotic study of this conditioned ν -random walk. We first prove a local limit theorem for the lifetime and the last positive value of the walk. Relying on this local limit, we next establish in Theorem 2.2.8 the scaling limit of that conditioned random walk dying at $-\ell$ towards a process that we call an $(a-1)$ -stable Lévy process conditioned to die at -1 . We hope that those results could be fruitful in other contexts.

2.1.4 Discussion

The gasket of a critical $O(n)$ loop-decorated planar map, which is obtained after deleting the edges and vertices in the outermost loops, is a particular example of an $(a-1/2)$ -stable map, where the case $n=2$ corresponds to $a=2$. This relation is depicted for instance in [LGM11, BBG12b]. The $O(n)$ loop-decorated maps undergo a phase transition described in [BBG12b, Bud18]. The critical case $n=2$ is often excluded in the literature. For the infinite Boltzmann $(a-1/2)$ -stable maps, in [BC17] a phase transition at $a=2$ is equally exhibited. Let us also mention the conjecture that for $n=2$ the gasket should converge in some sense to the CLE_4 ensemble (introduced in [She09, SW12]).

We are confident that some of our results such as Theorem 2.1.1 and Corollary 2.5.1 can be extended to the case $a \in (3/2, 2)$ with a $\log \ell$ scaling for the dual graph distance. The martingale (2.6.1) seems also adapted in this setting, so that Theorem 2.1.2 should have an analog in this case.

2.2 The random walk conditioned to die at $-\ell$

This section is devoted to the study the ν -random walk S conditioned to stay positive until it jumps and dies at $-\ell$, which will be denoted by $(P_\ell(n))_{n \geq 0}$ under $\mathbb{P}_\ell^{(1)}$. We establish in Lemma 2.2.5 a coupling between this conditioned random walk and the random walk conditioned to stay always positive. We then prove a local limit result for the hitting time $\tau_{-\ell}$ of $-\ell$ (Proposition 2.2.7) which we use to establish a scaling limit (Theorem 2.2.8).

2.2.1 The ν random walk, harmonic functions and conditionings

The criticality condition (2.1.1) imposes strong properties on the step distribution ν which are better stated in terms of harmonic functions. For all $\ell \geq 0$ and $p \geq 1$, let

$$h_p^\downarrow(\ell) = h^\downarrow(\ell)h^\downarrow(p)\frac{\ell}{\ell+p} \quad \text{and} \quad h_0^\downarrow(\ell) = h^\downarrow(\ell) = 2^{-2\ell} \binom{2\ell}{\ell}, \quad (2.2.1)$$

with by convention $h^\downarrow(\ell) = h_p^\downarrow(\ell) = 0$ for $\ell < 0$ except for $\ell = -p$ where we set $h_p(-p) = 1$. We also set $h^\uparrow(\ell) = 2\ell h^\downarrow(\ell)$ for all $\ell \in \mathbb{Z}$. Let $a \in (3/2, 5/2]$. In the rest of this section we only assume that ν is a probability measure on \mathbb{Z} such that

- (I) The functions h^\downarrow and h^\uparrow are harmonic for ν .
- (II) We have $\nu(-k) \sim p_{\mathbf{q}}/k^a$ as $k \rightarrow \infty$.

These two conditions are equivalent to the criticality of type a of the weight sequence \mathbf{q} associated with ν by (2.1.2) (see the paragraph 3.2 in [Bud16], Theorem 1 and Proposition 5 of [Bud17] or Theorem 5.4 and its proof in [Cur23] for (I), and Propositions 5.9 and 5.10 of [Cur23] for (II)). The admissibility of \mathbf{q} then entails the harmonicity of h_p^\downarrow for S for all $p \geq 1$ (see the proof of Lemma 5.2 in [Cur23]).

It is also known (see e.g. Proposition 10.1 in [Cur23]) that $(S_{\lfloor nt \rfloor} / n^{1/(a-1)})_{t \geq 0}$ converges in distribution towards $p_{\mathbf{q}}^{1/(a-1)} \Upsilon_a$ for the Skorokhod topology, where Υ_a is the $(a-1)$ -stable Lévy process starting at zero with positivity parameter $\rho = \mathbb{P}(S_t \geq 0)$ satisfying $(a-1)(1-\rho) = 1/2$, normalized so that its Lévy measure is $\cos(a\pi)(dx/x^a)\mathbf{1}_{x>0} + (dx/|x|^a)\mathbf{1}_{x<0}$.

Finally, from the harmonic functions, one can define the corresponding Doob h -transforms of the ν random walk S . For all $\ell \geq 1$, we denote by $(P_\infty(n))_{n \geq 0}$ under $\mathbb{P}_\infty^{(\ell)}$ the Doob h^\uparrow -transform of S starting at ℓ and for all $\ell \geq 1, p \geq 0$, we write $(P_p(n))_{n \geq 0}$ under $\mathbb{P}_p^{(\ell)}$ the Doob h_p^\downarrow -transform of S killed when it reaches $-p$. The notation is chosen to be consistent with the rest of the paper since these conditioned

random walks will be the perimeter processes in Section 2.3. The Doob h^\uparrow -transform of S can be seen as the random walk S conditioned to stay always positive while the Doob h_p^\downarrow -transform of S killed at $-p$ can be understood analogously as the random walk conditioned to stay positive until it is killed at $-p$ (see [Cur23] Proposition 5.3).

2.2.2 Preliminaries on the random walk conditioned to stay positive

Before delving into the study of the walk P_ℓ under $\mathbb{P}_\ell^{(1)}$, we record for future use some known results on the walks P_∞, P_p under $\mathbb{P}_\infty^{(1)}, \mathbb{P}_p^{(\ell)}$ for fixed p as $\ell \rightarrow +\infty$. One of the most useful results will be the scaling limit of the random walk conditioned to stay always positive P_∞ under $\mathbb{P}_\infty^{(1)}$. This scaling limit was introduced in [BC17] and [BCM18] using Theorem 1.1 from [CC08]. The limit is described informally as the Lévy process Υ_a , introduced in the above subsection, conditioned to stay positive and starting from zero. To do this, one can first define the process $(\mathcal{S}_x^\uparrow(t))_{t \geq 0}$ starting at some $x > 0$ as the Doob h -transform of $(x + \Upsilon_a(t))_{t \geq 0}$ starting at x , using the harmonic function $h : y \mapsto \mathbf{1}_{y \geq 0} \sqrt{y}$. The process $(\mathcal{S}_x^\uparrow(t))_{t \geq 0}$ then converges in distribution when $x \rightarrow 0$ towards a process written Υ_a^\uparrow which starts at zero. Recall the constant $p_{\mathbf{q}}$ which first appears in (2.1.1).

Theorem 2.2.1. (Proposition 10.3 of [Cur23]) *The following convergence holds for the J_1 -Skorokhod topology: under $\mathbb{P}_\infty^{(1)}$,*

$$\left(\frac{P_\infty(\lfloor nt \rfloor)}{n^{1/(a-1)}} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} (\Upsilon_a^\uparrow(p_{\mathbf{q}}t))_{t \geq 0} \stackrel{(d)}{=} \left(p_{\mathbf{q}}^{1/(a-1)} \Upsilon_a^\uparrow(t) \right)_{t \geq 0}.$$

The proof of Theorem 1.1 in [CC08] relies mostly on the scaling limit of the ν -random walk S conditioned to stay positive until some time towards a stable meander: let $\tilde{\Upsilon}_a$ be the meander of length 1 associated with Υ_a , which is informally the process Υ_a conditioned to remain positive up to time 1. Then the following theorem is a consequence of the main theorem of [Don85].

Theorem 2.2.2. ([Don85]) *For $S_0 = 1$, conditionally on $S_1, \dots, S_n > 0$, the process $(S(\lfloor nt \rfloor)/n^{1/(a-1)})_{t \geq 0}$ converges in distribution for the J_1 -Skorokhod topology towards $(p_{\mathbf{q}}^{1/(a-1)} \tilde{\Upsilon}_a(t))_{t \geq 0}$.*

The density of $\Upsilon_a^\uparrow(1)$ (resp. $\tilde{\Upsilon}_a(1)$) will be written f_a^\uparrow (resp. \tilde{f}_a). We know from [DS10] that \tilde{f}_a is continuous and bounded (and also bounded away from zero on any compact of $(0, \infty)$) and moreover that there exists $C_1 > 0$ such that $f_a^\uparrow(x) = C_1 x^{1/2} \tilde{f}_a(x)$.

Besides, so as to establish our local limit results, we state the following lemma which is an extension and a reformulation of Lemma 3 (i) of [BCM18]. Its proof is exactly the same as in [BCM18] and builds upon [VW09].

Lemma 2.2.3. (Lemma 3 (i) from [BCM18]) *Uniformly on $x \geq 1$ as $n \rightarrow \infty$,*

$$\mathbb{P}_1(S_1, \dots, S_n \geq 1 \text{ and } S_n = x) = \frac{\sqrt{\pi}}{2(np_{\mathbf{q}})^{1/(a-1)} \sqrt{x}} f_a^\uparrow \left(\frac{x}{(p_{\mathbf{q}}n)^{1/(a-1)}} \right) + o(n^{-3/(2(a-1))}).$$

Finally, the next proposition, which describes the scaling limit of P_p under $\mathbb{P}_p^{(\ell)}$ for fixed p , will be of some use in Section 2.5. The scaling limit is described by the Lévy process Υ_a^\downarrow conditioned to stay positive until it dies continuously at zero. The process Υ_a^\downarrow can be defined as a Doob transform of $\Upsilon_a + 1$ using the harmonic function $x \mapsto (1/\sqrt{x}) \mathbf{1}_{x > 0}$. This proposition slightly extends a particular case of Proposition 6.6 from [BBCK18]. Not only does its proof rely on the convergence in distribution of $(S_{\lfloor nt \rfloor}/n^{1/(a-1)})_{t \geq 0}$ towards $p_{\mathbf{q}}^{1/(a-1)} \Upsilon_a$ but it also relies on Theorem 1.3 of [CC08] in the case $p = 0$. In that case P_0 is a Doob h^\downarrow transform of the ν -random walk, i.e. a random walk conditioned to die at zero. The proof of Theorem 1.3 in [CC08] can be rewritten by replacing h^\downarrow (which is denoted by W in [CC08]) by h_p^\downarrow , i.e. by replacing the walk conditioned to die at zero by a walk conditioned to stay positive until it dies at $-p$.

Proposition 2.2.4. (Extension of Proposition 6.6 from [BBCK18]) *For all $p \geq 0$, the following convergence holds for the J_1 -Skorokhod topology: under $\mathbb{P}_p^{(\ell)}$,*

$$\left(\frac{P_p(\lfloor \ell^{a-1} t \rfloor)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (\Upsilon_a^\downarrow(p_{\mathbf{q}}t))_{t \geq 0}.$$

Moreover, jointly with the above convergence, we have the convergence of the lifetimes:

$$\frac{\tau_{-p}}{\ell^{a-1}} \xrightarrow[\ell \rightarrow \infty]{(d)} \frac{\zeta}{p_{\mathbf{q}}},$$

where $\tau_{-p} = \inf\{n \geq 0; P_p(n) = -p\}$ and $\zeta = \inf\{t \geq 0; \Upsilon_a^\downarrow(t) = 0\}$.

One may alternatively prove the first convergence of the above proposition by applying Theorem 2 of [BK16] while the second assertion can be seen as a consequence of Theorem 3 (i) of [BK16].

2.2.3 Coupling random walks conditioned to stay positive with random walks conditioned to die at $-\ell$

Let us denote by $\tau_{-\ell}$ the first hitting time of $-\ell$ by $(P_\ell(k))_{k \geq 0}$ the random walk conditioned to remain positive until it dies at $-\ell$ under $\mathbb{P}_\ell^{(1)}$. Let us first state a coupling lemma which will be useful in this section and in the next ones. In particular, it will enable us to take advantage of the results from [BCM18] in Section 2.4.

Lemma 2.2.5. *There exists a coupling of the random walks P_ℓ under $\mathbb{P}_\ell^{(1)}$ and P_∞ under $\mathbb{P}_\infty^{(1)}$ up to time n written $\mathbb{P}_{\ell, \infty}^{(1), n}$ such that: under $\mathbb{P}_{\ell, \infty}^{(1), n}$, for all $s_1, \dots, s_n \in \mathbb{N}^*$, conditionally on $(P_\infty(k))_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}$, the equality $(P_\ell(k))_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}$ holds with probability $\frac{1+\ell}{s_n+\ell}$ and $\tau_{-\ell} \leq n$ with probability $1 - \frac{1+\ell}{s_n+\ell}$ (see Figure 2.2).*

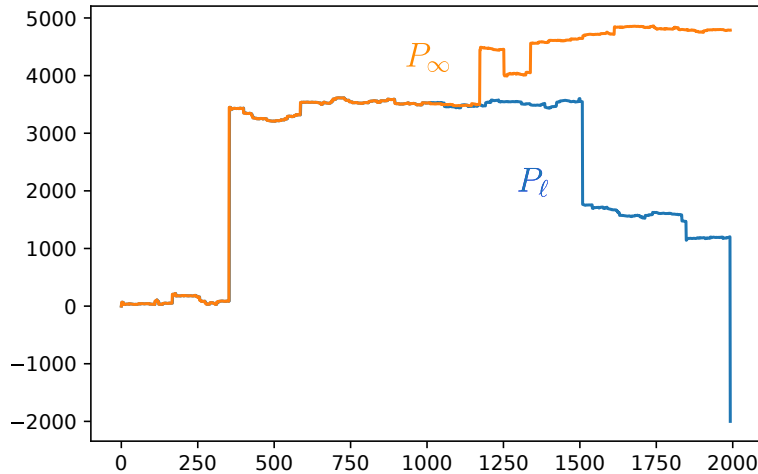


Figure 2.2 – Simulation of the conditioned random walks P_∞ and P_ℓ under the coupling $\mathbb{P}_{\ell, \infty}^{(1), n}$ in the case $\tau_{-\ell} > n$ for $n = 1000$ and $\ell = 2000$ (after time n , we let both processes evolve independently).

Proof. The key ingredient of the proof is the relation $h_\ell^\downarrow(k) = h^\downarrow(\ell)h^\uparrow(k)/(2(\ell + k))$ for $k, \ell \geq 1$ coming from the exact expression of the harmonic functions. From the definition of P_ℓ under $\mathbb{P}_\ell^{(1)}$ as a h_ℓ^\downarrow -Doob

transform of a ν -random walk S , we note that if $s_1, \dots, s_n \in \mathbb{N}^*$, then

$$\begin{aligned} \mathbb{P}_\ell^{(1)}((P_\ell(k))_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}) &= \frac{h_\ell^\downarrow(s_n)}{h_\ell^\downarrow(1)} \mathbb{P}_1((S_k)_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}) \\ &= \frac{h^\uparrow(s_n)}{h^\uparrow(1)} \frac{1 + \ell}{s_n + \ell} \mathbb{P}_1((S_k)_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}) \\ &= \frac{1 + \ell}{s_n + \ell} \mathbb{P}_\infty^{(1)}((P_\infty(k))_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}) \\ &\leq \mathbb{P}_\infty^{(1)}((P_\infty(k))_{1 \leq k \leq n} = (s_k)_{1 \leq k \leq n}). \end{aligned}$$

One can thus define such a coupling. \square

One can see from the proof that the above lemma only relies on condition (I). A simple consequence of this coupling is a limit theorem for $\tau_{-\ell}/\ell^{(a-1)}$ when \mathbf{q} has type $a \in (3/2, 5/2]$. Recall that Υ_a^\uparrow is the stable Lévy process Υ_a starting at zero conditioned to stay positive.

Proposition 2.2.6. *We have the convergence in law $\tau_{-\ell}/\ell^{a-1} \xrightarrow[\ell \rightarrow \infty]{d} \tilde{\tau}/p_{\mathbf{q}}$ where $\tilde{\tau}$ is a random variable such that for all $u \geq 0$, $\mathbb{P}(\tilde{\tau} \geq u) = \mathbb{E}(1/(1 + \Upsilon_a^\uparrow(u)))$.*

Proof. By Lemma 2.2.5 and Theorem 2.2.1, for every $u \geq 0$,

$$\mathbb{P}_\ell^{(1)}(\tau_{-\ell} > u\ell^{a-1}) = \mathbb{E}_\infty^{(1)} \frac{1 + \ell}{P_\infty(\lfloor u\ell^{a-1} \rfloor) + \ell} \xrightarrow{\ell \rightarrow \infty} \mathbb{E} \frac{1}{1 + \Upsilon_a^\uparrow(p_{\mathbf{q}}u)},$$

and the claim follows. \square

2.2.4 Local limit results for the random walk conditioned to die at $-\ell$

The result of Proposition 2.2.6 can be refined to obtain a joint local limit theorem for the lifetime $\tau_{-\ell}$ together with the last positive value of the conditioned random walk $P_\ell(\tau_{-\ell} - 1)$:

Proposition 2.2.7. *For all $\varepsilon > 0$ the convergence below holds uniformly in $u, s \in [\varepsilon, \varepsilon^{-1}]$:*

$$\ell^a \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell^{a-1} s \rfloor \text{ and } P_\ell(\tau_{-\ell} - 1) = \lfloor \ell u \rfloor) \xrightarrow{\ell \rightarrow \infty} \frac{\pi p_{\mathbf{q}}^{1-1/(a-1)}}{s^{1/(a-1)}} \frac{1}{\sqrt{u}(1+u)^a} f_a^\uparrow \left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right),$$

and the right-hand side is a probability density.

Proof. Let $\varepsilon > 0$. Let us write $n = \lfloor \ell^{a-1} s \rfloor$ and $x = \lfloor \ell u \rfloor$. Then,

$$\frac{1}{2} \frac{h^\downarrow(\ell)}{\ell + 1} \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell^{a-1} s \rfloor \text{ and } P_\ell(\tau_{-\ell} - 1) = \lfloor \ell u \rfloor) = \mathbb{P}_1(S_1, \dots, S_{n-1} \geq 1 \text{ and } S_{n-1} = x) \nu(-\ell - x).$$

But by Lemma 2.2.3 and since $\nu(-k) \sim p_{\mathbf{q}} k^{-a}$ as $k \rightarrow \infty$ by (II), uniformly in $u, s \in [\varepsilon, \varepsilon^{-1}]$,

$$\mathbb{P}_1(S_1, \dots, S_{n-1} \geq 1 \text{ and } S_{n-1} = x) \nu(-\ell - x) \underset{\ell \rightarrow \infty}{\sim} \frac{\sqrt{\pi} p_{\mathbf{q}}^{1-1/(a-1)}}{2s^{1/(a-1)} \ell^{a+3/2}} f_a^\uparrow \left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) \frac{1}{\sqrt{u}(1+u)^a}.$$

This proves the desired convergence using that $h^\downarrow(\ell) \sim 1/\sqrt{\pi\ell}$ as $\ell \rightarrow \infty$. Let us check that the right term indeed corresponds to a probability density, in other words that

$$\int_0^\infty \frac{\pi p_{\mathbf{q}}^{1-1/(a-1)}}{s^{1/(a-1)}} \int_0^\infty f_a^\uparrow \left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) \frac{1}{\sqrt{u}(1+u)^a} du ds = 1$$

By the change of variable $w = \frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}}$ at fixed u ,

$$\begin{aligned} \int_0^\infty \frac{\pi p_{\mathbf{q}}^{1-1/(a-1)}}{s^{1/(a-1)}} \int_0^\infty f_a^\uparrow \left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) \frac{1}{\sqrt{u}(1+u)^a} du ds &= \int_0^\infty \int_0^\infty \frac{\pi u^{a-2}(a-1)}{\sqrt{u}(1+u)^a} w^{1-a} f_a^\uparrow(v) dw du \\ &= \int_0^\infty \frac{\pi u^{a-2}(a-1)}{\sqrt{u}(1+u)^a} du \mathbb{E} \frac{1}{\Upsilon_a^\uparrow(1)^{a-1}}. \end{aligned}$$

Yet, by the change of variable $t = 1/(1+u)$, by using the link between the Beta function and the Gamma function and using that $\Gamma(3/2) = \sqrt{\pi}/2$, one can see that

$$\int_0^\infty \frac{\pi u^{a-2}(a-1)}{\sqrt{u}(1+u)^a} du = \frac{(a-1)\pi^{3/2}\Gamma(a-3/2)}{2\Gamma(a)}.$$

Finally, by exactly the same reasoning as in the end of Remark 3 in [BCM18], one can compute

$$\mathbb{E} \frac{1}{\Upsilon_a^\uparrow(1)^{a-1}} = \frac{2\Gamma(a)}{(a-1)\pi^{3/2}\Gamma(a-3/2)},$$

which is precisely the inverse of the above integral. Indeed, since Υ_a^\uparrow is a positive self-similar Markov process, let us denote by ξ^\uparrow the Lévy process arising in the Lamperti representation of Υ_a^\uparrow . Then by [BY02], we know that $\mathbb{E}1/\Upsilon_a^\uparrow(1)^{a-1} = 1/((a-1)\mathbb{E}\xi_1^\uparrow)$. But the Laplace exponent of $(\Gamma(a)/\pi)\xi^\uparrow$ is given by $\kappa_{a-1}(\omega_+ + q)$ where $\omega_+ = a + 1/2$ and κ_{a-1} is defined by Equation (19) of [BBCK18]. One could have made the computations using the characteristic exponent of ξ^\uparrow identified in Proposition 2 of [KPR10]. \square

From Proposition 2.2.7 one can get the following local estimates for $\tau_{-\ell}$ and $P_\ell(\tau_{-\ell} - 1)$. For $\tau_{-\ell}$ it suffices to sum on the values of $P_\ell(\tau_{-\ell} - 1)$ in an interval $[\ell\delta, \ell/\delta]$ for an arbitrary small $\delta > 0$. Thus we get that for all $s > 0$, when $\ell \rightarrow \infty$,

$$\mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell^{a-1}s \rfloor) \geq (1 + o(1)) \frac{\pi p_{\mathbf{q}}^{1-1/(a-1)}}{\ell^{a-1}s^{1/(a-1)}} \int_0^\infty \frac{1}{\sqrt{u}(1+u)^a} f_a^\uparrow\left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}}\right) du.$$

In the same vein, one can check the local estimate for $P_\ell(\tau_{-\ell} - 1)$: for all $u > 0$, when $\ell \rightarrow \infty$,

$$\mathbb{P}_\ell^{(1)}(P_\ell(\tau_{-\ell} - 1) = \lfloor \ell u \rfloor) \geq \frac{2\Gamma(a)}{\sqrt{\pi}\Gamma(a-3/2)\ell} \frac{u^{a-2}}{\sqrt{u}(1+u)^a} (1 + o(1)).$$

2.2.5 Scaling limit of the random walk conditioned to die at $-\ell$

The convergence in distribution of $(\tau_{-\ell}/\ell^{a-1}, P_\ell(\tau_{-\ell} - 1)/\ell)$ implied by Proposition 2.2.7 leads us to the following scaling limit: the rescaled random walk conditioned to die at $-\ell$, which is written $(P_\ell(\ell^{a-1}t)/\ell)_{t \geq 0}$ under $\mathbb{P}_\ell^{(1)}$ converges in distribution towards $(\Upsilon_{-1,a}^\downarrow(p_{\mathbf{q}}t))_{t \geq 0}$ where $\Upsilon_{-1,a}^\downarrow$ is the $(a-1)$ -stable Lévy process conditioned to stay positive until it dies at -1 . This process is defined as follows. Let $(\tilde{\tau}, Z)$ be a random variable of law

$$g(s, z) ds dz := \frac{\pi}{s^{1/(a-1)}} \frac{1}{\sqrt{z}(1+z)^a} f_a^\uparrow\left(\frac{z}{s^{1/(a-1)}}\right) ds dz.$$

We define $\Upsilon_{-1,a}^\downarrow$ so that, conditionally on $(\tilde{\tau}, Z)$, the path $(\Upsilon_{-1,a}^\downarrow(t))_{0 \leq t < \tilde{\tau}}$ is a stable meander associated with Υ_a of length $\tilde{\tau}$ conditioned on $\Upsilon_{-1,a}^\downarrow(\tilde{\tau}-) = Z$, and for all $t \geq \tilde{\tau}$, we set $\Upsilon_{-1,a}^\downarrow(t) = -1$.

Formally, the law of the process $\Upsilon_{-1,a}^\downarrow$ can be characterized with the law of the meander of length one $\tilde{\Upsilon}_a$. To express it, we introduce the density of $\tilde{\tau}$ that we write f . For all $s > 0$, we define the process $\tilde{\Upsilon}_a^s$ using $\tilde{\Upsilon}_a$ by setting for all $t \geq 0$,

$$\tilde{\Upsilon}_a^s(t) := \begin{cases} s^{1/(a-1)}\tilde{\Upsilon}_a(t/s) & \text{if } t < s, \\ -1 & \text{otherwise.} \end{cases}$$

Notice that $(s^{1/(a-1)}\tilde{\Upsilon}_a(t/s))_{t \geq 0}$ has the same law as the meander of length s evaluated at time t because Υ_a is $(a-1)$ -stable. We then define the law of $\Upsilon_{-1,a}^\downarrow$ by the relation

$$\mathbb{E} F\left(\left(\Upsilon_{-1,a}^\downarrow(t)\right)_{t \geq 0}\right) = \int_0^\infty f(s) \mathbb{E}\left(\frac{g(s, s^{1/(a-1)}\tilde{\Upsilon}_a(1))}{f(s)\tilde{f}_a(\tilde{\Upsilon}_a(1))} F\left(\tilde{\Upsilon}_a^s\right)\right) ds$$

for all bounded measurable function $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}$, where $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ is the space of càdlàg functions from \mathbb{R}_+ to \mathbb{R} equipped with the J_1 Skorokhod topology and \tilde{f}_a is the density of $\tilde{\Upsilon}_a(1)$. This is indeed a characterization of the law of $\Upsilon_{-1,a}^\downarrow$ as the meander of length $s > 0$ has the same law as $(s^{1/(a-1)}\tilde{\Upsilon}_a(t/s))_{t \geq 0}$.

Theorem 2.2.8. *With the convention that $P_\ell(k) = -\ell$ for $k \geq \tau_{-\ell}$, we have the convergence in distribution with respect to the Skorokhod topology:*

$$\text{Under } \mathbb{P}_\ell^{(1)}, \quad \left(\frac{P_\ell(\ell^{a-1}t)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (\Upsilon_{-1,a}^\downarrow(p_{\mathbf{q}}t))_{t \geq 0}.$$

The above theorem provides a first reason to see $\Upsilon_{-1,a}^\downarrow$ as the Lévy process Υ_a conditioned to stay positive until it jumps and dies at -1 .

Proof. The main ideas of the proof are the scaling limit result on random walks conditioned to stay positive up to time n of [Don85], which was recalled in Theorem 2.2.2, together with absolute continuity relations. Let $F : \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}$ be a bounded continuous real function. For conciseness, we set for all $\ell \geq 1$, for all $s, u \geq 0$,

$$g_\ell(s, u) = \ell^a \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell^{a-1}s \rfloor \text{ and } P_\ell(\lfloor \ell^{a-1}s \rfloor - 1) = \lfloor \ell u \rfloor), \quad f_\ell(s) = \ell^{a-1} \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell^{a-1}s \rfloor)$$

and

$$\tilde{f}_{\ell,s}(u) = \ell \mathbb{P}_1(S_{\lfloor \ell^{a-1}s \rfloor - 1} = \lfloor \ell u \rfloor | S_1, \dots, S_{\lfloor \ell^{a-1}s \rfloor - 1} \geq 1),$$

where we recall that $(S_i)_{i \geq 0}$ is a ν -random walk. We also write for all $s \geq 0, \ell \geq 1, t \geq 0$,

$$S^{s,\ell}(t) = \begin{cases} S_{\lfloor \ell^{a-1}t \rfloor} / \ell & \text{if } t < s, \\ -1 & \text{otherwise.} \end{cases}$$

Then, by the expression of the law of P_ℓ under $\mathbb{P}_\ell^{(1)}$, we get

$$\begin{aligned} & \mathbb{E}_\ell^{(1)} F \left(\left(\frac{P_\ell(\ell^{a-1}t)}{\ell} \right)_{t \geq 0} \right) \\ &= \sum_{k, n \geq 1} \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = k, P_\ell(k-1) = n) \mathbb{E}_1 \left(F \left(S^{k/\ell^{a-1}, \ell} \right) \middle| S_1, \dots, S_{k-1} \geq 1, S_{k-1} = n \right) \\ &= \int_0^\infty ds \sum_{n \geq 1} \frac{1}{\ell} g_\ell(s, \frac{n}{\ell}) \mathbb{E}_1 \left(F \left(S^{s, \ell} \right) \middle| S_1, \dots, S_{\lfloor \ell^{a-1}s \rfloor - 1} \geq 1, S_{\lfloor \ell^{a-1}s \rfloor - 1} = n \right) \\ &= \int_0^\infty ds f_\ell(s) \mathbb{E}_1 \left(\frac{g_\ell(s, S^{s, \ell}(s-1/\ell^{a-1}))}{f_\ell(s) \tilde{f}_{\ell,s}(S^{s, \ell}(s-1/\ell^{a-1}))} F \left(S^{s, \ell} \right) \middle| S_1, \dots, S_{\lfloor \ell^{a-1}s \rfloor - 1} \geq 1 \right). \end{aligned}$$

The factor inside the expectation can be computed as follows: for all $k, n \geq 1$,

$$\begin{aligned} \frac{g_\ell(k/\ell^{a-1}, n/\ell)}{f_\ell(k/\ell^{a-1}) \tilde{f}_{\ell, k/\ell^{a-1}}(n/\ell)} &= \frac{\frac{2(\ell+1)}{h^\downarrow(\ell)} \mathbb{P}_1(S_1, \dots, S_{k-1} \geq 1 \text{ and } S_{k-1} = n) \nu(-\ell - n)}{\mathbb{P}_\ell^{(1)}(\tau_{-\ell} = k) \mathbb{P}_1(S_{k-1} = n | S_1, \dots, S_{k-1} \geq 1)} \\ &= \frac{2(\ell+1) \nu(-\ell - n) \mathbb{P}_1(S_1, \dots, S_{k-1} \geq 1)}{h^\downarrow(\ell) \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = k)} \\ &= \frac{\nu(-\ell - n) \mathbb{P}_1(S_1, \dots, S_{k-1} \geq 1)}{\mathbb{P}_1(S_1, \dots, S_{k-1} \geq 1 \text{ and } S_k = -\ell)} \\ &= \left(\sum_{m \geq 1} \frac{\nu(-\ell - m)}{\nu(-\ell - n)} \mathbb{P}_1(S_{k-1} = m | S_1, \dots, S_{k-1} \geq 1) \right)^{-1} \\ &= \left(\mathbb{E}_1 \left(\frac{\nu(-\ell - S_{k-1})}{\nu(-\ell - n)} \middle| S_1, \dots, S_{k-1} \geq 1 \right) \right)^{-1}. \end{aligned}$$

Using Theorem 2.2.2, is easy to see that if $s, u > 0$, then

$$\mathbb{E}_1 \left(\frac{\nu(-\ell - S_{\lfloor \ell^{a-1}s \rfloor - 1})}{\nu(-\ell - \lfloor \ell u \rfloor)} \middle| S_1, \dots, S_{\lfloor \ell^{a-1}s \rfloor - 1} \geq 1 \right) \xrightarrow[\ell \rightarrow \infty]{} \int_0^\infty \left(\frac{1+u}{1+v} \right)^a \tilde{f}_a \left(\frac{v}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) dv,$$

where \tilde{f}_a is the density of $\tilde{\Upsilon}_a(1)$. Indeed, by (II),

$$\sup_{n,m \geq 1} \left| \frac{\nu(-\ell-m)}{\nu(-\ell-n)} - \left(\frac{\ell+n}{\ell+m} \right)^a \right| \xrightarrow{\ell \rightarrow \infty} 0.$$

Furthermore, since there exists a constant $C_1 > 0$ such that for all $x \geq 0$, $f_a^\uparrow(x) = C_1 x^{1/2} \tilde{f}_a(x)$, we have

$$\begin{aligned} \int_0^\infty \left(\frac{1+u}{1+v} \right)^a \tilde{f}_a \left(\frac{v}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) dv &= \frac{1}{C_1} (p_{\mathbf{q}}s)^{1/(2(a-1))} (1+u)^a \int_0^\infty \frac{1}{\sqrt{v}(1+v)^a} f_a^\uparrow \left(\frac{v}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) dv \\ &= \frac{(p_{\mathbf{q}}s)^{1/(2(a-1))} (1+u)^a}{C_1} \frac{(p_{\mathbf{q}}s)^{1/(a-1)}}{\pi} f(p_{\mathbf{q}}s) \\ &= \frac{f(p_{\mathbf{q}}s) \tilde{f}_a \left(\frac{u}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right)}{g(p_{\mathbf{q}}s, u)}. \end{aligned}$$

Thus, again using Theorem 2.2.2, we obtain that for all $s > 0$, conditionally on $S_1, \dots, S_{\lfloor \ell^{a-1} s \rfloor} \geq 1$, the random variable $g_\ell(s, S^{s,\ell}(s-1/\ell^{a-1})) / (f_\ell(s) \tilde{f}_{\ell,s}(S^{s,\ell}(s-1/\ell^{a-1})))$ converges in law towards

$$\frac{1}{\int_0^\infty \left(\frac{1+(p_{\mathbf{q}}s)^{1/(a-1)} \tilde{\Upsilon}_a(1)}{1+v} \right)^a \tilde{f}_a \left(\frac{v}{(p_{\mathbf{q}}s)^{1/(a-1)}} \right) dv} = \frac{g(p_{\mathbf{q}}s, (p_{\mathbf{q}}s)^{1/(a-1)} \tilde{\Upsilon}_a(1))}{f(p_{\mathbf{q}}s) \tilde{f}_a(\tilde{\Upsilon}_a(1))}.$$

Besides, if $s > 0$, one can upper bound uniformly in $n, \ell \geq 1$

$$\begin{aligned} \frac{g_\ell(s, n/\ell)}{f_\ell(s) \tilde{f}_{\ell,s}(n/\ell)} &= \frac{1}{\sum_{m \geq 1} \frac{\nu(-\ell-m)}{\nu(-\ell-n)} \mathbb{P}_1(S_{\lfloor \ell^{a-1} s \rfloor - 1} = m | S_1, \dots, S_{\lfloor \ell^{a-1} s \rfloor - 1} \geq 1)} \\ &\leq \frac{C}{\mathbb{P}_1(S_{\lfloor \ell^{a-1} s \rfloor - 1} \in [0, \ell] | S_1, \dots, S_{\lfloor \ell^{a-1} s \rfloor - 1} \geq 1)} \\ &\leq \frac{C'}{\mathbb{P}((p_{\mathbf{q}}s)^{1/(a-1)} \tilde{\Upsilon}_a(1) \in [0, 1])} \text{ by Theorem 2.2.2,} \end{aligned}$$

where C, C' are constants that only depend on ν . Therefore, by dominated convergence and using once again Theorem 2.2.2, for all $s > 0$,

$$\begin{aligned} G_\ell(s) &:= \mathbb{E}_1 \left(\frac{g_\ell(s, S^{s,\ell}(s-1/\ell^{a-1}))}{f_\ell(s) \tilde{f}_{\ell,s}(S^{s,\ell}(s-1/\ell^{a-1}))} F(S^{s,\ell}) \middle| S_1, \dots, S_{\lfloor \ell^{a-1} s \rfloor - 1} \geq 1 \right) \\ &\xrightarrow{\ell \rightarrow \infty} G(s) := \mathbb{E} \left(\frac{g(p_{\mathbf{q}}s, (p_{\mathbf{q}}s)^{1/(a-1)} \tilde{\Upsilon}_a(1))}{f(p_{\mathbf{q}}s) \tilde{f}_a(\tilde{\Upsilon}_a(1))} F((\tilde{\Upsilon}_a^{p_{\mathbf{q}}s}(p_{\mathbf{q}}t))_{t \geq 0}) \right). \end{aligned}$$

Furthermore, from Proposition 2.2.7, and using Scheffé's lemma, f_ℓ converges in $L^1(\mathbb{R})$ towards $p_{\mathbf{q}}f(p_{\mathbf{q}} \cdot)$. Therefore, again by dominated convergence, we obtain that for all $A > 0$,

$$\begin{aligned} \mathbb{E}_\ell^{(1)} \left(F \left(\left(\frac{P_\ell(\ell^{a-1}t)}{\ell} \right)_{t \geq 0} \right) \mathbf{1}_{\frac{\tau_\ell}{\ell^{a-1}} < A} \right) &= \int_0^A f_\ell(s) G_\ell(s) ds \\ &\xrightarrow{\ell \rightarrow \infty} \int_0^A p_{\mathbf{q}}f(p_{\mathbf{q}}s) G(s) ds \\ &= \mathbb{E} \left(F \left(\left(\Upsilon_{-1,a}^\downarrow(p_{\mathbf{q}}t) \right)_{t \geq 0} \right) \mathbf{1}_{\tau < p_{\mathbf{q}}A} \right), \end{aligned}$$

where the last equality stems from the definition of $\Upsilon_{-1,a}^\downarrow$ and from the change of variables $s' = p_{\mathbf{q}}s$. But since τ_ℓ/ℓ^{a-1} converges in law, and since F is bounded, we get that

$$\sup_{\ell \geq 1} \mathbb{E}_\ell^{(1)} \left(F \left(\left(\frac{P_\ell(\ell^{a-1}t)}{\ell} \right)_{t \geq 0} \right) \mathbf{1}_{\frac{\tau_\ell}{\ell^{a-1}} \geq A} \right) \xrightarrow{A \rightarrow \infty} 0,$$

hence the desired result. \square

The process $\Upsilon_{-1,a}^\downarrow$ can be linked to the stable Lévy process conditioned to stay positive Υ_a^\uparrow using Lemma 2.2.5 and Theorem 2.2.8:

Corollary 2.2.9. *Let $\Upsilon_{-1,a}^\downarrow$ as in Theorem 2.2.8. Let $\tilde{\tau}$ be the time at which $\Upsilon_{-1,a}^\downarrow$ dies at -1 . For all bounded measurable function $F : \mathbb{D}([0, t], \mathbb{R}) \rightarrow \mathbb{R}$,*

$$\mathbb{E} \mathbf{1}_{\tilde{\tau} > t} F\left(\left(\Upsilon_{-1,a}^\downarrow(s)\right)_{0 \leq s \leq t}\right) = \mathbb{E} \frac{1}{1 + \Upsilon_a^\uparrow(t)} F\left(\left(\Upsilon_a^\uparrow(s)\right)_{0 \leq s \leq t}\right).$$

One can also define for all $\lambda > 0$, the Lévy process conditioned to stay positive until it jumps and dies at $-\lambda$, denoted by $\Upsilon_{-\lambda,a}^\downarrow$. It satisfies the scaling limit:

$$\text{Under } \mathbb{P}_{[\lambda\ell]}^{(1)}, \left(\frac{P_{[\lambda\ell]}(\ell^{a-1}t)}{\ell}\right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\Upsilon_{-\lambda,a}^\downarrow(p_{\mathbf{q}}t)\right)_{t \geq 0}.$$

Let $\tilde{\tau}_{-\lambda,a}$ be the time at which $\Upsilon_{-\lambda,a}^\downarrow$ jumps at $-\lambda$. From the coupling $\mathbb{P}_{[\lambda\ell],\infty}^{(1),n}$, we get that conditionally on $\tilde{\tau}_{-\lambda,a} > t$, the process $(\Upsilon_{-\lambda,a}^\downarrow(s))_{0 \leq s \leq t}$ has the same law as $(\Upsilon_a^\uparrow(s))_{0 \leq s \leq t}$. Furthermore, from the scaling limit it is easy to see that $\tilde{\tau}_{-\lambda,a}$ has the same law as $\lambda^{a-1}\tilde{\tau}$. It is thus possible to show that

$$\Upsilon_{-\lambda,a}^\downarrow \xrightarrow[\lambda \rightarrow \infty]{(d)} \Upsilon_a^\uparrow$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ equipped with the Skorokhod J_1 -topology. This convergence echoes the local convergence of P_ℓ under $\mathbb{P}_\ell^{(1)}$ towards P_∞ under $\mathbb{P}_\infty^{(1)}$ when $\ell \rightarrow \infty$.

2.3 The peeling exploration “starting from a uniform random edge”

In this section, we recall the filled-in (edge) peeling exploration of a map with a distinguished face called the target and how this exploration enables to recover the distance from the root face to that target face. In Subsection 2.3.4, we show that for our purpose, we can start the exploration “from a uniform random edge” which will become the root while the former root face of degree 2ℓ will become the target face.

2.3.1 Filled-in exploration of a map with a target

The edge-peeling process was introduced by Budd in [Bud16] and the results presented here can be found in more details in Chapters III and IV of [Cur23] or in [Bud17].

A map with a target face $\mathbf{m}_\square = (\mathbf{m}, \square)$ is a (finite rooted bipartite planar) map \mathbf{m} given with a distinguished face \square different from the root face f_r . For all $\ell, p \geq 1$, we denote by $\mathcal{M}_p^{(\ell)}$ the set of finite maps of perimeter 2ℓ with a target face of perimeter $2p$. We set for all $\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}$:

$$w_{\mathbf{q}}(\mathbf{m}_\square) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r, \square\}} q_{\deg(f)/2}.$$

We then define also the partition function $W_p^{(\ell)} = \sum_{\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}_\square)$. Analogously, a map with a target vertex $\mathbf{m}_\bullet = (\mathbf{m}, \bullet)$ is a (finite rooted bipartite planar) map \mathbf{m} with a distinguished vertex \bullet . We denote by $\mathcal{M}_0^{(\ell)}$ the set of finite maps of perimeter 2ℓ with a target vertex. We set $w_{\mathbf{q}}(\mathbf{m}_\bullet) = w_{\mathbf{q}}(\mathbf{m})$ for all $\mathbf{m}_\bullet \in \mathcal{M}_0^{(\ell)}$, and $W_0^{(\ell)} = \sum_{\mathbf{m}_\bullet \in \mathcal{M}_0^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}_\bullet)$. The fact that \mathbf{q} is admissible implies that $W_p^{(\ell)} < \infty$ for all $\ell \geq 1, p \geq 0$. Moreover, we know that (see e.g. Equation (3.9) in [Cur23]) for all $p, \ell \geq 1$,

$$W_p^{(\ell)} = \frac{1}{2} h_p^\downarrow(\ell) c_{\mathbf{q}}^{\ell+p} \quad \text{and} \quad W_0^{(\ell)} = h^\downarrow(\ell) c_{\mathbf{q}}^\ell. \quad (2.3.1)$$

We write $\mathbb{P}_p^{(\ell)}$ the associated Boltzmann probability measure on $\mathcal{M}_p^{(\ell)}$ for all $p \geq 0$.

Let $\ell \geq 1$. By Theorem 2 of [Bud17], one can define a random infinite planar map $\mathfrak{M}_\infty^{(\ell)}$ as the local limit in distribution as $p \rightarrow \infty$ of random maps of law $\mathbb{P}_p^{(\ell)}$. The map $\mathfrak{M}_\infty^{(\ell)}$ is almost surely locally finite and one-ended, and is called a \mathfrak{q} -Boltzmann infinite planar random map of the plane. Its law is denoted by $\mathbb{P}_\infty^{(\ell)}$.

A (filled-in) peeling algorithm \mathcal{A} is a function which associates to a map $\bar{\mathfrak{e}}$ with a distinguished simple face (different from the root face) an edge on the boundary $\partial\bar{\mathfrak{e}}$ of this distinguished face. Let \mathfrak{m}_* be a map with a target face \square (“ $*$ = \square ”), a target vertex \bullet (“ $*$ = \bullet ”) or an infinite one-ended bipartite map (“ $*$ = ∞ ”). A filled-in peeling exploration of \mathfrak{m}_* with algorithm \mathcal{A} is an increasing sequence $(\bar{\mathfrak{e}}_n)_{n \geq 0}$ of sub-maps of \mathfrak{m} which contain the root face and which are maps with a distinguished simple face called a hole, where by sub-map we mean that gluing a well chosen map with perimeter corresponding with that of the hole of $\bar{\mathfrak{e}}_n$ will give back the map \mathfrak{m}_* . This sequence of sub-maps is constructed inductively using the algorithm \mathcal{A} : $\bar{\mathfrak{e}}$ is the map with only two faces which are the root face and a hole of the same perimeter and then for each $n \geq 0$, the sub-map $\bar{\mathfrak{e}}_{n+1}$ is obtained from $\bar{\mathfrak{e}}_n$ by peeling the edge $\mathcal{A}(\bar{\mathfrak{e}}_n)$ in \mathfrak{m}_* (see Figure 2.3). When peeling an edge, there are three cases:

- (i) The face in \mathfrak{m}_* on the other side of $\mathcal{A}(\bar{\mathfrak{e}}_n)$ is not a face of $\bar{\mathfrak{e}}_n$ and is not the target face \square when $*$ = \square ; then $\bar{\mathfrak{e}}_{n+1}$ is obtained by gluing that face to $\bar{\mathfrak{e}}_n$ onto the edge $\mathcal{A}(\bar{\mathfrak{e}}_n)$; we denote this case by C_k where k is the half-degree of the discovered face.
- (ii) The face in \mathfrak{m}_* on the other side of $\mathcal{A}(\bar{\mathfrak{e}}_n)$ is the target face \square (when $*$ = \square); in that case, denoted by C_p^{stop} where p is the half-degree of the target face, we set $\bar{\mathfrak{e}}_{n+1} = \mathfrak{m}_\square$ and the exploration stops.
- (iii) Or the other side of $\mathcal{A}(\bar{\mathfrak{e}}_n)$ corresponds to a face already discovered in $\bar{\mathfrak{e}}_n$ which must be on the boundary of the hole. In this case $\bar{\mathfrak{e}}_{n+1}$ is obtained by identifying the two edges in the hole of $\bar{\mathfrak{e}}_n$. This creates at most two holes. If $*$ \in $\{\square, \bullet\}$, we fill-in the hole which does not contain the target. If $*$ = ∞ , we fill-in the hole containing a finite part of \mathfrak{m}_∞ . This case is denoted by $G_{k,*}$ or $G_{*,k}$ where $2k$ is the degree of the hole which is filled-in, depending whether this hole is created on the left or on the right of the peeled edge. If $*$ = \bullet and $k = |\partial\bar{\mathfrak{e}}_n|/2 - 1$, then the exploration stops.

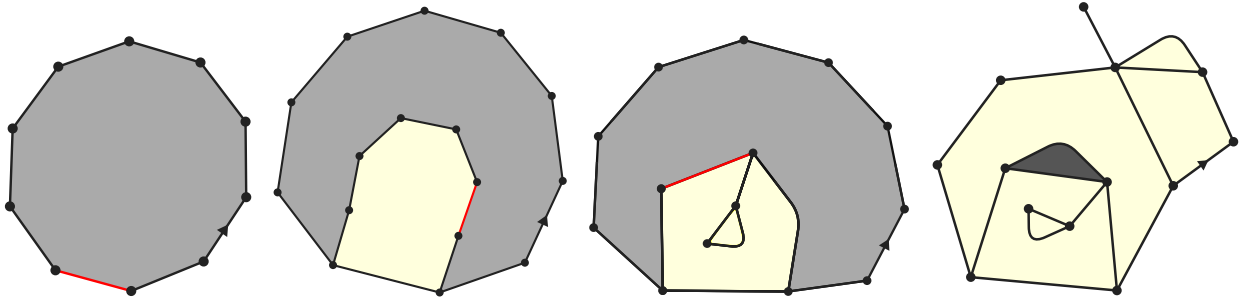


Figure 2.3 – A filled-in exploration of a map with a target face $\mathfrak{m}_\square \in \mathcal{M}_1^5$. The peeled edge is in red, the root face is in white, the target face is in black and the hole containing the target is in grey. At the first step, an event C_4 happens, then an event $G_{1,*}$ occurs and finally the event C_1^{stop} ends the exploration.

The peeling algorithm \mathcal{A} may be arbitrary, even random, but we require it to be Markovian with respect to the exploration in the sense that, conditionally on the explored region $\bar{\mathfrak{e}}_n$, the conditional law of $\mathcal{A}(\bar{\mathfrak{e}}_n)$ only depends on $\bar{\mathfrak{e}}_n$. For instance, the peeled edge may be chosen uniformly at random on the boundary of the hole as in Subsection 2.3.2 or in such a way that the dual distance from the peeled edge to the root is non-decreasing along the exploration as in Subsection 2.3.3. During the peeling exploration, one records the evolution of half of the number of edges on the boundary of the hole, which is denoted by $(P_p(n))_{n \geq 0}$ if $*$ \in $\{\bullet, \square\}$ and $(P_\infty(n))_{n \geq 0}$ if $*$ = ∞ and called the perimeter process associated with the exploration. One can express the law of the peeling process (and the law of the perimeter process) under $\mathbb{P}_p^{(\ell)}$ or under $\mathbb{P}_\infty^{(\ell)}$ using Doob h -transforms of the random walk $S = (S_n)_{n \geq 0}$ of step distribution ν .

Now we recall the law of the filled-in peeling exploration $(\bar{\mathfrak{e}}_n)_{n \geq 0}$ under $\mathbb{P}_p^{(\ell)}$ or under $\mathbb{P}_\infty^{(\ell)}$, as described by Proposition 4.7 from [Cur23] in the case $*$ = \square, \bullet and [Bud16] in the case $*$ = ∞ (see also Proposition 7.4 in [Cur23]).

Proposition 2.3.1. ([Cur23], [Bud16]) Let $\ell, p \geq 1$, let \mathcal{A} be a peeling algorithm and $(\bar{\epsilon}_n)_{n \geq 0}$ be the peeling process under $\mathbb{P}_0^{(\ell)}, \mathbb{P}_p^{(\ell)}$ or $\mathbb{P}_\infty^{(\ell)}$. Then $(\bar{\epsilon}_n)_{n \geq 0}$ is a Markov chain with the following transition probabilities: conditionally on $\bar{\epsilon}_n$, and $P(n) = m \geq 1$ (assuming that the exploration has not stopped yet), the events C_k, C_p^{stop} (for $* = \square$), $G_{k,*}$ and $G_{*,k}$ occur respectively with probabilities

$$\nu(k-1) \frac{h(m+k-1)}{h(m)}, \nu(-m-p) \frac{h(-p)}{h(m)}, \frac{1}{2} \nu(-k-1) \frac{h(m-k-1)}{h(m)} \text{ and } \frac{1}{2} \nu(-k-1) \frac{h(m-k-1)}{h(m)},$$

where $h = h_p^\downarrow$ for $* = \square$, $h = h^\downarrow$ for $* = \bullet$ and $h = h^\uparrow$ for $* = \infty$. Moreover, conditionally on each event, the holes filled-in during the exploration are independent finite \mathbf{q} -Boltzmann maps (without target) of corresponding perimeters.

As a consequence, for all $p \geq 0$, the perimeter process under $\mathbb{P}_p^{(\ell)}$ (resp. under $\mathbb{P}_\infty^{(\ell)}$) indeed corresponds to the Doob h_p^\downarrow -transform killed when it reaches $-p$ (resp. Doob h^\uparrow -transform) of the walk S started from ℓ , if we set by convention that the perimeter drops to $-p$ when the exploration stops.

2.3.2 Uniform peeling and first-passage percolation distance

It is possible to use the peeling process to study the first-passage percolation distance on a map with a target \mathbf{m}_* , or an infinite map of the plane. A very convenient algorithm is the uniform peeling algorithm $\mathcal{A}_{\text{uniform}}$ which selects an edge on the boundary of the hole uniformly at random at every step. We refer to [Cur23] 13.1 or [BC17] 2.4 for details.

This exploration can be related to the fpp distance on \mathbf{m}_* in the following way: for all $t \geq 0$, let $\overline{\text{Ball}}_t^{\text{fpp}}(\mathbf{m}_*)$ be the sub-map of \mathbf{m}_* obtained by keeping the connected subset of dual edges whose endpoints are at fpp distance at most t from the root face, then by gluing the corresponding faces according to those dual edges and finally by filling in the holes that do not contain the target (the finite holes in the case $* = \infty$). Then the process $(\overline{\text{Ball}}_t^{\text{fpp}}(\mathbf{m}_*))_{t \geq 0}$ admits jumps $0 = T_0 < T_1 < \dots$. One can check (see Proposition 2.3 in [BC17]) that for all $n \geq 0$, the sub-map $\overline{\text{Ball}}_{T_{n+1}}^{\text{fpp}}(\mathbf{m}_*)$ can be obtained from $\overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}_*)$ by the peeling of a uniform random edge on the boundary of the hole. Furthermore, conditionally on $(\overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}_*))_{n \geq 0}$, the random variables $T_{n+1} - T_n$ are independent and follow exponential laws of parameter $|\partial \overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}_*)|$ where $\partial \overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}_*)$ is the set of edges on the boundary of $\overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}_*)$.

From the above results, by coupling the uniform exploration with the exponential variables that define the fpp distance, the fpp distance from the root face to the submap \mathbf{u} filling the hole of boundary $\partial \bar{\epsilon}_n$ can be written as

$$d_{\text{fpp}}^\dagger(f_r, \mathbf{u}) = T_{n+1} = \sum_{i=0}^n \frac{\mathcal{E}_i}{|\partial \overline{\text{Ball}}_{T_i}^{\text{fpp}}(\mathbf{m}_*)|} = \sum_{i=0}^n \frac{\mathcal{E}_i}{2P(i)}, \quad (2.3.2)$$

where $P(i) = |\partial \overline{\text{Ball}}_{T_i}^{\text{fpp}}(\mathbf{m}_*)|/2$ is the perimeter process associated with the exploration and the \mathcal{E}_i for $i \geq 0$ are i.i.d. exponential random variables of parameter 1 which are independent from the exploration $(\bar{\epsilon}_n)_{n \geq 0}$, and a fortiori from $(P(n))_{n \geq 0}$. In particular, if $\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}$ for $\ell, p \geq 1$ and n is the time at which the target face \square is discovered, i.e. the event C_p^{target} happens at time n , or equivalently $\bar{\epsilon}_n = \mathbf{m}_\square$, then $d_{\text{fpp}}^\dagger(f_r, \square) = T_n$.

2.3.3 Peeling by layers and dual graph distance

In the same way, a particular peeling algorithm $\mathcal{A}_{\text{layers}}$ called the peeling by layers algorithm is suited to the study of the dual graph distance on a map with a target \mathbf{m}_* (or an infinite map of the plane). Recall that d_{gr}^\dagger denotes the dual graph distance on a map \mathbf{m} . If f is a face of \mathbf{m} the distance to the root face $d_{\text{gr}}^\dagger(f_r, f)$ is also called the height of f . We can also define the height of an edge e as the minimum of the heights of the faces on both sides of e .

The peeling algorithm $\mathcal{A}_{\text{layers}}$, described in [BC17] 2.3, see also [Cur23] p.185, is defined only on (finite) sub-maps $\bar{\epsilon}$ of \mathbf{m}_* with a hole (containing the target when $* = \square$) which satisfy the following hypothesis:

- (H) There exists an integer $h \geq 0$ such that all the edges on the boundary of the hole $\partial \bar{\epsilon}$ are at height h or $h + 1$, and such that the set of edges of $\partial \bar{\epsilon}$ at height h forms a (non empty) segment.

If $\bar{\epsilon}$ satisfies **(H)**, we then set $\mathcal{A}_{\text{layers}}(\bar{\epsilon})$ to be the unique edge of $\partial\bar{\epsilon}$ at height h such that the edge immediately on its left is at height $h + 1$. If all the edges of $\partial\bar{\epsilon}$ are at height h the edge $\mathcal{A}_{\text{layers}}(\bar{\epsilon})$ is then chosen deterministically.

One can check that on the events C_k , $G_{*,k}$ and $G_{k,*}$ the sub-map obtained after peeling $\mathcal{A}_{\text{layers}}(\bar{\epsilon})$ satisfies **(H)** again. Moreover, $\bar{\epsilon}_0$ satisfies **(H)** as well. Thus the peeling exploration with algorithm $\mathcal{A}_{\text{layers}}$ is well defined.

For every $n \geq 0$, let $H(n)$ be the minimum of the heights of the edges of $\partial\bar{\epsilon}_n$. Then $H(0) = 0$ and for all $n \geq 0$, we have $\Delta H(n) := H(n+1) - H(n) \in \{0, 1\}$. One has $\Delta H(n) = 1$ when all the edges of $\partial\bar{\epsilon}_{n+1}$ are at height $H(n) + 1$. The process $H(n)$ is called the height process associated with the peeling by layers exploration. The integer $H(n) + 1$ can also be seen as the height of the hole in $\bar{\epsilon}_n$. In particular, if $\mathfrak{m}_\square \in \mathcal{M}_p^{(\ell)}$ for some $\ell, p \geq 1$ and n is the first time such that $\bar{\epsilon}_n = \mathfrak{m}_\square$, then $d_{\text{gr}}^\dagger(f_r, \square) = H(n - 1) + 1$.

2.3.4 Exchanging the root and the target

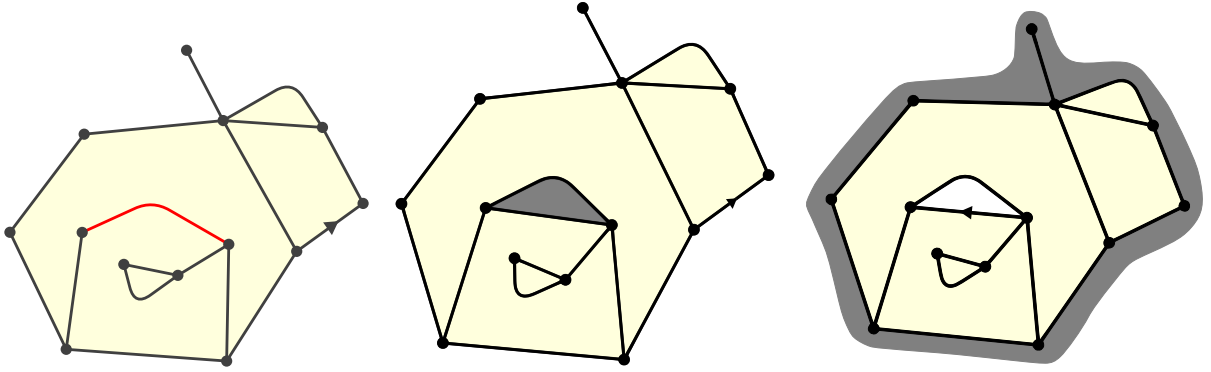


Figure 2.4 – From left to right: a map with a distinguished edge in red, the same map with the edge “unzipped” giving rise to a distinguished 2-face (in grey) and finally the same map which has been re-rooted.

In order to study the distance from the root face to a uniform random face of $\mathfrak{M}^{(\ell)}$, we will first look at the distance between the root face and a uniform random edge E in $\mathfrak{M}^{(\ell)}$. We will then come back to a uniform random face using a biasing argument in Subsection 2.4.3. This distance is defined as the minimum of the distances from the root face to a face next to this edge. By unzipping the uniform random edge and thus creating a distinguished face of degree 2, one can see that the map obtained from $\mathfrak{M}^{(\ell)}$ belongs to $\mathcal{M}_1^{(\ell)}$. We denote this map by $\widetilde{\mathfrak{M}}_1^{(\ell)}$.

Remark 2.3.2. *This unzipping operation does not modify the distances so much. More precisely, the graph distance $d_{\text{gr}}^\dagger(f_r, \square)$ in $\widetilde{\mathfrak{M}}_1^{(\ell)}$ equals $d_{\text{gr}}^\dagger(f_r, E) + 1$ in $\mathfrak{M}^{(\ell)}$ while $d_{\text{fpp}}^\dagger(f_r, \square)$ in $\widetilde{\mathfrak{M}}_1^{(\ell)}$ is lower bounded by $d_{\text{fpp}}^\dagger(f_r, E)$ and is upper bounded by $d_{\text{fpp}}^\dagger(f_r, E) + \mathcal{E}$ in $\mathfrak{M}^{(\ell)}$ where \mathcal{E} is an independent exponential variable of parameter 1.*

Besides, one can notice that exchanging the roles of the root face and of the target face by choosing a root edge on the former target face does not modify the distance between the root face and the target face. All the operations we did are illustrated by Figure 2.4. This swap between the root face and the target face will be useful insofar as in the next subsection we will introduce a coupling of $\mathbb{P}_\ell^{(1)}$ with $\mathbb{P}_\infty^{(1)}$ that will enable us to use the results of [BCM18]. Let $\widetilde{\mathfrak{M}}_\ell^{(1)}$ be the map obtained from $\widetilde{\mathfrak{M}}_1^{(\ell)}$ by exchanging the root face and the target face and choosing the root edge uniformly at random around the new root face.

So as to transfer our results under $\mathbb{P}_\ell^{(1)}$ to the map $\mathfrak{M}^{(\ell)}$ equipped with a random uniform edge E , we state the following lemma.

Lemma 2.3.3. *If $(\varphi_\ell)_{\ell \geq 1}$ is a sequence of functions from $\mathcal{M}_\ell^{(1)}$ to \mathbb{R}_+ uniformly bounded by a constant $C > 0$ such that*

$$\mathbb{E}_\ell^{(1)} \varphi_\ell(\mathfrak{m}_\square) \xrightarrow{\ell \rightarrow \infty} 0,$$

then

$$\mathbb{E}\varphi_\ell\left(\widetilde{\mathfrak{M}}_\ell^{(1)}\right) \xrightarrow{\ell \rightarrow \infty} 0.$$

In particular, if $(A_\ell)_{\ell \geq 1}$ is a sequence of subsets of $\mathcal{M}_\ell^{(1)}$ such that $\mathbb{P}_\ell^{(1)}(A_\ell) \rightarrow 0$ as $\ell \rightarrow \infty$, then $\mathbb{P}(\widetilde{\mathfrak{M}}_\ell^{(1)} \in A_\ell) \rightarrow 0$.

Proof. Observe first that the law of $\widetilde{\mathfrak{M}}_1^{(\ell)}$ can be characterized using $\mathbb{P}_1^{(\ell)}$. Indeed, if $\mathfrak{m}_\square \in \mathcal{M}_1^{(\ell)}$ is obtained by unzipping an edge of a map $\mathfrak{m}' \in \mathcal{M}^{(\ell)}$, then

$$\begin{aligned} \mathbb{P}(\widetilde{\mathfrak{M}}_1^{(\ell)} = \mathfrak{m}_\square) &= \frac{1}{\#\text{Edges}(\mathfrak{m}')} \mathbb{P}(\mathfrak{M}^{(\ell)} = \mathfrak{m}') = \frac{1}{\#\text{Edges}(\mathfrak{m}')} \frac{w_{\mathbf{q}}(\mathfrak{m}')}{W^{(\ell)}} \\ &= \frac{1}{\#\text{Edges}(\mathfrak{m}_\square) - 1} \frac{W_1^{(\ell)}}{W^{(\ell)}} \frac{w_{\mathbf{q}}(\mathfrak{m}_\square)}{W_1^{(\ell)}} = \frac{W_1^{(\ell)}}{W^{(\ell)}} \frac{1}{\#\text{Edges}(\mathfrak{m}_\square) - 1} \mathbb{P}_1^{(\ell)}(\mathfrak{m}_\square). \end{aligned}$$

Another well known and useful consequence of this unzipping operation is that

$$\mathbb{E} \#\text{Edges}(\mathfrak{M}^{(\ell)}) = \frac{W_1^{(\ell)}}{W^{(\ell)}} \underset{\ell \rightarrow \infty}{\sim} \frac{\frac{1}{2} h_1^\downarrow(\ell) c_{\mathbf{q}}^{\ell+1}}{\frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-2}} \underset{\ell \rightarrow \infty}{\sim} b_{\mathbf{q}} \ell^{3/2} \text{ where } b_{\mathbf{q}} = \frac{1}{2p_{\mathbf{q}}\sqrt{\pi}}, \quad (2.3.3)$$

applying successively (2.1.1), (2.3.1), (2.2.1) and the fact that $h^\downarrow(\ell) \sim 1/\sqrt{\pi\ell}$.

Next, the law of $\widetilde{\mathfrak{M}}_\ell^{(1)}$ can be expressed as follows. Let $\mathfrak{m}_\square \in \mathcal{M}_\ell^{(1)}$ which is obtained by re-rooting a map $\mathfrak{m}'_\square \in \mathcal{M}_1^{(\ell)}$. Then one can check that

$$\mathbb{P}(\widetilde{\mathfrak{M}}_\ell^{(1)} = \mathfrak{m}_\square) = \frac{2\ell}{2} \mathbb{P}(\widetilde{\mathfrak{M}}_1^{(\ell)} = \mathfrak{m}'_\square) = \frac{2\ell}{2} \frac{W_1^{(\ell)}}{W^{(\ell)}} \frac{\mathbb{P}_1^{(\ell)}(\mathfrak{m}'_\square)}{\#\text{Edges}(\mathfrak{m}'_\square) - 1} = \frac{W_1^{(\ell)}}{W^{(\ell)}} \frac{\mathbb{P}_\ell^{(1)}(\mathfrak{m}_\square)}{\#\text{Edges}(\mathfrak{m}_\square) - 1}. \quad (2.3.4)$$

We deduce that

$$\mathbb{E}\varphi_\ell\left(\widetilde{\mathfrak{M}}_\ell^{(1)}\right) = \sum_{\mathfrak{m}_\square \in \mathcal{M}_\ell^{(1)}} \frac{W_1^{(\ell)}}{W^{(\ell)}} \frac{\mathbb{P}_\ell^{(1)}(\mathfrak{m}_\square)}{\#\text{Edges}(\mathfrak{m}_\square) - 1} \varphi_\ell(\mathfrak{m}_\square) = \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_\ell^{(1)}\left(\frac{\varphi_\ell(\mathfrak{m}_\square)}{\#\text{Edges}(\mathfrak{m}_\square) - 1}\right). \quad (2.3.5)$$

Let $\varepsilon > 0$. We next distinguish whether the number of edges is larger than $\varepsilon\ell^{3/2} + 1$ or not. On the one hand, using (2.3.3), one can see that

$$\frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_\ell^{(1)}\left(\frac{\varphi_\ell(\mathfrak{m}_\square)}{\#\text{Edges}(\mathfrak{m}_\square) - 1} \mathbf{1}_{\#\text{Edges}(\mathfrak{m}_\square) \geq \varepsilon\ell^{3/2} + 1}\right) \leq \frac{W_1^{(\ell)}}{\varepsilon\ell^{3/2} W^{(\ell)}} \mathbb{E}_\ell^{(1)} \varphi_\ell(\mathfrak{m}_\square) \xrightarrow{\ell \rightarrow \infty} 0.$$

On the other hand,

$$\begin{aligned} \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_\ell^{(1)}\left(\frac{\varphi_\ell(\mathfrak{m}_\square) \mathbf{1}_{\#\text{Edges}(\mathfrak{m}_\square) \leq \varepsilon\ell^{3/2}}}{\#\text{Edges}(\mathfrak{m}_\square) - 1}\right) &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_\ell^{(1)}\left(C \frac{\mathbf{1}_{\#\text{Edges}(\mathfrak{m}_\square) \leq \varepsilon\ell^{3/2}}}{\#\text{Edges}(\mathfrak{m}_\square) - 1}\right) \\ &= C \mathbb{P}\left(\#\text{Edges}(\widetilde{\mathfrak{M}}_\ell^{(1)}) \leq \varepsilon\ell^{3/2}\right) \text{ by (2.3.5)} \\ &= C \mathbb{P}(\#\text{Edges}(\mathfrak{M}^{(\ell)}) \leq \varepsilon\ell^{3/2} - 1). \end{aligned}$$

Furthermore,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(\#\text{Edges}(\mathfrak{m}) \leq \varepsilon\ell^{3/2} - 1) = 0,$$

since $\#\text{Edges}(\mathfrak{M}^{(\ell)})\ell^{-3/2}$ converges in distribution towards a positive random variable as $\ell \rightarrow \infty$, see e.g. Proposition 10.4 in [Cur23]. \square

2.4 Scaling limit of the distance to a target

This section and the rest of this chapter are devoted to the study of the dual distances on the $3/2$ -stable map $\mathfrak{M}^{(\ell)}$ of perimeter 2ℓ . We thus henceforth assume that \mathbf{q} is non-generic critical of type $a = 2$. We will drop the subscript 2 in Υ_2 , Υ_2^\dagger , Υ_2^\downarrow and $\Upsilon_{-1,2}^\downarrow$.

As announced before, we first study the distance from the root face to a uniform random edge of $\mathfrak{M}^{(\ell)}$ and then transpose the result to the distance to a uniform random face and to a uniform random vertex. From Subsection 2.3.4, it suffices to study the distance from the root face f_r to the target face \square under $\mathbb{P}_\ell^{(1)}$. Thus, we may use the filled-in exploration of $\mathbb{P}_\ell^{(1)}$ presented in Section 2.3 with in particular the peeling algorithms $\mathcal{A}_{\text{uniform}}$ and $\mathcal{A}_{\text{layers}}$.

2.4.1 The first-passage percolation distance to a uniform random edge in $\mathfrak{M}^{(\ell)}$

From Subsection 2.3.2, in particular (2.3.2), we know that the fpp distance from the root face f_r to the target face \square under $\mathbb{P}_\ell^{(1)}$ is

$$d_{\text{fpp}}^\dagger(f_r, \square) = \sum_{k=0}^{\tau_\ell - 1} \frac{\mathcal{E}_k}{2P_\ell(k)}, \quad (2.4.1)$$

where the \mathcal{E}_k 's are i.i.d. exponential random variables of parameter one independent of the perimeter process. The properties of the perimeter process proved in Section 2.2 enable us to state the following proposition, taking us closer to the first convergence of Theorem 2.1.1.

Proposition 2.4.1. *Under $\mathbb{P}_\ell^{(1)}$,*

$$\frac{d_{\text{fpp}}^\dagger(f_r, \square)}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}}.$$

As a consequence, if $E^{(\ell)}$ is a uniform edge in $\mathfrak{M}^{(\ell)}$, then

$$\frac{d_{\text{fpp}}^\dagger(f_r, E^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}}.$$

Proof. For the first convergence, it suffices to prove that

$$\frac{1}{\log \ell} \sum_{k=0}^{\tau_\ell - 1} \frac{1}{2P_\ell(k)} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}}. \quad (2.4.2)$$

Indeed, assuming (2.4.2), like in the beginning of the proof of Proposition 3 of [BCM18], we can reason conditionally on P_ℓ :

$$\mathbb{E} \left(\left(\frac{1}{\log \ell} \sum_{k=0}^{\tau_\ell - 1} \frac{\mathcal{E}_k - 1}{2P_\ell(k)} \right)^2 \middle| P_\ell \right) = \frac{1}{(\log \ell)^2} \sum_{k=0}^{\tau_\ell - 1} \frac{1}{4P_\ell(k)^2} \leq \frac{1}{(\log \ell)^2} \sum_{k=0}^{\tau_\ell - 1} \frac{1}{4P_\ell(k)} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} 0,$$

where the convergence comes from (2.4.2). In order to prove (2.4.2), one can first see that by Theorem 2.2.8, for all $\varepsilon > 0$,

$$\sum_{k=\lfloor \varepsilon \ell \rfloor}^{\tau_\ell - 1} \frac{1}{2P_\ell(k)} \xrightarrow[\ell \rightarrow \infty]{(d)} \int_\varepsilon^{\tilde{\tau}} \frac{1}{2\Upsilon_{-1}^\downarrow(p_{\mathbf{q}} t)} dt. \quad (2.4.3)$$

Besides, under the coupling $\mathbb{P}_{\ell, \infty}^{(1), \lfloor \varepsilon \ell \rfloor}$ of Lemma 2.2.5, on the event $\tau_{-\ell} > \lfloor \varepsilon \ell \rfloor$ (which holds with probability greater than $1 - o(1)$ uniformly in ℓ as $\varepsilon \rightarrow 0$), for all $k \leq \lfloor \varepsilon \ell \rfloor$ we have $P_\ell(k) = P_\infty(k)$. Thus, on the event $\tau_{-\ell} > \lfloor \varepsilon \ell \rfloor$, we have

$$\frac{1}{\log \ell} \sum_{k=0}^{\lfloor \varepsilon \ell \rfloor} \frac{1}{2P_\ell(k)} = \frac{1}{\log \ell} \sum_{k=0}^{\lfloor \varepsilon \ell \rfloor} \frac{1}{2P_\infty(k)} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}},$$

by Equation (15) of [BCM18]. Consequently, combining this convergence with (2.4.3), we can conclude that (2.4.2) holds, which ends the proof of the first convergence.

The first convergence of the statement is immediately transferred to the map $\mathfrak{M}^{(\ell)}$ equipped with a uniform edge by Remark 2.3.2 and by Lemma 2.3.3, using the function φ_ℓ defined by

$$\forall \mathbf{m}_\square \in \mathcal{M}_\ell^{(1)}, \quad \varphi_\ell(\mathbf{m}_\square) = \mathbb{E} \left(\left| \frac{d_{\text{fpp}}^\dagger(f_r, \square)}{\log \ell} - \frac{1}{\pi^2 p_{\mathbf{q}}} \right| \wedge 1 \right),$$

where the expectation applies to the exponential weights defining the fpp distance. \square

2.4.2 The dual graph distance to a uniform random edge in $\mathfrak{M}^{(\ell)}$

The height process along the filled-in exploration using the peeling by layers algorithm of a map under $\mathbb{P}_\ell^{(1)}$ is denoted by H_ℓ . We recall that $H_\ell(n)$ is the smallest height of an edge in $\partial \bar{\mathbf{e}}_n$. From Subsection 2.3.3, we know that the dual graph distance from the root face f_r to the target face \square under $\mathbb{P}_\ell^{(1)}$ is $d_{\text{gr}}^\dagger(f_r, \square) = H_\ell(\tau_{-\ell} - 1) + 1$.

Now we can state and prove the main result of this section, which gives the asymptotic behavior of the dual graph distance from the root face to the target face under $\mathbb{P}_\ell^{(1)}$, and brings us closer to the second convergence of Theorem 2.1.1 by the arguments of Subsection 2.3.4.

Proposition 2.4.2. *Under $\mathbb{P}_\ell^{(1)}$,*

$$\frac{d_{\text{gr}}^\dagger(f_r, \square)}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2}.$$

As a consequence, if $E^{(\ell)}$ is a uniform random edge of $\mathfrak{M}^{(\ell)}$, then, in $\mathfrak{M}^{(\ell)}$,

$$\frac{d_{\text{gr}}^\dagger(f_r, E^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2}.$$

Proof. First of all, from Proposition 2.3.1, it is easy to see that for all $n \geq 1$, for any increasing sequence of maps with holes $\mathbf{u}_0 \subset \mathbf{u}_1 \subset \dots \subset \mathbf{u}_{n-1}$,

$$\begin{aligned} \mathbb{P}_\ell^{(1)}(\mathbf{e}_0 = \mathbf{u}_0, \dots, \mathbf{e}_{n-1} = \mathbf{u}_{n-1} | \tau_{-\ell} = n \text{ and } P_\ell(n-1) = |\partial \mathbf{u}_{n-1}|/2) \\ = \mathbb{P}_\infty^{(1)}(\mathbf{e}_0 = \mathbf{u}_0, \dots, \mathbf{e}_{n-1} = \mathbf{u}_{n-1} | P_\infty(n-1) = |\partial \mathbf{u}_{n-1}|/2). \end{aligned}$$

As a result, if $F : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous bounded function,

$$\mathbb{E}_\ell^{(1)} F \left(\frac{H_\ell(\tau_{-\ell} - 1)}{(\log \ell)^2} \right) = \sum_{n, k \geq 1} \mathbb{E}_\infty^{(1)} \left(F \left(\frac{H_\infty(\tau_{-\ell} - 1)}{(\log \ell)^2} \right) \middle| P_\infty(n-1) = k \right) \mathbb{P}_\ell^{(1)}(\tau_{-\ell} = n, P_\ell(n-1) = k)$$

We take $F : x \mapsto |x - 1/(2\pi^2)| \wedge 1$. Let $\varepsilon > 0$. Let $\delta > 0$ be small enough so that for all $\ell \geq 1$, under $\mathbb{P}_\ell^{(1)}$, we have $\tau_{-\ell}/\ell \in [\delta, 1/\delta]$ and $P_\ell(\tau_{-\ell} - 1)/\ell \in [\delta, 1/\delta]$ with probability at least $1 - \varepsilon$. Such a δ exists thanks to Proposition 2.2.7. Then, by Proposition 2.2.7 and Lemma 2.2.3, uniformly on $s, u \in [\delta, 1/\delta]$ as $\ell \rightarrow \infty$,

$$\begin{aligned} \frac{\mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell s \rfloor \text{ and } P_\ell(\lfloor \ell s \rfloor - 1) = \lfloor \ell u \rfloor)}{\mathbb{P}_\infty^{(1)}(P_\infty(\lfloor \ell s \rfloor - 1) = \lfloor \ell u \rfloor)} &= \frac{\mathbb{P}_\ell^{(1)}(\tau_{-\ell} = \lfloor \ell s \rfloor \text{ and } P_\ell(\lfloor \ell s \rfloor - 1) = \lfloor \ell u \rfloor)}{(h^\uparrow(\lfloor \ell u \rfloor)/h^\uparrow(1)) \mathbb{P}_1(S_1, \dots, S_{\lfloor \ell s \rfloor - 1} \geq 1 \text{ and } S_{\lfloor \ell s \rfloor - 1} = \lfloor \ell u \rfloor)} \\ &\sim \frac{\pi p_{\mathbf{q}}}{\ell \sqrt{u(1+u)^2}}, \end{aligned}$$

where we used that $h^\uparrow(\ell) \sim 2\sqrt{\ell/\pi}$. In particular, the above ratio is uniformly bounded in $\ell \geq 1$ and $u, s \in [\delta, 1/\delta]$ by a constant $C(\delta)$ times $1/\ell$. Thus

$$\begin{aligned} \mathbb{E}_\ell^{(1)} F \left(\frac{H_\ell(\tau_{-\ell} - 1)}{(\log \ell)^2} \right) &\leq \varepsilon + C(\delta) \ell^{-1} \int_\delta^{1/\delta} ds \int_\delta^{1/\delta} du \ell^2 \mathbb{E}_\infty^{(1)} \left(F \left(\frac{H_\infty(\lfloor \ell s \rfloor - 1)}{(\log \ell)^2} \right) \mathbf{1}_{P_\infty(\lfloor \ell s \rfloor - 1) = \lfloor \ell u \rfloor} \right) \\ &\leq \varepsilon + C(\delta) \int_\delta^{1/\delta} ds \mathbb{E}_\infty^{(1)} \left(F \left(\frac{H_\infty(\lfloor \ell s \rfloor - 1)}{(\log \ell)^2} \right) \right). \end{aligned}$$

The last integral tends to 0 as $\ell \rightarrow +\infty$ because of Proposition 4 from [BCM18], stating that under $\mathbb{P}_\infty^{(1)}$,

$$\frac{H_\infty(\ell)}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2}.$$

This entails the first convergence.

The second convergence is a straightforward consequence of the first one due to Remark 2.3.2 and to Lemma 2.3.3, using the function φ_ℓ defined by

$$\forall \mathbf{m}_\square \in \mathcal{M}_\ell^{(1)}, \quad \varphi_\ell(\mathbf{m}_\square) = \left| \frac{d_{\text{gr}}^\dagger(f_r, \square)}{(\log \ell)^2} - \frac{1}{2\pi^2} \right| \wedge 1.$$

This concludes the proof. \square

2.4.3 Replacing the uniform random edge by a vertex or a face

Let us explain how the results of the preceding subsections also give us the distance from the root face to a uniform random vertex or to a uniform random face, hence proving Theorem 2.1.1 at the end of this subsection. This subsection relies on classical arguments involving the peeling exploration and bijections with trees which may be skipped by the reader.

We first focus on the case of a uniform random vertex. Notice that $|\text{Vertices}(\mathfrak{M}^{(\ell)})| = \sum_{\vec{e}} \frac{1}{\deg v(\vec{e})}$, where the sum is over all the oriented edges \vec{e} of $\mathfrak{M}^{(\ell)}$. To get a uniform random vertex on $\mathfrak{M}^{(\ell)}$, conditionally on $\mathfrak{M}^{(\ell)}$, we pick a random oriented edge \vec{E} of $\mathfrak{M}^{(\ell)}$ of law

$$\forall \vec{e} \in \text{Edges}(\mathfrak{M}^{(\ell)}), \quad \mathbb{P}(\vec{E} = \vec{e} \mid \mathfrak{M}^{(\ell)}) := \frac{1}{\deg v(\vec{e})} \frac{1}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|}, \quad (2.4.4)$$

where $v(\vec{e})$ is the vertex from which starts \vec{e} . Then for all vertex v of $\mathfrak{M}^{(\ell)}$

$$\mathbb{P}(v(\vec{E}) = v \mid \mathfrak{M}^{(\ell)}) = \frac{1}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|}, \quad (2.4.5)$$

i.e. $V := v(\vec{E})$ is a random vertex taken uniformly on $\mathfrak{M}^{(\ell)}$.

Let us control the degree of the root vertex v_r (from which starts the root edge) under $\mathbb{P}_\ell^{(1)}$. The lemma below is an extension of Lemma 15.6 from [Cur23].

Lemma 2.4.3. *There exists $c \in (0, 1)$ such that for all $\ell, j \geq 1$,*

$$\mathbb{P}_\ell^{(1)}(\deg v_r \geq j) \leq c^{j-2}.$$

Proof. As in [Cur23] we use the peeling exploration “peeling around the root vertex”, defined as follows: as long as the root vertex belongs to the boundary of the hole $\partial \bar{\mathfrak{c}}_n$, we peel the edge on $\partial \bar{\mathfrak{c}}_n$ adjacent to the left of v_r until it becomes an interior vertex. More precisely, after an event of type G , we fill-in the hole which does not have v_r on its boundary and we continue the exploration. Thus, the hole may not contain the target after some time. We write G_{k_1, k_2} the event of an identification of two edges on the boundary of a hole of perimeter $2p = 2k_1 + 2k_2 + 2$ which does not contain the target, giving rise to a hole of perimeter $2k_1$ on the left of the peeled edge (which will be filled-in) and a hole of perimeter $2k_2$ on the right. This event has the transition probability $W^{(k_1)}W^{(k_2)}/W^{(p)}$. The exploration ends when v_r becomes an inner vertex, i.e. after an event of type $G_{*,0}$ or after an event of type $G_{k,0}$ for some $k \geq 0$.

At one step of this peeling exploration, let $2p$ be the perimeter of the hole whose boundary contains v_r . We distinguish the two possible cases:

- Either v_r is in the boundary of the hole containing the target face. In that case, the exploration stops if and only if the event $G_{*,0}$ occurs. And this happens with probability $\frac{1}{2}\nu(-1)\frac{h_\ell^\dagger(p-1)}{h_\ell^\dagger(p)}$ by Proposition 2.3.1 (and necessarily $p \geq 2$ so that the hole which contains the target has positive boundary length).
- Either v_r is in the boundary of a hole which does not contain the target face. In this case, the exploration stops when the event $G_{p-1,0}$ occurs. This event happens with probability $W^{(p-1)}W^{(0)}/W^{(p)}$.

But one can see by (2.1.1), (2.2.1) and since $h^\downarrow(\ell) \sim 1/\sqrt{\pi\ell}$ when $\ell \rightarrow \infty$ that

$$\frac{W^{(0)}W^{(0)}}{W^{(1)}} \wedge \inf_{\substack{p \geq 2 \\ \ell \geq 1}} \left(\frac{1}{2} \nu(-1) \frac{h_\ell^\downarrow(p-1)}{h_\ell^\downarrow(p)} \wedge \frac{W^{(p-1)}W^{(0)}}{W^{(p)}} \right) > 0.$$

Besides, in the first case, when $p = 1$ we automatically have $p \geq 1$ at the next step by an event of type C_k , and with positive probability we will have $p \geq 2$ at the next step. Moreover, at each step of the exploration, $\deg(v_r)$ only increases by zero or one, hence the statement of the lemma. \square

We now recall a lemma which gathers some well known consequences of the Bouttier-Di Francesco-Guitter bijection (from [BDFG04]) and Janson & Stefánsson's trick (from [JS15] Section 3), together with the Łukasiewicz path.

Lemma 2.4.4. ([BDFG04], [JS15]) *The measure μ on $\mathbb{Z}_{\geq -1}$ defined by*

$$\mu(-1) = \frac{4}{c_{\mathbf{q}}} \quad \text{and} \quad \mu(k) = q_{k+1} \binom{2k+1}{k+1} (c_{\mathbf{q}}/4)^k \text{ for all } k \geq 0.$$

is a probability measure. Moreover, to a map \mathbf{m}_\bullet under $\mathbb{P}_0^{(\ell)}$ we can associate a random walk $(Y_n)_{n \geq 0}$ of step distribution μ starting at zero and stopped when it attains $-\ell$ such that if we denote by $\tau_{-\ell}(Y)$ the time at which Y reaches $-\ell$, then

$$\begin{cases} \#\{n \in \llbracket 0, \tau_{-\ell}(Y) - 1 \rrbracket; Y_{n+1} - Y_n = -1\} + 1 = \#\text{Vertices}(\mathbf{m}_\bullet) \\ \#\{n \in \llbracket 0, \tau_{-\ell}(Y) - 1 \rrbracket; Y_{n+1} - Y_n \neq -1\} + 1 = \#\text{Faces}(\mathbf{m}_\bullet). \end{cases}$$

As a consequence $\tau_{-\ell}(Y) = \#\text{Edges}(\mathbf{m}_\bullet)$.

Using the previous lemmas, we are able to control the degree of a random uniform vertex in the map $\mathfrak{M}^{(\ell)}$. The degree of a uniform random vertex of the map $\mathfrak{M}^{(\ell)}$ is stochastically dominated by a random variable whose law does not depend on ℓ :

Lemma 2.4.5. *There exists $C > 0$ and $c \in (0, 1)$ such that for all $k, \ell \geq 1$, if V is a uniform random vertex of $\mathfrak{M}^{(\ell)}$, then*

$$\mathbb{P}(\deg(V) = k) \leq Cc^k.$$

Proof. Let $\ell \geq 1$ and $\varepsilon > 0$. Let $K > 0$ whose value will be chosen later. For $k \geq 1$,

$$\mathbb{P}(\deg(V) = k) = \mathbb{P}(\deg(V) = k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| \geq |\text{Edges}(\mathfrak{M}^{(\ell)})|) \quad (2.4.6)$$

$$+ \mathbb{P}(\deg(V) = k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| < |\text{Edges}(\mathfrak{M}^{(\ell)})|) \quad (2.4.7)$$

For the first term (2.4.6), if we denote by \vec{E}' a uniform oriented edge of $\mathfrak{M}^{(\ell)}$, one can see that for all $k \geq 1$,

$$\begin{aligned} & \mathbb{P}(\deg(V) = k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| \geq |\text{Edges}(\mathfrak{M}^{(\ell)})|) \\ &= \mathbb{E} \left(\mathbf{1}_{K|\text{Vertices}(\mathfrak{M}^{(\ell)})| \geq |\text{Edges}(\mathfrak{M}^{(\ell)})|} \sum_{\vec{e}} \frac{1}{\deg v(\vec{e})} \frac{1}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|} \mathbf{1}_{\deg(v(\vec{e}))=k} \right) \text{ by (2.4.5)} \\ &= \mathbb{E}^{(\ell)} \left(\mathbf{1}_{K|\text{Vertices}(\mathbf{m})| \geq |\text{Edges}(\mathbf{m})|} \frac{1}{k|\text{Vertices}(\mathbf{m})|} \#\{\vec{e}; \deg v(\vec{e}) = k\} \right) \\ &\leq \mathbb{E}^{(\ell)} \left(\mathbf{1}_{K|\text{Vertices}(\mathbf{m})| \geq |\text{Edges}(\mathbf{m})|} \frac{2|\text{Edges}(\mathbf{m})|}{k|\text{Vertices}(\mathbf{m})|} \mathbb{P}(\deg(v(\vec{E}')) = k | \mathfrak{M}^{(\ell)} = \mathbf{m}) \right) \\ &\leq \mathbb{E}^{(\ell)} \left(2K\mathbb{P}(\deg(v(\vec{E}')) = k | \mathfrak{M}^{(\ell)} = \mathbf{m}) \right) \\ &= 2K\mathbb{P}_\ell^{(1)}(\deg(v_r) = k) \\ &\leq 2Kc^{k-2} \text{ by Lemma 2.4.3,} \end{aligned}$$

with $C > 0$ and $c \in (0, 1)$ independent of ℓ . We now focus on the second term (2.4.7).

$$\begin{aligned} & \mathbb{P}\left(\deg(V) = k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| < |\text{Edges}(\mathfrak{M}^{(\ell)})|\right) \\ & \leq \mathbb{P}\left(|\text{Edges}(\mathfrak{M}^{(\ell)})| \geq k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| < |\text{Edges}(\mathfrak{M}^{(\ell)})|\right). \end{aligned}$$

Let Y be the μ -random walk stopped when it attains $-\ell$ associated with \mathbf{m}_\bullet under $\mathbb{P}_0^{(\ell)}$ by Lemma 2.4.4. We also know that $\tau_{-\ell}(Y) = |\text{Edges}(\mathbf{m})|$ and $D := \#\{i \in \llbracket 0, \tau_{-\ell}(Y) - 1 \rrbracket; Y_{i+1} - Y_i = -1\} + 1 = |\text{Vertices}(\mathbf{m})|$. Thus, by biasing by the number of vertices, we obtain that

$$\begin{aligned} & \mathbb{P}\left(|\text{Edges}(\mathfrak{M}^{(\ell)})| \geq k \text{ and } K|\text{Vertices}(\mathfrak{M}^{(\ell)})| < |\text{Edges}(\mathfrak{M}^{(\ell)})|\right) \\ & = \frac{1}{W^{(\ell)}} \sum_{\mathbf{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}) \mathbf{1}_{|\text{Edges}(\mathbf{m})| \geq k \text{ and } K|\text{Vertices}(\mathbf{m})| < |\text{Edges}(\mathbf{m})|} \\ & = \frac{W_0^{(\ell)}}{W^{(\ell)}} \mathbb{E}_0^{(\ell)} \left(\frac{1}{|\text{Vertices}(\mathbf{m}_\bullet)|} \mathbf{1}_{|\text{Edges}(\mathbf{m}_\bullet)| \geq k \text{ and } K|\text{Vertices}(\mathbf{m}_\bullet)| < |\text{Edges}(\mathbf{m}_\bullet)|} \right) \\ & = \frac{W_0^{(\ell)}}{W^{(\ell)}} \mathbb{E} \left(\frac{1}{D} \mathbf{1}_{KD \leq \tau_{-\ell}(Y) \text{ and } \tau_{-\ell}(Y) \geq k} \right) \\ & = \frac{W_0^{(\ell)}}{W^{(\ell)}} \mathbb{E} \left(\frac{1}{D} \mathbb{P}(KD \leq \tau_{-\ell}(Y) \text{ and } \tau_{-\ell}(Y) \geq k | D) \right). \end{aligned}$$

Let us upper bound the conditional probability inside the expectation. One can note that the random walk Y can be built by first sampling the non-zero steps and then inserting between these steps a geometric number (starting at zero, of parameter $1 - \mu(0)$) of zero steps (when $\mu(0) = 0$, we do not insert any such zero step). Moreover the number of positive steps before $\tau_{-\ell}(Y)$ is smaller than D . Thus one can write

$$\tau_{-\ell}(Y) \leq 2D + \sum_{i=1}^{2D} \mathcal{G}_i,$$

where the \mathcal{G}_i are geometric i.i.d. random variables of parameter $1 - \mu(0)$, where by convention $\mathcal{G}_i = 0$ when $\mu(0) = 0$. Let $\beta = (1 - \mu(0))/(2(3 - \mu(0)))$ so that $1 - 2\beta = 4\beta/(1 - \mu(0))$. We will distinguish whether $D \geq \beta k$ or not. On the one hand, if $D \leq \beta k$, then

$$\begin{aligned} \mathbb{P}(KD \leq \tau_{-\ell}(Y) \text{ and } \tau_{-\ell}(Y) \geq k | D) \mathbf{1}_{D \leq \beta k} & \leq \mathbb{P}(\tau_{-\ell}(Y) \geq k | D) \mathbf{1}_{D \leq \beta k} \\ & \leq \mathbb{P}\left(2D + \sum_{i=1}^{2D} \mathcal{G}_i \geq k \mid D\right) \mathbf{1}_{D \leq \beta k} \\ & \leq \mathbb{P}\left(\sum_{i=1}^{2\beta k} \mathcal{G}_i \geq (1 - 2\beta)k\right) \\ & \leq C'(c')^k, \end{aligned}$$

for $C' > 0$, $c' \in (0, 1)$ which do not depend on ℓ , where the last inequality comes from the large deviations for sums of i.i.d. geometric variables. On the other hand, when $D \geq \beta k$, by choosing $K \geq 2 + 4/(1 - \mu(0))$,

$$\begin{aligned} \mathbb{P}(KD \leq \tau_{-\ell}(Y) \text{ and } \tau_{-\ell}(Y) \geq k | D) \mathbf{1}_{D \geq \beta k} & = \mathbb{P}(KD \leq \tau_{-\ell}(Y) | D) \mathbf{1}_{D \geq \beta k} \\ & \leq \mathbb{P}\left((K - 2)D \leq \sum_{i=1}^{2D} \mathcal{G}_i \mid D\right) \mathbf{1}_{D \geq \beta k} \\ & \leq C'(c')^D \mathbf{1}_{D \geq \beta k} \leq C'(c')^{\beta k}, \end{aligned}$$

by the same large deviations. This ends the proof since $\mathbb{E}(1/D) = \mathbb{E}_0^{(\ell)}(1/|\text{Vertices}(\mathbf{m})|) = W^{(\ell)}/W_0^{(\ell)}$ and thus

$$\frac{W_0^{(\ell)}}{W^{(\ell)}} \mathbb{E} \left(\frac{1}{D} \mathbb{P}(KD \leq \tau_{-\ell}(Y) \text{ and } \tau_{-\ell}(Y) \geq k | D) \right) \leq C'(c')^{\beta k},$$

hence the upper bound of the statement. \square

If v is a vertex of \mathfrak{m} , and if d^\dagger is a distance on \mathfrak{m}^\dagger , we define $d^\dagger(f_r, v)$ as the smallest distance from f_r to a face f next to v .

Proposition 2.4.6. *If for all $\ell \geq 1$, $V^{(\ell)}$ is a random uniform vertex of $\mathfrak{M}^{(\ell)}$, then*

$$\frac{d_{\text{gr}}^\dagger(f_r, V^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2} \quad \text{and} \quad \frac{d_{\text{fpp}}^\dagger(f_r, V^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}}.$$

Proof. We do the proof for the dual graph distance (the same ideas work for the fpp distance using Proposition 2.4.1). First of all, we can work with $d_{\text{gr}}^\dagger(f_r, \vec{E})$ with \vec{E} defined as above by (2.4.4) since $d_{\text{gr}}^\dagger(f_r, \vec{E}) - \deg(V^{(\ell)}) \leq d_{\text{gr}}^\dagger(f_r, V^{(\ell)}) \leq d_{\text{gr}}^\dagger(f_r, \vec{E})$ and since by Lemma 2.4.5 $\deg V^{(\ell)}$ is stochastically dominated by a random variable which does not depend on ℓ . Let $\varepsilon, \delta > 0$. Let \vec{E}' be a uniform random oriented edge. By Proposition 2.4.2, for ℓ large, with probability at least $1 - \delta$,

$$\begin{aligned} \mathbb{P}\left(\frac{d_{\text{gr}}^\dagger(f_r, \vec{E})}{(\log \ell)^2} \in \left[\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon\right] \middle| \mathfrak{M}^{(\ell)}\right) &= \sum_{\vec{e}} \frac{1}{\deg v(\vec{e})} \frac{1}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|} \mathbf{1}_{\frac{d_{\text{gr}}^\dagger(f_r, \vec{E})}{(\log \ell)^2} \in [\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon]} \\ &= \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|} \mathbb{E}\left(\frac{1}{\deg v(\vec{E}')} \mathbf{1}_{\frac{d_{\text{gr}}^\dagger(f_r, \vec{E}')}{(\log \ell)^2} \in [\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon]} \middle| \mathfrak{M}^{(\ell)}\right) \\ &\geq \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|} \left(\mathbb{E}\left(\frac{1}{\deg v(\vec{E}')}\right) - \delta\right) \\ &= 1 - \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Vertices}(\mathfrak{M}^{(\ell)})|} \delta \\ &\geq 1 - C\delta, \end{aligned}$$

where $C > 0$ is a constant that does not depend on ℓ and δ . The last inequality comes from Lemma 2.4.4 and the law of large numbers. \square

The case of the distance from the root face to a uniform random face of $\mathfrak{M}^{(\ell)}$ is even simpler and relies on similar ideas.

Proof of Theorem 2.1.1. Let $\ell \geq 1$. Let F be a uniform random face on $\mathfrak{M}^{(\ell)}$ conditionally on $\mathfrak{M}^{(\ell)}$. Conditionally on F and $\mathfrak{M}^{(\ell)}$, let \vec{E} be uniform random oriented edge such that the face on its right is F . Then by definition of the distance to an edge,

$$d_{\text{gr}}^\dagger(f_r, \vec{E}) \leq d_{\text{gr}}^\dagger(f_r, F) \leq d_{\text{gr}}^\dagger(f_r, \vec{E}) + 1 \quad \text{and} \quad d_{\text{fpp}}^\dagger(f_r, \vec{E}) \leq d_{\text{fpp}}^\dagger(f_r, F) \leq d_{\text{fpp}}^\dagger(f_r, \vec{E}) + \mathcal{E},$$

where \mathcal{E} is an exponential random variable of parameter 1. Thus it is enough to show that

$$\frac{d_{\text{gr}}^\dagger(f_r, \vec{E})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2} \quad \text{and} \quad \frac{d_{\text{fpp}}^\dagger(f_r, \vec{E})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{2}{\pi^2 p_{\mathbf{q}}}.$$

We now focus on the dual graph distance. For the fpp distance, the same ideas apply using Proposition 2.4.1. If \vec{e} is an oriented edge, we write $f(\vec{e})$ the face on the right of \vec{e} . Conditionally on $\mathfrak{M}^{(\ell)}$, let \vec{E}' be a uniform oriented edge on $\mathfrak{M}^{(\ell)}$. Let $\varepsilon, \delta > 0$. Then by Proposition 2.4.2, for ℓ large, with probability at least $1 - \delta$,

$$\begin{aligned} \mathbb{P}\left(\frac{d_{\text{gr}}^\dagger(f_r, \vec{E})}{(\log \ell)^2} \in \left[\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon\right] \middle| \mathfrak{M}^{(\ell)}\right) &= \sum_{\vec{e}} \frac{1}{\deg f(\vec{e})} \frac{1}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \mathbf{1}_{\frac{d_{\text{gr}}^\dagger(f_r, \vec{E})}{(\log \ell)^2} \in [\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon]} \\ &= \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \mathbb{E}\left(\frac{1}{\deg f(\vec{E}')} \mathbf{1}_{\frac{d_{\text{gr}}^\dagger(f_r, \vec{E}')}{(\log \ell)^2} \in [\frac{1}{2\pi^2} - \varepsilon, \frac{1}{2\pi^2} + \varepsilon]} \middle| \mathfrak{M}^{(\ell)}\right) \\ &\geq \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \left(\mathbb{E}\left(\frac{1}{\deg f(\vec{E}')}\right) - \delta\right) \\ &= \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \frac{1}{2|\text{Edges}(\mathfrak{M}^{(\ell)})|} \left(\sum_{\vec{e}} \frac{1}{\deg f(\vec{e})}\right) - \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \delta. \end{aligned}$$

But by Lemma 2.4.4 and the law of large numbers, with probability at least $1 - \delta$ for ℓ large,

$$\frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \frac{1}{2|\text{Edges}(\mathfrak{M}^{(\ell)})|} \left(\sum_{\vec{e}} \frac{1}{\deg f(\vec{e})} \right) - \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \delta = 1 - \frac{2|\text{Edges}(\mathfrak{M}^{(\ell)})|}{|\text{Faces}(\mathfrak{M}^{(\ell)})|} \delta \geq 1 - C\delta,$$

where $C > 0$ is a constant that does not depend on ℓ (nor on δ, ε). \square

2.5 Distance between two uniform random faces

This section is devoted to the proof of the following corollary, which establishes the scaling limit in probability of the distance between two uniform random faces.

Corollary 2.5.1. *For all $\ell \geq 1$, let $F_1^{(\ell)}$ and $F_2^{(\ell)}$ be two independent uniform random faces in $\mathfrak{M}^{(\ell)}$. Then*

$$\frac{d_{\text{fpp}}^\dagger(F_1^{(\ell)}, F_2^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{2}{\pi^2 p_{\mathbf{q}}} \quad \text{and} \quad \frac{d_{\text{gr}}^\dagger(F_1^{(\ell)}, F_2^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2}.$$

Its statement can be extended to the case of uniform random edges or uniform random vertices, see Theorem 2.5.3 and Corollary 2.5.6. Actually, we will first prove the result for uniform random edges. The main idea will be to show that geodesics from the root to two uniform random edges “split near the root” with high probability. Hence, the map $\mathfrak{M}^{(\ell)}$ looks like a star with many branches as in Figure 2.1.

2.5.1 Two random edges are in “different branches” with high probability

We prove here that the branches of two uniform random edges split near the root face. More precisely, we would like to prove that the filled-in exploration targeted at the first random uniform edge swallows the second edge at a time relatively far from the end of the exploration with high probability. However, we do not know the law of the peeling exploration of $\mathfrak{M}^{(\ell)}$ targeted at a random uniform edge, but Lemma 2.3.3 gives us a way to transfer results about maps with target under $\mathbb{P}_1^{(\ell)}$ to $\mathfrak{M}^{(\ell)}$ equipped with a uniform edge. With this in mind, let $\mathfrak{M}_1^{(\ell)}$ be a random map with a target 2-face of law $\mathbb{P}_1^{(\ell)}$. The target 2-face \square will play the role of the first uniform edge. Let $E^{(\ell)}$ be a uniform random edge of $\mathfrak{M}_1^{(\ell)}$ (where \square is considered as one edge), which will play the role of the second uniform edge. Let $\tau_{-1}^{(\ell)}$ be the time at which the perimeter process $P_1^{(\ell)}$ associated with the filled-in exploration $(\bar{\mathbf{e}}_n^{(\ell)})_{n \geq 0}$ of $\mathfrak{M}_1^{(\ell)}$ dies at -1 , i.e. the duration of that filled-in exploration. Recall from Proposition 2.2.4 that $\tau_{-1}^{(\ell)}/\ell$ converges in law.

Lemma 2.5.2. *We have*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\ell \rightarrow \infty} \mathbb{P}(\square \text{ and } E^{(\ell)} \text{ lie in the same hole until time } (\tau_{-1}^{(\ell)} - \varepsilon \ell) \vee 0) = 0.$$

Proof. Let us consider the filled-in exploration $(\bar{\mathbf{e}}_n^{(\ell)})_{n \geq 0}$. The idea is to show that \square and $E^{(\ell)}$ are in different holes at a splitting event happening “not too late”. By Proposition 2.2.4, we know that

$$\left(\frac{P_1^{(\ell)}(\ell t)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (\Upsilon^\downarrow(p_{\mathbf{q}} t))_{t \geq 0}, \quad (2.5.1)$$

where Υ^\downarrow is the Cauchy process Υ starting from 1 and conditioned to die continuously at 0. Let $\zeta := \inf\{t \geq 0; \Upsilon^\downarrow(t) = 0\}$ be the lifetime of Υ^\downarrow .

First, we note that the number of negative jumps of size larger than half of Υ^\downarrow is a.s. infinite. More precisely

$$N := \#\left\{ t \in [0, \zeta[; \Upsilon^\downarrow(t) < \frac{\Upsilon^\downarrow(t-)}{2} \right\} = \infty \text{ a.s.}$$

Indeed, by Proposition 5.2 of [BBCK18] (in the case $\theta = 1, \rho = 1/2, \hat{\beta} = 1, \gamma = 1/2, \hat{\gamma} = 1/2$) up to a time change, Υ^\downarrow is the exponential of a Lévy process. Let Λ be its Lévy measure. The Laplace

exponent of that Lévy process is $\Phi^-(q) = \kappa_1(q - 3/2) = (q - 1/2) \tan(\pi q)$ by Section 4.3 of [BBCK18] with $\omega_- = 3/2$ for $q \in (-1/2, 3/2)$. One can check that Λ is the image measure of $\frac{1}{\pi\sqrt{x(1-x)^2}} dx$ by the mapping $x \mapsto \log x$. Then N is a Poisson random variable of expectation

$$\mathbb{E}N = \int_0^\infty dt \int_{-\infty}^{-\log 2} \Lambda(dy) = \infty \times \int_0^{1/2} \frac{1}{\pi\sqrt{x(1-x)^2}} dx = \infty,$$

hence a.s. $N = \infty$.

Moreover, by (2.5.1), the number $N_\varepsilon^{(\ell)}$ of negative jumps of size larger than half of $P_1^{(\ell)}(\cdot)$ until time $\tau_{-1}^{(\ell)} - \varepsilon\ell$ converges in distribution towards the number N_ε of negative jumps of Υ^\downarrow of size larger than half of $\Upsilon^\downarrow(\cdot)$ until time $\zeta - \varepsilon p_{\mathbf{q}}$ as $\ell \rightarrow \infty$, which further goes a.s. to $N = \infty$ as $\varepsilon \rightarrow 0$.

Now we fix such a step: let n be such that $P_1^{(\ell)}(n+1) < P_1^{(\ell)}(n)/2$. We write $k_1 = P_1^{(\ell)}(n+1)$ and $k_2 = P_1^{(\ell)}(n) - P_1^{(\ell)}(n+1) - 1$ the half-perimeters of the two holes created at this step. Let us show that the probability that the uniform random edge $E^{(\ell)}$ lies in the filled-in hole \mathfrak{h} , which is the hole that does not contain \square , conditionally on not being an internal edge of the explored region \mathfrak{e} obtained from $\bar{\mathfrak{e}}_n^{(\ell)}$ after creating the two holes but before filling in the hole \mathfrak{h} and conditionally on \mathfrak{e} , is greater than $1/3$ as soon as k_2 is sufficiently large. This condition is verified if $P_1^{(\ell)}(n)$ is large enough.

Indeed, by Proposition 2.3.1, conditionally on \mathfrak{e} , the maps filling the two holes are independent \mathbf{q} -Boltzmann planar maps of corresponding perimeter, and the one which contains \square has a target face of degree 2. Thus if we write for all $\ell \geq 1, p \geq 0, n \geq 1$,

$$W_p^{(\ell)}[n] = \sum_{\substack{\mathfrak{m}_\square \in \mathcal{M}_p^{(\ell)} \\ |\text{Edges}(\mathfrak{m})|=n}} w_{\mathbf{q}}(\mathfrak{m}_\square) \quad \text{and} \quad W^{(\ell)}[n] = \sum_{\substack{\mathfrak{m} \in \mathcal{M}^{(\ell)} \\ |\text{Edges}(\mathfrak{m})|=n}} w_{\mathbf{q}}(\mathfrak{m}),$$

then

$$\begin{aligned} \mathbb{P}(E^{(\ell)} \in \mathfrak{h} | \mathfrak{e} \text{ and } E^{(\ell)} \notin \mathfrak{e}) &= \sum_{\mathfrak{u}_1 \in \mathcal{M}_1^{(k_1)}, \mathfrak{u}_2 \in \mathcal{M}^{(k_2)}} \frac{|\text{Edges}(\mathfrak{u}_2)|}{|\text{Edges}(\mathfrak{u}_1)| - 1 + |\text{Edges}(\mathfrak{u}_2)|} \frac{w_{\mathbf{q}}(\mathfrak{u}_1)}{W_1^{(k_1)}} \frac{w_{\mathbf{q}}(\mathfrak{u}_2)}{W^{(k_2)}} \\ &= \sum_{n_1, n_2 \geq 1} \frac{n_2}{n_1 + n_2 - 1} \frac{W_1^{(k_1)}[n_1]}{W_1^{(k_1)}} \frac{W^{(k_2)}[n_2]}{W^{(k_2)}} \\ &= \sum_{n_1, n_2 \geq 1} \frac{1}{n_1 + n_2 - 1} \frac{W_1^{(k_1)}[n_1]}{W_1^{(k_1)}} \frac{W_1^{(k_2)}[n_2 + 1]}{W_1^{(k_2)}} \frac{W_1^{(k_2)}}{W^{(k_2)}}, \end{aligned}$$

where the last equality holds since by taking into account that zipping the target face gives a distinguished edge, we have $W_1^{(\ell)}[n+1] = nW^{(\ell)}[n]$ for all $n \geq 1$. Yet, it is known that

$$\forall n \geq 1, \quad W_1^{(\ell)}[n] = \frac{\ell}{\ell+1} \binom{2\ell}{\ell} (c_{\mathbf{q}}/4)^{\ell+1} \mathbb{P}(\tau_{-\ell-1}(Y) = n), \quad (2.5.2)$$

where Y is a μ -random walk starting from zero as in Lemma 2.4.4, see for instance Theorem 3.12 in [Cur23]. Thus

$$\begin{aligned} \sum_{n_1, n_2 \geq 1} \frac{1}{n_1 + n_2 - 1} \frac{W_1^{(k_1)}[n_1]}{W_1^{(k_1)}} \frac{W_1^{(k_2)}[n_2 + 1]}{W_1^{(k_2)}} \frac{W_1^{(k_2)}}{W^{(k_2)}} \\ &= \sum_{n_1, n_2 \geq 1} \frac{1}{n_1 + n_2 - 1} \mathbb{P}(\tau_{-k_1-1}(Y) = n_1) \mathbb{P}(\tau_{-k_2-1}(Y) = n_2 + 1) \frac{W_1^{(k_2)}}{W^{(k_2)}} \\ &= \mathbb{E} \frac{1}{\tau_{-k_1-k_2-2}(Y) - 2} \frac{\frac{1}{2} h_1^\downarrow(k_2) c_{\mathbf{q}}^{k_2+1}}{W^{(k_2)}} \text{ by (2.3.1)}. \end{aligned}$$

Furthermore, since we have $W_1^{(\ell)}[n+1] = nW^{(\ell)}[n]$ for all $n \geq 1$ so that using (2.5.2), one can see that for all $k \geq 1$,

$$\mathbb{E} \frac{1}{\tau_{-k-1}(Y) - 1} = \frac{W^{(k)}}{\frac{1}{2} h_1^\downarrow(k) c_{\mathbf{q}}^{k+1}}.$$

The same computation can be found e.g. in Equation (3.11) in [Cur23]. As a consequence, since $\mathbb{E}1/(\tau_{-k-1}(Y) - 2) \sim \mathbb{E}1/(\tau_{-k-1}(Y) - 1)$ when $k \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} \frac{1}{\tau_{-k_1-k_2-2}(Y) - 2} \frac{\frac{1}{2} h_1^\downarrow(k_2) c_{\mathbf{q}}^{k_2+1}}{W^{(k_2)}} &\underset{k_2 \rightarrow \infty}{\sim} \frac{W^{(k_1+k_2+1)}}{\frac{1}{2} h_1^\downarrow(k_1+k_2+1) c_{\mathbf{q}}^{k_1+k_2+2}} \frac{\frac{1}{2} h_1^\downarrow(k_2) c_{\mathbf{q}}^{k_2+1}}{W^{(k_2)}} \\ &\underset{k_2 \rightarrow \infty}{\sim} \frac{\frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{k_1+k_2+2} \frac{1}{(k_1+k_2)^2}}{\frac{1}{2} h_1^\downarrow(k_1+k_2+1) c_{\mathbf{q}}^{k_1+k_2+2}} \frac{\frac{1}{2} h_1^\downarrow(k_2) c_{\mathbf{q}}^{k_2+1}}{\frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{k_2+1} \frac{1}{k_2^2}} \text{ by (2.1.1)} \\ &\underset{k_2 \rightarrow \infty}{\sim} \frac{k_2^{3/2}}{(k_1+k_2)^{3/2}} \geq \frac{1}{2^{3/2}} > \frac{1}{3}, \end{aligned}$$

where we used that $k_1 < k_2 + 1$ by assumption. From the above computations, we obtain that

$$\mathbb{P}(E^{(\ell)} \in \mathfrak{h} \mid \mathfrak{e} \text{ and } E^{(\ell)} \notin \mathfrak{e}) > \frac{1}{3}$$

if $P_1^{(\ell)}(n)$ is large enough. Therefore, the probability that \square and $E^{(\ell)}$ lie in the same hole in the exploration until the k -th negative jump of $P_1^{(\ell)}$ larger than half of the perimeter is less than $(2/3)^k$ for large ℓ . We also use the fact that at the time of this k -th jump the perimeter of the distinguished hole is of order ℓ thanks to (2.5.1). Thus,

$$\limsup_{\ell \rightarrow \infty} \mathbb{P}(\square \text{ and } E^{(\ell)} \text{ lie in the same hole until time } (\zeta - \varepsilon)\ell) \leq \mathbb{E} \left(\left(\frac{2}{3} \right)^{N_\varepsilon} \right).$$

One concludes by dominated convergence. \square

2.5.2 Distance between two uniform random edges, vertices or faces

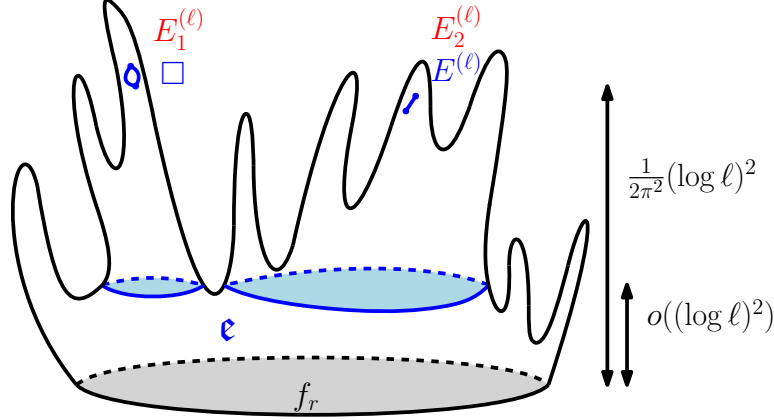


Figure 2.5 – Illustration of the proof of Theorem 2.5.3: the map $\mathfrak{M}^{(\ell)}$ (or $\mathfrak{M}_1^{(\ell)}$) is represented as a cactus where the height corresponds to the distance to the root face f_r . The blue holes are the two holes of the explored region \mathfrak{e} . The target 2-face \square and the uniform random edge $E^{(\ell)}$ on $\mathfrak{M}_1^{(\ell)}$, in blue, correspond respectively to the random uniform edges $E_1^{(\ell)}$ and $E_2^{(\ell)}$ in $\mathfrak{M}^{(\ell)}$, in red.

Theorem 2.5.3. *If $\mathfrak{M}^{(\ell)}$ follows the law $\mathbb{P}^{(\ell)}$ and $E_1^{(\ell)}, E_2^{(\ell)}$ are two uniform random edges of $\mathfrak{M}^{(\ell)}$, then*

$$\frac{d_{\text{gr}}^\dagger(E_1^{(\ell)}, E_2^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2} \quad \text{and} \quad \frac{d_{\text{fpp}}^\dagger(E_1^{(\ell)}, E_2^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{2}{\pi^2 p_{\mathbf{q}}}$$

The general idea of the proof is given in Figure 2.5. The key step is an intermediate result for the map $\mathfrak{M}_1^{(\ell)}$ of law $\mathbb{P}_1^{(\ell)}$.

Lemma 2.5.4. *If $\mathfrak{M}_1^{(\ell)}$ is a random map with target of law $\mathbb{P}_1^{(\ell)}$ and $E^{(\ell)}$ is a random uniform edge on $\mathfrak{M}_1^{(\ell)}$ (where the target 2-face has been zipped and is considered as an edge), then*

$$\frac{d_{\text{gr}}^\dagger(\square, E^{(\ell)}) - d_{\text{gr}}^\dagger(f_r, E^{(\ell)})}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{2\pi^2} \quad \text{and} \quad \frac{d_{\text{fpp}}^\dagger(\square, E^{(\ell)}) - d_{\text{fpp}}^\dagger(f_r, E^{(\ell)})}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2 p_{\mathbf{q}}}$$

Proof. We consider the filled-in peeling exploration of $\mathfrak{M}_1^{(\ell)}$ using the algorithm $\mathcal{A}_{\text{layers}}$ for the dual graph distance and the algorithm $\mathcal{A}_{\text{uniform}}$ for the fpp distance. Let n be the time at which during an event of type G , \square and $E^{(\ell)}$ are in two distinct holes \mathfrak{h}_1 and \mathfrak{h}_2 (when it exists). By Lemma 2.5.2, we know that for ε small, $n \leq \tau_{-1}^{(\ell)} - \varepsilon \ell$ with large probability. We also denote by \mathfrak{e} the sub-map obtained after identifying the edges but before filling-in the hole containing $E^{(\ell)}$.

We work with the dual graph distance, and thus with the peeling by layers algorithm. We point out that a path from \square to $E^{(\ell)}$ must go through \mathfrak{e} (see Figure 2.5), so that

$$d_{\text{gr}}^\dagger(\square, E^{(\ell)}) \geq d_{\text{gr}}^\dagger(\partial \mathfrak{e}, \square) + d_{\text{gr}}^\dagger(\partial \mathfrak{e}, E^{(\ell)}).$$

But at time n , the perimeter $P_1(n)$ is larger than some constant times ℓ with large probability thanks to Proposition 2.2.4 since $n \leq \tau_{-1} - \varepsilon \ell$. Consequently, by Proposition 2.4.2, and since the map filling the hole \mathfrak{h}_1 has the law $\mathbb{P}_1^{(P_1(n))}$ conditionally on the exploration up to time n by Proposition 2.3.1,

$$d_{\text{gr}}^\dagger(\partial \mathfrak{e}, \square) = \frac{(\log \ell)^2}{2\pi^2} (1 + o(1)) \quad \text{and} \quad d_{\text{gr}}^\dagger(f_r, \square) = \frac{(\log \ell)^2}{2\pi^2} (1 + o(1))$$

in probability on the event that n exists and $n \leq \tau_{-1} - \varepsilon \ell$. In particular, in probability and on the same event, $d_{\text{gr}}^\dagger(f_r, \partial \mathfrak{e}) = o((\log \ell)^2)$. As a consequence, $d_{\text{gr}}^\dagger(f_r, E^{(\ell)}) = d_{\text{gr}}^\dagger(\partial \mathfrak{e}, E^{(\ell)}) + o((\log \ell)^2)$. Thus, in probability,

$$d_{\text{gr}}^\dagger(\square, E^{(\ell)}) \geq \frac{(\log \ell)^2}{2\pi^2} (1 + o(1)) + d_{\text{gr}}^\dagger(f_r, E^{(\ell)}).$$

The converse inequality is a direct consequence of the triangle inequality and Proposition 2.4.2. For the fpp distance, the same reasoning holds using Proposition 2.4.1. \square

Remark 2.5.5. *In the above proof, we saw that $d_{\text{gr}}^\dagger(f_r, \partial \mathfrak{e}) = o((\log \ell)^2)$ and $d_{\text{fpp}}^\dagger(f_r, \partial \mathfrak{e}) = o(\log \ell)$. Nonetheless, it is easy to see that $d_{\text{fpp}}^\dagger(f_r, \partial \mathfrak{e}) = O(1)$ in probability using Proposition 2.2.4 and Lemma 2.5.2. Moreover, we point out that Lemma 2.6.6, together with Lemma 2.5.2, enables to show that $d_{\text{gr}}^\dagger(f_r, \partial \mathfrak{e}) = O(\log \ell)$ in probability.*

Theorem 2.5.3 follows by a judicious application of Lemma 2.3.3:

Proof of Theorem 2.5.3. For the dual graph distance, we apply Lemma 2.3.3 and Lemma 2.5.4 to the function φ_ℓ defined by

$$\forall \mathfrak{m}_\square \in \mathcal{M}_1^{(\ell)}, \quad \varphi_\ell(\mathfrak{m}_\square) = \mathbb{E} \left(\left| \frac{d_{\text{gr}}^\dagger(\square, E(\mathfrak{m}_\square)) - d_{\text{gr}}^\dagger(f_r, E(\mathfrak{m}_\square))}{(\log \ell)^2} - \frac{1}{2\pi^2} \right| \wedge 1 \right),$$

where $E(\mathfrak{m}_\square)$ is a random uniform edge on \mathfrak{m}_\square (where the target 2-face is considered as a single edge). As a result, we obtain that if $E_1^{(\ell)}$ and $E_2^{(\ell)}$ are two independent random uniform edges in $\mathfrak{M}^{(\ell)}$, then

$$\mathbb{E} \left(\left| \frac{d_{\text{gr}}^\dagger(E_1^{(\ell)}, E_2^{(\ell)}) - d_{\text{gr}}^\dagger(f_r, E_2^{(\ell)})}{(\log \ell)^2} - \frac{1}{2\pi^2} \right| \wedge 1 \right) \xrightarrow[\ell \rightarrow \infty]{} 0.$$

The first convergence of the theorem then follows by applying once again Proposition 2.4.2. For the fpp distance, the function φ_ℓ is defined similarly but the expectation also involves the exponential weights defining the fpp distance. \square

Using the same ideas as in the proof of Proposition 2.4.6 we also obtain the following result

Corollary 2.5.6. *If $\mathfrak{M}^{(\ell)}$ follows the law $\mathbb{P}^{(\ell)}$ and V_1, V_2 are two uniform vertices of $\mathfrak{M}^{(\ell)}$, then*

$$\frac{d_{\text{gr}}^{\dagger}(V_1, V_2)}{(\log \ell)^2} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{1}{\pi^2} \quad \text{and} \quad \frac{d_{\text{fpp}}^{\dagger}(V_1, V_2)}{\log \ell} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} \frac{2}{\pi^2 p_{\mathbf{q}}}.$$

Proof. We do the proof for the dual graph distance, the same ideas work for the fpp distance. We can work with $d_{\text{gr}}^{\dagger}(\vec{E}_1, \vec{E}_2)$ where \vec{E}_1, \vec{E}_2 are i.i.d. oriented edges whose law is the same as the law of \vec{E} in (2.4.4) since $d_{\text{gr}}^{\dagger}(\vec{E}_1, \vec{E}_2) - \deg(V_1^{(\ell)}) - \deg(V_2^{(\ell)}) \leq d_{\text{gr}}^{\dagger}(V_1^{(\ell)}, V_2^{(\ell)}) \leq d_{\text{gr}}^{\dagger}(\vec{E}_1, \vec{E}_2)$ and since by Lemma 2.4.5 $\deg V_1^{(\ell)}$ and $\deg V_2^{(\ell)}$ are stochastically dominated by a random variable which does not depend on ℓ . We then conclude by following the same lines as in the proof of Proposition 2.4.6 using Theorem 2.5.3. \square

We finally prove Corollary 2.5.1 using the same ideas as in the proof of Theorem 2.1.1 in Subsection 2.4.3 and using Theorem 2.5.3.

2.6 Diameter of $\mathfrak{M}^{(\ell, \dagger)}$

The main purpose of this section is to give bounds on the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the graph distance and for the fpp distance, proving Theorem 2.1.2. The upper bounds are achieved using a first moment method together with large deviations on the height of a uniform random edge, obtained using supermartingales. Two of these supermartingales (of Lemmas 2.6.2 and 2.6.9) are related to the martingale $(M_n^{(\ell)})_{n \geq 0}$ defined using the perimeter process P_ℓ under $\mathbb{P}_\ell^{(1)}$ as follows:

$$\forall n \geq 0, \quad M_n^{(\ell)} = \frac{1}{h_\ell^\downarrow(P_\ell(n))} \prod_{j=0}^{n-1} \left(\frac{1}{\nu([-P_\ell(j) + 1, \infty)) + \nu(-P_\ell(j) - \ell)} \vee \mathbf{1}_{P_\ell(j) = -\ell} \right). \quad (2.6.1)$$

It is indeed a martingale since for all $n \geq 0$, if $P_\ell(n) \geq 1$, then

$$\begin{aligned} \mathbb{E}_\ell^{(1)} \left(M_{n+1}^{(\ell)} \middle| \mathcal{F}_n \right) &= \frac{h_\ell^\downarrow(-\ell)}{h_\ell^\downarrow(P_\ell(n))} \nu(-P_\ell(n) - \ell) \frac{1}{h_\ell^\downarrow(-\ell)} \prod_{j=0}^n \frac{1}{\nu([-P_\ell(j) + 1, \infty)) + \nu(-P_\ell(j) - \ell)} \\ &\quad + \sum_{k \geq 1 - P_\ell(n)} \frac{h_\ell^\downarrow(P_\ell(n) + k)}{h_\ell^\downarrow(P_\ell(n))} \nu(k) \frac{1}{h_\ell^\downarrow(P_\ell(n) + k)} \prod_{j=0}^n \frac{1}{\nu([-P_\ell(j) + 1, \infty)) + \nu(-P_\ell(j) - \ell)} \\ &= M_n^{(\ell)}, \end{aligned}$$

while the indicator $\mathbf{1}_{P_\ell(j) = -\ell}$ enables to keep a martingale after P_ℓ is absorbed at $-\ell$. This martingale is particularly well suited to the study of the distances on 3/2-stable maps owing to the asymptotic behavior (2.1.3) of the step distribution ν : intuitively, since $\nu((-\infty, -P_\ell(n))) \approx p_{\mathbf{q}}/P_\ell(n)$, the above product can be approximated by the exponential of $\sum_{j=0}^{n-1} 1/P_\ell(j)$ (times an appropriate constant) and can thus be related to the fpp distance (2.4.1) via a concentration inequality.

One notable fact is that our bounds show that the diameter under the fpp distance heavily depends on the tails of ν , but also on the non-asymptotic behavior of ν . Hence, even though the fpp diameter and the distance between uniform points are of the same order, the ratio does not converge to a universal constant. On the contrary, in the case of the dual graph distance, our upper bound for the diameter is universal, echoing the universal scaling limit of the height of a random uniform face in Theorem 2.1.1.

2.6.1 Upper bound of the diameter for the fpp distance

In this subsection, we upper bound the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the first-passage percolation distance. Let $\gamma(\mathbf{q}) := \min_{k, \ell \geq 1} k(\nu((-\infty, -k]) - \nu(-\ell - k))$, which is well defined thanks to (2.1.3) and belongs to $(0, 1)$. The proposition below implies the first upper bound of Theorem 2.1.2.

Proposition 2.6.1. *Let $\text{Diam}_{\text{fpp}}^{\dagger}(\mathfrak{M}^{(\ell)})$ be the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the fpp distance. Then there exists a constant $C > 0$ which does not depend on \mathbf{q} such that with probability $1 - o(1)$ when $\ell \rightarrow \infty$,*

$$\frac{\text{Diam}_{\text{fpp}}^{\dagger}(\mathfrak{M}^{(\ell)})}{\log \ell} \leq C \left(1 + \frac{1}{\gamma(\mathbf{q})} \right).$$

So as to prove the above proposition, we introduce a first family of supermartingales associated with the perimeter process and related to the martingale (2.6.1):

Lemma 2.6.2. *Let $\lambda > 0$. For all $\ell \geq 1, n \geq 0$, let*

$$M_n^{\lambda, \ell} = \frac{1}{h_\ell^\downarrow(P_\ell(n))} \exp \left(\lambda \sum_{j=0}^{n-1} \frac{1}{P_\ell(j)} \mathbf{1}_{P_\ell(j) > 0} \right).$$

If $\lambda \leq \gamma(\mathbf{q})$, then for all $\ell \geq 1$, under $\mathbb{P}_\ell^{(1)}$, $(M_n^{\lambda, \ell})_{n \geq 0}$ is a supermartingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ associated with the peeling exploration.

Proof. If $\lambda \leq \gamma(\mathbf{q})$, then for all $\ell \geq 1, n \geq 0$,

$$\begin{aligned} \mathbb{E}_\ell^{(1)} \left(M_{n+1}^{\lambda, \ell} \middle| \mathcal{F}_n \right) &= \sum_{\substack{k > -P_\ell(n) \\ \text{or } k = -P_\ell(n) - \ell}} \frac{1}{h_\ell^\downarrow(P_\ell(n) + k)} \exp \left(\lambda \sum_{j=0}^n \frac{1}{P_\ell(j)} \mathbf{1}_{P_\ell(j) > 0} \right) \frac{h_\ell^\downarrow(P_\ell(n) + k)}{h_\ell^\downarrow(P_\ell(n))} \nu(k) \\ &= M_n^{\lambda, \ell} e^{\lambda \frac{1}{P_\ell(n)} \mathbf{1}_{P_\ell(n) > 0}} \left(\left(\sum_{k > -P_\ell(n)} \nu(k) \right) + \nu(-P_\ell(n) - \ell) \right) \\ &\leq M_n^{\lambda, \ell}, \end{aligned}$$

thanks to the definition of $\gamma(\mathbf{q}) < 1$ and using that $\exp((\lambda/P_\ell(n)) \mathbf{1}_{P_\ell(n) > 0}) \leq 1/(1 - (\lambda/P_\ell(n)) \mathbf{1}_{P_\ell(n) > 0})$. \square

The above supermartingales lead to a large deviation inequality for the distance from the root to a uniform random edge.

Corollary 2.6.3. *Let $c > 0$. Let E be a uniform random edge of $\mathfrak{M}^{(\ell)}$. If $K > 0$ is large enough, then*

$$\mathbb{P} \left(d_{\text{fpp}}^\dagger(f_r, E) \geq K \log \ell \right) = O(\ell^{-c}) \text{ when } \ell \rightarrow \infty.$$

One can choose $K = C'(1+c)(1+1/\gamma(\mathbf{q}))$, where C' is large enough and does not depend on \mathbf{q}, c .

Proof. Let $c > 0$. First of all, since the distance to an edge is smaller than the distance to the 2-face obtained by unzipping the edge, and by (2.3.4),

$$\begin{aligned} \mathbb{P} \left(d_{\text{fpp}}^\dagger(f_r, E) \geq K \log \ell \right) &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_1^{(\ell)} \left(\frac{1}{\#\text{Edges}(\mathbf{m}_\square) - 1} \mathbf{1}_{d_{\text{fpp}}^\dagger(f_r, \square) \geq K \log \ell} \right) \\ &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{P}_1^{(\ell)} \left(d_{\text{fpp}}^\dagger(f_r, \square) \geq K \log \ell \right). \end{aligned}$$

But, owing to (2.3.3), we know that $W_1^{(\ell)}/W^{(\ell)}$ is a $O(\ell^{3/2})$, so that it is enough to prove the analogous statement for $\mathbb{P}_1^{(\ell)}(d_{\text{fpp}}^\dagger(f_r, \square) \geq K \log \ell)$. Moreover, recalling (2.4.1), one can see that it suffices to show that there exists $K' > 0$ such that

$$\mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_\ell - 1} \frac{1}{P_\ell(k)} \geq K' \log \ell \right) = O(\ell^{-c}) \text{ when } \ell \rightarrow \infty. \quad (2.6.2)$$

Indeed, if we assume (2.6.2), then by conditioning on the perimeter process and applying Bernstein's inequality (the version we use comes from Corollary 2.10 of [Tal94]), we get that there exists a constant $C > 0$ such that for all $\ell \geq 1$ and for all $t > 0$,

$$\begin{aligned} \mathbb{P}_\ell^{(1)} \left(\left| \sum_{k=0}^{\tau_\ell - 1} \frac{\mathcal{E}_k - 1}{P_\ell(k)} \right| > t \log \ell \middle| P_\ell \right) &\leq 2 \exp \left(-C \min \left(\frac{t^2 (\log \ell)^2}{\sum_{k=0}^{\tau_\ell - 1} \frac{1}{P_\ell(k)^2}}, \frac{t \log \ell}{\max_{0 \leq k \leq \tau_\ell - 1} \frac{1}{P_\ell(k)}} \right) \right) \\ &\leq 2 \exp \left(-C \min \left(\frac{t^2 (\log \ell)^2}{\sum_{k=0}^{\tau_\ell - 1} \frac{1}{P_\ell(k)}}, t \log \ell \right) \right) \\ &\leq 2\ell^{-\left(\frac{Ct^2}{K'} \wedge Ct\right)} \text{ with probability at least } 1 - O(\ell^{-c}). \end{aligned}$$

One can then conclude by choosing $t > 0$ such that $\frac{Ct^2}{K'} \wedge Ct \geq c$ and by taking $K = \frac{K'+t}{2}$.

Now, let us prove (2.6.2). Let $\lambda \in (0, \gamma(\mathbf{q}))$ so that for all $\ell \geq 1$, $M^{\lambda, \ell}$ is a supermartingale (given by Lemma 2.6.2). Then, using Markov's inequality and Fatou's lemma,

$$\begin{aligned} \mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_\ell - 1} \frac{1}{P_\ell(k)} \geq K' \log \ell \right) &\leq \ell^{-\lambda K'} \mathbb{E}_\ell^{(1)} e^{\lambda \sum_{k=0}^{\tau_\ell - 1} \frac{1}{P_\ell(k)}} \\ &= \ell^{-\lambda K'} \mathbb{E}_\ell^{(1)} M_{\tau_\ell}^{\lambda, \ell} \\ &\leq \ell^{-\lambda K'} \mathbb{E}_\ell^{(1)} M_0^{\lambda, \ell} = \ell^{-\lambda K'} \frac{1}{h_\ell^\downarrow(1)} = O(\ell^{3/2 - \lambda K'}) = O(\ell^{-c}) \end{aligned}$$

for $K' \geq (c + 3/2)/\lambda$. □

Proof of Proposition 2.6.1. To begin with, since the diameter is upper bounded by twice the maximal distance to the root face, it suffices to show that there exists $K > 0$ such that

$$\mathbb{P} \left(\exists e \in \text{Edges}(\mathfrak{M}^{(\ell)}), d_{\text{fpp}}^\dagger(f_r, e) \geq K \log \ell \right) \xrightarrow{\ell \rightarrow \infty} 0,$$

using also that for all $f \in \text{Faces}(\mathfrak{M}^{(\ell)})$, if e is an edge surrounding f , then $d_{\text{fpp}}^\dagger(f_r, f) \leq d_{\text{fpp}}^\dagger(f_r, e) + \mathcal{E}$, where \mathcal{E} is an exponential random variable of parameter 1.

Then we distinguish whether the number of edges is too large or not: let $\varepsilon > 0$.

$$\begin{aligned} &\mathbb{P} \left(\exists e \in \text{Edges}(\mathfrak{M}^{(\ell)}), d_{\text{fpp}}^\dagger(f_r, e) \geq K \log \ell \right) \\ &\leq \mathbb{E} \left(\#\left\{ e \in \text{Edges}(\mathfrak{M}^{(\ell)}); d_{\text{fpp}}^\dagger(f_r, e) \geq K \log \ell \right\} \mathbf{1}_{\#\text{Edges}(\mathfrak{M}^{(\ell)}) \leq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2}} \right) \\ &\quad + \mathbb{P} \left(\#\text{Edges}(\mathfrak{M}^{(\ell)}) \geq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \right) \end{aligned}$$

and the last term is smaller than ε for ℓ large enough by (2.3.3). Henceforth we focus on the first term. Conditionally on $\mathfrak{M}^{(\ell)}$, let E be a uniform random edge of $\mathfrak{M}^{(\ell)}$.

$$\begin{aligned} &\mathbb{E} \left(\#\left\{ e \in \text{Edges}(\mathfrak{M}^{(\ell)}); d_{\text{fpp}}^\dagger(f_r, e) \geq K \log \ell \right\} \mathbf{1}_{\#\text{Edges}(\mathfrak{M}^{(\ell)}) \leq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2}} \right) \\ &\leq \mathbb{E} \left(\frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \mathbb{P}(d_{\text{fpp}}^\dagger(f_r, E) \geq K \log \ell | \mathfrak{M}^{(\ell)}) \right) \\ &= \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \mathbb{P}(d_{\text{fpp}}^\dagger(f_r, E) \geq K \log \ell). \end{aligned}$$

The expression on the last line tends to zero when $\ell \rightarrow \infty$ if K is chosen large enough according to Corollary 2.6.3. □

2.6.2 Lower bound for the diameter for the fpp distance

A lower bound for the diameter is already provided by Corollary 2.5.1. Nevertheless, this lower bound only depends on the tail behavior of ν . Actually, in the case of the fpp distance, the diameter depends heavily on ν . Let us show for instance the influence of q_1 , or equivalently $\nu(0)$, on the asymptotic behavior of the diameter, which implies in particular the first lower bound of Theorem 2.1.2, hence ending the proof of that theorem.

Proposition 2.6.4. *For all $\delta > 0$, with probability $1 - o(1)$ as $\ell \rightarrow \infty$,*

$$\frac{\text{diam}_{\text{fpp}}^\dagger(\mathfrak{M}^{(\ell)})}{\log \ell} \geq \frac{3}{4(1 - q_1^2)} - \delta.$$

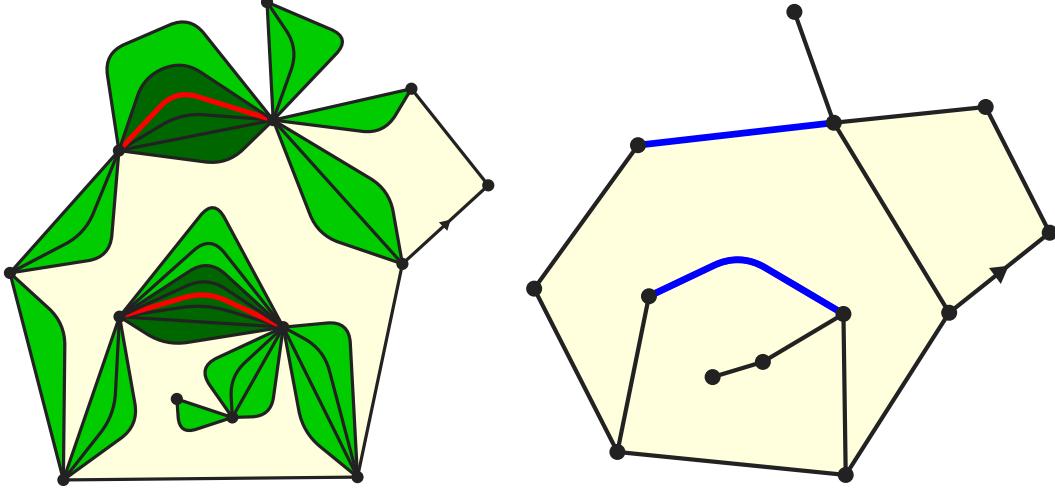


Figure 2.6 – On the left, the map $\mathfrak{M}^{(\ell)}$ with two random edges E_1 and E_2 in red. The 2-faces are in green and form “watermelons” and E_1 and E_2 are in the middle of their watermelons. The largest fpp-balls around E_1 and E_2 made of 2-faces only are in dark green. On the right the map $\widetilde{\mathfrak{M}}^{(\ell)}$ obtained from $\mathfrak{M}^{(\ell)}$ by deleting the 2-faces. The blue edges are the two random uniform edges \tilde{E}_1 and \tilde{E}_2 that correspond to the two watermelons containing E_1 and E_2 .

Proof. Let us prove that result with a second moment method. If e is an edge of $\mathfrak{M}^{(\ell)}$, let $R(e)$ denote the radius of the largest fpp-ball centred at e only constituted of faces of degree 2. Notice that this fpp-ball is included in the maximal sub-map made of 2-faces and edges which have the same endpoints as e , which is called the “watermelon” containing e (see Figure 2.6).

By Proposition 11.10 of [Cur23], we know that the map $\widetilde{\mathfrak{M}}^{(\ell)}$ obtained from $\mathfrak{M}^{(\ell)}$ by replacing each watermelon by one edge (see Figure 2.6) is a $\tilde{\mathbf{q}}$ -Boltzmann map of perimeter 2ℓ , where the weight sequence $\tilde{\mathbf{q}}$ is defined by

$$\begin{cases} \tilde{q}_1 = 0 \\ \tilde{q}_k = q_k / (1 - q_1)^k \quad \text{for } k \geq 2. \end{cases}$$

Moreover, conditionally on $\widetilde{\mathfrak{M}}^{(\ell)}$, the map $\mathfrak{M}^{(\ell)}$ is obtained by adding independently a geometric number of 2-faces on each edge of $\widetilde{\mathfrak{M}}^{(\ell)}$, with parameter q_1 . We denote by $M(\tilde{e})$ the number of edges of the watermelon corresponding to an edge \tilde{e} of $\widetilde{\mathfrak{M}}^{(\ell)}$ so that conditionally on $\widetilde{\mathfrak{M}}^{(\ell)}$, the random variables $M(\tilde{e}) - 1$ are i.i.d. geometric of parameter q_1 . By the paragraph just above Proposition 11.10 in [Cur23] we also know that $c_{\tilde{\mathbf{q}}} = c_{\mathbf{q}} / (1 - q_1)$ and that $\tilde{\mathbf{q}}$ is again a critical weight sequence of type 2.

If \tilde{E} is a random uniform edge in $\widetilde{\mathfrak{M}}^{(\ell)}$ and E is the edge in the middle of the watermelon corresponding to \tilde{E} (and if the watermelon has an even number of edges, we choose E uniformly among the two edges in the middle), then, conditionally on $\widetilde{\mathfrak{M}}^{(\ell)}$ and on \tilde{E} , the random variable $M(\tilde{E})$ is geometric of parameter q_1 starting at one. Hence, $R(E)$ is lower-bounded by a sum of $\lfloor M(\tilde{E})/2 \rfloor - 1$ exponential random variables of parameter 2 (i.e. of expectation $1/2$). As a consequence, for all $t \geq 0$,

$$\mathbb{P}\left(R(E) \geq t \mid \widetilde{\mathfrak{M}}^{(\ell)}, \tilde{E}\right) \geq \mathbb{P}\left(\sum_{i=1}^{\lfloor M(\tilde{E})/2 \rfloor - 1} \frac{\mathcal{E}_i}{2} \geq t \mid \widetilde{\mathfrak{M}}^{(\ell)}, \tilde{E}\right),$$

where the \mathcal{E}_i 's are i.i.d. exponential random variables of parameter 1 which are independent from $\widetilde{\mathfrak{M}}^{(\ell)}$

and $M(\tilde{E})$. But one can lower-bound

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{\lfloor M(\tilde{E})/2 \rfloor - 1} \frac{\mathcal{E}_i}{2} \geq t \mid \widetilde{\mathfrak{M}}^{(\ell)}, \tilde{E}\right) &= \sum_{k=4}^{\infty} q_1^{k-1} (1 - q_1) \int_{2t}^{\infty} \frac{x^{\lfloor k/2 \rfloor - 2} e^{-x}}{(\lfloor k/2 \rfloor - 2)!} dx \\
&\geq q_1^5 (1 - q_1) \int_{2t}^{\infty} \sum_{k=4}^{\infty} \frac{(q_1^2)^{\lfloor k/2 \rfloor - 2} x^{\lfloor k/2 \rfloor - 2} e^{-x}}{(\lfloor k/2 \rfloor - 2)!} dx \\
&= 2q_1^5 (1 - q_1) \int_{2t}^{\infty} e^{(q_1^2 - 1)x} dx \\
&= \frac{2q_1^5}{1 + q_1} e^{-2(1 - q_1^2)t}.
\end{aligned}$$

Henceforth, let $\tilde{E}_1, \dots, \tilde{E}_{\lfloor \ell^{3/2} \rfloor}$ be independent random uniform edges of $\widetilde{\mathfrak{M}}^{(\ell)}$ (conditionally on $\widetilde{\mathfrak{M}}^{(\ell)}$). For all $1 \leq i \leq \lfloor \ell^{3/2} \rfloor$, we write E_i the edge of $\mathfrak{M}^{(\ell)}$ in the middle of the watermelon corresponding to \tilde{E}_i (chosen uniformly at random between the two edges in the middle if the watermelon has an even number of edges). We again write $R(E_i)$ the radius of the largest fpp-ball centred at E_i containing only 2-faces. For conciseness we drop the $[\cdot]$. If $t > 0$, then by the second moment method,

$$\begin{aligned}
&\mathbb{P}\left(\exists i \leq \ell^{3/2}, R(E_i) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}\right) \\
&\geq \frac{\mathbb{E}(\#\{i \in [1, \ell^{3/2}], R(E_i) \geq t \log \ell\} \mid \widetilde{\mathfrak{M}}^{(\ell)})^2}{\mathbb{E}((\#\{i \in [1, \ell^{3/2}], R(E_i) \geq t \log \ell\})^2 \mid \widetilde{\mathfrak{M}}^{(\ell)})} \\
&= \frac{\ell^3 \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)})^2}{\ell^{3/2} \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) + \ell^{3/2} (\ell^{3/2} - 1) \mathbb{P}(R(E_1), R(E_2) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)})}.
\end{aligned}$$

Now, one can distinguish whether $\tilde{E}_1 = \tilde{E}_2$ or not. Then, in probability as $\ell \rightarrow \infty$,

$$\begin{aligned}
&\mathbb{P}(R(E_1), R(E_2) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) \\
&\leq \mathbb{P}(\tilde{E}_1 = \tilde{E}_2 \text{ and } R(E_1) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) + \mathbb{P}(\tilde{E}_1 \neq \tilde{E}_2 \text{ and } R(E_1), R(E_2) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) \\
&= \frac{1}{\#\text{Edges}(\widetilde{\mathfrak{M}}^{(\ell)})} \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) + \frac{\#\text{Edges}(\widetilde{\mathfrak{M}}^{(\ell)}) - 1}{\#\text{Edges}(\widetilde{\mathfrak{M}}^{(\ell)})} \mathbb{P}(R(E_1), R(E_2) \geq t \log \ell \mid \tilde{E}_1 \neq \tilde{E}_2, \widetilde{\mathfrak{M}}^{(\ell)}) \\
&= O(\ell^{-3/2}) \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) + \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)})^2,
\end{aligned}$$

where the last line is due to the convergence in distribution of $\#\text{Edges}(\widetilde{\mathfrak{M}}^{(\ell)}) \ell^{-3/2}$ as $\ell \rightarrow \infty$, see e.g. Proposition 10.4 in [Cur23]. By taking $t < 3/(4(1 - q_1^2))$ we ensure that $\ell^{3/2} \mathbb{P}(R(E) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}) \rightarrow \infty$ in probability as $\ell \rightarrow \infty$, hence

$$\mathbb{P}\left(\exists i \leq \ell^{3/2}, R(E_i) \geq t \log \ell \mid \widetilde{\mathfrak{M}}^{(\ell)}\right) \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} 1.$$

This entails what we wanted to show. \square

2.6.3 Upper bound of the diameter for the graph distance

We next upper bound the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the graph distance. The proposition below implies the second upper bound of Theorem 2.1.2. Unlike the case of the fpp distance, the upper bound for the graph distance is universal (i.e. it does not depend on \mathbf{q}).

Proposition 2.6.5. *Let $\text{Diam}_{\text{gr}}^{\dagger}(\mathfrak{M}^{(\ell)})$ be the diameter of $\mathfrak{M}^{(\ell, \dagger)}$ for the graph distance. Then for all $\delta > 0$, with probability $1 - o(1)$ when $\ell \rightarrow \infty$,*

$$\frac{\text{Diam}_{\text{gr}}^{\dagger}(\mathfrak{M}^{(\ell)})}{(\log \ell)^2} \leq 18 + \delta.$$

To prove this result, we will need to use the height process H_ℓ during the peeling by layers exploration, together with the interpolated height process. Let us first recall the definition of the interpolated height process from [BCM18]. At time n of the exploration, let D_n be the number of edges in the boundary $\partial\bar{\mathfrak{c}}_n$ that are at height $H_\ell(n)$, so that the other $2P_\ell(n) - D_n$ edges are at height $H_\ell(n) + 1$. If $f : [0, 1] \rightarrow [0, 1]$ is a non-increasing function, the interpolated height process $H_\ell^f(n)$ under $\mathbb{P}_\ell^{(1)}$ and its increment at time $n < \tau_{-\ell}$ are defined by

$$H_\ell^f(n) := H_\ell(n) + f(D_n/2P_n) \quad \text{and} \quad \Delta H_\ell^f(n) := H_\ell^f(n+1) - H_\ell^f(n),$$

where by convention, we set $H_\ell^f(n) = H_\ell^f(\tau_{-\ell} - 1)$ for $n \geq \tau_{-\ell}$. The next lemma is an analogue of Lemma 6 from [BCM18], bounding from above the mean increase of the interpolated height.

Lemma 2.6.6. *Let $0 < \varepsilon < 1$. If $f : [0, 1] \rightarrow [0, 1]$ is twice continuously differentiable with $f(0) = 1$, $f(1) = 0$, $f'(0) = f'(1) = f''(0) = f''(1) = 0$ and $0 \leq -f'(x) \leq 1 + \varepsilon$ for all $x \in [0, 1]$, then there exist $C(\varepsilon, \mathbf{q}), C'(\varepsilon, \mathbf{q}) > 0$ such that for all $\ell_0 \geq 1, \ell \geq 1, n \geq 0$,*

$$\mathbb{E}_\ell^{(\ell_0)} \left(\Delta H_\ell^f(n) \middle| \mathcal{F}_n \right) \leq \frac{(1 + 3\varepsilon)^2 p_{\mathbf{q}} \log(P_\ell(n)) + C(\varepsilon, \mathbf{q})}{1 - \varepsilon} \frac{1}{2P_\ell(n)} \quad \text{and} \quad \mathbb{E}_\ell^{(\ell_0)} \left(\left(\Delta H_\ell^f(n) \right)^2 \middle| \mathcal{F}_n \right) \leq \frac{C'(\varepsilon, \mathbf{q})}{P_\ell(n)},$$

where $(\mathcal{F}_n)_{n \geq 0}$ is the filtration associated with the exploration process.

Proof. The proof is essentially the same as the proof of Lemma 6 of [BCM18], but a little more complex. Indeed, the same terms appear except that they are multiplied by a factor $(p + \ell)/(p + k + \ell)$. Let us detail the proof.

We extend f to \mathbb{R} by setting $f(x) = 1$ for $x \leq 0$ and $f(x) = 0$ for all $x \geq 1$. As in [BCM18], by the Markov property of the exploration (Proposition 2.3.1), we claim that $(H_\ell(n), P_\ell(n), D_n)_{n \geq 0}$ is a Markov chain and that if $p \geq 1$ and $1 \leq d \leq 2p$, then

$$\mathbb{E}_\ell^{(\ell_0)} \left(\Delta H_\ell^f(n) \middle| P_\ell(n) = p, D_n = d \right) \tag{2.6.3}$$

$$= \sum_{k=0}^{\infty} \nu(k) \frac{p + \ell}{p + k + \ell} \frac{h^\uparrow(p + k)}{h^\uparrow(p)} \left[f\left(\frac{d-1}{2p+2k}\right) - f\left(\frac{d}{2p}\right) \right] \tag{2.6.4}$$

$$+ \sum_{k=1}^{p-1} \frac{\nu(-k)}{2} \frac{p + \ell}{p - k + \ell} \frac{h^\uparrow(p - k)}{h^\uparrow(p)} \left[f\left(\frac{d-2k}{2p-2k}\right) - f\left(\frac{d}{2p}\right) \right] \tag{2.6.5}$$

$$+ \sum_{k=1}^{p-1} \frac{\nu(-k)}{2} \frac{p + \ell}{p - k + \ell} \frac{h^\uparrow(p - k)}{h^\uparrow(p)} \left[f\left(\frac{d-1}{2p-2k}\right) - f\left(\frac{d}{2p}\right) \right]. \tag{2.6.6}$$

Indeed, if $d = 1$, then either $\Delta H_\ell(n) = 1$ and $D_{n+1} = 2P_\ell(n+1)$, and therefore $\Delta H_\ell^f(n) = 1 - f(1/(2p))$, with probability $\sum_{k=-p+1}^{\infty} \nu(k) (h_\ell^\downarrow(p+k)/h_\ell^\downarrow(p))$; or $\tau_{-\ell} = n+1$ and in that case $\Delta H_\ell^f(n) = 0$, with probability $\nu(-p-\ell)/h_\ell^\downarrow(p)$. This agrees with the sums since $f(x) = 1$ for $x \leq 0$ (and by using the definition of h^\uparrow and h_ℓ^\downarrow in (2.2.1)). Now, in the case $d \geq 2$, if an event C_{k+1} happens, then $P_\ell(n+1) = p+k$ and $D_{n+1} = d-1$, hence the first line (2.6.4) (again using (2.2.1)). If an event $G_{\square, k-1}$ happens for $1 \leq k \leq p-1$, i.e. if the peeled edge is identified with an edge on its right, then $P_\ell(n+1) = P_\ell(n) - k$ and either $D_{n+1} = d-2k$ and $\Delta H_\ell(n) = 0$, or $D_{n+1} = 2P_\ell(n+1)$ and $\Delta H_\ell(n) = 1$, depending on the sign of $d-2k$. These two cases are taken into account in the second line (2.6.5) since $f(x) = 1$ for all $x \leq 0$. If an event $G_{k-1, \square}$ happens for some $1 \leq k \leq p-1$, i.e. if the peeled edge is identified with an edge on its left, then again $P_\ell(n+1) = P_\ell(n) - k$ but $\Delta H_\ell(n) = 0$ and either $D_{n+1} = d-1$ or $D_{n+1} = 2P_\ell(n+1)$, depending on which is the smallest. Both situations are incorporated by the last line (2.6.6). Finally, if the event C_ℓ^{stop} happens, then $\Delta H_\ell^f(n) = 0$ by definition of the interpolated height process.

To deal with these three sums, we will use the notation $y = d/(2p) \in (0, 1]$. Let $c > 0$ given by Taylor's theorem such that $|f(x) - f(y) - f'(y)(x-y)| \leq c(x-y)^2$ for all $x \in \mathbb{R}$. We will also rely on the inequality

$$\forall k > -p, \quad \sqrt{1+k/p}(1-1/p) \leq \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \leq \sqrt{1+k/p}(1+1/p).$$

Let $N > 0$ such that $1 - \varepsilon < k^2\nu(-k)/p_{\mathbf{q}} < 1 + \varepsilon$ and $1 - \varepsilon < k\nu([k, \infty))/p_{\mathbf{q}} < 1 + \varepsilon$ for all $k \geq N$, given by (2.1.3). From the mean value inequality applied to f , the first N terms in each sum contribute $O(p^{-1})$ uniformly in d, ℓ , so it suffices to bound the sums restricted to $|k| \geq N$. Since $(p + \ell)/(p + k + \ell) \leq 1$, the aforementioned inequalities, the first sum from N to ∞ , which only contains positive terms, is smaller than

$$\sum_{k=N}^{\infty} \nu(k) \sqrt{1 + k/p} (1 + 1/p) \left[|f'(y)| \frac{1 + yk}{p + k} + c \left(\frac{1 + k}{p + k} \right)^2 \right]$$

where we used that

$$\frac{d}{2p} - \frac{d-1}{2p+2k} \leq \frac{1+yk}{p+k} \leq \frac{1+k}{p+k}.$$

Thus, by summation by parts, using that $1 - \varepsilon < k\nu([k, \infty))/p_{\mathbf{q}} < 1 + \varepsilon$ for all $k \geq N$, and then recognizing a Riemann sum, the first sum (2.6.4) is upper bounded by

$$(1 + \varepsilon) p_{\mathbf{q}} y \frac{1}{p} \int_{(N-1)/p}^{\infty} \frac{dx}{x \sqrt{1+x}} + O(p^{-1}) \leq (1 + \varepsilon)^2 p_{\mathbf{q}} y \frac{\log p}{p} + O(p^{-1}), \quad (2.6.7)$$

where the $O(p^{-1})$ is uniform in d and ℓ . For the second sum (2.6.5), which again has only positive terms, we distinguish whether $k \leq \varepsilon p$ or $k > \varepsilon p$. If $k \leq \varepsilon p$, $(p + \ell)/(p - k + \ell) \leq 1/(1 - \varepsilon)$, so that applying the inequalities and the fact that $1 - \varepsilon < k^2\nu(-k)/p_{\mathbf{q}} < 1 + \varepsilon$ for all $k \geq N$ gives us that

$$\begin{aligned} & \sum_{k=N}^{\varepsilon p} \frac{\nu(-k)}{2} \frac{p + \ell}{p - k + \ell} \frac{h^\uparrow(p - k)}{h^\uparrow(p)} \left[f\left(\frac{d - 2k}{2p - 2k}\right) - f\left(\frac{d}{2p}\right) \right] \\ & \leq \sum_{k=N}^{\varepsilon p} \frac{1 + \varepsilon}{1 - \varepsilon} \frac{p_{\mathbf{q}}}{2k^2} (1 + 1/p) \sqrt{1 - k/p} \left[|f'(y)| (1 - y) \frac{k}{p} \frac{1}{1 - \frac{k}{p}} + c \left(\frac{k}{p} \frac{1}{1 - \frac{k}{p}} \right)^2 \right], \end{aligned}$$

where we used that

$$\frac{d}{2p} - \frac{d - 2k}{2p - 2k} = (1 - y) \frac{k}{p} \frac{1}{1 - \frac{k}{p}} \leq \frac{k}{p} \frac{1}{1 - \frac{k}{p}}.$$

Hence, by recognizing a Riemann sum, we can upper bound the sum for the $k \leq \varepsilon p$ by

$$\frac{1 + \varepsilon}{1 - \varepsilon} p_{\mathbf{q}} \frac{1 - y}{2} |f'(y)| \frac{1}{p} \int_{(N-1)/p}^{\varepsilon+1/p} \frac{1}{x \sqrt{1-x}} dx + O(p^{-1}) \leq \frac{1 + \varepsilon}{1 - \varepsilon} p_{\mathbf{q}} \frac{1 - y}{2} |f'(y)| \frac{\log p}{p} + O(p^{-1}).$$

If $\varepsilon p \leq k \leq p - 1$, we have

$$1 \leq \frac{p + \ell}{p - k + \ell} \leq \frac{p}{p - k}$$

and

$$\frac{\nu(-k)}{2} \frac{h^\uparrow(p - k)}{h^\uparrow(p)} \left[f\left(\frac{d - 2k}{2p - 2k}\right) - f\left(\frac{d}{2p}\right) \right] \leq \nu(-k) \frac{h^\uparrow(p - k)}{h^\uparrow(p)} \leq \sqrt{(p - k)/p} O(p^{-2}).$$

So, using a Riemann sum, the sum for $k \geq \varepsilon p$ is upper bounded by $O(p^{-1})$. Therefore, the second sum (2.6.5) is bounded by

$$\frac{1 + \varepsilon}{1 - \varepsilon} p_{\mathbf{q}} \frac{1 - y}{2} |f'(y)| \frac{\log p}{p} + O(p^{-1}). \quad (2.6.8)$$

For the last sum (2.6.6), we first focus on the positive terms, i.e. those such that $kd \leq p$. If $d \geq 2$, then this entails that $(p + \ell)/(p - k + \ell) \leq 2$. Moreover, for all $1 \leq k \leq p - 1$,

$$\frac{d}{2p} - \frac{d-1}{2p-2k} \leq \frac{1}{2p},$$

so that the sum of the positive terms is a $O(p^{-1})$. If $d = 1$, then by Taylor's theorem $f(0) - f(1/(2p)) = O(p^{-2})$, so that the sum of the positive terms is a $O(p^{-2})$. In these two cases, the sum of the positive terms of (2.6.6) is upper bounded by a $O(p^{-1})$. In the case $0 < y < \varepsilon$, the upper bounds for the

positive terms already enable to conclude since by summing (2.6.7), (2.6.8) and the $O(p^{-1})$, we obtain the uniform bound

$$\frac{(1+\varepsilon)^3 p_{\mathbf{q}} \log p}{1-\varepsilon} \frac{1}{2} \frac{1}{p} + O(p^{-1}) \leq \frac{(1+3\varepsilon)^2 p_{\mathbf{q}} \log p}{1-\varepsilon} \frac{1}{2} \frac{1}{p} + O(p^{-1}).$$

In the case $y \geq \varepsilon$, we assume that we have chosen $N > 1/\varepsilon$, so that $N \geq 2p/d$ and thus all the terms of (2.6.6) for $N \leq k \leq \varepsilon p$ are negative. Using that $(p+\ell)/(p-k+\ell) \geq 1$, the sum of the negative terms of (2.6.6) with $N \leq k \leq \varepsilon p$ can be upper bounded by

$$-\frac{1-\varepsilon}{2} \sum_{k=N}^{\varepsilon p} \frac{p_{\mathbf{q}}}{k^2} (1-1/p) \sqrt{1-k/p} \left[|f'(y)| \left(y \frac{k}{p} - \frac{1}{2p} \right) - c(2k/p)^2 \right],$$

where we used that

$$\frac{d-1}{2p-2k} - \frac{d}{2p} \geq \frac{dk-p}{2p^2} = y \frac{k}{p} - \frac{1}{2p} \quad \text{and} \quad \left| \frac{d}{2p} - \frac{d-1}{2p-2k} \right| \leq \frac{k}{(1-\varepsilon)p}.$$

Therefore, using again a Riemann sum, the sum (2.6.6) is upper bounded by

$$-(1-\varepsilon) p_{\mathbf{q}} \frac{y}{2} |f'(y)| \frac{\log p}{p} + O(p^{-1}). \quad (2.6.9)$$

Finally, summing the upper bounds (2.6.7), (2.6.8), (2.6.9) and the $O(p^{-1})$, one recovers the desired result in the case $\varepsilon \leq y \leq 1$ as well for $\mathbb{E}_{\ell}^{(\ell_0)} \left(\Delta H_{\ell}^f(n) \middle| \mathcal{F}_n \right)$. For the expectation of the square, one can perform the same computation with the three sums (2.6.4), (2.6.5), (2.6.6), except that the expressions into brackets are squared. One can check using the same ideas that each sum is a $O(p^{-1})$ (uniformly in d, ℓ), hence the second upper bound. \square

We will rely on a direct consequence of Lemma 2.6.6.

Corollary 2.6.7. *For all $\varepsilon \in (0, 1)$, there exists $\lambda(\varepsilon) > 0$ (which does not depend on \mathbf{q}) and $C(\varepsilon, \mathbf{q}) > 0$ such that for all $0 < \lambda < \lambda(\varepsilon)$, for all $\ell \geq 1$ and $n \geq 0$,*

$$\mathbb{E}_{\ell}^{(1)} \left(e^{\lambda \Delta H_{\ell}^f(n)} \middle| \mathcal{F}_n \right) \leq 1 + \lambda \left(\frac{(1+3\varepsilon)^2 p_{\mathbf{q}} \log(P_{\ell}(n))}{1-\varepsilon} \frac{1}{2} \frac{1}{P_{\ell}(n)} + \frac{C(\varepsilon, \mathbf{q})}{P_{\ell}(n)} \right),$$

where $(\mathcal{F}_n)_{n \geq 0}$ is the filtration associated with the exploration (using the peeling by layers algorithm).

Proof. It suffices to take $\lambda(\varepsilon)$ small enough so that for all $0 < \lambda < c(\varepsilon)$, we have the inequality $e^{\lambda \Delta H_{\ell}^f(n)} \leq 1 + \lambda \Delta H_{\ell}^f(n) + \lambda^2 (\Delta H_{\ell}^f(n))^2$ (such a $\lambda(\varepsilon)$ does exist given that $-1 \leq \Delta H_{\ell}^f(n) \leq 1$). \square

We can therefore define another family of supermartingales.

Corollary 2.6.8. *Let $\varepsilon \in (0, 1)$. Let $\lambda < \lambda(\varepsilon)$ where $\lambda(\varepsilon)$ is given by Corollary 2.6.7. For all $\ell \geq 1$ for all $n \geq 0$, let*

$$M_n^{\varepsilon, \lambda, \ell} = \exp \left(\lambda H_{\ell}^f(n) - \lambda \sum_{k=0}^{n-1} \left(\frac{(1+3\varepsilon)^3 p_{\mathbf{q}} \log(P_{\ell}(k))}{1-\varepsilon} \frac{1}{2} \frac{1}{P_{\ell}(k)} + \frac{C(\varepsilon, \mathbf{q})}{P_{\ell}(k)} \right) \right).$$

Then, under $\mathbb{P}_{\ell}^{(1)}$, $(M_n^{\varepsilon, \lambda, \ell})_{n \geq 0}$ is a supermartingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ associated with the exploration.

We will also use a variant of Lemma 2.6.2, whose proof is similar and left to the reader.

Lemma 2.6.9. *If $\varepsilon > 0$, let k_0 such that $e^{(p_{\mathbf{q}} - \varepsilon)/k} \leq 1 + (\nu((-\infty, -k]) - \nu(-\ell - k))$ for all $k \geq k_0$ and $\ell \geq 1$. For all $\ell \geq 1, n \geq 0$, let*

$$M_n^{\varepsilon, k_0, \ell} = \frac{1}{h_{\ell}^{\downarrow}(P_{\ell}(n))} \exp \left((p_{\mathbf{q}} - \varepsilon) \sum_{j=0}^{n-1} \frac{1}{P_{\ell}(j)} \mathbf{1}_{P_{\ell}(j) \geq k_0} \right).$$

then for all $\ell \geq 1$, under $\mathbb{P}_{\ell}^{(1)}$, $(M_n^{\varepsilon, k_0, \ell})_{n \geq 0}$ is a supermartingale with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ associated with the peeling exploration.

Subsequently, from the above results, we obtain a large deviation inequality for the height of a random uniform edge in $\mathfrak{M}^{(\ell)}$.

Corollary 2.6.10. *Let $c > 0$. Let $\delta > 0$ and*

$$K = \frac{1}{2}(c+3)\left(c + \frac{5}{2}\right) + \delta.$$

Let $\mathfrak{M}^{(\ell)}$ of law $\mathbb{P}^{(\ell)}$, let E be a uniform random edge of $\mathfrak{M}^{(\ell)}$. Then,

$$\mathbb{P}(d_{\text{gr}}^{\dagger}(f_r, E) \geq K(\log \ell)^2) = O(\ell^{-c}) \text{ when } \ell \rightarrow \infty.$$

Proof. First of all, since the distance to an edge is smaller than the distance to the 2-face obtained by unzipping the edge, and by (2.3.4),

$$\begin{aligned} \mathbb{P}(d_{\text{gr}}^{\dagger}(f_r, E) \geq K(\log \ell)^2) &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_1^{(\ell)} \left(\frac{1}{\#\text{Edges}(\mathbf{m}_{\square}) - 1} \mathbf{1}_{d_{\text{gr}}^{\dagger}(f_r, \square) \geq K(\log \ell)^2} \right) \\ &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{P}_1^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq K(\log \ell)^2). \end{aligned}$$

But, since $W_1^{(\ell)}/W^{(\ell)}$ is a $O(\ell^{3/2})$ by (2.3.3), it is enough to prove that if $K' = \frac{1}{2}(c + \frac{3}{2})(c+1) + \delta$, then

$$\mathbb{P}_1^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq K'(\log \ell)^2) = O(\ell^{-c}).$$

Since $d_{\text{gr}}^{\dagger}(f_r, \square) \leq H_{\ell}(\tau_{-\ell} - 1) + 1$ and $H_{\ell}(n) \leq H_{\ell}^f(n)$ for all $n \leq \tau_{-\ell} - 1$, by taking δ a bit smaller it suffices to show that

$$\mathbb{P}_{\ell}^{(1)}\left(H_{\ell}^f(\tau_{-\ell} - 1) \geq K'(\log \ell)^2\right) = O(\ell^{-c}) \text{ when } \ell \rightarrow \infty.$$

So as to prove that, we can first see from Corollary 2.6.8 that if $\varepsilon \in (0, 1)$, if $0 < \lambda < \lambda(\varepsilon)$ and $\mu = \frac{(1+3\varepsilon)^3 p_{\mathbf{q}}}{1-\varepsilon}$ then for all $\ell \geq 1$,

$$\begin{aligned} \mathbb{E}_{\ell}^{(1)} \exp\left(\lambda H_{\ell}^f(\tau_{-\ell} - 1) - \lambda \mu \sum_{k=0}^{\tau_{-\ell}-2} \frac{\log(P_{\ell}(k))}{P_{\ell}(k)} - \lambda C(\varepsilon, \mathbf{q}) \sum_{k=0}^{\tau_{-\ell}-2} \frac{1}{P_{\ell}(k)}\right) &= \mathbb{E}_{\ell}^{(1)} M_{\tau_{-\ell}-1}^{\varepsilon, \lambda, \ell} \\ &\leq \mathbb{E}_{\ell}^{(1)} M_0^{\varepsilon, \lambda, \ell} \\ &= 1. \end{aligned}$$

As a consequence, if $A > 0$ is such that $\lambda A > c$, we have

$$\begin{aligned} \mathbb{P}_{\ell}^{(1)}\left(H_{\ell}^f(\tau_{-\ell}) \geq \mu \sum_{k=0}^{\tau_{-\ell}-1} \frac{\log(P_{\ell}(k))}{P_{\ell}(k)} + C(\varepsilon, \mathbf{q}) \sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_{\ell}(k)} + A \log \ell\right) \\ \leq \mathbb{P}_{\ell}^{(1)}\left(\exp\left(\lambda H_{\ell}^f(\tau_{-\ell}) - \lambda \mu \sum_{k=0}^{\tau_{-\ell}-1} \left(\frac{\log(P_{\ell}(k))}{P_{\ell}(k)} + \frac{1}{P_{\ell}(k)}\right)\right) \geq \ell^{A\lambda}\right) \leq \frac{1}{\ell^{A\lambda}} = O(\ell^{-c}). \end{aligned}$$

From (2.6.2), $\sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_{\ell}(k)}$ is well controlled by a constant times $\log \ell$. Thus it is sufficient to show that

$$\mathbb{P}_{\ell}^{(1)}\left(\sum_{k=0}^{\tau_{-\ell}-1} \frac{\log(P_{\ell}(k))}{P_{\ell}(k)} \mathbf{1}_{P_{\ell}(k) \geq k_0} \geq \frac{K'}{\mu}(\log \ell)^2\right) = O(\ell^{-c}), \quad (2.6.10)$$

where $k_0 > 0$ will be chosen later in the proof. However, one can note that since $x \mapsto x/\log x$ is increasing on $[e, \infty)$, we have

$$\sum_{k=3}^{\tau_{-\ell}-1} \frac{\log(P_{\ell}(k))}{P_{\ell}(k)} \mathbf{1}_{P_{\ell}(k) \geq k \log k} \leq \sum_{k=3}^{\tau_{-\ell}-1} \frac{\log k + \log \log k}{k \log k} \leq 2 \log \tau_{-\ell}$$

Moreover, using Lemma 2.2.5 and then that $\mathbb{E}_\infty^{(1)}(\ell/P_\infty(\ell)) = O(1)$ by Lemma 4 of [BCM18] as $\ell \rightarrow \infty$,

$$\mathbb{P}_\ell^{(1)}(\tau_{-\ell} > \ell^{c+1}) = \mathbb{E}_\infty^{(1)} \frac{1 + \ell}{P_\infty(\lfloor \ell^{1+c} \rfloor) + \ell} \leq 2\mathbb{E}_\infty^{(1)} \frac{\ell}{P_\infty(\lfloor \ell^{c+1} \rfloor)} = O(\ell^{-c}). \quad (2.6.11)$$

Consequently,

$$\mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_{-\ell}-1} \frac{\log(P_\ell(k))}{P_\ell(k)} \mathbf{1}_{P_\ell(k) \geq k \log k} \geq 2(c+1) \log \ell \right) = O(\ell^{-c}).$$

Hence, so as to prove (2.6.10), it suffices to show that

$$\mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_{-\ell}-1} \frac{\log(P_\ell(k))}{P_\ell(k)} \mathbf{1}_{k \log k \geq P_\ell(k) \geq k_0} \geq \frac{K'}{\mu} (\log \ell)^2 \right) = O(\ell^{-c}).$$

But one can upper bound

$$\begin{aligned} \sum_{k=0}^{\tau_{-\ell}-1} \frac{\log(P_\ell(k))}{P_\ell(k)} \mathbf{1}_{k \log k \geq P_\ell(k) \geq k_0} &\leq \sum_{k=0}^{\tau_{-\ell}-1} \frac{\log k + \log \log k}{P_\ell(k)} \mathbf{1}_{k \log k \geq P_\ell(k) \geq k_0} \\ &\leq (\log \tau_{-\ell} + \log \log \tau_{-\ell}) \sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_\ell(k)} \mathbf{1}_{k \log k \geq P_\ell(k) \geq k_0}. \end{aligned}$$

So, using again (2.6.11) and then taking ε small enough so that $\mu = \frac{(1+3\varepsilon)^3 p_{\mathbf{q}}}{1-\varepsilon}$ is close to $\frac{p_{\mathbf{q}}}{2}$, we just need to prove that for all $p_{\mathbf{q}} > \delta > 0$,

$$\mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_\ell(k)} \mathbf{1}_{P_\ell(k) \geq k_0} \geq \frac{c + \frac{3}{2}}{p_{\mathbf{q}} - \delta} (\log \ell)^2 \right) = O(\ell^{-c}). \quad (2.6.12)$$

In order to show that we will use the supermartingale $M^{\delta, k_0, \ell}$ of Lemma 2.6.9 for k_0 chosen large enough. Applying Markov's inequality and Fatou's lemma, we get that for $K'' = (c + 3/2)/(p_{\mathbf{q}} - \delta)$,

$$\begin{aligned} \mathbb{P}_\ell^{(1)} \left(\sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_\ell(k)} \mathbf{1}_{P_\ell(k) \geq k_0} \geq K'' \log \ell \right) &\leq \ell^{-(p_{\mathbf{q}} - \delta)K''} \mathbb{E}_\ell^{(1)} \exp \left((p_{\mathbf{q}} - \delta) \sum_{k=0}^{\tau_{-\ell}-1} \frac{1}{P_\ell(k)} \mathbf{1}_{P_\ell(k) \geq k_0} \right) \\ &= \ell^{-(p_{\mathbf{q}} - \delta)K''} \mathbb{E}_\ell^{(1)} M_{\tau_{-\ell}}^{\delta, k_0, \ell} \\ &\leq \ell^{-(p_{\mathbf{q}} - \delta)K''} \mathbb{E}_\ell^{(1)} M_0^{\delta, k_0, \ell} \\ &= \ell^{-(p_{\mathbf{q}} - \delta)K''} \frac{1}{h_\ell^\downarrow(1)} \\ &= O(\ell^{3/2 - (p_{\mathbf{q}} - \delta)K''}) = O(\ell^{-c}), \end{aligned}$$

hence (2.6.12). \square

The proof of Proposition 2.6.5 is then exactly the same as the proof of Proposition 2.6.1 using Corollary 2.6.10:

Proof of Proposition 2.6.5. Since the diameter is upper bounded by twice the maximal height, it suffices to show that for all $\delta > 0$

$$\mathbb{P} \left(\exists e \in \text{Edges}(\mathfrak{M}^{(\ell)}), d_{\text{gr}}^\dagger(f_r, e) \geq (9 + \delta)(\log \ell)^2 \right) \xrightarrow{\ell \rightarrow \infty} 0,$$

inasmuch as for all $f \in \text{Faces}(\mathfrak{M}^{(\ell)})$, if e is an edge surrounding f , then $d_{\text{gr}}^\dagger(f_r, f) \leq d_{\text{gr}}^\dagger(f_r, e) + 1$. Next, we distinguish whether the number of edges is too large or not: let $\varepsilon > 0$.

$$\begin{aligned} &\mathbb{P} \left(\exists e \in \text{Edges}(\mathfrak{M}^{(\ell)}), d_{\text{gr}}^\dagger(f_r, e) \geq (9 + \delta)(\log \ell)^2 \right) \\ &\leq \mathbb{E} \left(\#\left\{ e \in \text{Edges}(\mathfrak{M}^{(\ell)}); d_{\text{gr}}^\dagger(f_r, e) \geq (9 + \delta)(\log \ell)^2 \right\} \mathbf{1}_{\#\text{Edges}(\mathfrak{M}^{(\ell)}) \leq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2}} \right) \\ &\quad + \mathbb{P} \left(\#\text{Edges}(\mathfrak{M}^{(\ell)}) \geq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \right), \end{aligned}$$

and the last term is smaller than ε for ℓ large enough by (2.3.3). Henceforth we focus on the first term. Conditionally on $\mathfrak{M}^{(\ell)}$, let E be a uniform edge of $\mathfrak{M}^{(\ell)}$.

$$\begin{aligned} & \mathbb{E}\left(\#\left\{e \in \text{Edges}(\mathfrak{M}^{(\ell)}); d_{\text{gr}}^\dagger(f_r, e) \geq (9 + \delta)(\log \ell)^2\right\} \mathbf{1}_{\#\text{Edges}(\mathfrak{M}^{(\ell)}) \leq \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2}}\right) \\ & \leq \mathbb{E}\left(\frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \mathbb{P}(d_{\text{gr}}^\dagger(f_r, E) \geq (9 + \delta)(\log \ell)^2 | \mathfrak{M}^{(\ell)})\right) \\ & = \frac{2b_{\mathbf{q}}}{\varepsilon} \ell^{3/2} \mathbb{P}(d_{\text{gr}}^\dagger(f_r, E) \geq (9 + \delta)(\log \ell)^2). \end{aligned}$$

The expression on the last line tends to zero by Corollary 2.6.10 when $\ell \rightarrow \infty$ since for $c = 3/2$ we have $(c + 3)(c + 5/2)/2 = 9$. \square

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Chapitre 3

Distances on the CLE_4 , critical Liouville quantum gravity and $3/2$ -stable maps

Ce chapitre est issu de l'article [Kam23a] à paraître dans Probability and Mathematical Physics.

Abstract The purpose of this article is threefold. First, we show that when one explores a conformal loop ensemble of parameter $\kappa = 4$ (CLE_4) on an independent 2-Liouville quantum gravity (2-LQG) disk, the surfaces which are cut out are independent quantum disks. To achieve this, we rely on approximations of the explorations of a CLE_4 : we first approximate the $\text{SLE}_4^{(\mu)}(-2)$ explorations for $\mu \in \mathbb{R}$ using explorations of the CLE_κ as $\kappa \uparrow 4$ and then we approximate the uniform exploration by letting $\mu \rightarrow \infty$. Second, we describe the relation between the so-called natural quantum distance and the conformally invariant distance to the boundary introduced by Werner and Wu. Third, we establish the scaling limit of the distances from the boundary to the large faces of $3/2$ -stable maps and relate the limit to the CLE_4 -decorated 2-LQG.

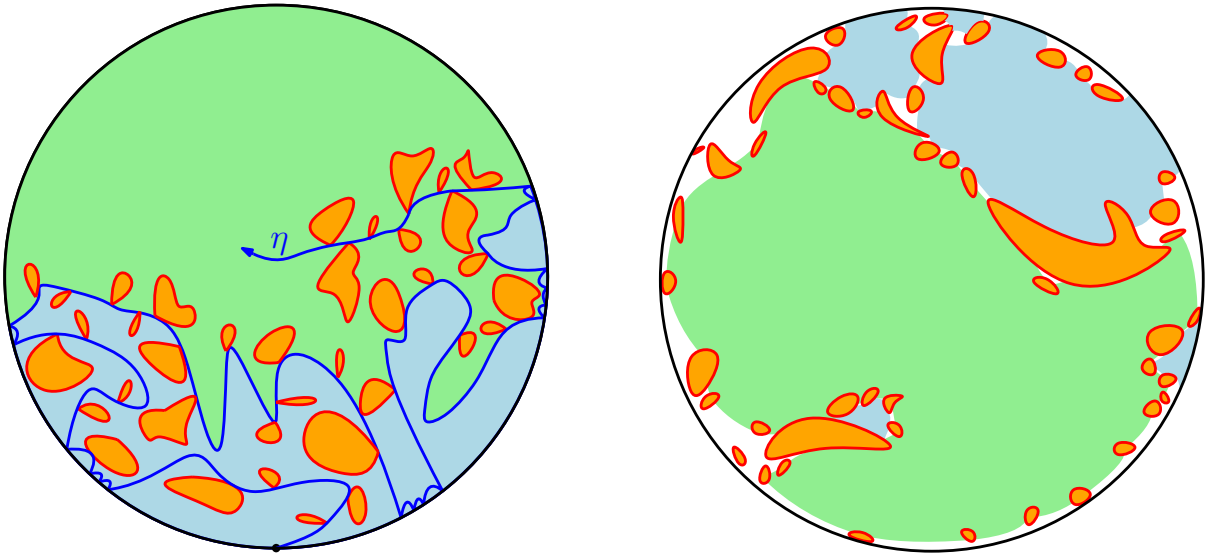


Figure 3.1 – Left: illustration of the $\text{SLE}_4^{(\mu)}(-2)$ exploration of the CLE_4 ensemble. Right: illustration of the uniform exploration of the CLE_4 ensemble. The blue curve η is the trunk of the exploration. The unexplored region is in green, the domains encircled by loops are in orange and the cut out domains are in blue.

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3.1 Introduction

3.1.1 Motivation

This article is motivated by the study of random 3/2-stable maps and aims to be a first step in the study of their scaling limit when their perimeter goes to ∞ . Informally, a 3/2-stable map is a random map such that every face of degree $k \geq 1$ has a weight of order k^{-2} , see Subsection 3.1.2 for the precise definition. Even though we proved in Chapter 2 that 3/2-stable maps, equipped with the graph distance on the dual map, do not satisfy a scaling limit in the usual sense of Gromov-Hausdorff or Gromov-Prokhorov, we show here that at a smaller scale their limiting behavior is related to the conformal loop ensemble CLE_4 , where we measure the lengths of the loops using an independent critical Liouville quantum gravity. In order to make sense of the scaling limit, one can order the faces of a 3/2-stable map in the non-increasing order of their degrees. Similarly, we order the loops of the CLE_4 on the unit disk \mathbb{D} in the non-increasing order of quantum boundary lengths, where the boundary $\partial\mathbb{D}$ is also viewed as a loop. We conjecture that the scaling limit of the 3/2-stable maps is the set of the loops of the CLE_4 equipped with a distance d_{Lamperti} which is defined as a Lamperti transform of the so-called quantum distance introduced in [AHPS23], in the sense that the rescaled infinite matrix of distances with the above ordering converges in distribution for the product topology. Such a random metric space which appears in random conformal geometry and which is proven to be the scaling limit of random planar maps has not been uncovered since the introduction of the Brownian map [MM06, Mie13, LG13, MS20, MS21a, MS21b]. However, the quantum distance of [AHPS23] is so far only defined as a distance from the loops to the boundary. In this work, we establish the scaling limit for the distances to the boundary (which provides

in particular the tightness for this topology) and we give some insight on the relation between several distances to the boundary in the CLE_4 .

3.1.2 Background

The conformal loop ensemble CLE_4 is a random collection of disjoint simple loops included in some simply connected domain. It was introduced and studied in [She09] as part of the one parameter family of simple conformal loop ensembles CLE_κ for $\kappa \in (8/3, 8)$, whose loops are no longer simple when $\kappa > 4$. In [She09], Sheffield also defined a family of branching explorations parametrized by some “drift” $\mu \in \mathbb{R}$ of the CLE_4 starting from the boundary and discovering the loops, called the $\text{SLE}_4^{(\mu)}(-2)$ exploration. According to [MSW17] this exploration can be described using a random simple continuous curve called the trunk of the exploration which hits the loops it discovers (see Figure 3.1). When the trunk hits the boundary of the explored region, the unexplored domain is split into two unexplored regions. One can choose for example to continue the exploration of the region which contains a particular point which is called a target point. The whole branching exploration can then be seen as a coupling of the explorations with target points. In [WW13], Werner and Wu introduced a uniform exploration of the CLE_4 which can be seen as the limit of the $\text{SLE}_4^{(\mu)}(-2)$ exploration as $\mu \rightarrow \infty$. Informally, it consists in discovering the loop next to a random point chosen according to the harmonic measure on the boundary of the explored region and then updating this random point. The uniform exploration of the CLE_4 is also a branching exploration, given that the unexplored domain is regularly split into two unexplored domains. The time parametrization of the uniform exploration in [WW13] is interpreted as a distance to the boundary. In an unfinished work [SWW], Sheffield, Watson and Wu developed a strategy to prove that this distance from the boundary to the loops of the CLE_4 , which will be denoted by d_{WW} , can be extended to a conformally invariant distance between all the loops of the CLE_4 , including the boundary of the domain. However, we will not use this result in this work.

More recently, the CLE_κ was shown to interact nicely with an independent Liouville quantum gravity surface of parameter $\sqrt{\kappa}$ on the same simply connected domain ($\sqrt{\kappa}$ -LQG). An LQG disk parametrized by a simply connected domain can be defined as a well-chosen version of a Gaussian free-field on this domain which provides a way to measure quantum lengths of some curves and quantum areas of sub-domains. In [MSW22], the law of the quantum boundary length of the unexplored region along the exploration of the CLE_κ for $\kappa < 4$ parametrized by the quantum length of the trunk is shown to behave as some branching Markov process called a growth-fragmentation process, which was introduced by Bertoin in [Ber17]. For the case $\kappa = 4$, building on the rich interplay between CLE ’s, LQG’s, and Brownian motion, and particularly on [DMS21], Aru, Holden, Powell and Sun identified in [AHPS23] a coupling between a uniform CLE_4 exploration, an independent 2-LQG disk and a Brownian excursion in the upper half-plane. Thanks to [AdS22], this coupling actually enables to describe the quantum boundary length of the explored region along the uniform exploration, parametrized by a so-called quantum natural distance to the boundary, as a growth-fragmentation process.

Independently, Boltzmann 3/2-stable maps were first studied in [BCM18] in the infinite volume setting. In this introduction, we will only talk about finite 3/2-stable maps. Recall that a planar map \mathbf{m} is a connected planar graph embedded in the sphere seen up to orientation preserving homeomorphism, with a distinguished oriented edge \vec{e}_r . The face on the right of the root edge is called the root face and is written f_r . The number of edges counted with multiplicity around a face f is called its degree and is denoted by $\deg(f)$. We focus on bipartite planar maps, which only have faces with even degrees. For all $\ell \in \mathbb{N}$, let us denote by $\mathcal{M}^{(\ell)}$ the set of bipartite planar maps of perimeter 2ℓ , i.e. such that the degree of the root face is 2ℓ . Let $\mathbf{q} = (q_k)_{k \geq 0}$ be a non-zero sequence of non-negative real numbers. The Boltzmann weight of a map $\mathbf{m} \in \mathcal{M}^{(\ell)}$ is defined by

$$w_{\mathbf{q}}(\mathbf{m}) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r\}} q_{\deg(f)/2}.$$

When the *partition function* $W^{(\ell)} := \sum_{\mathbf{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathbf{m})$ is finite, we say that the weight sequence \mathbf{q} is admissible. We further say that \mathbf{q} is *critical non-generic of type 2* when

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-2} \quad (3.1.1)$$

for some constants $c_{\mathbf{q}}, p_{\mathbf{q}} > 0$. Such a sequence \mathbf{q} does exist and an explicit example is given by Lemma 6.1 of [BC17]. Intuitively, (3.1.1) can be obtained by carefully choosing \mathbf{q} with $q_k \sim p_{\mathbf{q}} e^{-k+1} k^{-2}$ and such a polynomial decay favors the appearance of faces with high degrees. The Boltzmann probability measure on $\mathcal{M}^{(\ell)}$ is then defined by $\mathbb{P}^{(\ell)}(\mathbf{m}) = \frac{w_{\mathbf{q}}(\mathbf{m})}{W^{(\ell)}}$ for all $\mathbf{m} \in \mathcal{M}^{(\ell)}$. The associated expectation is written $\mathbb{E}^{(\ell)}$. A random variable of law $\mathbb{P}^{(\ell)}$ is called a *3/2-stable map* of perimeter 2ℓ . In what follows, \mathbf{q} will be assumed critical non-generic of type 2.

The distances that we will study on these maps are the graph distance on the dual map, called the *dual graph distance*, obtained by exchanging the vertices with the faces, written d_{gr}^{\dagger} , and the *first passage percolation distance* which is obtained by putting i.i.d. parameter 1 exponential random lengths on each edge of the dual map, written d_{fpp}^{\dagger} . The study of the distance d_{fpp}^{\dagger} is simpler than d_{gr}^{\dagger} and often gives an intuition on the behaviour of d_{gr}^{\dagger} . These distances behave in a dramatically different way from the (primal) graph distance on the map since faces with high degrees play the role of hubs. The scaling limit of stable maps equipped with the primal graph distance was first studied in [LGM11], then it was recently constructed in [CMR25] and it is conjectured to be related to a quantum metric on the CLE carpet/gasket which has not been defined yet.

The main technique in [BCM18] and [Kam23b] to study these distances is the peeling exploration first introduced in [Bud16]. It consists in a step by step Markovian exploration of the map where at each step, we “peel” an edge on the boundary of the unexplored region and discover what is behind it. In this setting, we can also observe the evolution of the perimeter of the unexplored region (which is the number of edges on its boundary).

3.1.3 Main results

Our main contributions can be informally summarized as follows:

- We describe the law of the unexplored regions during the uniform exploration (resp. $\text{SLE}_4^{(\mu)}(-2)$ exploration) of the CLE_4 -decorated 2-LQG disk when it is parametrized by the quantum natural distance $d_{\mathbf{q}}$ (resp. by the quantum length of the trunk of the $\text{SLE}_4^{(\mu)}(-2)$). In particular, we show that for all $t \geq 0$, conditionally on the quantum boundary lengths of the loops which are discovered before time t and conditionally on the quantum boundary lengths of the unexplored regions, the unexplored regions are independent 2-LQG disks (with prescribed boundary lengths). See Theorem 3.1.2.
- We relate the distance d_{WW} to the Lamperti transform of the quantum natural distance $d_{\mathbf{q}}$ with some continuous increasing process with stationary increments along a particular branch of the uniform exploration. See Theorem 3.1.3.
- We establish the scaling limit of the distances to the root in 3/2-stable maps. More precisely, we show that the scaling limit of the degrees of large faces corresponds to the quantum boundary lengths of the loops of the CLE_4 (this scaling limit can actually be seen as a simple consequence of the results of [BBCK18], [AHPS23] and [AdS22]), while the scaling limit of the distances from these faces to the boundary corresponds to the Lamperti transform of $d_{\mathbf{q}}$. However the quantum distance $d_{\mathbf{q}}$ is not the scaling limit of some distance in 3/2-stable maps, but corresponds to the scaling limit of the time of exploration in the peeling explorations of the maps. See Theorem 3.1.4.

Let us already state our main results more precisely even if the exact definitions will only arise in the next sections.

Exploration of CLE_4 and LQG. Recall that the uniform exploration of the CLE_4 is a branching exploration, in the sense that the unexplored domain is gradually split into different unexplored regions which are explored in a Markovian way. See Subsection 3.2.1 for a definition of the uniform exploration. Consider this uniform exploration on top of an independent 2-LQG disk (or 2-quantum disk), which is a simply connected domain equipped with a quantum area measure and a quantum boundary length measure which are both constructed using a Gaussian free field (see Subsection 3.2.2 for the definition of LQG disks). For simplicity, we choose to write the first results for a particular branch of the exploration of the CLE_4 on the unit disk \mathbb{D} equipped with an independent 2-LQG disk, namely the branch of the locally largest component: at each time the region remaining to be explored splits in two regions, we choose to explore the region which has the largest quantum boundary length and the other domain is

called a cut out domain (see Subsection 3.2.3 for the definition of the exploration branch of the locally largest component). See the right side of Figure 3.1. We parametrize the uniform exploration by the quantum natural distance to the boundary and the $\text{SLE}_4^{(\mu)}(-2)$ exploration by the quantum length of the trunk. See Subsection 3.3.2 for the definition of the $\text{SLE}_4^{(\mu)}(-2)$ exploration. These parametrizations are characterized as follows: if we denote by $(Y_t)_{t \geq 0}$ (resp. $(Y_t^\mu)_{t \geq 0}$) the càdlàg process of the quantum boundary length of the locally largest component along the uniform exploration (resp. $\text{SLE}_4^{(\mu)}(-2)$ exploration), then we have for all $t \geq 0$ the limit in probability

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \#\{s \in [0, t], |\Delta Y_s| \in [\varepsilon, 2\varepsilon]\} = \lim_{\varepsilon \rightarrow 0} \varepsilon \#\{s \in [0, t], |\Delta Y_s^\mu| \in [\varepsilon, 2\varepsilon]\} = t,$$

where for any càdlàg process $(X_t)_{t \geq 0}$ we write $\Delta X_t := X_t - X_{t-}$ for all $t \geq 0$. The positive jumps of $(Y_t)_{t \geq 0}$ or $(Y_t^\mu)_{t \geq 0}$ occur when the exploration draws a loop while the negative jumps happen when a part of the unexplored region is cut out by the exploration.

Let us recall the positive self-similar Markov process studied in [AdS22], which corresponds to a particular case in a family of such processes introduced in [BBCK18] for $\theta = 1$. Let Λ_1 be the image by $x \mapsto \log x$ of the measure $1/(\pi(x(1-x))^2) \mathbf{1}_{x \in (1/2, 1) \cup (1, \infty)} dx$. We denote by ξ the Lévy process with no Brownian part, with drift $-2/\pi$ and Lévy measure Λ_1 . More precisely, the Laplace exponent $\Psi : \mathbb{R}_+ \rightarrow (-\infty, \infty]$ of ξ , which is characterized by $\mathbb{E}(\exp(q\xi(t))) = \exp(t\Psi(q))$ for all $t, q \geq 0$, is given by

$$\forall q \geq 0, \quad \Psi(q) := \frac{-2}{\pi}q + \int_{\mathbb{R}} (e^{qy} - 1 + q(1 - e^y)) \Lambda_1(dy).$$

If $\alpha \in \mathbb{R}$, we define the Lamperti time-substitution

$$\tau(t) = \inf \left\{ r \geq 0 ; \int_0^r e^{-\alpha\xi(s)} ds \geq t \right\}. \quad (3.1.2)$$

For all $x > 0$, we write $P_x^{(\alpha)}$ the distribution of the time-changed process $X^{(\alpha)}(t) = xe^{\xi(\tau(tx^\alpha))}$ with the convention that $X^{(\alpha)}(t) = \partial$ for $t \geq \zeta := x^{-\alpha} \int_0^\infty e^{-\alpha\xi(s)} ds$, where ∂ is a cemetery point. Then $X^{(\alpha)}$ is a positive self-similar Markov process of index α , in the sense that for every $x > 0$, the law of $(xX^{(\alpha)}(x^\alpha t))_{t \geq 0}$ under $P_1^{(\alpha)}$ is $P_x^{(\alpha)}$. By the definition of ξ , we can see that either $X^{(\alpha)}$ is absorbed a.s. at the cemetery point ∂ after a finite time, or it converges a.s. exponentially fast to zero. In the case $\alpha = -1$, we have $\zeta < +\infty$.

The process $X^{(\alpha)}$ is the building block of a branching process: when $X^{(\alpha)}$ performs a negative jump at time $t \geq 0$, one can interpret this jump as the splitting event of a ‘‘cell’’ of size $X^{(\alpha)}(t-)$ into two cells of sizes $X^{(\alpha)}(t)$ and $X^{(\alpha)}(t-) - X^{(\alpha)}(t)$. Then one can run another self-similar process with the same law as $X^{(\alpha)}$ starting from $X^{(\alpha)}(t-) - X^{(\alpha)}(t)$. We will not delve into these branching processes until Subsection 3.6.3.

Remark 3.1.1. *The positive self-similar Markov process $X^{(\alpha)}$ can also be described using its jump kernel: at time $t \geq 0$ the process $X^{(\alpha)}$ makes a jump of size $y \in \mathbb{R}^*$ from $X^{(\alpha)}(t) = x$ according to the jump kernel*

$$K(x, dy) = x^\alpha \frac{1}{\pi((1+y/x)y/x)^2} \frac{dy}{x} \mathbf{1}_{y/x-1 \in (1/2, 1) \cup (1, \infty)} = \frac{x^{3+\alpha}}{\pi y^2 (x+y)^2} dy \mathbf{1}_{y > -x/2 \text{ and } y \neq 0} dy.$$

This is the way closely related self-similar Markov processes are described in Paragraph 1.1.3 of [MSW22].

Our first main result is the analogue of Theorem 1.2 of [MSW22] in the case $\kappa = 4$ for the $\text{SLE}_4^{(\mu)}(-2)$ exploration and for the uniform exploration.

Theorem 3.1.2. *Let $\mu \in \mathbb{R}$. The processes $(Y_t)_{t \geq 0}$ and $(Y_t^\mu)_{t \geq 0}$ have the same law as $(X^{(-1)}(\pi t))_{t \geq 0}$. Moreover, conditionally on $(Y_s)_{0 \leq s \leq t}$ (resp. $(Y_s^\mu)_{0 \leq s \leq t}$), if we denote by $(x_i)_{i \geq 1}$ the sizes of the jumps of $(Y_s)_{0 \leq s \leq t}$ (resp. $(Y_s^\mu)_{0 \leq s \leq t}$) up to time t ranked in the non-increasing order of absolute value, then the corresponding domains which are encircled by loops, associated with the positive jumps, or which are cut out, associated with the negative jumps, and the unexplored region are independent 2-quantum disks of boundary lengths $|x_i|$'s and Y_t (resp. Y_t^μ).*

Note that the fact that $(Y_t)_{t \geq 0}$ has the same law as $(X^{(-1)}(\pi t))_{t \geq 0}$ can be seen as a direct consequence of Theorem 5.5 of [AHPS23] and Theorem 1.1 of [AdS22]. This is not the case for the statement on the quantum disks since the half-plane Brownian excursion of [AHPS23] does not characterize the pair $\text{CLE}_4/2$ -quantum disk. Besides, the above theorem extends Theorem 1.1 of the recent article [AG23a] which focuses on the regions encircled by loops. Actually, in the proof of the above result, we will use a few results of [AG23a].

Quantum natural distance and the conformally invariant distance. For all $u \geq 0$, we denote by C_u the unexplored region at time u when following the locally largest component in the uniform exploration of [WW13] parametrized by the distance d_{WW} . We stress that this time parametrization is *different from the time parametrization by the quantum natural distance* used in the previous paragraph, which was denoted by t . In general, in this paper the letter $t \geq 0$ is related to the quantum natural distance. One can extend the definition of the quantum natural distance from loops to the boundary in order to measure also the quantum natural distance $d_{\text{q}}(\partial\mathbb{D}, C_u)$ from the unexplored region C_u to the boundary, so that if u is a time of discovery of a loop \mathcal{L} , then $d_{\text{q}}(\partial\mathbb{D}, C_u)$ corresponds to the quantum distance $d_{\text{q}}(\partial\mathbb{D}, \mathcal{L})$ from the boundary to \mathcal{L} . See Subsection 3.2.3 for the precise definition. Let $(\sigma(t))_{t \geq 0}$ be the right-continuous inverse of $u \mapsto d_{\text{q}}(\partial\mathbb{D}, C_u)$. The next result, a consequence of Theorem 3.1.2, gives some insight on $(\sigma(t))_{t \geq 0}$.

Theorem 3.1.3. *The process $(\sigma(t))_{t \geq 0}$ is continuous and increasing. Moreover, the continuous increasing process S characterized by*

$$\forall t \geq 0, \quad \sigma(t) = S\left(\int_0^t \frac{1}{Y_s} ds\right)$$

has stationary increments.

If one identifies Y with the process $(X^{(-1)}(\pi t))_{t \geq 0}$ using Theorem 3.1.2, then one can also define the process S by setting $S(t) = \sigma(\tau^{-1}(\pi t)/\pi)$, where τ^{-1} is the inverse bijection of τ defined in (3.1.2) with $\alpha = -1$, which can also be written $\tau^{-1}(t) = \int_0^t e^{\xi(s)} ds$ for all $t \geq 0$. The above result motivates the introduction of another distance to the boundary that we first define here only for the loops which are discovered by the branch of the locally largest component (see (3.1.5) for its definition for all the loops of the CLE_4): for every loop \mathcal{L} which is discovered by the exploration of the locally largest component, we set $d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}) := \int_0^{d_{\text{q}}(\partial\mathbb{D}, \mathcal{L})} ds/Y_s$. Then we have $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L}) = S(d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}))$. However, we do not believe that S has independent increments. The independence of the increments would entail that S is a deterministic linear function, hence d_{Lamperti} and d_{WW} would be equal up to some multiplicative constant and Subsection 3.7.2 provides some heuristics against it.

3/2-stable maps and CLE_4 -decorated LQG. Denote by $(P_*(n))_{n \geq 0}$ the perimeter process during the peeling exploration of the locally largest component (in terms of perimeter) under the law $\mathbb{P}^{(\ell)}$. Then it is known from Proposition 6.6 of [BBCK18] (which comes from Theorem 1 of [BCM18]) that under $\mathbb{P}^{(\ell)}$ the rescaled process $(P_*(\lfloor \ell t \rfloor)/\ell)_{t \geq 0}$ converges towards $(X^{(-1)}(\pi p_{\mathbf{q}} t))_{t \geq 0}$ where $p_{\mathbf{q}}$ is introduced in (3.1.1) and where $X^{(-1)}$ is defined using the Lamperti transform (3.1.2), so that by Theorem 3.1.2 the quantum length of the trunk of the $\text{SLE}_4^{(\mu)}(-2)$ exploration and the quantum natural distance correspond to the scaling limit of the time in the peeling exploration. The next results extend this relation between 3/2-stable maps and CLE_4 -decorated LQG by establishing the scaling limit of the distances to the boundary.

By using some particular peeling algorithm, we can explore the map in a way that follows the growth of the balls centred at the root face f_r or equivalently such that the distances from the edges on the boundary of the unexplored region to the root face are roughly the same. For example, for d_{gr}^\dagger , the idea is to explore the map layer by layer: at first the edges at distance 0 from the root face, then at distance 1, etc. This peeling algorithm is called the peeling by layers algorithm. For d_{fpp}^\dagger , we use the uniform peeling algorithm which takes at each step an edge uniformly at random on the boundary. The peeling explorations also have a branching nature since there are peeling steps when the unexplored region splits into two regions. A particular branch of the exploration, called the peeling exploration of the locally largest component, consists in choosing to explore the region which has the largest perimeter at each splitting event. See Subsection 3.6.1 for the definition of the peeling explorations. Let $\partial\bar{\mathfrak{e}}_n$ be the boundary of the explored region during the exploration of the locally largest component at time $n \geq 0$,

using the appropriate peeling exploration for each distance. Then we prove that the radius of the balls during the exploration evolves in the following way:

$$\text{under } \mathbb{P}^{(\ell)}, \quad (d_{\text{fpp}}^\dagger(f_r, \partial\bar{\mathfrak{e}}_{[\ell t]}))_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\int_0^t \frac{1}{2Y(p_{\mathbf{q}}s)} ds \right)_{t \geq 0}; \quad (3.1.3)$$

$$\text{under } \mathbb{P}^{(\ell)}, \quad (d_{\text{gr}}^\dagger(f_r, \partial\bar{\mathfrak{e}}_{[\ell t]}) / (\log \ell))_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\int_0^t \frac{p_{\mathbf{q}}}{2Y(p_{\mathbf{q}}s)} ds \right)_{t \geq 0}; \quad (3.1.4)$$

for the J_1 Skorokhod topology. Note that the above scaling limit can be seen as the Lamperti transform used to come back from the time parametrization of the self-similar Markov process $X^{(-1)}$ to the time parametrization of the Lévy process ξ in the definition of $X^{(-1)}$. In other words, the scaling limit of the exploration time is the time parametrization of $X^{(-1)}$ while the scaling limit of the distances to the root face is the time parametrization of ξ .

All our statements only focused so far on the locally largest component. But by the Markov property of Theorem 3.1.2 for the CLE_4 exploration, and by the spatial Markov property of the peeling exploration, they can be extended to the whole exploration tree. For each face f of a map \mathbf{m} , we may define the exploration time $n(f)$ of f as the number of peeling steps before discovering f which occurred in the peeling exploration branch of \mathbf{m} which discovers f . Note that $n(f)$ depends on the choice of the peeling algorithm. See Subsection 3.6.3 for a precise definition of $n(f)$. This enables us to state the scaling limit of the degree, the time of exploration and the distance from the boundary to the large faces of $3/2$ -stable maps towards the quantum boundary length, the quantum natural distance (or quantum length of the trunk depending on the exploration) and its Lamperti transform of a CLE_4 -decorated LQG disk. For every loop \mathcal{L} of the CLE_4 , recall that we denote by $d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})$ the quantum natural distance to \mathcal{L} . We also denote by $d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L})$ its Lamperti transform defined by

$$d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}) := \int_0^{d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})} \frac{dt}{Y_{\mathcal{L}}(t)}, \quad (3.1.5)$$

where $(Y_{\mathcal{L}}(t))_{0 \leq t \leq d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})}$ is the quantum boundary length of the unexplored region surrounding \mathcal{L} parametrized by the quantum natural distance. One can define it using an exploration with a target point z inside the loop. See Subsection 3.2.3 for a definition of $Y_{\mathcal{L}}$ and $d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L})$.

Theorem 3.1.4. *For all $\ell \geq 1$, let $\mathfrak{M}^{(\ell)}$ of law $\mathbb{P}^{(\ell)}$. Let $(f_i^{(\ell)})_{i \geq 1}$ be the collection of faces of $\mathfrak{M}^{(\ell)}$ ranked in the non-increasing order of degree. Let $(n(f_i^{(\ell)}))_{i \geq 1}$ be their exploration times using the peeling by layers algorithm. Let $(\mathcal{L}_i)_{i \geq 1}$ be the collection of loops of the CLE_4 ranked in the non-increasing order of quantum boundary length. Let us denote by $\nu_{h^2}^2(\mathcal{L})$ the quantum boundary length of a loop \mathcal{L} . Then, for the product topology,*

$$\left(\frac{\deg(f_i^{(\ell)})}{2\ell}, \frac{n(f_i^{(\ell)})}{p_{\mathbf{q}}\ell}, \frac{2d_{\text{gr}}^\dagger(f_r, f_i^{(\ell)})}{\log \ell} \right)_{i \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} (\nu_{h^2}^2(\mathcal{L}_i), d_{\mathbf{q}}(\partial\mathbb{D}, \mathcal{L}_i), d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}_i))_{i \geq 1}.$$

The same theorem can be stated for the $\text{SLE}_4^{(\mu)}(-2)$ exploration, replacing the quantum natural distance by the quantum length of the trunk. For the fpp distance, the factor $\log \ell$ is replaced by a factor $1/p_{\mathbf{q}}$ and the peeling by layers exploration is replaced by the uniform peeling exploration.

The scaling limit of the dual graph distances to the root was obtained in [BCK18, BBCK18] in the cases $a \in (2, 5/2]$, relying on the work [BC17] in the cases $a \in (2, 5/2)$ where the scale of the distances to the root under $\mathbb{P}^{(\ell)}$ is ℓ^{a-2} . See Theorem 6.8 in [BBCK18]. The proof of Theorem 3.1.4 makes use of the same kind of branching processes as in [BCK18, BBCK18]. However, dealing with the dual graph distance in the case $a = 2$ is much more delicate than in the case $a > 2$ since it involves Cauchy processes. To obtain Theorem 3.1.4 for the dual graph distance, we use techniques developed in [BCM18] for this special case.

3.1.4 Outline

In Section 3.2 we recall the definition of the uniform exploration of the CLE_4 from [WW13] and the definitions of γ -LQG disks for $\gamma \in (0, 2]$ from [AHPS23]. The definition of the $\text{SLE}_4^{(\mu)}(-2)$ exploration is postponed to the next section.

Then, in Section 3.3, after introducing the remaining definitions, we prove Theorem 3.1.2 for the $\text{SLE}_4^{(\mu)}(-2)$ exploration by approximating the exploration with some well-chosen $\text{SLE}_\kappa^\beta(\kappa-6)$ exploration, using the results of [Leh23] and by approximating the 2-LQG disks with $\sqrt{\kappa}$ -LQG disks relying on [AHPS23].

Next, in Section 3.4, so as to obtain the statement of Theorem 3.1.2 for the uniform exploration, we approximate the uniform exploration with the $\text{SLE}_4^{(\mu)}(-2)$ exploration as $\mu \rightarrow \infty$. This section relies partly on the same ideas as in Section 3.3.

On the basis of Theorem 3.1.2, using the Poissonian structure of the uniform exploration in [WW13], we prove Theorem 3.1.3 in Section 3.5.

In Section 3.6, we first establish the metric growth in the infinite volume setting using results of [BCM18]. By absolute continuity, we prove the convergences (3.1.3) and (3.1.4). We then describe the limit of the whole exploration tree and deduce Theorem 3.1.4. Our approximation results are informally summarized by the following diagram:

$$\begin{array}{ccccc} \text{CLE}_\kappa \text{ exploration} & \xrightarrow[\kappa \uparrow 4]{\text{Sec. 3.3}} & \text{SLE}_4^{(\mu)}(-2) & \xrightarrow[\mu \rightarrow \infty]{\text{Sec.3.4}} & \text{Uniform exploration} & \xleftarrow[\ell \rightarrow \infty]{\text{Sec.3.6}} & \text{3/2-stable map} \\ \text{on a } \sqrt{\kappa}\text{-LQG disk} & & \text{on a 2-LQG disk} & & \text{on a 2-LQG disk} & & \text{of perimeter } \ell \end{array}$$

Finally in Section 3.7, we describe the applications to $O(2)$ -loop decorated planar maps and we argue why we believe that the conformally invariant distance from [WW13] is not the Lamperti transform of the quantum natural distance.

3.2 Preliminaries on the uniform exploration of the CLE_4 and LQG

In this section, we first define the uniform exploration of the CLE_4 and then we give some background on Liouville quantum gravity and quantum disks.

3.2.1 The uniform exploration of the CLE_4

We recall here the uniform exploration of the CLE_4 introduced in [WW13]. Let $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ be the set of continuous functions from \mathbb{R}_+ to \mathbb{R}_+ . Let E be the space of positive excursions defined by

$$E = \{e \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+); e(0) = 0 \text{ and } \tau(e) := \sup\{s > 0; e(s) > 0\} \in (0, \infty)\}.$$

Let λ be the Itô excursion measure on E , which is such that for any $x > 0$, the measure of the subset E_x of excursions of height at least x is finite and if one renormalizes λ so that it is a probability measure on E_x , then after reaching x the excursion has the same law as a standard Brownian motion starting from x which is stopped when attaining zero. See e.g. [LG10] or Chapter XII of [RY05] for a definition of Itô's excursion measure of Brownian motion (see Chapter VI Section 8 of [RW00] for the general theory of excursions of Markov processes). The above description can be seen as a consequence of the point (iii) in Theorem (48.1) of Chapter VI in [RW00]. We denote by \mathbb{H} the upper half-plane. If $e \in E$ is an excursion of duration $\tau(e)$, we define the driving function w on $[0, \tau(e)]$ by

$$\forall t \in [0, \tau(e)], \quad w_t = 2e(t) - \int_0^t \frac{ds}{e(s)}.$$

It is shown in [SW12] that for λ -almost all $e \in E$, the conformal maps g_t characterized by the ODE

$$g_0(z) = z \quad \text{and} \quad \forall t \geq 0, \quad \partial_t g_t(z) = \frac{2}{g_t(z) - w_t}$$

define a simple loop $\gamma = \gamma(e)$ in the upper half plane starting at zero, in the sense that the domain of definition of g_t can be written $\mathbb{H} \setminus \gamma((0, t])$. The above ODE, called Loewner's differential equation, is classically used to define Schramm-Loewner evolutions (SLE). We define the measure μ^0 as the push-forward measure $\gamma_*\lambda$. For all $x \in \mathbb{R}$, we denote by μ^x the measure on loops rooted at x obtained by

translating μ^0 . The measure μ^x is often called the SLE₄ “bubble” measure as in [WW13] or the “one-point pinned” measure as in [SW12].

Let $z \in \mathbb{H}$. Let us first define the uniform exploration targeted at z . Let $(\widehat{\gamma}_u)_{u \geq 0}$ be a Poisson point process (PPP) of intensity $M = \int_{\mathbb{R}} dx \mu^x$. We denote by ∂ the cemetery point. For every loop $\widehat{\gamma}_u$, we denote by \widehat{f}_u the conformal transformation from the connected component of $\mathbb{H} \setminus \widehat{\gamma}_u$ containing z onto \mathbb{H} which fixes the point z and such that $\widehat{f}'_u(z) \in \mathbb{R}_+$. We define $\widehat{F}_u^z = \widehat{F}_u := \circ_{0 \leq v < u} \widehat{f}_v$ where the composition is done in the order of appearance of the maps \widehat{f}_u . This indeed defines a conformal map by p.19 of [WW13]. We also denote by τ^z the first time at which the loop $\widehat{\gamma}_u$ surrounds the point z . We define $(\widetilde{\gamma}_u^z)_{u \leq \tau^z} := (\widehat{F}_u^{-1}(\widehat{\gamma}_u))_{u \leq \tau^z}$.

Lemma 8 from [WW13] shows that it is possible to couple two processes $(\widetilde{\gamma}_u^z)_{u \leq \tau^z}$ and $(\widetilde{\gamma}_u^{z'})_{u \leq \tau^{z'}}$ so that they coincide until the time at which they disconnect z from z' . If we perform this coupling for a dense countable subset of \mathbb{H} then the union of the loops surrounding points of this subset of \mathbb{H} is the CLE₄. The processes $(\widetilde{\gamma}_u^z)_{u \leq \tau^z}$ for $z \in \mathbb{H}$ can then be seen as branches of the uniform exploration. There is also a nested version of the CLE₄ which is obtained recursively by putting independent CLE₄'s in the domains which are surrounded by loops. If we keep on drawing loops with $(\widehat{F}_u^{-1}(\widehat{\gamma}_u))_{u \geq 0}$ after the time τ^z , we then build the nested CLE₄. Except in Sections 3.4 and 3.7.1, we will always consider the non-nested version of the CLE₄ and it will simply be referred to the CLE₄.

For future use, we come back to the unit disk \mathbb{D} via a conformal transformation $\Phi : \mathbb{H} \rightarrow \mathbb{D}$ and we define the domain $D_u \subseteq \mathbb{D}$ of the unexplored points (during the uniform exploration of the CLE₄) at time $u \geq 0$ by $D_u = \Phi(\{z \in \mathbb{H}; \tau^z > u\})$. We also define for all $u \geq 0$ and $z \in \mathbb{D}$ the domain $\widetilde{D}_u^z \subseteq \mathbb{D}$ consisting in the unexplored points at time u in the uniform exploration of the nested CLE₄ targeted at z by $\widetilde{D}_u^z = \Phi((F_u^{\Phi^{-1}(z)})^{-1}(\mathbb{H}))$. The choice of Φ does not change the law of the exploration by Proposition 9 of [WW13]. The time $u \geq 0$ at which a loop \mathcal{L} is discovered can be seen as a distance from the loop to the boundary and we also denote it by $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L})$.

3.2.2 Critical Liouville quantum gravity and quantum disks

We recall here the definitions of the free boundary Gaussian free field (GFF) and of the γ -quantum disks for $\gamma \in (0, 2]$ following [AHP23], which are the surfaces that are expected to appear as scaling limits of discrete models such as random maps coupled with statistical mechanics models. The γ -quantum disks were first introduced in [HRV18] and in [DMS21]. See also [She07] for more details on GFF and [BP24] for more about GFF and Liouville quantum gravity. We follow exactly the definitions in Subsection 4.1 of [AHP23]. Let $D \subset \mathbb{C}$ be a simply connected domain with harmonically non-trivial boundary (i.e. for any point $z \in D$, a Brownian motion started at z almost surely hits ∂D). We denote by $H(D)$ the Hilbert space closure of the space of real functions f of class \mathcal{C}^∞ on D modulo constants which have finite Dirichlet energy (i.e. $\int_D |\nabla f|^2 < \infty$), equipped with the Dirichlet inner product defined for any pair f, g of such functions by

$$(f|g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2z,$$

where d^2z is the Lebesgue measure on \mathbb{C} . Let $(f_n)_{n \geq 0}$ be a Hilbert basis of $H(D)$. Let $(X_n)_{n \geq 0}$ be i.i.d. random variables of law $\mathcal{N}(0, 1)$. The *free boundary GFF* on D is defined as the sum

$$h = \sum_{n \geq 0} X_n f_n,$$

which converges in the space of distributions modulo constants. Even though the resulting random variable h lives in the space of distributions modulo constants, one can fix that constant by assigning some particular value to h on an arbitrary test function. The resulting random distribution is actually in the Sobolev space $H_{\text{loc}}^{-1}(D)$ which is the space of distributions such that their restrictions to any relatively compact open domain $U \subset D$ are in the Sobolev space $H^{-1}(U)$. We endow the space $H_{\text{loc}}^{-1}(D)$ with the corresponding limit topology which is metrizable and separable, such that a sequence converges in $H_{\text{loc}}^{-1}(D)$ if and only if the restrictions to any relatively compact open domain U converge in $H^{-1}(U)$.

Next, we recall that a γ -LQG surface for $\gamma \in (0, 2]$ can be seen as an equivalence class of domains $D \subset \mathbb{C}$ equipped with a variant of the GFF h (a free-boundary GFF plus a continuous random function on \overline{D}) under conformal maps. To that instance of the GFF, we associate an area and a boundary length

measures μ_h^γ and ν_h^γ . We define the *quantum area and quantum boundary length measures* μ_h^γ and ν_h^γ by the limits in probability: for $\gamma = 2$, for all bounded open set $A \subseteq D$ we set

$$\mu_h^2(A) = \lim_{\delta \rightarrow 0} \int_A \left(-h_\delta + \log\left(\frac{1}{\delta}\right) \right) e^{2h_\delta(z)} \delta d^2z$$

and for all bounded open set $B \subseteq \partial D$

$$\nu_h^2(B) = \lim_{\delta \rightarrow 0} \int_B \left(-\frac{h_\delta}{2} + \log\left(\frac{1}{\delta}\right) \right) e^{h_\delta(z)} \delta dz,$$

while for $\gamma \in (0, 2)$, for all bounded open set $A \subseteq D$, we set

$$\mu_h^\gamma(A) = \lim_{\delta \rightarrow 0} (8(2 - \gamma))^{-1} \int_A \exp(\gamma h_\delta(z)) \delta^{\gamma^2/2} d^2z$$

and for all bounded open set $B \subseteq \partial D$

$$\nu_h^\gamma(B) = \lim_{\delta \rightarrow 0} (8(2 - \gamma))^{-1} \int_B \exp\left(\frac{\gamma}{2} h_\delta(z)\right) \delta^{\gamma^2/4} dz,$$

where d^2z is the Lebesgue measure on \mathbb{C} , dz is the Lebesgue measure on \mathbb{R} when D is the upper half-plane \mathbb{H} and $h_\delta(z)$ is the circular mean of h on the circle of center z and radius δ .

We say that two pairs (D, h) and (\tilde{D}, \tilde{h}) , where D, \tilde{D} are domains and h, \tilde{h} are distributions, are equivalent (as γ -quantum surfaces) if there exists a conformal map $\varphi : D \rightarrow \tilde{D}$ such that

$$h = \tilde{h} \circ \varphi + Q_\gamma \log |\varphi'| \quad \text{where} \quad Q_\gamma = 2/\gamma + \gamma/2. \quad (3.2.1)$$

If h, \tilde{h} are absolutely continuous with respect to a GFF plus a continuous function, then $\mu_h^\gamma = \varphi_* \mu_{\tilde{h}}^\gamma$ and $\nu_h^\gamma = \varphi_* \nu_{\tilde{h}}^\gamma$.

Informally, a γ -quantum disk (also called γ -LQG disk) corresponds to the ‘‘domain encircled by a CLE_{γ^2} loop with the restriction of an independent GFF on \mathbb{C} ’’. A quantum disk has a finite boundary length and a finite volume. More precisely, our unit-boundary length γ -quantum disks are defined as in Definition 4.1 for $\gamma \in (0, 2)$ in [AHPS23] and Definition 4.3 for $\gamma = 2$ from [AHPS23]. They correspond up to a constant to the original definition in [DMS21]. Let us summarize their definition although we will not rely on it since we will use directly the results of [AHPS23]. The definition is easier to state when we parametrize the disk by $\mathcal{S} := \mathbb{R} \times (0, \pi) \subset \mathbb{C}$, the infinite strip. Let h_1^γ be a function on \mathcal{S} defined for all $s \in \mathbb{R}$ and $y \in (0, \pi)$ by $h_1^\gamma(s + iy) = \mathcal{B}_s^\gamma$, where \mathcal{B}^γ is defined by:

- (i) For $\gamma \in (0, 2)$, the function $(\mathcal{B}_s^\gamma)_{s \geq 0}$ has the same law as $B_{2s} - (2/\gamma - \gamma/2)s$ conditioned to stay negative for all $s \geq 0$, where $(B_s)_{s \geq 0}$ is a standard Brownian motion, while $(\mathcal{B}_s^\gamma)_{s \geq 0}$ is $(-\sqrt{2})$ times a Bessel process of dimension 3 (which corresponds in some sense to $(B_{2s})_{s \geq 0}$ conditioned to stay negative by a famous result of Williams).
- (ii) The process $(\mathcal{B}_{-s}^\gamma)_{s \geq 0}$ is independent of $(\mathcal{B}_s^\gamma)_{s \geq 0}$ and has the same law as $(\mathcal{B}_s^\gamma)_{s \geq 0}$.

Let h_2 be the orthogonal projection of a free boundary GFF on \mathcal{S} on the closure in $H(\mathcal{S})$ of the functions of mean zero along the vertical lines of \mathcal{S} . Set $h_s^\gamma = h_1^\gamma + h_2$ and let \hat{h}^γ be a random distribution which has the law of $h_s^\gamma - (2/\gamma) \log \nu_{h_s^\gamma}^\gamma(\partial \mathcal{S})$ reweighted by $\nu_{h_s^\gamma}^\gamma(\partial \mathcal{S})^{(4/\gamma^2)-1}$. Then the γ -quantum disk can be defined as the equivalence class of $(\mathcal{S}, \hat{h}^\gamma)$. We denote by (\mathbb{D}, h^γ) the parametrization by the unit disk \mathbb{D} of the unit boundary length γ -quantum disk.

One can define a marked quantum disk by sampling a marked boundary point according to the quantum boundary length measure, but we will not do this in this work. For $b > 0$, we define the b -boundary length γ -quantum disk to be the quantum surface parametrized by $(\mathbb{D}, h^\gamma + (2/\gamma) \log b)$. Note that $\nu_{h^\gamma + (2/\gamma) \log b}^\gamma = b \nu_{h^\gamma}^\gamma$ and $\mu_{h^\gamma + (2/\gamma) \log b}^\gamma = b^2 \mu_{h^\gamma}^\gamma$.

3.2.3 The quantum natural distance and the locally largest component

Let (\mathbb{D}, h^2) be a parametrization of a 2-quantum disk by \mathbb{D} and consider an independent uniform exploration of a CLE_4 on \mathbb{D} . For all $z \in \mathbb{Q}^2 \cap \mathbb{D}$, for all $u \in [0, \tau^z)$, let D_u^z be the connected component

of D_u containing z . The time u is also interpreted as a distance from the D_u^z to the boundary of the disk and is written $u = d_{\text{WW}}(\partial\mathbb{D}, D_u^z)$. For all $z \in \mathbb{Q}^2 \cap \mathbb{D}$, using the restriction of h^2 to D_u^z , one can define the quantum boundary length

$$\mathcal{Y}_u^z := \nu_{h^2|_{D_u^z}}(\partial D_u^z)$$

of D_u^z since it is encircled by SLE₄-type curves.

We recall the following result of [AHPS23], which comes from Equation (5.2) in Theorem 5.5 of [AHPS23].

Lemma 3.2.1. (consequence of Equation (5.2) in [AHPS23]) For all $z \in \mathbb{D}$, the quantity

$$\varepsilon \# \{s \in [0, \tau^z], |\Delta \mathcal{Y}_s^z| \in [\varepsilon, 2\varepsilon]\}$$

converges in probability as $\varepsilon \rightarrow 0$ to a quantity called the quantum natural distance to the loop surrounding z .

For all $u \in [0, \tau^z)$ such that $\widehat{\gamma}_u \neq \partial$, following Remark 2.3 of [MSW22], we define the quantum natural distance from $\partial\mathbb{D}$ to D_u^z as the limit in probability

$$d_q(\partial\mathbb{D}, D_u^z) = \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{s \in [0, u], |\Delta \mathcal{Y}_s^z| \in [\varepsilon, 2\varepsilon]\}. \quad (3.2.2)$$

The above convergence holds thanks to Lemma 3.2.1 and we extend the definition of $u \mapsto d_q(\partial\mathbb{D}, D_u^z)$ into a right-continuous non-decreasing function on the interval $[0, \tau^z]$ thanks to the fact that the set of times $u \geq 0$ such that $\widehat{\gamma}_u \neq \partial$, i.e. the set of times $u \geq 0$ of discovery of a loop, is a.s. dense. The above equality will actually hold for all $u \in [0, \tau^z]$ since we will see shortly that $u \mapsto d_q(\partial\mathbb{D}, D_u^z)$ is continuous.

Let \mathcal{L}^z be the loop surrounding z . We set

$$d_q(\partial\mathbb{D}, \mathcal{L}^z) := \lim_{u \uparrow \tau^z} d_q(\partial\mathbb{D}, D_u^z) = \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{s \in [0, \tau^z], |\Delta \mathcal{Y}_s^z| \in [\varepsilon, 2\varepsilon]\},$$

which is a limit in probability thanks to Lemma 3.2.1.

We denote by $(Y_t^z)_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ the process \mathcal{Y}^z reparametrized by the right-continuous inverse $(\sigma^z(t))_{t \geq 0}$ of $u \mapsto d_q(\partial\mathbb{D}, D_u^z)$. We have the following result from [AHPS23] which describes the jumps of the process Y^z :

Lemma 3.2.2. (consequence of Theorem 5.5 of [AHPS23]) The process Y^z is càdlàg. Each positive jump of $(Y^z(t))_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ at some time $t \geq 0$ corresponds to the quantum boundary length of the loop discovered at time $\sigma^z(t)$ by $(D_u^z)_{0 \leq u < \tau^z}$ (computed using the restriction of h^2 to the domain encircled by the loop). Moreover, each negative jump of $(Y^z(t))_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ at some time $t \geq 0$ corresponds to the splitting at time $u = \sigma^z(t)$ of D_{u-}^z into two components D_u^z and $D_{u-}^z \setminus D_u^z$ and the absolute value of the size of the negative jump corresponds to the quantum boundary length of the ‘‘cut out region’’ $D_{u-}^z \setminus D_u^z$. Besides, the set of times $t \geq 0$ of positive jumps of $(Y^z(t))_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ is dense.

The above lemma could also be seen as a consequence of our convergence results of CLE explorations and actually we will not rely on it for the proof of the convergence. Note that the times of positive jumps of $(Y_t^z)_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ are the quantum distances from $\partial\mathbb{D}$ to the corresponding loops which are discovered at those times. Notice also that if $z, z' \in \mathbb{Q}^2 \cap \mathbb{D}$ are encircled by the same loop \mathcal{L} , then $\mathcal{Y}^z = \mathcal{Y}^{z'}$, $Y^z = Y^{z'}$ and we can set $Y^{\mathcal{L}} := Y^z$.

Thanks to Lemma 3.2.2, we see that the quantum distances from the boundary to the loops discovered by $(D_u^z)_{0 \leq u < \tau^z}$ are distinct, so that $u \mapsto d_q(\partial\mathbb{D}, D_u^z)$ is increasing. But by density of the set of times of positive jumps of $(Y^z(t))_{0 \leq t < d_q(\partial\mathbb{D}, \mathcal{L}^z)}$ one can see that σ^z is increasing. Hence, $u \mapsto d_q(\partial\mathbb{D}, D_u^z)$ is also continuous. As a result, each positive jump $(\mathcal{Y}_u^z)_{0 \leq u < \tau^z}$ corresponds to the discovery of a loop of the CLE₄ and the size of the jump is exactly the quantum boundary length of the loop while each negative jump at some time u corresponds to the splitting of D_{u-}^z into two components D_u^z and $D_{u-}^z \setminus D_u^z$ and the size of the negative jump corresponds to the quantum boundary length of $D_{u-}^z \setminus D_u^z$.

Now, let us introduce the exploration of the locally largest component in terms of quantum boundary length, which is no longer an exploration towards a fixed point. Informally, at each discovery of a loop, it goes on exploring the exterior of the loop and at each splitting event, it explores the domain which has the largest quantum boundary length. Let us first define formally this locally largest component. Let $(z_j)_{j \geq 1}$ be an enumeration of $\mathbb{D} \cap \mathbb{Q}^2$.

The locally largest component $(C_u)_{u \geq 0}$ is defined as follows. Let

$$\tau_1 = \tau^{z_1} \wedge \inf\{u \geq 0; \mathcal{Y}_u^{z_1} < \mathcal{Y}_{u-}^{z_1}/2\}$$

be the first time u at which z_1 is encircled by a loop or $\mathcal{Y}_u^{z_1} < \mathcal{Y}_{u-}^{z_1}/2$. For all $u < \tau_1$, we set $C_u = D_u^{z_1}$. We then define by induction a sequence $(\tau_k)_{k \geq 1}$ and a subsequence $(z_{j_k})_{k \geq 1}$ by setting for all $k \geq 1$:

- If $\tau_k = \tau^{z_{j_k}}$, then τ_k is the time at which a loop \mathcal{L} encircles z_{j_k} . Let j_{k+1} be the first integer $j > j_k$ for which $z_j \in D_{\tau_k-}^{z_{j_k}}$ and such that z_j is not encircled by \mathcal{L} .
- If $\tau_k = \inf\{u \geq 0; \mathcal{Y}_u^{z_{j_k}} < \mathcal{Y}_{u-}^{z_{j_k}}/2\}$, then τ_k is the first splitting time at which z_{j_k} lies in the component with smallest quantum boundary length. Let j_{k+1} be the first integer $j > j_k$ for which $z_j \in D_{\tau_k-}^{z_{j_k}} \setminus D_{\tau_k}^{z_{j_k}}$.

In both cases we set

$$\tau_{k+1} = \tau^{z_{j_{k+1}}} \wedge \inf\{u \geq 0; \mathcal{Y}_u^{z_{j_{k+1}}} < \mathcal{Y}_{u-}^{z_{j_{k+1}}}/2\} \quad \text{and} \quad \forall u \in [\tau_k, \tau_{k+1}), C_u = D_u^{z_{j_{k+1}}}.$$

If $\lim_{k \rightarrow \infty} \tau_k < \infty$, then for all $u \geq \lim_{k \rightarrow \infty} \tau_k$, by convention we set $C_u = \emptyset$. Notice that by Lemma 8 of [WW13], the process $(C_u)_{u \geq 0}$ does not depend on the choice of the enumeration $(z_j)_{j \geq 1}$. For all $u \in [0, \lim_{k \rightarrow \infty} \tau_k)$, we set

$$\mathcal{Y}(u) := \nu_{h_{C_u}}^2(\partial C_u),$$

and by convention $\mathcal{Y}(u) = 0$ for all $u \geq \lim_{k \rightarrow \infty} \tau_k$ if $\lim_{k \rightarrow \infty} \tau_k < \infty$. By applying (3.2.2) to the z_{j_k} 's for all $k \geq 1$, for all $u \in [0, \lim_{k \rightarrow \infty} \tau_k)$, the quantum natural distance from $\partial \mathbb{D}$ to C_u is characterized as the limit in probability

$$d_q(\partial \mathbb{D}, C_u) = \lim_{\varepsilon \rightarrow 0} \varepsilon \#\{s \in [0, u], |\Delta \mathcal{Y}(s)| \in [\varepsilon, 2\varepsilon]\}. \quad (3.2.3)$$

It is clear that $u \mapsto d_q(\partial \mathbb{D}, C_u)$ is non-decreasing. Moreover, it is adapted with respect to the natural filtration of \mathcal{Y} . We denote by σ the right-continuous inverse of $u \mapsto d_q(\partial \mathbb{D}, C_u)$ and for all $t \geq 0$ we set $Y_t = \mathcal{Y}(\sigma(t))$. By combining Theorem 5.5 of [AHPS23] with Theorem 1.1 of [AdS22], one can directly see that $(Y_t)_{t \geq 0}$ has the same law as the self-similar Markov process $(X^{(-1)}(\pi t))_{t \geq 0}$ defined in Section 3.1.3. However, we will recover this result by transferring the results of [MSW22] which hold for $\kappa < 4$ by letting $\kappa \uparrow 4$.

3.3 The convergence $\kappa \uparrow 4$

In this section, we prove Theorem 3.1.2 for the $\text{SLE}_4^{(\mu)}(-2)$ exploration. To this end, we approximate the $\text{SLE}_4^{(\mu)}(-2)$ explorations of the CLE_4 decorated 2-quantum disk by explorations of a CLE_κ -decorated $\sqrt{\kappa}$ -quantum disk as $\kappa \uparrow 4$, relying on the convergence results of [Leh23] and [AHPS23] so as to transfer the analogous property of [MSW22] which describes the law of the quantum boundary lengths in the exploration of a CLE_κ -decorated $\sqrt{\kappa}$ quantum disk and states that, conditionally on the quantum boundary lengths, the encircled and cut out regions, as well as the unexplored region, are independent $\sqrt{\kappa}$ -quantum disks. But first, we recall the definition of the Carathéodory topology and the $\text{SLE}_\kappa^\beta(\kappa - 6)$ and $\text{SLE}_4^{(\mu)}(-2)$ explorations.

3.3.1 Carathéodory topology

We first provide some useful results on Carathéodory topology. Let $z \in \mathbb{C}$. For all $n \geq 0$, let $U_n \subset \mathbb{C}$ be a simply connected domain. Let $U \subset \mathbb{C}$ be a simply connected domain containing z . When $z \in U_n$, let f_n (resp. f) be the unique conformal map from \mathbb{D} to U_n (resp. U) such that $f_n(0) = z$ (resp. $f(0) = z$) and $f_n'(0) > 0$ (resp. $f'(0) > 0$). We recall that U_n converges to U as $n \rightarrow \infty$ in the *Carathéodory topology* if $z \in U_n$ for n large enough and f_n converges to f uniformly on compact subsets of \mathbb{D} . Note that the convergence does not depend on the choice of $z \in U$ (see e.g. Lemma 2.2 of [BRY19]). The Carathéodory topology on the space of simply connected open domains of \mathbb{C} containing z is clearly metrizable using the distance

$$d_{\text{Carathéodory}, z}(U_1, U_2) := \sum_{j \geq 1} 2^{-j} \sup_{(1-1/j)\mathbb{D}} |f_1 - f_2|, \quad (3.3.1)$$

where f_1, f_2 are defined as explained above.

An equivalent definition of Carathéodory convergence (see for instance Definition 2.1 of [BRY19]), in fact the original one, states that U_n converges to U for the Carathéodory topology if and only if for every compact $K \subset U$, we have $K \subset U_n$ for all n large enough and for every connected open set $V \ni z$, if $V \subset U_n$ for infinitely many n then $V \subset U$. The equivalence between the two definitions of the Carathéodory convergence comes from the Carathéodory kernel theorem (see e.g. p.3 of [BRY19] or Proposition 3.63 in [Law05]). From this equivalent definition, it is not hard to see that the space of simply connected open domains of \mathbb{C} containing z is separable. Still, with this definition, notice that a sequence U_n which converges to U in the sense of Carathéodory may also converge to another simply connected open domain V such that $V \cap U = \emptyset$. For instance, let U_n be the union of two disjoint non-empty open domains U, V and an ε_n -neighbourhood of a path Γ connecting the domains U and V with $\varepsilon_n := 1/n$. Then one can see that $U_n \rightarrow U$ and $U_n \rightarrow V$ as $n \rightarrow \infty$ in the sense of Carathéodory.

From this equivalent definition, one can see that for all compact $K \subset U$ the restriction of f_n^{-1} to K is well defined for n large enough. Actually, as a consequence of Arzelà-Ascoli's theorem, the convergence in the sense of Carathéodory of U_n to U implies the uniform convergence of f_n^{-1} towards f^{-1} on every compact subset of U . See Proposition 2.1 of [BRY19] for more details.

3.3.2 Background on CLE_κ explorations

We recall here the explorations of CLE_κ for $\kappa \in (8/3, 4]$ defined in [She09]. We will not rely on the precise definition of the explorations and we refer to [She09] for details. If $\beta \in [-1, 1]$, we define the skew Bessel process $(X_u)_{u \geq 0}$ of dimension $\delta > 0$ as the process such that $(|X_u|)_{u \geq 0}$ is a Bessel process of dimension δ and to each excursion of $(|X_u|)_{u \geq 0}$ we assign an independent sign: the corresponding excursion of $(X_u)_{u \geq 0}$ is positive with probability $\frac{1+\beta}{2}$ and negative with probability $\frac{1-\beta}{2}$. When $\delta \neq 1$, we associate a process $(Y_u)_{u \geq 0}$ to the skew Bessel process X which is characterized by Proposition 3.8 of [She09]: the process $(Y_u)_{u \geq 0}$ is the unique continuous process such that

$$\frac{dY_u}{du} = \frac{1}{X_u} \quad \text{on} \quad \{u \geq 0; X_u \neq 0\},$$

the pair (X, Y) is adapted with respect to the natural filtration of the Brownian motion appearing in the SDE which defines the Bessel process X , the law of (X, Y) is invariant under Brownian scaling and if T is a stopping time for (X, Y) such that $X_T = 0$ a.s., then $(X_{T+u}, Y_{T+u})_{u \geq 0}$ has the same law as (X, Y) . In the case $\delta = 1$, the process $(Y_u)_{u \geq 0}$ is only defined when $\beta = 0$ (and can be defined as the limit in law when $\delta \rightarrow 1$). Let $\mu \in \mathbb{R}$. One can obtain processes which satisfy the same properties as $(Y_u)_{u \geq 0}$ in Proposition 3.8 of [She09] for $\delta = 1$ by replacing $(Y_u)_{u \geq 0}$ by $(Y_u - \mu \ell_u^0)_{u \geq 0}$, where $(\ell_u^0)_{u \geq 0}$ is the local time at zero of $(X_u)_{u \geq 0}$. The term $\mu \ell_u^0$ can be interpreted as ‘‘a small drift each time X is not making an excursion’’. Actually, as explained in Section 3.2 of [She09], if one reparametrizes Y with the right-continuous inverse of the local time ℓ^0 , then one obtains a $(2 - \delta)$ -stable Lévy process. In the case $\delta = 1$, when we add this drift it remains a 1-stable Lévy process.

Let $\delta = 3 - 8/\kappa$. If $\delta \neq 1$, then let $\beta \in [-1, 1]$, otherwise let $\mu \in \mathbb{R}$. Let $(X_u)_{u \geq 0}$ and $(Y_u)_{u \geq 0}$ be the two processes defined above associated to δ, β when $\delta \neq 1$ and to δ, μ when $\delta = 1$. Let $(O_u)_{u \geq 0}$ and $(W_u)_{u \geq 0}$ be the processes given by $O_0 = W_0 = 0$ and for all $u \geq 0$,

$$O_u = -\frac{2}{\sqrt{\kappa}} Y_u \quad \text{and} \quad W_u = O_u + \sqrt{\kappa} X_u.$$

Then the *chordal* $\text{SLE}_\kappa^\beta(\kappa - 6)$ for $\kappa \in (8/3, 4)$ and the *chordal* $\text{SLE}_4^{\langle \mu \rangle}(-2)$ from 0 to ∞ in \mathbb{H} are defined as the growing family of closed sets $(K_u)_{u \geq 0}$ determined by the ODE

$$g_0(z) = z \quad \text{and} \quad \partial_u g_u(z) = \frac{2}{g_u(z) - W_u},$$

in the sense that the domain of definition of g_u is $\mathbb{H} \setminus K_u$. These processes can be seen as special cases of SLE processes with force points, with parameters κ and $\rho = \kappa - 6$. Such processes first appeared in [LSW03]. See e.g. Appendix A.3 of [BP24] for an overview. As explained in Remark A.6 of [BP24], the Bessel process X_u describes the distance between the driving function W_u and the marked point O_u .

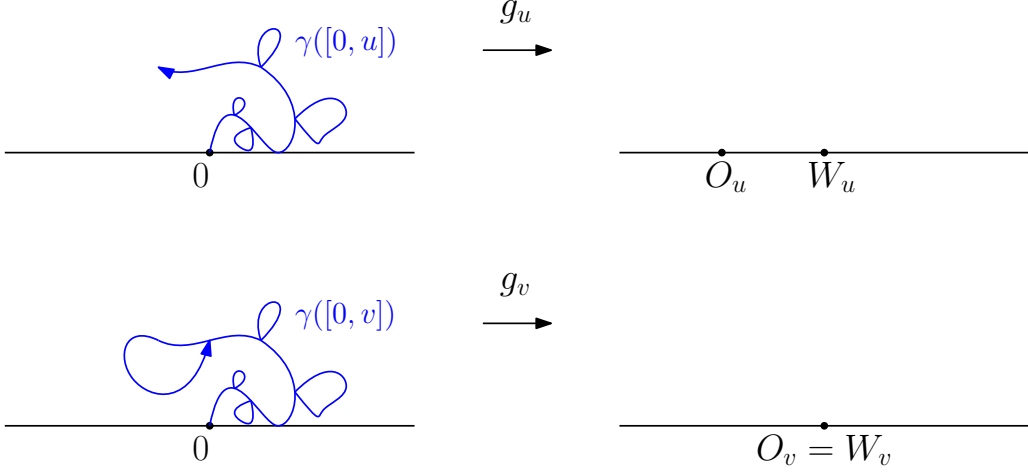


Figure 3.2 – Illustration of a chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ or $\text{SLE}_4^{(\mu)}(-2)$ from 0 to ∞ in \mathbb{H} . Above: at a time u where γ is drawing a loop, i.e. a time such that $O_u \neq W_u$. Below: at the time v where γ closes the loop, so that $O_v = W_v$.

However, given that in our case $\rho \leq -2$ and since the marked point O_u is sometimes infinitesimally close to the driving function W_u , the above definition is not so simple (see Remark A.7 of [BP24]).

It was shown in [MSW17] (in Theorem 7.4 for $\kappa \in (8/3, 4)$ and in Proposition 5.3 for $\kappa = 4$) that $\mathbb{H} \setminus K_u$ is the unbounded connected component of $\mathbb{H} \setminus \gamma([0, u])$ for some continuous path γ . On each excursion of $(X_u)_{u \geq 0}$, the path γ draws a simple loop (see Figure 3.2). We also recall from Proposition 3.10 of [She09] that $(K_u)_{u \geq 0}$ satisfies a “renewal property” and a “conformal Markov property” which are described as follows.

Proposition 3.3.1. (from Proposition 3.10 of [She09]) *The process $(K_u)_{u \geq 0}$ satisfies the following properties:*

- **Renewal property.** *Conditionally on W_u up to a stopping time T (for the natural filtration of (X, Y)) for which $O_T = W_T$ almost surely, the process $(W_{T+u} - W_T)_{u \geq 0}$ has the same law as $(W_u)_{u \geq 0}$.*
- **Conformal Markov property.** *Conditionally on W_u up to some stopping time T for which a.s. $O_T \neq W_T$, the growing family given by the closure of $g_T(K_{T+u})$ up to the first time $u \geq 0$ such that $W_{T+u} = O_{T+u}$ has the same law (up to a time-change) as a chordal SLE_κ in \mathbb{H} from W_T to O_T .*

The renewal property can also be rephrased as follows: if T is a stopping time at which a.s. the exploration is not drawing a loop (i.e. such that $X_T = 0$ a.s.), then conditionally on $(K_u)_{0 \leq u \leq T}$, the process $(g_T(K_{T+u}))_{u \geq 0}$ has the same law as $(K_u)_{u \geq 0}$.

A chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$) from a to b which are on the boundary of some simply connected domain $D \subset \mathbb{C}$ is just the image of the above $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$) by a conformal map sending 0 to a and ∞ to b . One can define the trunk η of γ , a chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$) as the path γ where we removed the loops. In Theorem 7.4 of [MSW17], the trunk is shown to be an $\text{SLE}_{\kappa'}$ -type curve for $\kappa' = 16/\kappa$. The path γ can be decomposed into the trunk η and the loops it traces along each excursion of X , see [MSW17] for details and the left-hand side of Figure 3.1 for an illustration.

One can also define a *radial* version of the $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$) from $-i$ to 0 in the unit disk \mathbb{D} , which is characterized by Proposition 3.13 of [She09] using the chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$). As explained in [WW13, p. 11] one can define this radial version relying only on chordal SLE’s as follows. Consider γ_1 a chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$) from $-i$ to i , up to the first time u_1 at which it finishes to draw a loop intersecting the circle of radius 1/2 centred at $-i$. Then, let F_1 be the conformal transformation from $D_{u_1}^0$ to \mathbb{D} that keeps 0 fixed and maps $\gamma_1(u_1)$ to $-i$ and we repeat the same procedure again: growing γ_2 another independent chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$

(or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$) from $-i$ to i until it draws a loop intersecting the circle of radius $1/2$ centred at $-i$, looking at its preimage by F_1 etc. We stop when we draw a loop surrounding 0 . The radial $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$) targeted at $z \in \mathbb{D}$ can be defined in a similar way.

Two radial $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$) are related by the following target-invariance property.

Proposition 3.3.2. (Proposition 3.14 of [She09]) *For all $z_1, z_2 \in \mathbb{D}$, one can couple two radial $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$), γ_1 starting from $-i$ and targeted at z_1 and γ_2 from $-i$ to some z_2 so that they coincide up to some time-change until the time that z_1 and z_2 fail to lie in the same connected component of $\mathbb{D} \setminus \gamma_1([0, u])$.*

By performing this coupling with targets in a countable dense subset of \mathbb{D} , one can thus build the branching $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ when $\kappa = 4$) starting at $-i$ in \mathbb{D} , hence exploring all the loops of a CLE_κ . It is also called the exploration tree. The radial and chordal SLE's can be seen as ‘‘branches’’ of this exploration tree. If we continue the radial explorations after the time they draw a loop surrounding their targets, then they explore the nested CLE_κ . Except in Sections 3.4 and 3.7.1, we will always consider the non-nested version of the CLE_κ and it will simply be referred as the CLE_κ .

3.3.3 Carathéodory convergence of the chordal exploration

We now describe an approximation result of the $\text{SLE}_4^{(\mu)}(-2)$ exploration in the sense of Carathéodory based on the results of [Leh23]. Let $\mu \in \mathbb{R}$. If $\kappa \in (8/3, 4)$, let $(\gamma^\kappa(u))_{u \geq 0}$ be a chordal $\text{SLE}_\kappa^{\mu((4/\kappa)-1)}(\kappa - 6)$ in \mathbb{D} from $-i$ to i . For all $u \geq 0$, if $X(u) = 0$, then let $v^\kappa(u) = u$ and otherwise let $v^\kappa(u)$ be the time of the beginning of the excursion X which is being drawn at time u . Let $(\gamma^4(u))_{u \geq 0}$ be a chordal $\text{SLE}_4^{(\mu)}(-2)$ in \mathbb{D} from $-i$ to i . Let $v^4(u)$ for $u \geq 0$ be defined as before using the corresponding driving function. For all $\kappa \in (8/3, 4]$, for all $u \geq 0$, for all $z \in \mathbb{D}$, let $D_u^{z, \kappa}$ be the connected component of $\mathbb{D} \setminus \gamma^\kappa([0, v^\kappa(u)])$ containing z . Note that we choose to restrict the definition of the $D_u^{z, \kappa}$'s to the times $v^\kappa(u)$ since the relation with quantum disks that we will recall in the next subsection is only valid for these times. Then a consequence of the results of [Leh23] is the following proposition.

Proposition 3.3.3. *We have the convergence in law*

$$(D_u^{z, \kappa})_{u \geq 0, z \in \mathbb{Q}^2 \cap \mathbb{D}} \xrightarrow[\kappa \uparrow 4]{(d)} (D_u^{z, 4})_{u \geq 0, z \in \mathbb{Q}^2 \cap \mathbb{D}} \quad (3.3.2)$$

in terms of finite-dimensional distributions for the Carathéodory topology.

The rest of this subsection is devoted to the proof of the above proposition. The outline of the proof is as follows: Proposition 3.10 of [Leh23] gives the convergence of the driving functions of the exploration. Next, Proposition 4.47 of [Law05] enables to deduce the convergence at the level of conformal maps. Finally, some technical work remains to deduce the convergence of the cut out and encircled regions in the desired topology.

We fix $\mu \in \mathbb{R}$. For all $\kappa \in (8/3, 4]$, we denote by $(W_u^\kappa)_{u \geq 0}$, $(O_u^\kappa)_{u \geq 0}$, $(g_u^\kappa)_{u \geq 0}$ and $(K_u^\kappa)_{u \geq 0}$ the processes with which the $\text{SLE}_\kappa^{\mu((4/\kappa)-1)}(\kappa - 6)$ for $\kappa < 4$ and $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$ is defined as described above. Let κ_n be an increasing sequence tending to 4. By Proposition 3.10 of [Leh23] we have the convergence of the driving functions:

$$(W^{\kappa_n}, O^{\kappa_n}) \xrightarrow[n \rightarrow \infty]{(d)} (W^4, O^4) \quad (3.3.3)$$

for the topology of uniform convergence on compact sets. By Skorokhod's representation theorem, we may suppose in the rest of the subsection that the above convergence is almost sure. In particular, by Proposition 4.47 of [Law05], for all $u_0 > 0$, for all $\varepsilon > 0$, we get a.s. the convergence

$$g^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} g^4 \quad (3.3.4)$$

uniformly on the set $[0, u_0] \times \{z \in \mathbb{H}; d(z, K_{u_0}^4) \geq \varepsilon\}$. As a result, we obtain the Carathéodory convergence of the unexplored region: a.s. for all $u \geq 0$, for any sequence of real numbers $u_n \geq 0$ converging to u ,

$$\mathbb{H} \setminus K_{u_n}^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \mathbb{H} \setminus K_u^4. \quad (3.3.5)$$

In particular, for all $u \geq 0$, we have a.s.

$$\mathbb{H} \setminus K_u^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \mathbb{H} \setminus K_u^4 \quad \text{and} \quad \mathbb{H} \setminus K_{v^{\kappa_n}(u)}^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \mathbb{H} \setminus K_{v^4(u)}^4. \quad (3.3.6)$$

We then look into the domains encircled by loops. The next lemma states the convergence of the domains encircled by the loops which are drawn by excursions of height larger than ε .

Lemma 3.3.4. *Let $\varepsilon > 0$ and $i \geq 1$. Let $\mathcal{B}_{i,\varepsilon}^n$ (resp. $\mathcal{B}_{i,\varepsilon}$) be the domain encircled by the loop which is drawn by the i -th excursion of $W^{\kappa_n} - O^{\kappa_n}$ (resp. $W^4 - O^4$) of height larger than ε . Then*

$$\mathcal{B}_{i,\varepsilon}^n \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mathcal{B}_{i,\varepsilon}$$

in the sense of Carathéodory.

In order to prove the above lemma, let us introduce some notation. Let $\varepsilon > 0$. For all $i \geq 1$, for all $\kappa \in (8/3, 4]$, let $e_{i,\varepsilon}^\kappa$ be the i -th excursion of $W^\kappa - O^\kappa$ of height larger than ε and let $S_{i,\varepsilon}^\kappa < T_{i,\varepsilon}^\kappa$ be its starting time and end time. First notice that by (3.3.3) we get readily the uniform convergence of the excursions $e_{i,\varepsilon}^{\kappa_n}$'s to the $e_{i,\varepsilon}^4$'s and the convergence of their endpoints $S_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}$ to $S_{i,\varepsilon}^4, T_{i,\varepsilon}^4$ as $n \rightarrow \infty$. Similarly, for all $\kappa \in (8/3, 4]$, let $R_{i,\varepsilon}^\kappa$ be the i -th time at which $W^\kappa - O^\kappa$ first reaches ε after hitting zero. Then the $R_{i,\varepsilon}^{\kappa_n}$'s converge a.s. to the $R_{i,\varepsilon}^4$'s as $n \rightarrow \infty$. Let us prove the following convergences:

Lemma 3.3.5. *We have the following convergences:*

— **Convergence of the beginning of the loop.** For all $u \in (S_{i,\varepsilon}^4, T_{i,\varepsilon}^4)$,

$$(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([S_{i,\varepsilon}^{\kappa_n}, u]) \xrightarrow[n \rightarrow \infty]{\text{(a.s.)}} (g_{S_{i,\varepsilon}^4}^4 \circ \gamma^4)([S_{i,\varepsilon}^4, u]) \quad (3.3.7)$$

with respect to the Hausdorff distance.

— **Convergence of the end of the loop.** for all $\delta > 0$ small enough,

$$(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([R_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]) \cap V_\delta \xrightarrow[n \rightarrow \infty]{} (g_{S_{i,\varepsilon}^4}^4 \circ \gamma^4)([R_{i,\varepsilon}^4, T_{i,\varepsilon}^4]) \cap V_\delta \quad (3.3.8)$$

for the Hausdorff distance, where

$$V_\delta := \left\{ z \in \mathbb{H}; d(z, g_{S_{i,\varepsilon}^4}^4(\gamma^4([S_{i,\varepsilon}^4, R_{i,\varepsilon}^4])) \cup \mathbb{R}) > \delta \right\}.$$

Proof. Let $u \in (S_{i,\varepsilon}^4, T_{i,\varepsilon}^4)$ and $\delta > 0$. Applying Proposition 4.47 of [Law05] to $(W_{(S_{i,\varepsilon}^{\kappa_n} + u) \wedge T_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})_{u \geq 0}$ which converges uniformly to $(W_{(S_{i,\varepsilon}^4 + u) \wedge T_{i,\varepsilon}^4}^4)_{u \geq 0}$ as $n \rightarrow \infty$, we know that for all n large enough,

$$\left((v, z) \mapsto g_{S_{i,\varepsilon}^{\kappa_n} + v}^{\kappa_n}((g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})^{-1}(z)) \right) \xrightarrow[n \rightarrow \infty]{} \left((v, z) \mapsto g_{S_{i,\varepsilon}^4 + v}^4((g_{S_{i,\varepsilon}^4}^4)^{-1}(z)) \right) \quad (3.3.9)$$

uniformly on $[0, u] \times \left\{ z \in \mathbb{H}; d(z, g_{S_{i,\varepsilon}^4}^4(K_u^4)) \geq \delta \right\}$. In particular, for n large enough,

$$\left\{ z \in \mathbb{H}; d(z, g_{S_{i,\varepsilon}^4}^4(K_u^4)) \geq \delta \right\} \subset \mathbb{H} \setminus g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(K_u^{\kappa_n}).$$

In other words, for all n large enough, the curve $(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([S_{i,\varepsilon}^{\kappa_n}, u])$ is included in the δ -neighbourhood of $(g_{S_{i,\varepsilon}^4}^4 \circ \gamma^4)([S_{i,\varepsilon}^4, u])$. Besides, let U be the open ball centred at $g_{S_{i,\varepsilon}^4}^4(\gamma^4(u))$ of radius δ . Then, by choosing δ small enough, U is an open connected subset of \mathbb{H} such that $U \cap (\mathbb{H} \setminus g_{S_{i,\varepsilon}^4}^4(K_u^4)) \neq \emptyset$. By the Carathéodory convergence of $\mathbb{H} \setminus g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(K_u^{\kappa_n})$ towards $\mathbb{H} \setminus g_{S_{i,\varepsilon}^4}^4(K_u^4)$ (which stems from (3.3.9)), we cannot have $U \subset \mathbb{H} \setminus g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(K_u^{\kappa_n})$ infinitely often, otherwise we would get that $\gamma^4(u) \in \mathbb{H} \setminus K_u^4$, which is not the case. Thus, for all n large enough, $U \cap g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(K_u^{\kappa_n}) \neq \emptyset$. By choosing δ small enough, by connectedness of $(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([S_{i,\varepsilon}^{\kappa_n}, u])$, we obtain that for n large enough, $(g_{S_{i,\varepsilon}^4}^4 \circ \gamma^4)([S_{i,\varepsilon}^4, u])$ is in the δ -neighbourhood of $(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([S_{i,\varepsilon}^{\kappa_n}, u])$. We have thus proven (3.3.7) that for all $u \in (S_{i,\varepsilon}^4, T_{i,\varepsilon}^4)$.

Next, let us prove (3.3.8). Recall that for all $\kappa \in (8/3, 4]$, $R_{i,\varepsilon}^\kappa$ is the i -th time at which $W^\kappa - O^\kappa$ first reaches ε after hitting zero. Then $R_{i,\varepsilon}^{\kappa_n}$ converges to $R_{i,\varepsilon}^4$ as $n \rightarrow \infty$. Moreover, for all $n \geq 0$, by the

conformal Markov property recalled in Proposition 3.3.1 the curve $g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma_{[[R_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]]}^{\kappa_n}$ has the same law as a chordal SLE(κ_n) in \mathbb{H} from $g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(\gamma^{\kappa_n}(R_{i,\varepsilon}^{\kappa_n})) = W_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}$ to $g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(\gamma^{\kappa_n}(T_{i,\varepsilon}^{\kappa_n})) = O_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}$. Furthermore, by taking Möbius transformations sending one of these two endpoints to ∞ , and by applying Theorem 1.10 of [KS17] (which states that the chordal SLE(κ_n) from 0 to ∞ converges uniformly on compact sets to the SLE(4) as $n \rightarrow \infty$), we obtain the convergence

$$(g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([R_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]) \xrightarrow[n \rightarrow \infty]{} (g_{R_{i,\varepsilon}^4}^4 \circ \gamma^4)([R_{i,\varepsilon}^4, T_{i,\varepsilon}^4]) \quad (3.3.10)$$

for the Hausdorff distance. Besides, by (3.3.9) and by the convergence of $R_{i,\varepsilon}^{\kappa_n}$ to $R_{i,\varepsilon}^4$, we know that

$$g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ (g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})^{-1} \xrightarrow[n \rightarrow \infty]{} g_{R_{i,\varepsilon}^4}^4 \circ (g_{S_{i,\varepsilon}^4}^4)^{-1}$$

uniformly on compact subsets of $\mathbb{H} \setminus (g_{S_{i,\varepsilon}^4}^4 \circ \gamma^4)([S_{i,\varepsilon}^4, R_{i,\varepsilon}^4])$. Since the limiting function is a conformal map, it is easy to see that $g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ (g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})^{-1}$ converges pointwise to $g_{S_{i,\varepsilon}^4}^4 \circ (g_{R_{i,\varepsilon}^4}^4)^{-1}$. By Montel's theorem, we deduce that

$$g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ (g_{R_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})^{-1} \xrightarrow[n \rightarrow \infty]{} g_{S_{i,\varepsilon}^4}^4 \circ (g_{R_{i,\varepsilon}^4}^4)^{-1}$$

uniformly on compact subsets of \mathbb{H} . Combining this with (3.3.10), we get (3.3.8). \square

We are now in position to prove the Carathéodory convergence of the domain encircled by the loop, i.e. to prove Lemma 3.3.4.

Proof of Lemma 3.3.4. For all $n \in \mathbb{N}$, let $\mathcal{L}_n = g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(\gamma^{\kappa_n}([S_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]))$ be the loop drawn by $u \mapsto W_{(S_{i,\varepsilon}^{\kappa_n} + u) \wedge T_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}$ and let $\mathcal{L} = g_{S_{i,\varepsilon}^4}^4(\gamma^4([S_{i,\varepsilon}^4, T_{i,\varepsilon}^4]))$ be the loop drawn by $u \mapsto W_{(S_{i,\varepsilon}^4 + u) \wedge T_{i,\varepsilon}^4}$. By the convergence of g^{κ_n} and $S_{i,\varepsilon}^{\kappa_n} \rightarrow \infty$, to prove the Carathéodory convergence of the open domain encircled by the loop $\gamma^{\kappa_n}([S_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}])$ towards the domain encircled by $\gamma^4([S_{i,\varepsilon}^4, T_{i,\varepsilon}^4])$, it suffices to prove the Carathéodory convergence of the domain \mathcal{B}_n encircled by \mathcal{L}_n towards the domain \mathcal{B} encircled by \mathcal{L} .

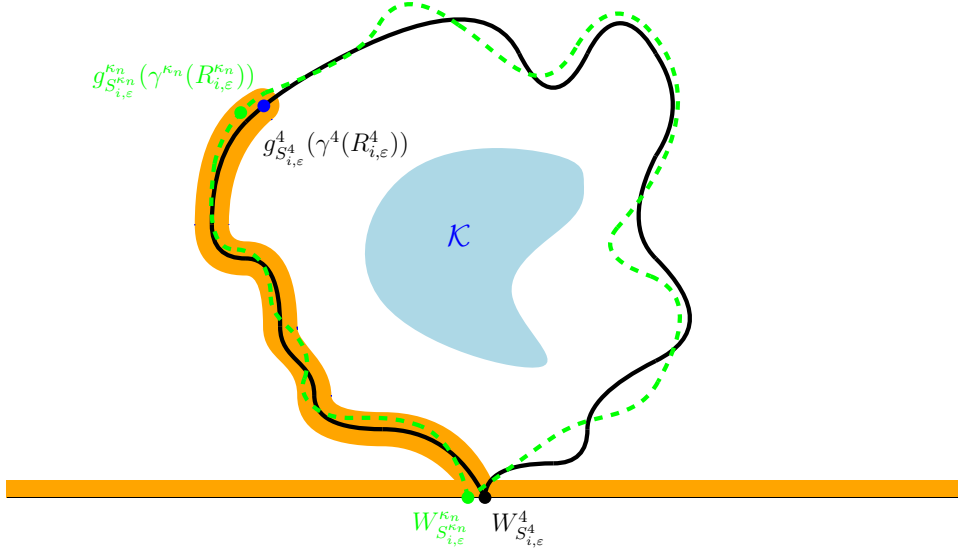


Figure 3.3 – Illustration of the proof of the Carathéodory convergence of the domain encircled by $\mathcal{L}_n = g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n}(\gamma^{\kappa_n}([S_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]))$, in green dashes, towards the domain encircled by $\mathcal{L} = g_{S_{i,\varepsilon}^4}^4(\gamma^4([S_{i,\varepsilon}^4, T_{i,\varepsilon}^4]))$, in black. The complement of V_δ is in orange. A compact \mathcal{K} encircled by \mathcal{L} is drawn in blue.

Let \mathcal{K} be a compact subset of the open domain \mathcal{B} . Then we may take $\delta > 0$ small enough such that $\mathcal{K} \subset V_\delta$, so that by (3.3.8), we obtain that $\mathcal{K} \cap (g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([R_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}]) = \emptyset$ for n large enough. Moreover, by (3.3.7), for $u > 0$ small enough, for all n large enough, we have $(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([S_{i,\varepsilon}^{\kappa_n}, R_{i,\varepsilon}^{\kappa_n} + u]) \subset V_\delta$. Therefore, by the convergence of $R_{i,\varepsilon}^{\kappa_n}$ to $R_{i,\varepsilon}^4$ we deduce that for all n large enough, $\mathcal{K} \cap \mathcal{L}_n = \emptyset$. One

can actually see that \mathcal{K} is encircled by \mathcal{L}_n for n large enough using that the endpoints of the path $(g_{S_{i,\varepsilon}^{\kappa_n}}^{\kappa_n} \circ \gamma^{\kappa_n})([R_{i,\varepsilon}^{\kappa_n}, T_{i,\varepsilon}^{\kappa_n}])$ are in $\mathbb{H} \setminus V_\delta$ and applying (3.3.8) once more (see Figure 3.3).

Let $z \in \mathcal{B}$. To end the proof of the Carathéodory convergence of \mathcal{B}_n towards \mathcal{B} , it suffices to show that if U is a connected open set containing z such that $U \subset \mathcal{B}_n$ infinitely often, then $U \subset \mathcal{B}$. Since U and \mathcal{B} are open it is enough to see that $U \subset \overline{\mathcal{B}}$. Assume by contradiction that there exists $w \in U \setminus \overline{\mathcal{B}}$. Take $\delta > 0$ small enough so that $B(w, \delta) \subset U \setminus \overline{\mathcal{B}}$. In particular, $w \in V_\delta$. As a result, due to the Hausdorff convergences (3.3.7) and (3.3.8), the point w is not encircled by \mathcal{L}_n for n large enough, hence a contradiction. We have thus proven that

$$\mathcal{B}_n \xrightarrow[n \rightarrow \infty]{} \mathcal{B}$$

in the sense of Carathéodory. This ends the proof of the lemma. \square

Proof of Proposition 3.3.3. By (3.3.6), we have the convergence of the unexplored region. Moreover, by Lemma 3.3.4, we have the convergence of the domain encircled by the loop $\gamma^{\kappa_n}([S_{\varepsilon,i}^{\kappa_n}, T_{\varepsilon,i}^{\kappa_n}])$ for $i \geq 1$ towards the domain encircled by $\gamma^4([S_{\varepsilon,i}^4, T_{\varepsilon,i}^4])$. By letting $\varepsilon \rightarrow 0$, we get the Carathéodory convergence of all the domains encircled by loops.

Finally, we focus on the domains which are cut out by the exploration. Let $u \geq 0$. Let $x \in \mathbb{R}$. For all n , let ζ_x^n be the first time that x is swallowed by $(g_u^{\kappa_n}(K_{u+v}^{\kappa_n}))_{v \geq 0}$ and let ζ_x be the first time that x is swallowed by $(g_u^4(K_{u+v}^4))_{v \geq 0}$. For all $n \geq 1$, let C_x^n be the interior of $g_u^{\kappa_n}(K_{u+\zeta_x^n}^{\kappa_n}) \setminus g_u^{\kappa_n}(K_{u+\zeta_x^n-}^{\kappa_n})$ and let C_x be the interior of $g_u^4(K_{u+\zeta_x}^4) \setminus g_u^4(K_{u+\zeta_x-}^4)$. By the proof of Theorem 1.2 of [Leh23] and by another application of Skorokhod's representation theorem, we know that ζ_x is the almost sure limit as $n \rightarrow \infty$ of ζ_x^n . To be more precise, the proof of Theorem 1.2 of [Leh23] gives the convergence of ζ_1^n , while the convergence of ζ_x^n follows by scale-invariance and reflection-invariance. As a consequence, by considering chordal $\text{SLE}_{\kappa_n}^{\mu((4/\kappa_n)-1)}(\kappa_n - 6)$'s and an $\text{SLE}_4^{(\mu)}(-2)$ targeted at x and coupled with the chordal $\text{SLE}_{\kappa_n}^{\mu((4/\kappa_n)-1)}(\kappa_n - 6)$'s and $\text{SLE}_4^{(\mu)}(-2)$ targeted at ∞ until x is separated from ∞ and by applying the convergence (3.3.4), we obtain the a.s. convergence

$$C_x^n \xrightarrow[n \rightarrow \infty]{} C_x$$

as $n \rightarrow \infty$ in the sense of Carathéodory. By (3.3.6), we deduce the convergence of the cut out domain

$$(g_u^{\kappa_n})^{-1}(C_x^n) \xrightarrow[n \rightarrow \infty]{} (g_u^4)^{-1}(C_x)$$

in the sense of Carathéodory. To conclude we just need to check that any cut out domain can be written in the form $(g_u^4)^{-1}(C_x)$ for some $u \geq 0$ and $x \in \mathbb{R}$. This is straightforward by taking $u \geq 0$ such that the frontier of the image by g_u^4 of the cut out domain has a non-trivial intersection with \mathbb{R} . \square

3.3.4 The $\sqrt{\kappa}$ -quantum disks and CLE_κ

Here, we describe the results of [MSW22] that we want to transfer to $\kappa = 4$. Recall Subsection 3.2.2. Let $\kappa \in (8/3, 4]$. Let $(\mathbb{D}, h^{\sqrt{\kappa}})$ be a parametrization of a unit boundary $\sqrt{\kappa}$ -quantum disk. By Lemma 4.5 of [AHPS23], we know that

$$(h^{\sqrt{\kappa}}, \mu_{h^{\sqrt{\kappa}}}^{\sqrt{\kappa}}, \nu_{h^{\sqrt{\kappa}}}^{\sqrt{\kappa}}) \xrightarrow[\kappa \uparrow 4]{} (h^2, \mu_{h^2}^2, \nu_{h^2}^2), \quad (3.3.11)$$

where the convergence of the first coordinate holds in the space $H_{\text{loc}}^{-1}(\mathbb{D})$ while the convergences of the other coordinates hold with respect to the weak topology of measures. In what follows, when we speak of the convergence of some quantum disks, it always means the convergence of the GFF and of the quantum area and boundary length measures associated to the parametrization by the unit disk as in the above equation.

When we run an independent chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ (or $\text{SLE}_4^{(\mu)}(-2)$ for $\kappa = 4$) γ on a $\sqrt{\kappa}$ -quantum disk $(\mathbb{D}, h^{\sqrt{\kappa}})$, the trunk of γ can be parametrized by its quantum length since it is an $\text{SLE}_{\kappa'}$ -type process by Theorem 7.4 of [MSW17] and Remark 2.3 of [MSW22]. Notice that this quantum length is defined up to some multiplicative constant. Besides, we can define the quantum boundary length of SLE_κ type curves, such as the loops drawn by γ or the boundary of the domains cut out by γ , via a

quantum boundary length measure induced by $h\sqrt{\kappa}$, using the quantum zipper properties first discovered in [She16]. The quantum boundary length measure on such a domain D is equal to $\nu_{h\sqrt{\kappa}|_D}^{\sqrt{\kappa}}$. We will see in the next proposition that they correspond exactly to the boundary length measures of some quantum disks.

We then state a result of [MSW22] for $\kappa \in (8/3, 4)$, which gives the law of the quantum boundary length of the unexplored region and a Markov property in terms of quantum disks. Let L_0 (resp. R_0) be the quantum boundary length along the clockwise (resp. counterclockwise) segment from $-i$ to i in $\partial\mathbb{D}$. Let $\beta \in (-1, 1)$. Let γ be a chordal $\text{SLE}_\kappa^\beta(\kappa - 6)$ from $-i$ to i taken independently from the quantum disk $(\mathbb{D}, h\sqrt{\kappa})$. We parametrize its trunk η by its quantum length. Let $T^{\kappa, \beta}$ be the total quantum length of the trunk. For each t (corresponding here to the quantum length of the trunk), let E_t be the event that the trunk has not yet attained i before having quantum length t , i.e. $E_t = \{T^{\kappa, \beta} > t\}$. For all $t < T^{\kappa, \beta}$, we set $L_t^{\kappa, \beta}$ and $R_t^{\kappa, \beta}$ to be the left and right quantum boundary lengths of the unexplored region \mathcal{D}_t^κ (\mathcal{D}_t^κ is the connected component of $\mathbb{D} \setminus \gamma([0, \tau_t])$ containing i in its closure, where τ_t is the time for γ at which the quantum length of the trunk η reaches t). More precisely, $L_t^{\kappa, \beta}$ is the quantum boundary length of the arc of $\partial\mathcal{D}_t^\kappa$ going from the tip of the trunk to i in the clockwise direction and $R_t^{\kappa, \beta}$ is the quantum boundary length of the arc of $\partial\mathcal{D}_t^\kappa$ going from the tip of the trunk to i in the anti-clockwise direction. Note that for all $t < T^{\kappa, \beta}$, we have $L_t^{\kappa, \beta} > 0$ and $R_t^{\kappa, \beta} > 0$. For all $t \geq T^{\kappa, \beta}$ we set $L_t^{\kappa, \beta} = R_t^{\kappa, \beta} = 0$. If we write $S_t^{\kappa, \beta} = L_t^{\kappa, \beta} + R_t^{\kappa, \beta}$ for all $t \geq 0$, then $S_t^{\kappa, \beta}$ is the boundary length of \mathcal{D}_t^κ for all $t < T^{\kappa, \beta}$. Let $a = 1 + 4/\kappa$. Let $\tilde{L}^{\beta, a}$ (resp. $\tilde{R}^{\beta, a}$) be the $(a - 1)$ -stable Lévy process of Lévy measure $\Lambda_L^{\beta, a} := \frac{1-\beta}{2} \cos(a\pi)(dx/x^a)\mathbf{1}_{x>0} + \frac{1}{2}(dx/|x|^a)\mathbf{1}_{x<0}$ (resp. $\Lambda_R^{\beta, a} := \frac{1+\beta}{2} \cos(a\pi)(dx/x^a)\mathbf{1}_{x>0} + \frac{1}{2}(dx/|x|^a)\mathbf{1}_{x<0}$). The two processes $\tilde{L}^{\beta, a}$ and $\tilde{R}^{\beta, a}$ are taken independent. Let $\tilde{S}^a = \tilde{L}^{\beta, a} + \tilde{R}^{\beta, a}$.

Proposition 3.3.6. (Proposition 5.1 of [MSW22]) *Up to some linear time-change, the process $(L_t^{\kappa, \beta}, R_t^{\kappa, \beta})_{t \geq 0}$ is a Doob h -transform of the process $(\tilde{L}^{\beta, a}, \tilde{R}^{\beta, a})$, in the sense that there exists a constant $c > 0$ such that for all $t > 0$, for all bounded measurable function $F : \mathbb{D}([0, t], \mathbb{R})^2 \rightarrow \mathbb{R}$, for all $x, y > 0$,*

$$\begin{aligned} & \mathbb{E}\left(F\left((L_s^{\kappa, \beta}, R_s^{\kappa, \beta})_{0 \leq s \leq t}\right) \mathbf{1}_{E_t} \mid (L_0^{\kappa, \beta}, R_0^{\kappa, \beta}) = (x, y)\right) \\ &= \mathbb{E}_{x, y}\left(\left(\frac{x+y}{\tilde{S}_{ct}^a}\right)^a F\left((\tilde{L}_{cs}^{\beta, a}, \tilde{R}_{cs}^{\beta, a})_{0 \leq s \leq t}\right) \mathbf{1}_{\forall s \in [0, t], \tilde{L}_{cs}^{\beta, a}, \tilde{R}_{cs}^{\beta, a} > 0}\right), \end{aligned}$$

where under $\mathbb{P}_{x, y}$ the Lévy process $\tilde{L}^{\beta, a}$ (resp. $\tilde{R}^{\beta, a}$) starts at x (resp. y). Moreover, conditionally on the quantum boundary lengths $(L_s^{\kappa, \beta}, R_s^{\kappa, \beta})_{0 \leq s \leq t}$, the quantum surfaces which are cut out to the left, cut out to the right, inside the left loops or inside the right loops are independent $\sqrt{\kappa}$ -quantum disks of quantum boundary lengths corresponding to the absolute values of the sizes of the negative and positive jumps of $L^{\kappa, \beta}$ and $R^{\kappa, \beta}$, while $(\mathcal{D}_t^\kappa, h_{|\mathcal{D}_t^\kappa}^{\sqrt{\kappa}})$ is an independent $\sqrt{\kappa}$ -quantum disk of quantum boundary length $S_t^{\kappa, \beta}$.

When we parametrize the trunk by its quantum length, the parametrization is chosen up to a multiplicative constant, so that for our purpose we choose to take $c = 1$ in the above proposition. This proposition will be a key ingredient to obtain analogous statements for $\kappa = 4$.

3.3.5 Joint convergence of the CLE_κ and the $\sqrt{\kappa}$ -quantum disks as $\kappa \uparrow 4$

In this subsection, we combine the convergences (3.3.2) and (3.3.11) to extend Proposition 3.3.6 to the case $\kappa = 4$ for the chordal $\text{SLE}_4^{(\mu)}(-2)$. Then the statement of Theorem 3.1.2 for the $\text{SLE}_4^{(\mu)}(-2)$ exploration will be a direct corollary.

Let (\mathbb{D}, h^2) be a unit boundary length 2-quantum disk. Let L_0^4 (resp. R_0^4) be the quantum boundary length along the clockwise (resp. counterclockwise) segment from $-i$ to i in $\partial\mathbb{D}$. Let $\mu \in \mathbb{R}$. Let γ be an independent chordal $\text{SLE}_4^{(\mu)}(-2)$ on \mathbb{D} from $-i$ to i . We again parametrize the trunk by its quantum length and we denote by $T^{4, \mu}$ its total quantum length. For each $t \geq 0$ (corresponding here to the quantum length of the trunk), let E_t be the event that the trunk has not yet attained i before having quantum length t , i.e. $E_t = \{T^{4, \mu} > t\}$. For all $t < T^{4, \mu}$, we set $L_t^{4, \mu}$ and $R_t^{4, \mu}$ to be the left and right quantum boundary lengths of the remaining to be explored region \mathcal{D}_t^4 (\mathcal{D}_t^4 is the connected component of $\mathbb{D} \setminus \gamma([0, \tau_t])$ containing i in its closure, where τ_t is the time for γ at which the quantum length of the

trunk reaches t). Note that for all $t < T^{4,\mu}$, we have $L_t^{4,\mu} > 0$ and $R_t^{4,\mu} > 0$. For all $t \geq T^{4,\mu}$, we set $L_t^{4,\mu} = R_t^{4,\mu} = 0$. If we write $S_t^{4,\mu} = L_t^{4,\mu} + R_t^{4,\mu}$ for all $t \geq 0$, then $S_t^{4,\mu}$ is the quantum boundary length of \mathcal{D}_t^4 when $t < T^{4,\mu}$. Let \tilde{L}^2 and \tilde{R}^2 be two independent symmetric Cauchy processes of Lévy measure $\frac{1}{2}(dx/x^2)\mathbf{1}_{x \neq 0}$. Let $\tilde{S}^2 = \tilde{L}^2 + \tilde{R}^2$.

Proposition 3.3.7. *The process $(S_t^{4,\mu})_{t \geq 0}$ satisfies the following absolute continuity relation: there exists a constant $c > 0$ such that for all $t > 0$, for all bounded measurable function $F : \mathbb{D}([0, t], \mathbb{R})^2 \rightarrow \mathbb{R}$, for all $x, y > 0$,*

$$\mathbb{E}(F((S_s^{4,\mu})_{0 \leq s \leq t})\mathbf{1}_{E_t} | (L_0^4, R_0^4) = (x, y)) = \mathbb{E}_{x,y} \left(\left(\frac{x+y}{\tilde{S}_{ct}^2} \right)^2 F((\tilde{S}_{cs}^2)_{0 \leq s \leq t})\mathbf{1}_{\forall s \in [0, t], \tilde{L}_{cs}^2, \tilde{R}_{cs}^2 > 0} \right), \quad (3.3.12)$$

where under $\mathbb{P}_{x,y}$ the Cauchy process \tilde{L}^2 (resp. \tilde{R}^2) starts at x (resp. y). Moreover, conditionally on $(S_s^{4,\mu})_{0 \leq s \leq t}$, the quantum surfaces which are cut out or inside discovered loops before time t are independent quantum disks of quantum boundary length corresponding to the absolute values of sizes of the negative and positive jumps of $(S_s^{4,\mu})_{0 \leq s \leq t}$, while $(\mathcal{D}_t^4, h_{|\mathcal{D}_t^4}^2)$ is an independent quantum disk of quantum boundary length $S_t^{4,\mu}$.

Here we also choose the multiplicative constant for the quantum length of the trunk so that $c = 1$ to be consistent with our choice of c for Proposition 3.3.6. Note that the quantum length of the trunk of the chordal $\text{SLE}_4^{(\mu)}(-2)$ can also be defined using the quantum boundary length measure since it is an SLE_4 -type curve, so that there should be a choice for c such that the quantum length of the trunk and the quantum boundary lengths of the cut out and encircled domains are defined using the same multiplicative constant. However, our approach does not enable us to identify this constant.

Remark 3.3.8. *The above proposition is not exactly the proper counterpart of Proposition 3.3.6 insofar as we do not describe the left and right quantum boundary lengths of the unexplored region, but only the whole quantum boundary length. To get the evolution of the left and right quantum boundary lengths, one has to identify the quantum surfaces which are cut out or encircled by a loop on the right side and on the left side of the trunk. For the cut out domains one can see from Proposition 3.3.3 that the domains which are cut out on the right (resp. left) side of the trunk are limits of domains which are cut out on the right (resp. left) side of the trunk. For the loops, one has to look at the direction the trunk takes after hitting the loops, using the proof of Proposition 3.3.3 once more. We do not go further in that direction since it is not the main purpose of this work. Note that in the case $\kappa = 4$, the quantum boundary lengths of the loops discovered on the right-hand side and on the left-hand side during the $\text{SLE}_4^{(\mu)}(-2)$ exploration should have the same intensity since the corresponding Bessel process here is symmetric ($\beta = 0$). The term μ only adds an “infinitesimal rotation at each time the process is not drawing a loop”. This explains why the processes \tilde{L}^2 and \tilde{R}^2 have the same law contrary to the case $\beta \neq 0$ and their law does not depend on μ .*

Using Proposition 3.3.7, one can deduce Theorem 3.1.2 for the $\text{SLE}_4^{(\mu)}(-2)$ exploration:

Proof of Theorem 3.1.2 for $(Y_t^\mu)_{t \geq 0}$ using Proposition 3.3.7. Using the $\text{SLE}_4^{(\mu)}(-2)$ exploration tree, one can define the branch which follows the locally largest component in terms of quantum boundary length. It can be defined through an approximation as in Subsection 5.3 of [MSW22]: let $\varepsilon > 0$, for each time of the form εn for $n \in \mathbb{N}$, one updates the target point for the chordal exploration and chooses the point such that the quantum boundary lengths on both sides of the unexplored region are equal. When $\varepsilon \rightarrow 0$, this exploration approximates the branch of the locally largest component. Denote by $(Y_t^\mu)_{t \geq 0}$ the quantum boundary length of the locally largest component parametrized by the quantum length of the trunk (which is given by a concatenation of parts of trunks of chordal $\text{SLE}_4^{(\mu)}(-2)$'s). Then, using exactly the same reasoning as in Subsection 5.3 of [MSW22], we deduce from Proposition 3.3.7 the analogous result for the branch of the locally largest component, which is no more than the statement of Theorem 3.1.2 for $(Y_t^\mu)_{t \geq 0}$. Let us still give the main idea for the identification of the law of $(Y_t^\mu)_{t \geq 0}$ for completeness.

By Proposition 3.3.7, while $L_t^4, R_t^4 > 0$, as a Doob h -transform of a Cauchy process, the process $S^{4,\mu}$

makes a jump of size $\ell \in \mathbb{R} \setminus \{0\}$ at rate

$$\frac{c}{\ell^2} \left(\frac{S_t^{4,\mu}}{S_0^{4,\mu}} \right)^2 \left(\frac{S_t^{4,\mu} + \ell}{S_0^{4,\mu}} \right)^{-2} = \frac{c}{\ell^2} \left(\frac{S_t^{4,\mu}}{S_t^{4,\mu} + \ell} \right)^2.$$

As a result, the process $(Y_t^\mu)_{t \geq 0}$ giving the quantum boundary length of the locally largest component makes a positive (resp. negative) jump of size $\ell > 0$ (resp. $-\ell < 0$) at rates

$$\frac{c}{\ell^2} \left(\frac{Y_t^\mu}{Y_t^\mu + \ell} \right)^2 \quad \text{and respectively} \quad \frac{c}{\ell^2} \left(\frac{Y_t^\mu}{Y_t^\mu - \ell} \right)^2 \mathbf{1}_{\ell < Y_t^\mu/2},$$

hence the desired result thanks to the description of $(X^{(-1)}(t))_{t \geq 0}$ given in Remark 3.1.1. \square

All this subsection and the next one are then devoted to the proof of Proposition 3.3.7. Let $(\mathbb{D}, h^{\sqrt{\kappa}})$ be a parametrization of a unit-boundary $\sqrt{\kappa}$ -quantum disk. We run an independent chordal $\text{SLE}_{\kappa}^{\mu((4/\kappa)-1)}(\kappa-6)$ in \mathbb{D} from $-i$ to i . We will prove the above result by letting $\kappa \uparrow 4$.

Short idea of proof of Proposition 3.3.7. Let us summarize the proof before entering the details. We first prove the convergence of the processes giving the quantum boundary length of the unexplored region. Next, we build on the convergence (3.3.11) of the quantum disk and on the Carathéodory convergence of the CLE_{κ} exploration to obtain the convergence of the restrictions of the field $h^{\sqrt{\kappa}}$ and of the quantum area measure to the cut out, encircled or unexplored domain containing a fixed point $z \in \mathbb{D}$. After proving this, the difficulty is to see that these domains appear in the ‘‘same order’’ in the exploration. To see this, we prove the convergence of the number of cut out or encircled domains of quantum boundary length at least $\varepsilon > 0$ for all $\varepsilon > 0$. Finally, we prove the convergence of the quantum length of the trunk. We deduce Proposition 3.3.7 from Proposition 3.3.6 by letting $\kappa \uparrow 4$ and using all the above-mentioned convergences.

The first step towards Proposition 3.3.7 is the convergence in law of the processes involved in Proposition 3.3.6. Let $(\bar{L}_0^4, \bar{R}_0^4) = (L_0^4, R_0^4)$ (see the beginning of Subsection 3.3.5). Next, conditionally on $(\bar{L}_0^4, \bar{R}_0^4) = (x, y)$, define a process $(\bar{L}_t^4, \bar{R}_t^4)_{t \geq 0}$ via the same Doob h -transform as in Proposition 3.3.6 for $\kappa = 4$, $a = 1 + 4/\kappa = 2$ and $\beta = 0$, replacing E_t by $\{\bar{L}_t^4, \bar{R}_t^4 > 0\}$ and setting $\bar{L}_t^4 = \bar{R}_t^4 = 0$ on the complement. Note that we choose to write \bar{L}^4, \bar{R}^4 the processes appearing in the following lemma instead of L^4, R^4 since we do not know yet whether they indeed describe the quantum boundary length of the unexplored region as in Proposition 3.3.7.

The following lemma establishes the convergence in law of the processes giving the left and right boundary lengths in Proposition 3.3.6. Its proof relies on their description as a Doob h -transform of Lévy processes and on the convergence of these Lévy processes.

Lemma 3.3.9. *We have the convergence*

$$(L_t^{\kappa, \mu((4/\kappa)-1)}, R_t^{\kappa, \mu((4/\kappa)-1)})_{t \geq 0} \xrightarrow[\kappa \uparrow 4]{(d)} (\bar{L}_t^4, \bar{R}_t^4)_{t \geq 0} \quad (3.3.13)$$

in the sense of Skorokhod.

Proof. The convergence of $(L_0^{\kappa, \mu((4/\kappa)-1)}, R_0^{\kappa, \mu((4/\kappa)-1)})$ stems from the convergence of the quantum boundary length measure in (3.3.11). Besides, one can easily see that for all continuous bounded function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ vanishing on a neighbourhood of zero, we have

$$\int_{\mathbb{R}} \varphi(x) \Lambda_L^{\mu((4/\kappa)-1), (4/\kappa)+1}(dx) \xrightarrow[\kappa \uparrow 4]{} \int_{\mathbb{R}} \varphi(x) \frac{dx}{2x^2}$$

and

$$\int_{\mathbb{R}} \varphi(x) \Lambda_R^{\mu((4/\kappa)-1), (4/\kappa)+1}(dx) \xrightarrow[\kappa \uparrow 4]{} \int_{\mathbb{R}} \varphi(x) \frac{dx}{2x^2}.$$

It is classical (see e.g. [JS87] VII.3.6) that this implies the convergence of Lévy processes

$$\left(\tilde{L}_t^{\mu((4/\kappa)-1), (4/\kappa)+1} \right)_{t \geq 0} \xrightarrow[\kappa \uparrow 4]{(d)} \left(\tilde{L}_t^2 \right)_{t \geq 0} \quad \text{and} \quad \left(\tilde{R}_t^{\mu((4/\kappa)-1), (4/\kappa)+1} \right)_{t \geq 0} \xrightarrow[\kappa \uparrow 4]{(d)} \left(\tilde{R}_t^2 \right)_{t \geq 0},$$

for the J_1 Skorokhod topology. We then reason by absolute continuity. Let $t, \varepsilon > 0$. By Proposition 3.3.6, if F is a bounded continuous function on $\mathbb{D}([0, t], \mathbb{R})^2$ vanishing on $\{(f, g); \exists s \in [0, t], f(s) < \varepsilon \text{ or } g(s) < \varepsilon\}$, we have

$$\begin{aligned} & \mathbb{E}\left(F((L_s^{\kappa, \beta}, R_s^{\kappa, \beta})_{0 \leq s \leq t}) \mathbf{1}_{E_t} \mid (L_0^{\kappa, \beta}, R_0^{\kappa, \beta}) = (x, y)\right) \\ &= \mathbb{E}_{x, y} \left(\left(\frac{x+y}{\tilde{S}_{ct}^a} \right)^a F\left((\tilde{L}_{cs}^{\beta, a}, \tilde{R}_{cs}^{\beta, a})_{0 \leq s \leq t}\right) \mathbf{1}_{\forall s \in [0, t], \tilde{L}_{cs}^{\beta, a}, \tilde{R}_{cs}^{\beta, a} > 0} \right), \end{aligned}$$

where we have set $\beta = \mu((4/\kappa) - 1)$ and $a = (4/\kappa) + 1$. One concludes by dominated convergence. \square

Another ingredient is the fact that the convergences (3.3.2) and (3.3.11) imply the convergence of the underlying fields and quantum area measure of the cut out, encircled and unexplored surfaces:

Lemma 3.3.10. *If for all $z \in \mathbb{D} \cap \mathbb{Q}^2$, the function $f_u^{z, \kappa}$ is the unique conformal map from \mathbb{D} to $D_u^{z, \kappa}$ which sends 0 to z such that $(f_u^{z, \kappa})'(0) > 0$, then jointly with (3.3.2) and (3.3.11),*

$$\left(\begin{array}{c} f_u^{z, \kappa} \\ h^{\sqrt{\kappa}} \circ f_u^{z, \kappa} + Q_{\sqrt{\kappa}} \log |(f_u^{z, \kappa})'| \\ \mu_{h^{\sqrt{\kappa}}}^{\sqrt{\kappa}} \circ f_u^{z, \kappa} \end{array} \right)_{u \geq 0, z \in \mathbb{Q}^2 \cap \mathbb{D}} \xrightarrow[\kappa \uparrow 4]{(d)} \left(\begin{array}{c} f_u^{z, 4} \\ h^2 \circ f_u^{z, 4} + Q_2 \log |(f_u^{z, 4})'| \\ \mu_{h^2}^2 \circ f_u^{z, 4} \end{array} \right)_{u \geq 0, z \in \mathbb{Q}^2 \cap \mathbb{D}}, \quad (3.3.14)$$

in terms of finite dimensional distributions, where the first coordinate is endowed with the topology of the uniform convergence on compact sets, the second one with the topology of the convergence of distributions and the third one with the weak convergence of measures on \mathbb{D} .

Proof. By Skorokhod's representation theorem we can assume that the convergences (3.3.2) and (3.3.11) hold a.s.. Let $z \in \mathbb{D} \cap \mathbb{Q}^2$ and $u \geq 0$. Let $f_\kappa = f_u^{z, \kappa}$. Then, from the Carathéodory convergence of $D_u^{z, \kappa}$ as $\kappa \uparrow 4$, we know that f_κ converges uniformly on compact sets towards f_4 . We also know that for every compact subset $K \subset D_u^{z, 4}$, the conformal map f_κ^{-1} converges uniformly on K towards f_4^{-1} as $\kappa \uparrow 4$. From Cauchy's formula, we know that the derivatives of f_κ^{-1} also converge uniformly on K to the derivatives of f_4^{-1} . Let us first show that $h^{\sqrt{\kappa}} \circ f_\kappa$ converges to $h^2 \circ f_4$ for the weak-* topology. Let $\varphi : \mathbb{D} \rightarrow \mathbb{R}$ be a C^∞ function with compact support. Since f_κ converges uniformly on compact sets to f_4 as $\kappa \uparrow 4$, there exists some relatively compact open subset $U \subset \mathbb{D}$ such that for κ close enough to 4, one has $\text{supp}(\varphi \circ f_\kappa^{-1}) = f_\kappa(\text{supp } \varphi) \subset U$. We thus have

$$\begin{aligned} |\langle h^{\sqrt{\kappa}} \circ f_\kappa - h^2 \circ f_4, \varphi \rangle| &= |\langle h^{\sqrt{\kappa}}, (\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 \rangle - \langle h^2, (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2 \rangle| \\ &= |\langle h^{\sqrt{\kappa}}, (\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 \rangle - \langle \varphi \circ f_4^{-1}, (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2 \rangle| \\ &\quad + |\langle h^{\sqrt{\kappa}} - h^2, (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2 \rangle| \\ &\leq \|h^{\sqrt{\kappa}}\|_{H^{-1}(U)} \|(\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 - (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2\|_{H^1(U)} \\ &\quad + \|h^{\sqrt{\kappa}} - h^2\|_{H^{-1}(U)} \|(\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2\|_{H^1(U)}. \end{aligned}$$

But we know that $\|h^{\sqrt{\kappa}} - h^2\|_{H^{-1}(U)} \rightarrow 0$ as $\kappa \uparrow 4$. Thus, to prove that $\langle h^{\sqrt{\kappa}} \circ f_\kappa - h^2 \circ f_4, \varphi \rangle \rightarrow 0$, it remains to check that $\|(\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 - (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2\|_{H^1(\mathbb{D})} \rightarrow 0$, in other words that

$$\|(\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 - (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2\|_{L^2(\mathbb{D})} + \|\nabla((\varphi \circ f_\kappa^{-1}) \cdot |(f_\kappa^{-1})'|^2 - (\varphi \circ f_4^{-1}) \cdot |(f_4^{-1})'|^2)\|_{L^2(\mathbb{D})}$$

tends to zero as $\kappa \uparrow 4$. The above expression indeed goes to zero as $\kappa \uparrow 4$ since φ is C^∞ with compact support and f_κ^{-1} and its derivatives converge uniformly on compact sets to f_4^{-1} and its derivatives. Thus $h^{\sqrt{\kappa}} \circ f_\kappa$ converges to $h^2 \circ f_4$ for the weak-* topology, so $h^{\sqrt{\kappa}} \circ f_\kappa + Q_{\sqrt{\kappa}} \log |f_\kappa'|$ converges to $h^2 \circ f_4 + Q_2 \log |f_4'|$ for the weak-* topology. The third convergence of (3.3.14) comes easily from the uniform convergence on compact sets of f_κ^{-1} and from the convergence of $\mu_{h^{\sqrt{\kappa}}}^{\sqrt{\kappa}}$. \square

3.3.6 Proof of Proposition 3.3.7

Now, let us prove Proposition 3.3.7 using the above lemmas.

We first introduce some notation. For all $i \geq 1$, for all $\varepsilon > 0$, let $\tau_{\varepsilon,i}^\kappa$ be the time of the i -th jump of $(S_t^{\kappa,\mu((4/\kappa)-1)})_{t \geq 0}$ of size larger than ε . Then we know that for all $\varepsilon > 0$, $\tau_{\varepsilon,i}^\kappa$ and the size of the jump at time $\tau_{\varepsilon,i}^\kappa$ converge in distribution by (3.3.13) but we do not know yet that the limit corresponds to a jump of $S^{4,\mu}$. Henceforth, in the rest of the subsection, we denote by $\text{ql}^\kappa(u)$ the quantum length of the trunk at time u for the original parametrization of the chordal $\text{SLE}_\kappa^{\mu((4/\kappa)-1)}(\kappa-6)$. The function ql^κ is non-decreasing and such that its limit $\lim_{u \rightarrow \infty} \text{ql}^\kappa(u)$ corresponds to the total quantum length of the trunk. Since the exploration runs for all $u > 0$, we have $\text{ql}^\kappa(u) < \lim_{v \rightarrow \infty} \text{ql}^\kappa(v)$ for all $u \geq 0$. From Remark 2.3 of [MSW22], we know that ql^κ is characterized as the limit in probability:

$$\text{ql}^\kappa(u) = C(\kappa) \lim_{\varepsilon \rightarrow 0} \varepsilon^{4/\kappa} \# \left\{ s \in [0, u]; -\Delta S_{\text{ql}^\kappa(s)}^{\kappa,\mu((4/\kappa)-1)} \in [\varepsilon, 2\varepsilon] \right\} \quad (3.3.15)$$

where one can compute $C(\kappa) = (4/\kappa)/(1 - 1/2^{4/\kappa})$ with our choice of the multiplicative constant below Proposition 3.3.6 (the value of $C(\kappa)$ is not important for our purpose, we only want it to converge to a positive number as $\kappa \uparrow 4$). It is clear from (3.3.15) that $(\text{ql}^\kappa(u))_{u \geq 0}$ is adapted with respect to the natural filtration of $(S_{\text{ql}^\kappa(u)}^{\kappa,\mu((4/\kappa)-1)})_{u \geq 0}$ and in particular to the filtration associated to the quantum disks discovered during the chordal exploration. For all $t \geq 0$, we denote by Cut_t^κ the domain which is cut out or encircled by a discovered loop at (quantum length) time t . By convention, we set $\text{Cut}_t^\kappa = \partial$ where ∂ is a cemetery point if no such domain is encircled or cut out at (quantum length) time t .

Similarly, for all $u \geq 0$, we denote by $\text{ql}^4(u)$ the quantum length of the trunk at time u for the original parametrization of the chordal $\text{SLE}_4^{\mu}(-2)$, which is the limit in probability of ε times the number of cut out regions until time t of quantum boundary length in $[\varepsilon, 2\varepsilon]$, up to a multiplicative constant. For consistency with (3.3.15), we take the constant equal to 2. The function ql^4 is then characterized as the limit in probability

$$\text{ql}^4(u) = 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{v \in [0, u]; -\Delta S_{\text{ql}^4(v)}^{4,\mu} \in [\varepsilon, 2\varepsilon]\}. \quad (3.3.16)$$

By Skorokhod's representation theorem, we may assume that (3.3.2), (3.3.5), (3.3.11) and (3.3.14) hold almost surely.

Let us denote by b_i^κ and $(b_i^\kappa)^2 a_i^\kappa$ the quantum boundary lengths and quantum areas of the domains which are encircled by loops or cut out by the trunk during the whole chordal exploration, ranked in the decreasing order of quantum boundary lengths. Note that by Proposition 3.3.6, conditionally on the b_i^κ 's, the a_i^κ 's are i.i.d. of the same law as $\mu_{h\sqrt{\kappa}}^{\sqrt{\kappa}}(\mathbb{D})$. We also have the relation

$$\mu_{h\sqrt{\kappa}}^{\sqrt{\kappa}}(\mathbb{D}) = \sum_{i \geq 1} (b_i^\kappa)^2 a_i^\kappa. \quad (3.3.17)$$

Moreover, by Lemma 3.3.9, the quantum boundary lengths of the cut out and encircled domains during the whole $\text{SLE}_\kappa^{\mu((4/\kappa)-1)}(\kappa-6)$ chordal exploration ranked in the non-increasing order converge in distribution as $\kappa \uparrow 4$. We also have the convergence in distribution of the a_i^κ 's by (3.3.11). Let $\kappa_n \uparrow 4$. We may find a subsequence again denoted by κ_n such that the families $(b_i^{\kappa_n})_{i \geq 1}$ and $(a_i^{\kappa_n})_{i \geq 1}$ converge a.s. by Skorokhod's theorem so that we have a.s. with the above convergences

$$\forall i \geq 1, \quad b_i^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} b_i \quad \text{and} \quad a_i^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} a_i. \quad (3.3.18)$$

Step 1. We first prove for all $\varepsilon > 0$ the tightness of the number of cut out and encircled domains of quantum boundary length larger than ε . Let $u \geq 0$. For all $\varepsilon > 0$, for all $\kappa \in (8/3, 4]$, let $N_\varepsilon^\kappa(u)$ be the number of cut out domains or domains encircled by loops of quantum boundary length larger than ε before time u , i.e.

$$N_\varepsilon^\kappa(u) = \# \left\{ t \leq \text{ql}^\kappa(u); \left| \Delta S_t^{\kappa,\mu((4/\kappa)-1)} \right| > \varepsilon \right\}.$$

Lemma 3.3.11. *For all $\varepsilon > 0$ and $u \geq 0$, the sequence of $(N_\varepsilon^{\kappa_n}(u))_{n \geq 0}$ is tight.*

Proof. One can first see that

$$\sum_{i \geq 1} \varepsilon^2 a_i^{\kappa_n} \mathbf{1}_{b_i^{\kappa_n} > \varepsilon} \leq \sum_{i \geq 1} (b_i^{\kappa_n})^2 a_i^{\kappa_n} \mathbf{1}_{b_i^{\kappa_n} > \varepsilon} \leq \mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(\mathbb{D}) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu_{h^2}^2(\mathbb{D}). \quad (3.3.19)$$

We deduce that $(\#\{i \geq 1; b_i^{\kappa_n} > \varepsilon\})_{n \geq 0}$ is tight, otherwise there would exist $\eta > 0$ such that for all integer $k \geq 1$, for infinitely many n , on an event of probability at least η the above sum stochastically dominates $\sum_{i=1}^k \varepsilon^2 a_i^{\kappa_n}$ which converges to $\sum_{i=1}^k \varepsilon^2 a_i$ by (3.3.18), where the a_i 's are i.i.d. random variables following the same law as $\mu_{h^2}^2(\mathbb{D})$ by (3.3.11), which contradicts (3.3.19) by taking k large. Thus $N_\varepsilon^{\kappa_n}(u)$ is tight as $N_\varepsilon^{\kappa_n}(u) \leq \#\{i \geq 1; b_i^{\kappa_n} > \varepsilon\}$. \square

By Lemma 3.3.11, we may take a subsequence such that jointly with the above convergences, for all $u \in \mathbb{Q} \cap \mathbb{R}_+$, for all $\varepsilon \in \mathbb{Q} \cap \mathbb{R}_+^*$, we have

$$N_\varepsilon^{\kappa_n}(u) \xrightarrow[n \rightarrow \infty]{} N_\varepsilon(u). \quad (3.3.20)$$

Step 2. Let $\varepsilon \in \mathbb{Q} \cap \mathbb{R}_+^*$. Let us prove the Carathéodory convergence of the cut out and encircled domains of quantum boundary length larger than ε along some subsequence:

Lemma 3.3.12. *Let $u \in \mathbb{Q} \cap \mathbb{R}_+$. Let $1 \leq i_0 \leq N_\varepsilon(u)$. The sequence $(\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n})_{n \geq 0}$ converges a.s. along some (deterministic) subsequence which is again denoted by $(\kappa_n)_{n \geq 0}$ in the sense of Carathéodory.*

Proof. First, note that $\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n})$ can be written in the form $(b_{\varepsilon, i_0}^{\kappa_n})^2 a_{\varepsilon, i_0}^{\kappa_n}$, where $b_{\varepsilon, i_0}^{\kappa_n}$ is the i_0 -th jump of $S^{\kappa_n, \mu((4/\kappa_n)-1)}$ of size larger than ε (in absolute value), and where $a_{\varepsilon, i_0}^{\kappa_n}$ is independent of $b_{\varepsilon, i_0}^{\kappa_n}$ and has the same law as the $a_i^{\kappa_n}$'s. Thus, by (3.3.11) and (3.3.13), by taking another subsequence again denoted by κ_n and by applying Skorokhod's representation theorem, we may assume that $b_{\varepsilon, i_0}^{\kappa_n}$ and $a_{\varepsilon, i_0}^{\kappa_n}$ converge a.s. to some independent random variables $b_{\varepsilon, i_0} \neq 0$ and a_{ε, i_0} , where a_{ε, i_0} has the same law as $\mu_{h^2}^2(\mathbb{D})$. In particular,

$$\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}) = (b_{\varepsilon, i_0}^{\kappa_n})^2 a_{\varepsilon, i_0}^{\kappa_n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} (b_{\varepsilon, i_0})^2 a_{\varepsilon, i_0}. \quad (3.3.21)$$

Let us then prove that a.s. there exists a random variable $\mathcal{Z} \in \mathbb{D} \cap \mathbb{Q}^2$ and a random subsequence \mathcal{K}_n of κ_n such that for all $n \geq 0$, we have $\mathcal{Z} \in \text{Cut}_{\tau_{\varepsilon, i_0}}^{\mathcal{K}_n}$, so that in particular $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\mathcal{K}_n}$ converges in the sense of Carathéodory towards $D_u^{\mathcal{Z}, 4}$ since $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\mathcal{K}_n} = D_u^{\mathcal{Z}, \mathcal{K}_n}$ and due to (3.3.2). Assume by contradiction that for all $z \in \mathbb{D} \cap \mathbb{Q}^2$, there exists $n_z \in \mathbb{N}$ such that for all $n \geq n_z$, we have $z \notin \text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}$. Let $(D_u^{z_i, 4})_{i \geq 1}$ be an enumeration of the domains of the form $D_u^{z, 4}$ for some $z \in \mathbb{D} \cap \mathbb{Q}^2$. By (3.3.11) we know that $\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(\mathbb{D})$ converges towards $\mu_{h^2}^2(\mathbb{D})$ as $\kappa \uparrow 4$. Moreover, by (3.3.14), for all $z \in \mathbb{D} \cap \mathbb{Q}^2$, we also have the convergence of $\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(D_u^{z, \kappa_n})$ towards $\mu_{h^2}^2(D_u^{z, 4})$ as $\kappa \uparrow 4$. As a consequence, we have for all $k \geq 1$,

$$\begin{aligned} \sum_{i=1}^k \mu_{h^2}^2(D_u^{z_i, 4}) &= \lim_{n \rightarrow \infty} \sum_{i=1}^k \mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(D_u^{z_i, \kappa_n}) \leq \lim_{n \rightarrow \infty} \mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(\mathbb{D}) - \min\{(b_j^{\kappa_n})^2 a_j^{\kappa_n}; j \geq 1, b_j^{\kappa_n} > \varepsilon\} \\ &= \mu_{h^2}^2(\mathbb{D}) - \min\{(b_j)^2 a_j; j \geq 1, b_j > \varepsilon\}. \end{aligned} \quad (3.3.22)$$

The above inequality in (3.3.22) comes from the fact that $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n} \neq D_u^{z_i, \kappa_n}$ for all $i \in \llbracket 1, k \rrbracket$ for all n large enough (since $z_i \notin \text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}$ for n large enough) and from the fact that for n large enough, the $D_u^{z_i, \kappa_n}$'s for $i \in \llbracket 1, k \rrbracket$ are distinct. Indeed, if $i \neq j$, then we cannot have $D_u^{z_i, \kappa_n} = D_u^{z_j, \kappa_n}$ infinitely often, otherwise we would have $D_u^{z_i, 4} = D_u^{z_j, 4}$. Moreover, the left-hand side of (3.3.22) converges to $\mu_{h^2}^2(\mathbb{D})$ as $k \rightarrow \infty$, hence a contradiction.

Finally, let us prove that a.s. $\mathcal{Z} \in \text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}$ for all n large enough. This will end the proof since in this case we will have $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n} = D_u^{\mathcal{Z}, \kappa_n}$ for all n large enough, which ensures the desired Carathéodory convergence. To see this, it suffices to notice that, on the one hand, by (3.3.14), we have the convergence of $\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(D_u^{\mathcal{Z}, \kappa_n})$ towards $\mu_{h^2}^2(D_u^{\mathcal{Z}, 4})$ and on the other hand, we have the convergence (3.3.21). Moreover, since the law of the a_i 's is the inverse of an exponential random variable thanks to Theorem 5.5 of

[AHPS23] and to Theorem 1.1 of [AdS22], it has a density, so that by (3.3.18), a.s. for all $z, z' \in \mathbb{D} \cap \mathbb{Q}^2$, if $D_u^{z,4} \neq D_u^{z',4}$ are distinct cut out domains, then $\mu_{h^2}^2(D_u^{z,4}) \neq \mu_{h^2}^2(D_u^{z',4})$. Thus, a.s. $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}$ cannot converge in the sense of Carathéodory to some other cut out domain $D_u^{z,4} \neq D_u^{z',4}$ along a subsequence. So by doing the same reasoning as above, one can see that a.s. we cannot have $\mathcal{Z} \notin \text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa_n}$ infinitely often (otherwise we would construct a random variable $\mathcal{Z}' \in \mathbb{D} \cap \mathbb{Q}^2$ a.s. not in $D_u^{z,4}$ and a random subsequence κ'_n such that for all $n \geq 0$ $\mathcal{Z}' \in \text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa'_n}$ so that $\text{Cut}_{\tau_{\varepsilon, i_0}}^{\kappa'_n}$ would converge in the sense of Carathéodory towards $D_u^{\mathcal{Z}',4} \neq D_u^{z,4}$). \square

Step 3. By applying Lemma 3.3.12, we assume that a.s. for all $i \in \llbracket 1, N_\varepsilon(u) \rrbracket$, the domain $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$ converges in the sense of Carathéodory. Let us prove that the convergence of the $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$'s also holds in terms of quantum disks. More precisely:

Lemma 3.3.13. *Let $i \in \llbracket 1, N_\varepsilon(u) \rrbracket$. Take z in the almost sure limit of $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$, then, if f_{κ_n} (resp. f_4) is the unique conformal map from \mathbb{D} to D_u^{z, κ_n} (resp. $D_u^{z,4}$) such that $f_{\kappa_n}(0) = z$ and $f'_{\kappa_n}(0) > 0$ (resp. $f_4(0) = z$ and $f'_4(0) > 0$), we have the convergence jointly with the above convergences*

$$\left(\begin{array}{c} h^{\sqrt{\kappa_n}} \circ f_{\kappa_n} + Q_{\sqrt{\kappa_n}} \log |f'_{\kappa_n}| \\ \mu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} \circ f_{\kappa_n} \\ \nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} \circ f_{\kappa_n} \end{array} \right) \xrightarrow[\kappa \uparrow 4]{(d)} \left(\begin{array}{c} h^2 \circ f_4 + Q_2 \log |f'_4| \\ \mu_{h^2}^2 \circ f_4 \\ \nu_{h^2}^2 \circ f_4 \end{array} \right) \quad (3.3.23)$$

where the first coordinate is endowed with the H_{loc}^{-1} topology and the two last ones with the weak convergence of measures respectively on \mathbb{D} and $\partial\mathbb{D}$.

Proof. Recall the convergence (3.3.14) and the fact that for n large enough, by the Carathéodory convergence of $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$, we have $D_u^{z, \kappa_n} = \text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$. Moreover, $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$ is one of the cut-out or encircled domains of quantum boundary length $b_i^{\kappa_n}$ so that there exists a random integer I_n , which is a.s. bounded due to (3.3.17), such that $\text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}$ is the quantum disk of quantum boundary length $b_{I_n}^{\kappa_n}$. Since for all $i \geq 1$ such that $b_i^{\kappa_n} > \varepsilon$, conditionally on $b_i^{\kappa_n}$, the corresponding domain is a $\sqrt{\kappa_n}$ -quantum disk, we deduce from (3.3.11) that

$$\tilde{h}^{\sqrt{\kappa_n}} := \mathbf{1}_{z \in \text{Cut}_{\tau_{\varepsilon, i}}^{\kappa_n}} \left(h^{\sqrt{\kappa_n}} \circ f_{\kappa_n} + Q_{\sqrt{\kappa_n}} \log |f'_{\kappa_n}| - \frac{2}{\sqrt{\kappa_n}} \log \left(\nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n}) \right) \right)$$

is tight as $n \rightarrow \infty$ in $H_{\text{loc}}^{-1}(\mathbb{D})$ (and also the quantum area and length measures $\mu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}$ and $\nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}$ for the weak convergence of measures). Thus $\log \nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n})$ is tight in \mathbb{R} as $n \rightarrow \infty$. So $h^{\sqrt{\kappa_n}} \circ f_{\kappa_n} + Q_{\sqrt{\kappa_n}} \log |f'_{\kappa_n}|$ is tight in $H_{\text{loc}}^{-1}(\mathbb{D})$ and hence converges in $H_{\text{loc}}^{-1}(\mathbb{D})$ towards $h^2 \circ f_4 + Q_2 \log |f'_4|$. Let us take a subsequence again denoted by $\kappa_n \uparrow 4$ such that

$$\left(\begin{array}{c} h^{\sqrt{\kappa_n}} \circ f_{\kappa_n} + Q_{\sqrt{\kappa_n}} \log |f'_{\kappa_n}| \\ (\tilde{h}^{\sqrt{\kappa_n}}, \mu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}, \nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}) \\ \log \left(\nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n}) \right) \end{array} \right) \xrightarrow[n \rightarrow \infty]{(a.s.)} \left(\begin{array}{c} h^2 \circ f_4 + Q_2 \log |f'_4| \\ (\tilde{h}^2, \mu_{h^2}^2, \nu_{h^2}^2) \\ Z \end{array} \right),$$

where Z is some real random variable and \tilde{h}^2 is the field corresponding to the parametrization by the unit disk of the quantum disk corresponding to b_I^4 for some random I integer such that $b_I^4 \geq \varepsilon$. Then $\tilde{h}^2 + Z = h^2 \circ f_4 + Q_2 \log |f'_4|$. Therefore

$$e^Z = \nu_{h^2 + Z}^2(\partial\mathbb{D}) = \nu_{h^2 \circ f_4 + Q_2 \log |f'_4|}^2(\partial\mathbb{D}) = \nu_{h^2}^2|_{D_u^{z,4}}(f_4(\partial\mathbb{D})) = \nu_{h^2}^2|_{D_u^{z,4}}(\partial D_u^{z,4}),$$

where the third equality comes by definition of the equivalence relation on quantum surfaces. Thus, we have the convergence in law of $\log \nu_{h^{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n})$ towards $\log \nu_{h^2}^2|_{D_u^{z,4}} (\partial D_u^{z,4})$. As a consequence,

$$\tilde{h}^2 = h^2 \circ f_4 + Q_2 \log |f'_4| - \log \nu_{h^2}^2|_{D_u^{z,4}} (\partial D_u^{z,4}).$$

The convergence of $\nu_{h_{|D_u^{z, \kappa_n}}}}^{\sqrt{\kappa_n}} \circ f_{\kappa_n}$ comes from the convergence of $\nu_{h_{|D_u^{z, \kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n})$ and from the fact that

$$\nu_{h_{|D_u^{z, \kappa_n}}}^{\sqrt{\kappa_n}} \circ f_{\kappa_n} = \nu_{h_{|D_u^{z, \kappa_n}}}^{\sqrt{\kappa_n}} (\partial D_u^{z, \kappa_n}) \nu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}.$$

This ends the proof of (3.3.23). \square

Step 4. Recall from (3.3.20) that $N_\varepsilon(u)$ is the almost sure limit of $N_\varepsilon^{\kappa_n}(u)$ for all $\varepsilon \in \mathbb{Q} \cap \mathbb{R}_+^*$ and $u \in \mathbb{Q} \cap \mathbb{R}_+$. Let us identify the limit. The main idea is to show that any cut out domain of quantum boundary length larger than ε is the limit of some cut out domain of quantum boundary length larger than ε so that $N_\varepsilon(u) \geq N_\varepsilon^4(u)$ and to use the convergence of the cut out domain of quantum boundary length larger than ε obtained in Lemma 3.3.13 to obtain the reverse inequality.

Lemma 3.3.14. *We have $N_\varepsilon(u) = N_\varepsilon^4(u)$ for all $u \in \mathbb{Q} \cap \mathbb{R}_+$.*

Proof. Recall that we have a sequence $\kappa_n \uparrow 4$ such that the convergences (3.3.2), (3.3.11), (3.3.14), (3.3.18), (3.3.20) and (3.3.23) from Lemma 3.3.13 hold almost surely.

By the proof of Proposition 3.3.3, we know that every cut out (resp. encircled) domain of the form $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ for some $i_0 \leq N_\varepsilon^4(u)$ is the limit in the sense of Carathéodory of some cut out (resp. encircled) domain. Moreover, by (3.3.14) we know that if $z \in \text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$, then for n large enough, we have

$$\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(D_u^{z, \kappa_n}) \geq \mu_{h^2}^2(\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4)/2. \quad (3.3.24)$$

To prove that D_u^{z, κ_n} converges in terms of quantum disks towards $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$, we distinguish two cases:

- If $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ is a domain encircled by a loop, then we know by Proposition 3.16 of [AG23a] that the quantum boundary length of $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ is the limit of the quantum boundary length of D_u^{z, κ_n} as $n \rightarrow \infty$.
- If $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ is a cut out domain, then we proceed as follows. Let $b_{i,+}^{\kappa_n}$'s for $i \geq 1$ (resp. the $(b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n}$'s) stand for the $b_i^{\kappa_n}$'s (resp. the $(b_i^{\kappa_n})^2 a_i^{\kappa_n}$'s) which correspond to a domain which is encircled by a loop, i.e. the $b_{i,+}^{\kappa_n}$'s are the positive jumps of $S^{\kappa_n, \mu((4/\kappa_n)-1)}$, ordered so that $(b_{i,+}^{\kappa_n})_{i \geq 1}$ is non-increasing. By Lemma 3.14 of [AG23a] we have the convergence in law of the quantum boundary length of a loop \mathcal{L}_n surrounding a point chosen at random according to $\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}/\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathbb{D})$ towards a positive random variable \tilde{b} . Besides, for all $\eta > 0$,

$$\sum_{i \geq k} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} \leq \sum_{i \geq 1} (b_{i,+}^{\kappa_n} \wedge \eta)^2 a_{i,+}^{\kappa_n} + \sum_{i \geq k} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} \mathbf{1}_{b_{i,+}^{\kappa_n} > \eta}. \quad (3.3.25)$$

The first sum on the right hand side of (3.3.25) can be rewritten

$$\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathbb{D}) \sum_{i \geq 1} \frac{a_{i,+}^{\kappa_n}}{\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathbb{D})} (b_{i,+}^{\kappa_n} \wedge \eta)^2 \leq \mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathbb{D}) \mathbb{E} \left(\left(\nu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathcal{L}_n) \wedge \eta \right)^2 \middle| (a_{i,+}^{\kappa_n})_{i \geq 1} \right)$$

Then note that by (3.3.14), $\mu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathbb{D})$ converges a.s. and that by the convergence in law of $\nu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathcal{L}_n)$ towards \tilde{b} we have

$$\mathbb{E} \left(\left(\nu_{h_{\sqrt{\kappa_n}}}^{\sqrt{\kappa_n}}(\mathcal{L}_n) \wedge \eta \right)^2 \right) \xrightarrow{n \rightarrow \infty} \mathbb{E} \left((\tilde{b} \wedge \eta)^2 \right),$$

which can be made arbitrarily small by letting $\eta \rightarrow 0$. Thus, a union bound and by Markov's inequality, we deduce that for all $\delta > 0$,

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sum_{i \geq 1} (b_{i,+}^{\kappa_n} \wedge \eta)^2 a_{i,+}^{\kappa_n} > \delta \right) = 0. \quad (3.3.26)$$

Next, let us consider the second sum on the right-hand side of (3.3.25). By the proof of Lemma 3.13 in [AG23a] we know that the number of loops with quantum boundary length larger than η is tight so that $N_{\eta,+}^{\kappa_n} := \#\{i \geq 1; b_{i,+}^{\kappa_n} > \eta\}$ is tight and note that

$$\sum_{i \geq k} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} \mathbf{1}_{b_{i,+}^{\kappa_n} > \eta} = \sum_{i=k}^{N_{\eta,+}^{\kappa_n}} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} \leq \sum_{i=k}^{N_{\eta,+}^{\kappa_n}} (b_i^{\kappa_n})^2 a_{i,+}^{\kappa_n}.$$

Therefore, using (3.3.18), we obtain that the right hand side in the above inequality is tight and if $L_{\eta,k}$ is a limit in distribution along a subsequence of the right-hand side of the above display, then one can see that $L_{\eta,k}$ goes to 0 in probability as $k \rightarrow \infty$. As a result, for all $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sum_{i \geq k} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} \mathbf{1}_{b_{i,+}^{\kappa_n} > \eta} > \delta \right) = 0. \quad (3.3.27)$$

Combining (3.3.25), (3.3.26) and (3.3.27), we get that for all $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sum_{i=k}^{\infty} (b_{i,+}^{\kappa_n})^2 a_{i,+}^{\kappa_n} > \delta \right) = 0.$$

As a consequence, by Proposition 3.3.6, and since the intensities of the Poisson point process of the positive and negative jumps of \tilde{S}^{1+4/κ_n} only differ by a multiplicative constant $\cos((1+4/\kappa_n)\pi)$ tending to 1 as $n \rightarrow \infty$, we also have for all $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sum_{i \geq k} (b_{i,-}^{\kappa_n})^2 a_{i,-}^{\kappa_n} > \delta \right) = 0,$$

where the $b_{i,-}^{\kappa_n}$'s (resp. the $(b_{i,-}^{\kappa_n})^2 a_{i,-}^{\kappa_n}$'s) stand for the $b_i^{\kappa_n}$'s (resp. the $(b_i^{\kappa_n})^2 a_i^{\kappa_n}$'s) which correspond to a cut out domain, i.e. the $b_{i,-}^{\kappa_n}$'s are the negative jumps of $S^{\kappa_n, \mu((4/\kappa_n)-1)}$. Therefore, for all $\delta \in (0, 1)$ we can choose $k \geq 0$ large enough so that for all n large enough, with probability at least $1 - \delta$,

$$\sum_{i \geq k} (b_i^{\kappa_n})^2 a_i^{\kappa_n} < \mu_{h^2}^2 (\text{Cut}_{\tau_{i_0, \varepsilon}^4}) / 2.$$

Thus, for all n large enough, with probability at least $1 - \delta$, we cannot have $\mu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(D_u^{z, \kappa_n}) = (b_i^{\kappa_n})^2 a_i^{\kappa_n}$ for some $i \geq k$ due to (3.3.24), so that it corresponds to an index $i < k$. In other words, since the $b_i^{\kappa_n}$'s are ranked in the non-increasing order, the quantum boundary length of D_u^{z, κ_n} is at least $b_k^{\kappa_n}$. Thus, given that $b_k^{\kappa_n}$ converges a.s. towards b_k by (3.3.18), for n large enough, with probability at least $1 - \delta$, the quantum boundary length of D_u^{z, κ_n} is larger than $b_k/2$.

In both cases, for all $\delta \in (0, 1)$, there exists $\eta > 0$ small enough such that for all $n \geq 0$, $\nu_{h\sqrt{\kappa_n}}^{\sqrt{\kappa_n}}(D_u^{z, \kappa_n}) \geq \eta$ with probability at least $1 - \delta$. In particular, for all $\delta \in (0, 1)$, there exists $\eta > 0$ small enough such that for all $n \geq 0$, with probability at least $1 - \delta$, there exists a random integer $I_{\eta, u, n}$ such that $D_u^{z, \kappa_n} = \text{Cut}_{\tau_{\eta, I_{\eta, u, n}}^{\kappa_n}}^{\kappa_n}$. Then, by the convergence (3.3.23) of the previous step (applied to $\text{Cut}_{\tau_{\eta, I_{\eta, u, n}}^{\kappa_n}}^{\kappa_n}$), we deduce that D_u^{z, κ_n} is tight as a quantum disk. In particular, after taking another subsequence of κ_n and applying Skorokhod's representation theorem once more, we may assume that D_u^{z, κ_n} converges a.s. in terms of quantum disks, i.e. for the topology of the convergence (3.3.23).

Since we already know by (3.3.14) (first coordinate) that $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ is the a.s. limit of D_u^{z, κ_n} in the sense of Carathéodory, we deduce that

$$D_u^{z, \kappa_n} \xrightarrow[n \rightarrow \infty]{\text{(a.s.)}} \text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$$

in terms of quantum disks. Since every cut out or encircled domain $\text{Cut}_{\tau_{i_0, \varepsilon}^4}^4$ for $i_0 \leq N_\varepsilon^4(u)$ is such a limit, we get that $N_\varepsilon(u) \geq N_\varepsilon^4(u)$. Moreover, since by the previous step the domain $\text{Cut}_{\tau_{i_0, \varepsilon}^{\kappa_n}}^{\kappa_n}$ converges in terms of quantum disks, and since the points of a Poisson point process of intensity dx/x^2 are a.s. different from ε , we obtain that $N_\varepsilon(u) \leq N_\varepsilon^4(u)$ and hence $N_\varepsilon(u) = N_\varepsilon^4(u)$. \square

Step 5. Let us prove that the cut-out and encircled domains converge in the right order.

Lemma 3.3.15. *We have a.s.*

$$\left(\text{Cut}_{\tau_{\varepsilon,i}^{\kappa_n}}^{\kappa_n} \right)_{i \in [1, N_{\varepsilon}^{\kappa_n}(u)]} \xrightarrow{n \rightarrow \infty} \left(\text{Cut}_{\tau_{\varepsilon,i}^4}^4 \right)_{i \in [1, N_{\varepsilon}^4(u)]}, \quad (3.3.28)$$

for the Carathéodory topology and in terms of quantum disks.

Proof. By the two previous steps, i.e. by Lemmas 3.3.13 and 3.3.14, we deduce that we have the a.s. convergence of the domains $(\text{Cut}_{\tau_{i,\varepsilon}^{\kappa_n}}^{\kappa_n})_{1 \leq i \leq N_{\varepsilon}^{\kappa_n}}$ towards some reordering of the family $(\text{Cut}_{\tau_{i,\varepsilon}^4}^4)_{1 \leq i \leq N_{\varepsilon}^4}$ for the Carathéodory topology and in terms of quantum disks.

Then (3.3.28) is easy to see by looking at rational times u_i 's for $i \geq 1$ such that $N_{\varepsilon}^4(u_i) = i$ for all $i \leq N_{\varepsilon}^4(u)$: at time u_1 there is only one possibility for the cut out or encircled domain, at time u_2 we have already identified the limit of the first region to be encircled or cut out, etc. Note that the convergence (3.3.28) holds for any $\varepsilon \in \mathbb{R}_+^* \cap \mathbb{Q}$. \square

Step 6. Here we prove the identity (3.3.12) of Proposition 3.3.7.

Proof of (3.3.12) in Proposition 3.3.7. Recall that we have the a.s. the convergences (3.3.2), (3.3.11), (3.3.14), (3.3.18), (3.3.20), (3.3.23) and also (3.3.28) by Lemma 3.3.15. We may also assume that the convergence (3.3.13) also holds almost surely by taking another subsequence again denoted by κ_n . By the definition of $(\bar{L}_t^4 + \bar{R}_t^4)_{t \geq 0}$ in Lemma 3.3.9, to prove (3.3.12), it is enough to check that for all $t \geq 0$, on the event E_t the process $(S_s^{4,\mu})_{0 \leq s \leq t}$ has the same law as $(\bar{S}_s^4)_{0 \leq s \leq t} := (\bar{L}_s^4 + \bar{R}_s^4)_{0 \leq s \leq t}$ on the event $\{\forall s \in [0, t], \bar{L}_s^4, \bar{R}_s^4 > 0\}$, where \bar{L}^4 and \bar{R}^4 are defined in Lemma 3.3.9. In what follows, we drop the exponents $\mu((4/\kappa_n) - 1)$ and μ above $S^{\kappa_n, \mu((4/\kappa_n) - 1)}$ and $S^{4,\mu}$ to simplify the notation. Note that for all $i_0 \geq 1$ for $\varepsilon > 0$ small enough, for all $i \leq i_0$ and $n \geq 1$, the time $\tau_{\varepsilon,i}^{\kappa_n}$ is finite and converges in law as $\kappa \uparrow 4$ by (3.3.13). Moreover, for all $u \geq 0$, by definition of $N_{\varepsilon}^{\kappa_n}(u)$, we obtain that

$$\tau_{\varepsilon, N_{\varepsilon}^{\kappa_n}(u)}^{\kappa_n} \leq \text{ql}^{\kappa_n}(u) < \tau_{\varepsilon, N_{\varepsilon}^{\kappa_n}(u)+1}^{\kappa_n}. \quad (3.3.29)$$

Thus, by (3.3.20) and (3.3.13) by letting $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ for all rational $u \geq 0$, there is a non-decreasing process $(\tilde{\text{ql}}(u))_{u \in \mathbb{Q} \cap \mathbb{R}_+}$ such that for all $u \in \mathbb{Q} \cap \mathbb{R}_+$,

$$\text{ql}^{\kappa_n}(u) \xrightarrow{n \rightarrow \infty} \tilde{\text{ql}}(u).$$

Actually, by density of the jumps of \bar{S}^4 , one can see that $\tilde{\text{ql}}$ extends to a continuous increasing function so that we have a.s. uniformly on compact sets

$$\text{ql}^{\kappa_n} \xrightarrow{n \rightarrow \infty} \tilde{\text{ql}}.$$

Then, by the definition of the Skorokhod J_1 convergence, the above convergence implies that

$$(S_{\text{ql}^{\kappa_n}(u)}^{\kappa_n})_{u \geq 0} \xrightarrow[n \rightarrow \infty]{\text{(a.s.)}} (\bar{S}_{\text{ql}(u)}^4)_{u \geq 0}$$

in the sense of Skorokhod. In particular, since the Skorokhod convergence implies the convergence of the jumps of size at least ε for all $\varepsilon > 0$, we deduce that a.s. for all $u \in \mathbb{Q} \cap \mathbb{R}_+$, for all $\varepsilon > 0$, by (3.3.28) we have

$$N_{\varepsilon,-}^4(u) := \#\{v \in [0, u]; -\Delta S_{\text{ql}^4(v)}^4 \geq \varepsilon\} = \#\{v \in [0, u]; -\Delta \bar{S}_{\text{ql}(v)}^4 \geq \varepsilon\}.$$

Since the left and the right terms are càdlàg in u , the above equality holds for all $u \geq 0$. We can check that $\tilde{\text{ql}} = \text{ql}^4$ using the characterization (3.3.16) of the quantum length of the trunk as a limit in probability: for all $u > 0$,

$$\begin{aligned} \text{ql}^4(u) &= 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \#\{v \in [0, u]; -\Delta S_{\text{ql}^4(v)}^4 \in [\varepsilon, 2\varepsilon]\} \\ &= 2 \lim_{\varepsilon \rightarrow 0} \varepsilon (N_{\varepsilon,-}^4(u) - N_{2\varepsilon,-}^4(u)) \\ &= 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \#\{t \in [0, \tilde{\text{ql}}(u)]; -\Delta \bar{S}_t^4 \in [\varepsilon, 2\varepsilon]\} \\ &= \tilde{\text{ql}}(u). \end{aligned}$$

The last equality of limits in probability stems from the definition of \overline{S}^4 as a Doob h -transform of Cauchy processes. Thus a.s. $\widetilde{ql} = ql^4$.

Therefore ql^{κ_n} converges towards ql^4 uniformly on compact sets. Using the fact that ql^4 is increasing, it is elementary to prove that $(ql^{\kappa_n})^{-1}$ converges towards $(ql^4)^{-1}$ uniformly on compact subsets of $[0, \lim_{v \rightarrow \infty} ql^4(v))$. But, by the convergence of the unexplored region for the Carathéodory topology (see (3.3.5)): for any sequence $u_n \rightarrow u$ of non-negative (possibly random) real numbers, a.s.

$$\mathcal{D}_{ql^{\kappa_n}(u_n)}^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \mathcal{D}_{ql^4(u)}^4.$$

We obtain a.s. the convergence for all $t \in [0, \lim_{v \rightarrow \infty} ql^4(v)) \cap \mathbb{Q}$ in the sense of Carathéodory,

$$\mathcal{D}_t^{\kappa_n} = \mathcal{D}_{ql^{\kappa_n}((ql^{\kappa_n})^{-1}(t))}^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \mathcal{D}_{ql^4((ql^4)^{-1}(t))}^4 = \mathcal{D}_t^4. \quad (3.3.30)$$

Besides, since

$$\nu_{h|\mathcal{D}^{\kappa_n}}^{\sqrt{\kappa_n}}(\partial \mathcal{D}_t^{\kappa_n}) = S_t^{\kappa_n} \xrightarrow[n \rightarrow \infty]{} \overline{S}_t^4,$$

and since for all n , $\mathcal{D}_t^{\kappa_n}$ is a quantum disk of quantum boundary length $S_t^{\kappa_n}$, we deduce that $\mathcal{D}_t^{\kappa_n}$ is tight in terms of quantum disk. Thus, after taking another subsequence and applying Skorokhod's representation theorem once again, $\mathcal{D}_t^{\kappa_n}$ converges towards \mathcal{D}_t^4 in terms of quantum disks. In particular, $S^4 = \overline{S}^4$, hence (3.3.12). \square

Step 7. Finally, let us prove the remaining part of Proposition 3.3.7.

Proof of Proposition 3.3.7: law of the cut out domains and unexplored region. In order to prove the statement on the cut out domains and on the region remaining to be explored, it suffices to check that for all $\varepsilon > 0$, if F is a continuous bounded function on $\mathbb{D}([0, t])$ and G is a continuous bounded function on the space of finite sequences of distributions on \mathbb{D} seen modulo the transformation (3.2.1) (formally it can be written as $\bigcup_{n>0} (\mathcal{D}'(\mathbb{D}))^n$ where $\mathcal{D}'(\mathbb{D})$ is the space of distributions on \mathbb{D} modulo the transformation (3.2.1) for all Möbius transformation φ), then

$$\mathbb{E}\left(F((S_s^4)_{0 \leq s \leq t})G\left(h_t, (h_{\varepsilon, i})_{i \geq 1, \tau_{\varepsilon, i}^4 < t}\right)\right) = \mathbb{E}\left(F((S_s^4)_{0 \leq s \leq t})G\left(h(S_t^4), (h(\Delta S_{\tau_{\varepsilon, i}^4}^4))_{i \geq 1, \tau_{\varepsilon, i}^4 < t}\right)\right),$$

where a representative of h_t is $h^2 \circ f_z + 2 \log |f'_z|$ where $z \in \mathcal{D}_t^4$ and $f_z : \mathbb{D} \rightarrow \mathcal{D}_t^4$ is the unique conformal map such that $f_z(0) = z$ and $f'_z(0) > 0$, where for all $i \geq 1$, we have defined $h_{\varepsilon, i}$ as the equivalence class of $h^2 \circ f_{z_i} + 2 \log |f'_{z_i}|$ where $z_i \in \text{Cut}_{\tau_{\varepsilon, i}^4}^4$ and $f_{z_i} : \mathbb{D} \rightarrow \text{Cut}_{\tau_{\varepsilon, i}^4}^4$ is the unique conformal map such that $f_{z_i}(0) = z_i$ and $f'_{z_i}(0) > 0$, and where $h(b)$ denotes the instance of the GFF of an independent quantum disk of quantum boundary length b (conditionally on b). But the above equality holds clearly thanks to the convergences (3.3.13), (3.3.28), (3.3.30) and thanks to the second point of Proposition 3.3.6 which states the same equality for $\kappa < 4$. \square

3.4 The $\text{SLE}_4^{(\mu)}(-2)$ converges to the uniform exploration as $\mu \rightarrow \infty$

Recall that we proved Theorem 3.1.2 in the case of the $\text{SLE}_4^{(\mu)}$ exploration for $\mu \in \mathbb{R}$ in Subsection 3.3.5. In this section, we prove Theorem 3.1.2 in the case of the uniform exploration, which was defined in Subsection 3.2.1. To this end, we let $\mu \rightarrow \infty$ to approximate the uniform exploration using the $\text{SLE}_4^{(\mu)}(-2)$ exploration. In this section, for convenience, we will explore the nested CLE_4 .

3.4.1 From the Carathéodory convergence to the Markov property for quantum disks

The fact that the uniform exploration can be constructed by letting $\mu \rightarrow \infty$ was first observed in the last paragraph of p.22 of [WW13]. However, we will need a convergence in terms of Carathéodory topology. Recall the radial $\text{SLE}_4^{(\mu)}(-2)$ defined at the end of Subsection 3.3.2. For all $\mu \in \mathbb{R}$, for all

$z \in \mathbb{D}$, let $(\mathbf{D}_s^{z,\mu})_{s \geq 0}$ be the radial exploration targeted at z using the $\text{SLE}_4^{(\mu)}(-2)$. More precisely, for every time $s \geq 0$, the domain $\mathbf{D}_s^{z,\mu}$ is the unexplored region containing z for the radial $\text{SLE}_4^{(\mu)}(-2)$. For convenience, we do not stop the exploration when the loop around z is discovered, hence discovering the loops of the nested CLE_4 . The parametrization is chosen so that if $f_s : \mathbb{D} \rightarrow \mathbf{D}_s^{z,\mu}$ is the unique conformal map such that $f_s(0) = z$ and $f'_s(0) > 0$, then $f'_s(0) = e^{-s}$. The positive number $f'_s(0)$ is called the conformal radius of $\mathbf{D}_s^{z,\mu}$ seen from z . For all $w \in \mathbb{D} \cap \mathbb{Q}^2 \setminus \{z\}$, we denote by $D_s^{z,w,\mu}$ the connected component containing w of the complement in \mathbb{D} of the path of the chordal $\text{SLE}_4^{(\mu)}(-2)$ towards z until time s . Note that there is a time $\sigma_{z,w}^\mu$ at which the radial $\text{SLE}_4^{(\mu)}(-2)$ towards z cuts w out from z or encircles w with a loop which does not encircle z , or encircles z and not w . Therefore, for all $s < \sigma_{z,w}^\mu$, we have $D_s^{z,w,\mu} = \mathbf{D}_s^{z,\mu}$ and for all $s \geq \sigma_{z,w}^\mu$, the domain $D_s^{z,w,\mu} = D_{\sigma_{z,w}^\mu}^{z,w,\mu}$ corresponds to the domain which is cut out or encircled by a loop. Similarly, using the uniform exploration of the nested CLE_4 defined in Subsection 3.2.1, let $(\mathbf{D}_s^z)_{s \geq 0}$ be the exploration targeted at z , again parametrized so that the conformal radius of \mathbf{D}_s^z is e^{-s} . More precisely, \mathbf{D}_s^z is the unexplored region containing z at time s . Note that for all $w \in \mathbb{D} \cap \mathbb{Q}^2 \setminus \{z\}$ there is a time $\sigma_{z,w}$ at which the exploration cuts w out from z or encircles w with a loop which does not encircle z . We then set $D_s^{z,w} = \mathbf{D}_s^z$ for all $s < \sigma_{z,w}$, and we define $D_s^{z,w}$ for $s \geq \sigma_{z,w}$ as the domain which is cut out or encircled by a loop at time $\sigma_{z,w}$.

Proposition 3.4.1. *One has the convergence in distribution of*

$$((\mathbf{D}_s^{z,\mu})_{s \geq 0, z \in \mathbb{D} \cap \mathbb{Q}^2}, (D_s^{z,w,\mu})_{s \geq 0, z \in \mathbb{D} \cap \mathbb{Q}^2, w \in \mathbb{D} \cap \mathbb{Q}^2 \setminus \{z\}})$$

to

$$((\mathbf{D}_s^z)_{s \geq 0, z \in \mathbb{D} \cap \mathbb{Q}^2}, (D_s^{z,w})_{s \geq 0, z \in \mathbb{D} \cap \mathbb{Q}^2, w \in \mathbb{D} \cap \mathbb{Q}^2 \setminus \{z\}})$$

as $\mu \rightarrow \infty$ in terms of finite dimensional distributions for the Carathéodory topology. The convergence holds also in a stronger sense: if $(s_\mu)_{\mu \geq 0}$ converges towards s as $\mu \rightarrow \infty$, then jointly with the above convergence, for all $z \in \mathbb{D} \cap \mathbb{Q}^2$, we have $\mathbf{D}_{s_\mu}^{z,\mu} \rightarrow \mathbf{D}_s^z$ as $\mu \rightarrow \infty$ in distribution.

The proof of Theorem 3.1.2 for $(Y_t)_{t \geq 0}$ then follows from roughly the same lines as the proof of Proposition 3.3.7, although here we deal with the radial exploration instead of the chordal one. Let us only give a sketch of the proof:

Sketch of the proof of Theorem 3.1.2 for $(Y_t)_{t \geq 0}$ using Proposition 3.4.1. For all $z \in \mathbb{D}$, for all $\mu \in \mathbb{R}$, let us write $\text{ql}_{z,\mu}(s)$ the quantum length of the trunk of the radial exploration $(\mathbf{D}_s^{z,\mu})_{s \geq 0}$. More precisely, it is characterized (with our choice of the multiplicative constant) by

$$\text{ql}_{z,\mu}(s) = 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \#\{v \in [0, s]; -\Delta \nu_{\mathbf{D}_v^{z,\mu}}^2 (\partial \mathbf{D}_v^{z,\mu}) \in [\varepsilon, 2\varepsilon]\}.$$

Let us write $\text{ql}_{z,\mu}^{-1}$ the right-continuous inverse of $\text{ql}_{z,\mu}$ and for all $t \geq 0$,

$$\widehat{S}_t^{z,\mu} := \nu_{\mathbf{D}_{\text{ql}_{z,\mu}^{-1}(t)}^{z,\mu}}^2 (\partial \mathbf{D}_{\text{ql}_{z,\mu}^{-1}(t)}^{z,\mu}).$$

Let Z_i for $i \geq 1$ be i.i.d. random points of \mathbb{D} of law $\mu_{h^2}^2 / \mu_{h^2}^2(\mathbb{D})$ which are independent of h^2 and of the explorations. One can see using Theorem 3.1.2 for $(Y_t^\mu)_{t \geq 0}$, the relation between the quantum area and the quantum boundary length and the renewal property (Proposition 3.3.1) that the process $(\widehat{S}_t^{Z_i,\mu})_{t \geq 0, i \geq 1}$ has the same distribution for all $\mu \in \mathbb{R}$ and that for all $i \geq 1$, conditionally on their quantum boundary lengths, $\mathbf{D}_{\text{ql}_{Z_i,\mu}^{-1}(t)}^{Z_i,\mu}$ is chosen at random among the quantum disks corresponding to the unexplored regions of the exploration tree at height t . More precisely, if we denote by $a_j^\mu(t)$ for $j \geq 1$ the quantum areas of the unexplored quantum disks when we stop the exploration tree at height t ranked in the non-increasing order, then conditionally on the exploration tree at height t and on the a_j^μ 's, the quantum surface $\mathbf{D}_{\text{ql}_{Z_i,\mu}^{-1}(t)}^{Z_i,\mu}$ corresponds to the quantum disk of area $a_j^\mu(t)$ with probability $a_j^\mu(t) / (\sum_{k \geq 1} a_k^\mu(t))$. In particular, the quantum surface $\mathbf{D}_{\text{ql}_{Z_i,\mu}^{-1}(t)}^{Z_i,\mu}$ is constant in distribution when μ varies.

By the same reasoning as in the proof of Lemma 3.3.10, we know that the convergence in Proposition 3.4.1 also holds for the underlying fields in terms of the convergence in the sense of distribution and for the quantum area measures with the weak convergence of measures.

Using that $\mu_{h^2}^2(\mathbb{D}) < \infty$, it is easy to see that the number of encircled or cut out disks of quantum boundary length larger than some $\varepsilon > 0$ during the exploration $\mathbf{D}^{Z_i, \mu}$ until time s is tight, as in Lemma 3.3.11. As a result, using that $\widehat{S}^{Z_i, \mu}$ is constant in law, we obtain as in (3.3.29) that for all $u \in \mathbb{Q} \cap \mathbb{R}_+$, the family $(\text{ql}_{Z_i, \mu}(u))_{\mu \geq 0}$ is tight. But note that $\text{ql}_{Z_i, \mu}$ is a non-decreasing function. Moreover, by density of the jumps of $\widehat{S}^{Z_i, \mu}$, we get that $\text{ql}_{Z_i, \mu}$ is continuous. As a result, $\text{ql}_{Z_i, \mu}$ is tight as $\mu \rightarrow \infty$ for the topology of the uniform convergence on compact sets and that the limit must be continuous and non-decreasing.

Let $\mu_n \rightarrow \infty$. By taking a subsequence again denoted by μ_n and by applying Skorokhod's theorem, we may assume that the convergence of Proposition 3.4.1 holds a.s. together with the convergences for all $i \geq 1$,

$$\widehat{S}^{Z_i, \mu_n} \xrightarrow[n \rightarrow \infty]{} \overline{S}^{Z_i} \quad \text{ql}_{Z_i, \mu_n} \xrightarrow[n \rightarrow \infty]{} \widetilde{\text{ql}}_{Z_i},$$

where the first convergence holds for the J_1 Skorokhod topology and the second one holds uniformly on compact sets, where \overline{S}^{Z_i} is a càdlàg process which has the same law as \widehat{S}^{Z_i, μ_n} for all n and where $\widetilde{\text{ql}}_{Z_i}$ is a continuous non-decreasing process from \mathbb{R}_+ to \mathbb{R}_+ . Let $\widetilde{\text{ql}}_{Z_i}^{-1}$ be its right-continuous inverse, which is increasing since $\widetilde{\text{ql}}_{Z_i}$ is continuous.

By the convergence of ql_{Z_i, μ_n} to $\widetilde{\text{ql}}_{Z_i}$, we deduce that a.s. for all $t \in \mathbb{R}_+$ such that $\widetilde{\text{ql}}_{Z_i}^{-1}$ is continuous at t , the random variable $\text{ql}_{Z_i, \mu_n}^{-1}(t)$ converges towards $\widetilde{\text{ql}}_{Z_i}^{-1}(t)$ as $n \rightarrow \infty$. As a consequence, by the convergence of Proposition 3.4.1, we also have a.s. for all $t \in \mathbb{R}_+$ such that $\widetilde{\text{ql}}_{Z_i}^{-1}$ is continuous at t ,

$$\mathbf{D}_{\text{ql}_{Z_i, \mu_n}^{-1}(t)}^{Z_i, \mu_n} \xrightarrow[n \rightarrow \infty]{} \mathbf{D}_{\widetilde{\text{ql}}_{Z_i}^{-1}(t)}^{Z_i}$$

in the sense of Carathéodory, and thus also in terms of quantum disks, since it is constant in law as a quantum surface. Therefore, a.s. for all $t \in \mathbb{R}_+$ such that $\widetilde{\text{ql}}_{Z_i}^{-1}$ is continuous at t ,

$$\widehat{S}_t^{Z_i, \mu_n} \xrightarrow[n \rightarrow \infty]{} \nu_{h^2}^2 \Big|_{\mathbf{D}_{\widetilde{\text{ql}}_{Z_i}^{-1}(t)}^{Z_i}} \left(\partial \mathbf{D}_{\widetilde{\text{ql}}_{Z_i}^{-1}(t)}^{Z_i} \right).$$

Hence for all $t \in \mathbb{R}_+$ such that $\widetilde{\text{ql}}_{Z_i}^{-1}$ is continuous at t ,

$$\nu_{h^2}^2 \Big|_{\mathbf{D}_{\widetilde{\text{ql}}_{Z_i}^{-1}(t)}^{Z_i}} \left(\partial \mathbf{D}_{\widetilde{\text{ql}}_{Z_i}^{-1}(t)}^{Z_i} \right) = \overline{S}_t^{Z_i}, \quad (3.4.1)$$

and the equality holds in fact for all $t \geq 0$ since both processes are càdlàg.

We see that the quantum length ql_{z, μ_n} converges to the quantum natural distance as $n \rightarrow \infty$ using the definition of the quantum natural distance $d_q(\partial \mathbb{D}, \mathbf{D}_s^z)$ as the limit in probability (which was given in (3.2.2))

$$d_q(\partial \mathbb{D}, \mathbf{D}_s^z) = 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{v \in [0, s], -\Delta \nu_{h^2}^2 \Big|_{\mathbf{D}_v^z} (\partial \mathbf{D}_v^z) \in [\varepsilon, 2\varepsilon]\}.$$

The above definition is indeed consistent with the definition (3.2.2) since the Cauchy process \widetilde{S}^2 in Proposition 3.3.7 is symmetric. From the above formula, it is easy to show, as in the end of the sixth step of the proof of Proposition 3.4.1, that for all $s \geq 0$, we have $d_q(\partial \mathbb{D}, \mathbf{D}_s^{Z_i}) = \widetilde{\text{ql}}_{Z_i}(s)$. Indeed, using the fact that \overline{S}^{Z_i} has the same law as $\widehat{S}^{Z_i, \mu}$ and the definition of $\widehat{S}^{Z_i, \mu}$, one has the convergence in probability

$$\begin{aligned} \widetilde{\text{ql}}_{Z_i}(s) &= 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{r \in [0, \widetilde{\text{ql}}_{Z_i}(s)], -\Delta \overline{S}_r^{Z_i} \in [\varepsilon, 2\varepsilon]\} \\ &= 2 \lim_{\varepsilon \rightarrow 0} \varepsilon \# \{v \in [0, s], -\Delta \nu_{h^2}^2 \Big|_{\mathbf{D}_v^{Z_i}} (\partial \mathbf{D}_v^{Z_i}) \in [\varepsilon, 2\varepsilon]\} \quad \text{by (3.4.1)} \\ &= d_q(\partial \mathbb{D}, \mathbf{D}_s^{Z_i}). \end{aligned}$$

For all $z \in \mathbb{D}$, if Ψ_z is the inverse of $s \mapsto d_q(\partial \mathbb{D}, \mathbf{D}_s^z)$, then we set for all $t \geq 0$,

$$\widehat{S}_t^z := \nu_{h^2}^2 \Big|_{\mathbf{D}_{\Psi_z(t)}^z} (\partial \mathbf{D}_{\Psi_z(t)}^z).$$

By (3.4.1) and since for all $s \geq 0$, we have $d_q(\partial\mathbb{D}, \mathbf{D}_s^{Z_i}) = \tilde{q}l_{Z_i}(s)$, we deduce that $\bar{S}^{Z_i} = \widehat{S}^{Z_i}$ and thus a.s.

$$(\widehat{S}_t^{Z_i, \mu_n})_{t \geq 0} \xrightarrow{n \rightarrow \infty} (\widehat{S}_t^{Z_i})_{t \geq 0} \quad (3.4.2)$$

in the sense of Skorokhod.

Next, we consider the locally largest component of the $\text{SLE}_4^{(\mu_n)}(-2)$ exploration parametrized by the quantum length of the trunk, that we denote by $(\mathcal{C}_t^{\mu_n})_{t \geq 0}$. Recall that its quantum boundary length is denoted by $(Y_t^{\mu_n})_{t \geq 0}$. By Theorem 3.1.2 for Y^{μ_n} , we know that the law of $(Y_t^{\mu_n})_{t \geq 0}$ does not depend on n and that conditionally on $(Y_{t'}^{\mu_n})_{0 \leq t' \leq t}$, the cut out or encircled domains, together with $\mathcal{C}_t^{\mu_n}$, are independent 2-quantum disks of boundary lengths corresponding to the negative and positive jumps of $(Y_{t'}^{\mu_n})_{0 \leq t' \leq t}$ and to $Y_t^{\mu_n}$.

Let us recall from Subsection 3.2.3 that the locally largest component along the uniform exploration of the CLE_4 parametrized by the quantum natural distance is denoted by $(C_{\sigma(t)})_{t \geq 0}$ and that its quantum boundary length is denoted by $(\mathcal{Y}(\sigma(t)))_{t \geq 0}$. We set $\mathcal{C}_t := C_{\sigma(t)}$ and recall that $Y_t := \mathcal{Y}(\sigma(t))$ for all $t \geq 0$.

Let $t \geq 0$. Let i such that $Z_i \in \mathcal{C}_t$. Then $\mathcal{C}_t = \mathbf{D}_{\Psi_{Z_i}(t)}^{Z_i}$. Therefore, for all $t' \in [0, t]$, we have

$$\mathbf{D}_{\Psi_{Z_i}(t')}^{Z_i} = \mathcal{C}_{t'} \quad \text{and} \quad \widehat{S}_{t'}^{Z_i} = Y_{t'}. \quad (3.4.3)$$

Besides, by the Skorokhod convergence (3.4.2), we know that for n large enough, the process $(\widehat{S}_{t'}^{Z_i, \mu_n})_{0 \leq t' \leq t}$ does not have any negative jump at any time $t' \leq t$ of size at least half of $\widehat{S}_{t'}^{Z_i, \mu_n}$. In other words, for n large enough, for all $t' \in [0, t]$,

$$\mathbf{D}_{\text{ql}_{Z_i, \mu_n}^{-1}(t')}^{Z_i, \mu_n} = \mathcal{C}_{t'}^{\mu_n} \quad \text{and} \quad \widehat{S}_{t'}^{Z_i, \mu_n} = Y_{t'}^{\mu_n}. \quad (3.4.4)$$

Therefore, from (3.4.2) we deduce that Y^{μ_n} converges almost surely towards Y for the J_1 topology of Skorokhod. In particular, Y has the same law as Y^{μ_n} . We obtain the law of the encircled, cut out and unexplored region by reasoning again as in Step 7 of the proof of Proposition 3.3.7 using the fact that (3.4.4) and (3.4.3) together with the convergence of the cut out and encircled regions stated in Proposition 3.4.1 entail the convergence of the cut out and encircled regions of quantum boundary length at least $\varepsilon > 0$ and the convergence of the unexplored region along the exploration of the locally largest component. \square

Outline of the proof of Proposition 3.4.1. Let us give a short outline of the proof of Proposition 3.4.1, which is performed in the next two subsections. Intuitively, the idea is that as $\mu \rightarrow \infty$ the small rotations at each time the exploration is not drawing a loop will be more and more important as $\mu \rightarrow \infty$, in a way that for each time the exploration starts to draw a loop, “the starting point is sampled uniformly at random on the boundary of the unexplored region”. We first prove in Subsection 3.4.2 the convergence of one branch of the exploration. By target-invariance, it suffices to prove the convergence towards the origin. This exploration is obtained using the same ideas as in Section 2 of [AHPS23]: we approximate the explorations by focusing on the “macroscopic loops”. Then, between each time of appearance of a macroscopic loop, we look at the impact of the “rotations” on the starting point of the next macroscopic loop. This starting point “becomes uniform” as $\mu \rightarrow \infty$. Next, in Subsection 3.4.3, we prove that the convergence of the exploration branches towards each point $z \in \mathbb{D} \cap \mathbb{Q}^2$ holds jointly, relying intuitively on the fact that “they draw the same loops”: the loops of a CLE_4 .

3.4.2 Convergence of the branch towards the origin

The rest of this section is devoted to the proof of Proposition 3.4.1 and follows the same lines as Section 2 of [AHPS23]. For this purpose, we recall a stronger topology on the space of families of domains as in [AHPS23] and [MS16]. Let \mathcal{D} be the space of families $(D_s)_{s \geq 0}$ of simply connected subdomains of \mathbb{D} which are increasing for the inclusion, such that for every $s \geq 0$, we have $0 \in D_s$, and if $f_s = f[D_s]$ is the unique conformal map from \mathbb{D} to D_s which sends 0 to 0 and such that $f'_s(0) > 0$, then $f'_s(0) = e^{-s}$. The space \mathcal{D} is endowed with the topology such that a sequence of domains $(D_s^n)_{s \geq 0} \in \mathcal{D}$ converges to $(D_s)_{s \geq 0}$ if and only if for all compact subset $K \subset \mathbb{D}$, for all $T > 0$,

$$\sup_{s \in [0, T]} \sup_{z \in K} |f_s^n(z) - f_s(z)| \xrightarrow{n \rightarrow \infty} 0,$$

where $f_s^n = f[D_s^n]$ and $f_s = f[\tilde{D}_s]$. The space \mathcal{D} is then metrizable and separable. We define a distance $d_{\mathcal{D}}$ on \mathcal{D} by setting for all $D, \tilde{D} \in \mathcal{D}$,

$$d_{\mathcal{D}}((D_s)_{s \geq 0}, (\tilde{D}_s)_{s \geq 0}) := \sum_{j,k \in \mathbb{N}} 2^{-j-k} \sup_{s \in [0, 2^k]} \sup_{z \in (1-1/j)\mathbb{D}} |f_s(z) - \tilde{f}_s(z)|, \quad (3.4.5)$$

where $f_s = f[D_s]$ and $\tilde{f}_s = f[\tilde{D}_s]$ for all $s \geq 0$. Note that the convergence of $(D_s^n)_{s \geq 0}$ to $(D_s)_{s \geq 0}$ in \mathcal{D} implies the convergence of D_s^n towards D_s in the Carathéodory topology for every $s \geq 0$. As in [AHPS23], we will sometimes consider evolving domains $(D_s)_{s \geq 0}$ such that the conformal radius of D_s is e^{-s} for all $s \in [0, S]$ for some finite $S > 0$ and such that for all $s \geq S$, $D_s = D_S$. In order to see $(D_s)_{s \geq 0}$ as an element of \mathcal{D} , we replace D_s for $s \geq S$ by $f_S(e^{-(s-S)}\mathbb{D})$ where $f_S = f[D_S]$.

Let us recall the definition of (measure-driven) radial Loewner chains. If λ is a measure on $[0, \infty) \times \partial\mathbb{D}$ whose marginal on $[0, \infty)$ is the Lebesgue measure, we define the radial Loewner equation driven by λ by setting for all $z \in \mathbb{D}$ and for all $s \geq 0$,

$$g_s(z) = \int_{[0, s] \times \partial\mathbb{D}} g_u(z) \frac{w + g_u(z)}{w - g_u(z)} d\lambda(u, w) + z.$$

It is known (see Proposition 6.1 of [MS16]) that for any such λ , this ODE has a unique solution $s \mapsto g_s(z)$ for each $z \in \mathbb{D}$, defined until time $s_z := \sup\{s \geq 0; g_s(z) \in \mathbb{D}\}$. Moreover, if one defines $D_s = \{z \in \mathbb{D}; s_z < s\}$ for all $s \geq 0$, then $(D_s)_{s \geq 0}$ is called the radial Loewner chain driven by λ .

If one restricts to a measure of the form $\lambda(ds, A) = \delta_{W(s)}(A)ds$ with $W : \mathbb{R}_+ \rightarrow \partial\mathbb{D}$ a piecewise continuous function, then one recovers the more classical notion of Loewner chains.

Remark 3.4.2. (Remark 2.1 of [AHPS23]) Using Proposition 6.1 of [MS16], we know that the weak convergence of the driving measure implies the convergence in \mathcal{D} of the radial Loewner chain. In particular, the convergence of the radial Loewner chain holds if we assume the convergence in the Skorokhod J_1 topology of the driving functions.

Let us then recall the direct construction of the radial $\text{SLE}_4^{(\mu)}(-2)$ (see [AHPS23] Subsection 2.1.3 for the analogous definition for $\kappa' > 4$ and see Subsection 4.1 of [ASW19] for the case $\mu = 0$). Let $(B_s)_{s \geq 0}$ be a standard Brownian motion and let $(\ell_s^0)_{s \geq 0}$ be its local time at 0 modulo π . Set, for all $s \geq 0$,

$$U_s = \int_0^s \cot(B_u) du + \mu \ell_s^0,$$

where the integral is not absolutely convergent but can be defined as the limit in probability of integrals of the form $\int_0^s \cot(B_u) \mathbf{1}_{d(B_u, \pi\mathbb{Z}) \geq \varepsilon} du$ as $\varepsilon \rightarrow 0$, in the same way as in [WW13] in the chordal case. Let $(W_0^\mu)_s = e^{i(2B_s - U_s)}$. Then, if one considers the radial Loewner chain $(\mathbf{D}_s^{0, \mu})_{s \geq 0}$ in \mathbb{D} driven by W_0^μ , it corresponds to the radial $\text{SLE}_4^{(\mu)}(-2)$ targeted at zero (and starting from 1). Let τ_0 be the first time that B reaches $\pm\pi$, which corresponds to the first time that 0 gets encircled by a loop during the $\text{SLE}_4^{(\mu)}(-2)$ exploration (see Subsection 4.1 of [ASW19] and Subsection 4.3 of [She09] for more details). Let

$$(D_s^{0, \mu})_{s \geq 0} = \left(\mathbf{D}_{s \wedge \tau_0}^{0, \mu} \right)_{s \geq 0}.$$

We also recall that, during each excursion of B away from $\pi\mathbb{Z}$, the radial Loewner chain draws a loop of the nested CLE_4 which does not contain 0 (see again for example Subsection 4.1 of [ASW19] in the less general case $\mu = 0$ and [She09] for more details). See Figure 3.4.

Let us present an approximation of this radial $\text{SLE}_4^{(\mu)}(-2)$. Let $n \in \mathbb{N}$. The idea is to remove intervals of time where B is making tiny excursions away from zero. We set $T_0^n := 0$ and for all $k \geq 0$,

$$\begin{cases} R_{k+1}^n &= \inf\{s \geq T_k^n; |B_s| \geq 1/2^n\}; \\ S_{k+1}^n &= \sup\{s \leq R_{k+1}^n; B_s = 0\}; \\ T_{k+1}^n &= \inf\{s \geq R_{k+1}^n; B_s = 0\}. \end{cases}$$

Call e_k^n the k -th excursion associated to the interval $[S_k^n, T_k^n]$. Set $\Lambda^n := \sup\{k; S_k^n \leq \tau_0\}$ and

$$l_k^n = T_k^n - S_k^n \quad \forall k < \Lambda^n; \quad l_{\Lambda^n}^n = \tau_0 - S_{\Lambda^n}^n; \quad L_k^n = \sum_{1 \leq j \leq k} l_j^n \quad \forall k \leq \Lambda^n.$$

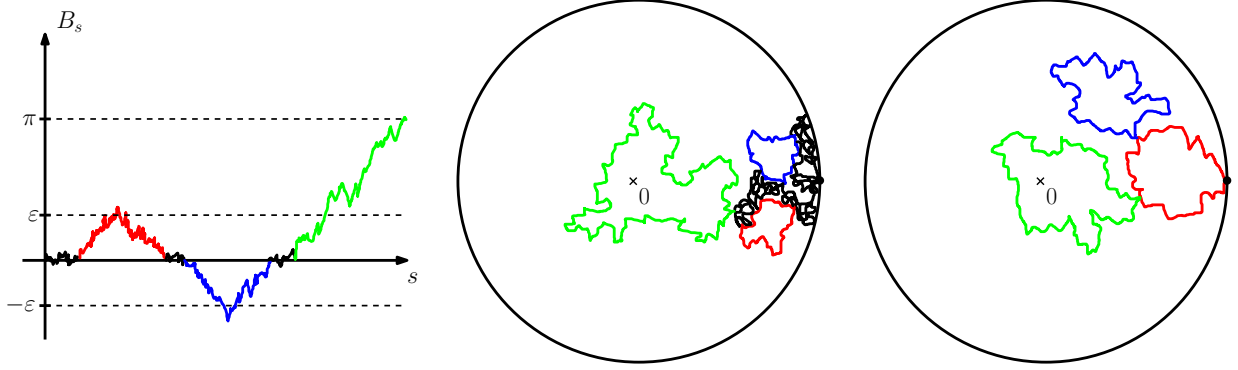


Figure 3.4 – Left: the Brownian motion $(B_s)_{s \geq 0}$. Center: a radial exploration of the CLE_4 . The red and blue excursions of height at least ε give rise to two loops which do not encircle zero and the green part of the Brownian motion generates a loop which encircles zero. Right: approximation of the radial exploration.

Now we define for all $k \in \llbracket 1, \Lambda^n \rrbracket$,

$$\forall k \in \llbracket 1, \Lambda^n \rrbracket, \forall s \in [L_{k-1}^n, L_k^n), \quad (W_0^{\mu, n})_s = (W_0^\mu)_{S_k^n + (s - L_{k-1}^n)},$$

and set $D^{0, \mu, n}$ to be the radial Loewner chain with driving function $W_0^{\mu, n}$. This is defined up to time $\tau_0^n = L_{\Lambda^n}^n$.

Let us recall an approximation of the uniform exploration of the nested CLE_4 . Sample $(X_k^n)_{k \geq 1}$ uniformly and independently on $\partial\mathbb{D}$. For all $s \in [0, \tau_0^n]$, set

$$\forall k \in \llbracket 1, \Lambda^n \rrbracket, \forall s \in [L_{k-1}^n, L_k^n), \quad (W_0^n)_s = X_k^n \exp \left(i \left(2B_{s+S_k^n} - \int_{S_k^n}^{s+S_k^n} \cot(B_u) du \right) \right). \quad (3.4.6)$$

Recall from Subsection 3.4.1 that $(\mathbf{D}_s^0)_{s \geq 0}$ is the uniform exploration targeted at 0, again parametrized so that the conformal radius of \mathbf{D}_s^0 is e^{-s} . More precisely, \mathbf{D}_s^0 is the unexplored region containing z at time s . Let $D^{0, n}$ be the radial Loewner chain with driving function W_0^n . By the same argument as in Section 4 of [WW13] we know that $D^{0, n}$ converges in \mathcal{D} in probability towards a uniform exploration D^0 targeted at 0 in \mathcal{D} and stopped at the time τ_0 defined by $D_s^0 = \mathbf{D}_{s \wedge \tau_0}^0$ for all $s \geq 0$. Note that the time τ_0 thus also corresponds to the first time at which 0 gets encircled by a loop for the uniform exploration \mathbf{D}^0 . Moreover, as in the beginning of Section 4 of [WW13], each loop drawn by \mathbf{D}^0 again corresponds to the excursion of B away from $\pi\mathbb{Z}$ at the same time.

Let us show the convergence in law jointly in $z \in \mathbb{Q}^2 \cap \mathbb{D}$ as $\mu \rightarrow \infty$ of the $\text{SLE}_4^{(\mu)}(-2)$ branch towards z to the uniform exploration branch towards z , together with the convergence of the CLE_4 loops surrounding z to the CLE_4 loops surrounding z . We first show the convergence of the branch $(\mathbf{D}_s^{0, \mu})_{s \geq 0}$ targeted at 0. To show this, it suffices to check the following statement:

Proposition 3.4.3. *The stopped exploration branch towards zero converges:*

$$(D^{0, \mu}, \tau_0, B) \xrightarrow[\mu \rightarrow \infty]{(d)} (D^0, \tau_0, B),$$

where the first coordinate converges in the space \mathcal{D} and where B is the Brownian motion from which the $D^{0, \mu}$'s are defined (and also D^0 via the approximation).

By the iterative definition of the exploration of the nested CLE_4 towards zero, the convergence for all time follows immediately from Proposition 3.4.3:

Proposition 3.4.4. *The whole exploration branch towards zero converges:*

$$(\mathbf{D}^{0, \mu}, \tau_0, B) \xrightarrow[\mu \rightarrow \infty]{(d)} (\mathbf{D}^0, \tau_0, B),$$

where the first coordinate converges in \mathcal{D} and B is the Brownian motion with which the $D^{0,\mu}$'s are defined. Notice that we also get in particular the convergence in the sense of Carathéodory of the domain $\mathbf{D}_{\tau_0}^{0,\mu}$ encircled by the outermost loop encircling 0 towards $\mathbf{D}_{\tau_0}^0$.

For all $z \in \mathbb{D} \cap \mathbb{Q}^2$, we can define the space \mathcal{D}_z of families of domains containing z which are in \mathcal{D} after applying the conformal map $w \mapsto (w - z)/(1 - \bar{z}w)$, equipped by the topology induced by the topology of \mathcal{D} . We also denote by τ_z^μ (resp. τ_z) the first time at which z is encircled by a loop in the radial SLE $_4^{(\mu)}(-2)$ targeted at z (resp. in the uniform exploration $(\mathbf{D}_s^z)_{s \geq 0}$). We also denote by $B^{z,\mu}$ (resp. B^z) the Brownian motion which defines $D^{z,\mu}$ (resp. D^z) composed with the conformal map $w \mapsto (w - z)/(1 - \bar{z}w)$. The time τ_z^μ (resp. τ_z) is then the first time at which $B^{z,\mu}$ (resp. B^z) reaches $\pm\pi$. Note that as in the case $z = 0$, we can build a coupling between the explorations using the same underlying Brownian motion so that $B^{z,\mu} = B^z$ and $\tau_z^\mu = \tau_z$ for all $\mu \in \mathbb{R}$. The convergence of the other branches then follows by conformal invariance as stated just below. We endow the space of continuous real functions $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ with the topology of the uniform convergence on compact sets.

Corollary 3.4.5. *For all $z \in \mathbb{D} \cap \mathbb{Q}^2$,*

$$(\mathbf{D}^{z,\mu}, \tau_z^\mu, B^{z,\mu}) \xrightarrow[\mu \rightarrow \infty]{(d)} (\mathbf{D}^z, \tau_z, B^z) \quad \text{in } \mathcal{D}_z \times \mathbb{R} \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}), \quad (3.4.7)$$

Moreover, we also have the convergence of the domain encircled by the outermost loop $D_{\tau_z^\mu}^{z,\mu}$ containing z towards $D_{\tau_z}^z$ in the sense of Carathéodory jointly with the above convergence.

Exactly as in [AHPS23], our proof of Proposition 3.4.3 goes through the approximations $D^{0,\mu,n}$ and $D^{0,n}$. Recall the definition of the distance $d_{\mathcal{D}}$ in (3.4.5). The following lemma states that the approximation $D^{0,\mu,n}$ converges to $D^{0,\mu}$ as $n \rightarrow \infty$ uniformly in n . Its proof relies on Remark 3.4.2 and on the convergence of the driving functions.

Lemma 3.4.6. *We have, a.s. $\tau_0^n \rightarrow \tau_0$ as $n \rightarrow \infty$ and*

$$\sup_{\mu \in \mathbb{R}} d_{\mathcal{D}}(D^{0,\mu,n}, D^{0,\mu}) \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Proof. The convergence of τ_0^n stems from elementary properties of the Brownian motion. To prove the uniform convergence in \mathcal{D} , let us define the driving functions using the same Brownian motion B . Let $T > 0$. Let us check that if $F : \partial\mathbb{D} \times [0, T] \rightarrow \mathbb{R}$ is a continuous function, bounded in absolute value by some constant $M > 0$, then a.s.

$$\int_0^T F((W_0^{\mu,n})_{s \wedge \tau_0^n}, s) ds \xrightarrow[n \rightarrow \infty]{} \int_0^T F((W_0^\mu)_{s \wedge \tau_0}, s) ds,$$

uniformly on $\mu \in \mathbb{R}$. This follows from the convergence of τ_0^n since

$$\left| \int_0^T F((W_0^{\mu,n})_{s \wedge \tau_0^n}, s) ds - \int_0^T F((W_0^\mu)_{s \wedge \tau_0}, s) ds \right| \leq 2M(\tau_0 - \tau_0^n).$$

By Remark 3.4.2, this gives the convergence for all $\mu \in \mathbb{R}$. To obtain the uniform convergence, we know by Remark 3.4.2 that the function which maps a measure λ on $[0, \infty) \times \partial\mathbb{D}$ whose marginal on $[0, \infty)$ is the Lebesgue measure and whose marginal on $\partial\mathbb{D}$ is a probability measure to the corresponding element of \mathcal{D} is continuous. Moreover, the space of such measures restricted to $\partial\mathbb{D} \times [0, T]$ is compact by compactness of $\partial\mathbb{D} \times [0, T]$. This enables to get the uniform convergence for the distance $d_{\mathcal{D}}$. \square

The lemma below is the key ingredient in the convergence towards the uniform exploration. It states the convergence of the approximation of the SLE $_4^{(\mu)}(-2)$ exploration to the approximation of the uniform exploration. The main idea in the proof of this lemma is that the starting points of each loop become i.i.d. uniform on the boundary when $\mu \rightarrow \infty$. This essentially stems from the fact that if E is an exponential random variable, then $e^{-i\mu E}$ converges in distribution as $\mu \rightarrow \infty$ to a uniform random variable on the circle $\partial\mathbb{D}$.

Lemma 3.4.7. Fix $n \in \mathbb{N}$. Then

$$(D^{0,\mu,n}, \tau_0^n, B) \xrightarrow[\mu \rightarrow \infty]{(d)} (D^{0,n}, \tau_0^n, B),$$

where the first coordinate converges in \mathcal{D} .

Proof. We set for all $k \in \llbracket 1, \Lambda^n \rrbracket$, $X_k^{\mu,n} = (W_0^\mu)_{S_k^n}$, and $\mathbf{X}^{\mu,n} = (X_k^{\mu,n})_{1 \leq k \leq \Lambda^n}$. For the uniform exploration we set $\mathbf{X}^n = (X_k^n)_{1 \leq k \leq \Lambda^n}$. We also write $(e_k^n)_{k \geq 1}$ the family of the excursions of B of height larger than $1/2^n$. Let us write also $(e_v)_{v \in \mathbb{R}_+}$ the Poisson point process of the excursions of B , which is indexed by the local time at 0 and we denote by $\tau(e_v)$ the duration of the excursion e_v . We define the point processes $(e_v^<)_{v \geq 0}$ and $(e_v^>)_{v \geq 0}$ of excursions of height smaller than $1/2^n$ and larger than $1/2^n$.

To prove the above lemma, it is enough to verify that

$$\mathbf{X}^{\mu,n} \xrightarrow[\mu \rightarrow \infty]{(d)} \mathbf{X}^n. \quad (3.4.8)$$

Indeed, we know that W_0^n is defined by (3.4.6) and moreover for all $k \in \llbracket 1, \Lambda^n \rrbracket$, for all $s \in [L_{k-1}^n, L_k^n)$,

$$\begin{aligned} (W_0^{\mu,n})_s &= (W_0^\mu)_{S_k^n} \exp \left(i \left(2B_{s+S_k^n} - 2B_{S_k^n} - \int_{S_k^n}^{s+S_k^n} \cot(B_u) du - \mu(\ell_s^0 - \ell_{S_k^n}^0) \right) \right) \\ &= X_k^{\mu,n} \exp \left(i \left(2B_{s+S_k^n} - \int_{S_k^n}^{s+S_k^n} \cot(B_u) du \right) \right). \end{aligned}$$

Thus, if one assumes (3.4.8), then we have clearly the convergence of the driving function $W_0^{\mu,n}$ towards W_0^n in the Skorokhod J_1 topology. This entails the convergence in \mathcal{D} of $(D_s^{0,\mu,n})_{s \geq 0}$ by Remark 3.4.2. Let us now prove the convergence (3.4.8). One can see that

$$\begin{aligned} (W_0^\mu)_{S_1^n} &= \exp \left(i \left(- \int_0^{S_1^n} \cot(B_s) ds - \mu \ell_{S_1^n}^0 \right) \right) \\ &= \exp \left(i \left(- \sum_{v < \ell_{S_1^n}^0} \int_0^{\tau(e_v)} \cot(e_v(s)) ds - \mu \ell_{S_1^n}^0 \right) \right) \end{aligned} \quad (3.4.9)$$

and that for all $k \geq 2$,

$$\begin{aligned} (W_0^\mu)_{S_k^n} &= (W_0^\mu)_{S_{k-1}^n} \exp \left(i \left(- \int_{S_{k-1}^n}^{S_k^n} \cot(B_s) ds - \mu(\ell_{S_k^n}^0 - \ell_{T_{k-1}^n}^0) \right) \right) \\ &= (W_0^\mu)_{S_{k-1}^n} \exp \left(i \left(- \int_0^{\tau(e_k^n)} \cot(e_k^n(s)) ds - \sum_{\ell_{T_{k-1}^n}^0 < v < \ell_{S_k^n}^0} \int_0^{\tau(e_v)} \cot(e_v(s)) ds - \mu(\ell_{S_k^n}^0 - \ell_{T_{k-1}^n}^0) \right) \right) \end{aligned} \quad (3.4.10)$$

By the restriction property of Poisson point processes, the point processes $(e_v^<)_{v \geq 0}$ and $(e_v^>)_{v \geq 0}$ of excursions of height smaller than $1/2^n$ and larger than $1/2^n$ are independent Poisson point processes. Note also that the atoms of $(e_v^>)_{v \geq 0}$ are the e_k^n 's for $k \geq 1$ which appear at the local times $\ell_{S_k^n}^0 = \ell_{T_k^n}^0$'s. In particular, the $\ell_{S_k^n}^0 - \ell_{T_{k-1}^n}^0$'s are i.i.d. exponential random variables whose parameter $\lambda_n > 0$ only depends on n (and not on μ), which are independent of $(e_v^<)_{v \geq 0}$ and of the excursions $(e_k^n)_{k \geq 1}$. But one can see that if E is an exponential random variable of parameter $\lambda_n > 0$, then

$$(E, e^{-i\mu E}) \xrightarrow[\mu \rightarrow \infty]{(d)} (E, U_1^n),$$

where U_1^n is independent from E and is a uniform random variable in $\partial\mathbb{D}$. Taking (3.4.9) and (3.4.10) into account, the above convergence yields

$$\left(X_1^{\mu,n}, \frac{X_2^{\mu,n}}{X_1^{\mu,n}}, \dots, \frac{X_{\Lambda^n}^{\mu,n}}{X_{\Lambda^n-1}^{\mu,n}} \right) \xrightarrow[\mu \rightarrow \infty]{(d)} (U_1^n, \dots, U_{\Lambda^n}^n),$$

where $(U_j^n)_{j \geq 1}$ is a family of i.i.d. uniform random variables in $\partial\mathbb{D}$ which is independent of $(B_s)_{s \geq 0}$, hence (3.4.8). \square

Proof of Proposition 3.4.3. This follows by combining Lemma 3.4.6, Lemma 3.4.7 and the fact that $D^{0,n}$ converges towards the uniform exploration of the nested CLE_4 targeted at zero. \square

3.4.3 Joint convergence of the exploration branches

Before proving Proposition 3.4.1, let us show the convergence of the interiors of the loops drawn along the exploration. The main idea of the proof of the lemma below is that the loops are drawn using the same excursions, the only difference being the “starting points”.

Lemma 3.4.8. *Let $\varepsilon > 0$. For all $i \geq 1$, let $e_{\varepsilon,i}$ be the i -th excursion of B of height at least ε away from $\pi\mathbb{Z}$ during an interval $[S_{\varepsilon,i}, T_{\varepsilon,i}]$. Let $\mathcal{L}_{\varepsilon,i}^{0,\mu}$ (resp. $\mathcal{L}_{\varepsilon,i}^0$) denote the loop drawn by $\mathbf{D}^{0,\mu}$ (resp. \mathbf{D}^0) during the interval $[S_{\varepsilon,i}, T_{\varepsilon,i}]$, so that the domain inside the loop is $\mathcal{B}_{\varepsilon,i}^{0,\mu} = \mathbf{D}_{S_{\varepsilon,i}}^{0,\mu} \setminus \overline{\mathbf{D}_{T_{\varepsilon,i}}^{0,\mu}}$ (resp. $\mathcal{B}_{\varepsilon,i}^0 = \mathbf{D}_{S_{\varepsilon,i}}^0 \setminus \overline{\mathbf{D}_{T_{\varepsilon,i}}^0}$). Then we have, jointly with the convergence of Proposition 3.4.4, for all $\varepsilon > 0$,*

$$\left(\mathcal{B}_{\varepsilon,i}^{0,\mu}\right)_{i \geq 1} \xrightarrow[\mu \rightarrow \infty]{(d)} \left(\mathcal{B}_{\varepsilon,i}^0\right)_{i \geq 1}$$

in terms of finite dimensional distributions for the Carathéodory topology and for the Hausdorff topology for their closure.

Proof. It suffices to prove the convergence in the sense of Carathéodory since the outermost loop surrounding a point z is constant in law as $\mu \rightarrow \infty$ given that it is the outermost loop of a CLE_4 surrounding z . Let $\mu_n \rightarrow \infty$. By Skorokhod’s representation theorem, we assume that the convergence of Proposition 3.4.3 holds almost surely along μ_n .

Let $\varepsilon > 0$. It is enough to prove the convergence for the excursions before the time τ_0 . Let $i \geq 1$ such that $T_{\varepsilon,i} < \tau_0$. By Proposition 3.4.3, we know that a.s.

$$D_{S_{\varepsilon,i}}^{0,\mu_n} \xrightarrow[n \rightarrow \infty]{} D_{S_{\varepsilon,i}}^0 \quad (3.4.11)$$

in the sense of Carathéodory.

Besides, let us denote by $g_{S_{\varepsilon,i}}^{0,\mu_n} : D_{S_{\varepsilon,i}}^{0,\mu_n} \rightarrow \mathbb{D}$ (resp. $g_{S_{\varepsilon,i}}^0 : D_{S_{\varepsilon,i}}^0 \rightarrow \mathbb{D}$) the unique conformal map such that $g_{S_{\varepsilon,i}}^{0,\mu_n}(0) = 0$ and $(g_{S_{\varepsilon,i}}^{0,\mu_n})'(0) = e^{-s}$ (resp. $g_{S_{\varepsilon,i}}^0(0) = 0$ and $(g_{S_{\varepsilon,i}}^0)'(0) = e^{-s}$). Then by definition of the radial Loewner chain, we have

$$\frac{1}{(W_0^{\mu_n})_{S_{\varepsilon,i}}} g_{S_{\varepsilon,i}}^{0,\mu_n}(\mathcal{L}_{\varepsilon,i}^{0,\mu_n}) = \frac{1}{X_{i,\varepsilon}} g_{S_{\varepsilon,i}}^0(\mathcal{L}_{\varepsilon,i}^0), \quad (3.4.12)$$

where $X_{i,\varepsilon}$ is a uniform random variable in $\partial\mathbb{D}$ which comes from the approximation of the uniform exploration: indeed, from the approximation of the uniform exploration (3.4.6), we know for all $i \geq 1$, that the loop $g_{S_{\varepsilon,i}}^0(\mathcal{L}_{\varepsilon,i}^0)$ is drawn using the radial Loewner chain driven by

$$s \mapsto X_{i,\varepsilon} \exp\left(i\left(2e_{i,\varepsilon}(s) - \int_0^s \cot(e_{i,\varepsilon}(s))\right)\right),$$

where the $X_{i,\varepsilon}$ ’s are i.i.d. uniform random variables in $\partial\mathbb{D}$ which are independent of B ; while by definition of the radial $\text{SLE}_4^{\langle \mu_n \rangle}(-2)$ exploration, using that the local time is constant on excursion time intervals, we know that the loop $g_{S_{\varepsilon,i}}^{0,\mu_n}(\mathcal{L}_{\varepsilon,i}^{0,\mu_n})$ is drawn using the radial Loewner chain driven by

$$s \mapsto (W_0^{\mu_n})_{S_{\varepsilon,i}} \exp\left(i\left(2e_{i,\varepsilon}(s) - \int_0^s \cot(e_{i,\varepsilon}(s))\right)\right),$$

hence (3.4.12) since the driving functions only differ by their starting point. One concludes using the fact that the $(W_0^{\mu_n})_{S_{\varepsilon,i}}$ ’s for $i \geq 1$ converge to independent uniform random variables on $\partial\mathbb{D}$ as in the end of the proof of Lemma 3.4.7, so that

$$g_{S_{\varepsilon,i}}^{0,\mu_n}(\mathcal{B}_{\varepsilon,i}^{0,\mu_n}) \xrightarrow[n \rightarrow \infty]{} g_{S_{\varepsilon,i}}^0(\mathcal{B}_{\varepsilon,i}^0)$$

in the sense of Carathéodory in distribution jointly in $i \geq 1$. By taking (3.4.11) into account, this ends the proof. \square

Let us conclude this section by ending the proof of Proposition 3.4.1.

Proof of Proposition 3.4.1. Let us first prove the convergence of the exploration branches. More precisely, we want the convergences (3.4.7) to hold jointly in $z \in \mathbb{D} \cap \mathbb{Q}^2$.

For all $\mu \in \mathbb{R}$, for all $z \neq w \in \mathbb{D} \cap \mathbb{Q}^2$, we define the time $\sigma_{z,w}^\mu$ (resp. $\sigma_{z,w}$) at which the radial $\text{SLE}_4^{(\mu)}(-2)$ exploration (resp. uniform exploration) towards z separates w from z by

$$\sigma_{z,w}^\mu = \inf\{s \geq 0; w \notin \mathbf{D}_s^{z,\mu}\} \quad (\text{resp. } \sigma_{z,w} = \inf\{s \geq 0; w \notin \mathbf{D}_s^z\}).$$

By the target invariance of the explorations (see Proposition 3.3.2 for the $\text{SLE}_4^{(\mu)}(-2)$) and by the renewal property (see Proposition 3.3.1 for the $\text{SLE}_4^{(\mu)}(-2)$), it is enough to show that for all $z \neq w \in \mathbb{D} \cap \mathbb{Q}^2$ we have the convergence of the separation time

$$\sigma_{z,w}^\mu \xrightarrow[\mu \rightarrow \infty]{(d)} \sigma_{z,w},$$

jointly with the convergence of the branch $\mathbf{D}^{z,\mu}$. Actually, by conformal invariance, it is enough to prove that for all $w \in (\mathbb{D} \cap \mathbb{Q}^2) \setminus \{0\}$,

$$\sigma_{0,w}^\mu \xrightarrow[\mu \rightarrow \infty]{(d)} \sigma_{0,w}, \quad (3.4.13)$$

jointly with the convergence of the branch $\mathbf{D}^{0,\mu}$. The convergence of the separation time (3.4.13) will be easier to prove in our context than in [AHPS23]. One can first see that $\sigma_{0,w}^\mu$ is tight as $\mu \rightarrow \infty$. Indeed, for all $\mu \in \mathbb{R}$, let I^μ be the smallest integer $i \geq 1$ such that the i -th loop surrounding 0 does not contain w . Then $\sigma_{0,w}^\mu$ is smaller than the time at which we draw the I^μ -th loop. If we assumed by contradiction that $\sigma_{0,z}^{\mu_n} \rightarrow \infty$ as $n \rightarrow \infty$ for some sequence $\mu_n \rightarrow \infty$, then we would have $I^{\mu_n} \rightarrow \infty$ and then w would be at distance zero from 0 by the Hausdorff convergence of the domains inside the loops encircling 0 of Lemma 3.4.8, absurd. Similarly, $\sigma_{w,0}^\mu$ is also tight as $\mu \rightarrow \infty$. Now let $\mu_n \rightarrow \infty$. By taking a subsequence, again denoted by μ_n , we may assume that

$$(\mathbf{D}^{0,\mu_n}, \tau_0^{\mu_n}, B^{0,\mu_n}, \mathbf{D}^{w,\mu_n}, \tau_w^{\mu_n}, B^{w,\mu_n}) \xrightarrow[n \rightarrow \infty]{(d)} (\mathbf{D}^0, \tau_0, B^0, \tilde{\mathbf{D}}^w, \tilde{\tau}_w, \tilde{B}^w) \quad (3.4.14)$$

in $\mathcal{D} \times \mathbb{R} \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times \mathcal{D}_w \times \mathbb{R} \times \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, where $(\tilde{\mathbf{D}}^w, \tilde{\tau}_w, \tilde{B}^w)$ has the same law as $(\mathbf{D}^w, \tau_w, B^w)$, jointly with the convergence of the separation times

$$(\sigma_{0,w}^{\mu_n}, \sigma_{w,0}^{\mu_n}) \xrightarrow[n \rightarrow \infty]{(d)} (\sigma_{0,w}^*, \sigma_{w,0}^*) \quad (3.4.15)$$

towards some random variables $\sigma_{0,w}^*, \sigma_{w,0}^*$.

In what follows, we prove that $\sigma_{0,w}^* = \sigma_{0,w}$ so that (3.4.13) holds. The main idea of the proof is to show that $\tilde{\mathbf{D}}^w$ coincides up to a time-change with \mathbf{D}^0 until the time $\sigma_{0,w}^*$, which will imply the difficult part of the equality $\sigma_{0,w}^* = \sigma_{0,w}$, which is the inequality $\sigma_{0,w}^* \leq \sigma_{0,w}$.

For all $n \geq 0$, let $\varphi_n : [0, \sigma_{0,w}^{\mu_n}] \rightarrow [0, \sigma_{w,0}^{\mu_n}]$ be the continuous increasing function such that for all $s \in [0, \sigma_{0,w}^{\mu_n}]$, we have $B_s^{0,\mu_n} = B_{\varphi_n(s)}^{w,\mu_n}$ (given by construction of the branching exploration). One can also express $\varphi_n(s)$ as minus the logarithm of the conformal radius of D_s^{0,μ_n} seen from w . Since $\sigma_{w,0}^{\mu_n}$ converges in distribution, and thanks to (3.4.14), we may take another subsequence again denoted by μ_n such that for all $s \in [0, \sigma_{0,w}^*] \cap \mathbb{Q}$, we have

$$\varphi_n(s) \xrightarrow[n \rightarrow \infty]{(d)} \varphi^*(s) \quad (3.4.16)$$

for some increasing process $(\varphi^*(t))_{t \in [0, \sigma_{0,w}^*] \cap \mathbb{Q}}$ jointly with the convergences (3.4.14) and (3.4.15). This implies that $B_t^0 = \tilde{B}_{\varphi^*(t)}^w$ for all $t \in [0, \sigma_{0,w}^*] \cap \mathbb{Q}$, hence φ^* is strictly increasing. Similarly, we can also assume that for all $s \in [0, \sigma_{w,0}^*] \cap \mathbb{Q}$, jointly with (3.4.14), (3.4.15) and (3.4.16), we have

$$\varphi_n^{-1}(s) \xrightarrow[n \rightarrow \infty]{(d)} \psi^*(s) \quad (3.4.17)$$

for some increasing process $(\psi^*(s))_{s \in [0, \sigma_{w,0}^*] \cap \mathbb{Q}}$. For all $s \in [0, \sigma_{w,0}^*] \cap \mathbb{Q}$, we also have $B_{\psi^*(s)}^0 = \tilde{B}_s^w$. So $B_{[0, \sigma_{0,w}^*]}^0$ is obtained as a time change of $\tilde{B}_{[0, \sigma_{w,0}^*]}^z$, which is itself a time change of $B_{[0, \sigma_{0,w}^*]}^0$. Thus, φ^* and ψ^* can be extended to continuous increasing bijections between $[0, \sigma_{0,w}^*]$ and $[0, \sigma_{w,0}^*]$ such that $\varphi^* = (\psi^*)^{-1}$. As a result, the convergences (3.4.16) and (3.4.17) hold uniformly on the corresponding intervals.

For all $\varepsilon > 0$, for all $i, n \geq 1$, let $[S_{\varepsilon,i}^{0,\mu_n}, T_{\varepsilon,i}^{0,\mu_n}]$ (resp. $[S_{\varepsilon,i}^0, T_{\varepsilon,i}^0]$, $[S_{\varepsilon,i}^{w,\mu_n}, T_{\varepsilon,i}^{w,\mu_n}]$, $[\tilde{S}_{\varepsilon,i}^w, \tilde{T}_{\varepsilon,i}^w]$) be the interval of the i -th excursion of B^{0,μ_n} (resp. B^0 , B^{w,μ_n} , \tilde{B}^w) of height larger than ε , during which the exploration \mathbf{D}^{0,μ_n} (resp. \mathbf{D}^0 , \mathbf{D}^{w,μ_n} , $\tilde{\mathbf{D}}^w$) draws a loop encircling a domain $\mathcal{B}_{\varepsilon,i}^{0,\mu_n}$ (resp. $\mathcal{B}_{\varepsilon,i}^0$, $\mathcal{B}_{\varepsilon,i}^{w,\mu_n}$, $\tilde{\mathcal{B}}_{\varepsilon,i}^w$). We may assume by Lemma 3.4.8 that these domains converge: for all $\varepsilon > 0$, jointly with the previous convergences,

$$\left(\mathcal{B}_{\varepsilon,i}^{0,\mu_n}, \mathcal{B}_{\varepsilon,i}^{w,\mu_n} \right)_{i \geq 1} \xrightarrow[n \rightarrow \infty]{(d)} \left(\mathcal{B}_{\varepsilon,i}^0, \tilde{\mathcal{B}}_{\varepsilon,i}^w \right)_{i \geq 1}$$

in the sense of Carathéodory and also for the Hausdorff distance when we consider their closure. Notice that, by the convergence of the Brownian motions, we also have the convergence of the times of start and end of the excursions of height larger than ε . By Skorokhod's representation theorem, we assume that all these convergences hold almost surely.

If $s < \sigma_{0,w}$, then $w \in \mathbf{D}_s^0$, so that for n large enough we have $w \in \mathbf{D}_s^{0,\mu_n}$, hence $s < \sigma_{0,w}^{\mu_n}$. Thus we have plainly $\sigma_{0,w}^* \geq \sigma_{0,w}$. Let us now prove that $\sigma_{0,w}^* \leq \sigma_{0,w}$. Let $\varepsilon > 0$ small enough so that we can find $i \geq 1$ such that $S_{\varepsilon,i}^0 < T_{\varepsilon,i}^0 < \sigma_{0,w}^*$. Let $s < S_{\varepsilon,i}^0$. Then we have by definition of the exploration $\mathcal{B}_{\varepsilon,i}^0 \subset \mathbf{D}_s^0$ and $\tilde{\mathcal{B}}_{\varepsilon,i}^w \subset \tilde{\mathbf{D}}_{\varepsilon,i}^w$. But since $T_{\varepsilon,i}^0 < \sigma_{0,w}^*$ we have for all n large enough $T_{\varepsilon,i}^{0,\mu_n} < \sigma_{0,w}^{\mu_n}$. Thus, for all n large enough, $\mathcal{B}_{\varepsilon,i}^{0,\mu_n} = \mathcal{B}_{\varepsilon,i}^{w,\mu_n}$ (given that $B_s^{0,\mu_n} = B_{\varphi_n(s)}^{w,\mu_n}$ for all $s \leq \sigma_{0,w}^{\mu_n}$). Hence $\mathcal{B}_{\varepsilon,i}^0 = \tilde{\mathcal{B}}_{\varepsilon,i}^w$ a.s. by the Hausdorff convergence. Therefore $\mathbf{D}_s^0 \cap \tilde{\mathbf{D}}_{\varphi^*(s)}^w \neq \emptyset$ almost surely. But then by taking some $z \in \mathbf{D}_s^0 \cap \tilde{\mathbf{D}}_{\varphi^*(s)}^w$, we see by the Carathéodory convergence that the limits of $\mathbf{D}_s^{0,\mu_n} = \mathbf{D}_{\varphi_n(s)}^{w,\mu_n}$ coincide, i.e. that $\mathbf{D}_s^0 = \tilde{\mathbf{D}}_{\varphi^*(s)}^w$ since $d_{\text{Carathéodory},z}$ defined in (3.3.1) is indeed a distance on simply connected open domains containing z . As a consequence, $s < \sigma_{0,w}$. By taking ε small enough and i such that $S_{\varepsilon,i}^0$ is close enough to $\sigma_{0,w}^*$ we obtain that $\sigma_{0,w}^* \leq \sigma_{0,w}$, hence (3.4.13). This concludes the proof since the domains D_s^{z,w,μ_n} for $s \geq 0$ correspond up to a time-change to the branch \mathbf{D}^{w,μ_n} stopped when z and w are separated. \square

3.5 Conformally invariant distance and quantum natural distance

This section is devoted to the relationships between the distance d_{WW} to the boundary of [WW13] recalled in Subsection 3.2.1 and the quantum natural distance d_q introduced by [AHPS23] which is characterized in (3.2.3). We prove Theorem 3.1.3 using Theorem 3.1.2, which relates these two distances via a Lamperti type transform modulo a process with stationary increments.

The main idea of the proof is that when one parametrizes the exploration by the Lamperti transform of the quantum distance, if we stop the exploration at time t , denote by x the quantum boundary length of the unexplored region and if we divide by x the process giving the future evolution of the quantum boundary length, then the process has the same law as the initial process. Moreover, once properly normalized, the unexplored region is again a unit-boundary quantum disk.

Proof of Theorem 3.1.3. Recall from Subsection 3.2.3 that $\mathcal{Y}(u) = \nu_{h_{C_u}^2}(\partial C_u)$ where $(C_u)_{u \geq 0}$ is the locally largest component in the uniform exploration parametrized by the distance d_{WW} to the boundary. Recall that σ is the right-continuous inverse of $(d_q(\partial \mathbb{D}, C_u))_{u \geq 0}$ and for all $t \geq 0$ we have $Y_t = \mathcal{Y}(\sigma(t))$. Let τ be defined by (3.1.2) with $\alpha = -1$. We know from Theorem 3.1.2 that

$$(\mathcal{Y}(\sigma(t)))_{t \geq 0} \stackrel{(d)}{=} (e^{\xi(\tau(\pi t))})_{t \geq 0},$$

so that $(\log(\mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)))_{t \geq 0}$ has the same law as ξ and we identify the two processes using an appropriate coupling. By (3.2.3) and by the fact that the Poisson point process (PPP) of loops $(\hat{\gamma}_u)_{u \geq 0}$ of intensity M introduced in Subsection 3.2.1 has an infinite intensity, we get that $u \mapsto d_q(\partial \mathbb{D}, C_u)$ is

increasing, so that $t \mapsto \sigma(t)$ is continuous. Similarly, since the Lévy measure of ξ is infinite, $t \mapsto \sigma(t)$ is increasing.

To end the proof, it is enough to show that the process $(\sigma(\tau^{-1}(t)/\pi))_{t \geq 0}$ has stationary increments.

Each positive jump of $(\mathcal{Y}(u))_{u \geq 0}$ at some time u corresponds to the discovery of a simple loop $F_u^{-1}(\widehat{\gamma}_u)$ where $(\widehat{\gamma}_u)_{u \geq 0}$ is a PPP which has the same law as described in Subsection 3.2.1 and $F_u : C_u \rightarrow \mathbb{D}$ is the conformal transformation which is given by $F_u = \Phi \circ \widehat{F}_u^{z_{j_k}} \circ \Phi^{-1}$ when $u \in [\tau_{k-1}, \tau_k)$ using the notation introduced in Subsections 3.2.1 and 3.2.3, and where $\Phi : \mathbb{H} \rightarrow \mathbb{D}$ is a fixed conformal map.

Let us denote by $(\mathcal{E}_t)_{t \geq 0}$ (resp. $(\mathcal{F}_u)_{u \geq 0}$) the augmented natural filtration of the process $(\mathcal{Y}(\sigma(t)))_{t \geq 0}$ (resp. $(h, \widehat{\gamma}_u, F_u)_{u \geq 0}$).

Furthermore, for all $u \geq 0$, we define the quantum surface

$$\text{QS}(u) := (\mathbb{D}, h \circ F_u^{-1} + 2 \log |(F_u^{-1})'| - \log(\nu_{h^2}^2(\partial C_u))).$$

The main idea of the proof is to show that for all $t \geq 0$, the pair $(\text{QS}(\sigma(\tau^{-1}(t)/\pi)), (\widehat{\gamma}_u)_{u \geq \sigma(\tau^{-1}(t)/\pi)})$ has the same law as $(\text{QS}(0), (\widehat{\gamma}_u)_{u \geq 0})$ and to show that this entails the fact that $(\sigma(\tau^{-1}(t)/\pi))_{t \geq 0}$ has stationary increments.

By Theorem 3.1.2, for all $t \geq 0$, conditionally on $(\mathcal{Y}(\sigma(s)))_{0 \leq s \leq t}$, the unexplored region $(C_{\sigma(t)}, h_{|_{C_{\sigma(t)}}}^2)$ is a 2-quantum disk of quantum boundary length $\nu_{h^2}^2(\partial C_{\sigma(t)})$. In other words, conditionally on $(\mathcal{Y}(\sigma(s)))_{0 \leq s \leq t}$, the quantum surface

$$\text{QS}(\sigma(t)) = (\mathbb{D}, h \circ F_{\sigma(t)}^{-1} + 2 \log |(F_{\sigma(t)}^{-1})'| - \log(\nu_{h^2}^2(\partial C_{\sigma(t)})))$$

is a unit boundary 2-quantum disk. But using the fact that $\tau^{-1}(t)/\pi$ is an $(\mathcal{E}_t)_{t \geq 0}$ -stopping time, we get that for all $t \geq 0$, conditionally on $\mathcal{E}_{\tau^{-1}(t)/\pi}$, the quantum surface

$$\text{QS}(\sigma(\tau^{-1}(t)/\pi)) = (\mathbb{D}, h \circ F_{\sigma(\tau^{-1}(t)/\pi)}^{-1} + 2 \log |(F_{\sigma(\tau^{-1}(t)/\pi)}^{-1})'| - \log(\nu_{h^2}^2(\partial C_{\sigma(\tau^{-1}(t)/\pi)}))) \quad (3.5.1)$$

is a unit boundary 2-quantum disk.

Besides, by (3.2.3) we know that $u \mapsto \sigma^{-1}(u) = d_q(\partial \mathbb{D}, C_u)$ is adapted with respect to $(\mathcal{F}_u)_{u \geq 0}$. Moreover, for all $u \geq 0$,

$$\tau(\pi \sigma^{-1}(u)) = \int_0^{\pi \sigma^{-1}(u)} \frac{dt}{\mathcal{Y}(\sigma(t/\pi))} = \int_0^{\sigma^{-1}(u)} \frac{\pi dt}{\mathcal{Y}(\sigma(t))} = \pi \int_0^u \frac{\sigma^{-1}(dv)}{\mathcal{Y}(v)}.$$

So $(\tau(\pi \sigma^{-1}(u)))_{u \geq 0}$ is $(\mathcal{F}_u)_{u \geq 0}$ -adapted. In particular, we obtain that for all $t \geq 0$, the random variable $\sigma(\tau^{-1}(t)/\pi)$ is an $(\mathcal{F}_u)_{u \geq 0}$ -stopping time. As a result, given that $(\widehat{\gamma}_u)_{u \geq 0}$ is independent of h^2 , we get that for all $t \geq 0$, conditionally on $\mathcal{F}_{\sigma(\tau^{-1}(t)/\pi)}$, the point process $(\widehat{\gamma}_{\sigma(\tau^{-1}(t)/\pi)+u})_{u \geq 0}$ has the same law as $(\widehat{\gamma}_u)_{u \geq 0}$ i.e. it is a PPP of intensity M . In addition to that, by definition of $(F_u)_{u \geq 0}$ from $(\widehat{\gamma}_u)_{u \geq 0}$, we have that conditionally on $\mathcal{F}_{\sigma(\tau^{-1}(t)/\pi)}$, the pair $((\widehat{\gamma}_{\sigma(\tau^{-1}(t)/\pi)+u})_{u \geq 0}, (F_{\sigma(\tau^{-1}(t)/\pi)} \circ F_{\sigma(\tau^{-1}(t)/\pi)+u}^{-1})_{u \geq 0})$ has the same law as $((\widehat{\gamma}_u)_{u \geq 0}, (F_u^{-1})_{u \geq 0})$.

But notice that $(\text{QS}(u))_{u \geq 0}$ is $(\mathcal{F}_u)_{u \geq 0}$ -adapted, so that the quantum disk $\text{QS}(\sigma(\tau^{-1}(t)/\pi))$ is $\mathcal{F}_{\sigma(\tau^{-1}(t)/\pi)}$ -measurable. Thus, by taking the above observation on the quantum disk (3.5.1) into account, we have that conditionally on $\mathcal{E}_{\tau^{-1}(t)/\pi} \subseteq \mathcal{F}_{\sigma(\tau^{-1}(t)/\pi)}$, the quantum disk $\text{QS}(\sigma(\tau^{-1}(t)/\pi))$ from (3.5.1) is a unit boundary length 2-LQG disk independent of the pair

$$((\widehat{\gamma}_{\sigma(\tau^{-1}(t)/\pi)+u})_{u \geq 0}, (F_{\sigma(\tau^{-1}(t)/\pi)} \circ F_{\sigma(\tau^{-1}(t)/\pi)+u}^{-1})_{u \geq 0}),$$

which has the same law as $((\widehat{\gamma}_u)_{u \geq 0}, (F_u^{-1})_{u \geq 0})$. In other words, conditionally on $\mathcal{E}_{\tau^{-1}(t)/\pi}$, the quantum disk $\text{QS}(\sigma(\tau^{-1}(t)/\pi))$ is a unit boundary length 2-LQG disk independent of the process $(F_{\sigma(\tau^{-1}(t)/\pi)}(C_{\sigma(\tau^{-1}(t)/\pi)+u}))_{u \geq 0}$, which has the same law as $(C_u)_{u \geq 0}$.

Besides, for all $u \geq 0$,

$$\begin{aligned} & \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi) + u) - \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)) \\ &= \log \left(\frac{\nu_{h^2}^2(F_{\sigma(\tau^{-1}(t)/\pi)}^{-1}(F_{\sigma(\tau^{-1}(t)/\pi)}(\partial C_{\sigma(\tau^{-1}(t)/\pi)+u})))}{\nu_{h^2}^2(\partial C_{\sigma(\tau^{-1}(t)/\pi)})} \right) \\ &= \log \left(\nu_{h \circ F_{\sigma(\tau^{-1}(t)/\pi)}^{-1}}^2 + 2 \log |(F_{\sigma(\tau^{-1}(t)/\pi)}^{-1})'| - \log(\nu_{h^2}^2(\partial C_{\sigma(\tau^{-1}(t)/\pi)})) (F_{\sigma(\tau^{-1}(t)/\pi)}(\partial C_{\sigma(\tau^{-1}(t)/\pi)+u})) \right), \end{aligned}$$

where we used (3.2.1) and the fact that $\nu_{h^2 - \log b}^2 = \nu_{h^2}^2/b$ for all $b > 0$. So conditionally on $\mathcal{E}_{\tau^{-1}(t)/\pi}$, the process $(\log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi) + u) - \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)))_{u \geq 0}$ has the same distribution as $(\log \mathcal{Y}(u))_{u \geq 0}$.

It is now a simple matter to check that $(\sigma(\tau^{-1}(t)/\pi))_{t \geq 0}$ has stationary increments. Indeed, let $t \geq 0$. For all $u, s \geq 0$, set

$$\tilde{\mathcal{Y}}(u) := \frac{\mathcal{Y}(\sigma(\tau^{-1}(t)/\pi) + u)}{\mathcal{Y}(\sigma(\tau^{-1}(t)/\pi))} \quad \text{and} \quad \tilde{\xi}(s) := \xi(t+s) - \xi(t).$$

Define $\tilde{\sigma}$ from $\tilde{\mathcal{Y}}$ as σ is defined from \mathcal{Y} . Notice that $\tilde{\xi}$ is characterized by the equality

$$\forall s \geq 0, \quad \log \left(\tilde{\mathcal{Y}} \left(\tilde{\sigma} \left((1/\pi) \int_0^s e^{\tilde{\xi}(r)} dr \right) \right) \right) = \tilde{\xi}(s). \quad (3.5.2)$$

Since $(\log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi) + u) - \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)))_{u \geq 0}$ has the same distribution as $(\log \mathcal{Y}(u))_{u \geq 0}$, we obtain that

$$\left((\log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi) + u) - \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)))_{u \geq 0}, \tilde{\sigma}, \tilde{\xi} \right)$$

has the same law as $(\mathcal{Y}, \sigma, \xi)$. Besides, by (3.5.2), we have for all $s \geq 0$,

$$\begin{aligned} \log \left(\tilde{\mathcal{Y}} \left(\tilde{\sigma} \left((1/\pi) \int_0^s e^{\tilde{\xi}(r)} dr \right) \right) \right) &= \tilde{\xi}(s) \\ &= \xi(t+s) - \xi(t) \\ &= \log \mathcal{Y}(\sigma(\tau^{-1}(t+s)/\pi)) - \log \mathcal{Y}(\sigma(\tau^{-1}(t)/\pi)) \\ &= \log \tilde{\mathcal{Y}}(\sigma(\tau^{-1}(t+s)/\pi) - \sigma(\tau^{-1}(t)/\pi)). \end{aligned}$$

As a consequence, by density of the jumps of $\tilde{\mathcal{Y}}$, by looking e.g. at the positive jumps of $\tilde{\mathcal{Y}}$ of size in $[\varepsilon, 2\varepsilon]$ and letting $\varepsilon \rightarrow 0$, we obtain that for all $s \geq 0$,

$$\sigma(\tau^{-1}(t+s)/\pi) - \sigma(\tau^{-1}(t)/\pi) = \tilde{\sigma} \left((1/\pi) \int_0^s e^{\tilde{\xi}(r)} dr \right).$$

Using the fact that for all $s \geq 0$, we have $\tau^{-1}(s) = \int_0^s e^{\xi(r)} dr$, we conclude that $\sigma \circ \tau^{-1}$ has stationary increments. \square

3.6 Scaling limit of the distances to the boundary in 3/2-stable maps

In this section, we describe the scaling limit of the distance between large faces and the boundary in 3/2 stable maps.

3.6.1 Background on 3/2-stable maps and the peeling exploration

Recall the definition of finite 3/2-stable maps from Subsection 3.1.2. If one conditions a 3/2-stable map of perimeter 2ℓ on having at least n edges and lets $n \rightarrow \infty$, then one obtains as a local limit an infinite 3/2-stable map of perimeter 2ℓ , whose law is written $\mathbb{P}_\infty^{(\ell)}$. The infinite 3/2-stable map was studied in [BCM18] and we will rely on this work in the next subsection.

Recall from Subsection 3.1.2 that if \mathfrak{m} is a planar map, d_{gr}^\dagger is the graph distance on the dual map of \mathfrak{m} and that d_{fpp}^\dagger is the first passage percolation distance, which is obtained by putting i.i.d. parameter 1 exponential random lengths on each edge of the dual map. To study the distances d_{gr}^\dagger and d_{fpp}^\dagger on Boltzmann planar maps, a convenient tool is the peeling exploration. We only give a brief overview of its definition and we refer the interested reader to the lecture notes [Cur23] and [Bud17]. It consists in discovering the faces of the map, starting from the root face f_r by looking at each step what is behind an edge on the boundary of the explored region which is chosen according to a peeling algorithm. More precisely, we say that a map \mathfrak{c} with a distinguished simple face different from f_r called a hole is a *submap*

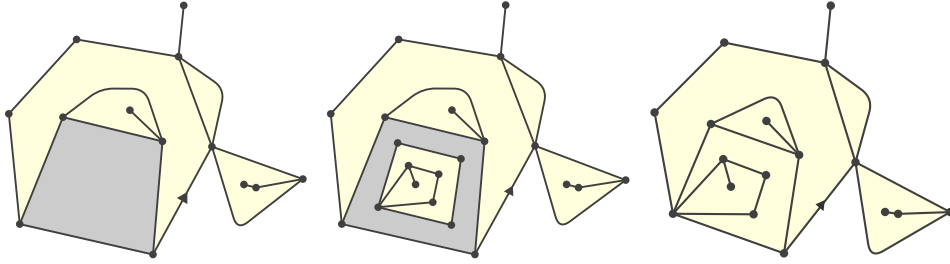


Figure 3.5 – Left: a map ϵ with one hole which is a submap of the map m on the right. The map u which is in the hole of ϵ in the centre enables to recover m after gluing the boundary of u to the boundary of the hole.

(with one hole) of m if one can obtain m by gluing a well chosen planar map in the hole of ϵ (see Figure 3.5).

A *peeling algorithm* \mathcal{A} is a function which takes a submap ϵ and gives back an edge in the boundary $\partial\epsilon$ of the hole of ϵ . A (filled-in) peeling exploration of m is an increasing sequence $(\bar{\epsilon}_n)_{n \geq 0}$ of submaps of m starting from the submap $\bar{\epsilon}_0$ with only two faces which are the root face f_r and a hole with the same degree, such that for all $n \geq 0$ the submap $\bar{\epsilon}_{n+1}$ is obtained from $\bar{\epsilon}_n$ as follows:

1. If the face f which is on the other side of the peeled edge $\mathcal{A}(\bar{\epsilon}_n)$ is not in $\bar{\epsilon}_n$, then $\bar{\epsilon}_{n+1}$ is obtained by gluing f to $\bar{\epsilon}_n$ onto the edge $\mathcal{A}(\bar{\epsilon}_n)$. We denote this case by C_k , where $k = \deg(f)/2$.
2. If the face f on the other side of $\mathcal{A}(\bar{\epsilon}_n)$ is already in $\bar{\epsilon}_n$, then f must be on the boundary of the hole of $\bar{\epsilon}_n$ and in this case $\bar{\epsilon}_{n+1}$ is obtained by identifying the two edges on the boundary of the hole. This creates at most two holes. If m is infinite, then we choose to fill in the hole which contains a finite part of the map. If m is finite, we choose here to fill in the hole which has the smallest degree and in case of equality we choose to fill in a hole chosen arbitrarily (so that in the finite case we explore the locally largest component). This case is written $G_{k,*}$ or $G_{*,k}$, where $2k$ is the degree of the hole which is filled-in, depending on whether the filled-in hole is on the left or on the right of the peeled edge (we can speak of left and right since the map has an orientation). If there is not any hole, then the exploration stops.

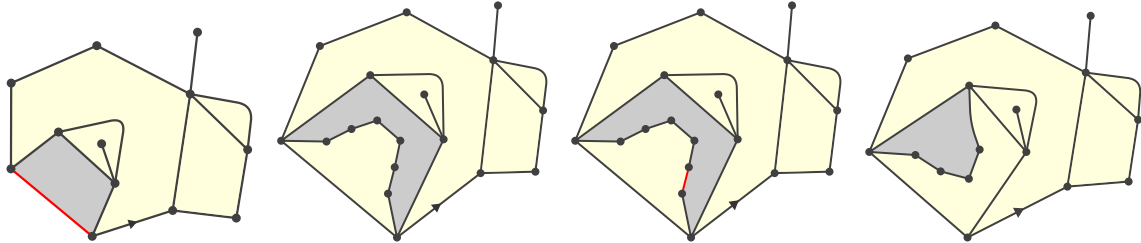


Figure 3.6 – From left to right: two peeling steps of the map m from Figure 3.5 starting from the submap ϵ on the left. The red edge on the left is the edge chosen by the peeling algorithm. The peeling of the red edge causes an event of type C_4 . The peeling algorithm then chooses another edge on the boundary which is again in red. When we peel this edge, an event of type $G_{3,1}$ happens and we fill-in the hole which has the smallest perimeter with a map made in this case of only one edge.

When performing the exploration, one can keep track of the *perimeter process* $(P(n))_{n \geq 0}$, where for all $n \geq 0$, the integer $P(n)$ is the half degree of the hole of $\bar{\epsilon}_n$. Notice that on an event of type C_k for $k \geq 1$, the perimeter process increases by $k - 1$ while on an event of type $G_{*,k}$ or $G_{k,*}$ for some $k \geq 0$, the perimeter process decreases by $k + 1$.

A peeling algorithm which is particularly well suited to the study of the distance d_{fpp}^\dagger is the *uniform peeling algorithm*. Given a submap ϵ , it chooses uniformly at random an edge in $\partial\epsilon$. Using this algorithm properly coupled with the exponential random lengths defining d_{fpp}^\dagger , by the elementary properties of exponential random variables, one can see that for all $n \geq 0$, assuming that the exploration has not

stopped before, the fpp distance from f_r to the hole u_n of $\bar{\epsilon}_n$ can be written

$$T(n+1) := d_{\text{fpp}}^\dagger(f_r, u_n) = \sum_{i=0}^n \frac{\mathcal{E}_i}{2P(i)}, \quad (3.6.1)$$

where the \mathcal{E}_i 's are i.i.d. exponential random variables of parameter 1. More precisely, $\bar{\epsilon}_n$ corresponds exactly to the closed ball around f_r of radius $T(n)$ where we filled-in the holes which were filled-in during the exploration, and where $T(0) = 0$ by convention. In particular, a face f which is discovered at time n (i.e. $f \in \text{Faces}(\bar{\epsilon}_n) \setminus \text{Faces}(\bar{\epsilon}_{n-1})$) is at distance $T(n)$ from the root face f_r . When the explored map has the law $\mathbb{P}^{(\ell)}$ or $\mathbb{P}_\infty^{(\ell)}$, the \mathcal{E}_i 's are independent from the perimeter process. See e.g. Section 13.1 of [Cur23] or Subsection 2.3.2 of Chapter 2 for details.

For the dual graph distance d_{gr}^\dagger , an appropriate peeling algorithm is the *peeling by layers algorithm* $\mathcal{A}_{\text{layers}}$. Given a submap ϵ of \mathfrak{m} , we define the height of an edge $e \in \partial\epsilon$ as the dual graph distance between f_r and the face next to e which is not in the hole. Assume that there exists an integer $h \geq 0$ such that all the edges of $\partial\epsilon$ have heights in $\{h, h+1\}$ and that the edges at height h form a connected interval of $\partial\epsilon$ (we denote this hypothesis by **(H)**). Then $\mathcal{A}_{\text{layers}}(\epsilon)$ is defined as the leftmost edge in $\partial\epsilon$ of height h (see Figure 3.7) and when all the edges of $\partial\epsilon$ are at the same height we choose one of them arbitrarily. Notice that $\bar{\epsilon}_0$ satisfies **(H)** and that if $\bar{\epsilon}_n$ satisfies **(H)** then with $\mathcal{A}_{\text{layers}}$ the submap $\bar{\epsilon}_{n+1}$ also satisfies **(H)** (with h increasing by zero or one). For all $n \geq 0$, let us denote by $H(n)$ the smallest height of the edges in $\partial\bar{\epsilon}_n$ when performing a peeling by layers exploration. The sequence $(H(n))_{n \geq 0}$ is called the *height process*. Notice that $H(0) = 0$ and that for all $n \geq 0$, we have $H(n+1) - H(n) \in \{0, 1\}$. See e.g. Section 13.2 of [Cur23] for more details.

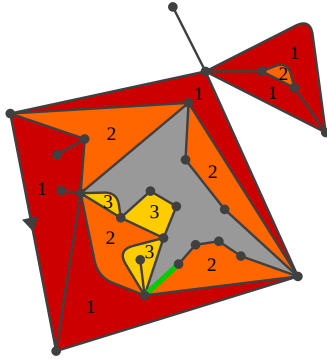


Figure 3.7 – Illustration of the peeling by layers algorithm. The faces are colored according to their distance to the root face f_r . The edge chosen by the algorithm $\mathcal{A}_{\text{layers}}$ is in green.

To be consistent with the notation of [Cur23], [BBCK18] and others, we add a subscript ∞ (resp. $*$) to the processes $(P(n))_{n \geq 0}$, $(T(n))_{n \geq 0}$, $(H(n))_{n \geq 0}$ under the law $\mathbb{P}_\infty^{(\ell)}$ (resp. $\mathbb{P}^{(\ell)}$). Recall that the peeling exploration under $\mathbb{P}^{(\ell)}$ satisfies a spatial Markov property in the sense that for all $n \geq 0$, conditionally on the explored region until time n , the maps which fill in the holes are independent Boltzmann maps, see e.g. Proposition 4.6 of [Cur23] for more details.

We finally recall from [BBCK18] the scaling limit of the perimeter process under $\mathbb{P}^{(\ell)}$ and $\mathbb{P}_\infty^{(\ell)}$ as $\ell \rightarrow \infty$. Let $(\Upsilon^\uparrow(t))_{t \geq 0}$ be a symmetric Cauchy process conditioned to stay positive and starting at 1. More precisely, $(\Upsilon^\uparrow(t))_{t \geq 0}$ is the Doob h -transform of the Lévy process with no drift and no Brownian part, of Lévy measure $(dx/|x|^2)\mathbf{1}_{x \neq 0}$ starting at one, using the function $h(x) = \sqrt{x}$. We recall from Proposition 6.6 of [BBCK18] the following convergences in distribution for the J_1 topology of Skorokhod:

$$\text{under } \mathbb{P}_\infty^{(\ell)}, \quad \left(\frac{P_\infty(\lfloor \ell t \rfloor)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (\Upsilon^\uparrow(p_{\mathbf{q}} t))_{t \geq 0}; \quad (3.6.2)$$

$$\text{under } \mathbb{P}^{(\ell)}, \quad \left(\frac{P_*(\lfloor \ell t \rfloor)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (X^{(-1)}(\pi p_{\mathbf{q}} t))_{t \geq 0}. \quad (3.6.3)$$

Notice here that the scaling limit of the perimeter of the locally largest component (parametrized by the exploration time) corresponds already, up to a linear time change, to the quantum boundary length of

the locally largest component during a uniform or $\text{SLE}_4^{(\mu)}(-2)$ exploration of a CLE_4 -decorated quantum disk, parametrized by the quantum natural distance or the quantum length of the trunk, by Theorem 3.1.2.

3.6.2 Metric growth for infinite 3/2-stable maps with increasing perimeter

Here we establish the scaling limit of T_∞ and H_∞ under $\mathbb{P}_\infty^{(\ell)}$ as $\ell \rightarrow \infty$. The scaling limit of T_∞ is very easy to obtain:

Proposition 3.6.1. *Using the uniform peeling exploration, we have jointly with (3.6.2) the convergence in distribution for the topology of the uniform convergence on compact sets:*

$$\text{under } \mathbb{P}_\infty^{(\ell)}, \quad (T_\infty(\lfloor \ell t \rfloor))_{t \geq 0} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left(\int_0^t \frac{1}{2\Upsilon^\uparrow(p_{\mathbf{q}}s)} ds \right)_{t \geq 0}.$$

Proof. Using (3.6.1) and the fact that the \mathcal{E}_n 's are independent from P_∞ , we have for all $t \geq 0$,

$$\mathbb{E}_\infty^{(\ell)} \left(\sum_{n=0}^{\lfloor \ell t \rfloor - 1} \left(\frac{\mathcal{E}_n - 1}{2P_\infty(n)} \right)^2 \middle| P_\infty \right) = \sum_{n=0}^{\lfloor \ell t \rfloor - 1} \frac{1}{(2P_\infty(n))^2} \xrightarrow[\ell \rightarrow +\infty]{(\mathbb{P})} 0,$$

where the convergence comes from (3.6.2). Moreover, again by (3.6.2),

$$\text{under } \mathbb{P}_\infty^{(\ell)}, \quad \left(\sum_{n=0}^{\lfloor \ell t \rfloor - 1} \frac{1}{2P_\infty(n)} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left(\int_0^t \frac{1}{2\Upsilon^\uparrow(p_{\mathbf{q}}s)} ds \right)_{t \geq 0}.$$

This ends the proof since $t \mapsto \int_0^t 1/(2\Upsilon^\uparrow(p_{\mathbf{q}}s))ds$ is a continuous and increasing process. \square

For the scaling limit of H_∞ , a $\log \ell$ factor appears:

Proposition 3.6.2. *For the peeling by layers exploration, we have the convergence in distribution jointly with (3.6.2) for the topology of the uniform convergence on compact sets:*

$$\text{under } \mathbb{P}_\infty^{(\ell)}, \quad \left(\frac{H_\infty(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left(\int_0^t \frac{p_{\mathbf{q}}}{2\Upsilon^\uparrow(p_{\mathbf{q}}s)} ds \right)_{t \geq 0}.$$

To prove the above result, we will use the inverse θ of H_∞ defined by

$$\forall k \geq 0, \quad \theta_k = \inf\{n \geq 0, H_\infty(n) \geq k\}.$$

We will rely on the following lemma from [BCM18] which upperbounds the number of peeling steps needed to complete k layers.

Lemma 3.6.3. *(Corollary 2 of [BCM18]) For all $\varepsilon \in (0, 1)$, for all $k \in \mathbb{N}$,*

$$\lim_{\ell \rightarrow \infty} \mathbb{P}_\infty^{(\ell)} \left(\frac{p_{\mathbf{q}} \log \ell \theta_k}{2\ell} \leq 1 + \varepsilon \right) = 1.$$

We then prove Proposition 3.6.2.

Proof of Proposition 3.6.2. This proof uses the same techniques as the proof of Proposition 6 in [BCM18], which gives a lower bound for $H_\infty(n)/(\log n)^2$ under $\mathbb{P}_\infty^{(1)}$ as $n \rightarrow \infty$, and will also rely on Lemma 6 of [BCM18], which bounds from above the mean increase of an interpolated version of H_∞ . Let $C > 0$. For all $\ell \geq 1$ and $k \geq 0$, set $t_k^{(\ell)} = \lfloor C \frac{k\ell}{\log \ell} \rfloor$. Notice that

$$\Delta t_k^{(\ell)} := t_{k+1}^{(\ell)} - t_k^{(\ell)} = (C + o(1)) \frac{\ell}{\log \ell} \tag{3.6.4}$$

when $\ell \rightarrow \infty$ where the $o(1)$ is uniform in k . Note also that for all real $s > 0$ we have

$$t_{\lfloor \frac{s \log \ell}{C} \rfloor}^{(\ell)} = \lfloor \ell s \rfloor (1 + o(1)) \quad (3.6.5)$$

as $\ell \rightarrow \infty$ where the $o(1)$ is non-positive.

Let $\delta \in (0, 1)$. For all $\ell \geq 1$ and $k \geq 0$, let

$$\eta_k^{(\ell)} = \left(H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)}) - C \frac{p_{\mathbf{q}} \ell}{2P_\infty(t_k^{(\ell)})} \right) \mathbf{1}_{P_\infty(t_k^{(\ell)})/\ell \in [\delta, 1/\delta]}.$$

Take $\varepsilon \in (0, 2\delta/Cp_{\mathbf{q}})$. Let $N_k^{(\ell)} = \lfloor (1 - \varepsilon)(Cp_{\mathbf{q}}\ell)/(2P_\infty(t_k^{(\ell)})) \rfloor$. Then one can see that

$$\left\{ \frac{P_\infty(t_k^{(\ell)})}{\ell} \notin [\delta, 1/\delta] \right\} \cup \left(\left\{ \frac{P_\infty(t_k^{(\ell)})}{\ell} \in [\delta, 1/\delta] \right\} \cap \left\{ N_k^{(\ell)} \leq H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)}) \right\} \right) \subset \left\{ \eta_k^{(\ell)} \geq -2 \right\}.$$

Besides, by (3.6.4), for all ℓ large enough, for all $k \geq 0$, on the event $\{P_\infty(t_k^{(\ell)})/\ell \in [\delta, 1/\delta]\}$, we have

$$\frac{\Delta t_k^{(\ell)}}{N_k^{(\ell)}} \geq \frac{2P_\infty(t_k^{(\ell)})}{(1 - \varepsilon/2)p_{\mathbf{q}} \log P_\infty(t_k^{(\ell)})}. \quad (3.6.6)$$

We define for all $\ell \geq 1$ and $k \geq 0$ the stopping times $\theta_{t_k^{(\ell)}, 0} = t_k^{(\ell)}$ and for all $i \geq 1$,

$$\theta_{t_k^{(\ell)}, i} = \inf \left\{ n \geq 0; H_\infty(n + t_k^{(\ell)}) \geq H_\infty(t_k^{(\ell)}) + i \right\}.$$

Then by Lemma 3.6.3, for all $\gamma \in (0, 1)$, for ℓ large enough, for all $k \geq 0$,

$$\begin{aligned} & \mathbb{P}_\infty^{(\ell)} \left(\left\{ \frac{P_\infty(t_k^{(\ell)})}{\ell} \in [\delta, 1/\delta] \right\} \cap \left\{ N_k^{(\ell)} \leq H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)}) \right\} \right) \\ &= \mathbb{P}_\infty^{(\ell)} \left(\left\{ \frac{P_\infty(t_k^{(\ell)})}{\ell} \in [\delta, 1/\delta] \right\} \cap \left\{ \frac{\theta_{t_k^{(\ell)}, N_k^{(\ell)}} - t_k^{(\ell)}}{N_k^{(\ell)}} \leq \frac{\Delta t_k^{(\ell)}}{N_k^{(\ell)}} \right\} \right) \\ &\geq \mathbb{P}_\infty^{(\ell)} \left(\left\{ \frac{P_\infty(t_k^{(\ell)})}{\ell} \in [\delta, 1/\delta] \right\} \cap \left\{ \frac{\theta_{t_k^{(\ell)}, N_k^{(\ell)}} - t_k^{(\ell)}}{N_k^{(\ell)}} \leq \frac{2P_\infty(t_k^{(\ell)})}{(1 - \varepsilon/2)p_{\mathbf{q}} \log P_\infty(t_k^{(\ell)})} \right\} \right) \quad \text{by (3.6.6)} \\ &\geq (1 - \gamma) \mathbb{P}_\infty^{(\ell)} \left(\frac{P_\infty(t_k^{(\ell)})}{\ell} \in [\delta, 1/\delta] \right), \end{aligned}$$

the last inequality being uniform in k since, under the event $\{P_\infty(t_k^{(\ell)})/\ell \in [\delta, 1/\delta]\}$, the random variable $N_k^{(\ell)}$ is bounded above by $(Cp_{\mathbf{q}})/(2\delta)$. Thus, by taking γ arbitrarily small,

$$\liminf_{\ell \rightarrow \infty} \inf_{k \geq 0} \mathbb{P}_\infty^{(\ell)} \left(\eta_k^{(\ell)} \geq -2 \right) = 1. \quad (3.6.7)$$

Let $s \geq 0$. Let

$$\mathcal{H}_{\delta, C, \ell}(s) = \frac{1}{\log \ell} \sum_{k=0}^{\lfloor (s \log \ell)/C \rfloor} (H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)})) \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_\infty(n)/\ell \in [\delta, 1/\delta]}.$$

By (3.6.5), we have

$$\mathcal{H}_{\delta, C, \ell}(s) \leq \frac{1}{\log \ell} H_\infty(\lfloor \ell s \rfloor) \quad (3.6.8)$$

Moreover, by (3.6.5) and (3.6.2), for all $v > 0$, with probability $1 - o(1)$ as $\delta \rightarrow 0$, for all ℓ large enough,

$$\frac{1}{\log \ell} H_\infty(\lfloor \ell s \rfloor) \leq \mathcal{H}_{\delta, C, \ell}(s + v). \quad (3.6.9)$$

One can write

$$\begin{aligned}\mathcal{H}_{\delta,C,\ell}(s) &= \frac{1}{\log \ell} \sum_{k=0}^{\lfloor (s \log \ell)/C \rfloor} C \frac{p_{\mathbf{q}} \ell}{2P_{\infty}(t_k^{(\ell)})} \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, 1/\delta]} \\ &\quad + \frac{1}{\log \ell} \sum_{k=0}^{\lfloor (s \log \ell)/C \rfloor} \eta_k^{(\ell)} \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, 1/\delta]}.\end{aligned}$$

Consider the first term.

$$\begin{aligned}X_{\delta,C,\ell}(s) &:= \frac{1}{\log \ell} \sum_{k=0}^{\lfloor (s \log \ell)/C \rfloor} C \frac{p_{\mathbf{q}} \ell}{2P_{\infty}(t_k^{(\ell)})} \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, 1/\delta]} \\ &= \frac{1}{\log \ell} \int_0^{\lfloor (s \log \ell)/C \rfloor + 1} C \frac{p_{\mathbf{q}} \ell}{2P_{\infty}(t_{\lfloor v \rfloor}^{(\ell)})} \mathbf{1}_{\forall n \in \llbracket t_{\lfloor v \rfloor}^{(\ell)}, t_{\lfloor v \rfloor + 1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, \frac{1}{\delta}]} dv \\ &= O\left(\frac{1}{\log \ell}\right) + \int_0^s \frac{p_{\mathbf{q}} \ell}{2P_{\infty}(t_{\lfloor (u \log \ell)/C \rfloor}^{(\ell)})} \mathbf{1}_{\forall n \in \llbracket t_{\lfloor (u \log \ell)/C \rfloor}^{(\ell)}, t_{\lfloor (u \log \ell)/C \rfloor + 1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, \frac{1}{\delta}]} du \\ &\stackrel{(d)}{\rightarrow}_{\ell \rightarrow +\infty} \mathcal{H}_{\delta}(s) := \int_0^s \frac{p_{\mathbf{q}}}{2\Upsilon^{\uparrow}(p_{\mathbf{q}} u)} \mathbf{1}_{\Upsilon^{\uparrow}(p_{\mathbf{q}} u) \in [\delta, 1/\delta]} du\end{aligned}\tag{3.6.10}$$

by (3.6.5) and by dominated convergence. By the $O(1/\log \ell)$ term in the third line, we mean a random variable which is bounded in absolute value by $\tilde{C}/\log \ell$, where \tilde{C} is a constant depending on C and δ .

For the second term, note that since H_{∞} is non-decreasing, the negative part of $\eta_k^{(\ell)}$ is upperbounded as follows: for all $\ell \geq 1, k \geq 0$,

$$\left(\eta_k^{(\ell)}\right)_- \leq 2 \mathbf{1}_{\eta_k^{(\ell)} \geq -2} + \mathbf{1}_{\eta_k^{(\ell)} < -2} C \frac{p_{\mathbf{q}} \ell}{2P_{\infty}(t_k^{(\ell)})} \mathbf{1}_{P_{\infty}(t_k^{(\ell)})/\ell \in [\delta, 1/\delta]} \leq 2 + \mathbf{1}_{\eta_k^{(\ell)} < -2} C \frac{p_{\mathbf{q}}}{2\delta},\tag{3.6.11}$$

hence for all $\ell \geq 1, k \geq 0$,

$$\mathbb{E}_{\infty}^{(\ell)}\left(\left(\eta_k^{(\ell)}\right)_-\right) \leq 2 + \mathbb{P}_{\infty}^{(\ell)}\left(\eta_k^{(\ell)} < -2\right) C \frac{p_{\mathbf{q}}}{2\delta}.$$

Hence, by (3.6.7),

$$\limsup_{\ell \rightarrow \infty} \sup_{k \geq 0} \mathbb{E}_{\infty}^{(\ell)}\left(\left(\eta_k^{(\ell)}\right)_-\right) \leq 2.$$

As a result, if we set

$$Y_{\delta,C,\ell}(s) := \frac{1}{\log \ell} \sum_{k=0}^{\lfloor (s \log \ell)/C \rfloor} \left(\eta_k^{(\ell)}\right)_- \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_{\infty}(n)/\ell \in [\delta, 1/\delta]},$$

then

$$\limsup_{\ell \rightarrow \infty} \mathbb{E}_{\infty}^{(\ell)}(Y_{\delta,C,\ell}(s)) \leq \frac{2s}{C} \quad \text{and} \quad \forall \ell \geq 1, \quad \mathcal{H}_{\delta,C,\ell}(s) \geq X_{\delta,C,\ell}(s) - Y_{\delta,C,\ell}(s).$$

Moreover, by (3.6.11), one can see that the sequence of non-negative random variables $(Y_{\delta,C,\ell}(s))_{\ell \geq 1}$ is upperbounded by $(s/C)(2 + Cp_{\mathbf{q}}/(2\delta))$. Let $(\ell_m)_{m \geq 0}$ be an increasing sequence of integers. By taking a subsequence again denoted by $(\ell_m)_{m \geq 0}$, we may assume that $Y_{\delta,C,\ell_m}(s)$ converges in distribution. By Skorokhod's representation theorem, we may assume that the random variables $\mathcal{H}_{\delta,C,\ell}(s)$, $X_{\delta,C,\ell}(s)$ and $Y_{\delta,C,\ell}(s)$ are defined on the same probability space and that the convergence (3.6.10) holds almost surely together with the convergence of $Y_{\delta,C,\ell_m}(s)$ towards a non-negative random variable $Y_{\delta,C}(s)$. Note that since Y_{δ,C,ℓ_m} is bounded it also converges in L^1 . As a result, a.s.

$$\liminf_{m \rightarrow \infty} \mathcal{H}_{\delta,C,\ell_m}(s) \geq \mathcal{H}_{\delta}(s) - Y_{\delta,C}(s) \quad \text{and} \quad \mathbb{E}(Y_{\delta,C}) \leq \frac{2s}{C}.\tag{3.6.12}$$

Let us then give an upper bound in expectation of $\mathcal{H}_{\delta,C,\ell}(s)$. We denote by $D_{\infty}(n) \in \llbracket 1, 2P_{\infty}(n) \rrbracket$ the number of edges on the boundary of the explored region at time n which are at height $H_{\infty}(n)$. Let

$\varepsilon' > 0$. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous non-increasing function such that $f(0) = 1$, $f(1) = 0$, $f'(0) = f'(1) = f''(0) = f''(1) = 0$ and $0 \leq -f'(x) \leq 1 + \varepsilon'$ for all $x \in [0, 1]$. For all $n \geq 0$, we set

$$\Delta H_\infty^f(n) := H_\infty(n+1) + f\left(\frac{D_\infty(n+1)}{2P_\infty(n+1)}\right) - H_\infty(n) - f\left(\frac{D_\infty(n)}{2P_\infty(n)}\right).$$

Note that $\Delta H_\infty^f(n) \geq 0$ since either the peeling by layers algorithm completes a layer and in this case $H_\infty(n+1) = H_\infty(n) + 1$ while $D_\infty(n+1) = 2P_\infty(n+1)$, or the height process remains constant and $D_\infty(n+1) \leq D_\infty(n)$. By Lemma 6 of [BCM18], for all $\varepsilon' > 0$, there exists a constant $C' > 0$ such that for all $\ell \geq 1$ and $n \geq 0$,

$$\mathbb{E}_\infty^{(\ell)}(\Delta H_\infty^f(n) | P_\infty(n)) \leq (1 + \varepsilon')^3 \frac{p_{\mathbf{q}} \log P_\infty(n) + C'}{2 P_\infty(n)}. \quad (3.6.13)$$

But by summing over $n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} - 1 \rrbracket$, using that $\Delta H_\infty^f(n) \geq 0$ for all $n \geq 0$, we get

$$\begin{aligned} & \mathbb{E}_\infty^{(\ell)} \left(\left(H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)}) + f\left(\frac{D_\infty(t_{k+1}^{(\ell)})}{2P_\infty(t_{k+1}^{(\ell)})}\right) - f\left(\frac{D_\infty(t_k^{(\ell)})}{2P_\infty(t_k^{(\ell)})}\right) \right) \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_\infty(n)/\ell \in [\delta, 1/\delta]} \right) \\ & \leq \sum_{n=t_k^{(\ell)}}^{t_{k+1}^{(\ell)}-1} \mathbb{E}_\infty^{(\ell)}(\Delta H_\infty^f(n) \mathbf{1}_{P_\infty(n)/\ell \in [\delta, 1/\delta]}). \end{aligned}$$

Thus, using (3.6.13) and the fact that $0 \leq f \leq 1$, we deduce that for all $\ell \geq 1$ and $k \geq 0$,

$$\begin{aligned} & \mathbb{E}_\infty^{(\ell)} \left(\left(H_\infty(t_{k+1}^{(\ell)}) - H_\infty(t_k^{(\ell)}) \right) \mathbf{1}_{\forall n \in \llbracket t_k^{(\ell)}, t_{k+1}^{(\ell)} \rrbracket, P_\infty(n)/\ell \in [\delta, 1/\delta]} \right) \\ & \leq 2 + (1 + \varepsilon')^3 \frac{p_{\mathbf{q}}}{2} \sum_{n=t_k^{(\ell)}}^{t_{k+1}^{(\ell)}-1} \mathbb{E}_\infty^{(\ell)} \left(\frac{\log P_\infty(n) + C'}{P_\infty(n)} \mathbf{1}_{P_\infty(n)/\ell \in [\delta, 1/\delta]} \right). \end{aligned}$$

As a result, by (3.6.5),

$$\mathbb{E}_\infty^{(\ell)}(\mathcal{H}_{\delta, C, \ell}(s)) \leq \frac{1}{\log \ell} \left(2 \left\lfloor \frac{s \log \ell}{C} \right\rfloor + 2 + (1 + \varepsilon')^3 \frac{p_{\mathbf{q}}}{2} \sum_{n=0}^{\lfloor s \ell \rfloor - 1} \mathbb{E}_\infty^{(\ell)} \left(\frac{\log P_\infty(n) + C'}{P_\infty(n)} \mathbf{1}_{P_\infty(n)/\ell \in [\delta, 1/\delta]} \right) \right).$$

Thus, using (3.6.2), by dominated convergence as $\ell \rightarrow \infty$, and then by letting $\varepsilon' \rightarrow 0$, we get that for all $\delta > 0$,

$$\limsup_{\ell \rightarrow \infty} \mathbb{E}_\infty^{(\ell)}(\mathcal{H}_{\delta, C, \ell}(s)) \leq \mathbb{E} \left(\int_0^s \frac{p_{\mathbf{q}}}{2\Upsilon^\dagger(p_{\mathbf{q}}v)} \mathbf{1}_{\Upsilon^\dagger(p_{\mathbf{q}}v) \in [\delta, 1/\delta]} dv \right) + \frac{2s}{C} = \mathbb{E}(\mathcal{H}_\delta(s)) + \frac{2s}{C}. \quad (3.6.14)$$

Besides, by the lower bound (3.6.12) and by Fatou's lemma,

$$\mathbb{E}\mathcal{H}_\delta(s) - \frac{2s}{C} \leq \mathbb{E}\mathcal{H}_\delta(s) - \mathbb{E}Y_{\delta, C}(s) \leq \mathbb{E} \left(\liminf_{m \rightarrow \infty} \mathcal{H}_{\delta, C, \ell_m}(s) \right) \leq \liminf_{m \rightarrow \infty} \mathbb{E}\mathcal{H}_{\delta, C, \ell_m}(s),$$

so that combining this with (3.6.14) we get

$$\mathbb{E}\mathcal{H}_\delta(s) - \frac{2s}{C} \leq \liminf_{m \rightarrow \infty} \mathbb{E}\mathcal{H}_{\delta, C, \ell_m}(s) \leq \limsup_{m \rightarrow \infty} \mathbb{E}\mathcal{H}_{\delta, C, \ell_m}(s) \leq \mathbb{E}\mathcal{H}_\delta(s) + \frac{2s}{C}. \quad (3.6.15)$$

Moreover, notice that for all $\varepsilon' > 0$, by dominated convergence and by (3.6.12),

$$\begin{aligned} & \mathbb{E}((\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s))_+) \\ & \leq \varepsilon' + \mathbb{E}((\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s)) \mathbf{1}_{\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s) > \varepsilon'}) \\ & \leq \varepsilon' + \mathbb{E}(\mathcal{H}_\delta(s) \mathbf{1}_{\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s) > \varepsilon'}) \xrightarrow{m \rightarrow +\infty} \varepsilon'. \end{aligned}$$

Now, since for all $x \in \mathbb{R}$, $|x| = -x + 2x_+$, using the second part of (3.6.12), the above upper bound and (3.6.15) we get that

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} \mathbb{E} |\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s)| \\
& \leq \mathbb{E}(Y_{\delta, C}(s)) + \limsup_{m \rightarrow \infty} \mathbb{E} |\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s)| \\
& \leq \frac{2s}{C} + 2 \limsup_{m \rightarrow \infty} \mathbb{E} \left((\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s))_+ \right) - \liminf_{m \rightarrow \infty} \mathbb{E} (\mathcal{H}_\delta(s) - \mathcal{H}_{\delta, C, \ell_m}(s) - Y_{\delta, C}(s)) \\
& \leq \frac{6s}{C}.
\end{aligned}$$

By taking C large and then δ small, using (3.6.8) and (3.6.9), we get that for any increasing sequence of integers $(\ell_m)_{m \geq 0}$ the sequence of random variables $H_\infty(\lfloor \ell_m s \rfloor) / \log \ell_m$ converges in distribution towards $\int_0^s p_{\mathbf{q}} / (2Y^\uparrow(p_{\mathbf{q}}u)) du$ along a subsequence. This concludes the proof. \square

3.6.3 Large 3/2-stable maps and CLE_4 -decorated quantum disks

We first derive the analogues of Propositions 3.6.1 and 3.6.2 for finite 3/2-stable maps by absolute continuity. Recall from Subsection 3.6.1 that H_* is the height process under $\mathbb{P}^{(\ell)}$ and that T_* records the fpp distance from the root face to the boundary of the unexplored region along the uniform peeling exploration under $\mathbb{P}^{(\ell)}$.

Corollary 3.6.4. *For the uniform peeling, jointly with (3.6.2) we have the convergence*

$$\text{under } \mathbb{P}^{(\ell)}, \quad (T_*(\lfloor \ell t \rfloor))_{t \geq 0} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left(\int_0^t \frac{1}{2X^{(-1)}(\pi p_{\mathbf{q}} s)} ds \right)_{t \geq 0}.$$

For the peeling by layers, jointly with (3.6.2) we have the convergence

$$\text{under } \mathbb{P}^{(\ell)}, \quad \left(\frac{H_*(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left(\int_0^t \frac{p_{\mathbf{q}}}{2X^{(-1)}(\pi p_{\mathbf{q}} s)} ds \right)_{t \geq 0}.$$

Then the convergences (3.1.3) and (3.1.4) are clearly a consequence of the above corollary.

Proof. Let us prove the statement for H_* (the same proof works for T_* using Proposition 3.6.1). We will mimic the proof of Proposition 6.6 of [BBCK18], but let us write the details.

By Equation (39) of Proposition 6.3 of [BBCK18], for all $\ell \geq 1$, the Radon-Nikodym derivative of the law of the processes (P_*, H_*) until some time $n \geq 0$ under $\mathbb{P}^{(\ell)}$ with respect to the law of (P_∞, H_∞) until time n under $\mathbb{P}^{(\ell)}$ is given by $f^\uparrow(\ell) / f^\uparrow(P_\infty(n))$ times the indicator function that P_∞ does not make any negative jump of size larger than its half, where $f^\uparrow(\ell) := h^\uparrow(\ell) / (W^{(\ell)} c_{\mathbf{q}}^{-\ell})$, where $W^{(\ell)}$ is the partition function of planar maps of perimeter 2ℓ , $c_{\mathbf{q}}$ appears in (3.1.1) and $h^\uparrow(\ell) = 2\ell 2^{-\ell} \binom{2\ell}{\ell}$.

More precisely, let $T > 0$. Denote by $\mathbb{D}([0, T])$ the set of real-valued càdlàg functions on $[0, T]$ equipped with the J_1 Skorokhod topology. Let $\varepsilon > 0$. Set

$$U_\varepsilon := \left\{ (x(t))_{t \in [0, T]} \in \mathbb{D}([0, T]); x(T) < \varepsilon \text{ or } \exists t \in (0, T], x(t) < \frac{x(t-)}{2} \right\}.$$

Note that the closure of U_ε satisfies the inclusion

$$\overline{U_\varepsilon} \subseteq \left\{ (x(t))_{t \in [0, T]} \in \mathbb{D}([0, T]); x(T) \leq \varepsilon \text{ or } \exists t \in (0, T], x(t) \leq \frac{x(t-)}{2} \right\}.$$

Let $F : \mathbb{D}([0, T], \mathbb{R})^2 \rightarrow \mathbb{R}$ be a continuous bounded function vanishing on the open set $\mathbb{D}([0, T]) \times U_\varepsilon$.

Then,

$$\begin{aligned}
& \mathbb{E}^{(\ell)} F \left(\left(\frac{H_*(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \in [0, T]}, \left(\frac{P_*(\lfloor \ell t \rfloor)}{\ell} \right)_{t \in [0, T]} \right) \\
&= \mathbb{E}_\infty^{(\ell)} \left(\frac{f^\uparrow(\ell)}{f^\uparrow(P_\infty(\lfloor \ell T \rfloor))} F \left(\left(\frac{H_\infty(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \in [0, T]}, \left(\frac{P_\infty(\lfloor \ell t \rfloor)}{\ell} \right)_{t \in [0, T]} \right) \right) \\
&\xrightarrow{\ell \rightarrow \infty} \mathbb{E} \left((\Upsilon^\uparrow(p_{\mathbf{q}} T))^{-5/2} F \left(\left(\int_0^t \frac{p_{\mathbf{q}}}{2\Upsilon^\uparrow(p_{\mathbf{q}} s)} ds \right)_{t \in [0, T]}, (\Upsilon^\uparrow(p_{\mathbf{q}} t))_{t \in [0, T]} \right) \right),
\end{aligned}$$

where the convergence comes from Proposition 3.6.2, together with the convergence in distribution of $f^\uparrow(\ell)/f^\uparrow(P_\infty(\lfloor \ell T \rfloor))$ towards $\Upsilon^\uparrow(p_{\mathbf{q}} t)^{-5/2}$ which comes from (3.1.1) and (3.6.2) and from the fact that $h^\uparrow(\ell) \sim 2\sqrt{\ell/\pi}$ as $\ell \rightarrow \infty$.

Finally, identifying the Radon-Nikodym derivative of the law of $(X^{(-1)}(\pi p_{\mathbf{q}} s))_{0 \leq t \leq T}$ with respect to the law of $(\Upsilon^\uparrow(p_{\mathbf{q}} t))_{0 \leq t \leq T}$ by Proposition 4.1 of [BBCK18], one obtains the identity

$$\begin{aligned}
& \mathbb{E} \left((\Upsilon^\uparrow(p_{\mathbf{q}} T))^{-5/2} F \left(\left(\int_0^t \frac{p_{\mathbf{q}}}{2\Upsilon^\uparrow(p_{\mathbf{q}} s)} ds \right)_{t \in [0, T]}, (\Upsilon^\uparrow(p_{\mathbf{q}} t))_{t \in [0, T]} \right) \right) \\
&= \mathbb{E} \left(F \left(\left(\int_0^t \frac{p_{\mathbf{q}}}{2X^{(-1)}(\pi p_{\mathbf{q}} s)} ds \right)_{t \in [0, T]}, (X^{(-1)}(\pi p_{\mathbf{q}} t))_{t \in [0, T]} \right) \right).
\end{aligned}$$

It is now elementary but tedious to get the desired convergence in law. Let us still give the details using the Portemanteau theorem. Let A_ε be the closed subset of $\mathbb{D}([0, T])$ defined by

$$A_\varepsilon := \left\{ (x(t))_{t \in [0, T]} \in \mathbb{D}([0, T]); x(T) \geq 2\varepsilon \text{ and } \forall t \in [0, T], x(t) \geq \frac{1+\varepsilon}{2} x(t-) \right\}.$$

Let B be a closed subset of $\mathbb{D}([0, T])^2$ and let B^δ be its δ -neighborhood in $\mathbb{D}([0, T])^2$ for some distance which induces the J_1 Skorokhod topology. Note that $(B \cap (\mathbb{D}([0, T]) \times A_\varepsilon)) \cap ((\mathbb{D}([0, T])^2 \setminus B^\delta) \cup (\mathbb{D}([0, T]) \times \overline{U_\varepsilon})) = \emptyset$, so that by Urysohn's lemma, there exists a continuous function $F : \mathbb{D}([0, T])^2 \rightarrow [0, 1]$ which vanishes on the closed set $((\mathbb{D}([0, T])^2 \setminus B^\delta) \cup (\mathbb{D}([0, T]) \times \overline{U_\varepsilon}))$ and is 1 on the closed set $B \cap (\mathbb{D}([0, T]) \times A_\varepsilon)$. Let us write

$$(X_\ell, Y_\ell) := \left(\left(\frac{H_*(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \in [0, T]}, \left(\frac{P_*(\lfloor \ell t \rfloor)}{\ell} \right)_{t \in [0, T]} \right)$$

and

$$(X, Y) := \left(\left(\int_0^t \frac{p_{\mathbf{q}}}{2X^{(-1)}(\pi p_{\mathbf{q}} s)} ds \right)_{t \in [0, T]}, (X^{(-1)}(\pi p_{\mathbf{q}} t))_{t \in [0, T]} \right).$$

Then, by the above convergence, one can see that

$$\begin{aligned}
\limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}((X_\ell, Y_\ell) \in B) &\leq \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(Y_\ell \notin A_\varepsilon) + \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}((X_\ell, Y_\ell) \in B \cap (\mathbb{D}([0, T]) \times A_\varepsilon)) \\
&\leq \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(Y_\ell \notin A_\varepsilon) + \limsup_{\ell \rightarrow \infty} \mathbb{E}^{(\ell)} F(X_\ell, Y_\ell) \\
&= \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(Y_\ell \notin A_\varepsilon) + \mathbb{E} F(X, Y) \\
&\leq \limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(Y_\ell \notin A_\varepsilon) + \mathbb{P}((X, Y) \in B^\delta \cap (\mathbb{D}([0, T]) \times (\mathbb{D}([0, T]) \setminus \overline{U_\varepsilon})))
\end{aligned}$$

Moreover, it is easy to see that $\sup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}(Y_\ell \notin A_\varepsilon) \rightarrow 0$ and $\mathbb{P}(Y \in \overline{U_\varepsilon}) \rightarrow 0$ as $\varepsilon \rightarrow 0$ so that

$$\limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}((X_\ell, Y_\ell) \in B) \leq \mathbb{P}((X, Y) \in B^\delta).$$

Finally, by letting $\delta \rightarrow 0$, we conclude that $\limsup_{\ell \rightarrow \infty} \mathbb{P}^{(\ell)}((X_\ell, Y_\ell) \in B) \leq \mathbb{P}((X, Y) \in B)$. This concludes the proof. \square

Let us now consider the whole exploration tree in order to prove Theorem 3.1.4: instead of filling-in the holes, we run new explorations inside. Let us focus on the dual graph distance d_{gr}^\dagger , using the peeling by layers algorithm.

Recall from Section 2.2 of [BBCK18] the recursive definition of the discrete perimeter cell-system

$$((P_u(k))_{k \geq 0})_{u \in \mathbb{U}}$$

associated with a branching peeling exploration of a finite map \mathbf{m} , where $\mathbb{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$ is the Ulam tree. The process P_\emptyset is the (half-)perimeter process of the locally largest cycle. We denote by $(N_j)_{j \geq 1}$ the times of negative jumps of P_\emptyset (usually ranked in the non-increasing order of jump sizes and in the decreasing order of jump times if the jump sizes are equal). Each one corresponds to an event of type G_{k_1, k_2} , i.e. to a splitting of the distinguished hole of half-perimeter $P_\emptyset(N_j)$ into two holes of half-perimeters $P_\emptyset(N_j + 1)$ and $\Delta_- P_\emptyset(N_j) := P_\emptyset(N_j) - P_\emptyset(N_j + 1) - 1 \leq P_\emptyset(N_j + 1)$. We then define $(P_j(k))_{k \geq 0}$ as the perimeter process of the locally largest cycle during the exploration of the submap filling the second hole. We define P_{u_j} from P_u and the time of negative jump N_{u_j} in the same way for all $u \in \mathbb{U}$. The birth time B_u of the cell $u \in \mathbb{U}$ is defined recursively by $B_\emptyset = 0$ and for all $u \in \mathbb{U}, j \in \mathbb{N}^*$, we set $B_{u_j} = B_u + N_{u_j}$. For all $u \in \mathbb{U}$, let \mathcal{C}_u be the cycle (i.e. the boundary of the hole) created during the birth of the cell u , with by convention \mathcal{C}_\emptyset being the boundary of the root face. For all $u \in \mathbb{U}$, each non-negative jump $P_u(k+1) - P_u(k) \geq 0$ of the perimeter of the cell u at some time k such that $P_u(k) \geq 1$ corresponds to the discovery of some face f in the branch u . The time of exploration of f is defined by

$$n(f) := B_u + k. \quad (3.6.16)$$

Let X be a self-similar Markov process such that 0 is an absorbing state which tends a.s. towards zero or is a.s. absorbed at zero after a finite time. Let P_x be the law of X starting from $x \in \mathbb{R}_+$. We say that $(X_u)_{u \in \mathbb{U}}$ is a (continuous) cell-system associated with X starting at $x \geq 0$ if X_\emptyset follows the law P_x and for all $u \in \mathbb{U}$, conditionally on X_u , if we write $((x_j, \beta_{u_j}))_{j \geq 1}$ the family of negative jumps of X_u , where $x_j > 0$ is the size of the jump and β_{u_j} is the time of jump, ordered in the non increasing lexicographic order ($x_j \geq x_{j+1}$ and if $x_j = x_{j+1}$ then $\beta_j > \beta_{j+1}$), then the X_{u_j} for $j \geq 1$ are independent of laws P_{x_j} . For all $u \in \mathbb{U}$, we set $\zeta_u = \inf\{t \geq 0, X_u(t) = 0\}$. Let $((\mathcal{X}_u(t))_{t \geq 0}, (\beta_{u_j})_{j \geq 1}, b_u, \zeta_u)_{u \in \mathbb{U}}$ be the continuous cell-system associated with the self-similar Markov process $X^{(-1)}$.

Recall the definition of the J_1 Skorokhod topology on the space of families of càdlàg functions indexed by \mathbb{U} denoted by $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^{\mathbb{U}} : (f_u^{(n)}(t))_{t \geq 0, u \in \mathbb{U}}$ converges towards $(f_u(t))_{t \geq 0, u \in \mathbb{U}}$ if for all finite subset $F \subset \mathbb{U}$, $(f_u^{(n)}(t))_{t \geq 0, u \in F}$ converges towards $(f_u(t))_{t \geq 0, u \in F}$ in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^F)$ for the J_1 Skorokhod topology.

The following lemma proves that the scaling limit of the discrete cell-system is given by the continuous cell-system.

Lemma 3.6.5. *The rescaled discrete cell-system associated with $\mathfrak{M}^{(\ell)}$ converges in distribution to the continuous cell-system as a random variable with values in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^{\mathbb{U}}$ equipped with the J_1 Skorokhod topology:*

$$\text{under } \mathbb{P}^{(\ell)}, \quad \left(\left(\frac{P_u(\ell t)}{\ell} \right)_{t \geq 0} \right)_{u \in \mathbb{U}} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left((\mathcal{X}_u(\pi p_{\mathbf{q}} t))_{t \geq 0} \right)_{u \in \mathbb{U}}. \quad (3.6.17)$$

Proof. The proof is the same as the proof of Lemma 17 in [BCK18]. The first step is to show that the convergence (3.6.3) holds in the space $\mathbb{D}([0, \infty])$ equipped with Skorokhod's J_1 topology. To prove this, it suffices to control $P_*(\ell t)/\ell$ for large $t \geq 0$ uniformly in ℓ using the supermartingale $(f^\dagger(P_*(n)))_{n \geq 0}$ under $\mathbb{P}^{(\ell)}$ coming from Proposition 6.4 of [BBCK18], where f^\dagger is defined in the proof of Corollary 3.6.4. Then (3.6.17) stems from the spatial Markov property and from the fact that the \mathcal{X}_u 's for $u \in \mathbb{U}$ have a.s. no common jump times, using Proposition 2.2 page 338 of [JS87]. \square

At this point, we are in position to prove the main result of this section.

Proof of Theorem 3.1.4. We first extend the convergence (3.6.17) to the height processes. For every $u \in \mathbb{U}$, we can define H_u the height process of the locally largest cycle during the (filled-in) exploration

of the submap of \mathbf{m} contained in the hole created at the birth time of the cell u . Notice that $H_\emptyset = H_*$. Analogously, in the continuous setting, we write for all $u \in \mathbb{U}$, for all $t \geq 0$,

$$\mathcal{H}_u(t) = \int_0^t \frac{p_{\mathbf{q}}}{2\mathcal{X}_u(\pi p_{\mathbf{q}}s)} ds.$$

A direct consequence of Corollary 3.6.4 and Lemma 3.6.5 is the following convergence towards a family of continuous increasing processes which holds jointly with (3.6.17):

$$\text{under } \mathbb{P}^{(\ell)}, \quad \left(\left(\frac{H_u(\lfloor \ell t \rfloor)}{\log \ell} \right)_{t \geq 0} \right)_{u \in \mathbb{U}} \xrightarrow[\ell \rightarrow +\infty]{(d)} \left((\mathcal{H}_u(t))_{t \geq 0} \right)_{u \in \mathbb{U}}. \quad (3.6.18)$$

The topology for which the above convergence holds is the product topology over \mathbb{U} where each factor is equipped with the topology of the uniform convergence on compact sets.

Yet, the processes H_u for $u \in \mathbb{U}$ only describe the height in the map encircled by \mathcal{C}_u . In order to describe the height in the whole map \mathbf{m} , we introduce some additional notation. In the discrete setting, for all $u \in \mathbb{U}$, let $H(\mathcal{C}_u)$ be the minimum of the heights (i.e. the distances to the root face) of the edges of the cycle \mathcal{C}_u . We also define \tilde{H}_u by induction: $\tilde{H}_\emptyset = 0$ and for all $u \in \mathbb{U}, j \geq 1$, $\tilde{H}_{uj} = \tilde{H}_u + H_u(N_{uj})$. In the continuous setting, we define the birth heights $(b_u^{\mathcal{H}})_{u \in \mathbb{U}}$ by $b_\emptyset^{\mathcal{H}} = 0$ and for all $u \in \mathbb{U}$, for all $j \geq 1$, $b_{uj}^{\mathcal{H}} = b_u^{\mathcal{H}} + \mathcal{H}_u(\beta_{uj}/(\pi p_{\mathbf{q}}))$.

Notice that by (3.6.17) and (3.6.18), we have jointly with (3.6.17) and (3.6.18) the convergence for the product topology

$$\text{under } \mathbb{P}^{(\ell)}, \quad \left(\frac{H_u(N_{uj})}{\log \ell} \right)_{u \in \mathbb{U}, j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\mathcal{H}_u \left(\frac{\beta_{uj}}{\pi p_{\mathbf{q}}} \right) \right)_{u \in \mathbb{U}, j \geq 1}. \quad (3.6.19)$$

Since the peeling algorithm which is used in the exploration is the peeling by layers, for all $u \in \mathbb{U}, j \geq 1$, every edge of \mathcal{C}_{uj} is at distance $H_u(N_{uj})$ or $H_u(N_{uj}) + 1$ from \mathcal{C}_u . Therefore, for all $u \in \mathbb{U}$, we have $\tilde{H}_u \leq H(\mathcal{C}_u) \leq \tilde{H}_u + |u|$, where $|u|$ denotes the length of the word u . Thus (3.6.19) entails that, jointly with (3.6.17) and (3.6.18),

$$\left(\frac{H(\mathcal{C}_u)}{\log \ell} \right)_{u \in \mathbb{U}} \xrightarrow[\ell \rightarrow +\infty]{(d)} (b_u^{\mathcal{H}})_{u \in \mathbb{U}} \quad (3.6.20)$$

Note that the positive jumps of the P_u 's for $u \in \mathbb{U}$ correspond to the (half) perimeters of the faces of \mathbf{m} minus one (without the root face).

Besides, notice that by the Markov property of Theorem 3.1.2, the cell system

$$((Y_u(t))_{t \geq 0})_{u \in \mathbb{U}} := \left((\mathcal{X}_u(\pi t))_{t \geq 0} \right)_{u \in \mathbb{U}}$$

describes the quantum boundary lengths along the exploration tree of the CLE_4 decorated 2-LQG disk, both for the uniform exploration and for the $\text{SLE}_4^{(\mu)}(-2)$ exploration, parametrized by the quantum natural distance for the first one and by the quantum length of the trunk for the second one. In particular, the positive jumps correspond the quantum boundary lengths of the loops, so that the quantum boundary lengths of the loops are all distinct (note that one can also see the fact that the quantum boundary lengths of the loops are distinct as a direct consequence of Theorem 1.2 of [AG23a] which gives the law of the quantum boundary lengths of the loops). Moreover, one can see that for all $\varepsilon > 0$ there is a finite number of loops of quantum boundary length larger than ε . It can be seen for instance as a consequence of Theorem 3.1.2 which shows that one can write $\mu_{h^2}^2(\mathbb{D}) \geq \sum_{j \geq 1} b_j^2 a_j$, where the a_j 's are i.i.d. of the same law as $\mu_{h^2}^2(\mathbb{D})$ and independent of the b_j 's which are the quantum boundary lengths of the loops.

Then the desired result is a straightforward consequence of the convergences (3.6.17), (3.6.18) and (3.6.20). Indeed, by Skorokhod's representation theorem, take a probability space where these convergences hold almost surely. Take $\varepsilon > 0$. Let $F \subset \mathbb{U}$ be the finite set of u 's such that Y_u has at least one jump of size larger than ε . The convergence (3.6.17) entails the scaling limit of the degrees of the faces to the quantum boundary lengths of the loops and the scaling limit of the exploration times $n(f_i^{(\ell)})$ for $i \geq 1$ towards the quantum natural distances or quantum lengths of the trunk. Finally, the convergences (3.6.18) and (3.6.20) give the scaling limit of the height towards the Lamperti transform. \square

Note that one can also obtain a “slicing at heights” of $3/2$ -stable maps as in [BCK18] or [BBCK18]. For all $r \geq 1$, let $\text{Ball}_r^\dagger(\mathbf{m})$ be the ball of radius r centred at f_r in \mathbf{m} . This ball is the submap of \mathbf{m} whose faces are the faces of \mathbf{m} which are at height at most r and the adjacency relation between these faces is the same as in \mathbf{m} , except that two faces at distance r to the root face which were adjacent in \mathbf{m} are no longer adjacent in the ball. This ball has several holes of boundaries $\mathfrak{h}_1(\mathbf{m}, r), \mathfrak{h}_2(\mathbf{m}, r), \dots$ ranked in the non-increasing order of perimeter. For all $i \geq 1$ and $r \geq 0$, let $L_i(r)$ be the (half-)perimeter of the hole $\mathfrak{h}_i(\mathbf{m}, r)$.

Let us recall the notions of self-similar growth-fragmentation. See Subsections 2.2 and 3.1 of [BBCK18] or Section 2 of [Ber17] for more details. If $(\mathcal{X}_u, b_u, \zeta_u)_{u \in \mathbb{U}}$ is a continuous cell system associated with a self-similar Markov process X , for all $t \geq 0$ we set

$$\mathbf{X}(t) = \{\{\mathcal{X}_u(t - b_u); u \in \mathbb{U}, b_u \leq t < b_u + \zeta_u\}\}$$

where the notation $\{\{\dots\}\}$ refers to multiset. By Theorem 2 in [Ber17], the elements of $\mathbf{X}(t)$ can be ranked in the non-increasing order and form a sequence that tends to 0, say $X_1(t) \geq X_2(t) \geq \dots \geq 0$. If $\mathbf{X}(t)$ has only finitely many elements, say n , then we set $X_{n+1}(t) = X_{n+2}(t) = \dots = 0$. The process $(\mathbf{X}_t)_{t \geq 0}$ is called the growth-fragmentation process associated with X .

When $p \geq 1$, let ℓ_p^\downarrow be the space of non-increasing sequences $(x_i)_{i \geq 0}$ such that $\sum_{i \geq 1} x_i^p < \infty$. Let $\mathbb{D}(\mathbb{R}_+, \ell_p^\downarrow)$ be the space of càdlàg functions from \mathbb{R}_+ to ℓ_p^\downarrow equipped with the J_1 Skorokhod topology. Let $\mathbf{X}^{(0)} = ((X_i^{(0)}(t))_{i \geq 1})_{t \geq 0}$ be the growth-fragmentation associated to the self-similar process $X^{(0)}$. Then, using exactly the same proof as the proof of Theorem 6.8 of [BBCK18], we obtain the following scaling limit:

Proposition 3.6.6. *In the space $\mathbb{D}(\mathbb{R}_+, \ell_{5/2}^\downarrow)$,*

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\left(\frac{L_i(s \log \ell)}{\ell} \right)_{i \geq 1} \right)_{s \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\mathbf{X}^{(0)}(2\pi s) \right)_{s \geq 0}.$$

3.7 Discussion

3.7.1 Application to $O(2)$ -decorated maps

Here we describe briefly how our results transfer to some critical $O(2)$ loop-decorated planar maps. Let us first recall the definition of a rigid loop-decorated map and the definition of Boltzmann $O(n)$ loop-decorated random planar maps for $n \in (0, 2]$.

A *loop-decorated map* $(\mathbf{m}, \mathcal{L})$ is a (rooted bipartite planar) map \mathbf{m} together with a loop configuration $\mathbf{L} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$ of disjoint unoriented simple closed paths on the dual map \mathbf{m}^\dagger . Furthermore, the loops \mathcal{L}_i do not go through the root face of \mathbf{m} and are rigid in the sense that they only visit quadrangles and they enter and exit the quadrangles through opposite sides. See Figure 3.8.

By convention, the distances d_{gr}^\dagger and d_{fpp}^\dagger are defined on a loop-decorated map by adding the rule that all the quadrangles of each loop are identified to one point, i.e. are at distance zero.

For all $\ell \geq 1$, we introduce the set $\mathcal{LM}^{(\ell)}$ of loop-decorated maps whose root face has degree 2ℓ . Let $\tilde{\mathbf{q}} = (\tilde{q}_k)_{k \geq 1}$ be a sequence of non-negative real numbers and two real numbers $h, n \geq 0$. We define the weight of a loop decorated map $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}$ by setting

$$w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}) = \left(\prod_{\mathcal{L} \in \mathbf{L}} n h^{|\mathcal{L}|} \right) \left(\prod_{f \in \text{Faces}(\mathbf{m}) \setminus (\{f_r\} \cup \bigcup_{\mathcal{L} \in \mathbf{L}} \mathcal{L})} \tilde{q}_{\deg(f)/2} \right),$$

where $|\mathcal{L}|$ is the length of the loop \mathcal{L} , i.e. its number of quadrangles. For all $\ell \geq 1$, we recall that the partition function of $O(n)$ loop-decorated maps with triple $(\tilde{\mathbf{q}}, h, n)$ of perimeter 2ℓ is defined by

$$F^{(\ell)}(\tilde{\mathbf{q}}, h, n) = \sum_{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}).$$

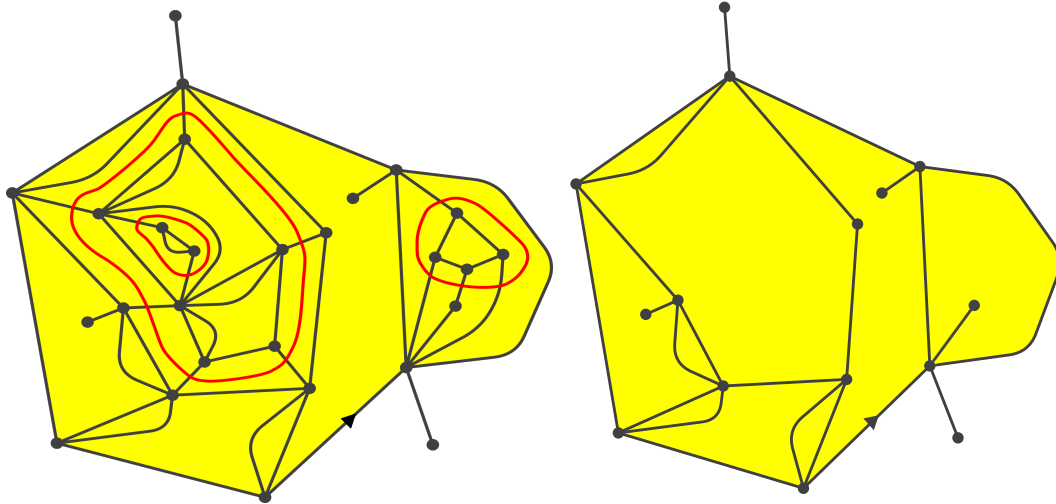


Figure 3.8 – Left: a loop-decorated planar map. The loops, which cross quadrangles, are in red. Right: its gasket, obtained by removing the interior of the outermost loops.

The triple $(\tilde{\mathbf{q}}, h, n)$ is said admissible if $F^{(\ell)}(\tilde{\mathbf{q}}, h, n) < \infty$ for all $\ell \geq 1$. The *Boltzmann probability measure* on $O(n)$ loop-decorated maps $\tilde{\mathbb{P}}^{(\ell)}$ on $\mathcal{LM}^{(\ell)}$ is then defined by setting

$$\forall (\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}, \quad \tilde{\mathbb{P}}^{(\ell)}(\{(\mathbf{m}, \mathbf{L})\}) = \frac{w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L})}{F^{(\ell)}(\tilde{\mathbf{q}}, h, n)}. \quad (3.7.1)$$

Let us now recall from [BBG12b] the gasket decomposition of a loop-decorated planar map. See also Sections 1.2.1 and 3.1 of [Bud18] for a presentation of this decomposition. The gasket of a loop-decorated map $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}$ is a planar map $\mathbf{g} \in \mathcal{M}^{(\ell)}$ obtained by removing the interior of the outermost loops of (\mathbf{m}, \mathbf{L}) . See the right-hand side of Figure 3.8. If we define the weight sequence $\mathbf{q} = (q_k)_{k \geq 1}$ by setting for all $k \geq 1$,

$$q_k = \tilde{q}_k + nh^{2k} F^{(k)}(\tilde{\mathbf{q}}, h, n), \quad (3.7.2)$$

then for all $\mathbf{g} \in \mathcal{M}^{(\ell)}$,

$$w_{\mathbf{q}}(\mathbf{g}) = \sum_{\substack{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)} \\ \mathbf{g} \text{ is the gasket of } (\mathbf{m}, \mathbf{L})}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}),$$

where we recall that $w_{\mathbf{q}}$ is defined in Subsection 3.1.2. Recall also from Subsection 3.1.2 that $W^{(\ell)}$ is the partition function of random maps with perimeter 2ℓ . In particular, summing over $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}$, we deduce that $W^{(\ell)} = F^{(\ell)}(\tilde{\mathbf{q}}, h, n)$ for all $\ell \geq 1$, and that \mathbf{q} is admissible if and only if the triple $(\tilde{\mathbf{q}}, h, n)$ is admissible.

As in [Bud18], we say that the triple $(\tilde{\mathbf{q}}, h, n)$ is *non-generic critical* if it is admissible, $n > 0$ and $h = 1/c_{\mathbf{q}}$. Note that by (3.7.2), a weight sequence $\mathbf{q} = (q_k)_{k \geq 1}$ is the weight sequence of the gasket of a non-generic critical $O(n)$ loop decorated map if and only if \mathbf{q} is admissible and

$$\forall k \geq 1, \quad q_k - nc_{\mathbf{q}}^{-2k} W^{(k)} \geq 0. \quad (3.7.3)$$

Moreover, the associated triple $(\tilde{\mathbf{q}}, h, n)$ of $O(n)$ loop-decorated maps is given by (3.7.2) and $h = 1/c_{\mathbf{q}}$. In Theorem 4.1.1 of Chapter 4 (see also Equation (1.7) in [ASH24] for the particular case of quadrangulations), we show that if $\mathbf{q} = (q_k)_{k \geq 1}$ is the weight sequence of the gasket of a non-generic critical $O(2)$ loop-decorated map, then there exists a slowly varying function $L_{\mathbf{q}}$ such that $1 = O(L_{\mathbf{q}}(\ell))$, $L_{\mathbf{q}}(\ell) = O(\log \ell)$ and

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} c_{\mathbf{q}}^{\ell+1} \frac{L_{\mathbf{q}}(\ell)}{2\ell^2}. \quad (3.7.4)$$

When this slowly varying function $L_{\mathbf{q}}$ is constant, we get (3.1.1). In other words, in our setting, when $L_{\mathbf{q}}$ is constant, the gasket of a critical $O(2)$ loop-decorated map is a 3/2-stable maps. We will make this

assumption in the rest of this subsection. Such a sequence does exist: in fact, as shown in Remark 1 of [Bud18] one can take for all $k \geq 1$

$$q_k = \frac{1}{(3\pi)^{k-1}} \left(\frac{2}{\pi} \frac{1}{(2k-3)(2k-1)} + \mathbf{1}_{k=1} \right).$$

Using the above-described gasket decomposition and more precisely (3.7.2), one can construct the critical $O(2)$ Boltzmann measure $\tilde{\mathbb{P}}^{(\ell)}$ on $\mathcal{LM}^{(\ell)}$ for all $\ell \geq 1$ defined in (3.7.1) in a recursive way as follows. Let $\mathfrak{M}^{(\ell)}$ be a random variable in $\mathcal{M}^{(\ell)}$ of law $\mathbb{P}^{(\ell)}$. Then, independently, on each face f of $\mathfrak{M}^{(\ell)}$, with probability $(2W^{(p)})/(c_{\mathbf{q}}^{2p}q_p)$ where $p = \deg(f)/2$, we glue quadrangles along the edges around f hence forming a rigid loop and we glue in this loop an independent loop-decorated map of law $\tilde{\mathbb{P}}^{(p)}$. The fact that $\tilde{\mathbb{P}}^{(\ell)}$ is well defined comes from the admissibility of \mathbf{q} , from (3.7.3) and from [BBG12b] or from Theorem 1 of [Bud18]. The map $\mathfrak{M}^{(\ell)}$ is the gasket of the obtained loop-decorated map. See Figure 3.8.

Exploration of loop-decorated maps and signed cell-systems As observed in [Bud18], we can extend the peeling exploration of the gasket to a peeling exploration of the loop-decorated map. More precisely, one can extend the exploration and the cell-system defined in Subsection 3.6.3 to explore the whole critical $O(2)$ loop-decorated map (not only its gasket) as follows: each time we observe a non-negative jump of size $p-1 \geq 0$ of the perimeter process, we know that with probability $(2W^{(p)})/(c_{\mathbf{q}}^{2p}q_p)$ it corresponds to the discovery of a loop \mathcal{L} of size $2p$ (and not of a face of degree $2p$). In this case, we run a new exploration of the map encircled by the loop \mathcal{L} . We thus obtain a discrete cell-system $((\tilde{P}_u(k))_{k \geq 0})_{u \in \mathbb{U}}$ which is defined as follows: $\tilde{P}_\emptyset = P_\emptyset$ and $\tilde{B}_\emptyset = 0$. Then, assume that the cell \tilde{P}_u and its birth time \tilde{B}_u are defined. Let \tilde{N}_{uj} for $j \geq 1$ be the times of negative jumps and also the non-negative jumps corresponding to the discovery of a loop, in the lexicographic order (using the non-increasing order of absolute value of the sizes of the jumps). We then set $\tilde{B}_{uj} = \tilde{B}_u + \tilde{N}_{uj}$ for all $j \geq 1$ and we define \tilde{P}_{uj} as the perimeter process of the exploration of the map filling the hole in the case of a negative jump or the map encircled by the loop in the case of a discovery of a loop. In the case of a discovery of a loop, by convention we change the sign of \tilde{P}_{uj} (i.e. if \tilde{P}_u is positive, then \tilde{P}_{uj} is minus the perimeter process, while if \tilde{P}_u is negative, then \tilde{P}_{uj} is the perimeter process).

Similarly, as in Subsection 3.1 of [DS23], we can define the continuous signed cell-system as follows. Let X be a self-similar Markov process such that 0 is an absorbing state which tends a.s. towards zero or is a.s. absorbed at zero after a finite time. Let P_x be the law of X starting from $x \in \mathbb{R}_+$. We say that $(\bar{X}_u)_{u \in \mathbb{U}}$ is a continuous signed cell-system associated with X starting at $x \geq 0$ if \bar{X}_\emptyset follows the law P_x and for all $u \in \mathbb{U}$, conditionally on \bar{X}_u , if we write $((\bar{x}_j, \bar{\beta}_{uj}))_{j \geq 1}$ the family of negative and positive jumps of \bar{X}_u , where $\bar{x}_j \in \mathbb{R}^*$ is the size of the jump and $\bar{\beta}_{uj}$ is the time of the jump, ordered in the non-increasing lexicographic order (the \bar{x}_j 's are compared using their absolute values), then the \bar{X}_{uj} 's are independent and have the law of $\pm X$ under $P_{|\bar{x}_j|}$, where the sign is determined as follows: the sign of \bar{X}_{uj} is minus the sign of \bar{X}_u when $\bar{x}_j > 0$ and is the same sign when $\bar{x}_j < 0$. We define inductively the birth times by setting $\bar{b}_\emptyset = 0$ and $\bar{b}_{uj} = \bar{b}_u + \bar{\beta}_{uj}$. The lifetime of \bar{X}_u is $\bar{\zeta}_u := \inf\{t \geq 0, \bar{X}_u(t) = 0\}$.

For all $\alpha \in \mathbb{R}$. Let $((\bar{\mathcal{X}}_u^{(\alpha)}(t))_{t \geq 0}, (\bar{\beta}_{uj})_{j \geq 1}, \bar{b}_u, \bar{\zeta}_u)_{u \in \mathbb{U}}$ be the continuous signed cell-system associated with the self-similar Markov process $X^{(\alpha)}$. Let us denote by $\bar{\mathbf{X}}^{(\alpha)}$ the self-similar signed growth-fragmentation process associated with this cell-system, which is defined for all $t \geq 0$ by

$$\bar{\mathbf{X}}^{(\alpha)}(t) = \left\{ \bar{\mathcal{X}}_u^{(\alpha)}(t - \bar{b}_u); u \in \mathbb{U}, \bar{b}_u \leq t < \bar{b}_u + \bar{\zeta}_u \right\}.$$

The self-similar signed growth-fragmentation $\bar{\mathbf{X}}^{(-1)}$ was studied in details in [AdS22]. As in Section 4 of [AdS22], we arrange the elements of $\bar{\mathbf{X}}^{(\alpha)}(t)$ in the non-increasing order of absolute values.

Scaling limit of the distances to the root. Using this gasket decomposition, one can see that Theorem 3.1.4 and Proposition 3.6.6 have their counterparts in terms of critical $O(2)$ loop-decorated planar maps.

In order to extend Theorem 3.1.4 to critical $O(2)$ loop-decorated maps, it suffices to replace the faces of the $3/2$ -stable map by the loops of the loop-decorated map and the non-nested CLE_4 by a nested CLE_4 . Recall the uniform exploration of the nested CLE_4 from Subsection 3.2.1. The definition of

the quantum natural distance defined in (3.2.2) can be extended beyond the time τ^z , which enables to define the quantum distance $d_q(\partial\mathbb{D}, \mathcal{L})$ to any loop \mathcal{L} of the nested CLE_4 . The distance d_{Lamperti} can be extended in the same way.

Corollary 3.7.1. *For all $\ell \geq 1$, let $(\mathfrak{M}^{(\ell)}, \mathbf{L}^{(\ell)})$ of law $\tilde{\mathbb{P}}^{(\ell)}$. Let $(\mathcal{L}_i^{(\ell)})_{i \geq 1}$ be the collection of loops of $(\mathfrak{M}^{(\ell)}, \mathbf{L}^{(\ell)})$ ranked in the non-increasing order of size. Let $(n(\mathcal{L}_i^{(\ell)}))_{i \geq 1}$ be their exploration times (i.e. the birth time of their corresponding cells in the discrete cell-system described above) using the peeling by layers algorithm. Let $(\mathcal{L}_i)_{i \geq 1}$ be the collection of loops of the nested CLE_4 ranked in the non-increasing order of quantum boundary length. Let us denote by $\nu_{h^2}^2(\mathcal{L})$ the quantum boundary length of a loop \mathcal{L} . Then, for the product topology,*

$$\left(\frac{|\mathcal{L}_i^{(\ell)}|}{2\ell}, \frac{n(\mathcal{L}_i^{(\ell)})}{p_{\mathbf{q}}\ell}, \frac{2d_{\text{gr}}^\dagger(f_r, \mathcal{L}_i^{(\ell)})}{\log \ell} \right)_{i \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} (\nu_{h^2}^2(\mathcal{L}_i), d_q(\partial\mathbb{D}, \mathcal{L}_i), d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L}_i))_{i \geq 1}.$$

The same theorem can be stated for the $\text{SLE}_4^{(\mu)}(-2)$ exploration, replacing the quantum natural distance by the quantum length of the trunk. For the fpp distance, the factor $\log \ell$ is replaced by a factor $1/p_{\mathbf{q}}$ and the peeling by layers exploration is replaced by the uniform peeling exploration.

The above corollary is a consequence of Theorem 3.1.4, of the fact that the maps encircled by the outermost loops are independent of law $\tilde{\mathbb{P}}^{(\ell)}$ conditionally on their perimeter, of the Markov property for the uniform exploration of the nested CLE_4 and of Theorem 3.1.2 which states in particular that the encircled domains are independent 2 quantum disks conditionally on their quantum boundary lengths.

The only difficulty is to see why in the limit all the positive jumps give rise to a loop while it is not the case in the exploration of the loop-decorated map. This comes from the fact that the jumps of the Cauchy process are symmetric. In other words, the terms q_k and $2c_{\mathbf{q}}^{-2k}W^{(k)}$ are “of the same order” so that the weights of the faces \tilde{q}_k are “negligible” compared with $2c_{\mathbf{q}}^{-2k}W^{(k)}$. More precisely, as explained in Subsection 6.4 of [BBCK18], the perimeter process is absolutely continuous with respect to the random walk with step distribution ν given by $\nu(k) = q_{k+1}c_{\mathbf{q}}^k$ and $\nu(-k-1) = 2W^{(k)}c_{\mathbf{q}}^{-k-1}$ for all $k \geq 0$. Here, by (4.1.1), each non-negative jump of size $k-1 \geq 0$ corresponds to the discovery of a loop with probability $2W^{(k)}/(c_{\mathbf{q}}^{2k}q_k) = \nu(-k-1)/\nu(k-1)$ but by Proposition 5.10 of [Cur23] we know that $\nu([k, \infty)) \sim p_{\mathbf{q}}/k \sim \nu((-\infty, -k])$ as $k \rightarrow \infty$. Thus, the PPP of the positive jumps of the Cauchy process appearing as the scaling limit of the random walk is the scaling limit of the non-negative jumps corresponding to the discovery of a loop

For Proposition 3.6.6, one has to add a sign $(-1)^{N(\mathfrak{h})}$ to the perimeter of each hole \mathfrak{h} , where $N(\mathfrak{h})$ is the number of loops surrounding \mathfrak{h} and one has to replace the growth-fragmentation $\mathbf{X}^{(0)}$ by the signed growth fragmentation $\overline{\mathbf{X}}^{(0)}$. Notice that this signed growth-fragmentation corresponds up to a Lamperti transform to the signed growth-fragmentation $\overline{\mathbf{X}}^{(-1)}$ studied in [Ads22].

For all $r \geq 1$, let $\text{Ball}_r^\dagger((\mathfrak{m}, \mathbf{L}))$ be the ball of radius r centred at f_r in the loop decorated map $(\mathfrak{m}, \mathbf{L})$. This ball has several holes of boundaries $\mathfrak{h}_1(\mathfrak{m}, r), \mathfrak{h}_2(\mathfrak{m}, r), \dots$ ranked in the non-increasing order of perimeter. For all $i \geq 1$ and $r \geq 0$, let $L_i(r)$ be $(-1)^{N(\mathfrak{h}_i)}$ times the (half-)perimeter of the hole $\mathfrak{h}_i(\mathfrak{m}, r)$. One obtains:

Proposition 3.7.2. *In the space $\mathbb{D}(\mathbb{R}_+, \ell_2^\downarrow)$,*

$$\text{Under } \tilde{\mathbb{P}}^{(\ell)}, \quad \left(\left(\frac{L_i(\lfloor s \log \ell \rfloor)}{\ell} \right)_{i \geq 1} \right)_{s \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} (\overline{\mathbf{X}}^{(0)}(2\pi s))_{s \geq 0}.$$

The proof goes along the same lines as the proof of Theorem 6.8 of [BBCK18]. The convergence towards the growth-fragmentation holds here in $\mathbb{D}(\mathbb{R}_+, \ell_2^\downarrow)$ since the harmonic functions associated to the exploration of Boltzmann planar maps which give rise to the martingales of Proposition 6.4 in [BBCK18] are replaced by the harmonic function $h_{\mathfrak{p}}^\downarrow$ for $\mathfrak{p} = 1$ in Equation (5) of [Bud18], which is constant equal to 1.

We believe the above results have their counterparts when the slowly varying function $L_{\mathbf{q}}$ in (3.7.4) is not constant but one has to adapt the techniques to deal with this case.

3.7.2 Why the author thinks that the conformally invariant distance is not the Lamperti transform of the quantum distance

We finally argue why we believe that the conformally invariant distance to the boundary d_{WW} of [WW13] is not the Lamperti transform d_{Lamperti} of the quantum distance.

Let us consider, as before, a unit boundary length 2-quantum disk (\mathbb{D}, h^2) on which we draw an independent (non-nested) CLE_4 using the uniform exploration. Let Z be an independent random point of law $\mu_{h^2}^2/\mu_{h^2}^2(\mathbb{D})$ and let \mathcal{L} be the loop surrounding Z . By definition of the uniform exploration, conditionally on h^2 and on Z , the distance $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L})$ is an exponential random variable of parameter λ for some constant $\lambda > 0$.

Indeed, since Z is independent from the uniform exploration, this distance has the same law as $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L}^z)$, where \mathcal{L}^z is the loop which encircles a deterministic point $z \in \mathbb{D}$. But coming back to the definition of the uniform exploration on \mathbb{H} in Subsection 3.2.1, the distance to the loop surrounding $z \in \mathbb{H}$ corresponds to the time τ^z when a loop of the PPP of loops $(\widehat{\gamma}_u)_{u \geq 0}$ encircles z , which is an exponential random variable since it is the time of the first atom of the restriction of the PPP $(\widehat{\gamma}_u)_{u \geq 0}$ to the loops surrounding z .

In particular, if we stop the exploration at distance $u > 0$, conditionally on not having encircled Z before time u , the distance $d_{\text{WW}}(\partial\mathbb{D}, \mathcal{L}) - u$ from \mathcal{L} to the boundary of the unexplored region is still an exponential random variable of parameter λ . On the contrary, if we stop the exploration at Lamperti transformed quantum distance $t > 0$, then, conditionally on not having encircled Z yet, the point Z lies in each unexplored quantum disk with probability proportional to its quantum area. So the quantum disk containing Z is in some sense weighted by its area, so that the absence of memory for $d_{\text{Lamperti}}(\partial\mathbb{D}, \mathcal{L})$ should be ruled out.

Nevertheless, one could start the uniform exploration from a loop of the CLE_4 and measure the quantum boundary lengths of the loops it discovers, which should provide a way to measure the d_{Lamperti} -distance from a given loop. Since d_{Lamperti} is the scaling limit of the distances to the boundary in $3/2$ -stable maps, it is natural to conjecture that such an extension of d_{Lamperti} is a measurable function of the CLE_4 -decorated 2-LQG and is a distance, which would be the scaling limit of $3/2$ -stable maps.

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Chapitre 4

Gaskets of $O(2)$ loop-decorated maps

Ce chapitre est issu de l'article [Kam24a], soumis.

Abstract We prove that for $n = 2$ the gaskets of critical rigid $O(n)$ loop-decorated random planar maps are $3/2$ -stable maps. The case $n = 2$ thus corresponds to the critical case in random planar maps. The proof relies on the Wiener-Hopf factorisation for random walks. Our techniques also provide a characterisation of weight sequences of critical $O(2)$ loop-decorated maps.

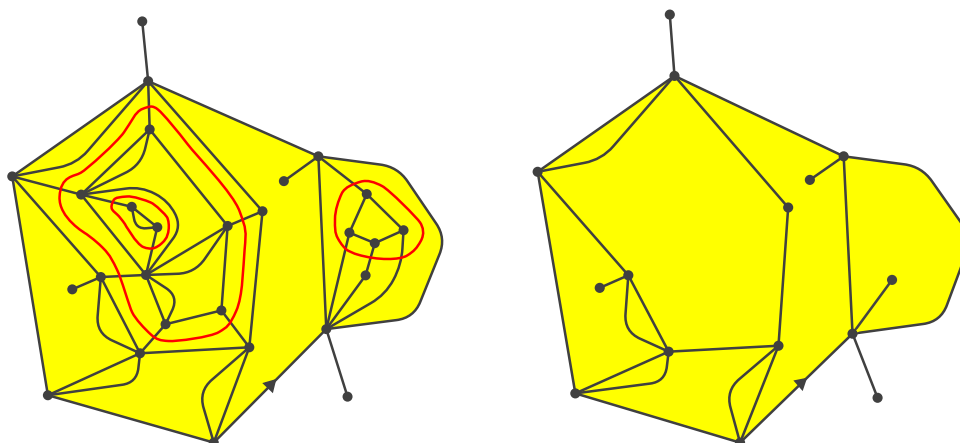


Figure 4.1 – Left: a rigid loop-decorated bipartite planar map. Right: its gasket.

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4.1 Introduction

4.1.1 $O(n)$ loop-decorated planar maps and their gaskets

Random $O(n)$ loop-decorated random planar maps are a classical model which couples a statistical mechanics model with quantum gravity. Although they have been studied for more than thirty years by physicists and then by mathematicians, their geometry remains elusive. At criticality, they are conjectured to satisfy scaling limits towards Liouville quantum gravity surfaces with parameter γ , where γ is related to the parameter $n \in (0, 2]$ by $\gamma = 2\sqrt{1 - \arccos(n/2)/\pi}$ or $\gamma = 2/\sqrt{1 + \arccos(n/2)/\pi}$, decorated by an independent conformal loop ensemble of parameter $\kappa = 4/(1 \pm \arccos(n/2)/\pi)$. See e.g. Conjecture 2.1 in [HL24] for a precise statement of the conjecture. In the nineties, $O(n)$ loop-decorated random planar maps were first studied in the physics literature using matrix model techniques, see e.g. [EK95, EK96] and the references therein. The precise model we deal with is the rigid $O(n)$ loop model on bipartite planar maps introduced in [BBG12b], and extensively studied in [BBG12a, BBD23, Bud18, CCM20, ASH24] using the gasket decomposition introduced in [BBG12b]. Actually, [BBG12b, BBG12a, BBD23] do not restrict themselves at all to the rigid case. See also [Kor22] for some results in the case where loops traverse triangles. However, the case $n = 2$ was often excluded, except in [EK95, EK96], in Theorem 5 of [Bud18], and in a new version of [ASH24] which deals with the particular case of $O(2)$ loop-decorated quadrangulations and appeared the same day on the arXiv. At first sight, the case $n = 2$ may be seen as a boundary case, but actually, in view of the scaling limit conjectures, it is the critical case since it is related to critical Liouville quantum gravity and to the conformal loop ensemble of parameter $\kappa = 4$.

A planar map \mathbf{m} is a finite connected planar graph which is embedded in the sphere and seen up to orientation-preserving homeomorphism. Our planar maps are equipped with a distinguished oriented edge, called the root edge. The face on the right of the root edge is called the root face and is denoted by f_r . For every face f , we denote by $\deg(f)$ and we call degree of f the number of edges surrounding f with multiplicity. The degree of f_r is called the perimeter of the map. We restrict ourselves to bipartite maps, i.e. whose face degrees are even. Let \mathcal{M} be the set of finite rooted bipartite planar maps and, for all $\ell \geq 1$, let $\mathcal{M}^{(\ell)}$ be the set of planar maps of perimeter 2ℓ . If \mathbf{m} is a map, we define the dual map \mathbf{m}^\dagger as the map obtained by exchanging the roles of the vertices and the faces. Two vertices of \mathbf{m}^\dagger are adjacent if the corresponding faces of \mathbf{m} share an edge on their boundary. A loop-decorated map (\mathbf{m}, \mathbf{L}) is a (finite rooted bipartite planar) map \mathbf{m} equipped with a loop configuration $\mathbf{L} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$ of disjoint unoriented simple closed paths on the dual map \mathbf{m}^\dagger . Furthermore, the loops \mathcal{L}_i do not go through the root face of \mathbf{m} and are rigid in the sense that they only visit quadrangles and they enter and exit quadrangles through opposite sides. See Figure 4.1¹. We denote the set of loop-decorated maps by \mathcal{LM} . For all $\ell \geq 1$, we also introduce the set $\mathcal{LM}^{(\ell)}$ of loop-decorated maps of perimeter 2ℓ .

Let $\mathbf{q} = (q_k)_{k \geq 1}$ be a non-zero sequence of non-negative real numbers. Define the weight of a planar map $\mathbf{m} \in \mathcal{M}$ by

$$w_{\mathbf{q}}(\mathbf{m}) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r\}} q_{\deg(f)/2}.$$

For all $\ell \geq 1$, we define the partition function of bipartite maps of weight sequence \mathbf{q} and perimeter 2ℓ

$$W^{(\ell)} = \sum_{\mathbf{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}).$$

When $W^{(\ell)} < \infty$ for all $\ell \geq 1$, the sequence \mathbf{q} is said admissible and we denote by $\mathbb{P}^{(\ell)}$ the associated Boltzmann probability measure characterized by $\mathbb{P}^{(\ell)}(\{\mathbf{m}\}) = w_{\mathbf{q}}(\mathbf{m})/W^{(\ell)}$ for all $\mathbf{m} \in \mathcal{M}^{(\ell)}$. By Lemma 3.13 of [Cur23], when \mathbf{q} is admissible, there exists a constant $c_{\mathbf{q}}$ such that

$$\frac{W^{(\ell+1)}}{W^{(\ell)}} \xrightarrow{\ell \rightarrow \infty} c_{\mathbf{q}}.$$

Similarly, let $\tilde{\mathbf{q}} = (\tilde{q}_k)_{k \geq 1}$ be a sequence of non-negative real numbers and two real numbers $h, n \geq 0$. We define the weight of a loop decorated map $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}$ by setting

$$w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}) = \left(\prod_{\mathcal{L} \in \mathbf{L}} nh^{|\mathcal{L}|} \right) \left(\prod_{f \in \text{Faces}(\mathbf{m}) \setminus (\{f_r\} \cup \bigcup_{\mathcal{L} \in \mathbf{L}} \mathcal{L})} \tilde{q}_{\deg(f)/2} \right),$$

1. All the planar maps drawn in this chapter were obtained using the planar map editor of Timothy Budd.

where $|\mathcal{L}|$ is the length of the loop \mathcal{L} , i.e. its number of quadrangles. For all $\ell \geq 1$, we define the partition function of $O(n)$ loop-decorated maps with triple $(\tilde{\mathbf{q}}, h, n)$ of perimeter 2ℓ

$$F^{(\ell)}(\tilde{\mathbf{q}}, h, n) = \sum_{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}).$$

We say that $(\tilde{\mathbf{q}}, h, n)$ is admissible if $F^{(\ell)}(\tilde{\mathbf{q}}, h, n) < \infty$ for all $\ell \geq 1$.

Let us now recall from [BBG12b] the gasket decomposition of a loop-decorated planar map. See also Sections 1.2.1 and 3.1 of [Bud18] for a presentation of this decomposition. The gasket of a loop-decorated map $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}$ is a planar map $\mathbf{g} \in \mathcal{M}$ obtained by removing the interior of the outermost loops of (\mathbf{m}, \mathbf{L}) . See the right-hand side of Figure 4.1. If we define the weight sequence $\mathbf{q} = (q_k)_{k \geq 1}$ by setting for all $k \geq 1$,

$$q_k = \tilde{q}_k + nh^{2k} F^{(k)}(\tilde{\mathbf{q}}, h, n), \quad (4.1.1)$$

then for all $\mathbf{g} \in \mathcal{M}$,

$$w_{\mathbf{q}}(\mathbf{g}) = \sum_{\substack{(\mathbf{m}, \mathbf{L}) \in \mathcal{LM} \\ \mathbf{g} \text{ is the gasket of } (\mathbf{m}, \mathbf{L})}} w_{\tilde{\mathbf{q}}, h, n}(\mathbf{m}, \mathbf{L}).$$

In particular, summing over $(\mathbf{m}, \mathbf{L}) \in \mathcal{LM}^{(\ell)}$, we deduce that $W^{(\ell)} = F^{(\ell)}(\tilde{\mathbf{q}}, h, n)$ for all $\ell \geq 1$, and that \mathbf{q} is admissible if and only if the triple $(\tilde{\mathbf{q}}, h, n)$ is admissible.

As in [Bud18], we say that the triple $(\tilde{\mathbf{q}}, h, n)$ is non-generic critical if it is admissible, $n > 0$ and $h = 1/c_{\mathbf{q}}$. Note that by (4.1.1), a weight sequence $\mathbf{q} = (q_k)_{k \geq 1}$ is the weight sequence of the gasket of a non-generic critical $O(n)$ loop decorated map if and only if \mathbf{q} is admissible and

$$\forall k \geq 1, \quad q_k - nc_{\mathbf{q}}^{-2k} W^{(k)} \geq 0. \quad (4.1.2)$$

Moreover, the associated triple $(\tilde{\mathbf{q}}, h, n)$ of $O(n)$ loop-decorated maps is given by (4.1.1) and $h = 1/c_{\mathbf{q}}$. It is known since [Kos89] that for $n \in (0, 2)$, if the triple $(\tilde{\mathbf{q}}, h, n)$ is non-generic critical, then we may have the following ‘‘perimeter exponent’’:

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-a} \quad (4.1.3)$$

where $p_{\mathbf{q}}$ is some positive constant, $c_{\mathbf{q}} = 1/h$ and $a = 2 \pm \arccos(n/2)/\pi$. See e.g. Equations (15) and (16) of [Kos89], Equation (2.9) of [BBD23] or Equation (3) of [Bud18]. This perimeter exponent (4.1.3) is central in the study of random planar maps. Indeed, (4.1.3) means that \mathbf{q} is non-generic critical of type a in the sense of [Cur23] (see Proposition 5.10 therein), or equivalently that the law of gasket $\mathbb{P}^{(\ell)}$ is the law of an α -stable map with $\alpha = a - 1/2$. The equivalent (4.1.3) actually determines the geometry of the gasket and thus many properties of critical $O(n)$ loop-decorated planar maps. The study of the geometry of α -stable maps was initiated in [LGM11] for the primal distances, then in [BC17, BCM18] for the dual distances in the local limit and then in [BCK18] for the dual distances to the root face under $\mathbb{P}^{(\ell)}$.

4.1.2 Main results

This work aims at establishing the asymptotic behaviour of the partition function, i.e. the analogue of (4.1.3), for critical $O(2)$ loop-decorated planar maps. Recall that a function $L : \mathbb{R}_+ \rightarrow \mathbb{R}$ is slowly varying if for all $\lambda > 0$, the ratio $L(\lambda x)/L(x)$ converges to 1 as $x \rightarrow \infty$.

Theorem 4.1.1. *Let \mathbf{q} be an admissible weight sequence satisfying (4.1.2) with $n = 2$. Then there exists a slowly varying function $L_{\mathbf{q}}$ such that $1 = O(L_{\mathbf{q}}(\ell))$, $L_{\mathbf{q}}(\ell) = O(\log \ell)$ and*

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} c_{\mathbf{q}}^{\ell+1} \frac{L_{\mathbf{q}}(\ell)}{2\ell^2}.$$

Note that, contrary to the equivalent (4.1.3) in the case $n \in (0, 2)$, in the above equivalent, a slowly-varying function appears. See Section 4.4 for some examples where the slowly-varying function is constant or of order $\log \ell$. This result shows that a weight sequence \mathbf{q} satisfying (4.1.2) with $n = 2$ is critical non-generic of type $a = 2$, not necessarily in the sense of [Cur23], but in the more general

setting of [Ric18, CR20] who allow slowly varying functions. See Subsection 2.1 of [CR20] for the exact definition. Equivalently the weight sequence of the gasket of a critical $O(2)$ loop-decorated map is critical non-generic of type $a = 2$, i.e. the gasket is a $3/2$ -stable map in the sense of [Ric18, CR20].

Actually, we will prove an analogous result on a class of random walks which will imply the above theorem. Recall that a function $h : \mathbb{Z} \rightarrow \mathbb{R}_+$ is ν -harmonic on a subset $A \subset \mathbb{Z}$ for a positive measure ν on \mathbb{Z} if for all $p \in A$, we have $h(p) = \sum_{k \in \mathbb{Z}} \nu(k)h(p+k)$. Let $h^\downarrow : \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by

$$\forall \ell \geq 0, \quad h^\downarrow(\ell) = 2^{-2\ell} \binom{2\ell}{\ell} \quad \text{and} \quad \forall \ell \leq -1, \quad h^\downarrow(\ell) = 0. \quad (4.1.4)$$

Theorem 4.1.2. *Let ν be a probability distribution on \mathbb{Z} such that h^\downarrow is ν -harmonic on $\mathbb{Z}_{\geq 1}$ and such that ν is not the Dirac mass at zero δ_0 . Assume that*

$$\forall k \geq 1, \quad \nu(k-1) \geq \nu(-k-1). \quad (4.1.5)$$

Then,

— Either $\sum_{k \geq 1} k(\nu(k-1) - \nu(-k-1)) \in [0, 1)$ and there exists $c > 0$ such that

$$\nu(-k) \underset{k \rightarrow \infty}{\sim} \frac{c \log k}{k^2};$$

— Or $\sum_{k \geq 1} k(\nu(k-1) - \nu(-k-1)) = 1$ and there exists a slowly varying function L such that $L(k) = o(\log k)$, $1 = O(L(k))$ and

$$\nu(-k) \underset{k \rightarrow \infty}{\sim} \frac{L(k)}{k^2}.$$

4.1.3 Applications and connections with other works

Here, we mention some consequences of Theorem 4.1.1 and we discuss connections with other works. In [LGM11], Le Gall and Miermont prove that α -stable maps with n vertices, when equipped with the primal graph distance, where each edge has length one, satisfy a scaling limit along subsequences towards a random metric space whose Hausdorff dimension is 2α . This result extends to the more general definition of α -stable maps from [Ric18, CR20] by adding a slowly varying function, and thus in particular to the gasket of critical (rigid) $O(2)$ loop-decorated maps.

Besides, one can also equip the gasket with some dual distances, where large faces become hubs so that the geometry lies in another universality class. In Chapter 2, under $\mathbb{P}^{(\ell)}$ as $\ell \rightarrow \infty$, we establish the scaling limit of the first passage percolation distance d_{fpp}^\dagger , obtained by putting i.i.d. exponential random lengths of parameter 1 on each dual edge, and the dual graph distance d_{gr}^\dagger , obtained by assigning a length 1 to each dual edge, between two uniform random vertices in a $3/2$ -stable map and show that their diameter is of the same order. Here, the slowly varying function $L_{\mathbf{q}}$ significantly modifies the asymptotic behaviour of these distances.

In Chapter 3, we express the scaling limit of the distances d_{gr}^\dagger and d_{fpp}^\dagger from the faces of high degree to the root face. There is no doubt that the analogues of Theorem 3.1.4 and Proposition 3.6.6 in Chapter 3 hold up to some slowly varying functions in the more general definition of $3/2$ -stable maps from [Ric18, CR20]. As a consequence, the results stated in Section 3.7.1 of Chapter 3 are valid for any model of critical (rigid) $O(2)$ loop-decorated planar maps, up to some slowly varying functions. This gives a new piece of evidence that the scaling limit of $O(2)$ loop-decorated maps is a critical Liouville quantum gravity disk decorated with an independent conformal loop ensemble of parameter $\kappa = 4$.

In [Bud18], Budd introduces a peeling exploration of $O(n)$ loop-decorated maps, which extends his peeling exploration of Boltzmann maps introduced in [Bud16], and obtains in Theorem 3 the scaling limit of the perimeter process as a positive self-similar Markov process. Thanks to our result, Theorem 3 of [Bud18] has an analogue in the case $n = 2$ but the ‘‘pointed’’ critical $O(2)$ loop-decorated map is actually infinite, and the limiting process becomes the absolute value of a Cauchy process.

Besides, the results of [CCM20] describe the scaling limit of the sizes of the nested loops in terms of multiplicative cascades in the particular case of $O(n)$ loop-decorated quadrangulations and [ASH24] give the scaling limit of the volume of $O(n)$ loop-decorated quadrangulations. The case $n = 2$ is treated by Aïdékon, Da Silva and Hu in a new version of [ASH24] for $O(2)$ loop-decorated quadrangulations. The techniques of [ASH24] in the case of $O(2)$ loop-decorated quadrangulations, relying on approximation

results as $n \uparrow 2$, were developed simultaneously with ours and are of independent interest. In view of Theorem 4.1.1, we conjecture that the analogous results hold for any model of (rigid) critical $O(2)$ loop-decorated maps.

4.1.4 Outline

In Section 4.2, we describe our key tool: the Wiener–Hopf factorisation. In Section 4.3, we prove Theorems 4.1.1 and 4.1.2. Finally, in Section 4.4 we show that the computations in the proofs of Theorems 4.1.1 and 4.1.2 enable to characterise weight sequences of critical $O(2)$ loop-decorated planar maps and we give examples.

4.2 Wiener–Hopf factorisation

We recall here the Wiener–Hopf factorisation of a random walk on \mathbb{Z} . See e.g. Section 2.3 of the lecture notes [Bud17] for more details on this tool and some applications to random planar maps, or Theorem 9.15 in the book [Kal02].

Let ν be a probability law on \mathbb{Z} . Let $(S_n)_{n \geq 0}$ be a random walk of step distribution ν starting at zero. The weak ascending ladder epochs $(T_i^\geq)_{i \geq 0}$ are the times at which S_n reaches its running maximum. In other words, $T_0^\geq = 0$ and for all $i \geq 0$,

$$T_{i+1}^\geq = \inf\{n > T_i^\geq, S_n = \max_{0 \leq k \leq n} S_k\},$$

where by convention $\inf \emptyset = \infty$. The weak ascending ladder heights $(H_i^\geq)_{i \geq 0}$ are defined by $H_i^\geq = S_{T_i^\geq}$ when $T_i^\geq < \infty$ and $H_i^\geq = \partial$ when $T_i^\geq = \infty$, where ∂ is a cemetery point. Similarly, the strict descending ladder epochs $(T_i^\leq)_{i \geq 0}$ are defined by $T_0^\leq = 0$ and for all $i \geq 0$,

$$T_{i+1}^\leq = \inf\{n > T_i^\leq, S_n < \min_{0 \leq k \leq n-1} S_k\}.$$

The strict descending ladder heights $(H_i^\leq)_{i \geq 0}$ are defined by $H_i^\leq = S_{T_i^\leq}$ when $T_i^\leq < \infty$ and $H_i^\leq = \partial$ when $T_i^\leq = \infty$.

We denote the generating functions of H_1^\geq and H_1^\leq by

$$G^\geq(z) = \mathbb{E}\left(z^{H_1^\geq}\right) = \sum_{k=0}^{\infty} z^k \mathbb{P}(H_1^\geq = k) \quad \text{and} \quad G^\leq(z) = \mathbb{E}\left(z^{H_1^\leq}\right) = \sum_{k=0}^{\infty} z^k \mathbb{P}(H_1^\leq = k)$$

Note that when S oscillates, $\mathbb{P}(H_1^\geq = \partial) = \mathbb{P}(H_1^\leq = \partial) = 0$ so that $G^\geq(1) = G^\leq(1) = 1$. We also denote by φ the characteristic function of S_1 defined by $\varphi(\theta) = \mathbb{E}(e^{iS_1\theta})$ for all $\theta \in \mathbb{R}$. The following proposition relates the functions φ, G^\geq and G^\leq .

Proposition 4.2.1. (*Wiener–Hopf factorisation*) For all $\theta \in \mathbb{R}$,

$$1 - \varphi(\theta) = (1 - G^\geq(e^{i\theta}))(1 - G^\leq(e^{-i\theta})).$$

The Wiener–Hopf factorisation was also used for the study of the peeling exploration of $O(n)$ loop-decorated planar maps by Timothy Budd in [Bud18].

4.3 Proof of the main results

4.3.1 Proof of Theorem 4.1.1 using Theorem 4.1.2

Proof of Theorem 4.1.1. Let \mathbf{q} be an admissible weight sequence satisfying (4.1.2). By Lemma 5.2 of [Cur23], the measure ν on \mathbb{Z} defined by setting for all $k \geq 0$,

$$\nu(k) = q_{k+1} c_{\mathbf{q}}^k \quad \text{and} \quad \nu(-k-1) = 2W^{(k)} c_{\mathbf{q}}^{-k-1} \quad (4.3.1)$$

is a probability measure. Moreover, by Proposition 5.3 of [Cur23], the function h^\downarrow is ν -harmonic on $\mathbb{Z}_{\geq 1}$. Furthermore, (4.1.2) can be rewritten

$$\forall k \geq 1, \quad \nu(k-1) \geq \nu(-k-1).$$

Finally, $\nu(-1) = 2/c_{\mathbf{q}} \neq 0$ so that $\nu \neq \delta_0$. Thus, Theorem 4.1.2 concludes the proof. \square

4.3.2 Proof of Theorem 4.1.2

Proof of Theorem 4.1.2. It is well known by [Don99] that, since h^\downarrow is ν -harmonic on $\mathbb{Z}_{\geq 1}$ and $h^\downarrow(0) = 1$, the function h^\downarrow corresponds to the pre-renewal function of the random walk S , i.e. for all $\ell \geq 0$,

$$h^\downarrow(\ell) = \sum_{p \geq 0} \mathbb{P}(H_p^< = \ell),$$

so that by the strong Markov property,

$$\sum_{\ell \geq 0} h^\downarrow(\ell) z^\ell = \sum_{\ell \geq 0} \sum_{p \geq 0} \sum_{\substack{\ell_1, \dots, \ell_p \geq 0 \\ \ell_1 + \dots + \ell_p = \ell}} \prod_{j=1}^p (\mathbb{P}(H_1^< = \ell_j) z^{\ell_j}) = \sum_{p \geq 0} G^<(z)^p = \frac{1}{1 - G^<(z)}.$$

Thus, in view of (4.1.4), one computes

$$G^<(z) = 1 - \sqrt{1 - z}. \quad (4.3.2)$$

Besides, let us define the power series with non-negative coefficients

$$g(z) = \sum_{k \geq 1} g_k z^k := \sum_{k \geq 1} (\nu(k-1) - \nu(-k-1)) z^k.$$

Step 1. Our aim is to express $\nu(-k)$ using the g_k 's. Note that for all $\theta \in \mathbb{R}$,

$$\varphi(\theta) = \sum_{k \leq -2} \nu(k) e^{ik\theta} + \nu(-1) e^{-i\theta} + \sum_{k \geq 0} \nu(k) e^{ik\theta} = e^{-i\theta} \nu(-1) + e^{-i\theta} g(e^{i\theta}) + e^{-i\theta} \sum_{k \geq 1} \nu(-k-1) (e^{ik\theta} + e^{-ik\theta}).$$

In particular, for all $\theta \in \mathbb{R}$,

$$e^{i\theta} \varphi(\theta) - e^{-i\theta} \varphi(-\theta) = g(e^{i\theta}) - g(e^{-i\theta}).$$

By taking (4.3.2) and Proposition 4.2.1 into account, we deduce that for all $\theta \in \mathbb{R}$,

$$g(e^{i\theta}) - g(e^{-i\theta}) = e^{i\theta} - e^{-i\theta} - e^{i\theta} \sqrt{1 - e^{-i\theta}} (1 - G^{\geq}(e^{i\theta})) + e^{-i\theta} \sqrt{1 - e^{i\theta}} (1 - G^{\geq}(e^{-i\theta})). \quad (4.3.3)$$

Let us define the power series

$$f(z) = \sum_{k \geq 1} f_k z^k := \frac{z}{\sqrt{1 - z}} (1 - G^{\geq}(z)). \quad (4.3.4)$$

We first compute the f_k 's and then express the $\nu(-k)$'s using the f_k 's. Note that $\theta \mapsto f(e^{i\theta}) \in L^1([0, 2\pi])$ and that (4.3.3) can be rewritten as

$$\frac{1}{\sqrt{1 - e^{i\theta}} \sqrt{1 - e^{-i\theta}}} (-g(e^{i\theta}) + g(e^{-i\theta}) + e^{i\theta} - e^{-i\theta}) = f(e^{i\theta}) - f(e^{-i\theta}),$$

hence, for all $k \geq 1$,

$$\begin{aligned} f_k &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{1 - e^{i\theta}} \sqrt{1 - e^{-i\theta}}} (-g(e^{i\theta}) + g(e^{-i\theta}) + e^{i\theta} - e^{-i\theta}) e^{-ik\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - e^{i\theta}|} \left(2i \sin \theta - \sum_{j \geq 1} g_j 2i \sin(j\theta) \right) e^{-ik\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sin(\theta/2)} \left(\sin \theta - \sum_{j \geq 1} g_j \sin(j\theta) \right) \sin(k\theta) d\theta. \end{aligned} \quad (4.3.5)$$

Next, for all integers $k, j \geq 1$,

$$\begin{aligned}
\int_0^{2\pi} \frac{\sin(k\theta)}{\sin(\theta/2)} \sin(j\theta) d\theta &= \int_0^{2\pi} \frac{e^{2ki\theta} - 1}{e^{i\theta} - 1} e^{i(1/2-k)\theta} \sin(j\theta) d\theta \\
&= \int_0^{2\pi} \sum_{m=0}^{2k-1} e^{i(m+1/2-k)\theta} \sin(j\theta) d\theta \\
&= 2 \sum_{m=0}^{k-1} \int_0^{2\pi} \cos((m+1/2)\theta) \sin(j\theta) d\theta \\
&= 2 \sum_{m=0}^{k-1} \int_0^{2\pi} \frac{1}{2} (\sin((m+j+1/2)\theta) - \sin((m-j+1/2)\theta)) d\theta \\
&= 2 \sum_{m=0}^{k-1} \left(\frac{1}{m+j+1/2} - \frac{1}{m-j+1/2} \right) \\
&= 2 \sum_{m=j}^{k+j-1} \frac{1}{m+1/2} - 2 \sum_{m=-j}^{k-j-1} \frac{1}{m+1/2} \\
&= 2 \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2}.
\end{aligned}$$

Therefore, (4.3.5) and the fact that $\sum_{j \geq 1} |g_j| \leq 1 < \infty$ yield, by Fubini-Tonelli, for all $k \geq 1$,

$$f_k = \frac{1}{\pi} \left(\frac{1}{k-1/2} + \frac{1}{k+1/2} \right) - \frac{1}{\pi} \sum_{j \geq 1} g_j \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2}. \quad (4.3.6)$$

Then, let us compute $\nu(-k)$ for $k \geq 1$ in terms of the f_k 's. By Proposition 4.2.1, (4.3.2) and (4.3.4), for all $\theta \in (0, 2\pi)$,

$$1 - \varphi(\theta) = \sqrt{1 - e^{-i\theta}} (1 - G^{\geq}(e^{i\theta})) = \frac{\sqrt{1 - e^{-i\theta}} \sqrt{1 - e^{i\theta}}}{e^{i\theta}} f(e^{i\theta}) = \frac{2|\sin(\theta/2)|}{e^{i\theta}} f(e^{i\theta}). \quad (4.3.7)$$

Consequently, assuming that $\theta \mapsto f(e^{i\theta}) \in L^2([0, 2\pi])$ for all $k \in \mathbb{Z}$,

$$\begin{aligned}
\nu(k) &= \mathbf{1}_{k=0} - \frac{1}{2\pi} \int_0^{2\pi} \frac{2\sin(\theta/2)}{e^{i\theta}} \sum_{\ell \geq 1} f_{\ell} e^{i\ell\theta} e^{-ik\theta} d\theta \\
&= \mathbf{1}_{k=0} - \frac{1}{\pi} \sum_{\ell \geq 1} \int_0^{2\pi} f_{\ell} \sin(\theta/2) \cos((\ell - k - 1)\theta) d\theta \\
&= \mathbf{1}_{k=0} - \frac{1}{2\pi} \sum_{\ell \geq 1} \int_0^{2\pi} f_{\ell} (\sin((\ell - k - 1/2)\theta) - \sin((\ell + k - 3/2)\theta)) d\theta \\
&= \mathbf{1}_{k=0} + \frac{1}{\pi} \sum_{\ell \geq 1} f_{\ell} \left(\frac{1}{\ell + k - 3/2} - \frac{1}{\ell - k - 1/2} \right),
\end{aligned}$$

where in the second equality we use the assumption that $\theta \mapsto f(e^{i\theta}) \in L^2([0, 2\pi])$ in order to exchange the series and the integral. Thus, for all $k \in \mathbb{Z}$,

$$\nu(k) = \mathbf{1}_{k=0} + \frac{1}{\pi} \sum_{\ell \geq 1} f_{\ell} \frac{4}{4(\ell - k - 1)^2 - 1}. \quad (4.3.8)$$

The above equality is also valid when we do not assume that $\theta \mapsto f(e^{i\theta}) \in L^2([0, 2\pi])$ by density of $L^2([0, 2\pi])$ in $L^1([0, 2\pi])$, since

$$\nu(k) = \mathbf{1}_{k=0} - \frac{1}{2\pi} \int_0^{2\pi} \frac{2|\sin(\theta/2)|}{e^{i\theta}} f(e^{i\theta}) e^{-ik\theta} d\theta \text{ and } f_{\ell} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-i\ell\theta} d\theta$$

In view of (4.3.6), we have completed the first step.

Step 2. Let us now prove that $\sum_{j \geq 1} jg_j \leq 1$. This will be useful in order to obtain the asymptotics of f_k as $k \rightarrow \infty$. Assume by contradiction that $\sum_{j \geq 1} jg_j \in (1, \infty]$. Then, by (4.3.6),

$$\forall \ell \geq 1, \quad f_\ell \leq \frac{2}{\pi} \left(\frac{1}{\ell - 1/2} - \sum_{j \geq 1} jg_j \frac{1}{j + \ell - 1/2} \mathbf{1}_{j \leq \ell} \right),$$

so that there exist $\varepsilon > 0$ and $\ell_0 \geq 1$ such that for all $\ell \geq \ell_0$, $f_\ell \leq -\varepsilon/\ell$. So

$$\sum_{\ell \geq \ell_0} f_\ell \frac{4}{4(\ell + k - 1)^2 - 1} \leq -\varepsilon \sum_{\ell \geq \ell_0} \frac{1}{\ell} \frac{4}{4(\ell + k - 1)^2 - 1} \leq -\varepsilon \int_{\ell_0}^{\infty} \frac{1}{x} \frac{4}{4(x + k - 1)^2 - 1} dx \sim -\varepsilon \frac{\log k}{k^2}$$

as $k \rightarrow \infty$, where the equivalent is obtained after taking the change of variable $y = x/k$. But the sum of the first ℓ_0 terms is a $O(1/k^2)$, so that by (4.3.8) for all k large enough, $\nu(-k) < 0$, absurd. As a result, $\sum_{j \geq 1} jg_j \leq 1$.

Step 3. Let us show that

$$\sum_{k \geq 1} \sum_{j \geq k/2} g_j \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2} < \infty \quad (4.3.9)$$

One can first see that for all $k \geq 1$,

$$\sum_{j \geq 2k} g_j \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2} = \sum_{j \geq 2k} g_j \sum_{m=j-k}^{k+j-1} \frac{1}{m+1/2} \leq \sum_{j \geq 2k} g_j 2k \frac{1}{k} \leq 2 \sum_{j \geq k} g_j$$

and $\sum_{k \geq 1} \sum_{j \geq k} g_j = \sum_{j \geq 1} jg_j < \infty$.

Moreover,

$$\begin{aligned} \sum_{k \geq 1} \sum_{j=k}^{2k} g_j \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2} &= \sum_{k \geq 1} \sum_{j=k}^{2k} g_j \sum_{m=j-k}^{k+j-1} \frac{1}{m+1/2} \\ &\leq C \sum_{k \geq 1} \sum_{j=k}^{2k} g_j \log \left(\frac{j+k+1/2}{j-k+1/2} \right) \\ &= C \sum_{j \geq 1} \sum_{j/2 \leq k \leq j} g_j \log \left(\frac{j+k+1/2}{j-k+1/2} \right) \\ &\leq C' \sum_{j \geq 1} g_j \int_{j/2}^j \log \left(\frac{j+x+1/2}{j-x+1/2} \right) dx, \end{aligned}$$

where C, C' are positive constants. The sum in the last line is finite since for all $j \geq 1$,

$$\begin{aligned} \int_{j/2}^j \log \left(\frac{j+x+1/2}{j-x+1/2} \right) dx &= \left(2j + \frac{1}{2} \right) \log \left(2j + \frac{1}{2} \right) - \left(2j + \frac{1}{2} \right) - \left(\frac{3}{2}j + \frac{1}{2} \right) \log \left(\frac{3}{2}j + \frac{1}{2} \right) \\ &\quad + \left(\frac{3}{2}j + \frac{1}{2} \right) - \left(\frac{j}{2} + \frac{1}{2} \right) \log \left(\frac{j}{2} + \frac{1}{2} \right) + \left(\frac{j}{2} + \frac{1}{2} \right) + \frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \\ &\leq C'' j, \end{aligned}$$

for some constant $C'' > 0$ and since $\sum_{j \geq 1} jg_j < \infty$.

Similarly, for some constants $C, C' > 0$,

$$\begin{aligned} \sum_{k \geq 1} \sum_{k/2 \leq j \leq k} g_j \sum_{m=k-j}^{k+j-1} \frac{1}{m+1/2} &\leq C \sum_{k \geq 1} \sum_{k/2 \leq j \leq k} g_j \log \left(\frac{j+k+1/2}{k-j+1/2} \right) \\ &= C \sum_{j \geq 1} \sum_{j \leq k \leq 2j} g_j \log \left(\frac{j+k+1/2}{k-j+1/2} \right) \\ &\leq C' \sum_{j \geq 1} g_j \int_j^{2j} \log \left(\frac{j+x+1/2}{x-j+1/2} \right) dx < \infty. \end{aligned}$$

This ends the proof of (4.3.9).

Step 4. Next, we show that

$$\sum_{\ell \geq 1} \left| f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right| < \infty. \quad (4.3.10)$$

Note that by (4.3.6),

$$\begin{aligned} \left| f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right| &\leq \frac{1}{\pi} \left| \frac{1}{\ell - 1/2} + \frac{1}{\ell + 1/2} - \frac{2}{\ell} \right| + \frac{1}{\pi} \sum_{1 \leq j \leq \ell/2} g_j \left| \frac{2j}{\ell} - \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2} \right| \\ &\quad + \frac{1}{\pi} \sum_{j \geq \ell/2} g_j \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2}. \end{aligned}$$

The first term is clearly summable. The last one is summable by (4.3.9). Finally, for the second term, we upperbound

$$\sum_{\ell \geq 1} \sum_{1 \leq j \leq \ell/2} g_j \left| \frac{2j}{\ell} - \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2} \right| = \sum_{\ell \geq 1} \sum_{1 \leq j \leq \ell/2} g_j \left| \sum_{m=\ell-j}^{\ell+j-1} \frac{m+1/2-\ell}{\ell(m+1/2)} \right| \leq C'' \sum_{\ell \geq 1} \sum_{1 \leq j \leq \ell/2} g_j \frac{j^2}{\ell^2},$$

where $C'' > 0$ is a constant, and

$$\sum_{\ell \geq 1} \sum_{1 \leq j \leq \ell/2} g_j \frac{j^2}{\ell^2} = \sum_{j \geq 1} g_j j^2 \sum_{\ell \geq 2j} \frac{1}{\ell^2} \leq \sum_{j \geq 1} g_j j^2 \frac{1}{j} < \infty.$$

This proves (4.3.10).

Step 5. Let us now deal with the case $\sum_{j \geq 1} jg_j \in [0, 1)$ so as to prove the first point of Theorem 4.1.2. By (4.3.8), for all $k \geq 1$,

$$\nu(-k) = \frac{1}{\pi} \sum_{\ell \geq 1} \left(f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right) \frac{4}{4(\ell+k-1)^2-1} + \sum_{\ell \geq 1} \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi^2} \frac{1}{\ell} \frac{4}{4(\ell+k-1)^2-1}.$$

The first term is a $O(1/k^2)$ as $k \rightarrow \infty$ by (4.3.10). For the second term, for all $\ell_0 \geq 1$,

$$\int_{\ell_0}^{\infty} \frac{1}{x} \frac{4}{4(x+k-1)^2-1} dx \leq \sum_{\ell \geq \ell_0} \frac{1}{\ell} \frac{4}{4(\ell+k-1)^2-1} \leq \frac{4}{4k^2-1} + \int_{\ell_0}^{\infty} \frac{1}{x} \frac{4}{4(x+k-1)^2-1} dx.$$

Moreover, after the change of variable $y = x/k$, the integral can be bounded by

$$\frac{1}{k^2} \int_{\ell_0/k}^{\infty} \frac{1}{y} \frac{4}{4(y+1)^2} dy \leq \frac{1}{k^2} \int_{\ell_0/k}^{\infty} \frac{1}{y} \frac{4}{4(y+1-1/k)^2-1/k^2} dy \leq \frac{1}{(1-1/k)^3} \frac{1}{k^2} \int_{\ell_0/k}^{\infty} \frac{1}{y} \frac{4}{4(y+1)^2} dy.$$

Hence, since ℓ_0 can be chosen arbitrarily large so that $\sum_{j=1}^{\ell_0/2} jg_j$ is close to $\sum_{j \geq 1} jg_j$,

$$\nu(-k) \underset{k \rightarrow \infty}{\sim} \frac{2(1 - \sum_{j \geq 1} jg_j) \log k}{\pi^2 k^2}. \quad (4.3.11)$$

Step 6. We finally deal with the case $\sum_{j \geq 1} jg_j = 1$ to prove the second point of Theorem 4.1.2. We first assume that

$$\sum_{\ell \geq 1} |f_\ell| = \infty,$$

which is equivalent by (4.3.10) to the fact that

$$\sum_{\ell \geq 1} \frac{1 - \sum_{j=1}^{\ell/2} jg_j}{\ell} = \infty. \quad (4.3.12)$$

We define the function L by setting

$$\forall x \geq 0, \quad L(x) = \sum_{\ell \geq 1} f_\ell \frac{4x^2}{4(\ell + x - 1)^2 - 1}.$$

Let us show that L is a slowly varying function in the sense that for all $\lambda > 0$, the ratio $L(\lambda x)/L(x)$ converges to 1 as $x \rightarrow \infty$. Let us also introduce the function \tilde{L} defined for all $x \geq 0$ by

$$\tilde{L}(x) = \sum_{\ell \geq 1} \frac{1 - \sum_{j=1}^{\ell/2} j g_j}{\ell} \frac{4x^2}{4(\ell + x - 1)^2 - 1}.$$

By (4.3.10) and since for all $x \geq 2, \ell \geq 1$, we have $4x^2/(4(\ell + x - 1)^2 - 1) \leq 2$,

$$L(x) = \tilde{L}(x) + O(1) \tag{4.3.13}$$

as $x \rightarrow \infty$.

Let $\lambda > 0$. One upperbounds for all x large enough,

$$\begin{aligned} & |\tilde{L}(\lambda x) - \tilde{L}(x)| \\ & \leq \sum_{\ell \geq 1} \frac{1 - \sum_{j=1}^{\ell/2} j g_j}{\ell} \left| \frac{4\lambda^2 x^2 (4(\ell + x - 1)^2 - 1) - 4x^2 (4(\ell + \lambda x - 1)^2 - 1)}{(4(\ell + \lambda x - 1)^2 - 1)(4(\ell + x - 1)^2 - 1)} \right| \\ & = \sum_{\ell \geq 1} \frac{1 - \sum_{j=1}^{\ell/2} j g_j}{\ell} \left| \frac{(4\lambda x(\ell + x - 1) - 4x(\ell + \lambda x - 1))(4\lambda x(\ell + x - 1) + 4x(\ell + \lambda x - 1)) - 4(\lambda^2 - 1)x^2}{(4(\ell + \lambda x - 1)^2 - 1)(4(\ell + x - 1)^2 - 1)} \right| \\ & = \sum_{\ell \geq 1} \frac{1 - \sum_{j=1}^{\ell/2} j g_j}{\ell} \left| \frac{4(\lambda - 1)(\ell - 1)x(4\lambda x(\ell + x - 1) + 4x(\ell + \lambda x - 1)) - 4(\lambda^2 - 1)x^2}{(4(\ell + \lambda x - 1)^2 - 1)(4(\ell + x - 1)^2 - 1)} \right| \\ & \leq C(\lambda) \sum_{\ell \geq 1} \frac{1}{\ell} \frac{x^2 \ell (\ell + x)}{(\ell + x)^4} \\ & \leq C(\lambda) \int_0^\infty \frac{x^2}{(x + y)^3} dy \\ & = C(\lambda) \int_0^\infty \frac{1}{(1 + z)^3} dz, \end{aligned}$$

where $C(\lambda)$ is a positive constant depending on λ . In particular,

$$\tilde{L}(\lambda x) - \tilde{L}(x) = O(1)$$

as $x \rightarrow \infty$. But by (4.3.12), we know that $\tilde{L}(x) \rightarrow \infty$ as $x \rightarrow \infty$. As a consequence, \tilde{L} is slowly varying. By (4.3.13), L is therefore slowly varying and $L(x) \rightarrow \infty$ as $x \rightarrow \infty$. Besides, since $\sum_{j \geq 1} j g_j = 1$, one can see that $\tilde{L}(x) = o(\log x)$ as $x \rightarrow \infty$ by a comparison between series and integrals, hence $L(x) = o(\log(x))$ as $x \rightarrow \infty$ by (4.3.13). This ends the proof of the second point in the case where $\sum_{\ell \geq 1} |f_\ell| = \infty$.

Step 7. In order to conclude, it remains to deal with the case where $\sum_{j \geq 1} j g_j = \bar{1}$ and

$$\sum_{\ell \geq 1} |f_\ell| < \infty \tag{4.3.14}$$

and to show that $\sum_{\ell \geq 1} f_\ell \neq 0$. Indeed, assuming (4.3.14), by dominated convergence and (4.3.8), we get

$$k^2 \nu(-k) \xrightarrow[k \rightarrow \infty]{} \frac{1}{\pi} \sum_{\ell \geq 1} f_\ell.$$

By (4.3.10), (4.3.14) and the fact that $\sum_{j \geq 1} j g_j = 1$, we have $\sum_{\ell \geq 1} (1/\ell) \sum_{j > \ell/2} j g_j < \infty$, hence by Fubini-Tonelli,

$$\sum_{j \geq 1} g_j j \log j < \infty. \tag{4.3.15}$$

Next, by (4.3.6) and since $\sum_{j \geq 1} j g_j = 1$, for all $n \geq 1$,

$$\sum_{\ell=1}^n f_\ell = \frac{1}{\pi} \sum_{j \geq 1} g_j \sum_{\ell=1}^n \left(\frac{j}{\ell-1/2} + \frac{j}{\ell+1/2} - \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2} \right),$$

and for all $j \geq 1$,

$$\begin{aligned} & \sum_{\ell=1}^n \left(\frac{j}{\ell-1/2} + \frac{j}{\ell+1/2} - \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2} \right) \\ &= \sum_{\ell=1}^n \sum_{m=0}^{j-1} \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} - \frac{1}{\ell+m+1/2} - \frac{1}{\ell-m-1/2} \right) \\ &= \sum_{m=0}^{j-1} \left(\sum_{\ell=1}^n \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} \right) - \sum_{\ell=1+m}^{n+m} \frac{1}{\ell+1/2} - \sum_{\ell=1-m}^{n-m} \frac{1}{\ell-1/2} \right) \\ &= \sum_{m=0}^{j-1} \left(\sum_{\ell=1}^m \frac{1}{\ell+1/2} - \sum_{\ell=n+1}^{n+m} \frac{1}{\ell+1/2} + \sum_{\ell=n-m+1}^n \frac{1}{\ell-1/2} - \sum_{\ell=1-m}^0 \frac{1}{\ell-1/2} \right) \\ &= \sum_{m=0}^{j-1} \left(\sum_{\ell=1}^m \left(\frac{1}{\ell+1/2} + \frac{1}{\ell-1/2} \right) + \sum_{\ell=1}^m \left(\frac{1}{n-m+\ell-1/2} - \frac{1}{n+\ell+1/2} \right) \right) \\ &= \sum_{m=0}^{j-1} \sum_{\ell=1}^m \left(\frac{1}{\ell+1/2} + \frac{1}{\ell-1/2} \right) + \sum_{m=0}^{j-1} \sum_{\ell=1}^m \frac{m+1}{(n+\ell+1/2)(n-m+\ell-1/2)}. \end{aligned}$$

Note that for all $n \geq 1$, $m \geq 0$ and $\ell \in \llbracket 1, m \rrbracket$,

$$\left| \frac{m+1}{(n+\ell+1/2)(n-m+\ell-1/2)} \right| \leq \begin{cases} 2/|n-m+\ell-1/2| & \text{if } n \geq (m+1)/2 \\ 4/(n+\ell+1/2) & \text{if } n \leq m/2 \text{ and } \ell \leq (m+1)/4 \\ 4/|n-m+\ell-1/2| & \text{if } n \leq m/2 \text{ and } \ell \geq (m+1)/4. \end{cases}$$

As a result, there exists $C > 0$ such that for all $j \geq 1$,

$$\left| \sum_{m=0}^{j-1} \sum_{\ell=1}^m \frac{m+1}{(n+\ell+1/2)(n-m+\ell-1/2)} \right| \leq C \sum_{m=1}^j \log m \leq C(j+1) \log(j+1).$$

Thus, in view of (4.3.15), by dominated convergence,

$$\sum_{\ell=1}^n f_\ell \xrightarrow{n \rightarrow \infty} \frac{1}{\pi} \sum_{j \geq 1} g_j \sum_{m=0}^{j-1} \sum_{\ell=0}^m \left(\frac{1}{\ell+1/2} + \frac{1}{\ell-1/2} \right),$$

and the limit is non-zero since $\sum_{j \geq 1} j g_j = 1$ so that there exists $j \geq 1$ such that $g_j > 0$ and we cannot have $g_1 = 1$ since $\nu \neq \delta_0$, hence there exists $j \geq 2$ such that $g_j > 0$. This ends the proof. \square

4.4 Weight sequences

In this section, we show how the computations in the proof of Theorem 4.1.2 enable to characterise weight sequences of critical $O(2)$ loop-decorated maps and we give two examples.

4.4.1 Characterisation of weight sequences of critical $O(2)$ decorated maps

Note that in the proof of Theorem 4.1.2 we computed $\nu(k)$ for all $k \in \mathbb{Z}$ in terms of the $g_j = \nu(j-1) - \nu(-j-1)$ for $j \geq 1$. Indeed, combining (4.3.6) and (4.3.8), one gets for all $k \in \mathbb{Z}$,

$$\nu(k) = \mathbf{1}_{k=0} + \frac{1}{\pi} \sum_{\ell \geq 1} \left(\frac{1}{\pi} \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} - \sum_{j \geq 1} g_j \sum_{m=\ell-j}^{\ell+j-1} \frac{1}{m+1/2} \right) \right) \frac{4}{4(\ell-k-1)^2-1}. \quad (4.4.1)$$

In order to obtain examples of weight sequences of critical $O(2)$ -decorated planar maps, an approach can be to choose a sequence $(g_k)_{k \geq 1}$ of non-negative real numbers such that $\sum_{j \geq 1} jg_j \leq 1$ and to see whether ν defined by (4.4.1) is indeed a probability measure on \mathbb{Z} such that h^\downarrow is ν -harmonic and $\nu \neq \delta_0$. The following proposition actually states that it suffices to check that $\nu(k) \geq 0$ for all $k \in \mathbb{Z}$.

Proposition 4.4.1. *Let $(g_j)_{j \geq 1}$ be a sequence of non-negative real numbers such that $\sum_{j \geq 1} jg_j \leq 1$. Let ν be defined by (4.4.1). Assume that $\nu(0) < 1$ and that for all $k \in \mathbb{Z}$, we have $\nu(k) \geq 0$. Then ν is a probability distribution and h^\downarrow is ν -harmonic on $\mathbb{Z}_{\geq 1}$. Moreover, for all $k \geq 1$,*

$$\nu(k-1) - \nu(-k-1) = g_k \geq 0.$$

In particular, the sequence $\mathbf{q} = (q_k)_{k \geq 1}$ defined by setting for all $k \geq 1$,

$$q_k = \nu(k-1)(\nu(-1)/2)^{k-1}$$

is the weight sequence of the gasket of a critical $O(2)$ loop-decorated map. The associated triple $(\tilde{\mathbf{q}}, 1/c_{\mathbf{q}}, 2)$ is given by $c_{\mathbf{q}} = 2/\nu(-1)$ and for all $k \geq 1$,

$$\tilde{q}_k = g_k(\nu(-1)/2)^{k-1}.$$

Proof. As in the previous section, let f_ℓ be the expression in the big parenthesis in (4.4.1). Note that the steps 3 and 4 of the proof of Theorem 4.1.2 only use the fact that $g_j \geq 0$ for all $j \geq 1$ and that $\sum_{j \geq 1} jg_j < \infty$ so that (4.3.10) holds. As a result, after writing

$$\nu(k) = \mathbf{1}_{k=0} + \frac{1}{\pi} \sum_{\ell \geq 1} \left(f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right) \frac{4}{4(\ell - k - 1)^2 - 1} + \sum_{\ell \geq 1} \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi^2} \frac{1}{\ell} \frac{4}{4(\ell - k - 1)^2 - 1},$$

one can apply Fubini-Tonelli theorem to obtain that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \nu(k) &= 1 + \frac{1}{\pi} \sum_{\ell \geq 1} \left(f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right) \sum_{k \in \mathbb{Z}} \frac{4}{4(\ell - k - 1)^2 - 1} \\ &\quad + \sum_{\ell \geq 1} \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi^2} \frac{1}{\ell} \sum_{k \in \mathbb{Z}} \frac{4}{4(\ell - k - 1)^2 - 1}. \end{aligned}$$

Moreover, after identifying a telescopic sum,

$$\sum_{k \in \mathbb{Z}} \frac{4}{4(\ell - k - 1)^2 - 1} = 2 \sum_{k \geq 1} \frac{4}{4k^2 - 1} - 4 = 2 \times 2 - 4 = 0.$$

This proves that ν is a probability measure.

Next, let us check that the function h^\downarrow defined in (4.1.4) is ν -harmonic on $\mathbb{Z}_{\geq 1}$. Let $p \geq 0$. Since h^\downarrow is non-negative and is bounded, we may apply again Fubini-Tonelli and obtain that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} h^\downarrow(p+k) \nu(k) &= h^\downarrow(p) + \frac{1}{\pi} \sum_{\ell \geq 1} \left(f_\ell - \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi \ell} \right) \sum_{k \in \mathbb{Z}} h^\downarrow(p+k) \frac{4}{4(\ell - k - 1)^2 - 1} \\ &\quad + \sum_{\ell \geq 1} \frac{2(1 - \sum_{j=1}^{\ell/2} jg_j)}{\pi^2} \frac{1}{\ell} \sum_{k \in \mathbb{Z}} h^\downarrow(p+k) \frac{4}{4(\ell - k - 1)^2 - 1}. \end{aligned}$$

So, in order to prove that h^\downarrow is ν -harmonic, it suffices to check that for all $p \geq 0$,

$$\sum_{k \in \mathbb{Z}} h^\downarrow(p+k) \frac{4}{4(\ell - k - 1)^2 - 1} = 0. \quad (4.4.2)$$

Let ν_{sym} be the probability distribution defined by

$$\forall k \in \mathbb{Z}, \quad \nu_{\text{sym}}(k) = \mathbf{1}_{k=0} + \frac{2}{\pi} \frac{1}{4k^2 - 1}.$$

One can check that it is indeed a probability distribution using a telescopic sum. This probability distribution appears in a particular weight sequence of $O(2)$ loop-decorated maps introduced in Remark 1 of [Bud18]. Its characteristic function is given for all $\theta \in \mathbb{R}$ by

$$\varphi_{\text{sym}}(\theta) = 1 - |\sin(\theta/2)|. \quad (4.4.3)$$

Indeed, one readily computes for all $k \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(\theta/2) e^{-ik\theta} d\theta = \frac{1}{4\pi} \int_0^{2\pi} (\sin((k+1/2)\theta) - \sin((k-1/2)\theta)) d\theta = \frac{1}{4\pi} \left(\frac{2}{k+1/2} - \frac{2}{k-1/2} \right).$$

We deduce from (4.4.3) the Wiener–Hopf factorisation $1 - \varphi_{\text{sym}}(\theta) = (1/2)\sqrt{1 - e^{-i\theta}}\sqrt{1 - e^{i\theta}}$, with the generating function of the strict descending ladder height $G_{\text{sym}}^<(z) = 1 - \sqrt{1 - z}$ (the uniqueness of the Wiener–Hopf factorisation was recently proven in Theorem 3 of [DSTW24]). Hence, by doing the same computations as in the beginning of the proof of Theorem 4.1.2, the function h^\downarrow is ν_{sym} -harmonic on $\mathbb{Z}_{\geq 1}$. In other words, for all $p \geq 0$,

$$\sum_{k \in \mathbb{Z}} h^\downarrow(p+k) \frac{4}{4k^2 - 1} = 0,$$

hence (4.4.2).

Then, let us prove that $\nu(k-1) - \nu(-k-1) = g_k$ for all $k \geq 1$. Let $g(z) := \sum_{k \geq 1} g_k z^k$. By (4.3.10), we deduce that the power series $f(z) := \sum_{k \geq 1} f_k z^k$ has radius one and that $\theta \mapsto f(e^{i\theta}) \in L^2([0, 2\pi])$. By doing the computations leading to (4.3.6) and (4.3.5) in reverse order, one gets that for all $k \geq 1$,

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{1 - e^{i\theta}}\sqrt{1 - e^{-i\theta}}} (-g(e^{i\theta}) + g(e^{-i\theta}) + e^{i\theta} - e^{-i\theta}) e^{-ik\theta} d\theta.$$

Moreover, for all $\theta \in \mathbb{R}$,

$$|g(e^{i\theta}) - g(e^{-i\theta})| = \left| \sum_{k \geq 1} g_k 2i \sin(k\theta) \right| \leq 2|\theta| \sum_{k \geq 1} k g_k,$$

so that the function $\theta \mapsto (1/(\sqrt{1 - e^{i\theta}}\sqrt{1 - e^{-i\theta}}))(-g(e^{i\theta}) + g(e^{-i\theta}) + e^{i\theta} - e^{-i\theta})$ is bounded.

As a consequence, we obtain the equality for Lebesgue almost all $\theta \in [0, 2\pi]$,

$$\frac{1}{\sqrt{1 - e^{i\theta}}\sqrt{1 - e^{-i\theta}}} (-g(e^{i\theta}) + g(e^{-i\theta}) + e^{i\theta} - e^{-i\theta}) = f(e^{i\theta}) - f(e^{-i\theta}).$$

Besides, let φ be the characteristic function of ν . Since $\theta \mapsto f(e^{i\theta}) \in L^2([0, 2\pi])$, we can also do the computation leading to (4.3.8) in reverse order and get (4.3.7) for Lebesgue almost every $\theta \in [0, 2\pi]$. Combining (4.3.7) with the above equality yields

$$e^{i\theta} \varphi(\theta) - e^{-i\theta} \varphi(-\theta) = g(e^{i\theta}) - g(e^{-i\theta})$$

for Lebesgue almost every $\theta \in [0, 2\pi]$. This proves that $\nu(k-1) - \nu(-k-1) = g_k$ for all $k \geq 1$. The next point of the proposition comes from exactly the same ideas as in the proof of Theorem 4.1.1. The last points come from the gasket decomposition. \square

4.4.2 Examples

Let us give two examples of models of critical $O(2)$ loop-decorated planar maps. We start with an example from Timothy Budd and then give a new example which only has faces of degree 4 which are traversed with loops.

The symmetric example of Budd

A key ingredient of the proof of Proposition 4.4.1 is the symmetric probability distribution ν_{sym} given by

$$\forall k \in \mathbb{Z}, \quad \nu_{\text{sym}}(k) = \mathbf{1}_{k=0} + \frac{2}{\pi} \frac{1}{4k^2 - 1}.$$

Here the slowly varying function of Theorem 4.1.2 is a constant. The associated weight sequence \mathbf{q} for the gasket is given by

$$\forall k \geq 1, \quad q_k = \left(\mathbf{1}_{k=1} + \frac{2}{\pi} \frac{1}{4(k-1)^2 - 1} \right) (3\pi)^{-k+1}$$

and $c_{\mathbf{q}} = 3\pi$, while the associated triple $(\tilde{\mathbf{q}}, 1/c_{\mathbf{q}}, 2)$ is given by Remark 1 of [Bud18]: using (4.1.1) and (4.3.1), one computes for all $k \geq 1$,

$$\tilde{q}_k = c_{\mathbf{q}}^{-k+1} (\nu(k-1) - \nu(-k-1)) = (3\pi)^{1-k} \frac{2k}{\pi(k-3/2)(k-1/2)(k+1/2)(k+3/2)} + \mathbf{1}_{k=1}.$$

From the above expression one can see that there are faces of arbitrary (even) degrees which are not traversed by loops. The partition function is given for all $\ell \geq 1$ by

$$W^{(\ell)} = \frac{c_{\mathbf{q}}^{\ell+1}}{2} \nu(-\ell-1) = \frac{(3\pi)^{\ell+1}}{\pi} \frac{1}{4(\ell+1)^2 - 1}.$$

Fully packed critical $O(2)$ -loop decorated quadrangulations

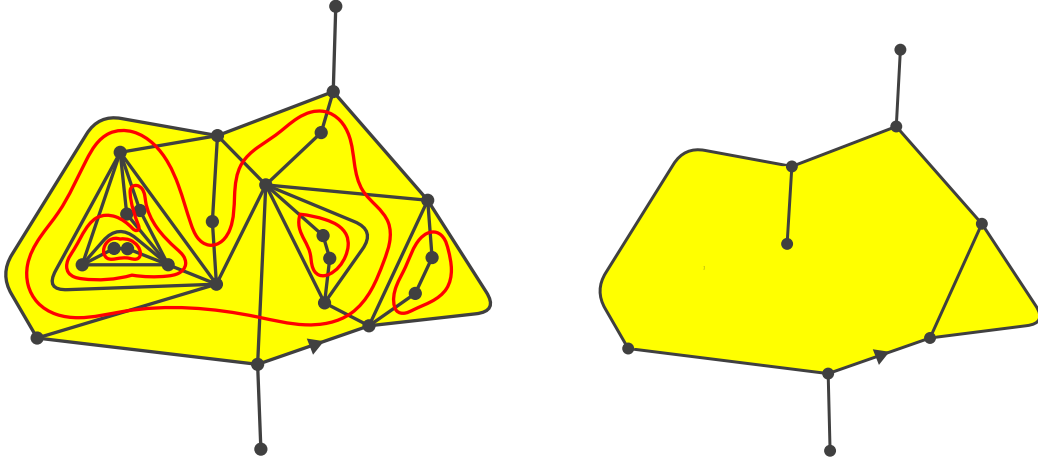


Figure 4.2 – Left: a fully packed loop-decorated quadrangulation. Right: its gasket.

A new simple example of weight sequence is obtained by taking $g_j = 0$ for all $j \geq 1$. By (4.4.1), for all $k \in \mathbb{Z}$,

$$\nu(k) = \mathbf{1}_{k=0} + \frac{1}{\pi^2} \sum_{\ell \geq 1} \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} \right) \frac{4}{4(\ell-k-1)^2 - 1}.$$

It clearly satisfies $\nu(k) \geq 0$ for all $k \leq -1$ and one computes for all $k \geq 1$,

$$\nu(k-1) - \nu(-k-1) = \mathbf{1}_{k=1} + \frac{1}{\pi^2} \sum_{\ell \geq 1} \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} \right) \left(\frac{4}{4(\ell-k)^2 - 1} - \frac{4}{4(\ell+k)^2 - 1} \right) = 0.$$

In particular, $\nu(k) \geq 0$ for all $k \in \mathbb{Z}$. Moreover, one computes $\nu(-1) = 4/\pi^2$. Thus, by Proposition 4.4.1, the weight sequence $\mathbf{q} = (q_k)_{k \geq 1}$ defined by

$$\forall k \geq 1, \quad q_k = \left(\mathbf{1}_{k=1} + \frac{1}{\pi^2} \sum_{\ell \geq 1} \left(\frac{1}{\ell-1/2} + \frac{1}{\ell+1/2} \right) \frac{4}{4(\ell-k)^2 - 1} \right) (\pi^2/2)^{1-k}$$

is the weight sequence of the gasket of a critical $O(2)$ -decorated planar map and $c_{\mathbf{q}} = 2/\nu(-1) = \pi^2/2$, while the associated triple is $(\tilde{\mathbf{q}}, 2/\pi^2, 2)$, where $\tilde{\mathbf{q}}$ is the constant sequence equal to zero. Here, the $O(2)$ -decorated map only has quadrangles and every quadrangle is traversed by a loop. See Figure 4.2. The partition function is given, for all $\ell \geq 1$, by

$$W^{(\ell)} = \frac{c_{\mathbf{q}}^{\ell+1}}{2} \nu(-\ell - 1) = \frac{(\pi^2/2)^{\ell+1}}{2\pi^2} \sum_{k \geq 1} \left(\frac{1}{k - 1/2} + \frac{1}{k + 1/2} \right) \frac{4}{4(k + \ell)^2 - 1}.$$

In this example, the slowly varying function $L_{\mathbf{q}}(\ell)$ is equivalent to $(2/\pi^2) \log(\ell)$ as $\ell \rightarrow \infty$.

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Chapter 5

Scaling limit of first passage percolation geodesics in planar maps

Ce chapitre est issu de l'article [Kam24b], soumis.

Abstract We establish the scaling limit of the geodesics to the root for the first passage percolation distance on random planar maps. We first describe the scaling limit of the number of faces along the geodesics. This result enables to compare the metric balls for the first passage percolation and the dual graph distance. It also enables to upperbound the diameter of large random maps. Then, we describe the scaling limit of the tree of first passage percolation geodesics to the root via a stochastic coalescing flow of pure jump diffusions. This stochastic flow also enables us to construct some random metric spaces which we conjecture to be the scaling limit of random planar maps with high degrees. The main tool in this work is a time-reversal of the uniform peeling exploration.

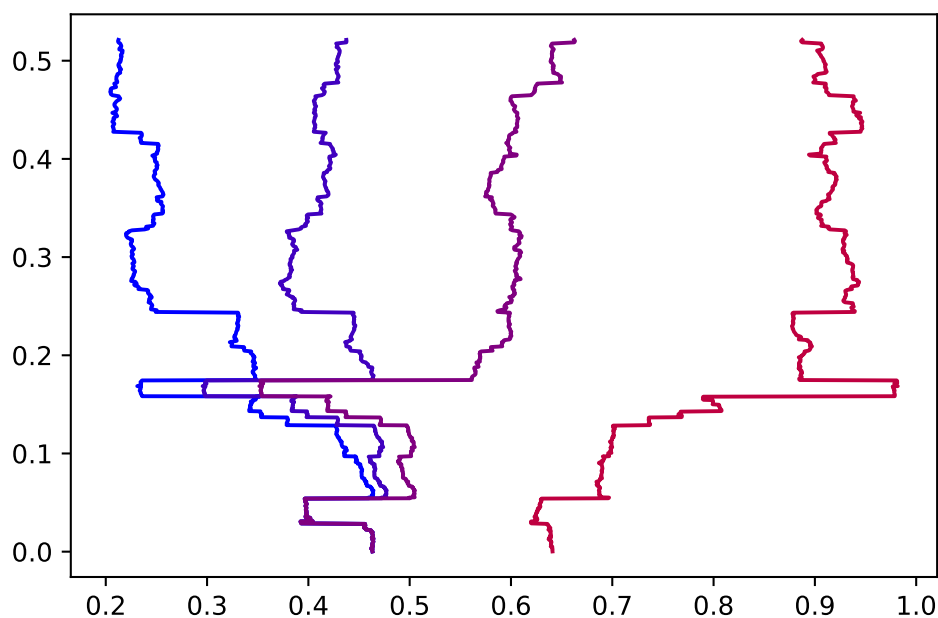


Figure 5.1 – Simulation of four trajectories of the stochastic coalescing flow of diffusions with jumps which arises as the scaling limit of first-passage percolation geodesics.

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5.1 Introduction

5.1.1 Random Boltzmann maps

We study the first-passage percolation distance on random Boltzmann planar maps of type $a \in (3/2, 5/2]$ which correspond to the dual of the stable maps of [LGM11] when $a \in (3/2, 5/2)$ and to random planar maps in the universality class of the Brownian sphere when $a = 5/2$. We give applications to the (dual) graph distance and establish the scaling limit of geodesics to the root.

A planar map is a connected planar graph which is embedded in the sphere and seen up to orientation-preserving transformations. Our planar maps are equipped with a distinguished oriented edge which is called the root edge. The face f_r which is on the right of the root edge is called the root face. We restrict our attention to bipartite planar maps, i.e. such that each face has an even degree. We denote by \mathcal{M} the set of finite bipartite planar maps and for all $\ell \geq 1$ we denote by $\mathcal{M}^{(\ell)}$ the set of bipartite planar maps whose root face has degree 2ℓ . See Figure 5.2 for an example.

A general way to pick a map at random consists in assigning positive weights to each face according to its degree. Following [MM07], let $\mathbf{q} = (q_k)_{k \geq 1}$ be a non-zero sequence of non-negative real numbers. The weight of a map $\mathbf{m} \in \mathcal{M}$ is defined by

$$w_{\mathbf{q}}(\mathbf{m}) = \prod_{f \in \text{Faces}(\mathbf{m}) \setminus \{f_r\}} q_{\deg(f)/2}.$$

We then define the partition function for all $\ell \geq 1$

$$W^{(\ell)} = \sum_{\mathbf{m} \in \mathcal{M}^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}).$$

When the partition function is finite, the sequence \mathbf{q} is called admissible. We require \mathbf{q} to be critical of type $a \in (3/2, 5/2]$, which is characterized by the fact that

$$W^{(\ell)} \underset{\ell \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{\ell+1} \ell^{-a}, \quad (5.1.1)$$

for some constants $p_{\mathbf{q}}, c_{\mathbf{q}} > 0$. See Proposition 5.10 of [Cur23]. The case $a = 5/2$ is called generic while the other values of a are called non-generic. Examples of such sequences \mathbf{q} are exhibited in Lemma 6.1 of [BC17]. We define the Boltzmann probability measure $\mathbb{P}^{(\ell)}$ on $\mathcal{M}^{(\ell)}$ by setting

$$\forall \mathbf{m} \in \mathcal{M}^{(\ell)}, \quad \mathbb{P}^{(\ell)}(\{\mathbf{m}\}) = \frac{w_{\mathbf{q}}(\mathbf{m})}{W^{(\ell)}}.$$

A random map of law $\mathbb{P}^{(\ell)}$ is called a random Boltzmann map of perimeter 2ℓ . If one conditions a random map of law $\mathbb{P}^{(\ell)}$ on having n vertices and let $n \rightarrow \infty$, then the law of the local limit given by Theorem 7.1 of [Cur23] is denoted by $\mathbb{P}_{\infty}^{(\ell)}$. A random map of law $\mathbb{P}_{\infty}^{(\ell)}$ is called an infinite random Boltzmann map of perimeter 2ℓ .

The object of our study is the dual map \mathbf{m}^{\dagger} associated with a map \mathbf{m} under $\mathbb{P}^{(\ell)}$ and $\mathbb{P}_{\infty}^{(\ell)}$, which is obtained by switching the roles of the faces and vertices and saying that two vertices of \mathbf{m}^{\dagger} are connected by an edge if the corresponding faces of \mathbf{m} are adjacent.

We focus on the first passage percolation (fpp) distance d_{fpp}^{\dagger} on the dual map \mathbf{m}^{\dagger} under $\mathbb{P}^{(\ell)}$ and $\mathbb{P}_{\infty}^{(\ell)}$ which is obtained by assigning i.i.d. exponential random edge lengths of parameter 1 to the dual edges. For every face f, f' of \mathbf{m} , the shortest path for the fpp distance from f to f' is unique since the exponential distribution has no atom. It does not necessarily correspond to a shortest path for the dual graph distance (see e.g. Figure 5.2). The main focus of this work is the study of the shortest paths to the root face f_r and their scaling limits. These shortest paths actually form a tree which is denoted by $\mathcal{T}(\mathbf{m})$. This study also has applications to the dual graph distance d_{gr}^{\dagger} , which corresponds to the graph distance on \mathbf{m}^{\dagger} , where each edge has length one.

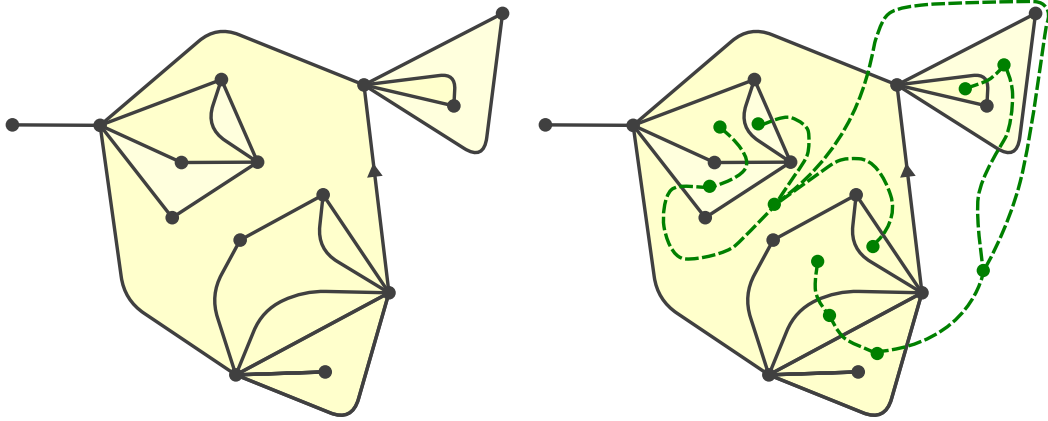


Figure 5.2 – Left: a planar map $\mathbf{m} \in \mathcal{M}^{(4)}$. The root edge is marked with an arrow and the root face corresponds to the outer face. Right: illustration for the tree $\mathcal{T}(\mathbf{m})$ in dashed green lines of fpp geodesics to the root face f_r . Note that $\mathcal{T}(\mathbf{m})$ is not a deterministic function of \mathbf{m} but depends on the random exponential edge lengths.

The geometry of infinite random maps under $\mathbb{P}_{\infty}^{(\ell)}$ was first studied in the case of the uniform infinite planar quadrangulation (which is a particular case of the case $a = 5/2$) in [CLG17] and [CC18], then in [BC17] in the cases $a \in (3/2, 5/2) \setminus \{2\}$ and in [BCM18] in the critical case $a = 2$. The scaling limit of the distances to the root under $\mathbb{P}^{(\ell)}$ or $\mathbb{P}_{\infty}^{(\ell)}$ as $\ell \rightarrow \infty$ was established using the framework of growth-fragmentations in [BCK18] in the case of triangulations (corresponding informally to the case $a = 5/2$), then in [BBCK18] in the case $a \in (2, 5/2)$ and in Chapter 3 in the case $a = 2$.

An important tool in the study of distances on random planar maps is the peeling exploration. See Subsection 5.2.1 for more details, as well as the lecture notes [Cur23] or [Bud17] for a detailed presentation

of this technique. It consists in a step by step exploration of the map \mathbf{m} starting from the root face, where at each step we discover what is behind an edge on the boundary of the unexplored region. This edge is called the peeled edge. The explored region at step $n \geq 0$ is denoted by $\bar{\epsilon}_n$. We denote by $(P(n))_{n \geq 0}$ the perimeter process which is defined as half of the number of edges on the boundary of the explored region $\bar{\epsilon}_n$ at time $n \geq 0$. Under $\mathbb{P}_\infty^{(\ell)}$, the perimeter process is a Markov chain which is a random walk on \mathbb{Z} of step distribution ν conditioned to stay positive, whose scaling limit is described by a stable Lévy process conditioned to stay positive Υ_a^\dagger . See Subsection 5.2.2 for the definition of ν and Subsection 5.3.1 for the definition of Υ_a^\dagger . The choice of the peeled edge enables to study different properties of the map. In this work, the peeled edge is chosen uniformly at random on the boundary of $\bar{\epsilon}_n$ for all $n \geq 0$ so that the explored region $\bar{\epsilon}_n$ follows the growth of the fpp balls of center f_r .

5.1.2 Main results

Our results can be divided into two parts. We first focus on the scaling limit of the number of faces along the geodesics, hence relating the fpp distance with the dual graph distance on the geodesics. This scaling limit enables to compare the growth of fpp and dual graph balls and also to obtain an upperbound for the diameter of large random planar maps for the dual graph distance. Then, we describe the scaling limit of the fpp geodesics to the root in large random planar maps.

Number of faces. We first give the scaling limit of the number of faces in the fpp geodesics to the root and give applications to the dual graph distance. We choose to state these results for infinite Boltzmann planar maps of law $\mathbb{P}_\infty^{(1)}$. When $a \in (2, 5/2]$, as in Definition 11.1 in [Cur23], we let $e_{\mathbf{q}} = \sum_{k \geq 0} (2k+1)\nu(k)$ be the so-called mean exposure. Recall the constant $p_{\mathbf{q}}$ from (5.1.1).

Theorem 5.1.1. *For all $n \geq 0$, let Γ_n be the shortest path for d_{fpp}^\dagger from f_r to some face of $\bar{\epsilon}_n$ adjacent to the boundary which is chosen arbitrarily. Then we have the following convergences:*

— *If $a \in (2, 5/2]$, then for the topology of uniform convergence on compact subsets,*

$$\text{Under } \mathbb{P}_\infty^{(1)}, \quad \left(n^{-\frac{a-2}{a-1}} \#\text{Faces}(\Gamma_{\lfloor nt \rfloor}) \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{e_{\mathbf{q}}}{2} \int_0^t \frac{ds}{\Upsilon_a^\dagger(p_{\mathbf{q}}s)} \right)_{t \geq 0};$$

— *If $a = 2$, then*

$$\text{Under } \mathbb{P}_\infty^{(1)}, \quad (\log n)^{-2} \#\text{Faces}(\Gamma_n) \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{1}{\pi^2};$$

— *If $a \in (3/2, 2)$, then*

$$\text{Under } \mathbb{P}_\infty^{(1)}, \quad (\log n)^{-1} \#\text{Faces}(\Gamma_n) \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{2}{\pi \tan((2-a)\pi)}.$$

This result has some applications to the dual graph distance d_{gr}^\dagger . Namely, in the cases $a \geq 2$ it enables to compare the distances d_{fpp}^\dagger and d_{gr}^\dagger , improving the results of [CM20] in the case of the uniform exploration. When \mathbf{m} is an infinite (one-ended) planar map, for all $r \geq 0$, we denote by $\overline{\text{Ball}}_r^{\text{fpp}}(\mathbf{m})$ the hull of the ball with center f_r of radius r for the fpp distance obtained from the ball of radius r by filling in the holes which contain finite portions of \mathbf{m} . Similarly, for all $r \geq 0$, we denote by $\overline{\text{Ball}}_r^\dagger$ the hull of the ball with center f_r of radius r for the dual graph distance obtained by filling in the holes which contain finite portions of \mathbf{m} .

Corollary 5.1.2. *Suppose \mathbf{q} of type $a \in (2, 5/2]$. Under $\mathbb{P}_\infty^{(1)}$, for all $\varepsilon > 0$, almost surely, for all r large enough,*

$$\overline{\text{Ball}}_r^{\text{fpp}}(\mathbf{m}) \subset \overline{\text{Ball}}_{\lfloor (1+\varepsilon)e_{\mathbf{q}}r \rfloor}^\dagger(\mathbf{m}).$$

An analogous result holds in the case $a = 2$.

Corollary 5.1.3. *Suppose \mathbf{q} of type $a = 2$. Under $\mathbb{P}_\infty^{(1)}$, for all $\varepsilon > 0$, with probability $1 - o(1)$ as $r \rightarrow \infty$,*

$$\overline{\text{Ball}}_r^{\text{fpp}}(\mathbf{m}) \subset \overline{\text{Ball}}_{\lfloor (1+\varepsilon)\pi^2 p_{\mathbf{q}}^2 r^2 \rfloor}^\dagger(\mathbf{m}).$$

A last consequence of Theorem 5.1.1 which is obtained using the martingale techniques introduced in Chapter 2 is an upperbound for the diameter $\text{Diam}_{\text{gr}}^\dagger(\mathbf{m})$ of (finite) random maps \mathbf{m} in the case $a \in (3/2, 2)$ when they are equipped with the dual graph distance. This completes the results of Part II of [BCR24] and Chapter 2 which were obtained in the respective cases $a \in (2, 5/2]$ and $a = 2$.

Theorem 5.1.4. *Suppose \mathbf{q} is of type $a \in (3/2, 2)$. Then, there exists a constant $C_{\mathbf{q}} > 0$ such that under $\mathbb{P}^{(\ell)}$ as $\ell \rightarrow \infty$, we have with probability $1 - o(1)$*

$$\text{Diam}_{\text{gr}}^\dagger(\mathbf{m}) \leq C_{\mathbf{q}} \log \ell.$$

The main tool for obtaining the above results is a backward exploration of the map, obtained by reversing time in the uniform peeling exploration, which enables to keep track of the geodesic Γ_n .

Scaling limit of the tree of geodesics. If \mathbf{m} is a planar map, recall that $\mathcal{T}(\mathbf{m})$ is the random tree of fpp geodesics from the root. Its vertices are the faces of \mathbf{m} and its edges are the dual edges which appear in the shortest paths for d_{fpp}^\dagger from each face to the root f_r . See e.g. Figure 5.2. We equip each edge of $\mathcal{T}(\mathbf{m})$ with the random fpp length of the associated dual edge of \mathbf{m} . We denote by $d_{\mathcal{T}(\mathbf{m})}$ the corresponding distance on $\mathcal{T}(\mathbf{m})$. Let $(f_i)_{i \geq 1}$ be the family of the faces of \mathbf{m} in the non-increasing order of degrees.

Theorem 5.1.5. *Assume \mathbf{q} is of type $a \in (3/2, 5/2)$. Then,*

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\ell^{2-a} d_{\mathcal{T}(\mathbf{m})}(f_i, f_j) \right)_{i, j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} d_{\mathcal{T}_a}(w_i, w_j) \right)_{i, j \geq 1},$$

for the product topology, where $c_a := \pi/\Gamma(a)$ and $\mathcal{T}_a = (\{w_i; i \geq 1\}, d_{\mathcal{T}_a})$ is a random countable metric space.

Let us give some insight into this scaling limit. Using the time reversal of the uniform peeling exploration, we describe the geodesics to the root using a family of coalescing processes. We prove that these processes converge towards a stochastic flow of coalescing diffusions with jumps. The metric space $\mathcal{T}_a = (\{w_i; i \geq 1\}, d_{\mathcal{T}_a})$ is then defined using this flow of diffusions and we show that it is indeed the scaling limit of $\mathcal{T}(\mathbf{m})$. The exact definition of the limit is in Section 5.5.1. The stochastic flow is driven by the jumps of the self-similar growth-fragmentation corresponding to the scaling limit of the slicing by heights as shown in [BCK18] in the generic case $a = 5/2$, [BBCK18] in the dilute case $a \in (2, 5/2)$ and Chapter 3 in the critical case $a = 2$ and by independent uniform random variables which arise from the uniform peeling exploration. We do not rely on the theory of coalescing flows of [LJR04]. The set $\{w_i; i \geq 1\}$ corresponds to the set of positive jumps of the growth-fragmentation, plus an additional point corresponding to the root face. One can then focus on the trajectories of the flow which “start from” the positive jumps and their coalescences give the genealogical structure of \mathcal{T}_a . Finally, the distances on \mathcal{T}_a are measured using a Lamperti transform.

The scaling limit of the coalescing processes also holds in the case $a = 5/2$. This should give a new description of the geodesics to the root in the Brownian disk as a flow of diffusions with jumps. However, due to the absence of high degree vertices, the scaling limit of the tree of fpp geodesics cannot be stated in the same way as in Theorem 5.1.5 in this case so that another approach is needed in the case $a = 5/2$. Actually, a related approach is carried out in Section 3 of [MS21a] for the axiomatic characterization of the Brownian sphere introducing a “Lévy net”.

Besides, one could define the stochastic flow on top of the self-similar Markov trees from [BCR24] in the case $a \in (2, 5/2]$ but we do not in order to cover the whole range $(3/2, 5/2]$.

5.1.3 Plan of the paper

We define the time reversal of the uniform peeling exploration in Section 5.2. Then, Section 5.3, independent of the next sections, is devoted to the scaling limit of the number of faces along the geodesics and its applications, i.e. to the proof of Theorem 5.1.1, Corollaries 5.1.2 and 5.1.3 and Theorem 5.1.4. Next, in Section 5.4, we define a coalescing flow of pure jump diffusions and prove that it arises as the scaling limit of a discrete coalescing flow which encodes the fpp geodesics to the root. Based on the coalescing flow, in Section 5.5, we define a random countable metric space and we prove that it is the

scaling limit of the tree of fpp geodesics in Theorem 5.1.5. Finally, in Section 5.6 we define a random countable metric space which we believe to be the scaling limit of large random planar maps with high degrees.

5.2 The time reversal of the uniform peeling exploration

In this section, we present a time-reversal of the uniform exploration of a random Boltzmann map conditionally on the perimeter process. This reversed uniform exploration is the central tool of this work. But first, we recall the definition of the uniform peeling exploration.

5.2.1 The uniform peeling exploration

We say that a map ϵ is a map with one hole if ϵ has a distinguished face with simple boundary which is different from f_r . Moreover, we say that ϵ is a submap of a map \mathfrak{m} if one can obtain \mathfrak{m} by gluing a map in the hole of ϵ . The boundary $\partial\epsilon$ of a map with one hole is the set of edges on the boundary of its hole. A peeling algorithm \mathcal{A} is a function which takes a map with one hole ϵ and returns an edge of $\partial\epsilon$. The algorithm \mathcal{A} may be random.

Let \mathfrak{m} be a map which is possibly infinite. A peeling exploration of \mathfrak{m} following the peeling algorithm \mathcal{A} is an increasing sequence of submaps $\bar{\epsilon}_0 \subset \bar{\epsilon}_1 \subset \dots \subset \bar{\epsilon}_n \subset \dots \subset \mathfrak{m}$ such that $\bar{\epsilon}_0$ is the map formed by gluing the root face f_r of \mathfrak{m} to a hole of the same degree, and such that for all $n \geq 0$, the submap $\bar{\epsilon}_{n+1}$ is obtained from $\bar{\epsilon}_n$ in the following way:

- If the face f behind the edge $\mathcal{A}(\bar{\epsilon}_n)$ in \mathfrak{m} is not in $\bar{\epsilon}_n$, then we “discover” this face by gluing f to $\bar{\epsilon}_n$ on the edge $\mathcal{A}(\bar{\epsilon}_n)$. See the two peeling steps on the right of the first row in Figure 5.3.
- If the face behind $\mathcal{A}(\bar{\epsilon}_n)$ in \mathfrak{m} is already in $\bar{\epsilon}_n$, then it is necessarily adjacent to the hole. Then we glue the edge $\mathcal{A}(\bar{\epsilon}_n)$ to the edge in $\partial\bar{\epsilon}_n$ following the adjacency in \mathfrak{m} . The hole then splits into two holes. When \mathfrak{m} is infinite, we choose to fill in the hole which contains only a finite number of faces in \mathfrak{m} . When \mathfrak{m} is finite, we choose to fill in the hole which has the smallest perimeter (and in case of equality we fill in one of the two holes chosen arbitrarily in a deterministic way). See the left of the first row and the second row in Figure 5.3 for such peeling steps.

We denote by $(P(n))_{n \geq 0}$ the perimeter process which is defined as half of the number of edges in $\partial\bar{\epsilon}_n$. When \mathfrak{m} is finite, the exploration stops when P reaches zero.

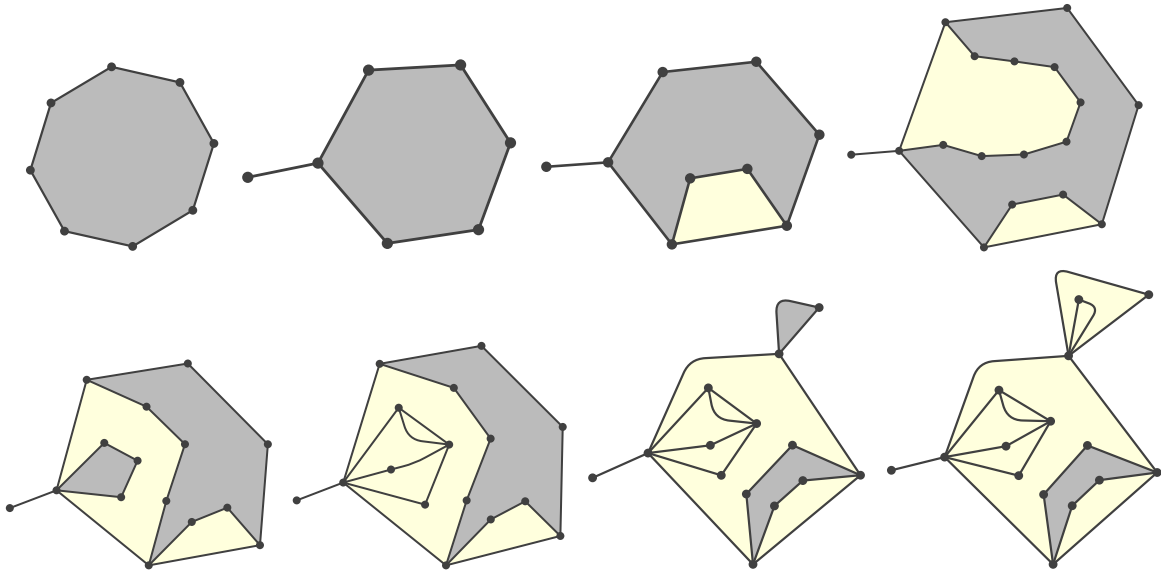


Figure 5.3 – The first five steps of the exploration of the map \mathfrak{m} from Figure 5.2. From left to right on the first row: $\bar{\epsilon}_0$, $\bar{\epsilon}_1$, $\bar{\epsilon}_2$, $\bar{\epsilon}_3$. From left to right on the second row: the submap obtained after gluing two edges on $\partial\bar{\epsilon}_3$, $\bar{\epsilon}_4$ after filling-in the hole with minimal perimeter, the submap obtained after gluing two edges on $\partial\bar{\epsilon}_4$, $\bar{\epsilon}_5$ after filling-in the hole with minimal perimeter.

Now, let us describe a peeling exploration which is very useful in the study of the fpp distance, which is the uniform peeling exploration. See Section 13.1 of [Cur23] for more details. For all $t \geq 0$, let $\text{Ball}_t^{\text{fpp}}(\mathbf{m})$ be the closed ball centered at f_r of radius $t \geq 0$ for the fpp distance d_{fpp}^\dagger . It is defined as the submap whose faces are at fpp distance at most t to the root face and containing only the dual edges whose points are at fpp distance at most t from the root.

Note that $\bar{\epsilon}_0 = \text{Ball}_0^{\text{fpp}}$. Next, for some step $n \geq 0$ of the exploration, assume that the submap $\bar{\epsilon}_n$ can be obtained from $\text{Ball}_{T_n}^{\text{fpp}}(\mathbf{m})$ for some $T_n \geq 0$ by filling in all the holes except one. By the absence of memory of the exponential distribution, we know that the lengths of the dual edges on the boundary of $\bar{\epsilon}_n$ are i.i.d. exponential random variables of parameter 1. We choose the next edge to peel to be the edge of $\partial\bar{\epsilon}_n$ whose dual edge has the smallest length. By well-known properties of the exponential distribution, this edge is actually chosen uniformly at random and the fpp length of the corresponding dual edge is an exponential random variable of expectation $1/\#\partial\bar{\epsilon}_n$ conditionally on $\bar{\epsilon}_n$ and on the fpp lengths of the dual edges of $\bar{\epsilon}_n$. This edge length can then be written $\mathcal{E}_n/(2P(n))$, where \mathcal{E}_n is an exponential r.v. of parameter 1. Let $T_{n+1} = T_n + \mathcal{E}_n/(2P(n))$. Then $\bar{\epsilon}_{n+1}$ is obtained from the ball $\text{Ball}_{T_{n+1}}^{\text{fpp}}(\mathbf{m})$ by filling in all the holes, except one.

In a word, using the uniform peeling algorithm which chooses at each step an edge on the boundary uniformly at random, we have for all $n \geq 0$,

$$\bar{\epsilon}_n = \overline{\text{Ball}}_{T_n}^{\text{fpp}}(\mathbf{m}) \quad \text{with} \quad T_n = \sum_{i=0}^{n-1} \frac{\mathcal{E}_i}{2P(i)},$$

where the \mathcal{E}_i 's for $i \geq 0$ are i.i.d. exponential r.v. of parameter 1 and where for all t , the submap $\overline{\text{Ball}}_t^{\text{fpp}}(\mathbf{m})$ is obtained by filling in all the holes of \mathbf{m} except one, which corresponds to the infinite hole when \mathbf{m} is infinite.

5.2.2 The law of the peeling exploration of Boltzmann planar maps

Under $\mathbb{P}^{(\ell)}$ (respectively $\mathbb{P}_\infty^{(\ell)}$) the peeling exploration has several nice features, which do not depend on the peeling algorithm. It is a Markov chain and its transitions can be described using the perimeter process.

First, the peeling exploration satisfies a spatial Markov property, stating that at each step $n \geq 0$, conditionally on $\bar{\epsilon}_n$, the map which fills the hole in \mathbf{m} is a random map of law $\mathbb{P}^{(p)}$ (resp. $\mathbb{P}_\infty^{(p)}$), where $p = P(n)$. Moreover, at each time n such that $P(n+1) < P(n)$, the map in the hole which is filled-in after gluing the two edges is a random map of law $\mathbb{P}^{(p)}$ with $p = -\Delta P(n) - 1$. See Propositions 4.6 and 6.3 of [Cur23] for a detailed statement.

Moreover, the perimeter process itself is a Markov chain, whose transition probabilities are better written using the distribution ν defined by

$$\forall k \in \mathbb{Z}, \quad \nu(k) = \begin{cases} q_{k+1} c_{\mathbf{q}}^k & \text{if } k \geq 0 \\ 2W^{(-1-k)} c_{\mathbf{q}}^k & \text{if } k < 0, \end{cases}$$

where we recall that $(q_k)_{k \geq 1}$ is the weight sequence, $W^{(\ell)}$ is the partition function of maps of perimeter 2ℓ for all $\ell \geq 1$, and $c_{\mathbf{q}}$ appears in (5.1.1). It is known that ν is a probability measure (see e.g. Lemma 5.2 of [Cur23]). Under $\mathbb{P}_\infty^{(\ell)}$, the perimeter process $(P(n))_{n \geq 0}$ is the Doob h -transform of the ν -random walk starting at ℓ , using the harmonic function h^\uparrow on $\mathbb{Z}_{\geq 1}$ defined by

$$\forall \ell \geq 1, \quad h^\uparrow(\ell) = 2\ell 2^{-2\ell} \binom{2\ell}{\ell}. \quad (5.2.1)$$

See e.g. Remark 7.5 of [Cur23]. This Doob h -transform can be interpreted as a conditioning of the ν -random to always stay positive. Besides, under $\mathbb{P}^{(\ell)}$, the perimeter process $(P(n))_{n \geq 0}$ is a Markov chain starting at ℓ , which is absorbed at zero with transition probabilities for all $n \geq 0, p \geq 1, k \in \mathbb{Z}$,

$$\mathbb{P}^{(\ell)}(P(n+1) = p+k | P(n) = p) = \nu(k) c_{\mathbf{q}}^{-k} \frac{W^{(p+k)}}{W^{(p)}} \left(\mathbf{1}_{p+k > (p-1)/2} + \frac{1}{2} \mathbf{1}_{p+k = (p-1)/2} \right). \quad (5.2.2)$$

See the second display of page 53 of [BBCK18] coming from Proposition 6.3 of [BBCK18].

The tail behavior of ν is well understood for \mathbf{q} of type a thanks to Propositions 5.9 and 5.10 of [Cur23]. We have, when $a \in (3/2, 5/2)$, as $k \rightarrow \infty$,

$$\nu(-k) \sim p_{\mathbf{q}} k^{-a} \quad \text{and} \quad \nu([k, \infty)) \sim p_{\mathbf{q}} \cos(a\pi) \frac{1}{a-1} k^{1-a}, \quad (5.2.3)$$

where the constant $p_{\mathbf{q}}$ comes from (5.1.1). When $a = 5/2$, only the first asymptotic holds and the second one becomes $\nu([k, \infty)) = o(k^{-3/2})$.

5.2.3 Reversing the uniform exploration

Let \mathbf{m} be a planar map. Let P be the perimeter process associated with the uniform peeling exploration of \mathbf{m} . Let us define the decreasing sequence of submaps $\mathbf{u}_0^n \supset \mathbf{u}_1^n \supset \dots \supset \mathbf{u}_n^n$ as follows: let \mathbf{u}_n^n be the map made of only one distinguished face of degree $2P(n)$ glued to one hole of the same degree. For all $k \in \llbracket 0, n-1 \rrbracket$, we obtain \mathbf{u}_k^n from \mathbf{u}_{k+1}^n by distinguishing two cases:

- If $\Delta P(k) < 0$ then \mathbf{u}_k^n is obtained by inserting $-2\Delta P(k)$ consecutive edges in the boundary of the distinguished face at a position which is chosen uniformly at random. The two extremal inserted edges are glued and we fill in the hole delimited by the $-2\Delta P(k) - 2$ remaining edges with a Boltzmann map of law $\mathbb{P}^{(p)}$, where $p := -\Delta P(k) - 1$. See the first row of Figure 5.4 and the rightmost figure in the second row for such exploration steps.
- If $\Delta P(k) \geq 0$, then \mathbf{u}_k^n is obtained by gluing a new face f of degree $2\Delta P(k) + 2$ to $2\Delta P(k) + 1$ consecutive edges where the position of the first of these consecutive edges is chosen uniformly at random on the boundary of the distinguished face. See the exploration steps \mathbf{u}_2^5 and \mathbf{u}_1^5 in the second row of Figure 5.4 for this case.

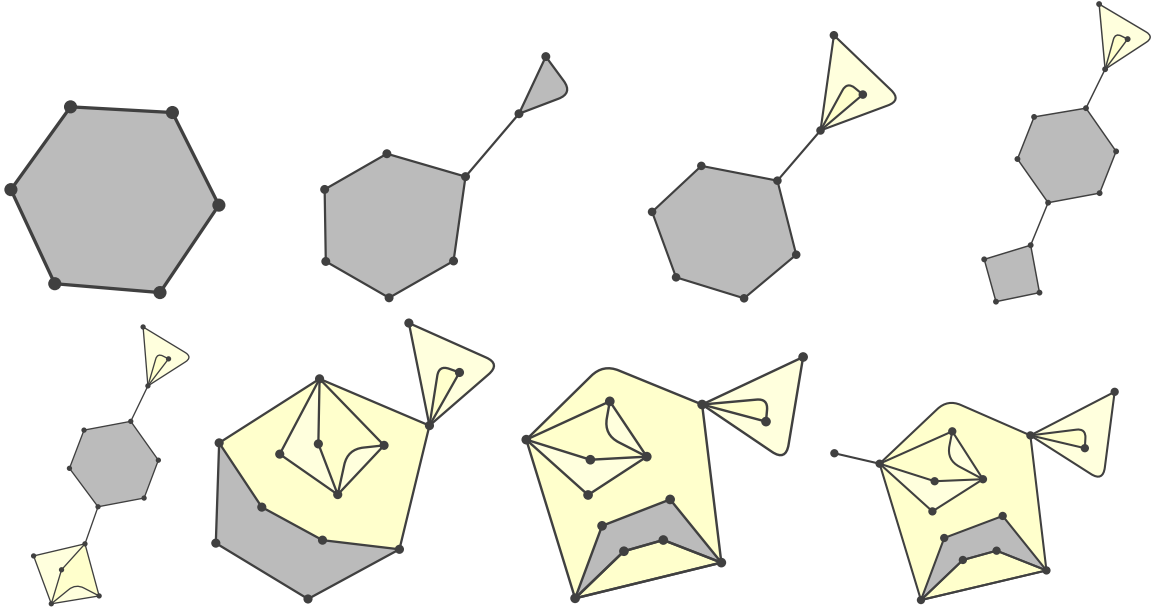


Figure 5.4 – The five steps of the reversed uniform exploration $\mathbf{u}_5^5 \subset \mathbf{u}_4^5 \subset \mathbf{u}_3^5 \subset \mathbf{u}_2^5 \subset \mathbf{u}_1^5 \subset \mathbf{u}_0^5$ of the map \mathbf{m} starting from time $n = 5$. They correspond to the time-reversal of the exploration depicted in Figure 5.3. From left to right on the first row: \mathbf{u}_5^5 , the map obtained by inserting 4 edges at a position chosen uniformly at random and gluing the two extremal edges, hence creating a hole of perimeter 2, the map \mathbf{u}_4^5 obtained by filling in the hole, the map obtained by inserting 6 edges at a position chosen uniformly at random and gluing the two extremal edges, hence creating a hole of perimeter 4. From left to right on the second row: the submap \mathbf{u}_3^5 obtained after filling in the hole, \mathbf{u}_2^5 , \mathbf{u}_1^5 and \mathbf{u}_0^5 .

Conditionally on the perimeter process, let us relate this backward exploration to the uniform peeling exploration $(\bar{\mathbf{e}}_k)_{k \geq 0}$. For all $n \geq 0$, for all $k \in \llbracket 0, n \rrbracket$, let $\bar{\mathbf{e}}_n \setminus \bar{\mathbf{e}}_k$ be the submap of $\bar{\mathbf{e}}_n$ obtained by identifying the explored region $\bar{\mathbf{e}}_k$ at time k to one (non-simple) distinguished face of degree $2P(k)$.

Proposition 5.2.1. *For all $n \geq 0$, under either $\mathbb{P}_\infty^{(\ell)}$ or $\mathbb{P}^{(\ell)}(\cdot | P(n) > 0)$, conditionally on $(P(k))_{0 \leq k \leq n}$, the sequence $(\mathbf{u}_k^n)_{0 \leq k \leq n}$ has the same distribution as $(\bar{\mathbf{e}}_n \setminus \bar{\mathbf{e}}_k)_{0 \leq k \leq n}$. In particular, conditionally on $(P(k))_{0 \leq k \leq n}$, the map \mathbf{u}_0^n has the same law as $\bar{\mathbf{e}}_n$.*

Proof. Let us prove the result by induction on n . The result is obvious for $n = 0$ since $\bar{\mathbf{e}}_0$ is the map consisting of the root face f_r and one hole. Assume that the statement holds for n . By definition of the uniform peeling exploration, $\bar{\mathbf{e}}_{n+1}$ is obtained after peeling an edge of $\partial\bar{\mathbf{e}}_n$ uniformly at random. Let us check that under $\mathbb{P}_\infty^{(\ell)}$ (or $\mathbb{P}^{(\ell)}$ under the event “ $P(n) > 0$ ”), conditionally on $(P(k))_{0 \leq k \leq n}$ and on $(\bar{\mathbf{e}}_k)_{0 \leq k \leq n}$, the map \mathbf{u}_n^{n+1} has the same law as $\bar{\mathbf{e}}_{n+1} \setminus \bar{\mathbf{e}}_n$ obtained by identifying all the explored region $\bar{\mathbf{e}}_n$ to one non-simple face. To do this we reason differently according to the sign of $\Delta P(n)$.

- If $\Delta P(n) < 0$, then the uniform exploration glues in the hole the peeled edge to another edge of $\partial\bar{\mathbf{e}}_n$ hence creating a hole of degree $-2\Delta P(n) - 2$ and then fills in this hole with a Boltzmann map of law $\mathbb{P}^{(p)}$, where $p := -\Delta P(n) - 1$.
- If $\Delta P(n) \geq 0$, then the uniform exploration glues in the hole a new face f of degree $2\Delta P(n) + 2$ to the peeled edge.

In both cases, it is clear that the map $\bar{\mathbf{e}}_{n+1} \setminus \bar{\mathbf{e}}_n$ has the same law as \mathbf{u}_n^{n+1} . To conclude, it suffices to notice that the \mathbf{u}_k^{n+1} for $k \in \llbracket 0, n \rrbracket$ are obtained by gluing the boundary of the hole of \mathbf{u}_k^n in the distinguished face of \mathbf{u}_n^{n+1} in a uniformly chosen way. Similarly, since the peeled edge is chosen uniformly at random, $\bar{\mathbf{e}}_{n+1} \setminus \bar{\mathbf{e}}_k$ is obtained by gluing the boundary of the hole of $\bar{\mathbf{e}}_n \setminus \bar{\mathbf{e}}_k$ in the distinguished face of $\bar{\mathbf{e}}_{n+1} \setminus \bar{\mathbf{e}}_n$ in a uniformly chosen way. \square

For all $n \geq 0$, let us denote by Γ_n the fpp geodesic from f_r to the already explored face next to the peeled edge at time n by the uniform peeling exploration. The geodesic Γ_n only contains faces corresponding to the positive jumps of $(P(i))_{0 \leq i \leq n}$. Moreover, conditionally on P , independently for each $i \in \llbracket 0, n-1 \rrbracket$ such that $\Delta P(i) \geq 0$, the face f_i discovered at time i belongs to Γ_n with probability

$$\theta_i := \mathbb{P}(f_i \in \Gamma_n | P) = \frac{2\Delta P(i) + 1}{2P(i) + 1} \mathbf{1}_{\Delta P(i) \geq 0}. \quad (5.2.4)$$

Indeed, this can be checked by induction using Proposition 5.2.1 and the fact that two adjacent faces are in the same fpp geodesic to the root if and only if they share an edge which has been peeled by the uniform exploration (and in that case there is a shared edge corresponding to the discovery of one of the two faces). The same holds for the fpp geodesic from the root face to any edge of \mathbf{u}_0^n (i.e. any edge of $\partial\bar{\mathbf{e}}_n$).

5.3 Number of faces in the geodesics

In this section, we prove Theorem 5.1.1 which gives the scaling limit of the number of faces along the fpp geodesics to the root for infinite Boltzmann planar maps of law $\mathbb{P}_\infty^{(1)}$. We start by stating some useful results on random walks conditioned to be positive. Then we enter the proof of the theorem. Finally, we prove the applications to the dual graph distance: the comparison between the dual graph distance and the fpp distance and the upperbound on the diameter.

5.3.1 Preliminaries on random walks conditioned to stay positive

We first state some useful scaling limit and local limit theorems for the perimeter process $(P(n))_{n \geq 0}$ under $\mathbb{P}_\infty^{(1)}$ which can be viewed as the ν -random walk starting at 1 and conditioned to stay always positive and whose law is described in Subsection 5.2.2. We will often add a subscript ∞ to P in order to underline the fact that we work under $\mathbb{P}_\infty^{(1)}$. Thanks to (5.2.3), the scaling limit of a ν -random walk starting at zero $(S_n)_{n \geq 0}$ is given by the $(a-1)$ -stable Lévy process Υ_a starting at zero with Lévy measure $\cos(a\pi)(dx/x^a)\mathbf{1}_{x>0} + (dx/|x|^a)\mathbf{1}_{x<0}$. It is known that $(S_{\lfloor nt \rfloor}/n^{1/(a-1)})_{t \geq 0}$ converges in distribution towards $(\Upsilon_a(p_{\mathbf{q}}t))_{t \geq 0}$. See e.g. Proposition 10.1 of [Cur23]. The Lévy process starting at $x > 0$ conditioned to stay always positive S_x^\uparrow is defined as the Doob h -transform of $x + \Upsilon_a$ using the harmonic function $y \mapsto \mathbf{1}_{y \geq 0} \sqrt{y}$. By letting $x \downarrow 0$, one obtains the Lévy process Υ_a starting at zero and conditioned to stay always positive which is denoted by Υ_a^\uparrow . The scaling limit of the perimeter process

is stated in Proposition 10.3 in [Cur23]:

$$\text{Under } \mathbb{P}_\infty^{(1)}, \quad \left(\frac{P_\infty(nt)}{n^{1/(a-1)}} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} (\Upsilon_a^\uparrow(p_{\mathbf{q}}t))_{t \geq 0} \quad (5.3.1)$$

for the Skorokhod J_1 topology. We then state a local limit theorem which is proven in the case $a = 2$ in [BCM18]. The same proof actually works for all $a \in (3/2, 5/2]$. Let f_a^\uparrow be the density of $\Upsilon_a^\uparrow(1)$.

Lemma 5.3.1. (Lemma 3 of [BCM18]) For all $n, x, y \in \mathbb{N}$, let

$$\eta_n(x, y) = \sqrt{\frac{n^{1/(a-1)}}{y}} \left((np_{\mathbf{q}})^{1/(a-1)} \frac{2\sqrt{y}}{h^\uparrow(y)\sqrt{\pi}} \mathbb{P}_\infty^{(x)}(P_\infty(n) = y) - f_a^\uparrow\left(\frac{y}{(p_{\mathbf{q}}n)^{1/(a-1)}}\right) \right).$$

Then

- Uniformly in $y \geq 1$, as $n \rightarrow \infty$ we have $\eta_n(1, y) \rightarrow 0$.
- For every $\varepsilon > 0$ and every sequence $\delta_n \rightarrow 0$, uniformly in $x \in [1, n^{1/(a-1)}\delta_n]$ and uniformly in $y \in [\varepsilon n^{1/(a-1)}, \varepsilon^{-1}n^{1/(a-1)}]$, as $n \rightarrow \infty$ we have $\eta_n(x, y) \rightarrow 0$.

The above lemma has the following direct corollary:

Corollary 5.3.2. For every $\varepsilon > 0$ and every sequence $\delta_n \rightarrow 0$, we have

$$n^{1/(a-1)} \left| \mathbb{P}_\infty^{(x)}(P_\infty(n) = y) - \mathbb{P}_\infty^{(x')} (P_\infty(n) = y) \right| \xrightarrow[n \rightarrow \infty]{} 0$$

uniformly in $x, x' \in [1, n^{1/(a-1)}\delta_n]$ and uniformly in $y \in [\varepsilon n^{1/(a-1)}, \varepsilon^{-1}n^{1/(a-1)}]$.

Besides, the following lemma is proven in the case $a = 2$ but the same proof works for all $a \in (3/2, 5/2]$.

Lemma 5.3.3. (Lemma 4 of [BCM18]) For all $0 < r < 3/2$, we have

$$\sup_{n \geq 1} \mathbb{E}_\infty^{(1)} \left(\left(\frac{n^{1/(a-1)}}{P_\infty(n)} \right)^r \right) < \infty.$$

Remark 5.3.4. By Lemma 5.3.3 and by (5.3.1), if \mathbf{q} is of type $a < 5/2$, then $(n/P_\infty(n)^{a-1})_{n \geq 0}$ is uniformly integrable and converges in distribution. Hence

$$\mathbb{E}_\infty^{(1)} \left(\frac{n}{P_\infty(n)^{a-1}} \right) \xrightarrow[n \rightarrow \infty]{} \mathbb{E} \left(\frac{1}{\Upsilon_a^\uparrow(p_{\mathbf{q}})^{a-1}} \right) = \mathbb{E} \left(\frac{1}{p_{\mathbf{q}} \Upsilon_a^\uparrow(1)^{a-1}} \right) = \frac{2\Gamma(a)}{p_{\mathbf{q}}(a-1)\pi^{3/2}\Gamma(a-3/2)}.$$

where the last equality comes from the end of the proof of Proposition 2.2.7 in Chapter 2.

From the previous lemma, as shown in [BCM18], one can deduce that the perimeter process decorrelates at different scales, in the following sense:

Lemma 5.3.5. (Lemma 5 of [BCM18]) Let A_n be a sequence tending to ∞ . For all $\varepsilon > 0$, for all $i \geq 1$, let $I(i, \varepsilon) = [\varepsilon i^{1/(a-1)}, \varepsilon^{-1}i^{1/(a-1)}]$ and let

$$X_i^{(\varepsilon)} := \frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i) \in I(i, \varepsilon)} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i) \in I(i, \varepsilon)} \right).$$

If \mathbf{q} is of type $a < 5/2$, then

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \sup_{A_n, i \leq j \leq n} \left| \mathbb{E}_\infty^{(1)} \left(iX_i^{(\varepsilon)} jX_j^{(\varepsilon)} \right) \right| = 0.$$

Proof. The proof follows the same lines as in the case $a = 2$ treated in [BCM18], but let us write it for completeness. We write

$$\begin{aligned} \left| \mathbb{E}_\infty^{(1)} \left(iX_i^{(\varepsilon)} jX_j^{(\varepsilon)} \right) \right| &\leq \left| \mathbb{E}_\infty^{(1)} \left(iX_i^{(\varepsilon)} jX_j^{(\varepsilon)} \mathbf{1}_{P_\infty(i) \notin I(i, \varepsilon)} \right) \right| + \left| \mathbb{E}_\infty^{(1)} \left(iX_i^{(\varepsilon)} jX_j^{(\varepsilon)} \mathbf{1}_{P_\infty(j) \notin I(j, \varepsilon)} \right) \right| \\ &\quad + \left| \mathbb{E}_\infty^{(1)} \left(iX_i^{(\varepsilon)} jX_j^{(\varepsilon)} \mathbf{1}_{P_\infty(i) \in I(i, \varepsilon)} \mathbf{1}_{P_\infty(j) \in I(j, \varepsilon)} \right) \right|. \end{aligned}$$

Let $K_i^{(\varepsilon)} = \mathbb{E}_\infty^{(1)}\left(\frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i) \in I(i, \varepsilon)}\right)$. Note that on the event $\{P_\infty(i) \notin I(i, \varepsilon)\}$, we have $X_i^{(\varepsilon)} = -K_i^{(\varepsilon)}$, so the first term can be written

$$\left| \mathbb{E}_\infty^{(1)}\left(iX_i^{(\varepsilon)}jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(i) \notin I(i, \varepsilon)}\right) \right| = (iK_i^{(\varepsilon)}) \left| \mathbb{E}_\infty^{(1)}\left(jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(i) \notin I(i, \varepsilon)}\right) \right|,$$

which converges to zero as $\varepsilon \rightarrow 0$ uniformly in $i, j \geq 1$ since $(iK_i^{(\varepsilon)})_{i \geq 1, \varepsilon > 0}$ is upperbounded by some constant $C > 0$ by Remark 5.3.4, since $(jX_j^{(\varepsilon)})_{j \geq 1, \varepsilon > 0}$ is uniformly integrable by Lemma 5.3.3 using that $a - 1 < 3/2$ and since $(P_\infty(i)/i^{1/(a-1)})_{i \geq 1}$ is tight by (5.3.1). The second term is handled in the same way. Thus, it only remains to show that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{1 \leq i \leq n} \sup_{A_n i \leq j \leq n} \left| \mathbb{E}_\infty^{(1)}\left(iX_i^{(\varepsilon)}jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(j)/j^{1/(a-1)}, P_\infty(i)/i^{1/(a-1)} \in [\varepsilon, 1/\varepsilon]}\right) \right| = 0. \quad (5.3.2)$$

Notice that for n large enough, for all $A_n i \leq j \leq n$ we have $j - i \geq j/2$, so that $(2(j - i))/A_{j-i} \geq i$. Therefore, by Corollary 5.3.2 with ε, n, δ_n replaced by $\varepsilon/2^{1/(a-1)}, j - i, (2/A_{j-i})^{1/(a-1)}/\varepsilon$, we know that for n large enough, for all $A_n i \leq j \leq n$,

$$\begin{aligned} & \sup_{\substack{1 \leq x \leq i^{1/(a-1)}/\varepsilon \\ \varepsilon j^{1/(a-1)} \leq y \leq j^{1/(a-1)}/\varepsilon}} \left| \mathbb{P}_\infty^{(x)}(P_\infty(j - i) = y) - \mathbb{P}_\infty^{(1)}(P_\infty(j - i) = y) \right| \\ & \leq \sup_{\substack{1 \leq x \leq (j-i)^{1/(a-1)}(2/A_{j-i})^{1/(a-1)}/\varepsilon \\ (\varepsilon/2^{1/(a-1)})(j-i)^{1/(a-1)} \leq y \leq (2^{1/(a-1)}/\varepsilon)(j-i)^{1/(a-1)}}} \left| \mathbb{P}_\infty^{(x)}(P_\infty(j - i) = y) - \mathbb{P}_\infty^{(1)}(P_\infty(j - i) = y) \right| \\ & \leq \varepsilon^a / j^{1/(a-1)} \end{aligned}$$

and so by the Markov property

$$\begin{aligned} & \mathbf{1}_{P_\infty(i) \leq i^{1/(a-1)}/\varepsilon} \left| \mathbb{E}_\infty^{(1)}\left(jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(j) \in I(j, \varepsilon)} \middle| P_\infty(i)\right) - \mathbb{E}_\infty^{(1)}\left(jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(j) \in I(j, \varepsilon)} \middle| P_\infty(i) = 1\right) \right| \\ & \leq \mathbf{1}_{P_\infty(i) \leq i^{1/(a-1)}/\varepsilon} \sum_{j^{1/(a-1)\varepsilon} \leq y \leq j^{1/(a-1)}/\varepsilon} \left| \frac{j}{y^{a-1}} - jK_j^{(\varepsilon)} \right| \frac{\varepsilon^a}{j^{1/(a-1)}} \\ & \leq \mathbf{1}_{P_\infty(i) \leq i^{1/(a-1)}/\varepsilon} \left(\frac{1}{\varepsilon^{a-1}} + C \right) \varepsilon^{a-1}. \end{aligned}$$

Moreover, for n large enough, for all $A_n i \leq j \leq n$,

$$\left| \mathbb{E}_\infty^{(1)}\left(jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(j) \in [j^{1/(a-1)\varepsilon}, j^{1/(a-1)}/\varepsilon]}\right) \middle| P_\infty(i) = 1 \right| \leq \mathbb{E}_\infty^{(1)}\left(|jX_j^{(\varepsilon)}|\right) \leq 4C.$$

Besides, using the fact that $\mathbb{E}_\infty^{(1)}X_i^{(\varepsilon)} = 0$, we have

$$\begin{aligned} \left| \mathbb{E}_\infty^{(1)}\left(iX_i^{(\varepsilon)}\mathbf{1}_{P_\infty(i) \in I(i, \varepsilon)}\right) \right| &= \left| \mathbb{E}_\infty^{(1)}\left(iX_i^{(\varepsilon)}\mathbf{1}_{P_\infty(i) \notin I(i, \varepsilon)}\right) \right| \\ &= iK_i^{(\varepsilon)} \mathbb{P}_\infty^{(1)}(P_\infty(i) \notin I(i, \varepsilon)) \\ &\leq C \mathbb{P}_\infty^{(1)}(\Upsilon_a^\uparrow(p_{\mathbf{q}}) \notin [\varepsilon, 1/\varepsilon]), \end{aligned}$$

for some constant $C > 0$. Therefore,

$$\begin{aligned} & \left| \mathbb{E}_\infty^{(1)}\left(iX_i^{(\varepsilon)}jX_j^{(\varepsilon)}\mathbf{1}_{P_\infty(i)/i^{1/(a-1)}, P_\infty(j)/j^{1/(a-1)} \in [\varepsilon, 1/\varepsilon]}\right) \right| \\ & \leq C \mathbb{P}_\infty^{(1)}(\Upsilon_a^\uparrow(p_{\mathbf{q}}) \notin [\varepsilon, 1/\varepsilon]) (4C + (1/\varepsilon^{a-1} + C)\varepsilon^{a-1}), \end{aligned}$$

which converges towards zero as $\varepsilon \rightarrow 0$. \square

5.3.2 Number of faces in the geodesics

We are now well equipped to deal with the proof of Theorem 5.1.1. For sake of readability, we treat separately the dilute case $a \in (2, 5/2]$, the critical case $a = 2$ and the dense case $a \in (3/2, 2)$.

Dilute case

Proof of Theorem 5.1.1 in the dilute case. For all $i \geq 0$, let f_i be the face of degree $2\Delta P_\infty(i) + 2$ which is discovered at time i of the exploration when $\Delta P_\infty(i) \geq 0$. By (5.2.4), we know that conditionally on P_∞ , the $X_i := \mathbf{1}_{f_i \in \Gamma_{\lfloor nt \rfloor}}$'s for $i \in [0, \lfloor nt \rfloor - 1]$ are independent Bernoulli random variables of parameters θ_i .

Let us start with the case $a \in (2, 5/2]$. Notice that it suffices to check that

$$\left(n^{-\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} \theta_i \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{e_{\mathbf{q}}}{2} \int_0^t \frac{ds}{\Upsilon_a^\uparrow(p_{\mathbf{q}}s)} \right)_{t \geq 0} \quad (5.3.3)$$

for the topology of uniform convergence on compact sets, which is equivalent to the convergence in terms of finite dimensional distributions since the limiting process is a continuous increasing process.

Indeed, if we suppose (5.3.3), then we deduce that for all $t \geq 0$,

$$\mathbb{E}_\infty^{(1)} \left(\left(n^{-\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} (X_i - \theta_i) \right)^2 \middle| P_\infty \right) = n^{-2\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} (\theta_i - \theta_i^2) \leq n^{-2\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} \theta_i \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0.$$

In order to prove (5.3.3), it suffices to show that

$$\left(n^{-\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} \left(\frac{e_{\mathbf{q}}}{2} \int_0^t \frac{ds}{\Upsilon_a^\uparrow(p_{\mathbf{q}}s)} \right)_{t \geq 0}. \quad (5.3.4)$$

Indeed, by a straightforward computation, for all $t \geq 0$,

$$\mathbb{E}_\infty^{(1)} \left(\left(\sum_{i=0}^{\lfloor nt \rfloor} (\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i))) \right)^2 \right) = \sum_{i=0}^{\lfloor nt \rfloor} \mathbb{E}_\infty^{(1)} \left((\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)))^2 \right) \leq \sum_{i=0}^{\lfloor nt \rfloor} \mathbb{E}_\infty^{(1)}(\theta_i^2),$$

and we have $\mathbb{E}_\infty^{(1)}(\theta_i^2) = O(i^{-1/(a-1)})$ due to the fact that, thanks to (5.2.3) and since $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$,

$$\mathbb{E}_\infty^{(1)}(\theta_i^2 | P_\infty(i) = p) = \sum_{k \geq 0} \left(\frac{2k+1}{2(p+k)} \right)^2 \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \nu(k) = O \left(\frac{1}{\sqrt{p}} \sum_{k \geq 0} \frac{k^{2-a}}{(p+k)^{3/2}} \right) = O \left(\frac{1}{p^{a-1}} \right) \quad (5.3.5)$$

and to the fact that $a-1 > 1$ and $\mathbb{E}_\infty^{(1)}(1/P_\infty(i)) = O(i^{-1/(a-1)})$ by Equation (18) of [BC17]. As a result,

$$n^{-2\frac{a-2}{a-1}} \mathbb{E}_\infty^{(1)} \left(\left(\sum_{i=0}^{\lfloor nt \rfloor} (\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i))) \right)^2 \right) = O \left(n^{-\frac{a-2}{a-1}} \right) \xrightarrow[n \rightarrow \infty]{} 0, \quad (5.3.6)$$

so that (5.3.4) implies (5.3.3).

Now, since $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$ and since $ph^\uparrow(p+k)/(h^\uparrow(p)(p+k))$ is upperbounded by a constant, we have

$$p \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i) = p) = \sum_{k \geq 0} \frac{2k+1}{2} \nu(k) \frac{p}{p+k} \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \xrightarrow[p \rightarrow \infty]{} \frac{e_{\mathbf{q}}}{2}. \quad (5.3.7)$$

Therefore, by (5.3.1), to show (5.3.4) it suffices to see that

$$n^{-\frac{a-2}{a-1}} \sum_{i=0}^{\lfloor nt \rfloor} \frac{1}{P_\infty(i)} \xrightarrow[n \rightarrow \infty]{(d)} \int_0^t \frac{1}{\Upsilon_a^\uparrow(p_{\mathbf{q}}s)} ds,$$

which is a consequence of the end of the proof of Proposition 4.1 in [BC17]. \square

Critical case

Proof of Theorem 5.1.1 in the case $a = 2$. By the same reasoning as in the dilute case, it is enough to prove that

$$\frac{1}{(\log n)^2} \sum_{i=1}^n \theta_i \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{1}{\pi^2}. \quad (5.3.8)$$

Let us first show that

$$\frac{1}{(\log n)^2} \sum_{i=1}^n \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{1}{\pi^2}. \quad (5.3.9)$$

For all $i \geq 1$, we have

$$\mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) = \sum_{k=0}^{\infty} \frac{2k+1}{2(P_\infty(i)+k)} \frac{h^\uparrow(P_\infty(i)+k)}{h^\uparrow(P_\infty(i))} \nu(k). \quad (5.3.10)$$

So, by (5.2.3) and the fact that $h^\uparrow(p) \sim 2\sqrt{p/\pi}$, as $p \rightarrow \infty$, the expectation $\mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i) = p)$ is upperbounded by

$$O\left(\frac{1}{p}\right) + O\left(\sum_{k=1}^{\infty} \frac{1}{k\sqrt{p(p+k)}}\right) = O\left(\frac{1}{p}\right) + O\left(\frac{1}{p} \int_{1/p}^{\infty} \frac{1}{x\sqrt{1+x}} dx\right) = O\left(\frac{\log p}{p}\right).$$

Moreover, by the end of the proof of Proposition 5 in [BCM18], we know that the sequence $((i \log P_\infty(i))/(P_\infty(i) \log i))_{i \geq 2}$ is uniformly integrable, so that $((i/\log i) \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)))_{i \geq 1}$ is also uniformly integrable. Besides, using (5.2.3), (5.3.1) and the fact that $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$, it is easy to see that $\mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i))$ is equivalent in probability to

$$\sum_{k=1}^{\infty} \frac{p_{\mathbf{q}}}{k\sqrt{P_\infty(i)(P_\infty(i)+k)}} \underset{i \rightarrow \infty}{\sim} \frac{1}{P_\infty(i)} \int_{1/P_\infty(i)}^{\infty} \frac{p_{\mathbf{q}}}{u\sqrt{1+u}} du \underset{i \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}} \log P_\infty(i)}{P_\infty(i)} \underset{i \rightarrow \infty}{\sim} \frac{p_{\mathbf{q}} \log i}{P_\infty(i)}.$$

Thus, by (5.3.1),

$$\frac{i \log P_\infty(i)}{P_\infty(i) \log i} - \frac{i}{\log i} \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \xrightarrow[i \rightarrow \infty]{(L^1)} 0.$$

As a consequence, in order to show (5.3.9) it suffices to prove that

$$\frac{1}{(\log n)^2} \sum_{i=1}^n \frac{p_{\mathbf{q}} \log i}{P_\infty(i)} \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{1}{\pi^2}.$$

By Remark 5.3.4 (or by Remark 3 of [BCM18] for this precise case), we know that

$$\mathbb{E}_\infty^{(1)} \frac{i}{P_\infty(i)} \xrightarrow{i \rightarrow \infty} \mathbb{E} \frac{1}{\Upsilon_2^\uparrow(p_{\mathbf{q}})} = \mathbb{E} \frac{1}{p_{\mathbf{q}} \Upsilon_2^\uparrow(1)} = \frac{2}{\pi^2 p_{\mathbf{q}}}. \quad (5.3.11)$$

Hence, since $\sum_{i=1}^n (\log i)/i \sim (\log n)^2/2$ as $n \rightarrow \infty$, to show (5.3.9) it remains to check that

$$\frac{1}{(\log n)^2} \sum_{i=1}^n (\log i) \left(\frac{1}{P_\infty(i)} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)} \right) \right) \xrightarrow[n \rightarrow \infty]{(L^1)} 0. \quad (5.3.12)$$

We prove (5.3.12) using the same techniques as in the proof of Proposition 3 in [BCM18]. For all $i \geq 1$ and $\varepsilon > 0$, as in [BCM18], we set

$$\begin{aligned} X_i^{(\varepsilon)} &:= \frac{1}{P_\infty(i)} \mathbf{1}_{\varepsilon \leq P_\infty(i)/i \leq 1/\varepsilon} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)} \mathbf{1}_{\varepsilon \leq P_\infty(i)/i \leq 1/\varepsilon} \right) \\ Y_i^{(\varepsilon)} &:= \frac{1}{P_\infty(i)} \mathbf{1}_{P_\infty(i)/i \notin [\varepsilon, 1/\varepsilon]} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)} \mathbf{1}_{P_\infty(i)/i \notin [\varepsilon, 1/\varepsilon]} \right). \end{aligned}$$

Then, to prove (5.3.12) it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_{\infty}^{(1)} \left(\left(\frac{1}{(\log n)^2} \sum_{i=1}^n (\log i) X_i^{(\varepsilon)} \right)^2 \right) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_{\infty}^{(1)} \left| \frac{1}{(\log n)^2} \sum_{i=1}^n (\log i) Y_i^{(\varepsilon)} \right| = 0 \quad (5.3.13)$$

The second term is zero since

$$\mathbb{E}_{\infty}^{(1)} |i Y_i^{(\varepsilon)}| \leq 2 \mathbb{E}_{\infty}^{(1)} \left(\frac{i}{P_{\infty}(i)} \mathbf{1}_{P_{\infty}(i)/i \notin [\varepsilon, 1/\varepsilon]} \right)$$

and since $(i/P_{\infty}(i))_{i \geq 1}$ is uniformly integrable by Lemma 5.3.3 and is tight in \mathbb{R}_+^* by (5.3.1). For the first term of (5.3.13), let A_n be an increasing sequence tending to infinity such that $\log A_n = o(\log n)$. We compute

$$\mathbb{E}_{\infty}^{(1)} \left(\left(\frac{1}{(\log n)^2} \sum_{i=1}^n (\log i) X_i^{(\varepsilon)} \right)^2 \right) \leq \frac{1}{(\log n)^4} \sum_{i=1}^n (\log i)^2 \mathbb{E}_{\infty}^{(1)} \left((X_i^{(\varepsilon)})^2 \right) \quad (5.3.14)$$

$$+ \frac{2}{(\log n)^4} \sum_{i=1}^n \sum_{j=i+1}^{A_n i} (\log i)(\log j) \left| \mathbb{E}_{\infty}^{(1)} (X_i^{(\varepsilon)} X_j^{(\varepsilon)}) \right| \quad (5.3.15)$$

$$+ \frac{2}{(\log n)^4} \sum_{i=1}^n \sum_{j=A_n i+1}^n (\log i)(\log j) \left| \mathbb{E}_{\infty}^{(1)} (X_i^{(\varepsilon)} X_j^{(\varepsilon)}) \right|. \quad (5.3.16)$$

The quantity on the right of the first line (5.3.14) is a $O(1/(\log n)^4)$ as $n \rightarrow \infty$ since $|X_i^{(\varepsilon)}| \leq 2/(\varepsilon i)$ by definition. Using that $\sum_{j=i+1}^{A_n i} (\log j)/j \leq C(\log(A_n i)^2 - \log(i)^2) = C((\log A_n)^2 + 2(\log A_n)(\log i))$ for some constant $C > 0$, the second line (5.3.15) is upperbounded by

$$O \left(\frac{1}{(\log n)^4} \sum_{i=1}^n \sum_{j=i+1}^{A_n i} (\log i)(\log j) \frac{1}{ij} \right) = O \left(\frac{\log A_n}{(\log n)^4} \sum_{i=1}^n \frac{2(\log i)^2 + (\log A_n)(\log i)}{i} \right) = O \left(\frac{\log A_n}{\log n} \right).$$

To deal with the third line (5.3.16), it suffices to combine Lemma 5.3.5 together with the fact that $\sum_{i=1}^n (\log i)/i = O((\log n)^2)$. This proves the convergence (5.3.13), hence (5.3.12) and thus the convergence (5.3.9).

To conclude the case $a = 2$ it remains to prove (5.3.8) which is implied by the following convergence:

$$\frac{1}{(\log n)^2} \sum_{i=1}^n \left(\theta_i - \mathbb{E}_{\infty}^{(1)}(\theta_i | P_{\infty}(i)) \right) \xrightarrow[n \rightarrow \infty]{(L^2)} 0. \quad (5.3.17)$$

To prove (5.3.17), by a straightforward computation, one can see that

$$\begin{aligned} \mathbb{E}_{\infty}^{(1)} \left(\left(\sum_{i=1}^n \left(\theta_i - \mathbb{E}_{\infty}^{(1)}(\theta_i | P_{\infty}(i)) \right) \right)^2 \right) &= \sum_{i=1}^n \mathbb{E}_{\infty}^{(1)} \left(\left(\theta_i - \mathbb{E}_{\infty}^{(1)}(\theta_i | P_{\infty}(i)) \right)^2 \right) \\ &= \sum_{i=1}^n \left(\mathbb{E}_{\infty}^{(1)}(\theta_i^2) - \mathbb{E}_{\infty}^{(1)} \left(\mathbb{E}_{\infty}^{(1)}(\theta_i | P_{\infty}(i))^2 \right) \right) \\ &\leq \sum_{i=1}^n \mathbb{E}_{\infty}^{(1)}(\theta_i^2) \\ &= O(\log n), \end{aligned} \quad (5.3.18)$$

where the last line comes from the fact that, by (5.2.3) and the fact that $h^\uparrow(p) \sim 2\sqrt{p/\pi}$, as $p \rightarrow \infty$,

$$\mathbb{E}_{\infty}^{(1)}(\theta_i^2 | P_{\infty}(i) = p) = \sum_{k \geq 0} \left(\frac{2k+1}{2(p+k)} \right)^2 \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \nu(k) = O \left(\frac{1}{\sqrt{p}} \sum_{k=1}^{\infty} \frac{1}{(p+k)^{3/2}} \right) = O \left(\frac{1}{p} \right),$$

and from the fact that $\mathbb{E}_{\infty}^{(1)}(1/P_{\infty}(i)) = O(1/i)$ as $i \rightarrow \infty$ by (5.3.11). \square

Dense case

Proof of Theorem 5.1.1 in the dense case. For the dense case $a \in (3/2, 2)$, the proof is analogous to the case $a = 2$. By the same reasoning as in the dilute case, it is enough to prove that

$$\frac{1}{\log n} \sum_{i=1}^n \theta_i \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{2}{\pi \tan((2-a)\pi)}. \quad (5.3.19)$$

Let us first show that

$$\frac{1}{\log n} \sum_{i=1}^n \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \xrightarrow[n \rightarrow \infty]{(L^1)} \frac{2}{\pi \tan((2-a)\pi)}. \quad (5.3.20)$$

By splitting the sum in (5.3.10) before and after $\log i$ and using the fact that $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$, we get that there exists a constant $C > 0$ such that for all $i \geq 0$ for all $p \geq 1$,

$$\left| \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i) = p) - \sum_{k=\lfloor \log i \rfloor}^{\infty} \frac{2k+1}{2(p+k)} \frac{h^\uparrow(p+k)}{h^\uparrow(p)} \nu(k) \right| \leq C \frac{(\log i)^2}{p}.$$

As a result, by Lemma 5.3.3, as $i \rightarrow \infty$,

$$\mathbb{E}_\infty^{(1)} \left| \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) - \sum_{k=\lfloor \log i \rfloor}^{\infty} \frac{2k+1}{2(P_\infty(i)+k)} \frac{h^\uparrow(P_\infty(i)+k)}{h^\uparrow(P_\infty(i))} \nu(k) \right| \leq C \mathbb{E}_\infty^{(1)} \left(\frac{(\log i)^2}{P_\infty(i)} \right) = O \left(\frac{(\log i)^2}{i^{1/(a-1)}} \right). \quad (5.3.21)$$

Moreover, by (5.2.3), since $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$, since $P_\infty(i)$ goes to ∞ as $i \rightarrow \infty$ in probability and using a Riemann sum, we have

$$\sum_{k=\lfloor \log i \rfloor}^{\infty} \frac{2k+1}{2(P_\infty(i)+k)} \frac{h^\uparrow(P_\infty(i)+k)}{h^\uparrow(P_\infty(i))} \nu(k) \stackrel{(\mathbb{P})}{\underset{i \rightarrow \infty}{\rightsquigarrow}} \frac{1}{P_\infty(i)^{a-1}} \int_0^\infty \frac{u}{\sqrt{1+u}} \frac{p_{\mathbf{q}} \cos(a\pi)}{u^a} du. \quad (5.3.22)$$

By (5.2.3) and since $h^\uparrow(p) \sim 2\sqrt{p/\pi}$ as $p \rightarrow \infty$, we can also upperbound for all $i, p \geq 1$,

$$\sum_{k=0}^{\infty} \frac{2k+1}{2(P_\infty(i)+k)} \frac{h^\uparrow(P_\infty(i)+k)}{h^\uparrow(P_\infty(i))} \nu(k) \leq \frac{C_{\mathbf{q}}}{P_\infty(i)^{a-1}} \int_0^\infty \frac{u}{\sqrt{1+u}} \frac{p_{\mathbf{q}} \cos(a\pi)}{u^a} du, \quad (5.3.23)$$

for some constant $C_{\mathbf{q}} > 0$, so that by applying Lemma 5.3.3 and (5.3.21) we obtain that $(i\mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)))_{i \geq 1}$ is uniformly integrable. Thus, by (5.3.22) and (5.3.1), we deduce that

$$i \left(\mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) - \frac{1}{P_\infty(i)^{a-1}} p_{\mathbf{q}} \cos(a\pi) \int_0^\infty \frac{1}{u^{a-1} \sqrt{1+u}} du \right) \xrightarrow[i \rightarrow \infty]{(L^1)} 0. \quad (5.3.24)$$

Moreover, by the change of variable $t = 1/(1+u)$, the relation between the Beta and the Gamma functions and the fact that $\Gamma(1/2) = \sqrt{\pi}$,

$$\int_0^\infty \frac{1}{u^{a-1} \sqrt{1+u}} du = \frac{\Gamma(2-a)\Gamma(a-3/2)}{\sqrt{\pi}}.$$

By Euler's reflection formula,

$$\frac{2\Gamma(a)}{(a-1)\pi^{3/2}\Gamma(a-3/2)} \frac{\Gamma(2-a)\Gamma(a-3/2)}{\sqrt{\pi}} = \frac{-2}{\pi \sin(\pi a)}.$$

So by Remark 5.3.4 and (5.3.24), we obtain that

$$i\mathbb{E}_\infty^{(1)}(\theta_i) \underset{i \rightarrow \infty}{\rightsquigarrow} i\mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)^{a-1}} p_{\mathbf{q}} \cos(a\pi) \int_0^\infty \frac{1}{u^{a-1} \sqrt{1+u}} du \right) \xrightarrow[i \rightarrow \infty]{\rightarrow} \frac{2}{\pi \tan((2-a)\pi)}. \quad (5.3.25)$$

Hence, by (5.3.24), to show (5.3.20) it remains to check that

$$\frac{1}{\log n} \sum_{i=1}^n \left(\frac{1}{P_\infty(i)^{a-1}} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)^{a-1}} \right) \right) \xrightarrow[n \rightarrow \infty]{(L^1)} 0, \quad (5.3.26)$$

which follows from truncations which are similar to the case $a = 2$. To prove (5.3.26), for all $i \geq 1$ and $\varepsilon > 0$, similarly to [BCM18], we set

$$\begin{aligned} X_i^{(\varepsilon)} &:= \frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{\varepsilon \leq P_\infty(i)^{a-1}/i \leq 1/\varepsilon} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{\varepsilon \leq P_\infty(i)^{a-1}/i \leq 1/\varepsilon} \right) \\ Y_i^{(\varepsilon)} &:= \frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i)^{a-1}/i \notin [\varepsilon, 1/\varepsilon]} - \mathbb{E}_\infty^{(1)} \left(\frac{1}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i)^{a-1}/i \notin [\varepsilon, 1/\varepsilon]} \right). \end{aligned}$$

Then, to prove (5.3.26) it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_\infty^{(1)} \left(\left(\frac{1}{\log n} \sum_{i=1}^n X_i^{(\varepsilon)} \right)^2 \right) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E}_\infty^{(1)} \left| \frac{1}{\log n} \sum_{i=1}^n Y_i^{(\varepsilon)} \right| = 0 \quad (5.3.27)$$

The second term of (5.3.27) is zero since

$$\mathbb{E}_\infty^{(1)} |i Y_i^{(\varepsilon)}| \leq 2 \mathbb{E}_\infty^{(1)} \left(\frac{i}{P_\infty(i)^{a-1}} \mathbf{1}_{P_\infty(i)^{a-1}/i \notin [\varepsilon, 1/\varepsilon]} \right)$$

and since $(i/P_\infty(i)^{a-1})_{i \geq 1}$ is uniformly integrable by Lemma 5.3.3 and is tight by (5.3.1). For the first term of (5.3.27), let A_n be an increasing sequence tending to infinity such that $\log A_n = o(\log n)$. One upper bounds

$$\mathbb{E}_\infty^{(1)} \left(\left(\frac{1}{\log n} \sum_{i=1}^n X_i^{(\varepsilon)} \right)^2 \right) \leq \frac{1}{(\log n)^2} \sum_{i=1}^n \mathbb{E}_\infty^{(1)} \left((X_i^{(\varepsilon)})^2 \right) \quad (5.3.28)$$

$$+ \frac{2}{(\log n)^2} \sum_{i=1}^n \sum_{j=i+1}^{A_n i} \left| \mathbb{E}_\infty^{(1)} (X_i^{(\varepsilon)} X_j^{(\varepsilon)}) \right| \quad (5.3.29)$$

$$+ \frac{2}{(\log n)^2} \sum_{i=1}^n \sum_{j=A_n i+1}^n \left| \mathbb{E}_\infty^{(1)} (X_i^{(\varepsilon)} X_j^{(\varepsilon)}) \right|. \quad (5.3.30)$$

Since for all $i \geq 1$ we have $|X_i^{(\varepsilon)}| \leq 2/(i\varepsilon)$, the first term (5.3.28) is a $O(1/(\log n)^2)$, the second line (5.3.29) is a $O((\log A_n)/(\log n))$ and the third one is controlled using Lemma 5.3.5. This proves (5.3.27), hence (5.3.26) and thus (5.3.20). To end the proof of Theorem 5.1.1, it is enough to check that

$$\frac{1}{\log n} \sum_{i=1}^n \left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \right) \xrightarrow[n \rightarrow \infty]{(L^2)} 0, \quad (5.3.31)$$

which implies (5.3.19). By (5.3.18) and by (5.3.25), we know that

$$\mathbb{E}_\infty^{(1)} \left(\left(\sum_{i=1}^n (\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i))) \right)^2 \right) \leq \sum_{i=1}^n \mathbb{E}_\infty^{(1)}(\theta_i^2) \leq \sum_{i=1}^n \mathbb{E}_\infty^{(1)} \theta_i = O(\log n),$$

hence (5.3.31), thus ending the proof. \square

5.3.3 Application: comparison of fpp and graph distance in the dilute case

We apply our results on the fpp geodesics to show Corollary 5.1.2 which states that in the dilute case, fpp balls are included in balls for the graph distance with a suitable radius.

Proof of Corollary 5.1.2. Let $\varepsilon \in (0, 1)$. We reason conditionally on the perimeter process P_∞ . Let e be an edge on $\partial \bar{\mathbf{e}}_n$, then, by (5.2.4), the number of faces in the fpp geodesic from e to the root face f_r in \mathbf{m}_∞ has the same law as $\sum_{i=0}^{n-1} X_i$ where the X_i 's are i.i.d. Bernoulli random variables of parameters θ_i . Thus, by a union bound on e ,

$$\begin{aligned} & \mathbb{P}_\infty^{(1)}\left(\bar{\mathbf{e}}_n \not\subseteq \overline{\text{Ball}}_{\lfloor (1+\varepsilon)\sum_{i=0}^{n-1}\theta_i \rfloor}^\dagger(\mathbf{m}_\infty) \mid P_\infty\right) \\ & \leq 2P_\infty(n)\mathbb{P}_\infty^{(1)}\left(\sum_{i=0}^{n-1} X_i \geq (1+\varepsilon)\sum_{i=0}^{n-1}\theta_i \mid P_\infty\right) \\ & \leq 4P_\infty(n)\exp\left(-c\varepsilon^2\sum_{i=0}^{n-1}\theta_i\right) \quad (\text{Chernoff}) \\ & \leq 4P_\infty(n)\exp\left(-c\varepsilon^2\sum_{i=0}^{n-1}\frac{1}{2P_\infty(i)+1}\right) \\ & \leq 4n^{\frac{1+\varepsilon}{a-1}}\exp\left(-\frac{c\varepsilon^2}{3}n^{\frac{a-2-\varepsilon}{a-1}}\right) \end{aligned}$$

a.s. for all n large enough, where c is some absolute constant from Chernoff's inequality and where the last inequality comes from the fact that for all $\varepsilon > 0$, a.s. for n large enough we have

$$n^{\frac{1-\varepsilon}{a-1}} \leq P_\infty(n) \leq n^{\frac{1+\varepsilon}{a-1}} \quad (5.3.32)$$

by Lemma 10.9 of [Cur23].

By summing over n , by Borel-Cantelli's lemma, we deduce that almost surely, for all $\varepsilon > 0$, for all n large enough, we have

$$\bar{\mathbf{e}}_n \subseteq \overline{\text{Ball}}_{\lfloor (1+\varepsilon)\sum_{i=0}^{n-1}\theta_i \rfloor}^\dagger(\mathbf{m}_\infty).$$

Next, let us prove that almost surely,

$$\sum_{i=0}^{n-1}\theta_i \underset{n \rightarrow \infty}{\sim} \frac{e_{\mathbf{q}}}{2} \sum_{i=0}^{n-1} \frac{1}{P_\infty(i)} \underset{n \rightarrow \infty}{\sim} e_{\mathbf{q}} \sum_{i=0}^{n-1} \frac{\mathcal{E}_i}{2P_\infty(i)}. \quad (5.3.33)$$

Let $\varepsilon > 0$. For the first equivalent of (5.3.33), one can see that we have uniformly in $i \geq 0$, for all $p \geq 1$,

$$\begin{aligned} \mathbb{E}_\infty^{(1)}\left(\exp\left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i \mid P_\infty(i))\right) \mid P_\infty(i) = p\right) & \leq 1 + C\mathbb{E}_\infty^{(1)}\left(\left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i \mid P_\infty(i))\right)^2 \mid P_\infty(i) = p\right) \\ & \leq 1 + C' \frac{1}{p^{a-1}} \end{aligned}$$

for some constants $C, C' > 0$, where the first inequality comes from the fact that $\theta_i \in [0, 1]$ and the second one stems from (5.3.5). As a result,

$$\mathbb{E}_\infty^{(1)}\left(\exp\left(\sum_{i=0}^{n-1}\left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i \mid P_\infty(i)) - C' \frac{1}{P_\infty(i)^{a-1}}\right)\right)\right) \leq 1$$

Thus, for all $\delta > 0$, we have

$$\mathbb{P}_\infty^{(1)}\left(\sum_{i=0}^{n-1}\left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i \mid P_\infty(i)) - C' \frac{1}{P_\infty(i)^{a-1}}\right) \geq \delta n^{\frac{a-2}{a-1}-\varepsilon}\right) \leq \exp\left(-\delta n^{\frac{a-2}{a-1}-\varepsilon}\right).$$

So, taking $\varepsilon < (a-2)/(a-1)$, by Borel-Cantelli's lemma, we have a.s. as $n \rightarrow \infty$,

$$\sum_{i=0}^{n-1}\left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i \mid P_\infty(i)) - C' \frac{1}{P_\infty(i)^{a-1}}\right) \leq o\left(n^{\frac{a-2}{a-1}-\varepsilon}\right).$$

Then, applying (5.3.32), we deduce that a.s. $\sum_{i=0}^n 1/P_\infty(i)^{a-1} = O(\sum_{i=1}^n 1/i^{1-\varepsilon}) = O(n^\varepsilon)$, so

$$\sum_{i=0}^{n-1} \left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \right) \leq o\left(n^{\frac{a-2}{a-1}-\varepsilon}\right).$$

By the same reasoning, we also get almost surely as $n \rightarrow \infty$

$$-\sum_{i=0}^{n-1} \left(\theta_i - \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \right) \leq o\left(n^{\frac{a-2}{a-1}-\varepsilon}\right).$$

Besides, by (5.3.32), we have a.s. $\sum_{i=0}^\infty 1/P_\infty(i) = \infty$ so that by (5.3.7), we deduce that

$$\sum_{i=0}^{n-1} \mathbb{E}_\infty^{(1)}(\theta_i | P_\infty(i)) \stackrel{\text{(a.s.)}}{\underset{n \rightarrow \infty}{\sim}} \sum_{i=0}^{n-1} \frac{e_{\mathbf{q}}}{2} \frac{1}{P_\infty(i)},$$

hence the first asymptotic in (5.3.33) using that $n^{\frac{a-2}{a-1}-\varepsilon} = o(\sum_{i=0}^{n-1} 1/P_\infty(i))$ a.s. by applying (5.3.32) once more. For the second asymptotic in (5.3.33), we apply Bernstein's concentration inequality for exponential random variables (see e.g. Corollary 2.10 of [Tal94]) and get that for $\varepsilon > 0$ small enough, a.s. for all $\delta > 0$, for n large enough,

$$\begin{aligned} \mathbb{P}_\infty^{(1)} \left(\left| \sum_{i=0}^{n-1} \frac{\mathcal{E}_i - 1}{P_\infty(i)} \right| > \delta n^{\frac{a-2}{a-1}-\varepsilon} \middle| P_\infty \right) &\leq 2 \exp \left(-C'' \min \left(\frac{\delta^2 n^{2\frac{a-2}{a-1}-2\varepsilon}}{\sum_{i=0}^{n-1} 1/P_\infty(i)^2}, \frac{\delta n^{\frac{a-2}{a-1}-\varepsilon}}{\max_{0 \leq i \leq n-1} 1/P_\infty(i)} \right) \right) \\ &\leq 2 \exp \left(-C'' \delta n^{\frac{a-2}{a-1}-\varepsilon} \right), \end{aligned}$$

where $C'' > 0$ is an absolute constant and where the second inequality comes from (5.3.32) and from the fact that $2/(a-1) \geq 4/3 > 1$ since $a \leq 5/2$ so that a.s. $\sum_{i=0}^\infty 1/P_\infty(i)^2 < \infty$, hence the second equivalent of (5.3.33) by Borel-Cantelli's lemma and by (5.3.32). The almost sure asymptotic (5.3.33) ends the proof since $\bar{\mathfrak{c}}_n$ is the hull of the fpp ball of radius $\sum_{i=0}^{n-1} \mathcal{E}_i / (2P_\infty(i))$. \square

We next turn to the proof of Corollary 5.1.3

Proof of Corollary 5.1.3. Recall that $\bar{\mathfrak{c}}_n$ is the hull of the fpp ball of radius $T_n = \sum_{i=0}^{n-1} \mathcal{E}_i / (2P_\infty(i))$. The result follows from the same lines as in the beginning of the proof of Corollary 5.1.2, using (5.3.8) and using the fact that $T_n / \log n$ converges in probability towards $1/(p_{\mathbf{q}} \pi^2)$ by Proposition 3 of [BCM18]. \square

5.3.4 Second application: the diameter of dense maps is logarithmic for the dual graph distance

We prove in this subsection Theorem 5.1.4 which states that in the case $a \in (3/2, 2)$, the diameter of random maps of law $\mathbb{P}^{(\ell)}$ grows at rate at most $\log \ell$. It is not a consequence of Theorem 5.1.1, but it relies on its proof, more precisely on the inequality (5.3.23). The proof relies on the martingales introduced in Chapter 2. Before entering the proof of the theorem, we recall the notion of maps with a target face. A planar map with a target face \mathbf{m}_\square is a planar map with a distinguished face \square which is not the root face f_r . For all $\ell, p \geq 1$, we denote by $\mathcal{M}_p^{(\ell)}$ the set of planar maps of perimeter 2ℓ with a target face of degree $2p$. The weight $w_{\mathbf{q}}(\mathbf{m}_\square)$ of a map $\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}$ is defined by

$$w_{\mathbf{q}}(\mathbf{m}_\square) = \prod_{f \in \text{Faces}(\mathbf{m}_\square) \setminus \{f_r, \square\}} q^{\deg(f)/2}.$$

The partition function of such maps is given by Equation (3.9) of [Cur23] for all $\ell, p \geq 1$:

$$W_p^{(\ell)} := \sum_{\mathbf{m}_\square \in \mathcal{M}_p^{(\ell)}} w_{\mathbf{q}}(\mathbf{m}_\square) = \frac{1}{2} h_p^\downarrow(\ell) c_{\mathbf{q}}^{\ell+p},$$

where

$$h_p^\downarrow(\ell) = h^\downarrow(\ell) h^\downarrow(p) \frac{\ell}{\ell+p} \quad \text{with} \quad h^\downarrow(\ell) = \frac{1}{2^{2\ell}} \binom{2\ell}{\ell}. \quad (5.3.34)$$

We also set by convention $h_p^\downarrow(\ell) = \mathbf{1}_{\ell=-p}$ for all $\ell \leq 0$. We denote by $\mathbb{P}_p^{(\ell)}$ the associated Boltzmann probability measure. The peeling exploration of a map with a target face is defined in the same way as for maps without target face except that at each gluing event, we choose to fill in the hole which does not contain the target face. When we discover the target face, the exploration ends and by convention, we say that the perimeter is absorbed at $-p$. The perimeter process under $\mathbb{P}_p^{(\ell)}$, which is denoted by $(P_p(n))_{n \geq 0}$ is then a Doob h_p^\downarrow -transform of the ν random walk starting at ℓ which is absorbed when it reaches $-p$ by Proposition 4.7 of [Cur23]. Its law can also be interpreted as a ν -random walk starting at ℓ conditioned to stay positive until it jumps and dies at $-p$ by Proposition 5.3 of [Cur23].

Next, we introduce two supermartingales.

Lemma 5.3.6. *Suppose \mathbf{q} is of type $a \in (3/2, 2)$. Let $\ell \geq 1$. Under $\mathbb{P}_\ell^{(1)}$, conditionally on P_ℓ , let $(X_i)_{i \geq 0}$ be a family of independent Bernoulli random variables of parameters $(2\Delta P_\ell(i) + 1)/(2(P_\ell(i) + 1))\mathbf{1}_{\Delta P_\ell(i) \geq 0}$ and $P_\ell(i) \geq 1$. Then, there exists a constant $C'_\mathbf{q}$ such that for all $\lambda \in (0, 1)$, the process $(M_n^{(\ell, \lambda)})_{n \geq 0}$ defined by*

$$\forall n \geq 0, \quad M_n^{(\ell, \lambda)} = \exp\left(\lambda \left(\sum_{i=0}^{n-1} X_i - C'_\mathbf{q} \sum_{i=0}^{n-1} \frac{1}{P_\ell(i)^{a-1}} \mathbf{1}_{P_\ell(i) \geq 1}\right)\right)$$

is a supermartingale with respect to the filtration $(\mathcal{G}_n)_{n \geq 0}$ defined by

$$\forall n \geq 0, \quad \mathcal{G}_n = \sigma((P_\ell(i))_{0 \leq i \leq n}, (X_i)_{0 \leq i \leq n-1}).$$

Proof. The process is clearly adapted. Moreover, by (5.3.34) and (5.2.1), for all $p \geq 1$, we have $h_\ell^\downarrow(p) = h^\downarrow(\ell)h^\uparrow(p)/(2(\ell + p))$, so that for all $n \geq 0$,

$$\begin{aligned} \mathbb{E}_\ell^{(1)}(\exp(\lambda X_n) | \mathcal{G}_n) &= \mathbb{E}_\ell^{(1)}\left(\left(e^\lambda - 1\right) \frac{2\Delta P_\ell(n) + 1}{2P_\ell(n) + 1} \mathbf{1}_{\Delta P_\ell(n) \geq 0 \text{ and } P_\ell(n) \geq 1} + 1 \mid \mathcal{G}_{n-1}\right) \\ &= 1 + (e^\lambda - 1) \sum_{k=0}^{\infty} \frac{2k + 1}{2P_\ell(n) + 2k} \frac{h_\ell^\downarrow(P_\ell(n) + k)}{h_\ell^\downarrow(P_\ell(n))} \nu^{(k)} \mathbf{1}_{P_\ell(n) \geq 1} \\ &\leq 1 + (e^\lambda - 1) \sum_{k=0}^{\infty} \frac{2k + 1}{2P_\ell(n) + 2k} \frac{h^\uparrow(P_\ell(n) + k)}{h^\uparrow(P_\ell(n))} \nu^{(k)} \mathbf{1}_{P_\ell(n) \geq 1} \\ &\leq 1 + C'_\mathbf{q} \frac{\lambda}{P_\ell(n)^{a-1}} \mathbf{1}_{P_\ell(n) \geq 1} \quad \text{by (5.3.23)} \\ &\leq \exp\left(C'_\mathbf{q} \frac{\lambda}{P_\ell(n)^{a-1}} \mathbf{1}_{P_\ell(n) \geq 1}\right), \end{aligned}$$

where $C'_\mathbf{q} > 0$ is a constant which depends on \mathbf{q} , hence the desired result. \square

We then introduce a second supermartingale, which is closely related to the martingale (2.6.1) in Chapter 2 and corresponds to the supermartingale of Lemma 2.6.2 in Chapter 2 in the case $a = 2$.

Lemma 5.3.7. *Suppose \mathbf{q} is of type $a \in (3/2, 5/2]$. Let $\lambda > 0$ such that $\lambda < \min_{k, \ell \geq 1} k^{a-1}(\nu((-\infty, -k]) - \nu(-k - \ell))$ (such a λ exists thanks to (5.2.3)). Then the process $(\widetilde{M}_n^{(\ell, \lambda)})_{n \geq 0}$ defined by*

$$\forall n \geq 0, \quad \widetilde{M}_n^{(\ell, \lambda)} = \frac{1}{h_\ell^\downarrow(P_\ell(n))} \exp\left(\lambda \sum_{i=0}^{n-1} \frac{1}{P_\ell(i)^{a-1}} \mathbf{1}_{P_\ell(i) \geq 1}\right)$$

is a supermartingale with respect to the natural filtration $(\mathcal{F}_n)_{n \geq 0}$ associated with the process P_ℓ .

Proof. The proof is the same as the proof of Lemma 2.6.2 in Chapter 2 and boils down to the fact that for all $n \geq 0$, by definition of λ and since $\exp(x) \leq 1/(1-x)$ for all $x \in [0, 1)$, on the event $\{P_\ell(n) \geq 1\}$, we have

$$\exp\left(\frac{\lambda}{P_\ell(n)^{a-1}}\right) \leq \frac{1}{1 - \lambda/P_\ell(n)^{a-1}} \leq (\nu([-P_\ell(n) + 1, \infty)) + \nu(-P_\ell(n) - \ell))^{-1}.$$

It then suffices to use that P_ℓ is a Doob h_ℓ^\downarrow -transform of the ν -random walk. \square

Proof of Theorem 5.1.4. It suffices to prove that there exists a constant $C_{\mathbf{q}} > 0$ large enough such that if E is a uniform random edge of the map, then

$$\mathbb{P}^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, E) \geq C_{\mathbf{q}} \log \ell) \leq \ell^{-2}, \quad (5.3.35)$$

where $d_{\text{gr}}^{\dagger}(f_r, E)$ is defined as the smallest dual graph distance from f_r to a face which is adjacent to E . Indeed,

$$\begin{aligned} \mathbb{P}^{(\ell)}(\exists e \in \text{Edges}(\mathbf{m}), d_{\text{gr}}^{\dagger}(f_r, e) \geq C_{\mathbf{q}} \log \ell) \\ \leq \mathbb{P}^{(\ell)}(\exists e \in \text{Edges}(\mathbf{m}), d_{\text{gr}}^{\dagger}(f_r, e) \geq C_{\mathbf{q}} \log \ell \text{ and } \#\text{Edges}(\mathbf{m}) \leq \ell^a) + \mathbb{P}^{(\ell)}(\#\text{Edges}(\mathbf{m}) \geq \ell^a) \\ \leq \mathbb{E}^{(\ell)}(\#\{e \in \text{Edges}(\mathbf{m}), d_{\text{gr}}^{\dagger}(f_r, e) \geq C_{\mathbf{q}} \log \ell\} \mathbf{1}_{\#\text{Edges}(\mathbf{m}) \leq \ell^a}) + \mathbb{P}^{(\ell)}(\#\text{Edges}(\mathbf{m}) \geq \ell^a) \\ \leq \ell^a \mathbb{P}^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, E) \geq C_{\mathbf{q}} \log \ell) + \mathbb{P}^{(\ell)}(\#\text{Edges}(\mathbf{m}) \geq \ell^a) \xrightarrow{\ell \rightarrow \infty} 0, \end{aligned}$$

where the convergence comes from (5.3.35) and from the fact that $\#\text{Edges}(\mathbf{m})/\ell^{a-1/2}$ converges in distribution under $\mathbb{P}^{(\ell)}$ as $\ell \rightarrow \infty$ by Proposition 10.4 of [Cur23]. In order to prove (5.3.35), one can see that by unzipping the uniform random edge, using Equation (2.3.4) of Chapter 2, we have

$$\begin{aligned} \mathbb{P}^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, E) \geq C_{\mathbf{q}} \log \ell) &= \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{E}_1^{(\ell)} \left(\frac{1}{\#\text{Edges}(\mathbf{m}_{\square}) - 1} \mathbf{1}_{d_{\text{gr}}^{\dagger}(f_r, \square) \geq C_{\mathbf{q}} \log \ell} \right) \\ &\leq \frac{W_1^{(\ell)}}{W^{(\ell)}} \mathbb{P}_1^{(\ell)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq C_{\mathbf{q}} \log \ell) \\ &= O(\ell^{a-1/2}) \mathbb{P}_{\ell}^{(1)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq C_{\mathbf{q}} \log \ell), \end{aligned}$$

where in the last line, we used that $W_1^{(\ell)}/W^{(\ell)} = \mathbb{E}^{(\ell)} \#\text{Edges}(\mathbf{m}) = O(\ell^{a-1/2})$ by Proposition 10.4 and Equation (10.8) of [Cur23]. Note also that in the last line, we have exchanged the root and the target face. Thus, it is enough to show that

$$\mathbb{P}_{\ell}^{(1)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq C_{\mathbf{q}} \log \ell) = O(\ell^{-4}). \quad (5.3.36)$$

By the reversed uniform exploration, we know that the length of the fpp geodesic from f_r to \square has $\sum_{i=0}^{\tau_{-\ell}-1} X_i$ faces, where the X_i 's are independent from the perimeter process and are independent Bernoulli random variables of parameters $\mathbf{1}_{\Delta P_{\ell}(i) \geq 0} (2\Delta P_{\ell}(i) + 1)/(2P_{\ell}(i + 1))$ by (5.2.4), hence

$$d_{\text{gr}}^{\dagger}(f_r, \square) \leq \sum_{i=0}^{\tau_{-\ell}-1} X_i.$$

Let $\lambda > 0$ satisfying the hypothesis of Lemma 5.3.7 and such that $\lambda/C'_{\mathbf{q}} < 1$, where $C'_{\mathbf{q}}$ appears in Lemma 5.3.6. Then, applying successively Markov's inequality, the fact that $h_{\ell}^{\downarrow}(-\ell) = 1$ and Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{P}_{\ell}^{(1)} \left(\sum_{i=0}^{\tau_{-\ell}-1} X_i \geq C_{\mathbf{q}} \log \ell \right) &\leq \ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}})} \mathbb{E}_{\ell}^{(1)} \exp \left(\frac{\lambda}{2C'_{\mathbf{q}}} \sum_{i=0}^{\tau_{-\ell}-1} X_i \right) \\ &= \ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}})} \mathbb{E}_{\ell}^{(1)} \sqrt{M_{\tau_{-\ell}}^{(\ell, \lambda/C'_{\mathbf{q}})} \widetilde{M}_{\tau_{-\ell}}^{(\ell, \lambda)}} \\ &\leq \ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}})} \sqrt{\mathbb{E}_{\ell}^{(1)} \left(M_{\tau_{-\ell}}^{(\ell, \lambda/C'_{\mathbf{q}})} \right) \mathbb{E}_{\ell}^{(1)} \left(\widetilde{M}_{\tau_{-\ell}}^{(\ell, \lambda)} \right)} \\ &\leq \ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}})} \sqrt{\mathbb{E}_{\ell}^{(1)} \left(M_0^{(\ell, \lambda/C'_{\mathbf{q}})} \right) \mathbb{E}_{\ell}^{(1)} \left(\widetilde{M}_0^{(\ell, \lambda)} \right)}, \end{aligned}$$

where the last inequality comes from Lemmas 5.3.6 and 5.3.7 and from the fact that $\tau_{-\ell}$ is a stopping time for the filtrations $(\mathcal{F}_n)_{n \geq 0}$ and $(\mathcal{G}_n)_{n \geq 0}$. Therefore,

$$\mathbb{P}_{\ell}^{(1)}(d_{\text{gr}}^{\dagger}(f_r, \square) \geq C_{\mathbf{q}} \log \ell) \leq \ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}})} \frac{1}{\sqrt{h_{\ell}^{\downarrow}(1)}} = O\left(\ell^{-\lambda C_{\mathbf{q}}/(2C'_{\mathbf{q}}) + 1/2}\right).$$

This proves (5.3.36) by taking $C_{\mathbf{q}}$ large enough and thus completes the proof. \square

5.4 Scaling limit of the coalescing flow of geodesics

This section and the next one describe the scaling limit of the fpp geodesics to the root under $\mathbb{P}^{(\ell)}$.

5.4.1 The self-similar Markov process arising as the scaling limit of the perimeter process

We recall the scaling limit of the perimeter process which is given in terms of positive self-similar Markov processes. These positive self similar Markov processes were first introduced and studied in [BBCK18]. As in p. 40 of [BBCK18], let Λ be the image by $x \mapsto \log x$ of the measure

$$\lambda(dx) := \frac{\Gamma(a)}{\pi} \left(\frac{1}{(x(1-x))^a} \mathbf{1}_{1/2 < x < 1} + \cos(a\pi) \frac{1}{(x(x-1))^a} \mathbf{1}_{x > 1} \right) dx.$$

Let ξ be the Lévy process with no Brownian part, with Lévy measure Λ and with drift

$$\frac{\Gamma(3-a)}{2\Gamma(4-2a)\sin(\pi(a-1))} + \frac{\Gamma(a)B_{1/2}(1-a, 3-a)}{\pi}, \quad (5.4.1)$$

where $B_{1/2}(x, y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1} dt$ is the incomplete Beta function. As in [BBCK18], for all $\alpha \in \mathbb{R}$, we define the Lamperti time substitution

$$\forall t \geq 0, \quad \tau_\alpha(t) = \inf \left\{ r \geq 0, \int_0^r e^{-\alpha\xi(s)} ds \geq t \right\}.$$

For all $x > 0$, the self-similar Markov process $X_a^{(\alpha)}$ associated with ξ of index α is defined by

$$\forall t \geq 0, \quad X_a^{(\alpha)}(t) = x e^{\xi(\tau_\alpha(tx^\alpha))}.$$

Note that $X_a^{(0)}(t) = x e^{\xi(t)}$. We can now describe the scaling limit of the perimeter process under $\mathbb{P}^{(\ell)}$ by Proposition 6.6 of [BBCK18]:

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{P(\ell^{a-1}t)}{\ell} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{} \left(X_a^{(1-a)}(c_a p_{\mathbf{q}} t) \right)_{t \geq 0}, \quad (5.4.2)$$

for the Skorokhod J_1 topology, where $c_a := \pi/\Gamma(a)$. Recall the notation $T_n = \sum_{i=0}^{n-1} \mathcal{E}_i / (2P(i))$ for all $n \geq 0$ from Subsection 5.2.1 where the \mathcal{E}_i 's are i.i.d. exponential random variables of parameter 1. Then, jointly with (5.4.2), for the topology of uniform convergence on compact sets,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{T_{\lfloor \ell^{a-1}t \rfloor}}{\ell^{a-2}} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} \int_0^{c_a p_{\mathbf{q}} t} \frac{1}{X_a^{(1-a)}(s)} ds \right)_{t \geq 0}. \quad (5.4.3)$$

In particular, combining (5.4.2) with (5.4.3) gives the scaling limit of the perimeter of the fpp balls. For all $r \geq 0$, let $\Theta(r) = \inf\{n \geq 0, T_n \geq r\}$. Then we have jointly with (5.4.2) and (5.4.3)

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{1}{2\ell} \left| \partial \overline{\text{Ball}}_{\ell^{a-2}r}^{\text{fpp}}(\mathbf{m}) \right| \right) = \left(\frac{P(\Theta(\ell^{a-2}r))}{\ell} \right)_{r \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_a^{(2-a)}(2c_a p_{\mathbf{q}} r) \right)_{r \geq 0}.$$

Actually, in order to introduce our coalescing flow of diffusions, we will also work at a different time-scale, so that the scaling limit of the perimeter process is $X_a^{(0)}$. Let $(\tilde{P}(t))_{t \geq 0}$ be the continuous time Markov process constructed from P under $\mathbb{P}^{(\ell)}$ where at each step n we wait a time $\mathcal{E}_n / (2P(n)^{a-1})$, where $\mathcal{E}_n / (2P(n)) = T_{n+1} - T_n$. We have the scaling limit jointly with the previous ones

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\sum_{n=0}^{\lfloor \ell^{a-1}t \rfloor} \frac{\mathcal{E}_n}{2P(n)^{a-1}} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} \int_0^{c_a p_{\mathbf{q}} t} \frac{1}{X_a^{(1-a)}(s)^{a-1}} ds \right)_{t \geq 0},$$

and therefore we have the convergence for the Skorokhod J_1 topology jointly with the previous ones

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{\tilde{P}(t)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_a^{(0)}(2c_a p_{\mathbf{q}} t) \right)_{t \geq 0} = \left(e^{\xi(2c_a p_{\mathbf{q}} t)} \right)_{t \geq 0}. \quad (5.4.4)$$

Note that in the case $a = 2$, the process \tilde{P} is parametrized by the fpp distance to the root.

5.4.2 A family of coalescing diffusions with jumps

We denote by $\{x\} := x - \lfloor x \rfloor$ the fractional part and by $x_+ = x \mathbf{1}_{x>0}$ the positive part of a real number x . For every integer $d \geq 1$, we denote by $C_1(\mathbb{R}^d)$ the set of continuous non-negative bounded functions f on \mathbb{R}^d which are zero on a neighbourhood of zero. For all $z \in \mathbb{R}_+^*$, $u \in [0, 1]$, $x \in \mathbb{R}$ we set

$$g(x, z, u) = \left(\{x - u\} - \left(\frac{z-1}{z} \right)_+ \right)_+ z - \{x - u\} + \frac{1}{2} \left(1 - \left(\frac{1}{z} \wedge z \right) \right).$$

Note that the function g is 1-periodic in x . Moreover, for all $z \in \mathbb{R}_+^*$, $x \in \mathbb{R}$, $u \in [0, 1]$ we have

$$g(x, z, u) \leq 3|1 - z| \quad (5.4.5)$$

and

$$\int_0^1 g(x, z, u) du = 0. \quad (5.4.6)$$

Let N be a Poisson point process (PPP) of intensity $2c_a p_{\mathbf{q}} dt \lambda(dz) du$. Let \tilde{N} be the associated compensated Poisson point process. For all $v \in \mathbb{R}$, let us consider the SDE with jumps

$$\forall t \geq 0, \quad X_t(v) = v + \int_0^t \int_0^\infty \int_0^1 g(X_{s-}(v), z, u) \tilde{N}(ds, dz, du). \quad (5.4.7)$$

The process $(X_t(v))_{t \geq 0}$ can also be seen as a time homogeneous jump diffusion with kernel K defined by for all $f \in C_1(\mathbb{R})$, for all $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} f(y) K(x, dy) = 2c_a p_{\mathbf{q}} \int_0^\infty \lambda(dz) \int_0^1 du f(g(x, z, u)).$$

See e.g. [JS87] Chapter III Paragraph 2c. We will only rely on [JS87] and on a general result of Li and Pu [LP12] but one can also see for instance the books [Jac79], [IW89] and [App09] for more details on the theory of stochastic differential equations with jumps.

Lemma 5.4.1. *The existence and uniqueness (up to undistinguishability) of strong solutions to (5.4.7) holds.*

Proof. In the case $a \in (3/2, 2] \cup \{5/2\}$, the above result could be shown using the classical result under Lipschitz type assumptions which can be found e.g. in Theorem 9.1 of Chapter IV of [IW89].

However, in order to cover the whole range $a \in (3/2, 5/2]$, we use Theorem 5.1 of [LP12]. To apply this theorem, we need to check the conditions (3.a), (3.b) and (5.a) of [LP12]. The condition (3.a) of [LP12] is automatically satisfied by our SDE (5.4.7) since the corresponding coefficients are zero. Moreover, for our SDE (5.4.7),

- The condition (3.b) of [LP12] is equivalent to the fact that $x \mapsto x + g(x, z, u)$ is non-decreasing for all $z \in \mathbb{R}_+^*$, $u \in [0, 1]$ and that for all $m \geq 1$, there exists $C_m \geq 0$ such that

$$\forall x, y \in [-m, m], \quad \int_0^\infty \lambda(dz) \int_0^1 du |g(x, z, u) - g(y, z, u)|^2 \leq C_m |x - y|; \quad (5.4.8)$$

- The condition (5.a) of [LP12] is equivalent to the fact that there exists a constant $C \geq 0$ such that for all $x \in \mathbb{R}$,

$$\int_{(0, \infty) \setminus \{1\}} \lambda(dz) \int_{[0, 1]} dug(x, z, u)^2 \leq C(1 + x^2).$$

Let us start with the condition (5.a). Using (5.4.5), we have for all $x \in \mathbb{R}$,

$$\int_{(0, \infty) \setminus \{1\}} \lambda(dz) \int_{[0, 1]} dug(x, z, u)^2 \leq 9 \int_{(1/2, 1) \cup (1, \infty)} (z-1)^2 \lambda(dz) < \infty \quad (5.4.9)$$

since the integral on the right is finite by definition of λ . Thus, the condition (5.a) of [LP12] is satisfied.

To conclude, it now remains to check condition (3.b) of [LP12].

It is easy to see that for all $z \in \mathbb{R}_+^*$, for all $u \in [0, 1]$, the function $x \mapsto x + g(x, z, u)$ is non-decreasing. Indeed, the points $n + u$ are fixed points of $x \mapsto x + g(x, z, u) - g(0, z, 0)$ for all $n \in \mathbb{Z}$. Moreover, on any interval of the form $[n + u, n + 1 + u]$ for some $n \in \mathbb{Z}$, the function $x \mapsto x + g(x, z, u)$ is constant on the interval $[n + u, n + u + ((z - 1)/z)_+]$ and is then linearly increasing on $[n + u + ((z - 1)/z)_+, n + 1 + u]$ with slope z .

To check condition (3.b), it remains to prove (5.4.8). Actually, it suffices to show that there exists $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$\int_0^\infty \lambda(dz) \int_0^1 du |g(x, z, u) - g(y, z, u)|^2 \leq C|x - y| \quad (5.4.10)$$

By symmetry, we may assume that $x < y$ and, by (5.4.9), we may assume that $y - x < 1/2$. Using the inequality $(r + s)^2 \leq 2(r^2 + s^2)$, for all $x, y \in \mathbb{R}$,

$$\int_0^\infty \lambda(dz) \int_0^1 du |g(x, z, u) - g(y, z, u)|^2 \leq 2 \int_0^\infty \lambda(dz) \int_0^1 du |(z - 1)(\{x - u\} - \{y - u\})|^2 \quad (5.4.11)$$

$$+ 2 \int_0^\infty \lambda(dz) \int_0^1 du \left| \left(\left(\frac{z - 1}{z} \right)_+ - \{x - u\} \right) - \left(\left(\frac{z - 1}{z} \right)_+ - \{y - u\} \right) \right|_+^2. \quad (5.4.12)$$

The integral on the right of the first line (5.4.11) is upperbounded by $8 \int_0^\infty \lambda(dz) (z - 1)^2 |x - y|$ using the fact that, since $x < y$ and $y - x < 1$, we have

$$\begin{aligned} \int_0^1 |\{x - u\} - \{y - u\}| du &= \int_0^1 |u - \{y - x + u\}| du \\ &= \int_0^1 |y - x| \mathbf{1}_{y - x + u \leq 1} du + \int_0^1 |1 - (y - x)| \mathbf{1}_{y - x + u \geq 1} du \\ &\leq y - x + \int_0^1 \mathbf{1}_{u \geq 1 - (y - x)} du \leq 2(y - x). \end{aligned} \quad (5.4.13)$$

For the second line (5.4.12) we split the integral between $z < 3/2$ and $z \geq 3/2$. The part of the integral in (5.4.12) for $z > 3/2$ is upperbounded by

$$4 \int_{3/2}^\infty \lambda(dz) \int_0^1 du |\{x - u\} - \{y - u\}| \leq C'|x - y|$$

for some constant $C' > 0$ which does not depend on x, y , using that $t \mapsto t_+$ is 1-Lipschitz, then (5.4.13) and that $\int_{3/2}^\infty \lambda(dz) < \infty$. The part of the integral in (5.4.12) for $z < 3/2$ can be rewritten

$$\int_1^{3/2} \lambda(dz) \int_0^1 du \left(\left(\frac{z - 1}{z} - u \right)_+ \right)^2 \quad (5.4.14)$$

$$- \int_1^{3/2} \lambda(dz) \int_0^1 du \left(\frac{z - 1}{z} - \{x - u\} \right)_+ \left(\frac{z - 1}{z} - \{y - u\} \right)_+. \quad (5.4.15)$$

The first line (5.4.14) equals $(1/3) \int_1^{3/2} \lambda(dz) ((z - 1)/z)^3$. The second line (5.4.15) can be rewritten

$$\begin{aligned} & - \int_1^{3/2} \lambda(dz) \int_0^{(z-1)/z} w \left(\frac{z - 1}{z} - \left\{ y - x + \frac{z - 1}{z} - w \right\}_+ \right) dw \\ &= - \int_1^{3/2} \lambda(dz) \int_0^{(z-1)/z} w(w + x - y)_+ dw \\ &\leq - \int_1^{3/2} \lambda(dz) \int_{y-x}^{(z-1)/z} (w + x - y)^2 dw \\ &= - \frac{1}{3} \int_1^{3/2} \lambda(dz) ((z - 1)/z + x - y)^3, \end{aligned}$$

where in the first equality we used that $0 < y - x < 1/2$ and $(z - 1)/z < 1/2$. Hence, by the mean value inequality,

$$\int_1^{3/2} \lambda(dz) \int_0^1 du \left| \left(\frac{z-1}{z} - \{x-u\} \right)_+ - \left(\frac{z-1}{z} - \{y-u\} \right)_+ \right|^2 \leq |x-y| \int_1^{3/2} \lambda(dz) ((z-1)/z)^2.$$

This proves (5.4.10). Thus, we can apply Theorem 5.1 of [LP12]. \square

Finally, note that the integral in (5.4.7) can be seen for all $t \geq 0$ as the limit of the martingale

$$\int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(X_{s-}(v), z, u) \tilde{N}(ds, dz, du) \xrightarrow[\varepsilon \rightarrow 0]{(L^2)} \int_0^t \int_0^\infty \int_0^1 g(X_{s-}(v), z, u) \tilde{N}(ds, dz, du),$$

and that, since for all $x \in \mathbb{R}$, the bijection $u \mapsto \{x-u\}$ preserves the uniform measure on $[0, 1]$, for all $v \in \mathbb{R}$, the process $(X_t(v))_{t \geq 0}$ is a Lévy process with no drift, no Brownian part, whose Lévy measure is the image by $(z, u) \mapsto g(0, z, u)$ of $\lambda(dz)du$.

5.4.3 The discrete coalescing flow of fpp geodesics

Recall that $(\tilde{P}(t))_{t \geq 0}$ is the continuous time Markov process constructed from P under $\mathbb{P}^{(\ell)}$ where at each step n we wait a time $\mathcal{E}_n/(2P(n)^{a-1})$. For every time $t \geq 0$ of jump of \tilde{P} we sample a random variable $V_t^{(\ell)}$ which is uniform in $\{0, 1/(2\tilde{P}(t)), \dots, (2\tilde{P}(t) - 1)/(2\tilde{P}(t))\}$. This random variable corresponds to the position where we glue a new face or where we insert edges in the reversed uniform exploration. For all $t \geq 0$, we set

$$R_t^{(\ell)} = - \sum_{\substack{0 \leq s < t \\ \Delta \tilde{P}(s) \neq 0}} \left(V_s^{(\ell)} + \frac{1}{2} \max \left(1 - \frac{\tilde{P}(s-)}{\tilde{P}(s)}, \left(1 - \frac{1}{2\tilde{P}(s)} \right) \left(1 - \frac{\tilde{P}(s)}{\tilde{P}(s-)} \right) \right) \right).$$

The quantity $R_t^{(\ell)}$ is introduced for technical reasons in order to compensate the jumps in the processes which are defined below. For every time $t \geq 0$ of jump of \tilde{P} , we write $U_t^{(\ell)} := \{V_t^{(\ell)} + R_t^{(\ell)}\}$. Let $T > 0$. We define the random measure $N^{(\ell), T}$ by

$$N^{(\ell), T} = \sum_{0 \leq t \leq T; \Delta \tilde{P}(t) > 0} \delta_{(T-t, \tilde{P}(t)/\tilde{P}(t-), U_t^{(\ell)})}.$$

The measure $N^{(\ell), T}$ gives the (translated) position at which we glue the face or insert edges and also the degree of the glued face or the number of inserted edges via the jump of \tilde{P} .

For all $t \leq 0$, for all $z \in \mathbb{R}_+^*$, let

$$g_t^{(\ell)}(x, z, u) = \left(\{x-u\} - \left(\frac{z-1}{z} \right)_+ \right)_+ z - \{x-u\} + \frac{1}{2} \max \left(1 - \frac{1}{z}, \left(1 - \frac{1}{2\tilde{P}(-t)} \right) (1-z) \right)$$

and for all $t \in [0, T]$, let $g_t^{(\ell), T} = g_{t-T}^{(\ell)}$. Note that g is 1-periodic in x . By construction, note that, by a simple computation, for all $t \in [0, T]$ such that $\Delta \tilde{P}(T-t) \neq 0$, for all $z \in \mathbb{R}_+^*$, $x \in \mathbb{R}$ such that $\tilde{P}(T-t)/z \in \mathbb{Z}$ and $2\tilde{P}(T-t)(x - R_{T-t}^{(\ell)}) \in \mathbb{Z}$, we have

$$\mathbb{E}^{(\ell)} \left(g_t^{(\ell), T}(x, z, U_{T-t}^{(\ell)}) \middle| \tilde{P} \right) = 0. \quad (5.4.16)$$

Indeed, if we write $p = \tilde{P}(T-t)$ and $k = p(z-1)/z$ then, when $k \geq 0$ (i.e. $z \geq 1$), we have

$$\begin{aligned} \mathbb{E}^{(\ell)} \left(\left(\{x - U_{T-t}^{(\ell)}\} - \frac{z-1}{z} \right)_+ z - \{x - U_{T-t}^{(\ell)}\} \middle| \tilde{P} \right) &= \frac{1}{2p} \sum_{i=0}^{2p-1} \left(\left(\frac{i}{2p} - \frac{k}{p} \right)_+ \frac{p}{p-k} - \frac{i}{2p} \right) \\ &= \frac{1}{2} \left(\frac{p-k}{p} - 1 \right) = \frac{1}{2} \left(\frac{1}{z} - 1 \right), \end{aligned}$$

and a similar calculation proves (5.4.16) when $k \leq 0$ (i.e. $z \leq 1$). Besides, we again have for all $x \in \mathbb{R}, z \in \mathbb{R}_+^*, u \in [0, 1]$,

$$g_t^{(\ell), T}(x, z, u) \leq 3|z - 1|. \quad (5.4.17)$$

The coalescing flow $(X_t^{(\ell), T}(v))_{v \in \mathbb{R}, 0 \leq t \leq T}$ associated with $N^{(\ell), T}$ is defined by

$$\forall v \in \mathbb{R}, \forall t \in [0, T], \quad X_t^{(\ell), T}(v) = v + \int_0^t \int_0^\infty \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T}(v), z, u) N^{(\ell), T}(ds, dz, du). \quad (5.4.18)$$

Notice that the above process is not a diffusion with jumps since the Markov process \tilde{P} is not reversible. This coalescing flow gives the genealogy of the fpp geodesics to the root face under $\mathbb{P}^{(\ell)}$ and, when $a = 2$, it is parametrized by the fpp distance.

Let us state the following useful lemma which is a direct consequence of (5.4.16).

Lemma 5.4.2. *Let $\mathcal{F}^{(\ell), T}$ be the filtration defined by*

$$\forall t \in [0, T], \quad \mathcal{F}_t^{(\ell), T} = \sigma\left(\tilde{P}((T-s)-), U_{T-s}^{(\ell)}; s \in [0, t]\right).$$

Then for all $\varepsilon > 0$, for all $v \in \mathbb{R}$, the process $(M_t^{(\ell), T, \varepsilon})_{t \in [0, T]}$ defined by

$$\forall t \in [0, T], \quad M_t^{(\ell), T, \varepsilon} = \int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T}(v), z, u) N^{(\ell), T}(ds, dz, du)$$

is a martingale with respect to the filtration $\mathcal{F}^{(\ell), T}$.

Finally, let us state a lemma which controls the small jumps of \tilde{P} .

Lemma 5.4.3. *We have the uniform convergence*

$$\sup_{\ell \geq 1} \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\tilde{P}(s)/\tilde{P}(s-) \in [1-\varepsilon, 1+\varepsilon] \setminus \{1\}} \left(\frac{\Delta \tilde{P}(s)}{\tilde{P}(s-)} \right)^2 \right) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Proof. Using the law of the peeling exploration described in Subsection 5.2.2, one can upperbound

$$\begin{aligned} & \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\tilde{P}(s)/\tilde{P}(s-) \in [1-\varepsilon, 1+\varepsilon] \setminus \{1\}} \left(\frac{\Delta \tilde{P}(s)}{\tilde{P}(s-)} \right)^2 \right) \\ & \leq C \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\frac{\tilde{P}(s)}{\tilde{P}(s-)} \neq 1} \sum_{-\varepsilon \tilde{P}(s-) \leq k \leq \varepsilon \tilde{P}(s-)} \frac{W^{(\tilde{P}(s-)+k)} c_{\mathbf{q}}^{-\tilde{P}(s-)-k}}{W^{(\tilde{P}(s-))} c_{\mathbf{q}}^{-\tilde{P}(s-)}} \nu(k) \left(\frac{k}{\tilde{P}(s-)} \right)^2 \right) \\ & \leq C' \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\frac{\tilde{P}(s)}{\tilde{P}(s-)} \neq 1} \sum_{1 \leq k \leq \varepsilon \tilde{P}(s-)} k^{-a} \left(\frac{k}{\tilde{P}(s-)} \right)^2 \right) \quad \text{by (5.1.1) and (5.2.3)} \\ & \leq \varepsilon^{3-a} C'' \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\frac{\tilde{P}(s)}{\tilde{P}(s-)} \neq 1} \frac{1}{\tilde{P}(s-)^{a-1}} \right) \\ & \leq \varepsilon^{3-a} C''', \end{aligned}$$

where C, C', C'', C''' are positive constants which do not depend on ℓ, ε . The last inequality comes from the fact that if we denote the exponential clocks defining \tilde{P} by $\mathcal{E}_n / (2P(n)^{a-1})$ for all integer $n \geq 0$, where the \mathcal{E}_n for $n \geq 0$ are i.i.d. exponential random variables of parameter 1 independent from P , then

$$\begin{aligned} \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\frac{\tilde{P}(s)}{\tilde{P}(s-)} \neq 1} \frac{1}{\tilde{P}(s-)^{a-1}} \right) & \leq \mathbb{E}^{(\ell)} \left(\sum_{n=0}^{\infty} \frac{1}{P(n)^{a-1}} \mathbf{1}_{\sum_{k=0}^{n-1} \mathcal{E}_k / (2P(k)^{a-1}) \leq T} \right) \\ & = \mathbb{E}^{(\ell)} \left(\sum_{n=0}^{\infty} \frac{\mathcal{E}_n}{P(n)^{a-1}} \mathbf{1}_{\sum_{k=0}^{n-1} \mathcal{E}_k / (2P(k)^{a-1}) \leq T} \right) \\ & \leq 2T + 1. \end{aligned}$$

This ends the proof of the lemma. \square

5.4.4 Recovering the fpp geodesics from the discrete coalescing flow

This subsection aims at making explicit the correspondence between the discrete coalescing flow and the fpp geodesics to the root via the reversed uniform exploration introduced in Subsection 5.2.3.

Let $T > 0$. Let $t_1 < \dots < t_n$ be the times of jumps of \tilde{P} in $[0, T]$. We set $t_0 = 0$. Let \mathbf{u}_n^n be the map made of one distinguished face of degree $2P(n)$ glued to one hole of the same degree. For all $k \in \llbracket 0, n \rrbracket$, the edges on the inner boundary of \mathbf{u}_k^n are encoded modulo \mathbb{Z} in the clockwise order by the real numbers

$$R_{t_k}^{(\ell), T} + \frac{i}{2P(k)}$$

for $i \in \llbracket 0, 2P(k) - 1 \rrbracket$. See Figure 5.5.

For all $k \in \llbracket 0, n - 1 \rrbracket$, we relate the edges on the inner boundary of \mathbf{u}_k^n to the edges on the inner boundary of \mathbf{u}_{k+1}^n as follows:

- If $\Delta P(k) \geq 0$, we glue a new face of degree $2\Delta P(k) + 2$ to the edges encoded by

$$U_{t_{k+1}}^{(\ell)}, U_{t_{k+1}}^{(\ell)} + \frac{1}{2P(k+1)}, \dots, U_{t_{k+1}}^{(\ell)} + \frac{2\Delta P(k)}{2P(k+1)},$$

and the non-glued edge of the face is encoded by

$$\begin{aligned} & U_{t_{k+1}}^{(\ell)} + \frac{i}{2P(k+1)} + g_{t_{k+1}}^{(\ell), T} \left(U_{t_{k+1}}^{(\ell)} + \frac{i}{2P(k+1)}, \frac{P(k+1)}{P(k)}, U_{t_{k+1}}^{(\ell)} \right) \\ &= U_{t_{k+1}}^{(\ell)} + \frac{i}{2P(k+1)} + \left(\frac{i}{2P(k+1)} - \frac{\Delta P(k)}{P(k+1)} \right) \frac{P(k+1)}{P(k)} - \frac{i}{2P(k+1)} \\ & \quad + \frac{1}{2} \max \left(1 - \frac{P(k)}{P(k+1)}, \left(1 - \frac{1}{2P(k+1)} \right) \left(1 - \frac{P(k+1)}{P(k)} \right) \right) \\ &= U_{t_{k+1}}^{(\ell)} + \frac{1}{2} \max \left(1 - \frac{P(k)}{P(k+1)}, \left(1 - \frac{1}{2P(k+1)} \right) \left(1 - \frac{P(k+1)}{P(k)} \right) \right) \\ &= R_{t_k}^{(\ell)}, \end{aligned}$$

where the result does not depend on $i \in \llbracket 0, 2\Delta P(k) \rrbracket$. Note that by this computation, the trajectories of $X^{(\ell), T}$ which were in $[U_{t_{k+1}}^{(\ell)}, U_{t_{k+1}}^{(\ell)} + \frac{2\Delta P(k)}{2P(k+1)}]$ in the time interval $[T - t_{k+1}, T - t_k]$ then coalesce in $R_{t_k}^{(\ell)}$ at time t_{k+1} . By a similar computation, the other edges of \mathbf{u}_{k+1}^n , which are encoded by $U_{t_{k+1}}^{(\ell), T} + (2\Delta P(k) + j)/(2P(k))$ modulo \mathbb{Z} for $j \in \llbracket 1, 2P(k) - 1 \rrbracket$ correspond in \mathbf{u}_k^n respectively to the edges $R_{t_k}^{(\ell), T} + j/(2P(k))$ modulo \mathbb{Z} for $j \in \llbracket 1, 2P(k) - 1 \rrbracket$, and the corresponding trajectories do not coalesce at time t_{k+1} . See Figure 5.5.

- If $\Delta P(k) < 0$, then we insert the edges

$$R_{t_k}^{(\ell)} - \frac{1}{2P(k)}, R_{t_k}^{(\ell)} - \frac{2}{2P(k)}, \dots, R_{t_k}^{(\ell)} - \frac{-2\Delta P(k)}{2P(k)},$$

we glue the first to the last edge and we fill in the hole which is hence created with a Boltzmann random map. For all $i \in \llbracket 0, 2P(k+1) - 1 \rrbracket$, the edge on the inner boundary of \mathbf{u}_{k+1}^n which is encoded by $U_{t_{k+1}}^{(\ell)} + i/(2P(k+1))$ then corresponds to the edge encoded by

$$\begin{aligned} & U_{t_{k+1}}^{(\ell)} + \frac{i}{2P(k+1)} + g_{t_{k+1}}^{(\ell), T} \left(U_{t_{k+1}}^{(\ell)} + \frac{i}{2P(k+1)}, \frac{P(k+1)}{P(k)}, U_{t_{k+1}}^{(\ell)} \right) = R_{t_k}^{(\ell)} + \frac{i}{2P(k+1)} \frac{P(k+1)}{P(k)} \\ &= R_{t_k}^{(\ell)} + \frac{i}{2P(k)}. \end{aligned}$$

Here, the trajectories do not coalesce at time t_{k+1} . See Figure 5.5.

5.4.5 Scaling limit of the flow of geodesics

In this subsection, we prove that the discrete coalescing flow of Subsection 5.4.3 converges to the family of coalescing diffusions with jumps introduced in Subsection 5.4.2 if one looks at a finite (or countable) number of trajectories. Let $T > 0$. Let us introduce the restriction N^T of N to subsets of $[0, T] \times \mathbb{R}_+^* \times [0, 1]$.

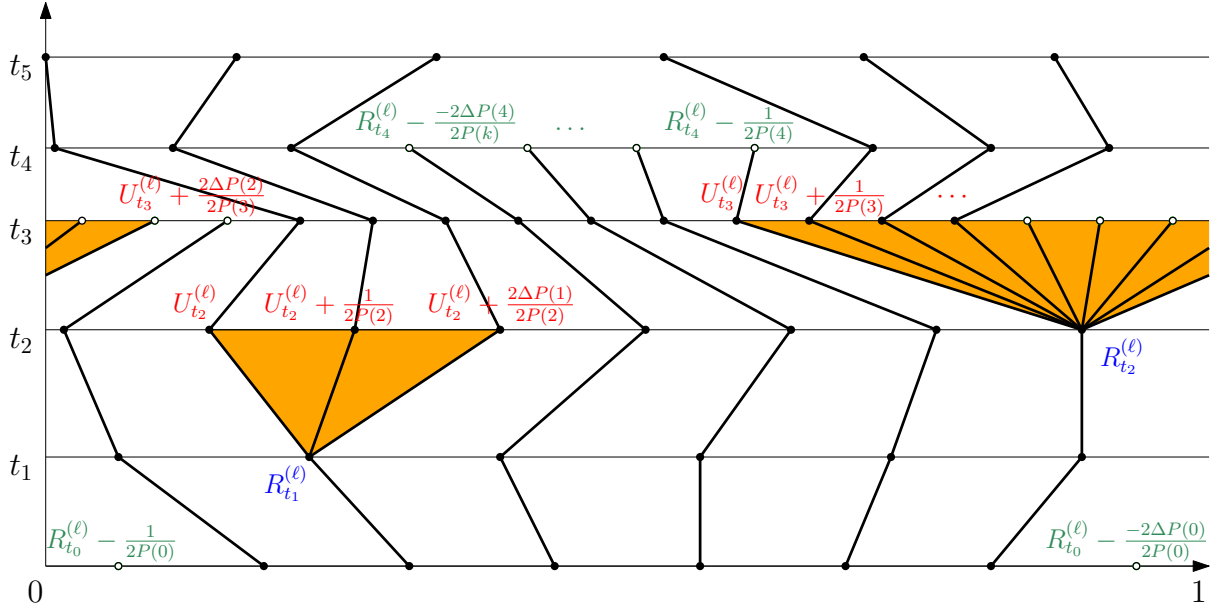


Figure 5.5 – The embedding of the reversed uniform exploration $u_3^5 \subset u_4^5 \subset u_3^5 \subset u_2^5 \subset u_1^5 \subset u_0^5$ of the map m depicted in Figure 5.4 into the discrete coalescing flow. Since the flow is 1-periodic in space, we only draw what happens on $[0, 1]$. The trajectories of the flow are drawn in black: for sake of visibility, they are interpolated by lines between each jump times. The orange triangles mark the coalescence events which correspond to the gluing of a face. The empty dots correspond to the edges which are inserted when \tilde{P} makes a negative jump. The coordinates of the trajectories before they coalesce are in red and the coordinate of the point where they coalesce is in blue.

Theorem 5.4.4. For all $\ell \geq 1$, let $v_1^{(\ell)}, v_2^{(\ell)}, \dots, v_d^{(\ell)} \in \mathbb{R}$ and let $v_1, \dots, v_d \in \mathbb{R}$ such that for all $i \in \llbracket 1, d \rrbracket$, we have $v_i^{(\ell)} \rightarrow v_i$ as $\ell \rightarrow \infty$. Then

$$\left(X_t^{(\ell), T}(v_i^{(\ell)}) \right)_{1 \leq i \leq d, t \in [0, T]} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_t(v_i) \right)_{1 \leq i \leq d, 0 \leq t \leq T},$$

in the space $\mathbb{D}([0, T], \mathbb{R})^d$, where each coordinate is equipped with the J_1 Skorokhod topology. Moreover, jointly with the above convergence, $N^{(\ell), T}$ converges in distribution towards N^T for the topology of weak convergence of measures on $[0, T] \times ((0, 1) \cup (1, \infty)) \times [0, 1]$.

In order to prove the above result, we denote by N_ε^T (resp. N_ε) the restriction of N to subsets of $[0, T] \times (\mathbb{R}_+^* \setminus (1 - \varepsilon, 1 + \varepsilon)) \times [0, 1]$ (resp. $\mathbb{R}_+ \times (\mathbb{R}_+^* \setminus (1 - \varepsilon, 1 + \varepsilon)) \times [0, 1]$). Note that the Poisson point process N_ε^T has a finite intensity. Similarly, for all $\ell \geq 1$, we denote by $N_\varepsilon^{(\ell), T}$ the restriction of $N^{(\ell), T}$ to the subsets of $[0, T] \times (\mathbb{R}_+^* \setminus (1 - \varepsilon, 1 + \varepsilon)) \times [0, 1]$.

Let $X^{(\ell), T, \varepsilon}$ be the approximation of the discrete coalescing flow defined using $N_\varepsilon^{(\ell), T}$ by

$$\forall v \in \mathbb{R}, \forall t \in [0, T], \quad X_t^{(\ell), T, \varepsilon}(v) = v + \int_0^t \int_0^\infty \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T, \varepsilon}(v), z, u) N_\varepsilon^{(\ell), T}(ds, dz, du),$$

and let X^ε be the process defined using N_ε by

$$\forall v \in \mathbb{R}, \forall t \in \mathbb{R}_+, \quad X_t^\varepsilon(v) = v + \int_0^t \int_0^\infty \int_0^1 g(X_{s-}^\varepsilon(v), z, u) N_\varepsilon(ds, dz, du).$$

The above integral is actually a finite sum since $\lambda(\mathbb{R}_+^* \setminus (1 - \varepsilon, 1 + \varepsilon)) < \infty$. Moreover, the integral with respect to the compensated PPP would be the same thanks to (5.4.6). We define $X^{T, \varepsilon}$ as the restriction of X^ε to the time interval $[0, T]$. Note that $X^{T, \varepsilon}$ can also be defined using N_ε^T .

By (5.4.4), we get the convergence of $g_t^{(\ell), T}$ towards g :

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \sup_{t \in [0, T], z \in \mathbb{R}_+^*, u \in [0, 1], x \in \mathbb{R}} \left| g_t^{(\ell), T}(x, z, u) - g(x, z, u) \right| \leq \frac{|1 - z|}{\inf_{t \in [0, T]} \tilde{P}(t)} \xrightarrow[\ell \rightarrow \infty]{(\mathbb{P})} 0. \quad (5.4.19)$$

We can then state the following scaling limit result:

Lemma 5.4.5. *Let $d \geq 1$. For all $\ell \geq 1$, let $v_1^{(\ell)}, v_2^{(\ell)}, \dots, v_d^{(\ell)} \in \mathbb{R}$ and let $v_1, \dots, v_d \in \mathbb{R}$ such that for all $i \in \llbracket 1, d \rrbracket$, we have $v_i^{(\ell)} \rightarrow v_i$ as $\ell \rightarrow \infty$. Then*

$$\left(X_t^{(\ell), T, \varepsilon}(v_i^{(\ell)}) \right)_{1 \leq i \leq d, t \in [0, T]} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_t^{T, \varepsilon}(v_i) \right)_{1 \leq i \leq d, 0 \leq t \leq T},$$

in the space $\mathbb{D}([0, T], \mathbb{R})^d$.

The above lemma is a direct consequence of (5.4.19), of the following lemma and of the fact that N_ε^T has a finite number of atoms.

Lemma 5.4.6. *For every continuous function on $[0, T] \times ((0, 1) \cup (1, \infty)) \times [0, 1]$ with compact support,*

$$\int_0^T \int_{(0,1) \cup (1,\infty)} \int_0^1 f(t, z, u) N^{(\ell), T}(dt, dz, du) \xrightarrow[\ell \rightarrow \infty]{(d)} \int_0^T \int_{(0,1) \cup (1,\infty)} \int_0^1 f(t, z, u) N^T(dt, dz, du).$$

In other words, $N^{(\ell), T}$ converges in distribution towards N^T for the topology of weak convergence of measures on $[0, T] \times ((0, 1) \cup (1, \infty)) \times [0, 1]$.

Proof. The convergence (5.4.4) implies that

$$\sum_{t \geq 0; \Delta \tilde{P}(t) > 0} \delta_{(t, \tilde{P}(t)/\tilde{P}(t-))} \xrightarrow[\ell \rightarrow \infty]{(d)} \text{PPP}(2c_a p_{\mathbf{q}} dt \lambda(dz)),$$

for the topology of weak convergence of measures on $\mathbb{R}_+ \times (\mathbb{R}_+^* \setminus \{1\})$. This concludes the proof by reversing time and using that $\ell / \inf_{[0, T]} \tilde{P}$ under $\mathbb{P}^{(\ell)}$ as $\ell \rightarrow \infty$ is tight by (5.4.4) so that the random variables $U_t^{(\ell)}$ converge in distribution towards independent uniform random variables in $[0, 1]$. \square

Next, we can approximate $X^{(\ell), T}$ using an appropriate coupling with $X^{(\ell), T, \varepsilon}$.

Lemma 5.4.7. *For every sequence of real numbers $(v^{(\ell)})_{\ell \geq 1}$, there exist couplings of $(X_t^{(\ell), T}(v^{(\ell)}))_{t \in [0, T]}$ with $(X_t^{(\ell), T, \varepsilon}(v^{(\ell)}))_{t \in [0, T]}$ for all $\ell \geq 1$ and for all $\varepsilon > 0$, denoted by $X^{(\ell), T}(v^{(\ell)})$, $\bar{X}^{(\ell), T, \varepsilon}(v^{(\ell)})$ such that*

$$\sup_{\ell \geq 1} \mathbb{E}^{(\ell)} \left(\sup_{t \in [0, T]} \left| X_t^{(\ell), T}(v^{(\ell)}) - \bar{X}_t^{(\ell), T, \varepsilon}(v^{(\ell)}) \right|^2 \right) \xrightarrow[\varepsilon \rightarrow 0]{} 0.$$

Proof. Let $(v^{(\ell)})_{\ell \geq 1}$ be a sequence of real numbers. The coupling of the two processes is defined using the same perimeter process P , the same exponential random variables for \tilde{P} but the random variables $\bar{U}^{(\ell), T}$ defining $\bar{X}_t^{(\ell), T, \varepsilon}(v^{(\ell)})$ are defined inductively by the relations

$$\bar{U}_{T-t}^{(\ell), T, \varepsilon} = \left\{ \bar{X}_{t-}^{(\ell), T, \varepsilon}(v^{(\ell)}) - X_{t-}^{(\ell), T}(v^{(\ell)}) + U_{T-t}^{(\ell)} \right\}, \quad (5.4.20)$$

$$\bar{N}_{[0, t] \times (\mathbb{R}_+^* \setminus \{1\}) \times [0, 1]}^{(\ell), T, \varepsilon} = \sum_{s \in [T-t, T]; \tilde{P}(s)/\tilde{P}(s-) \in \mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \delta_{(T-s, \tilde{P}(s)/\tilde{P}(s-), \bar{U}_s^{(\ell), T, \varepsilon})}, \quad (5.4.21)$$

$$\bar{X}_t^{(\ell), T, \varepsilon}(v^{(\ell)}) = v + \int_0^t \int_0^\infty \int_0^1 g_s^{(\ell), T}(\bar{X}_{s-}^{(\ell), T, \varepsilon}(v^{(\ell)}), z, u) \bar{N}_\varepsilon^{(\ell), T}(ds, dz, du), \quad (5.4.22)$$

for all $t \in [0, T]$ such that $\Delta \tilde{P}(t) \neq 0$. Then one can see that $\bar{X}^{(\ell), T, \varepsilon}(v^{(\ell)})$ has the same law as $X^{(\ell), T, \varepsilon}(v^{(\ell)})$ and that by construction, for all $t \in [0, T]$ such that $\Delta \tilde{P}(t) \neq 0$, for all $z \in \mathbb{R}_+^* \setminus \{1\}$,

$$g_t^{(\ell), T}(\bar{X}_{t-}^{(\ell), T, \varepsilon}(v^{(\ell)}), z, \bar{U}_{T-t}^{(\ell), T, \varepsilon}) = g_t^{(\ell), T}(X_{t-}^{(\ell), T}(v^{(\ell)}), z, U_{T-t}^{(\ell)}). \quad (5.4.23)$$

Using the coupling, we see that

$$\begin{aligned} & \mathbb{E}^{(\ell)} \left(\sup_{t \in [0, T]} \left| X_t^{(\ell), T}(v^{(\ell)}) - \bar{X}_t^{(\ell), T, \varepsilon}(v^{(\ell)}) \right|^2 \right) \\ &= \mathbb{E}^{(\ell)} \left(\sup_{t \in [0, T]} \left| \int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T}(v^{(\ell)}), z, u) N^{(\ell), T}(ds, dz, du) \right|^2 \right) \quad \text{by (5.4.23)}. \end{aligned}$$

Moreover, thanks to Lemma 5.4.2, the process

$$(M_t^{(\ell), T, \varepsilon})_{t \in [0, T]} = \left(\int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T}(v^{(\ell)}), z, u) N^{(\ell), T}(ds, dz, du) \right)_{t \in [0, T]}$$

is a martingale. So by Doob's martingale inequality,

$$\mathbb{E}^{(\ell)} \left(\sup_{t \in [0, T]} (M_t^{(\ell), T, \varepsilon})^2 \right) \leq 4 \sup_{t \in [0, T]} \mathbb{E}^{(\ell)} \left((M_t^{(\ell), T, \varepsilon})^2 \right).$$

Moreover, by (5.4.17), one can upperbound for all $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}^{(\ell)} \left((M_t^{(\ell), T, \varepsilon})^2 \right) &= \mathbb{E}^{(\ell)} \left(\int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 g_s^{(\ell), T}(X_{s-}^{(\ell), T}(v^{(\ell)}), z, u)^2 N^{(\ell), T}(ds, dz, du) \right) \quad (\text{quadratic variation}) \\ &\leq 9 \mathbb{E}^{(\ell)} \left(\int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 |z-1|^2 N^{(\ell), T}(ds, dz, du) \right) \\ &\leq 9 \mathbb{E}^{(\ell)} \left(\sum_{s \in [0, T]} \mathbf{1}_{\tilde{P}(s)/\tilde{P}(s-) \in [1-\varepsilon, 1+\varepsilon] \setminus \{1\}} \left(\frac{\Delta \tilde{P}(s)}{\tilde{P}(s-)} \right)^2 \right). \end{aligned}$$

Using Lemma 5.4.3 one concludes that

$$\sup_{\ell \geq 1} \sup_{t \in [0, T]} \mathbb{E}^{(\ell)} \left((M_t^{(\ell), T, \varepsilon})^2 \right) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (5.4.24)$$

This ends the proof. \square

We will need this last lemma:

Lemma 5.4.8. *We have for all $v_1, \dots, v_d \in \mathbb{R}$,*

$$(X_t^\varepsilon(v_i))_{1 \leq i \leq d, t \geq 0} \xrightarrow[\varepsilon \rightarrow 0]{(d)} (X_t(v_i))_{1 \leq i \leq d, t \geq 0}$$

in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R})^d$.

Proof. It is a direct consequence of Theorem 4.15 of Chapter IX of [JS87]. Indeed, for all $v_1, \dots, v_d \in \mathbb{R}$, the process $(X_t^\varepsilon(v_i))_{i \in [1, d], t \geq 0}$ (resp. $(X_t^\varepsilon(v_i))_{i \in [1, d], t \geq 0}$) is the time homogeneous jump diffusion starting from (v_1, \dots, v_d) with kernel $K^{d, \varepsilon}$ (resp. K^d) defined by the fact that for all $f \in C_1(\mathbb{R}^d)$, for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} f(y) K^{d, \varepsilon}(x, dy) = 2c_a p_{\mathbf{q}} \int_{(0, \infty) \setminus (1-\varepsilon, 1+\varepsilon)} \lambda(dz) \int_0^1 du f(g(x_1, z, u), \dots, g(x_d, z, u)),$$

and respectively

$$\int_{\mathbb{R}^d} f(y) K^d(x, dy) = 2c_a p_{\mathbf{q}} \int_{(0, \infty) \setminus \{1\}} \lambda(dz) \int_0^1 du f(g(x_1, z, u), \dots, g(x_d, z, u)).$$

We denote by $\|\cdot\|_\infty$ the infinite norm on \mathbb{R}^d . Then, by (5.4.9), for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |y|_\infty^2 K^{d, \varepsilon}(x, dy) \leq \int_{\mathbb{R}^d} |y|_\infty^2 K^d(x, dy) \leq 9 \int_{(1/2, 1) \cup (1, \infty)} \lambda(dz) |z-1|^2 < \infty,$$

and

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y|_{\infty}^2 \mathbf{1}_{|y|_{\infty} > b} K^d(x, dy) \leq 9 \int_{(1/2, 1) \cup (1, \infty)} \lambda(dz) |z - 1|^2 \mathbf{1}_{3|z-1| > b} \xrightarrow{b \rightarrow \infty} 0.$$

Besides, for all $x' \in \mathbb{R}^d$ for $\lambda(dz)du$ almost every $(z, u) \in \mathbb{R}_+^* \times [0, 1]$, the function $x \mapsto (g(x_i, z, u))_{1 \leq i \leq d}$ is continuous at x' , so that by dominated convergence, for all $f \in C_1(\mathbb{R}^d)$,

$$\int_0^{\infty} \lambda(dz) \int_0^1 du f(g(x_1, z, u), \dots, g(x_d, z, u)) \xrightarrow{x \rightarrow x'} \int_0^{\infty} \lambda(dz) \int_0^1 du f(g(x'_1, z, u), \dots, g(x'_d, z, u)),$$

where we use the domination

$$f(g(x_1, z, u), \dots, g(x_d, z, u)) \leq \mathbf{1}_{3|z-1| > r} \sup(f),$$

where $r > 0$ is such that $\{y \in \mathbb{R}^d; |y|_{\infty} < r\} \subset \mathbb{R}^d \setminus \text{Supp}(f)$. Therefore, the function $x \mapsto \int_{\mathbb{R}^d} K^d(x, dy) f(y)$ is continuous. By the same reasoning, it is also true for $x \mapsto \int_{\mathbb{R}^d} K^{d, \varepsilon}(x, dy) f(y)$.

Furthermore, by the Beppo Levi lemma, for all $x \in \mathbb{R}^d$, for all $f \in C_1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(y) K^{d, \varepsilon}(x, dy) \xrightarrow{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} f(y) K^d(x, dy),$$

and this convergence holds locally uniformly on x by Dini's theorem. Moreover, by (5.4.5), we have for all $i, j \in \llbracket 1, d \rrbracket$, uniformly on x ,

$$\left| \int_{\mathbb{R}^d} y_i y_j K^{d, \varepsilon}(x, dy) - \int_{\mathbb{R}^d} y_i y_j K^d(x, dy) \right| \leq \int_{1-\varepsilon}^{1+\varepsilon} \lambda(dz) \int_0^1 du \max_{1 \leq i \leq d} g(x_i, z, u)^2 \leq 9 \int_{1-\varepsilon}^{1+\varepsilon} |1 - z|^2 dz,$$

which goes to zero as $\varepsilon \rightarrow 0$. Finally, the last condition to check before applying Theorem 4.15 of Chapter IX is the ‘‘uniqueness-measurability hypothesis’’ 4.3 of Chapter IX. It is a consequence of the existence and uniqueness of strong solutions to (5.4.7) which is provided by Lemma 5.4.1. \square

Proof of Theorem 5.4.4. By combining the results of Lemmas 5.4.5, 5.4.7 and 5.4.8, we obtain that for every converging sequence $v^{(\ell)} \rightarrow v \in \mathbb{R}$,

$$\text{Under the law } \mathbb{P}^{(\ell)}, \quad \left(X_t^{(\ell), T}(v^{(\ell)}) \right)_{t \in [0, T]} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(X_t(v) \right)_{t \in [0, T]}.$$

Moreover, by Lemma 5.4.6, we know that jointly with (5.4.4),

$$\text{Under the law } \mathbb{P}^{(\ell)}, \quad N^{(\ell), T} \xrightarrow[\ell \rightarrow \infty]{(d)} N^T$$

for the topology of weak convergence of measures on $[0, T] \times (\mathbb{R}_+^* \setminus \{1\}) \times [0, 1]$.

Assume that, along some subsequence $(\ell_k)_{k \geq 0}$, we have

$$\text{Under } \mathbb{P}^{(\ell_k)}, \quad \left(X^{(\ell_k), T}(v^{(\ell_k)}), N^{(\ell_k), T}, \frac{\tilde{P}}{\ell_k} \right) \xrightarrow[\ell_k \rightarrow \infty]{(d)} \left(\check{X}^T(v), N^T, (\exp(\xi_a(2c_a p \mathbf{q} t)))_{t \geq 0} \right), \quad (5.4.25)$$

where $\check{X}^T(v)$ has the same law as $X^T(v)$. Then, to end the proof of Theorem 5.4.4, it suffices to show that

$$\forall t \in [0, T], \quad \check{X}_t^T(v) = v + \int_0^t \int_0^{\infty} \int_0^1 g(\check{X}_{s-}^T(v), z, u) \tilde{N}^T(ds, dz, du), \quad (5.4.26)$$

where \tilde{N}^T is the compensated Poisson process associated with N^T . By Skorokhod's representation theorem, we assume that (5.4.25) holds almost surely. For all $\ell \geq 1$, let ψ_{ℓ} be an increasing homeomorphism from $[0, T]$ to $[0, T]$ such that

$$\sup_{t \in [0, T]} \left| X_t^{(\ell_k), T}(v^{(\ell_k)}) - \check{X}_{\psi_{\ell_k}(t)}^T(v) \right| + \sup_{t \in [0, T]} |\psi_{\ell_k}(t) - t| \xrightarrow[\ell_k \rightarrow \infty]{(a.s.)} 0 \quad (5.4.27)$$

Using (5.4.19) and the fact that $(N^{(\ell),T}([0, T] \times ((1/2, 1 - \varepsilon) \cup (1 + \varepsilon, \infty)) \times [0, 1]))_{\ell \geq 1}$ is bounded for all $\varepsilon > 0$, we have for all $t \in [0, T]$, for all $\varepsilon > 0$, a.s. along the subsequence $(\ell_k)_{k \geq 0}$

$$\begin{aligned} & \int_0^t \int_{(1/2, 1-\varepsilon) \cup (1+\varepsilon, \infty)} \int_0^1 \left(g_s^{(\ell),T}(X_{s-}^{(\ell),T}(v^{(\ell)}), z, u) - g(\check{X}_{\psi_\ell(s)-}^T(v), z, u) \right) N^{(\ell),T}(ds, dz, du) \\ &= \int_0^t \int_{(1/2, 1-\varepsilon) \cup (1+\varepsilon, \infty)} \int_0^1 \left(g(X_{s-}^{(\ell),T}(v^{(\ell)}), z, u) - g(\check{X}_{\psi_\ell(s)-}^T(v), z, u) \right) N^{(\ell),T}(ds, dz, du) + o(1). \end{aligned}$$

and by taking (5.4.27) into account, using that for all $x_0 \in \mathbb{R}$, for $\lambda(dz)du$ almost all $(z, u) \in \mathbb{R}_+^* \times [0, 1]$, the function $x \mapsto g(x, z, u)$ is continuous at x_0 , we deduce by dominated convergence (with domination by a constant) that along $(\ell_k)_{k \geq 0}$,

$$\int_0^t \int_{(1/2, 1-\varepsilon) \cup (1+\varepsilon, \infty)} \int_0^1 \left(g_s^{(\ell),T}(X_{s-}^{(\ell),T}(v^{(\ell)}), z, u) - g(\check{X}_{\psi_\ell(s)-}^T(v), z, u) \right) N^{(\ell),T}(ds, dz, du) \xrightarrow[\ell \rightarrow \infty]{\text{(a.s.)}} 0.$$

In addition to that, thanks to the convergence of $N^{(\ell),T}$ towards N^T (and since λ has no atom), we have the convergence of finite sums for all $t \geq 0$.

$$\int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) N^{(\ell),T}(ds, dz, du) \xrightarrow[\ell \rightarrow \infty]{\text{(a.s.)}} \int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) N^T(ds, dz, du).$$

Thus,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g_s^{(\ell),T}(X_{s-}^{(\ell),T}(v^{(\ell)}), z, u) N^{(\ell),T}(ds, dz, du) \\ & \xrightarrow[\ell \rightarrow \infty]{\text{(a.s.)}} \int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) N^T(ds, dz, du). \end{aligned} \quad (5.4.28)$$

Moreover, by (5.4.24),

$$\sup_{\ell \geq 1} \mathbb{E}^{(\ell)} \left(\left| \int_0^t \int_{1-\varepsilon}^{1+\varepsilon} \int_0^1 g_s^{(\ell),T}(X_{s-}^{(\ell),T}(v^{(\ell)}), z, u) N^{(\ell),T}(ds, dz, du) \right|^2 \right) \xrightarrow[\varepsilon \rightarrow 0]{} 0. \quad (5.4.29)$$

Besides, using (5.4.6), one can see that we know that for all $t \geq 0$,

$$\int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) N^T(ds, dz, du) = \int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) \tilde{N}^T(ds, dz, du).$$

Finally, we have the convergence of martingale

$$\int_0^t \int_{\mathbb{R}_+^* \setminus (1-\varepsilon, 1+\varepsilon)} \int_0^1 g(\check{X}_{s-}^T(v), z, u) \tilde{N}^T(ds, dz, du) \xrightarrow[\varepsilon \rightarrow 0]{(L^2)} \int_0^t \int_{\mathbb{R}_+^* \setminus \{1\}} \int_0^1 g(\check{X}_{s-}^T(v), z, u) \tilde{N}^T(ds, dz, du).$$

Combining this with (5.4.28) and (5.4.29) gives (5.4.26). This concludes the proof. \square

5.5 Scaling limit of the tree of geodesics

In this section, we define a random infinite countable metric space \mathcal{T}_a associated with the stochastic flow arising from the coalescing diffusions with jumps and we prove that it is the scaling limit of the tree of fpp geodesics of random maps of type $a \in (3/2, 5/2)$.

5.5.1 The tree of the positive jumps via the stochastic flow

Let N be a PPP on $(-\infty, 0] \times (\mathbb{R}_+^* \setminus \{1\}) \times [0, 1]$ of intensity $dt\lambda(dz)du$. For all $v \in \mathbb{R}$ and $s \leq 0$, let $(X_{s,t}(v))_{v \in \mathbb{R}, t \in [s, 0]}$ be the solution starting from $v \in \mathbb{R}$ at time $s \leq 0$ of the SDE

$$\forall t \in [s, 0], \quad X_{s,t}(v) = v + \int_s^t \int_0^\infty \int_0^1 g(X_{s'-}(v), z, u) \tilde{N}(ds', dz, du), \quad (5.5.1)$$

where \tilde{N} is the compensated PPP associated with N . Note that $(X_{s,t}(v))_{v \in \mathbb{R}, s \leq t \leq 0}$ is a stochastic flow, and more precisely a Lévy flow in the terminology of [App09] Section 6.4.1 thanks to Lemma 5.4.1. It is also easy to see that X is a stochastic coalescing flow on the torus \mathbb{R}/\mathbb{Z} in the sense that for all $s_1, s_2 < 0$, for all $v_1, v_2 \in \mathbb{R}$, we have $\mathbb{P}(\exists t \in [s_1 \vee s_2, 0], \{X_{s_1,t}(v_1)\} = \{X_{s_2,t}(v_2)\}) > 0$ and if we set $T = \inf\{t \in [s_1 \vee s_2, 0], \{X_{s_1,t}(v_1)\} = \{X_{s_2,t}(v_2)\}\}$ then on the event $\{T < \infty\}$, for all $t \in [T, 0]$, we have $\{X_{s_1,t}(v_1)\} = \{X_{s_2,t}(v_2)\}$.

Let $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ be the Ulam tree, where \mathbb{N}^0 has only one element which is the empty word \emptyset . For all $v, w \in \mathbb{U}$, we write $v \prec w$ if v is a prefix of w . Let N_w for $w \in \mathbb{U}$ be independent copies of N . Let ∂ be a cemetery point. For convenience, we will also denote N_w in the form $(Z_t^w, U_t^w)_{t \leq 0}$, where $Z_t^w = \partial$ when there is no atom of N_w at time t , so that

$$N_w = \sum_{t \leq 0} \mathbf{1}_{Z_t^w \neq \partial} \delta_{(t, Z_t^w, U_t^w)}.$$

For all $w \in \mathbb{U}$, $s \leq 0$ and $v \in \mathbb{R}$, let $(X_{s,t}^w(v))_{t \in [s, 0]}$ be the solution of the SDE (5.5.1) where N is replaced by N_w . By analogy with the discrete coalescing flow, for all $t \leq 0$ such that $Z_t^w \neq \partial$ we set

$$R_t^w := U_t^w + g(U_t^w, Z_t^w, U_t^w)$$

and when $Z_t^w = \partial$ we set $R_t^w = \partial$. Then R_t^w represents the position at which the trajectories which were in $[U_t^w, U_t^w + (Z_t^w - 1)/Z_t^w]$ at time $t-$ coalesce when $Z_t^w > 1$ and R_t^w represents the position on the right of the empty hole of size $1 - Z_t^w$ created when $Z_t^w < 1$.

Let $(\mathcal{X}_w(t), b_w)_{w \in \mathbb{U}, t \geq 0}$ be the cell system which is constructed as in Section 2.2 of [BBCK18] as follows using the N_w 's:

- For all $w \in \mathbb{U}$, let ξ_w be the Lévy process with no Brownian part, whose jumps are given by the Poisson point process $\Delta \xi_w$ on $\mathbb{R}_+ \times \mathbb{R}^*$ which is defined as the image measure of N_w by $(t, z, u) \mapsto (-t, \log z)$, i.e. $(\Delta \xi_w(t))_{t \geq 0} = (\log Z_{-t}^w)_{t \geq 0}$, and whose drift is given by (5.4.1).
- We set $\mathcal{X}_\emptyset = \exp(\xi_\emptyset)$, with birth time $b_\emptyset = 0$.
- Assume by induction that we have defined \mathcal{X}_w and its birth time b_w for some $w \in \mathbb{U}$. Let $(x_{wi})_{i \in \mathbb{N}}$ be the collection of negative jumps of \mathcal{X}_w , ordered in the decreasing order of absolute value. Let $(\beta_{wi})_{i \geq 1}$ be the times at which they occur. For all $i \in \mathbb{N}$ we set $\mathcal{X}_{wi} = |x_{wi}| \exp(\xi_{wi})$, where wi is the finite sequence obtained by adding i at the end of the sequence w . We define the birth time of the cell wi by setting $b_{wi} = b_w + \beta_{wi}$.

For all $w \in \mathbb{U}$, let $(\Delta_{wi})_{i \in \mathbb{N}}$ be the collection of the positive jumps of \mathcal{X}_w in the decreasing order, with by convention $\Delta_\emptyset = 1$ and let $(\tau_{wi})_{i \in \mathbb{N}}$ be the times at which they occur. We define the times $(t_w)_{w \in \mathbb{U}}$ by setting $t_\emptyset = 0$ and for all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$,

$$t_{wi} = b_w + \tau_{wi}.$$

When $a \neq 2$, it is useful to introduce the Lamperti transformed birth times and jump times $(\tilde{b}_w)_{w \in \mathbb{U}}$, $(\tilde{t}_w)_{w \in \mathbb{U}}$, $(\tilde{\beta}_w)_{w \in \mathbb{U} \setminus \{\emptyset\}}$ and $(\tilde{\tau}_w)_{w \in \mathbb{U} \setminus \{\emptyset\}}$ by setting for all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$,

$$\tilde{\beta}_{wi} = \int_0^{\beta_{wi}} e^{(2-a)\xi_w(t)} dt \quad \text{and} \quad \tilde{\tau}_{wi} = \int_0^{\tau_{wi}} e^{(2-a)\xi_w(t)} dt,$$

then setting $\tilde{t}_\emptyset = \tilde{b}_\emptyset = 0$ and for all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$,

$$\tilde{b}_{wi} = \tilde{b}_w + \tilde{\beta}_{wi} \quad \text{and} \quad \tilde{t}_{wi} = \tilde{b}_w + \tilde{\tau}_{wi}.$$

Let \mathcal{T}_a be the countable metric space whose underlying set is $\mathcal{T}_a := \mathbb{U}$. Each $w \in \mathcal{T}_a$ corresponds to either a positive jump Δ_w when $w \neq \emptyset$ or to the root. To each $w = i_1 \cdots i_k \in \mathcal{T}_a \setminus \{\emptyset\}$, we associate a trajectory $(Y^{z,w})_{z \prec w}$ in the Lévy flows X^z for $z \prec w$. Let $\emptyset = w_0 \prec w_1 \prec \dots \prec w_k = w$ be the prefixes of w , so that for all $j \in \llbracket 0, k-1 \rrbracket$, we have $w_{j+1} = w_j i_{j+1}$. The idea is to follow the trajectory starting from the jump Δ_w (which corresponds to a coalescence event) in the Lévy flow $X^{w_{k-1}}$, and when we reach time 0 we continue the trajectory in the flow $X^{w_{k-2}}$ starting from the negative jump arising at time $-\beta_{w_{k-1}}$ in this flow, etc.

More precisely, we set

$$(Y_t^{w_{k-1}, w})_{-\tau_w \leq t \leq 0} := (X_{-\tau_w, t}^{w_{k-1}}(R_{-\tau_w}^{w_{k-1}}))_{-\tau_w \leq t \leq 0}.$$

Note that the starting point of the above process corresponds to the point where previous trajectories of the flow $X^{w_{k-1}}$ coalesce due to the positive jump Δ_w . See the left-hand side of Figure 5.6.

Next, for all $j \in \llbracket 1, k-1 \rrbracket$, we define inductively

$$(Y_t^{w_{j-1},w})_{-\beta_{w_j} \leq t \leq 0} := \left(X_{-\beta_{w_j},t}^{w_{j-1}} \left(R_{-\beta_{w_j}}^{w_{j-1}} - (1 - Z_{-\beta_{w_j}}^{w_{j-1}}) + (1 - Z_{-\beta_{w_j}}^{w_{j-1}}) Y_0^{w_{j-1},w} \right) \right)_{-\beta_{w_j} \leq t \leq 0}.$$

Here, the starting point corresponds to a point chosen in the ‘‘gap’’ created by the negative jump x_{w_j} , and it is chosen according to the ending point $Y_0^{w_{j-1},w}$ of the previous trajectory. See the right-hand side of Figure 5.6.

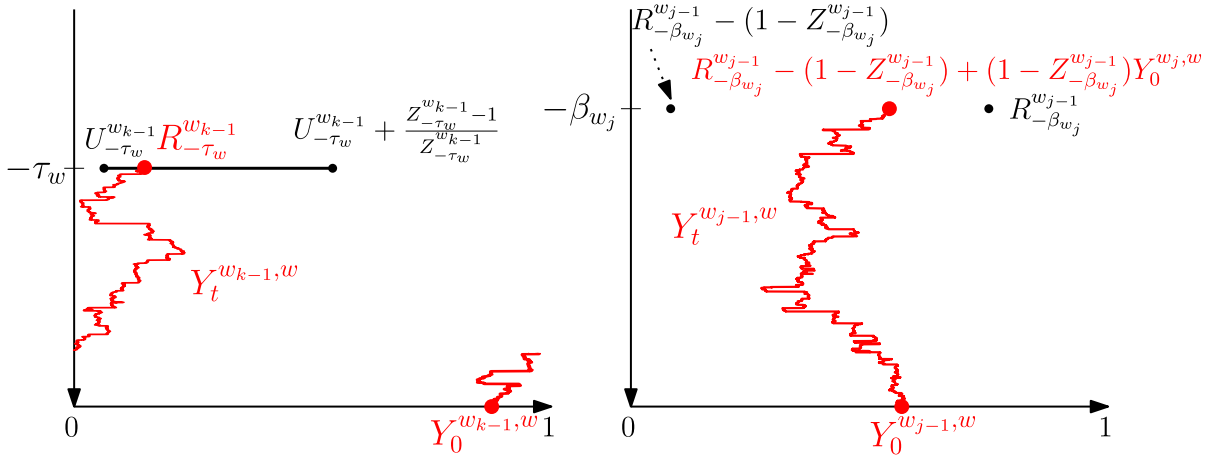


Figure 5.6 – The trajectory associated with a positive jump Δ_w . The trajectory is in red and is represented in $[0, 1]$ since the flow is 1-periodic in space. On the left, the trajectory starts at the point $R_{-\tau_w}^{w_{k-1}}$ where there is a coalescence in the flow at time $-\tau_w$ due to the positive jump. The segment of the positions at time $(-\tau_w)-$ which coalesce in $R_{-\tau_w}^{w_{k-1}}$ is drawn in black. The first part of the trajectory then ends at $Y_0^{w_{k-1},w}$. On the right the next parts of the trajectories for $j \in \llbracket 0, k-1 \rrbracket$ start at a position corresponding to the end of the previous part of the trajectory in the ‘‘gap’’ of size $1 - Z_{-\beta_{w_j}}^{w_{j-1}}$ created at time $-\beta_{w_j}$.

Let us denote by \mathcal{J}_a the (random) set of sequences of positive jumps $(\Delta_{w_{i_n}})_{n \geq 0}$ such that the sequence of times of jump $\tau_{w_{i_n}}$ is increasing and converges as $n \rightarrow \infty$. For all $u = (w_{i_n})_{n \geq 0} \in \mathcal{J}_a$, we define τ_u (resp. $t_u, \tilde{\tau}_u, \tilde{t}_u$) as the limit of $\tau_{w_{i_n}}$ (resp. $t_{w_{i_n}}, \tilde{\tau}_{w_{i_n}}, \tilde{t}_{w_{i_n}}$) as $n \rightarrow \infty$.

According to the value of a , we describe differently the way two trajectories coalesce modulo \mathbb{Z} .

Lemma 5.5.1. *Let $v \neq w \in \mathcal{T}_a \setminus \{\emptyset\}$. Let z be a common prefix of v and w . Assume that $\{Y_{t_0}^{z,v}\} \neq \{Y_{t_0}^{z,w}\}$ for some fixed $t_0 < 0$. Let*

$$\sigma := \inf\{t \in [t_0, 0], \{Y_t^{z,v}\} = \{Y_t^{z,w}\}\},$$

with $\inf \emptyset = \infty$ by convention. Then, on the event $\sigma < \infty$,

- If $a \in (3/2, 2)$, then a.s. $\Delta \xi_z(-\sigma) > 0$.
- If $a \in [2, 5/2)$, then there exists a sequence of positive jumps $u = (\Delta_{z_{i_n}})_{n \geq 0} \in \mathcal{J}_a$ such that $\tau_u = -\sigma$ and such that for all $n \geq 0$, we have $\{Y_{-\tau_{z_{i_n}}}^{z,w} - U_{-\tau_{z_{i_n}}}^z\} \leq (Z_{-\tau_{z_{i_n}}}^z - 1)/Z_{-\tau_{z_{i_n}}}^z$.

Proof. Let us start with the case $a \in (3/2, 2)$. Note that $\int_0^\infty x^{1-a} dx < \infty$ and $\int_0^\infty x^{1-2a} dx < \infty$, so that

$$\int_{(1/2, \infty) \setminus \{1\}} |x - 1| \lambda(dx) < \infty.$$

Therefore, almost surely we have

$$\sum_{t \in [t_0, 0]} \mathbf{1}_{Z_t^z \neq 1} |Z_t^z - 1| < \infty. \quad (5.5.2)$$

By Borel-Cantelli lemma, we deduce that almost surely there is a finite number of times $t \in [t_0, 0]$ such that $Z_t^z > 1$ and $\{Y_t^{z,w} - U_t^z\} \leq (Z_t^z - 1)/Z_t^z$ or $\{Y_t^{z,v} - U_t^z\} \leq (Z_t^z - 1)/Z_t^z$. We denote those times by $s_1 < \dots < s_N$. Furthermore, by (5.5.2), we have a.s.

$$\sum_{t \in [t_0, 0]} \mathbf{1}_{1/2 < Z_t^z < 1} \log Z_t^z > -\infty,$$

so that between two consecutive coalescence times $s_i < s_{i+1} < 0$ such that $\{Y_{s_i}^{z,w}\} \neq \{Y_{s_i}^{z,v}\}$, the distance between $Y_t^{z,w}$ and $Y_t^{z,v}$ in the torus \mathbb{R}/\mathbb{Z} cannot converge to zero since it is lower bounded by $\prod_{s_i < t < s_{i+1}; 1/2 < Z_t^z < 1} Z_t^z > 0$ times the distance between $\{Y_{s_i}^{z,w}\}$ and $\{Y_{s_i}^{z,v}\}$ in \mathbb{R}/\mathbb{Z} . Thus, the coalescence can only take place during a time s_i for some $i \in \llbracket 1, N \rrbracket$. This means that $Z_\sigma^z > 1$, i.e. $\Delta \xi_z(-\sigma) > 0$.

Now let us treat the case $a \in [2, 5/2)$. Note that $\int_0 x^{1-a} dx = \infty$ and hence

$$\int_{(1, \infty)} \frac{x-1}{x} \lambda(dx) = \infty.$$

Therefore, by Campbell's theorem, a.s.

$$\sum_{t \in [t_0, 0]} \mathbf{1}_{Z_t^z > 1} \frac{Z_t^z - 1}{Z_t^z} = \infty.$$

By the second Borel-Cantelli lemma, using the independence of the U_t^z 's, we deduce that a.s. almost surely the set of times $t \in [\sigma, 0]$ such that $Z_t^z > 1$ and $\{Y_t^{z,w} - U_t^z\} \leq (Z_t^z - 1)/Z_t^z$ is dense in $[\sigma, 0]$. In particular, there exists a sequence $(\Delta_{z_{i_n}})_{n \geq 0} \in \mathcal{J}_a$ of positive jumps such that $\tau_{z_{i_n}} \uparrow -\sigma$ as $n \rightarrow \infty$ and such that for all $n \geq 0$, we have $\{Y_{-\tau_{z_{i_n}}}^{z,w} - U_{-\tau_{z_{i_n}}}^z\} \leq (Z_{-\tau_{z_{i_n}}}^z - 1)/Z_{-\tau_{z_{i_n}}}^z$. \square

For all $v, w \in \mathcal{T}_a$, their nearest common ancestor $c(v, w)$ in $\mathcal{T}_a \cup \mathcal{J}_a$ is defined as follows:

- If $v = \emptyset$ or $w = \emptyset$, then $c(v, w) = \emptyset$.
- If $v = w$, then $c(v, w) = v = w$.
- If $v \neq w \in \mathcal{T}_a \setminus \{\emptyset\}$, then let z be their largest common prefix. Let $\emptyset = z_0 \prec z_1 \prec \dots \prec z_k = z$ be the sequence of prefixes of z . Let

$$\sigma_{v,w}(z_k) := \inf\{t \leq 0, Y_t^{z_k,v} \text{ and } Y_t^{z_k,w} \text{ are defined and } \{Y_t^{z_k,v}\} = \{Y_t^{z_k,w}\}\}.$$

If $\sigma_{v,w}(z_k) < \infty$, then the trajectories $Y^{z_k,w}$ and $Y^{z_k,v}$ coalesce at time $\sigma_{v,w}(z_k)$. We distinguish two subcases:

- If $a \in (3/2, 2)$, then by Lemma 5.5.1, we have $\Delta \xi_{z_k}(-\sigma_{v,w}(z_k)) > 0$, so that there exists a unique $i \in \mathbb{N}$ such that $\Delta \mathcal{X}_{z_k}(-\sigma_{v,w}(z_k)) = \Delta_{z_k i}$. Then we set $c(v, w) = z_k i$.
- Otherwise, when $a \in [2, 5/2)$, by Lemma 5.5.1 there exists a sequence $(\Delta_{z_k i_n})_{n \geq 0} \in \mathcal{J}_a$ of positive jumps such that $\tau_{z_k i_n} \uparrow -\sigma_{v,w}(z_k)$ as $n \rightarrow \infty$ and such that for all $n \geq 0$, we have $\{Y_{-\tau_{z_k i_n}}^{z_k,w} - U_{-\tau_{z_k i_n}}^{z_k}\} \leq (Z_{-\tau_{z_k i_n}}^{z_k} - 1)/Z_{-\tau_{z_k i_n}}^{z_k}$. Then we set $c(v, w) = (\Delta_{z_k i_n})_{n \geq 0}$.

When $\sigma_{v,w}(z_k) = \infty$, we set

$$\sigma_{v,w}(z_{k-1}) := \inf\{t \leq 0, \{Y_t^{z_{k-1},v}\} = \{Y_t^{z_{k-1},w}\}\}$$

and we do as above. If $\sigma_{v,w}(z_{k-1}) < \infty$, then the trajectories $Y^{z_{k-1},w}$ and $Y^{z_{k-1},v}$ coalesce at time $\sigma_{v,w}(z_{k-1})$. Then when $a \in (3/2, 2)$, we have $\Delta \xi_{z_{k-1}}(-\sigma_{v,w}(z_{k-1})) > 0$, so that there exists a unique $i \in \mathbb{N}$ such that $\Delta \mathcal{X}_{z_{k-1}}(-\sigma_{v,w}(z_{k-1})) = \Delta_{z_{k-1} i}$. Then we set $c(v, w) = z_{k-1} i$. When $a \in [2, 5/2)$ there exists a sequence $(\Delta_{z_{k-1} i_n})_{n \geq 0}$ such that $\tau_{z_{k-1} i_n} \uparrow -\sigma_{v,w}(z_{k-1})$ as $n \rightarrow \infty$ and such that for all $n \geq 0$, we have $\{Y_{-\tau_{z_{k-1} i_n}}^{z_{k-1},w} - U_{-\tau_{z_{k-1} i_n}}^{z_{k-1}}\} \leq (Z_{-\tau_{z_{k-1} i_n}}^{z_{k-1}} - 1)/Z_{-\tau_{z_{k-1} i_n}}^{z_{k-1}}$. Then we set $c(v, w) = (\Delta_{z_{k-1} i_n})_{n \geq 0}$, etc. Finally, if for all $j \in \llbracket 0, k \rrbracket$ the trajectories do not coalesce, i.e. $\sigma_{v,w}(z_j) = \infty$ (this happens with positive probability), then we set $c(v, w) = \emptyset$.

The space \mathcal{T}_a is equipped with the distance $d_{\mathcal{T}_a}$ defined as follows.

For all $w \in \mathcal{J}_a \cup \mathcal{T}_a$, we set

$$d_{\mathcal{T}_a}(\emptyset, w) := \tilde{t}_w \tag{5.5.3}$$

Note that in the case $a = 2$, the above expression can also be written $d_{\mathcal{T}}(\emptyset, w) = t_w$. Besides, for future use, we stress that, thanks to the absence of atom in the intensity of the PPPs N_w 's, the function $w \mapsto d_{\mathcal{T}}(\emptyset, w)$ is a.s. injective. For all $w, v \in \mathcal{T}_a$, we set

$$d_{\mathcal{T}_a}(w, v) := d_{\mathcal{T}_a}(\emptyset, w) + d_{\mathcal{T}_a}(\emptyset, v) - 2d_{\mathcal{T}_a}(\emptyset, c(v, w)). \quad (5.5.4)$$

It is easy to check that $d_{\mathcal{T}_a}$ is a distance on \mathcal{T}_a . Indeed, positivity and symmetry are obvious. For the triangle inequality, by (5.5.4) it suffices to check that for all $u, v, w \in \mathcal{T}_a$, we have

$$d_{\mathcal{T}_a}(\emptyset, v) + d_{\mathcal{T}_a}(\emptyset, c(u, w)) \geq d_{\mathcal{T}_a}(\emptyset, c(u, v)) + d_{\mathcal{T}_a}(\emptyset, c(v, w)).$$

This holds clearly since:

- If the trajectories of u and v coalesce before those of u and w , then $c(v, w) = c(u, w)$, and one can see that $d_{\mathcal{T}_a}(\emptyset, v) \geq d_{\mathcal{T}_a}(\emptyset, c(u, v))$;
- Otherwise, $d_{\mathcal{T}_a}(\emptyset, c(u, w)) \geq d_{\mathcal{T}_a}(\emptyset, c(u, v))$ and one can see that $d_{\mathcal{T}_a}(\emptyset, v) \geq d_{\mathcal{T}_a}(\emptyset, c(v, w))$.

5.5.2 The tree of geodesics via the discrete coalescing flow

We recall the branching peeling exploration of a map \mathbf{m} via discrete cell-systems from [BCM18, BBCK18]. One can then perform the time-reversed uniform exploration on each branch via the discrete coalescing flow thanks to the encoding presented in Subsection 5.4.4.

The branching peeling exploration of a map \mathbf{m} , contrary to the filled-in peeling exploration, does not fill-in the holes which are created when we identify two edges (i.e. when the perimeter process has a negative jump) but instead starts another process which explores this hole. Thanks to the spatial Markov property, under $\mathbb{P}^{(\ell)}$, this new exploration is independent conditionally on the perimeter of the hole and has the same law as the exploration of a Boltzmann map of perimeter corresponding to the perimeter of the hole.

Under $\mathbb{P}^{(\ell)}$, the perimeter processes of this branching exploration are described by a cell-system

$$((P_w(n))_{n \geq 0})_{w \in \mathbb{U}}$$

which is defined as follows: $P_\emptyset = P$ is the perimeter process of the exploration of \mathbf{m} following the locally largest component. Recall that it is a Markov chain whose probability transitions are given by (5.2.2). We define the birth-time of the cell \emptyset by $B_\emptyset = 0$. Then, assume inductively that $(P_w(n))_{n \geq 0}$ and its birth-time B_w are defined for some $w \in \mathbb{U}$. Let ℓ_{wi} for $i \geq 1$ be the sizes in absolute value of the negative jumps, ranked in the non-increasing order and let $(n_{wi})_{i \geq 1}$ be the times at which they occur. For all $i \geq 1$ we define $(P_{wi}(n))_{n \geq 0}$ as the exploration of the independent Boltzmann map of law $\mathbb{P}^{(\ell_{wi})}$ which fills the hole created by the associated negative jump. Then conditionally on the ℓ_{wi} 's (and on all the branches $P_{w'}$ for $w' \in \mathbb{U}$ such that w is not a strict prefix of w'), the processes $(P_{wi}(n))_{n \geq 0}$'s for $i \geq 1$ are independent Markov chains whose transitions are given by (5.2.2). This property is called the branching Markov property. For all $i \geq 1$, birth-time of the cell wi is defined by $B_{wi} = B_w + n_i$.

On each branch corresponding to a cell P_w for $w \in \mathbb{U}$, we define the associated discrete coalescing flow which gives the fpp geodesics in \mathbf{m} . Let us denote by $(\mathcal{E}_n^w)_{n \geq 0}$ the i.i.d. exponential random variables arising in the uniform peeling exploration of the map filling in the corresponding hole (note that $\mathcal{E}_n^\emptyset = \mathcal{E}_n$ for all $n \geq 0$). Let $(\tilde{P}_w(t))_{t \geq 0}$ be the continuous time process obtained by waiting a time $\mathcal{E}_n^w / (2P_w(n)^{a-1})$ at $P_w(n)$ before jumping at $P_w(n+1)$ for all $n \geq 0$. We define $U_t^{(\ell), w}, g_t^{(\ell), w}$ as in Subsection 5.4.3 using \tilde{P}_w in place of \tilde{P} . We define the random measure

$$N_w^{(\ell)} = \sum_{t \geq 0} \mathbf{1}_{\Delta \tilde{P}_w(t) \neq 0} \delta_{(-t, \tilde{P}_w(t) / \tilde{P}_w(t-), U_t^{(\ell), w})}.$$

We denote by $(X_{s,t}^{(\ell), w}(v))_{v \in \mathbb{R}, s \leq t \leq 0}$ the discrete coalescing flow defined by setting

$$\forall v \in \mathbb{R}, \forall s \leq t \leq 0, \quad X_{s,t}^{(\ell), w}(v) = v + \int_s^t g_{s'}^{(\ell), w}(X_{s,s'}^{(\ell), w}(v), z, u) N_w^{(\ell)}(ds', dz, du).$$

Notice that at each time of creation of a hole, conditionally on the already explored region and on the fpp lengths of the inner dual edges, the dual edges on the boundary of the hole are i.i.d. exponential

random variables of parameter one. Therefore, the fpp geodesic in the map filling the hole from a face f to the boundary of the hole is a part of the fpp geodesic from f to f_r in \mathbf{m} .

For all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$, let $\partial\mathfrak{h}_w$ be the boundary of the hole corresponding to the cell P_w and let f_{wi} be the face created by the i -th largest positive jump of P_w which occurs at time m_i . By convention, we set $f_\emptyset = f_r$. We can thus record the fpp distances

$$d_{\text{fpp}}^\dagger(\partial\mathfrak{h}_w, \partial\mathfrak{h}_{wi}) = \sum_{n=0}^{n_i-1} \frac{\mathcal{E}_n^w}{2P_w(n)} \quad \text{and} \quad d_{\text{fpp}}^\dagger(\partial\mathfrak{h}_w, f_{wi}) = \sum_{n=0}^{m_i-1} \frac{\mathcal{E}_n^w}{2P_w(n)} \quad (5.5.5)$$

and then we have

$$d_{\text{fpp}}^\dagger(f_r, \partial\mathfrak{h}_{wi}) = d_{\text{fpp}}^\dagger(f_r, \partial\mathfrak{h}_w) + d_{\text{fpp}}^\dagger(\partial\mathfrak{h}_w, \partial\mathfrak{h}_{wi}) \quad \text{and} \quad d_{\text{fpp}}^\dagger(f_r, f_{wi}) = d_{\text{fpp}}^\dagger(f_r, \partial\mathfrak{h}_w) + d_{\text{fpp}}^\dagger(\partial\mathfrak{h}_w, f_{wi}). \quad (5.5.6)$$

We also denote the associated times of negative and positive jumps in \tilde{P}_w by

$$\beta_{wi}^{(\ell)} := \sum_{n=0}^{n_i-1} \frac{\mathcal{E}_n^w}{2P_w(n)^{a-1}} \quad \text{and} \quad \tau_{wi}^{(\ell)} := \sum_{n=0}^{m_i-1} \frac{\mathcal{E}_n^w}{2P_w(n)^{a-1}}$$

and we set

$$b_{wi}^{(\ell)} := b_w^{(\ell)} + \beta_{wi}^{(\ell)} \quad \text{and} \quad t_{wi}^{(\ell)} := b_w^{(\ell)} + \tau_{wi}^{(\ell)}.$$

In order to understand the distances between two faces in the tree of geodesics to the root, we describe the trajectories associated with the fpp geodesics in the discrete coalescing flows. Let $w = i_1 \cdots i_k \in \mathbb{U} \setminus \{\emptyset\}$. Let $\emptyset = w_0 \prec w_1 \prec \dots \prec w_k = w$ be the prefixes of w . The face f_w corresponds to the i_k -th largest positive jump of $\tilde{P}_{w_{k-1}}$. We follow the fpp geodesic from f_w to the root f_r in the flows $X^{(\ell), w_k}, X^{(\ell), w_{k-1}}, \dots, X^{(\ell), w_0}$. More precisely, we set

$$(Y_t^{(\ell), w_{k-1}, w})_{-\tau_w^{(\ell)} \leq t \leq 0} := \left(X_{-\tau_w^{(\ell)}, t}^{(\ell), w_{k-1}} \left(R_{\tau_w^{(\ell)}}^{(\ell), w} \right) \right)_{-\tau_w^{(\ell)} \leq t \leq 0}$$

Note that the starting point of the above process corresponds to the point where previous trajectories of the flow $X^{(\ell), w_{k-1}}$ coalesce due to the positive jump $\Delta\tilde{P}_{w_{k-1}}(\tau_w^{(\ell)})$.

Next, for all $j \in [1, k-1]$, we define inductively $Y_t^{(\ell), w_{j-1}, w}$ of every $t \in [-\beta_{w_j}^{(\ell)}, 0]$ by

$$Y_t^{(\ell), w_{j-1}, w} := X_{-\beta_{w_j}^{(\ell)}, t}^{(\ell), w_{j-1}} \left(R_{\beta_{w_j}^{(\ell)}}^{(\ell), w} - \frac{-\Delta\tilde{P}_{w_{j-1}}(\beta_{w_j}^{(\ell)})}{\tilde{P}_{w_{j-1}}(\beta_{w_j}^{(\ell)})} + \frac{-\Delta\tilde{P}_{w_{j-1}}(\beta_{w_j}^{(\ell)})}{\tilde{P}_{w_{j-1}}(\beta_{w_j}^{(\ell)})} Y_0^{(\ell), w_j, w} \right)$$

Let us then describe the nearest common ancestor of two faces f and f' of \mathbf{m} in the tree of fpp geodesics to the root using the above-defined trajectories. When $f \neq f_r$ (resp. $f' \neq f_r$), the face f (resp. f') corresponds to the i -th (resp. i' -th) largest positive jump of P_w (resp. $P_{w'}$), i.e. $f = f_{wi}$ (resp. $f' = f_{w'i'}$). Note that the fpp geodesics from f and f' to the root cannot coalesce until they lie in the same branch. See Figure 5.7.

The nearest common ancestor $c^{(\ell)}(f, f')$ in the tree of fpp geodesics to the root is characterized as follows:

- If $f = f_r$ or $f' = f_r$, then $c^{(\ell)}(f, f') = f_r$;
- If $f = f'$, then $c^{(\ell)}(f, f') = f = f'$;
- If $f \neq f'$ and $f, f' \neq f_r$, then let z be the largest prefix of $wi, w'i'$. Let $\emptyset = z_0 \prec z_1 \prec \dots \prec z_k = z$ be the sequence of the prefixes of z . Let

$$\sigma_{wi, w'i'}^{(\ell)}(z_k) := \inf\{t \leq 0, Y_t^{(\ell), z_k, wi}, Y_t^{(\ell), z_k, w'i'} \text{ are defined and } \{Y_t^{(\ell), z_k, wi}\} = \{Y_t^{(\ell), z_k, w'i'}\}\}.$$

If $\sigma_{wi, w'i'}^{(\ell)}(z_k) < \infty$, then the trajectories $Y^{(\ell), z_k, wi}$ and $Y^{(\ell), z_k, w'i'}$ coalesce at time $\sigma_{wi, w'i'}^{(\ell)}(z_k)$. Then $\Delta\tilde{P}_{z_k}(-\sigma_{wi, w'i'}^{(\ell)}(z_k)) > 0$, so that there exists a unique $j \in \mathbb{N}$ such that $\Delta\tilde{P}_{z_k}(-\sigma_{wi, w'i'}^{(\ell)}(z_k))$ is the j -th largest positive jump of P_{z_k} . In other words, the fpp geodesics coalesce at the face $f_{z_k j}$. Then we have $c^{(\ell)}(f, f') = f_{z_k j}$. Otherwise, we set

$$\sigma_{wi, w'i'}^{(\ell)}(z_{k-1}) := \inf\{t \leq 0, \{Y_t^{(\ell), z_{k-1}, wi}\} = \{Y_t^{(\ell), z_{k-1}, w'i'}\}\}.$$

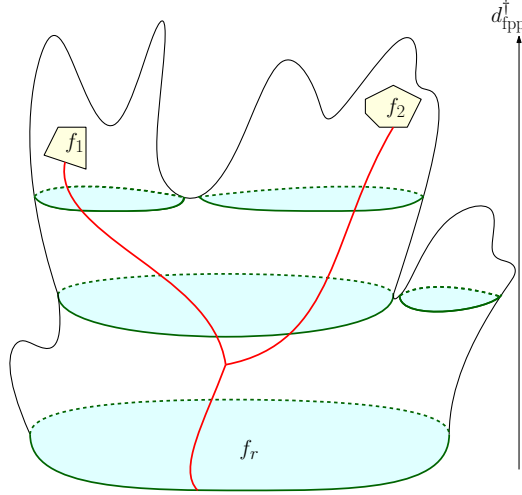


Figure 5.7 – Coalescence of the fpp geodesics from f_1 and f_2 to the root face f_r .

If $\sigma_{w_i, w'_i}^{(\ell)}(z_{k-1}) < \infty$, then the trajectories $Y^{(\ell), z_{k-1}, w_i}$ and $Y^{(\ell), z_{k-1}, w'_i}$ coalesce at time $\sigma_{w_i, w'_i}^{(\ell)}(z_{k-1})$, so that there exists a unique $j \geq 1$ such that $\Delta \tilde{P}_{z_{k-1}}(-\sigma_{w_i, w'_i}^{(\ell)}(z_{k-1}))$ is the j -th largest positive jump of $P_{z_{k-1}}$. In other words, the fpp geodesics coalesce at the face $f_{z_{k-1}j}$. Then we have $c^{(\ell)}(f, f') = f_{z_{k-1}j}$, etc. Finally, if for all $m \in \llbracket 0, k \rrbracket$ the trajectories do not coalesce, i.e. $\sigma_{w_i, w'_i}^{(\ell)}(z_m) = \infty$, then the fpp geodesics coalesce at $c^{(\ell)}(f, f') = f_r$.

Furthermore, the distance $d_{\mathcal{T}(\mathbf{m})}$ induced by the fpp lengths on the tree $\mathcal{T}(\mathbf{m})$ of fpp geodesics to the root is characterized by the relation

$$d_{\mathcal{T}(\mathbf{m})}(f, f') = d_{\text{fpp}}^\dagger(f_r, f) + d_{\text{fpp}}^\dagger(f_r, f') - 2d_{\text{fpp}}^\dagger(f_r, c^{(\ell)}(f, f')). \quad (5.5.7)$$

5.5.3 Scaling limit of the tree of geodesics to the root

In this subsection, assuming $a \in (3/2, 5/2)$, we prove Theorem 5.1.5.

Proof of Theorem 5.1.5. By (5.4.4) and by the branching Markov property, one can check the joint convergence for all $w \in \mathbb{U}$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{\tilde{P}_w(t)}{\ell} \right)_{t \geq 0} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(e^{\xi_w(2c_a p_{\mathbf{q}} t)} \right)_{t \geq 0}. \quad (5.5.8)$$

Notice that the above convergence entails the scaling limit of the degrees of the faces, i.e. jointly with the above convergence, $\deg(f_w)/\ell$ converges in law towards Δ_w jointly for all $w \in \mathbb{U}$. Furthermore, the fact that one can enumerate the Δ_w 's in the decreasing order comes from Proposition 3 of [CCM20], which, combined with (5.5.8), shows in particular that the law of the family of the Δ_w 's is absolutely continuous with respect to the jumps of a stable Lévy process with no negative jump stopped when it reaches -1 .

By (5.4.4), jointly with the above convergence, we have the joint convergence of the jump times for all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\beta_{w_i}^{(\ell)}, \tau_{w_i}^{(\ell)} \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \frac{1}{2c_a p_{\mathbf{q}}} (\beta_{w_i}, \tau_{w_i}),$$

hence the convergence of the times

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(b_w^{(\ell)}, t_{w_i}^{(\ell)} \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \frac{1}{2c_a p_{\mathbf{q}}} (b_w, t_{w_i}).$$

Moreover, thanks to (5.4.3) and (5.5.5), using a Lamperti transform, jointly with the previous convergences, for all $w \in \mathbb{U}$, for all $i \in \mathbb{N}$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{1}{\ell^{a-2}} d_{\text{fpp}}^\dagger(\partial \mathfrak{h}_w, \partial \mathfrak{h}_{w_i}), \frac{1}{\ell^{a-2}} d_{\text{fpp}}^\dagger(\partial \mathfrak{h}_w, f_{w_i}) \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \frac{1}{2c_a p_{\mathbf{q}}} (\tilde{\beta}_{w_i}, \tilde{\tau}_{w_i}),$$

Therefore, by (5.5.6), we have for all $w \in \mathbb{U}, i \in \mathbb{N}$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{1}{\ell^{a-2}} d_{\text{fpp}}^\dagger(f_r, \partial \mathfrak{h}_w), \frac{1}{\ell^{a-2}} d_{\text{fpp}}^\dagger(f_r, f_{w_i}) \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \frac{1}{2c_a p_{\mathbf{q}}} (\tilde{b}_w, \tilde{t}_{w_i}), \quad (5.5.9)$$

Moreover, for all $d \geq 1$, for every sequences $s_i^{(\ell)} \rightarrow s_i < 0$ and $v_i^{(\ell)} \rightarrow v_i \in \mathbb{R}$ for $i \in \llbracket 1, d \rrbracket$, for all $w_1, \dots, w_d \in \mathbb{U}$, jointly with the previous convergences, a consequence of Theorem 5.4.4 is the joint convergence for all $i \in \llbracket 1, d \rrbracket$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\left(X_{s_i^{(\ell)}, t}^{(\ell), w_i}(v_i^{(\ell)}) \right)_{t \in [s_i^{(\ell)}, 0]}, \left(\frac{\tilde{P}_{w_i}(t)}{\tilde{P}_{w_i}(t^-)}, U_t^{(\ell), w_i} \right)_{t \geq 0} \right) \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\left(X_{2c_a p_{\mathbf{q}} s_i, 2c_a p_{\mathbf{q}} t}^{w_i}(v_i) \right)_{t \in [s_i, 0]}, \left(Z_{-t}^{w_i}, U_{-t}^{w_i} \right)_{t \geq 0} \right), \quad (5.5.10)$$

where the first coordinate converges for the Skorokhod J_1 topology while the second coordinate converges for the topology of weak convergence of measures on $\mathbb{R}_+ \times \mathbb{R}_+^* \setminus \{1\} \times [0, 1]$. The above convergence for the Skorokhod topology is written on an interval $[s_i^{(\ell)}, 0]$ whose length varies as ℓ varies but is simply defined as the convergence for the Skorokhod topology on $\mathbb{D}(\mathbb{R}_-, \mathbb{R})$ where we extend the process on $(-\infty, s_i^{(\ell)})$ with its value at $s_i^{(\ell)}$. In particular, we get the convergence of the trajectories jointly with the previous ones: for all $w \in \mathbb{U}$,

$$\text{Under } \mathbb{P}^{(\ell)}, \quad (Y^{(\ell), z, w})_{z \prec w} \xrightarrow[\ell \rightarrow \infty]{(d)} (Y^{z, w})_{z \prec w}. \quad (5.5.11)$$

By Skorokhod's representation theorem, we may assume that the above convergences hold almost surely and we add a superscript (ℓ) to \tilde{P}_w and to the faces f_w to distinguish them in the same probability space. Let $w \neq w' \in \mathbb{U}$.

Let us first focus on the dense case $a \in (3/2, 2)$, where by definition $c(w, w')$ is an element of \mathbb{U} . By Equations (5.5.7) and (5.5.4), it suffices to prove that almost surely for ℓ large enough,

$$c^{(\ell)}(f_w^{(\ell)}, f_{w'}^{(\ell)}) = f_{c(w, w')}^{(\ell)}. \quad (5.5.12)$$

In order to check this, we use the convergence (5.5.10).

If $c(w, w') = \emptyset$, then the trajectories $(Y^{z, w})_{z \prec w}$ and $(Y^{z, w'})_{z \prec w'}$ do not coalesce. Then by (5.5.11) we know that for all ℓ large enough the trajectories $(Y^{(\ell), z, w})_{z \prec w}$ and $(Y^{(\ell), z, w'})_{z \prec w'}$ do not coalesce, hence (5.5.12).

Otherwise, if $c(w, w') \neq \emptyset$, then one can write $c(w, w') = zj$ for some $z \in \mathbb{U}, j \in \mathbb{N}$. Then, by (5.5.10), we have a.s. the convergence of the jump

$$\left(\frac{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)})}{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)} -)}, U_{\tau_{zj}^{(\ell)}}^{(\ell), z} \right) \xrightarrow[\ell \rightarrow \infty]{(a.s.)} \left(Z_{-\tau_{zj}}^z, U_{-\tau_{zj}}^z \right). \quad (5.5.13)$$

But by taking (5.5.11) into account, one obtains (5.5.12). Indeed, by the convergence of the trajectories, we know that a.s. for ℓ large enough, the two trajectories will not coalesce until the jump associated with the face $f_{c(w, w')}^{(\ell)}$. But at the same time, by the convergence of the trajectories and by the convergence of the jump, a.s. for ℓ large enough they have to coalesce due to this jump. See Figure 5.8. Let us turn the idea on Figure 5.8 into a rigorous proof. Let $T < -\tau_{zj}$. By (5.5.11), for all $\ell \geq 1$, let $\psi_w^{(\ell)}$ and $\psi_{w'}^{(\ell)}$ be increasing homeomorphisms from $[T, 0]$ to $[T, 0]$ such that

$$\sup_{t \in [T, 0]} \left| Y_{\psi_w^{(\ell)}(t)}^{(\ell), z, w} - Y_t^{z, w} \right| + \sup_{t \in [T, 0]} \left| \psi_w^{(\ell)}(t) - t \right| + \sup_{t \in [T, 0]} \left| Y_{\psi_{w'}^{(\ell)}(t)}^{(\ell), z, w'} - Y_t^{z, w'} \right| + \sup_{t \in [T, 0]} \left| \psi_{w'}^{(\ell)}(t) - t \right| \xrightarrow[\ell \rightarrow \infty]{(a.s.)} 0. \quad (5.5.14)$$

Almost surely for ℓ large enough, we have

$$\psi_w^{(\ell)}(-\tau_{zj}) = \psi_{w'}^{(\ell)}(-\tau_{zj}) = -\tau_{zj}^{(\ell)}. \quad (5.5.15)$$

Indeed, by (5.5.14),

$$\left(Y_{\psi_w^{(\ell)}(-\tau_{zj})}^{(\ell),z,w}, Y_{\psi_{w'}^{(\ell)}(-\tau_{zj})}^{(\ell),z,w}, Y_{\psi_w^{(\ell)}(-\tau_{zj})}^{(\ell),z,w'}, Y_{\psi_{w'}^{(\ell)}(-\tau_{zj})}^{(\ell),z,w'} \right) \xrightarrow[\ell \rightarrow \infty]{\text{(a.s.)}} \left(Y_{-\tau_{zj}}^{z,w}, Y_{-\tau_{zj}-}^{z,w}, Y_{-\tau_{zj}}^{z,w'}, Y_{-\tau_{zj}-}^{z,w'} \right),$$

and using that the jumps of $Y^{z,w}$ are a.s. distinct given that the image of $\lambda(dz)du$ by $(z, u) \mapsto g(0, z, u)$ has no atom (and the same holds for $Y^{z,w'}$), we deduce (5.5.15) from (5.5.13). Therefore,

$$\left(Y_{-\tau_{zj}^{(\ell)}}^{(\ell),z,w}, Y_{-\tau_{zj}^{(\ell)}-}^{(\ell),z,w}, Y_{-\tau_{zj}^{(\ell)}}^{(\ell),z,w'}, Y_{-\tau_{zj}^{(\ell)}-}^{(\ell),z,w'} \right) \xrightarrow[\ell \rightarrow \infty]{\text{(a.s.)}} \left(Y_{-\tau_{zj}}^{z,w}, Y_{-\tau_{zj}-}^{z,w}, Y_{-\tau_{zj}}^{z,w'}, Y_{-\tau_{zj}-}^{z,w'} \right),$$

But since $Y^{z,w}$ and $Y^{z,w'}$ coalesce, and since the law of $U_{-\tau_{zj}}^z$ has no atom,

$$\{Y_{-\tau_{zj}}^{z,w} - U_{-\tau_{zj}}^z\} < (Z_{-\tau_{zj}}^z - 1)/Z_{-\tau_{zj}}^z \quad \text{and} \quad \{Y_{-\tau_{zj}-}^{z,w'} - U_{-\tau_{zj}-}^z\} < (Z_{-\tau_{zj}-}^z - 1)/Z_{-\tau_{zj}-}^z.$$

Thus, by (5.5.13), a.s. for ℓ large enough,

$$\{Y_{-\tau_{zj}^{(\ell)}-}^{(\ell),z,w} - U_{\tau_{zj}^{(\ell)}}^{(\ell),z}\} \vee \{Y_{-\tau_{zj}^{(\ell)}-}^{(\ell),z,w'} - U_{\tau_{zj}^{(\ell)}}^{(\ell),z}\} \leq \left(\frac{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)})}{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)}-)} - 1 \right) / \left(\frac{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)})}{\tilde{P}_z^{(\ell)}(\tau_{zj}^{(\ell)}-)} \right).$$

So, a.s. for ℓ large enough, the trajectories $Y^{(\ell),z,w}, Y^{(\ell),z,w'}$ coalesce at time $-\tau_{zj}^{(\ell)}$, hence (5.5.12).

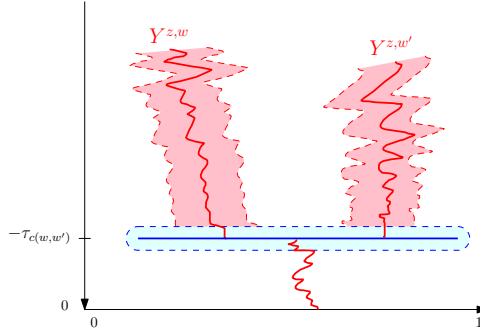


Figure 5.8 – Sketch of the end of the proof of Theorem 5.1.5 in the case $a \in (3/2, 2)$: the two trajectories $Y^{z,w}$ and $Y^{z,w'}$ for some $z \prec w, w'$ in red coalesce due to the jump $\Delta_{c(w,w')}$ in blue. A neighbourhood of the trajectories is drawn in pink and a neighbourhood of the jump is in light blue.

Next, we deal with the case $a \in [2, 5/2)$. If $c(w, w') = \emptyset$, then we conclude as before. When $c(w, w') \neq \emptyset$, then by definition $c(w, w') = (\Delta_{z_i n})_{n \geq 0} \in \mathcal{J}_a$ for some common prefix $z \prec w, w'$, where $\tau_{z_i n}$ is an increasing sequence which converges towards the time $-\sigma_{w, w'}(z)$ which is the time at which the trajectories coalesce. Let $\varepsilon > 0$. Let n large enough such that $\tilde{t}_{z_i n} \geq \tilde{t}_{c(w, w')} - \varepsilon$. Then, by the same reasoning as in the case $a \in (3/2, 2)$, by (5.5.11) we know that a.s. for ℓ large enough, $f_{z_i n}^{(\ell)}$ is a common ancestor of $f_w^{(\ell)}, f_{w'}^{(\ell)}$. As a result,

$$d_{\mathcal{T}(\mathfrak{m})}(f_w^{(\ell)}, f_{w'}^{(\ell)}) \leq d_{\text{fpp}}^\dagger(f_r^{(\ell)}, f_w^{(\ell)}) + d_{\text{fpp}}^\dagger(f_r^{(\ell)}, f_{w'}^{(\ell)}) - 2d_{\text{fpp}}^\dagger(f_r^{(\ell)}, f_{z_i n}^{(\ell)}).$$

Consequently, a.s.

$$\limsup_{\ell \rightarrow \infty} \frac{1}{\ell^{a-2}} d_{\mathcal{T}(\mathfrak{m})}(f_w^{(\ell)}, f_{w'}^{(\ell)}) \leq \frac{1}{2c_a p_{\mathbf{q}}} (\tilde{t}_w + \tilde{t}_{w'} - 2\tilde{t}_{z_i n}) \leq \frac{1}{2c_a p_{\mathbf{q}}} (d_{\mathcal{T}_a}(w, w') + 2\varepsilon).$$

One can then let $\varepsilon \rightarrow 0$. Finally the almost sure inequality

$$\liminf_{\ell \rightarrow \infty} \frac{1}{\ell^{a-2}} d_{\mathcal{T}(\mathfrak{m})}(f_w^{(\ell)}, f_{w'}^{(\ell)}) \geq \frac{1}{2c_a p_{\mathbf{q}}} d_{\mathcal{T}_a}(w, w')$$

comes directly from (5.5.11) since if at some time the trajectories $(Y^{z,w})_{z \prec w}$ and $(Y^{z,w'})_{z \prec w'}$ have not yet coalesced, then for ℓ large enough the same holds for the trajectories $(Y^{(\ell),z,w})_{z \prec w}$ and $(Y^{(\ell),z,w'})_{z \prec w'}$. This ends the proof. \square

5.6 Conjectures

Our choice of topology for Theorem 5.1.5 comes from the fact that \mathcal{T}_a is non-compact as soon as $a \leq 2$. Still, in the dilute case $a \in (2, 5/2)$, we expect that the scaling limit of Theorem 5.1.5 holds in the sense of Gromov-Hausdorff, after taking the completion of \mathcal{T}_a . We also expect a Gromov-Hausdorff-Prokhorov scaling limit in the dilute case when we equip the map with the uniform measure on the vertices/faces/edges.

Next, we state a few conjectures on the scaling limit of the map itself. We first define a random countable metric space which is conjecturally the scaling limit of large planar maps with high degrees. Let us define another distance by inserting shortcuts in the metric space \mathcal{T}_a . To introduce our shortcuts, it will be more convenient to focus on the macroscopic positive jumps. For all $\varepsilon \in (0, 1)$, we define $(\mathcal{T}_a^\varepsilon, d_{\mathcal{T}_a^\varepsilon})$ as the subspace of \mathcal{T}_a consisting of the elements $w \in \mathcal{T}_a$ such that $\Delta_w > \varepsilon$. We now modify the distance $d_{\mathcal{T}_a^\varepsilon}$ using the negative jumps of the ξ_w 's for $w \in \mathbb{U}$.

Let $w \in \mathbb{U} \setminus \{\emptyset\}$. Let us denote by $\emptyset = w_0 \prec w_1 \prec \dots \prec w_k = w$ its prefixes. From the negative jump x_w of $\mathcal{X}_{w_{k-1}}$, we define two trajectories: one starting on the left of the ‘‘gap’’ created by the jump, one starting from the right. Set $\mathcal{Y}_L^{w,w}(0) = 0$ and $\mathcal{Y}_R^{w,w}(0) = 1$. For all $i \in \{L, R\}$, for all $j \in \llbracket 1, k \rrbracket$, we define inductively

$$(\mathcal{Y}_i^{w_{j-1},w}(t))_{-\beta_{w_j} \leq t \leq 0} := \left(X_{-\beta_{w_j},t}^{w_{j-1}} \left(R_{-\beta_{w_j}}^{w_{j-1}} - (1 - Z_{-\beta_{w_j}}^{w_{j-1}}) + (1 - Z_{-\beta_{w_j}}^{w_{j-1}}) \mathcal{Y}_i^{w_j,w}(0) \right) \right)_{-\beta_{w_j} \leq t \leq 0}.$$

For all $\varepsilon \in (0, 1)$, $i \in \{L, R\}$, let $J_i^{w,\varepsilon}$ be the largest $j \in \llbracket 1, k \rrbracket$ such that there exists $t \in [-\beta_{w_j}, 0]$ satisfying $\Delta \mathcal{X}_{w_{j-1}}(-t) > \varepsilon$ and $\{\mathcal{Y}_i^{w_{j-1},w}(t-) - U_t^{w_{j-1}}\} \leq (Z_t^{w_{j-1}} - 1)/Z_t^{w_{j-1}}$. When it exists, we also define $V_i^{w,\varepsilon} \in \mathcal{T}_a^\varepsilon \setminus \{\emptyset\}$ as the index of the positive jump such that $-\tau_{V_i^{w,\varepsilon}}$ is the smallest $t \in [-\beta_{w_j}, 0]$ to satisfy the above conditions. If $J_i^{w,\varepsilon}$ does not exist, then we set $J_i^{w,\varepsilon} = 0$ and $V_i^{w,\varepsilon} = \emptyset$. In other words, $V_L^{w,\varepsilon}$ (resp. $V_R^{w,\varepsilon}$) corresponds to the first coalescence corresponding to a positive jump of size larger than ε happening in the trajectory $(\mathcal{Y}_L^{w',w})_{w' \prec w}$ (resp. $(\mathcal{Y}_R^{w',w})_{w' \prec w}$).

For all $w \in \mathbb{U} \setminus \{\emptyset\}$, we define the length $\mathcal{L}_w^\varepsilon$ of the shortcut between $V_L^{w,\varepsilon}$ and $V_R^{w,\varepsilon}$ associated with the negative jump x_w by $\mathcal{L}_w^\varepsilon = \tilde{b}_w - \tilde{t}_{V_L^{w,\varepsilon}} + \tilde{b}_w - \tilde{t}_{V_R^{w,\varepsilon}}$. In the case $a = 2$, the expression simplifies to $\mathcal{L}_w^\varepsilon = b_w - t_{V_L^{w,\varepsilon}} + b_w - t_{V_R^{w,\varepsilon}}$. Moreover, for all $u, v \in \mathcal{T}_a^\varepsilon$, we define the length of the smallest shortcut between u and v by setting

$$\mathcal{L}^\varepsilon(u, v) := \inf\{\mathcal{L}_w^\varepsilon; w \in \mathbb{U} \setminus \{\emptyset\} \text{ s.t. } (u, v) = (V_L^{w,\varepsilon}, V_R^{w,\varepsilon}) \text{ or } (u, v) = (V_R^{w,\varepsilon}, V_L^{w,\varepsilon})\} \in [0, \infty].$$

We set $\mathcal{D}^\varepsilon := \mathcal{T}_a^\varepsilon$ and we equip \mathcal{D}^ε with the distance $d_{\mathcal{D}^\varepsilon}$ defined for all $v, w \in \mathcal{D}^\varepsilon$ by

$$d_{\mathcal{D}^\varepsilon}(v, w) = \inf \left\{ d_{\mathcal{T}_a}(v, v_1) + \sum_{k=1}^n (\mathcal{L}^\varepsilon(v_k, w_k) + d_{\mathcal{T}_a}(w_k, v_{k+1})); n \in \mathbb{N}, v_1, \dots, v_{n+1}, w_1, \dots, w_n \in \mathcal{D}^\varepsilon, v_{n+1} = w \right\}.$$

Note that by definition of the shortcuts we have $d_{\mathcal{D}^\varepsilon}(v, w) \geq |d_{\mathcal{T}_a}(\emptyset, v) - d_{\mathcal{T}_a}(\emptyset, w)|$, so that positivity is preserved since $d_{\mathcal{T}_a}(\emptyset, v) \neq d_{\mathcal{T}_a}(\emptyset, w)$ as soon as $v \neq w$.

Lemma 5.6.1. *For all $\varepsilon \in (0, 1)$, $\varepsilon' \in (0, \varepsilon)$ and $v, w \in \mathcal{D}^\varepsilon$, $d_{\mathcal{D}^{\varepsilon'}}(v, w) \leq d_{\mathcal{D}^\varepsilon}(v, w)$.*

Proof. It suffices to check that for all $w \in \mathbb{U} \setminus \{\emptyset\}$, we have $d_{\mathcal{D}^{\varepsilon'}}(V_L^{w,\varepsilon}, V_R^{w,\varepsilon}) \leq \mathcal{L}_w^\varepsilon$. But

$$\begin{aligned} d_{\mathcal{D}^{\varepsilon'}}(V_L^{w,\varepsilon}, V_R^{w,\varepsilon}) &\leq d_{\mathcal{D}^{\varepsilon'}}(V_L^{w,\varepsilon}, V_L^{w,\varepsilon'}) + \mathcal{L}_w^{\varepsilon'} + d_{\mathcal{D}^{\varepsilon'}}(V_R^{w,\varepsilon}, V_R^{w,\varepsilon'}) \\ &= (\tilde{t}_{V_L^{w,\varepsilon'}} - \tilde{t}_{V_L^{w,\varepsilon}}) + (\tilde{b}_w - \tilde{t}_{V_L^{w,\varepsilon'}} + \tilde{b}_w - \tilde{t}_{V_R^{w,\varepsilon'}}) + (\tilde{t}_{V_R^{w,\varepsilon'}} - \tilde{t}_{V_R^{w,\varepsilon}}) \\ &= \mathcal{L}_w^\varepsilon, \end{aligned}$$

where in the second line we used the fact that $V_L^{w,\varepsilon}$ (resp. $V_R^{w,\varepsilon}$) is an ancestor of $V_L^{w,\varepsilon'}$ (resp. $V_R^{w,\varepsilon'}$) in \mathcal{T}_a , in the sense that $c(V_L^{w,\varepsilon}, V_L^{w,\varepsilon'}) = V_L^{w,\varepsilon}$ (resp. $c(V_R^{w,\varepsilon}, V_R^{w,\varepsilon'}) = V_R^{w,\varepsilon}$). \square

Notice also that the sets $\mathcal{D}^{\varepsilon'}$'s increase as ε decreases. One can thus define $\mathcal{D}_a = \mathcal{T}_a$ and the distance $d_{\mathcal{D}_a}$ on \mathcal{D}_a by writing for all $v, w \in \mathcal{D}_a$,

$$d_{\mathcal{D}_a}(v, w) = \lim_{\varepsilon \rightarrow 0} d_{\mathcal{D}^\varepsilon}(v, w),$$

where the term on the right is well defined for ε small enough. The function $d_{\mathcal{D}_a}$ satisfies the triangle inequality and symmetry since it is a limit of distances. Moreover, it also satisfies positivity since $d_{\mathcal{D}^\varepsilon}(v, w) \geq |d_{\mathcal{T}_a}(\emptyset, v) - d_{\mathcal{T}_a}(\emptyset, w)|$.

Conjecture 5.6.2. *Suppose $a \in (3/2, 5/2)$. Then, for the product topology,*

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\ell^{2-a} d_{\text{fpp}}^\dagger(f_i, f_j) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a p_{\mathbf{q}}} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1}.$$

Conjecture 5.6.3. *Suppose $a \in (2, 5/2)$. Then, for the product topology,*

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\ell^{2-a} d_{\text{gr}}^\dagger(f_i, f_j) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1 + g_{\mathbf{q}}}{2c_a p_{\mathbf{q}}} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1},$$

where $g_{\mathbf{q}} = (1/2) \sum_{k \geq 1} \nu(-k)(2k-1)$. Moreover, \mathcal{D}_a has a compact completion and this convergence of metric spaces also holds in the sense of Gromov-Hausdorff for this completion of \mathcal{D}_a .

Conjecture 5.6.4. *Suppose $a = 2$. Then, for the product topology,*

$$\text{Under } \mathbb{P}^{(\ell)}, \quad \left(\frac{1}{\log \ell} d_{\text{gr}}^\dagger(f_i, f_j) \right)_{i,j \geq 1} \xrightarrow[\ell \rightarrow \infty]{(d)} \left(\frac{1}{2c_a} d_{\mathcal{D}_a}(w_i, w_j) \right)_{i,j \geq 1}.$$

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Deuxième partie

Arbres aléatoires et coalescence

Chapitre 6

Uniform attachment with freezing

*Ce chapitre est issu d'un travail [BBRKK23a] avec Étienne Bellin, Arthur Blanc-Renaudie et Igor Kortchemski à paraître dans *Annals of Applied Probability*.*

Abstract In the classical model of random recursive trees, trees are recursively built by attaching new vertices to old ones. What happens if vertices are allowed to freeze, in the sense that new vertices cannot be attached to already frozen ones? We are interested in the impact of freezing on the height of such trees.

Figure 6.1 – Simulation of a tree of size 10000 built by uniform attachment with freezing, when the number of active (i. e. non-frozen) vertices roughly evolves as a positive fraction of the total number of vertices. Frozen vertices are blue; active vertices are red. The animation (played with Acrobat Reader) shows the resulting process as the number of steps increases.

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6.1 Introduction

Random graphs are instrumental in the study of real-world networks. Uniform recursive trees (sometimes also called uniform attachment trees) are one of such models. They are constructed recursively by starting with one single vertex, and successively attaching new vertices to a previous existing vertex, chosen uniformly at random. This model has been introduced in [NR70] in the context of system generation, and has received considerable interest since, starting with graph-theoretical properties such as the height, number of leaves, etc. (see e.g. [Pit94] and references therein, as well as the survey [SM95]). Recursive trees have been proposed as models for the spread of epidemics [Moo74], the family trees of preserved copies of ancient or medieval texts [NH82], pyramid schemes [Gas77], internet interface maps [JKZVM02] and appear in the study of Hopf algebras [GL89]. Very interesting connections have been made with other probabilistic objects such as the Bolthausen-Sznitman coalescent [GM05] and elephant random walks [Kĭ6]. Uniform recursive trees have also been extended in several directions, for instance by introducing deterministic weights [KR01] or random weights with a random environment [BV06].

In the present work, we introduce and study a modification of this model by introducing *freezing*, in that existing vertices can freeze and new vertices cannot be attached to frozen vertices.

Our motivation is twofold. First, in the context of real-world networks such mechanisms are natural: for instance, on the social network Twitter a user can choose to set their account to “private” which prevents strangers from “following” them; also performing an infection-tracing of an SIR epidemics falls within this framework (see Section 6.6). Second, it is natural to investigate from a mathematical point of view the impact of freezing in dynamically-built random graph models. In particular, in Chapter 7, we investigate the regime where the number of active vertices roughly evolves as the total number of vertices to the power α , for fixed $\alpha \in (0, 1]$, we describe a phase transition where the macroscopic geometry of our model drastically changes according to the value of α . Let us also mention the work [BGY23], which considers a growth-fragmentation-isolation process on random recursive trees in the context of contact tracing, see also [Ber23].

Uniform attachment with freezing. Let us first define our model. In order to incorporate freezing in the model of uniform attachment trees and to obtain results in a rather general setup, we shall

fix beforehand the steps where an attachment takes place and the steps where freezing takes place. Specifically, our input is a deterministic sequence $\mathbf{x} = (x_i)_{i \geq 1}$ of elements of $\{-1, +1\}$. Starting from a sole active vertex, we recursively build random trees by reading the elements of the sequence one after the other, by applying a “freezing” step when reading -1 (which amounts to freezing an active vertex chosen uniformly at random) and a “uniform attachment” step when reading $+1$ (which amounts to attaching a new vertex to an active vertex chosen uniformly at random).

More precisely, given a sequence $\mathbf{x} = (x_i)_{i \geq 1} \in \{-1, +1\}^{\mathbb{N}}$, we set $S_0(\mathbf{x}) := 1$ and for every $n \geq 1$

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i \quad \text{then} \quad \tau(\mathbf{x}) := \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}. \quad (6.1.1)$$

Observe that $\tau(\mathbf{x})$, if it is finite, is the first time when all the vertices are frozen, so that afterwards the tree does not evolve any more. For every $0 \leq n \leq \tau(\mathbf{x})$, let $\mathcal{T}_n(\mathbf{x})$ be the random tree recursively built in this fashion after reading the first n elements of \mathbf{x} (see Algorithm 1 in Sec. 6.2.1 for a precise definition and Fig. 6.3 for an example).

Let us comment on our choice of parametrization. It would have been possible to define the model starting with a random sequence \mathbf{x} , but the choice of a deterministic sequence defines a more general model, for which our results can then be applied.

Local limits. When $\tau(\mathbf{x}) = \infty$, the next result allows to give a meaning to $\mathcal{T}_\infty(\mathbf{x})$ as a local limit of finite trees (see Sec. 6.3 for background on the local topology).

Theorem 6.1.1. *Let $(\mathbf{x}^n)_{n \geq 0}$ be a sequence of elements of $\{-1, 1\}^{\mathbb{N}}$. Suppose that there exists \mathbf{x} such that $\tau(\mathbf{x}) = \infty$ and such that for all $i \geq 1$, $x_i^n = x_i$ for all n large enough. Then the following assertions are equivalent:*

- (I) *The sequence of trees $(\mathcal{T}_n(\mathbf{x}^n))_{n \geq 0}$ converges locally, in distribution.*
- (II) *The sequence of trees $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$ converges locally, almost surely, towards $\mathcal{T}_\infty(\mathbf{x})$.*
- (III) *The sum $\sum_{i \geq 1} \frac{1}{S_i(\mathbf{x})} \mathbb{1}_{\{x_i = -1\}}$ diverges.*

In this case $\mathcal{T}_\infty(\mathbf{x})$ is the local limit of $(\mathcal{T}_n(\mathbf{x}^n))_{n \geq 0}$.

Examples. It is interesting to note that this model encompasses the two classical models of random recursive trees and random uniform plane trees:

- when $x_i = 1$ for every $i \geq 1$, then $\mathcal{T}_n(\mathbf{x})$ is a random recursive tree on n vertices built by uniform attachment;
- when $\mathbf{X} = (X_i)_{i \geq 1}$ is a sequence of non constant i.i.d. uniform random variables on $\{-1, +1\}$, then for every $n \geq 1$, conditionally given $\tau(\mathbf{X}) = n$, $\mathcal{T}_n(\mathbf{X})$ is a uniform plane tree when the plane order among the vertices of $\mathcal{T}_n(\mathbf{X})$ is chosen uniformly at random.

More generally we have the following result, in which we consider the case of random sequences. In this case we will write \mathbf{X} in upper case rather than \mathbf{x} to emphasize the fact that the sequence is random. And, the law of $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ given the random sequence \mathbf{X} follows the description above where \mathbf{X} is considered to be fixed. In other words, we first choose \mathbf{X} randomly, then, conditionally on \mathbf{X} , we construct $\mathcal{T}_n(\mathbf{X})$ following Algorithm 1 where we consider the sequence \mathbf{X} to be deterministic.

Theorem 6.1.2. *Let $p \in [0, 1)$ and let $\mathbf{X} = (X_i)_{i \geq 1}$ be a sequence of i.i.d random variables such that $\mathbb{P}(X_1 = +1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p$. Then $\mathcal{T}_\infty(\mathbf{X})$ has the law of a Bienaymé tree with offspring distribution μ given by $\mu(k) = (1 - p)p^k$ for $k \geq 0$.*

Roughly speaking, this comes from the fact that the subtrees grafted on the initial vertex evolve in an i.i.d. fashion.

Height of uniform attachment trees with freezing. Since we consider large trees, let us consider for $n \geq 1$, a sequence $\mathbf{x}^n \in \{-1, 1\}^{\mathbb{N}}$ such that $\tau(\mathbf{x}^n) > n$, and let $\mathcal{T}_n := \mathcal{T}_n(\mathbf{x}^n)$. Also, to simplify notation, let $S_k^n := S_k(\mathbf{x}^n)$ for $0 \leq k \leq n$.

Our next main theorem is that the height of \mathcal{T}_n is of order

$$h_n^+ = \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n = 1\}}.$$

Theorem 6.1.3. *The following results hold.*

(I) *Let U^n be an active vertex of \mathcal{T}_n chosen uniformly at random, and denote by $H(U^n)$ its height. Then for all $p \geq 1$:*

$$\frac{H(U^n)}{h_n^+} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} 1.$$

(II) *For all $\varepsilon \in (0, 1)$:*

$$\mathbb{P}\left(1 - \varepsilon \leq \frac{\text{Height}(\mathcal{T}_n)}{h_n^+} \leq e + \varepsilon\right) \xrightarrow[n \rightarrow \infty]{} 1.$$

(III) *Assume that $\log(n)/h_n^+ \rightarrow 0$ as $n \rightarrow \infty$. Then for all $p \geq 1$:*

$$\frac{\text{Height}(\mathcal{T}_n)}{h_n^+} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} 1.$$

The proof of Theorem 6.1.3 is based on an alternative construction of \mathcal{T}_n , based on time-reversal, through a growth-coalescent process of rooted forests. This construction can be roughly described as follows (see Algorithm 2 for a precise definition): start with a forest made of S_n^n rooted one-vertex trees. Read successively $x_n^n, x_{n-1}^n, \dots, x_1^n$; when reading $+1$ add a new one-vertex tree to the forest; when reading -1 , choose successively two different forests uniformly at random, connect their roots by an edge, and root this new tree at the root of the first forest. It turns out that, in the end, one gets a tree having the same distribution as \mathcal{T}^n (see Theorem 6.2.4 for a precise statement). This is a generalization of the connection between random recursive trees and Kingman's coalescent which first appeared in [DR76] (see [Dev87, Sec. 6], [Pit94, Sec. 3], [AB15, Sec. 2.2], [ABE18] for applications), in the context of union-find data structures (which are data structures that store a collection of disjoint sets where merging sets and finding a representative member of a set), see e.g. [KS78]. The main difference is that a growth feature must be added when freezing is introduced.

We are next interested in the specific regime where roughly speaking the number of active vertices increases linearly compared to the size of the tree.

Theorem 6.1.4. *Let $c \in (0, 1]$. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $A_n \rightarrow \infty$ and $A_n = o(\log n)$ as $n \rightarrow \infty$. Assume that*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{A_n \leq i \leq \varepsilon n} \left| \frac{S_i^n}{i} - c \right| = 0 \quad \text{and} \quad \forall \varepsilon \in (0, 1), \liminf_{n \rightarrow \infty} \min_{\varepsilon n \leq i \leq n} \frac{S_i^n}{n} > 0. \quad (6.1.2)$$

The following assertions hold.

(I) *We have* $\frac{h_n^+}{\log n} \xrightarrow[n \rightarrow \infty]{} \frac{c+1}{2c}$.

(II) *For all $n \geq 1$, conditionally given \mathcal{T}_n , let V_1^n and V_2^n be independent uniform vertices of \mathcal{T}_n . Then for every $p \geq 1$*

$$\frac{H(V_1^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} \frac{c+1}{2c} \quad \text{and} \quad \frac{d^n(V_1^n, V_2^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} \frac{c+1}{c}, \quad (6.1.3)$$

where $H(V_1^n)$ is the height of V_1^n in \mathcal{T}_n and d^n denotes the graph distance in \mathcal{T}_n .

(III) *We have*

$$\frac{\text{Height}(\mathcal{T}_n)}{\log(n)} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \frac{c+1}{2c} f(c),$$

where $f(c)$ is the unique solution with $f(c) > 1$ to $f(c)(\log f(c) - 1) = (c-1)/(c+1)$.

Let us make some comments on these results. Roughly speaking, \mathcal{T}_n looks like a ‘‘tentacular bush’’ in the sense that two typical vertices are always at the same distance of order $2h_n^+ \sim \frac{c+1}{c} \cdot \log(n)$, but the total height is of order $f(c) \cdot h_n^+$. A typical example where the assumptions (6.1.2) are satisfied is e.g. when $\max_{A_n \leq i \leq n} |S_i^n/i - c| \rightarrow 0$. This is for instance the case when $p \in (1/2, 1)$, $\mathbf{X} = (X_i)_{i \geq 1}$ is a sequence of i.i.d random variables such that $\mathbb{P}(X_1 = +1) = p$ and $\mathbb{P}(X_1 = -1) = 1 - p$, setting $S_i^n = S_i(\mathbf{X})$, conditionally given $\tau(\mathbf{X}) > n$, Theorem 6.1.4 applies almost surely with $c = 2p - 1$ thanks

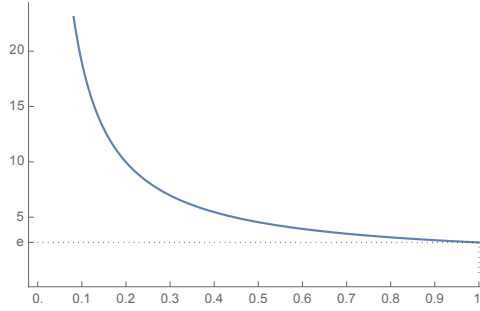


Figure 6.2 – Plot of the function $(\frac{c+1}{2c} f(c) : 0 < c \leq 1)$.

to the laws of large numbers. The reason we rather use (6.1.2) is for applications to contact-tracing in the SIR model (see Sec. 6.6).

Theorem 6.1.4 (III) gives a more precise result than Theorem 6.1.3 (II) under an additional assumption. It generalizes the well known result that when \mathcal{T}_n is a random recursive tree (which is obtained by taking $x_n = 1$ for every $n \geq 1$), we have [Dev87, Theorem 10]: $\text{Height}(\mathcal{T}_n)/\log(n) \rightarrow e$ in probability (see also [Pit94, Theorem 1] for a different approach based on continuous-time branching processes).

The proof of Theorem 6.1.4 (III) is rather delicate: the main difficulty is that the presence of freezing impedes the direct use of branching process techniques. The alternative growth-coalescent process construction previously mentioned lies at the heart of our proof, which is different from how corresponding results would be proved for the random recursive tree.

Let us mention that it is possible to check that $(\mathcal{T}_n/\log(n))_{n \geq 1}$ is not tight for the so-called Gromov–Hausdorff–Prokhorov topology. However, for the topology considered in [ET22, Jan21], $\mathcal{T}_n/\log(n)$ converges to the so-called long dendron Υ_ν (we use the notation of [Jan21, Example 3.12]) with ν being the Dirac mass $\nu = \delta_{(c+1)/2c}$.

Application to contact-tracing in a stochastic SIR dynamics. Our results may be applied to analyze the geometry of the so-called “infection tree” of a stochastic SIR dynamics, in which the vertices are individuals and where edges connect two individuals if one has infected the other. To keep this introduction at reasonable length, we refer to Section 6.6 for details and to Theorem 6.6.2 for a precise statement.

Plan of the paper. The rest of the paper is organized as follows: In Section 6.2, we present our alternative construction, and describe a few of its properties. In Section 6.3, we show Theorem 6.1.1 and 6.1.2 about local limits. In Sections 6.4 and 6.5, we prove respectively Theorems 6.1.3 and 6.1.4 about the height of our trees. We then apply our results in Section 6.6 to a stochastic SIR dynamics. Finally, in Section 6.7 we give a few open problems.

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6.2 Trees constructed by uniform attachment with freezing

We start by defining our model. We also provide for future use a table of notation below (Table 6.1).

6.2.1 Uniform attachment with freezing: recursive construction

Let $\mathbf{x} = (x_i)_{i \geq 1} \in \{-1, +1\}^{\mathbb{N}}$. In what follows, we formally construct the random trees $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$. These trees will be rooted, edge-labelled, and vertex-labelled. The label of an edge is the time it appears. The label of a vertex in $\mathcal{T}_n(\mathbf{x})$ is either the time it froze, or the label “ a ” if it is still active at time n .

Table 6.1 – Table of the main notation and symbols introduced in Section 6.2 and used later.

\mathbb{N}	$= \{1, 2, 3, \dots\}$ positive integers
$\llbracket i, j \rrbracket$	$= \{i, i+1, \dots, j-1, j\}$ all integers between i and j
$\#A$	cardinality of a finite set A
$\mathbf{x} = (x_n)_{n \in \mathbb{N}}$	a sequence of elements of $\{-1, 1\}$
$S_n(\mathbf{x})$	$= 1 + \sum_{i=1}^n x_i$
$\tau(\mathbf{x})$	$= \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}$
$\mathcal{T}_n(\mathbf{x})$	tree built at time n by Algorithm 1; $S_n(\mathbf{x})$ is its number of active vertices
$N_n(\mathbf{x})$	total number of vertices in $\mathcal{T}_n(\mathbf{x})$; $N_n(\mathbf{x}) = (S_n(\mathbf{x}) + n + 1)/2$ when $n \leq \tau(\mathbf{x})$
$\mathcal{F}_n^n(\mathbf{x}), \mathcal{F}_{n-1}^n(\mathbf{x}), \dots, \mathcal{F}_0^n(\mathbf{x})$	the forest of trees built by Algorithm 2
$\mathcal{T}^n(\mathbf{x})$	$= \mathcal{F}_0^n(\mathbf{x})$ the -output of Algorithm 2
$\mathbb{F}_n(\mathbf{x})$	$= \{i \in \llbracket 1, n \rrbracket : x_i = -1\}$ the set of labels of frozen vertices of $\mathcal{T}^n(\mathbf{x})$
$\mathbb{A}_n(\mathbf{x})$	$= \{a_1, \dots, a_{S_n(\mathbf{x})}\}$ the set of labels of active vertices of $\mathcal{T}^n(\mathbf{x})$
$\mathbb{V}_n(\mathbf{x})$	$= \mathbb{F}_n(\mathbf{x}) \cup \mathbb{A}_n(\mathbf{x})$ the set of labels of all vertices of $\mathcal{T}^n(\mathbf{x})$
$b_n(u)$	the birth time of $u \in \mathbb{V}_n(\mathbf{x})$ in the construction of $\mathcal{T}^n(\mathbf{x})$ by Algorithm 2
$c_n(u, v)$	the coalescence time between $u, v \in \mathbb{V}_n(\mathbf{x})$ in the construction of $\mathcal{T}^n(\mathbf{x})$ by Algorithm 2
$H_i^n(u)$	the height of vertex u in $\mathcal{F}_i^n(\mathbf{x})$

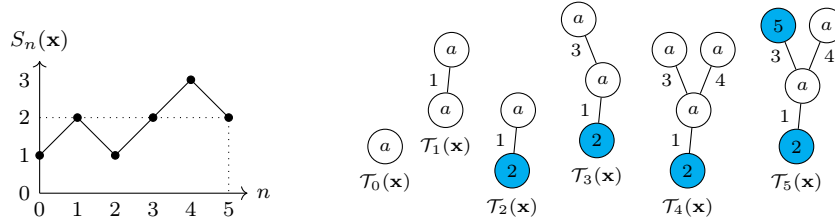


Figure 6.3 – On the left is represented the walk $(S_n(\mathbf{x}))_{n \geq 0}$ up to time $n = 5$ associated with the sequence $\mathbf{x} = +1, -1, +1, +1, -1, \dots$. On the right, a possible realisation of the trees $\mathcal{T}_0(\mathbf{x})$ to $\mathcal{T}_5(\mathbf{x})$ given this sequence. Frozen vertices have been colored in blue.

Algorithm 1.

- Start with the tree $\mathcal{T}_0(\mathbf{x})$ made of a single root vertex labelled a .
- For every $n \geq 1$, if $\mathcal{T}_{n-1}(\mathbf{x})$ has no vertices labelled a , then set $\mathcal{T}_n(\mathbf{x}) = \mathcal{T}_{n-1}(\mathbf{x})$. Otherwise let V_n be a random uniform active vertex of $\mathcal{T}_{n-1}(\mathbf{x})$, chosen independently from the previous ones. Then:
 - if $x_n = -1$, build $\mathcal{T}_n(\mathbf{x})$ from $\mathcal{T}_{n-1}(\mathbf{x})$ simply by replacing the label a of V_n with the label n ;
 - if $x_n = 1$, build $\mathcal{T}_n(\mathbf{x})$ from $\mathcal{T}_{n-1}(\mathbf{x})$ by adding an edge labelled n between V_n and a new vertex labelled a .

For $n \geq 0$, we view $\mathcal{T}_n(\mathbf{x})$ as a rooted, double-labelled tree (that is, edge-labelled and vertex-labelled). If $N_n(\mathbf{x})$ represents the total number of vertices of $\mathcal{T}_n(\mathbf{x})$, observe that by construction $N_n(\mathbf{x}) = (S_n(\mathbf{x}) + n + 1)/2$ for $0 \leq n \leq \tau(\mathbf{x})$.

6.2.2 Uniform attachment with freezing: growth-coalescent construction

As before, let $\mathbf{x} = (x_n)_{n \geq 1} \in \{-1, 1\}^{\mathbb{N}}$. We introduce in this section an alternative time-reversed construction of uniform attachment trees with freezing. It may be seen as a growth coalescence process of forests, and as previously mentioned, is a generalization of the connection between random recursive trees and Kingman's coalescent introduced in [DR76]. Most of our proofs are based on this construction.

Algorithm 2. Fix $0 \leq n \leq \tau(\mathbf{x})$. We construct a sequence $(\mathcal{F}_n^n(\mathbf{x}), \mathcal{F}_{n-1}^n(\mathbf{x}), \dots, \mathcal{F}_0^n(\mathbf{x}))$ of forests of rooted, edge-labelled, vertex-labelled, unoriented trees by induction as follows.

- Let $\mathcal{F}_n^n(\mathbf{x})$ be a forest made of $S_n(\mathbf{x})$ one-vertex trees labelled $a_1, \dots, a_{S_n(\mathbf{x})}$.

- For every $1 \leq i \leq n$, if $\mathcal{F}_i^n(\mathbf{x})$ has been constructed, define $\mathcal{F}_{i-1}^n(\mathbf{x})$ as follows:
 - if $x_i = -1$, $\mathcal{F}_{i-1}^n(\mathbf{x})$ is obtained by adding to $\mathcal{F}_i^n(\mathbf{x})$ a new one-vertex tree labelled i ;
 - if $x_i = 1$, let (T_1, T_2) be a pair of different random trees in $\mathcal{F}_i^n(\mathbf{x})$ chosen uniformly at random, independently of the previous choices; then $\mathcal{F}_{i-1}^n(\mathbf{x})$ is obtained from $\mathcal{F}_i^n(\mathbf{x})$ by adding an edge labelled i between the roots $r(T_1)$ and $r(T_2)$ of respectively T_1 and T_2 , and rooting the tree thus obtained at $r(T_1)$;
- Let $\mathcal{T}^n(\mathbf{x})$ be the only tree of $\mathcal{F}_0^n(\mathbf{x})$.

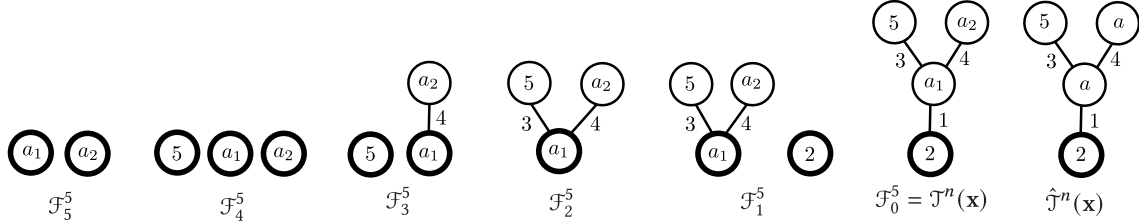


Figure 6.4 – An illustration of Algorithm 2 with $n = 5$ and $(x_5, x_4, x_3, x_2, x_1) = (-1, 1, 1, -1, 1)$ (this is the same sequence as in Fig. 6.3). For example, since $x_2 = -1$, \mathcal{F}_1^5 is obtained from \mathcal{F}_2^5 by adding a new tree made of a vertex labeled 2. Since $x_4 = 1$, to build \mathcal{F}_3^5 from \mathcal{F}_4^5 we have chosen in \mathcal{F}_4^5 two trees (T_1, T_2) with T_1 being the vertex a_1 and T_2 being the vertex a_2 , and we have added an edge labelled 4 between the roots $r(T_1) = a_1$ and $r(T_2) = a_2$ of T_1 and T_2 , and rooted the tree thus obtained at $r(T_1) = a_1$.

Figure 6.5 – Simulation of Algorithm 2: the animation (played with Acrobat Reader) shows the resulting process as the number of steps increases. When a new vertex appears, its color is chosen at random. When two trees merge, the resulting tree keeps the color of the largest of the two trees.

We shall soon see in Sec. 6.2.4 the equivalence between Algorithm 1 and Algorithm 2, but before that, we introduce some important notation and consequences which will be useful in the analysis of the height of trees built by uniform attachment with freezing.

Important convention Throughout this article, when the context is clear, we will sometimes drop the parameter \mathbf{x} to lighten notation: in particular, we shall write τ , S_n , \mathcal{T}_n , \mathcal{F}_i^n instead of $\tau(\mathbf{x})$, $S_n(\mathbf{x})$, $\mathcal{T}_n(\mathbf{x})$, $\mathcal{F}_i^n(\mathbf{x})$, etc.

6.2.3 Laws of the birth and coalescence times

Let $\mathbf{x} = (x_n)_{n \geq 1} \in \{-1, 1\}^{\mathbb{N}}$. First, define

$$\mathbb{F}_n(\mathbf{x}) = \{i \in [1, n] : x_i = -1\}, \quad \mathbb{A}_n(\mathbf{x}) = \{a_1, \dots, a_{S_n(\mathbf{x})}\}, \quad \mathbb{V}_n(\mathbf{x}) = \mathbb{F}_n(\mathbf{x}) \cup \mathbb{A}_n(\mathbf{x}). \quad (6.2.1)$$

It is crucial to observe that while \mathcal{T}^n is a random tree, the labels of its vertices are deterministic quantities that only depend on \mathbf{x} : \mathbb{A}_n are the labels of the active vertices of \mathcal{T}^n and \mathbb{F}_n are the labels of the frozen vertices of \mathcal{T}^n . In particular, the elements of \mathbb{V}_n will be called vertices of \mathcal{T}^n .

Next, for every $u \in \mathbb{V}_n$, let $\mathbf{b}_n(u)$ be the largest $i \in \{0, 1, \dots, n\}$ such that u belongs to the forest \mathcal{F}_i^n . Explicitly, if $u \in \mathbb{A}_n$ is an active vertex then $\mathbf{b}_n(u) = n$, and if $u \in \mathbb{F}_n$ (note that u is then an integer) then $\mathbf{b}_n(u) = u - 1$ (see Fig. 6.4 for an example). We say that $\mathbf{b}_n(u)$ is the *birth time* of u , since it encodes the first time when vertex u appears in Algorithm 2. For $0 \leq i \leq \mathbf{b}_n(u)$, let $\mathbf{H}_i^n(u)$ be the height of vertex u in \mathcal{F}_i^n (that is, the graph distance between u and the root of whichever tree of \mathcal{F}_i^n contains u).

Finally, for $u, v \in \mathbb{V}_n$, let $\mathbf{c}_n(u, v)$ be the largest $i \in \{0, 1, \dots, n\}$ such that u and v belong to the same tree in the forest \mathcal{F}_i^n obtained when building \mathcal{T}^n in Algorithm 2. We say that $\mathbf{c}_n(u, v)$ is the *coalescence time* between u and v , since it encodes the first time u and v belong to the same tree in Algorithm 2 (observe that while $\mathbf{b}_n(u)$ is deterministic, $\mathbf{c}_n(u, v)$ is random).

We now state several simple consequences of Algorithm 2, which will be useful to study the geometry of \mathcal{T}^n .

Lemma 6.2.1. *Fix $1 \leq n \leq \tau$ and $v \in \mathbb{V}_n$. For every $1 \leq i \leq \mathbf{b}_n(v)$:*

$$\mathbb{P}(\mathbf{H}_{i-1}^n(v) - \mathbf{H}_i^n(v) = 1 | \mathcal{F}_n^n, \dots, \mathcal{F}_i^n) = \frac{1}{S_i} \mathbb{1}_{\{x_i=1\}}.$$

This is clear from the definition of Algorithm 2. It follows from Lemma 6.2.1 that if $(Y_i)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i)_{1 \leq i \leq n}$, we have for every $u \in \mathbb{V}_n$

$$\mathbf{H}_0^n(u) \stackrel{(d)}{=} \sum_{i=1}^{\mathbf{b}_n(u)} Y_i \mathbb{1}_{\{x_i=1\}}. \quad (6.2.2)$$

The next result identifies the law of the birth time of a vertex of \mathcal{T}^n chosen uniformly at random.

Lemma 6.2.2. *Fix $1 \leq n \leq \tau$. Let V be an element of \mathbb{V}_n chosen uniformly at random. For every $1 \leq m \leq n$,*

$$\mathbb{P}(\mathbf{b}_n(V) < m) = \frac{m+1 - S_m}{n+1 + S_n}.$$

Proof. We noted that $\mathbf{b}_n(u) = n$ if $u \in \mathbb{A}_n$ and $\mathbf{b}_n(u) = u - 1$ if $u \in \mathbb{F}_n$. Also, conditionally given the fact that $V \in \mathbb{F}_n$, $\mathbf{b}_n(V)$ is uniform on \mathbb{F}_n . Since $\#\mathbb{A}_n = S_n$ and $\#\mathbb{F}_n = \sum_{i=1}^n \mathbb{1}_{\{x_i=-1\}}$, it follows by the definition of \mathbb{F}_n that

$$\mathbb{P}(\mathbf{b}_n(V) < m) = \frac{\sum_{i=1}^m \mathbb{1}_{\{x_i=-1\}}}{S_n + \sum_{i=1}^n \mathbb{1}_{\{x_i=-1\}}} = \frac{m+1 - S_m}{n+1 + S_n},$$

where for the last equality we have used the fact that $\sum_{i=1}^k \mathbb{1}_{\{x_i=-1\}} = (k - S_k + 1)/2$ for every $1 \leq k \leq n$. \square

The last useful result identifies the law of the coalescence times between two vertices.

Lemma 6.2.3. *Fix $1 \leq n \leq \tau$ and consider $u, v \in \mathbb{V}_n$. Then for every $0 \leq c < \mathbf{b}_n(u) \wedge \mathbf{b}_n(v)$ with $x_{c+1} = 1$:*

$$\mathbb{P}(\mathbf{c}_n(u, v) = c) = \frac{1}{\binom{S_{c+1}}{2}} \prod_{\substack{i=c+2 \\ \text{s.t. } x_i=1}}^{\mathbf{b}_n(u) \wedge \mathbf{b}_n(v)} \left(1 - \frac{1}{\binom{S_i}{2}}\right).$$

Proof. When running Algorithm 2, the factors in the product correspond to the probability that the two vertices born at times $\mathbf{b}_n(u)$ and $\mathbf{b}_n(v)$ do not coalesce until time c and the factor $1/\binom{S_{c+1}}{2}$ corresponds to the conditional probability that the two vertices coalesce at time c , given that they have not previously done so. \square

6.2.4 Equivalence between the two constructions

Table 6.2 – Table of the main notation and symbols introduced in Section 6.2.4.

$\hat{\mathcal{T}}^n(\mathbf{x})$	the tree obtained from $\mathcal{T}^n(\mathbf{x})$ by relabelling all its active vertices by “ a ”
$\mathfrak{T}_{k,n}(A)$, $n \geq 1$ $0 \leq k \leq n$, $A \subset \mathbb{N}$ $\#A = 2n - k - 1$	set of rooted trees with n vertices, such that all the $n - 1$ edges and $n - k$ of the vertices are labelled in a one-to-one manner with the integers of A , while the other k vertices are labelled with the letter “ a ”
$\mathfrak{T}_{k,n}^+(A)$	set of all trees in $\mathfrak{T}_{k,n}(A)$ such that edge-labels increase along paths directed away from the root, and all vertices labelled with an integer (i.e. not “ a ”) have a label which is larger than the labels of their adjacent edges
$\mathfrak{T}_{k,n}^+$	$\mathfrak{T}_{k,n}^+(A)$ with $A = \llbracket 1, 2n - k - 1 \rrbracket$

Observe that the active vertices of \mathcal{T}_n are all labelled a while the active vertices of \mathcal{T}^n are labelled a_1, \dots, a_{S_n} . It turns out that \mathcal{T}_n is equal in law to \mathcal{T}^n when its active vertices are all relabelled a . More precisely, let $\hat{\mathcal{T}}^n$ be the tree obtained from \mathcal{T}^n by relabeling its S_n active vertices by a (see the right-most part of Fig. 6.4 for an illustration).

Theorem 6.2.4. *The two trees \mathcal{T}_n and $\hat{\mathcal{T}}^n$ have the same distribution.*

To prove this result, we first identify the law of \mathcal{T}_n . Let us determine the range of \mathcal{T}_n . For $n \in \mathbb{N}$, $0 \leq k \leq n$ and $A \subset \mathbb{N}$ with $\#A = 2n - k - 1$, let $\mathfrak{T}_{k,n}(A)$ be the set of rooted trees with n vertices, such that all the $n - 1$ edges and $n - k$ of the vertices are labelled in a one-to-one manner with the integers of A , while the other k vertices are labelled with the letter a . We then define $\mathfrak{T}_{k,n}^+(A)$ to be the set of all trees in $\mathfrak{T}_{k,n}(A)$ such that edge labels increase along paths directed from the root, and all vertices labelled with an integer (i.e. not a) have a label which is larger than the labels of their adjacent edges. We set $\mathfrak{T}_{k,n}^+ = \mathfrak{T}_{k,n}^+(\llbracket 1, 2n - k - 1 \rrbracket)$. Note that \mathcal{T}_n is an element of $\mathfrak{T}_{S_n, N_n}^+$ for $0 \leq n \leq \tau$ (see the left-most tree in Fig. 6.7 for an example) where $N_n = (S_n + n + 1)/2$ is the total number of vertices in \mathcal{T}_n . In particular, $\mathfrak{T}_{k,n}^+$ is the set of all possible trees with k active vertices and $n - k$ frozen vertices which can be obtained by Algorithm 1.

Lemma 6.2.5. *Fix $0 \leq n \leq \tau$ and $T_n \in \mathfrak{T}_{S_n, N_n}^+$. Then*

$$\mathbb{P}(\mathcal{T}_n = T_n) = \prod_{1 \leq i \leq n} \frac{1}{S_{i-1}}.$$

Proof. For two trees T, T' with $T \in \mathfrak{T}_{k,n}^+$, we write $T \rightsquigarrow T'$ if either $T' \in \mathfrak{T}_{k+1,n}^+$ and T' is obtained by replacing in T the label of a vertex labeled a by $2n - k$, or $T' \in \mathfrak{T}_{k,n+1}^+$ and T' is obtained from T by adding an edge labelled by $2n - k$ between a new vertex labeled a and an existing a -labeled vertex.

Since $T_n \in \mathfrak{T}_{S_n, N_n}^+$, observe that there is a unique sequence (T_0, \dots, T_{n-1}) such that $T_i \in \mathfrak{T}_{S_i, N_i}^+$ for every $0 \leq i \leq n - 1$ and such that $T_i \rightsquigarrow T_{i+1}$ for every $0 \leq i \leq n - 1$. In particular,

$$\mathbb{P}(\mathcal{T}_n = T_n) = \mathbb{P}((\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_n) = (T_0, T_1, \dots, T_n)) = \prod_{i=1}^n \mathbb{P}(\mathcal{T}_i = T_i | \mathcal{T}_{i-1} = T_{i-1}).$$

Then observe that $\mathbb{P}(\mathcal{T}_i = T_i | \mathcal{T}_{i-1} = T_{i-1}) = \frac{1}{S_{i-1}}$. Indeed, T_{i-1} has S_{i-1} active vertices. Thus, if $x_i = 1$, then the probability of attaching a new vertex to a given active vertex is $1/S_{i-1}$; if $x_i = -1$, then the probability of freezing a given active vertex is $1/S_{i-1}$. \square

Proof of Theorem 6.2.4. First of all, by Lemma 6.2.5, for every $T_n \in \mathfrak{T}_{S_n, N_n}^+$ we have

$$\mathbb{P}(\mathcal{T}_n = T_n) = \prod_{1 \leq i \leq n} \frac{1}{S_{i-1}}.$$

Now let $\tilde{\mathfrak{T}}_{k,n}^+$ be the set of all trees obtained from $\mathfrak{T}_{k,n}^+$ by labelling a_1, \dots, a_k their k active vertices. It is enough to show that for every $\tilde{T}_n \in \tilde{\mathfrak{T}}_{k,n}^+$ we have

$$\mathbb{P}(\mathcal{T}^n = \tilde{T}_n) = \frac{1}{S_n!} \prod_{1 \leq i \leq n} \frac{1}{S_{i-1}}. \quad (6.2.3)$$

Indeed, since there are $S_n!$ ways to relabel by a_1, \dots, a_k the S_n active vertices of any given tree $T_n \in \mathfrak{T}_{k,n}^+$, this will imply that $\mathbb{P}(\mathcal{T}_n = T_n) = \mathbb{P}(\hat{\mathcal{T}}^n = T_n)$.

To establish (6.2.3) we need to introduce some notation. Let $\tilde{\mathfrak{F}}_k^n$ be the set of all forests of rooted, edge-labelled and vertex-labelled trees, such that:

- S_n vertices are labelled a_1, \dots, a_{S_n} ;
- all the other vertices are labelled in a one-to-one fashion by $\llbracket k+1, n \rrbracket \cap \{1 \leq i \leq n : x_i = -1\}$;
- the edges are labelled in a one-to-one fashion by $\llbracket k+1, n \rrbracket \cap \{1 \leq i \leq n : x_i = 1\}$.

In addition, the labelling satisfies the following condition: edge-labels increase along paths directed from the roots, and all vertices labelled with an integer (i.e. not a) have a label which is larger than the labels of their adjacent edges. Notice that forests in $\tilde{\mathfrak{F}}_k^n$ are made of S_k trees.

Observe that the random forest \mathcal{F}_i^n satisfies $\mathcal{F}_i^n \in \tilde{\mathfrak{F}}_i^n$ for every $0 \leq i \leq n$. If $F_i \in \tilde{\mathfrak{F}}_i^n$ and $F_{i-1} \in \tilde{\mathfrak{F}}_{i-1}^n$, we write $F_i \rightsquigarrow F_{i-1}$ if F_{i-1} is obtained from F_i by either adding a new tree made of a sole vertex labeled i , or choosing two different trees T_1, T_2 in F_i and adding an edge labeled i between the roots $r(T_1)$ and $r(T_2)$ of respectively T_1 and T_2 , and rooting the tree thus obtained at $r(T_1)$.

Now, let F_0^n be the forest made of the tree \tilde{T}_n . Observe that there is a unique sequence (F_n^n, \dots, F_1^n) such that $F_i^n \in \tilde{\mathfrak{F}}_i^n$ for every $0 \leq i \leq n-1$ and such that $F_i^n \rightsquigarrow F_{i-1}^n$ for every $1 \leq i \leq n$. Notice that F_i^n is made of S_i trees. In particular,

$$\begin{aligned} \mathbb{P}(\mathcal{T}^n = \tilde{T}_n) &= \mathbb{P}((\mathcal{F}_n^n, \mathcal{F}_{n-1}^n, \dots, \mathcal{F}_0^n) = (F_n^n, F_{n-1}^n, \dots, F_0^n)) \\ &= \prod_{i=1}^n \mathbb{P}(\mathcal{F}_{i-1}^n = F_{i-1}^n | \mathcal{F}_i^n = F_i^n). \end{aligned}$$

Then observe that if $x_i = -1$ we have $\mathbb{P}(\mathcal{F}_{i-1}^n = F_{i-1}^n | \mathcal{F}_i^n = F_i^n) = 1$ since to build \mathcal{F}_{i-1}^n from \mathcal{F}_i^n with probability one we add a new one-vertex tree labelled i to \mathcal{F}_{i-1}^n , and since $x_i = -1$, the forest F_{i-1}^n is also obtained from F_i^n by adding a new one-vertex tree labelled i . If $x_i = 1$ we have the equality $\mathbb{P}(\mathcal{F}_{i-1}^n = F_{i-1}^n | \mathcal{F}_i^n = F_i^n) = \frac{1}{S_i(S_i-1)}$; indeed, there are $S_i(S_i-1)$ ways of choosing a pair (T_1, T_2) of different trees in F_i^n , and one of them gives F_{i-1}^n when adding an edge labeled i between the roots $r(T_1)$ and $r(T_2)$ of respectively T_1 and T_2 , and rooting the tree thus obtained at $r(T_1)$.

As a consequence,

$$\mathbb{P}(\mathcal{T}^n = \tilde{T}_n) = \prod_{\substack{1 \leq i \leq n \\ x_i = 1}} \frac{1}{S_i(S_i-1)} = \frac{1}{S_n!} \prod_{1 \leq i \leq n} \frac{1}{S_{i-1}},$$

where the last equality is readily checked by induction. This establishes (6.2.3) and completes the proof. \square

6.2.5 A connection with increasing binary trees

Table 6.3 – Table of the main notation and symbols introduced in Section 6.2.5.

$\mathfrak{B}_{k,n}(A)$	set of rooted binary plane trees with $2n-1$ vertices, such that k of the leaves are labelled with the letter a and the $2n-k-1$ other vertices are labelled in a one-to-one manner with the integers of A
$\mathfrak{B}_{k,n}^+(A)$	set of trees in $\mathfrak{B}_{k,n}(A)$ such that the vertices have increasing labels paths directed from the root.
$\mathfrak{B}_{k,n}^+$	$\mathfrak{B}_{k,n}^+(A)$ with $A = \llbracket 1, 2n-k-1 \rrbracket$

It turns out that $\mathfrak{T}_{k,n}(A)$ is in bijection with a certain set of increasing plane binary trees, which are rooted trees, with an order among the children of internal vertices (see Sec. 6.3.2 for a precise setting), and where every vertex has either zero or two children.

Such a bijection is well-known in the case $k = 0$ (see e.g. [KS78] and [Dev87, Sec. 6]). To make this connection explicit, we need some further notation. For $n \geq 1$, $0 \leq k \leq n$ and $A \subset \mathbb{N}$ with cardinality $\#A = 2n - k - 1$ we define $\mathfrak{B}_{k,n}(A)$ to be the set of rooted binary plane trees with $2n - 1$ vertices, such that k of the leaves are labelled with the letter a and the $2n - k - 1$ other vertices are labelled in a one-to-one manner with the integers of A . We also define $\mathfrak{B}_{k,n}^+(A)$ to be the set of trees in $\mathfrak{B}_{k,n}(A)$ such that the vertices have increasing labels along paths directed away from the root. We set $\mathfrak{B}_{k,n}^+ = \mathfrak{B}_{k,n}^+(\llbracket 1, 2n - k - 1 \rrbracket)$ (see the right of Figure 6.7 for an example of a tree in $\mathfrak{B}_{4,15}^+$).

We are now ready to define a function Φ from $\bigcup \mathfrak{T}_{k,n}(A)$ to $\bigcup \mathfrak{B}_{k,n}(A)$ where the unions are over all possible triplets (k, n, A) . The definition is done by induction. If τ is a trivial tree with one vertex then set $\Phi(\tau) = \tau$. If τ has at least 2 vertices, then let u be the child of the root ρ such that the edge between ρ and u has the smallest label, denoted by ℓ , among all the edges adjacent to ρ . Denote by τ_1 the sub-tree rooted in u composed of all the descendants of u and τ_2 the sub-tree rooted in ρ composed of all the vertices that are not in τ_1 . Then $\Phi(\tau)$ is the binary tree starting with a root, labelled ℓ , with two children: the first child on the left gives birth to $\Phi(\tau_1)$ and the second one on the right gives birth to $\Phi(\tau_2)$ (see Figure 6.6 and 6.7).

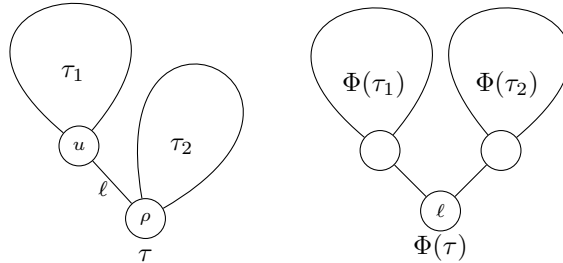


Figure 6.6 – Illustration of the construction of the bijection Φ by induction.

Proposition 6.2.6. *The function Φ defined above is a bijection between $\mathfrak{T}_{k,n}^+$ and $\mathfrak{B}_{k,n}^+$.*

Proof. It suffices to construct explicitly the inverse bijection Ψ of Φ . We proceed by induction. For a tree τ with a single vertex let $\Psi(\tau) = \tau$. If τ is a binary tree with at least 2 children and a root labelled ℓ , let τ_1 be the tree on the left and let τ_2 be the tree on the right of the root. Then $\Psi(\tau)$ is obtained by attaching the root of $\Psi(\tau_1)$ to the root of $\Psi(\tau_2)$ with an edge labelled ℓ ; the root of $\Psi(\tau_2)$ becomes the root of $\Psi(\tau)$. It is a simple matter to check that Ψ is indeed the inverse bijection of Φ . \square

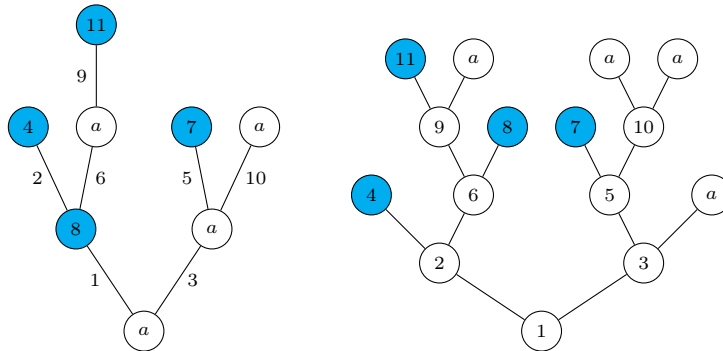


Figure 6.7 – An example of a tree $\tau \in \mathfrak{T}_{4,8}^+$ on the left and its image by the bijection $\Phi(\tau) \in \mathfrak{B}_{4,15}^+$ on the right.

As a consequence, we can identify the cardinality of $\mathfrak{T}_{0,n}^+$, which is the set of all possible trees with 0 active vertices and n frozen vertices which can be obtained by Algorithm 1, using the fact that $|\mathfrak{B}_{0,n}^+|$ is counted by the so-called tangent numbers [Don75].

Corollary 6.2.7. *The cardinality of $\mathfrak{T}_{0,n}^+$ is given by*

$$\#\mathfrak{T}_{0,n}^+ = \frac{4^n(4^n - 1)|B_{2n}|}{2n} = \tan^{(2n-1)}(0)$$

where B_n is the n -th Bernoulli number and $\tan^{(n)}$ stands for the n -th derivative of the tangent function. This sequence of numbers is given by OEIS A000182.

6.3 Local limits

Let \mathfrak{T} be the set of all rooted, vertex-labelled, edge-labelled, and locally finite (meaning that every vertex has finite degree) trees. For every $t \in \mathfrak{T}$ and $h \in \mathbb{N}$ let $\llbracket t \rrbracket_h$ be the finite rooted vertex-labelled edge-labelled tree obtained from t by keeping only the vertices at distance at most h from the root vertex together with the edges between them, and their labels. It is standard to construct a metric d_{loc} on \mathfrak{T} such that the space $(\mathfrak{T}, d_{\text{loc}})$ is Polish (i.e. separable and complete) and $t_n \rightarrow t$ for d_{loc} if and only if for every $h \geq 1$, we have $\llbracket t_n \rrbracket_h = \llbracket t \rrbracket_h$ (as rooted, vertex-labelled, edge-labelled trees) for n large enough. Similar metrics can be defined for other families of trees (plane and unlabelled, plane and vertex-labelled and edge-labelled) and will be also denoted by d_{loc} . We refer to the associated topologies as “the local topology”.

6.3.1 Proof of Theorem 6.1.1

We begin by showing that conditions (II) and (III) are equivalent. Since the sequence of trees $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$ is increasing, it converges almost surely for d_{loc} , if and only if, almost surely, the degree of each vertex converges (to a finite value).

First, we show that every vertex degree converges a.s. if and only if every vertex freezes a.s. at some finite time. Once a vertex is frozen its degree does not change, so if every vertex freezes a.s. at some finite time then a.s. every vertex degree converges. For the other implication, we show that if with positive probability a vertex does not eventually freeze, then with positive probability its degree does not converge. Set $\mathbf{P} = \{n \geq 1 : x_n = +1\}$ and $\mathbf{M} = \{n \geq 1 : x_n = -1\}$. Observe that

$$\sum_{n \in \mathbf{P}} \frac{1}{S_n(\mathbf{x})} = +\infty. \quad (6.3.1)$$

Indeed, since the walk $(S_n(\mathbf{x}))_{n \geq 0}$ never touches 0 we have $n^{-1}\#\{\mathbf{P} \cap \{1, \dots, n\}\} \geq 1/2$ for all $n \geq 1$. Thus $\sum_{n \in \mathbf{P}} 1/S_n(\mathbf{x}) \geq \sum_{n \in \mathbf{P}} 1/(n+1)$ diverges.

Now fix $n_0 \in \mathbf{P}$ and denote by v the vertex that appears at time n_0 . Let E be the event “ v never freezes”. We show that $\mathbb{P}(E) > 0$ implies that with positive probability the degree of v does not converge. To this end, for $n \in \mathbf{P}$, $n \geq n_0$, let E_n denote the event “the degree of v increases by one at time n ”. Then, since the event E_n involves times in \mathbf{P} while the event E involves times in \mathbf{M} , conditionally given E the events $(E_n)_{n \geq 1}$ are independent and $\mathbb{P}(E_n|E) = 1/S_n(\mathbf{x})$. Thus, by (6.3.1) and Borel-Cantelli, $\mathbb{P}(\text{the degree of } v \text{ diverges as } n \rightarrow \infty | E) = 1$. Thus with positive probability the degree of v does not converge.

Now, to show that (II) and (III) are equivalent, it remains to show that, almost surely, the fact that all the vertices freeze at some point is equivalent to the divergence of the sum $\sum_{n \in \mathbf{M}} 1/S_n(\mathbf{x})$. Fix a vertex v appearing at time $n_0 \geq 0$. The probability that it never freezes is $\prod_{n \in \mathbf{M}, n > n_0} (1 - 1/S_n(\mathbf{x}))$ which converges towards 0 if and only if $\sum_{n \in \mathbf{M}} 1/S_n(\mathbf{x}) = +\infty$.

Now we prove that (II) and (III) imply (I). Let t be a finite tree with no vertex labelled a (so all vertices are frozen). If m is the biggest label of t then for all h and n the event “ $\llbracket \mathcal{T}_n(\mathbf{x}) \rrbracket_h = t$ ” depends only on the first m steps of \mathbf{x} . Therefore for all n large enough $\mathbb{P}(\llbracket \mathcal{T}_n(\mathbf{x}) \rrbracket_h = t) = \mathbb{P}(\llbracket \mathcal{T}_n(\mathbf{x}^n) \rrbracket_h = t)$. If $\mathcal{T}_n(\mathbf{x})$ converges locally to $\mathcal{T}_\infty(\mathbf{x})$, then, by (III), all the vertices of $\mathcal{T}_\infty(\mathbf{x})$ are frozen so we deduce that $\mathcal{T}_n(\mathbf{x}^n)$ also converges locally in distribution to $\mathcal{T}_\infty(\mathbf{x})$.

Finally we show that (I) implies (II). Assume (I) and suppose that $\mathcal{T}_n(\mathbf{x})$ does not converge locally almost surely. Then there exists i such that $x_i = +1$ and the vertex v_i added at time i in $\mathcal{T}_n(\mathbf{x})$ has a degree converging towards $+\infty$ with positive probability. In other words

$$\mathbb{P}(\deg(v_i, \mathcal{T}_n(\mathbf{x})) \rightarrow +\infty) = \lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{n \geq i} \{\deg(v_i, \mathcal{T}_n(\mathbf{x})) \geq m\} \right) > 0.$$

Let φ be an increasing function such that for all $k \leq n$, $\mathbf{x}_k^{\varphi(n)} = \mathbf{x}_k$. There is an obvious coupling such that $\mathcal{T}_n(\mathbf{x}) = \mathcal{T}_n(\mathbf{x}^{\varphi(n)})$ for all n . Therefore, for all m ,

$$\bigcup_{n \geq i} \{\deg(v_i, \mathcal{T}_n(\mathbf{x})) \geq m\} = \bigcup_{n \geq i} \{\deg(v_i, \mathcal{T}_n(\mathbf{x}^{\varphi(n)})) \geq m\},$$

and

$$\lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{n \geq i} \{\deg(v_i, \mathcal{T}_{\varphi(n)}(\mathbf{x}^{\varphi(n)})) \geq m\} \right) \geq \lim_{m \rightarrow \infty} \mathbb{P} \left(\bigcup_{n \geq i} \{\deg(v_i, \mathcal{T}_n(\mathbf{x}^{\varphi(n)})) \geq m\} \right) > 0.$$

We deduce that under this coupling the degree of v_i in $\mathcal{T}_{\varphi(n)}(\mathbf{x}^{\varphi(n)})$ tends to ∞ with positive probability so $\mathcal{T}_n(\mathbf{x}^n)$ can't converge locally. \square

6.3.2 A connection with Geometric Bienaymé trees: Proof of Theorem 6.1.2

For the proof of Theorem 6.1.2, we need the Ulam–Harris formalism of plane trees. Let \mathcal{U} be the set of labels defined by

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where, by convention, $(\mathbb{N})^0 = \{\emptyset\}$. In other words, an element of \mathcal{U} is a (possibly empty) sequence $u = u_1 \cdots u_j$ of positive integers. When $u = u_1 \cdots u_j$ and $v = v_1 \cdots v_k$ are elements of \mathcal{U} , we let $uv = u_1 \cdots u_j v_1 \cdots v_k$ be the concatenation of u and v . In particular, $u\emptyset = \emptyset u = u$. Finally, a (Ulam–Harris) *plane tree* is a subset of \mathcal{U} satisfying the following three conditions:

- (i) $\emptyset \in \tau$,
- (ii) if $v \in \tau$ and $v = uj$ for some $j \in \mathbb{N}$, then $u \in \tau$,
- (iii) for every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ such that for every $j \in \mathbb{N}$, $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

If τ is a plane tree and $u \in \tau$ let $\theta_u \tau = \{v \in \mathcal{U} : uv \in \tau\}$ be the subtree of descendants of u . We finally let \mathbb{T} be the set of all plane trees (observe that such trees may be infinite, but are always locally finite).

Given a probability distribution μ on \mathbb{Z}_+ , the law \mathbb{P}_μ of a Bienaymé tree with offspring distribution μ (sometimes known as a Galton–Watson tree, but we prefer to use the terminology suggested in [ABBHK22]) is a probability measure on \mathbb{T} that can be characterized by the following two properties (see e.g. [Nev86] for more general statements):

- (i) $\mathbb{P}_\mu(k_\emptyset = j) = \mu(j)$ for $j \in \mathbb{Z}_+$,
- (ii) for every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $\theta_1 \tau, \dots, \theta_j \tau$ are independent under the conditional probability $\mathbb{P}_\mu(\cdot | k_\emptyset = j)$ and their conditional distribution is \mathbb{P}_μ .

The following simple result gives a characterization of Bienaymé trees with geometric offspring distribution in terms of an invariance property involving subtrees (since we have not managed to find a reference in the literature we include the short proof for completeness).

Lemma 6.3.1. *Let $p \in [0, 1)$. Let \mathcal{T} be a random plane tree such that $\mathbb{P}(k_\emptyset(\mathcal{T}) = 0) = 1 - p$ and under the conditional probability $\mathbb{P}(\cdot | k_\emptyset(\mathcal{T}) > 0)$ the two trees $\theta_1 \mathcal{T}$ and $\mathcal{T} \setminus \theta_1 \mathcal{T}$ are independent with conditional distribution equal to the law of \mathcal{T} . Then \mathcal{T} is a Bienaymé tree with offspring distribution μ given by $\mu(k) = (1 - p)p^k$ for $k \geq 0$.*

Proof. First, for $j \geq 1$ we have $\mathbb{P}(k_\emptyset(\mathcal{T}) \geq j) = \mathbb{P}(k_\emptyset(\mathcal{T}) > 0)\mathbb{P}(k_\emptyset(\mathcal{T} \setminus \theta_1 \mathcal{T}) \geq j-1 | k_\emptyset(\mathcal{T}) > 0)$, which implies $\mathbb{P}(k_\emptyset(\mathcal{T}) \geq j) = p\mathbb{P}(k_\emptyset(\mathcal{T}) \geq j-1)$. Thus $\mathbb{P}(k_\emptyset(\mathcal{T}) = k) = (1-p)p^k$ for $k \geq 0$.

Second, we argue by induction. Take $n \geq 1$ and assume that the shifted trees $\theta_1 \mathcal{T}, \dots, \theta_n \mathcal{T}$ are independent under the conditional probability $\mathbb{P}(\cdot | k_\emptyset(\mathcal{T}) = n)$ with conditional distribution equal to the law of \mathcal{T} . Then for $f_1, \dots, f_{n+1} \geq 0$ measurable

$$\mathbb{E}[f_1(\theta_1 \mathcal{T}) \cdots f_{n+1}(\theta_{n+1} \mathcal{T}) | k_\emptyset(\mathcal{T}) = n+1] = \mathbb{E}[f_1(\theta_1 \mathcal{T}) g_n(\mathcal{T} \setminus \theta_1 \mathcal{T}) | k_\emptyset(\mathcal{T}) > 0]$$

with $g_n(\tau) = \mathbb{1}_{k_\emptyset(\tau)=n} f_2(\theta_1 \tau) \cdots f_{n+1}(\theta_n \tau) \mathbb{P}(k_\emptyset(\mathcal{T}) > 0) / \mathbb{P}(k_\emptyset(\mathcal{T}) = n+1)$. Thus, by assumption, we have $\mathbb{E}[f_1(\theta_1 \mathcal{T}) \cdots f_{n+1}(\theta_{n+1} \mathcal{T}) | k_\emptyset(\mathcal{T}) = n+1] = \mathbb{E}[f_1(\mathcal{T})] \mathbb{E}[g_n(\mathcal{T})]$. But

$$\mathbb{E}[g_n(\mathcal{T})] = \frac{\mathbb{P}(k_\emptyset(\mathcal{T}) = n) \mathbb{P}(k_\emptyset(\mathcal{T}) > 0)}{\mathbb{P}(k_\emptyset(\mathcal{T}) = n+1)} \mathbb{E}[f_2(\theta_1 \mathcal{T}) \cdots f_{n+1}(\theta_n \mathcal{T}) | k_\emptyset(\mathcal{T}) = n],$$

which is equal to $\mathbb{E}[f_2(\theta_1 \mathcal{T}) \cdots f_{n+1}(\theta_n \mathcal{T}) | k_\emptyset(\mathcal{T}) = n]$ thanks to the first step. The desired result follows from the induction hypothesis. \square

For every tree $\tau \in \mathfrak{T}_{k,n}^+(A)$, we define $\tilde{\tau}$ to be the unique element of $\mathfrak{T}_{k,n}^+$ obtained by shifting the labels of τ to make them consecutive; specifically if $f : \{1, \dots, 2n-k-1\} \rightarrow A$ denotes the unique increasing bijection, an edge/vertex is labeled $f(i)$ in τ if and only if its corresponding edge/vertex is labeled i in $\tilde{\tau}$ (see Figure 6.8).

We are now ready to establish Theorem 6.1.2.

Proof of Theorem 6.1.2. First, we claim that almost surely the sequence $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ converges locally towards a tree denoted by \mathcal{T}_∞ . Observe that if $(S_n)_{n \geq 0}$ touches 0, then the sequence of trees $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ is almost surely eventually constant and thus converges. If $p \leq 1/2$, $(S_n)_{n \geq 0}$ touches 0 almost surely at some point, so the sequence of trees $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ converges almost surely. If $p > 1/2$, conditionally on the fact that $(S_n)_{n \geq 0}$ never touches 0, this walk satisfies the condition of Theorem 6.1.1, so $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ has a local limit. We conclude that, for every value of $p \in [0, 1)$, almost surely the sequence $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ converges locally towards a tree \mathcal{T}_∞ .

We shall first consider \mathcal{T}_∞ conditioned on having at least 2 vertices (which amounts to conditioning on the event $\{X_1 = +1\}$). Conditionally given $\{X_1 = +1\}$, $\mathcal{T}_n(\mathbf{X})$ is composed of two trees: the tree $\mathcal{T}_n^1(\mathbf{X})$ representing all the descendants of the first child of the root of $\mathcal{T}_n(\mathbf{X})$, and the tree $\mathcal{T}_n^2(\mathbf{X})$ representing all the other vertices, including the root of $\mathcal{T}_n(\mathbf{X})$ (see Figure 6.8).

The main idea is then to check that if one of the trees $\mathcal{T}_n^1(\mathbf{X})$ or $\mathcal{T}_n^2(\mathbf{X})$ has at least one active vertex, then it will eventually almost surely evolve (i.e. its degree changes or it freezes). This will show that the two trees $\tilde{\mathcal{T}}_n^1(\mathbf{X})$ or $\tilde{\mathcal{T}}_n^2(\mathbf{X})$ evolve independently with the same transition probabilities, which in turn will entail that conditionally given $\{X_1 = 1\}$, $(\tilde{\mathcal{T}}_n^1(\mathbf{X}))_{n \geq 1}$ and $(\tilde{\mathcal{T}}_n^2(\mathbf{X}))_{n \geq 1}$ converge locally, jointly, towards two independent trees distributed like \mathcal{T}_∞ .

Let us now make this discussion more formal. Notice that $(\mathcal{T}_n(\mathbf{X}))_{n \geq 0}$ is a Markov chain in the state space $E := \bigcup_{n \geq 1} \bigcup_{0 \leq k \leq n} \mathfrak{T}_{k,n}^+$ starting at the trivial tree, denoted by t_\circ , composed of a single active vertex. Denote by α its transition matrix. The absorbing states are exactly the trees in $\bigcup_{n \geq 1} \mathfrak{T}_{0,n}^+$ with no active vertices. For $n \geq 2$, let I_n be the random variable which is equal to 1 when $S_k = 0$ for some $k \leq n-1$ or when the n -th action (corresponding to the step X_n) occurs in the tree $\mathcal{T}_{n-1}^1(\mathbf{X})$ and 2 if it occurs in the tree $\mathcal{T}_{n-1}^2(\mathbf{X})$. For example, when $X_n = -1$ and $S_k > 0$ for all $k \leq n-1$, an active vertex of $\mathcal{T}_{n-1}(\mathbf{X})$, chosen uniformly at random, will freeze. If this vertex is chosen in $\mathcal{T}_{n-1}^1(\mathbf{X})$, then $I_n = 1$, otherwise, it is chosen in $\mathcal{T}_{n-1}^2(\mathbf{X})$ and $I_n = 2$. By convention we also set $I_1 = 1$. Denote by $a(t)$ the number of active vertices of a tree $t \in E$.

The previous discussion shows that, conditionally given the event $\{X_1 = 1\}$, $(\tilde{\mathcal{T}}_n^1(\mathbf{X}), \tilde{\mathcal{T}}_n^2(\mathbf{X}), I_n)_{n \geq 1}$ is a Markov chain in the state space $E^2 \times \{1, 2\}$, starting at $(t_\circ, t_\circ, 1)$, with transitions given by

$$\begin{aligned} (x, y, i) &\rightarrow (x', y, 1) && \text{with probability } k(x, y)\alpha(x, x') \\ (x, y, i) &\rightarrow (x, y', 2) && \text{with probability } (1-k(x, y))\alpha(y, y'). \end{aligned}$$

for every $x, x', y, y' \in E$ and $i \in \{1, 2\}$ where

$$k(x, y) := \frac{a(x)}{a(x) + a(y)} \quad \text{if } a(x) + a(y) > 0 \quad \text{and} \quad k(x, y) := 1 \quad \text{otherwise.}$$

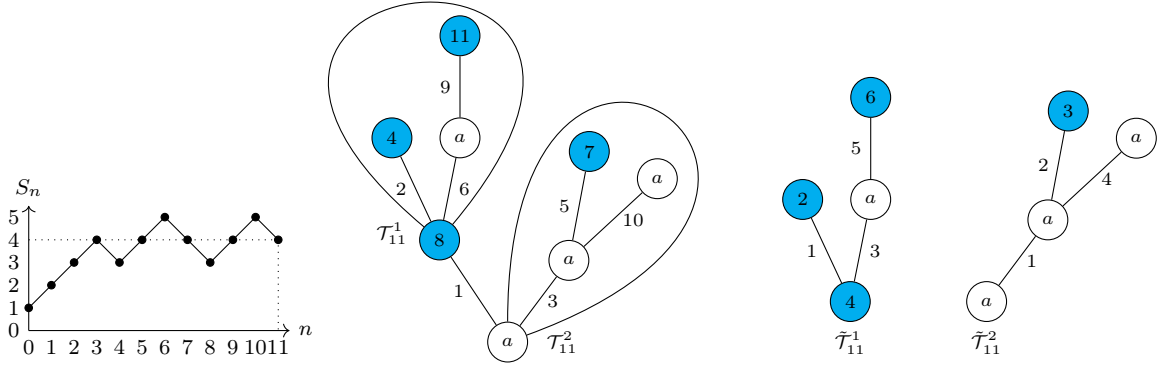


Figure 6.8 – Illustration of the proof of Theorem 6.1.2

Let $x \in E$ be a non-absorbing state (i.e. $a(x) > 0$). Let $y \in E$ and $n \geq 1$. We show that conditionally given $\{\tilde{\mathcal{T}}_n^1(\mathbf{X}) = x$ and $\tilde{\mathcal{T}}_n^2(\mathbf{X}) = y\}$ the probability that $I_k = 1$ for some $k > n$ is 1 and that the probability that $I_k = 2$ for some $k > n$ is 1. The forthcoming Lemma 6.3.2 then entails that conditionally given $\{X_1 = 1\}$, the trees $(\tilde{\mathcal{T}}_n^1(\mathbf{X}))_{n \geq 1}$ and $(\tilde{\mathcal{T}}_n^2(\mathbf{X}))_{n \geq 1}$ converge locally, jointly, towards two independent trees distributed like \mathcal{T}_∞ .

Now, we show that that conditionally given $\{\tilde{\mathcal{T}}_n^1(\mathbf{X}) = x$ and $\tilde{\mathcal{T}}_n^2(\mathbf{X}) = y\}$ the probability that $I_k = 1$ for some $k > n$ is 1 (for $I_k = 2$ the argument is the same). Observe that since $a(x) > 0$, we have $S_k > 0$ for every $1 \leq k \leq n$. If $S_k = 0$ for some $k > n$, this implies that both $\mathcal{T}_k^1(\mathbf{X})$ and $\mathcal{T}_k^2(\mathbf{X})$ have no active vertices left. So in particular, there must be an integer $n < \ell \leq k$ such that $I_\ell = 1$. Conditionally given $(S_k)_{k \geq n}$, if $S_k > 0$ for all $k > n$, then the probability that $I_k = 2$ for all $k > n$ is

$$\prod_{k \geq n} \frac{S_k - a(x)}{S_k} \leq \prod_{k \geq n} \left(1 - \frac{1}{S_k}\right) \leq \prod_{k \geq n} \left(1 - \frac{1}{k+1}\right) = 0.$$

Therefore we can apply the forthcoming Lemma 6.3.2 and deduce that, conditionally given $\{X_1 = 1\}$, the trees $(\tilde{\mathcal{T}}_n^1(\mathbf{X}))_{n \geq 1}$ and $(\tilde{\mathcal{T}}_n^2(\mathbf{X}))_{n \geq 1}$ converge locally, jointly, towards two independent trees distributed like \mathcal{T}_∞ .

To conclude, for a vertex and edge labelled locally finite tree τ denote by $\bar{\tau}$ to be the plane tree obtained from τ by first ordering the vertices in such a way that the labels of edges connecting vertices to their children are increasing from left to right, and second erasing all the labels (for example the vertices of the tree on the left of Fig. 6.7 are ordered in such a way). By continuity of the operation $\tau \mapsto \bar{\tau}$ on the set of labelled locally finite trees with respect to the local topology, we conclude that conditionally on the fact that $\bar{\mathcal{T}}_\infty$ is not a single vertex, the two trees $\bar{\mathcal{T}}_\infty^1$ and $\bar{\mathcal{T}}_\infty^2$ are independent copies of $\bar{\mathcal{T}}_\infty$. Since $\bar{\mathcal{T}}_\infty$ has only one vertex with probability $1 - p$ (this happens if and only if $X_1 = -1$), the desired result follows from Lemma 6.3.1. \square

Lemma 6.3.2. *Let $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ be two Markov chains on the countable state space E with the same transition matrix α . Denote by $A := \{x \in E : \alpha(x, x) = 1\}$ their set of absorbing states. Let $k : E \times E \rightarrow [0, 1]$. Define a Markov chain $(U_n, V_n, I_n)_{n \geq 0}$ on the state space $E^2 \times \{1, 2\}$ with transition matrix given by, for all $x, x', y, y' \in E$ and $i \in \{1, 2\}$:*

$$\begin{aligned} (x, y, i) &\rightarrow (x', y, 1) && \text{with probability } k(x, y)\alpha(x, x') \\ (x, y, i) &\rightarrow (x, y', 2) && \text{with probability } (1 - k(x, y))\alpha(y, y'). \end{aligned}$$

Suppose that, for all $(x, y, i) \in (E \setminus A) \times E \times \{1, 2\}$, $\mathbb{P}_{(x, y, i)}(\exists n \geq 1, I_n = 1) = 1$. Similarly, suppose that for all $(x, y, i) \in E \times (E \setminus A) \times \{1, 2\}$, $\mathbb{P}_{(x, y, i)}(\exists n \geq 1, I_n = 2) = 1$. Let $n_0 = m_0 = 0$. For all $k \geq 1$, if the set $\{n : n > n_{k-1} \text{ and } I_n = 1\}$ is non-empty, we define n_k as its minimum, otherwise we set $n_k = n_{k-1}$.

Similarly, if the set $\{m : m > m_{k-1} \text{ and } I_m = 2\}$ is non-empty, we define m_k as its minimum, otherwise we set $m_k = m_{k-1}$. Then, for every $(x, y) \in E^2$, $((U_{n_k})_{k \geq 0}, (V_{m_k})_{k \geq 0})$ under $\mathbb{P}_{(x, y, 1)}$ has the same law as $((X_k)_{k \geq 0}, (Y_k)_{k \geq 0})$ under $\mathbb{P}_x \otimes \mathbb{P}_y$.

Proof of Lemma 6.3.2. We show that $((U_{n_k})_{0 \leq k \leq r}, (V_{m_k})_{0 \leq k \leq s})$ under $\mathbb{P}_{(x, y, 1)}$ has the same law as $((X_k)_{0 \leq k \leq r}, (Y_k)_{0 \leq k \leq s})$ under $\mathbb{P}_x \otimes \mathbb{P}_y$ by double induction on (r, s) . If $r = s = 0$, the result is obvious. Suppose it is true for some pair (r, s) . We show it for $(r, s + 1)$ (the case $(r + 1, s)$ is similar). Let $\mathbf{x} = (x_1, \dots, x_r) \in E^r$, $\mathbf{y} = (y_1, \dots, y_s) \in E^s$ and $y_{s+1} \in E$. Write $\mathbf{U} := (U_{n_k})_{1 \leq k \leq r}$, $\mathbf{V} := (V_{m_k})_{1 \leq k \leq s}$, $\mathbf{X} := (X_k)_{1 \leq k \leq s}$, $\mathbf{Y} := (Y_k)_{1 \leq k \leq s}$ and $\mathbb{P} := \mathbb{P}_{(x, y, 1)}$. Suppose that $y_s \notin A$, then $m_{s+1} > m_s$ and

$$\mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}, V_{m_{s+1}} = y_{s+1}) = \mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}) \mathbb{P}_{(x_r, y_s, 1)}(\exists n \geq 1, I_1 = \dots = I_{n-1} = 1, I_n = 2, V_n = y_{s+1}).$$

By induction hypothesis, $\mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}) = \mathbb{P}_x(\mathbf{X} = \mathbf{x}) \mathbb{P}_y(\mathbf{Y} = \mathbf{y})$. Moreover

$$\begin{aligned} \mathbb{P}_{(x_r, y_s, 1)}(\exists n \geq 1, I_1 = \dots = I_{n-1} = 1, I_n = 2, V_n = y_{s+1}) &= \mathbb{P}_{(x_r, y_s, 1)}(\exists n \geq 1, I_n = 2) \alpha(y_s, y_{s+1}) \\ &= \alpha(y_s, y_{s+1}). \end{aligned}$$

This shows the formula for $(r, s + 1)$ in the case $y_s \notin A$. Now, if $y_s \in A$ then

$$\begin{aligned} \mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}, V_{m_{s+1}} = y_{s+1}) &= \mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}) \mathbb{P}_{(x_r, y_s, 1)}(\forall m \geq 1, I_m = 1) \mathbb{1}_{y_s = y_{s+1}} \\ &\quad + \mathbb{P}(\mathbf{U} = \mathbf{x}, \mathbf{V} = \mathbf{y}) \mathbb{P}_{(x_r, y_s, 1)}(\exists m \geq 1, I_m = 2) \alpha(y_s, y_{s+1}) \\ &= \mathbb{P}_x(\mathbf{X} = \mathbf{x}) \mathbb{P}_y(\mathbf{Y} = \mathbf{y}, Y_{s+1} = y_{s+1}). \end{aligned}$$

This completes the proof. \square

6.4 The height of uniform attachment trees with freezing

We shall now study the geometry of uniform attachment trees with freezing. It will be convenient to work with $\mathcal{T}^n(\mathbf{x})$ as built using Algorithm 2. Recall from Theorem 6.2.4 that: the only difference between $\mathcal{T}_n(\mathbf{x})$ and $\mathcal{T}^n(\mathbf{x})$ is that all the active vertices of $\mathcal{T}^n(\mathbf{x})$ are labeled $a_1, \dots, a_{S_n(\mathbf{x})}$, while all the active vertices of $\mathcal{T}_n(\mathbf{x})$ are labelled a . In particular the graph structure of both trees is the same in law, so it is equivalent to establish our main results with $\mathcal{T}_n(\mathbf{x})$ replaced with $\mathcal{T}^n(\mathbf{x})$.

Also recall from (6.2.1) the definition of $\mathbb{V}_n(\mathbf{x})$, which is a deterministic set representing the labels of the vertices of $\mathcal{T}^n(\mathbf{x})$, and that by a slight abuse of notation we view elements of $\mathbb{V}_n(\mathbf{x})$ as vertices of $\mathcal{T}^n(\mathbf{x})$.

An important ingredient in our proofs is Bennett's concentration inequality. Since it will be used multiple times, we state it here, tailored for our purpose:

Proposition 6.4.1 (Bennett's inequality). *Let $(Y_i)_{1 \leq i \leq n}$ be independent Bernoulli random variables of respective parameters $(p_i)_{1 \leq i \leq n}$. Set $m_n = \sum_{i=1}^n p_i$ and assume that $m_n > 0$. Then for every $t > 0$:*

$$\mathbb{P}\left(\sum_{i=1}^n Y_i > m_n + t\right) \leq \exp\left(-m_n g\left(\frac{t}{m_n}\right)\right) \quad \text{and} \quad \mathbb{P}\left(\sum_{i=1}^n Y_i < m_n - t\right) \leq \exp\left(-m_n g\left(\frac{t}{m_n}\right)\right),$$

where $g(u) = (1 + u) \log(1 + u) - u$ for $u > 0$.

6.4.1 Uniform bounds on the height

We keep the notation introduced before (see in particular Table 6.1). In particular, $\mathbf{x} = (x_n)_{n \geq 1}$ is a fixed sequence of elements of $\{-1, 1\}$, $S_0(\mathbf{x}) := 1$ and for every $n \geq 1$

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i; \quad \tau(\mathbf{x}) := \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}. \quad (6.4.1)$$

To lighten notation we will drop again the \mathbf{x} in parenthesis (e.g. we will write S_n instead of $S_n(\mathbf{x})$, τ instead of $\tau(\mathbf{x})$, \mathcal{T}^n instead of $\mathcal{T}^n(\mathbf{x})$, $\mathbb{V}_n(\mathbf{x})$ instead of \mathbb{V}_n , etc.). We first start with an estimate concerning the height of \mathcal{T}^n (recall its construction using Algorithm 2 in Section 6.2.2).

Proposition 6.4.2. Fix $n \geq 1$ and assume that $\tau > n$. Let $\text{Height}(\mathcal{T}^n)$ denote the height of \mathcal{T}^n . Then, for every $t > 0$,

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) - h_n^+ > t) \leq \exp\left(-h_n^+ g\left(\frac{t}{h_n^+}\right) + \log(n)\right)$$

and

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) - h_n^+ < -t) \leq \exp\left(-h_n^+ g\left(\frac{t}{h_n^+}\right)\right)$$

where $g(u) = (1+u)\log(1+u) - u$ for $u > 0$.

Proof. Fix $u \in \mathbb{V}_n$. From (6.2.2), if $(Y_i)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i)_{1 \leq i \leq n}$, we have

$$H_0^n(u) \stackrel{(d)}{=} \sum_{i=1}^{b_n(u)} Y_i \mathbb{1}_{\{x_i=1\}}.$$

Thus, by a union bound over all vertices of \mathcal{T}^n we get

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) > h_n^+ + t) \leq n \mathbb{P}\left(\sum_{i=1}^n \left(Y_i - \frac{1}{S_i}\right) \mathbb{1}_{\{x_i=1\}} > t\right).$$

In addition, when $u \in \mathbb{V}_n$ is an active vertex (so that $b_n(u) = n$), using the fact that $H_0^n(u) \leq \text{Height}(\mathcal{T}_n)$, we get

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) < h_n^+ - t) \leq \mathbb{P}\left(\sum_{i=1}^n \left(Y_i - \frac{1}{S_i}\right) \mathbb{1}_{\{x_i=1\}} < -t\right).$$

The desired bounds then both follow from Bennett's inequality. \square

6.4.2 A simple lower bound for h_n^+

In view of Theorem 6.1.3, which tells us that h_n^+ is the order of magnitude of $\text{Height}(\mathcal{T}^n)$, it is useful to have estimates on h_n^+ . We give here a simple lower bound on h_n^+ .

Lemma 6.4.3. Let $n \geq 1$. If $\tau > n$ then $h_n^+ \geq \log(n/4)$.

Proof. Set $p_n := \#\{1 \leq i \leq n : x_i = +1\}$ and $m_n := \#\{1 \leq i \leq n : x_i = -1\}$. If there exists $1 \leq i \leq n-1$ such that $x_i = -1$ and $x_{i+1} = +1$, set $\mathbf{x}' = (x_1, \dots, x_{i-1}, +1, -1, x_{i+2}, x_{i+3}, \dots)$. Then it is a simple matter to check that $h_n^+(\mathbf{x}) \geq h_n^+(\mathbf{x}')$. By iteration, it readily follows that $h_n^+(\mathbf{x}) \geq h_n^+((+1, \dots, +1, -1, \dots, -1, \mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \dots))$ where $+1$ appears p_n times and -1 appears m_n times. Since $\tau(\mathbf{x}) > n$ entails $p_n \geq n/2$, it follows that $h_n^+(\mathbf{x}) \geq \sum_{i=1}^{p_n} 1/(i+1) \geq \log((p_n+1)/2) \geq \log(n/4)$. \square

6.4.3 Limit theorems for the height

We are now in position to establish Theorem 6.1.3.

Proof of Theorem 6.1.3. We start with (I). Recall that for an active vertex $u \in \mathbb{A}_n$ we always have $b_n(u) = n$. Therefore, by (6.2.2)

$$H(U^n) \stackrel{(d)}{=} \sum_{i=1}^n Y_i^n \mathbb{1}_{\{x_i=1\}}.$$

Using Bennett's inequality (Proposition 6.4.1), for all $\varepsilon > 0$

$$\mathbb{P}\left(\left|\frac{H(U^n)}{h_n^+} - 1\right| > \varepsilon\right) \leq 2 \exp(-h_n^+ g(\varepsilon)).$$

This upper bound goes to 0 as n tends to infinity since h_n^+ goes to infinity by Lemma 6.4.3. We deduce that the sequence $(H(U^n)/h_n^+ - 1)_n$ converges to 0 in probability. To conclude it suffices to show that it is bounded in \mathbb{L}^p for all $p \geq 1$. Let $p \geq 1$, we have

$$\mathbb{E}\left[\left|\frac{H(U^n)}{h_n^+} - 1\right|^p\right] = p \int_0^\infty \mathbb{P}\left(\left|\frac{H(U^n)}{h_n^+} - 1\right| \geq u\right) u^{p-1} du.$$

Using Bennett's inequality again and by splitting the integral in half at $u = 7$, the above quantity is bounded by

$$7^p + 2p \int_7^\infty \exp(-h_n^+ g(u)) u^{p-1} du.$$

Notice that for $u \geq 7 \geq e^2 - 1$, we have $g(u) \geq u$ thus

$$\int_7^\infty \exp(-h_n^+ g(u)) u^{p-1} du \leq \int_7^\infty \exp(-h_n^+ u) u^{p-1} du = \frac{1}{(h_n^+)^p} \int_{7h_n^+}^\infty \exp(-u) u^{p-1} du.$$

By an iterated integration by parts, the last quantity is $\mathcal{O}(\exp(-7h_n^+)/h_n^+)$. Therefore the boundedness in \mathbb{L}^p follows.

We now establish (II) and (III). Fix $C > 1$. By Proposition 6.4.2, for all $n \geq 1$:

$$\mathbb{P}\left(\frac{\text{Height}(\mathcal{T}^n)}{h_n^+} > C\right) \leq \exp(-h_n^+ g(C-1) + \log(n)).$$

This upper bound goes to 0 when $\log(n) = o(h_n^+)$ since $g(C-1) > 0$. It also goes to 0 when $C = e + \varepsilon$ since $g(e-1+\varepsilon) > 1$ and by Lemma 6.4.3, $h_n^+ \geq \log(n/4)$. Similarly,

$$\mathbb{P}\left(\frac{\text{Height}(\mathcal{T}^n)}{h_n^+} < 1 - \varepsilon\right) \leq \exp(-h_n^+ g(\varepsilon)) \xrightarrow[n \rightarrow \infty]{} 0.$$

This completes the proof of (II). It also shows that the convergence

$$\frac{\text{Height}(\mathcal{T}^n)}{h_n^+} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1$$

holds in probability when $\log(n) = o(h_n^+)$. To fully show (III) we just need to prove that the sequence $(\text{Height}(\mathcal{T}^n)/h_n^+ - 1)_n$ is bounded in \mathbb{L}^p for all $p \geq 1$ when $\log(n) = o(h_n^+)$. It is done using the same method as in the proof of (I) by using the fact that for n sufficiently large we have

$$\mathbb{P}\left(\frac{\text{Height}(\mathcal{T}^n)}{h_n^+} > C\right) \leq \exp(-h_n^+ g(C-1)/2)$$

for every $C \geq 2$. □

6.5 Regime with a linear amount of active vertices

We shall now investigate the regime where the number of active vertices grows roughly linearly in the size of the trees: the main goal of this section is to establish Theorem 6.1.4. Here we assume that (6.1.2) is in force, namely that there exist $c \in (0, 1]$ and a sequence $(A_n)_{n \in \mathbb{N}}$ of positive numbers such that $A_n = o(\log n)$ as $n \rightarrow \infty$ and such that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{A_n \leq i \leq \varepsilon n} \left| \frac{S_i^n}{i} - c \right| = 0 \quad \text{and} \quad \forall \varepsilon > 0, \liminf_{n \rightarrow \infty} \min_{\varepsilon n \leq i \leq n} \frac{S_i^n}{n} > 0.$$

6.5.1 An asymptotic equivalent for h_n^+

Here we show the convergence of Theorem 6.1.4 (I), namely:

$$\frac{h_n^+}{\log n} \xrightarrow[n \rightarrow \infty]{} \frac{c+1}{2c}. \tag{6.5.1}$$

To simplify notation, for all $n \in \mathbb{N}$ and for all $\varepsilon > 0$, set

$$\eta_n(\varepsilon) := \max_{A_n \leq i \leq \varepsilon n} \left| \frac{S_i^n}{i} - c \right|.$$

By (6.1.2), we know that $\limsup \eta_n(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Lemma 6.5.1. *Suppose that the assumptions (6.1.2) hold with $A_n \rightarrow \infty$ when $n \rightarrow \infty$. Then, for all $C > 1$*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{\substack{A_n \leq a < b \leq \varepsilon n \\ aC < b}} \left| \frac{1}{\log(b/a)} \sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} - \frac{c+1}{2c} \right| = 0. \quad (6.5.2)$$

Proof of Lemma 6.5.1. We prove (6.5.2) with a double bound. First, let us start with the upper bound. Let $\varepsilon \in (0, 1)$ and $\beta > 1$. For all $A_n \leq a < b \leq \varepsilon n$:

$$\sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} = \sum_{a < i \leq \beta^\ell} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} + \sum_{k=1+\ell}^m \sum_{\beta^{k-1} < i \leq \beta^k} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} + \sum_{\beta^m < i \leq b} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}$$

where $\ell := \lceil \log a / \log \beta \rceil$, $m := \lfloor \log b / \log \beta \rfloor$ and where, in all the sums over i , it is implicit that i is an integer. Let $P_n := \{1 \leq i \leq n : x_i = 1\}$ and $M_n := \{1 \leq i \leq n : x_i = -1\}$. Note that for all $1 \leq i < j \leq n$, we have

$$\#(P_n \cap [i+1, j]) - \#(M_n \cap [i+1, j]) = S_j^n - S_i^n, \quad \#(P_n \cap [i+1, j]) + \#(M_n \cap [i+1, j]) = j - i.$$

Moreover, when $A_n \leq i < j \leq \varepsilon n$, we have $(j-i)c - 2j\eta_n(\varepsilon) \leq S_j^n - S_i^n \leq (j-i)c + 2j\eta_n(\varepsilon)$. As a result,

$$\frac{c+1}{2}(j-i) - j\eta_n(\varepsilon) \leq \#(P_n \cap [i+1, j]) = \frac{S_j^n - S_i^n + j - i}{2} \leq \frac{c+1}{2}(j-i) + j\eta_n(\varepsilon). \quad (6.5.3)$$

Therefore, for every $k \in \llbracket 1 + \ell, m \rrbracket$,

$$\begin{aligned} \sum_{\beta^{k-1} < i \leq \beta^k} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} &\leq \sum_{\beta^{k-1} < i \leq \beta^k} \frac{1}{i(c - \eta_n(\varepsilon))} \mathbb{1}_{\{x_i^n=1\}} \leq \#\{\beta^{k-1} < i \leq \beta^k : x_i^n = 1\} \frac{1}{\beta^{k-1}(c - \eta_n(\varepsilon))} \\ &\leq \left(\frac{c+1}{2}(\beta^k - \beta^{k-1} + 1) + \beta^k \eta_n(\varepsilon) \right) \frac{1}{\beta^{k-1}(c - \eta_n(\varepsilon))} \\ &\leq \left(\frac{c+1}{2}(\beta - 1 + 1/A_n) + \beta \eta_n(\varepsilon) \right) \frac{1}{(c - \eta_n(\varepsilon))} =: U_n(\varepsilon) \end{aligned}$$

where the third inequality comes from (6.5.3) and where for the fourth inequality we used that $\beta^{k-1} \geq \beta^\ell \geq a \geq A_n$. Similarly, since $\beta^{\ell-1} \leq a$,

$$\sum_{a < i \leq \beta^\ell} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq \#\{a < i \leq \beta^\ell : x_i^n = 1\} \frac{1}{\beta^{\ell-1}(c - \eta_n(\varepsilon))} \leq \frac{\beta^\ell - \beta^{\ell-1}}{\beta^{\ell-1}(c - \eta_n(\varepsilon))},$$

and since $\beta^{m+1} \geq b$,

$$\sum_{\beta^m < i \leq b} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq \#\{\beta^m < i \leq b : x_i^n = 1\} \frac{1}{\beta^{\ell-1}(c - \eta_n(\varepsilon))} \leq \frac{\beta^{m+1} - \beta^m}{\beta^m(c - \eta_n(\varepsilon))}.$$

Thus

$$\sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq (m - \ell)U_n(\varepsilon) + \frac{\beta^\ell - \beta^{\ell-1}}{\beta^{\ell-1}(c - \eta_n(\varepsilon))} + \frac{\beta^{m+1} - \beta^m}{\beta^m(c - \eta_n(\varepsilon))} \leq \frac{\log(b/a)}{\log \beta} U_n(\varepsilon) + \frac{2(\beta - 1)}{c - \eta_n(\varepsilon)}.$$

For the lower bound, the same ideas apply. Indeed, for every integer $k \in \llbracket 1 + \ell, m \rrbracket$ we have

$$\begin{aligned} \sum_{\beta^{k-1} < i \leq \beta^k} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} &\geq \#\{\beta^{k-1} < i \leq \beta^k : x_i^n = 1\} \frac{1}{\beta^k(c + \eta_n(\varepsilon))} \\ &\geq \left(\frac{c+1}{2}(\beta^k - \beta^{k-1} - 2) - \beta^k \eta_n(\varepsilon) \right) \frac{1}{\beta^k(c + \eta_n(\varepsilon))} \\ &\geq \left(\frac{c+1}{2} \left(1 - \frac{1}{\beta} - \frac{2}{\beta A_n} \right) - \eta_n(\varepsilon) \right) \frac{1}{(c + \eta_n(\varepsilon))} =: L_n(\varepsilon) \end{aligned}$$

where the second inequality comes from (6.5.3) and the third one comes from the bound $\beta^k \geq \beta^{\ell+1} \geq \beta a \geq \beta A_n$. Thus

$$\sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \geq (m-\ell)L_n(\varepsilon) \geq \frac{\log(b/a)}{\log \beta} L_n(\varepsilon) - 2L_n(\varepsilon) \geq \frac{\log(b/a)}{\log \beta} L_n(\varepsilon) - \frac{c+1}{c} \left(1 - \frac{1}{\beta} - \frac{2}{\beta A_n}\right).$$

Using the assumption that $aC < b$, we end up with the following inequalities

$$\frac{L_n(\varepsilon)}{\log \beta} - \frac{c+1}{c \log C} \left(1 - \frac{1}{\beta} - \frac{2}{\beta A_n}\right) \leq \frac{1}{\log(b/a)} \sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq \frac{U_n(\varepsilon)}{\log \beta} + \frac{2(\beta-1)}{(c-\eta_n(\varepsilon)) \log C}.$$

Using the fact that $A_n \rightarrow \infty$, notice that $\limsup_{n \rightarrow \infty} U_n(\varepsilon) \rightarrow (\beta-1)(c+1)/(2c)$ and $\liminf_{n \rightarrow \infty} L_n(\varepsilon) \rightarrow (1-1/\beta)(c+1)/(2c)$ as $\varepsilon \rightarrow 0$. Therefore

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{\substack{A_n \leq a < b \leq \varepsilon n \\ aC < b}} \left| \frac{1}{\log(b/a)} \sum_{i=a+1}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} - \frac{c+1}{2c} \right| \leq \frac{c+1}{2c} \left| \frac{\beta-1}{\log \beta} - 1 \right| + \frac{c+1}{c \log C} \left(1 - \frac{1}{\beta}\right) + \frac{2(\beta-1)}{c \log C}.$$

This bound tends to 0 as $\beta \rightarrow 1$; since $\beta > 1$ was arbitrary, the result follows. \square

Proof of Theorem 6.1.4 (I). Let $(A'_n)_{n \in \mathbb{N}}$ be a sequence of positive integers such that $A_n \leq A'_n$, $A'_n = o(\log n)$ and $A'_n \rightarrow \infty$ as $n \rightarrow \infty$. Applying Lemma 6.5.1 we see that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left| \frac{1}{\log n} \sum_{A'_n < i \leq \varepsilon n} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} - \frac{c+1}{2c} \right| = 0$$

Since $S_i^n \geq 1$ for all $1 \leq i \leq n$ and $A'_n = o(\log n)$, we have

$$\sum_{i=1}^{A'_n} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} = o(\log n).$$

Finally, for all $0 < \varepsilon < 1$, by the second assumption of (6.1.2),

$$\sum_{\varepsilon n \leq i \leq n} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} = \mathcal{O}(1).$$

Combining everything gives the desired convergence (6.5.1). \square

6.5.2 Distances between uniform vertices

Here we prove Theorem 6.1.4 (II). We keep the notation introduced in Section 6.2.3: for a vertex $v \in \mathbb{V}_n$, let $\mathbf{b}_n(v)$ be the largest $i \in \{0, 1, \dots, n\}$ such that v belongs to the forest \mathcal{F}_i^n obtained when building \mathcal{T}^n using Algorithm 2. We first focus on $\mathbf{H}(V_1^n)$ and introduce some notation. Set

$$Z_n := \sum_{i=1}^{\mathbf{b}_n(V_1^n)} Y_i^n \mathbb{1}_{\{x_i^n=1\}} \quad \text{and} \quad M_n := \sum_{i=1}^{\mathbf{b}_n(V_1^n)} \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}$$

where $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n)_{1 \leq i \leq n}$ (independent from $\mathbf{b}_n(V_1^n)$). Observe that $M_n = \mathbb{E}[Z_n | \mathbf{b}_n(V_1^n)]$. By (6.2.2), we have

$$\mathbf{H}(V_1^n) \stackrel{(d)}{=} Z_n. \tag{6.5.4}$$

Fix $p \geq 1$. First we establish that M_n/h_n^+ converges to 1 in \mathbb{L}^p . Actually, since $0 \leq M_n \leq h_n^+$ almost surely it is enough to show the convergence in probability.

Let $\varepsilon > 0$ be small enough so that $\eta_n(\varepsilon) \leq c/2$ for all n large enough. Let $\gamma_\varepsilon > 0$ be such that for all n large enough we have

$$\gamma_\varepsilon \leq \min_{\varepsilon n \leq i \leq n} \frac{S_i^n}{n}.$$

Such a γ_ε exists thanks to (6.1.2). Besides, since $A_n = o(\log n) = o(n)$, by Lemma 6.2.2 we have $b_n(V_1^n) \geq A_n$ with high probability. Using this, as well as the fact that $h_n^+ \geq \log(n/4)$ (see Lemma 6.4.3), we obtain, with high probability as $n \rightarrow \infty$,

$$\begin{aligned} 0 \leq 1 - \frac{M_n}{h_n^+} &= \frac{1}{h_n^+} \sum_{i=b_n(V_1^n)}^n \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq \frac{1}{h_n^+} \sum_{i=b_n(V_1^n)}^n \left(\frac{1}{i(c-\eta_n(\varepsilon))} + \frac{1}{\gamma_\varepsilon n} \right) \mathbb{1}_{\{x_i^n=1\}} \\ &\leq \frac{1}{\log(n/4)} \frac{2}{c} \log\left(\frac{n}{b_n(V_1^n)}\right) + \frac{1}{\gamma_\varepsilon h_n^+}. \end{aligned}$$

Now, fix $0 < \delta < 1$ and take $m_n = n^{1-\delta}2^\delta$. Then by Lemma 6.2.2

$$\mathbb{P}\left(\log\left(\frac{n}{b_n(V_1^n)}\right) > \delta \log\left(\frac{n}{2}\right)\right) = \frac{m_n + 1 - S_{m_n}^n}{n + 1 + S_n^n} \xrightarrow{n \rightarrow \infty} 0.$$

We conclude that M_n/h_n^+ converges towards 1 in probability (and therefore in \mathbb{L}^p for all $p \geq 1$). Now we show that Z_n/M_n converges to 1 in \mathbb{L}^p following the same spirit as in the proof of Theorem 6.1.3. We have

$$\mathbb{E}\left[\left|\frac{Z_n}{M_n} - 1\right|^p \mid \mathbf{b}_n(V_1^n)\right] = p \int_0^\infty \mathbb{P}\left(\left|\frac{Z_n}{M_n} - 1\right| \geq u \mid \mathbf{b}_n(V_1^n)\right) u^{p-1} du.$$

By Bennett's inequality (Proposition 6.4.1)

$$\mathbb{P}\left(\left|\frac{Z_n}{M_n} - 1\right| \geq u \mid \mathbf{b}_n(V_1^n)\right) \leq 2 \exp(-M_n g(u))$$

where $g(u) = (u+1)\log(u+1) - u$. Since M_n is of order $\log n$ it is a simple matter to check that

$$\mathbb{E}\left|\frac{Z_n}{M_n} - 1\right|^p \leq 2p \mathbb{E}\left[\int_0^\infty \exp(-M_n g(u)) u^{p-1} du\right] \xrightarrow{n \rightarrow \infty} 0.$$

Since $M_n \leq h_n^+$ we deduce that

$$\mathbb{E}\left|\frac{Z_n - M_n}{h_n^+}\right|^p \leq \mathbb{E}\left|\frac{Z_n}{M_n} - 1\right|^p \xrightarrow{n \rightarrow \infty} 0.$$

So $(Z_n - M_n)/h_n^+$ converges to 0 in \mathbb{L}^p . Since M_n/h_n^+ converges to 1 in \mathbb{L}^p , Z_n/h_n^+ goes to 1 as well. Finally, using Theorem 6.1.4 (I), we conclude that $Z_n/\log n$ converges to $(c+1)/(2c)$ in \mathbb{L}^p . This proves the first convergence of Theorem 6.1.4 (II).

So as to deduce the asymptotic behaviour of the distance between two uniform points, we next show that two uniform random vertices coalesce near the root.

Lemma 6.5.2. *Under the assumptions of Theorem 6.1.4 the following holds. For all $n \geq 1$, let V_1^n, V_2^n be two independent uniform vertices of \mathbb{V}_n . Then*

$$\frac{c_n(V_1^n, V_2^n)}{n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0.$$

Proof. Let $0 < \varepsilon < 1$. Let $i, j \leq n-1$ distinct with $x_{i+1}^n = x_{j+1}^n = -1$ and $k \leq i \wedge j$ with $x_{k+1}^n = 1$. Denote by B the event " $\mathbf{b}_n(V_1^n) = i$ and $\mathbf{b}_n(V_2^n) = j$ " and by C the event " $c_n(V_1^n, V_2^n) = k$ ". By Lemma 6.2.3, it is clear that

$$\mathbb{P}(C \mid B) = \mathbb{P}(C \text{ and } c_n(V_1^n, V_2^n) \leq k \mid B) \leq \mathbb{P}(C \mid c_n(V_1^n, V_2^n) \leq k, B) = \left(\frac{S_{k+1}^n}{2}\right)^{-1}.$$

Therefore

$$\mathbb{P}(c_n(V_1^n, V_2^n) \geq \varepsilon n \mid B) \leq \sum_{k=\lceil \varepsilon n \rceil}^{i \wedge j} \left(\frac{S_{k+1}^n}{2}\right)^{-1} \mathbb{1}_{\{x_{k+1}^n=1\}} \leq \sum_{k=\lceil \varepsilon n \rceil}^{n-1} \left(\frac{S_{k+1}^n}{2}\right)^{-1} \mathbb{1}_{\{x_{k+1}^n=1\}}.$$

By averaging over i and j it follows that

$$\mathbb{P}(\mathbf{c}_n(V_1^n, V_2^n) \geq \varepsilon n) \leq \sum_{k=\lceil \varepsilon n \rceil}^{n-1} \binom{S_{k+1}^n}{2}^{-1} \mathbb{1}_{\{\mathbf{x}_{k+1}^n=1\}}.$$

Finally, by the second hypothesis of (6.1.2), we know that

$$\sum_{k=\lceil \varepsilon n \rceil}^{n-1} \binom{S_{k+1}^n}{2}^{-1} \mathbb{1}_{\{\mathbf{x}_{k+1}^n=1\}} = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty,$$

hence the desired result. \square

The second convergence of the point (II) of Theorem 6.1.4 is then an immediate consequence of the first convergence of the same point using dominated convergence and Lemma 6.5.2.

6.5.3 Limit theorem for the height

Here we show Theorem 6.1.4 (III). The proof of the upper bound is readily obtained by a union bound, while the lower bound is much more delicate.

Proof of the upper bound in Theorem 6.1.4 (III). Fix $\eta > 0$. First observe that by definition of $f(c)$ we have

$$g(f(c) - 1) = f(c) \log f(c) - f(c) + 1 = 1 + \frac{c-1}{c+1} = \frac{2c}{c+1},$$

so that $g(f(c) - 1 + \eta) > 2c/(c+1)$. By Theorem 6.1.4 (I) we have $h_n^+ \sim (c+1)/(2c) \log(n)$ as $n \rightarrow \infty$. As a consequence, there exists $\delta \in (0, 1)$ such that $h_n^+ g(f(c) - 1 + \eta) \geq (1 + \delta) \log(n)$ for n sufficiently large. Hence, by Proposition 6.4.2, for n sufficiently large,

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) > h_n^+(f(c) + \eta)) \leq \exp(-h_n^+ g(f(c) - 1 + \eta) + \log n) \leq \exp(-\delta \log(n)) \xrightarrow[n \rightarrow \infty]{} 0.$$

This completes the proof of the upper bound. \square

It therefore remains to show the lower bound on $\text{Height}(\mathcal{T}_n)$, which is the delicate part of the proof. Let us explain the main idea of our approach. Since the upper bound has been obtained by using a union bound over all vertices, one would hope to obtain the lower bound from the fact that the height of the vertices are "almost independent". However, their heights are highly correlated. To overcome this issue, we use the so-called chaining technique and estimate by induction on k the height of the subtrees $\text{Height}(\mathcal{T}_k(\mathbf{x}^n))$. More precisely, for every increasing sequence of integers (chain) $0 = R_0 < R_1 < \dots < R_m = n$, we have

$$\text{Height}(\mathcal{T}_n) = \sum_{0 \leq i < m} \text{Height}(\mathcal{T}_{R_{i+1}}(\mathbf{x}^n)) - \text{Height}(\mathcal{T}_{R_i}(\mathbf{x}^n)).$$

We shall choose an appropriate chain of integers $(R_i)_{0 \leq i \leq m}$ and then bound the differences appearing in the last display. To this end, the main idea is that if R_{i+1}/R_i is close to some large constant $C > 0$, then $\mathcal{T}_{R_{i+1}}(\mathbf{x}^n)$ can be approximated by a forest of independent trees attached on $\mathcal{T}_{R_i}(\mathbf{x}^n)$. We then show by induction on i that many vertices have a large height in this forest, and among them many are attached on vertices of $\mathcal{T}_{R_i}(\mathbf{x}^n)$ with large height.

To make this intuition rigorous we first need to define some objects. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, recall the definition of $\mathcal{T}_k^n := \mathcal{T}_k(\mathbf{x}^n)$ from Algorithm 1. We denote by \mathcal{A}_k^n the set of all vertices in \mathcal{T}_k^n that were active at time k when running Algorithm 1 (these are vertices of \mathcal{T}_k^n that are either labelled with a and have an adjacent edge labelled by $j \leq k$, or labelled with $i > k$ and have an adjacent edge labelled by $j \leq k$). For $0 \leq k \leq k' \leq n$, let $\mathcal{F}_{k,k'}^n$ be the forest obtained as follows:

- (i) Consider $\mathcal{T}_{k'}^n$ and relabel by \star all the vertices of $\mathcal{T}_{k'}^n$ belonging to \mathcal{A}_k^n (see Fig. 6.9 for an example);
- (ii) then remove all the edges of $\mathcal{T}_{k'}^n$ that belongs to \mathcal{T}_k^n (i.e. all the edges with labels in $\llbracket 1, k \rrbracket$);
- (iii) finally remove all the vertices with labels in $\llbracket 1, k \rrbracket$.

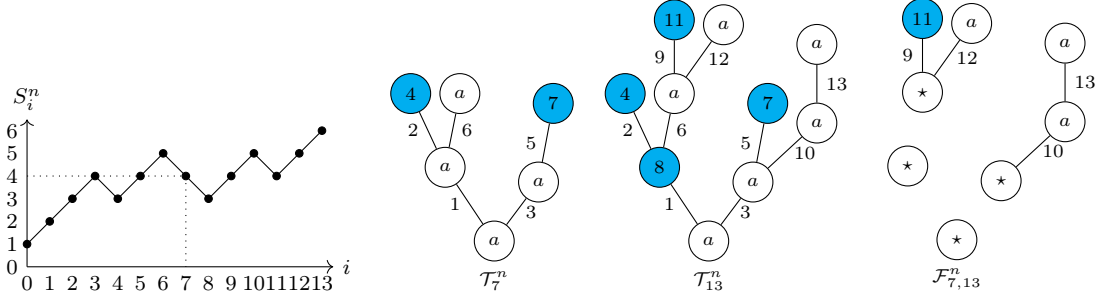


Figure 6.9 – Illustration of the construction of $\mathcal{F}_{k,k'}^n$ with $k = 7$ and $k' = 13$. From left to right: the sequence \mathbf{x}^n , a realization of \mathcal{T}_k^n , a realization of $\mathcal{T}_{k'}^n$ and the forest $\mathcal{F}_{k,k'}^n$ composed of four trees all rooted at their unique vertex labelled \star . If one matches and merges uniformly at random every \star vertex of $\mathcal{F}_{k,k'}^n$ with every active vertex of \mathcal{T}_k^n , one obtains a tree having the same distribution as $\mathcal{T}_{k'}^n$.

Note that all the trees in $\mathcal{F}_{k,k'}^n$ have exactly one vertex labelled \star (see Fig. 6.9 for an example), which by convention is considered to be active. We will thus consider the vertices labelled \star as the roots of the trees in $\mathcal{F}_{k,k'}^n$. Observe that one cannot deterministically reconstruct $\mathcal{T}_{k'}^n$ from $\{\mathcal{F}_{k,k'}^n, \mathcal{T}_k^n\}$, but in distribution it is possible: if one merges uniformly at random every active vertex of \mathcal{T}_k^n with one vertex labelled \star from $\mathcal{F}_{k,k'}^n$, then one obtains a tree having the same distribution as $\mathcal{T}_{k'}^n$. This comes from the fact that conditionally given \mathcal{T}_k^n , at any future time the subtrees grafted on active vertices of \mathcal{T}_k^n after time k are exchangeable. Furthermore, it will be crucial to keep in mind that \mathcal{T}_k^n and $\mathcal{F}_{k,k'}^n$ are independent. Indeed $\mathcal{F}_{k,k'}^n$ may be constructed following Algorithm 1, independently of the steps $1 \leq i \leq k$.

For every $0 \leq k \leq k' \leq n$, for every vertex $v \in \mathcal{F}_{k,k'}^n$ we write $H_{k,k'}^n(v)$ for the height of v in the forest $\mathcal{F}_{k,k'}^n$ minus its height in $\mathcal{F}_{k',n}^n$ (where, by convention, the height of v in $\mathcal{F}_{k',n}^n$ is 0 if v is not in this forest). In words, if v belongs to a certain tree τ of $\mathcal{F}_{k',n}^n$, the quantity $H_{k,k'}^n(v)$ represents the height in $\mathcal{F}_{k,k'}^n$ of the vertex associated with the root τ .

For $\eta > 0$, for every integers $n, k, C \in \mathbb{N}$ such that $C^k \leq n/C$, set

$$N_k^n(C, \eta) = \#\left\{v \in \mathcal{A}_{C^{k+1}}^n : H_{C^k, C^{k+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta)\right\}.$$

The set of vertices involved in the definition of N_k^n will play an important role in our approach. Roughly speaking, they correspond to vertices active at time C^{k+1} which are “quite far” from a vertex active at time C^k . The first main input in the proof of the lower bound of the height is the following result, which shows that $C > 0$ can be chosen such that if R_{i+1}/R_i is close to $C > 0$, then there are many active vertices with a “large” height in the forest $\mathcal{F}_{R_i, R_{i+1}}^n$. Its proof is deferred to Section 6.5.4.

Lemma 6.5.3. *For every $\eta \in (0, 1)$, there exists an integer $C \geq 2$, there exist $\lambda > 0$ and $\varepsilon \in (0, 1)$ such that for every n large enough, for every $k \in \mathbb{N}$, with $A_n \leq C^k \leq \varepsilon n/C$, we have*

$$\mathbb{P}(N_k^n(C, \eta) \leq 4C^k) \leq \frac{\lambda}{C^k}$$

For every $n, k \in \mathbb{N}$, with $C^k \leq n/C$, let M_k^n be the maximal number of active vertices of a tree in the forest $\mathcal{F}_{C^k, C^{k+1}}^n$. The second main input in the proof of the lower bound of the height is the following result, which shows that this quantity cannot be too large. Its proof is deferred to Section 6.5.4.

Lemma 6.5.4. *For every n large enough, for every $k \in \mathbb{N}$, with $A_n \leq C^k \leq n/C$,*

$$\mathbb{P}(M_k^n > k^3) \leq \frac{1}{k^2}.$$

Let us now explain how the lower bound for the height follows from Lemmas 6.5.3 and 6.5.4.

Proof of the lower bound for Theorem 6.1.4 (III). Fix $\eta \in (0, 1)$ and take $C \geq 2, \lambda > 0, \varepsilon \in (0, 1)$ such that the conclusion of Lemma 6.5.3 holds and write N_k^n instead of $N_k^n(C, \eta)$ to simplify notation. For all integers $n, k \in \mathbb{N}$ such that $A_n \leq C^k \leq n/C$, set

$$\bar{N}_k^n = \#\left\{v \in \mathcal{A}_{C^k}^n, \quad \forall \ell \in [1, k-1] \text{ s.t. } A_n \leq C^\ell, \quad H_{C^\ell, C^{\ell+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta)\right\}.$$

We shall show that if K_n is the largest integer k such that $C^k \leq \varepsilon n/C$, then

$$\mathbb{P}\left(\bar{N}_{K_n}^n \geq 1\right) \xrightarrow{n \rightarrow \infty} 1. \quad (6.5.5)$$

Indeed, if (6.5.5) holds, then with probability tending to 1 there exists a vertex $v_{K_n} \in \mathcal{A}_{C^{K_n}}^n$ and vertices $v_i \in \mathcal{A}_{C^i}^n$ for every $i \in [1, K_n - 1]$ such that $A_n \leq C^i$, v_i is an ancestor of v_{i+1} and $d^n(v_i, v_{i+1}) > \frac{c+1}{2c} \log(C)(f(c) - \eta)$. Thus the height of v_{K_n} in \mathcal{T}^n is at least

$$\#\{i \in \mathbb{N} : \log(A_n)/\log(C) \leq i < K_n\} \cdot \frac{c+1}{2c} \log(C)(f(c) - \eta),$$

which is equal to

$$\left(\left\lfloor \frac{\log(n)}{\log(C)} + \frac{\log(\varepsilon/C)}{\log(C)} \right\rfloor - \left\lfloor \frac{\log(A_n)}{\log(C)} \right\rfloor\right) \cdot \frac{c+1}{2c} \log(C) \cdot (f(c) - \eta) \underset{n \rightarrow \infty}{\sim} \log(n) \frac{c+1}{2c} (f(c) - \eta).$$

Since $\eta \in (0, 1)$ was chosen arbitrarily, this will imply the desired result.

The main step of the proof to prove (6.5.5) is to establish that for every $k \in \mathbb{N}$ with $A_n \leq C^k \leq \varepsilon n/C$ we have

$$\mathbb{P}\left(\bar{N}_{k+1}^n < 2^{k+1} \mid \bar{N}_k^n \geq 2^k\right) = \mathcal{O}\left(\frac{1}{k^2}\right), \quad (6.5.6)$$

where here, and in the rest of the proof, the \mathcal{O} is uniform in $A_n \leq C^k \leq \varepsilon n/C$. This means that there exists a constant $\Delta > 0$ such that the term $\mathcal{O}(1/k^2)$ can be bounded from above by Δ/k^2 for every $A_n \leq C^k \leq \varepsilon n/C$ and n large enough. Indeed, we will then have

$$\mathbb{P}\left(\exists k \text{ with } A_n \leq C^k \leq \varepsilon n/C \text{ and } \bar{N}_k^n < 2^k\right) \leq \mathbb{P}(N_{k_n}^n < 2^{k_n}) + \mathcal{O}\left(\sum_{k=k_n}^{\infty} \frac{1}{k^2}\right)$$

with $k_n = \lceil \log(A_n)/\log(C) \rceil$. By Lemma 6.5.3,

$$\mathbb{P}(N_{k_n}^n < 2^{k_n}) \leq \mathbb{P}(N_{k_n}^n \leq 4C^{k_n}) \leq \frac{\lambda}{C^{k_n}} \leq \frac{\lambda}{A_n} \xrightarrow{n \rightarrow \infty} 0,$$

where the first inequality holds because $C \geq 2$. As a consequence, with probability tending to 1 as $n \rightarrow \infty$, for every integer k such that $A_n \leq C^k \leq \varepsilon n/C$, we have $\bar{N}_k^n \geq 2^k$, which implies (6.5.5).

It thus remains to establish (6.5.6). Note that the random variable \bar{N}_k^n is $\mathcal{T}_{C^k}^n$ measurable, and so is independent of the random variables N_k^n and M_k^n which are $\mathcal{F}_{C^k, C^{k+1}}^n$ measurable. As a consequence, we get for every $k \in \mathbb{N}$ with $A_n \leq C^k \leq \varepsilon n/C$:

$$\begin{aligned} & \mathbb{P}\left(\bar{N}_{k+1}^n < 2^{k+1} \mid \bar{N}_k^n \geq 2^k\right) \\ & \leq \mathbb{P}(N_k^n < 4C^k) + \mathbb{P}(M_k^n > k^3) + \mathbb{P}\left(\bar{N}_{k+1}^n < 2^{k+1}, N_k^n \geq 4C^k, M_k^n \leq k^3 \mid \bar{N}_k^n \geq 2^k\right) \\ & \leq \mathcal{O}\left(\frac{1}{C^k}\right) + \mathcal{O}\left(\frac{1}{k^2}\right) + \mathbb{P}\left(\bar{N}_{k+1}^n < 2^{k+1}, N_k^n \geq 4C^k, M_k^n \leq k^3 \mid \bar{N}_k^n \geq 2^k\right), \end{aligned} \quad (6.5.7)$$

where we have used Lemmas 6.5.3 and 6.5.4.

To bound the third term, we shall estimate \bar{N}_k^n by induction on k by using a second moment technique. To this end, we recall that one may reconstruct a tree with the same distribution as $\mathcal{T}_{C^{k+1}}^n$, by merging uniformly at random each active vertex from $\mathcal{T}_{C^k}^n$ with one vertex labelled \star from $\mathcal{F}_{C^k, C^{k+1}}^n$. With this in mind, it follows that

$$\mathbb{E}\left[\bar{N}_{k+1}^n \mid \mathcal{T}_{C^k}^n, \mathcal{F}_{C^k, C^{k+1}}^n\right] = \frac{\bar{N}_k^n N_k^n}{S_{C^k}^n}, \quad (6.5.8)$$

where we recall that $S_{C^k}^n = S_{C^k}(\mathbf{x}^n)$ represents the number of active vertices of $\mathcal{T}_{C^k}^n$.

Next, denote by $\mathcal{A}_{C^k, C^{k+1}}^n$ the set of all active vertices of the forest $\mathcal{F}_{C^k, C^{k+1}}^n$. For $v \in \mathcal{A}_{C^k, C^{k+1}}^n$, let $E_k^n(v)$ be the event defined by

$$E_k^n(v) = \left\{ \forall \ell \in \llbracket 1, k-1 \rrbracket \text{ s.t. } A_n \leq C^\ell, \quad H_{C^\ell, C^{\ell+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta) \right\};$$

if this event holds say that v is k -good, and k -bad otherwise. Observe that by definition, for every tree of $\mathcal{F}_{C^k, C^{k+1}}^n$, either all its active vertices are k -good, or all its vertices are k -bad. Note that for all $v, v' \in \mathcal{A}_{C^k, C^{k+1}}^n$, conditionally on $\mathcal{T}_{C^k}^n$ and $\mathcal{F}_{C^k, C^{k+1}}^n$, the events $E_{k+1}^n(v)$ and $E_{k+1}^n(v')$ are negatively correlated unless v, v' are in the same tree in $\mathcal{F}_{C^k, C^{k+1}}^n$.

Indeed, on the one hand, if the trees of $\mathcal{F}_{C^k, C^{k+1}}^n$ containing v and v' are different, then when we attach the tree containing v to an active vertex of $\mathcal{A}_{C^k}^n$ in such a way that v is k -good then the tree containing v' will have less chance to be attached to another vertex of $\mathcal{A}_{C^k}^n$ in such a way that v' is k -good, and vice versa. On the other hand, if v and v' are in the same tree and if we have the inequalities $H_{C^k, C^{k+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta)$ and $H_{C^k, C^{k+1}}^n(v') > \frac{c+1}{2c} \log(C)(f(c) - \eta)$, if we denote by w the (random) active vertex of $\mathcal{T}_{C^k}^n$ on which this tree is attached, then v and v' are both $k+1$ -good if and only if w is k -good.

As a result, writing $\widetilde{\mathbb{P}}$ and $\widetilde{\mathbb{E}}$ for the conditional probability and expectation conditionally given $\mathcal{T}_{C^k}^n$ and $\mathcal{F}_{C^k, C^{k+1}}^n$, we get

$$\begin{aligned} \widetilde{\text{Var}}[\overline{N}_{k+1}^n] &= \widetilde{\text{Var}} \left[\sum_{v \in \mathcal{A}_{C^k, C^{k+1}}^n} \mathbb{1}_{E_{k+1}^n(v)} \right] = \sum_{v, v' \in \mathcal{A}_{C^k, C^{k+1}}^n} \widetilde{\text{Cov}}(\mathbb{1}_{E_{k+1}^n(v)}, \mathbb{1}_{E_{k+1}^n(v')}) \\ &\leq M_k^n \sum_{v \in \mathcal{A}_{C^k, C^{k+1}}^n} \widetilde{\mathbb{P}}(E_{k+1}^n(v)) = M_k^n \cdot \widetilde{\mathbb{E}}[\overline{N}_{k+1}^n]. \end{aligned} \quad (6.5.9)$$

Then, by (6.5.8) on the event $\{\overline{N}_k^n \geq 2^k, N_k^n \geq 4C^k, M_k^n \leq k^3\}$ (which is measurable with respect to $\mathcal{T}_{C^k}^n, \mathcal{F}_{C^k, C^{k+1}}^n$), using the bound $S_{C^k}^n \leq C^k$, we have

$$\widetilde{\mathbb{E}}[\overline{N}_{k+1}^n] \geq \frac{\overline{N}_k^n N_k^n}{S_{C^k}^n} \geq 2^{k+2}.$$

Thus, by (6.5.9), still on the event $\{\overline{N}_k^n \geq 2^k, N_k^n \geq 4C^k, M_k^n \leq k^3\}$, we have

$$\begin{aligned} \widetilde{\mathbb{P}}(\overline{N}_{k+1}^n < 2^{k+1}) &\leq \widetilde{\mathbb{P}}\left(\overline{N}_{k+1}^n < \frac{1}{2} \widetilde{\mathbb{E}}[\overline{N}_{k+1}^n]\right) \leq 4 \frac{\widetilde{\text{Var}}[\overline{N}_{k+1}^n]}{\widetilde{\mathbb{E}}[\overline{N}_{k+1}^n]^2} \leq 4 \frac{M_k^n}{\widetilde{\mathbb{E}}[\overline{N}_{k+1}^n]} \\ &= \frac{4M_k^n S_{C^k}^n}{N_k^n N_k^n} \leq \frac{4k^3 C^k}{2^k C^k} = \mathcal{O}\left(\frac{1}{k^2}\right). \end{aligned}$$

As a consequence,

$$\begin{aligned} &\mathbb{P}(\overline{N}_{k+1}^n < 2^{k+1}, N_k^n \geq 4C^k, M_k^n \leq k^3 \mid \overline{N}_k^n \geq 2^k) \\ &= \mathbb{E}\left[\widetilde{\mathbb{P}}(\overline{N}_{k+1}^n < 2^{k+1}) \mathbb{1}_{N_k^n \geq 4C^k, M_k^n \leq k^3} \mid \overline{N}_k^n \geq 2^k\right] = \mathcal{O}\left(\frac{1}{k^2}\right), \end{aligned}$$

and (6.5.6) follows from (6.5.7). This completes the proof. \square

6.5.4 Proof of Lemmas 6.5.3 and 6.5.4

We keep the notation introduced in Section 6.5.3. The following estimate is the first step for the proof of Lemma 6.5.3.

Lemma 6.5.5. *For every $\eta \in (0, 1)$, there exists an integer $C > 1$ and $\varepsilon \in (0, 1)$ such that for every n large enough for every $k \in \mathbb{N}$, with $A_n \leq C^k \leq \varepsilon n/C$, we have*

$$\mathbb{E}[N_k^n(C, \eta)] \geq 5C^k.$$

Proof of Lemma 6.5.5. Recall that $g(x) = (1+x)\log(1+x) - x$. Set $\delta = 1 - \frac{c+1}{2c} \cdot g(f(c) - 1 - \eta)$, which belongs to $(0, 1)$ since $g(f(c) - 1 - \eta) < (2c)/(c+1)$, and then choose an integer $C > 1$ large enough such that

$$\frac{1}{\log(C)^4} \exp(-(1-\delta)\log(C)) \geq \frac{6}{cC} \quad \text{and} \quad \log(C) \geq \frac{c+1}{2c}(f(c) - \eta). \quad (6.5.10)$$

It remains to check that this value of C satisfies the desired conclusion. By Lemma 6.2.1 for every $v \in \mathcal{A}_{C^{k+1}}^n$,

$$H_{C^k, C^{k+1}}^n(v) \stackrel{(d)}{=} \sum_{C^k < i \leq C^{k+1}} Y_i^n,$$

where $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n \mathbb{1}_{x_i^n=1})_{1 \leq i \leq n}$. Lemma 6.5.1 yields

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{A_n \leq C^k \leq \varepsilon n} \left| \sum_{C^k < i \leq C^{k+1}} \frac{1}{S_i^n} \mathbb{1}_{x_i^n=1} - \frac{c+1}{2c} \log(C) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Hence, by [BH84, Theorem 1], we have

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{A_n \leq C^k \leq \varepsilon n} d_{\text{TV}} \left(H_{C^k, C^{k+1}}^n(v), \mathcal{Poi} \left(\frac{c+1}{2c} \log(C) \right) \right) = 0,$$

where $\mathcal{Poi}(\lambda)$ denotes a Poisson random variable with parameter λ and d_{TV} stands for the total variation distance. As a consequence, setting

$$P(C) := \mathbb{P} \left(\mathcal{Poi} \left(\frac{c+1}{2c} \log(C) \right) > \frac{c+1}{2c} \log(C)(f(c) - \eta) \right),$$

we have

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{A_n \leq C^k \leq \varepsilon n} \left| \mathbb{P} \left(H_{C^k, C^{k+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta) \right) - P(C) \right| = 0. \quad (6.5.11)$$

Then, for $\lambda > 2$ and $t > 1$, using the bound $[u]! \leq u^3(u/e)^u$ for $u \geq 2$, observe that

$$\mathbb{P}(\mathcal{Poi}(\lambda) > \lambda t) \geq e^{-\lambda} \frac{\lambda^{[\lambda t]}}{[\lambda t]!} \geq e^{-\lambda} \lambda^{\lambda t} \left(\frac{e}{\lambda t} \right)^{\lambda t} \frac{1}{(\lambda t)^3} = e^{-\lambda g(t-1)} \frac{1}{(\lambda t)^3},$$

Taking into account our choice of C (see (6.5.10)), it follows that

$$P(C) \geq \frac{1}{\log(C)^4} \exp \left(-\frac{c+1}{2c} \log(C) g(f(c) - 1 - \eta) \right) \geq \frac{6}{cC}. \quad (6.5.12)$$

Then observe that by linearity we have

$$\mathbb{E}[N_k^n(C, \eta)] = \#\mathcal{A}_{C^{k+1}}^n \mathbb{P} \left(H_{C^k, C^{k+1}}^n(v) > \frac{c+1}{2c} \log(C)(f(c) - \eta) \right).$$

But $|\#\mathcal{A}_{C^{k+1}}^n / (cC^{k+1}) - 1| = |S_{C^{k+1}}^n / (cC^{k+1}) - 1| \leq \eta_n(\varepsilon)/c$ for $C^{k+1} \leq \varepsilon n$, where we recall that

$$\eta_n(\varepsilon) = \max_{A_n \leq i \leq \varepsilon n} \left| \frac{S_i^n}{i} - c \right|.$$

The desired result follows by using (6.5.11) and (6.5.12). \square

Observe that by (6.1.2) we may choose $\varepsilon_0 \in (0, 1)$ such that

$$\limsup_{n \rightarrow \infty} \max_{A_n \leq i \leq \varepsilon_0 n} \left| \frac{S_i^n}{i} - c \right| < \frac{c}{2} \quad \text{and} \quad \liminf_{n \rightarrow \infty} \min_{\varepsilon_0 n \leq i \leq n} \frac{S_i^n}{n} > 0,$$

which implies the existence of a constant $c_0 > 0$ such that:

$$\text{for every } n \text{ sufficiently large, for every } 1 \leq i \leq n, \quad c_0 i \leq S_i^n \leq i + 1, \quad (6.5.13)$$

where the upper bound comes from the fact that $x_i^n \leq 1$.

We are now ready to prove Lemma 6.5.3.

Proof of Lemma 6.5.3. Fix $\eta \in (0, 1)$. Consider $C > 1$ given by Lemma 6.5.5. Take $\varepsilon \in (0, 1)$ small enough such that the conclusion of Lemma 6.5.5 holds. The main idea of the proof is to show that for some fixed $\lambda > 0$ we have uniformly for every $k \in \mathbb{N}$ with $A_n \leq C^k \leq \varepsilon n / C$

$$\mathcal{V}\text{ar}(N_k^n(C, \eta)) \leq \lambda C^k. \quad (6.5.14)$$

Indeed, it will then directly follow from Lemma 6.5.5 and Bienaymé-Chebyshev's inequality that

$$\mathbb{P}(N_k^n(C, \eta) < 4C^k) \leq \mathbb{P}(|N_k^n(C, \eta) - \mathbb{E}[N_k^n(C, \eta)]| > C^k) \leq \mathcal{V}\text{ar}(N_k^n(C, \eta)) / C^{2k} \leq \lambda / C^k.$$

We shall show that

$$\mathbb{E}[(N_k^n(C, \eta))^2] \leq (\mathbb{E}[N_k^n(C, \eta)])^2 + \mathcal{O}(C^k). \quad (6.5.15)$$

This indeed implies (6.5.14). Observe that by definition of $N_k^n(C, \eta)$, setting $t = \frac{c+1}{2c} \log(C)(f(c) - \eta)$, we have

$$N_k^n(C, \eta) = \sum_{v \in \mathcal{A}_{C^{k+1}}^n} \mathbb{1}_{H_{C^k, C^{k+1}}^n(v) > t}.$$

Note from Algorithm 2, that for every $v, w \in \mathcal{A}_{C^{k+1}}^n$ with $v \neq w$

$$\left(H_{C^k, C^{k+1}}^n(v), H_{C^k, C^{k+1}}^n(w) \right) \stackrel{(d)}{=} \left(\sum_{C^k < i \leq C^{k+1}} Y_i^n, \sum_{C^k < i \leq C^{k+1}} Z_i^n \right)$$

where $(Z_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n \mathbb{1}_{x_i^n=1})_{1 \leq i \leq n}$. The problem is that $(Z_i^n)_{1 \leq i \leq n}$ and $(Y_i^n)_{1 \leq i \leq n}$ are not independent. The idea is to say that, conditionally on the fact that v and w do not coalesce (i.e they do not belong to the same tree) before time C^{k+1} , then, when the height of v goes up by 1 in the coalescence process, the height of w doesn't change and vice versa. Therefore their heights are negatively correlated. One can see that for all $v, w \in \mathcal{A}_{C^{k+1}}^n$,

$$\begin{aligned} & \mathbb{P}\left(H_{C^k, C^{k+1}}^n(v) > t \text{ and } H_{C^k, C^{k+1}}^n(w) > t\right) \\ & \leq \mathbb{P}(C^k \leq \mathbf{c}_n(v, w) \leq C^{k+1}) + \mathbb{P}\left(\mathbf{c}_n(v, w) < C^k \text{ and } H_{C^k, C^{k+1}}^n(v) > t \text{ and } H_{C^k, C^{k+1}}^n(w) > t\right). \end{aligned} \quad (6.5.16)$$

The first probability of (6.5.16) is bounded from above as follows:

$$\mathbb{P}(C^k \leq \mathbf{c}_n(v, w) \leq C^{k+1}) \leq 1 - \prod_{C^k \leq i \leq C^{k+1}} \left(1 - \mathbb{1}_{x_i^n=1} \frac{1}{\binom{S_i^n}{2}}\right) \leq 1 - \exp\left(\sum_{C^k \leq i \leq C^{k+1}} \log\left(1 - \frac{1}{\binom{S_i^n}{2}}\right)\right) = \mathcal{O}\left(\frac{1}{C^k}\right), \quad (6.5.17)$$

where the last equality comes from (6.1.2) and where the $\mathcal{O}(1/C^k)$ is uniform in v, w and in $n \geq C^{k+1}$.

We then focus on the event involved in the second probability of (6.5.16). One can write

$$H_{C^k, C^{k+1}}^n(v) = \sum_{C^k < i \leq C^{k+1}} Y_i^n,$$

where $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n \mathbb{1}_{x_i^n=1})_{1 \leq i \leq n}$. Moreover, we note that if we know that $Y_i^n = 1$ and that $c_n(v, w) \leq i - 1$, then it implies that there is no coalescence with the tree containing w . More precisely, one can write

$$H_{C^k, C^{k+1}}^n(w) = \sum_{C^k < i \leq C^{k+1}} Z_i^n.$$

with

$$Z_i^n = Y_i^n \mathbb{1}_{c_n(v, w) \geq i} + \xi_i^n \mathbb{1}_{c_n(v, w) < i \text{ and } Y_i^n = 0},$$

where the ξ_i^n 's are independent Bernoulli r.v. of parameter $(1/(S_i^n - 1)) \mathbb{1}_{x_i^n=1}$ respectively and are taken independently from the Y_i^n 's. As a result, the second probability in (6.5.16) is bounded from above by

$$\begin{aligned} & \mathbb{P} \left(c_n(v, w) < C^k, \sum_{C^k < i \leq C^{k+1}} Y_i^n > t \text{ and } \sum_{C^k < i \leq C^{k+1}} Z_i^n > t \right) \\ &= \mathbb{P} \left(c_n(v, w) < C^k, \sum_{C^k < i \leq C^{k+1}} Y_i^n > t \text{ and } \sum_{C^k < i \leq C^{k+1}} \xi_i^n \mathbb{1}_{Y_i^n=0} > t \right) \\ &\leq \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} Y_i^n > t \text{ and } \sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \right) \\ &= \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} Y_i^n > t \right) \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \right) \leq \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \right)^2. \end{aligned} \quad (6.5.18)$$

Now, thanks to the fact that $1/S_i^n \leq 1/(S_i^n - 1)$, one can define some random variables \tilde{Y}_i^n 's for $i \geq 1$ such that for all $i \geq 1$, we have $\tilde{Y}_i^n \leq \xi_i^n$ and $(\tilde{Y}_i^n)_{i \geq 1}$ is a family of independent Bernoulli random variables of parameter $(1/S_i^n) \mathbb{1}_{x_i^n=1}$, having thus the same law as $(Y_i^n)_{i \geq 1}$. It follows that

$$\begin{aligned} \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \right) - \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} Y_i^n > t \right) &= \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \geq \sum_{C^k < i \leq C^{k+1}} \tilde{Y}_i^n \right) \\ &\leq \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} (\xi_i^n - \tilde{Y}_i^n) > 0 \right). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} \xi_i^n > t \right) - \mathbb{P} \left(\sum_{C^k < i \leq C^{k+1}} Y_i^n > t \right) &\leq \mathbb{E} \left[\sum_{C^k < i \leq C^{k+1}} (\xi_i^n - \tilde{Y}_i^n) \right] \\ &= \sum_{C^k < i \leq C^{k+1}} \frac{1}{S_i^n (S_i^n - 1)} \mathbb{1}_{x_i^n=1} \\ &= \mathcal{O} \left(\frac{1}{C^k} \right), \end{aligned}$$

where the $\mathcal{O}(1/C^k)$ is uniform in $n \geq C^{k+1}$. By combining this estimate with (6.5.16) and (6.5.18), we conclude that

$$\mathbb{P} \left(H_{C^k, C^{k+1}}^n(v) > t \text{ and } H_{C^k, C^{k+1}}^n(w) > t \right) \leq \mathbb{P} \left(H_{C^k, C^{k+1}}^n(v) > t \right) \mathbb{P} \left(H_{C^k, C^{k+1}}^n(w) > t \right) + \mathcal{O} \left(\frac{1}{C^k} \right)$$

Therefore, since $\#\mathcal{A}_{C^{k+1}}^n \leq C^{k+1} + 1$,

$$\begin{aligned} & \mathbb{E}[(N_k^n(C, \eta))^2] \\ &= \sum_{v \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t) + \sum_{v \neq w \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t \text{ and } H_{C^k, C^{k+1}}^n(w) > t) \\ &= \mathcal{O}(C^k) + \sum_{v \neq w \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t) \mathbb{P}(H_{C^k, C^{k+1}}^n(w) > t) + (\#\mathcal{A}_{C^{k+1}}^n)^2 \mathcal{O}\left(\frac{1}{C^k}\right) \\ &= \sum_{v \neq w \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t) \mathbb{P}(H_{C^k, C^{k+1}}^n(w) > t) + \mathcal{O}(C^k). \end{aligned}$$

But observe that

$$\begin{aligned} \mathbb{E}[N_k^n(C, \eta)]^2 &= \sum_{v \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t)^2 + \sum_{v \neq w \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t) \mathbb{P}(H_{C^k, C^{k+1}}^n(w) > t) \\ &= \sum_{v \neq w \in \mathcal{A}_{C^{k+1}}^n} \mathbb{P}(H_{C^k, C^{k+1}}^n(v) > t) \mathbb{P}(H_{C^k, C^{k+1}}^n(w) > t) + \mathcal{O}(C^k). \end{aligned}$$

This implies (6.5.14) and completes the proof. \square

To establish Lemma 6.5.4 and bound M_k^n we use the following estimate on Pólya urns, which may be shown by following verbatim the proof of Lemma A.1 from [BR23].

Lemma 6.5.6. *Fix an integer $z_0 > 0$ and set $U_0 = 1$. Let $(z_n)_{n \geq 1} \in \{-1, 1\}^{\mathbb{N}}$, set $Z_n = \sum_{i=0}^n z_i$. Assume that $Z_n > 0$ for every $n \geq 1$ and $\sum_{n=0}^{\infty} 1/Z_n^2 < \infty$. Let $(U_n)_{n \geq 1}$ be a sequence of random non-negative integers such that for every $n \geq 0$,*

$$\mathbb{P}(U_{n+1} = U_n + z_{n+1} | U_n) = \frac{U_n}{Z_n} \quad ; \quad \mathbb{P}(U_{n+1} = U_n | U_n) = \frac{Z_n - U_n}{Z_n}.$$

Almost surely for every $t \geq 0$

$$\mathbb{P}\left(\sup_{i \geq 1} \left| \frac{U_i}{Z_i} - \frac{1}{z_0} \right| > t \frac{1}{z_0}\right) \leq 2 \exp\left(-\frac{(t^2/4)(1/z_0)}{\sum_{n \geq 1} 1/Z_n^2 + t \max\left(\sum_{n \geq 1} 1/Z_n^2, \max_{n \geq 1} 1/Z_n\right)}\right).$$

Proof of Lemma 6.5.4. We apply Lemma 6.5.6 with $t = k^2$, $z_0 = S_{C^k}^n$, $z_i = x_{C^k+i}^n$ for $1 \leq i \leq C^{k+1} - C^k$ and $z_i = 1$ for all $i > C^{k+1} - C^k$, and U_i being the number of active vertices in a given tree of $\mathcal{F}_{C^k, C^{k+i}}^n$. Thus, writing X_k for the number of active vertices of a any fixed tree of the forest $\mathcal{F}_{C^k, C^{k+1}}^n$, writing $\delta_k^n := \sum_{C^k < n \leq C^{k+1}} 1/(S_i^n)^2 + \sum_{j \geq 1} 1/(S_{C^{k+1}+j}^n)^2$, and $\varepsilon_k^n := \max_{C^k < i \leq C^{k+1}} 1/S_i^n$,

$$\mathbb{P}(X_k/S_{C^{k+1}}^n > (1+k^2)/S_{C^k}^n) \leq 2 \exp\left(-\frac{(k^4/4)/S_{C^k}^n}{\delta_k^n + k^2 \max(\delta_k^n, \varepsilon_k^n)}\right).$$

And by (6.5.13), as $k, n \rightarrow \infty$ with $A_n \leq C^k \leq n/C$, we have $S_{C^k}^n \leq C^k + 1$, $1/S_{C^k}^n = \mathcal{O}(1/C^k)$, $\varepsilon_k^n = \mathcal{O}(1/C^k)$ and $\delta_k^n = \mathcal{O}(1/C^k)$, thus for every k large enough,

$$\mathbb{P}(X_k > k^3) \leq 2e^{-k^2/\mathcal{O}(1)}.$$

Then, since there are at most $S_{C^k}^n = \mathcal{O}(C^k)$ trees in $\mathcal{F}_{C^k, C^{k+1}}^n$, for n large enough, for every $k \in \mathbb{N}$, with $A_n \leq C^k \leq n/C$,

$$\mathbb{P}(M_k^n > k^3) \leq S_{C^k}^n 2e^{-k^2/\mathcal{O}(1)} \leq 1/k^2. \quad \square$$

6.6 Application: contact-tracing in a stochastic SIR dynamics

Our results can be applied to study the so-called “infection tree” of a stochastic SIR dynamics, which is a classical model for the evolution of epidemics (for background on stochastic epidemic models, see [AB12, BPB⁺19]). We assume that initially there is 1 infectious individual (that has just become infected) and n susceptible individuals. The infectious periods of different infectives are i.i.d distributed according to an exponential random variable of parameter 1. During its infectious period an infective makes contacts with a given individual at the time points of a time homogeneous Poisson process with intensity λ_n . If a contacted individual is still susceptible, then it becomes infectious and is immediately able to infect other individuals. An individual is considered “removed” once its infectious period has terminated, and is then immune to new infections, playing no further part in the epidemic spread. The epidemic ceases as soon as there are no more infectious individual present in the population. All Poisson processes are assumed to be independent of each other; they are also independent of the infectious periods.

We call a “step” an event where either a susceptible individuals becomes infective, or where an individual’s infectious period terminates. Denote by τ_n the number of steps made when the epidemic ceases. For $0 \leq k \leq \tau_n$, let \mathcal{F}_k^n be the infection tree after k steps, in which the vertices are individuals and where an edge is present between two individuals if one has infected the other. We are interested in the evolution of the associated “infection tree” $(\mathcal{F}_k^n)_{0 \leq k \leq \tau_n}$, as well as in the shape of the full infection tree $\mathcal{F}_{\tau_n}^n$ when the epidemic ceases.

Let $(U_k^n, I_k^n)_{k \geq 0}$ be a Markov chain with initial state $(U_0^n, I_0^n) = (n, 1)$ and transition probabilities given by

$$(U_{k+1}^n, I_{k+1}^n) = \begin{cases} (U_k^n - 1, I_k^n + 1) & \text{with probability } \frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \\ (U_k^n, I_k^n - 1) & \text{with probability } \frac{1}{1 + \lambda_n U_k^n} \end{cases}$$

with $\{(k, 0) : 0 \leq k \leq n\}$ as absorbing states (we use “U” for “uninfected”). Observe that evolution of the number of susceptible individuals and the number of infectious individuals in the infection tree evolves according to this Markov chain. Then define the random sequence $\mathbf{X}^n = (X_i^n)_{1 \leq i \leq \tau_n'}$ of ± 1 as follows: let τ_n' be the absorption time of the Markov chain, and for $1 \leq i \leq \tau_n'$ set $X_i^n = I_i^n - I_{i-1}^n$.

Then by construction, it is clear that:

$$(\mathcal{F}_k^n)_{0 \leq k \leq \tau_n} \stackrel{(d)}{=} (\mathcal{T}_k(\mathbf{X}^n))_{0 \leq k \leq \tau_n'}.$$

To simplify notation, let $\mathcal{F}^n = \mathcal{F}_{\tau_n}^n$ be the full infection tree when the epidemic ceases. We also identify the random variables in the above equality in distribution. For all $p \in [0, 1)$, we denote by $\mathcal{G}(1-p)$ the geometric distribution μ given by $\mu(k) = (1-p)p^k$ for every integer $k \geq 0$.

Theorem 6.6.1. *The following assertions hold.*

- (I) *Assume that $\lambda_n \sim \lambda/n$ for some $\lambda > 0$. Then \mathcal{F}^n converges in distribution locally towards a Bienaymé tree with offspring distribution $\mathcal{G}(1/(1+\lambda))$.*
- (II) *Assume that $\lambda_n \gg 1/n$. Then \mathcal{F}^n is not tight for the topology of the local convergence.*

Proof. For (I), assume that $\lambda_n \sim \lambda/n$ as $n \rightarrow \infty$ for some $\lambda > 0$. We first prove that $(I_k^n)_{k \geq 0}$ converges in distribution for the product topology towards the random walk $(S_k)_{k \geq 0}$ such that $S_{k+1} - S_k = 1$ with probability $\lambda/(1+\lambda)$ and $S_{k+1} - S_k = -1$ with probability $1/(1+\lambda)$. Indeed, the local convergence of $\mathcal{T}_n(\mathbf{X}^n)$ implies the local convergence of \mathcal{F}^n . So as to achieve this, one can first see that since $U_k^n \leq n$ for all $k \geq 0$, we have

$$\frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \leq \frac{n \lambda_n}{1 + n \lambda_n} \xrightarrow{n \rightarrow \infty} \frac{\lambda}{1 + \lambda}.$$

But we also have

$$\inf_{k \leq \sqrt{n}} \frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \geq \frac{\lambda_n (n - \sqrt{n})}{1 + n \lambda_n} \xrightarrow{n \rightarrow \infty} \frac{\lambda}{1 + \lambda}.$$

For all $k_0 \geq 0$, one can thus build a coupling between $(I_k^n)_{0 \leq k \leq k_0}$ and $(S_k)_{0 \leq k \leq k_0}$ such that the two walks coincide w.h.p. until time k_0 as $n \rightarrow \infty$.

Thus, by Skorokhod’s representation theorem we may assume that almost surely, for every $k_0 \geq 1$, $(I_k^n)_{0 \leq k \leq k_0} = (S_k)_{0 \leq k \leq k_0}$ for n sufficiently large. By Theorem 6.1.1, \mathcal{F}^n converges locally in distribution

to a uniform attachment tree with freezing built using the sequence $(S_{k+1} - S_k)_{k \geq 0}$. By Theorem 6.1.2, this is a Bienaymé tree with offspring distribution $\mathcal{G}(1/(1 + \lambda))$. This proves (I).

For (II), we argue by contradiction and assume that $(\mathcal{T}^n)_{n \geq 0}$ is tight. By Prokhorov's theorem, let φ be an extraction such that $(\mathcal{T}^{\varphi(n)})_{n \geq 0}$ converges in distribution. Similarly to the proof of (I), observe that $(I_k^{\varphi(n)})_{k \geq 0}$ converges in distribution with respect to the product topology towards $(k+1)_{k \geq 0}$ since for all fixed $k \geq 0$, the transition probability $\lambda_n U_k^{\varphi(n)} / (1 + \lambda_{\varphi(n)} U_k^{\varphi(n)})$ converges to 1 as $n \rightarrow \infty$. Thus, by Skorokhod's representation theorem we may assume that almost surely, for every $k_0 \geq 1$, $(I_k^{\varphi(n)})_{0 \leq k \leq k_0} = (k+1)_{0 \leq k \leq k_0}$ for n sufficiently large. But if $\mathbf{x}_i = 1$ for every $i \geq 1$, the sum $\sum_{i \geq 1} \frac{1}{S_i(\mathbf{x})} \mathbb{1}_{\{x_i = -1\}}$ converges, so by Theorem 6.1.1 the sequence $(\mathcal{T}^{\varphi(n)})_{n \geq 0}$ does not converge locally in distribution, a contradiction. \square

Theorem 6.6.2. *The following assertions hold.*

(I) *Assume that $\lambda_n \sim \lambda/n$ for some $\lambda > 1$. Let A be the event of survival of a Bienaymé process with offspring distribution $\mathcal{G}(1/(1 + \lambda))$.*

(a) *The convergence*

$$\left(\frac{I_{\lfloor nt \rfloor}^n}{n} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(d)} (\max(2 - 2g_\lambda(t) - t, 0) \mathbb{1}_A)_{t \geq 0}, \quad (6.6.1)$$

holds in distribution for the topology of uniform convergence on compact sets, where g_λ is the solution of the ordinary differential equation $g'_\lambda(t) = -\lambda g_\lambda(t)/(1 + \lambda g_\lambda(t))$ with $g_\lambda(0) = 1$.

(b) *For all $n \geq 1$, conditionally given \mathcal{T}^n , let V_1^n and V_2^n be independent uniform vertices of \mathcal{T}^n . Then*

$$\frac{H(V_1^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(d)} \frac{\lambda}{\lambda - 1} \mathbb{1}_A \quad \text{and} \quad \frac{d^n(V_1^n, V_2^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(d)} \frac{2\lambda}{\lambda - 1} \mathbb{1}_A, \quad (6.6.2)$$

where $H(V_1^n)$ is the height of V_1^n in \mathcal{T}^n and d^n denotes the graph distance in \mathcal{T}^n .

(II) *Assume that $\lambda_n \gg 1/n$.*

(a) *The convergence*

$$\left(\frac{I_{\lfloor nt \rfloor}^n}{n} \right)_{0 \leq t \leq 2} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (\min(t, 2 - t))_{0 \leq t \leq 2}, \quad (6.6.3)$$

holds in probability for the topology of uniform convergence.

(b) *We have*

$$\frac{H(V_1^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1, \quad \frac{d^n(V_1^n, V_2^n)}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 2 \quad \text{and} \quad \frac{\text{Height}(\mathcal{T}^n)}{\log(n)} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} e. \quad (6.6.4)$$

We observe that the solution g_λ of the differential equation can be expressed using the principal branch of the Lambert W function defined as the inverse function $W : [-1/e, \infty) \rightarrow [-1, \infty)$ of the function $x \mapsto xe^x$ which is increasing on $[-1, \infty)$. See e.g. [CGH⁺96] for more details about this function. Explicitly, using the fact that $W'(x) = W(x)/(x(1 + W(x)))$, we check that for all $t \geq 0$,

$$g_\lambda(t) = \frac{1}{\lambda} W(\lambda e^\lambda e^{-\lambda t}). \quad (6.6.5)$$

In particular, g_λ is infinitely differentiable and one can see that g_λ is convex, decreasing and goes to 0 at ∞ .

Remark 6.6.3. *It would be very interesting to obtain a limit theorem for $\text{Height}(\mathcal{T}^n)$ in the setting of Theorem 6.6.2 (I) (b). This requires more work, because the behavior of the number of infected/active vertices near the extinction time τ'_n should give a non-negligible contribution to the height.*

Proof of Theorem 6.6.2. For (I) (a), we assume $\lambda > 1$ and we determine the fluid limit of the chain (U^n, I^n) . Let $(\tilde{U}^n, \tilde{I}^n)$ be a Markov chain which has the same initial conditions as (U^n, I^n) and the same transition probabilities but which is not stopped when \tilde{I}^n reaches zero. We check that (6.6.1) holds using the classical theory of fluid limits of Markov chains (see e.g. [Kur81] or [DN08]), and more precisely a combination of Grönwall's inequality with Doob's maximal inequality.

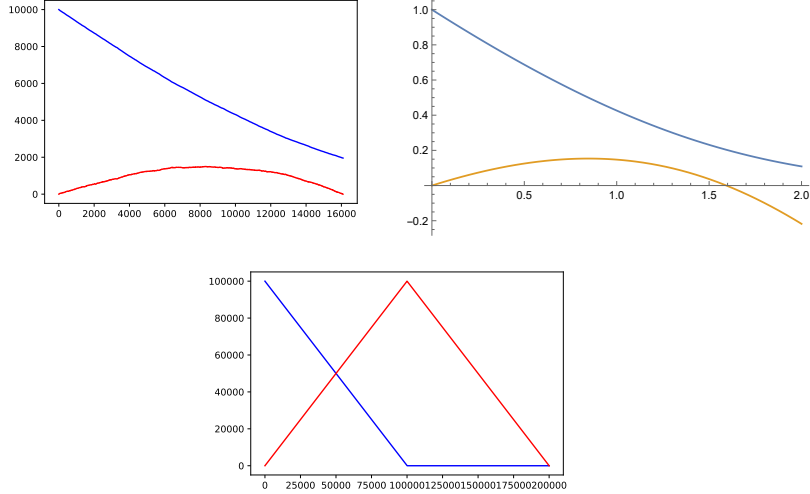


Figure 6.10 – Left: simulation of (U^n, I^n) for $\lambda_n = 2/n$ and $n = 10000$ (U^n in blue and I^n in red). Centre: the corresponding fluid limit. Right: simulation of (U^n, I^n) for $\lambda_n = 2$ and $n = 100000$.

We claim that it is enough to show that

$$\left(\frac{\tilde{U}_{[nt]}^n}{n} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (g_\lambda(t))_{t \geq 0}, \quad (6.6.6)$$

uniformly on compacts. Indeed, it is easy to see that $\tilde{I}_k^n = 2(n - \tilde{U}_k^n) - k + 1$ so that (6.6.6) entails the convergence

$$\left(\frac{\tilde{I}_{[nt]}^n}{n} \right)_{t \geq 0} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (2 - 2g_\lambda(t) - t)_{t \geq 0}, \quad (6.6.7)$$

uniformly on compacts. Let us check that this implies (6.6.1). Since $t \mapsto 2 - 2g_\lambda(t) - t$ is concave, starts from zero at time zero, stays positive and then goes below zero, it is thus enough to identify the limit of the probability that the chain \tilde{I}^n reaches zero before time εn as first $n \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ as the probability of extinction of a $\mathcal{G}(1/(1 + \lambda))$ -Bienaymé process. To this end, for all $\varepsilon > 0$, for n large enough, for all $k \leq \varepsilon n$, since $n \geq \tilde{U}_k^n \geq (1 - \varepsilon)n$ we have

$$\frac{\lambda}{1 + \lambda} - \varepsilon \leq (1 - \varepsilon) \frac{n\lambda_n}{1 + n\lambda_n} \leq \frac{\lambda_n \tilde{U}_k^n}{1 + \lambda_n \tilde{U}_k^n} \leq \frac{n\lambda_n}{1 + n\lambda_n} \leq \frac{\lambda}{1 + \lambda} + \varepsilon.$$

Thus, one can couple (I^n, U^n) until time εn with two simple random walks $(\bar{Y}_k^\varepsilon)_{k \geq 0}$ and $(\underline{Y}_k^\varepsilon)_{k \geq 0}$ starting from one such that

$$\mathbb{P}(\bar{Y}_1^\varepsilon = 1) = \frac{\lambda}{1 + \lambda} + \varepsilon \quad \text{and} \quad \mathbb{P}(\underline{Y}_1^\varepsilon = 1) = \frac{\lambda}{1 + \lambda} - \varepsilon$$

so that for all n large enough, for all $k \leq \varepsilon n$,

$$\underline{Y}_k^\varepsilon \leq \tilde{I}_k^n \leq \bar{Y}_k^\varepsilon. \quad (6.6.8)$$

Since the probabilities that the random walks $\underline{Y}^\varepsilon$ and \bar{Y}^ε starting from 1 reach the negative integers both converge as $\varepsilon \rightarrow 0$ to the probability of extinction of a $\mathcal{G}(1/(1 + \lambda))$ -Bienaymé process, we get the desired result.

It remains to check (6.6.6). This is a rather direct application of Lemma 6.5 in [Cur24] (which remains clearly true if we add a superscript n to all the objects). More precisely, let $t > 0$, for all $k \geq 0$ let

$X_k^n = \tilde{U}_k^n - ng_\lambda(k/n)$ and let $(\mathcal{F}_k^n)_{k \geq 0}$ be the natural filtration associated with X^n . This lemma states (in greater generality) that if there exists a constant $C > 0$ such that for all $n \geq 1$, for all $0 \leq k \leq nt$,

$$|\mathbb{E}(X_{k+1}^n - X_k^n | \mathcal{F}_k^n)| \leq \frac{C}{n} (1 + \sup_{0 \leq j \leq nt} |X_j^n|) \quad \text{a.s. and} \quad \mathbb{E}\left((X_{k+1}^n - X_k^n)^2\right) \leq C, \quad (6.6.9)$$

then $\sup_{0 \leq k \leq nt} |X_k^n|/n \rightarrow 0$ in probability as $n \rightarrow \infty$, which implies (6.6.6).

To show (6.6.9), we first compute

$$\mathbb{E}(X_{k+1}^n - X_k^n | \mathcal{F}_k^n) = -\frac{\lambda \tilde{U}_k^n / n}{1 + \lambda \tilde{U}_k^n / n} - n \left(g_\lambda \left(\frac{k+1}{n} \right) - g_\lambda \left(\frac{k}{n} \right) \right).$$

Moreover, since the function g_λ is twice differentiable, using a Taylor expansion at order 2 and using the definition of g_λ one gets that

$$\begin{aligned} \left| \mathbb{E} \left(X_{k+1}^n - X_k^n + \frac{\lambda \tilde{U}_k^n / n}{1 + \lambda \tilde{U}_k^n / n} - \frac{\lambda g_\lambda(k/n)}{1 + \lambda g_\lambda(k/n)} \middle| \mathcal{F}_k^n \right) \right| &= n \left| g_\lambda \left(\frac{k+1}{n} \right) - g_\lambda \left(\frac{k}{n} \right) + \frac{1}{n} \frac{\lambda g_\lambda(k/n)}{1 + \lambda g_\lambda(k/n)} \right| \\ &\leq n \frac{1}{n^2} \sup_{0 \leq s \leq t+1} |g_\lambda''(s)| \\ &\leq \frac{C}{n}, \end{aligned}$$

for some constant $C > 0$. Moreover, the function $x \mapsto x/(1+x)$ is 1-Lipschitz on $[0, \infty)$. Thus, the first inequality of (6.6.9) is satisfied. The second one is clearly satisfied since $\tilde{U}_{k+1}^n - \tilde{U}_k^n \in \{0, -1\}$ and since g_λ is differentiable and g_λ' is bounded on $[0, t+1]$.

We now turn to (I) (b). We essentially apply Theorem 6.1.4 with $c = (\lambda-1)/(\lambda+1)$ and $A_n = \log \log n$. However, the rub is that the fluid limit $t \mapsto \max(2 - 2g_\lambda(t) - t, 0)$ of I^n reaches zero at some time t_0 which is the unique positive time at which $2 - 2g_\lambda(t_0) - t_0 = 0$ so that we can not apply it directly. To overcome this issue, we shall check that we are in position to apply Theorem 6.1.4 to the subtree $\mathcal{T}_{\lfloor (t_0 - \varepsilon)n \rfloor}(\mathbf{X}^n)$. This will indeed enable us to conclude since if V_1^n and V_2^n are two independent uniform random vertices of \mathcal{T}^n , then by Lemma 6.2.2,

$$\mathbb{P}(\mathbf{b}_{\tau'_n}(V_1^n) \leq (t_0 - \varepsilon)n) = \mathbb{P}(\mathbf{b}_{\tau'_n}(V_2^n) \leq (t_0 - \varepsilon)n) \geq 1 - \varepsilon + o(1) \quad \text{as } n \rightarrow \infty.$$

By Skorokhod's representation theorem, we may assume that the convergence (6.6.1) holds almost surely. Therefore, the right-hand side of (6.1.2) then holds almost surely on the event that $I_k^n > 0$ for all $k \leq (t_0 - \varepsilon)n$ thanks to (6.6.1). To check that Theorem 6.1.4 applies to the subtree $\mathcal{T}_{\lfloor (1-\varepsilon)\tau'_n \rfloor}(\mathbf{X}^n)$ with $c = (\lambda-1)/(\lambda+1)$, it thus suffices to show that for the same coupling, the left hand side of (6.1.2) holds almost surely on the event that $I_k^n > 0$ for all $k \leq (t_0 - \varepsilon)n$ with $A_n = \log \log n$. By Skorokhod's representation theorem it is enough to check that for all $\varepsilon > 0$, there exists $\delta > 0$ such that on the event that $I_k^n > 0$ for all $k \leq (t_0 - \varepsilon)n$,

$$\max_{\log \log n \leq k \leq \delta n} \left| \frac{I_k^n}{k} - \frac{\lambda-1}{\lambda+1} \right| \leq \varepsilon \quad (6.6.10)$$

with high probability as $n \rightarrow \infty$. In turn, it is enough to prove that for every $\varepsilon > 0$ we have

$$\max_{\log \log n \leq k \leq \varepsilon n} \left| \frac{\tilde{I}_k^n}{k} - \frac{\lambda-1}{\lambda+1} \right| \leq 2\varepsilon \quad (6.6.11)$$

with high probability as $n \rightarrow \infty$. Indeed, note that by definition of g_λ , we have $g_\lambda'(0) = -\lambda/(\lambda+1)$ so that as $t \rightarrow 0$,

$$2 - 2g_\lambda(t) - t = \frac{\lambda-1}{\lambda+1}t + o(t).$$

Let $\varepsilon > 0$. Take $\delta > 0$ small enough that for all $t \in [0, \delta]$, we have $|2 - 2g_\lambda(t) - t - ((\lambda-1)/(\lambda+1))t| \leq \varepsilon t$. On the event that $I_k^n > 0$ for all $k \leq (t_0 - \varepsilon)n$,

$$\max_{\log \log n \leq k \leq \delta n} \left| \frac{I_k^n}{k} - \frac{\lambda-1}{\lambda+1} \right| \leq \max_{\log \log n \leq k \leq \varepsilon n} \left| \frac{\tilde{I}_k^n}{k} - \frac{\lambda-1}{\lambda+1} \right| + \max_{\varepsilon n \leq k \leq \delta n} \left| \frac{\tilde{I}_k^n}{k} - \frac{\lambda-1}{\lambda+1} \right|.$$

The second term is upperbounded by 2ε with high probability as $n \rightarrow \infty$ thanks to (6.6.7). Thus, (6.6.11) implies (6.6.10). The upper bound (6.6.11) follows by combining the coupling (6.6.8) with the law of large numbers.

We now turn to (II) (a) and assume $\lambda_n \gg 1/n$. Let us first establish that if $\delta > 0$, then

$$\left(\frac{I^n_{\lfloor nt \rfloor}}{n}\right)_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (t)_{0 \leq t \leq 1} \quad \text{and} \quad \sup_{\log \log n \leq k \leq (1-\delta)n} \left| \frac{I_k^n}{k} - 1 \right| \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0, \quad (6.6.12)$$

where the first convergence holds for the topology of the uniform convergence. Let $\delta > 0$. Note that for all $k \leq (1-\delta)n$ we have $U_k^n \geq \delta n$. As a result, for all $\varepsilon > 0$, for n large enough, for all $k \leq (1-\delta)n$.

$$1 - \varepsilon \leq \frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \leq 1.$$

One can thus obtain the right-hand side of (6.6.12) by coupling I_k^n with a simple random walk (whose step is $+1$ with probability $1 - \varepsilon$). The left-hand side of (6.6.12) stems from the right-hand side and from the fact that I^n only makes ± 1 steps, so that for all $k \in \llbracket [(1-\delta)n], n \rrbracket$, we have $|I_k^n - I_{\lfloor (1-\delta)n \rfloor}^n| \leq n\delta$.

We now explain why (6.6.12) implies (6.6.3). By (6.6.12) and by definition of the Markov chain (I^n, U^n) , we have $U_n^n \leq n + 1 - I_n^n = o(n)$ with high probability, so that after time n , w.h.p. there are $o(n)$ remaining steps of $+1$ for I^n and all the other steps are -1 . Hence, for the topology of the uniform convergence,

$$\left(\frac{I^n_{\lfloor nt \rfloor}}{n}\right)_{1 \leq t \leq 2} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} (2-t)_{1 \leq t \leq 2},$$

which concludes the proof of (6.6.3).

We finally turn to (II) (b). As in the proof of (I) (a), the first two convergences of (6.6.4) are proved by applying Theorem 6.1.4 to the tree $\mathcal{T}_{\lfloor (2-\varepsilon)n \rfloor}(\mathbf{X}^n)$ for some $\varepsilon > 0$ small enough, with $c = 1$ and $A_n = \log \log n$. Indeed, by (6.6.12),

$$\sup_{\log \log n \leq k \leq n} \left| \frac{I_k^n}{k} - 1 \right| \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (6.6.13)$$

After applying Skorokhod's representation theorem, (6.6.13) implies the left hand side of (6.1.2) and the right hand side follows from (6.6.3).

The last convergence of (II) (b) is more delicate. The idea is to apply Theorem 6.1.4 to the tree $\mathcal{T}_n(\mathbf{X}^n)$ and then to show that the vertices added between times n and τ'_n , in a first approximation, do not affect the height.

More precisely, one can first see that by Theorem 6.1.4 the height of $\mathcal{T}_n(\mathbf{X}^n)$ divided by $\log n$ converges in probability towards e . Indeed, by (6.6.13), after applying Skorokhod's representation theorem, entails that (6.1.2) holds almost surely for $\mathcal{T}_n(\mathbf{X}^n)$ with $c = 1$ and $A_n = \log \log n$. Next, to understand how the height of \mathcal{T}^n behaves compared to $\mathcal{T}_n(\mathbf{X}^n)$ we need to estimate the quantity

$$h^n := \sum_{i=n}^{\tau'_n-1} \frac{1}{I_i^n} \mathbb{1}_{\{X_i^n=1\}}.$$

Indeed, as we have already seen multiple times, roughly speaking this quantity represents “the height” added between times n and τ'_n .

Let us prove that

$$\sup_{3n/2 \leq k \leq 2n} \frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \leq \frac{\lambda_n U_{\lfloor 3n/2 \rfloor}^n}{1 + \lambda_n U_{\lfloor 3n/2 \rfloor}^n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (6.6.14)$$

The inequality holds since $(U_k^n)_{k \geq 0}$ is non-increasing. For the above convergence, assume by contradiction that there exists $\varepsilon > 0$ and an increasing sequence of integers $(n_j)_{j \geq 1}$ such that for all $j \geq 1$,

$$\mathbb{P} \left(\frac{\lambda_{n_j} U_{\lfloor 3n_j/2 \rfloor}^{n_j}}{1 + \lambda_{n_j} U_{\lfloor 3n_j/2 \rfloor}^{n_j}} \geq \varepsilon \right) \geq \varepsilon.$$

Moreover, on this event, by monotonicity for every $k \in \llbracket n_j, \lfloor 3n_j/2 \rrbracket$ we have

$$\frac{\lambda_{n_j} U_k^{n_j}}{1 + \lambda_{n_j} U_k^{n_j}} \geq \varepsilon.$$

Then, using a coupling between $(I_k^{n_j})_{k \geq n}$ and a simple random walk $(\tilde{Y}_k^\varepsilon)_{k \geq n_j}$ starting at $I_{n_j}^{n_j}$ at time n_j whose step is $+1$ with probability ε and -1 with probability $1 - \varepsilon$, by the law of large numbers and since $I_n^n/n \rightarrow 1$ in probability as $n \rightarrow \infty$ by (6.6.3), one obtains that on the same event,

$$\frac{I_{\lfloor 3n_j/2 \rfloor}^{n_j}}{n_j} \geq \frac{\tilde{Y}_{\lfloor 3n_j/2 \rfloor}^\varepsilon}{n_j} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 1 + \frac{2\varepsilon - 1}{2},$$

which is absurd in view of (6.6.3). This proves (6.6.14).

Next, let us prove that

$$\frac{\mathbf{h}^n}{\log n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (6.6.15)$$

Define $\hat{X}_i^n = -X_{\tau'_n - i + 1}^n$ for all $i \in \llbracket 1, \tau'_n \rrbracket$ and $\hat{S}_i^n = I_{\tau'_n - i}^n = \hat{X}_1^n + \dots + \hat{X}_i^n$ for all $i \in \llbracket 0, \tau'_n \rrbracket$. Then

$$\mathbf{h}^n = \sum_{i=1}^{\tau'_n - n} \frac{1}{\hat{S}_i^n} \mathbb{1}_{\{\hat{X}_i^n = -1\}}.$$

By (6.6.3), to prove (6.6.15), it suffices to prove that

$$\frac{1}{\log n} \sum_{i=1}^{\lfloor n/2 \rfloor \wedge (\tau'_n - n)} \frac{1}{\hat{S}_i^n} \mathbb{1}_{\{\hat{X}_i^n = -1\}} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (6.6.16)$$

For all $\varepsilon > 0$, for all $n \geq 1$, define the event

$$\mathcal{A}_\varepsilon^n = \left\{ \sup_{3n/2 \leq k \leq 2n} \frac{\lambda_n U_k^n}{1 + \lambda_n U_k^n} \leq \varepsilon \right\}.$$

By (6.6.14), we know that

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\mathcal{A}_\varepsilon^n) = 1. \quad (6.6.17)$$

Besides, on the event $\mathcal{A}_\varepsilon^n$, one can couple the $(\hat{X}_i^n)_{1 \leq i \leq n/2 \wedge (\tau'_n - n)}$'s with i.i.d. random variables $(\xi_i^\varepsilon)_{i \geq 1}$ such that $\mathbb{P}(\xi_i^\varepsilon = -1) = \varepsilon$ and $\mathbb{P}(\xi_i^\varepsilon = 1) = 1 - \varepsilon$ so that for all $i \leq n/2 \wedge (\tau'_n - n)$, we have $\xi_i^\varepsilon \leq \hat{X}_i^n \leq 1$. For all $i \geq 0$, let $\hat{Y}_i^\varepsilon = \xi_1^\varepsilon + \dots + \xi_i^\varepsilon$. Then, for all $i \leq n/2 \wedge (\tau'_n - n)$, we have $\hat{Y}_i^\varepsilon \leq \hat{S}_i^n$. In particular, on the event $\mathcal{A}_\varepsilon^n \cap \{\forall i \in \llbracket 1, n/2 \rrbracket, \hat{Y}_i^\varepsilon > 0\}$,

$$\sum_{i=1}^{\lfloor n/2 \rfloor \wedge (\tau'_n - n)} \frac{1}{\hat{S}_i^n} \mathbb{1}_{\{\hat{X}_i^n = -1\}} \leq \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{\hat{Y}_i^\varepsilon} \mathbb{1}_{\{\xi_i^\varepsilon = -1\}}.$$

But, by the strong law of large numbers, we have

$$\mathbb{P}\left(\forall i \geq 0, \hat{Y}_i^\varepsilon \geq i/2\right) \xrightarrow[\varepsilon \rightarrow 0]{} 1. \quad (6.6.18)$$

For all $\varepsilon > 0$, on the event $\mathcal{A}_\varepsilon^n \cap \{\forall i \geq 0, \hat{Y}_i^\varepsilon \geq i/2\}$, we obtain that

$$\sum_{i=1}^{\lfloor n/2 \rfloor \wedge (\tau'_n - n)} \frac{1}{\hat{S}_i^n} \mathbb{1}_{\{\hat{X}_i^n = -1\}} \leq \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{2}{i} \mathbb{1}_{\{\xi_i^\varepsilon = -1\}}.$$

Besides,

$$\mathbb{E} \left[\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{2}{i} \mathbb{1}_{\{\xi_i^\varepsilon = -1\}} \right] = 2\varepsilon \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i}.$$

Therefore, we obtain that

$$\mathbb{E} \left[\mathbb{1}_{\mathcal{A}_\varepsilon^n \cap \{\forall i \geq 0, \hat{Y}_i^\varepsilon \geq i/2\}} \sum_{i=1}^{\lfloor n/2 \rfloor \wedge (\tau'_n - n)} \frac{1}{\hat{S}_i^n} \mathbb{1}_{\{\hat{X}_i^n = -1\}} \right] \leq 2\varepsilon \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i}.$$

After letting $\varepsilon \rightarrow 0$, using (6.6.17) and (6.6.18), we deduce (6.6.16).

It remains to show how (6.6.15) implies that $\text{Height}(\mathcal{T}^n) = \text{Height}(\mathcal{T}_n(\mathbf{X}^n)) + o(\log n)$. To do so we will use, once again, the coalescence construction and Bennett's inequality. By Algorithm 2, conditionally on τ'_n and $(I_i^n)_{0 \leq i \leq \tau'_n}$, the height of any new vertex added between time n and τ'_n is stochastically dominated by $\text{Height}(\mathcal{T}_n(\mathbf{X}^n)) + \sum_{i=n}^{\tau'_n-1} Y_i$ where $(Y_i^n)_{n \leq i \leq \tau'_n-1}$ are i.i.d Bernoulli random variables of respective parameters $(1/I_i^n \mathbb{1}_{\{X_i^n=1\}})_{n \leq i \leq \tau'_n-1}$. Using Bennett's inequality and a union bound, as in Sec. 6.4.3, for all $\varepsilon > 0$ we obtain that

$$\mathbb{P}(\text{Height}(\mathcal{T}^n) - \text{Height}(\mathcal{T}_n(\mathbf{X}^n)) \geq h^n + \varepsilon \log n \mid \tau'_n, (I_i^n)_{0 \leq i \leq \tau'_n}) \leq n \exp\left(-h^n g\left(\frac{\varepsilon \log n}{h^n}\right)\right)$$

where $g(u) = (u+1) \log(u+1) - u \sim u \log u$ as $u \rightarrow +\infty$. Since $h^n = o(\log n)$ in probability, the above upper bound goes to 0 when n goes to ∞ and thus $\text{Height}(\mathcal{T}^n)$ is of order $e \log n$. \square

6.7 Perspectives and extensions

We mention here some perspectives for future research in the direction of studying the impact of freezing in random graph models.

- (1) What is the number of ends of $\mathcal{T}_\infty(\mathbf{x})$?
- (2) It would be interesting to study the evolution of the sizes of the trees in the forests obtained by Algorithm 2 (in the particular case where $\mathbf{x}_n = 1$ for every $n \geq 1$ this corresponds to Kingman's coalescent), as well as the evolution of degree vertices (in the particular case where $\mathbf{x}_n = 1$ for every $n \geq 1$ there are nice connections with Pólya urns).
- (3) How small can be the typical height of $\mathcal{T}_n(\mathbf{x})$ when $\tau(\mathbf{x}) \geq n$? We conjecture that for any sequence $(\mathbf{x}^n)_{n \geq 1}$ such that $\tau(\mathbf{x}^n) \geq n$ for every $n \geq 1$, for every $\varepsilon > 0$ we have

$$\mathbb{P}(\text{Height}(\mathcal{T}_n(\mathbf{x}^n)) \geq (e - \varepsilon) \log(n)) \xrightarrow[n \rightarrow \infty]{} 1.$$

- (4) A natural question is to obtain a limit theorem for the height of the epidemic tree \mathcal{T}^n in the setting of Theorem 6.6.2 (I) (b) (see Remark 6.6.3).
- (5) It would be very interesting to extend our results to other attachment mechanisms, such as preferential attachment, where new vertices attach to existing vertices proportional to their degree.
- (6) Similarly one may wonder what happens when the frozen vertices are not chosen uniformly, notably when vertices with high degrees are more or less likely to freeze.
- (7) An important question arising in the context of growing real-world networks or in the context of infection tracing is how to find the root, or patient zero [SDP+20]. This is a very active area of research, sometimes called *network archeology* [NK11]. Such questions have been for instance considered for uniform attachment and preferential attachment trees [Hai70, BDL17, LP19], as well as Bienaymé trees [BDG22].

Chapter 7

Uniform attachment with freezing: Scaling limits

Ce chapitre est issu d'un travail [BBRKK23b] avec Étienne Bellin, Arthur Blanc-Renaudie et Igor Kortchemski, à paraître dans les Annales de l'Institut Henri Poincaré Probabilités et Statistiques.

Abstract. We investigate scaling limits of trees built by uniform attachment with freezing, which is a variant of the classical model of random recursive trees introduced in a companion paper. Here vertices are allowed to freeze, and arriving vertices cannot be attached to already frozen ones. We identify a phase transition when the number of non-frozen vertices roughly evolves as the total number of vertices to a given power. In particular, we observe a critical regime where the scaling limit is a random compact real tree, closely related to a time non-homogenous Kingman coalescent process identified by Aldous. Interestingly, in this critical regime, a condensation phenomenon can occur.

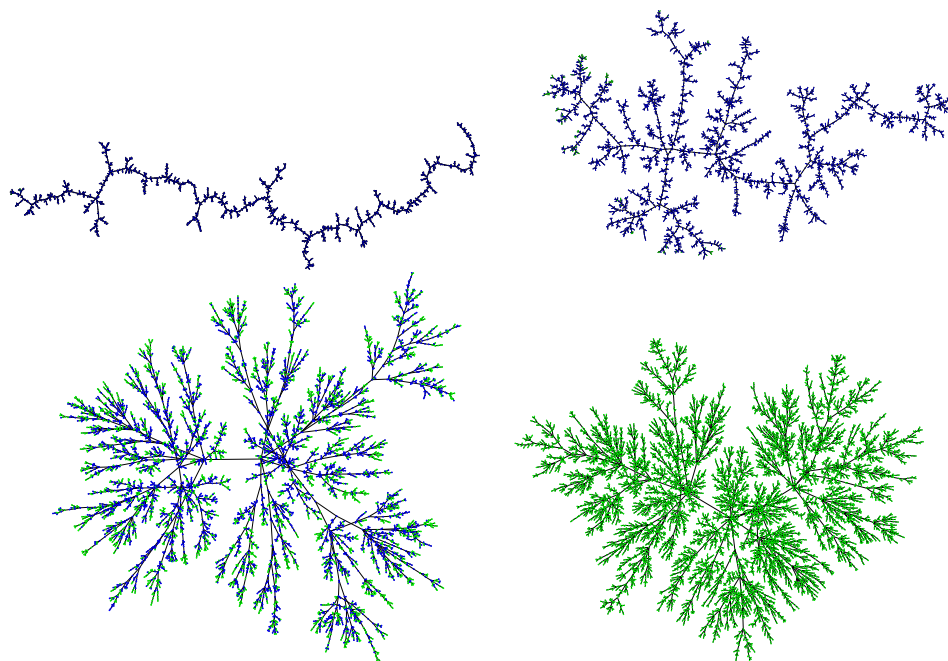


Figure 7.1 – Simulations of the model of uniform attachment with freezing, when the number of active (i.e. non-frozen) vertices roughly evolves as the total number of vertices to the power α ; top: $\alpha = 0.2, 0.5$, bottom: $\alpha = 0.8, 1$. Frozen vertices are blue; active vertices are green.

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7.1 Introduction

We are interested in the geometry of uniform attachment trees with freezing, which we introduced in Chapter 6. In the classical model of uniform recursive trees (sometimes also called uniform attachment trees), trees are constructed recursively starting with one single vertex, and successively attaching new vertices to a previous existing vertex, chosen uniformly at random. In the model of uniform attachment trees with freezing, existing vertices can freeze and new vertices cannot be attached to frozen vertices.

The motivations in introducing this model were multiple. In the context of real-world networks such mechanisms naturally appear: for instance, on social media (such as Twitter) a user can choose to set their account to “private” which prevents strangers from “following” them; also performing an infection-tracing of an SIR epidemics falls within this framework (see Chapter 6). Second, this model presents interesting mathematical features: for instance, it extends both uniform recursive trees and uniform plane trees, and in Chapter 6 universal bounds on the height have been obtained.

The main purpose of this paper is to study scaling limits of uniform attachment trees with freezing, in the specific regime where the number of non-frozen vertices roughly evolves as the total number of vertices to a given power.

Uniform attachment with freezing. Let us recall the model, which is parametrized by a deterministic sequence $\mathbf{x} = (x_i)_{i \geq 1}$ of elements of $\{-1, +1\}$. Starting from a unique active vertex, we recursively build random trees by reading the elements of the sequence one after the other, by applying a “freezing” step when reading -1 (which amounts to freezing an active vertex chosen uniformly at random) and a “uniform attachment” step when reading $+1$ (which amounts to attaching a new vertex to an active vertex chosen uniformly at random). For every $n \geq 1$ we denote by $\mathcal{T}_n(\mathbf{x})$ the random tree recursively built in this fashion after reading the first n elements of \mathbf{x} (see Sec. 6.2.1 for a precise definition and Fig. 6.3 for an example).

Let us comment on our choice of parametrization. It would have been possible to define the model starting with a random sequence \mathbf{x} , but the choice of a deterministic sequence defines a more general model. Our results can then be applied in the context of a random sequence.

It is interesting to note that this model encompasses the two classical models of random recursive trees (when $x_i = 1$ for every $i \geq 1$) and random uniform plane trees (when the sequence $\mathbf{X} = (X_i)_{i \geq 1}$ is a sequence of non constant i.i.d. uniform random variables on $\{-1, +1\}$, see Theorem 6.1.2 in Chapter 6).

Scaling limits. In order to make explicit the regime we are interested in, we need to introduce some notation. Given a sequence $\mathbf{x} = (x_i)_{i \geq 1}$ of elements of $\{-1, +1\}$, we set $S_0(\mathbf{x}) := 1$ and for every $n \geq 1$

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i; \quad \tau(\mathbf{x}) := \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}. \quad (7.1.1)$$

Observe that $S_n(\mathbf{x})$ represents the number of active vertices of $\mathcal{T}_n(\mathbf{x})$. In particular, $\tau(\mathbf{x})$, if it is finite, is the first time when all the vertices are frozen, so that the tree does not evolve anymore. In order to consider large trees, we thus need this time to be large. For this reason, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$ (if $\tau(\mathbf{x}) = \infty$ we can of course take $\mathbf{x}^n = \mathbf{x}$).

We investigate scaling limits for a particular class of sequences $(\mathbf{x}^n)_{n \geq 1}$ where the number of active vertices, that is $S_n(\mathbf{x}^n)$, roughly evolves as the total number of vertices to the power α , for fixed $\alpha \in (0, 1]$. More precisely, we require $(n^\alpha/S_{nt}(\mathbf{x}^n))_{0 \leq t \leq 1}$ to converge in $L^1([0, 1])$ towards some function $1/f$ together with an L^2 -type condition (see Sec. 7.2 for precise conditions). It turns out that the geometry of $\mathcal{T}_n(\mathbf{x}^n)$ can be understood within this class, through scaling limits for which we identify different regimes according to the value of α . In order to keep this introduction at reasonable length, we postpone the precise statements to Sec. 7.2, and summarize here the essence of our results.

- *Subcritical regime* $\alpha \in (0, 1/2)$: $\mathcal{T}_n(\mathbf{x}^n)/n^{1-\alpha}$ converges in distribution to a deterministic line-segment for the Gromov–Hausdorff–Prokhorov topology (Theorem 7.2.2);
- *critical regime* $\alpha = 1/2$: $\mathcal{T}_n(\mathbf{x}^n)/\sqrt{n}$ converges in distribution to a random compact measured real tree for the Gromov–Hausdorff–Prokhorov topology, with a condensation phenomenon when $1/f^2$ is integrable near 0 (Theorem 7.2.3);
- *supercritical regime* $\alpha \in (1/2, 1)$: $\mathcal{T}_n(\mathbf{x}^n)$ looks like a “star” with branches of random lengths of order $n^{1-\alpha}$ (Theorem 7.2.6).

Here the terminology “ X looks like Y ” means that the joint laws of the graph-distances of k points chosen in an i.i.d. way from X and from Y are “close”.

Motivation. Let us give some further motivation:

- it is natural to delve into the mathematical aspects of freezing’s impacts in dynamically-built random graph models. Our goal is to explore whether a regime exists where scaling limits can be established for the GHP topology (for recursive trees, without freezing, there are notably no scaling limits for the GHP topology);
- The choice to focus on the quantity S_n is natural, since it represents the number of active vertices at time n . Also, in the context of the study of the so-called “infection tree” of a stochastic SIR dynamics, in which the vertices are individuals and where edges connect two individuals if one has infected the other, the quantity S_n represents the number of infected vertices at time n ;
- our assumption that $S_n(\mathbf{x}^n)$ roughly evolves as the total number of vertices to the power α is motivated by the following three compelling reasons: it includes the case where $(S_k(\mathbf{x}^n))_{0 \leq k \leq 2n+1}$ is an excursion of a simple random walk of length $2n+1$ (in this case $\alpha = 1/2$), when $\alpha = 1/2$ the scaling limit closely related to a time non-homogeneous Kingman coalescent process identified by Aldous, and finally it involves a one-parameter family exhibiting a phase transition;
- one of the motivations to consider a “triangular array” setting, where we consider $S_n(\mathbf{x}^n)$ instead of $S_n(\mathbf{x})$, due to the need to accommodate cases involving excursions of a simple random walk with lengths dependent on n (see Sec. 7.5.7);

- the assumption that $(n^\alpha/S_{nt}(\mathbf{x}^n))_{0 \leq t \leq 1}$ converges, in a certain sense, towards some function $1/f$ is motivated by its applicability in cases where \mathbf{x}^n is an excursion of a simple random walk (in this case $\alpha = 1/2$ and f is a normalized Brownian excursion). Additionally, it exhibits a universal behavior.

Main ideas. Our proof is based on an alternative construction of uniform attachment trees with freezing, which we introduced in Chapter 6: by reversing time, uniform attachment trees with freezing can be built using a growth-coalescent process of rooted forests. This extends the well-known connection between recursive trees and the Kingman coalescent which first appeared in [DR76] (see [Dev87, Sec. 6], [Pit94, Sec. 3], [AB15, Sec. 2.2], [ABE18, Esl21, Esl22] for applications). This construction enables us to control distances using classical concentration tools combined with the chaining method. This method is an important technique in concentration theory [Tal05] whose goal is to estimate the maximum of a given function on a given space using a sequence of increasing subspaces. It has found many applications in the study of random metric spaces, see e.g. [Ald91, ADGO17, CH17, Sén19, BR23, BR22]. Moreover, in the case $\alpha = 1/2$, the continuous analogue of the alternative construction can be viewed as a time-change of a time non-homogeneous Kingman coalescent process introduced by Aldous.

Plan of the chapter. We start in Sec.7.2 by stating our main results concerning scaling limits in various regimes. Sec. 6.2 then recalls an alternative construction of \mathcal{T}_n , based on time-reversal, through a growth-coalescent process of rooted forests, which we have introduced in Chapter 6. We then obtain in Sec. 7.3 general estimates for distances in \mathcal{T}_n , and introduce in particular another distance based on coalescence times. This enables us to establish our scaling limit results, first in the non-critical cases $\alpha \neq 1/2$ in Section 7.4 and then in the critical case $\alpha = 1/2$ in Section 7.5 (in particular Sec. 7.5.1 defines the limiting object and gives a more detailed description of the proof approach). Finally, Section 7.6 contains background on different topologies that we use and Section 7.7 contains a tightness criterion.

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7.2 Scaling limits: main results

We consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$ (if $\tau(\mathbf{x}) = \infty$ we can of course take $\mathbf{x}^n = \mathbf{x}$ for every $n \geq 1$), and set $\mathcal{T}_n = \mathcal{T}_n(\mathbf{x}^n)$. We keep the notation introduced in Sec. 7.1, recall in particular (7.1.1). To simplify the notation, we set $S_k^n = S_k(\mathbf{x}^n)$ for every $0 \leq k \leq n$ and set $S_t^n = S_{\lfloor nt \rfloor}^n$ for all real number $t \in [0, n+1)$. We also set

$$h_n^+ = \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}.$$

The reason why the quantity $\frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}$ appears is essentially that it represents the probability that at step i the degree of a given active vertex changes.

We study the geometry of \mathcal{T}_n under certain assumptions, which we now state. Given $\alpha \in (0, 1)$ and a measurable function $f : [0, 1] \rightarrow \mathbb{R}_+$ such that $1/f \in L^1([0, 1]) \cap L_{\text{loc}}^2((0, 1))$, we consider the following set of conditions:

(Lim $_{1/f}^\alpha$) $(n^\alpha/S_{nt}^n)_{0 \leq t \leq 1}$ converges in $L^1([0, 1])$ towards $1/f$;

(Lim $_{1/f^2}^\alpha$) For all $0 < a < b < 1$ we have $\lim_{n \rightarrow \infty} \int_a^b \left(\frac{n^\alpha}{S_{nt}^n}\right)^2 dt = \int_a^b \frac{dt}{f(t)^2}$.

Notice that given (Lim $_{1/f}^\alpha$), the condition (Lim $_{1/f^2}^\alpha$) is equivalent to the convergence of $((n^\alpha/S_{nt}^n)^2)_{t \in [0, 1]}$ towards $1/f^2$ in $L_{\text{loc}}^1((0, 1))$. Roughly speaking, the condition (Lim $_{1/f}^\alpha$) is related to the convergence of

the heights of vertices, while the condition $(\text{Lim}_{1/f^2}^\alpha)$ is related to the behavior of distances between pairs of vertices. Besides, one can show that when $\alpha \geq 1/2$, the condition $(\text{Lim}_{1/f^2}^\alpha)$ implies that for all $0 < a < b < 1$, the minimum $\min_{an \leq k \leq bn} S_k^n$ tends to ∞ as $n \rightarrow \infty$. It is for instance a consequence of Equation (7.3.5) in the proof of Lemma 7.3.1.

We first identify the height of \mathcal{T}_n under these assumptions:

Theorem 7.2.1. *Assume that $\alpha \in (0, 1)$ and that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$. Then for all $p \geq 1$:*

$$\frac{\text{Height}(\mathcal{T}_n)}{n^{1-\alpha}} \xrightarrow[n \rightarrow \infty]{(\mathbb{L}^p)} \int_0^1 \frac{dt}{2f(t)}.$$

This follows from Theorem 6.1.3 (III), in virtue of which the convergence $\text{Height}(\mathcal{T}_n)/h_n^+ \rightarrow 1$ holds in \mathbb{L}^p , combined with estimating the asymptotic behavior of h_n^+ .

7.2.1 Subcritical regime $\alpha \in (0, 1/2)$

Theorem 7.2.2. *Assume that $\alpha \in (0, 1/2)$ and that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$. Then the convergence*

$$\frac{1}{n^{1-\alpha}} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \left[0, \int_0^1 \frac{dt}{2f(t)} \right]$$

holds in distribution for the Gromov–Hausdorff–Prokhorov topology, where the segment $\left[0, \int_0^1 1/(2f(t))dt \right]$ is equipped with the image measure of the uniform measure on $[0, 1]$ by the mapping $x \mapsto \int_0^x \frac{dt}{2f(t)}$.

We refer to Sec. 7.6 for details concerning the Gromov–Hausdorff–Prokhorov topology. See Theorem 7.4.2 for the case where $\|S^n\|_\infty$ is bounded (which roughly speaking may be viewed as the case $\alpha = 0$).

7.2.2 Critical regime $\alpha = 1/2$

Theorem 7.2.3. *Assume that $\alpha = 1/2$, that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$ and that $\max_{1 \leq k \leq n} S_k^n = \mathcal{O}(\sqrt{n})$. Then the convergence*

$$\frac{1}{\sqrt{n}} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}(f)$$

holds in distribution for the Gromov–Hausdorff–Prokhorov topology, where $\mathcal{T}(f)$ is a random measured compact real tree.

The proof is divided in two steps: first the identification of the limit through finite-dimensional distributions by constructing a continuous analog of Algorithm 2, second establishing tightness for the Gromov–Hausdorff–Prokhorov topology by relying on the chaining method. Let us mention that the convergence of finite-dimensional distributions holds without the assumption $\max_{1 \leq k \leq n} S_k^n = \mathcal{O}(\sqrt{n})$, which is only used in the proof of tightness. It is unclear whether this assumption can be removed.

We show that this tree is similar in many aspects to the Brownian tree introduced by Aldous [Ald91] (see Sec. 7.5 for background concerning real trees). Specifically, we establish the following properties concerning the geometry of $\mathcal{T}(f)$:

Theorem 7.2.4. *The following assertions hold almost surely.*

- (1) *The mass measure on $\mathcal{T}(f)$ has full support and gives full measure to the set of its leaves.*
- (2) *$\mathcal{T}(f)$ is binary if and only if $\int_0^{1/2} \frac{1}{f^2} = \infty$. When $\int_0^{1/2} \frac{1}{f^2} < \infty$, the root has infinite degree and all the other branchpoints are binary.*

Observe that, quite surprisingly, a condensation phenomenon may occur; in particular the topological class of $\mathcal{T}(f)$ depends on the integrability of $1/f^2$ near 0. This behavior is new among random trees with “large” degrees, such as stable trees [LGLJ98, DLG02] (where there are infinitely many vertices with “large” degrees, with existence of GHP scaling limits), non-generic Bienaymé trees [JS11, Jan12, Kor15] and Cauchy–Bienaymé-trees [KR19] (where there is a unique vertex with macroscopic degree, but without existence of nontrivial GHP scaling limits), simply generated trees with superexponential

branching weights [JJS11] (where there is a unique vertex with macroscopic degree, but without existence of nontrivial GHP scaling limits).

Let us mention that it is not too hard to check that almost surely the Minkowski, packing and Hausdorff dimensions of $\mathcal{T}(f)$ are at least equal to 2. However, the upper dimensions depend on the fine behavior of f near 0, and will be investigated in future work.

It turns out that by appropriately choosing f one can recover the Brownian CRT.

Corollary 7.2.5. *If e is a normalized Brownian excursion then $\mathcal{T}(e)$ has the same law as the Brownian CRT.*

Let us comment on the fact that by $\mathcal{T}(e)$ we mean a random variable whose law is characterized in the following ‘‘annealed’’ way:

$$\mathbb{E}[F(\mathcal{T}(e))] = \int \mathbb{E}[F(\mathcal{T}(f))] \mathbb{P}_e(df),$$

for every nonnegative functional F , where \mathbb{P}_e is the law of e . Corollary 7.2.5 comes from Theorem 7.2.3 applied with a carefully chosen random sequence $(\mathbf{x}^n)_{n \geq 1}$ (see Sec. 7.5.7 for details). This result can also be seen as a consequence of [Ald98, Theorem 2].

7.2.3 Supercritical regime $\alpha \in (1/2, 1)$

We denote by d^n the graph distance on the vertices of \mathcal{T}_n .

Theorem 7.2.6. *Assume $\alpha \in (1/2, 1)$ and $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$. For every $k \geq 2$, conditionally given \mathcal{T}_n , let $V_1^n, V_2^n, \dots, V_k^n$ be independent uniform vertices in \mathcal{T}_n and let U_1, \dots, U_n be i.i.d. uniform random variables in $[0, 1]$. Also let V_0^n be the root of \mathcal{T}_n and let $U_0 = 0$. Then the following convergence holds in distribution.*

$$\left(\frac{1}{n^{1-\alpha}} \cdot d^n(V_i^n, V_j^n) \right)_{0 \leq i \neq j \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_0^{U_i} \frac{dt}{2f(t)} + \int_0^{U_j} \frac{dt}{2f(t)} \right)_{0 \leq i \neq j \leq k}$$

where d^n denotes the graph distance in the tree \mathcal{T}_n .

Intuitively speaking, this tells us that the tree \mathcal{T}_n roughly looks like a ‘‘star’’ with branches of length $n^{1-\alpha} \int_0^1 1/(2f(t))dt$. It is worth noting that this case gives a natural example of trees growing polynomially in their size, without scaling limits. Indeed, $(\mathcal{T}_n/n^{1-\alpha})_{n \geq 1}$ is not tight for the GHP topology (since for $\varepsilon > 0$ the number of balls of radius ε needed to cover this space tends to infinity in probability). However, for another topology closely related to the Gromov–Prokhorov topology (see [ET22, Jan21]), $\mathcal{T}_n/n^{1-\alpha}$ converges to the so-called long dendron Υ_ν (we use the notation of [Jan21, Example 3.12]) with ν being the image measure of the uniform measure on $[0, 1]$ by the mapping $x \mapsto \int_0^x 1/(2f(t))dt$.

7.3 Distances in uniform attachment trees with freezing

We recall Table 6.1 from the previous chapter which contains the notation that will be used in the future sections. In what follows, we will denote by $[n]$ the set of integers $\{1, \dots, n\}$.

We shall now study the geometry of uniform attachment trees with freezing. It will be convenient to work with $\mathcal{T}^n(\mathbf{x})$ as built using Algorithm 2. Recall Theorem 6.2.4: the only difference between $\mathcal{T}_n(\mathbf{x})$ and $\mathcal{T}^n(\mathbf{x})$ is that all the active vertices of $\mathcal{T}^n(\mathbf{x})$ are labelled $a_1, \dots, a_{S_n(\mathbf{x})}$, while all the active vertices of $\mathcal{T}_n(\mathbf{x})$ are labelled a . In particular the graph structure of both trees is the same in law, so it is equivalent to establish our main results with $\mathcal{T}_n(\mathbf{x})$ replaced with $\mathcal{T}^n(\mathbf{x})$.

Table 7.1 – Table of the main notation and symbols introduced in Section 7.3 and used later.

d^n	graph distance on the set of all vertices $\mathbb{V}_n(\mathbf{x})$ of $\mathcal{T}^n(\mathbf{x})$ modified distance on $\mathbb{V}_n(\mathbf{x})$ defined for $u, v \in \mathbb{V}_n(\mathbf{x})$ by
D^n	$D^n(u, v) = \frac{1}{2} \sum_{\mathbf{c}_n(u, v) \leq i \leq \mathbf{b}_n(u)} \frac{1}{S_i^n} + \frac{1}{2} \sum_{\mathbf{c}_n(u, v) \leq i \leq \mathbf{b}_n(v)} \frac{1}{S_i^n}$

Recall from (6.2.1) the definition of $\mathbb{V}_n(\mathbf{x})$, which is a deterministic set representing the labels of the vertices of $\mathcal{T}^n(\mathbf{x})$, and that by a slight abuse of notation we view elements of $\mathbb{V}_n(\mathbf{x})$ as vertices of $\mathcal{T}^n(\mathbf{x})$. Also recall Bennett's inequality from Proposition 6.4.1.

7.3.1 Deterministic estimates

From now on, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$.

To simplify notation, in the sequel, we drop the dependence in \mathbf{x}^n , so for instance we will write \mathcal{T}^n for the tree $\mathcal{T}^n(\mathbf{x}^n)$ built from \mathbf{x}^n using Algorithm 2. We also set $S_k^n := S_k(\mathbf{x}^n)$ for all $k \leq n$, and define for every $1 \leq a \leq b < \tau(\mathbf{x}^n)$:

$$h_n(a, b) := \sum_{i=a}^b \frac{1}{S_i^n}, \quad h_n := h_n(1, n), \quad h_n^+(a, b) := \sum_{i=a}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}, \quad h_n^+ := h_n^+(1, n). \quad (7.3.1)$$

Our goal is now to study the geometry of \mathcal{T}^n . To this end, it is useful to estimate h_n^+ , since it is connected with the evolution of distances in the growth-coalescent construction (see Lemma 6.2.1). The following result will allow to replace h_n^+ with h_n up to a factor 2 (this comes from the fact that in the regime under consideration there are roughly as much $+1$'s as -1 's and that their locations are roughly uniformly distributed), and in turn the asymptotic behavior of h_n can be related to the integral of $1/f$ by $(\text{Lim}_{1/f}^\alpha)$.

Lemma 7.3.1. *Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$ are in force. Then the following assertions hold:*

(i) *For all $a \in (0, 1]$, we have $S_{na}^n = o(n)$.*

(ii) *The following convergences*

$$\frac{2}{n} \sum_{i=1}^n \mathbb{1}_{x_i^n=1} \delta_{i/n} \xrightarrow{n \rightarrow \infty} \text{Leb}_{[0,1]} \quad \text{and} \quad \frac{2}{n} \sum_{i=1}^n \mathbb{1}_{x_i^n=-1} \delta_{i/n} \xrightarrow{n \rightarrow \infty} \text{Leb}_{[0,1]}$$

hold weakly, where $\text{Leb}_{[0,1]}$ is the Lebesgue measure on $[0, 1]$.

(iii)

$$2n^{\alpha-1} h_n^+ \sim n^{\alpha-1} h_n \sim \int_0^1 \frac{1}{f(t)} dt.$$

(iv)

$$\frac{1}{n^{1-\alpha}} \max_{1 \leq a \leq b \leq n} \left| h_n^+(a, b) - \frac{1}{2} h_n(a, b) \right| \xrightarrow{n \rightarrow \infty} 0.$$

(v) *For all $\varepsilon > 0$,*

$$\max_{\varepsilon n \leq a \leq b \leq (1-\varepsilon)n} \left| \frac{1}{n} \sum_{i=a}^b \frac{n^{2\alpha}}{S_i^n (S_i^n - 1)} \mathbb{1}_{x_i^n=1} - \int_{a/n}^{b/n} \frac{dt}{2f(t)^2} \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. For (i), let $a \in (0, 1]$. We assume by contradiction that there exists $\varepsilon > 0$ such that $S_{an}^n \geq \varepsilon n$ for infinitely many n . Then, since the steps are ± 1 , we have for infinitely many n , for all $k \leq (\varepsilon \wedge a)n$,

$$S_{an-k}^n \geq S_{an}^n - k \geq \varepsilon n - k.$$

Hence, by summing over $(\varepsilon \wedge a)n/2 \leq k \leq (\varepsilon \wedge a)n$, for infinitely many n we have

$$\sum_{(\varepsilon \wedge a)n/2 \leq k \leq (\varepsilon \wedge a)n} \frac{1}{S_{an-k}^n} = O(1),$$

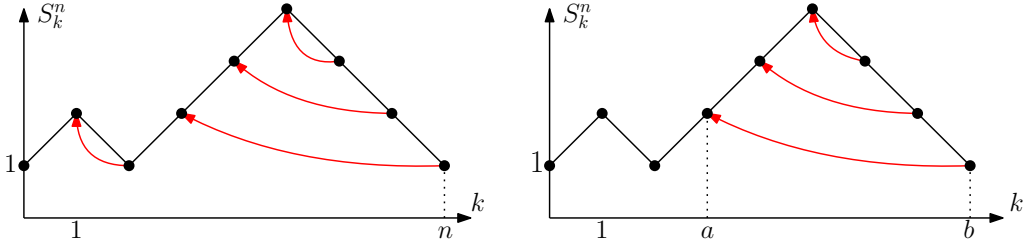


Figure 7.2 – Illustration of the pairings which are used in the proof of the assertions (iii), on the left, and (iv),(v), on the right, of Lemma 7.3.1.

which is absurd by $(\text{Lim}_{1/f}^\alpha)$. Thus (i) holds.

For $0 \leq a < b \leq 1$,

$$\#\{[an] + 1 \leq i \leq [bn] : x_i = 1\} = \frac{[bn] - [an] + S_{[bn]}^n - S_{[an]}^n}{2} \underset{n \rightarrow \infty}{\sim} \frac{(b-a)n}{2} \quad (7.3.2)$$

since $S_{[an]}^n = o(n)$ and $S_{[bn]}^n = o(n)$, hence (ii) for the +1 steps by the Portmanteau theorem. The same reasoning works for the -1 steps.

Let us show (iii). The second equivalence is obvious from $(\text{Lim}_{1/f}^\alpha)$. So, we want to show that

$$2n^{\alpha-1} h_n^+ \underset{n \rightarrow \infty}{\sim} n^{\alpha-1} h_n.$$

But we have

$$h_n^+ + \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1} = h_n.$$

Therefore, it is enough to show that

$$h_n^+ \underset{n \rightarrow \infty}{\sim} \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1}.$$

However, by mapping each -1 step $x_i^n = -1$ to the last time $j < i$ such that $S_{j-1}^n = S_i^n$, which is thus such that $x_j^n = +1$ (see Figure 7.2), we see that

$$\begin{aligned} h_n^+ &= \sum_{i=1}^n \frac{1}{S_i^n + 1} \mathbb{1}_{x_i^n = -1} + \sum_{s=2}^n \frac{1}{s} = \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1} - \sum_{i=1}^n \frac{1}{(S_i^n + 1)S_i^n} \mathbb{1}_{x_i^n = -1} + \log(S_n^n) + O(1) \\ &= \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1} + o(n^{1-\alpha}), \end{aligned}$$

where the last equality comes from the fact that $S_n^n \leq n + 1$ and from the convergence

$$\frac{1}{n} \sum_{i=1}^n \frac{n^\alpha}{(S_i^n + 1)S_i^n} \xrightarrow{n \rightarrow \infty} 0. \quad (7.3.3)$$

The above convergence can be shown as follows: if U is a uniform random variable in $[0, 1]$, then the sequence $(n^\alpha / (S_{U_n}^n (S_{U_n}^n + 1)))_{n \geq 1}$ is uniformly integrable since it is upper bounded by $(n^\alpha / S_{U_n}^n)_{n \geq 1}$, which converges in \mathbb{L}^1 by $(\text{Lim}_{1/f}^\alpha)$, and $n^\alpha / (S_{U_n}^n (S_{U_n}^n + 1))$ converges in probability towards zero, again by $(\text{Lim}_{1/f}^\alpha)$. This proves (iii).

For (iv), let $1 \leq a \leq b \leq n$. We use the same idea: by mapping each -1 step $x_i^n = -1$ for $a \leq i \leq b$ to the last time j for $a \leq j < i$ such that $S_{j-1}^n = S_i^n$ when it exists (see Figure 7.2), one can see that

$$h_n^+(a, b) = \sum_{i=a}^b \frac{1}{S_i^n + 1} \mathbb{1}_{x_i^n = -1} - \sum_{s=\min_{a \leq j \leq b} S_j^n}^{S_a^n - 1} \frac{1}{s + 1} + \sum_{s=\min_{a \leq j \leq b} S_j^n + 1}^{S_b^n} \frac{1}{s},$$

so that for all $1 \leq a \leq b \leq n$

$$\left| h_n^+(a, b) - \sum_{i=a}^b \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1} \right| \leq \sum_{i=1}^n \frac{1}{S_i^n (S_i^n + 1)} \mathbb{1}_{x_i^n = -1} + 2 \sum_{s=1}^{n+1} \frac{1}{s} = o(n^{1-\alpha}),$$

where the $o(n^{1-\alpha})$ is uniform in $1 \leq a \leq b \leq n$ and comes from (7.3.3). But

$$h_n^+(a, b) - \frac{1}{2} h_n(a, b) = \frac{1}{2} \left(h_n^+(a, b) - \sum_{i=a}^b \frac{1}{S_i^n} \mathbb{1}_{x_i^n = -1} \right),$$

hence (iv).

For the last point (v), let $\varepsilon \in (0, 1)$. We use the same trick, so that for all $1 \leq a \leq b \leq n$,

$$\sum_{i=a}^b \frac{1}{S_i^n (S_i^n - 1)} \mathbb{1}_{x_i^n = 1} = \sum_{i=a}^b \frac{1}{(S_i^n + 1) S_i^n} \mathbb{1}_{x_i^n = -1} + O\left(\frac{1}{\min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n}\right), \quad (7.3.4)$$

where the $O(1/\min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n)$ is uniform in $\varepsilon n \leq a \leq b \leq (1-\varepsilon)n$.

Moreover, note that the condition $(\text{Lim}_{1/f^2}^\alpha)$ implies that

$$\frac{1}{\min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n} = o(n^{1-2\alpha}). \quad (7.3.5)$$

Indeed, let $(k_n)_{n \geq 0}$ be a sequence of integers such that for all $n \geq 0$, we have $S_{k_n}^n = \min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n$. Let $\eta_n > 0$ such that $\eta_n \rightarrow 0$ and $S_{(1-\varepsilon)n}^n = o(n\eta_n)$ (such a sequence exists thanks to point (i)). Then we have as $n \rightarrow \infty$,

$$\int_{k_n/n}^{k_n/n + \eta_n} \left(\frac{n^\alpha}{S_{nt}^n} \right)^2 dt \geq \frac{1}{n} \sum_{j=1}^{\lfloor \eta_n n \rfloor - 1} \frac{n^{2\alpha}}{(\min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n + j)^2} \geq (1 + o(1)) \frac{n^{2\alpha-1}}{\min_{\varepsilon n \leq k \leq (1-\varepsilon)n} S_k^n + 1}.$$

By $(\text{Lim}_{1/f^2}^\alpha)$, the integral on the left tends to zero, hence (7.3.5).

Besides, using $(\text{Lim}_{1/f^2}^\alpha)$ and the same idea as for (7.3.3), one can check that

$$\frac{1}{n} \sum_{i=a}^b \frac{n^{2\alpha}}{S_i^n (S_i^n - 1)} \mathbb{1}_{x_i^n = 1} - \frac{1}{n} \sum_{i=a}^b \frac{n^{2\alpha}}{S_i^n (S_i^n + 1)} \mathbb{1}_{x_i^n = -1} = o(1). \quad (7.3.6)$$

Thus, combining (7.3.4), (7.3.5) and (7.3.6),

$$\begin{aligned} & \max_{\varepsilon n \leq a \leq b \leq (1-\varepsilon)n} \left| \frac{1}{n} \sum_{i=a}^b \frac{n^{2\alpha}}{S_i^n (S_i^n - 1)} \mathbb{1}_{x_i^n = 1} - \int_{a/n}^{b/n} \frac{dt}{2f(t)^2} \right| \\ & \leq o(1) + \max_{\varepsilon n \leq a \leq b \leq (1-\varepsilon)n} \left| \frac{1}{n} \sum_{i=a}^b \frac{n^{2\alpha}}{2S_i^n (S_i^n + 1)} - \int_{a/n}^{b/n} \frac{dt}{2f(t)^2} \right| \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the convergence is due to $(\text{Lim}_{1/f^2}^\alpha)$. \square

7.3.2 An alternative distance using coalescence times

Recall that d^n denotes the graph distance on the vertices \mathbb{V}_n of the tree \mathcal{T}_n . In this section, we introduce a new distance on \mathbb{V}_n which will be more convenient to study, based on the coalescence time between vertices. For every $n \in \mathbb{N}$, and $u, v \in \mathcal{T}^n$, set

$$D^n(u, v) := \frac{1}{2} \sum_{c_n(u, v) \leq i \leq b_n(u)} \frac{1}{S_i^n} + \frac{1}{2} \sum_{c_n(u, v) \leq i \leq b_n(v)} \frac{1}{S_i^n} = \frac{1}{2} h_n(c_n(u, v), b_n(u)) + \frac{1}{2} h_n(c_n(u, v), b_n(v)),$$

where we use the notation h introduced in (7.3.1).

Proposition 7.3.2. *Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$ are in force. Then for every $\varepsilon > 0$,*

$$\mathbb{P}\left(\max_{u, v \in \mathbb{V}_n} |d^n(u, v) - D^n(u, v)| > \varepsilon n^{1-\alpha}\right) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. Recall that, for $v \in \mathbb{V}_n$ and $0 \leq i \leq \mathbf{b}_n(v)$, $\mathbf{H}_i^n(v)$ is the height of vertex v in \mathcal{F}_i^n . First note that by the definition of Algorithm 2, for every $u, v \in \mathbb{V}_n$, the nearest common ancestor of u and v in \mathcal{T}^n is the root of the tree in the forest $\mathcal{F}_{\mathbf{c}_n(u, v)}^n$ which contains u and v . As a result, $d^n(u, v) = \mathbf{H}_{\mathbf{c}_n(u, v)}^n(u) + \mathbf{H}_{\mathbf{c}_n(u, v)}^n(v)$. So

$$|d^n(u, v) - D^n(u, v)| = \left| \mathbf{H}_{\mathbf{c}_n(u, v)}^n(u) + \mathbf{H}_{\mathbf{c}_n(u, v)}^n(v) - \frac{1}{2} \mathbf{h}_n(\mathbf{c}_n(u, v), \mathbf{b}_n(u)) - \frac{1}{2} \mathbf{h}_n(\mathbf{c}_n(u, v), \mathbf{b}_n(v)) \right|.$$

Hence

$$\max_{u, v \in \mathbb{V}_n} |d^n(u, v) - D^n(u, v)| \leq 2 \max_{\substack{v \in \mathbb{V}_n \\ 0 \leq a < \mathbf{b}_n(v)}} \left| \mathbf{H}_a^n(v) - \frac{1}{2} \mathbf{h}_n(a, \mathbf{b}_n(v)) \right|. \quad (7.3.7)$$

We next estimate the right hand side of (7.3.7). It follows from Lemma 6.2.1 that for every $v \in \mathbb{V}_n$ and $0 \leq a \leq \mathbf{b}_n(v)$,

$$\mathbf{H}_a^n(v) \stackrel{(d)}{=} \sum_{i=a+1}^{\mathbf{b}_n(v)} Y_i^n \mathbb{1}_{\mathbf{x}_i^n=1},$$

where $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n)_{1 \leq i \leq n}$. As before, set $g(u) = (1+u) \log(1+u) - u$ for $u > 0$. Then, by Bennett's inequality, recalled in Proposition 6.4.1, and using the fact that $x \mapsto xg(t/x)$ is decreasing, for every $t > 0$, we see that $\mathbb{P}(|\mathbf{H}_a^n(v) - \mathbf{h}_n^+(a+1, \mathbf{b}_n(v))| > t)$ is smaller than $2 \exp(-\mathbf{h}_n^+ g(t/\mathbf{h}_n^+))$. Therefore, by a union bound,

$$\mathbb{P}\left(\max_{v \in \mathbb{V}_n, 0 \leq a < \mathbf{b}_n(v)} |\mathbf{H}_a^n(v) - \mathbf{h}_n^+(a+1, \mathbf{b}_n(v))| > t\right) \leq 2n^2 \exp(-\mathbf{h}_n^+ g(t/\mathbf{h}_n^+)). \quad (7.3.8)$$

By Lemma 7.3.1 (iii) it follows that for every $\varepsilon > 0$,

$$\mathbb{P}\left(\max_{v \in \mathbb{V}_n, 0 \leq a < \mathbf{b}_n(v)} |\mathbf{H}_a^n(v) - \mathbf{h}_n^+(a+1, \mathbf{b}_n(v))| > \varepsilon n^{1-\alpha}\right) \leq 2n^2 \exp(-\mathbf{h}_n^+ g(\varepsilon n^{1-\alpha}/\mathbf{h}_n^+)) \rightarrow 0.$$

In addition, by Lemma 7.3.1 (iv), and since $1/S_i \leq 1$ for every $0 \leq i \leq n$, we have

$$\frac{1}{n^{1-\alpha}} \max_{1 \leq a \leq b \leq n} \left| \mathbf{h}_n^+(a+1, b) - \frac{1}{2} \mathbf{h}_n(a, b) \right| \xrightarrow{n \rightarrow \infty} 0,$$

which completes the proof. \square

7.4 Scaling limits: non-critical regimes

As before, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$.

7.4.1 Subcritical regime $\alpha \in (0, 1/2)$

We first show that, roughly speaking, vertices coalesce quickly.

Lemma 7.4.1. *Let $\alpha \in (0, 1/2)$. Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$ are in force. Then,*

$$\frac{1}{n} \max_{u, v \in \mathbb{V}_n} |\mathbf{b}_n(u) \wedge \mathbf{b}_n(v) - \mathbf{c}_n(u, v)| \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0.$$

Proof. Recall that the total number of vertices of \mathcal{T}^n is $(n + S_n^n + 1)/2$. Using Lemma 6.2.3, by a union bound and then by $(\text{Lim}_1^\alpha/f^2)$, for n large enough

$$\begin{aligned} \mathbb{P}(\exists u, v \in \mathbb{V}_n : |\mathbf{b}_n(u) \wedge \mathbf{b}_n(v) - \mathbf{c}_n(u, v)| \geq \varepsilon n) &\leq \left(\frac{n + S_n^n + 1}{2}\right)^2 \max_{\lfloor \varepsilon n \rfloor + 1 \leq b \leq n} \prod_{\substack{i=b-\lfloor \varepsilon n \rfloor + 1 \\ \text{s.t. } x_i^n = 1}}^b \left(1 - \frac{1}{\binom{S_i^n}{2}}\right) \\ &\leq (n+1)^2 \exp\left(\max_{\lfloor \varepsilon n \rfloor + 1 \leq b \leq n} \sum_{i=b-\lfloor \varepsilon n \rfloor + 1}^b \frac{-2}{S_i^n(S_i^n - 1)} \mathbb{1}_{x_i^n = 1}\right) \\ &\leq (n+1)^2 e^{-(n^{1-2\alpha}/2) \min \varepsilon \leq y \leq 1 \int_{y-\varepsilon/2}^y dt/f(t)^2} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the last inequality comes from Lemma 7.3.1 (v). This completes the proof. \square

Proof of Theorem 7.2.2. Let $k \geq 1$. For all $n \geq 1$, let V_1^n, \dots, V_k^n be uniform random vertices of \mathcal{T}^n . Let U_1, \dots, U_k be i.i.d. uniform random variables in $[0, 1]$. Let us first show that

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{n^{1-\alpha}}\right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{U_j \wedge U_\ell}^{U_j \vee U_\ell} \frac{dt}{2f(t)}\right)_{1 \leq j, \ell \leq k}.$$

By Proposition 7.3.2, it suffices to prove that

$$\left(\frac{1}{n^{1-\alpha}} \left(\sum_{i=\mathbf{c}_n(V_j^n, V_\ell^n)}^{\mathbf{b}_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=\mathbf{c}_n(V_j^n, V_\ell^n)}^{\mathbf{b}_n(V_\ell^n)} \frac{1}{2S_i^n} \right)\right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{U_j \wedge U_\ell}^{U_j \vee U_\ell} \frac{dt}{2f(t)}\right)_{1 \leq j, \ell \leq k}. \quad (7.4.1)$$

However, one can write

$$\sum_{i=\mathbf{c}_n(V_j^n, V_\ell^n)}^{\mathbf{b}_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=\mathbf{c}_n(V_j^n, V_\ell^n)}^{\mathbf{b}_n(V_\ell^n)} \frac{1}{2S_i^n} = \sum_{i=\mathbf{c}_n(V_j^n, V_\ell^n)}^{\mathbf{b}_n(V_j^n) \wedge \mathbf{b}_n(V_\ell^n)} \frac{1}{S_i^n} + \sum_{(\mathbf{b}_n(V_j^n) \vee \mathbf{b}_n(V_\ell^n)) + 1}^{\mathbf{b}_n(V_j^n) \vee \mathbf{b}_n(V_\ell^n)} \frac{1}{2S_i^n}.$$

Besides, $S_n^n = o(n)$, so that with high probability the vertices V_j^n 's are frozen. Moreover, conditionally on being frozen, the $\mathbf{b}_n(V_j^n)$'s are independent and uniform in $\{i \in [n]; x_i^n = -1\}$. Therefore, by Lemma 7.3.1 (ii), we know that the $\mathbf{b}_n(V_j^n)/n$'s converge in distribution towards the U_j 's. Finally, combining this remark with Lemma 7.4.1 and (Lim_1^α/f) , we obtain (7.4.1).

So as to end the proof of the GHP convergence, by Lemma 7.6.2, it suffices to establish the convergence in the sense Gromov-Hausdorff since we already have the Gromov-Prokhorov convergence. Let us define the correspondence

$$\mathcal{C}_n = \left\{ \left(u, \int_0^t \frac{1}{2f(s)} ds \right); u \in \mathbb{V}_n \text{ and } \left| t - \frac{\mathbf{b}_n(u)}{n} \right| = \min_{v \in \mathbb{V}_n} \left| t - \frac{\mathbf{b}_n(v)}{n} \right| \right\}.$$

By Theorem 7.3.25 in [BBI01] it then suffices to show that

$$\text{dis}(\mathcal{C}_n) := \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \frac{d^n(u, v)}{n^{1-\alpha}} - \int_{t \wedge r}^{t \vee r} \frac{1}{2f(s)} ds \right| \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (7.4.2)$$

We upper bound

$$\text{dis}(\mathcal{C}_n) \leq \sup_{u, v \in \mathbb{V}_n} \frac{|d^n(u, v) - D^n(u, v)|}{n^{1-\alpha}} \quad (7.4.3)$$

$$+ \sup_{u, v \in \mathbb{V}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=\mathbf{c}_n(u, v)}^{\mathbf{b}_n(u) \wedge \mathbf{b}_n(v)} \frac{1}{S_i^n} \right| \quad (7.4.4)$$

$$+ \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=\mathbf{b}_n(u) \wedge \mathbf{b}_n(v) + 1}^{\mathbf{b}_n(u) \vee \mathbf{b}_n(v)} \frac{1}{2S_i^n} - \int_{t \wedge r}^{t \vee r} \frac{1}{2f(s)} ds \right| \quad (7.4.5)$$

The first line (7.4.3) converges to zero in probability as $n \rightarrow \infty$ thanks to Proposition 7.3.2. For the second line (7.4.4),

$$\sup_{u,v \in \mathbb{V}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=c_n(u,v)}^{b_n(u) \wedge b_n(v)} \frac{1}{S_i^n} \right| \leq \int_0^1 \left| \frac{n^\alpha}{S_{[ns]}^n} - \frac{1}{f(s)} \right| ds + \sup_{u,v \in \mathbb{V}_n} \int_{c_n(u,v)/n}^{(b_n(u) \wedge b_n(v)+1)/n} \frac{ds}{f(s)} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0,$$

by (Lim_1^α) and by Lemma 7.4.1. Thus the second line (7.4.4) converges to zero in probability as $n \rightarrow \infty$.

We can upper-bound the quantity appearing in (7.4.5) by

$$\int_0^1 \left| \frac{n^\alpha}{2S_{[ns]}^n} - \frac{1}{2f(s)} \right| ds + \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \int_{\frac{b_n(u) \wedge b_n(v)+1}{n}}^{\frac{b_n(u) \vee b_n(v)+1}{n}} \frac{ds}{2f(s)} - \int_{t \wedge r}^{t \vee r} \frac{ds}{2f(s)} \right|.$$

Observe that, by definition of \mathcal{C}_n and thanks to Lemma 7.3.1 (ii), we have

$$\sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left(\left| t \wedge r - \frac{b_n(u) \wedge b_n(v)}{n} \right| + \left| t \vee r - \frac{b_n(u) \vee b_n(v)}{n} \right| \right) \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0,$$

and combined with the integrability of $1/f$ this completes the proof. \square

We complete the study of the subcritical regime by establishing a scaling limit result when the sequence $\|S^n\|_\infty$ is bounded (which roughly speaking may be thought of as the case $\alpha = 0$). The main difference with Theorem 7.2.2 is that now the limit is for the Gromov–Hausdorff topology, and not in the Gromov–Hausdorff–Prokhorov sense (in general there is no universal behavior for the latter topology).

Theorem 7.4.2. *Let $(\mathbf{x}^n)_n$ be such that $\tau(\mathbf{x}^n) > n$ for all n and $S_k^n \leq M$ for all $k \leq n$ where M is some positive constant. Then*

$$\frac{1}{h_n^+} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} [0, 1]$$

holds in probability for the Gromov–Hausdorff topology.

Proof. For all n , let us call a *side sub-tree* of \mathcal{T}_n , a sub-tree rooted at a vertex of the longest branch of \mathcal{T}_n and containing only vertices that are not on the longest branch (except the root). The height of the longest branch of \mathcal{T}_n is of order h_n^+ . Indeed, it is easy to check that $h_n^+ = \Theta(n)$. Therefore, by Theorem 6.1.3 (III) we deduce that $\text{Height}(\mathcal{T}_n)/h_n^+$ converges to 1 in \mathbb{L}^p for all $p \geq 1$. To show the proposition, it is enough to show that, for all $\varepsilon > 0$, the probability to see a side sub-tree of height larger than εn in \mathcal{T}_n goes to 0 when n goes to infinity.

Since for ε small enough, the longest branch has a length at least εn with high probability, it suffices to show that the probability that “there exist two vertices $u, v \in \mathbb{V}_n$ such that their nearest common ancestor w is at distance at least εn from each of them” goes to zero as $n \rightarrow \infty$. But for all $u, v \in \mathbb{V}_n$, if w is their common ancestor, then one can directly upperbound

$$d^n(u, w) \leq b_n(u) - c_n(u, v) \quad \text{and} \quad d^n(v, w) \leq b_n(v) - c_n(u, v).$$

Finally, by a union bound on $u, v \in \mathbb{V}_n$ and using Lemma 6.2.3,

$$\begin{aligned} \mathbb{P}(\exists u, v \in \mathbb{V}_n, b_n(u) \wedge b_n(v) - c_n(u, v) \geq \varepsilon n) &\leq \left(\frac{n + S_n^n + 1}{2} \right)^2 \max_{[\varepsilon n]+1 \leq b \leq n} \prod_{\substack{i=b-[\varepsilon n]+1 \\ \text{s.t. } x_i^n=1}}^b \left(1 - \frac{1}{\binom{S_i^n}{2}} \right) \\ &\leq (n+1)^2 \left(1 - \frac{1}{\binom{M}{2}} \right)^{[\varepsilon n]/2-M} \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

where the last inequality stems from the fact that $S_k^n \leq M$ for all $k \leq n$ and from (7.3.2) which also holds in this case. \square

7.4.2 Supercritical regime $\alpha \in (1/2, 1)$

We first show that, roughly speaking, vertices coalesce near the origin.

Lemma 7.4.3. *Let $\alpha \in (1/2, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$ are in force. For all $n \geq 1$, let V_1^n, V_2^n be two independent uniform vertices of \mathbb{V}_n . Then*

$$\frac{c_n(V_1^n, V_2^n)}{n} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0.$$

Proof. Let $\varepsilon, \delta \in (0, 1)$ with $\varepsilon < 1 - \delta$. By Lemma 6.2.3,

$$\mathbb{P}(c_n(V_1^n, V_2^n) \geq \varepsilon n) \leq \mathbb{P}(\mathbf{b}_n(V_1^n) \wedge \mathbf{b}_n(V_2^n) \geq (1 - \delta)n) + \sum_{\substack{\varepsilon n \leq i \leq (1 - \delta)n \\ x_i^n = 1}} \frac{1}{\binom{S_i^n}{2}}.$$

By Lemma 6.2.2 and the fact that $S_n^n = o(n)$,

$$\mathbb{P}(\mathbf{b}_n(V_1^n) \wedge \mathbf{b}_n(V_2^n) \geq (1 - \delta)n) \xrightarrow[n \rightarrow \infty]{} \delta^2.$$

The remaining sum is a $O(n^{1-2\alpha})$ by Lemma 7.3.1 (v). Making $\delta \rightarrow 0$ we conclude that $\mathbb{P}(c_n(V_1^n, V_2^n) \geq \varepsilon n) \rightarrow 0$ when $n \rightarrow \infty$. \square

Proof of Theorem 7.2.6. By Proposition 7.3.2 it suffices to show that

$$\frac{1}{n^{1-\alpha}} \sum_{i=1}^{b_n(V_1^n)} \frac{1}{2S_i^n} \xrightarrow[n \rightarrow \infty]{(d)} \int_0^{U_1} \frac{dt}{2f(t)} \quad (7.4.6)$$

and that

$$\left(\frac{1}{n^{1-\alpha}} \left(\sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_\ell^n)} \frac{1}{2S_i^n} \right) \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_0^{U_j} \frac{dt}{2f(t)} + \int_0^{U_\ell} \frac{dt}{2f(t)} \right)_{1 \leq j, \ell \leq k}. \quad (7.4.7)$$

We know that $S_n^n = o(n)$, so that $(V_i^n)_{1 \leq i \leq k}$ are frozen with probability tending to 1. In addition, conditionally on being frozen, $\mathbf{b}_n(V_1^n)$ is uniform in $\{i \in [n]; x_i^n = -1\}$. Therefore, by Lemma 7.3.1 (ii), we know that $\mathbf{b}_n(V_1^n)/n$ converges in distribution towards U_1 . The convergence (7.4.6) then comes from $(\text{Lim}_{1/f}^\alpha)$. For (7.4.7), we again use $(\text{Lim}_{1/f}^\alpha)$, using also that $c_n(V_j^n, V_\ell^n)/n$ converges in probability towards zero by Lemma 7.4.3. \square

7.5 Scaling limits: critical regime

The goal of Section 7.5 is to establish Theorem 7.2.3. We first explain in Section 7.5.1 the strategy of the proof of Theorem 7.2.3. Roughly speaking, the identification of the “finite dimensional marginal distributions” (Gromov–Prokhorov convergence) is based on a continuous-time coalescent process, defined in Section 7.5.2, and which is closely related to a non-homogeneous analogue of Kingman’s coalescent introduced by Aldous in [Ald98]. The law of the genealogy of clusters in this continuous-time coalescent process can be explicitly described (Section 7.5.3), which allows us to establish the Gromov–Prokhorov convergence of Theorem 7.2.3. Tightness is then established in Section 7.5.5.

Table 7.2 – Table of the main notation and symbols introduced in Section 7.5 and used later.

$\mathfrak{F}_{r,k}^+$	the set of ordered forests of $r + 1$ plane binary trees with $2k - 1 - r$ vertices with a vertex-labelling from $r + 1$ to $2k - 1$ which increases along the branches.
Δ^n	quantity defined for $u, v \in \mathbb{V}_n(\mathbf{x})$ by $\Delta^n(u, v) = \frac{1}{2}(\mathbf{b}_n(u) + \mathbf{b}_n(v) - 2c_n(u, v))$.

From now on we assume that the assumptions of Theorem 7.2.3 are in force.

7.5.1 Definition of the limit

The way we define the limiting compact measured tree $\mathcal{T}(f)$ is rather indirect. For every $k \geq 1$, let V_1^n, \dots, V_k^n be k i.i.d. vertices in \mathbb{V}_n chosen uniformly at random (conditionally given \mathcal{T}^n). Our strategy is to establish the following two facts:

- Gromov–Prokhorov convergence:

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{\sqrt{n}} \right)_{1 \leq j, \ell \leq k} \quad (7.5.1)$$

converges in distribution as $n \rightarrow \infty$, where we recall that d^n denotes the graph distance on \mathcal{T}^n .

This is established in Sec. 7.5.4

- “Leaf-tightness”: for every $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\max_{v \in \mathbb{V}_n} \min_{1 \leq i \leq k} \frac{d^n(v, V_i^n)}{\sqrt{n}} > \varepsilon \right) = 0. \quad (7.5.2)$$

This is established in Sec. 7.5.5.

Indeed, by Lemma 7.7.1, (7.5.1) and (7.5.2) entail the existence of a limiting compact metric space $\mathcal{T}(f)$ equipped with a probability measure μ with full support such that the convergence of Theorem 7.2.3 holds. The compact metric space $\mathcal{T}(f)$ is a real tree, being the Gromov–Hausdorff limit of real trees.

7.5.2 A continuous-time coalescent

Here we introduce a continuous-time coalescent process, which appears in the limit of the quantity (7.5.1). Let $f : [0, 1] \rightarrow \mathbb{R}_+$ be a measurable function satisfying the following conditions:

$$\int_0^1 \frac{1}{f(t)} dt < \infty, \quad ; \quad \forall a, b \in (0, 1), \quad \int_a^b \frac{1}{f(t)^2} dt < \infty.$$

The continuous coalescent construction proceeds as follows (see Figure 7.3): let each of k particles be born at independent uniform random times B_1, \dots, B_k in $[0, 1]$. Particles coalesce into clusters according to the following rule: in time $[t, t - dt]$, each pair of clusters merges with rate $\frac{1}{f(t)^2}$. If we furthermore assume

$$\forall a \in (0, 1), \quad \int_0^a \frac{1}{f(t)^2} dt = \infty \quad (7.5.3)$$

this ensures that all the particles will eventually merge into one cluster (Aldous assumes this condition in [Ald98] but it will not be necessary for our purpose). Since we will not assume this condition, two clusters may not merge with positive probability. If there are r clusters which have not coalesced at time 0, we denote by $0 < C_{r+1} < \dots < C_{k-1}$ the times of coalescence into clusters in the increasing order (they are a.s. distinct) and by convention, we also set $C_1 = \dots = C_r = 0$.

This process is actually a time-change of a non-homogeneous analogue of Kingman’s coalescent introduced by Aldous in [Ald98], see Remark 7.5.3 below.

7.5.3 The genealogy of clusters in the continuous-time coalescent

The genealogy of the clusters is described by an increasing binary forest \mathcal{F} . The trees forming the forest are associated with the clusters which have not merged at time 0. Each internal vertex in one of these trees corresponds to a coalescence of two clusters and the leafs correspond to the particles’ births. We also give a uniform random order to the children of each internal node, so that the trees of \mathcal{F} can be viewed as plane trees. We equip \mathcal{F} with a labelling from $r + 1$ to $2k - 1$ of its vertices according to the order of the times of coalescence or birth. If $k > r \geq 0$, we denote by $\mathfrak{F}_{r,k}^+$ the set of ordered forests of $r + 1$ plane binary trees with $2k - 1 - r$ vertices with a vertex-labelling from $r + 1$ to $2k - 1$ which increases along the branches. If there are $r + 1$ clusters at time 0, then the forest \mathcal{F} belongs to $\mathfrak{F}_{r,k}^+$. To express the law of $(\mathcal{F}, (C_j)_{j \in [k-1]}, (B_j)_{j \in [k]})$, we first describe what will be its support.

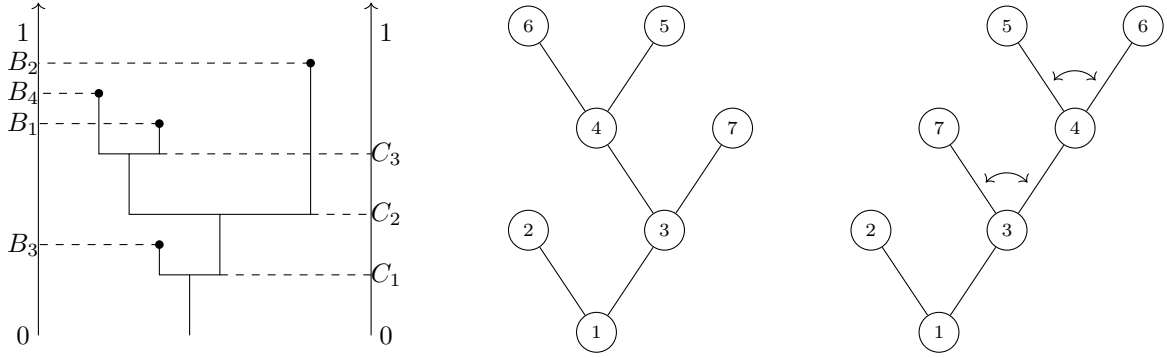


Figure 7.3 – Illustration of the continuous coalescence construction on the left for $k = 4$. In the middle is represented the binary forest \mathcal{F} composed here of only one tree associated with this construction. On the right is the binary forest where the vertices of each internal node have been exchanged uniformly at random. In this case, the child of the vertices 3 and 4 have been exchanged but those of 1 have not.

Definition 7.5.1. Let $k > r \geq 0$. Let $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$. Let $b_1, \dots, b_k \in \mathbb{R}_+$ and $0 \leq c_{r+1} < \dots < c_{k-1}$ be distinct real numbers. The triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is called *admissible* if there is an increasing bijection $\varphi : \{c_j; j \in \{r+1, \dots, k-1\}\} \cup \{b_j; j \in [k]\} \rightarrow \{r+1, \dots, 2k-1\}$ such that for all $j \in [k]$, the integer $\varphi(b_j)$ is the label of a leaf in \mathcal{F}_0 and for all $j \in \{r+1, \dots, k-1\}$, the integer $\varphi(c_j)$ is the label of an internal vertex of \mathcal{F}_0 .

Roughly speaking, $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is admissible if it is possible to associate the times (c_j) with cluster coalescence times and $(b_j)_{j \in [k]}$ with particle birth times in such a way that their relative order is compatible with \mathcal{F}_0 . In the example of Fig. 7.3 we have $r = 0$, $B_1 = 7/10$, $B_2 = 9/10$, $B_3 = 3/10$, $B_4 = 8/10$, $C_1 = 2/10$, $C_2 = 4/10$, $C_3 = 6/10$ and φ is defined by $\varphi(2/10) = 1$, $\varphi(3/10) = 2$, $\varphi(4/10) = 3$, $\varphi(6/10) = 4$, $\varphi(7/10) = 5$, $\varphi(8/10) = 6$, $\varphi(9/10) = 7$.

Proposition 7.5.2. For all $k > r \geq 0$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$, for all $b_1, \dots, b_k \in [0, 1]$ and $0 < c_{r+1} < \dots < c_{k-1} < 1$ distinct real numbers such that the triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is admissible, we set

$$g_{r,k}(\mathcal{F}_0, (b_j)_{1 \leq j \leq k}, (c_j)_{j \in \{r+1, \dots, k-1\}}) = \frac{1}{2^{k-1-r}} \exp\left(-\int_0^1 \frac{a(t)}{f(t)^2} dt\right) \prod_{j=r+1}^{k-1} \frac{1}{f(c_j)^2}$$

where $a(t) := \#\{j \in [k]; b_j > t\} - \#\{j \in \{r+1, \dots, k-1\}; c_j > t\}$ corresponds to the number of clusters at time t . Then for all $k \geq 1$, the joint law of the increasing binary forest \mathcal{F} associated with the genealogy (where the order of the children of each inner vertex is chosen uniformly at random), of the coalescing times $C_1 \leq \dots \leq C_{k-1}$ and birth times B_1, \dots, B_k is given by: for all $r \in \llbracket 0, k-1 \rrbracket$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$,

$$\begin{aligned} \mathbb{P}(\mathcal{F} = \mathcal{F}_0, B_1 \in db_1, \dots, B_k \in db_k, C_1 = 0, \dots, C_r = 0, C_{r+1} \in dc_{r+1}, \dots, C_{k-1} \in dc_{k-1}) \\ = g_{r,k}(\mathcal{F}_0, (b_j)_{j \in [k]}, (c_j)_{j \in \{r+1, \dots, k-1\}}) db_1 \cdots db_k dc_{r+1} \cdots dc_{k-1}. \end{aligned}$$

Proof. The proof is the same as the proof of the density formula (2) in [Ald98] except that we use that the probability that none of the $r+1$ clusters coalesce between time c_{r+1} and 0 is

$$\exp\left(-\int_0^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right).$$

The factor $1/2^{k-1-r}$ comes from the uniform order assigned to the children of each internal node of \mathcal{F} . \square

Remark 7.5.3. The change of time relating the continuous coalescent in [Ald98] and the continuous coalescent presented here is constructed as follows: for all $s \in [0, 1]$, let $G(s) = \int_0^s \frac{1}{2f(t)} dt$, which is an increasing continuous function from $[0, 1]$ to \mathbb{R}_+ . We define F as the inverse of G , extended to \mathbb{R}_+ by

setting $F(t) = 1$ if $t > G(1)$. Let $\ell(t) = 2f(F(t))$. Then one can show that if we change the time by G , the continuous coalescent becomes the one introduced in [Ald98]. Besides, if we assume condition (7.5.3), then the law of the coalescent is fully characterized by the density $g_{0,k}$ by Proposition 7.5.2, which is nothing else but the time-change of the density (2) in [Ald98].

7.5.4 Gromov–Prokhorov convergence

Our goal is now to establish (7.5.1). Recall from (6.2.1) the notation $\mathbb{F}_n(\mathbf{x}^n) = \{i \in [n] : x_i^n = -1\}$, which denotes the labels of the frozen vertices of \mathcal{T}^n . Let V_1^n, \dots, V_k^n be k vertices in \mathcal{T}^n chosen uniformly (conditionally to the construction of \mathcal{T}^n). Let \mathcal{B}^n be the (rooted non-plane) binary tree associated with the genealogy of those k vertices in the coalescent construction: its leaves correspond to the birth-times $B_1^n := \mathbf{b}_n(V_1^n), \dots, B_k^n := \mathbf{b}_n(V_k^n)$, of V_1^n, \dots, V_k^n and its internal vertices correspond to a coalescence of two trees which contain each one a V_i^n for $i \in [k]$.

Note that $B_1^n + 1, \dots, B_k^n + 1 \in \mathbb{F}_n(\mathbf{x}^n)$ with large probability when $n \rightarrow \infty$, since the number of non frozen vertices at time n is of order \sqrt{n} . Moreover, conditionally on the fact that they belong to $\mathbb{F}_n(\mathbf{x}^n)$, the integers $B_1^n + 1, \dots, B_k^n + 1$ are taken uniformly in $\{i \in [n]; x_i^n = -1\}$. In the sequel, for the sake of simplicity, we will thus assume implicitly that they belong to $\mathbb{F}_n(\mathbf{x}^n)$.

Each coalescence between two rooted trees both containing vertices among V_1^n, \dots, V_k^n in the coalescent construction is associated with an internal node v of \mathcal{B}^n and we write those coalescence times in the increasing order $C_1^n < \dots < C_{k-1}^n$. We also label the vertices of \mathcal{B}^n from 1 to $2k - 1$ in the increasing order of the associated coalescence times or birth times. We then equip \mathcal{B}^n with an ordering of the children of each internal nodes, taking into account which one of the two roots remains the root of the tree after the coalescence, so that \mathcal{B}^n is a plane tree. With these additional structures, $\mathcal{B}^n \in \mathfrak{F}_{0,k}^+$. If $r \in \{0, \dots, k - 1\}$, we denote by \mathcal{F}_r^n the increasing binary forest in $\mathfrak{F}_{r,k}^+$ which describes the genealogy but forgetting the r last coalescences.

Let us state a local limit estimate for $(\mathcal{F}^n, (B_j^n)_{j \in [k]}, (C_j^n)_{j \in [k-1]})$.

Lemma 7.5.4. *Assume $(\text{Lim}_{1/f}^\alpha)$ and $(\text{Lim}_{1/f^2}^\alpha)$. Let $k > r \geq 0$. Let $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$. For all $n \geq 1$, independently of \mathcal{T}^n , let $\tilde{B}_1^n, \dots, \tilde{B}_k^n$ be i.i.d. uniform random variables in $\{0 \leq i < n, x_{i+1}^n = -1\}$ and let $\tilde{C}_{r+1}^n, \dots, \tilde{C}_{k-1}^n$ be an independent sequence of i.i.d. uniform random variables in $\{0 \leq i < n, x_{i+1}^n = 1\}$. Let $\tilde{B}_1, \dots, \tilde{B}_k$ and $\tilde{C}_{r+1}, \dots, \tilde{C}_{k-1}$ be i.i.d. uniform random variables in $[0, 1]$. Then we have the convergence in distribution*

$$\begin{aligned} \left(\frac{n}{2}\right)^{2k-1-r} \mathbb{P} \left(\mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = \tilde{B}_j^n, \forall j \in [r], C_j^n \leq \varepsilon n \right. \\ \left. \text{and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = \tilde{C}_j^n \middle| (\tilde{B}_j^n)_{j \in [r]}, (\tilde{C}_j^n)_{r+1 \leq j \leq k-1} \right) \\ \xrightarrow[n \rightarrow \infty]{(d)} \xrightarrow[\varepsilon \rightarrow 0]{(d)} g_{r,k} \left(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}} \right), \end{aligned} \quad (7.5.4)$$

where by convention the function $g_{r,k}$ vanishes on non-admissible triples. In particular, if, for all $i, j \in [k]$ we denote by $C_{i,j}$ the coalescing time between the cluster containing the i -th particle with the cluster containing the j -th particle, then we have

$$\left((B_j^n/n)_{j \in [k]}, (\mathbf{c}_n(V_i^n, V_j^n)/n)_{i,j \in [k]} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left((B_j)_{j \in [k]}, (C_{i,j})_{i,j \in [k]} \right). \quad (7.5.5)$$

Proof. By Lemma 7.3.1 (ii) and by Skorokhod's representation theorem we may assume that

$$\forall j \in [k], \frac{\tilde{B}_j^n}{n} \xrightarrow[n \rightarrow \infty]{(a.s.)} \tilde{B}_j \quad \text{and} \quad \forall j \in \{r+1, \dots, k-1\}, \frac{\tilde{C}_j^n}{n} \xrightarrow[n \rightarrow \infty]{(a.s.)} \tilde{C}_j.$$

For all $i \in [n]$, we set

$$a_i^n = \#\{j \in [k]; \tilde{B}_j^n \geq i\} - \#\{j \in \{r+1, \dots, k-1\}; \tilde{C}_j^n \geq i\}.$$

Notice that if we replace the \tilde{C}_j^n by C_j^n and \tilde{B}_j^n by B_j^n , then for $i \geq C_r^n + 1$ we have that a_i^n is the number of trees in $\mathcal{F}_i^n(\mathbf{x}^n)$ containing vertices among V_1^n, \dots, V_k^n .

Let us denote by $\tilde{\mathbb{P}}$ the conditional probability $\mathbb{P}(\cdot | (\tilde{B}_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}})$. Then, by the definition of Algorithm 2, when the triple $(\mathcal{F}_0, (\tilde{B}_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}})$ is admissible, we have

$$\begin{aligned} & \tilde{\mathbb{P}}\left(C_1^n \leq \varepsilon n, \dots, C_r^n \leq \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = \tilde{B}_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = \tilde{C}_j^n\right) \\ &= \tilde{\mathbb{P}}\left(\mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = \tilde{B}_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = \tilde{C}_j^n\right) \\ &\quad - \tilde{\mathbb{P}}\left(C_r^n > \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = \tilde{B}_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = \tilde{C}_j^n\right) \\ &= \frac{1}{k! \binom{(n-S_r^n+1)/2+S_r^n}{k}} \left(\prod_{j=r+2}^k \prod_{\substack{i=\tilde{C}_{j-1}^n+2 \\ \text{s.t. } \mathbf{x}_i^n=1}}^{\tilde{C}_j^n} \left(1 - \frac{\binom{a_i^n}{2}}{\binom{S_i^n}{2}}\right) \right) \prod_{j=r+1}^{k-1} \frac{1}{2 \binom{S_{\tilde{C}_j^n+1}^n}{2}} \\ &\quad - \sum_{\varepsilon n < \tilde{C}_r^n \leq n} \binom{r+1}{2} \frac{1}{k! \binom{(n-S_r^n+1)/2+S_r^n}{k}} \left(\prod_{j=r+1}^k \prod_{\substack{i=\tilde{C}_{j-1}^n+2 \\ \text{s.t. } \mathbf{x}_i^n=1}}^{\tilde{C}_j^n} \left(1 - \frac{\binom{a_i^n}{2}}{\binom{S_i^n}{2}}\right) \right) \prod_{j=r}^{k-1} \frac{1}{2 \binom{S_{\tilde{C}_j^n+1}^n}{2}}, \end{aligned}$$

where by convention $\tilde{C}_k^n = n$. Indeed, the factor $1/(k! \binom{(n-S_r^n+1)/2+S_r^n}{k})$ comes from the choice of the uniform (distinct) birth-times B_j^n , the factor $1/(2 \binom{S_{\tilde{C}_j^n+1}^n}{2})$ corresponds to the coalescence at time \tilde{C}_j^n of a pair of trees containing some V_j^n but the choice of the trees is prescribed by \mathcal{F}_0 and its planar ordering. The factor $1 - \binom{a_i^n}{2} / \binom{S_i^n}{2}$ encodes the fact that there is no coalescence between the trees containing V_1^n, \dots, V_k^n at times different from $\tilde{C}_{r+1}^n, \dots, \tilde{C}_{k-1}^n$. Finally, the binomial coefficient $\binom{r+1}{2}$ comes from the different possibilities for the coalescence at time C_r^n (since the forest \mathcal{F}_0 does not impose the genealogical structure for this coalescence).

By $(\text{Lim}_{1/f}^\alpha)$, by (Lim_{1/f^2}) , by (7.3.5) and using Lemma 7.3.1 (ii),(iv),(v), it is easy to derive the convergence in probability

$$\begin{aligned} & \left(\frac{n}{2}\right)^{2k-1-r} \tilde{\mathbb{P}}\left(C_1^n \leq \varepsilon n, \dots, C_r^n \leq \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = \tilde{B}_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = \tilde{C}_j^n\right) \\ & \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \frac{1}{2^{k-1-r}} \left(\prod_{j=r+2}^k \exp\left(-\int_{\tilde{C}_{j-1}^n}^{\tilde{C}_j^n} \frac{a(t)}{f(t)^2} dt\right) \right) \left(\prod_{j=r+1}^{k-1} \frac{1}{f(\tilde{C}_j^n)^2} \right) \left(1 - \binom{r+1}{2} \int_\varepsilon^{\tilde{C}_{r+1}^n} \exp\left(-\int_c^{\tilde{C}_{r+1}^n} \frac{(r+1)}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc\right) \\ &= \frac{1}{2^{k-1-r}} \exp\left(-\int_{\tilde{C}_{r+1}^n}^1 \frac{a(t)}{f(t)^2} dt\right) \left(\prod_{j=r+1}^{k-1} \frac{1}{f(\tilde{C}_j^n)^2} \right) \left(1 - \binom{r+1}{2} \int_\varepsilon^{\tilde{C}_{r+1}^n} \exp\left(-\int_c^{\tilde{C}_{r+1}^n} \frac{(r+1)}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc\right), \end{aligned}$$

where by convention $\tilde{C}_k^n = 1$, where $a(t)$ is defined in Proposition 7.5.2 and where the convergence in probability of $n/(S_{\tilde{C}_j^n+1}^n)^2$ towards $1/f(\tilde{C}_j^n)^2$ comes from the points (ii) and (iv) of Lemma 7.3.1. Besides,

$$1 - \binom{r+1}{2} \int_\varepsilon^{\tilde{C}_{r+1}^n} \exp\left(-\int_c^{\tilde{C}_{r+1}^n} \frac{(r+1)}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc \xrightarrow[\varepsilon \rightarrow 0]{} \exp\left(-\int_0^{\tilde{C}_{r+1}^n} \frac{(r+1)}{f(t)^2} dt\right),$$

hence (7.5.4).

For all $n \geq 1, \varepsilon > 0$, let $R^n(\varepsilon)$ be the smallest $r' \geq 0$ such that $C_{r'}^n > \varepsilon n$. For all $(b_j^n)_{j \in [k]}, (c_j^n)_{j \in \{r+1, \dots, k-1\}}$, let

$$\begin{aligned} & g_{\varepsilon, n, r, k}(\mathcal{F}_0, (b_j^n)_{j \in [k]}, (c_j^n)_{j \in \{r+1, \dots, k-1\}}) \\ &= (n - \#\mathbb{F}_n(\mathbf{x}^n))^{k-1-r} (\#\mathbb{F}_n(\mathbf{x}^n))^k \mathbb{P}\left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right). \end{aligned}$$

Then, since $\#\mathbb{F}_n(\mathbf{x}^n) \sim n/2$ as $n \rightarrow \infty$, (7.5.4) can be rewritten as the convergence in probability

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} g_{\varepsilon, n, r, k}(\mathcal{F}_0, (\tilde{B}_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}}) = g_{r, k}(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}}). \quad (7.5.6)$$

Let us check that this implies the convergence in law

$$\left(\mathcal{F}_{R^n(\varepsilon)}^n, \left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{C_j^n}{n} \right)_{j \in [k-1]} \right) \xrightarrow[n \rightarrow \infty, \varepsilon \rightarrow 0]{(d)} \left(\mathcal{F}, (B_j)_{j \in [k]}, (C_j)_{j \in [k-1]} \right). \quad (7.5.7)$$

But by definition of $g_{\varepsilon, n, r, k}$ we have

$$\mathbb{P}\left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0\right) = \mathbb{E}\left[g_{\varepsilon, n, r, k}\left(\mathcal{F}_0, \left(\frac{\tilde{B}_j^n}{n}\right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n}\right)_{j \in [r+1, k-1]}\right)\right].$$

In addition,

$$\mathbb{E}\left[g_{r, k}\left(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}}\right)\right] = \mathbb{P}(\mathcal{F} = \mathcal{F}_0).$$

Therefore, using (7.5.6) with Fatou's lemma and then using the fact that if we sum over all possible $\mathcal{F}_0 \in \bigcup_{0 \leq r \leq k-1} \mathfrak{F}_{r, k}^+$ then we get one, one can see that

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}\left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0\right) = \mathbb{P}(\mathcal{F} = \mathcal{F}_0),$$

i.e.

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E}\left[g_{\varepsilon, n, r, k}\left(\mathcal{F}_0, (\tilde{B}_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}}\right)\right] = \mathbb{E}\left[g_{r, k}\left(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}}\right)\right].$$

Here we use the following result: if I is a finite set, $(X_n^k)_{n \geq 1, k \in I}, (X^k)_{k \in I}$ are nonnegative random variables, if $X_n^k \rightarrow X^k$ almost surely as $n \rightarrow \infty$ for every $k \in I$, if for every $n \geq 1$ we have $\sum_{k \in I} \mathbb{E}[X_n^k] = \sum_{k \in I} \mathbb{E}[X^k] = 1$, then $\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k]$ as $n \rightarrow \infty$ for every $k \in I$. Thus, by Scheffé's lemma, the convergence in (7.5.6) holds also in \mathbb{L}^1 . Let $\varepsilon_0 > 0$. Let $F : \bigcup_{0 \leq r < k} \mathfrak{F}_{r, k} \times [0, 1]^k \times [0, 1]^{k-1} \rightarrow \mathbb{R}$ be a continuous bounded function which is constant on the elements which have one coordinate in $[0, \varepsilon_0]$. Then for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} & \mathbb{E}\left[F\left(\mathcal{F}_{R^n(\varepsilon)}^n, \left(\frac{B_j^n}{n}\right)_{j \in [k]}, \left(\frac{C_j^n}{n}\right)_{j \in [k-1]}\right)\right] \\ &= \sum_{0 \leq r < k, \mathcal{F}_0 \in \mathfrak{F}_{r, k}} \mathbb{E}\left[F\left(\mathcal{F}_0, \left(\frac{\tilde{B}_j^n}{n}\right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n}\right)_{j \in [k-1]}\right) g_{\varepsilon, n, r, k}\left(\mathcal{F}_0, \left(\frac{\tilde{B}_j^n}{n}\right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n}\right)_{j \in [r+1, k-1]}\right)\right] \\ &\xrightarrow[n \rightarrow \infty]{\varepsilon \rightarrow 0} \sum_{0 \leq r < k, \mathcal{F}_0 \in \mathfrak{F}_{r, k}} \mathbb{E}\left[F\left(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in [k-1]}\right) g_{r, k}\left(\mathcal{F}_0, (\tilde{B}_j)_{j \in [k]}, (\tilde{C}_j)_{j \in [r+1, k-1]}\right)\right] \\ &= \mathbb{E}\left[F(\mathcal{F}, (B_j)_{j \in [k]}, (C_j)_{j \in [k-1]})\right], \end{aligned}$$

where by convention for all $j \in [r]$ we have set $\tilde{C}_j^n = 0$ and $\tilde{C}_j = 0$; here we use the following generalized version of the dominated convergence theorem: if X_n converges almost surely to X , if $|X_n| \leq Y_n$ almost surely, if Y_n converges in \mathbb{L}^1 to Y , then X_n converges in \mathbb{L}^1 to X . The convergence (7.5.7) follows.

Now, one can see that given the forest $\mathcal{F}_{R^n(\varepsilon)}^n$ and given the order of the births, one can recover the order of the coalescences occurring at times greater than εn , hence identifying to which C_j^n 's the $c_n(V_i^n, V_\ell^n)$'s which are greater than εn are equal. More precisely, for all $r \in \{0, \dots, k-1\}$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r, k}^+$, for all permutation σ of $\{1, \dots, k\}$ encoding a labelling by $1, \dots, k$ of the leafs of \mathcal{F}_0 , we associate the subset $I \subseteq [k]^2$ of couples of labelled leafs which have a common ancestor in \mathcal{F}_0 and the unique continuous function $\varphi_{\mathcal{F}_0, \sigma}$ from $\{(c_{r+1}, \dots, c_{k-1}) \in [0, 1]^{k-r-1}, \varepsilon < c_{r+1} < \dots < c_{k-1}\}$ to $[0, 1]^I$ which sends $(c_{r+1}, \dots, c_{k-1})$ to the unique family $(c_{i, j})_{(i, j) \in I}$ such that for all $(i, j) \in I$, the entry $c_{i, j}$ equals c_ℓ where the nearest common ancestor of the leafs labelled i, j by σ has the ℓ -th smallest label among the internal nodes of the labelled forest \mathcal{F}_0 .

Let $\varepsilon > 0$ and $n \geq 1$. Let π^n (resp. π) be the random uniform permutation associated with the ordering of B_1^n, \dots, B_k^n (resp. B_1, \dots, B_k), which defines a labelling of the leafs of $\mathcal{F}_{R^n(\varepsilon)}^n$ (resp. \mathcal{F}). Let $I_\varepsilon^n = \{(i, j) \in [k]^2, c_n(V_i^n, V_j^n) > \varepsilon n\}$. Then

$$\left(\frac{c_n(V_i^n, V_j^n)}{n} \right)_{(i, j) \in I_\varepsilon^n} = \varphi_{\mathcal{F}_{R^n(\varepsilon)}^n, \pi^n} \left(\left(\frac{C_j^n}{n} \right)_{j \in \{R^n(\varepsilon)+1, \dots, k-1\}} \right),$$

hence the convergence in law (7.5.5) by (7.5.7). \square

Proof of (7.5.1). Now, let us prove the convergence in law of $(d^n(V_j^n, V_\ell^n)/\sqrt{n})_{1 \leq j, \ell \leq k}$. For all $j, \ell \in [k]$,

$$D^n(V_j^n, V_\ell^n) = \sum_{i=c_n(V_j^n, V_\ell^n)}^{B_j^n} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{B_\ell^n} \frac{1}{2S_i^n}.$$

Using (Lim_1^α/f) , Lemma 7.3.1 (ii),(iv), Proposition 7.3.2 and the convergence (7.5.5), it is henceforth straightforward to conclude that

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{\sqrt{n}} \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{C_{j,\ell}}^{B_j} \frac{1}{2f(t)} dt + \int_{C_{j,\ell}}^{B_\ell} \frac{1}{2f(t)} dt \right)_{1 \leq j, \ell \leq k}.$$

\square

7.5.5 Gromov–Hausdorff tightness

We now establish (7.5.2). We first introduce some notation: for $n \geq 1$, let V^n and $(V_i^n)_{i \geq 1}$ be independent uniform vertices in \mathbb{V}_n . For simplicity, we drop the superscript and write $V, (V_i)_{i \geq 1}$. For $u, v \in \mathbb{V}_n$ and $B \subset \mathbb{V}_n$ we write

$$\Delta^n(u, v) := \frac{1}{2}(\mathbf{b}_n(u) + \mathbf{b}_n(v) - 2c_n(u, v)), \quad \Delta^n(u, B) := \min_{v \in B} \Delta^n(u, v).$$

Observe that Δ^n satisfies the triangle inequality. Finally, we set

$$M_n := \max_{0 \leq i \leq n} S_i^n.$$

Our main technical input is the following uniform estimate.

Proposition 7.5.5. *For every $k, n \geq 1$ large enough with $k \leq n^{1/3}$,*

$$\mathbb{P}\left(\Delta^n(V, \{V_1, \dots, V_k\}) > \frac{\log(k)^2}{\sqrt{k}} n\right) \leq \frac{1}{k^4}.$$

The idea underlining the proof of Proposition 7.5.5 is the following: on every time-interval of length n/\sqrt{k} , we can find roughly \sqrt{k} vertices among V_1, \dots, V_k and the probability that \sqrt{k} vertices do not coalesce with a given vertex during n/\sqrt{k} time steps is strictly positive. Let us first explain how tightness follows from this estimate.

Proof of Theorem 7.2.3 (tightness) using Proposition 7.5.5. Step 1. We first check that for every $\varepsilon > 0$ we have

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_k\})/n > \varepsilon\right) = 0. \quad (7.5.8)$$

To this end, we apply the so-called chaining method. Specifically, for $k, n \geq 1$ we set

$$\varepsilon_k^n := \max_{2^k < i \leq 2^{k+1}} \Delta^n(V_i, \{V_1, \dots, V_{2^k}\}),$$

and roughly speaking, we show that $\varepsilon_k^n/n \lesssim 2^{-k/2}$, where \lesssim denotes an informal upper bound. Indeed, this entails

$$\frac{1}{n} \max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2^k}\}) \lesssim \frac{1}{n} \sum_{i \geq k} \varepsilon_i^n \lesssim 2^{-k/2}$$

and (7.5.8) will follow. Let us now be more precise. For $n \geq 1$, let N_n be the largest integer with $2^{N_n} \leq n^{1/3}$ and set $\varepsilon_\infty^n := \max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2^{N_n}}\})$. By Proposition 7.5.5, for every k, n large enough with $k < N_n$,

$$\mathbb{P}\left(\varepsilon_k^n > k^2 2^{-k/2} n\right) \leq 2^k \mathbb{P}\left(\Delta^n(V, \{V_1, \dots, V_{2^k}\}) > k^2 2^{-k/2} n\right) \leq 2^{-3k}.$$

Similarly, since there are at most n vertices in \mathcal{T}^n ,

$$\mathbb{P}\left(\varepsilon_\infty^n > (N_n)^2 2^{-N_n/2n}\right) \leq n\mathbb{P}\left(\Delta^n(V, \{V_1, \dots, V_{2N_n}\}) > (N_n)^2 2^{-N_n/2n}\right) \leq n2^{-4N_n} = o(1).$$

Hence, for every k large enough, for every n large enough, with probability at least $1 - \sum_{i=k}^\infty 2^{-3i} + o(1) \geq 1 - 2^{-k} + o(1)$, using the triangle inequality,

$$\max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2k}\}) \leq \sum_{i=k}^{N_n-1} \varepsilon_i^n + \varepsilon_\infty^n \leq \sum_{i=k}^{N_n} i^2 2^{-i/2} n \leq c 2^{-k/3} n$$

for some constant $c > 0$. The convergence (7.5.8) readily follows.

Step 2. We now show (7.5.2) using (7.5.8). By Proposition 7.3.2 (applied with $\alpha = 1/2$), it suffices to show that for every $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{v \in \mathbb{V}_n} \min_{1 \leq i \leq k} D^n(v, V_i) / \sqrt{n} > \varepsilon\right) = 0. \quad (7.5.9)$$

Now, observe that we may fix $\delta > 0$ such that for n sufficiently large:

$$\sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{S_i^n \leq \delta \sqrt{n}} \leq \varepsilon \sqrt{n}. \quad (7.5.10)$$

Indeed, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{S_i^n \leq \delta \sqrt{n}} = \int_0^1 \frac{\sqrt{n}}{S_{nt}} \mathbb{1}_{1/\delta \leq \frac{\sqrt{n}}{S_{nt}}} dt$$

and (7.5.10) follows from $(\text{Lim}_{1/f}^\alpha)$ by using the fact that convergence in L^1 implies uniform integrability.

Then, by the definition of D^n , write for every $\delta > 0$ and $u, v \in \mathbb{V}_n$:

$$\frac{D^n(u, v)}{\sqrt{n}} \leq \sum_{i=1}^n \frac{1}{\sqrt{n} S_i^n} \mathbb{1}_{S_i^n \leq \delta \sqrt{n}} + \frac{\Delta^n(u, v)}{n\delta} \leq \varepsilon + \frac{\Delta^n(u, v)}{n\delta}.$$

The estimate (7.5.9) then follows from (7.5.8). \square

We now turn to the proof of Proposition 7.5.5. Recall that for $n \in \mathbb{N}$, $M_n := \max_{0 \leq i \leq n} S_i^n$. Let $M > 0$ be such that $M_n \leq M\sqrt{n}$ for all $n \geq 1$ (this is possible by assumption). For $k, n \in \mathbb{N}$ let $\delta_{k,n} := 4\lceil Mn/\sqrt{k} \rceil$. Observe that $\delta_{k,n} > M_n$ for k, n sufficiently large with $k \leq n^{1/3}$.

Lemma 7.5.6. *For every $k, n \geq 1$ large enough with $k \leq n^{1/3}$, for every $0 \leq i \leq n/\delta_{k,n}$, for every $v_0 \in \mathbb{V}_n$ with $\mathbf{b}_n(v_0) > (i+1)\delta_{k,n}$, for every $K \geq 1$ and $v_1 \neq \dots \neq v_K \in \mathbb{V}_n$ such that $i\delta_{k,n} < \mathbf{b}_n(v_j) \leq (i+1)\delta_{k,n}$, we have*

$$\mathbb{P}\left(\text{the tree of } \mathcal{F}_{(i-1)\delta_{k,n}}^n \text{ containing } v_0 \text{ does not contain any vertex among } v_1, \dots, v_K\right) \leq \frac{3M\sqrt{k}}{4K}.$$

Proof. For every $0 \leq t \leq n$, we define N_t^n by induction as follows: $N_n^n = K + 1$, and then for $1 \leq t \leq n$ we set $N_{t-1}^n = N_t^n - 1$ if $\mathbf{x}_t^n = 1$ and two trees of \mathcal{F}_t^n which contain a vertex in $\{v_0, \dots, v_K\}$ coalesce at time $t - 1$ and $N_{t-1}^n = N_t^n$ otherwise. In words, N_t^n represents the number of trees in \mathcal{F}_t^n which contain a vertex in $\{v_0, \dots, v_K\}$, taking also into account vertices which are not born yet.

Step 1. We first show that

$$\begin{aligned} \mathbb{P}\left(\text{the tree of } \mathcal{F}_{(i-1)\delta_{k,n}}^n \text{ containing } v_0 \text{ does not contain any vertex among } v_1, \dots, v_K\right) \\ \leq \frac{1}{K+1} \mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right]. \end{aligned} \quad (7.5.11)$$

To this end, we say that $u \in \{v_0, \dots, v_K\}$ stays alone at time t if either $t > \mathbf{b}_n(u)$, or if the tree of \mathcal{F}_t^n which contains u does not contain any other vertex of $\{v_0, \dots, v_K\} \setminus \{u\}$. Plainly, by definition N_t^n is at

least equal to the number of vertices which stay alone at time t . Moreover, since $\mathbf{b}_n(v_0) > \mathbf{b}_n(v_i)$ for $1 \leq i \leq K$, v_0 is the vertex with the less chance to stay alone at time t among $\{v_0, \dots, v_K\}$. Hence

$$\frac{1}{K+1} \mathbb{E}[N_t^n] \geq \frac{1}{K+1} \sum_{i=0}^{K+1} \mathbb{P}(v_i \text{ stays alone at time } t) \geq \mathbb{P}(v_0 \text{ stays alone at time } t).$$

The bound (7.5.11) follows by noting that if v_0 stays alone at time t then the tree of \mathcal{F}_t^n containing v_0 does not contain any vertex among v_1, \dots, v_K , with $t = (i-1)\delta_{k,n}$.

Step 2. We next show that

$$\mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right] \leq 1 + \frac{2(M_n)^2}{\delta_{k,n} - M_n - 1} \quad (7.5.12)$$

for $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$. For every $(i-1)\delta_{k,n} < t \leq i\delta_{k,n}$, depending on whether two trees which contain a vertex in $\{v_0, \dots, v_k\}$ coalesce or not at time $t-1$ in Algorithm 2, we get

$$\mathbb{E}[N_{t-1}^n] = \mathbb{E}[N_t^n] - \mathbb{1}_{\mathbf{x}_t^n=1} \frac{\mathbb{E}[N_t^n(N_t^n - 1)]}{S_t^n(S_t^n - 1)} \leq \mathbb{E}[N_t^n] - \mathbb{1}_{\mathbf{x}_t^n=1} \frac{\mathbb{E}[N_t^n]\mathbb{E}[N_t^n - 1]}{(M_n)^2}.$$

Then, for every $(i-1)\delta_{k,n} < t \leq i\delta_{k,n}$, write

$$\frac{1}{\mathbb{E}[N_{t-1}^n] - 1} - \frac{1}{\mathbb{E}[N_t^n] - 1} = \frac{\mathbb{E}[N_t^n] - \mathbb{E}[N_{t-1}^n]}{(\mathbb{E}[N_{t-1}^n] - 1)(\mathbb{E}[N_t^n] - 1)} \geq \frac{\mathbb{1}_{\mathbf{x}_t^n=1} \mathbb{E}[N_t^n]\mathbb{E}[N_t^n - 1]/(M_n)^2}{(\mathbb{E}[N_{t-1}^n] - 1)(\mathbb{E}[N_t^n] - 1)} \geq \frac{\mathbb{1}_{\mathbf{x}_t^n=1}}{(M_n)^2}.$$

As a consequence, for $k, n \geq 1$,

$$\frac{1}{\mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right] - 1} \geq \frac{1}{\mathbb{E}\left[N_{i\delta_{k,n}}^n\right] - 1} + \sum_{j=(i-1)\delta_{k,n}+1}^{i\delta_{k,n}} \frac{\mathbb{1}_{\mathbf{x}_j^n=1}}{(M_n)^2} \geq \frac{S_{i\delta_{k,n}}^n - S_{(i-1)\delta_{k,n}}^n + i\delta_{k,n} - (i-1)\delta_{k,n}}{2(M_n)^2}$$

and (7.5.12) follows $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$ (so that $\delta_{k,n} > M_n$).

Step 3. To complete the proof of Lemma 7.5.6, it just remains to note that for $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$ we have $\delta_{k,n} - M_n - 1 \geq 4Mn/\sqrt{k} - M_n - 1 \geq 3Mn/\sqrt{k}$ so that

$$\frac{1}{K+1} \left(1 + \frac{2(M_n)^2}{\delta_{k,n} - M_n - 1}\right) \leq \frac{1}{K+1} \left(1 + \frac{2M^2n}{3Mn/\sqrt{k}}\right) \leq \frac{M\sqrt{k}}{K+1} \left(\frac{1}{M\sqrt{k}} + \frac{2}{3}\right) \leq \frac{3M\sqrt{k}}{4K}.$$

This completes the proof. \square

Proof of Proposition 7.5.5. Recall that $M_n \leq M\sqrt{n}$ for every $n \geq 1$ and that $\delta_{k,n} = 4\lceil Mn/\sqrt{k} \rceil$.

To simplify notation, set $\mathbf{b}_n(V_j) = \mathbf{b}_n^j$ for $1 \leq j \leq n$ and $\mathbf{b}_n = \mathbf{b}_n(V)$ and define the event E by

$$E = \left\{ \forall 0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1, \#\{\mathbf{b}_n^j : 1 \leq j \leq k \text{ and } i\delta_{k,n} < \mathbf{b}_n^j \leq (i+1)\delta_{k,n}\} > M\sqrt{k} \right\}.$$

Step 1. We first check that

$$\mathbb{P}(E^c) = o\left(\frac{1}{k^4}\right). \quad (7.5.13)$$

To this end, first note that for every $0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1$,

$$\sum_{i\delta_{k,n} < t \leq (i+1)\delta_{k,n}} \mathbb{1}_{\mathbf{x}_t^n=-1} = (\delta_{k,n} - S_{(i+1)\delta_{k,n}}^n + S_{i\delta_{k,n}}^n)/2 \geq \delta_{k,n}/2 - M\sqrt{n}/2 \geq \frac{3Mn}{2\sqrt{k}}.$$

for k, n sufficiently large with $k \leq n^{1/3}$. Since there are at most n vertices in \mathcal{T}^n , it follows that we have $\mathbb{P}(i\delta_{k,n} < \mathbf{b}_n \leq (i+1)\delta_{k,n}) \geq 3M/(2\sqrt{k}) - O(1/\sqrt{n})$. Next, if $\text{Bin}(k, p)$ denotes a Binomial random variable with parameter (k, p) , by Bennett's inequality (Proposition 6.4.1) we have for $0 < u < pk$:

$$\mathbb{P}(\text{Bin}(k, p) < u) \leq \exp\left(-pk g\left(1 - \frac{u}{pk}\right)\right).$$

Since $\text{Bin}(k, p)$ is stochastically increasing in p , it follows that

$$\mathbb{P}\left(\#\{1 \leq j \leq k : \mathbf{b}_n^j \in (i\delta_{k,n}, (i+1)\delta_{k,n}]\} \leq 4M\sqrt{k}\right) \leq \exp\left(-6M\sqrt{k}g(1/3)\right) \leq \exp\left(-\frac{\sqrt{k}}{100}\right),$$

since M can be chosen large. So, by a union bound,

$$\mathbb{P}\left(\exists 0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1, \#\{1 \leq j \leq k : \mathbf{b}_n^j \in (i\delta_{k,n}, (i+1)\delta_{k,n}]\} \leq 4M\sqrt{k}\right) \leq \frac{n}{\delta_{k,n}} e^{-\sqrt{k}/100} \leq \frac{\sqrt{k}}{4M} e^{-\sqrt{k}/100},$$

which is $o(k^{-4})$. To establish (7.5.13) it thus remains to check that

$$\mathbb{P}(\exists 1 \leq i_1 < i_2 < i_3 < i_4 \leq k : \mathbf{b}_n^{i_1} = \mathbf{b}_n^{i_2} = \mathbf{b}_n^{i_3} = \mathbf{b}_n^{i_4}) = o\left(\frac{1}{k^4}\right).$$

To this end, using the fact that there are at least $n/3$ vertices in \mathcal{T}^n for n sufficiently large, a union bound yields $\mathbb{P}(\exists 1 \leq i_1 < i_2 < i_3 < i_4 \leq k : \mathbf{b}_n^{i_1} = \mathbf{b}_n^{i_2} = \mathbf{b}_n^{i_3} = \mathbf{b}_n^{i_4}) \leq 27k^4/n^3 = o(1/k^4)$ since $k \leq n^{1/3}$.

Step 2. Next we need some more notation. Set

$$\mathbb{V}_n(i) = \{V_j : 1 \leq j \leq k \text{ and } i\delta_{k,n} < \mathbf{b}_n^j \leq (i+1)\delta_{k,n}\}.$$

Observe that under E we have $\#\mathbb{V}_n(i) \geq M\sqrt{k}$ for every $0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1$. Setting $p_k = \lfloor 100 \log(k) \rfloor$, we show that

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k\delta_{k,n} \mid E) = o(k^{-4}), \quad (7.5.14)$$

and the desired result will follow

First case: \mathbf{b}_n is small. First, if $\mathbf{b}_n \leq 2p_k\delta_{k,n}$, under E there exists $1 \leq j \leq k$ such that $\mathbf{b}_n^j \leq 2\delta_{k,n}$, and then $\Delta^n(V, \{V_1, \dots, V_k\}) \leq \Delta^n(V, V_j) \leq 2p_k\delta_{k,n}$. Thus

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k\delta_{k,n} \mid E, \mathbf{b}_n \leq 2p_k\delta_{k,n}) = 0$$

Second case. \mathbf{b}_n is large. For $i \geq 1$, let A_i be the event:

“the tree of $\mathcal{F}_{(i-1)\delta_{k,n}}^n$ containing V does not contain any vertex of $\mathbb{V}_n(i)$.”

By definition of Algorithm 2, for $i_0 \geq 2p_k$, conditionally given E and $i_0\delta_{k,n} < \mathbf{b}_n \leq (i_0+1)\delta_{k,n}$, the events $(A_i)_{2 \leq i \leq i_0-1, i \text{ even}}$ are independent and by Lemma 7.5.6 (applied with $K = \lceil M\sqrt{k} \rceil$) their probability is at most $3/4$. As a consequence, for every $i_0 \geq 2p_k$:

$$\mathbb{P}\left(\bigcap_{i=i_0-p_k-1}^{i_0-1} A_i \mid E, i_0\delta_{k,n} < \mathbf{b}_n \leq (i_0+1)\delta_{k,n}\right) \leq (3/4)^{(p_k-1)/2}.$$

But then observe that when $i_0\delta_{k,n} < \mathbf{b}_n \leq (i_0+1)\delta_{k,n}$, if for $i_0-p_k-1 \leq i \leq i_0-1$, if the tree of $\mathcal{F}_{i\delta_{k,n}}^n$ containing V contains a vertex of $\mathbb{V}_n(i)$, say V_j , then

$$\Delta^n(V, V_j) \leq \frac{1}{2}((i_0+1)\delta_{k,n} + (i+1)\delta_{k,n} - 2(i-1)\delta_{k,n}) \leq \left(\frac{i_0-i}{2} + 3\right)\delta_{k,n} + 1 \leq 2p_k\delta_{k,n}$$

for k, n sufficiently large. We conclude from the previous discussion that

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k\delta_{k,n} \mid E, \mathbf{b}_n > 2p_k\delta_{k,n}) = o\left((3/4)^{(p_k-1)/2}\right) = o(k^{-4}),$$

where the last equality comes from our choice of p_k . This completes the proof. \square

7.5.6 Properties of $\mathcal{T}(f)$

Here we study some properties of $\mathcal{T}(f)$ by establishing Theorem 7.2.4.

Proof of Theorem 7.2.4 (1). The fact that the mass measure on $\mathcal{T}(f)$ has full support was already mentioned in Section 7.5.1 as a consequence of Lemma 7.7.1, taking into account (7.5.1) and (7.5.2). The fact that it gives full support to the leaves follows from the property the coalescence time of two particles of the continuous-time coalescent is strictly smaller than the birth times, combined with (7.5.5) and (7.5.4): this implies that for two vertices sampled uniformly at random in $\mathcal{T}(f)$, one of them is not a descendant of the other. \square

Proof of Theorem 7.2.4 (3). Let us first assume that $\int_0 \frac{1}{f^2} < \infty$. To show that the degree of the root of $\mathcal{T}(f)$ is a.s. infinite, it suffices to show that a.s. there exist infinitely many clusters which do not coalesce with each other. First, observe that since $\int_0 \frac{1}{f^2} < \infty$, we can build a decreasing sequence $(\varepsilon_k)_{k \geq 0}$ of positive numbers such that

$$\int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{1}{f^2} \leq \frac{2^{-k}}{\binom{k}{2}}.$$

Next, since $(B_i)_{i \geq 1}$ are i.i.d. uniform random variables, we know that a.s. there exists a subsequence $(B_{i_k})_{k \geq 1}$ such that for all $k \geq 1$, we have $B_{i_k} \in [\varepsilon_{k+1}, \varepsilon_k]$. But then, conditionally on the B_{i_k} 's, for all $k_0 \geq 1$, the probability that none of the clusters containing a B_{i_k} for $k \geq k_0$ coalesce with each other is

$$\prod_{k \geq k_0} \exp\left(-\int_{B_{i_{k+1}}}^{B_{i_k}} \frac{\binom{k}{2}}{f(t)^2} dt\right) \geq \prod_{k \geq k_0} \exp\left(-\int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{\binom{k}{2}}{f(t)^2} dt\right) \geq \exp\left(-\sum_{k \geq k_0} 2^{-k}\right) \xrightarrow{k_0 \rightarrow \infty} 1.$$

Thus, with probability one there exist infinitely many clusters which do not coalesce with each other. Conversely, when $\int_0 \frac{1}{f^2} = \infty$, the probability that two clusters which have not merged yet at time $t > 0$ do not coalesce until time zero is $\exp\left(-\int_0^t \frac{1}{f(t)^2} dt\right) = 0$, which proves that the root has a.s. degree 1. For the other branchpoints, the fact that they have degree two is due to the fact that a.s. the coalescences happen at distinct times and thus at distinct heights. \square

7.5.7 Application: the Brownian CRT

Here we establish Corollary 7.2.5.

Proof of Corollary 7.2.5. The idea is to apply Theorem 7.2.3 with a random sequence $(\mathbf{x}^{2n+1})_{n \geq 1}$ chosen in such a way that $(S_k^{2n+1})_{0 \leq k \leq 2n+1}$ has the law of a simple random walk started at 1 and conditioned to hit 0 for the first time at time $2n+1$.

First, by Theorem 6.1.2, as a random compact metric space, \mathcal{T}_{2n+1} has the same law as a uniform plane tree with $n+1$ vertices, so that $\frac{1}{\sqrt{2n+1}} \cdot \mathcal{T}_{2n+1}$ converges in distribution to the Brownian CRT [Ald93].

Second, we shall show that $\frac{1}{\sqrt{2n+1}} \cdot \mathcal{T}_{2n+1}$ converges in distribution to $\mathcal{T}(\mathfrak{e})$, which will imply that $\mathcal{T}(\mathfrak{e})$ has the law of the Brownian CRT. To this end, it is standard (see e.g. [Kai76]) that the convergence

$$\left(\frac{S_{(2n+1)t}^{2n+1}}{\sqrt{2n+1}} : 0 \leq t \leq 1\right) \xrightarrow[n \rightarrow \infty]{(d)} \mathfrak{e} \quad (7.5.15)$$

holds in distribution for the J_1 -Skorokhod topology and also for the uniform convergence since the limit is continuous, where \mathfrak{e} is the Brownian excursion.

By Skorokhod's representation theorem, we may assume that the convergence (7.5.15) holds almost surely. We shall check that the conditions of application of Theorem 7.2.3 are satisfied almost surely with the sequence $(\mathbf{x}^{2n+1})_{n \geq 1}$ and $f = \mathfrak{e}$.

Notice that the conditions $(\text{Lim}_{1/f^2}^\alpha)$ and $\max_{1 \leq k \leq 2n+1} S_k^{2n+1} = \mathcal{O}(\sqrt{n})$ are satisfied thanks to (7.5.15) and to the continuity of f . We claim that $(\text{Lim}_{1/f}^\alpha)$ follows if we check that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}} = 0. \quad (7.5.16)$$

Indeed, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}} = \int_0^1 \frac{\sqrt{n}}{S_{nt}^{2n+1}} \mathbb{1}_{1/\delta \leq \frac{\sqrt{n}}{S_{nt}^{2n+1}}} dt,$$

so (7.5.16) is equivalent to the uniform integrability of $(\sqrt{2n+1}/S_{(2n+1)t}^{2n+1})_{0 \leq t \leq 1}$. Combined with (7.5.15), this implies convergence in L^1 and thus $(\text{Lim}_{1/f}^\alpha)$.

It remains to check (7.5.16). To simplify notation, for $n \geq 1$ and $0 \leq \eta < \delta \leq 1$ set

$$X_n(\delta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}}, \quad X(\eta, \delta) = \int_0^1 \frac{1}{\mathfrak{e}(t)} \mathbb{1}_{\eta \leq \mathfrak{e}(t) \leq \delta} dt.$$

It suffices to check that for every $\varepsilon > 0$, for a well chosen coupling, almost surely there exists $\delta > 0$ such that every $n \geq 1$ sufficiently large we have $X_n(\delta) \leq \varepsilon$.

Step 1. We show that

$$\mathbb{P}(X_n(\delta) \geq \varepsilon) \leq \frac{C}{\varepsilon} \delta^2 \quad (7.5.17)$$

for all $n \geq 1$ for a certain constant $C > 0$. This is essentially [CMY00, Lemma 3.6], but let us give a different proof based on the local limit theorem which gives a slightly better upper bound. To this end, denote by $(S_i)_{i \geq 0}$ a simple random walk started from 1. By Kemperman's formula (see e.g. [Pit06, Sec. 6.1]), we have for $1 \leq i \leq 2n+1$

$$\mathbb{P}(S_i^{2n+1} = k) = \frac{2n+1}{\mathbb{P}(S_{2n+1} = 0)} \cdot \frac{k-1}{i} \mathbb{P}(S_i = k) \cdot \frac{k}{2n-i+1} \mathbb{P}(S_{2n-i+1} = -k+1). \quad (7.5.18)$$

By the local limit theorem (see e.g. [IL71, Theorem 4.2.1]), we get for all $n \geq 1$, for $1 \leq i \leq n$,

$$\mathbb{P}(S_i^{2n+1} = k) \leq C n^{3/2} \cdot \frac{k}{i} \cdot \frac{1}{\sqrt{i}} \cdot \frac{k}{n} \cdot \frac{1}{\sqrt{n}} = C \frac{k^2}{i^{3/2}},$$

where C is a constant that may change from line to line. It follows that

$$\mathbb{E} \left[\frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}} \right] \leq \sum_{k=1}^{\delta \sqrt{n}} C \frac{k}{i^{3/2}} \leq C \frac{\delta^2 n}{i^{3/2}}.$$

Thus

$$\mathbb{P} \left(\sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}} \geq \varepsilon \sqrt{n} \right) \leq \frac{1}{\varepsilon \sqrt{n}} \sum_{i=1}^n \mathbb{E} \left[\frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}} \right] \leq \frac{C}{\varepsilon} \delta^2 \sqrt{n} \sum_{i=1}^n \frac{1}{i^{3/2}} \leq \frac{C}{\varepsilon} \delta^2$$

and (7.5.17) follows.

Step 2. We show that

$$X_n(\delta) \xrightarrow[n \rightarrow \infty]{(d)} X(0, \delta) \quad (7.5.19)$$

jointly with (7.5.15). Observe that by continuity, for $\eta \in (0, \delta)$,

$$X_n(\delta) - X_n(\eta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{\eta \sqrt{n} \leq S_i^{2n+1} \leq \delta \sqrt{n}} \xrightarrow[n \rightarrow \infty]{(d)} X(\eta, \delta)$$

jointly with (7.5.15). Then we have:

- $X_n(\eta) \rightarrow 0$ in probability uniformly in n as $\eta \rightarrow 0$ by (7.5.17);
- $X(\eta, \delta) \rightarrow X(0, \delta)$ as $\eta \rightarrow 0$;
- for every $\eta \in (0, \delta)$ we have $X_n(\delta) - X_n(\eta) \rightarrow X(\eta, \delta)$ in distribution as $n \rightarrow \infty$.

This entails (7.5.19).

Step 3. By Skorokhod's representation theorem, we may assume that the convergences (7.5.15) and (7.5.19) both hold almost surely. Choose $\delta > 0$ such that $X(0, \delta) \leq \varepsilon$. Then for every n sufficiently large we have $X_n(\delta) \leq 2\varepsilon$. This completes the proof. \square

Remark 7.5.7. *By combining Theorem 7.2.1 and Corollary 7.2.5 we get the following well-known distributional identity (see e.g. [CMY00]):*

$$\frac{1}{2} \int_0^1 \frac{dt}{\mathfrak{e}(t)} \stackrel{(d)}{=} \sup \mathfrak{e}.$$

7.6 Background on the GP, GH and GHP topologies

We give background on various topologies that we use, based on [BR21, BR23].

7.6.1 The Gromov–Prokhorov (GP) topology

A measured metric space is a triple (X, d, μ) such that (X, d) is a Polish space and μ is a Borel probability measure on X . Two such spaces $(X, d, \mu), (X', d', \mu')$ are called GP-isometry-equivalent if and only if there exists an isometry $f : \text{supp}(X) \rightarrow \text{supp}(X')$ such that if $f_*\mu$ is the image of μ by f then $f_*\mu = \mu'$. Let \mathbb{K}_{GP} be the set of GP-equivalent classes of measured metric space. Given a measured metric space (X, d, μ) , we write $[X, d, \mu]$ for the GP-isometry-equivalence class of (X, d, μ) and frequently use the notation X for either (X, d, μ) or $[X, d, \mu]$.

We now recall the definition of the Prokhorov distance. Consider a metric space (X, d) . For every $A \subset X$ and $\varepsilon > 0$ let $A^\varepsilon := \{x \in X, d(x, A) < \varepsilon\}$ be the open ε -neighborhood of A . Then given two (Borel) probability measures μ, ν on X , the Prokhorov distance between μ and ν is defined by

$$d_P(\mu, \nu) := \inf\{\varepsilon > 0: \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \text{ for all Borel set } A \subset X\}.$$

The Gromov–Prokhorov (for short GP) distance is an extension of the Prokhorov’s distance: For every $[X, d, \mu], [X', d', \mu'] \in \mathbb{K}_{\text{GP}}$ the Gromov–Prokhorov distance between X and X' is defined by

$$d_{\text{GP}}([X, d, \mu], [X', d', \mu']) := \inf_{S, \phi, \phi'} d_P(\phi_*\mu, \phi'_*\mu'),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S, \phi' : X' \rightarrow S$. d_{GP} is indeed a distance on \mathbb{K}_{GP} and $(\mathbb{K}_{\text{GP}}, d_{\text{GP}})$ is a Polish space (see e.g. [ADH13]).

We use another convenient characterization of the GP topology using the convergence of distance matrices: For every measured metric space (X, d^X, μ^X) let $(x_i^X)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of common distribution μ^X and let $M^X := (d^X(x_i^X, x_j^X))_{i, j \in \mathbb{N}}$. We have the following result from [Loe13],

Lemma 7.6.1. *Let $(X^n)_{n \in \mathbb{N}} \in \mathbb{K}_{\text{GP}}^{\mathbb{N}}$ and let $X \in \mathbb{K}_{\text{GP}}$ then $X^n \xrightarrow{\text{GP}} X$ as $n \rightarrow \infty$ if and only if M^{X^n} converges in distribution toward M^X .*

7.6.2 The Gromov–Hausdorff (GH) topology

Let \mathbb{K}_{GH} be the set of isometry-equivalent classes of compact metric space. For every metric space (X, d) , we write $[X, d]$ for the isometry-equivalent class of (X, d) , and frequently use the notation X for either (X, d) or $[X, d]$.

For every metric space (X, d) , the Hausdorff distance between $A, B \subset X$ is given by

$$d_H(A, B) := \inf\{\varepsilon > 0, A \subset B^\varepsilon, B \subset A^\varepsilon\}.$$

The Gromov–Hausdorff distance between $[X, d], [X', d'] \in \mathbb{K}_{\text{GH}}$ is given by

$$d_{\text{GH}}([X, d], [X', d']) := \inf_{S, \phi, \phi'} (d_H(\phi(X), \phi'(X'))),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S, \phi' : X' \rightarrow S$. d_{GH} is indeed a distance on \mathbb{K}_{GH} and $(\mathbb{K}_{\text{GH}}, d_{\text{GH}})$ is a Polish space (see e.g. [ADH13]).

7.6.3 The Gromov–Hausdorff–Prokhorov (GHP) topology

Two measured metric spaces $(X, d, \mu), (X', d', \mu')$ are called GHP-isometry-equivalent if and only if there exists an isometry $f : X \rightarrow X'$ such that if $f_*\mu$ is the image of μ by f then $f_*\mu = \mu'$. Let $\mathbb{K}_{\text{GHP}} \subset \mathbb{K}_{\text{GP}}$ be the set of isometry-equivalent classes of compact measured metric space.

The Gromov–Hausdorff–Prokhorov distance between $[X, d, \mu], [X', d', \mu'] \in \mathbb{K}_{\text{GHP}}$ is given by

$$d_{\text{GHP}}([X, d, \mu], [X', d', \mu']) := \inf_{S, \phi, \phi'} (d_P(\phi_*\mu, \phi'_*\mu') + d_H(\phi(X), \phi'(X'))),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S$, $\phi' : X' \rightarrow S$. d_{GHP} is indeed a distance on \mathbb{K}_{GHP} and $(\mathbb{K}_{\text{GHP}}, d_{\text{GHP}})$ is a Polish space (see [ADH13]).

Note that random variables which are GHP measurable are also GH measurable. For every $[X, d, p] \in \mathbb{K}_{\text{GHP}}$, let $[X, d]_{\text{GH}}$ denote its natural projection on \mathbb{K}_{GH} . Note that GHP convergence implies GH convergence of the projections on \mathbb{K}_{GH} , then that the projection on \mathbb{K}_{GH} is a measurable function. The same statements hold for the GP topology. We will need the following statement.

Lemma 7.6.2. [Lemma 4 in [BR21]] *Let $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ and $[X, d, p]$ be GHP measurable random variables in \mathbb{K}_{GHP} . Assume that almost surely $[X, d, p]$ has full support. Assume that $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d, p]$ in a GP sense, and that $([X^n, d^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d]$ in a GH sense. Then $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d, p]$ in a GHP sense.*

7.7 The leaf-tightness criterion

In this section, $\mathbf{X} = ((X^n, d^n, p^n))_{n \in \mathbb{N}}$ denotes a sequence of random compact measured metric spaces GHP-measurable. For every $n \in \mathbb{N}$ let $(x_i^n)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of common distribution p^n , then let $M^n := (d^n(x_i^n, x_j^n))_{i, j \in \mathbb{N}}$. We say that \mathbf{X} is leaf-tight if and only if

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_H(X^n, \{x_1^n, x_2^n, \dots, x_k^n\}) > \delta) = 0. \quad (7.7.1)$$

Although the criterion is unrelated with leaves, its name ‘‘leaf-tight’’ stayed as the original name from Aldous [Ald93] which first introduced this criterion to study random trees with random leaves.

Proposition 7.7.1. *If $(M^n)_{n \in \mathbb{N}}$ converges weakly toward a random matrix M , and \mathbf{X} is leaf-tight, then \mathbf{X} converges weakly for the GHP topology toward a random compact measured metric space (X, d, p) . Furthermore, if $(x_i)_{i \in \mathbb{N}}$ are i.i.d. random variables of law p then $M^X := (d(x_i, x_j))_{i, j \in \mathbb{N}} \stackrel{(d)}{=} M$. In addition, a.s. p has full support.*

Proof. We focus on the first statement, as the other two will naturally follow alongside the proof. First, it directly follows from the convergence of $(M_n)_{n \in \mathbb{N}}$ and from (7.7.1) that:

- $(\text{Diam}(X^n))_{n \in \mathbb{N}}$ is tight.
- For every $\delta > 0$, $(N_\delta(X^n))_{n \in \mathbb{N}}$ is tight, where N_δ stands for the usual δ -covering number (see [ADH13])

As a result, by [ADH13, Theorem 2.4], \mathbf{X} is tight for the weak GHP-topology.

Let $[X, d, p]$, $[X', d', p']$ be two GHP subsequential limits of \mathbf{X} . Let us show $[X, d, p] \stackrel{(d)}{=} [X', d', p']$. Let $(x_i)_{i \in \mathbb{N}}$ be i.i.d. random variables of law p and let $M^X := (d(x_i, x_j))_{i, j \in \mathbb{N}}$. Define similarly $(x'_i)_{i \in \mathbb{N}}$, $M^{X'}$. By Lemma 7.6.1, jointly with the above convergence, along the proper subsequence, we have weakly $M^n \rightarrow M^X$ and $M^n \rightarrow M^{X'}$. Note that this implies $M^X \stackrel{(d)}{=} M$. By the Skorokhod representation theorem, and up to further extract other subsequences, we may assume that those two convergences hold almost surely, and that almost surely $M^X = M^{X'}$. This implies that almost surely,

$$d_{\text{GH}}([\{x_i\}_{i \in \mathbb{N}}, d], [\{x'_i\}_{i \in \mathbb{N}}, d]) = 0.$$

And by the leaf-tightness criterion (7.7.1), almost surely $\{x_i\}_{i \in \mathbb{N}}$ and $\{x'_i\}_{i \in \mathbb{N}}$ are dense on X , X' so

$$d_{\text{GH}}([X, d], [X', d']) = 0.$$

Hence $[X, d]_{\text{GH}} \stackrel{(d)}{=} [X', d']_{\text{GH}}$. Also by Lemma 7.6.1, since $(M^n)_{n \in \mathbb{N}}$ converges weakly, we have the equality in distribution $[X, d, p]_{\text{GP}} \stackrel{(d)}{=} [X', d', p']_{\text{GP}}$. Moreover, since almost surely $\{x_i\}_{i \in \mathbb{N}}$ and $\{x'_i\}_{i \in \mathbb{N}}$ are dense on X and X' , almost surely p and p' have full support on X and X' . Therefore, by Lemma 7.6.2 we have $[X, d, p] \stackrel{(d)}{=} [X', d', p']$. This shows that there is only one subsequential possible limit for \mathbf{X} , and thus concludes the proof. \square

Chapter 8

Scaling limit of trees with vertices of fixed degrees and heights

Ce chapitre est issu d'un travail [BRK24] avec Arthur Blanc-Renaudie, soumis.

Abstract. We consider large uniform random trees where we fix for each vertex its degree and height. We prove, under natural conditions of convergence for the profile, that those trees properly renormalized converge. To this end, we study the paths from random vertices to the root using coalescent processes. As an application, we obtain scaling limits of Bienaymé-Galton-Watson trees in varying environment.

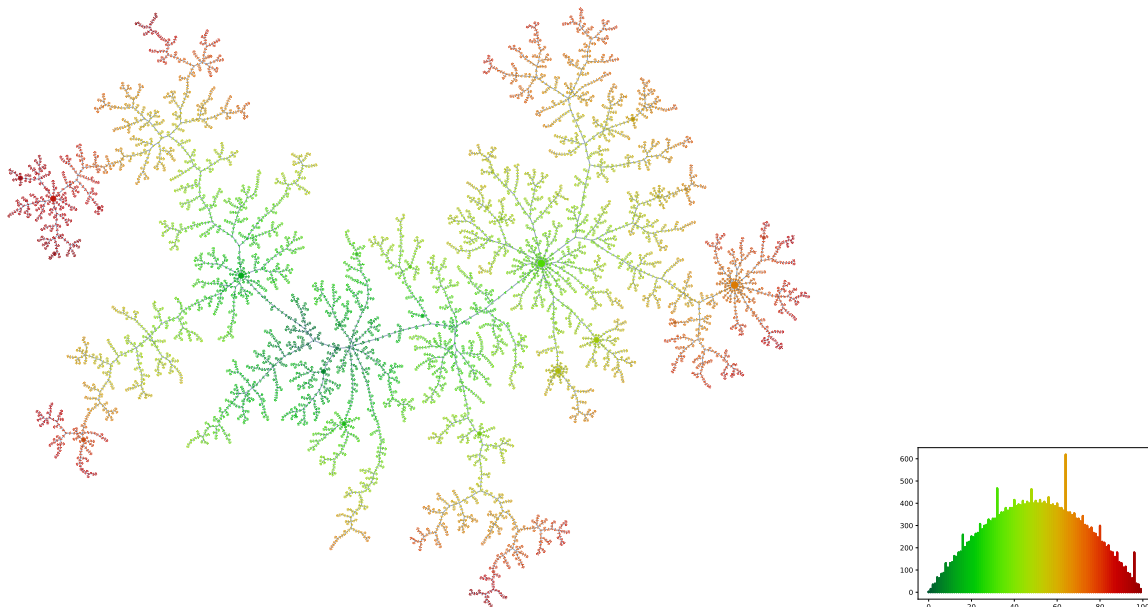


Figure 8.1 – Simulation of a uniform random tree of height 100 whose profile scales towards $t \mapsto \sin(\pi t)$ with vertices of degrees 0 and 4 with some additional vertices with large degrees. The profile is drawn on the right. The low vertices are green while the high vertices are red.

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8.1 Introduction

8.1.1 Model

Trees with fixed degree sequence are universal models of random trees and are related to many other objects: Galton–Watson trees, configuration model, bipartite planar maps with fixed face degrees. . . The aim of this paper is to study a variant of this model where we fix for each vertex its degree and also its height. More precisely, we are interested in the shortest-path distance on the rooted labelled trees defined below.

Consider a set of vertices $\{(i, j)\}_{i \geq 0, j \geq 1}$. We consider (i, j) to have height i and degree (number of children) $d_{i,j}^n \geq 0$. We use a superscript $n \in \mathbb{N}$ as we study the behaviour of the trees as the degrees vary. For convenience, we assume that on every line $i \geq 0$ the sequence $(d_{i,j}^n)_{j \geq 1}$ is non-increasing. Also, to ensure that every vertex (i, j) of degree $d_{i,j}^n \geq 1$ is connected to the root $(0, 1)$, we assume that $d_{0,2}^n = 0$ and

$$\forall i \geq 0, \quad \#\{j \geq 1; d_{i+1,j}^n > 0\} \leq D_i^n := \sum_{j \geq 1} d_{i,j}^n < \infty. \quad (8.1.1)$$

We construct the plane tree \mathcal{T}^n height by height: $(0, 1)$ is the root, and independently and uniformly for every height $i \geq 0$ we attach each vertex in $\{(i+1, j)\}_{1 \leq j \leq D_i^n}$ to a vertex in $\{(i, j); j \geq 1, d_{i,j}^n \geq 1\}$ so that every vertex (i, j) has exactly $d_{i,j}^n$ children (see Figure 8.2). Note that \mathcal{T}^n has height $h^n := \inf\{i \geq 0; d_{i,1}^n = 0\}$.

8.1.2 Main results

We now present our assumptions on the degrees for the scaling limit of \mathcal{T}^n . Without loss of generality, our assumptions are written so that the random metric space \mathcal{T}^n/n obtained after dividing the distances by n converges in distribution. First we assume that the profile converges weakly: for a non-atomic probability measure ν with compact support $\text{supp } \nu = [0, 1]$, we assume, writing δ_x for the Dirac measure on x ,

$(\text{Lim}_\nu) \quad \sum_{i \geq 0} \delta_{i/n} D_i^n / (\sum_{i \geq 0} D_i^n) \text{ converges weakly towards } \nu.$

Note that the above assumption is equivalent to asking a scaling limit for the height of random vertices. It also implies that $\liminf h^n/n \geq 1$

Then, to show the convergence of (\mathcal{T}^n/n) , we look at the **genealogies** of random vertices, *that is their paths to the root*. The rate at which genealogies merge actually only depends on the degrees. We

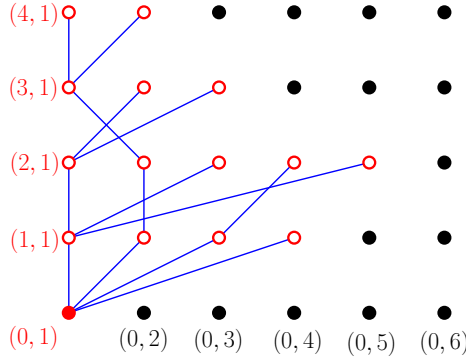


Figure 8.2 – An example of tree \mathcal{T} such that $d_{0,1} = 4$, $d_{1,1} = 3$, $d_{1,2} = 1$, $d_{1,3} = 1$, $d_{2,1} = 2$, $d_{2,2} = 1$, $d_{3,1} = 2$ and all the other degrees are null. The vertices of \mathcal{T} are represented in red, the root as a red dot, while the other vertices as black dots. Here, $D_0 = 4, D_1 = 5, D_2 = 3, D_3 = 2, D_4 = 0$.

assume that it converges. We distinguish two cases for such merging: either the paths merge at a vertex with large degree, or they merge among the numerous vertices of small degrees.

Concretely, we use a locally finite measure ρ on $(0, 1)$ to describe the merging rate due to small vertices. We also describe the limit of large degrees, using non negative parameters $\Theta = ((\theta_j(t))_{j \geq 1})_{t \in (0, 1)}$ satisfying:

- For every height $t \in (0, 1)$, the sequence $(\theta_j(t))_{j \geq 1}$ is non-increasing.
- For every $t \in (0, 1)$, we have $\sum_{j \geq 1} \theta_j(t) \leq 1$.
- The set $\{(j, t) : \theta_j(t) \neq 0\}$ is countable. Moreover, for every real numbers $0 < x < y < 1$:

$$\sum_{x \leq t \leq y} \sum_{j \geq 1} \theta_j(t)^2 < \infty. \quad (8.1.2)$$

We assume as $n \rightarrow \infty$:

(Lim $_{\rho}$) The measure $\sum_{1 \leq i < n, j: d_{i,j} > 0} \delta_{i/n} \binom{d_{i,j}^n}{2} / \binom{D_i^n}{2}$ converges weakly toward $\rho + \sum_{j,t: \theta_j(t) > 0} \theta_j(t)^2 \delta_t$.

(Lim $_{\Theta}$) We have the following weak convergence on $((0, 1) \times \mathbb{R}_+^*, \mathcal{B}((0, 1) \times \mathbb{R}_+^*))$

$$\sum_{1 \leq i < n, j: d_{i,j} > 0} \delta_{(i/n, d_{i,j}^n / D_i^n)} \xrightarrow{n \rightarrow \infty} \sum_{t, j: \theta_j(t) > 0} \delta_{(t, \theta_j(t))}.$$

(Lim $_{\neq}$) For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $n > 0$, i, i' with $\varepsilon n < i, i' < (1 - \varepsilon)n$, such that $d_{i,1}^n > \varepsilon D_i^n$ and $d_{i',1}^n > \varepsilon D_{i'}^n$ we either have $i = i'$ or $|i - i'| > \delta(\varepsilon)n$.

We require the technical condition **(Lim $_{\neq}$)** because the merging rate behaves really differently if vertices of large degree are at the same height or not. Morally, we use **(Lim $_{\neq}$)** to ensure that in the limit tree vertices of large degree that are at the same height correspond to vertices at the same height in \mathcal{T}^n .

Lastly, while the previous assumptions are sufficient to prove the convergence of the distances between random vertices, we want the following tightness assumption to ensure that the limit space is well defined:

(Tight $_{\text{GP}}$) ν has support $[0, 1]$ and for all $0 < a < b < 1$, $\rho([a, b]) > 0$ or $\sum_{j \in \mathbb{N}, t \in [a, b]: \theta_j(t) > 0} \theta_j(t) = \infty$.

Remark 8.1.1. Aldous, Camarri, and Pitman [AP00, CP00] used a similar condition to define inhomogeneous continuum random trees. Namely, with their notation $\theta_0 > 0$ or $\sum_{i \geq 1} \theta_i = \infty$.

Theorem 8.1.2. Assume that for some ν, ρ, Θ the conditions **(Lim $_{\nu}$)**, **(Lim $_{\rho}$)**, **(Lim $_{\Theta}$)**, **(Lim $_{\neq}$)**, **(Tight $_{\text{GP}}$)** hold. Then there exists a random measured metric space $\mathcal{T}(\nu, \rho, \Theta)$ such that the following weak conver-

gence holds for the Gromov–Prokhorov topology (see Section 7.6 for more details)

$$\mathcal{T}^n/n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}(\nu, \rho, \Theta),$$

where \mathcal{T}_n is equipped with the shortest path distance and the uniform probability measure on its vertices.

The Gromov–Prokhorov topology essentially describes the distances between random vertices in \mathcal{T}^n . To study geometric variables that depend on all the vertices such as the diameter of \mathcal{T}^n , one needs a stronger topology. To this end, we as usual strengthen the convergence to the Gromov–Hausdorff–Prokhorov topology (see Appendix 7.6). We use the following tightness assumptions.

First, for every $i \in \mathbb{N}, k > 0$, let

$$\tau_i^n(k) := \sum_{j \geq 1} \frac{d_{i,j}^n}{D_i^n} \min\left(k \frac{d_{i,j}^n - 1}{D_i^n - 1}, 1\right), \quad (8.1.3)$$

with the convention $\tau_i^n(k) = \infty$ when $D_i^n \in \{0, 1\}$. Then, for every $a < b$ let $\tau_{a,b}^n(k) := \sum_{i \in [a,b]} \tau_i^n(k)$.

(Lim_h) h^n/n converges towards 1.

(Tight_{||D||}) Let $\|D^n\| := \max_{0 \leq i \leq h^n} D_i^n$. As $n \rightarrow \infty$, $\log \log \|D^n\| = o(n)$.

(Tight_{GHP}) For every $0 < \alpha < \beta < 1$, for every $n \in \mathbb{N}, k \leq \|D^n\|$ large enough and $i \in [\alpha n, \beta n]$,

$$\tau_{i, i+n/(\log \log k)^2}^n(k) \geq \log(k).$$

Theorem 8.1.3. *Under the setting of Theorem 8.1.2 if furthermore the conditions (Lim_h), (Tight_{||D||}), and (Tight_{GHP}) hold then $\mathcal{T}^n/n \rightarrow \mathcal{T}(\nu, \rho, \Theta)$ weakly for the Gromov–Hausdorff–Prokhorov topology.*

Remark 8.1.4. *Writing $\psi_i^n : k \mapsto k\tau_i^n(k)$, one may see ψ as the analog for our model of the Laplace exponent of Lévy trees and Bienaymé–Galton–Watson trees. Also, (Tight_{GHP}) may be seen as an integrability condition for $1/\psi$. The main difference is that we morally need this integrability condition at all heights. Due to that, although condition (Tight_{GHP}) is sub-optimal, it seems strong enough to deal with the applications below.*

8.1.3 Connections with previous works and applications

When one only prescribes the degrees of the vertices but not their heights, one obtains the well studied model of uniform trees with fixed degree sequence, whose scaling limit is studied for example in [BM14], [BR21] and also in [Mar18b] with applications to Bienaymé–Galton–Watson trees and random planar maps. This model is the analogue of the configuration model for random trees. Note that if one conditions on the heights of the vertices in a uniform trees with fixed degree sequence, then one obtains a uniform random tree with fixed degrees and heights. Moreover, the paper [AU20] gives conditions for the scaling limit of the profile of uniform trees with fixed degree sequence. Checking our assumptions in order to give a new proof of the scaling limit of uniform trees with fixed degree sequence thus seems feasible.

Besides, if one considers a Galton–Watson process in varying environment (GWVE) and conditions on the evolution of the total population, i.e. the profile of the tree, and on the number of children of each individual, i.e. the degrees, then the corresponding tree is a uniform random tree with fixed degrees and heights. As a result, our result provides the scaling limit of GWVE trees as soon as the hypothesis are satisfied by the profile and the degrees. The first scaling limit of the profile for GWVE trees was obtained by Kurtz [Kur78] under a third moment assumption. These results were subsequently extended and general conditions for the convergence of the profile can be found in [BS15] or in [FLL22]. See section 8.5 for an application of our results to GWVE trees using the scaling limit of the profile of [BS15].

In the same direction, the scaling limit of the size of the n -th generation conditioned on survival up to time n was obtained in [Ker20] for GWVE under mild assumptions. Under a second moment assumption, the genealogy of the individuals at this n -th generation in GWVE trees was studied in [HPP22] in the

critical case, and in [BFRS22] in the near critical case with a GHP scaling limit of the genealogy of the n -th generation in this latter work. In [CKKM22], for i.i.d. environments, in the critical case and under a second moment assumption, a GHP scaling limit towards the Brownian continuous random tree was obtained. Although we do not wish to pursue in that direction, our results should be useful to study the stable case.

Finally, let us briefly mention that, our approach is similar to the one we recently developed with Bellin and Kortchemski in Chapters 6 and 7 to study (critical) uniform attachment trees with freezing. However, as we no longer work in a Brownian case only, we need to use new techniques and ideas.

Plan of the paper: The rest of the paper is organized as follows. In section 8.2 we define the limit tree appearing in Theorems 8.1.2 and 8.1.3. In section 8.3 we prove theorem 8.1.2. In section 8.4 we show theorem 8.1.3. Finally in section 8.5, which can be read right after the introduction, we discuss an application of our result to Bienaymé–Galton–Watson trees in varying environment. See Section 7.6 for the standard notions about the GP, GH, and GHP topologies. Section 8.6 recalls the classical notion of leaf-tightness.

8.2 Definition of the limit

In this section, we define the limit trees appearing in Theorem 8.1.2 in a rather implicit way: we first construct, using a growth-coalescent process, a random infinite matrix which describes the distances between i.i.d. random points. We then prove, under the condition (Tight_{GP}), leaf-tightness for this matrix, hence defining the limiting random measured metric space (see Proposition 8.6.1).

8.2.1 Description of the distance matrix between uniform vertices in the limit tree

We go downward in the tree from random vertices to the root, and describe how their genealogies merge. This description gives both their heights and the heights of their most recent common ancestors, and so the distances between those vertices. We start with an informal construction of the genealogies' coalescent:

Let k particles be born at i.i.d. times (heights) of law ν . On each interval $[t, t - dt]$, two particles merge into a cluster at rate $\rho(dt)$. Moreover, at every time t such that $\theta_1(t) > 0$, the clusters merge according to the sub-probability measure $(\theta_j(t))_{j \geq 1}$ as follows: independently, each cluster is associated with some random integer j in $\mathbb{N} \cup \{0\}$ with probability $\theta_j(t)$ when $j \geq 1$ and $\theta_0(t) := 1 - \sum_{j' \geq 1} \theta_{j'}(t)$ when $j = 0$. Then, the clusters associated to 0 do not merge whereas for all $j \geq 1$, the clusters associated to j merge together into one cluster. Finally, at time zero, all the particles which have not merged yet coalesce.

Here the birth time of the particle should be seen as the height of random vertices in the tree, and the merging time of two clusters as the height of their most recent common ancestor. And, depending on the degree of those ancestors (3 or ∞), the coalescences occur differently (using respectively ρ or Θ).

Formally, let us use the same notation as for coalescent processes where each particle is associated with an integer, where clusters are described by sets of integers and where each coalescence is performed by replacing two clusters by their union. See e.g. [Pit06] or [Ber06] for the use of this notation in the classical theory of coalescent and fragmentation processes. We consider three types of independent random events:

- **Birth:** For every $r \in \mathbb{N}$, let H_r be a random variable in $[0, 1]$ of law ν ;
- **Small merge:** For every $q < r \in \mathbb{N}$, let $\Gamma_{q,r}$ be a Poisson point process on $[0, 1]$ of intensity $\rho(dt)$;
- **Large merge:** For every t such that $\theta_1(t) > 0$ and $r \in \mathbb{N}$, let $\Theta_{t,r}$ be a random variables of law given by $\mathbb{P}(\Theta_{t,r} = j) = \theta_j(t)$ for every $j \in \mathbb{N} \cup \{0\}$. We write $S^\Theta(t, j) := \{r \in \mathbb{N}, \Theta_{t,r} = j\}$ for all $j \geq 1$.

From there we describe a càglàd (left continuous with right limit) jump process $((\mathcal{P}_r^k(t))_{1 \leq r \leq k})_{1 \geq t \geq 0}$ which is called a growth-coalescent process. The process starts at $\mathcal{P}_1^k(1) = \mathcal{P}_2^k(1) = \dots = \mathcal{P}_k^k(1) = \emptyset$ and its jumps occur only at some time t satisfying one of the following conditions:

- **Birth:** If $t = H_r$ for some $1 \leq r \leq k$, then $\mathcal{P}_r^k(t^+) = \{\emptyset\}$, and $\mathcal{P}_r^k(t) = \{r\}$;

- **Small merge:** If $t \in \Gamma_{q,r}$ for some $q < r$, then whenever $\mathcal{P}_q^k(t^+) \neq \emptyset$ and $\mathcal{P}_r^k(t^+) \neq \emptyset$ we have $\mathcal{P}_q^k(t) = \mathcal{P}_q^k(t^+) \cup \mathcal{P}_r^k(t^+)$ and $\mathcal{P}_r^k(t) = \emptyset$;
- **Large merge:** If $\theta_1(t) > 0$, then for every $j \in \mathbb{N}$ such that $\#(S^\Theta(t, j) \cap [k]) \geq 2$, we write $m^k(t, j)$ for the smallest integer $r \in S^\Theta(t, j) \cap [k]$ such that $\mathcal{P}_r^k(t^+) \neq \emptyset$. We have:

$$\forall r \in (S^\Theta(t, j) \cap [k]) \setminus \{m^k(t, j)\}, \quad \mathcal{P}_r^k(t) = \emptyset, \quad \text{and} \quad \mathcal{P}_{m^k(t, j)}^k(t) = \bigcup_{r \in S^\Theta(t, j) \cap [k]} \mathcal{P}_r^k(t^+).$$

There is a final jump at time 0 to reach the configuration $\mathcal{P}_1^k(0) = \{1, \dots, k\}$ and $\mathcal{P}_2^k(0) = \dots = \mathcal{P}_k^k(0) = \emptyset$.

Lemma 8.2.1. *The process $((\mathcal{P}_r^k(t))_{1 \leq r \leq k})_{0 \leq t \leq 1}$ is a.s. well defined.*

Proof. To see that such a process is well defined, it suffices to focus on the number of possible jumps. Births only occur at $\{H_r\}_{1 \leq r \leq k}$ which is a finite subset of $(0, 1)$ since ν has no atom at zero or one. Note that for $1 \leq r \leq k$ and $t > H_r$ we must have $\mathcal{P}_r^k(t) = \emptyset$ so that the highest jump occurs at time $\max_{1 \leq r \leq k} H_r$.

Then, for every $a > 0$, we prove that there is only a finite amount of possible times for small merges and large merges in $[a, \max_{1 \leq r \leq k} H_r]$, so that the process $((\mathcal{P}_r^k(t))_{1 \leq r \leq k})_{0 \leq t \leq 1}$ is well defined on $[a, 1]$. Since a.s. $\mathcal{P}_1^k(0) = \{1, \dots, k\}$ and $\mathcal{P}_2^k(0) = \dots = \mathcal{P}_k^k(0) = \emptyset$, and since a is arbitrary this will conclude the proof.

Small merges can only occur when $t \in \Gamma_{q,r}$ for some $1 \leq q < r \leq k$. And a.s. $\Gamma_{q,r}$ is locally finite on $(0, 1)$ as a Poisson point process of rate ρ which is locally finite on $(0, 1)$.

For the large merges, note for a jump to occur at time t there must exist $j \in \mathbb{N}$ with $\#S^\Theta(t, j) \cap [k] \geq 2$. This occurs for every time t with probability upperbounded by $k^2 \sum_{j=1}^{\infty} \theta_j(t)^2$. Thus thanks to (8.1.2), a.s. the number of possible times $t \in [a, \max_{1 \leq r \leq k} H_r]$ where large merges can occur is finite. \square

Lemma 8.2.2. *A.s. for every $0 \leq t \leq 1$ and $1 \leq r \leq k < k'$ we have $\mathcal{P}_r^{k'}(t) \cap [1, \dots, k] = \mathcal{P}_r^k(t)$.*

Proof. Since the number of jumps is locally finite on $(0, 1)$ the result follows by a simple (but tedious) downward induction on t . We omit the details. \square

The above consistency condition allows us to define for every $r \in \mathbb{N}$, $0 \leq t \leq 1$, $\mathcal{P}_r(t) := \bigcup_{k \in \mathbb{N}} \mathcal{P}_r^k(t)$.

Also, note that below $\min_{1 \leq r \leq k} H_r$, there are no birth times anymore for the process $((\mathcal{P}_r^k(t))_{1 \leq r \leq k})_{0 \leq t \leq 1}$, so that the number of $r \in [k]$ such that $\mathcal{P}_r^k(t) \neq \emptyset$ can not increase when t decreases. Therefore, the process $((\mathcal{P}_r^k(t))_{1 \leq r \leq k})_{0 \leq t \leq 1}$ undergoes only a finite number of jumps in $[0, 1]$. The genealogy of the k particles is then described by a finite tree \mathcal{T}_k whose leaves, denoted by V_1, \dots, V_k correspond to the births of each particle and whose internal vertices correspond to the coalescences between clusters. Each vertex v of \mathcal{T}_k is associated with a time t_v corresponding to the birth time if v is a leaf or the coalescence time if v is an internal vertex. We equip this tree with the distance induced by the times of birth and coalescence: for every pair u, v of vertices of \mathcal{T}_k , the distance $d(u, v)$ between u and v is given by

$$d(u, v) = t_u + t_v - 2c(u, v),$$

where $c(u, v)$ is the first time at which the clusters of u and v coalesce. Note that if V_q, V_r are leaves of \mathcal{T}_k corresponding to the particles born at times H_q, H_r for $1 \leq q < r \leq k$, then

$$d(V_q, V_r) = H_q + H_r - 2 \max\{t \in [0, 1], \exists m \in [k], q, r \in \mathcal{P}_m^k(t)\}. \quad (8.2.1)$$

Since the above equation can also be written

$$d(V_q, V_r) = H_q + H_r - 2 \max\{t \in [0, 1], \exists m \geq 1, q, r \in \mathcal{P}_m(t)\},$$

the random variable $d(V_q, V_r)$ does not depend on k .

8.2.2 Leaf-tightness

While the previous section defines the matrix of distances between random vertices of the limit trees, it is not enough to properly define the limit. This difference is quite subtle and has been extensively studied by Aldous in [Ald93]. As recalled in Appendix 8.6, to properly define the limit trees it is enough to prove the following weak leaf tightness criterion:

$$\forall \delta > 0, \quad \lim_{k \rightarrow \infty} \mathbb{P}(d(V_{k+1}, \{V_1, V_2, \dots, V_k\}) > 3\delta) = 0.$$

Since H_{k+1} (the height of V_{k+1}) is sampled according to the probability measure ν which satisfy $\nu(\{0\}) = 0$, we may assume for some $a > 0$ that $a < H_{k+1} < a + \delta$. Also, since, under $(\text{Tight}_{\text{GP}})$, ν has support $[0, 1]$ by taking k large enough, we may assume that $a - \delta < H_1, H_2, \dots, H_k < a$. (To detail, by considering enough vertices, we can have among them as many as we want at height between $a - \delta$ and a .) In other words, it suffices to check that for every $0 < a \leq 1$ and $\delta > 0$,

$$\lim_{k \rightarrow \infty} \mathbb{P}(d(V_{k+1}, \{V_1, V_2, \dots, V_k\}) > 3\delta | H_{k+1} \in [a, a + \delta], \forall 1 \leq i \leq k, H_i \in [a - \delta, a]) = 0.$$

Then we write $\mathbb{P}^k(\cdot) = \mathbb{P}(\cdot | H_{k+1} \in [a, a + \delta], \forall 1 \leq i \leq k, H_i \in [a - \delta, a])$. Define similarly the associated expectation \mathbb{E}^k for later. And we recall that for all $i \neq j$, the random variable $c(V_i, V_j)$ is the coalescence time between (the genealogies of) V_i and V_j , that is the height of their most recent common ancestor. So that by (8.2.1), to prove the above it suffices to show

$$\lim_{k \rightarrow \infty} \mathbb{P}^k(\forall 1 \leq i \leq k, c(V_i, V_{k+1}) < a - 2\delta) = 0. \quad (8.2.2)$$

To that end we adapt an argument from Section 7.5.5 of Chapter 7. First for every $1 \leq i \leq k, x \geq 0$, consider the event

$$E_{i,k}(x) := \{H_i > x, \forall j \in [k] \setminus \{i\}, c(V_i, V_j) < x\},$$

that is the event that V_i has height at least x and has not coalesced with any other vertices among V_1, \dots, V_k above height x . Then let

$$N_k(x) := \#\{1 \leq i \leq k, E_{i,k}(x)\}.$$

To prove (8.2.2), we rely on the simple but crucial remark: under $H_{k+1} > H_1, \dots, H_k, V_{k+1}$ has the most chance among V_1, V_2, \dots, V_{k+1} to have coalesced with one of the others. In other words,

$$\mathbb{P}^k(E_{k+1,k+1}(a - 2\delta)) \leq \frac{1}{k+1} \sum_{i=1}^{k+1} \mathbb{P}^k(E_{i,k+1}(a - 2\delta)) = \frac{1}{k+1} \mathbb{E}^k[N_{k+1}(a - 2\delta)].$$

Note that the left term above is exactly the probability in (8.2.2), so to define the limit it suffices to prove:

Lemma 8.2.3. *For any choice of $a, \delta > 0$ as $k \rightarrow \infty$, $\mathbb{E}^k[N_{k+1}(a - 2\delta)] = o(k)$.*

Proof. We look for each small interval $[t, t + \Delta t]$, given $N_{k+1}(t + \Delta t)$ at the expected number of vertices in V_1, \dots, V_{k+1} that have not yet merged at time $t + \Delta t$ that merge with another vertex between time t and $t + \Delta t$. As in the definition of our coalescent, we distinguish two cases: either there is a large merge at time t or there is a small merge between time t and $t + \Delta t$. The second case is easy to count as the merging occurs by pair at rate given by ρ . For the first case we look for every V_r not merged at the value of $\Theta_{t,r}$ and check if there is no other $V_{r'}$ not yet merged that have the same value. This, together with the fact that $N_{k+1}(s) \geq N_{k+1}(t)$ for all $s \in [t, t + \Delta t]$, gives us the next formula: for every $a - 2\delta < t \leq t + \Delta t < a - \delta$,

$$\mathbb{E}^k[N_{k+1}(t + \Delta t) - N_{k+1}(t)] \geq \mathbb{E}^k \left[\sum_{j \geq 1: \theta_j(t) > 0} N_{k+1}(t) \theta_j(t) (1 - (1 - \theta_j(t))^{N_{k+1}(t) - 1}) + 2 \binom{N_{k+1}(t)}{2} \rho(t, t + \Delta t) \right].$$

Using the fact that for all $x \in (0, 1), y > 0$ with $xy > 1$ we have $(1 - x)^y \leq 1/e$, this implies that

$$\mathbb{E}^k[N_{k+1}(t + \Delta t) - N_{k+1}(t)] \geq \mathbb{E}^k \left[\sum_{j \geq 1: \theta_j(t) > 1/(N_{k+1}(t) - 1)} \theta_j(t) \left(1 - \frac{1}{e}\right) N_{k+1}(t) \right] + 2 \mathbb{E}^k \left[\binom{N_{k+1}(t)}{2} \right] \rho(t, t + \Delta t),$$

Then, by integrating between $a - 2\delta$ and $a - \delta$, and using $0 \leq N_{k+1} \leq k + 1$ for the left hand side, we get:

$$k + 1 \geq \sum_{t \in (a-2\delta, a-\delta), j \geq 1} \theta_j(t) \left(1 - \frac{1}{e}\right) \mathbb{E}^k [N_{k+1}(t) \mathbb{1}_{\theta_j(t) > 1/(N_{k+1}(t)-1)}] + \int_{a-2\delta}^{a-\delta} 2\mathbb{E}^k \left[\binom{N_{k+1}(t)}{2} \right] \rho(dt).$$

Next, by fixing $\varepsilon > 0$ and considering only the case where $N_{k+1}(t) > \varepsilon k$, we obtain

$$k+1 \geq \sum_{t \in (a-2\delta, a-\delta), j \geq 1} \theta_j(t) \left(1 - \frac{1}{e}\right) \varepsilon k \mathbb{P}^k(N_{k+1}(t) > \varepsilon k) \mathbb{1}_{\theta_j(t) > 1/(\varepsilon k-1)} + \int_{a-2\delta}^{a-\delta} (\varepsilon k-1)^2 \mathbb{P}^k(N_{k+1}(t) > \varepsilon k) \rho(dt),$$

or even by using that for all $t \geq a - 2\delta$, we have $N_{k+1}(t) \geq N_{k+1}(a - 2\delta)$, so that

$$k + 1 \geq \mathbb{P}^k(N_{k+1}(a - 2\delta) > \varepsilon k) \left(\left(1 - \frac{1}{e}\right) \varepsilon k \sum_{\substack{t \in (a-2\delta, a-\delta), j \geq 1 \\ \theta_j(t) > 1/(\varepsilon k-1)}} \theta_j(t) + (\varepsilon k - 1)^2 \rho([a - 2\delta, a - \delta]) \right).$$

Using the second part of [\(Tight_{GP}\)](#), k is negligible compared to the inside of the huge parenthesis above, so

$$\mathbb{P}^k(N_{k+1}(a - 2\delta) > \varepsilon k) \xrightarrow[k \rightarrow \infty]{} 0.$$

Since ε is arbitrary and since $N_{k+1} \leq k+1$ this implies the desired result, and so concludes the section. \square

8.3 GP convergence

8.3.1 Coalescent associated with the genealogies of random vertices in \mathcal{T}^n

In this section, we introduce a discrete growth-coalescent process, which describes how the genealogies of random vertices in \mathcal{T}^n merge. In the next section, we will prove a scaling limit for this coalescent toward the similar continuous coalescent introduced in section 8.2.1. At the end of this section, we explain why this implies the scaling limit of the distance matrix of random vertices in \mathcal{T}^n , and so Theorem 8.1.2.

Beforehand, let us formally split high degrees and small degrees.

Lemma 8.3.1. *There exists a sequence $(\varepsilon_n)_{n \geq 0}$ decreasing to 0 slowly enough so that for all $n \geq 0$, there exists a bijection $I_n : \{t, \theta_1(t) > \varepsilon_n \text{ and } \varepsilon_n < t < 1 - \varepsilon_n\} \rightarrow \{i, d_{i,1}^n/D_i^n > \varepsilon_n \text{ and } \varepsilon_n < i/n < 1 - \varepsilon_n\}$ such that for all $\varepsilon > 0$,*

$$\sum_{j \geq 1, t \in (\varepsilon, 1-\varepsilon)} \left(\frac{d_{I_n(t),j}^n}{D_{I_n(t)}^n} - \theta_j(t) \right)^2 \mathbb{1}_{\theta_j(t) > \varepsilon_n} \xrightarrow[n \rightarrow \infty]{} 0 \quad (8.3.1)$$

and such that for all $t \geq 0$ such that $\theta_1(t) > 0$,

$$I_n(t)/n \xrightarrow[n \rightarrow \infty]{} t. \quad (8.3.2)$$

Proof. We construct the sequence ε_n as follows. For all $k \geq 1$, let $\eta_k \in (2^{-k-1}, 2^{-k})$ such that $\eta_k \notin \{\theta_j(t); t \in (0, 1), j \geq 1\}$. By [\(Lim \$_{\Theta}\$ \)](#) and [\(Lim \$_{\neq}\$ \)](#), there exists $n_k \geq 1$ such that for all $n \geq n_k$, there exists a bijection $I_{k,n} : \{t \in (\eta_k, 1 - \eta_k), \theta_1(t) > \eta_k\} \rightarrow \{i, d_{i,1}^n/D_i^n > \eta_k \text{ and } \eta_k < i/n < 1 - \eta_k\}$ such that for all $t \in (\eta_k, 1 - \eta_k)$, for all $j \geq 1$ such that $\theta_j(t) > \eta_k$,

$$\left| \frac{d_{I_{k,n}(t),j}^n}{D_{I_{k,n}(t)}^n} - \theta_j(t) \right| \leq \eta_k \quad \text{and} \quad \left| \frac{I_{k,n}(t)}{n} - t \right| \leq \eta_k.$$

Then it suffices to take $\varepsilon_n := \min\{\eta_k; k \geq 1 \text{ and } n_k \leq n\}$ and define I_n as the $I_{k,n}$ where k is the largest integer such that $n_k \leq n$. Then [\(8.3.2\)](#) holds clearly. For [\(8.3.1\)](#), note that for all $\varepsilon > 0$,

$$\sum_{j \geq 1, t \in (\varepsilon, 1-\varepsilon)} \left(\frac{d_{I_n(t),j}^n}{D_{I_n(t)}^n} - \theta_j(t) \right)^2 \mathbb{1}_{\theta_j(t) > \varepsilon_n} \leq \sum_{j \geq 1, t \in (\varepsilon, 1-\varepsilon)} \varepsilon_n^2 \mathbb{1}_{\theta_j(t) > \varepsilon_n} \leq \sum_{j \geq 1, t \in (\varepsilon, 1-\varepsilon)} \varepsilon_n^2 \wedge \theta_j(t)^2 \xrightarrow[n \rightarrow \infty]{} 0,$$

where the convergence holds by dominated convergence thanks to [\(8.1.2\)](#). \square

Then, let us prove a direct consequence of (Lim_{\neq}) which will be very convenient:

Lemma 8.3.2. *We have for all $\varepsilon \in (0, 1)$,*

$$\min_{\varepsilon n \leq i \leq (1-\varepsilon)n} D_i^n \xrightarrow{n \rightarrow \infty} \infty.$$

Proof. Let $\varepsilon \in (0, 1)$. Let us prove that for all n large enough for all $\varepsilon n < i < (1-\varepsilon)n$, we have $D_i^n \geq 1/\varepsilon$. For all $i \leq h^n - 1$, if $d_{i,1}^n \leq \varepsilon D_i^n$, then, since $d_{i,1}^n \geq 1$, we deduce that $D_i^n \geq 1/\varepsilon$.

Besides, note that for all n large enough, if $d_{i,1}^n > \varepsilon D_i^n$ for some $\varepsilon n < i < (1-\varepsilon)n$, then by (Lim_{\neq}) we have $d_{i+1,1}^n \leq \varepsilon D_{i+1}^n$. But we also know that $D_i^n d_{i+1,1}^n \geq D_{i+1}^n$ since there are exactly D_i^n vertices of \mathcal{T}^n at height $i+1$, so that

$$D_{i+1}^n \leq d_{i+1,1}^n D_i^n \leq \varepsilon D_{i+1}^n D_i^n.$$

Finally, we also get $D_i^n \geq 1/\varepsilon$ by dividing both sides by D_{i+1}^n (that can be done since since by (Lim_ν) , $\liminf h^n/n \geq 1$ and so $D_{i+1}^n \geq 1$ for all n large enough and for all $i < (1-\varepsilon)n$). \square

Now, to prove the convergence of the rescaled matrix of distances between k vertices V_1^n, \dots, V_k^n taken uniformly at random, we introduce the associated growth-coalescent process and prove that it satisfies a scaling limit towards the coalescent process defined in Section 8.2.1.

In order to get some additional independence, we add some vertices at each height to the ancestors of V_1^n, \dots, V_k^n in order to have a constant number of vertices at each height. For all $i > \max_{r \in [k]} h(V_r^n)$:

- When $D_{i-1}^n \geq k$, let $(V_{i,j}^n)_{1 \leq j \leq k}$ be distinct vertices in $\{(i, j); j \leq D_i^n\}$ taken uniformly at random.
- When $D_{i-1}^n < k$, we set $V_{i,r}^n = (i, 1)$ for all $r \in [k]$.

Then, we define $V_{i,1}^n, \dots, V_{i,k}^n$ for $i \leq \max_{r \in [k]} h(V_r^n)$ by downwards induction on i . Given $(V_{i+1,j}^n)_{1 \leq j \leq k}$ define $(V_{i,j}^n)_{1 \leq j \leq k}$ as follows:

- **Position of V_1^n, \dots, V_k^n :** For all $r \in [k]$, if $h(V_r^n) = i$, then we set $V_{i,r}^n = V_r^n$.
- **Position of the ancestors:** For all $p \geq 1$, for all $1 \leq r_1 < \dots < r_p \leq k$, if the vertices $V_{i+1,r_1}^n, \dots, V_{i+1,r_p}^n$ are the vertices in $\{V_{i+1,1}^n, \dots, V_{i+1,k}^n\}$ which are attached to a vertex (i, j) in \mathcal{T}^n , and if one of them is an ancestor of V_r^n for some $r \in [k]$, then:
 - If $(i, j) = V_{r'}^n$ for some $r' \in [k]$, we set as explained before $V_{i,r'}^n = V_{r'}^n$.
 - Otherwise, if $(i, j) \notin \{V_1^n, \dots, V_k^n\}$, let r be the smallest integer in $[k]$ such that $r \in \{r_1, \dots, r_p\}$ and such that $V_{i+1,r}^n$ is the ancestor of V_r^n for some $r' \in [k]$. We then set $V_{i,r}^n = (i, j)$.
- **List completion:** When $D_{i-1}^n \geq k$, we complete the list by taking distinct vertices uniformly at random among the vertices $\{(i, j)\}_{1 \leq j \leq D_{i-1}^n}$ not yet taken. By convention, when $D_{i-1}^n < k$, all the $V_{i,r}^n$ for $r \in [k]$ not yet defined are equal to $(i, 1)$. By Lemma 8.3.2, this latter case is excluded on any interval $[\varepsilon n, (1-\varepsilon)n]$ for every $\varepsilon > 0$ for every n large enough.

We look at the fathers of $V_{i+1,1}^n, \dots, V_{i+1,k}^n$ in \mathcal{T}^n . If $V_{i+1,q}^n$ and $V_{i+1,r}^n$ for some $q < r \in [k]$ have the same father, then we say that they coalesce at this vertex. For all $q < r \in [k]$ and $i \geq 0$, let

$$X_{q,r,i}^n = \mathbb{1}_{V_{i+1,q}^n, V_{i+1,r}^n \text{ coalesce in a vertex of degree at most } \varepsilon_n D_i^n}. \quad (8.3.3)$$

For all $n, i \geq 0$, and $r \in [k]$, let $\Theta_{i,r}^n$ be the integer j such that the vertex $V_{i+1,r}^n$ is attached to the vertex of degree $d_{i,j}^n$ when $d_{i,j}^n > \varepsilon_n D_i^n$ and $\Theta_{i,r}^n = 0$ otherwise. We set for all $i, j \geq 1$,

$$S_{i,j}^{n,k} = \{r \in [k], \Theta_{i,r}^n = j\}.$$

We are now in position to define the growth-coalescent process. Let $((\mathcal{P}_r^{n,k}(i))_{r \in [k]})_{i \geq 0}$ be the process defined as follows. We set for all $i > \max_{r \in [k]} h(V_r^n)$, and for all $r \in [k]$, the initial values $\mathcal{P}_r^{n,k}(i) = \emptyset$. Then, for all $i \geq 0$,

- **Birth:** If $i = h(V_r^n)$ for some $r \in [k]$, then $\mathcal{P}_r^{n,k}(i) = \{r\}$;
- **Small merge:** If $X_{q,r,i}^n = 1$ for some $1 \leq q < r \leq k$ then whenever $\mathcal{P}_q^{n,k}(i+1) \neq \emptyset$ and $\mathcal{P}_r^{n,k}(i+1) \neq \emptyset$ we have $\mathcal{P}_q^{n,k}(i) = \mathcal{P}_q^{n,k}(i+1) \cup \mathcal{P}_r^{n,k}(i+1)$ and $\mathcal{P}_r^{n,k}(i) = \emptyset$;
- **Large merge:** If $\#S_{i,j}^{n,k} \geq 2$ for some $j \geq 1$, we write $m^{n,k}(i, j)$ for the smallest integer $r \in S_{i,j}^{n,k}$ such that $\mathcal{P}_r^{n,k}(i+1) \neq \emptyset$. We set:

$$\forall r \in S_{i,j}^{n,k} \setminus \{m^{n,k}(i, j)\}, \quad \mathcal{P}_r^{n,k}(i) = \emptyset, \quad \text{and} \quad \mathcal{P}_{m^{n,k}(i, j)}^{n,k}(i) = \bigcup_{r \in S_{i,j}^{n,k}} \mathcal{P}_r^{n,k}(i+1);$$

and for the other $r \in [k]$ we set $\mathcal{P}_r^{n,k}(i) = \mathcal{P}_r^{n,k}(i+1)$.

This discrete growth-coalescent is well-defined and gives the genealogy of V_1^n, \dots, V_r^n in \mathcal{T}^n .

Proposition 8.3.3. *With high probability, the discrete growth coalescent process $((\mathcal{P}_q^{n,k}(i))_{q \in [k]})_{i \geq 0}$ is well-defined. Moreover, for all $0 \leq i \leq h^n$, the set $\mathcal{P}_q^{n,k}(i)$ is the set of $r \in [k]$ such that the vertex $V_{i,q}^n$ is an ancestor of V_r^n .*

Proof. The above result is straightforward after checking that for all $\varepsilon \in (0, 1)$, with high probability, for all $\varepsilon n \leq i \leq (1 - \varepsilon)n$, at most two vertices coalesce in vertices of degree at most $\varepsilon_n D_i^n$. This is a direct consequence of Lemma 8.3.6 which is stated and proven in the next subsection. One can then use that by (Lim_ν) ,

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\forall r \in [k], \varepsilon n \leq h(V_r^n) \leq (1 - \varepsilon)n) = 1. \quad \square$$

We then state the scaling limit result towards the continuous time growth-coalescent process defined in Section 8.2.1. Its proof is postponed to the next subsections.

Proposition 8.3.4. *We have the convergence*

$$\left((\mathcal{P}_r^{n,k}(\lceil n(1-t) \rceil))_{r \in [k]} \right)_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} \left((\mathcal{P}_r^k(1-t))_{r \in [k]} \right)_{t \in [0,1]}$$

for the J_1 -Skorokhod topology.

The convergence in the sense of Gromov-Prokhorov readily follows from the above proposition:

Proof of Theorem 8.1.2 using Proposition 8.3.4. Thanks to Lemma 7.6.1, it suffices to prove the convergence for all $k \geq 1$

$$(d^n(V_q^n, V_r^n)/n)_{1 \leq q < r \leq k} \xrightarrow[n \rightarrow \infty]{(d)} (d(V_q, V_r))_{1 \leq q < r \leq k}, \quad (8.3.4)$$

where d^n is the graph distance on \mathcal{T}^n . Let $k \geq 1$. Let $q < r \in [k]$. By construction of \mathcal{T}^n and by definition of the processes $\mathcal{P}_m^{n,k}$ for $m \in [k]$, we have

$$D^n(V_q^n, V_r^n) = h(V_q^n) + h(V_r^n) - 2 \max\{i \geq 0, \exists m \in [k], q, r \in \mathcal{P}_m^{n,k}(i)\}.$$

Therefore, by (8.2.1), in order to show (8.3.4), it is enough to check the convergence

$$\frac{1}{n} \max\{i \geq 0, \exists m \in [k], q, r \in \mathcal{P}_m^{n,k}(i)\} \xrightarrow[n \rightarrow \infty]{(d)} \max\{t \geq 0, \exists m \in [k], q, r \in \mathcal{P}_m^k(t)\},$$

jointly with

$$((h(V_r^n)/n)_{r \in [k]}) \xrightarrow[n \rightarrow \infty]{(d)} (H_r)_{r \in [k]}. \quad (8.3.5)$$

But note that $h(V_r^n) = \max\{i \geq 0, r \in \mathcal{P}_r^{n,k}(i)\}$ and $H_r = \max\{t \geq 0, r \in \mathcal{P}_r^k(t)\}$. Thus, the two convergences stem from Proposition 8.3.4. \square

The rest of this section is dedicated to the proof of Proposition 8.3.4.

8.3.2 Convergence of the coalescences between random vertices at the same height

In this subsection, we prove Proposition 8.3.4. We first prove some useful lemmas.

A consequence of (Lim_ρ) , (8.3.1) and (8.3.2) is the weak convergence of measures on $(0, 1)$

$$n \sum_{j \geq 1} \frac{\binom{d_{nt,j}^n}{2}}{\binom{D_{nt}^n}{2}} \mathbb{1}_{d_{nt,j}^n \leq \varepsilon_n D_{nt}^n} dt \xrightarrow[n \rightarrow \infty]{} \rho. \quad (8.3.6)$$

We set for all $n, i \geq 0$,

$$p_{n,i} := \sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{2}}{\binom{D_i^n}{2}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n}.$$

Then Equation (8.3.6) entails that for all $\varepsilon > 0$, we have

$$\max_{\varepsilon n \leq i \leq (1-\varepsilon)n} p_{n,i} \xrightarrow[n \rightarrow \infty]{} 0 \quad (8.3.7)$$

since ρ has no atom. Another consequence of (8.3.6) is the following lemma:

Lemma 8.3.5. *For all $k \geq 2$, for any integer n large enough let B_i^n for $i \geq 0$ be independent Bernoulli random variables of parameter $p_{n,i}$. Then we have the weak convergence of measures on $(0, 1)$ in distribution*

$$\sum_{1 \leq i < n} B_i^n \delta_{i/n} \xrightarrow[n \rightarrow \infty]{(d)} \Pi_\rho,$$

where Π_ρ is a Poisson random measure of intensity ρ .

Proof. It is a consequence of the Poisson paradigm. By the independence of the B_i^n 's and the fact that ρ has no atom, it suffices to show that for all $0 < s < t < 1$, for all $x \in \mathbb{R}$, we have

$$\mathbb{E} \left[e^{ix \sum_{ns \leq j \leq nt} B_j^n} \right] \xrightarrow[n \rightarrow \infty]{} \exp(\rho([s, t])(e^{ix} - 1)),$$

which is straightforward since

$$\mathbb{E} \left[e^{ix \sum_{ns \leq j \leq nt} B_j^n} \right] = \prod_{ns \leq j \leq nt} (p_{n,j} e^{ix} + 1 - p_{n,j})$$

and then using (8.3.6) and (8.3.7). \square

Let us focus on the times of small merges. For all $n, i \geq 0$, let $X_{i,k}^n$ be the number of pairs $\{r, s\}$ for $r < s \in [k]$ such that $V_{i+1,r}^n$ and $V_{i+1,s}^n$ coalesce in a vertex of degree at most $\varepsilon_n D_i^n$, i.e.

$$X_{i,k}^n = \sum_{1 \leq r < s \leq k} \mathbb{1}_{V_{i+1,r}^n, V_{i+1,s}^n \text{ coalesce in a vertex of degree at most } \varepsilon_n D_i^n}.$$

Lemma 8.3.6. *For all $n \geq 0$ large enough, for all $k \geq 2$, let $(B_{i,k}^n)_{i \geq 0}$ a family of independent Bernoulli random variables of parameters $\binom{k}{2} p_{n,i}$. Then for all $\varepsilon > 0$,*

$$d_{\text{TV}} \left((X_{i,k}^n)_{\varepsilon n < i < (1-\varepsilon)n}, (B_{i,k}^n)_{\varepsilon n < i < (1-\varepsilon)n} \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where d_{TV} denotes the total variation distance.

Proof. Let $\varepsilon > 0$. By definition of $X_{i,k}^n$, it is enough to show that there exists a coupling between $X_{i,k}^n$ and $B_{i,k}^n$ such that

$$\sum_{\varepsilon n \leq i \leq (1-\varepsilon)n} \mathbb{P}(X_{i,k}^n \neq B_{i,k}^n) \xrightarrow[n \rightarrow \infty]{} 0. \quad (8.3.8)$$

Note that by Lemma 8.3.2, for n large enough, for all $\varepsilon n < i < (1-\varepsilon)n$, we have $D_i^n \geq k$, so that the vertices $V_{i+1,1}^n, \dots, V_{i+1,k}^n$ are distinct. First of all, we see that

$$\mathbb{E}[X_{i,k}^n] = \sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{2} \binom{D_i^n - 2}{k-2}}{\binom{D_i^n}{k}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} = \binom{k}{2} p_{n,i}. \quad (8.3.9)$$

Besides, we have

$$\mathbb{E}[(X_{i,k}^n)^2] = \mathbb{E}[X_{i,k}^n] \quad (8.3.10)$$

$$+ 6 \mathbb{E} \left[\sum_{1 \leq r_1 < r_2 < r_3 \leq k} \mathbb{1}_{V_{i+1,r_1}^n, V_{i+1,r_2}^n, V_{i+1,r_3}^n \text{ coalesce in a vertex of degree at most } \varepsilon_n D_i^n} \right] \quad (8.3.11)$$

$$+ \mathbb{E} \left[\sum_{\substack{1 \leq r_1 < s_1 \leq k \\ 1 \leq r_2 < s_2 \leq k}} \mathbb{1}_{\{r_1, s_1\} \cap \{r_2, s_2\} = \emptyset} \mathbb{1}_{\substack{V_{i+1,r_1}^n, V_{i+1,s_1}^n \text{ coalesce in a vertex of degree at most } \varepsilon_n D_i^n \\ \text{and } V_{i+1,r_2}^n, V_{i+1,s_2}^n \text{ coalesce in a vertex of degree at most } \varepsilon_n D_i^n}} \right] \quad (8.3.12)$$

The first term (8.3.10) is $\binom{k}{2}p_{n,i}$ by (8.3.9). The second term (8.3.11) is

$$\sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{3} \binom{D_i^n - 3}{k-3}}{\binom{D_i^n}{k}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} = \mathbb{1}_{D_i^n \geq 3} \sum_{j \geq 1} \frac{d_{i,j}^n - 2}{3} \frac{k-2}{D_i^n - 2} \frac{\binom{d_{i,j}^n}{2} \binom{D_i^n - 2}{k-2}}{\binom{D_i^n}{k}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \leq \varepsilon_n \frac{k-2}{3} \binom{k}{2} p_{n,i}, \quad (8.3.13)$$

while the third term (8.3.12) is

$$\begin{aligned} & \sum_{\substack{j_1, j_2 \geq 1 \\ j_1 \neq j_2}} \mathbb{1}_{d_{i,j_1}^n, d_{i,j_2}^n \leq \varepsilon_n D_i^n} \frac{\binom{d_{i,j_1}^n}{2} \binom{d_{i,j_2}^n}{2} \binom{D_i^n - 4}{k-4}}{\binom{D_i^n}{k}} + 6 \sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{4} \binom{D_i^n - 4}{k-4}}{\binom{D_i^n}{k}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \\ &= \mathbb{1}_{D_i^n \geq 4} \binom{k}{2} \binom{k-2}{2} \sum_{\substack{j_1, j_2 \geq 1 \\ j_1 \neq j_2}} \frac{\binom{d_{i,j_1}^n}{2} \binom{d_{i,j_2}^n}{2}}{\binom{D_i^n}{2} \binom{D_i^n - 2}{2}} \mathbb{1}_{d_{i,j_1}^n, d_{i,j_2}^n \leq \varepsilon_n D_i^n} + 6 \mathbb{1}_{D_i^n \geq 4} \binom{k}{4} \sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{4}}{\binom{D_i^n}{4}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \\ &\leq \mathbb{1}_{D_i^n \geq 4} \frac{D_i^n (D_i^n - 1)}{(D_i^n - 2)(D_i^n - 3)} \binom{k}{2} \binom{k-2}{2} \sum_{\substack{j_1, j_2 \geq 1 \\ j_1 \neq j_2}} \frac{\binom{d_{i,j_1}^n}{2} \binom{d_{i,j_2}^n}{2}}{\binom{D_i^n}{2} \binom{D_i^n}{2}} \mathbb{1}_{d_{i,j_1}^n, d_{i,j_2}^n \leq \varepsilon_n D_i^n} + 6 \mathbb{1}_{D_i^n \geq 4} \binom{k}{4} \sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{2}^2}{\binom{D_i^n}{2}^2} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \\ &\leq \mathbb{1}_{D_i^n \geq 4} \left(\frac{D_i^n (D_i^n - 1)}{(D_i^n - 2)(D_i^n - 3)} \binom{k}{2} \binom{k-2}{2} + 6 \binom{k}{4} \right) \left(\sum_{j \geq 1} \frac{\binom{d_{i,j}^n}{2}}{\binom{D_i^n}{2}} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \right)^2 \\ &\leq 12 \binom{k}{2} \binom{k-2}{2} (p_{n,i})^2. \end{aligned} \quad (8.3.14)$$

For all n large enough, let $B_{i,k}^n$ be a Bernoulli random variable of parameter $\binom{k}{2}p_{n,i}$ such that $B_{i,k}^n \geq \mathbb{1}_{X_{i,k}^n \geq 1}$. One can indeed perform such a coupling given that $\mathbb{P}(X_{i,k}^n \geq 1) \leq \mathbb{E}[X_{i,k}^n] = \binom{k}{2}p_{n,i} \leq 1$ for n large enough. As a result,

$$\begin{aligned} \sum_{\varepsilon_n \leq i \leq (1-\varepsilon)n} \mathbb{P}(X_{i,k}^n \neq B_{i,k}^n) &\leq \sum_{\varepsilon_n \leq i \leq (1-\varepsilon)n} \left(\mathbb{P}(X_{i,k}^n \neq \mathbb{1}_{X_{i,k}^n \geq 1}) + \mathbb{P}(B_{i,k}^n \neq \mathbb{1}_{X_{i,k}^n \geq 1}) \right) \\ &\leq \sum_{\varepsilon_n \leq i \leq (1-\varepsilon)n} \left(\mathbb{E} \left[\left| X_{i,k}^n - \mathbb{1}_{X_{i,k}^n \geq 1} \right|^2 \right] + \mathbb{E}[X_{i,k}^n] - \mathbb{P}(X_{i,k}^n \geq 1) \right) \\ &= \sum_{\varepsilon_n \leq i \leq (1-\varepsilon)n} \left(\mathbb{E}[(X_{i,k}^n)^2] - \mathbb{E}[X_{i,k}^n] \right) \\ &\leq \sum_{\varepsilon_n \leq i \leq (1-\varepsilon)n} \left(\varepsilon_n \frac{k-2}{3} \binom{k}{2} p_{n,i} + 12 \binom{k}{2} \binom{k-2}{2} (p_{n,i})^2 \right) \text{ by (8.3.13) and (8.3.14),} \\ &\xrightarrow{n \rightarrow \infty} 0 \text{ by (8.3.6) and (8.3.7),} \end{aligned}$$

hence (8.3.8). \square

Combining Lemmas 8.3.5 and 8.3.6 we obtain the scaling limit of the point processes induced by the $X_{q,r,i}^n$'s which are defined in (8.3.3).

Corollary 8.3.7. *We have the weak convergence of measures on $(0, 1)$ in distribution*

$$\left(\sum_{i \geq 0} X_{q,r,i}^n \delta_{i/n} \right)_{1 \leq q < r \leq k} \xrightarrow[n \rightarrow \infty]{(d)} (\Gamma_{q,r})_{1 \leq q < r \leq k},$$

where the $\Gamma_{q,r}$'s for $1 \leq q < r \leq k$ are independent Poisson point processes of intensity $\rho(dt)$.

Proof. By Lemma 8.3.6, we know that with high probability, at each height we have at most one coalescence in a vertex of degree at most $\varepsilon_n D_i^n$. Moreover, conditionally on $(X_{i,k}^n)_{i \geq 1}$, the indices $q < r$ of the vertices $V_{i+1,q}^n, V_{i+1,r}^n$ which coalesce at each height are uniform. For all $n, i \geq 1$ let (Q_i^n, R_i^n) be

independent random variables which are uniform in $\{(q, r), 1 \leq q < r \leq k\}$ and taken independently of the B_i^n 's. Then,

$$d_{\text{TV}}\left(\left(X_{q,r,i}^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}, \left(\mathbb{1}_{(Q_i^n, R_i^n)=(q,r)} B_i^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}\right) \xrightarrow{n \rightarrow \infty} 0.$$

Besides, one can see that

$$\begin{aligned} & d_{\text{TV}}\left(\left(\mathbb{1}_{(Q_i^n, R_i^n)=(q,r)} B_i^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}, \left(B_{q,r,i}^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}\right) \\ & \leq \sum_{\varepsilon n < i < (1-\varepsilon)n} \left(\mathbb{P}\left(\sum_{1 \leq q < r \leq k} B_{q,r,i}^n \geq 2\right) + \left| \mathbb{P}(B_i^n = 1) - \mathbb{P}\left(\sum_{1 \leq q < r \leq k} B_{q,r,i}^n = 1\right) \right| \right) \\ & \quad + \left| \mathbb{P}(B_i^n = 0) - \mathbb{P}\left(\sum_{1 \leq q < r \leq k} B_{q,r,i}^n = 0\right) \right| \\ & = 2 \sum_{\varepsilon n < i < (1-\varepsilon)n} \mathbb{P}\left(\sum_{1 \leq q < r \leq k} B_{q,r,i}^n \geq 2\right) \\ & \leq 2 \binom{k}{2} \sum_{\varepsilon n < i < (1-\varepsilon)n} p_{n,i}^2 \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where the convergence comes from (8.3.6) and (8.3.7). Thus we have the stronger result

$$d_{\text{TV}}\left(\left(X_{q,r,i}^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}, \left(B_{q,r,i}^n\right)_{\varepsilon n < i < (1-\varepsilon)n, 1 \leq q < r \leq k}\right) \xrightarrow{n \rightarrow \infty} 0,$$

where the $(B_{q,r,i}^n)_{i \geq 1}$ for $1 \leq q < r \leq k$ are i.i.d. copies of the family $(B_i^n)_{i \geq 1}$ defined in Lemma 8.3.5. One concludes using Lemma 8.3.5. \square

Now, let us study the coalescences in high degree vertices. Recall that in the continuous coalescent, the large merges are described by the independent random variables $\Theta_{t,r}$ of law $(\theta_j(t))_{j \geq 0}$ for $r \in [k]$ and t such that $\theta_1(t) > 0$ which are defined in Section 8.2.1. Recall also that $S^\Theta(t, j) \cap [k] = \{r \in [k], \Theta_{t,r} = j\}$.

By (Lim \neq), we may also assume that ε_n decreases slowly enough and that there exists a sequence $(\eta_n)_{n \geq 0}$ such that $n\eta_n \rightarrow \infty$ and for all $n \geq 0$, for all $(i, j), (i', j')$ with $\varepsilon_n n < i, i' < (1 - \varepsilon_n)n$ such that $d_{i,j}^n > \varepsilon_n D_i^n$ and $d_{i',j'}^n > \varepsilon_n D_{i'}^n$, then we have either $i = i'$ or $|i - i'| > \eta_n n$.

Lemma 8.3.8. *For all $\varepsilon' > 0$, we have the weak convergence of measures on $(0, 1) \times (\mathbb{N} \cup \{0\})^k$ in distribution*

$$\sum_{1 \leq i \leq n} \mathbb{1}_{d_{i,1}^n > \varepsilon' D_i^n} \delta_{(i/n, (\Theta_{t,r}^n)_{r \in [k]})} \xrightarrow[n \rightarrow \infty]{(d)} \sum_{t \geq 0} \mathbb{1}_{\theta_1(t) > \varepsilon'} \delta_{(t, (\Theta_{t,r})_{r \in [k]})}.$$

Proof. For all $n, i \geq 0$, for all $j \geq 1$, let $N_{i,j,k}^n$ be the number of vertices among $V_{i+1,1}^n, \dots, V_{i+1,k}^n$ with father (i, j) .

Note that by Lemma 8.3.2, for n large enough, for all $\varepsilon n < i < (1 - \varepsilon)n$, we have $D_i^n \geq k$ so that the vertices $V_{i+1,1}^n, \dots, V_{i+1,k}^n$ are distinct. Conditionally on the $N_{i,j,k}^n$'s for $\varepsilon n < i < (1 - \varepsilon)n$ and $j \geq 1$ such that $d_{i,j}^n > \varepsilon_n D_i^n$, the $N_{i,j,k}^n$ vertices which coalesce at (i, j) are chosen uniformly at random among $V_{i+1,1}^n, \dots, V_{i+1,k}^n$. More precisely, the sets $S_{i,j}^{n,k} := \{r \in [k], \Theta_{i,r}^n = j\}$ for $j \in \{j \geq 1, d_{i,j}^n > \varepsilon_n D_i^n\} \cup \{0\}$ form a random partition of $[k]$ with prescribed sizes which is chosen uniformly at random independently for all $i \geq 1$ and conditionally on the $N_{i,j,k}^n$'s.

The statement of the lemma is thus implied by the convergence for all $t \geq 0$ such that $\theta_1(t) > 0$,

$$\left(N_{T_n(t),j,k}^n\right)_{j \geq 1} \xrightarrow[n \rightarrow \infty]{(d)} (\#(S^\Theta(t, j) \cap [k]))_{j \geq 1}. \quad (8.3.15)$$

Let $t \geq 0$ such that $\theta_1(t) > 0$. Let $(m_j)_{j \geq 1}$ be a family of non-negative integers such that $\sum_{j \geq 1} m_j \leq k$.

Then, for n large enough,

$$\begin{aligned} \mathbb{P}(\forall j \geq 1, N_{I_n(t),j,k}^n = m_j) &= \left(\binom{D_{I_n(t)}^n - \sum_{j \geq 1} d_{I_n(t),j}^n \mathbb{1}_{\theta_j(t) > \varepsilon_n}}{k - \sum_{j \geq 1} m_j} \prod_{j \geq 1} \binom{d_{I_n(t),j}^n}{m_j} \right) \left(\binom{D_{I_n(t)}^n}{k} \right)^{-1} \\ &\xrightarrow{n \rightarrow \infty} \left(\frac{\theta_0(t)^{k - \sum_{j \geq 1} m_j}}{(k - \sum_{j \geq 1} m_j)!} \prod_{j \geq 1} \frac{\theta_j(t)^{m_j}}{m_j!} \right) k! = \mathbb{P}(\forall j \geq 1, \#(S^\Theta(t, j) \cap [k]) = m_j), \end{aligned}$$

by the convergence (8.3.1). \square

Proof of Proposition 8.3.4. Since the processes take their values in a finite set and since the number of jumps of $((\mathcal{P}_r^k(t))_{r \in [k]})_{t \in [0,1]}$ is finite, it suffices to prove the convergence of the times of jumps and of the jumps.

Actually, by definition of the two processes, it suffices to prove the joint convergence of the heights of the uniform vertices (8.3.5), of the small coalescences on $(0, 1)$

$$\left(\sum_{1 \leq i < n} X_{q,r,i}^n \delta_{i/n} \right)_{1 \leq q < r \leq k} \xrightarrow[n \rightarrow \infty]{(d)} (\Gamma_{q,r})_{1 \leq q < r \leq k} \quad (8.3.16)$$

and of the coalescence at vertices of high degrees on $(0, 1)$

$$\forall \varepsilon' > 0, \quad \sum_{1 \leq i < n} \mathbb{1}_{d_{i,1}^n \geq \varepsilon' D_i^n} \delta_{(i/n, (\Theta_{i,r}^n)_{r \in [k]})} \xrightarrow[n \rightarrow \infty]{(d)} \sum_{t \in (0,1)} \mathbb{1}_{\theta_1(t) > \varepsilon'} \delta_{(t, (\Theta_{t,r})_{r \in [k]})}. \quad (8.3.17)$$

The convergence (8.3.16) comes from Corollary 8.3.7. The convergence (8.3.17) holds thanks to Lemma 8.3.8 and holds jointly with the second one since with high probability coalescences in small degrees do not happen at heights i such that $d_{i,1}^n \geq \varepsilon' D_i^n$. Indeed, for all $\eta > 0$, by a union bound, the probability of having a small coalescence at a height $\eta n \leq i \leq (1 - \eta)n$ such that $d_{i,1}^n \geq \varepsilon' D_i^n$ is upperbounded by

$$\sum_{\eta n \leq i \leq (1-\eta)n} \mathbb{1}_{d_{i,1}^n > \varepsilon' D_i^n} p_{n,i} \leq \left(\max_{\eta n \leq i \leq (1-\eta)n} p_{n,i} \right) \sum_{\eta n \leq i \leq (1-\eta)n} \frac{1}{(\varepsilon')^2} \frac{\binom{d_{i,1}^n}{2}}{\binom{D_i^n}{2}} \xrightarrow[n \rightarrow \infty]{} 0,$$

where the convergence comes from (8.3.1) and (8.3.7). Finally, (8.3.5) comes from (Lim_ν) and holds jointly with (8.3.16) and (8.3.17) since the conditional law of $((\Theta_{i,r}^n)_{r \in [k], i \geq 0}, (X_{q,r,i}^n)_{1 \leq q < r \leq k, i \geq 0})$ does not depend on $(\mathfrak{h}(V_r^n))_{r \in [k]}$. \square

8.4 GHP convergence

To show the GHP convergence of \mathcal{T}^n , by proposition 8.6.2, it suffices to prove that \mathcal{T}^n is close to the genealogies of a fixed large number of random vertices (that is their paths to the root). We would like to use the same method as in Chapter 7. However, we cannot as we have poor estimate on the number of genealogies that still have not yet merged at given height.

To avoid this issue, we look instead at the genealogies of vertices chosen such that at each height exactly k of them have not yet merged. We call the union of the genealogies a k -trail as those sets have at each height k vertices and consists of paths going to the root (see Figure 8.3). We construct inductively k -trails (lemma 8.4.1), and show that k^2 -trails tend to be close to k -trails (lemma 8.4.3). By a chaining argument this gives us that \mathcal{T}^n is close to k -trails (proposition 8.4.2). Finally, we prove that k -trails stay close to the genealogies of a large number of random vertices, which allows us to conclude.

8.4.1 Using k -trails to approximate \mathcal{T}^n

For every $n, i \in \mathbb{N}, j \leq D_{i-1}^n$ we denote by $\mathfrak{f}^n(i, j) = \mathfrak{f}^n((i, j))$ the father of the vertex (i, j) in the tree \mathcal{T}^n . Also, for $i \geq 0$ let \mathfrak{f}_i^n be the i -th iterate of \mathfrak{f}^n . In other words, $\mathfrak{f}_i^n(x)$ is the ancestor of x with height i below x .

In this whole section, $(X_{i,j}^n)_{i \leq h^n, j \in \mathbb{N}}$ is a family of random variables satisfying the following properties:

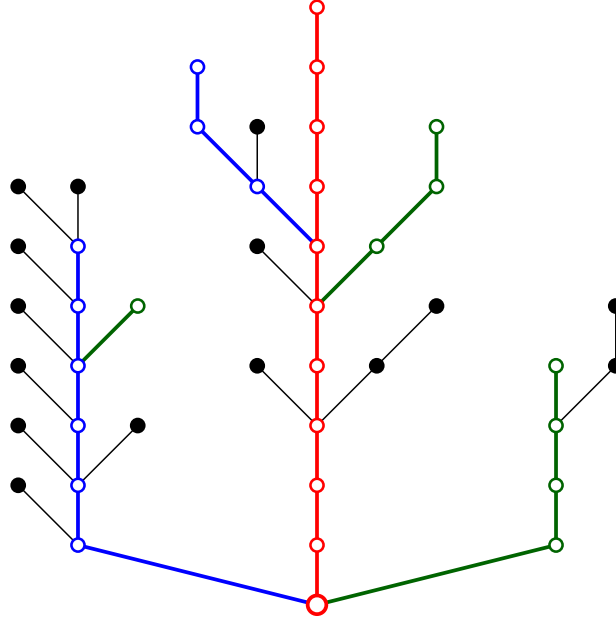


Figure 8.3 – An example of 3-trail. The root is at the bottom. The vertices of the 3-trail are represented as empty circle. The vertices of the first trail $\{X_{i,1}\}_{1 \leq i \leq h^n}$ with their edge to their father are represented in red. The second trail $\{X_{i,2}\}_{1 \leq i \leq h^n}$ is in blue. And the third $\{X_{i,3}\}_{1 \leq i \leq h^n}$ is in green. Note that at each height there are exactly 3 vertices in the 3-trail, except when in the tree there are less than k vertices at this height in which case all of them are in the 3-trail. Note also that the fathers of the vertices in the 3-trail are also in the 3-trail.

- (h) For every $i \leq h^n$, for every $j \in \mathbb{N}$, the random variable $X_{i,j}^n$ is a vertex of \mathcal{T}^n of height i .
- (\neq) For every $i \leq h^n$, for every $j \neq j' \leq D_{i-1}^n$, we have a.s. $X_{i,j}^n \neq X_{i,j'}^n$.
- (f) For every $1 \leq i \leq h^n$, $j \leq D_{i-1}^n$, a.s. $f^n(X_{i,j}^n) \in \{X_{i-1,j'}^n\}_{j' \leq j}$.
- (\perp) For every $1 \leq H \leq h^n$, the σ -algebra generated by $(X_{i,j}^n)_{H \leq i \leq h^n, j \in \mathbb{N}}$ and $(f^n(i, j))_{i > H, j \leq D_{i-1}^n}$ is independent from $(f^n(i, j))_{1 \leq i \leq H, j \leq D_{i-1}^n}$.

For $k \in \mathbb{N}$, we call the set $\mathcal{S}_k^n := \{X_{i,j}^n\}_{i \leq h^n, j \leq k'}$ the k -trail (associated to $(X_{i,j}^n)_{i,j}$). Note that in \mathcal{S}_k^n there are exactly $\inf(k, D_{i-1}^n)$ vertices of height i .

Lemma 8.4.1. *There exist some random variables $(X_{i,j}^n)_{i \leq h^n, j \in \mathbb{N}}$ satisfying (h), (\neq), (f), (\perp).*

Proof. We construct the sequence by induction on j and downward induction on i . For $j = 1$ it suffices to take a uniform random vertex at height h^n and its ancestors. Assume that for every $1 \leq i \leq h^n$ and $j \leq k$, the vertex $X_{i,j}^n$ is defined. Then we let $X_{h^n, k+1}^n$ be

- If possible, a uniform random vertex of \mathcal{T}^n of height h^n not in $\{X_{h^n, j}^n\}_{1 \leq j \leq k}$;
- Else, any vertex of height h^n in \mathcal{T}^n .

Then for every $i < h^n$ given $(X_{h, j}^n)_{h \leq h^n, 1 \leq j \leq k}$ and $X_{i+1, k+1}^n$ we define $X_{i, k+1}^n$ as:

- $f^n(X_{i+1, k+1}^n)$ if it is not in $\{X_{i, j}^n\}_{1 \leq j \leq k}$;
- Else, if possible, any vertex in \mathcal{T}^n not in $\{X_{i, j}^n\}_{1 \leq j \leq k}$ of height i ;
- Else, if not possible, any vertex in \mathcal{T}^n of height i .

A quick induction then shows that this sequence is well defined and satisfies the desired properties. \square

The aim of this section is to prove that k -trails approximate well \mathcal{T}^n .

Proposition 8.4.2. *For any choice of $(X_{i,j}^n)$ we have*

$$\lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_H(\mathcal{S}_k^n, \mathcal{T}^n) > \varepsilon n) = 0.$$

First, we prove that k -trails and k^2 -trails are close. For every $n \in \mathbb{N}$, we write $\|D^n\|_\infty := \max_{0 \leq i \leq h^n} D_i^n$ and let $\Gamma_{<\beta n}$ be the set of vertices of \mathcal{T}^n which are at height smaller than βn .

Lemma 8.4.3. *For every $0 < \alpha < \beta < 1$, for every $n \in \mathbb{N}$, $k \leq \|D^n\|_\infty$ large enough, writing $L = \lceil n/(\log \log k)^2 \rceil$, with probability at least $1 - 1/k$, we have*

$$\forall x \in \mathcal{S}_{k^2}^n, \quad \alpha n + 9L < h(x) < \beta n \implies d^n(x, \mathcal{S}_k^n \cap \Gamma_{<\beta n}) \leq 9L.$$

Proof. First note that if $d^n(X_{i,j}^n, \mathcal{S}_k^n \cap \Gamma_{\beta n}) > 9L$ for some $\alpha n + 9L < i < \beta n$ and $j \in [k^2]$, then for every $1 \leq a \leq L$, we have $d^n(\mathfrak{f}_a^n(X_{i,j}^n), \mathcal{S}_k^n \cap \Gamma_{\beta n}) > 8L$, so by (f),

$$\#\{\alpha n + 8L < i' < \beta n, 1 \leq j' \leq k^2 \text{ such that } d^n(\mathfrak{f}^n(X_{i',j'}^n), \mathcal{S}_k^n \cap \Gamma_{<\beta n}) > 8L\} \geq L.$$

Thus, by Markov's inequality it is enough to prove that for every $\alpha n + 8L < i < \beta n$, $1 \leq j \leq k^2$ we have

$$\mathbb{P}(d^n(X_{i,j}^n, \mathcal{S}_k^n \cap \Gamma_{<\beta n}) > 8L) \leq \frac{1}{k} \frac{L}{nk^2}. \quad (8.4.1)$$

Fix such i, j . To this end, we can first compute explicitly for every $1 \leq a \leq 4L$,

$$\mathbb{P}(\mathfrak{f}_{a+1}^n(X_{i,j}^n) \notin \{X_{i-a-1,j}^n\}_{1 \leq j \leq k} | \mathfrak{f}_a^n(X_{i,j}^n) \notin \{X_{i-a,j}^n\}_{1 \leq j \leq k}). \quad (8.4.2)$$

Indeed, by (\perp) , given the event $\mathfrak{f}_a^n(X_{i,j}^n) \notin \{X_{i-a,j}^n\}_{1 \leq j \leq k}$, the fathers of $\mathfrak{f}_a^n(X_{i,j}^n)$ and of the $X_{i-a,j}^n$'s for $1 \leq j \leq k$ have the same law as the (unconditioned) father of any $k+1$ different vertices of height exactly $i-a$. In other words, writing $i' = i-a-1$, if $D_{i'}^n > k$ (note that by (\neq) when $D_{i-1-a}^n \leq k$, a.s. $\mathfrak{f}_{a+1}^n(X_{i,j}^n) \in \{X_{i-a-1,j}^n\}_{1 \leq j \leq k}$) then (8.4.2) equals, dividing according to (i', ℓ) the value of $\mathfrak{f}_{a+1}^n(X_{i,j}^n)$,

$$\sum_{\ell \geq 1} \frac{d_{i',\ell}^n}{D_{i'}^n} \prod_{1 \leq b \leq k} \left(1 - \frac{d_{i',\ell}^n - 1}{D_{i'}^n - b}\right),$$

which we may upper-bound, using that $b \geq 1$ and then that $(1-x)^k \leq 1 - \min(1, kx)/2$ for all $x \in [0, 1]$, by

$$\sum_{\ell \geq 1} \frac{d_{i',\ell}^n}{D_{i'}^n} \left(1 - \frac{d_{i',\ell}^n - 1}{D_{i'}^n - 1}\right)^k \leq \sum_{\ell \geq 1} \frac{d_{i',\ell}^n}{D_{i'}^n} \left(1 - \min\left(1, k \frac{d_{i',\ell}^n - 1}{D_{i'}^n - 1}\right)/2\right).$$

The right-hand side is then equal to $1 - \tau_{i'}^n(k)/2$ since $\sum_{\ell \geq 1} d_{i',\ell}^n = D_{i'}^n$ and by definition of τ (see (8.1.3) in the introduction), which we may again upper-bound by $e^{-\tau_{i'}^n(k)/2}$ since for every $x > 0$, we have $(1-x) \leq e^{-x}$. Thus,

$$\begin{aligned} \mathbb{P}(d^n(X_{i,j}^n, \mathcal{S}_k^n \cap \Gamma_{<\beta n}) > 8L) &\leq \mathbb{P}(\mathfrak{f}_{8L}^n(X_{i,j}^n) \notin \{X_{i-8L,j}^n\}_{1 \leq j \leq k}) \\ &\leq \prod_{a=0}^{8L-1} \mathbb{P}(\mathfrak{f}_{a+1}^n(X_{i,j}^n) \notin \{X_{i-a-1,j}^n\}_{1 \leq j \leq k} | \mathfrak{f}_a^n(X_{i,j}^n) \notin \{X_{i-a,j}^n\}_{1 \leq j \leq k}) \\ &\leq \prod_{i-8L < h \leq i} e^{-\tau_h^n(k)/2} \\ &= \exp(-\tau_{i-8L+1,i}^n(k)/2) \\ &\leq k^{-7/2}, \end{aligned}$$

using for the last inequality $L = \lceil n/(\log \log k)^2 \rceil$ and $(\text{Tight}_{\text{GHP}})$. This implies the desired bound (8.4.1). \square

Proof of Proposition 8.4.2. We deduce Proposition 8.4.2 from Lemma 8.4.3 with a chaining argument. Fix $0 < \varepsilon < 0.01$. Fix K large enough depending only on ε such that $\log \log K > 2/\varepsilon$, and such that for every $n \in \mathbb{N}$ large enough, Lemma 4.3 holds with $\alpha = \varepsilon/2$, $\beta = 1 - \varepsilon$, and every $K \leq k \leq \|D^n\|_\infty$. Let $N = N(n, K)$ be the smallest integer such that $K^{2^N} \geq \|D^n\|_\infty$. Note already that by $(\text{Tight}_{\|\text{D}\|})$ we have $N = o(n)$.

By Lemma 8.4.3, for every n large enough with probability at least $1 - \sum_{i=0}^{N-1} 1/K^{2^i} \geq 1 - 2/K$:

$$\forall 0 \leq i \leq N-1, \quad \forall x \in \mathcal{S}_{K^{2^{i+1}}}^n, \quad \varepsilon n < h(x) < (1-\varepsilon)n \implies d^n(x, \mathcal{S}_{K^{2^i}}^n \cap \Gamma_{<(1-\varepsilon)n}) \leq 9\lceil n/(\log \log K^{2^i})^2 \rceil. \quad (8.4.3)$$

Then writing $\Gamma_{\leq \varepsilon n}$ for the set of vertices with height at most εn , (8.4.3) implies

$$\forall 0 \leq i \leq N-1, \quad d_H\left(\left(\Gamma_{\leq \varepsilon n} \cup \mathcal{S}_{K^{2^i}}^n\right) \cap \Gamma_{<(1-\varepsilon)n}, \left(\Gamma_{\leq \varepsilon n} \cup \mathcal{S}_{K^{2^{i+1}}}^n\right) \cap \Gamma_{<(1-\varepsilon)n}\right) \leq 9\lceil n/(\log \log K^{2^i})^2 \rceil,$$

which by the triangle inequality implies for every n, K large enough,

$$\begin{aligned} d_H\left(\left(\Gamma_{\leq \varepsilon n} \cup \mathcal{S}_K^n\right) \cap \Gamma_{<(1-\varepsilon)n}, \left(\Gamma_{\leq \varepsilon n} \cup \mathcal{S}_{K^{2^N}}^n\right) \cap \Gamma_{<(1-\varepsilon)n}\right) &\leq \sum_{i=0}^{N-1} 9\lceil n/(\log \log K^{2^i})^2 \rceil \\ &\leq N + 9n \sum_{i=1}^{\infty} \frac{1}{(i \log(2) + \log \log K)^2} \\ &\leq N + \frac{40n}{\log \log K}. \end{aligned}$$

Then note that by (≠) and since $K^{2^N} \geq \|D^n\|_{\infty}$, we know that $\mathcal{S}_{K^{2^N}}^n$ contains all vertices of \mathcal{T}^n . Moreover, $(0, 1) \in \mathcal{S}_K^n$ so every $x \in \Gamma_{\leq \varepsilon n}$ is at distance at most εn of \mathcal{S}_K^n . Finally, if x is a vertex of \mathcal{T}^n such that $h(x) \geq (1-\varepsilon)n$ then by considering the highest ancestor x' of x belonging to $\Gamma_{<(1-\varepsilon)n}$, we obtain that $d(x, \Gamma_{\leq \varepsilon n} \cup \mathcal{S}_K^n) \leq 1 + n\varepsilon + d(x', \Gamma_{\leq \varepsilon n} \cup \mathcal{S}_K^n)$. Hence, the last line implies

$$d_H(\mathcal{S}_K^n, \mathcal{T}^n) \leq 1 + 2\varepsilon n + N + \frac{40n}{\log \log K}.$$

Since ε can be chosen arbitrarily small, K arbitrarily large, and since $N = o(n)$ this concludes the proof. \square

8.4.2 Approximating k -trails from genealogies of random vertices

By Proposition 8.6.2, to extend the GP convergence in Theorem 8.1.2 to the GHP convergence in Theorem 8.1.3 it suffices to prove the following strong leaf tightness criterion:

$$\forall \delta > 0, \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_H(\mathcal{T}^n, \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n) = 0.$$

where V_1^n, V_2^n, \dots are i.i.d. uniform random vertices in \mathcal{T}^n . To that end, by Proposition 8.4.2, it is enough to prove that

$$\forall \delta > 0, \forall K \in \mathbb{N}, \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\max_{x \in \mathcal{S}_K^n} d^n(x, \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n\right) = 0.$$

Or even, by noting that if there exists an $x \in \mathcal{S}_K^n$ such that $d^n(x, \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n$ then either:

- $h(x) < (\delta/2)n$ and $\inf_{1 \leq i \leq k} h(V_i^n) \geq (\delta/2)n$. For every $\delta > 0$, the probability of this event goes to 0 as $k \rightarrow \infty, n \rightarrow \infty$ by (Lim $_{\nu}$) and since ν has support $[0, 1]$.
- Or $h(x) \geq (\delta/2)n$, so that by (f) there are at least $\lfloor (\delta/4)n \rfloor$ different vertices $y \in \mathcal{S}_K^n$ such that $d^n(y, \{V_1^n, V_2^n, \dots, V_k^n\}) \geq \lfloor (\delta/4)n \rfloor$.

Thus, by Markov's inequality, since $\#\mathcal{S}_K^n \leq Kh^n \sim Kn$ it is enough to show that

$$\forall \delta > 0, \forall K > 0, \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq a \leq h^n, 1 \leq b \leq K} \mathbb{P}(d^n(X_{a,b}^n, \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n) = 0,$$

or even by considering the different possible values of $X_{a,b}^n$ it suffices to prove the following result:

Proposition 8.4.4. *Under the setting of Theorem 8.1.3, for every $\delta > 0$,*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{1 \leq a \leq h^n, 1 \leq b \leq D_{a-1}^n} \mathbb{P}(d^n((a, b), \{V_1^n, V_2^n, \dots, V_k^n\}) > 3\delta n) = 0.$$

Proof. We follow directly the same approach as in Section 8.2.2 which proves an analogous result for the limit tree. Again by (Lim_ν) and since $\text{supp}(\nu) = [0, 1]$ we may restrict to the case where $a > 2\delta n$. And as in Section 8.2.2, since we may consider a larger but still bounded number of random vertices $V_1^n, V_2^n, \dots, V_k^n$ and since ν has support $[0, 1]$, we can assume them to have height between $a - \delta n$ and a . In other words, it suffices to check that for every $\delta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{2\delta n < a \leq h^n, 1 \leq b \leq D_{a-1}^n} \mathbb{P}(d^n((a, b), \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n | \forall 1 \leq i \leq k, a - \delta n < h(V_i^n) < a) = 0.$$

To ease the writing, with a slight abuse of notation, let $\mathbb{P}^k(\cdot) := \mathbb{P}(\cdot | \forall 1 \leq i \leq k, a - \delta n < h(V_i^n) < a)$ (define similarly \mathbb{E}^k) and let $V_0^n = (a, b)$. Again we may consider the number of vertices that have not coalesced up to time x that is:

$$N_k(x) := \#\{1 \leq i \leq k \text{ such that } h(V_i) \geq x \text{ and } \forall 1 \leq j \neq i \leq k, h(V_i^n \wedge V_j^n) < x\},$$

where $V_i^n \wedge V_j^n$ is the nearest common ancestor of V_i^n and V_j^n in \mathcal{T}^n . And, for exactly the same reasons as in Section 8.2.2, one can see that

$$\mathbb{P}^k(d^n(V_0^n, \{V_1^n, V_2^n, \dots, V_k^n\}) > \delta n) \leq \frac{\mathbb{E}^k[N_k(a - 2\delta n)]}{k}.$$

The proof of Proposition 8.4.4 is thus complete as long as we prove the following result. \square

Lemma 8.4.5. *For every $\delta > 0$,*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{2\delta n < a \leq h^n} \frac{\mathbb{E}^k[N_k(a - 2\delta n)]}{k} = 0.$$

Proof. The proof is essentially the same as for Lemma 8.2.3 with slightly different computations. Let $\delta > 0$. Let $k \geq 1$, $n \geq 1$ and $2\delta n < a \leq h^n$. Recall the definition of the sequence (ε_n) from Subsection 8.3.1. For all $a - 2\delta n \leq i \leq a - \delta n$, we lowerbound

$$\begin{aligned} & \mathbb{E}^k[N_k(i+1) - N_k(i) | N_k(i+1)] \\ & \geq \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n} N_k(i+1) \frac{d_{i,j}^n}{D_i^n} \left(1 - \prod_{r=1}^{N_k(i+1)-1} \left(1 - \frac{d_{i,j}^n - 1}{D_i^n - r} \right) \right) + 2 \binom{N_k(i+1)}{2} \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)} \\ & \geq \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n} N_k(i+1) \frac{d_{i,j}^n}{D_i^n} \left(1 - \left(1 - \frac{d_{i,j}^n - 1}{D_i^n} \right)^{N_k(i+1)-1} \right) + 2 \binom{N_k(i+1)}{2} \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)} \\ & \geq \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n \vee (1 + D_i^n / (N_k(i+1) - 1))} N_k(i+1) \frac{d_{i,j}^n}{D_i^n} \left(1 - \frac{1}{e} \right) + 2 \binom{N_k(i+1)}{2} \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)}, \end{aligned}$$

where in the last inequality we used that for all $x \in (0, 1)$, $y > 0$ such that $xy > 1$ we have $(1-x)^y \leq 1/e$.

Next, fix $\eta > 0$. Considering only the case where $N_k(i+1) \geq \eta k$ and taking the expectation we get

$$\begin{aligned} & \mathbb{E}^k[N_k(i+1) - N_k(i)] \\ & \geq \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n \vee (1 + D_i^n / (\eta k - 1))} \mathbb{E}^k[N_k(i+1)] \frac{d_{i,j}^n}{D_i^n} \left(1 - \frac{1}{e} \right) + \mathbb{E}^k \left[2 \binom{N_k(i+1)}{2} \right] \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)}. \end{aligned}$$

We then sum over $a - 2\delta n \leq i \leq a - \delta n$ and use the fact that $N_k \leq k$, which gives

$$\begin{aligned} k & \geq \sum_{a-2\delta n \leq i \leq a-\delta n} \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n \vee (1 + D_i^n / (\eta k - 1))} \mathbb{E}^k[N_k(i+1)] \frac{d_{i,j}^n}{D_i^n} \left(1 - \frac{1}{e} \right) \\ & \quad + \sum_{a-2\delta n \leq i \leq a-\delta n} \mathbb{E}^k \left[2 \binom{N_k(i+1)}{2} \right] \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)}. \end{aligned}$$

But since for all $a - 2\delta n \leq i \leq a - \delta n$ we have $N_k(i) \geq N_k(a - 2\delta n)$, using Markov's inequality we obtain that

$$k \geq \mathbb{P}^k(N_k(a - 2\delta n) \geq \eta k)(\eta k - 1) \\ \times \sum_{a - 2\delta n \leq i \leq a - \delta n} \left(\sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n \vee (1 + D_i^n / (\eta k - 1))} \frac{d_{i,j}^n}{D_i^n} \left(1 - \frac{1}{e}\right) + (\eta k - 1) \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)} \right). \quad (8.4.4)$$

Finally, by (8.3.6), by (Lim_{Θ}) and (Lim_h) , we deduce that

$$\liminf_{n \rightarrow \infty} \min_{2\delta n < a \leq hn} \sum_{a - 2\delta n \leq i \leq a - \delta n} \left(\sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n > \varepsilon_n D_i^n \vee (1 + D_i^n / (\eta k - 1))} \frac{d_{i,j}^n}{D_i^n} \left(1 - \frac{1}{e}\right) + (\eta k - 1) \sum_{j=1}^{\infty} \mathbb{1}_{d_{i,j}^n \leq \varepsilon_n D_i^n} \frac{d_{i,j}^n (d_{i,j}^n - 1)}{D_i^n (D_i^n - 1)} \right) \\ \geq \min_{1 \leq \ell \leq \lceil 2/\delta \rceil} \left(\left(1 - \frac{1}{e}\right) \sum_{(\ell-1)\delta/2 < t < \ell\delta/2} \sum_{j=1}^{\infty} \mathbb{1}_{\theta_j(t) > 1/(\eta k - 1)} \theta_j(t) + (\eta k - 1) \rho([\ell - 1]\delta/2, \ell\delta/2) \right) \\ \xrightarrow[k \rightarrow \infty]{} \infty,$$

where the last convergence stems from $(\text{Tight}_{\text{GP}})$. Taking (8.4.4) into account, we have thus proven that for all $\eta > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \max_{2\delta n < a < hn} \mathbb{P}^k(N_k(a - 2\delta n) \geq \eta k) = 0.$$

This concludes the proof since $N_k \leq k$. \square

8.5 Application to BGW trees in varying environment

Let us describe here how our result gives general conditions under which Bienaymé–Galton–Watson trees in varying environment stopped at height n satisfy a scaling limit for the GP and the GHP topology. We rely on the scaling limit of the profile obtained by Bansaye and Simatos in [BS15]. This section can be read just after the introduction. We refer to [BS15] for examples.

As in [BS15], for all $n \geq 1$, independently for all $1 \leq i \leq n$, let $(\xi_{i,n}(j))_{j \geq 1}$ be a family of i.i.d. random variables in the set of non-negative integers. We recall that the Bienaymé–Galton–Watson process in varying environment $(Z_{i,n})_{0 \leq i \leq n}$ is defined by setting $Z_{0,n} = 1$ and for all $i \in [n]$,

$$Z_{i,n} = \sum_{j=1}^{Z_{i-1,n}} \xi_{i,n}(j).$$

Note that the environment, i.e. the law of the $\xi_{i,n}(j)$'s, depends on i . The Bienaymé–Galton–Watson tree in varying environment \mathcal{T}_n of height n may be constructed conditionally on the $(\xi_{i,n}(j))_{j \geq 1, 0 \leq i \leq n}$ as the tree with vertices of fixed degrees and heights obtained by letting for all $0 \leq i \leq n - 1$ the sequence $(d_{i,j}^n)_{1 \leq j \leq Z_{i,n}}$ be a reordering of the $\xi_{i+1,n}(j)$'s for $1 \leq j \leq Z_{i,n}$ in the non-increasing order, by setting for all $j > Z_{i,n}$, $d_{i,j}^n = 0$ and for all $i \geq n$ and $j \geq 1$ by setting $d_{i,j}^n = 0$.

We mainly follow the notation of [BS15] except that here n is the scale of the height (i.e. the time scale for the GWVE process). Let $(\ell_n)_{n \geq 1}$ be a sequence of positive real numbers tending to ∞ which will be the scale of the profile (i.e. the space scale). Let $(\gamma_n)_{n \geq 1}$ be a sequence of increasing bijections from $[0, 1]$ to $[0, 1]$ which converges uniformly to $t \mapsto t$. To simplify the notation, we write $\xi_{i,n} = \xi_{i,n}(1)$ for all $i \in [n]$. As in [BS15], we define the rescaled quantities for all integers $n \geq 1, i \in [n]$ and for all $t \in [0, 1]$,

$$\bar{\xi}_{i,n} = \frac{1}{\ell_n} (\xi_{i,n} - 1), \quad \alpha_{i,n} = \mathbb{E} \left[\frac{\bar{\xi}_{i,n}}{1 + \bar{\xi}_{i,n}} \right], \quad \beta_{i,n} = \mathbb{E} \left[\frac{\bar{\xi}_{i,n}^2}{1 + \bar{\xi}_{i,n}} \right] \quad \text{and} \quad X_n(t) = \frac{1}{\ell_n} Z_{\lfloor n\gamma_n(t) \rfloor, n}.$$

Intuitively, $\alpha_{i,n}$ is a proxy for the drift and $\beta_{i,n}$ plays the role of a ‘‘variance’’ of $\bar{\xi}_{i,n}$. We then set for all $n \geq 1, t \in [0, 1], x > 0$,

$$\alpha_n(t) = \ell_n \sum_{i=0}^{\lfloor n\gamma_n(t) \rfloor - 1} \alpha_{i,n}, \quad \beta_n(t) = \frac{1}{2} \ell_n \sum_{i=0}^{\lfloor n\gamma_n(t) \rfloor - 1} \beta_{i,n} \quad \text{and} \quad \mu_n([x, \infty) \times (0, t]) = \ell_n \sum_{i=0}^{\lfloor n\gamma_n(t) \rfloor - 1} \mathbb{P}(\bar{\xi}_{i,n} \geq x).$$

Next, we transpose Assumption A of [BS15] in our framework. We assume that there exist a càdlàg function of finite variation α , an increasing càdlàg function β and a positive measure μ on $(0, \infty) \times (0, 1)$ such that

(A1) For all $t \in [0, 1], x > 0$ such that $\mu(\{x\} \times (0, t]) = 0$,

$$\begin{aligned} \alpha_n(t) &\xrightarrow{n \rightarrow \infty} \alpha(t), & \|\alpha_n\|(t) &\xrightarrow{n \rightarrow \infty} \|\alpha\|(t), & \beta_n(t) &\xrightarrow{n \rightarrow \infty} \beta(t) \\ & & \text{and } \mu_n([x, \infty) \times (0, t]) &\xrightarrow{n \rightarrow \infty} \mu([x, \infty) \times (0, t]), \end{aligned}$$

where $\|\cdot\|$ is the total variation.

(A2) For every $t \in [0, 1]$ such that $\Delta\alpha(t) \neq 0, \Delta\beta(t) \neq 0$ or $\mu((0, \infty) \times \{t\}) \neq 0$ and for all $x > 0$ such that $\mu(\{x\} \times \{t\}) = 0$,

$$\alpha_{\lfloor n\gamma_n(t) \rfloor, n} \xrightarrow{n \rightarrow \infty} \Delta\alpha(t), \beta_{\lfloor n\gamma_n(t) \rfloor, n} \xrightarrow{n \rightarrow \infty} \Delta\beta(t) \quad \text{and} \quad \ell_n \mathbb{P}(\bar{\xi}_{\lfloor n\gamma_n(t) \rfloor, n} \geq x) \xrightarrow{n \rightarrow \infty} \mu([x, \infty) \times \{t\}).$$

We further assume the following condition coming from Proposition 2.2 of [BS15]:

(A3) For every $C > 0$,

$$\liminf_{n \rightarrow \infty} \left(\inf_{1 \leq i \leq n} \mathbb{E}[\xi_{i,n} \mathbb{1}_{\xi_{i,n} \leq C\ell_n}] \right) > 0.$$

Then by Theorem 2.1 of [BS15], we know that there exists a càdlàg process $(X(t))_{t \in [0,1]}$ taking its values in $[0, \infty]$ such that

$$(X_n(t))_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{(d)} (X(t))_{t \in [0,1]} \tag{8.5.1}$$

for the Skorokhod J1 topology on the space $\mathbb{D}([0, 1], [0, \infty])$, where $[0, \infty]$ is equipped with the distance $d(x, y) = |e^{-x} - e^{-y}|$, and ∞ is an absorbing state of the process $(X(t))_{t \in [0,1]}$. As in the first point of Theorem 2.1 of [BS15], we define the function $\tilde{\beta}$ by setting for all $t \in [0, 1]$,

$$\tilde{\beta}(t) = \beta(t) - \frac{1}{2} \int_{(0, \infty) \times (0, t)} \frac{x^2}{1 + x^2} \mu(dx, dt).$$

On the event ‘‘ $X(1) \in (0, \infty)$ ’’, we define the measures on $(0, 1)$

$$\nu(dt) = \frac{X(t)dt}{\int_0^1 X(s)ds} \quad \text{and} \quad \rho(dt) = \frac{2\tilde{\beta}(dt)}{X(t)},$$

where $\tilde{\beta}(dt)$ is the Lebesgue-Stieltjes measure associated with $\tilde{\beta}$. We also let $(\Delta_+ X(t))_{t \in (0,1)}$ be the collection of the positive jumps of X in $(0, 1)$ and we set for all $t \in (0, 1), \theta_1(t) = \Delta_+ X(t)^2 / X(t)^2$ and for all $j \geq 2, \theta_j(t) = 0$. We set $\Theta = ((\theta_j(t))_{j \geq 1})_{t \in (0,1)}$.

By Skorokhod’s representation theorem, we may assume that (8.5.1) holds almost surely. Then the consequence of Theorems 8.1.2 and 8.1.3 is the following:

Corollary 8.5.1. *Suppose that α, β, μ satisfy the conditions (A1), (A2) and (A3). Assume that $\mu((0, \infty) \times dt)$ has no atom and that $\mathbb{P}(\forall t \in (0, 1], 0 < X(t) < \infty) > 0$.*

- *If for all $0 < a < b < 1$, we have almost surely $\sum_{a \leq t \leq b} \Delta_+ X(t) / X(t) = \infty$ or $\tilde{\beta}(b) - \tilde{\beta}(a) > 0$, then on the event ‘‘ $\forall t \in (0, 1], X(t) \in (0, \infty)$ ’’, the tree \mathcal{T}_n/n converges in distribution to the random metric space $\mathcal{T}(\nu, \rho, \Theta)$ for the GP topology.*

- If $\log \log \ell_n = o(n)$ and for all $\delta > 0$, for all $0 < a < b < 1$, there exist $n_0, k_0 \geq 0$ such that for all $n \geq n_0$, on the event “ $\forall t \in (0, 1]$, $X(t) \in (0, \infty)$ ”, with probability at least $1 - \delta$, for all $k_0 \leq k \leq \max_{1 \leq i \leq n} Z_{i,n}$, for all $an \leq h \leq bn$,

$$\sum_{i=h}^{h+n/(\log \log k)^2} \sum_{j=1}^{Z_{i-1,n}} \frac{\xi_{i,n}(j)}{Z_{i,n}} \min\left(k \frac{\xi_{i,n}(j) - 1}{Z_{i,n} - 1}, 1\right) \geq \log k, \quad (8.5.2)$$

then the same convergence holds for the GHP topology.

Remark 8.5.2. Note that the assumption that $\mu((0, \infty) \times dt)$ has no atom rules out the case where two high degree vertices are at the same height. If one removes this assumption, then the $\theta_j(t)$'s could not be described using only the process X .

Proof. The proof boils down to checking that the conditions (8.1.2), (Lim_ν) , (Lim_\neq) , (Lim_Θ) , (Lim_ρ) , with $(\text{Tight}_{\text{GP}})$ and $(\text{Tight}_{\text{GHP}})$ for the GP convergence and the GHP convergence, are satisfied in probability. The assumption (Lim_ν) comes clearly from (8.5.1).

Besides, note that for all $A > 0$, if $\delta_n \rightarrow 0$, then

$$\begin{aligned} & \mathbb{P}(\exists i_1, i_2 \in [n], \exists j_1, j_2 \leq A\ell_n, (i_1, j_1) \neq (i_2, j_2), |i_1 - i_2| \leq n\delta_n \text{ and } \xi_{i_1,n}(j_1), \xi_{i_2,n}(j_2) \geq \varepsilon\ell_n) \\ & \leq A^2 \ell_n^2 \sum_{i_1=1}^n \mathbb{P}(\xi_{i_1,n} \geq \varepsilon\ell_n) \sum_{i_1 - n\delta_n \leq i_2 \leq i_1 + n\delta_n} \mathbb{P}(\xi_{i_2,n} \geq \varepsilon\ell_n) \\ & \leq A^2 \mu_n([\varepsilon, \infty) \times (0, 1)) \max_{1 \leq i_1 \leq n} \ell_n \sum_{i_1 - n\delta_n \leq i_2 \leq i_1 + n\delta_n} \mathbb{P}(\xi_{i_2,n} \geq \varepsilon\ell_n), \end{aligned}$$

and therefore

$$\mathbb{P}(\exists i_1, i_2 \in [n], \exists j_1, j_2 \leq A\ell_n, (i_1, j_1) \neq (i_2, j_2), |i_1 - i_2| \leq n\delta_n \text{ and } \xi_{i_1,n}(j_1), \xi_{i_2,n}(j_2) \geq \varepsilon\ell_n) \xrightarrow[n \rightarrow \infty]{} 0, \quad (8.5.3)$$

since $\mu((0, \infty) \times dt)$ has no atom and since $\mu_n \rightarrow \mu$ by (A1). This gives directly (Lim_\neq) with high probability and (Lim_Θ) in probability by (8.5.1).

Then, let us check the assumption (8.1.2) for Θ . In other words, we will prove that for all $0 < a < b < 1$, almost surely

$$\sum_{t \in [a,b]} \frac{\Delta_+ X(t)^2}{X(t)^2} < \infty. \quad (8.5.4)$$

In order to do this, let $0 < a < b < 1$ and let $\varepsilon > 0$ and $\delta > 0$. On the event

$$\begin{aligned} \mathcal{A}_n(\varepsilon, \delta) = \{ & \forall t \in [a, b], \delta \leq X_n(t) \leq 1/\delta \text{ and } \forall i, i' \in [n], \forall j, j' \leq \ell_n/\delta, \\ & ((i, j) \neq (i', j') \text{ and } \bar{\xi}_{i,n}(j), \bar{\xi}_{i',n}(j') \geq \varepsilon) \implies |i - i'| \geq \delta n \}, \end{aligned}$$

we have

$$\begin{aligned} \sum_{a \leq t \leq b} \frac{\Delta_+ X_n(t)^2}{X_n(t)^2} \mathbb{1}_{\Delta_+ X_n(t)/X_n(t) \geq \varepsilon} & \leq \sum_{i=1}^n \sum_{j=1}^{\ell_n/\delta} \left(\frac{\xi_{i,n}(j)}{Z_{i,n}} \right)^2 \mathbb{1}_{\xi_{i,n}(j)/Z_{i,n} \geq \varepsilon} \\ & \leq \frac{1}{\delta^2} \sum_{i=1}^n \sum_{j=1}^{\ell_n/\delta} (\bar{\xi}_{i,n}(j)^2 \wedge 1) \mathbb{1}_{\bar{\xi}_{i,n}(j) \geq \varepsilon\delta}. \end{aligned}$$

Furthermore, using (A1) which entails that μ_n converges weakly towards μ , we obtain that

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^n \sum_{j=1}^{\ell_n/\delta} (\bar{\xi}_{i,n}(j)^2 \wedge 1) \mathbb{1}_{\bar{\xi}_{i,n}(j) \geq \varepsilon\delta} \right] & = \int_{(0, \infty) \times (0, 1)} \frac{1}{\delta} (x^2 \wedge 1) \mathbb{1}_{x \geq \varepsilon\delta} \mu_n(dx, dt) \\ & \xrightarrow[n \rightarrow \infty]{} \frac{1}{\delta} \int_{(0, \infty) \times (0, 1)} (x^2 \wedge 1) \mathbb{1}_{x \geq \varepsilon\delta} \mu(dx, dt). \end{aligned}$$

Similarly,

$$\begin{aligned} \text{Var} \left(\sum_{i=1}^n \sum_{j=1}^{\ell_n/\delta} (\bar{\xi}_{i,n}(j)^2 \wedge 1) \mathbb{1}_{\bar{\xi}_{i,n}(j) \geq \varepsilon \delta} \right) &\leq \sum_{i=1}^n \sum_{j=1}^{\ell_n/\delta} \mathbb{E} \left[(\bar{\xi}_{i,n}(j)^2 \wedge 1)^2 \mathbb{1}_{\bar{\xi}_{i,n}(j) \geq \varepsilon \delta} \right] \\ &\leq \frac{1}{\delta} \int_{(0,\infty) \times (0,1)} (x^2 \wedge 1) \mathbb{1}_{x \geq \varepsilon \delta} \mu_n(dx, dt) \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{\delta} \int_{(0,\infty) \times (0,1)} (x^2 \wedge 1) \mathbb{1}_{x \geq \varepsilon \delta} \mu(dx, dt). \end{aligned}$$

As a result, the sequence

$$\left(\mathbb{1}_{\mathcal{A}_n(\varepsilon, \delta)} \sum_{a \leq t \leq b} \frac{\Delta_+ X_n(t)^2}{X_n(t)^2} \mathbb{1}_{\Delta_+ X_n(t)/X_n(t) \geq \varepsilon} \right)_{n \geq 1}$$

is uniformly integrable and it converges a.s. by (8.5.1). Thus, for all $0 < a < b < 1$, for all $\delta, \varepsilon > 0$,

$$\mathbb{E} \left[\mathbb{1}_{\forall t \in [a,b], \delta \leq X(t) \leq 1/\delta} \sum_{a \leq t \leq b} \frac{\Delta_+ X(t)^2}{X(t)^2} \mathbb{1}_{\Delta_+ X(t)/X(t) \geq \varepsilon} \right] \leq \frac{1}{\delta^3} \int_{(0,\infty) \times (0,1)} (x^2 \wedge 1) \mu(dx, dt),$$

hence (8.5.4) since the upper-bound does not depend on ε and since $\int_{(0,\infty) \times (0,1)} x^2 \wedge 1 \mu(dx, dt) < \infty$ by the first point of Theorem 2.1 of [BS15].

Next, let us show (Lim_ρ) . By (A1), (A2), it is elementary to see that we have the weak convergences $\beta_n(dt) \rightarrow \beta(dt)$ and $\alpha_n(dt) \rightarrow \alpha(dt)$. For all non-negative continuous function $f : [0, 1] \rightarrow \mathbb{R}$ with compact support $[a, b] \subset (0, 1)$,

$$\begin{aligned} \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\binom{\xi_{i,n}(j)}{2}}{\binom{Z_{i,n}}{2}} &= \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{(\xi_{i,n}(j) - 1)^2 + \xi_{i,n}(j) - 1}{2 \binom{Z_{i,n}}{2}} \\ &= \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2 + \bar{\xi}_{i,n}(j)/\ell_n}{2 \binom{Z_{i,n}}{2} / \ell_n^2}. \end{aligned}$$

Let $\varepsilon, \delta > 0$. Let us separate the large jumps from the small jumps. We first focus on the large jumps. By (8.5.1) and (8.5.3), and on the event “ ε is not the value of a positive jump of X ”, we have

$$\sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2 + \bar{\xi}_{i,n}(j)/\ell_n}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) \geq \varepsilon} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \sum_{t \in (0,1)} f(t) \frac{\Delta_+ X(t)^2}{X(t)^2} \mathbb{1}_{\Delta_+ X(t) \geq \varepsilon}. \quad (8.5.5)$$

We then focus on the small jumps. We first focus on

$$\sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon}$$

Let us write

$$G_{n,\varepsilon} := \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2 / (1 + \bar{\xi}_{i,n}(j)^2)}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon}.$$

Then,

$$G_{n,\varepsilon} \leq \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} \leq (1 + \varepsilon^2) G_{n,\varepsilon}.$$

Moreover, if for all $i \in [n]$, we set $\tilde{\beta}_{i,n,\varepsilon} = \mathbb{E}[\bar{\xi}_{i,n}^2/(1 + \bar{\xi}_{i,n}^2) \mathbb{1}_{\bar{\xi}_{i,n} < \varepsilon}]$, then

$$\mathbb{E} \left[\left(G_{n,\varepsilon} - \sum_{i=1}^n f(i/n) Z_{i-1,n} \frac{\tilde{\beta}_{i,n,\varepsilon}}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \right)^2 \mathbb{1}_{\forall t \in [a,b], \delta \leq X_n(t) \leq 1/\delta} \right] \leq \max(f) \frac{\ell_n}{\delta} \sum_{i=1}^n \frac{\beta_{i,n} \varepsilon^2}{\delta^2} \\ \xrightarrow{n \rightarrow \infty} \max(f) \frac{\varepsilon^2}{\delta^3} \beta(1),$$

where the convergence comes from (A1).

Furthermore, if for all $t \in [0, 1]$, we set

$$\tilde{\beta}_{n,\varepsilon}(t) = \frac{1}{2} \ell_n \sum_{i=1}^{\lfloor nt \rfloor} \tilde{\beta}_{i,n,\varepsilon} \quad \text{and} \quad \tilde{\beta}_\varepsilon(t) = \beta(t) - \frac{1}{2} \int_{(\varepsilon, \infty) \times (0,t)} \frac{x^2}{1+x^2} \mu(dx, dt),$$

then by (A1) and since γ_n converges uniformly towards the identity, we know that $\tilde{\beta}_{n,\varepsilon}(dt)$ converges weakly towards $\tilde{\beta}_\varepsilon(dt)$. As a consequence, by (8.5.1),

$$\sum_{i=1}^n f(i/n) Z_{i-1,n} \frac{\tilde{\beta}_{i,n,\varepsilon}}{2 \binom{Z_{i,n}}{2} / \ell_n^2} = \int_0^1 f(t) \frac{Z_{\lfloor nt \rfloor - 1, n}}{\ell_n} \frac{2 \tilde{\beta}_{n,\varepsilon}(dt)}{2 \binom{Z_{\lfloor nt \rfloor, n}}{2} / \ell_n^2} \xrightarrow[n \rightarrow \infty]{\text{(a.s.)}} \int_0^1 f(t) \frac{2 \tilde{\beta}_\varepsilon(dt)}{X(t)}.$$

Thus, since $\tilde{\beta}_\varepsilon$ converges weakly towards $\tilde{\beta}$ as $\varepsilon \rightarrow 0$, we deduce that, on the event “ $\forall t \in (0, 1], X(t) \in (0, \infty)$ ”,

$$\sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} \xrightarrow[n \rightarrow \infty, \varepsilon \rightarrow 0]{\mathbb{P}} \int_0^1 f(t) \frac{2 \tilde{\beta}(dt)}{X(t)}. \quad (8.5.6)$$

Let us then show that

$$R_{n,\varepsilon} := \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j) / \ell_n}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} 0. \quad (8.5.7)$$

Using that for all $x \leq \varepsilon$, we have $x \leq x/(1+x^2) + \varepsilon x^2/(1+x^2)$, we upper-bound and lower-bound

$$\frac{1 + 1/\ell_n^2}{\ell_n} \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)/(1 + \bar{\xi}_{i,n}(j)^2)}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} \leq R_{n,\varepsilon} \\ \leq \frac{1}{\ell_n} \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)/(1 + \bar{\xi}_{i,n}(j)^2)}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} + \frac{\varepsilon}{\ell_n} \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)^2/(1 + \bar{\xi}_{i,n}(j)^2)}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon}.$$

By (8.5.6) and since $\ell_n \rightarrow \infty$, we know that the above term on the right goes to zero as $n \rightarrow \infty$ in probability.

Moreover,

$$\mathbb{E} \left[\mathbb{1}_{\forall t \in [a,b], \delta \leq X_n(t) \leq 1/\delta} \frac{1}{\ell_n} \sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\bar{\xi}_{i,n}(j)/(1 + \bar{\xi}_{i,n}(j)^2)}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \mathbb{1}_{\bar{\xi}_{i,n}(j) < \varepsilon} \right] \\ \leq \mathbb{E} \left[\mathbb{1}_{\forall t \in [a,b], \delta \leq X_n(t) \leq 1/\delta} \frac{1}{\ell_n} \sum_{i=1}^n f(i/n) Z_{i-1,n} \frac{\alpha_{i,n}}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \right] \\ \leq \mathbb{E} \left[\mathbb{1}_{\forall t \in [a,b], \delta \leq X_n(t) \leq 1/\delta} \frac{1}{\ell_n} \sum_{i=1}^n f(i/n) Z_{i-1,n} \frac{|\alpha_{i,n}|}{2 \binom{Z_{i,n}}{2} / \ell_n^2} \right] \\ \leq \frac{1}{\ell_n} \max(f) \frac{1}{\delta} \frac{\|\alpha_n(1)\|}{\delta^2} \xrightarrow{n \rightarrow \infty} 0,$$

where the convergence stems from (A1) and from the fact that $\ell_n \rightarrow \infty$. This proves (8.5.7) by taking δ small enough.

Combining (8.5.5), (8.5.6), (8.5.7), we conclude that on the event “ $\forall t \in (0, 1], X(t) \in (0, \infty)$ ”,

$$\sum_{i=1}^n f(i/n) \sum_{j=1}^{Z_{i-1,n}} \frac{\binom{\xi_{i,n}(j)}{2}}{\binom{Z_{i,n}}{2}} \xrightarrow[n \rightarrow \infty]{(\mathbb{P})} \sum_{t \in (0,1)} f(t) \frac{\Delta_+ X(t)^2}{X(t)^2} + \int_0^1 f(t) \frac{2\tilde{\beta}(dt)}{X(t)}.$$

This proves (Lim_ρ) .

The assumption that for all $0 < a < b < 1$, we have $\int_{(0,\infty) \times [a,b]} x\mu(dx, dt) = \infty$ or $\beta(b) - \beta(a) > 0$ entails readily $(\text{Tight}_{\text{GP}})$. Finally, under the assumption (8.5.2), for all $\delta \in (0, 1)$, the condition $(\text{Tight}_{\text{GHP}})$ is satisfied with probability $1 - \delta$. \square

8.6 Leaf-tightness criteria

In this section, $\mathbf{X} = ((X^n, d^n, p^n))_{n \in \mathbb{N}}$ denotes a sequence of random compact measured metric spaces GHP-measurable. For every $n \in \mathbb{N}$, let $(x_i^n)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of common distribution p^n , then let $M^n := (d^n(x_i^n, x_j^n))_{i,j \in \mathbb{N}}$.

We say that \mathbf{X} is weak-leaf-tight if and only if

$$\forall \delta > 0, \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d^n(x_{k+1}^n, \{x_1^n, x_2^n, \dots, x_k^n\}) > \delta) = 0. \quad (8.6.1)$$

We say that \mathbf{X} is strong-leaf-tight if and only if

$$\forall \delta > 0, \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_H(X^n, \{x_1^n, x_2^n, \dots, x_k^n\}) > \delta) = 0. \quad (8.6.2)$$

Those criteria were first introduced by Aldous [Ald91, Ald93].

Proposition 8.6.1. *If $(M^n)_{n \in \mathbb{N}}$ converges weakly toward a random matrix M , \mathbf{X} is weak-leaf-tight, and for every $n \in \mathbb{N}$, (X^n, d^n) is a random tree equipped with random edge lengths, then \mathbf{X} converges weakly for the GP topology toward a random measured \mathbb{R} -tree (X, d, p) . Furthermore, if $(x_i)_{i \in \mathbb{N}}$ are i.i.d. random variables of law p then $M^X := (d(x_i, x_j))_{i,j \in \mathbb{N}} \stackrel{(d)}{=} M$.*

Proof. The result is directly adapted from Theorem 3 of Aldous [Ald93] using modern formalism. \square

Proposition 8.6.2 (Proposition 7.7.1 from Chapter 7). *If $(M^n)_{n \in \mathbb{N}}$ converges weakly toward a random matrix M , and \mathbf{X} is strong-leaf-tight, then \mathbf{X} converges weakly for the GHP topology toward a random compact measured metric space (X, d, p) . Furthermore, if $(x_i)_{i \in \mathbb{N}}$ are i.i.d. random variables of law p then $M^X := (d(x_i, x_j))_{i,j \in \mathbb{N}} \stackrel{(d)}{=} M$. In addition, a.s. p has full support.*

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Titre : Cartes aléatoires $3/2$ -stables

Mots clés : cartes aléatoires, probabilités, graphes aléatoires, arbres aléatoires, géométrie aléatoire, gravité quantique

Résumé : Cette thèse se concentre sur l'étude du comportement asymptotique des cartes planaires aléatoires et des arbres aléatoires. Dans une première partie, nous étudions la géométrie de grandes cartes aléatoires avec des sommets de grands degrés, appelées cartes stables, avec un intérêt tout particulier pour le cas critique : celui des cartes $3/2$ -stables. Nous démontrons que ces cartes apparaissent naturellement dans l'étude des cartes aléatoires couplées au modèle de boucles $O(n)$, modèle classique de mécanique statistique, dans le cas $n = 2$. Nous obtenons le comportement asymptotique des distances entre deux sommets uniformes et démontrons que le diamètre est du même ordre de grandeur. Une conséquence est que les cartes $3/2$ -stables ne satisfont pas de limite d'échelle au sens habituel de Gromov-Hausdorff. Cependant, nous obtenons la limite d'échelle des distances entre les sommets de grands degrés et la racine. Cette limite d'échelle correspond à une distance entre les boucles d'un ensemble conforme de boucles et le bord du disque, mesurée à l'aide d'une gravité quantique de Liouville indépendante, qui est reliée à la distance

quantique au bord introduite par Aru, Holden, Powell et Sun. Nous étudions ensuite les géodésiques à la racine pour la distance de percolation de premier passage. Nous construisons la limite d'échelle des géodésiques au moyen d'un flot coalescent de diffusions à sauts.

Dans une seconde partie, nous généralisons le modèle des arbres récursifs uniformes en ajoutant un mécanisme de gel. À certaines étapes, un sommet choisi au hasard est gelé et plus aucun nouveau sommet ne peut s'attacher à celui-ci. L'arbre des infections dans un modèle SIR sur le graphe complet rentre dans le cadre de ce modèle. Nous obtenons les limites locales de ces arbres et les limites d'échelle des distances entre la racine et un sommet typique, de la distance entre deux sommets typiques et de la hauteur totale. Dans le cas sous-linéaire où le nombre de sommets non gelés évolue comme une puissance donnée du nombre de sommets, nous identifions une transition de phase. En utilisant des techniques similaires, nous obtenons enfin la limite d'échelle d'arbres aléatoires uniformes dont on fixe le degré et la hauteur de chaque sommet.

Title : On random $3/2$ -stable maps

Keywords : random maps, probability, random graphs, random trees, random geometry, quantum gravity

Abstract : This thesis focuses on the asymptotic behaviour of random planar maps and random trees. In a first part, we study the geometry of large random planar maps with high degree vertices, also called stable maps, with a particular interest in the critical case: the case of $3/2$ -stable maps. We prove that these maps appear naturally in the study of random planar maps coupled with a loop $O(n)$ model, a classical model of statistical mechanics, in the case $n = 2$. We obtain the asymptotic behaviour of the distances between two uniform random vertices and we prove that the diameter is of the same order. A consequence of these results is that $3/2$ -stable maps do not satisfy scaling limits in the usual sense of Gromov-Hausdorff. However, we get the scaling limit of the distances between high degree vertices and the root. This scaling limit corresponds to a distance between the loops of a conformal loop ensemble and the boundary of the disc, measured using an independent Liouville quantum gravity, which is the Lamperti transform of the "quantum distance" to the boundary introduced by Aru, Holden, Powell and Sun. We then study the geo-

desics to the root for the first passage percolation distance on stable maps. We construct the scaling limit of the geodesics by means of a coalescing flow of pure jump diffusions.

In a second part, we introduce a generalisation of the model of random recursive trees by adding a freezing mechanism. At some steps, a uniform random vertex is frozen and new vertices can no longer be attached to this vertex. The infection tree in an SIR model on the complete graph falls within this framework. We obtain local limits of these trees and scaling limits of the distances between the root and a typical vertex, of the distance between two uniform typical vertices and of the total height of the tree obtained after n steps. In the sub-linear case where the number of non-frozen vertices evolves as a given power of n , we identify a phase transition. Lastly, we consider large uniform random trees where we fix for each vertex its degree and height. We prove, under natural conditions of convergence for the profile, that those trees properly renormalised converge.