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# Existence, instability and energy partition for linear or nonlinear evolution equations issued from fluid mechanics

Haocheng Yang

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# Existence, instabilité et répartition de l'énergie pour des équations d'évolution linéaires ou non linéaires issues de la mécanique des fluides

*Existence, instability and energy partition for linear or  
nonlinear evolution equations issued from fluid  
mechanics*

**Thèse de doctorat de l'université Paris-Saclay**

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Thèse préparée dans les unités de recherche **Centre Borelli** (Université Paris-Saclay, CNRS, ENS Paris-Saclay) et **LAGA** (Université Sorbonne Paris Nord), sous la direction de **Thomas ALAZARD**, Directeur de recherche, et la co-direction de **Jean-Marc DELORT**, Professeur.

**Thèse soutenue à Paris-Saclay, le 16 Juin 2025, par**

**Haocheng YANG**

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**Titre :** Existence, instabilité et répartition de l'énergie pour des équations d'évolution linéaires ou non linéaires issues de la mécanique des fluides

**Mots clés :** Analyse microlocale, Équation dispersive, Répartition de l'énergie, Théorème du viriel, Instabilité de Rayleigh-Taylor, Écoulements de jet

**Résumé :** Cette thèse contient trois thèmes : la partition microlocale de l'énergie, le théorème du viriel, et le caractère bien posé des écoulements de jet. Le premier thème concerne la manière dont l'énergie des solutions des équations dispersives est distribuée également à l'intérieur et à l'extérieur du cône de lumière. Ce phénomène a été fréquemment utilisé pour étudier le comportement à long terme des équations d'ondes (non) linéaires. Nous l'étendons à une grande classe d'équations dispersives linéaires, en remplaçant le cône de lumière par une hypersurface dans l'espace des phases et en caractérisant la partition de l'énergie via des opérateurs pseudo-différentiels.

Le deuxième thème examine l'équipartition

des énergies cinétique et potentielle pour les systèmes de vagues d'eau à deux phases. Cette étude conduit à une description quantitative de l'instabilité du système lorsque le fluide supérieur est plus dense que le fluide inférieur, ce qui est connu comme l'instabilité de Rayleigh-Taylor et l'instabilité de Kelvin-Helmoltz.

Le troisième thème traite du problème de Cauchy pour l'évolution des fluides cylindriques à frontière libre, à savoir les jets. Nous établissons le caractère localement bien posé pour cette équation d'évolution dans les espaces de Sobolev sans aucune hypothèse de symétrie. Ce résultat constitue une étape fondamentale vers la compréhension du comportement à long terme des écoulements de jets.

**Title :** Existence, instability and energy partition for linear or nonlinear evolution equations issued from fluid mechanics

**Keywords :** Microlocal analysis, Dispersive equation, Partition of energy, Virial theorem, Rayleigh-Taylor instability, Jets

**Abstract :** This thesis contains three themes : microlocal partition of energy, the virial theorem, and the well-posedness of jet flows. The first theme explores how the energy of solutions to dispersive equations is equally distributed inside and outside the light cone. This phenomenon has been frequently used to study the long-term behavior of (non)linear wave equations. We will extend it to a large class of linear dispersive equations, replacing the light cone by a hypersurface in phase space and characterizing the energy partition via pseudo-differential operators.

The second theme examines the equipartition of kinetic and potential energies for two-

phase water-wave systems. This study leads to a quantitative description of the instability of the system when the lower fluid is no denser than the upper one, which is known as the Rayleigh-Taylor instability and the Kelvin-Helmoltz instability.

The third theme addresses the Cauchy problem of the evolution of cylindrical fluids with free boundary, namely jets. We establish the local well-posedness theory for this evolution equation in Sobolev spaces without any symmetric assumptions. This result constitutes a fundamental step towards understanding the long-term behaviour of jet flows.

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# 1 - Introduction

Cette thèse est consacrée à l'étude de trois problèmes d'équations aux dérivées partielles linéaires ou non linéaires dispersives : la **répartition de l'énergie** pour certaines équations d'évolution linéaires, l'**instabilité de Rayleigh-Taylor** en mécanique des fluides et l'existence locale de solutions pour le **problème des jets**.

Pour les équations d'ondes (non) linéaires, il est bien connu qu'à long terme, l'énergie se concentre près du cône de lumière et se répartit de manière égale à l'intérieur et à l'extérieur de celui-ci. Cette propriété constitue un outil fondamental dans l'analyse des solitons des équations d'ondes non linéaires. Le premier travail de cette thèse étend cette propriété de concentration asymptotique de l'énergie à une large classe d'équations dispersives linéaires, notamment l'équation de Schrödinger, l'équation de (demi-)Klein-Gordon et l'équation linéarisée des vagues gravitationnelles ou capillaires. Une différence majeure avec les équations d'ondes réside dans le fait que, dans ces cas, le cône de lumière doit être remplacé par une hypersurface dans l'espace des phases. De plus, la caractérisation précise de la localisation de l'énergie à l'intérieur et à l'extérieur du cône de lumière nécessite l'usage d'opérateurs pseudo-différentiels. Dans le cadre de cette analyse, nous établissons la bornitude d'une classe d'opérateurs pseudo-différentiels dont les symboles sont des fonctions caractéristiques de certains domaines convexes non bornés dans l'espace des phases.

Un autre type de répartition de l'énergie découle du **théorème du viriel**, qui énonce qu'en moyenne temporelle, l'énergie cinétique et l'énergie potentielle deviennent asymptotiquement égales. Ce phénomène a été observé dans nombreux systèmes hamiltoniens issus de la physique, des mathématiques, de la biologie et de l'économie. Le système des vagues à deux phases, qui décrit le mouvement de deux couches de fluides potentiels, immiscibles, non visqueux et incompressibles, séparées par une interface libre, admet une formulation hamiltonienne, rendant possible l'établissement d'un théorème du viriel dans cette thèse. Lorsque la couche supérieure possède une densité plus élevée que celle de la couche inférieure, l'énergie potentielle du système est négative, tandis que l'énergie cinétique reste non négative. Dans ce cas, le théorème du viriel implique une croissance polynomiale de l'énergie cinétique et potentielle, entraînant une augmentation correspondante de la pente et de la courbure de l'interface. Cette instabilité, connue sous les noms d'**instabilité de Rayleigh-Taylor** et d'**instabilité de Kelvin-Helmholtz**, a été largement étudiée par expériences physiques et simulations numériques, et des descriptions quantitatives rigoureusement justifiées seront développées dans cette thèse.

Dans le cas des fluides cylindriques à frontière libre, à savoir **jets**, un phénomène physique d'instabilité provient d'un mécanisme différent. Plus précisément, l'effet de la tension superficielle entraîne une rupture (c'est-à-dire, le jet devient plusieurs gouttes séparables) en temps fini, un phénomène connu sous le nom d'**instabilité de Rayleigh-Plateau**. Avant d'analyser cette instabilité, il est nécessaire d'établir le caractère bien posé pour l'évolution de ces systèmes en trois dimensions. En supposant que le fluide est axi-symétrique, le problème se ramène à un système en deux dimensions, et le problème de Cauchy peut être résolu comme le système des vagues planaires en dimension deux. En l'absence d'une telle hypothèse de symétrie, des difficultés techniques supplémentaires apparaissent à proximité de l'axe. Dans cette thèse, nous abordons ces défis et établissons un caractère localement bien posé dans les espaces de Sobolev, sans imposer des conditions de symétrie.

## 1.1 . Répartition microlocale de l'énergie

Pour les équations d'évolution, la **répartition de l'énergie** désigne la concentration de l'énergie dans certains domaines spécifiques. Par exemple, considérons l'équation des ondes linéaire sur  $\mathbb{R}^d$  :

$$(1.1.1) \quad \begin{cases} (\partial_t^2 - \Delta)w = 0, \\ w|_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d). \end{cases}$$

Il est bien connu qu'à long terme, l'énergie de ce système se concentre près du cône de lumière  $\{|x| = |t|\}$ . Dans [DKM11 ; DKM12], les auteurs montrent que l'énergie se répartit équitablement à l'intérieur et à l'extérieur du cône de lumière. Plus précisément, étant donnée la troncature critique de l'énergie à l'intérieur et à l'extérieur du cône de lumière,

$$E^{\text{int}}(w_0, w_1, t) := \int_{|x| < |t|} (|\nabla w|^2 + |\partial_t w|^2) dx,$$

$$E^{\text{ext}}(w_0, w_1, t) := \int_{|x| > |t|} (|\nabla w|^2 + |\partial_t w|^2) dx,$$

en dimension *impaire*  $d$ , nous avons

$$(1.1.2) \quad \begin{aligned} \text{soit } E^{\text{int}}(w_0, w_1, t) &\geq \frac{1}{2} (\|w_1\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2), \quad \forall t \geq 0, \\ \text{soit } E^{\text{ext}}(w_0, w_1, t) &\geq \frac{1}{2} (\|w_1\|_{L^2}^2 + \|\nabla w_0\|_{L^2}^2), \quad \forall t \geq 0. \end{aligned}$$

Grâce aux techniques de l'analyse non linéaire, cette propriété peut être étendue à certaines équations d'ondes non linéaires. Puisque la coupure critique permet une localisation exacte de l'énergie, cette répartition d'énergie peut être utilisée pour localiser les solitons des équations d'ondes non linéaires. Par exemple, cette stratégie est employée

dans [DKM11 ; DKM12] pour étudier les solitons de l'équation des ondes non linéaire focalisante critique en énergie en dimension  $d = 3, 5$ . Nous pouvons également renvoyer à [Côt+15a ; Côt+15b] pour l'application aux applications d'ondes équivariantes.

Notons que, en dimension *paire*  $d$ , la propriété (1.1.2) est vérifiée avec certaines corrections supplémentaires, qui ont été calculées en détail pour les solutions radiales dans [CKS14] et pour des données générales dans [CL21 ; Del22]. Dans ces mêmes articles, les auteurs ont aussi découvert des données spéciales telles que la correction supplémentaire s'annule. Une application de ce résultat à l'équation des ondes focalisante critique en énergie en dimension 4 se trouve dans [Côt+18].

La démonstration de (1.1.2) s'appuie sur les estimations dispersives de l'équation des ondes linéaire, ce qui laisse présager un analogue pour d'autres équations dispersives. Dans [Del22], Delort a établi la première répartition de l'énergie pour l'équation de Schrödinger et l'équation des demi-ondes, ce qui implique le résultat (1.1.2) ci-dessus. Dans le cas de l'équation de Schrödinger, dont la relation de dispersion est  $P(\xi) = \xi^2/2$ , le cône de lumière est donnée par l'hypersurface  $|x| = |t||\xi|$  dans l'espace des phases, et la troncature critique de l'énergie est réalisée par des opérateurs pseudo-différentiels de symbole  $\mathbb{1}_{|x| < |t||\xi|}$  ou  $\mathbb{1}_{|x| > |t||\xi|}$ . Étant donné que l'équation de Schrödinger ne présente aucune particularité dans ce problème, le premier objectif de cette thèse est d'étendre ce résultat aux équations dispersives de type fractionnaire, couvrant la majorité des équations dispersives fréquemment étudiées.

### 1.1.1 . Les équations dispersives de type fractionnaire

Considérons l'équation dispersive sur  $\mathbb{R}^d$  :

$$(1.1.3) \quad \begin{cases} \left( \frac{\partial_t}{i} - P(D_x) \right) u = 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

où la relation de dispersion  $P$  est radiale et lisse sauf en  $\xi = 0$ . Nous supposons en outre que  $P$  est de **type fractionnaire**, à savoir qu'elle se comporte comme une fonction fractionnaire au voisinage de zéro et à l'infini. Cette condition effectivement couvre la plupart des équations dispersives connues.

Notons que l'équation de Korteweg–de Vries et de Benjamin–Ono sont définies en dimension 1 et peuvent être étudiées séparément en fréquence positive et négative, ce qui ramène leurs relations de dispersion au cas  $P(\xi) = |\xi|^3$  et  $P(\xi) = |\xi|^2$ , respectivement.

Dans ce cadre général, le cône de lumière est donné par

$$|x| = |t||P'(\xi)|,$$

qui définit une hypersurface dans l'espace des phases. Il est facile de vérifier que l'énergie  $L^2$  de la solution  $u$  à l'équation dispersive (1.1.3) se concentre près du cône de lumière. En effet, pour toute fonction lisse à support compact  $\chi \in C_c^\infty(\mathbb{R}^d)$  valant 1 au voisinage

Équation	Relation de dispersion $P(\xi)$
Schrödinger	$ \xi ^2/2$
Demi-Klein-Gordon	$\langle \xi \rangle = \sqrt{1 +  \xi ^2}$
Korteweg-de Vries	$\xi^3$
Benjamin-Ono	$ \xi \xi$
Vague de gravité	$ \xi ^{1/2},  \xi ^{1/2} \tanh( \xi )$
Vague capillaire	$ \xi ^{3/2},  \xi ^{3/2} \tanh( \xi )$

Tableau 1.1 – Exemples d'équations dispersives de type fractionnaire

de zéro, nous avons, pour tout  $\delta > 0$ ,

$$(1.1.4) \quad \text{Op} \left( \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \right) (u(t)) = u(t) + o_{L^2}(1), \quad \text{lorsque } t \rightarrow \pm\infty,$$

où l'opérateur pseudo-différentiel est défini par la quantification standard

$$\text{Op}(a)u := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi,$$

avec  $\hat{u}$  désignant la transformée de Fourier de  $u$ .

À partir de (1.1.4), la troncature critique en dehors du cône de lumière peut être naturellement définie par

$$(1.1.5) \quad E_\delta(u_0, t) = \|\text{Op}(a_\delta(t))(u(t))\|_{L^2}^2 = \left\| \text{Op}(a_\delta(t)) \left( e^{itP(D_x)} u_0 \right) \right\|_{L^2}^2,$$

avec le symbole  $a_\delta$  défini par

$$(1.1.6) \quad a_\delta(t, x, \xi) := \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|}.$$

Par souci de simplicité, nous nous concentrons sur la troncature en dehors du cône de lumière dans la suite, et le cas intérieur peut être traité de même manière en remplaçant  $|x| > |t| |P'(\xi)|$  par  $|x| < |t| |P'(\xi)|$ .

Le problème de la répartition de l'énergie peut alors être formulé ainsi :

**Problème 1.1.1.** *Existe-t-il une constante  $c_0 > 0$  telle que la limite*

$$(1.1.7) \quad \lim_{t \rightarrow \pm\infty} E_\delta(u_0, t) = c_0 \|u_0\|_{L^2}^2$$

*soit satisfaite pour tout état initial  $u_0 \in L^2(\mathbb{R}^d)$  ?*

Comparé au cas des équations des ondes, la troncature utilisée ici introduit une coupure supplémentaire au voisinage du cône de lumière, fournissant ainsi des informations plus précises sur la localisation de l'énergie  $L^2$ . En fait, il est possible de supprimer cette coupure supplémentaire en utilisant la concentration de l'énergie (1.1.4).

### 1.1.2. La répartition de l'énergie

La réponse au Problème 1.1.1 est positive lorsque la relation de dispersion  $P$  est strictement convexe ou concave, ce qui est vérifié pour de nombreuses équations dispersives, y compris celles listées dans le tableau 1.1. De plus, la constante  $c_0$  dans le membre de droite de (1.1.7) dépend de  $\delta$ , qui décrit la taille de la région tronquée.

Lorsque  $\delta < 0$ , nous pouvons vérifier que le membre de droite de (1.1.4) devient  $o_{L^2}(1)$ , ce qui signifie que la région tronquée est choisie si petite qu'aucune énergie ne reste. À l'inverse, pour  $\delta > 0$ , nous nous attendons à ce que (1.1.7) soit vérifiée, en tenant compte de (1.1.4). Le cas critique  $\delta = 0$  doit être traité séparément, car la limite (1.1.7) pourrait ne pas prendre la forme  $c_0 \|u_0\|_{L^2}^2$ , même en l'absence de la troncature stricte  $\mathbb{1}_{|x| > |t| |P'(\xi)|}$ .

**Théorème 1.1.2.** *Supposons que le symbole de type fractionnaire  $P$  vérifie  $P'' > 0$  ou  $P'' < 0$ . Alors, en toute dimension  $d \geq 1$ , la limite (1.1.7) est satisfaite pour tout  $u_0 \in L^2(\mathbb{R}^d)$ , avec  $c_0$  égal à  $1/4$  lorsque  $0 < \delta < 1/2$  et égal à  $0$  lorsque  $\delta < 0$ .*

*Dans le cas critique  $\delta = 0$ , l'énergie tronquée  $E_\delta(u_0, t)$  admet une limite lorsque  $t \rightarrow \pm\infty$  pour tout  $u_0 \in L^2(\mathbb{R}^d)$ . De plus, il existe des choix de fonction  $\chi$  telle que cette limite soit strictement positive pour tout  $u_0 \neq 0$ .*

Rappelons que nous avons utilisé la troncature  $|x + tP'(\xi)| \ll |t|^{1/2+\delta}$ , dont le membre de gauche est, en approximation, de taille  $t$ . Ainsi, cette troncature devient trivial lorsque  $\delta \geq 1/2$ , et ce cas est donc omis dans le théorème ci-dessus.

Pour  $0 < \delta < 1/2$ , le résultat précédent indique qu'un quart de l'énergie est distribuée en dehors (ou à l'intérieur) du cône de lumière, tandis que un demi de l'énergie est réparti dans l'interaction entre les composantes intérieure et extérieure au cône. En comparaison, l'énergie de l'équation des ondes est également distribuée à l'intérieur et à l'extérieur du cône de lumière. Cette différence est due à la vitesse de propagation constante. Dans le cas de l'équation de Klein-Gordon, qui possède une vitesse de propagation finie comme l'équation des ondes, le Théorème 1.1.2 implique qu'elle se comporte comme l'équation de Schrödinger, avec un quart de l'énergie à l'intérieur et à l'extérieur du cône de lumière.

**Corollaire 1.1.3.** *Soit  $w$  l'unique solution de l'équation de Klein-Gordon sur  $\mathbb{R}^d$ ,*

$$\begin{cases} (\partial_t^2 - \Delta + 1)w = 0, \\ w|_{t=0} = w_0 \in H^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d), \end{cases}$$

*où  $w_0, w_1$  sont réels, et donc  $w$  l'est aussi. Alors, l'énergie tronquée en dehors du cône de lumière*

$$\begin{aligned} E^{KG}(w_0, w_1, t) := & \| \text{Op}(a^{KG}(t)) \partial_t w(t) \|_{L^2}^2 + \| \text{Op}(a^{KG}(t)) \nabla w(t) \|_{L^2}^2 \\ & + \| \text{Op}(a^{KG}(t)) w(t) \|_{L^2}^2 \end{aligned}$$

*vérifie*

$$\lim_{t \rightarrow \pm\infty} E^{KG}(w_0, w_1, t) = \frac{1}{4} (\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2),$$

où le symbole de l'opérateur de troncature est donné par

$$a^{KG}(t, x, \xi) := \mathbb{1}_{|x| > |t \frac{\xi}{\langle \xi \rangle}|} \chi \left( \frac{\xi}{|t|^\epsilon} \right),$$

avec  $0 < \epsilon < 1$ , et  $\chi \in C_c^\infty(\mathbb{R}^d)$  est une fonction réelle, radiale, et égale à 1 près de zéro.

Remarquons que le facteur  $\chi(\xi/|t|^\epsilon)$  dans le symbole de troncature  $a^{KG}$  est ajouté uniquement pour des raisons techniques, la partie principale étant la coupure critique  $|x| > |t| |\xi| / \langle \xi \rangle$ .

La preuve du Théorème 1.1.2 se déroule en deux étapes. Tout d'abord, nous vérifions que l'opérateur de troncature  $\text{Op}(a_\delta)$  est borné sur  $L^2$ , uniformément en temps  $t \in \mathbb{R}$ . Ensuite, nous approchons  $u_0$  par des fonctions régulières et calculons directement la limite (1.1.7). Étant donné que cette seconde étape a déjà été étudiée dans [Del22], la tâche restante consiste à démontrer la bornitude uniforme en temps de  $\text{Op}(a_\delta)$  sur  $L^2$ , ce qui est une question technique approfondie qui sera traitée au Chapitre 1.

### 1.1.3. Un sous-produit

Comme mentionné dans le paragraphe précédent, nous devons prouver la bornitude sur  $L^2$  de l'opérateur pseudo-différentiel  $\text{Op}(a_\delta)$ . Dans le cas où la relation de dispersion est donnée par  $P(\xi) = |\xi|^{p+1}/(p+1)$  avec  $p \neq 0$ , un changement d'échelle permet de réduire cette question à la proposition suivante.

**Proposition 1.1.4.** *En dimension générale  $d \geq 1$ , pour tout  $p \in \mathbb{R}$ , l'opérateur pseudo-différentiel  $\text{Op}(\mathbb{1}_{|x| > |\xi|^p})$  est borné sur  $L^2(\mathbb{R}^d)$ .*

En général, la bornitude sur  $L^2$  des opérateurs pseudo-différentiels peut être démontrée si le symbole appartient à certaines classes, par exemple la classe de Hörmander  $S_{\rho, \delta}^0$  (cf. [Hör07]), où les symboles sont supposés être lisses. Grâce au calcul paradifférentiel, il est aussi possible de traiter des symboles ayant une régularité limitée en  $x$ , mais il n'existe pas de théorie générale pour les symboles discontinus comme  $\mathbb{1}_{|x| > |\xi|^p}$ .

Cependant, ce type de problème apparaît naturellement en mécanique quantique. Soit  $u$  la fonction d'état d'une particule (c'est-à-dire une solution de l'équation de Schrödinger). Pour un ensemble mesurable  $E$  de l'espace des phases  $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$ , la probabilité que cette particule se trouve dans  $E$  est donnée par

$$(1.1.8) \quad \frac{1}{(2\pi)^{\frac{d}{2}}} \int_E \mathcal{W}(u, u)(x, \xi) dx d\xi.$$

Ici,  $\mathcal{W}(u, u)(x, \xi)$ , appelée **distribution de Wigner**, est définie par

$$\mathcal{W}(u, v)(x, \xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{iy \cdot \xi} u\left(x + \frac{y}{2}\right) \overline{v\left(x - \frac{y}{2}\right)} dy,$$

ce qui décrit la distribution de la particule dans l'espace des phases. Afin que la probabilité (1.1.8) soit bien définie à une normalisation près, il faut prouver l'intégrabilité de la distribution de Wigner. De manière équivalente, il suffit de montrer que l'opérateur pseudo-différentiel  $\text{Op}^w(\mathbb{1}_E)u$  (quantification de Weyl) défini ci-dessous est borné sur  $L^2$  :

$$\langle \text{Op}^w(\mathbb{1}_E)u, v \rangle_{L^2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_E(x, \xi) \mathcal{W}(u, v)(x, \xi) dx d\xi, \quad \forall v \in L^2.$$

Si  $E$  est un ensemble mesurable arbitraire, la réponse est négative (cf. [Ler23]). Cependant, il a été démontré que les propriétés de l'opérateur  $\text{Op}^w(\mathbb{1}_E)$  (bornitude, positivité, spectre, etc.) sont liées à la géométrie de  $E$ . Par exemple, lorsque  $E$  est une ellipsoïde, dans [Fla88] et [LO10] les auteurs ont obtenu des estimations précises de la norme  $L^2$  de  $\text{Op}^w(\mathbb{1}_E)$ , en fonction de la taille de l'ellipsoïde. Un autre exemple concerne le cas où  $E$  est un polygone dans  $\mathbb{R}^2$  avec  $N$  côtés. Il a été démontré dans [Ler23] que la norme  $\mathcal{L}(L^2)$  de  $\text{Op}^w(\mathbb{1}_E)$  peut être contrôlée par  $\sqrt{N/2}$  pour  $N \geq 3$ . Le lecteur pourra consulter [Ler23] pour plus de résultats sur ce sujet.

Une question naturelle est de savoir si des résultats similaires s'appliquent à la quantification standard. Bien que la quantification standard soit souvent utilisée en analyse mathématique, ce type de résultat reste rare. Or, la Proposition 1.1.4 fournit une réponse positive pour des domaines  $E = \{|x| > |\xi|^p\}$ . Avec la méthode développée dans le Chapitre 1, nous pouvons également espérer établir la bornitude de  $\text{Op}(\mathbb{1}_{|x| > f(|\xi|)})$  pour des fonctions convexes ou concaves  $f$ , au moins celles qui se comportent comme des fonctions fractionnaires.

## 1.2 . L'instabilité des vagues

La dynamique des vagues à deux phases décrit le mouvement de deux couches de fluide séparées par une interface libre. Nous supposons que les fluides sont potentiels et gouvernés par les équations d'Euler incompressibles, avec des conditions cinématiques et dynamiques garantissant l'absence de vide entre les couches et l'équilibre des pressions. Lorsque le fluide inférieur est moins dense que le fluide supérieur, il est bien connu que le système possède un effet dispersif, et la relation de dispersion est donnée par

$$\omega(\xi)^2 = Ag|\xi| + \frac{\sigma}{\rho^+ + \rho^-} |\xi|^3,$$

où  $A$  est le **nombre d'Atwood**, défini par

$$(1.2.1) \quad A := \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-},$$

où  $\rho^+$  et  $\rho^-$  représentent respectivement les densités du fluide inférieur et supérieur. Ici,  $g > 0$  désigne l'accélération de gravité et  $\sigma \geq 0$  est le coefficient de tension superficielle. Une question naturelle qui se pose est de déterminer les conditions de stabilité de ce système.

Suite aux travaux pionniers de Rayleigh [Ray83], de nombreuses recherches ont été menées sur cette problématique. Dans les années 1990, Wu a rigoureusement établi le caractère bien posé pour le système avec un seul fluide [Wu97; Wu99], sous l'hypothèse d'une condition de stabilité connue sous le nom de **condition de signe de Taylor** ou **condition de Rayleigh-Taylor**, exprimée par

$$(1.2.2) \quad -\partial_y P|_{\text{interface}} > 0.$$

Ici,  $P$  désigne la pression et  $y$  est la coordonnée verticale. Cette condition, initialement mise en évidence dans l'analyse linéaire de Rayleigh [Ray83] et Taylor [Tay50], a été étendue aux systèmes non linéaires à deux fluides (voir [Lan13a] et les références associées) :

$$(1.2.3) \quad \llbracket -\partial_y P \rrbracket > \frac{1}{4\sigma} \left( \frac{\rho^+ \rho^-}{\rho^+ + \rho^-} \right)^2 \mathfrak{c}(\eta) \|\llbracket V \rrbracket\|_{L^\infty}^4,$$

où  $\llbracket \cdot \rrbracket$  désigne le saut à l'interface et  $V$  est la composante horizontale de vitesse.

Un contre-exemple notable à cette condition de stabilité est l'**instabilité de Rayleigh-Taylor**, notamment le cas  $A < 0$ , c'est-à-dire lorsque le fluide supérieur est plus dense que le fluide inférieur. Dans ce cas, le système linéarisé prend la forme

$$\partial_t u = \pm \left( |A|g|D_x| - \frac{\sigma}{\rho^+ + \rho^-} |D_x|^3 \right)^{\frac{1}{2}} u,$$

indiquant la présence de modes à croissance exponentielle. De telles instabilités ont été largement observées dans des expériences physiques [Ku91] et d'autres études connexes. Cependant, les investigations quantitatives de ce phénomène, en particulier dans le cadre non linéaire, restent limitées. La deuxième partie de cette thèse est consacrée à une description quantitative de l'instabilité de Rayleigh-Taylor.

L'approche développée dans le Chapitre 3 repose sur le **théorème du viriel** pour l'équation des vagues, introduit par Alazard et Zuily [AZ24]. De manière informelle, ce théorème énonce que si la trajectoire du système reste dans un ensemble compact, nous pouvons en déduire une égalité asymptotique entre l'énergie cinétique et l'énergie potentielle au sens de moyenne en temps. Dans le cas de l'instabilité de Rayleigh-Taylor, l'énergie cinétique est non négative par définition, tandis que l'énergie potentielle peut être négative, à cause de sa composante gravitationnelle, entraînant ainsi une contradiction. Par conséquent, la solution du système doit croître, et la vitesse de cette croissance peut être déterminée par une version quantitative du théorème du viriel à développer dans le Chapitre 3. Cette méthode, qui prédit une croissance polynomiale (plus faible que celle obtenue par analyse linéaire), s'applique à une large classe de solutions régulières, avec ou sans effet de tension superficielle.

En adoptant une stratégie similaire, on obtient des résultats analogues dans le cas critique  $A = 0$ . Notons qu'il existe un autre type d'instabilité, connue sous le nom d'**instabilité de Kelvin-Helmholtz** (KHI), ou **problème des nappes de tourbillons**, qui provient du saut de vitesse à l'interface. Cette instabilité n'est pas directement liée à la gravité et peut se manifester pour tout  $A \in \mathbb{R}$ . Dans notre formulation suivante, la continuité de

vélocité est autorisée, ce qui rend peu plausible d'attribuer la croissance polynomiale à une instabilité de type KHI. En fait, lorsque  $A = 0$ , le système est hyperbolique dégénéré, ce qui peut naturellement engendrer une croissance polynomiale en temps.

### 1.2.1 . Formulation du système

Soient  $\Omega^\pm(t)$  deux domaines connexes, séparés par une hypersurface  $\Sigma(t)$  sans auto-intersection, paramétrée par

$$(1.2.4) \quad \gamma(t): \mathbb{T}^d \rightarrow \mathbb{T}^d \times ]-H^+, H^-[ , \quad x \mapsto \gamma(t, x).$$

Ici,  $H^\pm \in ]0, +\infty]$  désigne la profondeur, et nous notons

$$(1.2.5) \quad \Gamma^\pm = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} \mid y = \mp H^\pm\}$$

les frontières solides (ou fonds). Lorsque  $H^+ = +\infty$  ou  $H^- = +\infty$ , la frontière solide correspondante est ignorée. La dimension  $d$  est généralement égale à 1 ou 2 pour des motivations physiques, bien que la plupart des arguments restent valables en toute dimension.

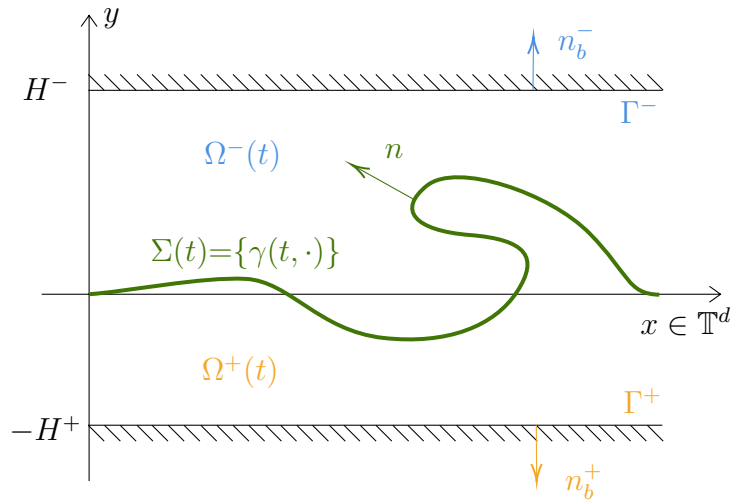


Figure 1.1 – Vagues à deux phases

Les domaines  $\Omega^\pm(t)$  représentent des régions de fluide supposées parfaites, potentielles et incompressibles. Par conséquent, le champ de vélocité admet une représentation potentielle  $u^\pm = \nabla_{x,y} \phi^\pm$ , où le potentiel  $\phi^\pm$  évolue selon l'équation de Bernoulli :

$$(1.2.6) \quad \rho^\pm \left( \partial_t \phi^\pm + \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) + P^\pm + \rho^\pm g y = 0, \quad \text{dans } \Omega^\pm(t).$$

Ici,  $\rho^\pm > 0$  est la densité constante,  $P^\pm$  la pression, et  $g \geq 0$  l'accélération gravitationnelle. L'incompressibilité du fluide impose que le potentiel  $\phi^\pm$  soit harmonique :

$$(1.2.7) \quad \Delta_{x,y} \phi^\pm = 0, \quad \text{dans } \Omega^\pm(t).$$

Les conditions aux frontières (libres et solides) complètent la description du mouvement des vagues. Sur les fonds  $\Gamma^\pm$ , l'imperméabilité du fond impose :

$$(1.2.8) \quad u^\pm \cdot n_b^\pm = 0, \quad \text{sur } \Gamma^\pm,$$

où  $n_b^\pm$  est la normale extérieure. En cas de profondeur infinie  $H^+ = +\infty$  ou  $H^- = +\infty$ , cette condition devient :

$$(1.2.9) \quad \lim_{y \rightarrow -\infty} u^+ = 0, \quad \text{ou} \quad \lim_{y \rightarrow +\infty} u^- = 0.$$

Deux conditions supplémentaires sont imposées à l'interface libre. La première est la condition cinématique, qui garantit que la vitesse normale de l'interface et des fluides coïncident :

$$(1.2.10) \quad \partial_t \gamma \cdot n = u^+ \cdot n = u^- \cdot n, \quad \text{sur } \Sigma(t),$$

où  $n$  est la normale unitaire (vers le haut) à l'interface  $\Sigma(t)$ . La seconde condition exprime l'équilibre des pressions avec la tension superficielle, appelée condition dynamique :

$$(1.2.11) \quad P^+ - P^- = \sigma \kappa, \quad \text{sur } \Sigma(t),$$

où  $\sigma \geq 0$  est le coefficient de tension superficielle et  $\kappa$  la courbure moyenne.

Ainsi, l'évolution des vagues à deux phases est décrite par le système suivant :

$$(1.2.12) \quad \begin{cases} \rho^\pm (\partial_t \phi^\pm + \frac{1}{2} |\nabla_{x,y} \phi^\pm|^2) + P^\pm + \rho^\pm g y = 0, & \text{dans } \Omega^\pm(t), \\ \Delta_{x,y} \phi^\pm = 0, & \text{dans } \Omega^\pm(t), \\ n \cdot \partial_t \gamma = n \cdot \nabla_{x,y} \phi^+ = n \cdot \nabla_{x,y} \phi^-, & \text{sur } \Sigma(t), \\ P^+ - P^- = \sigma \kappa, & \text{sur } \Sigma(t), \\ (\text{profondeur finie}) n_b^\pm \cdot \nabla_{x,y} \phi^\pm = 0, & \text{sur } \Gamma^\pm, \\ (\text{profondeur infinie}) \lim_{y \rightarrow \mp \infty} \nabla_{x,y} \phi^\pm = 0. \end{cases}$$

Grâce à la *formulation de Craig-Sulem-Zakharov*, ce système peut être reformulé de manière autonome, éliminant explicitement les pressions  $P^\pm$  (voir [Zak68; CS93; BB97]). D'une manière imprécise, nous prenons la différence entre la partie + et - de la première équation dans (1.2.12) limitée à l'interface  $\Sigma(t)$ , où une annulation concernant les pressions  $P^\pm$  peut être effectuée par la condition dynamique (1.2.11). Le système reformulé de cette manière contient seulement deux inconnues, l'interface  $\gamma$  et les traces des potentiels  $\phi^\pm$ , d'où le potentiel se retrouve en résolvant  $\Delta_{x,y} \phi^\pm = 0$  sous certaines conditions au bord. Cependant, bien que cette formulation soit compacte, elle introduit des difficultés techniques qui ne seront pas abordées ici (voir [Cra17] pour une discussion détaillée).

En particulier, nous nous concentrons sur deux configurations importantes :

- (1) Le **cas du graphe** : en dimension générale  $d \geq 1$ ,  $\Sigma(t)$  est représenté par le graphe d'une fonction  $\eta(t) : \mathbb{T}^d \rightarrow ]-H^+, H^-[$ ;
- (2) Le **cas de chevauchement** : en dimension  $d = 1$ ,  $\Sigma(t)$  est une courbe sans auto-intersection.

Ces deux cas jouent un rôle fondamental dans l'étude des vagues à deux phases.

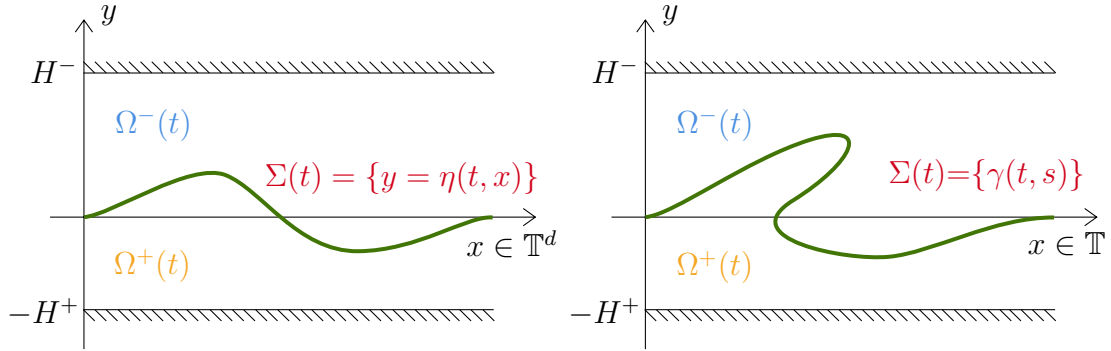


Figure 1.2 - Le cas du graphe & le cas de chevauchement

### 1.2.2 . Lois de conservation et la structure hamiltonienne

Dans l'évolution de vague à deux phases (1.2.12), il est bien connu que la masse et l'énergie sont conservées en temps. La masse est définie par

$$(1.2.13) \quad M := \int_{\Sigma} n \cdot (0, y) dS,$$

où  $dS$  désigne l'élément de surface (voir également (3.2.10) pour une formulation alternative). Pour la simplicité, nous supposons que  $M = 0$  pour tout temps. L'énergie totale du système (3.1.10)

$$(1.2.14) \quad E = E_k + E_p$$

est composée par l'énergie cinétique  $E_k$  et l'énergie potentielle  $E_p$ , qui sont définies par

$$(1.2.15) \quad E_k := \frac{1}{2} \sum_{\pm} \rho^{\pm} \iint_{\Omega^{\pm}} |\nabla_{x,y} \phi^{\pm}|^2 dx dy,$$

$$(1.2.16) \quad E_p := \frac{Ag}{2} \int_{\Sigma} n \cdot (0, y^2) dS + \frac{\sigma}{\rho^+ + \rho^-} [\text{Aire}(\Sigma(t)) - (2\pi)^d],$$

où  $\rho^{\pm} = \rho^{\pm} / (\rho^+ + \rho^-)$  est la densité relative,  $A$  est le nombre d'Atwood (1.2.1), et  $\text{Aire}(\Sigma(t))$  représente l'aire de l'hypersurface  $\Sigma(t)$ . Remarquons que cette énergie n'est pas définie positive lorsque le nombre d'Atwood  $A < 0$ .

Considérons la trace unifiée du potentiel,

$$(1.2.17) \quad \psi := \rho^+ \phi^+|_{\Sigma} - \rho^- \phi^-|_{\Sigma}.$$

Si, de plus, l'interface  $\Sigma(t)$  est le graphe d'une fonction  $\eta$ , l'énergie  $E$  devient une fonctionnelle non linéaire de  $(\eta, \psi)$  et le système (1.2.12) admet la formulation hamiltonienne suivante,

$$(1.2.18) \quad \eta_t = \frac{\delta E}{\delta \psi}, \quad \psi_t = -\frac{\delta E}{\delta \eta}.$$

Une dérivation formelle de cette formulation est donnée dans [KL95; BB97] pour le cas du graphe et dans [CGK05] pour le cas de chevauchement. En considérant cette structure hamiltonienne (formelle), nous pouvons espérer l'établissement d'un théorème du viriel pour le système (1.2.12), ce qui sera discuté dans la section suivante.

### 1.2.3 . Théorème du viriel

Le **théorème du viriel**, ou **équipartition de l'énergie**, décrit un phénomène selon lequel, dans de nombreux systèmes hamiltoniens, l'énergie cinétique et potentielle sont égales au sens de la moyenne temporelle, si la trajectoire du système reste dans un domaine compact. Proclamé par Clausius [Cla70] en 1870, il est rapidement devenu un outil puissant en physique [Zwi33; Ray05; Ein22]. Les liens entre le théorème du viriel et de nombreux modèles importants en physique classique ont également été découverts au cours des dernières décennies [Han95; GG99; Sho12; AZ24]. De plus, l'idée de ce théorème a également été appliquée à certains sujets en biologie [Pod19] et en économie [LP23].

En mathématiques, le théorème du viriel joue un rôle important dans les problèmes d'explosion des EDP. Dans les années 1970, Zakharov-Sobolev-Synakh [ZSS71] et Glassey [Gla77] ont montré que le théorème du viriel peut être utilisé pour étudier les solutions explosives des équations de Schrödinger non linéaires. Cette approche a depuis été utilisée dans divers contextes, notamment en mécanique des fluides et en physique des plasmas. Cette technique est également utilisée par Merle [Mer96], Sideris [Sid85], Levine [Lev74; Lev90], Keel-Tao [KT99], et Kenig-Merle [KM08]. Cette propriété est aussi montrée pour l'équation des ondes et de Maxwell [CS81], l'équation des ondes non linéaires [Gla73; WV08], et le thermo-élasticité [DG84]. Récemment, le théorème du viriel a également été établi pour l'équation des vagues avec gravité par Alazard-Zuily [AZ24], et les auteurs ont en déduit l'instabilité de Rayleigh-Taylor.

Le présent travail s'inscrit dans cette lignée. Vu que le système de vague à deux phases admet une formulation hamiltonienne, nous espérons une identité du viriel ci-dessous, qui sera prouvée dans le Chapitre 3.

**Théorème 1.2.1.** *Soit  $(\gamma, \phi^\pm, P^\pm)$  une solution régulière à (1.2.12). Alors, il existe une fonctionnelle  $I$  de  $(\gamma, \phi^\pm|_\Sigma)$ , telle que*

$$(1.2.19) \quad \frac{1}{2} \frac{d}{dt} I = \tilde{E}_k - E_p + R,$$

où  $\tilde{E}_k$  désigne l'énergie cinétique modifiée

$$(1.2.20) \quad \tilde{E}_k := \sum_{\pm} \rho^\pm \iint_{\Omega^\pm} \left( \frac{1}{4} |\nabla_x \phi^\pm|^2 + \frac{3}{4} |\partial_y \phi^\pm|^2 \right) dx dy,$$

$E_p$  est l'énergie potentielle, et le reste  $R$  est non-négatif.

Remarquons que, dans [AZ24], les auteurs ont traité le cas du graphe avec une couche de fluide et choisi  $I$  comme le produit scalaire de  $\eta$  (l'interface) et  $\psi$  (la trace du potentiel), ce qui forme les inconnus dans la formulation hamiltonienne (1.2.18). Dans cette thèse, nous allons étendre cette quantité au cas à deux phases et au cas de chevauchement.

#### 1.2.4. L'instabilité du système

Avant examiner l'instabilité, nous commençons par un rappel sur le caractère bien posé. En l'absence de tension superficielle, le caractère localement bien posé peut être établie dans l'espace des fonctions analytiques. Par exemple, dans [Sul+81; SS85], les auteurs ont utilisé le théorème de Cauchy-Kowalewski pour prouver que si les données initiales admettent une extension à la bande complexe de largeur  $r_0 > 0$ , alors il existe une solution unique dans l'espace des fonctions analytiques dans la bande de largeur  $r \in ]0, r_0[$ , et le temps d'existence est de taille  $r_0 - r$ . Ce résultat n'a été prouvé que pour de petites données initiales, où l'interface a un petit gradient et est plate à l'infini. Cette condition de petitesse n'est pas essentielle dans le cas du tore 2D grâce à Xie [Xie07] en ramenant le système sur le disque unité via une transformation conforme.

En effet, le cadre analytique est optimal au sens où la solution du problème KHI ou RTI est soit excessivement rugueuse, soit analytique. Plus précisément, pour le problème KHI en deux dimensions, Lebeau a prouvé dans [Leb02] que, une fois que l'interface a la régularité de Hölder  $C^{1,\epsilon}$  avec  $\epsilon > 0$ , la solution doit être analytique en temps et en espace. Ce résultat a également été prouvé indépendamment par Wu dans [Wu06], où l'interface est supposée avoir seulement une régularité Lipschitz. Et pour le problème RTI en deux dimensions, grâce à [KL05], la régularité de Hölder  $C^{1,\epsilon}$  de l'interface garantit que la solution est lisse, ou analytique dans l'espace avec l'hypothèse supplémentaire que le saut à l'interface n'a pas de zéro. Remarquons que la régularité minimale supposée pour l'interface doit être Lipschitz afin de donner un sens approprié au système (1.2.12). En outre, si le tourbillon est non trivial, Ebin a prouvé dans [Ebi88] que les problèmes RTI et KHI sont mal posés (au sens de Hadamard) dans les espaces de Sobolev. Un résultat similaire a également été prouvé par Guo-Tice [GT11] pour les fluides compressibles.

Pour RTI avec tension de surface, les cas où le caractère bien posé est connu sont rares sauf si la viscosité est non nulle. Lorsque le mouvement des fluides est gouverné par l'équation de Navier-Stokes, les auteurs de [WTK14] ont montré le caractère localement bien posé dans les espaces de Sobolev d'ordre élevé pour les petites données et leur décroissance presque exponentielle en temps. Si les fluides sont de plus supposés compressibles, il est prouvé dans [JTW16a; JTW16b; JTW16c] que le problème est localement bien posé pour de petites données initiales dans des espaces de Sobolev d'ordre élevé et qu'il est exponentiellement stable près de l'équilibre lorsque la tension superficielle est suffisamment grande.

Dans cette thèse, au lieu du caractère bien posé, nous nous concentrons sur l'instabilité du système (1.2.12). Grâce aux discussion ci-dessus, les solution à étudier peuvent être supposées assez régulière. Les résultats existants concernant l'instabilité sont principale-

ment basés sur les méthodes du mode croissant ou de la régularité maximale [HG03; GT10; WT12; JTW16b; JJJ23; PS10; Wil17]. Même si elles peuvent être appliquées aux contextes plus généraux, ces méthodes soulignent la comparaison entre systèmes linéaires et non linéaires, et l'instabilité ne peut être montrée que pour certaines perturbations proches de zéro en un temps court. En raison de la nature physique de ce problème, il faut s'attendre à une explosion (ou, au moins, à une croissance rapide) pour la plupart des solutions régulières, ce qui pourrait être extrêmement difficile en raison de la non-linéarité totale du système. Comme indiqué dans [AZ24], une solution possible pourrait être l'identité du viriel (voir Théorème 1.2.1), impliquant les résultats principaux suivants.

**Théorème 1.2.2.** *Soit  $(\gamma, \phi^\pm, P^\pm)$  une solution régulière à (1.2.12). Supposons que l'énergie totale  $E$  vérifie*

$$(1.2.21) \quad E \neq 0 \text{ si } \sigma = 0, \quad \text{et } E < 0 \text{ si } \sigma > 0.$$

(1) *En cas du graphe, notamment  $\gamma(t, x) = (x, \eta(t, x))$  pour certaine fonction  $\eta$ , il existe une constante  $C > 0$  qui dépend des données initiales, tel que, pour tout temps d'existence,*

$$(1.2.22) \quad |E|t + I(0) \leq C (1 + \|\nabla_x \eta(t)\|_{L^\infty})^{\frac{5}{2}};$$

(2) *En cas de chevauchement, l'inégalité ci-dessus deviens*

$$(1.2.23) \quad |E|t + I(0) \leq C (1 + \|\kappa(t)\|_{L^\infty})^{\frac{9}{2}}.$$

*Ici,  $I$  est la fonctionnelle construite dans Théorème 1.2.1, et  $\kappa(t)$  est la courbure.*

Remarquons que la condition (1.2.21) peut être vérifiée par la plupart des données régulières lorsque  $\sigma = 0$ . En considérant l'effet de la tension superficielle, la condition  $E < 0$ , nécessaire afin que le membre de droite de l'identité du viriel (1.2.19) soit positive, est satisfaite par un nombre limité des données comme  $\phi^\pm(0, x, y) = 0$ ,  $\eta(0, x) = M \sin(x)$ , avec  $M \in \mathbb{R}$  suffisamment grand. La caractérisation du cas  $E > 0$ ,  $\sigma > 0$  est encore ouverte.

Le Théorème (1.2.2) fournit une singularité possible du système (1.2.12), la croissance de la pente ou de la courbure. Cependant, d'autres types de singularité existent. Une possibilité est l'auto-intersection, qui est observée déjà dans l'équation des vagues [Cas+12] et reste ouverte dans le contexte de RTI et KHI. Une autre possibilité est l'intersection entre l'interface libre et les fonds. Dans le cas de RTI, il est bien connu que le largeur de l'interface, à savoir le largeur de la région de mélange, admet une croissance  $O(t^2)$ , ce qui entraîne l'intersection en temps fini entre l'interface libre et les fonds dont le profondeur est finie.

La démonstration du Théorème 1.2.1 suit de l'identité du viriel (1.2.19). Sous la condition (1.2.21) de l'énergie, nous pouvons montrer que le membre de droite de (1.2.19) est minoré par une constante strictement positive dépendant que l'énergie totale, qui est

conservée en temps. C'est-à-dire  $I'(t) \geq C(E) > 0$ , ce qui entraîne que  $I$  admet une croissance plus vite que  $O(t)$  :

$$I(t) \geq I(0) + C(E)t.$$

Dans le cas du graphe, d'après la construction de la fonctionnelle  $I$  dans Théorème 1.2.1, il est contrôlé par la norme  $L^2$  de la trace du potentiel  $\phi^\pm$  et donc par l'énergie cinétique  $E_k$  grâce au théorème de la trace.

$$I(t) \leq C(\eta) \sum_{\pm} \|\phi^\pm\|_{L^2(\Sigma)} \leq C(\eta) \sum_{\pm} \|\nabla_{x,y}\phi^\pm\|_{L^2(\Omega^\pm)} \sim C(\eta)E_k = C(\eta)(E - E_p)$$

Notons que l'énergie totale  $E$  est conservée en temps (donc dépend que la donnée initiale) et l'énergie potentielle  $E_p$  dépend que la forme  $\eta$  de l'interface. Alors, la majoration de  $I$  peut être réalisée par une fonction des normes de  $\eta$ , plus précisément une fonction de la pente  $\|\nabla_x\eta\|_{L^\infty}$  de l'interface. Pour le cas de chevauchement, l'idée du preuve est similaire et la seule différence est que, pour montrer le théorème de la trace, il faut faire un aplatissement de l'interface, une courbe générale sans auto-intersection. De cette raison, nous devons introduire les coordonnées géodésiques normales, d'où la courbure apparait au lieu de la pente.

### 1.3 . L'évolution des jets

L'écoulement de **jet** se réfère au mouvement d'une fluide occupant un domaine cylindrique avec une interface libre ou solide. L'étude des jets possède une longue histoire en physique. Ce n'est qu'au début du XIXe siècle que les chercheurs ont découvert que la singularité des jets, notamment la **rupture** de jets, est indépendante des forces extérieures, comme la gravité, et qu'elle doit être attribuée à la tension de surface, une propriété intrinsèque du fluide. En 1873, Plateau [Pla73] a observé pour la première fois que cette instabilité est liée à l'aire des jets. Peu après, en 1879, Rayleigh [Ray79] a développé une méthode de stabilité linéaire permettant d'expliquer ce phénomène d'un point de vue théorique. Si le jet est périodique en direction de l'axe, le mouvement du jet est gouverné, au moins au niveau linéaire, par une équation dispersive dont le relation de dispersion est

$$\omega(\xi)^2 = \frac{\sigma}{\rho R^3} \frac{I_1(|\xi|R)}{I_0(|\xi|R)} |\xi|R (|\xi|^2 R^2 - 1),$$

où  $\sigma > 0$  désigne le coefficient de la tension superficielle,  $\rho > 0$  est la densité, et  $I_0, I_1$  sont fonctions de Bessel. Ici  $R > 0$  est le rayon de référence, à savoir que la frontière libre est supposé être proche du cylindre de rayon  $R$ . Pour les jets de fréquence grande, d'après l'effet dispersif, nous pouvons espérer sa stabilité. Si la fréquence est assez petite ( $|\xi| < R^{-1}$ ), la relation de dispersion ci-dessus indique la possibilité d'une croissance exponentiel, qui est connue comme le **instabilité de Rayleigh-Plateau**. Du point de vue physique, ce mode d'instabilité exponentiel est la raison pour la rupture des jets.

Cette étude linéaire est parfaitement en accord avec les expériences physiques, cf. [GY70], bien que l'effet non linéaire soit peu pris en compte dans les calculs. Une question naturelle est de refaire ce calcul pour le système non linéaire et en déduire la rupture des jets. Un aperçu historique et un état de l'art en physique se trouvent dans [EV08].

Les résultats mathématiques concernant des jets se limitent principalement au régime stationnaire (indépendant du temps), en particulier l'écoulement dans des tuyères. Pour des travaux sur les tuyères infiniment longues, nous renvoyons à [BL58; XX07; XX09; DXY11]. Dans le cas des tuyères semi-infinies (présentant une frontière libre partielle), les premières études utilisant la transformation hodographique et les applications conformes remontent à [BZ90; Gil60], tandis qu'une approche plus puissante, reposant sur une formulation variationnelle, a été développée par Alt, Caffarelli et Friedman [CA81; ACF82; ACF83]. Des avancées récentes sur ce sujet sont disponibles dans [CDW18; WX19; Li24] et les références citées. Outre les problèmes liés aux tuyères, d'autres résultats sur les jets stationnaires peuvent être trouvés dans [GN18; GMN20].

Afin d'établir la théorie de stabilité/instabilité pour le mouvement des jets, il faut établir un caractère bien posé pour les systèmes décrivant le mouvement des jets. Récemment, le premier résultat d'existence et d'unicité a été obtenu par Huang-Karakhanyan [HK23], qui ont démontré que les jets symétriques par rapport à l'axe sont localement bien posés dans les espaces de Sobolev. Comme les perturbations des états stationnaires ne sont pas forcément symétriques, il est nécessaire d'étendre l'existence locale pour des données sans conditions de symétrie, ce qui est l'objectif de Chapitre 4 de cette thèse.

### 1.3.1 . Formulation du problème

La forme des jets à étudier est représentée par une fonction scalaire  $\eta$ , au sens où

$$(1.3.1) \quad \Omega(t) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : x = 0 \text{ or } |x| < \eta(t, x/|x|, z)\},$$

et l'interface libre associée est désignée par

$$(1.3.2) \quad \Sigma(t) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : |x| = \eta(t, x/|x|, z)\}.$$

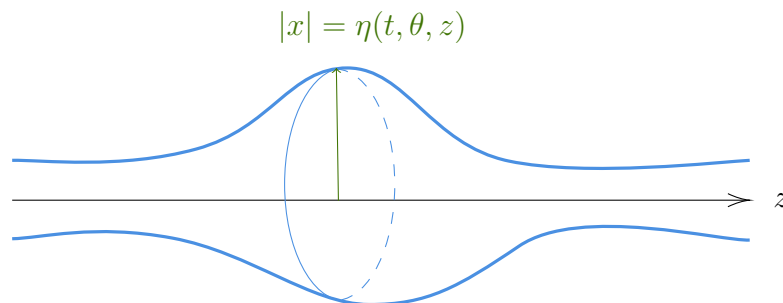


Figure 1.3 – Les jets sans conditions de symétrie

Pour la motivation physique, nous considérons deux situations :

- (1) Le jet est une petite perturbation du cylindre de référence (c'est-à-dire un cylindre de rayon  $R > 0$ ), et la perturbation tend vers zéro à l'infinie;
- (2) Le jet est périodique en direction de l'axe, et sa différence avec le cylindre de référence peut être grande.

Supposons que le fluide est incompressible, parfait, et de rotationnel nul,

$$(1.3.3) \quad \begin{cases} \partial_t u + u \cdot \nabla_{x,z} u + \nabla_{x,z} P = 0, & \text{dans } \Omega(t), \\ \operatorname{div}_{x,z} u = 0, \quad \operatorname{curl}_{x,z} u = 0, & \text{dans } \Omega(t), \end{cases}$$

où  $P$  désigne la pression associée à la tension superficielle.

Comme dans le problème des vagues introduit dans la section précédente, des conditions cinématique et dynamique sont requises pour la description complète du mouvement des jets. La condition cinématique, déterminant l'évolution de la frontière libre, est donnée par

$$(1.3.4) \quad n \cdot \gamma_t = n \cdot u, \quad \text{sur } \Sigma(t),$$

où,  $n$  désigne la direction normale à  $\Sigma$  vers l'extérieur et  $\gamma$  est la paramétrisation de  $\Sigma$  via coordonnée cylindrique et la fonction  $\eta$ .

A propos de la condition dynamique, nous ne considérons que l'effet de la tension de surface,

$$(1.3.5) \quad P|_{\Sigma} = \sigma \left( H - \frac{1}{2R} \right),$$

où  $\sigma > 0$  est le coefficient de la tension superficielle et  $H$  désigne la courbure moyenne. Notons que  $1/(2R)$  est une normalisation pour que  $P|_{\Sigma}$  tende vers zéro à l'infinie.

Remarquons que nous nous intéressons uniquement un système sans gravité, bien que celle-ci soit un facteur essentiel dans les problèmes de vagues. Ce choix est motivé par notre objectif final : l'étude de l'instabilité de Rayleigh-Plateau, pour laquelle la tension de surface constitue la seule source d'instabilité. Il est important de noter que ce modèle sans gravité peut être considéré comme une approximation des systèmes avec un grand nombre de *Froude* — un régime typique des jets liquides fins et allongés (voir l'expérience physique présentée dans [RJ70]).

Pour réécrire le système (1.3.3)-(1.3.5) sous une forme compacte sur un domaine fixé en temps, nous introduisons la **formulation de Craig-Sulem-Zakharov** [Zak68; CS93]. Comme  $u$  est de divergence et rotationnel nuls, il admet l'expression  $u = \nabla_{x,z} \phi$  pour un potentiel scalaire  $\phi$  harmonique dans le domaine de fluide  $\Omega$ . En notant par  $\psi$  la trace du potentiel  $\phi$ , qui peut être récupéré à partir de  $\psi$  en résolvant l'équation de Poisson

$$(1.3.6) \quad \begin{cases} \Delta_{x,z} \phi = 0, & \text{dans } \Omega(t), \\ \phi|_{\Sigma(t)} = \psi, \end{cases}$$

nous avons la définition suivante de l'**opérateur de Dirichlet-à-Neumann**,

$$(1.3.7) \quad G(\eta)\psi := \sqrt{1 + (\eta_\omega/\eta)^2 + \eta_z^2} n \cdot \nabla_{x,z} \phi|_{r=\eta},$$

où  $(r, \omega, z)$  est la coordonnée cylindrique. En termes de cet opérateur, nous pouvons reformuler le système (1.3.3)-(1.3.5) comme un système autonome de  $(\eta, \psi)$  sur le domaine  $\mathbb{T} \times \mathbb{R}$  (fixé en temps),

$$(1.3.8) \quad \begin{cases} \eta_t = G(\eta)\psi, \\ \psi_t + \sigma \left( H - \frac{1}{2R} \right) + N = 0, \end{cases}$$

avec le terme non linéaire  $N$  donné par

$$N = \frac{1}{2} |\bar{\nabla} \psi|^2 - \frac{1}{2} \frac{(G(\eta)\psi + \bar{\nabla} \psi \cdot \bar{\nabla} \eta)^2}{1 + |\bar{\nabla} \eta|^2}, \quad \bar{\nabla} = (\eta^{-1} \partial_\omega, \partial_z).$$

### 1.3.2. Problème de Cauchy

En utilisant une stratégie similaire à [HK23], qui traite le système (1.3.8) comme les équations des vagues planeaire [ABZ11], nous pouvons montrer l'existence et l'unicité du système (1.3.8) localement en temps dans des espaces de Sobolev.

**Théorème 1.3.1.** *Étant donné  $R > 0$ , soient  $(\eta_0 - R, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R})$  avec  $s > 3$ . Alors, pour tous données initiales  $(\eta_0, \psi_0)$  vérifiant  $\eta_0 > 0$ , il existe  $T > 0$ , tel que le système (1.3.8) admet une unique solution*

$$(\eta - R, \psi) \in C \left( [0, T]; H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R}) \right).$$

En outre, pour tous temps d'existence  $t \in [0, T[$ , nous avons  $\eta(t) > 0$ .

De plus, le flot associé est continue dans l'espace métrique  $\left( R + H^{s+\frac{1}{2}} \right) \times H^s$ .

De manière analogue à [ABZ11], l'idée principale de la démonstration est d'étudier la nature hyperbolique du système (1.3.8). Plus précisément, nous reformulons le système via calcul paradifférentiel sous la forme :

$$(1.3.9) \quad \partial_t Y + A(Y)Y = F(Y),$$

où  $Y$  est une nouvelle variable définie comme un opérateur elliptique agissant sur  $(\eta, \psi)$  et  $F(Y)$  est un terme de source relativement régulier, dépendant implicitement de  $Y$ . L'opérateur linéaire d'ordre élevé  $A(Y)$  dépendant également implicitement de  $Y$  est antisymétrique. Les méthodes classiques des équations hyperboliques permettent alors de résoudre le problème de Cauchy par les systèmes approchés et les estimations de l'énergie. Plus précisément, la démonstration du Théorème 1.3.1 est décomposée par cinq étapes ci-dessous :

*Étape 1* (Paralinéarisation de  $G(\eta)$ ) Afin d'exprimer l'opérateur de Dirichlet-à-Neumann  $G(\eta)$  comme un opérateur paradifférentiel dont le symbole dépend explicitement de  $\eta$ , nous devons reformuler l'équation de Poisson (1.3.6) sous forme d'une équation elliptique sur un cylindre fixé. L'opérateur de Laplace  $\Delta_g$  qui intervient

dépend alors de la métrique  $g$ , ou de manière équivalente, de la forme  $\eta$ . En décomposant le Laplacien en parties normale et tangentielle, nous montrerons que la différence entre la dérivée normale de  $\phi$  et un opérateur tangentiel agissant sur  $\phi$  satisfait une équation hyperbolique, ce qui implique la paralinéarisation de  $G(\eta)$ .

*Étape 2* (Paralinéarisation des termes non linéaires) Le traitement du terme non linéaire  $N$  est relativement simple puisqu'il s'exprime comme une fonction régulière de  $G(\eta)\psi$ ,  $\eta$ ,  $\psi$  et de leurs dérivées, d'où la paralinéarisation découle directement. La même technique s'applique à la courbure moyenne  $H$ , qui est une fonction régulière de  $\eta$  et de ses dérivées.

*Étape 3* (Symétrisation) Grossièrement, les résultats précédents permettent de réécrire l'équation principale  $A(Y)$  dans (1.3.9) comme

$$A(Y) = V \cdot \bar{\nabla} + \mathcal{L},$$

où le premier terme est un opérateur de transport et le deuxième est un opérateur paradifférentiel dont le symbole est une matrice  $2 \times 2$  dont les valeurs propres sont purement imaginaires. Donc, après des transformées simples,  $A(Y)$  est antisymétrique modulo un opérateur régularisant.

*Étape 4* (Problème de Cauchy) Nous construisons une suite de solutions approchées de l'équation :

$$\partial_t Y_\epsilon + A_\epsilon(Y_\epsilon)Y_\epsilon = F(Y_\epsilon),$$

où  $A_\epsilon(Y)$  est une version régularisée de  $A(Y)$ . Pour assurer la propriété  $A_\epsilon(Y)^* = -A_\epsilon(Y)$ , cette régularisation ne peut être un simple lissage en variable de Fourier, mais doit être définie par insertion d'un opérateur paradifférentiel bien choisi dépendant de  $\eta$  dans  $A(Y)$ .

*Étape 5* (Continuité de la solution) Enfin, nous devons vérifier que la solution obtenue comme limite des solutions approchées est continue en temps et en données initiales. Au lieu des méthodes classiques, nous utilisons un résultat récent de [Ala+24], qui établit un théorème d'interpolation non linéaire pour les systèmes avec des estimations douces, qui est une conséquence du calcul paradifférentiel.

Dans ce schéma de preuve, il y a deux difficultés :

- (1) Le calcul paradifférentiel de l'opérateur de Dirichlet-à-Neumann nécessite l'étude paradifférentielle de l'équation de Poisson (1.3.6). Si le système est symétrique par rapport à l'axe, elle se réduit à un problème sur un demi-plan de dimension deux et les méthodes pour les vagues planaires s'appliquent ici. Sinon, la même méthode ne marche plus à cause des singularités sur l'axe. Donc, il faut directement étudier la régularité de (1.3.6) dans un domaine presque cylindrique.
- (2) Afin de clarifier la dépendance de la forme  $\eta$ , il faut étudier la *dérivée de la forme*, qui est bien connue dans le cas planaire, cf. [Lan13b]. Dans le cas cylindrique, il faut traiter les problèmes techniques issus de la géométrie. Nous allons suivre les méthodes de

Beyer-Günther [BG98], développées pour étudier le mouvement des gouttes (dont la forme est presque une sphère).

## 1.4 . Perspectives : Interaction fluide-structure

Notre travail ouvre des perspectives quant à la résolution d'autres problèmes issus de la mécanique des fluides, pour lesquels certains des idées et méthodes que nous avons développé pourraient s'avérer fructueuses. Nous pensons en particulier avec questions d'**interaction fluide-structure**, qui décrivent le comportement d'un fluide entouré par une structure élastique, tel le sang s'écoulant dans une veine. Nous avons entamé un tel travail, dans le prolongement de cette thèse, dont nous décrivons ci-dessus les idées.

Rappelons d'abord que les effets dispersive peuvent être utilisés pour abaisser le niveau de régularité minimal à imposer aux données initiales pour pouvoir prouver un résultat d'existence locale. Par exemple, le mouvement des vagues ou des jets peut être décrit par une équation dispersive de la forme

$$\partial_t u + ip(D_x)u = N(u),$$

avec  $p(\xi) = |\xi|^{1/2}$  (resp.  $p(\xi) = |\xi|^{3/2}$ ) pour les vagues de gravité (resp. de capillarité),  $N(u)$  désignant le terme non linéaire. Si l'utilisation du caractère hyperbolique de la partie linéaire de l'équation permet en général d'obtenir le caractère bien posé local de l'équation non linéaire, les hypothèses de régularité des données initiales à introduire sont loin d'être optimales. Par contre, l'utilisation du caractère dispersif de l'équation peut permettre d'affaiblir les conditions de régularité de la donnée sous lesquelles nous pouvons prouver l'existence locale. Toutefois, en règle générale, nous ne pouvons abaisser le seuil de régularité jusqu'au niveau de l'espace d'énergie de l'équation.

Plus spécifiquement, considérons l'équation des vagues capillaires en dimension 2. Il est montré dans [ABZ11] que ce système est localement bien posé pour des données  $(\eta, \psi) \in H^{s+1/2} \times H^s$  avec  $s > 5/2$ , en utilisant uniquement des méthodes hyperboliques. Si on fait appel en outre à des méthodes dispersives, telles les inégalités de Strichartz, on peut affaiblir l'hypothèse  $s > 5/2$  en  $s > 9/4$ . Ce seuil reste largement supérieur à celui correspondant à l'espace d'énergie  $s = 1/2$ . Néanmoins, dans le cas de l'interaction fluide-structure que nous allons décrire ci-après, nous espérons, en utilisant les méthodes dispersives, atteindre le niveau de régularité de l'espace d'énergie (ce qui permet alors, par conservation de l'énergie, de déduire d'un résultat local en temps un théorème d'existence globale).

Considérons un fluide occupant le domaine

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 : y < \eta(t, x)\},$$

pour certaine fonction scalaire  $\eta(t, x)$ . Le champ de vitesse  $v$  du fluide dans  $\Omega(t)$  est supposé être gouverné par l'équation d'Euler incompressible sans rotationnel,

$$(1.4.1) \quad \begin{cases} \partial_t v + (v \cdot \nabla_{x,y})v + \nabla_{x,y}P + gy = 0, & \text{dans } \Omega(t), \\ \operatorname{div}_{x,y} v = 0, \quad \operatorname{curl}_{x,y} v = 0, & \text{dans } \Omega(t), \end{cases}$$

où  $P$  désigne la pression et  $g > 0$  est l'accélération de gravité. Le mouvement de la frontière libre  $\Sigma(t) = \{y = \eta(t, x)\}$  est contrôlé par la condition cinématique

$$(1.4.2) \quad n \cdot \partial_t(x, \eta) = n \cdot u|_{\Sigma(t)},$$

où  $n$  est la direction normale à l'interface  $\Sigma(t)$ , ainsi que l'équation des plaques (cf. [Red06] pour la théorie générale des plaques)

$$(1.4.3) \quad P|_{y=\eta} = \partial_t^2 \eta + \Delta_x^2 \eta.$$

Le système (1.4.1)-(1.4.3) admet aussi la formulation de Craig-Sulem-Zakharov. En écrivant la vélocité sous la forme potentielle  $v = \nabla_{x,y} \phi$ , nous notons par  $\psi$  la trace du potentiel scalaire  $\phi$ , et, comme dans (1.3.7) ci-dessus, l'**opérateur de Dirichlet-à-Neumann** est donnée par

$$G(\eta)\psi := \sqrt{1 + \eta_x^2} n \cdot \nabla_{x,y} \phi|_{y=\eta}.$$

En terme de la nouvelle inconnue

$$u = \psi + G(\eta)\psi,$$

le système (1.4.1)-(1.4.3) se réduit à un système sur  $\mathbb{R}$ ,

$$(1.4.4) \quad \begin{cases} \partial_t \eta = G(\eta)(1 + G(\eta))^{-1} u, \\ \partial_t u + \partial_x^4 \eta + \eta + N(\eta, u) = 0, \end{cases}$$

où  $N$  est le terme non linéaire et nous prenons  $g = 1$  par simplicité.

Il est connu que l'opérateur de Dirichlet-à-Neumann est égal au multiplicateur de Fourier  $|D_x|$  modulo un opérateur régularisant. Donc, la linéarisation du système (1.4.4) est donnée par

$$\begin{aligned} \partial_t U + ip(D_x)U &= F(U), \\ U &= |D_x|^{-1/2}(1 + |D_x|)^{1/2}(1 + |D_x|^4)^{1/2} \eta + iu, \end{aligned}$$

où la relation de dispersion est égale à

$$p(\xi) = |\xi|^{1/2}(1 + |\xi|)^{-1/2}(1 + |\xi|^4)^{1/2},$$

et  $F(U)$  désigne le terme non linéaire. Remarquons que la relation de dispersive  $p$  est similaire à laquelle de l'équation de Schrödinger.

En répétant l'argument pour l'équation des vagues, nous pouvons obtenir le caractère bien posé dans l'espace  $(\eta, u) \in H^{s+2} \times H^s$  avec  $s > 1/2$ . Et, via l'estimation de Strichartz, cette régularité peut être amélioré à  $s = 0$ , ce qui correspond à la régularité de l'énergie

$$E = \frac{1}{2} \int_{\mathbb{R}} u G(\eta)(1 + G(\eta))^{-1} u dx + \frac{1}{2} \int_{\mathbb{R}} ((\partial_x^2 \eta)^2 + \eta^2) dx.$$

Comme une conséquence, la conservation de l'énergie implique l'existence et l'unicité globales en temps. Pour la démonstration, les difficultés se concentrent sur les estimation de terme non linéaire, dont la dépendance en  $\eta$  (ou la partie réelle de  $U$ ) est implicite. Pour étudier précisément cette dépendance, il faut établir une paralinéarisation précise de l'opérateur Dirichlet-à-Neumann et une formule pour la dérivée de la forme dans un régime de basse régularité.

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## 2 - Microlocal Partition of Energy for Fractional-type Dispersive Equations

This chapter is devoted to the proof of microlocal partition of energy for fractional-type dispersive equations including Schrödinger equation, linearized gravity or capillary water-wave equation and half-Klein-Gordon equation. Roughly speaking, a quarter of the  $L^2$  energy lies inside or outside the “light cone”  $|x| = |tP'(\xi)|$  for large time. In addition, based on the study of half-Klein-Gordon equation, the microlocal partition of energy will also be proved for Klein-Gordon equation.

### 2.1 . Introduction

#### 2.1.1 . Background

The classical partition of energy states that the energy of the solution  $w$  to linear wave equation

$$(W) \quad \begin{cases} (\partial_t^2 - \Delta)w = 0, \\ w|_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d), \end{cases}$$

inside and outside the light cone  $|x| = |t|$  satisfies, in *odd* dimension  $d$ ,

$$(2.1.1) \quad \begin{aligned} \lim_{t \rightarrow +\infty} (E^{\text{in}}(w_0, w_1, t) + E^{\text{in}}(w_0, w_1, -t)) &= \|\partial_t w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2, \\ \lim_{t \rightarrow +\infty} (E^{\text{out}}(w_0, w_1, t) + E^{\text{out}}(w_0, w_1, -t)) &= \|\partial_t w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2, \end{aligned}$$

where

$$\begin{aligned} E^{\text{in}}(w_0, w_1, t) &:= \int_{|x| < |t|} (|\nabla w|^2 + |\partial_t w|^2) dx, \\ E^{\text{out}}(w_0, w_1, t) &:= \int_{|x| > |t|} (|\nabla w|^2 + |\partial_t w|^2) dx. \end{aligned}$$

A proof of this can be found in [DKM11] and [DKM12], where the authors applied this result to study the soliton of focusing energy-critical nonlinear wave equation in dimension  $d = 3, 5$  via some nonlinear analysis on small data solutions. One may also refer to

[Côt+15a] and [Côt+15b] for the application in equivariant wave maps. In *even* dimension  $d$ , the limits (2.1.1) do not hold in general settings. It is essential to add some extra corrections, which have been calculated in detail for radial solutions in [CKS14] and for general data in [CL21]. In these references, the authors also discovered some special data such that the corrections vanish, an application of which to 4-dimensional focusing energy-critical wave equation can be found in [Côt+18].

The above results have been recently revisited in [Del22] using the tools of microlocal analysis. Consider first a solution  $u$  of the half-wave equation

$$(HW) \quad \begin{cases} \left( \frac{\partial_t}{i} - |D_x| \right) u = 0, \\ u|_{t=0} = u_0 \in L^2. \end{cases}$$

Since  $u = e^{it|D_x|}u_0$ , stationary phase formula shows that one expects the microlocalized energy of the solution outside a convenient neighborhood of  $\{(x, \xi) : x = t \frac{\xi}{|\xi|}\}$  at time  $t$  to vanish when  $t$  goes to infinity. Because of that, it is natural to ask whether the microlocalized energy close to the preceding point, truncated outside the wave cone, gives rise to lower bound of the form (2.1.1). More precisely, if one defines this microlocalized truncated energy as

$$(2.1.2) \quad E_{\chi, \tilde{\chi}, \delta}^{HW}(u_0, t) := \|\text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW}(t)) u(t)\|_{L^2}^2,$$

$$(2.1.3) \quad a_{\chi, \tilde{\chi}, \delta}^{HW}(t, x, \xi) := \chi \left( \frac{x + t \frac{\xi}{|\xi|}}{|t|^{\frac{1}{2} + \delta}} \right) \tilde{\chi} \left( \frac{|x| - |t|}{|t|^\delta} \right) \mathbb{1}_{|x| > |t|},$$

where  $\chi, \tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$  are chosen to be real, radial and equal to 1 near zero with  $\delta \in ]0, \frac{1}{2}]$ , then it has been proved in [Del22] that in any dimension  $d$ ,

$$(2.1.4) \quad \lim_{t \rightarrow +\infty} (E_{\chi, \tilde{\chi}, \delta}^{HW}(u_0, t) + E_{\chi, \tilde{\chi}, \delta}^{HW}(u_0, -t)) = \|u_0\|_{L^2}^2.$$

This result may be used to recover (2.1.1) in odd dimension by taking  $u = (-i\partial_t + |D_x|)w$ . Actually, the truncated energy in (2.1.2) may be expressed from the microlocalized truncated energy for the solution  $u$  of the half wave equation and from extra terms. These extra terms give a zero contribution at the limit  $t$  tending to infinity in *odd* dimensions, but not in even ones.

The heuristics underlying estimate (2.1.4) for the half wave equation are as follows. Define the quantization  $\text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW})$  of the symbol (2.1.3) by

$$(\text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW}) f)(t, x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} a_{\chi, \tilde{\chi}, \delta}^{HW}(t, x, \xi) \hat{f}(t, \xi) d\xi.$$

Then (2.1.2) may be written as

$$(2.1.5) \quad \begin{aligned} E_{\chi, \tilde{\chi}, \delta}^{HW}(u_0, t) &= \left\langle \text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW}) e^{it|D_x|} u_0, \text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW}) e^{it|D_x|} u_0 \right\rangle_{L^2} \\ &= \left\langle u_0, e^{-it|D_x|} \text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW})^* \text{Op}(a_{\chi, \tilde{\chi}, \delta}^{HW}) e^{it|D_x|} u_0 \right\rangle_{L^2}. \end{aligned}$$

If the symbols were smooth ones, so that symbolic calculus (whose details can be found in [Zwo22]) could be used, one would expect the composition  $\text{Op} \left( a_{\chi, \tilde{\chi}, \delta}^{\text{HW}} \right)^* \text{Op} \left( a_{\chi, \tilde{\chi}, \delta}^{\text{HW}} \right)$  to be equal, modulo negligible remainders, to  $\text{Op} (b)$ , with  $b = \left| a_{\chi, \tilde{\chi}, \delta}^{\text{HW}} \right|^2$ , and the conjugation  $e^{-it|D_x|} \text{Op} (b) e^{it|D_x|}$  to be equal, up to remainders, to  $\text{Op} (c)$  where  $c(x, \xi) = b(x - t \frac{\xi}{|\xi|})$ . Applying this to (2.1.5), one could write this quantity as  $\langle u_0, \text{Op} (e) u_0 \rangle$  modulo a term tending to zero when  $t$  goes to infinity, where  $e$  is given by

$$e(t, x, \xi) = \chi^2 \left( \frac{x}{|t|^{\frac{1}{2} + \delta}} \right) \tilde{\chi}^2 \left( \frac{|x - t \frac{\xi}{|\xi|}| - |t|}{|t|^\delta} \right) \mathbb{1}_{|x - t \frac{\xi}{|\xi|}| > |t|}$$

This symbol  $e$  roughly cuts-off the phase space on the domain

$$\left\{ (x, \xi) : |x| \lesssim |t|^{\frac{1}{2} + \delta}, \left| x - t \frac{\xi}{|\xi|} \right| > |t| \right\}.$$

When  $t \rightarrow +\infty$ , the truncated domain tends to the half-space

$$(2.1.6) \quad \left\{ (x, \xi) : \text{sgn}(t)x \cdot \frac{\xi}{|\xi|} < 0 \right\}.$$

As a consequence, the sum of truncated energy at time  $t$  and  $-t$  covers the whole phase space.

We emphasize that all the arguments above are merely formal. They hold only if all the involved functions are regular enough. In fact, it has also been proved in [Del22] that the cut-off operator  $\text{Op} \left( a_{\chi, \tilde{\chi}, \delta}^{\text{HW}} \right)$  may not even be bounded if the singular cut-off  $|x| > |t|$  is replaced by  $x \cdot \frac{\xi}{|\xi|} > t$ , which seems to work formally.

The above formal point of view, though purely heuristic, suggests that the classical result (2.1.1) might be extended to a large class of dispersive equations. In general system

$$\left( \frac{\partial_t}{i} - P(D_x) \right) u = 0,$$

as in half-wave equation, one may expect that the energy concentrates in the phase space around  $x + tP'(\xi) = 0$  and the partition of energy holds with generalized "light cone"

$$|x| = |tP'(\xi)|.$$

The first result for Schrödinger equation has been given in [Del22] with truncation

$$(2.1.7) \quad E_{\chi, \delta}^{\text{Schr}}(u_0, t) := \left\| \text{Op} \left( a_{\chi, \delta}^{\text{Schr}}(t) \right) u(t) \right\|_{L^2}^2,$$

$$(2.1.8) \quad a_{\chi, \delta}^{\text{Schr}}(t, x, \xi) := \chi \left( \frac{x + t\xi}{|t\xi| \langle \sqrt{|t||\xi|} \rangle^{-\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| > |t\xi|}.$$

The result is similar :

$$(2.1.9) \quad \lim_{t \rightarrow +\infty} \left( E_{\chi, \delta}^{\text{Schr}}(u_0, t) + E_{\chi, \delta}^{\text{Schr}}(u_0, -t) \right) = \frac{1}{2} \|u_0\|_{L^2}^2.$$

Here the extra factors in the cut-off  $\chi$  is only for technical use, and the loss of half of total energy  $\|u_0\|_{L^2}^2$  is due to the convexity of  $P(\xi) = \frac{|\xi|^2}{2}$ .

The goal of this chapter is to examine if the microlocal partition of energy results (2.1.4), (2.1.9) may be extended to a large class of dispersive equations. In particular, this generalized result covers the system of linearized gravity or capillary water-wave with infinite depth

$$(LGWW) \quad \frac{\partial_t u}{i} - |D_x|^{\frac{1}{2}} u = 0,$$

$$(LCWW) \quad \frac{\partial_t u}{i} - |D_x|^{\frac{3}{2}} u = 0,$$

or with finite and constant depth  $h$ ,

$$(LGWW_h) \quad \frac{\partial_t u}{i} - |D_x|^{\frac{1}{2}} \tanh(h|D_x|) u = 0,$$

$$(LCWW_h) \quad \frac{\partial_t u}{i} - |D_x|^{\frac{3}{2}} \tanh(h|D_x|) u = 0.$$

Moreover, the system of half-Klein-Gordon

$$(HKG) \quad \frac{\partial_t u}{i} - \langle D_x \rangle u = 0$$

can also be covered by the generalized result, and the associated conclusion will further imply the microlocal partition of energy for standard Klein-Gordon equation

$$(KG) \quad \begin{cases} (\partial_t^2 - \Delta + 1)w = 0, \\ w|_{t=0} = w_0 \in H^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d). \end{cases}$$

One may have noticed that (2.1.4) and (2.1.9) are proved only for  $\delta \in ]0, \frac{1}{2}]$  or  $]0, \frac{1}{2}[$ . The case  $\delta \geq \frac{1}{2}$  seems useless since with such  $\delta$  the cut-off  $\chi$  gives no information in the concentration of energy. The critical case  $\delta = 0$ , however, leads to some interesting results in the limit of truncated energy. The related results will be presented in detail in the next part.

### 2.1.2 . Main results

We consider the fractional-type dispersive equation :

$$(E) \quad \begin{cases} \left( \frac{\partial_t}{i} - P(D_x) \right) u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where  $P$  is radial and smooth except at  $\xi = 0$ . For simplicity,  $P$  will be identified as a function of  $\rho = |\xi|$  in what follows. We further assume that  $P$  is a fractional-type symbol.

Namely, the following hypothesis hold for some  $p_0, p_1 \neq 0$ .

$$\begin{aligned}
 & (1) P^{(1)} \text{ is strictly positive and monotone on } ]0, +\infty[; \\
 & (2.0) \exists P_0 \geq 0, \rho \rightarrow 0+, |P^{(1)}(\rho) - P_0| \sim \rho^{p_0}, |P^{(2)}(\rho)| \sim \rho^{p_0-1}; \\
 (H_{p_0, p_1}) \quad & (2.1) \exists P_1 \geq 0, \rho \rightarrow +\infty, |P^{(1)}(\rho) - P_1| \sim \rho^{p_1}, |P^{(2)}(\rho)| \sim \rho^{p_1-1}; \\
 & (3.0) \forall j \in \mathbb{N}^*, j \geq 3, \forall \rho \in ]0, 1[, |P^{(j)}(\rho)| \lesssim \rho^{p_0+1-j}; \\
 & (3.1) \forall j \in \mathbb{N}^*, j \geq 3, \forall \rho \in ]1, \infty[, |P^{(j)}(\rho)| \lesssim \rho^{p_1+1-j}.
 \end{aligned}$$

We introduce the symbol

$$(2.1.10) \quad a(t, x, \xi) = a_{\chi, \delta}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|},$$

where  $\delta \in \mathbb{R}$ ,  $\chi \in C_c^\infty(\mathbb{R}^d)$  is real with  $\chi(0) = 1$ . The corresponding truncated energy is defined as

$$(2.1.11) \quad \begin{aligned} E(u_0, t) &= E_{\chi, \delta}(u_0, t) = \| \text{Op}(a_{\chi, \delta}(t))(u(t)) \|_{L^2}^2 \\ &= \| \text{Op}(a_{\chi, \delta}(t))(e^{itP(D_x)}u_0) \|_{L^2}^2, \end{aligned}$$

where

$$(2.1.12) \quad \text{Op}(a_{\chi, \delta}(t))(e^{itP(D_x)}u_0)(x) := \frac{1}{(2\pi)^d} \int e^{ix\xi + itP(\xi)} a_{\chi, \delta}(t, x, \xi) \hat{u}_0(\xi) d\xi.$$

In Proposition 2.2.1, 2.2.2, and 2.3.1, we shall prove that, under some extra conditions on  $P$ , operator  $\text{Op}(a(t))$ , together with its variations to be introduced later, is bounded on  $L^2$ , uniformly in  $|t| \gg 1$ .  $E(u_0, t)$  is therefore a well-defined truncated energy, at least when time  $|t|$  is sufficiently large.

We first state the fundamental result which is available for fractional equations, namely equation (E) with  $P'(\xi) = |\xi|^p \frac{\xi}{|\xi|}$ , such as Schrödinger equation ( $p = 1$ ), linearized gravity water-wave equation ( $p = -\frac{1}{2}$ ), and linearized capillary water-wave equation ( $p = \frac{1}{2}$ ).

**Theorem 2.1.1.** *Let  $\chi \in C_c^\infty(\mathbb{R}^d)$  be a real function such that  $\chi(0) = 1$ . We further assume that  $P$  satisfies hypothesis  $(H_{p_0, p_1})$  with  $P_0 = P_1 = 0$ .*

(i) If  $\delta < 0$ ,

$$(2.1.13) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}(u_0, t) = 0.$$

(ii) If  $\delta = 0$ ,

$$(2.1.14) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}(u_0, t) = \frac{1}{(2\pi)^d} \int G_\chi(\rho, \omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega,$$

where  $(\rho, \omega)$  is polar coordinate. The function  $G_\chi(\rho, \omega)$  is defined, when  $P$  is convex, by

$$(2.1.15) \quad G_\chi(\rho, \omega) := \frac{1}{(2\pi)^d} \left| \int_0^\infty \int_{y \cdot \omega = 0} e^{i \frac{r^2 + |y|^2}{2}} \chi \left( \sqrt{P''(\rho)} r \omega + \sqrt{\rho^{-1} P'(\rho)} y \right) dy dr \right|^2,$$

and, when  $P$  is concave, by

$$(2.1.16) \quad G_\chi(\rho, \omega) := \frac{1}{(2\pi)^d} \left| \int_0^\infty \int_{y \cdot \omega = 0} e^{i \frac{-r^2 + |y|^2}{2}} \chi \left( \sqrt{-P''(\rho)} r \omega + \sqrt{\rho^{-1} P'(\rho)} y \right) dy dr \right|^2.$$

(iii) If  $0 < \delta < \frac{1}{2}$ , we further assume that  $\chi$  is radial. Then

$$(2.1.17) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}(u_0, t) = \frac{1}{4} \|u_0\|_{L^2}^2.$$

**Remark 2.1.2.** In (2.1.17), we manage to calculate the limit of  $E_{\chi, \delta}(u_0, t)$  and  $E_{\chi, \delta}(u_0, -t)$  as  $t \rightarrow +\infty$ , instead of their sum as in (2.1.1) and (2.1.9). Notice that the heuristics discussed after (2.1.6) in the case of half-wave equations do not predict this fact. This shows the limitation of this formal reasoning when sharp cut-offs are involved in the symbols.

**Remark 2.1.3.** For Schrödinger equation, the special structure of  $P'(\xi) = \xi$  allows us to reduce the regularity required for  $\chi$ . In Appendix 2.C, we will show that limits (2.1.13) and (2.1.14) hold for all  $\chi \in L^1$ .

We will see later that our proof of Theorem 2.1.1 does not hold for  $P$  with nonzero  $P_0, P_1$ , such as half-Klein-Gordon equation (HKG), where  $P_1 = 1$ . In order to deal with this difficulty, one way is to add some cut-off in frequency  $\xi$ . To be precise, we introduce the modified truncated symbol

$$(2.1.18) \quad a^{\text{mod}}(t, x, \xi) = a_{\chi, \delta}^{\text{mod}}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|} (1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right),$$

where  $\epsilon_0, \epsilon_1$  satisfies

$$(2.1.19) \quad 0 < \epsilon_0 \leq \frac{1}{p_0 + 1}, \quad 0 < \epsilon_1 \leq \begin{cases} +\infty, & \text{if } -1 \leq p_1 < 0, \\ \frac{1}{-(p_1 + 1)}, & \text{if } p_1 < -1, \end{cases}$$

and  $\chi_l, \chi_h \in C_c^\infty$  are radial and equal to 1 near zero. This symbol will be concerned only when  $p_1 < 0 < p_0$ , the reason of which will be explained later. The corresponding truncated energy is denoted by

$$E^{\text{mod}}(u_0, t) = E_{\chi, \delta}^{\text{mod}}(u_0, t) := \|\text{Op} \left( a_{\chi, \delta}^{\text{mod}}(t) \right) u(t)\|_{L^2}^2.$$

**Theorem 2.1.4.** Let  $\chi \in C_c^\infty(\mathbb{R}^d)$  be a real function such that  $\chi(0) = 1$ . We further assume that  $P$  satisfies hypothesis  $(H_{p_0, p_1})$  with  $p_1 < 0 < p_0$ ,  $P_0, P_1 > 0$ , and  $\epsilon_0, \epsilon_1$  satisfy condition (2.1.19).

(i) If  $\delta < 0$ ,

$$(2.1.20) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}^{\text{mod}}(u_0, t) = 0.$$

(ii) If  $\delta = 0$ ,

$$(2.1.21) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}^{mod}(u_0, t) = \frac{1}{(2\pi)^d} \int G_{\chi}(\rho, \omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega,$$

where  $(\rho, \omega)$  is polar coordinate. The function  $G_{\chi}(\rho, \omega)$  is the same one defined by (2.1.15) and (2.1.16).

(iii) If  $0 < \delta < \frac{1}{2}$ , we further assume that  $\chi$  is radial. Then

$$(2.1.22) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}^{mod}(u_0, t) = \frac{1}{4} \|u_0\|_{L^2}^2.$$

**Remark 2.1.5.** In this theorem, we only consider the case  $p_1 < 0 < p_0$ , which is enough to cover all  $P_0, P_1 \neq 0$ . Actually, when  $p_0 < 0$  (resp.  $p_1 > 0$ ), the hypothesis  $(H_{p_0, p_1})$  with  $P_0 > 0$  (resp.  $P_1 > 0$ ) is equivalent to  $(H_{p_0, p_1})$  with  $P_0 = 0$  (resp.  $P_1 = 0$ ), which has been studied in Theorem 2.1.1.

Another way to deal with nonzero  $P_0, P_1$  is to add extra factor in the cut-off  $\chi$ , namely, considering an alternative truncated symbol

$$(2.1.23) \quad a^{\text{alt}}(t, x, \xi) = a_{\chi, \delta, \Lambda}^{\text{alt}}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta} \Lambda(|t|^{\frac{1}{2}} \xi)} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|},$$

where  $\Lambda \in C^\infty(\mathbb{R}^d \setminus \{0\})$  is strictly positive, radial and satisfies the following condition for some  $\sigma_0, \sigma_1 \in \mathbb{R}$ :

$$(C_{\sigma_0, \sigma_1}) \quad \begin{aligned} (1.0) \quad & \rho \rightarrow 0+, \quad \Lambda(\rho) \sim \rho^{\sigma_0}; \\ (1.1) \quad & \rho \rightarrow +\infty, \quad \Lambda(\rho) \sim \rho^{\sigma_1}; \\ (2.0) \quad & \forall \rho \in ]0, 1[, \quad \Lambda^{(j)}(\rho) \lesssim \rho^{\sigma_0 - j}; \\ (2.1) \quad & \forall \rho \in ]1, \infty[, \quad \Lambda^{(j)}(\rho) \lesssim \rho^{\sigma_1 - j}; \\ (3) \quad & \lim_{\rho \rightarrow +\infty} \frac{\Lambda(\rho)}{\rho^{\sigma_1}} = \lambda_1 > 0. \end{aligned}$$

The associated truncated energy is denoted by

$$E^{\text{alt}}(u_0, t) = E_{\chi, \delta, \Lambda}^{\text{alt}}(u_0, t) := \|\text{Op} \left( a_{\chi, \delta, \Lambda}^{\text{alt}}(t) \right) u(t)\|_{L^2}^2,$$

and the result becomes

**Theorem 2.1.6.** Let  $\chi \in C_c^\infty(\mathbb{R}^d)$  be a real function such that  $\chi(0) = 1$ . We assume that  $P$  satisfies hypothesis  $(H_{p_0, p_1})$  and  $\Lambda$  satisfies condition  $(C_{\sigma_0, \sigma_1})$  with  $\sigma_0 \geq p_0, \sigma_1 \leq p_1$ .

(i) If  $\delta + \frac{\sigma_1}{2} < 0$ ,

$$(2.1.24) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta, \Lambda}^{\text{alt}}(u_0, t) = 0.$$

(ii) If  $\delta + \frac{\sigma_1}{2} = 0$ ,

$$(2.1.25) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta, \Lambda}^{\text{alt}}(u_0, t) = \frac{1}{(2\pi)^d} \int G_{\chi}^{\text{alt}}(\rho, \omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega,$$

where  $(\rho, \omega)$  is polar coordinate. The function  $G_\chi^{alt}(\rho, \omega)$  is defined, when  $P$  is convex, by

$$(2.1.26) \quad G_\chi^{alt}(\rho, \omega) := \frac{1}{(2\pi)^d} \left| \int_0^\infty \int_{y \cdot \omega = 0} e^{i \frac{r^2 + |y|^2}{2}} \chi \left( \frac{\sqrt{P''(\rho)} r \omega + \sqrt{\rho^{-1} P'(\rho)} y}{\lambda_1 \rho^{\sigma_1}} \right) dy dr \right|^2,$$

and, when  $P$  is concave, by

$$(2.1.27) \quad G_\chi^{alt}(\rho, \omega) := \frac{1}{(2\pi)^d} \left| \int_0^\infty \int_{y \cdot \omega = 0} e^{i \frac{-r^2 + |y|^2}{2}} \chi \left( \frac{\sqrt{-P''(\rho)} r \omega + \sqrt{\rho^{-1} P'(\rho)} y}{\lambda_1 \rho^{\sigma_1}} \right) dy dr \right|^2.$$

(iii) If  $0 < \delta + \frac{\sigma_1}{2} < \frac{1}{2}$ , we further assume that  $\chi$  is radial. Then

$$(2.1.28) \quad \lim_{t \rightarrow \pm\infty} E_{\chi, \delta, \Lambda}^{alt}(u_0, t) = \frac{1}{4} \|u_0\|_{L^2}^2.$$

**Remark 2.1.7.** The proof of Theorem 2.1.1 fails when  $|x/t|$  is close to  $P_0$  or  $P_1$ . The extra factor  $\Lambda$  together with condition  $\sigma_0 \geq p_0, \sigma_1 \leq p_1$  allows us to eliminate this case, and a demonstration similar to Theorem 2.1.1 will work when  $|x/t|$  is away from  $P_0, P_1$ .

As a byproduct of Proposition 2.2.1 and 2.2.2, where the uniform-in- $t$  boundedness on  $L^2$  of  $\text{Op}(a(t))$  and  $\text{Op}(a^{\text{mod}}(t))$  will be proved, these operators are also uniformly bounded with  $\chi$  identically equal to 1, namely

**Theorem 2.1.8.** Let  $p_0, p_1 \neq 0$  and  $P$  satisfy the hypothesis  $(H_{p_0, p_1})$ . There exists a constant  $C > 0$  independent of  $t$ , such that,

(i) when  $P_0 = P_1 = 0$ ,

$$\|\text{Op}(\mathbb{1}_{|x| > |tP'(\xi)|})\|_{\mathcal{L}(L^2)} \leq C$$

holds for all  $|t| > 0$ ;

(ii) when  $P_0, P_1 > 0$ ,

$$\|\text{Op} \left( \mathbb{1}_{|x| > |tP'(\xi)|} (1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right) \right)\|_{\mathcal{L}(L^2)} \leq C$$

holds for all  $|t| > t_0 \gg 1$ , where  $\epsilon_0, \epsilon_1$  are arbitrary parameters satisfying (2.1.19) and  $\chi_l, \chi_h \in C_c^\infty$  are equal to 1 near zero.

The boundedness of  $\text{Op}(\mathbb{1}_E)$  for measurable sets  $E \subset \mathbb{R}^{2d}$  is of great concern in microlocal analysis, and the results above give a positive answer to some  $E$  defined via convex functions. If one changes to Weyl quantization, this problem is known as localization of Wigner distribution. In fact, in this case, we have

$$\langle \text{Op}^w(\mathbb{1}_E) u, v \rangle_{L^2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_E \mathcal{W}(u, v)(x, \xi) dx d\xi,$$

where

$$\mathcal{W}(u, v)(x, \xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{iy \cdot \xi} u\left(x + \frac{y}{2}\right) \overline{v\left(x - \frac{y}{2}\right)} dy$$

is the Wigner distribution of  $(u, v)$ . It has been found that the operator properties of  $\text{Op}^w(\mathbb{1}_E)$  (boundedness, positivity, spectrum, etc) is related to the geometry of  $E$ . For example, when  $E$  is an ellipsoid, in [Fla88] and [LO10], the authors gave some sharp estimates of the  $L^2$ -norm of  $\text{Op}^w(\mathbb{1}_E)$ , which is related to the size of the ellipsoid. As another example, when  $E$  is a polygon on  $\mathbb{R}^2$  with  $N$  sides, it was proved in [Ler23] that the norm of  $\text{Op}^w(\mathbb{1}_E)$  can be controlled by  $\sqrt{N/2}$  for  $N \geq 3$ . In the same paper, the author also proved that there exists open set  $E$  such that  $\text{Op}^w(\mathbb{1}_E)$  is not even bounded on  $L^2$ . The readers may refer to [Ler23] for more results on this topic.

By applying the results of Theorem 2.1.4 and 2.1.8 to half-Klein-Gordon equation (HKG), we are able to obtain the following microlocal partition of energy for Klein-Gordon equation, which is an analogue of partition of energy for wave equation.

**Theorem 2.1.9.** *Let  $w$  be the unique solution to Klein-Gordon equation (KG), namely*

$$\begin{cases} (\partial_t^2 - \Delta + 1)w = 0, \\ w|_{t=0} = w_0 \in H^1(\mathbb{R}^d), \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^d), \end{cases}$$

where  $w_0, w_1$  are real, so as  $w$ . Then, the truncated energy

$$(2.1.29) \quad E_\epsilon^{KG}(w_0, w_1, t) := \|\text{Op}(a_\epsilon^{KG}(t)) \partial_t w(t)\|_{L^2}^2 + \|\text{Op}(a_\epsilon^{KG}(t)) \nabla w(t)\|_{L^2}^2 + \|\text{Op}(a_\epsilon^{KG}(t)) w(t)\|_{L^2}^2$$

satisfies

$$(2.1.30) \quad \lim_{t \rightarrow \pm\infty} E_\epsilon^{KG}(w_0, w_1, t) = \frac{1}{4} (\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2),$$

where

$$(2.1.31) \quad a_\epsilon^{KG}(t, x, \xi) = a_\epsilon^{KG}(t, x, \xi) := \mathbb{1}_{|x| > |t| \frac{\xi}{\epsilon}} \chi\left(\frac{\xi}{|t|^\epsilon}\right),$$

$0 < \epsilon < 1$ , and  $\chi \in C_c^\infty(\mathbb{R}^d)$  is a real and radial function equal to 1 near zero.

**Remark 2.1.10.** *In view of the similarity between wave equation and Klein-Gordon equation, one may ask what (2.1.30) will become if we apply the same truncation  $\mathbb{1}_{|x| > |t|}$  as in classical result (2.1.1). The answer is, for all  $0 \leq r_0 \leq r_1$ ,*

$$(2.1.32) \quad \lim_{t \rightarrow \pm\infty} \int_{r_0 < \frac{x}{|t|} < r_1} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2) dx = \|\mathbb{1}_{]r_0, \rho_1[}(|D_x|)w_0\|_{H^1}^2 + \|\mathbb{1}_{]r_0, \rho_1[}(|D_x|)w_1\|_{L^2}^2,$$

where  $]r_0, \rho_1[ = P'^{-1}(]r_0, r_1[)$ . Since  $P'$  takes value in  $[0, 1[$ , we have in particular,

$$\lim_{t \rightarrow \pm\infty} \int_{|x| > |t|} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2) dx = 0.$$

A detailed discussion of (2.1.32) will be given in Section 2.D.

### 2.1.3. Non-nullity of the limit in the critical case

In Theorem 2.1.1, 2.1.4, and 2.1.6, we calculate the limit of energy in three cases. In the sub-critical case  $\delta < 0$  (or  $\delta + \frac{\sigma_1}{2} < 0$ ), the truncated energy tends to 0, no matter which  $\chi$  we choose. This phenomenon also exists in the super-critical case  $0 < \delta < \frac{1}{2}$  (or  $0 < \delta + \frac{\sigma_1}{2} < \frac{1}{2}$ ), where the limit is always half of total energy  $\|u_0\|_{L^2}$ . In the critical case  $\delta = 0$  (or  $\delta + \frac{\sigma_1}{2} = 0$ ), however, the limit does depend on our choice of  $\chi$ . If we further assume  $\chi$  to be radial, it is not difficult to check that the limits (2.1.14), (2.1.21), and (2.1.25) are bounded and non-negative.

In fact, when  $\chi$  is radial, the function  $G_\chi, G_\chi^{\text{alt}}$  can be written in the form

$$\frac{1}{4} \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{i \frac{\pm x_1^2 + |x'|^2}{2}} \chi \left( \frac{\sqrt{\pm P''(\rho)} x_1 + \sqrt{P'(\rho)/\rho} x'}{\lambda \rho^\sigma} \right) dx \right|^2,$$

where  $+$  and  $-$  stand for convex and concave case, respectively,  $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$ , and  $\lambda > 0, \sigma \in \mathbb{R}$ . By Plancherel Theorem, it equals to

$$\frac{1}{4} \frac{1}{(2\pi)^{2d}} \left| \int_{\mathbb{R}^d} e^{i \frac{\pm \xi_1^2 + |\xi'|^2}{2(\lambda \rho^\sigma)^2}} \hat{\chi} \left( \frac{\xi_1}{\sqrt{\pm P''(\rho)}} + \sqrt{\frac{\rho}{P'(\rho)}} \xi' \right) \frac{1}{\sqrt{\pm P''(\rho)}} \left( \frac{\rho}{P'(\rho)} \right)^{\frac{d-1}{2}} d\xi \right|^2,$$

which, after a change of variable, reads

$$(2.1.33) \quad \frac{1}{4} \frac{1}{(2\pi)^{2d}} \left| \int_{\mathbb{R}^d} e^{i \frac{P''(\rho) \rho \xi_1^2 + P'(\rho) |\xi'|^2}{2\rho(\lambda \rho^\sigma)^2}} \hat{\chi}(\xi) d\xi \right|^2.$$

Therefore,  $G_\chi$  can be estimated by

$$0 \leq G_\chi(\rho, \omega) \leq \frac{1}{4} \frac{1}{(2\pi)^{2d}} \|\hat{\chi}\|_{L^1}^2.$$

A natural question is then whether limits (2.1.14), (2.1.21), and (2.1.25) are nonzero for nontrivial initial data  $u_0$ . The answer is positive for fractional equation, i.e. with  $P'(\xi) = |\xi|^{p-1}\xi, p \neq 0$ . More precisely,

**Proposition 2.1.11.** *Under the assumption  $P'(\xi) = |\xi|^{p-1}\xi$ , (2.1.33) can be written, up to some multiple with constants, as*

$$\tilde{G}(\rho) = \left| \int_{\mathbb{R}^d} e^{i \frac{1}{2\lambda^2} \rho^{p-1-2\sigma} (p\xi_1^2 + |\xi'|^2)} \hat{\chi}(\xi) d\xi \right|^2.$$

If  $p \neq 2\sigma + 1$  and  $\chi \in \mathcal{S}(\mathbb{R}^d)$  with  $\chi(0) \neq 0$ ,  $\tilde{G}(\rho)$  is nonzero except on a set of null Lebesgue measure.

*Proof.* Since  $\chi$  is a Schwartz function, the complex function

$$F(z) := \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} e^{i \frac{z}{2\lambda^2} (p\xi_1^2 + |\xi'|^2)} \hat{\chi}(\xi) d\xi$$

is analytic on upper half plane  $\{z \in \mathbb{C} : \text{Im } z > 0\}$  and continuous on its closure. In [LP25], the authors proved that either the real zeros of such function form a set of zero Lebesgue measure, or it is identically zero. The same result holds thus for  $\tilde{G}(\rho) = |F(\rho^{p-1-2\sigma})|^2$ . Due to the fact that  $\chi(0) \neq 0$ ,  $\tilde{G}$  is nonzero as  $\rho^{p-1-2\sigma_1}$  is small enough, and  $\tilde{G}$  is therefore nonzero almost everywhere.  $\square$

As a consequence, the limits (2.1.14), (2.1.21), and (2.1.25) are strictly positive for all nontrivial  $u_0 \in L^2$  under the assumption  $p \neq 1$  (or  $p \neq 2\sigma_1 + 1$ ). If  $p = 1$  (or  $p = 2\sigma_1 + 1$ ), the function  $G_\chi(\rho)$  (or  $G_\chi^{\text{alt}}$ ) will no more depend on  $\rho$  and the limits (2.1.14), (2.1.21), (2.1.25) will take the form  $c_0(\chi)\|u_0\|_{L^2}^2$ , where

$$c_0(\chi) = \left| \frac{1}{2(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\frac{p}{2}\xi_1^2 + \frac{1}{2}|\xi'|^2)} \hat{\chi}(\xi) d\xi \right|^2.$$

To obtain a nonzero limit, it suffices to choose  $\chi$  such that the quantity above is nonzero. For example, one may take

- $\chi$  to be positive and supported in a sufficiently small ball centered at zero;
- $\chi$  to be Gaussian;
- $\chi$  of the form  $\chi = \tilde{\chi}(\cdot/R)$ , with  $\tilde{\chi} \in C_c^\infty$ ,  $\tilde{\chi}(0) \neq 0$ , and  $R \gg 1$ .

### 2.1.4 . Plan of this chapter

The proof of Theorem 2.1.1, 2.1.4, and 2.1.6 will be divided into two parts : uniform boundedness of truncated operator and calculation of limit. In Section 2.2, we will prove that  $\text{Op}(a(t))$  and  $\text{Op}(a^{\text{mod}}(t))$  are uniformly bounded on  $L^2$  in three steps. The first two steps are exactly the same, while the difference arises in the last step where one may see the difficulties caused by nonzero  $P_0, P_1$ . As a byproduct of this proof, Theorem 2.1.8 can be shown easily. Section 2.3 is devoted to the uniform boundedness of  $\text{Op}(a^{\text{alt}}(t))$ , which is much simpler than that of  $\text{Op}(a(t))$  and  $\text{Op}(a^{\text{mod}}(t))$  thanks to the extra factor  $\Lambda$ . The uniform boundedness of truncated operators allows us to calculate the limits stated in Theorem 2.1.1, 2.1.4, and 2.1.6 only for some regular data  $u_0$ , which will be precised in Section 2.4. In Section 2.5, we will prove the microlocal partition of energy for Klein-Gordon equation by studying the half-Klein-Gordon equation.

In Appendix 2.A, we collect technical inequalities which are frequently used in this chapter, as well as some criteria of  $L^2$ -boundedness for pseudo-differential operators. Several stationary phase lemmas are presented in Appendix 2.B which is a key technique in calculating the limit of truncated energy. As mentioned before, our main result Theorem 2.1.1 can be refined for Schrödinger equation, whose rigorous statement and proof will be given in Appendix 2.C. Appendix 2.D is devoted to the discussion on the classical partition of energy for Klein-Gordon equation due to the study of asymptotic behavior of solution to half-Klein-Gordon equation. The last part, Appendix 2.E, contains some details omitted in Section 2.4, especially for concave  $P$ .

### 2.1.5 . Notations and conventions

To end this section, we clarify some notations and conventions used in this chapter.

- We say  $a$  is a symbol on  $\mathbb{R}^d$ , if  $a$  is a function on  $\{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\}$ . The corresponding (pseudo-differential) operator is defined by

$$\text{Op}(a)f(x) := \frac{1}{(2\pi)^d} \int e^{ix\xi} a(x, \xi) \hat{f}(\xi) d\xi.$$

To make this definition meaningful, we will assume in this chapter that  $a$  is a measurable function with at most polynomial growth in  $\xi$  and that  $f$  belongs to the class of Schwartz functions.

- For any function  $P : \mathbb{R}^d \rightarrow \mathbb{C}$ , which can be regarded as a symbol independent of  $x$ , the corresponding operator will be denoted by  $P(D_x)$ .

- The kernel (or kernel function) of a linear operator  $A : \mathcal{S}(\mathbb{R}^d) \mapsto \mathcal{S}'(\mathbb{R}^d)$  is defined as (if it exists) a tempered distribution  $K$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , such that

$$Au(x) = \int K(x, y)u(y)dy.$$

For the simplicity of notation, in this chapter, we will use the symbol of kernel function to represent the operator, i.e.

$$Ku(x) = \int K(x, y)u(y)dy.$$

- A function  $F : \mathbb{R}^d \rightarrow \mathbb{C}$  is said to be radial, if there exists a function  $f : [0, \infty[ \mapsto \mathbb{C}$ , such that  $F(x) = f(|x|)$  for all  $x \in \mathbb{R}^d$ . In this case, we will not distinguish function  $F$  and  $f$ . That is, we will write instead  $F(x) = F(|x|)$  or  $f(x) = f(|x|)$  for  $x \in \mathbb{R}^d$ .

- We will use  $c, C$ , sometimes equipped with superscripts and subscripts, to represent all the small and large constants respectively.

- For non-zero quantities  $\rho, r$ , the notation  $\rho \sim r$  means that there exists constants  $c, C > 0$ , such that  $c < \frac{\rho}{r} < C$ .

## 2.2 . $L^2$ -boundedness of microlocal truncation operators

The goal of this section is the demonstration of following proposition, which eventually implies Theorem 2.1.8.

**Proposition 2.2.1.** *Let  $p_0, p_1 \neq 0$ ,  $\delta \in \mathbb{R}$ , and  $\chi \in C_c^\infty(\mathbb{R}^d)$ . There exists a constant  $C > 0$  independent of  $t$ , such that, for all  $|t| > 0$ ,*

$$\|\text{Op}(a_{\chi, \delta}(t))\|_{\mathcal{L}(L^2)} \leq C,$$

where the symbol  $a_{\chi, \delta}(t)$  is defined in (2.1.10) and  $P$  satisfies the hypothesis  $(H_{p_0, p_1})$  with  $P_0 = P_1 = 0$ .

In parallel, we shall also prove the following result :

**Proposition 2.2.2.** *Let  $p_0 > 0 > p_1$ ,  $\delta \in \mathbb{R}$ , and  $\chi \in C_c^\infty(\mathbb{R}^d)$ . We assume that  $P$  satisfies the hypothesis  $(H_{p_0, p_1})$  with  $P_0, P_1 > 0$ . Then the modified truncated symbol*

$$a_{\chi, \delta}^{\text{mod}}(t, x, \xi) = a_{\chi, \delta}(t, x, \xi)(1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right),$$

which has already been defined in (2.1.18), corresponds to a bounded operator on  $L^2$ , uniformly in  $|t| > t_0 \gg 1$ . Here  $\chi_l, \chi_h \in C_c^\infty$  are radial and equal to 1 near zero and  $\epsilon_0, \epsilon_1$  satisfy the condition (2.1.19).

One can see in the following proof that our demonstration cannot eliminate the truncation in  $\xi$  in the definition (2.1.18) of  $a^{\text{mod}}$ . In fact, after some change of scaling, we will decompose the symbol  $a$  (or  $a^{\text{mod}}$ ) into three components, two of which are bounded for all  $P_0, P_1 \geq 0$ , while our treatment for the last component does not hold for nonzero  $P_0, P_1$ . The complementary cut-off in  $\xi$  is used to solve this problem. Remark that it is still unknown whether such restriction is essential.

To begin with, one observes that, it is equivalent to study the cut-off inside the cone, namely

$$a^{\text{in}}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| < |t| |P'(\xi)|},$$

since the operator with symbol

$$\chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right)$$

is bounded uniformly in  $t$  and  $\delta \in \mathbb{R}$ , due to Lemma 2.A.10 together with Lemma 2.A.5. In this section, we will not distinguish  $a$  and  $a^{\text{in}}$  and denote both of them as  $a$ .

With a reflection in  $\xi$ ,  $t$  can be assumed to be positive. The application of Lemma 2.A.4 allows us to replace  $a$  by

$$\tilde{a}(t, x, \xi) = a(t, \sqrt{t}x, \frac{\xi}{\sqrt{t}}) = \chi \left( \frac{\frac{x}{\sqrt{t}} + P' \left( \frac{\xi}{\sqrt{t}} \right)}{t^{\delta - \frac{1}{2}}} \right) \mathbb{1}_{\left| \frac{x}{\sqrt{t}} \right| < |P' \left( \frac{\xi}{\sqrt{t}} \right)|}.$$

Now, we split  $\tilde{a}$  into high and low frequency, namely  $\tilde{a} = \tilde{a}^b + \tilde{a}^\sharp$ , where

$$\tilde{a}^b(t, x, \xi) = \tilde{a}(t, x, \xi) \tilde{\chi}(\xi),$$

and  $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$  is a radial function which equals 1 near zero. In the following, we shall treat high and low frequency part at the same time. Before entering the next step, we introduce some notations which will be frequently used in this section. In all cases, we set

$$\mu = t^{\delta - \frac{1}{2}} \in ]0, +\infty[.$$

With  $j = 0$  for low frequency part and  $j = 1$  for high frequency part, we set

-when  $P_j = 0$ ,

$$X(t, x) := \frac{|x|}{\sqrt{t}}, \quad \Xi(t, \xi) := P' \left( \frac{|\xi|}{\sqrt{t}} \right), \quad \nu_j = +;$$

-when  $P_j > 0, P' > P_j$

$$X(t, x) := \frac{|x|}{\sqrt{t}} - P_j, \quad \Xi(t, \xi) := P' \left( \frac{|\xi|}{\sqrt{t}} \right) - P_j, \quad \nu_j = +;$$

-when  $P_j > 0, P' < P_j$

$$X(t, x) := P_j - \frac{|x|}{\sqrt{t}}, \quad \Xi(t, \xi) := P_j - P' \left( \frac{|\xi|}{\sqrt{t}} \right), \quad \nu_j = -.$$

Remark that for all nonzero  $\xi$ ,  $\Xi$  is strictly positive. With these notations, our problem can be reduced to the uniform-in- $\mu, t$  boundedness of

$$\begin{aligned} b^b(t, \mu, x, \xi) &= \chi \left( \frac{(P_0 + \nu_0 X) \frac{x}{|x|} + (P_0 + \nu_0 \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \mathbb{1}_{0 < \frac{X}{\Xi} < 1} \tilde{\chi} \left( \frac{\xi}{\sqrt{t}} \right), \\ b^\sharp(t, \mu, x, \xi) &= \chi \left( \frac{(P_1 + \nu_1 X) \frac{x}{|x|} + (P_1 + \nu_1 \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \mathbb{1}_{0 < \frac{X}{\Xi} < 1} (1 - \tilde{\chi}) \left( \frac{\xi}{\sqrt{t}} \right). \end{aligned}$$

We emphasize that our definition of  $X$  does not ensure its strict positivity, but one may always eliminate the part  $X < 0$ , due to the uniform boundedness of operator with symbol

$$\chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{\mu} \right),$$

which is also a consequence of Lemma 2.A.10 and Lemma 2.A.5.

Now, we decompose  $b^l$  ( $l = b, \sharp$ ) as the sum of  $b_0^l, \tilde{b}^l, b_1^l$  with cut-off  $0 < \frac{X}{\Xi} \ll 1$  and  $\frac{X}{\Xi} \sim 1$  and  $0 < 1 - \frac{X}{\Xi} \ll 1$ , respectively. To be precise,

$$\begin{aligned} b^l(t, \mu, x, \xi) &= b_0^l(t, \mu, x, \xi) + \tilde{b}^l(t, \mu, x, \xi) + b_1^l(t, \mu, x, \xi), \\ b_0^l(t, \mu, x, \xi) &= b^l(t, \mu, x, \xi) \chi_0 \left( \frac{X}{\Xi} \right), \\ \tilde{b}^l(t, \mu, x, \xi) &= b^l(t, \mu, x, \xi) \Psi \left( \frac{X}{\Xi} \right), \\ b_1^l(t, \mu, x, \xi) &= b^l(t, \mu, x, \xi) \chi_1 \left( 1 - \frac{X}{\Xi} \right), \end{aligned}$$

where  $\chi_0, \chi_1$  and  $\Psi$  are radial, smooth and compactly supported.  $\chi_0, \chi_1$  are supported in a small neighborhood of zero and equal to 1 near zero, while  $\Psi$  is compactly supported in  $]0, 1[$ . By regarding  $\mu$  as a  $t$ -independent parameter, we can reduce Proposition 2.2.1 and 2.2.2 to the following proposition :

**Proposition 2.2.3.** *There exists  $t, \mu$ -independent constants  $C > 0$ , such that,*

(i) *if  $P_0, P_1 \geq 0$ , for all  $t, \mu > 0$ ,*

$$(2.2.1) \quad \|\text{Op}(b_1^t(t, \mu))\|_{\mathcal{L}(L^2)} \leq C,$$

$$(2.2.2) \quad \|\text{Op}(\tilde{b}^t(t, \mu))\|_{\mathcal{L}(L^2)} \leq C,$$

(ii) *if  $P_0 = P_1 = 0$ , for all  $t, \mu > 0$ .*

$$(2.2.3) \quad \|\text{Op}(b_0^t(t, \mu))\|_{\mathcal{L}(L^2)} \leq C;$$

(iii) *if  $P_0, P_1 > 0$  and  $p_1 < 0 < p_0$ , for all  $\mu > 0, t > 1$ ,*

$$(2.2.4) \quad \left\| \text{Op} \left( b_0^t(t, \mu)(1 - \chi_l) \left( \frac{\xi}{t^{\frac{1}{2} - \epsilon_0}} \right) \chi_h \left( \frac{\xi}{t^{\frac{1}{2} + \epsilon_1}} \right) \right) \right\|_{\mathcal{L}(L^2)} \leq C.$$

Before giving the proof, we indicate below the consequence of this proposition, which implies Proposition 2.2.1, 2.2.2, and will be used in the end of this section to conclude Theorem 2.1.8.

**Corollary 2.2.4.** *Let  $\chi_l, \chi_b, \chi_h$  be defined as before.*

(i) *If  $P$  verifies  $(H_{p_0, p_1})$  with  $p_0, p_1 \neq 0$  and  $P_0 = P_1 = 0$ , the operator*

$$\text{Op} \left( \chi \left( \frac{x + tP'(\xi)}{|t|\mu} \right) \mathbb{1}_{|x| > |tP'(\xi)|} \right)$$

*is bounded on  $L^2$  uniformly in  $t \neq 0$  and  $\mu > 0$ .*

(ii) *If  $P$  verifies  $(H_{p_0, p_1})$  with  $p_0, p_1 \neq 0$  and  $\epsilon_0, \epsilon_1$  satisfy conditions (2.1.19), the operator*

$$\text{Op} \left( \chi \left( \frac{x + tP'(\xi)}{|t|\mu} \right) \mathbb{1}_{|x| > |tP'(\xi)|} (1 - \chi_l) \left( \frac{\xi}{t^{\frac{1}{2} - \epsilon_0}} \right) \chi_h \left( \frac{\xi}{t^{\frac{1}{2} + \epsilon_1}} \right) \right)$$

*is bounded on  $L^2$  uniformly in  $t > 1$  and  $\mu > 0$ .*

### 2.2.1 . Study of symbol $b_1$ and $\tilde{b}$

In this part, we shall prove (2.2.1) and (2.2.2). One observes that both  $b_1^t$  and  $\tilde{b}^t$  are supported for  $X \sim \Xi$ , which allows us to reduce our problem via dyadic decomposition.

*Proof of (2.2.2).* We start with a homogeneous dyadic decomposition

$$1 = \sum_{k \in \mathbb{Z}} \varphi \left( \frac{\eta}{2^k} \right),$$

where  $\varphi \in C_c^\infty(\mathbb{R}^d)$  is radial and supported away from zero. In this way, we may decompose  $\tilde{b}^\#$  as  $\sum_{k \geq 0} \tilde{b}_k$ , and  $\tilde{b}^\flat$  as  $\sum_{k < 0} \tilde{b}_k$  where

$$(2.2.5) \quad \tilde{b}_k(t, \mu, x, \xi) = \chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \Psi \left( \frac{X}{\Xi} \right) \psi \left( 2^{-kp_j} X \right) \varphi \left( \frac{\xi}{2^k \sqrt{t}} \right),$$

where  $\psi \in C_c^\infty(\mathbb{R}^d)$  is also radial and supported away from zero. The extra factor  $\psi$  comes from the truncation  $X \sim \Xi \sim (t^{-\frac{1}{2}}|\xi|)^{p_j} \sim 2^{kp_j}$ . This factors implies that the  $\tilde{b}_k$ 's are almost orthogonal so that it suffices to prove the uniform (in  $k, t, \mu$ ) boundedness of  $\text{Op}(\tilde{b}_k)$ .

Remark that, due to the compact support of  $\chi$  and the fact that  $0 < c < \frac{X}{\Xi} < 1 - c$  for some small  $c > 0$ , we have

$$(2.2.6) \quad \begin{aligned} 2^{kp_j} \sim \Xi &\lesssim |X - \Xi| = \left| |(P_j + \nu_j X) \frac{x}{|x|}| - |(P_j + \nu_j \Xi) \frac{\xi}{|\xi|}| \right| \\ &\leq \left| (P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|} \right| \lesssim \mu. \end{aligned}$$

When  $t2^{k(p_j+1)} \geq 1$ , we shall apply Calderón-Vaillancourt Theorem (see Lemma 2.A.11). For each derivative in  $x$ , if it acts on  $\chi$ , one gains  $t^{-\frac{1}{2}}\mu^{-1} \lesssim t^{-\frac{1}{2}}2^{-kp_j}$  by (2.2.6). If  $\partial_x$  acts on  $\Psi$  or  $\psi$ , one obtains factors of size  $t^{-\frac{1}{2}}2^{-kp_j}$ . As for the derivatives in  $\xi$ , similarly, it leads to factors of size  $P''(\xi/\sqrt{t})t^{-\frac{1}{2}}\mu^{-1}$ ,  $\Xi^{-1}P''(\xi/\sqrt{t})t^{-\frac{1}{2}}$  or  $t^{-\frac{1}{2}}2^{-k}$ , which are all controlled by  $t^{-\frac{1}{2}}2^{-k}$ , as  $|\xi| \sim 2^k\sqrt{t}$  and  $2^{kp_j} \lesssim \mu$ . Since

$$t^{-\frac{1}{2}}2^{-kp_j} \times t^{-\frac{1}{2}}2^{-k} = t^{-1}2^{-k(p_j+1)} \leq 1,$$

we may conclude by a change of scaling (Lemma 2.A.4).

When  $t2^{k(p+1)} \leq 1$ , we shall use Lemma 2.A.9. We first check the assumption (2.A.6) of this lemma with  $\mu_k \in ]0, +\infty[$  defined below

$$\mu_k = \begin{cases} \mu 2^{-kp_j}, & \text{if } P_j + \nu_j X \sim 2^{kp_j}, \\ \mu, & \text{if } P_j + \nu_j X \sim 1. \end{cases}$$

Note that we have either  $P_j + \nu_j X \sim P_j + \nu_j \Xi \sim 1$  or  $P_j + \nu_j X \sim P_j + \nu_j \Xi \sim 2^{kp_j}$ . In fact, by definition,  $P_j + \nu_j X$  and  $P_j + \nu_j \Xi$  are both strictly positive. Thus, it is sufficient to consider  $|k| \gg 1$ . When  $P_j$  is nonzero and  $kp_j < 0$ , we have  $X \sim \Xi \sim 2^{kp_j} \ll 1$  and then  $P_j + \nu_j X \sim P_j + \nu_j \Xi \sim 1$ . While  $P_j$  is nonzero and  $kp_j > 0$ , we have similarly  $X \sim \Xi \sim 2^{kp_j} \gg 1$  and  $P_j + \nu_j X \sim P_j + \nu_j \Xi \sim 2^{kp_j}$ . Otherwise,  $P_j$  equals to zero, which implies trivially  $P_j + \nu_j X = X \sim 2^{kp_j}$  and  $P_j + \nu_j \Xi = \Xi \sim 2^{kp_j}$ . Due to observation  $X \sim \Xi \sim 2^{kp_j}$ , it is easy to obtain that, for all  $\alpha, \beta \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\theta^\beta \tilde{b}_k(t, \mu, r\omega, \rho\theta)| \leq C_{\alpha, \beta, N} g_k(r, \rho) \mu_k^{-|\alpha| - |\beta|} \left\langle \frac{d(\omega, -\theta)}{\mu_k} \right\rangle^{-N},$$

where  $x = r\omega, \xi = \rho\theta$  are polar coordinates and

$$g_k(r, \rho) = \mathbb{1}_{\rho \sim 2^k \sqrt{t}} \mathbb{1}_{r \sim t^{\frac{1}{2}} 2^{kp_j}}.$$

The operator of kernel  $g_k$  is controlled by

$$\|g_k\|_{L^2(dr d\rho)} \lesssim (\sqrt{t} 2^k \times \sqrt{t} 2^{kp_j})^{\frac{1}{2}} = (t 2^{k(p_j+1)})^{\frac{1}{2}} \leq 1,$$

which is no more than the assumption (2.A.5) of Lemma 2.A.9. As a result, we may conclude (2.2.2) by (2.A.7).  $\square$

The idea of proof of (2.2.1) is similar. The only difficulty is that  $b_1$  has a singularity near  $X = \Xi$ . We may treat the part away from  $X = \Xi$  as above and study the area near  $X = \Xi$  by convexity (or concavity) of  $P$ .

*Proof of (2.2.1).* As before, we begin with the homogeneous dyadic decomposition in  $\frac{\xi}{\sqrt{t}}$ , namely  $b_1^\sharp = \sum_{k \geq 0} b_{1,k}$  and  $b_1^\flat = \sum_{k < 0} b_{1,k}$ , with

$$b_{1,k} = \chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \mathbb{1}_{\frac{X}{\Xi} < 1} \chi_1 \left( 1 - \frac{X}{\Xi} \right) \psi \left( 2^{-kp_j} X \right) \varphi \left( \frac{\xi}{2^k \sqrt{t}} \right).$$

It suffices to prove that  $\text{Op}(b_{1,k})$  is bounded on  $L^2$ , uniformly in  $k, t, \mu$ . In comparison with  $\tilde{b}_k$  defined by (2.2.5), the main difficulty is that the non-smooth term cannot be deleted. In the case  $t2^{k(p_j+1)} \leq 1$ , we may repeat exactly the same argument as the study of  $\tilde{b}_k$  since this argument does not require any regularity in  $|x|, |\xi|$ .

When  $t2^{k(p_j+1)} > 1$ , we will separate the singularity near  $\frac{X}{\Xi} = 1$ . Consider the following decomposition :

$$\begin{aligned} b_{1,k} &= b'_{1,k} + b''_{1,k}, \\ b'_{1,k} &= b_{1,k} \tilde{\chi}_1(\sqrt{t} 2^{k \frac{1-p_j}{2}} (\Xi - X)), \end{aligned}$$

where  $\tilde{\chi}_1 \in C_c^\infty(\mathbb{R}^d)$  is radial and equal to 1 near zero.

The proof of boundedness of  $b'_{1,k}$  is similar to that of the case  $t2^{k(p_j+1)} \leq 1$ . By setting  $\mu_k \in ]0, +\infty[$  as before, namely

$$\mu_k = \begin{cases} \mu 2^{-kp_j}, & \text{if } P_j + \nu_j X \sim 2^{kp_j}, \\ \mu, & \text{if } P_j + \nu_j X \sim 1, \end{cases}$$

we may have, for all  $\alpha, \beta \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\theta^\beta b'_{1,k}(t, \mu, r\omega, \rho\theta)| \leq C_{\alpha, \beta, N} h_k(r, \rho) \mu_k^{-|\alpha| - |\beta|} \left\langle \frac{d(\omega, -\theta)}{\mu_k} \right\rangle^{-N},$$

where

$$(2.2.7) \quad h_k(r, \rho) = \sum_{n \sim \sqrt{t} 2^{k \frac{p_j+1}{2}}} \mathbb{1}_{I_n}(r) \mathbb{1}_{J_n}(\rho),$$

with

$$\begin{aligned} J_n &:= [2^{k \frac{1-p_j}{2}} n, 2^{k \frac{1-p_j}{2}} (n+1)], \\ I_n &:= \begin{cases} \left[ \left[ \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} n \right) - c 2^{k \frac{p_j-1}{2}}, \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+1) \right) + c 2^{k \frac{p_j-1}{2}} \right], \right. \\ \quad \text{if } P'' > 0, \\ \left. \left[ \left[ \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+1) \right) - c 2^{k \frac{p_j-1}{2}}, \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} n \right) + c 2^{k \frac{p_j-1}{2}} \right], \right. \right. \\ \quad \left. \left. \text{if } P'' < 0. \right. \right. \end{cases} \end{aligned}$$

Note that by writing in polar system  $r = |x|$   $\rho = |\xi|$ ,  $b'_{1,k}$  is supported for  $\rho \sim 2^k \sqrt{t}$  and  $\sqrt{t} 2^{k \frac{1-p_j}{2}} |\Xi - X| \ll 1$  due to cut-off  $\tilde{\chi}_1$ . We first make a decomposition in  $\rho$ , namely

$$\rho \in [C^{-1} 2^k \sqrt{t}, C 2^k \sqrt{t}] \subset \bigcup_{n \sim \sqrt{t} 2^{k \frac{p_j+1}{2}}} J_n,$$

and then the support  $\sqrt{t} 2^{k \frac{1-p_j}{2}} |\Xi - X| \ll 1$  ensures that  $r$  lies in  $I_n$  defined above, once  $\rho$  belongs to  $J_n$ . This gives the control  $h_k$  defined in (2.2.7).

In order to apply Lemma 2.A.9, it suffices to check that the operator with kernel  $h_k$  is uniformly bounded on  $L^2(\mathbb{R}_+)$ , which can be reduced to  $|I_n| |J_n| \lesssim 1$  and that  $\{I_n\}$  forms a uniformly finite cover. In fact, since  $I_n$ 's are pairwise disjoint (except for end points), one observes that, for all  $w \in L^2(\mathbb{R}_+)$ ,

$$\begin{aligned} \left\| \int h_k(\cdot, \rho) w(\rho) d\rho \right\|_{L^2(\mathbb{R}_+)}^2 &= \left\| \sum_n \mathbb{1}_{I_n}(\cdot) \int \mathbb{1}_{J_n}(\rho) w(\rho) d\rho \right\|_{L^2(\mathbb{R}_+)}^2 \\ &= \sum_n \left\| \mathbb{1}_{I_n}(\cdot) \int \mathbb{1}_{J_n}(\rho) w(\rho) d\rho \right\|_{L^2(\mathbb{R}_+)}^2 \\ &\leq \sum_n |I_n| |J_n| \|\mathbb{1}_{J_n} w\|_{L^2(\mathbb{R}_+)}^2 \lesssim \sum_n \|\mathbb{1}_{J_n} w\|_{L^2(\mathbb{R}_+)}^2, \end{aligned}$$

where the last inequality follows from the first assertion  $|I_n| |J_n| \lesssim 1$ . The second assertion guarantees that each point of  $\mathbb{R}_+$  belongs to at most  $N$  intervals in  $\{J_n\}$  for some  $N \in \mathbb{N}$ . This implies that  $\sum_n \|\mathbb{1}_{J_n} w\|_{L^2(\mathbb{R}_+)}^2 \leq N \|w\|_{L^2(\mathbb{R}_+)}^2$ , which proves the uniform-in- $t, k$   $L^2(\mathbb{R}_+)$ -boundedness of operator with kernel  $h_k$ .

The first assertion is obvious since  $n \sim \sqrt{t} 2^{k \frac{p_j+1}{2}}$  implies that

$$\begin{aligned} |J_n| |I_n| &\lesssim 2^{k \frac{1-p_j}{2}} \left( \left| \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+1) \right) - \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} n \right) \right| + 2c 2^{k \frac{p_j-1}{2}} \right) \\ &\lesssim 2^{k \frac{1-p_j}{2}} \left( \sqrt{t} 2^{k(p_j-1)} t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} + 2c 2^{k \frac{p_j-1}{2}} \right) \lesssim 1. \end{aligned}$$

As for the second one, we observe that  $I_n \cap I_{n+l} \neq \emptyset$  if and only if

$$\left| \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} n \right) - \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+l) \right) \right| \leq 2c 2^{k \frac{p_j-1}{2}}.$$

Without loss of generality, we may assume  $l \geq 0$ . Actually, the left hand side has the following equivalence :

$$\begin{aligned} &\left| \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} n \right) - \sqrt{t} P' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+l) \right) \right| \\ &= \left| 2^{k \frac{1-p_j}{2}} l P'' \left( t^{-\frac{1}{2}} 2^{k \frac{1-p_j}{2}} (n+sl) \right) \right|, \text{ for some } s \in [0, 1], \end{aligned}$$

$$\begin{aligned} &\sim 2^{k\frac{1-p_j}{2}} l \times \left( t^{-\frac{1}{2}} 2^{k\frac{1-p_j}{2}} (n+sl) \right)^{p_j-1}, \text{ since } |P''(\rho)| \sim \rho^{p_j-1} \\ &\sim 2^{k\frac{1-p_j}{2}} l \times 2^{k(p_j-1)} = 2^{k\frac{p_j-1}{2}} l. \end{aligned}$$

To prove the last equivalence, we may use the fact that  $n, n+l \sim \sqrt{t} 2^{k\frac{p_j+1}{2}}$ , which implies that

$$c\sqrt{t} 2^{k\frac{p_j+1}{2}} \leq n \leq n+sl \leq n+l \leq C\sqrt{t} 2^{k\frac{p_j+1}{2}}.$$

In conclusion, we have that  $I_n \cap I_{n+l} \neq \emptyset$  holds for finitely many  $l$ . As a result,  $\text{Op}(b'_{1,k})$  is bounded uniformly in  $t, k$ .

It remains to study the smooth symbol  $b''_{1,k}$ , which reads

$$\begin{aligned} b''_{1,k} = &\chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \psi \left( 2^{-kp} X \right) \varphi \left( \frac{\xi}{2^k \sqrt{t}} \right) \\ &\times \mathbb{1}_{\frac{X}{\Xi} < 1} \chi_1 \left( 1 - \frac{X}{\Xi} \right) (1 - \tilde{\chi}_1) (\sqrt{t} 2^{k\frac{1-p_j}{2}} (\Xi - X)). \end{aligned}$$

Remark that this symbol is smooth, since the singularity  $\frac{X}{\Xi} = 1$  is removed by  $(1 - \tilde{\chi}_1)$  factor. Under the condition  $t 2^{k(p_j+1)} > 1$ , it verifies the condition of Calderón-Vaillancourt Theorem (see Lemma 2.A.11). In fact, each derivative in  $x$  leads to a factor of size  $t^{-\frac{1}{2}} \mu^{-1}$  (from  $\chi$ ),  $t^{-\frac{1}{2}} 2^{-kp_j}$  (from  $\psi$  and  $\chi_1$ ), or  $2^{k\frac{1-p_j}{2}}$  (from  $(1 - \tilde{\chi}_1)$ ). The condition  $t 2^{k(p_j+1)} > 1$  implies that  $t^{-\frac{1}{2}} 2^{-kp_j} \leq 2^{k\frac{1-p_j}{2}}$ , while the compact support of  $\chi$  and support of  $(1 - \tilde{\chi}_1)$  ensures that

$$t^{-\frac{1}{2}} 2^{k\frac{p_j-1}{2}} \lesssim |X - \Xi| \lesssim \mu,$$

i.e.  $t^{-\frac{1}{2}} \mu^{-1} \lesssim 2^{\frac{1-p_j}{2}k}$ . The same argument for  $\partial_\xi$  gives that each derivative in  $\xi$  leads to a factor of size  $2^{-k\frac{1-p_j}{2}}$ . The desired result thus follows from a change of scaling (Lemma 2.A.4).  $\square$

### 2.2.2 . Study of symbol $b_0$ with $P_0 = P_1 = 0$

In the case  $P_0 = P_1 = 0$ , due to the lack of almost orthogonality as  $\tilde{b}_k$ 's and  $b_{1,k}$ 's, the remaining symbol  $b'_0$  will be treated via Cotlar-Stein Lemma (Lemma 2.A.3). As before, we start with homogeneous dyadic decomposition in  $\xi$ , namely  $b_0^\sharp = \mathbb{1}_{X>0} \sum_{k \geq 0} c_k$  and  $b_0^\flat = \mathbb{1}_{X>0} \sum_{k < 0} c_k$ , with

$$(2.2.8) \quad c_k = \chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \chi_0 \left( \frac{X}{\Xi} \right) \varphi \left( \frac{\xi}{2^k \sqrt{t}} \right).$$

It suffices to prove the (uniform in  $t$ ) boundedness of  $\sum_{k \in \mathbb{Z}} c_k$  as the multiplication with  $\mathbb{1}_{X>0}$  is trivially bounded on  $L^2$ .

We first check that  $\text{Op}(c_k)$ 's are bounded uniformly in  $k, t, \mu$ . More precisely, all the  $c_k$ 's satisfy the following estimate :

**Lemma 2.2.5.** *There exists  $C > 0$  independent of  $k, t$ , such that for all  $t > 0$  and  $k \in \mathbb{Z}$ ,*

$$(2.2.9) \quad \|\text{Op}(c_k)\|_{\mathcal{L}(L^2)} \leq C \min\left(\max\left(1, (t2^{k(p_j+1)})^{-N_d}\right), (t2^{k(p_j+1)})^{\frac{d}{2}}\right) \leq C,$$

where  $N_d \in \mathbb{N}$  depends only on dimension  $d$ .

*Proof.* We observe that  $c_k$  is supported for  $X \ll \Xi \sim 2^{kp_j}$ , which implies, on the one hand, as in (2.2.6),

$$2^{kp_j} \sim |X - \Xi| \leq \left| (P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|} \right| \lesssim \mu,$$

and, on the another hand,

$$|x| \ll \sqrt{t}2^{kp_j}, \quad |\xi| \sim \sqrt{t}2^k.$$

As a consequence of the second result,  $\|\text{Op}(c_k)\|_{\mathcal{L}(L^2)}$  can be trivially bounded by

$$\|\text{Op}(c_k)\|_{\mathcal{L}(L^2)} \lesssim \|c_k\|_{L^2(\mathbb{R}^{2d})} \lesssim (t2^{k(p_j+1)})^{\frac{d}{2}}.$$

It remains to check that

$$\|\text{Op}(c_k)\|_{\mathcal{L}(L^2)} \leq C \max\left(1, (t2^{k(p_j+1)})^{-N_d}\right),$$

which can be proved via Calderón-Vaillancourt Theorem (see Lemma 2.A.11). In fact, from each derivative in  $x$ , we may obtain extra factors of size  $t^{-\frac{1}{2}}\mu^{-1}$  (action on  $\chi$ ) or  $t^{-\frac{1}{2}}2^{-kp_j}$  (action on  $\chi_0$ ). As we have seen that  $t^{-\frac{1}{2}}\mu^{-1} \lesssim t^{-\frac{1}{2}}2^{-kp_j}$ , each derivative in  $x$  leads to a factor of size  $(t^{\frac{1}{2}}2^{kp_j})^{-1}$ . Similarly, the action of  $\partial_\xi$  on  $\chi, \chi_0, \varphi$  gives factors of size  $t^{-\frac{1}{2}}\mu^{-1}2^{k(p_j-1)}$ ,  $t^{-\frac{1}{2}}2^{-k}$ , and  $t^{-\frac{1}{2}}2^{-k}$ , respectively. We may also check that  $t^{-\frac{1}{2}}\mu^{-1}2^{k(p_j-1)} \lesssim 2^{-k}t^{-\frac{1}{2}}$ . To sum up,  $c_k$  is smooth and satisfies

$$(2.2.10) \quad \left| \partial_x^\alpha \partial_\xi^\beta c_k(x, \xi) \right| \leq C_{\alpha, \beta} \left( \frac{1}{\sqrt{t}2^{kp_j}} \right)^{|\alpha|} \left( \frac{1}{\sqrt{t}2^k} \right)^{|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^d.$$

By Lemma 2.A.4, it is equivalent to consider rescaled symbol

$$\tilde{c}_k(x, \xi) = c_k\left(2^{k\frac{p_j-1}{2}}x, 2^{-k\frac{p_j-1}{2}}\xi\right),$$

which, as a result of (2.2.10), satisfies for all  $\gamma \in \mathbb{N}^{2d}$ ,

$$\|\partial_{x, \xi}^\gamma \tilde{c}_k\|_{L^\infty(\mathbb{R}^{2d})} \lesssim (t2^{k(p_j+1)})^{-|\gamma|}.$$

By applying Calderón-Vaillancourt Theorem (Lemma 2.A.11) to  $\tilde{c}_k$ , we have, due to estimate (2.A.8), that

$$\|\text{Op}(c_k)\|_{\mathcal{L}(L^2)} = \|\text{Op}(\tilde{c}_k)\|_{\mathcal{L}(L^2)} \leq C \max\left(1, (t2^{k(p_j+1)})^{-N_d}\right).$$

□

In order to conclude (2.2.3) by Cotlar-Stein Lemma (Lemma 2.A.3), it is sufficient to check condition (2.A.2) and (2.A.3), namely

**Lemma 2.2.6.** *There exists  $t, \mu$ -independent constants  $C$ , such that for all  $t, \mu > 0$ ,*

$$(2.2.11) \quad \sup_{k \in \mathbb{Z}_+} \sum_{l \in \mathbb{Z}_+} \|\text{Op}(c_k) \text{Op}(c_l)^*\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leq C, \quad \sup_{k \in \mathbb{Z}_-} \sum_{l \in \mathbb{Z}_-} \|\text{Op}(c_k) \text{Op}(c_l)^*\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leq C,$$

$$(2.2.12) \quad \sup_{k \in \mathbb{Z}_+} \sum_{l \in \mathbb{Z}_+} \|\text{Op}(c_k)^* \text{Op}(c_l)\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leq C, \quad \sup_{k \in \mathbb{Z}_-} \sum_{l \in \mathbb{Z}_-} \|\text{Op}(c_k)^* \text{Op}(c_l)\|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \leq C,$$

where  $\mathbb{Z}_- = \mathbb{Z} \cap ]-\infty, 0[$  corresponds to low frequency part and  $\mathbb{Z}_+ = \mathbb{Z} \cap [0, +\infty[$  corresponds to high frequency part.

*Proof of (2.2.11).* By symbolic calculus,  $\text{Op}(c_k) \text{Op}(c_l)^*$  is an operator of symbol

$$c_k \sharp c_l^*(x, \xi) = \frac{1}{(2\pi)^d} \int e^{-iy\eta} c_k(x, \xi + \eta) \overline{c_l(x + y, \xi + \eta)} d\eta.$$

By definition (2.2.8),  $c_l(x, \xi)$  is supported for  $|\xi| \sim 2^l \sqrt{t}$ . Thus,  $c_k \sharp c_l^*$  is nonzero only if  $|l - k| < N_0$  for some large  $N_0 \in \mathbb{N}^*$ . As a consequence, (2.2.11) can be reduced to the uniform boundedness of  $\text{Op}(c_l)$ , which has already been proved in Lemma 2.2.5.  $\square$

*Proof of (2.2.12).* We apply again the symbolic calculus to obtain the following expression of symbol of  $\text{Op}(c_k)^* \text{Op}(c_l)$ ,

$$c_k^* \sharp c_l(x, \xi) = \frac{1}{(2\pi)^d} \int e^{i(x-y)\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) d\eta dy.$$

Recall that  $c_l$  is supported for  $|x| \ll \sqrt{t} 2^{lp_j}$  and  $|\xi| \sim t^{\frac{1}{2}} 2^l$  with the estimate (2.2.10). We shall check that for all  $l, k \in \mathbb{N}$ ,

$$(2.2.13) \quad \|\text{Op}(c_k)^* \text{Op}(c_l)\|_{\mathcal{L}(L^2)} \lesssim 2^{-\frac{d}{2}|k-l|},$$

which is enough to conclude (2.2.12). Due to (2.2.9), we may ignore the case  $|k - l| \leq N_0$ , for some fixed large  $N_0 \in \mathbb{N}$ . Remark that it is sufficient to prove (2.2.13) only for  $l \geq k$ , since for terms with  $l < k$ , we have

$$\begin{aligned} \|\text{Op}(c_k)^* \text{Op}(c_l)\|_{\mathcal{L}(L^2)} &= \|(\text{Op}(c_k)^* \text{Op}(c_l))^*\|_{\mathcal{L}(L^2)} \\ &= \|\text{Op}(c_l)^* \text{Op}(c_k)\|_{\mathcal{L}(L^2)} \lesssim 2^{-\frac{d}{2}|l-k|}. \end{aligned}$$

One observes that the bound of operator with symbol  $c_k^* \sharp c_l$  can be controlled by

$$\|\text{Op}(c_k^* \sharp c_l)\|_{\mathcal{L}(L^2)} \lesssim \|c_k^* \sharp c_l\|_{L^2(dx d\xi)} \lesssim \left\| \int e^{-iy\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) dy \right\|_{L^2(d\eta d\xi)}.$$

The integrand of the last integral is supported for

$$(2.2.14) \quad |\xi + \eta| \sim 2^k \sqrt{t}, |\xi| \sim 2^l \sqrt{t}, \text{ and } |y| \lesssim \min(2^{kp_j} \sqrt{t}, 2^{lp_j} \sqrt{t}).$$

Moreover, we may apply integration by parts in  $y$  to obtain some extra bounds in the estimate. To be precise, for all  $N_1 \in \mathbb{N}$

$$\begin{aligned}
& \int e^{-iy\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) dy \\
&= \int \left( \frac{-\Delta_y}{|\eta|^2} \right)^{N_1} e^{-iy\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) dy \\
&= \int e^{-iy\eta} (-\Delta_y)^{N_1} \left( \overline{c_k(y, \xi + \eta)} c_l(y, \xi) \right) |\eta|^{-2N_1} dy \\
&= \sum_{|\alpha|+|\beta|=2N_1} C_{\alpha, \beta} \int e^{-iy\eta} \overline{\partial_y^\alpha c_k(y, \xi + \eta)} \partial_y^\beta c_l(y, \xi) |\eta|^{-2N_1} dy.
\end{aligned}$$

Since we have reduced our problem to the case  $l \geq k + N_0$ , the integral above is supported for  $|\eta| \sim 2^l \sqrt{t}$ . Together with (2.2.10) and (2.2.14), we have

$$\begin{aligned}
& \left| \int e^{-iy\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) dy \right| \\
&\leq \sum_{|\alpha|+|\beta|=2N_1} C_{\alpha, \beta} \int |\partial_y^\alpha c_k(y, \xi + \eta)| |\partial_y^\beta c_l(y, \xi)| |\eta|^{-2N_1} dy \\
&\lesssim \sum_{|\alpha|+|\beta|=2N_1} \mathbb{1}_{|\xi+\eta| \sim 2^k \sqrt{t}} \mathbb{1}_{|\xi| \sim 2^l \sqrt{t}} \left( 2^{kp_j} \sqrt{t} \right)^{-|\alpha|} \left( 2^{lp_j} \sqrt{t} \right)^{-|\beta|} (2^l \sqrt{t})^{-2N_1} \int \mathbb{1}_{|y| \lesssim \min(2^{kp_j} \sqrt{t}, 2^{lp_j} \sqrt{t})} dy \\
&\lesssim \mathbb{1}_{|\xi+\eta| \sim 2^k \sqrt{t}} \mathbb{1}_{|\xi| \sim 2^l \sqrt{t}} (2^l \sqrt{t})^{-2N_1} \min \left( 2^{kp_j} \sqrt{t}, 2^{lp_j} \sqrt{t} \right)^{d-2N_1}.
\end{aligned}$$

The estimate above holds for all  $N_1 \in \mathbb{N}$ , thus for all  $N_1 \in [0, \infty[$ . In particular, we choose  $N_1 = \frac{d}{2}$ , which gives that

$$\begin{aligned}
\| \text{Op}(c_k^* \sharp c_l) \|_{\mathcal{L}(L^2)} &\lesssim \left\| \int e^{-iy\eta} \overline{c_k(y, \xi + \eta)} c_l(y, \xi) dy \right\|_{L^2(d\eta d\xi)} \\
&\lesssim (2^l \sqrt{t})^{-d} \| \mathbb{1}_{|\xi+\eta| \sim 2^k \sqrt{t}} \mathbb{1}_{|\xi| \sim 2^l \sqrt{t}} \|_{L^2(d\eta d\xi)} \\
&\lesssim (2^l \sqrt{t})^{-d} \times \left( 2^k \sqrt{t} \times 2^l \sqrt{t} \right)^{\frac{d}{2}} = 2^{\frac{d}{2}(k-l)} = 2^{-\frac{d}{2}|k-l|}.
\end{aligned}$$

As a conclusion, we have managed to prove that

$$\sup_{k \in \mathbb{Z}_\pm} \sum_{l \in \mathbb{Z}_\pm} \| \text{Op}(c_k)^* \text{Op}(c_l) \|_{\mathcal{L}(L^2)}^{\frac{1}{2}} \lesssim \sup_{k \in \mathbb{Z}_\pm} \sum_{l \in \mathbb{Z}_\pm} 2^{-\frac{d}{4}|k-l|} < \infty,$$

which completes the proof.  $\square$

### 2.2.3 . Study of symbol $b_0$ with $P_0, P_1 > 0$

Till now, we have finished the proof of Proposition 2.2.1. To complete the proof of Proposition 2.2.2, it remains to check (2.2.4). Remark that the argument above relies on

the fact that  $|X| \ll 2^{kp_j}$  implies  $x$  is supported in a region of area  $(\sqrt{t}2^{kp_j})^d$ , which is not true in the case where  $P_0, P_1$  are non zero. To overcome this problem we need the extra truncation in  $\xi$ .

*Proof of (2.2.4).* As above, we may ignore the non-smooth factor  $\mathbb{1}_{0 < \frac{x}{\Xi} < 1}$ . It remains smooth symbols

$$\begin{aligned}\tilde{c}^\sharp(t, x, \xi) &= \chi \left( \frac{(P_1 + \nu_1 X) \frac{x}{|x|} + (P_1 \nu_1 \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \chi_0 \left( \frac{X}{\Xi} \right) (1 - \tilde{\chi}) \left( \frac{\xi}{\sqrt{t}} \right) \chi_h \left( \frac{\xi}{t^{\frac{1}{2} + \epsilon_1}} \right), \\ \tilde{c}^\flat(t, x, \xi) &= \chi \left( \frac{(P_0 + \nu_0 X) \frac{x}{|x|} + (P_0 \nu_0 \Xi) \frac{\xi}{|\xi|}}{\mu} \right) \chi_0 \left( \frac{X}{\Xi} \right) \tilde{\chi} \left( \frac{\xi}{\sqrt{t}} \right) (1 - \chi_l) \left( \frac{\xi}{t^{\frac{1}{2} - \epsilon_0}} \right).\end{aligned}$$

We shall first check that  $\tilde{c}^\sharp$  belongs uniformly to the Hörmander class  $S_{1, \kappa}^0$  for some  $\kappa \in ]0, 1[$ , namely the collection of smooth symbols  $c(x, \xi)$  such that for all  $\alpha, \beta \in \mathbb{N}^d$ ,

$$|\partial_x^\alpha \partial_\xi^\beta c(x, \xi)| \lesssim \langle \xi \rangle^{-|\beta| + \kappa|\alpha|}.$$

It is well-known that the operators of symbol in this class is bounded on  $L^2$ , a proof of which can be found in [Hör07]. We begin with the observation that high frequency symbol  $\tilde{c}^\sharp$  is supported for  $t^{\frac{1}{2}} \lesssim |\xi| \lesssim t^{\frac{1}{2} + \epsilon_1}$ . Before calculating the bounds of derivatives in  $x$  and  $\xi$ , recall that our goal is to show (2.2.4) under the condition  $t > 1$ .

For each derivative in  $\xi$ , we obtain from  $\chi$  a factor of size

$$t^{-\frac{1}{2}} \mu^{-1} P''(|\xi|/\sqrt{t}) \sim \mu^{-1} \left( \frac{|\xi|}{\sqrt{t}} \right)^{p_1} |\xi|^{-1} \lesssim \langle \xi \rangle^{-1}.$$

The last inequality is due the support of  $\tilde{c}^\sharp$ . More precisely,

$$\left( \frac{|\xi|}{\sqrt{t}} \right)^{p_1} \sim \Xi \lesssim |X - \Xi| \lesssim \left| (P_1 + \nu_1 X) \frac{x}{|x|} + (P_1 + \nu_1 \Xi) \frac{\xi}{|\xi|} \right| \lesssim \mu.$$

From the factor  $\chi_0$ , one gains

$$\frac{X}{\Xi} \frac{P''(\xi/\sqrt{t})}{\Xi \sqrt{t}} \sim \frac{X}{\Xi} \frac{1}{|\xi|} \lesssim \langle \xi \rangle^{-1}.$$

Trivially, we will also obtain  $\langle \xi \rangle^{-1}$  from derivative on  $(1 - \tilde{\chi})$  and  $\chi_h$ .

As for derivatives in  $x$ , if  $\partial_x$  acts on  $\chi$ , one gains  $t^{-\frac{1}{2}} \mu^{-1} \lesssim (\sqrt{t}\Xi)^{-1}$ . When it acts on  $\chi_0$ , the resulting factor is of size

$$\frac{1}{\sqrt{t}\Xi} \sim \frac{1}{t} \left( \frac{|\xi|}{\sqrt{t}} \right)^{-p_1 - 1} |\xi|.$$

When  $0 > p_1 \geq -1$ , we have

$$\frac{1}{t} \left( \frac{|\xi|}{\sqrt{t}} \right)^{-p_1 - 1} |\xi| = |\xi|^{-p_1} t^{\frac{p_1 - 1}{2}} \lesssim \langle \xi \rangle t^{-\frac{1}{2}} \lesssim \langle \xi \rangle^\kappa t^{(\frac{1}{2} + \epsilon_1)(1 - \kappa) - \frac{1}{2}}.$$

Thus,  $\tilde{c}^\sharp \in S_{1,\kappa}^0$  for any  $\kappa \in ]0, 1[$  such that

$$\left(\frac{1}{2} + \epsilon_1\right)(1 - \kappa) - \frac{1}{2} \leq 0,$$

which is possible by choosing  $\kappa$  close to 1. When  $p_1 < -1$ , the estimate above becomes

$$\frac{1}{t} \left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1} |\xi| = \frac{|\xi|^{1-\kappa}}{t} \left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_1-1} |\xi|^\kappa \lesssim t^{-1+(\frac{1}{2}+\epsilon_1)(1-\kappa)-(p_1+1)\epsilon_1} \langle \xi \rangle^\kappa.$$

To conclude  $\tilde{c}^\sharp \in S_{1,\kappa'}^0$ , it suffices to choose  $\kappa \in ]0, 1[$  such that

$$-1 + \left(\frac{1}{2} + \epsilon_1\right)(1 - \kappa) - (p_1 + 1)\epsilon_1 \leq 0,$$

which is equivalent to

$$\epsilon_1 \leq \frac{1}{-(p_1 + 1)} \left[ 1 - \left(\frac{1}{2} + \epsilon_1\right)(1 - \kappa) \right].$$

This can be realized by choosing  $\kappa$  close to 1, due to the definition (2.1.19) of  $\epsilon_1$ .

To prove the uniform boundedness of  $\tilde{c}^\flat$ , which is supported for  $t^{\frac{1}{2}-\epsilon_0} \lesssim |\xi| \lesssim t^{\frac{1}{2}}$ , we shall apply Calderón-Vaillancourt Theorem (Lemma 2.A.11). As above one may check easily that each  $\partial_\xi$  gives

$$|\xi|^{-1} \lesssim t^{-\frac{1}{2}+\epsilon_0};$$

while each  $\partial_x$  gives

$$\frac{1}{\sqrt{t}} \left(\frac{|\xi|}{\sqrt{t}}\right)^{-p_0} \lesssim t^{-\frac{1}{2}+p_0\epsilon_0}.$$

As a consequence, the desired result follows from Lemma 2.A.4 and Calderón-Vaillancourt Theorem (Lemma 2.A.11) once we have

$$t^{-\frac{1}{2}+\epsilon_0} \times t^{-\frac{1}{2}+p_0\epsilon_0} \lesssim 1, \quad \forall t > 1,$$

equivalently,  $(1 + p_0)\epsilon_0 \leq 1$ , which is exactly the definition (2.1.19) of  $\epsilon_0$ .  $\square$

#### 2.2.4 . Proof of Theorem 2.1.8

In all the proof above, we regard  $\mu = t^{\delta-\frac{1}{2}} \in ]0, +\infty[$  as a time-independent parameter. This allows us to take the limit  $\mu \rightarrow +\infty$  with all the uniform estimates remaining true. Rigorously, due to Corollary 2.2.4, for all  $f, g \in \mathcal{S}(\mathbb{R}^d)$  and  $t \in \mathbb{R}, \mu > 0$ ,

$$(2.2.15) \quad \left| \left\langle f, \text{Op} \left( \chi \left( \frac{x + tP'(\xi)}{|t|\mu} \right) \mathbb{1}_{|x|>|tP'(\xi)|} \Omega(t, \xi) \right) g \right\rangle \right| \leq C \|f\|_{L^2} \|g\|_{L^2},$$

where  $\Omega = 1$  when  $P_0 = P_1 = 0$ , and

$$\Omega(t, \xi) = (1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right)$$

when  $P_0, P_1 > 0$ .

The left hand side of (2.2.15) is equal to

$$\left| \frac{1}{(2\pi)^d} \int f(x) e^{-ix\xi} \chi \left( \frac{x + tP'(\xi)}{|t|^\mu} \right) \mathbb{1}_{|x| > |tP'(\xi)|} \Omega(t, \xi) \overline{\hat{g}(\xi)} d\xi dx \right|,$$

which, when  $\mu \rightarrow +\infty$ , due to Dominated Convergence Theorem, tends to

$$\left| \frac{1}{(2\pi)^d} \int f(x) e^{-ix\xi} \mathbb{1}_{|x| > |tP'(\xi)|} \Omega(t, \xi) \overline{\hat{g}(\xi)} d\xi dx \right|,$$

where we take  $\chi(0) = 1$  without loss of generality. In conclusion, for all  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$|\langle f, \text{Op}(\mathbb{1}_{|x| > |tP'(\xi)|} \Omega(t, \xi)) g \rangle| \leq C \|f\|_{L^2} \|g\|_{L^2}.$$

Theorem 2.1.8 follows from the density of  $\mathcal{S}(\mathbb{R}^d)$  in  $L^2$ .

### 2.3 . $L^2$ -boundedness of microlocal truncation operators - An alternative symbol

In this section, we will treat those  $P$  with nonzero  $P_0, P_1$  in an alternative way. Instead of adding extra truncation in  $\xi$ , we shall add some extra factor in the main truncation  $\chi$ . To be precise,

**Proposition 2.3.1.** *Let  $P, \Lambda$  satisfy conditions  $(H_{p_0, p_1})$  and  $(C_{\sigma_0, \sigma_1})$  of Section 2.1.2 respectively, with  $p_1 < 0 < p_0$ ,  $\sigma_0 \geq p_0$ , and  $\sigma_1 \leq p_1$ . We further assume that  $\delta + \frac{\sigma_j}{2} < \frac{1}{2}$ ,  $j = 0, 1$ . Then there exist time-independent constants  $C > 0$ ,  $t_0 \gg 1$ , such that, for all  $|t| > t_0$*

$$\| \text{Op} \left( a_{\chi, \delta, \Lambda}^{alt}(t) \right) \|_{\mathcal{L}(L^2)} \leq C,$$

where  $a_{\chi, \delta, \Lambda}^{alt}$  is defined in (2.1.23)

Without loss of generality, we may assume  $t > 0$ . Meanwhile, the change of scaling (Lemma 2.A.4) allows us to reduce to the symbol

$$(2.3.1) \quad b(t, x, \xi) = \chi \left( \frac{\frac{x}{\sqrt{t}} + P' \left( \frac{\xi}{\sqrt{t}} \right)}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) H \left( \frac{\frac{|x|}{\sqrt{t}} - P' \left( \frac{|\xi|}{\sqrt{t}} \right)}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right),$$

where  $H \in C_b^\infty(\mathbb{R} \setminus \{0\}) \cap L^\infty(\mathbb{R})$ . To recover the desired result in Proposition 2.3.1, it suffices to take  $H = \mathbb{1}_{]0, +\infty[}$ .

As in previous section, we set

-when  $P_j = 0$ ,

$$X := \frac{|x|}{\sqrt{t}}, \quad \Xi := P' \left( \frac{|\xi|}{\sqrt{t}} \right), \quad \nu_j = +;$$

-when  $P_j > 0, P' > P_j$

$$X := \frac{|x|}{\sqrt{t}} - P_j, \quad \Xi := P' \left( \frac{|\xi|}{\sqrt{t}} \right) - P_j, \quad \nu_j = +;$$

-when  $P_j > 0, P' < P_j$

$$X := P_j - \frac{|x|}{\sqrt{t}}, \quad \Xi := P_j - P' \left( \frac{|\xi|}{\sqrt{t}} \right), \quad \nu_j = -.$$

With these notations, via homogeneous dyadic decomposition, we may rewrite symbol  $b$  as

$$b(t, x, \xi) = \sum_{k \in \mathbb{Z}} b_k(t, x, \xi),$$

$$b_k(t, x, \xi) = \chi \left( \frac{(P_1 + \nu_1 X) \frac{x}{|x|} + (P_1 + \nu_1 \Xi) \frac{\xi}{|\xi|}}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) H \left( \frac{X - \Xi}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) \varphi \left( \frac{\xi}{\sqrt{t} 2^k} \right), \quad \forall k \geq 0,$$

$$b_k(t, x, \xi) = \chi \left( \frac{(P_0 + \nu_0 X) \frac{x}{|x|} + (P_0 + \nu_0 \Xi) \frac{\xi}{|\xi|}}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) H \left( \frac{X - \Xi}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) \varphi \left( \frac{\xi}{\sqrt{t} 2^k} \right), \quad \forall k < 0,$$

where  $\varphi \in C_c^\infty(\mathbb{R}^d)$  is radial and supported in an annulus centered at zero.

One observes that, due to factors  $\chi$  and  $\varphi$ ,  $b_k$  is supported for

$$|X - \Xi| \lesssim t^{\delta - \frac{1}{2}} \Lambda(\xi) \sim t^{\delta - \frac{1}{2}} (\sqrt{t} 2^k)^{\sigma_j},$$

where  $j = 0$  when  $k < 0$ ,  $j = 1$  when  $k \geq 0$ . By using the fact that  $\Xi \sim \left| \frac{\xi}{\sqrt{t}} \right|^{p_j} \sim 2^{kp_j}$ , we obtain

$$\left| \frac{X}{\Xi} - 1 \right| \lesssim t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k(\sigma_j - p_j)} \leq t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}}.$$

The last inequality is the consequence of our assumptions  $\sigma_0 \geq p_0$  and  $\sigma_1 \leq p_1$ . Since  $\delta + \frac{\sigma_j}{2} - \frac{1}{2} < 0$ , if we further assume that  $t \geq t_0 \gg 1$ , the inequality above implies that  $X \sim \Xi$ , which allows us to add a complementary factor  $\psi(2^{-kp_j} X)$  to the definition of  $b_k$ , where  $\psi \in C_c^\infty(\mathbb{R})$  is radial and supported in an annulus centered at zero. In this way, we may reduce Proposition 2.3.1 to

**Proposition 2.3.2.** *Under the same assumptions as in Proposition 2.3.1, the operator  $\text{Op}(b_k)$  is bounded, uniformly in  $t$  and  $k$ .*

In what follows, we keep using the subscript  $j$ , where  $j = 0$  for  $k < 0$  and  $j = 1$  for  $k \geq 0$ . By definition,

$$b_k(t, x, \xi) = \chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) H \left( \frac{X - \Xi}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) \varphi \left( \frac{\xi}{\sqrt{t} 2^k} \right) \psi \left( \frac{X}{2^{kp_j}} \right).$$

When  $P_j = 0$ ,  $b_k$  is supported for

$$\left| \frac{x}{|x|} + \frac{\xi}{|\xi|} \right| \leq \frac{1}{X} \left| X \frac{x}{|x|} + \Xi \frac{\xi}{|\xi|} \right| + \frac{1}{X} \left| (X - \Xi) \frac{\xi}{|\xi|} \right|$$

$$\lesssim \frac{1}{X} t^{\delta - \frac{1}{2}} \Lambda(\xi) \sim 2^{-kp_j} t^{\delta - \frac{1}{2}} (\sqrt{t} 2^k)^{\sigma_j} = t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k(\sigma_j - p_j)}.$$

When  $P_j \neq 0$ , we observe that  $2^{k\sigma_j} \leq 2^{kp_j} \leq 1$ , due to the choice  $p_1 < 0 < p_0$ . By choosing  $\text{Supp } \varphi$  small, which allows us to take  $\text{Supp } \psi$  small, we have  $P_j + \nu_j X \sim 1$ . Thus,  $b_k$  is supported for

$$\begin{aligned} \left| \frac{x}{|x|} + \frac{\xi}{|\xi|} \right| &\leq \frac{1}{P_j + \nu_j X} \left| (P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|} \right| + \frac{1}{P_j + \nu_j X} \left| (X - \Xi) \frac{\xi}{|\xi|} \right| \\ &\lesssim \frac{1}{P_j + \nu_j X} t^{\delta - \frac{1}{2}} \Lambda(\xi) \sim t^{\delta - \frac{1}{2}} (\sqrt{t} 2^k)^{\sigma_j} = t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k\sigma_j}. \end{aligned}$$

If  $t 2^{k(p_j+1)} \leq 1$ , by setting  $\mu := t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k(\sigma_j - p_j)} \in ]0, 1[$  in the case  $P_j = 0$ , and  $\mu := t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k\sigma_j} \in ]0, 1[$  in the case  $P_j \neq 0$ , one may check that, for all  $\alpha, \beta \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$ ,

$$\left| \partial_\omega^\alpha \partial_\theta^\beta b_k(r\omega, \rho\theta) \right| \lesssim \mu^{-|\alpha| - |\beta|} \mathbb{1}_{r \lesssim \sqrt{t} 2^{kp_j}} \mathbb{1}_{\rho \sim \sqrt{t} 2^k} \left\langle \frac{d(\omega, -\theta)}{\mu} \right\rangle^{-N}.$$

By applying Lemma 2.A.9, we have

$$\| \text{Op}(b_k) \|_{\mathcal{L}(L^2)} \lesssim \| \mathbb{1}_{r \lesssim \sqrt{t} 2^{kp_j}} \mathbb{1}_{\rho \sim \sqrt{t} 2^k} \|_{L^2(\mathbb{R}_+^2)} = C \sqrt{t 2^{k(p_j+1)}} \leq C.$$

If  $t 2^{k(p_j+1)} > 1$ , we decompose  $b_k$  as the sum of  $b'_k, b''_k$ , which are defined by

$$\begin{aligned} b_k(t, x, \xi) &= b'_k(t, x, \xi) + b''_k(t, x, \xi), \\ b'_k(t, x, \xi) &= b_k \chi_0 \left( \sqrt{t} 2^{k \frac{1-p_j}{2}} (X - \Xi) \right), \end{aligned}$$

where  $\chi_0 \in C_c^\infty(\mathbb{R})$  is radial, supported in a neighborhood of zero, and equal to 1 near zero.

Clearly,  $b''_k$  is smooth on  $\mathbb{R}^{2d}$ . Thus, to prove the boundedness of  $\text{Op}(b''_k)$ , we may apply Calderón-Vaillancourt Theorem (Lemma 2.A.11). By definition,  $b''_k$  reads

$$\begin{aligned} &\chi \left( \frac{(P_j + \nu_j X) \frac{x}{|x|} + (P_j + \nu_j \Xi) \frac{\xi}{|\xi|}}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) H \left( \frac{X - \Xi}{t^{\delta - \frac{1}{2}} \Lambda(\xi)} \right) \\ &\quad \times \varphi \left( \frac{\xi}{\sqrt{t} 2^k} \right) \psi \left( \frac{X}{2^{kp_j}} \right) (1 - \chi_0) \left( \sqrt{t} 2^{k \frac{1-p_j}{2}} (X - \Xi) \right), \end{aligned}$$

which is supported for

$$t^{-\frac{1}{2}} 2^{k \frac{p_j-1}{2}} \lesssim |X - \Xi| \lesssim t^{\delta - \frac{1}{2}} \Lambda(\xi) \sim t^{\delta + \frac{\sigma_j}{2} - \frac{1}{2}} 2^{k\sigma_j}.$$

As a consequence,  $t^{-(\delta + \frac{\sigma_j}{2})} \lesssim 2^{k(\sigma_j + \frac{1-p_j}{2})}$ .

For each  $\partial_x$ , when it acts on  $\chi$ , one obtains in its bound an extra factor of size

$$\frac{1}{\sqrt{t}} t^{\frac{1}{2} - \delta} \Lambda(\xi)^{-1} \sim t^{-(\delta + \frac{\sigma_j}{2})} 2^{-kp_j} \lesssim 2^{k \frac{1-p_j}{2}} 2^{k(\sigma_j - p_j)} \leq 2^{k \frac{1-p_j}{2}}.$$

When  $\partial_x$  acts on  $\psi$  and  $(1 - \chi_0)$  factors, we gain  $t^{-\frac{1}{2}}2^{-kp_j} < 2^{k\frac{p_j+1}{2}-kp_j} = 2^{k\frac{1-p_j}{2}}$  and  $2^{k\frac{1-p_j}{2}}$ , respectively. For each  $\partial_\xi$ , similarly, one gains  $2^{k\frac{p_j-1}{2}}$  in its bound. Namely, for all  $\alpha, \beta \in \mathbb{N}^d$ ,

$$\left| \partial_x^\alpha \partial_\xi^\beta b_k''(x, \xi) \right| \lesssim 2^{k\frac{1-p_j}{2}(|\alpha|-|\beta|)}.$$

The uniform boundedness of  $\text{Op}(b_k'')$  follows from Lemma 2.A.4 and Calderón-Vaillancourt Theorem (Lemma 2.A.11).

It remains to study the symbol  $b_k'$ . To overcome the singularity near  $X = \Xi$ , we will apply Lemma 2.A.9 with the same setting of  $\mu$  as in previous paragraph, namely  $\mu := t^{\delta+\frac{\sigma_j}{2}-\frac{1}{2}}2^{k(\sigma_j-p_j)} \in ]0, 1[$  in the case  $P_j = 0$ , and  $\mu := t^{\delta+\frac{\sigma_j}{2}-\frac{1}{2}}2^{k\sigma_j} \in ]0, 1[$  in the case  $P_j \neq 0$ . It is easy to check that, for all  $\alpha, \beta \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$ ,

$$\left| \partial_\omega^\alpha \partial_\theta^\beta b_k(r\omega, \rho\theta) \right| \lesssim \mu^{-|\alpha|-|\beta|} h_k(r, \rho) \left\langle \frac{d(\omega, -\theta)}{\mu} \right\rangle^{-N},$$

where

$$h_k(r, \rho) = \sum_{n \sim \sqrt{t}2^{k\frac{p_j+1}{2}}} \mathbb{1}_{I_n}(r) \mathbb{1}_{J_n}(\rho),$$

which is exactly the same one defined in (2.2.7). We have seen that the operator with symbol  $h_k$  is bounded on  $L^2(\mathbb{R}_+)$  uniformly in  $k$ . By Lemma 2.A.9, we may conclude the uniform-in- $k$  boundedness of  $b_k'$  and the proof of Proposition 2.3.2, hence Proposition 2.3.1 is completed.

## 2.4 . Limit of truncated energy

In this section, we will complete the proof of Theorem 2.1.1, 2.1.4, and 2.1.6 by calculating the limit of truncated energy for some regular initial data  $u_0$ . These three results will follow from the following proposition.

**Proposition 2.4.1.** *Let  $a_{\chi, \delta, \Lambda}(t)$  be the symbol defined by*

$$(2.4.1) \quad a = a_{\chi, \delta, \Lambda}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta} \Lambda(|t|^{\frac{1}{2}} \xi)} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|},$$

where  $P \in C^\infty(\mathbb{R}^d \setminus \{0\})$  is assumed to be a real radial function satisfying that  $P''(\rho) \neq 0$  for all  $\rho \in ]0, \infty[$ . Furthermore, we assume that  $\chi \in C_c^\infty(\mathbb{R}^d)$  is real and radial with  $\chi(0) = 1$  and that  $\Lambda$  verifies condition  $(C_{\sigma_0, \sigma_1})$  without any restriction in  $\sigma_0, \sigma_1$ .

With these settings, if there exists  $t_0 \gg 1$ , such that  $\text{Op}(a(t))$  is bounded on  $L^2$  uniformly in  $|t| > t_0$ , for all  $u_0 \in L^2$ , the limits (2.1.24), (2.1.25), and (2.1.28) hold true.

**Corollary 2.4.2.** *We consider the same symbol  $a$  with an extra truncation in  $\xi$ , i.e.*

$$\tilde{a} = \tilde{a}_{\chi, \delta, \Lambda}(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2}+\delta} \Lambda(|t|^{\frac{1}{2}} \xi)} \right) \mathbb{1}_{|x| > |t| |P'(\xi)|} (1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right)$$

where  $\chi_l, \chi_h \in C_c^\infty(\mathbb{R}^d)$  are equal to 1 in a neighborhood of zero, and  $\epsilon_0, \epsilon_1 > 0$ .

If  $P, \Lambda$  satisfy the same conditions as in Proposition 2.4.1 and  $\text{Op}(\tilde{a}_{\delta, \chi, \Lambda}(t))$  is bounded on  $L^2$  uniformly in  $|t| > t_0 \gg 1$ , then, for all  $u_0 \in L^2$ , the limits (2.1.24), (2.1.25), and (2.1.28) hold true with  $E_{\delta, \chi, \Lambda}^{alt}(t)$  replaced by  $\|\text{Op}(\tilde{a}_{\delta, \chi, \Lambda}(t)) u(t)\|_{L^2}^2$ .

In order to complete the proof of Theorem 2.1.1 and 2.1.6, we may combine Proposition 2.2.1, 2.3.1 with Proposition 2.4.1, where we need to take  $\Lambda \equiv 1$  in the proof of Theorem 2.1.1. In the same way, Theorem 2.1.4 follows from Proposition 2.2.2 and Corollary 2.2.4 with  $\Lambda \equiv 1$ .

Before calculating the limit of truncated energy, we remark that, in the hypothesis of Proposition 2.4.1 and Corollary 2.4.2,  $\text{Op}(a(t))$  (or  $\text{Op}(\tilde{a}(t))$ ) is assumed to be bounded uniformly in  $|t| > t_0 \gg 1$ , which allows us to replace general  $u_0 \in L^2$  by those belonging to some dense subset of  $L^2$ . In what follows, we may assume that  $\hat{u}_0 \in C_c^\infty(\mathbb{R} \setminus \{0\})$ . As a consequence, by taking  $t_0 \gg 1$ ,

$$(1 - \chi_l) \left( \frac{\xi}{|t|^{-\epsilon_0}} \right) \chi_h \left( \frac{\xi}{|t|^{\epsilon_1}} \right) \hat{u}_0(\xi) = \hat{u}_0(\xi),$$

which proves Corollary 2.4.2 from Proposition 2.4.1.

### 2.4.1 . Super-critical case $0 < \delta + \frac{\sigma_1}{2} < \frac{1}{2}$

In this part, we will study the case  $\delta + \frac{\sigma_1}{2} \in ]0, \frac{1}{2}[$  (associated to limit (2.1.17), (2.1.22), or (2.1.28)) by following the same method introduced in [Del22].

By definition, the truncated energy introduced in (2.1.11) is

$$\begin{aligned} E_{\chi, \delta, \Lambda}(\epsilon t) &= \|\text{Op}(a_{\chi, \delta, \Lambda}(\epsilon t)) u(\epsilon t)\|_{L^2}^2 \\ &= \frac{1}{(2\pi)^{2d}} \int e^{ix \cdot (\xi - \xi')} e^{i\epsilon t(P(\xi) - P(\xi'))} \chi \left( \frac{x + \epsilon t P'(\xi)}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \xi)} \right) \\ &\quad \times \chi \left( \frac{x + \epsilon t P'(\xi')}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \xi')} \right) \mathbb{1}_{\frac{|x|}{t} > |P'(\xi)|, |P'(\xi')|} \hat{u}_0(\xi) \overline{\hat{u}_0(\xi')} dx d\xi d\xi', \end{aligned}$$

where  $t \gg 1$  and  $\epsilon = \pm$ . In polar system  $x = r\omega$ ,  $\xi = \rho\theta$ ,  $\xi' = \rho'\theta'$ , the integral above can be written as

$$\begin{aligned} &\frac{1}{(2\pi)^{2d}} \int e^{i(r\rho\omega \cdot \theta - r\rho'\omega \cdot \theta')} e^{i\epsilon t(P(\rho) - P(\rho'))} \chi \left( \frac{r\omega + \epsilon t P'(\rho)\theta}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \chi \left( \frac{r\omega + \epsilon t P'(\rho')\theta'}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho')} \right) \\ &\quad \times \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \hat{u}_0(\rho\theta) \overline{\hat{u}_0(\rho'\theta')} (r\rho\rho')^{d-1} d\theta d\theta' d\omega dr d\rho d\rho'. \end{aligned}$$

We firstly focus on the integral in  $\theta$ , with integral in  $\theta'$  treated in exactly the same way,

$$(2.4.2) \quad \int e^{ir\rho\omega \cdot \theta} \chi \left( \frac{r\omega + \epsilon t P'(\rho)\theta}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \hat{u}_0(\rho\theta) d\theta.$$

In this part, we always set

$$\mu = t^{\delta + \frac{\sigma_1}{2} - \frac{1}{2}} \in ]0, 1[,$$

which is strictly positive and small, since we may choose  $t > t_0 \gg 1$ . Due to Lemma 2.B.5, (2.4.2) can be written as the sum of

$$(2\pi)^{\frac{d-1}{2}} e^{i\epsilon \frac{\pi}{4}(d-1)} e^{-i\epsilon r \rho} (r\rho)^{-\frac{d-1}{2}} \chi \left( \frac{r - tP'(\rho)}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \hat{u}_0(-\epsilon \rho \omega) \kappa \left( \frac{r}{t} \right)$$

and a remainder

$$e^{-i\epsilon r \rho} \mu^{d-1} S_{-\frac{d+1}{2}}(\omega, \mu, \rho, \frac{r}{t} - P'(\rho), t; r\rho\mu^2) \kappa \left( \frac{r}{t} \right),$$

where  $\kappa \in C_c^\infty(]0, \infty[)$  equals to 1 in a neighborhood of 1, and  $S_m(\omega, \mu, \rho, r', t; \zeta)$  is supported for  $\zeta > c > 0$ ,  $\rho \sim 1$  and  $|r'| \lesssim \mu$  and satisfies for all  $\alpha \in \mathbb{N}^{d-1}$ ,  $j, k, l, \gamma \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\mu^j \partial_\rho^k \partial_{r'}^l \partial_\zeta^\gamma S_m| \leq C \mu^{-(|\alpha|+j+l)} \langle \zeta \rangle^{m-\gamma}.$$

Remark that it is harmless to add an extra factor  $\kappa_r$ , since the integrand of (2.4.2) is supported for  $r \sim t$ , which is a consequence of the cut-off  $\chi$  together with  $t \gg 1$ ,  $\delta + \frac{\sigma_1}{2} < \frac{1}{2}$ , and  $\rho \sim 1$ . We may repeat this argument for the integral in  $\theta'$  and the truncated energy  $E_{\chi, \delta, \Lambda}(\epsilon t)$  can be decomposed into a principal part

$$(2.4.3) \quad \frac{1}{(2\pi)^{d+1}} \int e^{-i\epsilon r(\rho - \rho')} e^{i\epsilon t(P(\rho) - P(\rho'))} \chi \left( \frac{r - tP'(\rho)}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \chi \left( \frac{r - tP'(\rho')}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho')} \right) \\ \times \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \hat{u}_0(-\epsilon \rho \omega) \overline{\hat{u}_0(-\epsilon \rho' \omega)} (\rho \rho')^{\frac{d-1}{2}} \kappa^2 \left( \frac{r}{t} \right) d\omega dr d\rho d\rho',$$

and remainders

$$\frac{1}{(2\pi)^{d+1}} \int e^{-i\epsilon r(\rho - \rho')} e^{i\epsilon t(P(\rho) - P(\rho'))} \mu^{2(d-1)} \kappa^2 \left( \frac{r}{t} \right) S_m(\omega, \mu, \rho, \frac{r}{t} - P'(\rho), t; r\rho\mu^2) \\ \times S_{m'}(\omega, \mu, \rho', \frac{r}{t} - P'(\rho'), t; r\rho'\mu^2) \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} (r\rho\rho')^{d-1} d\omega dr d\rho d\rho',$$

where  $(m, m')$  takes values among  $(-\frac{d-1}{2}, -\frac{d+1}{2})$ ,  $(-\frac{d+1}{2}, -\frac{d-1}{2})$ ,  $(-\frac{d+1}{2}, -\frac{d+1}{2})$ . Note that due to the condition  $\delta + \frac{\sigma_1}{2} > \frac{1}{2}$ , we have

$$r\rho\mu^2 = r\rho t^{2(\delta + \frac{\sigma_1}{2}) - 1} \sim t^{2(\delta + \frac{\sigma_1}{2})} > c > 0.$$

The sum of these remainders can be simplified as

$$(2.4.4) \quad \int e^{-i\epsilon r(\rho - \rho')} e^{i\epsilon t(P(\rho) - P(\rho'))} \frac{1}{\mu^2 r} \Sigma(\omega, \mu, \rho, \rho', r, t; \frac{r}{t} - P'(\rho), \frac{r}{t} - P'(\rho')) \\ \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} d\omega dr d\rho d\rho',$$

where  $\Sigma(\omega, \mu, \rho, \rho', r, t; s, s')$  is supported for  $r \sim t$ ,  $\rho, \rho' \sim 1$  and  $|s|, |s'| \lesssim \mu$  and satisfies for all  $\alpha \in \mathbb{N}^{d-1}$ ,  $j, k, k', l, \gamma, \gamma' \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\mu^j \partial_\rho^k \partial_{\rho'}^{k'} \partial_r^l \partial_s^\gamma \partial_{s'}^{\gamma'} \Sigma| \lesssim \mu^{-(|\alpha|+j+\gamma+\gamma')} t^{-l}.$$

Before stepping further, we introduce the following integral :

$$(2.4.5) \quad I(t, \epsilon_1, \epsilon'_1, \epsilon_2, \epsilon'_2; F) := \int e^{i[r(\epsilon_1\rho + \epsilon'_1\rho') - t(\epsilon_2 P(\rho) + \epsilon'_2 P(\rho'))]} \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \\ \times F(\rho, \rho', r, t; r - \epsilon_1\epsilon_2 t P'(\rho), r - \epsilon'_1\epsilon'_2 t P'(\rho')) dr d\rho d\rho'.$$

The limit of such integral has been studied in [Del22] for strictly convex  $P$ , while the concave case can be studied with almost the same argument. To be precise,

**Proposition 2.4.3.** *Let  $F(\rho, \rho', r, t; \zeta, \zeta')$  be a smooth function on  $\mathbb{R}_+^4 \times \mathbb{R}^2$  and  $\delta' \in ]\frac{1}{2}, 1[$ . Assume that  $F$  is supported for*

$$\rho, \rho' \sim 1, r \sim t, |\zeta|, |\zeta'| \lesssim t^{\delta'},$$

and for all  $j, j', k, \gamma, \gamma' \in \mathbb{N}$ ,

$$|\partial_\rho^j \partial_{\rho'}^{j'} \partial_r^k \partial_\zeta^\gamma \partial_{\zeta'}^{\gamma'} F(\rho, \rho', r, t; \zeta, \zeta')| \lesssim t^{-\delta'(k+\gamma+\gamma')}.$$

We assume further that the following point-wise limit exists

$$\lim_{t \rightarrow +\infty} F(\rho, \rho', r\sqrt{t} + tP'(\rho'), t; \zeta\sqrt{t}, \zeta'\sqrt{t}) = F_0(\rho, \rho').$$

Under all the assumptions above, we have

$$\lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) = \frac{\pi}{2} \int_0^\infty F_0(\rho, \rho) d\rho,$$

for all  $\epsilon = \pm$  and  $P \in C^\infty$  with  $P'' > 0$  or  $P'' < 0$ .

The proof for strictly convex  $P$  follows from that of Proposition 3.1.3 in [Del22]. As for concave case, we will give a brief proof in Appendix 2.E. Remark that we compute the limit of  $I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F)$  for both signs  $\epsilon = \pm 1$ , while in [Del22] only the limit of the sum of these two terms was determined. The proof of our stronger result is not essentially different from the one in [Del22] and we shall explain the modification one has to make to the argument in Appendix 2.E.

With the notations above, the truncated energy  $E_{\chi, \delta, \Lambda}(\epsilon t)$  given by the sum of (2.4.3) and (2.4.4) equals to

$$E_{\chi, \delta, \Lambda}(\epsilon t) = I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) + I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; \bar{F}_R),$$

where

$$F(\rho, \rho', r, t; \zeta, \zeta') = \frac{1}{(2\pi)^{d+1}} \kappa^2 \left(\frac{r}{t}\right) \int \chi \left(\frac{\zeta}{t^{\frac{1}{2}+\delta} \Lambda(t^{\frac{1}{2}} \rho)}\right) \chi \left(\frac{\zeta'}{t^{\frac{1}{2}+\delta} \Lambda(t^{\frac{1}{2}} \rho')}\right) \\ \times \hat{u}_0(-\epsilon \rho \omega) \overline{\hat{u}_0(-\epsilon \rho' \omega)} (\rho \rho')^{\frac{d-1}{2}} d\omega, \\ F_R(\rho, \rho', r, t; \zeta, \zeta') = t^{-2(\delta + \frac{\sigma_1}{2})} \int \frac{t}{r} \Sigma(\omega, t^{\delta + \frac{\sigma_1}{2} - \frac{1}{2}}, \rho, \rho', r, t; \frac{\zeta}{t}, \frac{\zeta'}{t}) d\omega.$$

It is easy to check that  $F, F_R$  satisfy the conditions of Proposition 2.4.3 with

$$\delta' = \delta + \frac{\sigma_1}{2} + \frac{1}{2}.$$

Note that due to condition  $\delta + \frac{\sigma_1}{2} \in ]0, \frac{1}{2}[$ , we have  $\delta' \in ]\frac{1}{2}, 1[$ , which is required by Proposition 2.4.3. The corresponding limit is

$$F_0(\rho, \rho') = \frac{1}{(2\pi)^{d+1}} \int \hat{u}_0(-\epsilon\rho\omega) \overline{\hat{u}_0(-\epsilon\rho'\omega)} (\rho\rho')^{\frac{d-1}{2}} d\omega$$

and 0, respectively. Therefore, we may conclude (2.1.17), (2.1.22), and (2.1.28) by Proposition 2.4.3. In fact, with  $\epsilon = \pm$ , one has

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}(t) &= \lim_{t \rightarrow +\infty} E_{\chi, \delta}(\epsilon t) = \lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) + \lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F_R) \\ &= \frac{\pi}{2} \int_0^\infty F_0(\rho, \rho) d\rho + 0 \\ &= \frac{1}{4} \frac{1}{(2\pi)^d} \int \hat{u}_0(-\epsilon\rho\omega) \overline{\hat{u}_0(-\epsilon\rho\omega)} \rho^{d-1} d\omega d\rho = \frac{1}{4} \|u_0\|_{L^2}^2. \end{aligned}$$

### 2.4.2. Sub-critical and critical case $\delta + \frac{\sigma_1}{2} \leq 0$

In the rest of this section, we will study, under the condition  $\delta + \frac{\sigma_1}{2} \leq 0$ , the truncated energy  $E_{\chi, \delta, \Lambda}(u_0, \epsilon t)$ , with  $\epsilon = \pm$ ,  $t > t_0 \gg 1$ . Here, we only write the proof of the case  $P'' > 0$ , while the case  $P'' < 0$  can be calculated in exactly the same way.

By definition (2.1.11), the truncated energy  $E_{\chi, \delta, \Lambda}(u_0, \epsilon t)$  equals to

$$\begin{aligned} \frac{1}{(2\pi)^{2d}} \int e^{ix \cdot (\xi - \xi')} e^{i\epsilon t(P(\xi) - P(\xi'))} \chi \left( \frac{x + \epsilon t P'(\xi)}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \xi)} \right) \chi \left( \frac{x + \epsilon t P'(\xi')}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \xi')} \right) \\ \times \mathbb{1}_{\frac{|x|}{t} > |P'(\xi)|, |P'(\xi')|} \hat{u}_0(\xi) \overline{\hat{u}_0(\xi')} d\xi d\xi' dx, \end{aligned}$$

which can be rewritten in polar system  $x = r\omega$ ,  $\xi = \rho\theta$ ,  $\xi' = \rho'\theta'$  as

$$\begin{aligned} \frac{1}{(2\pi)^{2d}} \int e^{ir\omega \cdot (\rho\theta - \rho'\theta')} e^{i\epsilon t(P(\rho) - P(\rho'))} \chi \left( \frac{r\omega + \epsilon t P'(\rho)\theta}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \chi \left( \frac{r\omega + \epsilon t P'(\rho')\theta'}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho')} \right) \\ \times \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \hat{u}_0(\rho\theta) \overline{\hat{u}_0(\rho'\theta')} (r\rho\rho')^{d-1} dr d\theta d\theta' d\omega d\rho d\rho'. \end{aligned}$$

We decompose  $E_{\chi, \delta, \Lambda}(u_0, \epsilon t)$  as the sum of  $E_{\pm}(\epsilon t)$ , where  $E_+$ ,  $E_-$  are defined as integral over  $\rho > \rho'$ ,  $\rho' > \rho$ , respectively, and the dependence on  $\chi, \delta, \Lambda, u_0$  is omitted for the simplicity of notation. Since  $E_- = \overline{E_+}$ , it is enough to focus on the study of  $E_+(\epsilon t)$ . We first check that

**Lemma 2.4.4.** *The integral*

$$\begin{aligned} E_+(\epsilon t) &= \frac{1}{(2\pi)^{2d}} \int e^{ir\omega \cdot (\rho\theta - \rho'\theta')} e^{i\epsilon t(P(\rho) - P(\rho'))} \chi \left( \frac{r\omega + \epsilon t P'(\rho)\theta}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho)} \right) \chi \left( \frac{r\omega + \epsilon t P'(\rho')\theta'}{t^{\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}} \rho')} \right) \\ &\quad \times \mathbb{1}_{\frac{r}{t} > P'(\rho)} \mathbb{1}_{\rho > \rho'} \hat{u}_0(\rho\theta) \overline{\hat{u}_0(\rho'\theta')} (r\rho\rho')^{d-1} dr d\theta d\theta' d\omega d\rho d\rho' \end{aligned}$$

equals, up to some  $O(t^{-\frac{1}{2}})$  terms, to

$$(2.4.6) \quad \frac{1}{(2\pi)^{2d}} \int_{\theta, \theta' \in \sqrt{t}(\epsilon w + \mathbb{S}^{d-1})} e^{i[\sqrt{t}P'(\rho)\rho\omega \cdot (\theta - \theta') + r\rho\omega \cdot (\theta - \theta') + wP'(\rho)\theta' \cdot \omega]} e^{-\epsilon i[rw + \frac{1}{2}P''(\rho)w^2]} \\ \times \chi\left(\frac{r\omega + \epsilon P'(\rho)\theta}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \chi\left(\frac{(r + P''(\rho)w)\omega + \epsilon P'(\rho)\theta'}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \\ \times \mathbb{1}_{r>0} \mathbb{1}_{\rho > \frac{w}{\sqrt{t}} > 0} |\hat{u}_0(-\epsilon\rho\omega)|^2 (P'(\rho))^{d-1} \rho^{2(d-1)} d\theta d\theta' dr dw d\omega d\rho.$$

*Proof.* To begin with, via change of variable,

$$r \rightarrow rt^{\frac{1}{2}} + tP'(\rho), \quad \rho' \rightarrow \rho - \frac{w}{t^{\frac{1}{2}}},$$

the integral  $E_+(\epsilon t)$  can be rewritten as

$$\frac{t^{d-1}}{(2\pi)^{2d}} \int e^{i[tP'(\rho)\rho\omega \cdot (\theta - \theta') + \sqrt{t}r\rho\omega \cdot (\theta - \theta') + \sqrt{t}wP'(\rho)\theta' \cdot \omega + rw\theta' \cdot \omega]} \\ \times e^{\epsilon it(P(\rho) - P(\rho - \frac{w}{\sqrt{t}}))} \chi\left(\frac{r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta)}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \\ \times \chi\left(\frac{r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta') - \epsilon\sqrt{t}(P'(\rho) - P'(\rho - t^{-\frac{1}{2}}w))\theta'}{t^\delta \Lambda(t^{\frac{1}{2}}\rho - w)}\right) \\ \times \mathbb{1}_{r>0} \mathbb{1}_{\rho > \frac{w}{\sqrt{t}} > 0} \hat{u}_0(\rho\theta)\hat{u}_0((\rho - t^{-\frac{1}{2}}w)\theta') \\ \times \left(\frac{r}{\sqrt{t}} + P'(\rho)\right)^{d-1} \rho^{d-1} (\rho - t^{-\frac{1}{2}}w)^{d-1} d\theta d\theta' dr dw d\omega d\rho.$$

Due to  $\chi, \hat{u}_0$  factors, the integrand is supported for

$$(2.4.7) \quad 0 < r, w \lesssim t^{\delta + \frac{\sigma_1}{2}}, \quad \rho \sim 1, \\ |\omega + \epsilon\theta|, |\omega + \epsilon\theta'| \lesssim t^{\delta + \frac{\sigma_1}{2} - \frac{1}{2}}.$$

In fact,  $\rho \sim 1$  follows directly from the fact that  $\hat{u}_0$  is compactly supported away from zero. Since  $\chi$  is compactly supported, the first  $\chi$  factor implies that

$$r = \left| |(r + \sqrt{t}P'(\rho))\omega| - |\sqrt{t}P'(\rho)\theta| \right| \\ \leq \left| r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta) \right| \lesssim t^\delta \Lambda(t^{\frac{1}{2}}\rho) \sim t^{\delta + \frac{\sigma_1}{2}},$$

which further implies that

$$\sqrt{t}|\omega + \epsilon\theta| \lesssim |\sqrt{t}P'(\rho)(\omega + \epsilon\theta)| \\ \leq \left| r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta) \right| + |r\omega| \lesssim t^\delta \Lambda(t^{\frac{1}{2}}\rho) \sim t^{\delta + \frac{\sigma_1}{2}}.$$

By applying a similar argument to the second  $\chi$  factor, we obtain

$$\begin{aligned} w &\sim \left| \sqrt{t}P'(\rho) - \sqrt{t}P'(\rho - t^{-\frac{1}{2}}w) \right| \\ &\leq \left| (r + \sqrt{t}P'(\rho)) - \sqrt{t}P'(\rho - t^{-\frac{1}{2}}w) \right| + r \\ &\leq \left| (r + \sqrt{t}P'(\rho))\omega + \epsilon\sqrt{t}P'(\rho - t^{-\frac{1}{2}}w)\theta' \right| + r \lesssim t^{\delta + \frac{\sigma_1}{2}}, \end{aligned}$$

and that

$$\begin{aligned} \sqrt{t}|\omega + \epsilon\theta'| &\sim \left| \sqrt{t}P'(\rho)(\omega + \epsilon\theta') \right| \\ &\leq \left| r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta') - \epsilon\sqrt{t}(P'(\rho) - P'(\rho - t^{-\frac{1}{2}}w))\theta' \right| \\ &\quad + r + \left| \sqrt{t}(P'(\rho) - P'(\rho - t^{-\frac{1}{2}}w)) \right| \\ &\lesssim t^{\delta + \frac{\sigma_1}{2}} + r + w \lesssim t^{\delta + \frac{\sigma_1}{2}}. \end{aligned}$$

As a result, the boundedness of integrand implies that

$$E_+(\epsilon t) \lesssim t^{d-1} \times t^{2(\delta + \frac{\sigma_1}{2})} \times \left( \frac{1}{\sqrt{t}} t^{\delta + \frac{\sigma_1}{2}} \right)^{2(d-1)} = t^{2(\delta + \frac{\sigma_1}{2})d},$$

which tends to zero as  $t \rightarrow +\infty$  when  $\delta + \frac{\sigma_1}{2} < 0$ , and the limit (2.1.13), (2.1.20), and (2.1.24) follow. In the remaining of this section, we take  $\delta + \frac{\sigma_1}{2} = 0$ .

The support of integrand also allows us to simplify  $E_+(\epsilon t)$ , up to some  $O(t^{-\frac{1}{2}})$  terms, as

$$\begin{aligned} &\frac{t^{d-1}}{(2\pi)^{2d}} \int e^{i[tP'(\rho)\rho\omega \cdot (\theta - \theta') + \sqrt{t}r\rho\omega \cdot (\theta - \theta') + \sqrt{t}wP'(\rho)\theta' \cdot \omega + rw\theta' \cdot \omega]} \\ &\quad \times e^{\epsilon i[\sqrt{t}P'(\rho)w - \frac{1}{2}P''(\rho)w^2]} \chi \left( \frac{r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta)}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)} \right) \\ &\quad \times \chi \left( \frac{r\omega + \sqrt{t}P'(\rho)(\omega + \epsilon\theta') - \epsilon P''(\rho)w\theta'}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)} \right) \\ &\quad \times \mathbb{1}_{r>0} \mathbb{1}_{\rho > \frac{w}{\sqrt{t}} > 0} \hat{u}_0(\rho\theta) \overline{\hat{u}_0(\rho\theta')} (P'(\rho))^{d-1} \rho^{2(d-1)} d\theta d\theta' dr dw d\omega d\rho. \end{aligned}$$

Here, we use the approximations

$$\begin{aligned} t \left( P(\rho) - P \left( \rho - \frac{w}{t^{\frac{1}{2}}} \right) \right) &= t \left( -P'(\rho) \left( -\frac{w}{t^{\frac{1}{2}}} \right) - \frac{P''(\rho)}{2} \frac{w^2}{t} + O \left( \frac{w^3}{t^{\frac{3}{2}}} \right) \right) \\ &= \sqrt{t}P'(\rho)w - \frac{1}{2}P''(\rho)w^2 + O \left( \frac{1}{\sqrt{t}} \right), \end{aligned}$$

to simplify the phase and

$$\begin{aligned} \sqrt{t} \left( P'(\rho) - P' \left( \rho - \frac{w}{\sqrt{t}} \right) \right) &= \sqrt{t} \left( -P''(\rho) \left( -\frac{w}{\sqrt{t}} \right) + O \left( \frac{w^2}{t} \right) \right) \\ &= P''(\rho)w + O \left( \frac{1}{\sqrt{t}} \right) \end{aligned}$$

to simplify the argument of the second  $\chi$ .

By applying a change of variable in  $\theta, \theta'$ ,

$$\begin{aligned}\theta &\mapsto t^{-\frac{1}{2}}\theta - \epsilon\omega, \\ \theta' &\mapsto t^{-\frac{1}{2}}\theta' - \epsilon\omega,\end{aligned}$$

we can rewrite  $E_+(\epsilon t)$  as

$$\begin{aligned}&\frac{1}{(2\pi)^{2d}} \int_{\theta, \theta' \in \sqrt{t}(\epsilon\omega + \mathbb{S}^{d-1})} e^{i[\sqrt{t}P'(\rho)\rho\omega \cdot (\theta - \theta') + r\rho\omega \cdot (\theta - \theta') + wP'(\rho)\theta' \cdot \omega + t^{-\frac{1}{2}}rw\theta' \cdot \omega]} \\ &\quad \times e^{-\epsilon i[rw + \frac{1}{2}P''(\rho)w^2]} \chi\left(\frac{r\omega + \epsilon P'(\rho)\theta}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \\ &\quad \times \chi\left(\frac{(r + P''(\rho)w)\omega + \epsilon P'(\rho)\theta' - \epsilon t^{-\frac{1}{2}}P''(\rho)w\theta'}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \\ &\quad \times \mathbb{1}_{r>0} \mathbb{1}_{\rho > \frac{w}{\sqrt{t}} > 0} \hat{u}_0(t^{-\frac{1}{2}}\rho\theta - \epsilon\rho\omega) \overline{\hat{u}_0(t^{-\frac{1}{2}}\rho\theta' - \epsilon\rho\omega)} \\ &\quad \times (P'(\rho))^{d-1} \rho^{2(d-1)} d\theta d\theta' dr dw d\rho + O(t^{-\frac{1}{2}}),\end{aligned}$$

where the integrand is supported for  $0 < r, w \lesssim 1$ ,  $\rho \sim 1$ , and  $|\theta|, |\theta'| \lesssim 1$  due to (2.4.7) together with  $\delta + \frac{\sigma_1}{2} = 0$ , which allows us to do another simplification and write  $E_+(\epsilon t)$  as (2.4.6).  $\square$

Till now, we have managed to write  $E_+(\epsilon t)$ , up to some admissible terms, as (2.4.6), namely

$$\begin{aligned}&\frac{1}{(2\pi)^{2d}} \int_{\theta, \theta' \in \sqrt{t}(\epsilon\omega + \mathbb{S}^{d-1})} e^{i[\sqrt{t}P'(\rho)\rho\omega \cdot (\theta - \theta') + r\rho\omega \cdot (\theta - \theta') + wP'(\rho)\theta' \cdot \omega]} e^{-\epsilon i[rw + \frac{1}{2}P''(\rho)w^2]} \\ &\quad \times \chi\left(\frac{r\omega + \epsilon P'(\rho)\theta}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \chi\left(\frac{(r + P''(\rho)w)\omega + \epsilon P'(\rho)\theta'}{t^\delta \Lambda(t^{\frac{1}{2}}\rho)}\right) \\ &\quad \times \mathbb{1}_{r>0} \mathbb{1}_{\rho > \frac{w}{\sqrt{t}} > 0} |\hat{u}_0(-\epsilon\rho\omega)|^2 (P'(\rho))^{d-1} \rho^{2(d-1)} d\theta d\theta' dr dw d\rho.\end{aligned}$$

In the rest of this section, we will calculate the limit of this integral and conclude Proposition 2.4.1. Since the integral in  $\theta, \theta'$  is over a sphere centered at  $\epsilon\omega$  with radius  $\sqrt{t}$ , we can write  $\theta, \theta'$  in local coordinate

$$\begin{aligned}\theta &= h\epsilon\omega + y, \quad h = \sqrt{t} - \sqrt{t - |y|^2}, \\ \theta' &= h'\epsilon\omega + y', \quad h' = \sqrt{t} - \sqrt{t - |y'|^2},\end{aligned}$$

where  $h, h' \in \mathbb{R}$ ,  $y, y' \in \omega^\perp := \{z \in \mathbb{R}^d : z \cdot \omega = 0\}$ . The condition of support implies that  $|h|, |h'|, |y|, |y'| \lesssim 1$ . It is easy to check that, as  $t \rightarrow +\infty$

$$\sqrt{t}\omega \cdot (\theta - \theta') = \epsilon\sqrt{t} \left( \sqrt{t - |y|^2} - \sqrt{t - |y'|^2} \right) \rightarrow \epsilon \left( \frac{|y|^2}{2} - \frac{|y'|^2}{2} \right),$$

$$\theta \cdot \omega = \epsilon \left( \sqrt{t} - \sqrt{t - |y|^2} \right) \rightarrow 0,$$

$$\theta' \cdot \omega = \epsilon \left( \sqrt{t} - \sqrt{t - |y'|^2} \right) \rightarrow 0.$$

Therefore, by Dominated Convergence Theorem, as  $t$  tends to infinity, the limit of  $E_+(\epsilon t)$  equals to

$$\begin{aligned} & \frac{1}{(2\pi)^{2d}} \int_{y, y' \in \omega^\perp, r, w > 0} e^{i \left[ P'(\rho) \rho \left( \frac{|y|^2}{2} - \frac{|y'|^2}{2} \right) - r w - \frac{1}{2} P''(\rho) w^2 \right]} \\ & \quad \times \chi \left( \frac{r\omega + \epsilon P'(\rho) y}{\lambda_1 \rho^{\sigma_1}} \right) \chi \left( \frac{(r + P''(\rho) w) \omega + \epsilon P'(\rho) y'}{\lambda_1 \rho^{\sigma_1}} \right) \\ & \quad \times |\hat{u}_0(-\epsilon \rho \omega)|^2 (P'(\rho))^{d-1} \rho^{2(d-1)} dy dy' dr dw d\omega d\rho, \end{aligned}$$

which, after a change of variable, is equal to

$$\begin{aligned} & \frac{1}{(2\pi)^{2d}} \int_{y, y' \in \omega^\perp, r, w > 0} e^{i \left[ \left( \frac{|y|^2}{2} - \frac{|y'|^2}{2} \right) - r w - \frac{1}{2} w^2 \right]} \chi \left( \frac{\sqrt{P''(\rho)} r \omega + \epsilon \sqrt{\rho^{-1} P'(\rho)} y}{\lambda_1 \rho^{\sigma_1}} \right) \\ (2.4.8) \quad & \quad \times \chi \left( \frac{\sqrt{P''(\rho)} (r + w) \omega + \epsilon \sqrt{\rho^{-1} P'(\rho)} y'}{\lambda_1 \rho^{\sigma_1}} \right) \\ & \quad \times |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} dy dy' dr dw d\omega d\rho. \end{aligned}$$

In order to give a compact form, we introduce the following functions,

$$\begin{aligned} H(r, \omega) & := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{y \cdot \omega = 0} e^{i \frac{r^2 + |y|^2}{2}} \chi \left( \frac{\sqrt{P''(\rho)} r \omega + \epsilon \sqrt{\rho^{-1} P'(\rho)} y}{\lambda_1 \rho^{\sigma_1}} \right) dy, \\ F(r, \omega) & := \int_r^\infty H(s, \omega) ds, \end{aligned}$$

Remark that since  $\chi \in \mathcal{S}(\mathbb{R}^d)$ ,  $H$  decay rapidly at infinity, uniformly in  $\omega$ . Thus,  $F$  is well-defined. With these functions, we can rewrite the integral (2.4.8) as

$$\begin{aligned} & \frac{1}{(2\pi)^d} \int_0^\infty \int_0^\infty H(r, \omega) \int_0^\infty \overline{H(r+w, \omega)} dw dr |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} d\omega d\rho \\ & = - \frac{1}{(2\pi)^d} \int_0^\infty \int_0^\infty \partial_r F(r, \omega) \overline{F(r, \omega)} dr |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} d\omega d\rho. \end{aligned}$$

As a consequence,

$$\begin{aligned} \lim_{t \rightarrow \infty} E_{\chi, \delta, \Lambda}(u_0, \epsilon t) & = \lim_{t \rightarrow \infty} 2 \operatorname{Re} E_+(\epsilon t) \\ & = - \frac{2}{(2\pi)^d} \operatorname{Re} \int_0^\infty \int_0^\infty \partial_r F(r, \omega) \overline{F(r, \omega)} dr |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} d\omega d\rho \\ & = - \frac{1}{(2\pi)^d} \int_0^\infty \int_0^\infty \frac{\partial}{\partial r} |F|^2(r, \omega) dr |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} d\omega d\rho \\ & = \frac{1}{(2\pi)^d} \int |F(0, \omega)|^2 |\hat{u}_0(-\epsilon \rho \omega)|^2 \rho^{d-1} d\omega d\rho, \end{aligned}$$

where

$$|F(0, \omega)|^2 = \frac{1}{(2\pi)^d} \left| \int_0^\infty \int_{y \cdot \omega = 0} e^{\epsilon i \frac{r^2 + |y|^2}{2}} \chi \left( \frac{\sqrt{P''(\rho)} r \omega + \epsilon \sqrt{\rho^{-1} P'(\rho)} y}{\lambda_1 \rho^{\sigma_1}} \right) dy dr \right|^2,$$

which is exactly  $G_\chi^{\text{alt}}(\rho, \omega)$  defined in (2.1.26), or (2.1.15) with  $\sigma_1 = 0$  and  $\lambda_1 = 1$ . The limits (2.1.14), (2.1.21), and (2.1.25) thus follow.

## 2.5 . Study of Klein-Gordon equation

In this section, we shall prove Theorem 2.1.9 via a study of half-Klein-Gordon equation, i.e. (E) with  $P(\xi) = \langle \xi \rangle$ . Let  $w$  be the (real) solution to Klein-Gordon equation (KG). We have then

$$0 = (\partial_t^2 - \Delta + 1)w = - \left( \frac{\partial_t}{i} - P(D_x) \right) \left( \frac{\partial_t}{i} + P(D_x) \right) w.$$

Thus, the complex-valued function

$$u := \left( \frac{\partial_t}{i} + P(D_x) \right) w,$$

is the unique solution to half-Klein-Gordon equation with initial data

$$u_0 := u|_{t=0} = \frac{w_1}{i} + P(D_x)w_0 \in L^2.$$

Due to the fact that  $w$  is real-valued, we have the following relations

$$\partial_t w = -\text{Im } u, \quad w = \langle D_x \rangle^{-1} \text{Re } u.$$

In this section, we denote the truncation  $\text{Op}(a_\epsilon^{\text{KG}}(t))$ , whose symbol is defined in (2.1.31), as  $A(t)$ , and define operators  $A_\pm(t)$  as  $\text{Op}(a_\pm^{\text{KG}}(t))$ , where

$$(2.5.1) \quad a_\pm^{\text{KG}}(t) = \chi \left( \frac{x \pm tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{|x| > |tP'(\xi)|} \chi \left( \frac{\xi}{|t|^\epsilon} \right),$$

$0 < \delta < \frac{1}{2}$ ,  $0 < \epsilon < 1$ , and  $\chi \in C_c^\infty(\mathbb{R}^d)$  are the same as in (2.1.31). Since we are interested in the behavior as  $t \rightarrow +\infty$ , it is harmless to assume  $t \gg 1$ . Before continuing the proof, we clarify that all the involved operators are bounded uniformly in  $t \gg 1$ .

**Lemma 2.5.1.** *There exists time-independent constant  $C > 0$  and  $t_0 \gg 1$ , such that for all  $t > t_0$ ,*

$$\|A(t)\|_{\mathcal{L}(L^2)}, \|A_\pm(t)\|_{\mathcal{L}(L^2)} \leq C.$$

*Proof.* It is obvious that  $P(\xi) = \langle \xi \rangle$  satisfies hypothesis  $(H_{p_0, p_1})$  with  $p_0 = 1$ ,  $P_0 = 0$ , and  $p_1 = -2$ ,  $P_1 = 1$ . In what follows, we shall focus on symbols  $a_\pm^{\text{KG}}(t)$  (with  $a_\epsilon^{\text{KG}}(t)$  treated in the same way) and decompose them in high and low frequency. Let  $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$  be

a radial function which is equal to 1 in the ball  $B(0, 1)$  and vanishes outside a larger ball  $B(0, 2)$ . We may write

$$a_{\pm, l}^{\text{KG}}(t) := a_{\pm}^{\text{KG}}(t)\tilde{\chi}(\xi), \quad a_{\pm, h}^{\text{KG}}(t) := a_{\pm}^{\text{KG}}(t)(1 - \tilde{\chi})(\xi).$$

For low frequency part  $a_{\pm, l}^{\text{KG}}(t)$ , it is easy to construct a symbol  $P_l(\xi)$  such that  $P_l$  verifies hypothesis  $(H_{p_0, p_1})$  with  $p_0 = p_1 = 1$  and  $P_0 = P_1 = 0$ , and that  $P_l(\xi) = P(\xi)$  for all  $|\xi| \leq 2$ . In this way, we have

$$a_{\pm, l}^{\text{KG}}(t, x, \xi) = \chi\left(\frac{x \pm tP'_l(\xi)}{|t|^{\frac{1}{2} + \delta}}\right) \mathbb{1}_{|x| > |tP'(\xi)|} \tilde{\chi}(\xi),$$

where the factor involving  $t^\epsilon$  disappears since the symbol is supported for  $|\xi| \leq 2 \ll t^\epsilon$ , by choosing  $t \gg 1$ . Now, we may apply Proposition 2.2.1 to obtain the uniform-in- $t$  boundedness of operator with symbol

$$\chi\left(\frac{x \pm tP'_l(\xi)}{|t|^{\frac{1}{2} + \delta}}\right) \mathbb{1}_{|x| > |tP'(\xi)|}$$

and hence boundedness of the operator  $\text{Op}\left(a_{\pm, l}^{\text{KG}}(t)\right)$ .

The boundedness of  $\text{Op}\left(a_{\pm, h}^{\text{KG}}(t)\right)$  is actually an immediate consequence of Proposition 2.2.2. More precisely, by choosing arbitrary  $\epsilon'$  such that  $(\epsilon', \epsilon)$  satisfies conditions (2.1.19) associated to  $P$  (i.e. with  $p_0 = 1$  and  $p_1 = -2$ ), namely

$$0 < \epsilon' \leq \frac{1}{p_0 + 1} = \frac{1}{2}, \quad 0 < \epsilon \leq \frac{1}{-(p_1 + 1)} = 1,$$

we are able to apply Proposition 2.2.2 to obtain the uniform boundedness of operator with symbol

$$\chi\left(\frac{x \pm tP'(\xi)}{|t|^{\frac{1}{2} + \delta}}\right) \mathbb{1}_{|x| > |tP'(\xi)|} (1 - \chi)\left(\frac{\xi}{t^{-\epsilon'}}\right) \chi\left(\frac{\xi}{t^\epsilon}\right).$$

If we add the high frequency truncation  $(1 - \tilde{\chi})(\xi)$ , the symbol will be supported in  $|\xi| \geq 1 \gg t^{-\epsilon'}$  since we have chosen  $t \gg 1$ . As a result, the truncation  $(1 - \chi)$  equals 1 and the uniform boundedness of  $\text{Op}\left(a_{\pm, h}^{\text{KG}}(t)\right)$  follows.

We have shown that  $A_\pm(t) = \text{Op}\left(a_{\pm, l}^{\text{KG}}(t)\right) + \text{Op}\left(a_{\pm, h}^{\text{KG}}(t)\right)$  is uniformly bounded on  $L^2$ . By repeating the same argument and replacing Proposition 2.2.1, 2.2.2 by Theorem 2.1.8, we may also obtain the uniform boundedness of  $A(t)$ .  $\square$

Now we turn back to the proof of Theorem 2.1.9. By definition, the truncated energy (2.1.30) can be expressed as

$$E_\epsilon^{\text{KG}}(\pm t) = \|A(t) \text{Im } u(\pm t)\|_{L^2}^2 + \left\| A(t) \frac{D_x}{\langle D_x \rangle} \text{Re } u(\pm t) \right\|_{L^2}^2 + \left\| A(t) \frac{1}{\langle D_x \rangle} \text{Re } u(\pm t) \right\|_{L^2}^2.$$

The three terms on the right hand side takes the form

$$(2.5.2) \quad Q_{\pm}^{KG}(t, \epsilon_0, R) = \frac{1}{4} \left\| A(t) \left( Ru(\pm t) + \epsilon_0 \overline{Ru(\pm t)} \right) \right\|_{L^2}^2,$$

where  $\epsilon_0 \in \{+, -\}$  and  $R$  is a bounded Fourier multiplier taking values among  $1, D_x \langle D_x \rangle^{-1}$ , and  $\langle D_x \rangle^{-1}$ . Due to the uniform boundedness of truncation operators  $A(t), A_{\pm}(t)$ , (2.5.2) can be written as

$$\begin{aligned} & \frac{1}{4} \left\| A_{\pm}(t) Ru(\pm t) + \epsilon_0 A_{\mp}(t) \overline{Ru(\pm t)} \right\|_{L^2}^2 + O(\|(A(t) - A_{\pm}(t))Ru(\pm t)\|_{L^2}) \\ & + O\left(\|(A(t) - A_{\mp}(t))\overline{Ru(\pm t)}\|_{L^2}\right) \\ = & \frac{1}{4} \|A_{\pm}(t)Ru(\pm t)\|_{L^2}^2 + \frac{1}{4} \|A_{\mp}(t)\overline{Ru(\pm t)}\|_{L^2}^2 + \epsilon_0 \operatorname{Re} \left\langle A_{\pm}(t)Ru(\pm t), A_{\mp}(t)\overline{Ru(\pm t)} \right\rangle_{L^2} \\ & + O(\|(A(t) - A_{\pm}(t))Ru(\pm t)\|_{L^2}) + O\left(\|(A(t) - A_{\mp}(t))\overline{Ru(\pm t)}\|_{L^2}\right) \\ = & \frac{1}{2} \|A_{\pm}(t)Ru(\pm t)\|_{L^2}^2 + \epsilon_0 \operatorname{Re} \left\langle A_{\pm}(t)Ru(\pm t), A_{\mp}(t)\overline{Ru(\pm t)} \right\rangle_{L^2} \\ & + O(\|(A(t) - A_{\pm}(t))Ru(\pm t)\|_{L^2}), \end{aligned}$$

where we use the fact that for all complex-valued function  $f \in L^2$

$$A(t)\bar{f} = \overline{A(t)f}, \quad A_{\pm}(t)\bar{f} = \overline{A_{\mp}(t)f}.$$

In order to conclude the desired limit (2.1.30), it suffices to prove

**Proposition 2.5.2.** *Let  $v_0, v_{1,0}$  be two functions in  $L^2$ . With  $A(t), A_{\pm}(t)$  as above, we have following limits :*

$$(2.5.3) \quad \lim_{t \rightarrow +\infty} \left\| A_{\pm}(t) e^{\pm itP(D_x)} v_0 \right\|_{L^2}^2 = \frac{1}{4} \|v_0\|_{L^2}^2,$$

$$(2.5.4) \quad \lim_{t \rightarrow +\infty} \left\langle A_{\pm}(t) e^{\pm itP(D_x)} v_0, A_{\mp}(t) e^{\mp itP(D_x)} v_{0,1} \right\rangle_{L^2} = 0,$$

$$(2.5.5) \quad \lim_{t \rightarrow +\infty} \left\| (A(t) - A_{\pm}(t)) e^{\pm itP(D_x)} v_0 \right\|_{L^2}^2 = 0.$$

Once Proposition 2.5.2 is proved, we may apply the three limits with  $v_0 = Ru_0$  and  $v_{0,1} = \overline{Ru_0}$  to obtain that

$$\lim_{t \rightarrow +\infty} Q_{\pm}^{KG}(t, \epsilon_0, R) = \frac{1}{8} \|Ru_0\|_{L^2}^2,$$

since, due to the definition of  $u$ , we have, for all  $t \in \mathbb{R}$ ,

$$Ru(t) = e^{itP(D_x)} Ru_0, \quad \overline{Ru(t)} = e^{-itP(D_x)} \overline{Ru_0}.$$

As a result, the limit (2.1.30) follows from

$$\lim_{t \rightarrow \pm\infty} E_{\epsilon}^{KG}(w_0, w_1, t) = \lim_{t \rightarrow +\infty} E_{\epsilon}^{KG}(w_0, w_1, \pm t)$$

$$\begin{aligned}
&= \lim_{t \rightarrow +\infty} Q_{\pm}^{\text{KG}}(t, -1, 1) + Q_{\pm}^{\text{KG}}(t, -1, D_x \langle D_x \rangle^{-1}) + Q_{\pm}^{\text{KG}}(t, 1, \langle D_x \rangle^{-1}) \\
&= \frac{1}{8} \left( \|u_0\|_{L^2}^2 + \left\| \frac{D_x}{\langle D_x \rangle} u_0 \right\|_{L^2}^2 + \left\| \frac{1}{\langle D_x \rangle} u_0 \right\|_{L^2}^2 \right) = \frac{1}{4} \|u_0\|_{L^2}^2 = \frac{1}{4} (\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2).
\end{aligned}$$

In the rest of this section, we shall prove the limits (2.5.3), (2.5.4), and (2.5.5). We have seen that operators  $A(t)$ ,  $A_{\pm}(t)$  are bounded uniformly in  $t > t_0 \gg 1$ . As a result, it suffices to calculate these limits for those  $v_0, v_{0,1}$  belonging to some dense subspace of  $L^2$ . In what follows, we assume  $\hat{v}_0, \hat{v}_{0,1}$  are smooth and supported in a annulus centered at zero.

The first limit (2.5.3) is no more than a consequence of (2.1.22) with  $P(\xi) = \langle \xi \rangle$ ,  $\epsilon_1 = \epsilon$ ,  $\chi_h = \chi$ , and any  $\epsilon_0 \in ]0, \frac{1}{2}]$ . The exceptional truncation  $\chi_l$  is not a problem, since it disappears when  $|t|$  is large enough. For the remaining results (2.5.4) and (2.5.5), we will apply a similar argument as in the proof of Proposition 2.4.1 for super-critical case.

### 2.5.1 . Limit of interaction term

We first calculate the limit (2.5.4). Clearly, via conjugation, it is enough to study the limit of

$$(2.5.6) \quad \left\langle A_+(t) e^{itP(D_x)} v_0, A_-(t) e^{-itP(D_x)} v_{0,1} \right\rangle_{L^2},$$

since the other one can be recovered by relation

$$\begin{aligned}
\overline{\left\langle A_-(t) e^{-itP(D_x)} v_0, A_+(t) e^{itP(D_x)} v_{0,1} \right\rangle_{L^2}} &= \left\langle \overline{A_-(t) e^{-itP(D_x)} v_0}, \overline{A_+(t) e^{itP(D_x)} v_{0,1}} \right\rangle_{L^2} \\
&= \left\langle A_+(t) e^{itP(D_x)} \overline{v_0}, A_-(t) e^{-itP(D_x)} \overline{v_{0,1}} \right\rangle_{L^2},
\end{aligned}$$

where the Fourier transform of  $\overline{v_0}, \overline{v_{0,1}}$  still belongs to the class  $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ .

By definition (2.5.1) of  $A_{\pm}(t) = \text{Op}(a_{\pm}^{\text{KG}}(t))$ , (2.5.6) equals, in polar system, to

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}} \int e^{i(r\rho\omega\theta - r\rho'\omega'\theta')} e^{it(P(\rho) + P(\rho'))} \chi\left(\frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}}\right) \chi\left(\frac{r\omega - tP'(\rho')\theta'}{t^{\frac{1}{2} + \delta}}\right) \\
&\quad \times \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \hat{v}_0(\rho\theta) \overline{\hat{v}_{0,1}(\rho'\theta')}(r\rho\rho')^{d-1} d\theta d\theta' d\omega d\omega' d\rho d\rho'.
\end{aligned}$$

Here we may omit the truncation in  $\xi = \rho\theta$  and  $\xi' = \rho'\theta'$  by taking  $t \gg 1$ . As in previous section, we focus on the integral in  $\theta$  and  $\theta'$ , which are equal to, by Lemma 2.B.5,

$$\begin{aligned}
&\int e^{ir\rho\omega\theta} \chi\left(\frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}}\right) \overline{\hat{v}_0(\rho\theta)} d\theta \\
&\quad = e^{-ir\rho} \mu^{d-1} S_{-\frac{d-1}{2}}^+(\omega, \mu, \rho, \frac{r}{t} - P'(\rho), t; r\rho\mu^2) \kappa\left(\frac{r}{t}\right), \\
&\int e^{-ir\rho'\omega'\theta'} \chi\left(\frac{r\omega - tP'(\rho')\theta'}{t^{\frac{1}{2} + \delta}}\right) \hat{v}_{0,1}(\rho'\theta') d\theta' \\
&\quad = e^{-ir\rho'} \mu'^{d-1} S_{-\frac{d-1}{2}}^-(\omega, \mu, \rho', \frac{r}{t} - P'(\rho'), t; r\rho'\mu'^2) \kappa\left(\frac{r}{t}\right),
\end{aligned}$$

respectively, where  $\mu = t^{\delta - \frac{1}{2}}$  and  $S_m^\pm(\omega, \mu, \rho, r', t; \zeta)$  is supported for  $\zeta > c > 0$ ,  $\rho \sim 1$  and  $|r'| \lesssim \mu$  and satisfies for all  $\alpha \in \mathbb{N}^{d-1}$ ,  $j, k, l, \gamma \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\mu^j \partial_\rho^k \partial_{r'}^l \partial_\zeta^\gamma S_m| \leq C \mu^{-(|\alpha|+j+l)} \langle \zeta \rangle^{m-\gamma}.$$

Here we add extra factor  $\kappa \in C_c^\infty(]0, +\infty[)$ , which equals to 1 in a neighborhood of 1, due to the support of integrand.

As a consequence, (2.5.6) reads

$$\begin{aligned} & \frac{1}{(2\pi)^{2d}} \int e^{i[r(-\rho-\rho')-t(-P(\rho)-P(\rho'))]} S_0^+(\omega, \mu, \rho, \frac{r}{t} - P'(\rho), t; r\rho\mu^2) \\ & \quad \times S_0^-(\omega, \mu, \rho', \frac{r}{t} - P'(\rho'), t; r\rho'\mu^2) \kappa^2\left(\frac{r}{t}\right) \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \\ & \quad \times \hat{v}_0(\rho\theta) \overline{\hat{v}_0(\rho'\theta')} (\rho\rho')^{\frac{d-1}{2}} d\omega dr d\rho d\rho', \end{aligned}$$

which can be rewritten as  $I(t, -, -, -, -; F)$  defined in (2.4.5), with (2.5.7)

$$F(\rho, \rho', r, t; \zeta, \zeta') = S_0^+(\omega, t^{\delta - \frac{1}{2}}, \rho, \frac{\zeta}{t}, t; r\rho t^{2\delta-1}) S_0^-(\omega, t^{\delta - \frac{1}{2}}, \rho', \frac{\zeta'}{t}, t; r\rho' t^{2\delta-1}) \kappa^2\left(\frac{r}{t}\right).$$

In [Del22], the author has proved in Proposition 3.1.1 that

**Proposition 2.5.3.** *Let  $F(\rho, \rho', r, t; \zeta, \zeta')$  be a smooth function on  $\mathbb{R}_+^4 \times \mathbb{R}^2$  and  $\delta' \in ]\frac{1}{2}, 1[$ . Assume that  $F$  is supported for*

$$\rho, \rho' \sim 1, r \sim t, |\zeta|, |\zeta'| \lesssim t^{\delta'},$$

and for all  $j, j', k, \gamma, \gamma' \in \mathbb{N}$ ,

$$|\partial_\rho^j \partial_{\rho'}^{j'} \partial_r^k \partial_\zeta^\gamma \partial_{\zeta'}^{\gamma'} F(\rho, \rho', r, t; \zeta, \zeta')| \lesssim t^{-\delta'(k+\gamma+\gamma')}.$$

Under all the assumptions above, we have

$$\lim_{t \rightarrow +\infty} I(t, \pm, \pm, \pm, \pm; F) = 0.$$

It is easy to check that the function  $F$  defined in (2.5.7) satisfies the conditions above with  $\delta' = \delta + \frac{1}{2}$  and limit (2.5.4) follows.

## 2.5.2. Limit of energy outside the truncation area

It remains to prove (2.5.5), which requires a study of the  $L^2$ -norm of  $(A(t) - A_\pm(t))e^{\pm itP(D_x)}v_0$ . As in previous part, by conjugation, it suffices to focus on

$$(A(t) - A_+(t))e^{itP(D_x)}v_0(x) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{itP(\xi)} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \mathbb{1}_{|x| > |tP'(\xi)|} \hat{v}_0(\xi) d\xi,$$

where we omit again the truncation in  $\xi$  by assuming  $t \gg 1$ . Actually, we have

$$\overline{(A(t) - A_+(t))e^{itP(D_x)}v_0} = (A(t) - A_+(t))e^{itP(D_x)}\overline{v_0},$$

with  $\overline{v_0}$  belonging to the same subspace  $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ .

We first check that the  $L^2$ -norm of  $(A(t) - A_+(t))e^{itP(D_x)}v_0$  concentrates near  $|x| = t$ , i.e.

**Lemma 2.5.4.** *There exists a radial function  $\kappa \in C_c^\infty(\mathbb{R}^d)$  supported in an annulus centered at zero, such that, when  $t \rightarrow +\infty$ ,*

$$(A(t) - A_+(t))e^{itP(D_x)}v_0 = \kappa\left(\frac{x}{t}\right)(A(t) - A_+(t))e^{itP(D_x)}v_0 + O_{L^2}(t^{-N}),$$

for any  $N \in \mathbb{N}$ .

*Proof.* Let  $\kappa_0 \in C_c^\infty(\mathbb{R}^d)$  be a radial function supported near zero and equal to 1 near zero. By definition,

$$(A(t) - A_+(t))e^{itP(D_x)}v_0(x) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{itP(\xi)} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \mathbb{1}_{|x| > |tP'(\xi)|} \hat{v}_0(\xi) d\xi.$$

Due to the support of integrand, namely  $0 < c < |\xi| < C$  and  $|x| > tP'(|\xi|)$ , the function above is supported for  $|x| > tc_0$ , where  $0 < c_0 < P'(c)$ . That is to say, when  $\text{Supp } \kappa_0$  is chosen small enough,

$$\kappa_0\left(\frac{x}{t}\right)(A(t) - A_+(t))e^{itP(D_x)}v_0(x) = 0, \quad \forall t, x.$$

Let  $\kappa_1 \in C_c^\infty(\mathbb{R}^d)$  be a radial function supported outside the unit ball. We further assume that  $\kappa_1$  equals to 1 away from zero. By observing that  $|P'(\xi)| < 1$  for all  $\xi \in \mathbb{R}^d$ , we have

$$\kappa_1\left(\frac{x}{t}\right)(A(t) - A_+(t))e^{itP(D_x)}v_0(x) = \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d} \int e^{i(x\xi + tP(\xi))} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \hat{v}_0(\xi) d\xi.$$

Remark that the non-smooth term  $\mathbb{1}_{|x| > |tP'(\xi)|}$  is identically 1 as we add the cut-off  $\kappa_1$ . By integration by parts in  $\xi$ , we may rewrite the quantity above as

$$\begin{aligned} & \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d} \int e^{i(x\xi + tP(\xi))} \frac{-\partial_\xi}{i} \cdot \left[ \frac{x + tP'(\xi)}{|x + tP'(\xi)|^2} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \hat{v}_0(\xi) \right] d\xi \\ &= \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d} \int e^{i(x\xi + tP(\xi))} q_1(t, x, \xi) \hat{v}_0(\xi) d\xi + \frac{\kappa_1\left(\frac{x}{t}\right)}{(2\pi)^d} \int e^{i(x\xi + tP(\xi))} q_0(t, x, \xi) \cdot \hat{v}_1(\xi) d\xi, \end{aligned}$$

where  $v_1(x) = xv_0(x)$ , and

$$\begin{aligned} q_0(t, x, \xi) &= \frac{-\partial_\xi}{i} \left[ \frac{x + tP'(\xi)}{|x + tP'(\xi)|^2} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \right], \\ q_1(t, x, \xi) &= \frac{x + tP'(\xi)}{|x + tP'(\xi)|^2} (1 - \chi)\left(\frac{x + tP'(\xi)}{t^{\frac{1}{2} + \delta}}\right) \end{aligned}$$

are smooth symbols satisfying for all  $\alpha, \beta \in \mathbb{N}^d$ ,

$$\left| \partial_x^\alpha \partial_\xi^\beta q_k(t, x, \xi) \right| \lesssim t^{-2\delta} t^{-(\frac{1}{2} + \delta)|\alpha|} t^{(\frac{1}{2} - \delta)|\beta|}, \quad k = 0, 1.$$

By Calderón-Vaillancourt Theorem (Lemma 2.A.11) and Lemma 2.A.4, the  $\mathcal{L}(L^2)$ -norm of operators  $\text{Op}(q_k)$ 's is bounded by  $t^{-2\delta}$ , which implies that

$$\left\| \kappa_1\left(\frac{x}{t}\right)(A(t) - A_+(t))e^{itP(D_x)}v_0 \right\|_{L^2} \lesssim t^{-2\delta} (\|v_0\|_{L^2} + \|v_1\|_{L^2}) \sim t^{-2\delta} \|\langle x \rangle v_0\|_{L^2}.$$

By repeating this procedure for  $M$  times, we obtain

$$\left\| \kappa_1 \left( \frac{x}{t} \right) (A(t) - A_+(t)) e^{itP(D_x)} v_0 \right\|_{L^2} \lesssim t^{-2M\delta} \|\langle x \rangle^M v_0\|_{L^2} \lesssim t^{-2M\delta}.$$

The last estimate follows from the fact that  $v_0$  is a Schwartz function. The proof is completed by choosing  $\kappa = 1 - \kappa_0 - \kappa_1$ .  $\square$

Thanks to Lemma 2.5.4, it remains to estimate the  $L^2(dx)$ -norm of

$$\kappa \left( \frac{x}{t} \right) (A(t) - A_+(t)) e^{itP(D_x)} v_0(x),$$

or equivalently, in polar system, the  $L^2(r^{d-1} dr d\omega)$ -norm of  $I(t, r, \omega)$  defined by

$$(2.5.8) \quad I(t, r, \omega) = \kappa \left( \frac{r}{t} \right) \int e^{ir\rho\omega \cdot \theta} e^{itP(\rho)} (1 - \chi) \left( \frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{r > tP'(\rho)} \hat{v}_0(\rho\theta) \rho^{d-1} d\rho d\theta$$

To begin with, we decompose (2.5.8) into three parts  $I_+(t, r, \omega)$ ,  $I_-(t, r, \omega)$ ,  $I_0(t, r, \omega)$  by inserting

$$\chi_0 \left( \frac{\omega + \theta}{t^{\delta - \frac{1}{2}}} \right), \quad \chi_0 \left( \frac{\omega - \theta}{t^{\delta - \frac{1}{2}}} \right), \quad 1 - \chi_0 \left( \frac{\omega + \theta}{t^{\delta - \frac{1}{2}}} \right) - \chi_0 \left( \frac{\omega - \theta}{t^{\delta - \frac{1}{2}}} \right)$$

to the integral respectively, where  $\chi_0 \in C_c^\infty(\mathbb{R}^d)$  is radial, supported in a small ball centered at zero, and equal to 1 near zero. The desired result (2.5.5) thus follows from the lemma below :

**Lemma 2.5.5.** *For all  $N \in \mathbb{N}$ , there exists constants  $C, C_N$  which is independent of  $t, r, \omega$ , such that*

$$(2.5.9) \quad \|I_+(t, r, \omega)\|_{L^2(r^{d-1} dr d\omega)}^2 \leq C_N t^{-N},$$

$$(2.5.10) \quad \|I_-(t, r, \omega)\|_{L^2(r^{d-1} dr d\omega)}^2 \leq C t^{-1},$$

$$(2.5.11) \quad \|I_0(t, r, \omega)\|_{L^2(r^{d-1} dr d\omega)}^2 \leq C_N t^{-N}.$$

*Proof of (2.5.9).* The integral  $I_+(t, r, \omega)$ , by definition, reads

$$\kappa \left( \frac{r}{t} \right) \int e^{i[r\rho\omega \cdot \theta + tP(\rho)]} (1 - \chi) \left( \frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}} \right) \chi_0 \left( \frac{\omega + \theta}{t^{\delta - \frac{1}{2}}} \right) \mathbb{1}_{r > tP'(\rho)} \hat{v}_0(\rho\theta) \rho^{d-1} d\rho d\theta.$$

The intergand of  $I_+$  is supported for

$$|r - tP'(\rho)| = |-r\theta + tP'(\rho)\theta| \geq |r\omega + tP'(\rho)\theta| - r|\theta + \omega| \geq ct^{\frac{1}{2} + \delta} - Cc_0 t \times t^{\delta - \frac{1}{2}} = (c - Cc_0) t^{\frac{1}{2} + \delta},$$

where  $0 < c_0 \ll 1$  is the radius of  $\text{Supp } \chi_0$ . This implies that

$$|r - tP'(\rho)| \geq c' t^{\frac{1}{2} + \delta}.$$

As a consequence, the integrand of  $I_+$  is smooth. This allows us to apply integration by parts in  $\rho$ , since  $\rho \sim 1$  and

$$|r\omega \cdot \theta + tP'(\rho)| \geq |-r + tP'(\rho)| - r|\omega + \theta| \geq (c - 2Cc_0)t^{\frac{1}{2}+\delta} \geq c''t^{\frac{1}{2}+\delta}.$$

More precisely, by using

$$e^{i[r\rho\omega \cdot \theta + tP(\rho)]} = \frac{-i}{r\omega \cdot \theta + tP'(\rho)} \partial_\rho e^{i[r\rho\omega \cdot \theta + tP(\rho)]},$$

one may gain  $t^{-(\frac{1}{2}+\delta)}$  from  $|r\omega \cdot \theta + tP'(\rho)|^{-1}$  and  $t^{\frac{1}{2}-\delta}$  from each  $\partial_\rho$ . In conclusion, after  $M$  times integration by parts in  $\rho$  we have

$$|I_+(t, r, \omega)| \lesssim t^{-2M\delta}.$$

Due to factor  $\kappa$ ,  $I_+$  is supported for  $r \sim t$ , which implies that

$$\|I_+(t, r, \omega)\|_{L^2(r^{d-1}drd\omega)}^2 \lesssim t^{-4M\delta+d} \leq t^{-N},$$

where  $M$  is large enough so that  $4M\delta - d \geq N$ .  $\square$

*Proof of (2.5.10).* The integral  $I_-$  will be treated as in Section 2.4. We first observe that the integrand of  $I_-(t, r, \omega)$ , which reads

$$\kappa \left(\frac{r}{t}\right) \int e^{i[r\rho\omega \cdot \theta + tP(\rho)]} (1 - \chi) \left(\frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2}+\delta}}\right) \chi_0 \left(\frac{\omega - \theta}{t^{\delta-\frac{1}{2}}}\right) \mathbb{1}_{r > tP'(\rho)} \hat{v}_0(\rho\theta) \rho^{d-1} d\rho d\theta,$$

is supported for, as  $t \gg 1$ ,

$$|r\omega + tP'(\rho)\theta| \geq |r\theta + tP'(\rho)\theta| - r|\omega - \theta| \geq r + tP'(\rho) - Cc_0t^{\frac{1}{2}+\delta} \geq ct,$$

where  $0 < c_0 \ll 1$  is the radius of  $\text{Supp } \chi_0$ . That is to say, the factor  $(1 - \chi)$  is identically 1, and  $I_-$  becomes

$$\kappa \left(\frac{r}{t}\right) \int_{\rho_0}^{\min(\rho_1, (P')^{-1}(r/t))} e^{itP(\rho)} \int_{\mathbb{S}^d} e^{ir\rho\omega \cdot \theta} \chi_0 \left(\frac{\omega - \theta}{t^{\delta-\frac{1}{2}}}\right) \hat{v}_0(\rho\theta) d\theta \rho^{d-1} d\rho,$$

where  $\hat{v}_0(\rho\theta)$  is supported for  $\rho \in [\rho_0, \rho_1]$  and we use the convention  $(P')^{-1}(s) = +\infty$  when  $s \geq 1$ . Now, we may apply Lemma 2.B.1 with  $\lambda = r\rho$ ,  $\mu = t^{\delta-\frac{1}{2}}$ , and

$$F(x, y, z, \mu; \rho) = \chi_0 \left(\frac{y - z}{\mu}\right) \tilde{\chi}_0 \left(\frac{x - y}{\mu}\right) \hat{v}_0(\rho y).$$

Here  $\tilde{\chi}_0 \in C_c^\infty(\mathbb{R}^d)$  is chosen to be equal to 1 on the support of  $\chi_0$ . Note that  $F$  has to be taken at  $(x, y, z) = (\omega, \theta, \omega)$  in order to recover the above integral. This corresponds in the statement of Lemma 2.B.1 to  $x = \theta$ ,  $y = \theta'$ , and  $z = \omega$  (and no variable  $\omega'$ ). The extra term  $\rho$  is an extra parameter staying in a compact subset. Due to our choice of  $\mu$ , it is clear that  $r \sim t$  and  $\rho \sim 1$  imply  $\lambda\mu^2 \geq ct^{2\delta} \gg 1$ . As a result, the integral in  $\theta$  equals to

$$\int_{\mathbb{S}^d} e^{ir\rho\omega \cdot \theta} \chi_0 \left(\frac{\omega - \theta}{t^{\delta-\frac{1}{2}}}\right) \hat{v}_0(\rho\theta) d\theta = (2\pi)^{\frac{d-1}{2}} e^{ir\rho} \mu^{d-1} S_{-\frac{d-1}{2}}(\omega, \rho, \mu; r\rho\mu^2),$$

with  $S_m(\omega, \rho, \mu; \zeta)$  smooth, supported for  $\rho \sim 1, \zeta > 1$ , and satisfying for all  $\alpha, \in \mathbb{N}, j, k, \gamma \in \mathbb{N}$

$$|\partial_\omega^\alpha \partial_\rho^j \partial_\mu^k \partial_\zeta^\gamma S_m^\pm(\omega, \rho, \mu; \zeta)| \leq C \mu^{-|\alpha| - k} \langle \zeta \rangle^{m - \gamma}.$$

This formulation allows us to rewrite  $I_- r^{\frac{d-1}{2}}$ , up to multiplication with constants, as

$$I_-(t, r, \omega) r^{\frac{d-1}{2}} = \kappa \left( \frac{r}{t} \right) \int_{\rho_0}^{\min(\rho_1, (P')^{-1}(r/t))} e^{i[r\rho + tP(\rho)]} S_0(\omega, \rho, \mu; r\rho\mu^2) \rho^{\frac{d-1}{2}} d\rho.$$

Now, we may apply integration by parts in  $\rho$ . Due to the fact that  $r + tP'(\rho) \sim t$  and  $\min(\rho_1, (P')^{-1}(r/t)) \sim 1$ , the boundary terms and remaining term are all bounded by  $t^{-1}$ , namely

$$|I_-(t, r, \omega) r^{\frac{d-1}{2}}| \lesssim \mathbb{1}_{r \sim t} t^{-1},$$

which implies (2.5.10).  $\square$

*Proof of (2.5.11).* Unlike the study of  $I_\pm$ ,  $I_0$  will be estimated via integration by parts in  $\theta$  variable. It is clear that the integrand of

$$\begin{aligned} I_0(t, r, \omega) = & \kappa \left( \frac{r}{t} \right) \int e^{ir\rho\omega \cdot \theta} e^{itP(\rho)} (1 - \chi) \left( \frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{r > tP'(\rho)} \hat{v}_0(\rho\theta) \rho^{d-1} \\ & \times \left[ 1 - \chi_0 \left( \frac{\omega + \theta}{t^{\delta - \frac{1}{2}}} \right) - \chi_0 \left( \frac{\omega - \theta}{t^{\delta - \frac{1}{2}}} \right) \right] d\rho d\theta \end{aligned}$$

is supported away from  $\{\theta \pm \omega = 0\}$ , which allows us to rewrite the integral above in local coordinate

$$\theta = h\omega + \sqrt{1 - h^2}y, \quad h \in [-1, 1], y \in \omega^\perp \cap \mathbb{S}^{d-1},$$

where  $\omega^\perp$  is defined as the hyperplane  $\{y \in \mathbb{R}^d : y \cdot \omega = 0\}$ . In this way, we may rewrite  $I_0$  as

$$\begin{aligned} (2.5.12) \quad & \kappa \left( \frac{r}{t} \right) \int e^{ir\rho h} F_0(t, r, \rho, \omega, h\omega + \sqrt{1 - h^2}y) (1 - h^2)^{\frac{d}{2} - 1} \\ & \times \left[ 1 - \chi_0 \left( \frac{(h+1)\omega + \sqrt{1 - h^2}y}{t^{\delta - \frac{1}{2}}} \right) - \chi_0 \left( \frac{(h-1)\omega + \sqrt{1 - h^2}y}{t^{\delta - \frac{1}{2}}} \right) \right] dh dy d\rho, \end{aligned}$$

where

$$F_0(t, r, \rho, \omega, \theta) = e^{itP(\rho)} (1 - \chi) \left( \frac{r\omega + tP'(\rho)\theta}{t^{\frac{1}{2} + \delta}} \right) \mathbb{1}_{r > tP'(\rho)} \hat{v}_0(\rho\theta) \rho^{d-1}.$$

Remark that due to the cut-off away from  $\pm\omega$ , the integrand of (2.5.12) is supported for

$$|(h \pm 1)\omega + \sqrt{1 - h^2}y|^2 \geq ct^{2\delta - 1}.$$

By developing the inequality above we obtain

$$\sqrt{1 - h^2} \geq c't^{\delta - \frac{1}{2}}.$$

Thus,  $F_0$  is supported for  $\rho \sim 1$  and satisfies for all  $k \in \mathbb{N}$ ,

$$\left| \partial_h^k \left( F_0(t, r, \rho, \omega, h\omega + \sqrt{1 - h^2}y) \right) \right| \lesssim t^{(1-2\delta)k}.$$

The same estimates hold for  $(1 - h^2)^{\frac{d}{2}-1}$  and cut-off  $\chi_0$ 's in dimension  $d \geq 2$ , while in trivial case  $d = 1$ ,  $I_0$  is identically zero.

Now, we may apply  $M$  times integration by parts in  $h$  for (2.5.12). As  $r \sim t$ ,  $\rho \sim 1$ , each  $\partial_h$  in the amplitude gives  $t^{1-2\delta}$ , we have

$$|I_0(t, r, \omega)| \lesssim \mathbb{1}_{r \sim t} t^{-2M\delta},$$

and (2.5.11) follows from

$$\|I_0(t, r, \omega)\|_{L^2(r^{d-1} dr d\omega)}^2 \lesssim t^{-4M\delta+d} \lesssim t^{-N},$$

by choosing  $4M\delta - d \geq N$ . □

## 2.A . Technical lemmas

This appendix is a collection of technical lemmas which are used in previous sections.

### 2.A.1 . Some technical inequalities

**Lemma 2.A.1.** *For any real number  $m > d - 1$ , there exists constant  $C = C(m, d) > 0$ , such that*

$$\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \langle R\omega - \xi \rangle^{-m} d\omega \leq CR^{-(d-1)}.$$

*Proof.* By change of variable, it is equivalent to study the boundedness of

$$\int_{R\mathbb{S}^{d-1}} K(\omega - \xi) d\omega,$$

where  $K(x) = (1 + |x|^m)^{-1}$ . This integral can be regarded as a convolution of  $K$  and the Borel measure  $\mu_R$ , which is defined by

$$\forall \phi \in C_c(\mathbb{R}^d), \quad \langle \mu_R, \phi \rangle := \int_{R\mathbb{S}^{d-1}} \phi(\omega) d\omega.$$

We introduce the  $\alpha$ -dimensional density of a Borel measure  $\mu$  :

$$M_\mu^{(\alpha)}(x) := \sup_{r>0} \frac{\mu(B(x, r))}{r^\alpha}.$$

Remark that  $M_{\mu_R}^{(d-1)}(\xi) = M_{\mu_1}^{(d-1)}(\xi/R)$  and that  $M_{\mu_1}^{(d-1)}$  is a bounded function. Thus,  $M_{\mu_R}^{(d-1)}(\xi)$  is bounded uniformly in  $R > 0$  and  $\xi \in \mathbb{R}^d$ .

Now, it suffices to show that there exists some constant  $c = c(m, d)$ , such that for any Borel measure  $\mu$  on  $\mathbb{R}^d$ ,

$$K * \mu(\xi) \leq c M_\mu^{(d-1)}(\xi).$$

By applying a translation, the problem can be reduced to the case  $\xi = 0$ .

$$\begin{aligned} K * \mu(0) &= \int_{\mathbb{R}^d} K(-y) d\mu(y) \\ &= \int_{\mathbb{R}^d} K(y) d\mu(y) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mu(B(0, r_j^{(n)})), \end{aligned}$$

where that last equality follows from dominated convergence theorem, with  $r_j^{(n)}$  defined by  $K^{-1}(1 - \frac{j}{n}) = r_j^{(n)} \mathbb{S}^{d-1}$ . We may calculate  $r_j^{(n)}$  explicitly  $r_j^{(n)} = \left( \frac{j/n}{1-j/n} \right)^{\frac{1}{m}}$ . Therefore, by definition of  $(d-1)$ -dimensional density, we have

$$\begin{aligned} K * \mu(0) &\leq M_\mu^{(d-1)}(0) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left( r_j^{(n)} \right)^{d-1} \\ &= M_\mu^{(d-1)}(0) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left( \frac{j/n}{1-j/n} \right)^{\frac{d-1}{m}} \\ &= M_\mu^{(d-1)}(0) \int_0^1 \left( \frac{x}{1-x} \right)^{\frac{d-1}{m}} dx. \end{aligned}$$

The last quantity is finite since  $m > d - 1$ . □

**Lemma 2.A.2.** *Let  $S_{-1} \in C^\infty(\mathbb{R})$  with  $|S_{-1}^{(\alpha)}(\xi)| \leq C_\alpha \langle \xi \rangle^{-1-\alpha}$ , for any  $\alpha \in \mathbb{N}$ . Then, for any  $N \in \mathbb{N}$ , there exists some constant  $C$ , such that*

$$(2.A.1) \quad \left| \int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi) d\xi \right| \leq C \langle \lambda \rangle^{-N} (1 + \log_-(|\lambda|)), \quad \forall \lambda \neq 0,$$

where the integral on the left hand side should be understood as oscillatory integral and the function  $\log_-$  is defined by :

$$\log_-(t) := \begin{cases} |\log(t)| & \text{if } t \in ]0, 1[ \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* For simplicity, we may assume  $\lambda > 0$  and that  $S_{-1}$  is supported in  $[c_0, \infty[$  for some  $c_0 > 0$ .

When  $\lambda \geq c$  for some small constant  $c > 0$ , we may apply integration by part on  $\xi$ , then the left hand side of (2.A.1) can be controlled by

$$\frac{C}{\lambda} \int_{\mathbb{R}} \langle \xi \rangle^{-2} d\xi \lesssim \langle \lambda \rangle^{-1}.$$

To obtain arbitrary polynomial decrease for large  $\lambda$ , we only need to apply integration by parts several times.

When  $\lambda < c$ , we introduce a cut-off  $\chi \in C_c^\infty(\mathbb{R})$ , such that  $\chi = 1$  in a large neighbourhood of 0. For the part where  $|\lambda\xi|$  is small :

$$\left| \int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi) \chi(\lambda\xi) d\xi \right| \leq C \int_{c_1}^{\frac{c_2}{\lambda}} \frac{d\xi}{\xi} \leq C'(1 + \log_-(\lambda)).$$

For the remaining part where  $|\lambda\xi|$  is large, we need to estimate

$$\int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi) (1 - \chi)(\lambda\xi) d\xi.$$

By applying integration by parts in  $\xi$ , we will obtain two integrals. The first one is of the form

$$\int_{\mathbb{R}} e^{i\lambda\xi} S_{-1}(\xi) \chi'(\lambda\xi) d\xi,$$

which can be treated as above. The second one take the form

$$\frac{1}{\lambda} \int_{\mathbb{R}} e^{i\lambda\xi} S_{-2}(\xi) (1 - \chi)(\lambda\xi) d\xi,$$

which is bounded by

$$\frac{1}{\lambda} \int_{|\xi| > \frac{c'}{\lambda}} \langle \xi \rangle^{-2} d\xi \leq C.$$

□

**Lemma 2.A.3** (Cotlar-Stein Lemma). *Let  $H$  be a Hilbert space and  $\{T_j\}_{j \in \mathbb{N}}$  be a series of bounded operators on  $H$ . If*

$$(2.A.2) \quad A = \sup_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \|T_j T_k^*\|_{\mathcal{L}(H)}^{\frac{1}{2}} < +\infty,$$

$$(2.A.3) \quad B = \sup_{j \in \mathbb{N}} \sum_{k \in \mathbb{N}} \|T_j^* T_k\|_{\mathcal{L}(H)}^{\frac{1}{2}} < +\infty,$$

the operator  $T = \sum_{j \in \mathbb{N}} T_j$  is well-defined via point-wise limit

$$\forall u \in H, \quad Tu := \lim_{J \rightarrow +\infty} \sum_{j=0}^J T_j u \text{ in } H.$$

Moreover,  $T$  is bounded on  $H$  with estimate

$$(2.A.4) \quad \|T\|_{\mathcal{L}(H)} \leq \sqrt{AB}.$$

The inequality (2.A.4) was first given in [Cot56] for finite sum with  $\|T_j T_k^*\|_{\mathcal{L}(H)}$  and  $\|T_j^* T_k\|_{\mathcal{L}(H)}$  decreasing exponentially in  $|j - k|$ . The generalized version stated above and its proof can be found, for example, in Theorem 1, Chapter VII of [Ste93].

**2.A.2 . Criteria on  $L^2$ -boundedness of pseudo-differential operators**

**Lemma 2.A.4.** *Let  $a$  be a symbol on  $\mathbb{R}^d$  and  $\lambda > 0$ . The rescaled symbol*

$$a_\lambda(x, \xi) := a(\lambda x, \frac{\xi}{\lambda})$$

*satisfies*

$$\| \text{Op}(a_\lambda) \|_{\mathcal{L}(L^2)} = \| \text{Op}(a) \|_{\mathcal{L}(L^2)}.$$

**Lemma 2.A.5.** *Let  $a$  be a symbol on  $\mathbb{R}^d$  and symbol  $\tilde{a}$  is defined as*

$$\tilde{a}(x, \xi) := a(\xi, x).$$

*Then we have*

$$\| \text{Op}(a) \|_{\mathcal{L}(L^2)} = \| \text{Op}(\tilde{a}) \|_{\mathcal{L}(L^2)}.$$

**Lemma 2.A.6.** *Let  $K$  be a kernel function of operator  $\mathcal{K}$  defined as*

$$\mathcal{K}u(x) := \int K(x, x - y)u(y)dy.$$

*If there exists  $K_0 \in L^1(\mathbb{R}^d)$  such that*

$$|K(x, z)| \leq K_0(z), \forall x, z \in \mathbb{R}^d,$$

*we have*

$$\| \mathcal{K} \|_{\mathcal{L}(L^2)} \leq \| K_0 \|_{L^1}.$$

**Lemma 2.A.7.** *Let  $a$  and  $b$  be symbols satisfying*

$$|a(x, \xi)| \leq b(x, \xi),$$

*and  $B \in \mathcal{L}(L^2)$  be an operator defined by*

$$Bu(x) = \int b(x, \xi)v(\xi)d\xi.$$

*Then*

$$\| \text{Op}(a) \|_{\mathcal{L}(L^2)} \leq \| B \|_{\mathcal{L}(L^2)}.$$

**Lemma 2.A.8.** Let  $m : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \times ]0, \infty[ \rightarrow \mathbb{C}$  be a smooth function satisfying for all  $\alpha, \alpha' \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$

$$|\partial_\omega^\alpha \partial_{\omega'}^{\alpha'} m(\omega, \omega', \mu)| \leq C_{\alpha, \alpha', N} A \mu^{-|\alpha| - |\alpha'|} \left\langle \frac{d(\omega, -\omega')}{\mu} \right\rangle^{-N},$$

where  $A$  is a quantity independent of  $\omega$  and  $\omega'$ ,  $d$  is the distance on the sphere  $\mathbb{S}^{d-1}$ . For all  $\lambda > 0$ , the operator  $T_\lambda$  is defined by

$$T_\lambda u(\omega') = \int_{\mathbb{S}^{d-1}} e^{i\lambda\omega\omega'} m(\omega, \omega', \mu) u(\omega) d\omega.$$

Then there exists a constant  $C > 0$ , such that

$$\|T_\lambda\|_{\mathcal{L}(L^2(\mathbb{S}^{d-1}))} \leq C A \lambda^{-\frac{d-1}{2}}.$$

When  $\mu \in [1, \infty[$ , the condition on  $m$  becomes  $m(\cdot, \cdot, \mu) \in C_b^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$  uniformly in  $\mu \geq 1$ . By using the local coordinate of the unit sphere  $\mathbb{S}^{d-1}$  (which is compact and thus can be covered by finite local patches), one can reduce the problem to  $\mathbb{R}^{d-1}$  case without considering the parameter  $\mu$ . The proof of this reduced case can be found in, for example, [Sog17], Theorem 2.1.1. As for the case  $\mu \in ]0, 1]$ , one may refer to [Del22], Proposition A.1.7. A direct consequence is stated as following :

**Lemma 2.A.9.** Let  $a(x, \xi, \mu)$  be a smooth symbol on  $\mathbb{R}^d \times \mathbb{R}^d$  depending on a parameter  $\mu \in ]0, \infty[$ . Let  $b$  be a function on  $\mathbb{R}_+^2$ , such that the associated operator  $B$  defined below is bounded on  $L^2(\mathbb{R}_+)$ , namely

$$(2.A.5) \quad \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} < \infty, \quad Bf(r) = \int_0^\infty b(r, \rho) f(\rho) d\rho.$$

If, in polar system  $x = r\omega$ ,  $\xi = \rho\theta$ ,  $a(x, \xi, \mu)$  satisfies for all  $\alpha, \beta \in \mathbb{N}^{d-1}$  and  $N \in \mathbb{N}$ ,

$$(2.A.6) \quad |\partial_\omega^\alpha \partial_\theta^\beta a(r\omega, \rho\theta, \mu)| \leq C_{\alpha, \beta, N} b(r, \rho) \mu^{-|\alpha| - |\beta|} \left\langle \frac{d(\omega, -\theta)}{\mu} \right\rangle^{-N},$$

then we have

$$(2.A.7) \quad \|\text{Op}(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))},$$

where  $C > 0$  is a universal constant.

*Proof.* By definition of  $\text{Op}(a)$ , we have

$$\text{Op}(a)u(r\omega) = \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) \hat{u}(\rho\theta) \rho^{d-1} d\theta d\rho.$$

It is easy to check that  $\|f(\rho, \theta)\|_{L^2(d\rho d\theta)} = (2\pi)^{\frac{d}{2}} \|u\|_{L^2(\mathbb{R}^d)}$ , with  $f(\rho, \theta) = \hat{u}(\rho\theta) \rho^{\frac{d-1}{2}}$ . Thus,

$$\begin{aligned} \|\text{Op}(a)u\|_{L^2(\mathbb{R}^d)} &= \|\text{Op}(a)u(r\omega) r^{\frac{d-1}{2}}\|_{L^2(dr d\omega)} \\ &= \left\| \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta) (r\rho)^{\frac{d-1}{2}} d\theta d\rho \right\|_{L^2(dr d\omega)} \\ &\leq C_0 \left\| \int_0^\infty (r\rho)^{\frac{d-1}{2}} \left\| \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta) d\theta \right\|_{L^2(d\omega)} d\rho \right\|_{L^2(dr)}. \end{aligned}$$

By applying previous lemma with  $m(\omega, \theta, \mu) = a(r\omega, \rho\theta, \mu)$ , where  $r, \rho$  should be regarded as parameters,  $A = b(r, \rho)$ , and  $\lambda = r\rho$ , we obtain

$$\left\| \int_{\mathbb{S}^{d-1}} e^{ir\rho\omega\theta} a(r\omega, \rho\theta) f(\rho, \theta) d\theta \right\|_{L^2(d\omega)} \leq C_1 (r\rho)^{-\frac{d-1}{2}} b(r, \rho) \|f(\theta, \rho)\|_{L^2(d\theta)}.$$

Thus,

$$\begin{aligned} \|\text{Op}(a)u\|_{L^2(\mathbb{R}^d)} &\leq C_0 C_1 \left\| \int_0^\infty b(r, \rho) \|f(\theta, \rho)\|_{L^2(d\theta)} d\rho \right\|_{L^2(dr)} \\ &\leq C_0 C_1 \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|f(\theta, \rho)\|_{L^2(d\theta d\rho)} = C \|B\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \|u\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

□

**Lemma 2.A.10.** *Let  $a$  be a symbol on  $\mathbb{R}^d$ , depending on some parameter  $\lambda \in ]0, \infty[$ . If for all  $\alpha \in \mathbb{N}^d$ ,*

$$\sup_{x \in \mathbb{R}^d} \|\partial_\xi^\alpha a(x, \cdot)\|_{L_\xi^1} \leq C_\alpha \lambda^{d-|\alpha|},$$

*the operator  $\text{Op}(a)$  is bounded on  $L^2$ , uniformly in  $\lambda$ .*

*Proof.* Due to Lemma 2.A.4, the problem can be reduced to the case  $\lambda = 1$ . The kernel of operator  $\text{Op}(a)$  is  $K(x, y) = J(x, x - y)$ , where

$$J(x, z) = \frac{1}{(2\pi)^d} \int e^{iz \cdot \xi} a(x, \xi) d\xi.$$

We first observe that, for all  $x \in \mathbb{R}^d$ ,  $|J(x, z)| \leq (2\pi)^{-d} \|a(x, \cdot)\|_{L_\xi^1}$ , which is uniformly bounded. Then, by integration by part, we have, for all  $N \in \mathbb{N}$ ,

$$\begin{aligned} ||z|^{2N} J(x, z)| &\lesssim \sum_{|\alpha|=2N} \left| \int e^{iz \cdot \xi} \partial_\xi^\alpha a(x, \xi) d\xi \right| \\ &\lesssim \sum_{|\alpha|=2N} \sup_{x \in \mathbb{R}^d} \|\partial_\xi^\alpha a(x, \cdot)\|_{L_\xi^1} \leq C_\alpha. \end{aligned}$$

That is to say,  $J$  is bounded and has any polynomial decay in  $z$ , uniformly in  $x$ . In particular,

$$|K(x, y)| = |J(x, x - y)| \lesssim \langle x - y \rangle^{-(d+1)}.$$

The conclusion follows from Schur's Lemma. □

**Lemma 2.A.11** (Calderón-Vaillancourt Theorem). *For smooth symbol  $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ , the following estimate holds :*

$$(2.A.8) \quad \|\text{Op}(a)\|_{\mathcal{L}(L^2)} \lesssim \sup_{|\alpha|, |\beta| \leq N_d} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)},$$

*where  $N_d$  is a universal constant depending only on dimension  $d$ .*

The earliest version of (2.A.8) was given in [CV71] where  $\alpha_j, \beta_j$  are required to be no more than 3 for all  $j = 1, 2, \dots, d$ . Then, in [CM78], the authors optimized it to  $N_d = \lfloor \frac{d}{2} \rfloor + 1$ , while some other assumptions in  $\alpha, \beta$  are given in the same paper. The readers may also find an alternative proof via Gabor transform in [Hwa87].

## 2.B . Stationary phase lemmas

**Lemma 2.B.1.** *Let*

$$\begin{aligned} F : (\mathbb{S}^{d-1})^4 \times ]0, 1] &\rightarrow \mathbb{C} \\ (\theta, \theta', \omega, \omega', \mu) &\mapsto F(\theta, \theta', \omega, \omega', \mu) \end{aligned}$$

be a smooth function supported for  $d(\theta', \theta) + d(\theta', \omega') < \delta'$ , where  $d$  is the metric on sphere  $\mathbb{S}^{d-1}$  and  $\delta' > 0$  is a small constant. We assume that  $F$  satisfies for all  $\alpha, \alpha', \beta, \beta' \in \mathbb{N}, j, N \in \mathbb{N}$

$$|\partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j F| \leq C \mu^{-|\alpha| - |\alpha'| - |\beta| - j} \left\langle \frac{d(\theta', \omega)}{\mu} \right\rangle^{-N}.$$

For any parameter  $\lambda > 0$ , we define the integral

$$I_\pm(\theta, \omega, \omega'; \lambda, \mu) = \int_{\mathbb{S}^{d-1}} e^{\pm i \lambda \theta \theta'} F(\theta, \theta', \omega, \omega', \mu) d\theta'.$$

If  $\delta'$  is small enough (depending on the sphere  $\mathbb{S}^{d-1}$ ), and  $\lambda \mu^2 \geq c > 0$ , then we can write

$$I_\pm(\theta, \omega, \omega'; \lambda, \mu) = e^{\pm i \lambda \mu^{d-1} S_{-\frac{d-1}{2}}^\pm(\theta, \omega, \omega', \mu; \lambda \mu^2)},$$

where  $S_{-\frac{d-1}{2}}^\pm(\theta, \omega, \omega', \mu; \zeta)$  is a smooth function supported for  $d(\theta, \omega') \leq 2\delta'$ , satisfying for all  $\alpha, \beta, \beta' \in \mathbb{N}, j, \gamma, N \in \mathbb{N}$

$$(2.B.1) \quad |\partial_\theta^\alpha \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j \partial_\zeta^\gamma S_{-\frac{d-1}{2}}^\pm| \leq C \mu^{-|\alpha| - |\beta| - j} \langle \zeta \rangle^{-\frac{d-1}{2} - \gamma} \left\langle \frac{d(\theta, \omega)}{\mu} \right\rangle^{-N}.$$

Moreover, for  $|\zeta| \geq c > 0$ ,  $S_{-\frac{d-1}{2}}^\pm(\theta, \omega, \omega', \mu; \zeta)$  can be decomposed as

$$(2.B.2) \quad (2\pi)^{\frac{d-1}{2}} e^{\mp i \frac{\pi}{4}(d-1)} F(\theta, \theta, \omega, \omega', \mu) \zeta^{-\frac{d-1}{2}} + S_{-\frac{d+1}{2}}^\pm(\theta, \omega, \omega', \mu; \zeta),$$

where  $S_{-\frac{d+1}{2}}^\pm(\theta, \omega, \omega', \mu; \zeta)$  is smooth and supported for  $d(\theta, \omega') \leq 2\delta'$ , satisfying the estimate

(2.B.1) with  $d - 1$  replaced by  $d + 1$ .

**Lemma 2.B.2.** *Let*

$$\begin{aligned} F_0 : (\mathbb{S}^{d-1})^4 \times ]0, 1] &\rightarrow \mathbb{C} \\ (\theta, \theta', \omega, \omega', \mu) &\mapsto F(\theta, \theta', \omega, \omega', \mu) \end{aligned}$$

be a smooth function supported for  $\min(d(\theta', \theta), d(\theta', -\theta)) > \delta' > 0$ , where  $d$  is the metric on the sphere  $\mathbb{S}^{d-1}$  and  $\delta'$  is constant. We assume that  $F$  satisfies for all  $\alpha, \alpha', \beta, \beta' \in \mathbb{N}, j, N \in \mathbb{N}$

$$|\partial_\theta^\alpha \partial_{\theta'}^{\alpha'} \partial_\omega^\beta \partial_{\omega'}^{\beta'} \partial_\mu^j F_0| \leq C \mu^{-|\alpha| - |\alpha'| - |\beta| - j} \left\langle \frac{d(\theta', \omega)}{\mu} \right\rangle^{-N}.$$

For any parameter  $\lambda > 0$ , the integral

$$I_\pm(\theta, \omega, \omega'; \lambda, \mu) = \int_{\mathbb{S}^{d-1}} e^{\pm i \lambda \theta \theta'} F_0(\theta, \theta', \omega, \omega', \mu) d\theta'$$

can be written as

$$I_{\pm}(\theta, \omega, \omega'; \lambda, \mu) = e^{\pm i\lambda\theta\omega} \mu^{d-1} R^{\pm}(\theta, \omega, \omega', \mu; \lambda\mu),$$

where  $R^{\pm}(\theta, \omega, \omega', \mu; \zeta)$  is a smooth function satisfying for all  $\alpha, \beta, \beta' \in \mathbb{N}$ ,  $j, \gamma, N \in \mathbb{N}$

$$(2.B.3) \quad |\partial_{\theta}^{\alpha} \partial_{\omega}^{\beta} \partial_{\omega'}^{\beta'} \partial_{\mu}^j \partial_{\zeta}^{\gamma} R^{\pm}| \leq C \mu^{-|\alpha|-|\beta|-j} \langle \zeta \rangle^{-N}.$$

For the proof of Lemma 2.B.1 and 2.B.2, one may find a general result in [Del22], Proposition A.1.1.

**Remark 2.B.3.** In Lemma 2.B.1 and Lemma 2.B.2, the dependence on  $\omega'$  is not crucial in the proof, we may eliminate the conditions and results involving  $\omega'$ . Meanwhile, if  $F$  (resp.  $F_0$ ) depends on some other parameters, the same condition can be inherited by  $S_{-\frac{d-1}{2}}$  (resp.  $R$ ) from  $F$  (resp.  $F_0$ ).

Lemma 2.B.1 and 2.B.2, together with Remark 2.B.3, result in the following lemmas, which are important techniques used in the main text.

**Lemma 2.B.4.** Let

$$\begin{aligned} G : (\mathbb{S}^{d-1})^2 \times ]0, 1] &\rightarrow \mathbb{C} \\ (\theta', \omega, \mu) &\mapsto G(\theta', \omega, \mu) \end{aligned}$$

be a smooth function supported for  $d(\theta', \omega) < \delta'$ , where  $d$  is the metric on  $\mathbb{S}^{d-1}$  and  $\delta' > 0$  is a small constant. We further assume that, for all  $\alpha', \beta \in \mathbb{N}^{d-1}$ ,  $j, N \in \mathbb{N}$ ,

$$|\partial_{\theta'}^{\alpha'} \partial_{\omega}^{\beta} \partial_{\mu}^j G| \lesssim \mu^{-|\alpha'|-|\beta|-j} \left\langle \frac{d(\theta', \omega)}{\mu} \right\rangle^{-N}.$$

For  $\lambda > 0$ ,  $\epsilon \in \{\pm\}$ , we define

$$I(\theta, \omega, \mu; \lambda) := \int_{\mathbb{S}^{d-1}} e^{i\lambda\theta \cdot \theta'} G(\theta', \omega, \mu) d\theta'.$$

Then, under the conditions that  $\delta'$  is small enough and that  $\lambda\mu^2 > c > 0$  for some constant  $c$ , the integral  $I$  may be written as the sum of principal terms

$$e^{\pm i\lambda\theta \cdot \omega} \mu^{d-1} S_{-\frac{d-1}{2}}^{\pm}(\theta, \omega, \mu; \lambda\mu^2),$$

and a reminder

$$e^{i\lambda\theta \cdot \omega} \mu^{d-1} R(\theta, \omega, \mu; \lambda\mu),$$

where  $S_m^{\pm}(\theta, \omega, \mu; \zeta)$  is a smooth function supported on  $d(\theta, \pm\omega) < 2\delta'$ ,  $\zeta > c > 0$ , satisfying for all  $\alpha, \beta \in \mathbb{N}^{d-1}$ ,  $j, n, N \in \mathbb{N}$ ,

$$|\partial_{\theta}^{\alpha} \partial_{\omega}^{\beta} \partial_{\mu}^j \partial_{\zeta}^n S_m^{\pm}| \lesssim \mu^{-|\alpha|-|\beta|-j} \langle \zeta \rangle^{m-n} \left\langle \frac{d(\theta, \pm\omega)}{\mu} \right\rangle^{-N};$$

and  $R(\theta, \omega, \mu; \zeta)$  is a smooth function satisfying for all  $N \in \mathbb{N}$ ,

$$|R| \lesssim \langle \zeta \rangle^{-N}.$$

Moreover, if  $G$  depends on some extra parameters, the same bound for  $G$  (and its derivatives in parameters) can be inherited by  $S_{-\frac{d-1}{2}}^{\pm}$  and  $R$ .

*Proof.* We only give the proof for  $\epsilon = +$ , while the other one can be treated in the same way. To begin with, we introduce the functions

$$\begin{aligned} F_{\pm}(\theta, \theta', \omega, \mu) &:= G(\pm\theta', \pm\omega, \mu)\chi(\theta - \theta'), \\ F_0(\theta, \theta', \omega, \mu) &:= G(\theta', \omega, \mu) (1 - \chi(\theta - \theta') - \chi(\theta + \theta')), \end{aligned}$$

where  $\chi \in C_c^\infty(\mathbb{R}^d)$  is radial and supported in a small neighbor of zero, with value 1 near zero. By using these functions, we may rewrite the integral  $I$  into  $I_+ + I_- + I_0$  with

$$\begin{aligned} I_{\pm} &= \int e^{\pm i\lambda\theta \cdot \theta'} F_{\pm}(\theta, \theta', \pm\omega, \mu) d\theta', \\ I_0 &= \int e^{i\lambda\theta \cdot \theta'} F_0(\theta, \theta', \omega, \mu) d\theta'. \end{aligned}$$

It is clear that  $F_{\pm}, F_0$  verify the conditions in Lemma 2.B.1 and 2.B.2, respectively. Then, by applying these lemmas, we can obtain the desired expressions and corresponding estimates. There remain two points to check : the support of  $S_{\frac{d-1}{2}}^{\pm}$  and the dependence on extra parameters. The latter is merely an application of Remark 2.B.3, while the former can be shown by observing that, when  $\text{Supp } \chi$  is taken to be small enough, the integrand of integrals  $I_{\pm}$  is supported on  $d(\pm\theta', \omega) < \delta'$  and  $d(\theta, \theta') < \delta'$ .  $\square$

**Lemma 2.B.5.** *Let  $\chi \in C_c^\infty(\mathbb{R}^d)$  and  $f \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ . Assume that  $P$  is smooth on  $]0, \infty[$ , and  $\Lambda$  is a positive function, such that*

$$\begin{aligned} \Lambda(\rho) &\sim \rho^\sigma, \text{ as } \rho \rightarrow +\infty, \\ \left| \Lambda^{(j)}(\rho) \right| &\lesssim \rho^{\sigma-j}, \quad \forall \rho > \rho_0 > 0, j \in \mathbb{N}, \\ \frac{\Lambda(\rho)}{\rho^\sigma} &\rightarrow \lambda_0, \text{ as } \rho \rightarrow +\infty, \end{aligned}$$

hold for some  $\sigma \in \mathbb{R}$  and  $\lambda_0 > 0$ .

Then there exists  $t_0 \gg 1$  depending on  $f, P'$ , and  $\Lambda$ , such that for  $\epsilon, \epsilon' \in \{+1, -1\}$ ,  $\delta + \frac{\sigma}{2} \in [0, \frac{1}{2}[$ ,  $r > c > 0$ , and  $t > t_0$ , the integral

$$\int_{\mathbb{S}^{d-1}} e^{i\epsilon' r \rho \omega \theta} \chi \left( \frac{r\omega + \epsilon t P'(\rho)\theta}{t^{\frac{1}{2}+\delta} \Lambda(t^{\frac{1}{2}}\rho)} \right) f(\rho\theta) d\theta$$

can be decomposed as a principal term

$$(2\pi)^{\frac{d-1}{2}} e^{i\epsilon\epsilon' \frac{\pi}{4}(d-1)} e^{-i\epsilon\epsilon' r \rho} (r\rho)^{-\frac{d-1}{2}} \chi \left( \frac{r - t P'(\rho)}{t^{\frac{1}{2}+\delta} \Lambda(t^{\frac{1}{2}}\rho)} \right) f(-\epsilon\rho\omega)$$

and a remainder

$$e^{-i\epsilon\epsilon' r \rho} \mu^{d-1} S_{-\frac{d+1}{2}}(\omega, \mu, \rho, \frac{r}{t} - P'(\rho), t; r\rho\mu^2),$$

where  $\mu = t^{\delta + \frac{\sigma}{2} - \frac{1}{2}}$ ,  $S_m(\omega, \mu, \rho, r', t; \zeta)$  is supported for  $\zeta > c > 0$ ,  $\rho \sim 1$  and  $|r'| \lesssim \mu$  and satisfies for all  $\alpha \in \mathbb{N}^{d-1}$ ,  $j, k, l, \gamma \in \mathbb{N}$ ,

$$|\partial_\omega^\alpha \partial_\mu^j \partial_\rho^k \partial_{r'}^l \partial_\zeta^\gamma S_{-\frac{d+1}{2}}| \leq C \mu^{-(|\alpha|+j+l)} \langle \zeta \rangle^{m-\gamma}.$$

*Proof.* Using the notation  $r = t(r' + P'(\rho))$ , where  $|r'| \lesssim \mu = t^{\delta + \frac{\sigma}{2} - \frac{1}{2}}$ , we can rewrite the integral as

$$(2.B.4) \quad \int_{\mathbb{S}^{d-1}} e^{i\epsilon t(r' + P'(\rho))\rho\omega\theta} \chi \left( \frac{r'\omega + P'(\rho)(\omega + \epsilon\theta)}{t^{-\frac{1}{2} + \delta} \Lambda(t^{\frac{1}{2}}\rho)} \right) f(\rho\theta) d\theta.$$

Consider the function

$$F(x, y, z, \mu; \rho, r', t) = \chi \left( \frac{r'x + P'(\rho)(x - y)}{\mu t^{-\frac{\sigma}{2}} \Lambda(t^{\frac{1}{2}}\rho)} \right) \tilde{\chi} \left( \frac{x - y}{\mu} \right) \tilde{\chi} \left( \frac{y - z}{\mu} \right) f(\rho y),$$

with  $\tilde{\chi} \in C_c^\infty(\mathbb{R}^d)$  taking value 1 in a large ball centered at the origin. By setting  $\lambda = t(r' + P'(\rho))\rho > c > 0$ , we may rewrite the integral (2.B.4) as

$$\int_{\mathbb{S}^{d-1}} e^{-i\epsilon\epsilon'\lambda(-\epsilon\omega)\theta} F(-\epsilon\omega, \theta, -\epsilon\omega, \mu; \rho, r', t) d\theta.$$

Remark that the integrand in (2.B.4) ensures that  $|\omega + \epsilon\theta| \lesssim \mu \ll 1$ , which allows us to add  $\tilde{\chi}$  factors in the definition of  $F$ .

By using the fact that  $\rho \sim 1$ , it is easy to verify that when  $x, y, z \in \mathbb{S}^{d-1}$ , for all  $\alpha, \alpha', \beta \in \mathbb{N}^{d-1}$ ,  $j, k, l, N \in \mathbb{N}$ ,

$$|\partial_x^\alpha \partial_y^{\alpha'} \partial_z^\beta \partial_\mu^j \partial_\rho^k \partial_{r'}^l F(x, y, z, \mu; \rho, r')| \leq C \mu^{-|\alpha| - |\alpha'| - |\beta| - j - l} \left\langle \frac{d(y, z)}{\mu} \right\rangle^{-N},$$

and that  $F$  is supported for  $\rho \sim 1$ ,  $|r'| \lesssim \mu$ ,  $d(x, y) \leq C_0 \mu \ll 1$ , which ensures that  $\lambda \mu^2 = t^{2(\delta + \frac{\sigma}{2})} (r' + P'(\rho))\rho > c > 0$ . Therefore, the conclusion follows from Lemma 2.B.1 with extra parameters  $\rho, r', t$  but without  $\omega'$  variable.  $\square$

## 2.C . A refined result for Schrödinger equation

In the case of Schrödinger equation  $P(\xi) = \frac{|\xi|^2}{2}$ , the structure of  $P'(\xi) = \xi$  allows us to prove the parts (i) and (ii) of Theorem 2.1.1 for non-smooth  $\chi$ , namely

**Theorem 2.C.1.** *Let  $a_{\chi, \delta}$  and  $E_{\chi, \delta}$  be as defined in (2.1.10) and (2.1.11), respectively. We assume, as in Theorem 2.1.1, that  $\Lambda$  is identically equal to 1. Then, for any  $u_0 \in L^2$ , we have :*

(i) *if  $\chi \in L^1$  and  $\delta < 0$ ,*

$$(2.C.1) \quad \lim_{t \rightarrow +\infty} E_{\chi, \delta}(u_0, t) = \lim_{t \rightarrow +\infty} E_{\chi, \delta}(u_0, -t) = 0;$$

(ii) *if  $\chi \in L^1$  and  $\delta = 0$ ,*

$$(2.C.2) \quad \lim_{t \rightarrow +\infty} E_{\chi, \delta}(u_0, t) = \lim_{t \rightarrow +\infty} E_{\chi, \delta}(u_0, -t) = \frac{1}{(2\pi)^d} \int G_\chi(\omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega,$$

where  $(\rho, \omega)$  is polar coordinate, and function  $G_\chi(\omega)$  is defined as

$$(2.C.3) \quad G_\chi(\omega) := \frac{1}{(2\pi)^d} \left| \int_{x \cdot \omega > 0} e^{i \frac{|x|^2}{2}} \chi(x) dx \right|^2;$$

In order to prove the limit (2.C.1) and (2.C.2), we indicate that, compared with Theorem 2.1.1, the only difficulty is the loss of regularity in  $\chi$ , which can be overcome by finding a bound of operator  $\text{Op}(a(t))$  depending merely on  $\|\chi\|_{L^1}$ .

### 2.C.1 . An alternative bound of truncated operator

The goal of this subsection is to find a uniform bound of  $\text{Op}(a(t))$ , which involves less regularity of  $\chi$  than Proposition 2.2.1. Via Lemma 2.A.4, it suffices to study the symbol

$$b(x, \xi) = \chi\left(\frac{x + \xi}{\lambda}\right) \mathbb{1}_{|x| > |\xi|},$$

where  $\lambda = |t|^\delta \in ]0, \infty[$ . To be precise, we shall prove the following proposition

**Proposition 2.C.2.** *There exists constant  $C > 0$  independent of  $\lambda$ , such that, for all  $\lambda > 0$ ,*

$$\|\text{Op}(b)\|_{\mathcal{L}(L^2)} \leq C\lambda^d \|\chi\|_{L^1}.$$

*Proof of Proposition 2.C.2.* It is easy to calculate that

$$\begin{aligned} \int |b(x, \xi)| dx &\leq \int \left| \chi\left(\frac{x + \xi}{\lambda}\right) \right| dx = \lambda^d \|\chi\|_{L^1}, \\ \int |b(x, \xi)| d\xi &\leq \int \left| \chi\left(\frac{x + \xi}{\lambda}\right) \right| d\xi = \lambda^d \|\chi\|_{L^1}. \end{aligned}$$

Thus, the desired estimate follows from Schur's Lemma and Lemma 2.A.7.  $\square$

### 2.C.2 . Calculation of limit

By Proposition 2.C.2, when  $\delta < 0$ , we have

$$\|\text{Op}(a)\|_{\mathcal{L}(L^2)} \leq C t^{\delta d} \|\chi\|_{L^1} \rightarrow 0, \text{ as } t \rightarrow \infty,$$

which implies (2.C.1).

In the critical case  $\delta = 0$ , when  $\chi$  is smooth, compactly supported, and constant near zero, limit (2.C.2) follows from the limit (2.1.14). If  $\chi$  is no more than an  $L^1$  function, we may approximate  $\chi$  by some regular function in  $L^1$ . To be precise, for all  $n \in \mathbb{N}$ , there exists  $\chi_n \in C_c^\infty(\mathbb{R}^d)$  which are constant near zero, such that

$$\|\chi - \chi_n\|_{L^1} < \frac{1}{n}.$$

To highlight the dependence in  $\chi$ , in the rest of this section, we will add subscript  $\chi$  for concerning terms, for example,

$$a_\chi(t, x, \xi) = \chi\left(\frac{x + t\xi}{|t|^{\frac{1}{2}}}\right) \mathbb{1}_{|x| > |t||\xi|}.$$

Proposition 2.C.2 implies that, for all  $u_0 \in L^2$  and  $t \neq 0$ ,

$$\begin{aligned} \|\text{Op}(a_\chi) e^{itP(D_x)} u_0 - \text{Op}(a_{\chi_n}) e^{itP(D_x)} u_0\|_{L^2} &= \|\text{Op}(a_{\chi - \chi_n}) e^{itP(D_x)} u_0\|_{L^2} \\ &\leq C \|\chi - \chi_n\|_{L^1} \|e^{itP(D_x)} u_0\|_{L^2} \\ &< C \frac{1}{n} \|u_0\|_{L^2}. \end{aligned}$$

Therefore, for fixed  $u_0 \in L^2$ , the limit

$$\text{Op}(a_\chi) e^{itP(D_x)} u_0 \rightarrow \text{Op}(a_{\chi_n}) e^{itP(D_x)} u_0 \text{ in } L^2, \text{ as } n \rightarrow \infty$$

is uniform in  $t$ . We may conclude (2.C.2) by passing to the limit  $t \rightarrow \pm\infty$  :

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} E_{\chi, \delta}(u_0, t) &= \lim_{t \rightarrow \pm\infty} \|\text{Op}(a_\chi) e^{itP(D_x)} u_0\|_{L^2}^2 \\ &= \lim_{t \rightarrow \pm\infty} \lim_{n \rightarrow \infty} \|\text{Op}(a_{\chi_n}) e^{itP(D_x)} u_0\|_{L^2}^2 \\ &= \lim_{n \rightarrow \infty} \lim_{t \rightarrow \pm\infty} \|\text{Op}(a_{\chi_n}) e^{itP(D_x)} u_0\|_{L^2}^2 \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2\pi)^d} \int G_{\chi_n}(\omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega \\ &= \frac{1}{(2\pi)^d} \int G_\chi(\omega) |\hat{u}_0(\rho\omega)|^2 \rho^{d-1} d\rho d\omega \end{aligned}$$

The last equality follows from Dominated Convergence Theorem and the continuity of  $G$  on  $\chi \in L^1$ , which is obvious due to the definition (2.C.3) of  $G_\chi$ .

## 2.D . Partition of energy for Klein-Gordon equation : a classical setting

In this part, we shall give a proof of (2.1.32) via an alternative study of asymptotic behavior of solution to half-Klein-Gordon equation (HKG), namely

$$\begin{cases} \left( \frac{\partial_t}{i} - P(D_x) \right) u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where  $P(\xi) = \langle \xi \rangle$  is a smooth symbol. As in Section 2.5, instead of studying the solution  $w$  to Klein-Gordon equation (KG), we turn to

$$u = \left( \frac{\partial_t}{i} + P(D_x) \right) w,$$

which is a solution to half-Klein-Gordon equation with initial data

$$u_0 = u|_{t=0} = \frac{w_1}{i} + P(D_x)w_0 \in L^2.$$

Using this notation, we may rewrite the integral on the left hand side of (2.1.32) as

$$\begin{aligned}
& \int_{r_0 < |\frac{x}{t}| < r_1} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2) dx \\
&= \int \left( \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \partial_t w \right|^2 + \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \nabla w \right|^2 + \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} w \right|^2 \right) dx \\
&= \int \left( \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \operatorname{Im} u \right|^2 + \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} i \frac{D_x}{\langle D_x \rangle} \operatorname{Re} u \right|^2 + \left| \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \langle D_x \rangle^{-1} \operatorname{Re} u \right|^2 \right) dx \\
&= \int \left( \left| \operatorname{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} u \right|^2 + \left| \operatorname{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \frac{D_x}{\langle D_x \rangle} u \right|^2 + \left| \operatorname{Re} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \langle D_x \rangle^{-1} u \right|^2 \right) dx \\
&= \int \left( \left| \operatorname{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} e^{itP(D_x)} u_0 \right|^2 + \left| \operatorname{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} e^{itP(D_x)} \frac{D_x}{\langle D_x \rangle} u_0 \right|^2 \right. \\
&\quad \left. + \left| \operatorname{Re} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} e^{itP(D_x)} \langle D_x \rangle^{-1} u_0 \right|^2 \right) dx
\end{aligned}$$

Notice that the term on the right hand takes the form of

$$(2.D.1) \quad \int \left| \mathcal{A}g \left( \frac{x}{t} \right) v(t, x) \right|^2 dx,$$

where  $\mathcal{A} \in \{\operatorname{Re}, \operatorname{Im}\}$ ,  $g(y) = \mathbb{1}_{r_0 < |y| < r_1}$ , and  $v$  is a solution to half-Klein-Gordon equation, which can be written as  $v(t) = e^{itP(D_x)} v_0$ , with

$$v_0 = u_0, \frac{D_x}{\langle D_x \rangle} u_0, \langle D_x \rangle^{-1} u_0,$$

which all belong to  $L^2$ , since  $u_0 \in L^2$ . In order to calculate the limit of such quantity, we need to study the asymptotic behavior of  $v$ , which will be given in the next part.

### 2.D.1 . Asymptotic behavior

In this part, we shall state our problem in a general setting. Let  $u$  be the unique solution to (E) with initial data  $u_0 \in L^2$ . We assume the symbol  $P$  is smooth except at zero, which covers all the fractional-type equations. Since  $P'$  is well-defined except at zero, we may introduce  $\nu$  the push forward of Lebesgue measure under  $P'$ , i.e.

$$(2.D.2) \quad \nu(E) := \operatorname{Leb}(P'^{-1}(E)), \quad \forall E \subset \mathbb{R}^d \text{ measurable.}$$

For all function  $g \in L^\infty$ , we are interested in the following asymptotic formula :

$$(2.D.3) \quad g \left( \frac{x}{t} \right) u(t) = g(-P'(D_x))u(t) + o_{L^2}(1), \quad \text{as } t \rightarrow \pm\infty.$$

We denote by  $\mathcal{G}$  the collection of those function  $g$  satisfying this formula for all  $u_0 \in L^2$ , namely

$$\begin{aligned}
\mathcal{G} &:= \{g \in L^\infty : (2.D.3) \text{ holds for all } u_0 \in L^2\} \\
&:= \{g \in L^\infty : (2.D.3) \text{ holds for all } u_0 \text{ with } \hat{u}_0 \in C_c^\infty(\mathbb{R}^d)\}.
\end{aligned}$$

The two definitions given above are equivalent since the multiplication with  $g(x/t)$  and Fourier multiplier  $g(-P'(D_x))$  are both bounded on  $L^2$  uniformly in time. The equivalence then follows from the fact that the subspace  $\mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \setminus \{0\})$  is dense in  $L^2$ , where  $\mathcal{F}$  is the Fourier transform.

In order to prove (2.1.32), we may use the following lemma

**Lemma 2.D.1.** *Let  $E \subset \mathbb{R}^d$  be any measurable set whose boundary has null Lebesgue measure, namely*

$$\text{Leb}(\partial E) = 0.$$

*If the measure  $\nu$  defined in (2.D.2) is absolutely continuous w.r.t. Lebesgue measure, we have*

$$\mathbb{1}_E \in \mathcal{G}.$$

We assume this lemma is true and prove it later. It is easy to check that the assumptions in Lemma 2.D.1 hold true for  $P(\xi) = \langle \xi \rangle$  and  $E = \{r_0 < |y| < r_1\}$ . As a result, (2.D.1) can be written as

$$\begin{aligned} \int \left| \mathcal{A} \mathbb{1}_E \left( \frac{x}{t} \right) v(t, x) \right|^2 dx &= \int \left| \mathcal{A} \mathbb{1}_E (-P'(D_x)) v(t, x) \right|^2 dx + o(1) \\ &= \int \left| \mathcal{A} \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) v(t, x) \right|^2 dx + o(1), \end{aligned}$$

when  $t \rightarrow \pm\infty$ . Recall that  $] \rho_0, \rho_1[ = P'^{-1}(]r_0, r_1[)$ . Actually, this formula implies (2.1.32), since as  $t \rightarrow \pm\infty$ , we have

$$\begin{aligned} & \int_{r_0 < |\frac{x}{t}| < r_1} (|\partial_t w|^2 + |\nabla w|^2 + |w|^2) dx \\ &= \int \left( \left| \text{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} u(t, x) \right|^2 + \left| \text{Im} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \frac{D_x}{\langle D_x \rangle} u(t, x) \right|^2 \right. \\ & \quad \left. + \left| \text{Re} \mathbb{1}_{r_0 < |\frac{x}{t}| < r_1} \langle D_x \rangle^{-1} u(t, x) \right|^2 \right) dx \\ &= \int \left( \left| \text{Im} \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) u(t, x) \right|^2 + \left| \text{Im} \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) \frac{D_x}{\langle D_x \rangle} u(t, x) \right|^2 \right. \\ & \quad \left. + \left| \text{Re} \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) \langle D_x \rangle^{-1} u(t, x) \right|^2 \right) dx + o(1) \\ &= \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) \partial_t w \right\|_{L^2}^2 + \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) \nabla w \right\|_{L^2}^2 + \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) w \right\|_{L^2}^2 + o(1) \\ &= \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) \partial_t w \right\|_{L^2}^2 + \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) w \right\|_{H^1}^2 + o(1) \\ &= \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) w_1 \right\|_{H^1}^2 + \left\| \mathbb{1}_{] \rho_0, \rho_1[} (|D_x|) w_0 \right\|_{L^2}^2 + o(1). \end{aligned}$$

In order to complete the proof of (2.1.32), we give now the proof of Lemma 2.D.1.

*Proof of Lemma 2.D.1.* Without loss of generality, we may assume in what follows that  $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$ . Now, we fix  $\chi \in C_c^\infty(\mathbb{R}^d)$  which equals to 1 in a ball centered at zero and consider the symbol

$$(2.D.4) \quad a_0(t, x, \xi) = \chi \left( \frac{x + tP'(\xi)}{|t|^{\frac{1}{2} + \delta}} \right),$$

where  $\delta \in ]0, \frac{1}{2}[$ . We have seen that Lemma 2.A.5 and 2.A.10 imply the uniform-in- $t$  boundedness of  $\text{Op}(a_0(t))$ . Moreover, by integration by parts, we can prove that for all  $u_0 \in L^2$

$$\|u(t) - \text{Op}(a_0(t))u(t)\|_{L^2} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

By writing

$$\begin{aligned} E_t &:= E + B(0, ct^{-\frac{1}{2} + \delta}), \\ \tilde{E}_t &:= E^c + B(0, ct^{-\frac{1}{2} + \delta}), \end{aligned}$$

where  $c > 0$  is a constant determined by  $\text{Supp } \chi$ , we may apply the uniform-in-time  $L^2$ -boundedness of multiplication with  $\mathbb{1}_E(x/t)$  and Fourier multiplier  $\mathbb{1}_E(-P'(D_x))$  to obtain that, as  $t \rightarrow \pm\infty$ ,

$$\begin{aligned} & \mathbb{1}_E \left( \frac{x}{t} \right) u(t, x) - \mathbb{1}_E(-P'(D_x))u(t, x) \\ &= \mathbb{1}_E \left( \frac{x}{t} \right) \text{Op}(a_0(t))u(t, x) - \text{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t, x) + o_{L^2}(1) \\ &= \mathbb{1}_E \left( \frac{x}{t} \right) \text{Op}(a_0(t))\mathbb{1}_{E_t}(-P'(D_x))u(t, x) - \text{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t, x) + o_{L^2}(1) \\ &= \mathbb{1}_E \left( \frac{x}{t} \right) \text{Op}(a_0(t))\mathbb{1}_{E_t \setminus E}(-P'(D_x))u(t, x) \\ & \quad - \mathbb{1}_{E^c} \left( \frac{x}{t} \right) \text{Op}(a_0(t))\mathbb{1}_E(-P'(D_x))u(t, x) + o_{L^2}(1) \\ &= \mathbb{1}_E \left( \frac{x}{t} \right) \text{Op}(a_0(t))\mathbb{1}_{E_t \setminus E}(-P'(D_x))u(t, x) \\ & \quad - \mathbb{1}_{E^c} \left( \frac{x}{t} \right) \text{Op}(a_0(t))\mathbb{1}_{E \cap \tilde{E}_t}(-P'(D_x))u(t, x) + o_{L^2}(1) \end{aligned}$$

Note that in the calculation above, it is possible to add extra cut-off in  $-P'(D_x)$  since the symbol  $a_0(t)$  is supported for

$$\left| \frac{x}{t} - (-P'(\xi)) \right| \leq ct^{-\frac{1}{2} + \delta}.$$

As a consequence, the uniform boundedness of multiplication with  $\mathbb{1}_E(x/t)$ ,  $\mathbb{1}_{E^c}$  and  $\text{Op}(a_0(t))$  implies that

$$\begin{aligned} & \limsup_{t \rightarrow +\infty} \left\| \mathbb{1}_E \left( \frac{x}{t} \right) u(t) - \mathbb{1}_E(-P'(D_x))u(t) \right\|_{L^2} \\ & \lesssim \limsup_{t \rightarrow +\infty} \left( \left\| \mathbb{1}_{E_t \setminus E}(-P'(D_x))u(t) \right\|_{L^2} + \left\| \mathbb{1}_{E \cap \tilde{E}_t}(-P'(D_x))u(t) \right\|_{L^2} \right) \\ & = (2\pi)^{-\frac{d}{2}} \limsup_{t \rightarrow +\infty} \left( \left\| \mathbb{1}_{E_t \setminus E}(-P')\hat{u}_0 \right\|_{L^2} + \left\| \mathbb{1}_{E \cap \tilde{E}_t}(-P')\hat{u}_0 \right\|_{L^2} \right) \quad (\text{Plancherel Theorem}) \end{aligned}$$

$$\begin{aligned}
 &= (2\pi)^{-\frac{d}{2}} \left( \left\| \mathbb{1}_{\cap_{t>0} E_t \setminus E} (-P') \hat{u}_0 \right\|_{L^2} + \left\| \mathbb{1}_{\cap_{t>0} E \cap \tilde{E}_t} (-P') \hat{u}_0 \right\|_{L^2} \right) \\
 &\lesssim \left\| \mathbb{1}_{\partial E} (-P') \hat{u}_0 \right\|_{L^2}
 \end{aligned} \tag{DCT}$$

The last quantity is actually zero since  $\partial E$  has zero Lebesgue measure and  $\nu$  is absolutely continuous w.r.t. Lebesgue measure, which ensures that  $\mathbb{1}_{\partial E}(-P'(\xi))$  vanishes almost everywhere.  $\square$

## 2.D.2 . Further remarks on $\mathcal{G}$

In the proof of (2.1.32), we only study the function

$$g(y) = \mathbb{1}_{r_0 < |y| < r_1}.$$

A natural question is whether (2.1.32) holds true for other function  $g$ , or equivalently, which type of function is contained in class  $\mathcal{G}$ . In this part, we shall indicate an error in a classical result and prove that  $\mathcal{G}$  contains at least the continuous functions.

The asymptotic formula (2.D.3) is, to our knowledge, firstly studied in [Str81], where the author claimed in Corollary 2.2 of [Str81] that, when  $\nu$  is absolutely continuous w.r.t Lebesgue measure,

$$(2.D.5) \quad L^\infty = \mathcal{G}.$$

However, the original proof given in [Str81] is false. Actually, the author managed to prove in Theorem 2.1 that for general  $\nu$ ,

$$\{\text{Fourier transform of finite measure}\} \subset \mathcal{G},$$

and reduce the conjecture to

$$(2.D.6) \quad \mathbb{1}_E \in \mathcal{G}, \text{ for bounded and measurable } E \subset \mathbb{R}^d.$$

The method used in [Str81] is that, by regularity of Lebesgue measure, we may choose a series of compact sets  $\{K_n\}_{n \in \mathbb{N}}$  and bounded open sets  $\{U_n\}_{n \in \mathbb{N}}$ , such that for all  $n \in \mathbb{N}$

$$K_n \subset K_{n+1}, \quad U_{n+1} \subset U_n, \quad K_n \subset E \subset U_n;$$

$$\lim_{n \rightarrow +\infty} \text{Leb}(U_n \setminus K_n) = 0.$$

Then it is easy to find smooth functions  $g_n \in C_c^\infty(U_n)$  which are non-negative, range in  $[0, 1]$ , and equal to 1 on  $K_n$ . Clearly,  $\mathbb{1}_E - g_n$  is supported in  $U_n \setminus K_n$ , whose measure tends to zero. By Dominated Convergence Theorem and the fact that  $0 \leq \mathbb{1}_E - g_n \leq 1$ , we have

$$\limsup_{t \rightarrow \pm\infty} \left\| (\mathbb{1}_E - g_n)(-P'(D_x))u \right\|_{L^2}^2 = (2\pi)^{-d} \left\| (\mathbb{1}_E - g_n)(-P'(\xi))\hat{u}_0(\xi) \right\|_{L_\xi^2}^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Remark that, to apply Dominated Convergence Theorem, we need  $(\mathbb{1}_E - g_n)(-P'(\xi))$  converges to zero almost everywhere, which is a consequence of absolute continuity of  $\nu$  w.r.t Lebesgue measure. Therefore, in order to prove (2.D.6), it suffices to check that

$$(2.D.7) \quad \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L_x^2}^2 \xrightarrow{n \rightarrow +\infty} 0.$$

Once it holds true, one has, for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} & \limsup_{t \rightarrow \pm\infty} \|\mathbb{1}_E(x/t)u - \mathbb{1}_E(-P'(D_x))u\|_{L_x^2} \\ & \leq \limsup_{t \rightarrow \pm\infty} \|g_n(x/t)u - g_n(-P'(D_x))u\|_{L_x^2} + \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L_x^2} \\ & \quad + \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(-P'(D_x))u\|_{L^2} \\ & = \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u\|_{L_x^2} + \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(-P'(D_x))u\|_{L^2}, \end{aligned}$$

since  $g_n$  is obviously the Fourier transform of some finite measure. The two quantities on the right hand side vanish as  $n$  tends to infinity, and the desired conclusion  $\mathbb{1}_E \in \mathcal{G}$  follows.

The proof of (2.D.7) given in [Str81] is to find  $h_n \in \mathcal{G}$  such that

$$C \geq h_n \geq \mathbb{1}_{O_n} \text{ and } h_n \rightarrow 0, \text{ a.e.,}$$

where  $O_n = U_n \setminus K_n$  is an open set. If such  $h_n$  exists, one obtains immediately that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \limsup_{t \rightarrow \pm\infty} \|(\mathbb{1}_E - g_n)(x/t)u(t)\|_{L^2}^2 & \leq \lim_{n \rightarrow +\infty} \limsup_{t \rightarrow \pm\infty} \|h_n(x/t)u(t)\|_{L^2}^2 \\ & \leq \lim_{n \rightarrow +\infty} \limsup_{t \rightarrow \pm\infty} \|h_n(-P'(D_x))u(t)\|_{L^2}^2 \\ & = \lim_{n \rightarrow +\infty} (2\pi)^{-d} \int h_n^2(-\nabla P(\xi)) |\hat{u}_0(\xi)|^2 d\xi = 0. \end{aligned}$$

Note that the second inequality is a consequence of  $h_n \in \mathcal{G}$  and the last equality follows from Dominated Convergence Theorem.

Since we only know that  $\mathcal{G}$  contains a subset of continuous function (Fourier transform of finite measures), it is essential to assume  $h_n$ 's to be continuous. However, for general decreasing bounded open sets  $O_n$ , even if their measures decrease to zero, such continuous  $h_n$ 's do NOT exist. Otherwise, it is harmless to assume  $h_n$ 's are supported in the same large ball. Then, for one thing by Dominated Convergence Theorem,

$$\lim_{n \rightarrow +\infty} \int h_n(y) dy = 0;$$

and for another thing, we have

$$\int h_n(y) dy \geq \int_{O_n} h_n(y) dy = \text{Leb}(\overline{O_n}) \geq 0,$$

where  $\overline{O_n}$  is the closure of open set  $O_n$ . Here the first inequality is due to the continuity of  $h_n$ 's. As a result,

$$\lim_{n \rightarrow +\infty} \text{Leb}(\overline{O_n}) = 0.$$

The contradiction arises from the fact that  $\text{Leb}(O_n)$  tends to zero does not imply that  $\text{Leb}(\overline{O_n})$  tends to zero. For example, Let  $\{r_j\}_{j \in \mathbb{N}}$  be a numeration of rational numbers in the unit ball  $B = B(0, 1)$  of  $\mathbb{R}^d$  centered at zero. Consider the series of open sets

$$O_n := \cup_{j \in \mathbb{N}} B(r_j, 2^{-j-n}).$$

Clearly  $O_n$ 's are open as the union of open sets and

$$\text{Leb}(O_n) \leq \sum_{j \in \mathbb{N}} \text{Leb}(B(r_j, 2^{-j-n})) \lesssim \sum_{j \in \mathbb{N}} 2^{-(j+n)d} \sim 2^{-nd} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Since  $\{r_j\}_{j \in \mathbb{N}}$  is dense in  $B = B(0, 1)$ , the closure of each  $O_n$  contains at least the unit ball  $B$ . As a consequence,

$$\lim_{n \rightarrow +\infty} \text{Leb}(\overline{O_n}) \geq \lim_{n \rightarrow +\infty} \text{Leb}(B) = C_d > 0.$$

We emphasize that the argument above does not falsify (2.D.5) and it is still unknown whether this conjecture is true. Here we shall prove rigorously that all bounded continuous functions belong to  $\mathcal{G}$ .

**Proposition 2.D.2.** *If the measure  $\nu$  defined in (2.D.2) is absolutely continuous w.r.t. Lebesgue measure, we have*

$$C_b^0(\mathbb{R}^d) \subset \mathcal{G}.$$

*Proof.* As in the proof of Lemma 2.D.1, we assume that  $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$ . To begin with, we check that the Schwartz class  $\mathcal{S} \subset \mathcal{G}$ . For any  $g \in \mathcal{S}$ ,

$$\begin{aligned} & \left( g\left(\frac{x}{t}\right) - g(-P'(D_x)) \right) u(t, x) \\ &= \frac{1}{(2\pi)^d} \int e^{i(x \cdot \xi + tP(\xi))} \left( g\left(\frac{x}{t}\right) - g(-P'(\xi)) \right) \hat{u}_0(\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int e^{i(x \cdot \xi + tP(\xi))} \left( \frac{x}{t} + P'(\xi) \right) \int_0^1 g' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right) d\tau \hat{u}_0(\xi) d\xi \\ &= \frac{i}{(2\pi)^d t} \int e^{i(x \cdot \xi + tP(\xi))} \partial_\xi \left[ \int_0^1 g' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right) d\tau \hat{u}_0(\xi) \right] d\xi \\ &= \frac{i}{(2\pi)^d t} \int e^{i(x \cdot \xi + tP(\xi))} \int_0^1 g'' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right) (\tau - 1) d\tau P''(\xi) \hat{u}_0(\xi) d\xi \\ &+ \frac{i}{(2\pi)^d t} \int e^{i(x \cdot \xi + tP(\xi))} \int_0^1 g' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right) d\tau \partial_\xi \hat{u}_0(\xi) d\xi. \end{aligned}$$

Notice that, by Lemma 2.A.5 and 2.A.10, the operator of symbol

$$g'' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right), \quad g' \left( \frac{\tau}{t} x - (1-\tau)P'(\xi) \right)$$

is bounded uniformly in  $t$  and  $\tau$  and that functions

$$P''(\xi)\hat{u}_0(\xi), \quad \partial_\xi \hat{u}_0(\xi)$$

belong to  $L^2$ , since we have assumed  $\hat{u}_0 \in C_c^\infty(\mathbb{R}^d)$ . As a consequence,

$$\left\| \left( g\left(\frac{x}{t}\right) - g(-P'(D_x)) \right) u(t, x) \right\|_{L_x^2} \lesssim |t|^{-1} \xrightarrow{t \rightarrow \pm\infty} 0.$$

By noticing that  $\mathcal{G}$  is closed under  $L^\infty$ -norm, we have

$$C_0^0 := \{g \in C^0 : \lim_{|y| \rightarrow +\infty} g(y) = 0\} = \bar{\mathcal{S}} \subset \mathcal{G},$$

where  $\bar{\mathcal{S}}$  is the closure of  $\mathcal{S}$  w.r.t.  $L^\infty$ -norm.

It remains to pass to general continuous function  $g$ . In fact, we only need to consider those  $g \geq 0$ , since once may always write  $g$  as the difference of two non-negative continuous functions, which are both bounded. Let us fix  $\chi \in C_c^\infty$  which equals to 1 near zero and define, for all  $R > 0$ ,

$$\chi_R(y) := \chi\left(\frac{y}{R}\right).$$

For arbitrary  $R > 0$ , we have

$$\begin{aligned} & \limsup_{t \rightarrow \pm\infty} \left\| \left( g(x/t) - g(-P'(D_x)) \right) u(t) \right\|_{L^2} \\ & \leq \limsup_{t \rightarrow \pm\infty} \left\| \left( (g\chi_R)(x/t) - (g\chi_R)(-P'(D_x)) \right) u(t) \right\|_{L^2} \\ & \quad + \limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(x/t)u(t) \right\|_{L^2} + \limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(-P'(D_x))u(t) \right\|_{L^2} \\ & = \limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(x/t)u(t) \right\|_{L^2} + \limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(-P'(D_x))u(t) \right\|_{L^2}, \end{aligned}$$

since  $g\chi_R \in C_0^0$ . The second term on the right hand side can be calculated as follows

$$\limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(-P'(D_x))u(t) \right\|_{L^2} = (2\pi)^{-\frac{d}{2}} \left\| (g(1 - \chi_R))(-P')\hat{u}_0 \right\|_{L^2},$$

which, due to Dominated Convergence Theorem and the absolute continuity of  $\nu$  w.r.t. Lebesgue measure, converges to zero as  $R \rightarrow +\infty$ . As for the cut-off in  $x$ , we observe that,

$$\limsup_{t \rightarrow \pm\infty} \left\| (g(1 - \chi_R))(x/t)u(t) \right\|_{L^2} \leq \|g\|_{L^\infty} \limsup_{t \rightarrow \pm\infty} \left\| (1 - \chi_R)(x/t)u(t) \right\|_{L^2}.$$

Note that since  $\chi_R \in \mathcal{S} \subset \mathcal{G}$ ,  $(1 - \chi_R)(x/t)u(t)$  can be written, when  $t \rightarrow \pm\infty$ , as

$$u(t) - \chi_R(-P'(D_x))u(t) + o_{L^2}(1).$$

As a result,

$$\limsup_{t \rightarrow \pm\infty} \left\| (1 - \chi_R)(x/t)u(t) \right\|_{L^2} = \limsup_{t \rightarrow \pm\infty} \left\| (1 - \chi_R)(-P'(D_x))u(t) \right\|_{L^2}$$

$$=(2\pi)^{-\frac{d}{2}} \|(1 - \chi_R)\hat{u}_0\|_{L^2}.$$

We have seen that the last quantity tends to zero as  $R \rightarrow +\infty$ . In conclusion, we have proved that, for all  $u_0 \in \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \setminus \{0\})$ ,

$$\lim_{R \rightarrow +\infty} \limsup_{t \rightarrow \pm\infty} \|(g(x/t) - g(-P'(D_x)))u(t)\|_{L^2} = 0,$$

and thus  $g \in \mathcal{G}$ . □

### 2.D.3 . Proof of (2.D.5) for dispersive system

Before ending this section, we give a proof of (2.D.5) for dispersive system. To be precise, we assume that there exists some dense subspace  $\mathcal{D}_0 \subset L^2$ , such that

$$(2.D.8) \quad \|e^{itP(D_x)}u_0\|_{L^\infty} \leq C(u_0)|t|^{-\frac{d}{2}}, \quad \forall u_0 \in \mathcal{D}_0, \quad \forall |t| > 1,$$

where  $C(u_0)$  is a constant depending on  $u_0$ .

**Theorem 2.D.3.** *Assume that  $P$  is smooth except at zero and  $\nu$  defined in (2.D.2) is absolutely continuous w.r.t. Lebesgue measure. Then the dispersion estimate (2.D.8), associated with some dense subspace  $\mathcal{D}_0 \subset L^2$ , implies (2.D.5).*

*Proof.* Recall that, as mentioned in previous section, it has been proved in [Str81] that, when  $\nu$  is absolutely continuous w.r.t. Lebesgue measure, (2.D.5) is equivalent to (2.D.6). Therefore, it suffices to check that, for all  $u_0 \in \mathcal{D}_0$ ,

$$\left\| \mathbb{1}_E \left( \frac{x}{t} \right) u(t) - \mathbb{1}_E(D_x)u(t) \right\|_{L^2} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty,$$

where  $E$  is any bounded measurable set.

In Lemma 2.D.1, we have proved this result for those  $E$  whose boundary has zero Lebesgue measure. The idea of treatment of general  $E$  is to approximate it by finite union of cubes and control the remaining part via dispersion estimate. To begin with, we fix an arbitrarily small  $\epsilon > 0$ . By outer regularity of Lebesgue measure, there exists open set  $\tilde{E}_\epsilon \supset E$ , such that

$$\text{Leb}(\tilde{E}_\epsilon \setminus E) < \frac{\epsilon}{2}.$$

Since any open set can be expressed as the union of almost disjoint closed cubes, we may find finitely many closed cubes  $\{K_j\}_{j=1}^N$ , such that  $K_j \subset \tilde{E}_\epsilon$  and

$$\text{Leb}(\tilde{E}_\epsilon \setminus E_\epsilon) < \frac{\epsilon}{2}, \quad \text{where } E_\epsilon = \bigcup_{j=1}^N K_j.$$

One may observe that

$$\text{Leb}(E \setminus E_\epsilon \sqcup E_\epsilon \setminus E) < \epsilon.$$

Now, for any  $u_0 \in \mathcal{D}_0$ , we have

$$\left\| \mathbb{1}_E \left( \frac{x}{t} \right) u(t) - \mathbb{1}_E(D_x)u(t) \right\|_{L^2}$$

$$\begin{aligned}
&\leq \left\| (\mathbb{1}_{E_\epsilon} - \mathbb{1}_E) \left( \frac{x}{t} \right) u(t) \right\|_{L^2} + \left\| \mathbb{1}_{E_\epsilon} \left( \frac{x}{t} \right) u(t) - \mathbb{1}_{E_\epsilon} (D_x) u(t) \right\|_{L^2} \\
&\quad + \left\| (\mathbb{1}_{E_\epsilon} - \mathbb{1}_E) (D_x) u(t) \right\|_{L^2} \\
&\leq C(u_0) \left\| (\mathbb{1}_{E_\epsilon} - \mathbb{1}_E) \left( \frac{x}{t} \right) \right\|_{L^2} |t|^{-\frac{d}{2}} + \left\| \mathbb{1}_{E_\epsilon} \left( \frac{x}{t} \right) u(t) - \mathbb{1}_{E_\epsilon} (D_x) u(t) \right\|_{L^2} \\
&\quad + (2\pi)^{-\frac{d}{2}} \left\| (\mathbb{1}_{E_\epsilon} - \mathbb{1}_E) \hat{u}_0 \right\|_{L^2} \\
&\leq C(u_0) \text{Leb}(E \setminus E_\epsilon \sqcup E_\epsilon \setminus E)^{\frac{1}{2}} + \left\| \mathbb{1}_{E_\epsilon} \left( \frac{x}{t} \right) u(t) - \mathbb{1}_{E_\epsilon} (D_x) u(t) \right\|_{L^2} \\
&\quad + (2\pi)^{-\frac{d}{2}} \left\| \mathbb{1}_{E \setminus E_\epsilon \sqcup E_\epsilon \setminus E}(\xi) \hat{u}_0(\xi) \right\|_{L^2_\xi} \\
&< C(u_0) \epsilon^{\frac{1}{2}} + \left\| \mathbb{1}_{E_\epsilon} \left( \frac{x}{t} \right) u(t) - \mathbb{1}_{E_\epsilon} (D_x) u(t) \right\|_{L^2} + (2\pi)^{-\frac{d}{2}} \left\| \mathbb{1}_{E \setminus E_\epsilon \sqcup E_\epsilon \setminus E}(\xi) \hat{u}_0(\xi) \right\|_{L^2_\xi}.
\end{aligned}$$

Due to the fact that  $E_\epsilon$  is the union of finitely many closed cubes, the boundary of  $E_\epsilon$  has zero Lebesgue measure. As a result, the second term on the right hand side tends to zero as  $t \rightarrow \pm\infty$ , i.e.

$$\limsup_{t \rightarrow \pm\infty} \left\| \mathbb{1}_{E_\epsilon} \left( \frac{x}{t} \right) u(t) - \mathbb{1}_{E_\epsilon} (D_x) u(t) \right\|_{L^2} \lesssim \epsilon^{\frac{1}{2}} + \left\| \mathbb{1}_{E \setminus E_\epsilon \sqcup E_\epsilon \setminus E}(\xi) \hat{u}_0(\xi) \right\|_{L^2_\xi}.$$

Since  $\hat{u}_0 \in L^2$  and  $\text{Leb}(E \setminus E_\epsilon \sqcup E_\epsilon \setminus E) < \epsilon$ , the right hand side becomes arbitrarily small, if  $\epsilon > 0$  is taken small enough. The desired result thus follows.  $\square$

The dispersion estimate (2.D.8) holds for all symbols  $P$  we deal with in the present chapter, as a consequence of stationary phase lemma.

**Proposition 2.D.4.** *Let  $P$  be radial and smooth except at zero. If  $P'' > 0$  or  $P'' < 0$ , the dispersion estimate (2.D.8) holds with  $\mathcal{D}_0 = \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \setminus \{0\})$ .*

*Proof.* Without loss of generality, we assume that  $t > 1$ . Let  $u_0$  be any function in  $\mathcal{D}_0 = \mathcal{F}^{-1}C_c^\infty(\mathbb{R}^d \setminus \{0\})$ . To prove the inequality (2.D.8), it is equivalent by definition to prove that

$$\left\| \int e^{i(x \cdot \xi + tP(\xi))} \hat{u}_0(\xi) d\xi \right\|_{L^\infty(dx)} \lesssim t^{-\frac{d}{2}}.$$

Since  $P$  is radial, we may write the inequality above in polar system  $\xi = \rho\theta$  as

$$\left\| \int e^{it \left( \frac{x \cdot \theta}{t} \rho + P(\rho) \right)} \hat{u}_0(\rho\theta) \rho^{d-1} d\rho d\theta \right\|_{L^\infty(dx)} \lesssim t^{-\frac{d}{2}}.$$

By denoting  $s = x \cdot \theta / t \in \mathbb{R}$ , it suffices to prove that

$$\sup_{s \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}} \left| \int e^{it(s\rho + P(\rho))} \hat{u}_0(\rho\theta) \rho^{d-1} d\rho \right| \lesssim t^{-\frac{d}{2}}.$$

Since  $\rho$  stays between two positive constants and  $P'' \neq 0$  on  $]0, +\infty[$ , this inequality follows from stationary phase lemma.  $\square$

**Corollary 2.D.5.** (2.D.5) holds for all  $P$  satisfying condition  $(H_{p_0, p_1})$  with  $p_0, p_1 \neq 0$ .

## 2.E . Proof of Proposition 2.4.3

In [Del22], the author has proved Proposition 2.4.3 for strictly convex  $P$ . In this part, we will explain how the same argument works for strictly concave  $P$  and how to calculate the limit for  $\epsilon = \pm$  respectively. Since most of calculations has been done in Section 3 of [Del22], we will omit these details.

By definition  $I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F)$  equals to

$$\int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{\frac{r}{t} > P'(\rho), P'(\rho')} \\ \times F(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')) dr d\rho d\rho',$$

which can be split into  $I_+$  and  $I_-$ , with domain of integral  $\rho - \rho' > 0$  and  $\rho - \rho' < 0$ , respectively. Namely,

$$I_+ = \int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{r > tP'(\rho')} \mathbb{1}_{\rho - \rho' > 0} \\ \times F(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')) dr d\rho d\rho', \\ I_- = \int e^{i\epsilon[r(-\rho+\rho')-t(-P(\rho)+P(\rho'))]} \mathbb{1}_{r > tP'(\rho')} \mathbb{1}_{\rho - \rho' < 0} \\ \times F(\rho, \rho', r, t; r - tP'(\rho), r - tP'(\rho')) dr d\rho d\rho'.$$

In what follows, we study mainly integral  $I_+$  with  $I_-$  manipulated in the same way. By change of variable  $r \rightarrow tr + tP'(\rho')$ ,  $\rho' \rightarrow \rho - w$ ,  $I_+$  reads

$$t \int e^{i\epsilon t[-(r+P'(\rho-w))w - (-P(\rho)+P(\rho-w))]} \mathbb{1}_{r > 0} \mathbb{1}_{w > 0} \\ \times F(\rho, \rho - w, t(r + P'(\rho - w)), t; tr - tP'(\rho) + tP'(\rho - w), tr) dr d\rho dw.$$

We introduce the following notation :

$$P(\rho') - P(\rho) = P'(\rho)(\rho' - \rho) + g(\rho, \rho')(\rho' - \rho)^2,$$

where  $g$  is strictly negative since  $P$  is concave;

$$\tilde{F}(\rho, \rho', r, t; \zeta, \zeta') = F(\rho, \rho', tr, t; t\zeta, t\zeta'),$$

which is smooth, supported for

$$\rho, \rho', r \sim 1, |\zeta|, |\zeta'| \lesssim t^{\delta'-1},$$

and satisfies for all  $j, j', k, \gamma, \gamma' \in \mathbb{N}$ ,

$$|\partial_\rho^j \partial_{\rho'}^{j'} \partial_r^k \partial_\zeta^\gamma \partial_{\zeta'}^{\gamma'} \tilde{F}(\rho, \rho', r, t; \zeta, \zeta')| \lesssim t^{(1-\delta')(k+\gamma+\gamma')}.$$

Moreover, we have the point-wise limit

$$\lim_{t \rightarrow \infty} \tilde{F}(\rho, \rho', \frac{r}{\sqrt{t}} + P'(\rho'), t; \frac{\zeta}{\sqrt{t}}, \frac{\zeta'}{\sqrt{t}}) = F_0(\rho, \rho').$$

With these notations,  $I_+$  can be expressed as

$$t \int e^{-i\epsilon t[rw - g(\rho - w, \rho)w^2]} \mathbb{1}_{r>0} \mathbb{1}_{w>0} \\ \times \tilde{F}(\rho, \rho - w, r + P'(\rho - w), t; r - P'(\rho) + P'(\rho - w), r) dr d\rho dw.$$

The same calculus as in Lemma 3.1.4 of [Del22] shows that, up to some terms tending to zero,  $I_+$  equals to

$$t \int e^{-i\epsilon trw} \mathbb{1}_{r>0} e^{i\epsilon tg(\rho - w, \rho)w^2} \mathbb{1}_{w>0} \tilde{F}(\rho, \rho - w, r + P'(\rho), t; r, r) dr d\rho dw \\ = \int e^{-i\epsilon rw} \mathbb{1}_{r>0} e^{i\epsilon g(\rho - \frac{w}{\sqrt{t}}, \rho)w^2} \mathbb{1}_{w>0} \tilde{F}(\rho, \rho - \frac{w}{\sqrt{t}}, \frac{r}{\sqrt{t}} + P'(\rho), t; \frac{r}{\sqrt{t}}, \frac{r}{\sqrt{t}}) dr d\rho dw,$$

whose formal limit by taking point-wise limit of integrand is

$$\int e^{-i\epsilon rw} \mathbb{1}_{r>0} e^{i\epsilon g(\rho, \rho)w^2} \mathbb{1}_{w>0} F_0(\rho, \rho) dr d\rho dw.$$

The error between this integral and  $I_+$  is actually  $o(1)$ , due the calculation of the proof of Proposition 3.1.3 of [Del22]. In conclusion, we have

$$I_+ = \int e^{-i\epsilon rw} \mathbb{1}_{r>0} e^{i\epsilon g(\rho, \rho)w^2} \mathbb{1}_{w>0} F_0(\rho, \rho) dr d\rho dw + o(1), \\ I_- = \int e^{-i\epsilon rw} \mathbb{1}_{r>0} e^{-i\epsilon g(\rho, \rho)w^2} \mathbb{1}_{w<0} F_0(\rho, \rho) dr d\rho dw + o(1).$$

As a consequence, the limit of  $I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F)$  is

$$\int e^{-i\epsilon rw} \mathbb{1}_{r>0} dr \left( e^{i\epsilon g(\rho, \rho)w^2} \mathbb{1}_{w>0} + e^{-i\epsilon g(\rho, \rho)w^2} \mathbb{1}_{w<0} \right) dw F_0(\rho, \rho) d\rho \\ = \int -i\epsilon(w - i\epsilon 0)^{-1} (\cos(\epsilon g(\rho, \rho)w^2) + \text{sgn}(w) i \sin(\epsilon g(\rho, \rho)w^2)) dw F_0(\rho, \rho) d\rho \\ = \int -i\epsilon(w - i\epsilon 0)^{-1} \cos(g(\rho, \rho)w^2) dw F_0(\rho, \rho) d\rho \\ + \int \frac{\sin(g(\rho, \rho)w^2)}{|w|} dw F_0(\rho, \rho) d\rho,$$

where the integrals in  $r$  and  $w$  should be understood in the sense of oscillatory integral and distribution, respectively.

Due to the assumption that  $P$  is strictly concave, or equivalently  $P'' < 0$ ,  $g$  is negative. Since  $(w - i\epsilon 0)^{-1}$  is homogeneous of degree  $-1$ , the right hand side of equality above can be rewritten as

$$\int -i\epsilon(w - i\epsilon 0)^{-1} \cos(w^2) dw F_0(\rho, \rho) d\rho \\ - \int \frac{\sin(w^2)}{|w|} dw F_0(\rho, \rho) d\rho,$$

For one thing, the distribution  $(w \pm i0)^{-1}$  can be expressed as

$$(w \pm i0)^{-1} = \mp \pi i \delta_0 + \text{P. V.} \frac{1}{w}.$$

And for another thing, due to Dirichlet integral  $\int_0^\infty \frac{\sin y}{y} dy = \frac{\pi}{2}$ , we have

$$\int \frac{\sin(w^2)}{|w|} dw = \frac{\pi}{2}.$$

These identities imply that

$$\begin{aligned} \lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) &= \int \left( \pi \delta_0 - i\epsilon \text{P. V.} \frac{1}{w} \right) \cos(w^2) dw F_0(\rho, \rho) d\rho \\ &\quad + \text{sgn}(g) \frac{\pi}{2} \int F_0(\rho, \rho) d\rho \\ &= \pi \int F_0(\rho, \rho) d\rho - \frac{\pi}{2} \int F_0(\rho, \rho) d\rho \\ &= \frac{\pi}{2} \int F_0(\rho, \rho) d\rho. \end{aligned}$$

In the case of strictly convex  $P$ , it has been proved in [Del22] that

$$\begin{aligned} \lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) &= \int i\epsilon (w + i\epsilon 0)^{-1} \cos(g(\rho, \rho)w^2) dw F_0(\rho, \rho) d\rho \\ &\quad - \int \frac{\sin(g(\rho, \rho)w^2)}{|w|} dw F_0(\rho, \rho) d\rho, \end{aligned}$$

where  $g$  is positive. We may repeat the argument above and conclude that

$$\begin{aligned} \lim_{t \rightarrow +\infty} I(t, -\epsilon, \epsilon, -\epsilon, \epsilon; F) &= \int \left( \pi \delta_0 + i\epsilon \text{P. V.} \frac{1}{w} \right) \cos(w^2) dw F_0(\rho, \rho) d\rho \\ &\quad - \frac{\pi}{2} \int F_0(\rho, \rho) d\rho \\ &= \pi \int F_0(\rho, \rho) d\rho - \frac{\pi}{2} \int F_0(\rho, \rho) d\rho \\ &= \frac{\pi}{2} \int F_0(\rho, \rho) d\rho. \end{aligned}$$

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## 3 - Virial Theorem and Its Applications in Instability of Two-Phase Water-Wave

In this chapter, we analyze the dynamics of two layers of immiscible, inviscid, and incompressible potential fluids through a fully nonlinear system. Our goal is to establish a virial theorem and prove the polynomial growth of slope and curvature of the interface over time when the fluid below is no denser than the one above. These phenomena, known as Rayleigh-Taylor instability, will be proved for a broad class of regular initial data, including the case of 2D overlapping interface.

### 3.1 . Introduction

#### 3.1.1 . Two-phase water-wave

Consider two fluids occupying the domains  $\Omega^\pm(t) \subset \mathbb{T}^d \times ]-H^+, H^-[$  separated by a hypersurface  $\Sigma(t)$  without self-intersection, parametrized by

$$(3.1.1) \quad \gamma(t): \mathbb{T}^d \rightarrow \mathbb{T}^d \times ]-H^+, H^-[, \quad x \mapsto \gamma(t, x).$$

Here  $H^\pm \in ]0, +\infty]$  stands for the depth and we denote by

$$(3.1.2) \quad \Gamma^\pm = \{(x, y) \in \mathbb{T}^d \times \mathbb{R} : y = \mp H^\pm\}$$

the solid boundaries (or the "bottoms"). When the depth  $H^+ = +\infty$  or  $H^- = +\infty$ , we do not consider the corresponding solid boundary. Note that, in this chapter, the dimension  $d$  is chosen to be 1 or 2 for physical relevance, while most of the arguments hold true for general dimensions.

Since the fluids are assumed to be potential, inviscid, and incompressible, their velocity field  $u^\pm$  is expressed as  $u^\pm = \nabla_{x,y} \phi^\pm$  for some scalar potential  $\phi^\pm$  and satisfies the Euler equation with divergence-free condition,

$$(3.1.3) \quad \begin{cases} \rho^\pm (\partial_t u^\pm + u^\pm \cdot \nabla_{x,y} u^\pm) + \nabla_{x,y} (P^\pm + \rho^\pm g y) = 0, & \text{in } \Omega^\pm(t), \\ \operatorname{div}_{x,y} u^\pm = 0, & \text{in } \Omega^\pm(t), \\ u^\pm = \nabla_{x,y} \phi^\pm, & \text{in } \Omega^\pm(t), \end{cases}$$

where  $\rho^\pm \geq 0$  (with  $\rho^+ + \rho^- > 0$  to avoid trivial case) is constant density,  $P^\pm$  is the pressure, and  $g \geq 0$  is the gravity acceleration. In order to complete the description of the system,

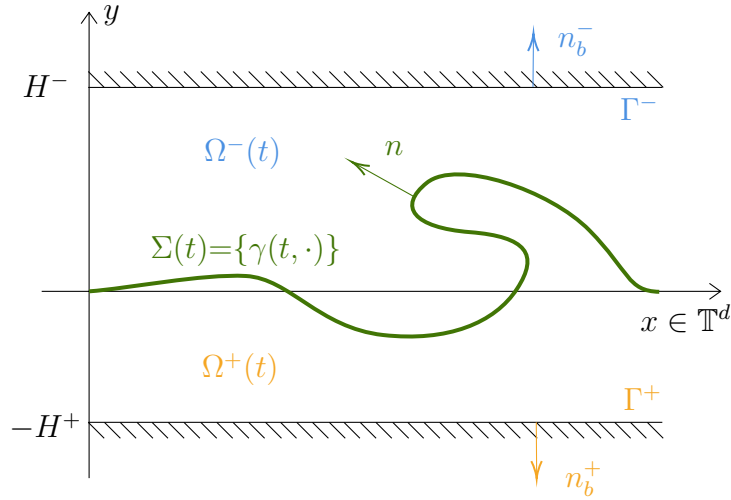


Figure 3.1 – Two phase water-wave

we need to determine the boundary conditions on  $\Sigma(t)$  and  $\Gamma^\pm$ . Since the fluids cannot go through the outer boundary  $\{y = \mp H^\pm\}$ , one has

$$(3.1.4) \quad u^\pm \cdot n_b^\pm = 0, \quad \text{on } \Gamma^\pm,$$

where  $n_b^\pm = (0, \mp 1)$  is the unit outward normal vector of the solid boundary  $\Gamma^\pm$ . Remark that, in infinite depth case  $H^+ = +\infty$  or  $H^- = +\infty$ , this condition should be replaced by

$$(3.1.5) \quad \lim_{y \rightarrow -\infty} u^+ = 0, \quad \text{or} \quad \lim_{y \rightarrow +\infty} u^- = 0,$$

respectively. As for the interface  $\Sigma(t)$ , we assume that the normal component of the velocity field is continuous to avoid vacuum area, which is known as *kinematic boundary condition*,

$$(3.1.6) \quad \partial_t \gamma \cdot n = u^+ \cdot n = u^- \cdot n, \quad \text{on } \Sigma(t),$$

where  $n$  is the unit normal vector of interface. The last boundary condition is about the pressure  $P^\pm$ ,

$$(3.1.7) \quad P^+ - P^- = \sigma \kappa, \quad \text{on } \Sigma(t),$$

where  $\sigma \geq 0$  is the surface tension constant ( $\sigma = 0$  represents the case without surface tension), and  $\kappa$  is the mean curvature.

Since the velocity field  $u^\pm$  can be expressed as the gradient  $\nabla_{x,y} \phi^\pm$  of scalar potential  $\phi^\pm$ , the divergence-free condition implies

$$(3.1.8) \quad \Delta_{x,y} \phi^\pm = 0, \quad \text{in } \Omega^\pm(t).$$

Consequently, the Euler equation (3.1.3) yields the *Bernoulli's equation*,

$$(3.1.9) \quad \rho^\pm \left( \partial_t \phi^\pm + \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) + P^\pm + \rho^\pm g y = 0, \quad \text{in } \Omega^\pm(t).$$

Note that the right-hand side is generally a constant depending only on time and could be taken as zero by absorbing the constant in the definition of the pressure  $P^\pm$ . Now, we can state the main system of  $(\gamma, \phi^\pm, P^\pm)$  to be studied throughout this chapter,

$$(3.1.10) \quad \begin{cases} \rho^\pm (\partial_t \phi^\pm + \frac{1}{2} |\nabla_{x,y} \phi^\pm|^2) + P^\pm + \rho^\pm g y = 0, & \text{in } \Omega^\pm(t), \\ \Delta_{x,y} \phi^\pm = 0, & \text{in } \Omega^\pm(t), \\ n \cdot \partial_t \gamma = n \cdot \nabla_{x,y} \phi^+ = n \cdot \nabla_{x,y} \phi^-, & \text{on } \Sigma(t), \\ P^+ - P^- = \sigma \kappa, & \text{on } \Sigma(t), \\ (\text{finite depth}) n_b^\pm \cdot \nabla_{x,y} \phi^\pm = 0, & \text{on } \Gamma^\pm, \\ (\text{infinite depth}) \lim_{y \rightarrow \mp \infty} \nabla_{x,y} \phi^\pm = 0. & \end{cases}$$

In fact, it is possible to determine the pressure  $P^\pm$  from  $(\gamma, \phi^\pm)$ , via *Craig-Sulem-Zakharov formulation*, cf. [Zak68; CS93; BB97]. Roughly speaking, we compute the difference of + and - parts of the first equation in (3.1.10) and restrict this difference at the free boundary  $\Sigma(t)$ . Then the fourth equation in (3.1.10) could cancel the pressures. In this way, we obtain a self-contained system involving only  $\gamma$  and the traces of  $\phi^\pm$ , from which we can restore  $\phi^\pm$  by solving the Poisson equation  $\Delta_{x,y} \phi^\pm = 0$  with proper boundary conditions. We refer to [Cra17] for the detailed computations. To avoid technical details, this formulation will not be used in this chapter.

There are two special cases to be studied in this chapter. One is the *graph case* where  $d \geq 1$  and  $\Sigma(t)$  is a graph of some function  $\eta(t): \mathbb{T}^d \rightarrow ]-H^+, H^-[$ . The parametrization  $\gamma(t)$  can be fixed as  $\gamma(t, x) = (x, \eta(t, x))$ .

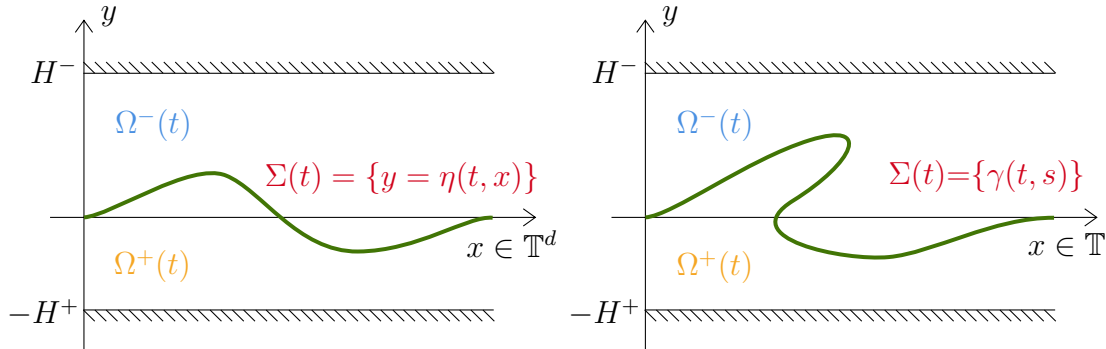


Figure 3.2 – Graph case & 2D overlapping case

Another one is the *2D overlapping case*, where  $d = 1$  and the interface is a curve in 2D. In this case, we only consider the arc-length parameter  $\gamma(t): \mathbb{R}/L(t) \rightarrow \mathbb{T} \times ]-H^+, H^-[$ , where  $L(t)$  denotes the total length of  $\Sigma(t)$ . To avoid self-intersection, one may add the *chord-arc condition* (see (4) of Definition 3.1.2 below).

### 3.1.2 . Conservation laws and Hamiltonian structure

For the two-phase water-wave system (3.1.10), there are two frequently used conservation laws, the conservation of mass and energy. The mass is defined as

$$(3.1.11) \quad M := \int_{\Sigma} n \cdot (0, y) dS,$$

where  $dS$  denotes the surface element (see also (3.2.10) for an alternative formulation). For simplicity, we shall assume that  $M = 0$  for all time (see (3) of Definition 3.1.2 below). The total energy of the system (3.1.10) reads

$$(3.1.12) \quad E = E_k + E_p$$

with kinetic energy  $E_k$  and potential energy  $E_p$  defined by

$$(3.1.13) \quad E_k := \frac{1}{2} \sum_{\pm} \underline{\rho}^{\pm} \iint_{\Omega^{\pm}} |\nabla_{x,y} \phi^{\pm}|^2 dx dy,$$

$$(3.1.14) \quad E_p := \frac{Ag}{2} \int_{\Sigma} n \cdot (0, y^2) dS + \frac{\sigma}{\rho^+ + \rho^-} \left[ \text{Area}(\Sigma(t)) - (2\pi)^d \right],$$

where  $\underline{\rho}^{\pm} = \rho^{\pm}/(\rho^+ + \rho^-)$  is the relative density, the constant

$$(3.1.15) \quad A := \frac{\rho^+ - \rho^-}{\rho^+ + \rho^-}$$

is *Atwood number*, and  $\text{Area}(\Sigma(t))$  is the area of the hypersurface  $\Sigma(t)$ . This energy is preserved in time, while it is not positively definite when the Atwood number  $A < 0$ . A detailed justification of these conservation laws will be left to Section 3.2.2.

Consider the unified trace of scalar potential,

$$(3.1.16) \quad \psi := \underline{\rho}^+ \phi^+|_{\Sigma} - \underline{\rho}^- \phi^-|_{\Sigma}.$$

If, in addition, the interface  $\Sigma(t)$  can be represented as  $y = \eta(t, x)$  for some function  $\eta$ , the energy  $E$  will be a nonlinear functional of  $(\eta, \psi)$  and the system (3.1.10) admits the following Hamiltonian formulation,

$$(3.1.17) \quad \eta_t = \frac{\delta E}{\delta \psi}, \quad \psi_t = -\frac{\delta E}{\delta \eta}.$$

A formal derivation of this can be found in [KL95; BB97] for graph case and in [CGK05] for 2D overlapping case. In view of this (formal) Hamiltonian structure, one may expect to establish a *virial theorem* for the system (3.1.10), which will be discussed in the next subsection.

### 3.1.3. Virial theorem

The *virial theorem* describes the phenomenon that, in many Hamiltonian systems, the kinetic and potential energy are equal in the time-averaged sense. Enunciated by Clausius

[Cla70] in 1870, it quickly became a powerful tool in physics [Zwi33; Ray05; Ein22]. The connection between virial theorem and numerous important models in classical physics has also been discovered in recent decades [Han95; GG99; Sho12; AZ24]. In addition, the idea of this theorem could also be applied to some topics in biology [Pod19] and economics [LP23].

In mathematics, virial theorem serves as an important method in the blow-up problem of PDEs. In the 1970s, Zakharov-Sobolev-Synakh [ZSS71] and Glassey [Gla77] proved the virial theorem for nonlinear Schrödinger equation and deduced the existence of singular solution in defocusing settings. Such technique is also used by Merle [Mer96], Sideris [Sid85], Levine [Lev74; Lev90], Keel-Tao [KT99], and Kenig-Merle [KM08]. Recently, Alazard-Zuily [AZ24] proved the first virial theorem for water-wave equation and established the Rayleigh-Taylor instability for a large class of initial data, which will be discussed in the next part.

Additionally, the virial theorem is often expressed as *equipartition of energy* which asserts that the kinetic and potential energy are equal, asymptotically in time. This phenomenon itself has also attracted lots of attentions in the mathematics and physics community. We refer to [CS81] for the case of wave equation and Maxwell's equation, [Gla73] for nonlinear wave equation, [DG84] for thermoelasticity, and [VV08] for more results on nonlinear wave equation.

Inspired by [AZ24], we shall prove in this chapter a virial theorem for the two-phase water-wave problem (3.1.10).

**Theorem 3.1.1.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to (3.1.10) on  $[0, T[$  (see Definition 3.1.2 below) for some  $T \in ]0, +\infty]$ . Then the following identity holds for all time  $t \in [0, T[$ ,*

$$(3.1.18) \quad \frac{1}{2} \frac{d}{dt} \int_{\Sigma} n \cdot (0, y\psi) dS = \tilde{E}_k - E_p + R,$$

where  $\psi$  is defined in (3.1.16),  $\tilde{E}_k$  is the modified kinetic energy

$$(3.1.19) \quad \tilde{E}_k := \sum_{\pm} \underline{\rho}^{\pm} \iint_{\Omega^{\pm}} \left( \frac{1}{4} |\nabla_x \phi^{\pm}|^2 + \frac{3}{4} |\partial_y \phi^{\pm}|^2 \right) dx dy,$$

$E_p$  is the potential energy defined in (3.1.14), and  $R$  is a non-negative remainder.

The remainder  $R$  reads

$$(3.1.20) \quad R = R_b^+ + R_b^- + R_s,$$

with

$$(3.1.21) \quad R_b^{\pm} = \begin{cases} \frac{1}{4} \underline{\rho}^{\pm} H^{\pm} \int_{\Gamma^{\pm}} |\nabla_x \phi^{\pm}|^2 dx, & \text{when } H^{\pm} < +\infty, \\ 0, & \text{when } H^{\pm} = +\infty, \end{cases}$$

$$(3.1.22) \quad R_s = \frac{\sigma}{2(\rho^+ + \rho^-)} \left[ \text{Area}(\Sigma) + \int_{\Sigma} |n \cdot (0, 1)|^2 dS - 2(2\pi)^d \right].$$

Here  $R_b^{\pm}$  and  $R_s$  are both non-negative.

Note that, in [AZ24] where the authors focus on the one-phase graph case (namely,  $\rho^+ = 1$  and  $\rho^- = 0$ ), the integral on the left-hand side of (3.1.18) is the inner product of  $\eta$  and  $\psi$ , which are variables in the Hamiltonian formulation (3.1.17). Our choice here is the general version of the same quantity.

This theorem implies that the integral on the left-hand of (3.1.1) grows at least faster than  $O(t)$ , so as its upper bounds. Thus, we will attempt to control this integral by a function of some geometrical quantities (for instance, slope and curvature) and deduce their growth. The detailed results will be presented in the next paragraph.

### 3.1.4 . Instability of the system

#### Well-posedness of the problem

In the two-phase water-wave problem (3.1.10), if we assume that the *Atwood number*  $A$  defined by (3.1.15) is negative, meaning that the denser fluid is placed above the other one, the system will become highly unstable, which is known as the *Rayleigh-Taylor Instability* (RTI), due to the pioneer work of Rayleigh [Ray83] and Taylor [Tay50]. A comprehensive physical description of this problem can be found in [Kul91]. In the degenerate case  $A = 0$ , or equivalently  $\rho^+ = \rho^-$ , another instability arises from the jump of tangential velocity instead of the gravity, known as the *Kelvin-Helmholtz Instability* (KHI), which was firstly studied by Kelvin [Tho71] and Helmholtz [Hel68]. Let us first review the well-posedness of RTI and KHI problems.

In the absence of surface tension, the local well-posedness theory can be established in the space of analytical functions. For example, in [Sul+81 ; SS85], the authors used the Cauchy-Kowalewski theorem to prove that if the initial data could be extended to a complex strip of width  $r_0 > 0$ , then there exists a unique solution in the space formed by analytic functions in a strip of width  $r \in ]0, r_0[$ , and the existence time is of size  $r_0 - r$ . This result has only been proved for small initial data, where the interface has small gradient and is flat at infinity. This smallness condition is not essential in 2D torus case due to Xie [Xie07] by pulling the system to unit disc via conformal mapping.

Indeed, the analytic framework is optimal, since the solution to KHI or RTI problem is either excessively rough or analytic. More precisely, for 2D KHI problem, Lebeau proved in [Leb02] that, once the interface has Hölder regularity  $C^{1,\epsilon}$  with  $\epsilon > 0$ , the solution must be analytic in time and space. This result was also proved independently by Wu in [Wu06], where the interface is assumed to have only Lipschitz regularity. And for 2D RTI problem, due to [KL05], Hölder regularity  $C^{1,\epsilon}$  of interface guarantees that the solution is smooth, or analytic in space with the additional assumption that the jump at interface is non-vanishing. Remark that the minimal regularity assumed for the interface should be Lipschitz in order to give a proper sense to the system (3.1.10). In addition, if the irrotational condition is eliminated, Ebin proved in [Ebi88] that both RTI and KHI problems are ill-posed (Hadamard sense) in Sobolev spaces. A similar result was also proved by Guo-Tice [GT11] for compressible fluids.

In the presence of surface tension, most of the mathematical researches are dedicated to KHI, or equivalently the *vortex sheet problem*. In this case, the surface tension can be shown to stabilize the system and better local well-posedness results could be expected, compared with the case without surface tension. For example, in [ITT97] the authors proved the local well-posedness in high order Sobolev spaces for small initial data where the interface is almost flat and the velocity is close to zero. The smallness condition is then eliminated by Ambrose [Amb03] in 2D and by Ambrose-Masmoudi [AM07] in 3D. We also refer to [SZ08; CSC08] for similar results.

For RTI with surface tension, seldom results are obtained for inviscid fluids. However, when the viscosity is considered, the well-posedness theory will be available, as in the case of water-wave problem [GT13c; GT13b; GT13a]. For incompressible viscous fluids, in [WTK14], the authors proved local well-posedness in high order Sobolev spaces for small data and their almost exponential decay. In compressible viscous setting, it is proved in [JTW16a; JTW16b; JTW16c] that the problem is locally well-posed for small initial data in high order Sobolev spaces and is exponentially stable near equilibrium when the surface tension is large enough.

In this chapter, we will not concern the well-posedness problem, but concentrate on the characterization of instability for the system (3.1.10). As discussed above, it is reasonable to assume that there exist non-trivial solutions to the main system (3.1.10) and that these solutions are smooth enough (see (1) of Definition 3.1.2 below), for the sake of simplicity. Before stating our main results, we precise the solutions to be considered throughout this chapter.

**Definition 3.1.2** (Regular solutions). *Given  $T \in ]0, +\infty]$ ,  $g, \sigma \in [0, +\infty[$ ,  $H^\pm \in ]0, +\infty]$ , and  $\rho^\pm \in [0, +\infty[$  with  $\rho^+ + \rho^- > 0$ , we say  $(\gamma, \phi^\pm, P^\pm)$  is a regular solution to the system (3.1.10) on  $[0, T[$ , if it solves (3.1.10) on the time interval  $[0, T[$  with the following conditions :*

(1) (Smoothness) *The parametrization  $\gamma$ , the scalar potential  $\phi^\pm$ , and the pressure  $P^\pm$ ,*

$$\gamma: [0, T[ \times \mathbb{T}^d \rightarrow \Omega, \quad \phi^\pm, P^\pm: \{(t, x, y) : (x, y) \in \Omega(t)\} \rightarrow \mathbb{R},$$

*are all  $C^1$  in time. In space variable,  $\gamma(t)$  is  $C^2$ ,  $\phi^\pm(t)$  is  $C^2$  up to boundary, and  $P^\pm(t)$  is  $C^1$  up to boundary.*

(2) (Finite energy) *The kinetic energy  $E_k = E_k(t)$  defined by (3.1.13) is finite for all time  $t \in [0, T[$  with  $E_k \in L^\infty_{\text{loc}}([0, T[)$ .*

(3) (Zero mass) *The mass  $M$  defined by (3.1.11) is zero for all time  $t \in [0, T[$ .*

(4) (No self-intersection) *For all time  $t \in [0, T[$ , the parametrization  $\gamma(t)$  is a diffeomorphism from  $\mathbb{T}^d$  to  $\Sigma(t)$ . In particular, for 2D case (namely  $d = 1$ ), we take  $\gamma(t)$  as the arc-length parametrization and add the chord-arc condition : there exists  $0 < c_0 \leq 1$ , such that, for all  $s_1, s_2 \in \mathbb{R}/L(t)$  and all time  $t \in [0, T[$ ,*

$$(3.1.23) \quad c_0 |s_1 - s_2| \leq |\gamma(t, s_1) - \gamma(t, s_2)| \leq |s_1 - s_2|,$$

*where  $L(t)$  is the length of  $\Sigma(t)$  and the absolute value  $|s_1 - s_2|$  should be understood as the distance in  $\mathbb{R}/L(t)$ .*

- (5) (*Distance between free and solid boundaries*) If  $H^+ < +\infty$  (resp.  $H^- < +\infty$ ), there exists  $d_0 > 0$  such that, for all time  $t \in [0, T[$ , the distance between the free boundary  $\Sigma(t)$  and the solid boundary  $\Gamma^+$  (resp.  $\Gamma^-$ ) is larger than  $d_0$ . Moreover, when  $H^+ = H^- = +\infty$ , we take  $d_0 = +\infty$ .
- (6) (*Growth of pressure*) If  $H^+ = +\infty$  (resp.  $H^- = +\infty$ ), the pressure  $P^+$  (resp.  $P^-$ ) has at most polynomial growth for all time  $t \in [0, T[$ . Namely, for arbitrary  $0 < T' < T$ , there exists  $N \in \mathbb{N}$  such that

$$(3.1.24) \quad \sup_{x \in \mathbb{T}^d, t \in [0, T']} |P^\pm(t, x, y)| \lesssim (1 + |y|)^N.$$

Moreover,  $(\gamma, \phi^\pm, P^\pm)$  is said to be *global-in-time* if  $T = +\infty$ .

**Remark 3.1.3.** In infinite depth case where  $H^+ = +\infty$  (resp.  $H^- = +\infty$ ), one could deduce from the conditions (1) and (2), that the derivatives of the scalar potential  $\phi^+$  (resp.  $\phi^-$ ) is  $C^\infty$  in space and decays rapidly at infinity. Rigorously, for arbitrary sub-interval  $[0, T'] \subset [0, T[$ , we have, for all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^{d+1}$  with  $|\alpha| \geq 1$ ,

$$(3.1.25) \quad \begin{aligned} & \lim_{y \rightarrow -\infty} \sup_{x \in \mathbb{T}^d, t \in [0, T']} |y^k \nabla_{x,y}^\alpha \phi^+(t, x, y)| = 0, \\ \text{resp. } & \lim_{y \rightarrow +\infty} \sup_{x \in \mathbb{T}^d, t \in [0, T']} |y^k \nabla_{x,y}^\alpha \phi^-(t, x, y)| = 0. \end{aligned}$$

The proof of this decay is based on classical elliptic theories and will be detailed in Section 3.2.1.

**Remark 3.1.4.** The conditions (4) in Definition 3.1.2 is assumed so that the system (3.1.10) is valid. However, it is unclear whether conditions (4) can hold for large time. In fact, the self-intersection, also known as splash singularity has already been proved for 2D water-wave problem ( $\rho^+ = 1$  and  $\rho^- = 0$ ), cf. [Cas+12], while the corresponding result for two-phase water-wave is still open.

**Remark 3.1.5.** The conditions (5) in Definition 3.1.2 is also proposed for the validation of the system (3.1.10). One should note that it fails in finite time for RTI with finite depths. Indeed, it is well-known that the free boundary  $\Sigma(t)$  is contained in a strip  $\mathbb{T}^d \times [-\alpha^+(t), \alpha^-(t)]$ , known as mixing area, with upper and lower heights  $\alpha_\pm(t)$  of  $\Sigma(t)$  growing like  $O(t^2)$ , cf. [Cha13; KMW24; GKS21]. Therefore, the growth of mixing area could invalidate the system (3.1.10), which is not the subject of this chapter.

**Remark 3.1.6.** The purpose of the condition (6) in Definition 3.1.2 is to avoid exponential growth of  $P^\pm$  at infinite. One can see from the first equation of (3.1.10) that  $P^\pm$  solves the elliptic equation

$$(3.1.26) \quad \Delta_{x,y} P^\pm = -\frac{\rho^\pm}{2} \Delta_{x,y} (|\nabla_{x,y} \phi^\pm|^2)$$

at least in the sense of distribution, where the right-hand side decays rapidly at infinity (see (3.1.25)). Through separation of variables,  $P^\pm$  can be written as the sum of a term with at most polynomial growth in  $y$  and an exponentially increasing (in  $y$ ) harmonic function. With the condition (6) in Definition 3.1.2, this extra harmonic function will not appear.

### Stability criteria and the description of instability

In the one-phase case ( $\rho^+ > 0$  and  $\rho^- = 0$ ), a well-known stability criteria is *Taylor sign condition* :

$$-\partial_y P^+|_{\Sigma} > 0,$$

which is rigorously justified by Wu in 1990s [Wu97; Wu99]. For two-phase fluid, an analogue is given by Kelvin for the linearized system :

$$g(\rho^+ - \rho^-) > \frac{1}{4\sigma} \left( \frac{\rho^+ \rho^-}{\rho^+ + \rho^-} \right)^2 \llbracket u \rrbracket^4,$$

where  $\llbracket u \rrbracket$  denotes the jump of the horizontal velocity at the interface (see for example Ch.X and Ch.XI of [Cha13]). This stability condition is then generalized to the nonlinear problem (graph case) by Lannes [Lan13] :

$$\llbracket -\partial_y P \rrbracket > \frac{1}{4\sigma} \left( \frac{\rho^+ \rho^-}{\rho^+ + \rho^-} \right)^2 \mathfrak{c}(\eta) \llbracket V \rrbracket_{L^\infty}^4,$$

where  $\llbracket -\partial_y P \rrbracket := -\partial_y P^+|_{\Sigma} + \partial_y P^-|_{\Sigma}$  is the jump of  $y$ -derivative of pressure at the interface,  $\mathfrak{c}(\eta) > 0$  is a constant depending on the shape of the interface,  $\llbracket V \rrbracket$  is the jump of the horizontal velocity at the interface.

According to these stability criteria, two potential sources of instability can be identified : gravity and the velocity jump at the interface. Evidently, the case with a negative Atwood number  $A < 0$ —or equivalently  $\rho^+ < \rho^-$ —corresponds to an unstable regime, known as the Rayleigh–Taylor instability. In contrast, the Kelvin–Helmholtz instability is driven by a discontinuity in velocity across the interface. When there is no density jump, i.e.,  $\rho^+ = \rho^-$ , the RTI is absent, and the only remaining mechanism of instability is the velocity jump (KHI). It is also noteworthy that even when the Atwood number is positive ( $A > 0$ , i.e.  $\rho^+ > \rho^-$ ), a sufficiently large velocity jump can still lead to instability, as indicated by the aforementioned stability criteria.

In this chapter, we focus on the instable regime  $A < 0$  (namely,  $\rho^+ < \rho^-$ ) and attempt to obtain a quantitative description of the instability. From the linear analysis (see for example the summary [BL12]), one may expect an exponential growth in short time. In fact, the existing results, mainly based on growing mode or maximal regularity [HG03; GT10; WT12; JTW16b; JJJ23; PS10; Wil17] manage to find some data with such growth. Even if these methods allow us to treat more general contexts (compressible, rotational, viscous, etc), they do lay emphasis on the comparison between linear and nonlinear systems, from which one can only deduce the existence of unstable perturbation near zero in short time. Due to the physical nature of this problem, one should expect the blow-up (or, at least, rapid growth) for most regular solutions, which could be extremely challenging due to the full nonlinearity of the system. As indicated in [AZ24], a possible solution could be the virial identity (see Theorem 3.1.1), which implies the following main results.

### Main results

For graph case, we shall prove that the slope of the interface  $\Sigma(t)$  grows faster than  $O(t^{2/5})$  (or  $O(t^{2/3})$  when  $\rho^+ = \rho^-$ ). Rigorously, we have

**Theorem 3.1.7** (Instability of graph case). *Consider  $d \geq 1$ ,  $H^\pm \in ]0, +\infty]$ ,  $\rho^+ \leq \rho^-$ ,  $\sigma \geq 0$ , and  $T \in ]0, +\infty]$ . Let  $(\gamma, \phi^\pm, P^\pm)$  be any regular solution to (3.1.10) on  $[0, T[$  (see Definition 3.1.2) with  $\gamma(t, x) = (x, \eta(t, x))$  for some real-valued function  $\eta$ . We assume that the total energy  $E$  defined by (3.1.12) satisfies*

$$(3.1.27) \quad E \neq 0 \text{ when } \sigma = 0, \text{ and } E < 0 \text{ when } \sigma > 0.$$

Then there exists a constant  $C > 0$  depending on  $d_0$  (see (5) in Definition 3.1.2), such that, for all time  $t \in [0, T[$ ,

$$(3.1.28) \quad |E|t + \int_{\mathbb{T}^d} \eta \psi dx \Big|_{t=0} \leq C \|\nabla_x \eta\|_{L^\infty} \sqrt{1 + \|\nabla_x \eta\|_{L^\infty}} \sqrt{E + |A|g \|\nabla_x \eta\|_{L^\infty}^2},$$

where  $\psi$  is defined in (3.1.16) and  $A \leq 0$  is the Atwood number (see (3.1.15)). Note that, due to Poincaré's inequality, the quantity  $E + |A|g \|\nabla_x \eta\|_{L^\infty}^2$  is non-negative.

**Remark 3.1.8.** *When  $\rho^+ < \rho^-$ , the polynomial growth observed in (3.1.28) may be due to Rayleigh–Taylor instability, as the condition  $\rho^+ < \rho^-$  is critical in the proof. However, this explanation does not apply in the case  $\rho^+ = \rho^-$ . Recall that in the formulation of the problem, zero velocity jump across the interface is permitted, which precludes the direct connection with Kelvin–Helmholtz instability. In fact, as shown in the formulation via vorticity in [BL12], the system is hyperbolic with degeneracy when  $\rho^+ = \rho^-$ , which may naturally lead to polynomial growth in time.*

Among the conditions of Theorem 3.1.7, except for the reasonable assumptions listed in Definition 3.1.2, the only non-trivial one is (3.1.27) about the total energy, depending only on the initial data. Without surface tension,  $E \neq 0$  contains already a large class of solutions. However, in the case with surface tension ( $\sigma > 0$ ), we have to make the assumption  $E < 0$  so that the right-hand side of the virial identity (3.1.18) is positive. Note that  $E < 0$  can be verified only by a few solutions such as those with initial data  $\phi^\pm(0, x, y) = 0$ ,  $\eta(0, x) = M \sin(x)$ , where  $M \in \mathbb{R}$  is large enough. The characterization of instability for  $E > 0$ ,  $\sigma > 0$  is still open.

For 2D overlapping case, instead of the slope, we shall consider the curvature of the interface  $\Sigma(t)$ . In a similar manner, the  $L^\infty$ -norm of the curvature can be shown to grow faster than  $O(t^{2/9})$  (or  $O(t^{1/3})$  when  $\rho^+ = \rho^-$ ), provided that the depths are finite.

**Theorem 3.1.9** (Instability of 2D overlapping case). *Consider  $d = 1$ ,  $H^\pm < +\infty$ ,  $\rho^+ \leq \rho^-$ ,  $\sigma \geq 0$ , and  $T \in ]0, +\infty]$ . Let  $(\gamma, \phi^\pm, P^\pm)$  be any regular solution to (3.1.10) on  $[0, T[$  (see Definition 3.1.2) with  $\gamma$  being the arc-length parametrization.*

If the total energy  $E$  defined by (3.1.12) satisfies the condition (3.1.27), then there exists a universal constant  $N_0 > 0$  and another constant  $C > 0$  depending on  $H^\pm$ , such that, for all time  $t \in [0, T]$ ,

$$(3.1.29) \quad |E|t + \int_{\Sigma} n \cdot (0, y\psi) dS \Big|_{t=0} \leq C\epsilon(t)^{-3} \sqrt{|E| + |A|g\epsilon(t)^{-3}},$$

with  $\epsilon(t) > 0$  defined by

$$(3.1.30) \quad \epsilon(t) := \min \left( \frac{c_0}{N_0(\|\kappa(t)\|_{L^\infty} + 1)}, d_0 \right),$$

where  $\psi$  is defined in (3.1.16),  $\kappa$  is the curvature,  $c_0$  is the constant from the chord-arc condition (3.1.23), and  $d_0$  is the constant appearing in the condition (5) of Definition 3.1.2.

Note that if the maximal time of existence  $T$  is large enough, the right-hand side of (3.1.29) will be large enough so that  $\epsilon^{-1} > d_0^{-1}$ , which implies the growth of  $\|\kappa(t)\|_{L^\infty}$ . This result should be understood as : if the self-intersection and the intersection between free and solid boundaries never occur, then the curvature  $\|\kappa(t)\|_{L^\infty}$  grows faster than  $O(t^{2/9})$  (or  $O(t^{1/3})$  for KHI). That is to say, in general, there are at least three possible singularities in large time,

- (1) The curvature  $\|\kappa\|_{L^\infty}$  tends to infinity (faster than  $O(t^{2/9})$  for RTI and  $O(t^{1/3})$  for KHI);
- (2) There exists self-intersection (see Remark 3.1.4);
- (3) The free boundary  $\Sigma(t)$  intersects with the solid boundaries  $\Gamma^\pm$  (see Remark 3.1.5).

The proof of Theorem 3.1.7 and 3.1.9 are based on the virial identity (3.1.18). Noticing that the right-hand side of (3.1.18) can be bounded from below by  $|E|/2$  (see Corollary 3.4.1), it suffices to control the integral of  $n \cdot (0, y\psi)$  on  $\Sigma(t)$ , where the main difficulty is to bound  $\psi$  by quantities depending only on slope or curvature. Since  $\psi$  (see (3.1.16)) is defined using the trace of scalar potential  $\phi^\pm$ , one may expect to apply the trace estimate. Namely, the  $L^2$ -norm of  $\psi$  can be bounded by the  $L^2$ -norm of  $\nabla_{x,y}\phi^\pm$ , or equivalently, the kinematic energy  $E_k$  (see (3.1.13)). Then the conservation of energy allows us to write  $E_k = E - E_p$  with  $E_p$  depending only on the geometry of the interface  $\Sigma(t)$ , which can be controlled by slope or curvature via some direct calculus.

### 3.1.5 . Plan of the chapter and conventions

In Section 3.2, we shall clarify the decay of scalar potential (in infinite depth case) and a proof of conservation laws will be given. Then the virial theorem 3.1.1 will be proved in Section 3.3. Based on this, we conclude the instability results, Theorem 3.1.7 and 3.1.9, in Section 3.4, where the graph case and 2D overlapping case are treated separately. In Appendix 3.A, we will show that the normal geodesic coordinate realizes a global diffeomorphism, which is useful in the proof of Theorem 3.1.9.

Throughout this chapter, we shall frequently use the following notations and conventions :

1.  $\mathbb{T}$  represents the periodic domain  $\mathbb{R}/(2\pi)$ .
2. For non-negative quantities  $A, B$  and some parameter  $\alpha$ , we write  $A \lesssim_\alpha B$ , if there exists a constant  $C_\alpha > 0$  depending on  $\alpha$  such that  $A \leq C_\alpha B$ . Furthermore, we write  $A \sim_\alpha B$  when  $A \lesssim_\alpha B$  and  $B \lesssim_\alpha A$ .
3. For non-negative quantities  $A, B$ , we write  $A \ll B$  if there exists a large universal constant  $C > 0$  such that  $A \leq CB$ .
4. In infinite depth case, the integral of a function  $f$  on  $\Omega^\pm$  should be understood as the limit

$$\iint_{\Omega^\pm} f dx dy = \lim_{h^\pm \rightarrow +\infty} \iint_{\Omega^\pm \cap \{\mp y < h^\pm\}} f dx dy.$$

The functions  $f$  to be studied in this chapter decay to zero as  $y \rightarrow \pm\infty$ , uniformly in  $x \in \mathbb{T}^d$ . The precised behavior of  $f$  at infinity will be clarified in the context.

## 3.2 . Preliminaries

The aim of this section is to rigorously prove some results well-known in physics, including the decay of scalar potential  $\phi^\pm$  and the conservation of mass and energy.

### 3.2.1 . Decay of scalar potential

In this part, we clarify how to deduce the rapid decay (3.1.25) of the scalar potentials in infinite depth cases. For simplicity, we will focus on the lower fluid domain  $\Omega^+$  with  $H^+ = +\infty$  (the argument for the upper domain is the same). To conclude the decay (3.1.25), we choose a large depth  $-h^+$  that the free boundary  $\Sigma$  does not reach. By using the data at  $y = -h^+$  and the condition  $E_k < +\infty$ , we will find an explicit formula for  $\phi^+$ , which simply gives the desired decay. The detailed result is stated in the following lemma.

**Lemma 3.2.1.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to (3.1.10) on the time interval  $[0, T[$ . If the depth  $H^+ = +\infty$  (resp.  $H^- = +\infty$ ), then, for all time  $t \in [0, T[$ ,  $\phi^\pm(t)$  is  $C^\infty$  inside  $\Omega^\pm(t)$ . Moreover, for all  $0 < T' < T$ ,  $k \in \mathbb{N}$ , and  $\alpha \in \mathbb{N}^{d+1}$  with  $|\alpha| \geq 1$ , there holds*

$$(3.2.1) \quad \begin{aligned} & \lim_{y \rightarrow -\infty} \sup_{x \in \mathbb{T}^d, t \in [0, T']} \left| y^k \nabla_{x,y}^\alpha \phi^+(t, x, y) \right| = 0, \\ \text{resp. } & \lim_{y \rightarrow +\infty} \sup_{x \in \mathbb{T}^d, t \in [0, T']} \left| y^k \nabla_{x,y}^\alpha \phi^-(t, x, y) \right| = 0. \end{aligned}$$

*Proof.* The smoothness of  $\phi^\pm(t)$  is a consequence of elliptic regularity since  $\Delta_{x,y} \phi^\pm = 0$ . For simplicity, in what follows, we focus on the decay of  $\phi^+$ , while  $\phi^-$  can be treated in the same way.

Since the parametrization  $\gamma$  is continuous on  $[0, T'] \times \mathbb{T}^d$ , there exists  $h^+ > 0$  such that  $\mathbb{T}^d \times ]-\infty, -h^+]$  is contained in  $\Omega^+(t)$ , for all  $t \in [0, T']$ . Since  $\phi^+$  is smooth in space variables, we defined two functions  $f, g \in C^\infty(\mathbb{T}^d)$  by  $f(x) = \phi^+(x, -h^+)$  and  $g(x) =$

$\partial_y \phi^+(x, -h^+)$ . Here the time dependence is omitted for simplicity, since all the arguments below are uniform in time  $t \in [0, T']$ . The smoothness of  $f, g$  guarantees that their Fourier coefficients have arbitrary polynomial decay. Namely, for all  $N \in \mathbb{N}$ ,

$$(3.2.2) \quad |\hat{f}(\xi)| + |\hat{g}(\xi)| \lesssim_N (1 + |\xi|)^{-N}.$$

Thanks to  $\Delta_{x,y} \phi^+ = 0$  (the second equation in (3.1.10)), the standard Fourier method gives the following explicit formula.

$$\begin{aligned} \hat{\phi}^+(\xi, y) &= \frac{1}{2} e^{(y+h^+)|\xi|} \left( \hat{f}(\xi) + |\xi|^{-1} \hat{g}(\xi) \right) + \frac{1}{2} e^{-(y+h^+)|\xi|} \left( \hat{f}(\xi) - |\xi|^{-1} \hat{g}(\xi) \right), \quad \text{if } \xi \neq 0 \\ \hat{\phi}^+(0, y) &= \hat{g}(0)(y + h^+) + \hat{f}(0), \end{aligned}$$

where  $\hat{\phi}^+(\xi, y)$  is the Fourier coefficient of  $\phi^+(y)$  at frequency  $\xi$ . We claim that  $\hat{g}(\xi) = |\xi| \hat{f}(\xi)$  for all  $\xi \in \mathbb{Z}^d$ . Otherwise, the  $L^2(\mathbb{T}^d)$ -norm of  $\nabla_{x,y} \phi^+(\cdot, y)$  admits a strictly positive lower bound as  $y \rightarrow -\infty$ , contradicting the condition (2) in Definition 3.1.2 which implies  $\nabla_{x,y} \phi^+ \in L^2(\Omega^+)$ . Consequently, we have

$$\hat{\phi}^+(\xi, y) = \frac{1}{2} e^{(y+h^+)|\xi|} \left( \hat{f}(\xi) + |\xi|^{-1} \hat{g}(\xi) \right), \quad \text{if } \xi \neq 0, \quad \text{and } \hat{\phi}^+(0, y) = \hat{f}(0).$$

Based on this, we compute, for all  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^{d+1}$  with  $|\alpha| \geq 1$ ,

$$\left| y^k \nabla_{x,y}^\alpha \phi^+(x, y) \right| \lesssim |y|^k \sum_{\xi \in \mathbb{Z}^d} e^{-(y+h^+)|\xi|} |\xi|^{|\alpha|-1} \left| |\xi| \hat{f}(\xi) + \hat{g}(\xi) \right|.$$

Due to the rapid decay (3.2.2) of  $\hat{f}(\xi), \hat{g}(\xi)$ , the right-hand side, independent of  $x$ , converges to zero as  $y \rightarrow -\infty$ , which completes the proof of (3.2.1).  $\square$

From this proof, one can see that the derivatives of  $\phi^+$  has actually exponential decay at infinity, while the polynomial decay is enough for the arguments in this chapter.

### 3.2.2. Conservation laws

In this paragraph, we will rigorously prove the conservation of mass and energy introduced in Section 3.1.2. These conservation laws are well-known for graph case where  $\Sigma(t) = \{y = \eta(t, x)\}$ , while the proof of general cases requires some geometrical arguments. To begin with, we introduce a technical lemma which will be frequently used in the proof of conservation laws and the virial identity (3.1.18) to be studied in Section 3.3.

**Lemma 3.2.2.** *Consider two domains  $\Omega^\pm(t)$  filling the domain  $\mathbb{T}^d \times ]-H^-, H^+[$  and varying in time  $t \in [0, T[$  with  $H^\pm < +\infty$  and  $T > 0$ . These domains are assumed to be separated by an interface  $\Sigma(t)$  that admits a parametrization  $\gamma : [0, T[ \times \mathbb{T}^d \rightarrow \mathbb{T}^d \times ]-H^-, H^+[$ . We denote by  $\tilde{\Omega}^\pm$  the time-space domains  $\tilde{\Omega}^\pm = \{(t, x, y) \in [0, T[ \times \mathbb{T}^d \times \mathbb{R} : (x, y) \in \Omega^\pm(t)\}$ .*

*If  $\gamma$  is  $C^1$  in space and time, then, for all functions  $f^\pm \in C^1(\tilde{\Omega}^\pm)$ , there holds*

$$(3.2.3) \quad \frac{d}{dt} \iint_{\Omega^\pm(t)} f^\pm dx dy = \iint_{\Omega^\pm(t)} \partial_t f^\pm dx dy \pm \int_{\Sigma(t)} n \cdot \gamma_t f^\pm dS.$$

*Note that  $dS$  is the surface element on  $\Sigma(t)$ .*

*Proof.* To prove the formula (3.2.3), we will prove that both side are equal as distributions in time, namely  $\mathcal{D}'(]0, T[)$ . This allows us to work on time-space domains, which is fixed. Then the desired equality follows from divergence theorem.

Consider the time-space interface  $\tilde{\Sigma}^\pm = \{(t, x, y) \in [0, T[ \times \mathbb{T}^d \times \mathbb{R} : (x, y) \in \Sigma(t)\}$ , which can be parameterized as  $(t, x) \mapsto (t, \gamma(t, x))$ . Therefore, the tangent space of  $\tilde{\Sigma}$  is spanned by  $(1, \gamma_t)$  and  $(0, \tau_j)$ , where  $(\tau_j)$  is arbitrary basis of the tangent space of  $\Sigma(t)$ . Consequently, the unit normal vector and surface element of  $\tilde{\Sigma}$  read

$$(3.2.4) \quad \tilde{n} = \frac{(-n \cdot \gamma_t, n)}{\sqrt{1 + (n \cdot \gamma_t)^2}}, \quad d\tilde{S} = \sqrt{1 + (n \cdot \gamma_t)^2} dS dt,$$

respectively. Here  $n = n(t)$  denotes the unit normal vector of  $\Sigma(t)$ .

For any test function  $\varphi = \varphi(t) \in C_c^\infty(]0, T[)$ , we have

$$(3.2.5) \quad \begin{aligned} \int_0^T \left( \frac{d}{dt} \iint_{\Omega^\pm(t)} f^\pm dx dy \right) \varphi dt &= - \iiint_{\tilde{\Omega}^\pm} f^\pm \partial_t \varphi dt dx dy \\ &= \iiint_{\tilde{\Omega}^\pm} \partial_t f^\pm \varphi dt dx dy - \iiint_{\tilde{\Omega}^\pm} \partial_t (f^\pm \varphi) dt dx dy \\ &= \int_0^T \iint_{\Omega^\pm(t)} \partial_t f^\pm \varphi dx dy dt - \iiint_{\tilde{\Omega}^\pm} \partial_t (f^\pm \varphi) dt dx dy. \end{aligned}$$

To handle the second term on the right-hand side, we apply the divergence theorem in time-space,

$$(3.2.6) \quad \iiint_{\tilde{\Omega}^\pm} \partial_t (f^\pm \varphi) dt dx dy = \pm \iint_{\tilde{\Sigma}} \tilde{n} \cdot (f^\pm \varphi, 0, 0) d\tilde{S}.$$

Note that the solid boundaries  $\{y = \mp H^\pm\}$  does not appear since the time-space version of its normal vector equals  $(0, \mp 1, 0)$ , whose scalar product with  $(f\varphi, 0, 0)$  vanishes. By applying the formula (3.2.4) of  $\tilde{n}$  and  $d\tilde{S}$ , we have

$$(3.2.7) \quad \pm \iint_{\tilde{\Sigma}} \tilde{n} \cdot (f^\pm \varphi, 0, 0) d\tilde{S} = \mp \int_0^T \int_{\Sigma(t)} n \cdot \gamma_t f^\pm \varphi dS dt.$$

By combining (3.2.5), (3.2.6), and (3.2.7), one concludes

$$\int_0^T \left( \frac{d}{dt} \iint_{\Omega^\pm(t)} f^\pm dx dy \right) \varphi dt = \int_0^T \left( \iint_{\Omega^\pm(t)} \partial_t f^\pm dx dy \pm \int_{\Sigma(t)} n \cdot \gamma_t f^\pm dS \right) \varphi dt$$

for arbitrary  $\varphi = \varphi(t) \in C_c^\infty(]0, T[)$ , which implies the desired formula (3.2.3).  $\square$

Now, we are in a position to prove the conservation laws.

**Proposition 3.2.3.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to the system (3.1.10) with depth  $H^\pm \in ]0, +\infty]$ . Then the following conservation laws hold true :*

$$(3.2.8) \quad \frac{d}{dt} M = 0, \quad \frac{d}{dt} E = 0,$$

where the mass  $M$  and the total energy  $E$  are defined in (3.1.11) and (3.1.12), respectively.

In order to overcome the difficulties caused by infinite depths, we will work on truncated domains defined as below.

$$(3.2.9) \quad \begin{aligned} \Omega_h^+ &:= \Omega^+ \cap \{y > -h^+\}, & \Omega_h^- &:= \Omega^- \cap \{y < h^-\}; \\ h^\pm &= \begin{cases} H^\pm, & \text{if } H^\pm < +\infty; \\ \text{arbitrary number such that } \Sigma \subset \{\mp y < h^\pm\}, & \text{if } H^\pm = +\infty. \end{cases} \end{aligned}$$

In the proof of the infinite depth case, we will work on these bounded domains and take the limit  $h^\pm \rightarrow +\infty$  in the end.

### Conservation of mass

Before proving the conservation of mass  $M$  defined in (3.1.11), we give an alternative formulation of  $M$ . An application of divergence theorem in the domains  $\Omega_h^\pm$  gives

$$\begin{aligned} \iint_{\Omega_h^\pm} dx dy &= \iint_{\Omega_h^\pm} \nabla_{x,y} \cdot (0, y) dx dy = \pm \int_{\Sigma} n \cdot (0, y) dS \mp \int_{y=\mp h^\pm} y dx \\ &= \pm \int_{\Sigma} n \cdot (0, y) dS + (2\pi)^d h^\pm = \pm M + (2\pi)^d h^\pm, \end{aligned}$$

where the last equality follows from the definition (3.1.11) of  $M$ . Thus, we conclude that

$$(3.2.10) \quad M = \pm \left( \iint_{\Omega_h^\pm} dx dy - (2\pi)^d h^\pm \right).$$

From this formula, the time derivative of  $M$  can be obtained via Lemma 3.2.2 with  $f = 1$  and  $\Omega^\pm$  replaced by  $\Omega_h^\pm$ .

$$\frac{d}{dt} M = \pm \frac{d}{dt} \iint_{\Omega_h^\pm} dx dy = \int_{\Sigma} n \cdot \gamma_t dS.$$

Due to the third equation of (3.1.10),  $n \cdot \gamma_t$  can be replaced by  $n \cdot \nabla_{x,y} \phi^\pm$ . Then we apply the divergence theorem and use  $\Delta_{x,y} \phi^\pm = 0$  to compute

$$\frac{d}{dt} M = \int_{\Sigma} n \cdot \nabla_{x,y} \phi^\pm dS = \int_{y=\mp h^\pm} \partial_y \phi^\pm dx.$$

In finite depth case, by choosing  $h^\pm = H^\pm$  and using the boundary condition (the fifth equation in (3.1.10)), we conclude that the right-hand side is zero. In infinite depth case, as  $h^\pm \rightarrow +\infty$ , the right-hand side tends to zero due to the rapid decay of  $\partial_y \phi^\pm$  (see (3.1.25)), which completes the proof of conservation law (3.2.8) for the mass  $M$ .

### Conservation of energy.

Let us begin with the time derivative of potential energy  $E_p$  (see (3.1.14)), which contains two components concerning gravity and surface tension, respectively.

$$\begin{aligned} E_p &= E_p^{\text{grav}} + E_p^{\text{surf}}, \\ E_p^{\text{grav}} &= \frac{Ag}{2} \int_{\Sigma} n \cdot (0, y^2) dS, & E_p^{\text{surf}} &= \frac{\sigma}{\rho^+ + \rho^-} \left[ \text{Area}(\Sigma(t)) - (2\pi)^d \right]. \end{aligned}$$

**Lemma 3.2.4.** *Under the hypothesis of Proposition 3.2.3, the following formulas hold.*

$$(3.2.11) \quad \frac{d}{dt} E_p^{\text{grav}} = Ag \int_{\Sigma} n \cdot \gamma_t y dS,$$

$$(3.2.12) \quad \frac{d}{dt} E_p^{\text{surf}} = \frac{\sigma}{\rho^+ + \rho^-} \int_{\Sigma} n \cdot \gamma_t \kappa dS.$$

Recall that  $\kappa$  is the mean curvature of the interface  $\Sigma$ .

*Proof.* The time derivative (3.2.12) of  $E_p^{\text{surf}}$  can be simply obtained by the first variation of area formula (see, for instance, Theorem 5.19 of [GHL90]),

$$\frac{d}{dt} \text{Area}(\Sigma(t)) = \int_{\Sigma} n \cdot \gamma_t \kappa dS.$$

To calculate the time derivative of  $E_p^{\text{grav}}$ , as in the proof of the conservation of mass, we could rewrite it as the linear combination of integrals on  $\Omega_h^{\pm}$  (see (3.2.9)) so that the formula (3.2.3) can be applied. The divergence theorem in  $\Omega_h^{\pm}$  yields

$$\begin{aligned} 2 \iint_{\Omega_h^{\pm}} y dx dy &= \iint_{\Omega_h^{\pm}} \nabla_{x,y} \cdot (0, y^2) dx dy \\ &= \pm \int_{\Sigma} n \cdot (0, y^2) dS \mp \int_{y=\mp h^{\pm}} y^2 dx = \pm \int_{\Sigma} n \cdot (0, y^2) dS \mp (2\pi)^d (h^{\pm})^2 \end{aligned}$$

which implies

$$(3.2.13) \quad E_p^{\text{grav}} = \sum_{\pm} \underline{\rho}^{\pm} \left[ \iint_{\Omega_h^{\pm}} y dx dy \pm \frac{1}{2} (2\pi)^d (h^{\pm})^2 \right].$$

Recall that  $\underline{\rho}^{\pm} = \rho^{\pm} / (\rho^+ + \rho^-)$ . Then, via Lemma 3.2.2, we can conclude (3.2.11) by

$$\frac{d}{dt} E_p^{\text{grav}} = \sum_{\pm} \pm \underline{\rho}^{\pm} \int_{\Sigma} n \cdot \gamma_t y dS = Ag \int_{\Sigma} n \cdot \gamma_t y dS,$$

where we used the fact that  $A = \underline{\rho}^+ - \underline{\rho}^-$  (see (3.1.15)). □

**Remark 3.2.5.** *Note that the contribution  $E_p^{\text{grav}}$  of gravity in potential energy has the same signature as the Atwood number  $A$ . To see this, we consider the following real numbers*

$$M^{\pm} := \iint_{\Omega_h^{\pm}} y dx dy \pm \frac{1}{2} (2\pi)^d (h^{\pm})^2.$$

It is clear that  $M^+ + M^- = 0$ . Besides, we have

$$M^+ \geq \iint_{\Omega_{h^+}^+ \cap \{y < 0\}} y dx dy + \frac{1}{2} (2\pi)^d (h^+)^2 \geq \iint_{\{y < 0\}} y dx dy + \frac{1}{2} (2\pi)^d (h^+)^2 = 0.$$

These yield that there exists  $M \geq 0$  such that  $M^{\pm} = \pm M$ . Thus,  $E_p^{\text{grav}}$  equals  $AM$ , whose signature is the same as  $A$ . In particular, when  $A \leq 0$ ,  $E_p^{\text{grav}}$  becomes non-positive.

It remains to calculate the time derivative of kinetic energy  $E_k$  (see (3.1.13)). Since it is already written in terms of integrals on  $\Omega^\pm$ , one may expect to apply the formula (3.2.3), which is not valid for unbounded domains (namely, infinite depth case). To overcome this difficulty, we introduce a truncated kinetic energy,

$$(3.2.14) \quad E_k^h := \frac{1}{2} \sum_{\pm} \rho^\pm \iint_{\Omega_h^\pm} |\nabla_{x,y} \phi^\pm|^2 dx dy,$$

where  $\Omega_h^\pm$  are bounded domains defined by (3.2.9). Then we can compute the time derivative of  $E_k^h$  by the following lemma.

**Lemma 3.2.6.** *Under the hypothesis of Proposition 3.2.3, the following formula holds.*

$$(3.2.15) \quad \frac{d}{dt} E_k^h = -\frac{\sigma}{\rho^+ + \rho^-} \int_{\Sigma} n \cdot \gamma_t \kappa dS - Ag \int_{\Sigma} n \cdot \gamma_t y dS + \sum_{\pm} \mp \rho^\pm \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx.$$

*Proof.* Through Lemma 3.2.2, the time derivative of truncated kinetic energy  $E_k^h$ , defined by (3.2.14), reads

$$\frac{d}{dt} E_k^h = \sum_{\pm} \rho^\pm \left[ \iint_{\Omega_h^\pm} \nabla_{x,y} \phi^\pm \cdot \nabla_{x,y} \partial_t \phi^\pm dx dy \pm \int_{\Sigma} n \cdot \gamma_t \frac{|\nabla_{x,y} \phi^\pm|^2}{2} dS \right].$$

Then we apply divergence theorem to the integral in  $\Omega_h^\pm$  together with  $\Delta_{x,y} \phi^\pm = 0$  (the second equation in (3.1.10)) and obtain

$$\begin{aligned} \iint_{\Omega_h^\pm} \nabla_{x,y} \phi^\pm \cdot \nabla_{x,y} \partial_t \phi^\pm dx dy &= \pm \int_{\Sigma} n \cdot \nabla_{x,y} \phi^\pm \partial_t \phi^\pm dS \mp \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx \\ &= \pm \int_{\Sigma} n \cdot \gamma_t \partial_t \phi^\pm dS \mp \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx, \end{aligned}$$

where the second equality follows from the third equation in (3.1.10). By combing the computations above, we can conclude by using the first equation in (3.1.10) that

$$\begin{aligned} \frac{d}{dt} E_k^h &= \sum_{\pm} \rho^\pm \left[ \pm \int_{\Sigma} n \cdot \gamma_t \left( \phi_t^\pm + \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) dS \mp \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx \right] \\ &= \sum_{\pm} \left[ \mp \int_{\Sigma} n \cdot \gamma_t \left( \frac{P^\pm}{\rho^+ + \rho^-} + \rho^\pm g y \right) dS \mp \rho^\pm \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx \right] \\ &= -\frac{\sigma}{\rho^+ + \rho^-} \int_{\Sigma} n \cdot \gamma_t \kappa dS - Ag \int_{\Sigma} n \cdot \gamma_t y dS + \sum_{\pm} \mp \rho^\pm \int_{y=\mp h^\pm} \phi_y^\pm \phi_t^\pm dx. \end{aligned}$$

Here, we use the fourth equation in (3.1.10) to obtain the last equality.  $\square$

In the rest of this section, we will finish the proof of Proposition 3.2.3, which is a consequence of Lemma 3.2.4 and 3.2.6.

*Proof of Proposition 3.2.3.* Taking the sum of (3.2.11), (3.2.12), and (3.2.15), we have, for all time  $t \in [0, T[$ ,

$$E_p(t) + E_k^h(t) = E_p(0) + E_k^h(0) + \sum_{\pm} \mp \rho^{\pm} \int_0^t \int_{\mathbb{T}^d} \phi_y^{\pm}(t', x, \mp h^{\pm}) \phi_t^{\pm}(t', x, \mp h^{\pm}) dx dt'.$$

Recall that, in finite depth case  $H^{\pm} < +\infty$ , we choose  $h^{\pm} = H^{\pm}$  (see (3.2.9)). Thus, the truncated kinetic energy  $E_k^h$  equals the real one  $E_k$  and the last integral on the right-hand side vanishes due to the fifth equation in (3.1.10). The conservation of energy then follows.

The main difficulty is the infinite depth case  $H^+ = +\infty$  or  $H^- = +\infty$ . Notice that, from the fifth equation of (3.1.10) and the condition (2) in Definition 3.1.2, one could see that, as  $h^{\pm} \rightarrow H^{\pm}$ , the truncated kinetic energy  $E_k^h$  tends to the real one  $E_k$ . Thus, it remains to show that, uniformly in time  $t' \in [0, t]$ , the integral

$$\int_{\mathbb{T}^d} \phi_y^{\pm}(t', x, \mp h^{\pm}) \phi_t^{\pm}(t', x, \mp h^{\pm}) dx$$

tends to zero as  $h^{\pm} \rightarrow +\infty$ . Recall that the derivative  $\phi_y^{\pm}$  of scalar potential has arbitrary polynomial decay at infinity (3.1.25) (see also Lemma 3.2.1), which reduces our problem to the uniform-in-time polynomial growth of  $\phi_t^{\pm}$ . From the first equation in (3.1.10), we can rewrite  $\phi_t^{\pm}$  as

$$\phi_t^{\pm} = -\frac{|\nabla_{x,y} \phi^{\pm}|^2}{2} - \frac{P^{\pm}}{\rho^+ + \rho^-} - gy,$$

where the first term has polynomial decay thanks to (3.1.25), the last term is polynomial, and the growth of  $P^{\pm}$  is assumed in the condition (6) in Definition 3.1.2. In conclusion,  $\phi_y^{\pm}(\cdot, \mp h^{\pm}) \phi_t^{\pm}(\cdot, \mp h^{\pm})$  decays polynomially at infinity, uniformly in time  $t' \in [0, t]$ , and its integral in  $x$  tends to zero as  $h^{\pm} \rightarrow +\infty$ , which proves the conservation of energy in infinite depth case.  $\square$

### 3.3 . Virial theorem

The goal of this section is to complete the proof of Theorem 3.1.1 via a direct calculus. Recall that the virial identity reads

$$(3.3.1) \quad \frac{1}{2} \frac{d}{dt} \int_{\Sigma} n \cdot (0, y\psi) dS = \tilde{E}_k - E_p + R, \quad R = R_b^+ + R_b^- + R_s,$$

where the modified kinetic energy  $\tilde{E}_k$  is defined by (3.1.19) and the formula of reminders  $R_b^{\pm}$  and  $R_s$  is given by (3.1.21) and (3.1.22), respectively. The main idea is similar to the proof of conservation laws in Section 3.2.2. We will rewrite the left-hand side of (3.3.1) in terms of integrals on  $\Omega^{\pm}$  (see Lemma 3.3.1) and apply the formula (3.2.3). The only difficulty is the term concerning curvature in (3.3.7), which will be studied in Lemma 3.3.3. To conclude Theorem 3.1.1, we also need to check that the reminders  $R_b^{\pm}$  and  $R_s$  are non-negative. The fact  $R_b^{\pm} \geq 0$  is obvious, while  $R_s \geq 0$  is not trivial and we shall use the same method as in Lemma 3.3.3 to prove it.

As in Section 3.2.2, to cover the infinite depth case  $H^+ = +\infty$  or  $H^- = +\infty$ , we will work on the truncated domains  $\Omega_h^\pm$  defined by (3.2.9), which we recall below

$$(3.3.2) \quad \begin{aligned} \Omega_h^+ &:= \Omega^+ \cap \{y > -h^+\}, \quad \Omega_h^- := \Omega^- \cap \{y < h^-\}; \\ h^\pm &= \begin{cases} H^\pm, & \text{if } H^\pm < +\infty; \\ \text{arbitrary number such that } \Sigma \subset \{\mp y < h^\pm\}, & \text{if } H^\pm = +\infty, \end{cases} \end{aligned}$$

and take the limit  $h^\pm \rightarrow +\infty$  in the end.

**Lemma 3.3.1.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to (3.1.10) (see Definition 3.1.2) and  $\Omega_h^\pm$  be truncated domains defined by (3.3.2). Then there holds*

$$(3.3.3) \quad \int_\Sigma n \cdot (0, y\psi) dS = \sum_{\pm} \rho^\pm \left[ \iint_{\Omega_h^\pm} \partial_y(y\phi^\pm) dx dy - h^\pm \int_{y=\mp h^\pm} \phi^\pm dx \right].$$

*Proof.* Recall that  $\psi$  is defined by (3.1.16). By divergence theorem on  $\Omega_h^\pm$ , we have

$$\begin{aligned} \int_\Sigma n \cdot (0, y\psi) dS &= \sum_{\pm} \rho^\pm \int_\Sigma n \cdot (0, y\phi^\pm) dS \\ &= \sum_{\pm} \rho^\pm \left[ \iint_{\Omega_h^\pm} \nabla_{x,y} \cdot (0, y\phi^\pm) dx dy \pm \int_{y=\mp h^\pm} y\phi^\pm dx \right], \end{aligned}$$

which completes the proof.  $\square$

Based on Lemma 3.3.1, it suffices to compute the time derivative of the terms on the right-hand side of (3.3.3).

**Lemma 3.3.2.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to (3.1.10) (see Definition 3.1.2) and  $\Omega_h^\pm$  be the truncated domains defined by (3.3.2). Then there hold*

$$(3.3.4) \quad \begin{aligned} \frac{d}{dt} \iint_{\Omega_h^\pm} \partial_y(y\phi^\pm) dx dy &= \iint_{\Omega_h^\pm} \left( \frac{1}{2} |\nabla_x \phi^\pm|^2 + \frac{3}{2} |\partial_y \phi^\pm|^2 \right) dx dy \\ &\quad - \int_\Sigma n \cdot (0, y) \left( \pm \frac{P^\pm}{\rho^\pm} \right) dS - h^\pm \int_{y=\mp h^\pm} \frac{P^\pm}{\rho^\pm} dx \\ &\quad - 2g \iint_{\Omega_h^\pm} y dx dy \pm \int_{y=\mp h^\pm} \phi^\pm \partial_y(y\phi^\pm) dx, \end{aligned}$$

$$(3.3.5) \quad \frac{d}{dt} \int_{y=\mp h^\pm} \phi^\pm dx = - \int_{y=\mp h^\pm} \frac{|\nabla_{x,y} \phi^\pm|^2}{2} dx - \int_{y=\mp h^\pm} \frac{P^\pm}{\rho^\pm} dx \pm gh^\pm (2\pi)^d.$$

*Proof.* The second equality (3.3.5) can be obtained directly from the first equation of (3.1.10). In what follows, we focus on the proof of the first equality (3.3.4). An application of the formula (3.2.3) gives

$$\frac{d}{dt} \iint_{\Omega_h^\pm} \partial_y(y\phi^\pm) dx dy = \iint_{\Omega_h^\pm} \partial_y(y\phi_t^\pm) dx dy \pm \int_\Sigma n \cdot \gamma_t \partial_y(y\phi^\pm) dS$$

$$\begin{aligned}
&= - \iint_{\Omega_h^\pm} \partial_y \left( y \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) dx dy - \iint_{\Omega_h^\pm} \partial_y \left( y \frac{P^\pm}{\rho^\pm} \right) dx dy \\
&\quad - \iint_{\Omega_h^\pm} \partial_y (gy^2) dx dy \pm \int_{\Sigma} n \cdot \nabla_{x,y} \phi^\pm \partial_y (y \phi^\pm) dS,
\end{aligned}$$

where the second equality follows from the system (3.1.10). Now, we apply divergence theorem to the second and the last term on the right-hand side,

$$\begin{aligned}
\iint_{\Omega_h^\pm} \partial_y \left( y \frac{P^\pm}{\rho^\pm} \right) dx dy &= \pm \int_{\Sigma} n \cdot (0, y) \frac{P^\pm}{\rho^\pm} dS \mp \int_{y=\mp h^\pm} y \frac{P^\pm}{\rho^\pm} dx \\
&= \int_{\Sigma} n \cdot (0, y) \left( \pm \frac{P^\pm}{\rho^\pm} \right) dS + h^\pm \int_{y=\mp h^\pm} \frac{P^\pm}{\rho^\pm} dx, \\
\pm \int_{\Sigma} n \cdot \nabla_{x,y} \phi^\pm \partial_y (y \phi^\pm) dS &= \iint_{\Omega_h^\pm} \nabla_{x,y} \cdot (\nabla_{x,y} \phi^\pm \partial_y (y \phi^\pm)) dx dy \\
&\quad \pm \int_{y=\mp h^\pm} \phi_y^\pm \partial_y (y \phi^\pm) dx,
\end{aligned}$$

and deduce that

$$\begin{aligned}
\frac{d}{dt} \iint_{\Omega_h^\pm} \partial_y (y \phi^\pm) dx dy &= \iint_{\Omega_h^\pm} \left[ \nabla_{x,y} \cdot (\nabla_{x,y} \phi^\pm \partial_y (y \phi^\pm)) - \partial_y \left( y \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) \right] dx dy \\
&\quad - \int_{\Sigma} n \cdot (0, y) \left( \pm \frac{P^\pm}{\rho^\pm} \right) dS - h^\pm \int_{y=\mp h^\pm} \frac{P^\pm}{\rho^\pm} dx \\
&\quad - 2g \iint_{\Omega_h^\pm} y dx dy \pm \int_{y=\mp h^\pm} \phi_y^\pm \partial_y (y \phi^\pm) dx.
\end{aligned}$$

Then the desired equality (3.3.6) follows from the following identity,

$$\nabla_{x,y} \cdot (\nabla_{x,y} \phi^\pm \partial_y (y \phi^\pm)) - \partial_y \left( y \frac{|\nabla_{x,y} \phi^\pm|^2}{2} \right) = \frac{1}{2} |\nabla_x \phi^\pm|^2 + \frac{3}{2} |\partial_y \phi^\pm|^2,$$

which can be computed directly by using  $\Delta_{x,y} \phi^\pm = 0$  (the second equation in (3.1.10)).  $\square$

By combining Lemma 3.3.1 and 3.3.2, we conclude that

$$\begin{aligned}
(3.3.6) \quad \frac{d}{dt} \int_{\Sigma} n \cdot (0, y \psi) dS &= 2(\tilde{E}_k^h - E_p^{\text{grav}} + R_{b,+}^h + R_{b,-}^h) \\
&\quad \pm \int_{y=\mp h^\pm} \phi_y^\pm \partial_y (y \phi^\pm) dx - \frac{\sigma}{\rho^+ + \rho^-} \int_{\Sigma} n \cdot (0, y) \kappa dS,
\end{aligned}$$

where we used the fourth equation of (3.1.10) to make appear the mean curvature  $\kappa$  and used (3.2.13) to obtain  $E_p^{\text{grav}}$ . Here  $\tilde{E}_k^h$  and  $R_{b,\pm}^h$  represent the truncated quantities

$$\tilde{E}_k^h = \sum_{\pm} \rho^\pm \iint_{\Omega_h^\pm} \left( \frac{1}{4} |\nabla_x \phi^\pm|^2 + \frac{3}{4} |\partial_y \phi^\pm|^2 \right) dx dy, \quad R_{b,\pm}^h = \frac{1}{4} \rho^\pm h^\pm \int_{y=\mp h^\pm} |\nabla_x \phi^\pm|^2 dx,$$

whose limit as  $h^\pm \rightarrow H^\pm$  equals  $\tilde{E}_k$  (see (3.1.19)) and  $R_b^\pm$  (see (3.1.21)), respectively, according to the fifth equation in (3.1.10) and the decay (3.1.25) of scalar potentials. With the same reason, the second term on the right-hand side of (3.3.6) tends to zero as  $h^\pm \rightarrow H^\pm$ . As a conclusion, we have

$$(3.3.7) \quad \frac{1}{2} \frac{d}{dt} \int_{\Sigma} n \cdot (0, y\psi) dS = \tilde{E}_k - E_p^{\text{grav}} + R_b^+ + R_b^- - \frac{\sigma}{2(\rho^+ + \rho^-)} \int_{\Sigma} n \cdot (0, y) \kappa dS.$$

In comparison with the target (3.3.1), it remains to compute the last integral on the right-hand side of (3.3.7).

**Lemma 3.3.3.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to (3.1.10) (see Definition 3.1.2). Then there holds*

$$(3.3.8) \quad \int_{\Sigma} n \cdot (0, y) \kappa dS = \text{Area}(\Sigma) - \int_{\Sigma} |n \cdot (0, 1)| dS.$$

To prove the identity (3.3.8), we introduce another method to represent the fluid domains  $\Omega^\pm$ . For simplicity, we omit the dependence in time. There exists a  $C^2$  function  $\chi$  defined on  $\Omega := \mathbb{T}^d \times ]-H^+, H^-[$ , such that

$$\Omega^+ = \{\chi > 0\}, \quad \Omega^- = \{\chi < 0\}, \quad \Sigma = \{\chi = 0\}.$$

Such  $\chi$  is not unique, while our argument works for any choice of  $\chi$ . Let  $F$  be an arbitrary continuous function defined on  $\Omega$ . Then the integral of  $f := F|_{\Sigma}$  on the interface  $\Sigma$  equals

$$\int_{\Sigma} f dS = \iint_{\Omega} F \delta(\chi) |\nabla_{x,y} \chi| dx dy := \lim_{\epsilon \rightarrow 0} \iint_{\Omega} F \varphi_{\epsilon}(\chi) |\nabla_{x,y} \chi| dx dy,$$

where  $\varphi_{\epsilon} := \epsilon^{-1} \varphi(\cdot/\epsilon)$  with  $\varphi \in C_c^\infty(\mathbb{R})$  and  $\varphi(0) = 1$ . Note that the normal direction  $n$  and the mean curvature  $\kappa$ , which are both defined on  $\Sigma$ , could be rewritten as restrictions

$$n = - \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \Big|_{\Sigma} \quad \text{and} \quad \kappa = - \nabla_{x,y} \cdot \left( \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \right) \Big|_{\Sigma},$$

respectively.

*Proof of Lemma 3.3.3.* Through the representation above, the left-hand side of (3.3.8) reads

$$(3.3.9) \quad \begin{aligned} \int_{\Sigma} n \cdot (0, y) \kappa dS &= \iint_{\Omega} \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \cdot (0, y) \nabla_{x,y} \cdot \left( \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \right) \delta(\chi) |\nabla_{x,y} \chi| dx dy \\ &= \iint_{\Omega} y \chi_y \nabla_{x,y} \cdot \left( \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \right) \delta(\chi) dx dy \\ &= \lim_{\epsilon \rightarrow 0} \iint_{\Omega} y \chi_y \nabla_{x,y} \cdot \left( \frac{\nabla_{x,y} \chi}{|\nabla_{x,y} \chi|} \right) \varphi_{\epsilon}(\chi) dx dy. \end{aligned}$$

Then we apply divergence theorem to the right-hand side (note that the integrands vanish near  $y = \mp H^\pm$  due to the cut-off  $\varphi_\epsilon$ ) and deduce that

$$\begin{aligned}
& \iint_{\Omega} y\chi_y \nabla_{x,y} \cdot \left( \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \right) \varphi_\epsilon(\chi) dx dy \\
(3.3.10) \quad &= - \iint_{\Omega} \nabla_{x,y}(y\chi_y) \cdot \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \varphi_\epsilon(\chi) dx dy - \iint_{\Omega} y\chi_y \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \cdot (\varphi'_\epsilon(\chi) \nabla_{x,y}\chi) dx dy \\
&= - \iint_{\Omega} \frac{\chi_y^2}{|\nabla_{x,y}\chi|} \varphi_\epsilon(\chi) dx dy - \iint_{\Omega} \partial_y (|\nabla_{x,y}\chi|) y \varphi_\epsilon(\chi) dx dy \\
&\quad - \iint_{\Omega} y\chi_y |\nabla_{x,y}\chi| \varphi'_\epsilon(\chi) dx dy.
\end{aligned}$$

Via integration by parts, the second term on the right-hand side equals

$$(3.3.11) \quad - \iint_{\Omega} \partial_y (|\nabla_{x,y}\chi|) y \varphi_\epsilon(\chi) dx dy = \iint_{\Omega} |\nabla_{x,y}\chi| \varphi_\epsilon(\chi) dx dy + \iint_{\Omega} y\chi_y |\nabla_{x,y}\chi| \varphi'_\epsilon(\chi) dx dy.$$

By combing (3.3.9), (3.3.10), and (3.3.11) above, we have

$$\begin{aligned}
\int_{\Sigma} n \cdot (0, y) \kappa dS &= \lim_{\epsilon \rightarrow 0} \left[ - \iint_{\Omega} \frac{\chi_y^2}{|\nabla_{x,y}\chi|} \varphi_\epsilon(\chi) dx dy + \iint_{\Omega} |\nabla_{x,y}\chi| \varphi_\epsilon(\chi) dx dy \right] \\
&= - \iint_{\Omega} \left| \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \cdot (0, 1) \right|^2 \delta(\chi) |\nabla_{x,y}\chi| dx dy + \iint_{\Omega} \delta(\chi) |\nabla_{x,y}\chi| dx dy,
\end{aligned}$$

where, in term of integrals on  $\Sigma$ , the right-hand side can be written as

$$- \int_{\Sigma} |n \cdot (0, 1)|^2 dS + \int_{\Sigma} dS,$$

which completes the proof of (3.3.8).  $\square$

*Proof of Theorem 3.1.1.* By inserting (3.3.8) into (3.3.7) and using the definition (3.1.14) of potential energy  $E_p$  together with the definition (3.1.22) of  $R_s$ , we are able to conclude the virial identity (3.3.1).

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Sigma} n \cdot (0, y\psi) dS \\
&= \tilde{E}_k - E_p^{\text{grav}} + R_b^+ + R_b^- - \frac{\sigma}{2(\rho^+ + \rho^-)} \left[ \text{Area}(\Sigma(t)) - \int_{\Sigma} |n \cdot (0, 1)| dS \right] \\
&= \tilde{E}_k - E_p^{\text{grav}} - E_p^{\text{surf}} + R_b^+ + R_b^- \\
&\quad + \frac{\sigma}{2(\rho^+ + \rho^-)} \left[ \text{Area}(\Sigma(t)) + \int_{\Sigma} |n \cdot (0, 1)| dS - 2(2\pi)^d \right] \\
&= \tilde{E}_k - E_p + R_b^+ + R_b^- + R_s.
\end{aligned}$$

The only remaining point is to show that  $R_s \geq 0$ , which can be proved by using again the method in the proof of Lemma 3.3.3. Recall that the remainder  $R_s$  equals

$$R_s = \frac{\sigma}{2(\rho^+ + \rho^-)} \left[ \int_{\Sigma} (1 + |n \cdot (0, 1)|^2) dS - 2(2\pi)^d \right],$$

where the integral on  $\Sigma$  can be expressed as an integral on  $\Omega = \mathbb{T}^d \times ]-H^+, H^-[$ . Namely,

$$\begin{aligned} \int_{\Sigma} (1 + |n \cdot (0, 1)|^2) dS &= \iint_{\Omega} \left( 1 + \left| \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \cdot (0, 1) \right|^2 \right) \delta(\chi) |\nabla_{x,y}\chi| dx dy \\ &= \iint_{\Omega} \left( |\nabla_{x,y}\chi| + \frac{\chi_y^2}{|\nabla_{x,y}\chi|} \right) \delta(\chi) dx dy. \end{aligned}$$

Noticing that

$$|\nabla_{x,y}\chi| + \frac{\chi_y^2}{|\nabla_{x,y}\chi|} \geq 2|\chi_y|,$$

we have

$$\begin{aligned} \int_{\Sigma} (1 + |n \cdot (0, 1)|^2) dS &\geq 2 \iint_{\Omega} |\chi_y| \delta(\chi) dx dy \\ &= 2 \iint_{\Omega} \left| \frac{\nabla_{x,y}\chi}{|\nabla_{x,y}\chi|} \cdot (0, 1) \right| \delta(\chi) |\nabla_{x,y}\chi| dx dy. \end{aligned}$$

In terms of integrals on  $\Sigma$ , this inequality can be rewritten as

$$\int_{\Sigma} (1 + |n \cdot (0, 1)|^2) dS \geq 2 \int_{\Sigma} |n \cdot (0, 1)| dS \geq 2 \left| \int_{\Sigma} n \cdot (0, 1) dS \right|.$$

Since  $\nabla_{x,y} \cdot (0, 1) = 0$ , through divergence theorem on  $\Omega_h^+$  (see (3.3.2)), the last absolute value equals

$$2 \left| \int_{\Sigma} n \cdot (0, 1) dS \right| = 2 \left| \int_{y=-h^+} dx \right| = 2(2\pi)^d,$$

which proves the desired result  $R_s \geq 0$ .  $\square$

## 3.4 . Instability of system

The purpose of this section is to prove the instability of the system (3.1.10) in graph case (Theorem 3.1.7) and 2D overlapping case (Theorem 3.1.9). The idea of the proof is similar to that in [AZ24]. Based on the virial identity (3.1.18), we first show that the right-hand side can be bounded from below by the absolute value  $|E|$  of the total energy  $E$  (see (3.1.12)), which is preserved in time. Then we apply the trace estimate to control the integral on the left-hand side of (3.1.18) by the geometrical quantities such as slope and curvature.

### 3.4.1 . The consequence of the virial identity

In this part, we aim to prove the following corollary of the virial identity (3.1.18), stating that, in the setting of RTI or KHI problem, the integral on the left-hand side of (3.1.18) grows at least like  $O(t)$ .

**Corollary 3.4.1.** Consider  $d \geq 1$ ,  $H^\pm \in ]0, +\infty]$ ,  $\rho^+ \leq \rho^-$ ,  $\sigma \geq 0$ , and  $T \in ]0, +\infty]$ . Let  $(\gamma, \phi^\pm, P^\pm)$  be any regular solution to (3.1.10) (see Definition 3.1.2) on  $[0, T[$ . If the condition (3.1.27) recalled below holds

$$E \neq 0 \text{ when } \sigma = 0, \text{ and } E < 0 \text{ when } \sigma > 0,$$

then we have, for all time  $t \in [0, T[$ ,

$$(3.4.1) \quad I(t) \geq I(0) + |E|t, \quad I(t) = \int_{\Sigma} n \cdot (0, y\psi) dS,$$

where  $\psi$  is defined by (3.1.16),

$$\psi = \underline{\rho}^+ \phi^+|_{\Sigma} - \underline{\rho}^- \phi^-|_{\Sigma},$$

and  $E$  is the total energy defined by (3.1.12), which is preserved in time.

*Proof.* By integrating in time on both sides of the virial identity (3.1.18), we have

$$I(t) = I(0) + 2 \int_0^t \left( \tilde{E}_k(\tau) - E_p(\tau) + R(\tau) \right) d\tau \geq I(0) + 2 \int_0^t \left( \frac{E_k(\tau)}{2} - E_p(\tau) \right) d\tau,$$

where the inequality follows from the fact that  $R \geq 0$  and  $\tilde{E}_k \geq E_k/2$ , which can be seen by comparing the definition (3.1.13) and (3.1.19) of kinetic energy  $E_k$  and modified kinetic energy  $\tilde{E}_k$ , respectively. Then the desired inequality (3.4.1) can be reduced to the following one,

$$\frac{E_k(t)}{2} - E_p(t) \geq \frac{|E|}{2}, \quad \forall t \in [0, T[$$

where  $E$  is the total energy preserved in time. To prove this, we divide the condition (3.1.27) into two cases : (1)  $E < 0$ ; (2)  $E > 0$  with  $\sigma = 0$ . For simplicity, we omit the dependence in time in the sequel.

In the first case  $E < 0$ , the proof is direct

$$\frac{E_k}{2} - E_p = \frac{E_k}{2} - (E - E_k) = |E| + \frac{3}{2}E_k \geq |E|.$$

In the second case  $E > 0$  with  $\sigma = 0$ , we observe from the definition (3.1.14) that the potential energy  $E_p$  is non-negative since  $\rho^+ \leq \rho^-$ . Thus,

$$\frac{E_k}{2} - E_p = \frac{1}{2}(E - E_p) - E_p = \frac{E}{2} - \frac{3}{2}E_p \geq \frac{E}{2} = \frac{|E|}{2},$$

which completes the proof. □

### 3.4.2 . The graph case

In this section, we focus on the graph case, where the interface  $\Sigma(t)$  is represented as  $\{y = \eta(t, x)\}$  for some real-valued function  $\eta$ , and complete the proof of Theorem 3.1.7. By comparing the inequality (3.4.1) above and the target (3.1.28), it suffices to control the integral  $I(t)$  defined in (3.4.1) above by a function of  $\|\nabla_x \eta\|_{L^\infty}$ , which will be achieved by trace estimate (see Proposition 3.4.2). Since the time dependence has no impact in the arguments below, we will omit it for the sake of simplicity.

To begin with, we reformulate  $I(t)$  defined by (3.4.1) above in terms of  $\eta$ . By fixing the parametrization  $\gamma$  as  $\gamma(t, x) = (x, \eta(t, x))$ , the normal direction  $n$  and the surface element  $dS$  read

$$n = \frac{(-\nabla_x \eta, 1)}{\sqrt{1 + |\nabla_x \eta|^2}}, \quad dS = \sqrt{1 + |\nabla_x \eta|^2} dx,$$

allowing us to express  $I(t)$  as

$$(3.4.2) \quad I = \int_{\mathbb{T}^d} \frac{(-\nabla_x \eta, 1)}{\sqrt{1 + |\nabla_x \eta|^2}} \cdot (0, \eta \psi) \sqrt{1 + |\nabla_x \eta|^2} dx = \int_{\mathbb{T}^d} \eta \psi dx.$$

Recall that  $\psi$  is defined by (3.1.16), which can be written as,

$$(3.4.3) \quad \psi(x) = \underline{\rho}^+ \phi^+(x, \eta(x)) - \underline{\rho}^- \phi^-(x, \eta(x)).$$

The main difficulty is to control  $\psi$  by some quantities depending only on  $\eta$ . To do so, we introduce the following trace estimate, whose detailed proof can be found in Appendix B of [AZ24].

**Proposition 3.4.2.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to the system (3.1.10) (see Definition 3.1.2) with depths  $H^\pm \in ]0, +\infty]$ . Then the following estimate holds,*

$$(3.4.4) \quad \int_{\mathbb{T}^d} |\phi^\pm(x, \eta(x)) - m^\pm|^2 dx \leq C(1 + \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}) \int_{\Omega^\pm} |\nabla_{x,y} \phi^\pm|^2 dx dy,$$

where  $m^\pm \in \mathbb{R}$  is the average of  $\phi^\pm(\cdot, \eta(\cdot))$  on  $\mathbb{T}^d$ , namely

$$(3.4.5) \quad m^\pm := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \phi^\pm(x, \eta(x)) dx.$$

Here the constant  $C > 0$  depends on  $d_0 > 0$  appearing in the condition (5) of Definition 3.1.2.

**Remark 3.4.3.** *Indeed, the left-hand side of (3.4.4) can be refined as the Sobolev norm  $H^{1/2}(\mathbb{T}^d)$ , while  $L^2(\mathbb{T}^d)$ -norm is enough for the proof of Theorem 3.1.7.*

*Proof of Theorem 3.1.7.* By using the reformulation (3.4.2) of  $I(t)$ , the inequality (3.4.1) reads

$$(3.4.6) \quad |E|t + \int_{\mathbb{T}^d} \eta \psi dx \Big|_{t=0} \leq \int_{\mathbb{T}^d} \eta(t, x) \psi(t, x) dx.$$

Recall that, in the condition (2) of Definition 3.1.2, we have assumed that  $M = 0$ , where the mass  $M$  is defined by (3.1.11) and can be expressed, in the graph case, as

$$(3.4.7) \quad M(t) = \int_{\mathbb{T}^d} \eta(t, x) dx \equiv 0.$$

Therefore, from the formula (3.4.3) of  $\psi$ , the right-hand side of (3.4.6) equals

$$\int_{\mathbb{T}^d} \eta(t, x) \psi(t, x) dx = \sum_{\pm} \pm \rho^{\pm} \int_{\mathbb{T}^d} \eta(t, x) (\phi^{\pm}(t, x, \eta(x)) - m^{\pm}(t)) dx,$$

where the real number  $m^{\pm}(t)$  (independent of  $x$ ) is defined by (3.4.5). Thanks to the trace estimate (3.4.4), we can control the inner product of  $\eta$ ,  $\psi$  as

$$(3.4.8) \quad \begin{aligned} \int_{\mathbb{T}^d} \eta \psi dx &\leq \sum_{\pm} \rho^{\pm} \|\eta\|_{L^2(\mathbb{T}^d)} \sqrt{\int_{\mathbb{T}^d} |\phi^{\pm}(x, \eta(x)) - m^{\pm}|^2 dx} \\ &\lesssim_{d_0} \sum_{\pm} \|\eta\|_{L^2(\mathbb{T}^d)} \sqrt{(1 + \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}) \rho^{\pm} \int_{\Omega^{\pm}} |\nabla_{x,y} \phi^{\pm}|^2 dx dy} \\ &\lesssim \|\eta\|_{L^2(\mathbb{T}^d)} \sqrt{1 + \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}} \sqrt{E_k}, \end{aligned}$$

where the last inequality follows from the definition (3.1.13) of kinetic energy  $E_k$ . Then, we replace the  $\sqrt{E_k}$  on the right-hand side of (3.4.8) by  $\sqrt{E - E_p}$ . The potential energy  $E_p$  defined by (3.1.14) can be expressed, in the graph case, as

$$E_p = \frac{Ag}{2} \|\eta\|_{L^2(\mathbb{T}^d)}^2 + \frac{\sigma}{\rho^+ + \rho^-} [\text{Area}(\Sigma(t)) - (2\pi)^d],$$

satisfying

$$E - E_p \leq |E| - \frac{Ag}{2} \|\eta\|_{L^2(\mathbb{T}^d)}^2 \leq E + |A|g \|\eta\|_{L^2(\mathbb{T}^d)}^2.$$

Note that the Atwood number  $A = (\rho^+ - \rho^-)/(\rho^+ + \rho^-) \in [-1, 1]$  is negative when  $\rho^+ \leq \rho^-$ . Therefore, by combining (3.4.6) and (3.4.8), we have

$$(3.4.9) \quad |E|t + \int_{\mathbb{T}^d} \eta \psi dx \Big|_{t=0} \lesssim_{d_0} \|\eta\|_{L^2(\mathbb{T}^d)} \sqrt{1 + \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}} \sqrt{E + |A|g \|\eta\|_{L^2(\mathbb{T}^d)}^2}.$$

This inequality concludes the desired result (3.1.28) by observing  $\|\eta\|_{L^2(\mathbb{T}^d)} \lesssim \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}$ , which is a consequence of the zero mass condition 3.4.7 (namely, the condition (3) in Definition 3.1.2).  $\square$

To end this part, let us assume that the regular solution  $(\gamma, \phi^{\pm}, P^{\pm})$  studied in Theorem 3.1.7 is global-in-time (we do not know whether such solution exists). Directly from the inequality (3.1.28), we have

$$t^{2/5} \lesssim \|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)}, \text{ as } t \rightarrow +\infty.$$

In the absence of surface tension ( $\sigma = 0$ ), this growing rate of the slope  $\|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)}$  can be refined to  $O(t^{2-\epsilon})$  for any  $0 < \epsilon \ll 1$ . More precisely, we have

**Proposition 3.4.4.** *Under the same assumption of Theorem 3.1.7, we further assume that  $\sigma = 0$  and the regular solution  $(\gamma, \phi^\pm, P^\pm)$  is global-in-time. Then, for all  $0 < \epsilon \ll 1$ , there holds*

$$(3.4.10) \quad \limsup_{t \rightarrow +\infty} \frac{\|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)}}{t^{2-\epsilon}} = +\infty.$$

*Proof.* The proof is based on a refined version of Corollary 3.4.1. We claim that, for all time  $t \geq 0$ , the right-hand side of the virial identity (3.1.18) can be bounded from below by

$$(3.4.11) \quad \tilde{E}_k - E_p + R \geq \frac{|E| - E_p}{2}.$$

To prove this, we first use the fact that  $\tilde{E}_k \geq E_k/2$  and  $R \geq 0$  and obtain

$$\tilde{E}_k - E_p + R \geq \frac{E_k}{2} - E_p.$$

When  $E > 0$ , the right-hand side reads

$$\frac{E}{2} - \frac{3}{2}E_p = \frac{|E|}{2} - \frac{3}{2}E_p \geq \frac{|E| - E_p}{2},$$

where we used the fact that  $E_p \leq 0$  since there is no surface tension  $\sigma = 0$ . When  $E < 0$ , we have

$$\frac{E}{2} - \frac{3}{2}E_p = \frac{E}{2} - (E - E_k) - \frac{E_p}{2} = \frac{|E|}{2} + E_k - \frac{E_p}{2} \geq \frac{|E| - E_p}{2},$$

which completes the proof of (3.4.11).

By inserting (3.4.11) into the arguments in previous paragraphs, (3.4.9) becomes

$$\int_0^t \frac{|E| - E_p(\tau)}{2} d\tau + I(0) \lesssim \|\eta\|_{L^2(\mathbb{T}^d)} \sqrt{1 + \|\nabla_x \eta\|_{L^\infty(\mathbb{T}^d)}} \sqrt{E + |A|g\|\eta\|_{L^2(\mathbb{T}^d)}^2}.$$

Recall that  $I(0)$ , defined in (3.4.1), depends only on initial data and  $E_p = Ag\|\eta\|_{L^2(\mathbb{T}^d)}/2$  is non-positive. Therefore, the inequality above implies that

$$(3.4.12) \quad \int_0^t F(\tau) d\tau \leq C \left( \sqrt{1 + \|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)}} F(t) + 1 \right), \quad F(t) = |E| + |A|g\|\eta(t)\|_{L^2(\mathbb{T}^d)},$$

for some the constant  $C > 0$  independent of time.

Assume that (3.4.10) is false. Namely, there exists time-independent constant  $M > 0$  such that  $\|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)}$  can be bounded by  $Mt^{2-\epsilon}$  for all time  $t \geq 1$ . Then (3.4.12) implies

$$(3.4.13) \quad \int_0^t F(\tau) d\tau \lesssim_M t^{1-\frac{\epsilon}{2}} (F(t) + 1).$$

We claim that this will lead to arbitrary polynomial growth of  $F(t)$ . Namely, for all  $N \in \mathbb{N}$ ,

$$(3.4.14) \quad t^{\frac{N\epsilon}{2}} \lesssim F(t), \quad \text{as } t \rightarrow +\infty,$$

which is a contradiction since there exists time-independent constant  $C_0 > 0$  such that, for all  $N \in \mathbb{N}$ ,

$$t^{\frac{N\epsilon}{2}} \lesssim F(t) \leq |E| + |A|gC_0 \|\nabla_x \eta(t)\|_{L^\infty(\mathbb{T}^d)} \leq |E| + |A|gC_0 M t^{2-\epsilon},$$

as  $t \rightarrow +\infty$ .

It remains to prove (3.4.14) for all  $N \in \mathbb{N}$  by induction. Firstly, the case  $N = 0$  is trivial from the definition (3.4.12) and the fact  $E \neq 0$ . Once (3.4.14) holds true for some  $N \in \mathbb{N}$ , the inequality (3.4.13) gives,

$$t^{\frac{N\epsilon}{2}+1} \lesssim \int_0^t \tau^{\frac{N\epsilon}{2}} d\tau \lesssim \int_0^t F(\tau) d\tau \lesssim t^{1-\frac{\epsilon}{2}}(F(t) + 1),$$

which yields

$$t^{\frac{(N+1)\epsilon}{2}} \lesssim F(t) + 1, \quad \text{as } t \rightarrow +\infty.$$

This is equivalent to (3.4.14) with  $N$  replaced by  $N + 1$ . Then the desired estimate (3.4.14) follows from an induction in  $N$ .  $\square$

### 3.4.3 . The 2D overlapping case

In this part, we shall focus on the 2D overlapping case and complete the proof of Theorem 3.1.9. The strategy of the proof is the same as that in Section 3.4.2. The only difference is the trace estimate (3.4.4), while, in 2D overlapping case, we need to replace the slope  $\|\nabla_x \eta\|_{L^\infty}$  appearing on the right-hand side of (3.4.4) by some quantities depending on the curvature  $\kappa$ .

In 2D overlapping case, the interface can be expressed via arc-length parameter  $s$ . In what follows, we fix the parametrization  $\gamma(t)$  as

$$(3.4.15) \quad \begin{aligned} \gamma(t): \mathbb{R}/L(t) &\rightarrow \mathbb{T} \times ]-H^+, H^-[ \\ s &\mapsto \gamma(t, s) = (\alpha(t, s), \beta(t, s)), \end{aligned}$$

where  $L(t)$  is the length of the interface  $\Sigma(t)$  and  $\alpha, \beta$  are horizontal and vertical components of  $\gamma(t)$ , respectively. Recall that the interface is assumed to satisfy the *chord-arc* condition (the condition (4) of Definition 3.1.2),

$$(3.4.16) \quad c_0 |s - s'| \leq |\gamma(t, s) - \gamma(t, s')| \leq |s - s'|, \quad \forall s, s' \in \mathbb{R}/L(t).$$

Here  $|s - s'|$  should be understood as the distance in  $\mathbb{R}/L(t)$  and  $c_0 > 0$  is a universal constant.

In terms of arc-length parametrization  $\gamma(t, s)$ , the normal direction reads  $n = (-\beta_s, \alpha_s)$ , and the curvature  $\kappa$  is given by

$$(3.4.17) \quad \tau_s(s) = -\kappa(s)n(s),$$

where  $\tau := \gamma_s = (\alpha_s, \beta_s)$  is the unit tangent direction. Besides, the surface element  $dS = ds$ , yielding that the mass  $M$  defined by (3.1.11), the potential energy  $E_p$  defined by (3.1.14), and the integral  $I$  defined in (3.4.1) can be written as

$$(3.4.18) \quad \begin{aligned} M &= \int_0^L \beta \alpha_s ds, & E_p &= \frac{Ag}{2} \int_0^L \beta^2 \alpha_s ds + \frac{\sigma}{\rho^+ + \rho^-} (L - 2\pi), \\ I &= \int_0^L \beta \alpha_s \psi ds. \end{aligned}$$

Recall that  $\psi$  is defined by (3.1.16)

$$\psi(t, s) = \underline{\rho}^+ \phi^+(t, \gamma(t, s)) - \underline{\rho}^- \phi^-(t, \gamma(t, s))$$

in terms of the traces of scalar potentials  $\phi^\pm$  at the interface  $\Sigma$ .

Now we introduce the trace estimate in 2D overlapping case, which is an analogue of Proposition 3.4.2. Since this result is independent of time, we will omit the time dependence for simplicity.

**Proposition 3.4.5.** *Let  $(\gamma, \phi^\pm, P^\pm)$  be a regular solution to the system (3.1.10) with depths  $H^\pm \in ]0, +\infty]$  (see Definition 3.1.2). We define  $\epsilon > 0$  as in (3.1.30),*

$$(3.4.19) \quad \epsilon := \min \left( \frac{c_0}{N_0(\|\kappa\|_{L^\infty} + 1)}, d_0 \right),$$

where  $0 < c_0 \leq 1$  is the chord-arc constant defined in (3.4.16),  $d_0 > 0$  is the constant appearing in the condition (5) of Definition 3.1.2 (in infinite depth case,  $d_0 = +\infty$ ), and  $N_0 \gg 1$  is a large enough universal constant. For all  $C^1$  function  $F$  on  $\Omega^\pm$  with  $\|\nabla_{x,y} F\|_{L^2(\Omega^\pm)} < +\infty$ , if its trace  $f = F|_\Sigma$  has zero mean

$$\int_\Sigma f ds = 0,$$

then there exists a universal constant  $C > 0$  such that

$$(3.4.20) \quad \|f\|_{L^2(\Sigma)} \leq C \frac{L}{\sqrt{\epsilon}} \|\nabla_{x,y} F\|_{L^2(\Omega^\pm)}.$$

Recall that  $L$  is the length of the interface  $\Sigma$ .

In particular, when the depths  $H^\pm < +\infty$ , the estimate (3.4.20) can be refined as

$$(3.4.21) \quad \|f\|_{L^2(\Sigma)} \leq C(H^+ + H^-) \epsilon^{-3/2} \|\nabla_{x,y} F\|_{L^2(\Omega^\pm)}$$

*Proof.* We focus on the domain  $\Omega^+$ , while the other one  $\Omega^-$  can be treated in the same way. The proof of this proposition is based on flattening by *normal geodesic coordinate* defined by

$$(3.4.22) \quad \begin{aligned} \Phi: \mathbb{R}/L \times [-\epsilon, \epsilon] &\rightarrow \mathbb{T} \times ]-H^+, H^-[ \\ (s, r) &\mapsto \gamma(s) + rn(s) \end{aligned}$$

It is clear that, for  $N_0$  large enough,  $\Phi$  is diffeomorphic (see Proposition 3.A.1) and its Jacobian is of determinant  $j(s, r) = 1 + r\kappa(s) \in [1/2, 3/2]$ , since

$$|r\kappa(s)| \leq \epsilon \|\kappa\|_{L^\infty} \leq \frac{c_0}{N_0} \leq \frac{1}{N_0} \ll 1.$$

Remark that  $\{r < 0\} \subset \Omega^+$ ,  $\{r = 0\} = \Sigma$ , and  $\{r > 0\} \subset \Omega^-$ . In the sequel, we shall use the restriction

$$-\epsilon \leq r \leq 0.$$

To begin with, we truncate  $F$  inside  $\{-\epsilon < r \leq 0\}$  using

$$G(s, r) := F \circ \Phi(s, r) \chi\left(\frac{r}{\epsilon}\right),$$

where  $\chi \in C_c^\infty(\mathbb{R})$  is supported inside  $] -1, 1[$  and equals to 1 near zero. One may observe that  $G$  has the same trace as  $F$  and

$$\begin{aligned} \iint |\partial_r G(s, r)|^2 ds dr &= \iint \left| n(s) \cdot (\nabla_{x,y} F) \circ \Phi(s, r) \chi\left(\frac{r}{\epsilon}\right) + \frac{1}{\epsilon} F \circ \Phi(s, r) \chi'\left(\frac{r}{\epsilon}\right) \right|^2 ds dr \\ &\lesssim \iint_{\Omega^+} |\nabla_{x,y} F|^2 dx dy + \frac{1}{\epsilon^2} \int_{-\epsilon}^0 \int_0^L |F \circ \Phi|^2 ds dr. \end{aligned}$$

Since  $G$  is supported in  $\{-\epsilon < r \leq 0\}$ , we have trivially,

$$f(s) = G(s, 0) = \int_{-\epsilon}^0 \partial_r G(s, r) dr,$$

thus

$$|f(s)|^2 \leq \epsilon \int_{-\epsilon}^0 |\partial_r G(s, r)|^2 dr.$$

By integrating in  $s$  on both sides, one obtains

$$\|f\|_{L^2(\Sigma)}^2 \leq \epsilon \iint |\partial_r G|^2 ds dr \lesssim \epsilon \iint_{\Omega^+} |\nabla_{x,y} F|^2 dx dy + \frac{1}{\epsilon} \int_{-\epsilon}^0 \int_0^L |F \circ \Phi|^2 ds dr.$$

In order to conclude (3.4.20), we try to apply Poincaré's inequality for the last term on right-hand side. The difficulty is that the mean value  $m$  of  $F \circ \Phi$  in  $[0, L] \times [-\epsilon, 0]$ , namely

$$m = \frac{1}{\epsilon L} \int_{-\epsilon}^0 \int_0^L F \circ \Phi(s, r) ds dr,$$

is non zero in general. To handle this problem, it suffices to check that this mean value  $m$  can also be bounded by  $\|\nabla_{x,y} F\|_{L^2(\Omega)}$ . In fact, by repeating the argument above and applying the zero mean condition for  $f = F \circ \Phi(\cdot, 0)$ , one may write  $m$  as an integral of  $\partial_r(F \circ \Phi)$ ,

$$m = \frac{1}{\epsilon L} \int_{-\epsilon}^0 \int_0^L F \circ \Phi(s, r) ds dr$$

$$\begin{aligned}
 &= \frac{1}{\epsilon L} \int_{-\epsilon}^0 \int_0^L \left( F \circ \Phi(s, 0) + \int_0^r \partial_r(F \circ \Phi)(s, r') dr' \right) ds dr \\
 &= \frac{1}{\epsilon L} \int_{-\epsilon}^0 \int_0^L \int_0^r \partial_r(F \circ \Phi)(s, r') dr' ds dr,
 \end{aligned}$$

which implies the desired control of  $m$  :

$$\begin{aligned}
 |m| &\leq \frac{\sqrt{\epsilon}}{\epsilon L} \int_{-\epsilon}^0 \int_0^L \sqrt{\int_{-\epsilon}^0 \int_0^L |\partial_r(F \circ \Phi)(s, r')|^2 dr' ds dr} \\
 &\leq \frac{\sqrt{\epsilon} \sqrt{\epsilon L}}{\epsilon L} \sqrt{\int_{-\epsilon}^0 \int_0^L \int_{-\epsilon}^0 |\partial_r(F \circ \Phi)(s, r')|^2 dr' ds dr} \\
 &= \sqrt{\frac{\epsilon}{L}} \|\partial_r(F \circ \Phi)\|_{L^2([0, L] \times [-\epsilon, 0])} \lesssim \sqrt{\frac{\epsilon}{L}} \|\nabla_{x,y} F\|_{L^2(\Omega^+)}
 \end{aligned}$$

Now we are able to apply Poincaré's inequality to  $F \circ \Phi - m$  on  $[0, L] \times [-\epsilon, 0]$  :

$$\begin{aligned}
 \int_{-\epsilon}^0 \int_0^L |F \circ \Phi|^2 ds dr &\leq 2 \int_{-\epsilon}^0 \int_0^L |F \circ \Phi - m|^2 ds dr + 2\epsilon L m^2 \\
 &\lesssim \max\{L, \epsilon\}^2 \int_{-\epsilon}^0 \int_0^L |\nabla_{s,r}(F \circ \Phi)|^2 ds dr + \epsilon^2 \|\nabla_{x,y} F\|_{L^2(\Omega^+)}^2 \\
 &\lesssim L^2 \|\nabla_{x,y} F\|_{L^2(\Omega^+)}^2,
 \end{aligned}$$

where the last inequality follows from the fact that  $L \geq 2\pi$  and  $\epsilon \ll 1$  by choosing  $N_0 \gg 1$ .

As for the refined version (3.4.21) in finite depth case  $H^\pm < +\infty$ , it suffices to apply Lemma 3.4.6 below.  $\square$

**Lemma 3.4.6.** *Under the assumptions of Proposition 3.4.5, if we further assume that  $H^\pm < +\infty$ , then the following inequality holds true.*

$$(3.4.23) \quad L\epsilon \leq 2\pi(H^+ + H^-).$$

*Proof.* To prove this inequality, we shall use the change of variable  $\Phi$  defined in (3.4.22),

$$\Phi(s, r) = \gamma(s) + rn(s).$$

Let us denote by  $\mathcal{O}_\epsilon$  the image of  $[0, L] \times [-\epsilon, \epsilon]$  by  $\Phi$ , which is a subdomain of  $\Omega^+ \cup \Sigma \cup \Omega^-$ . Therefore,

$$2\pi(H^+ + H^-) = \text{Vol}(\Omega^+ \cup \Sigma \cup \Omega^-) \geq \text{Vol}(\mathcal{O}_\epsilon) = \int_0^L \int_{-\epsilon}^\epsilon j(s, r) ds dr \geq L\epsilon,$$

where  $j$  is the determinant of the Jacobian of  $\Phi$  and, due to the definition (3.4.22) of  $\epsilon$ ,

$$j(s, r) = 1 + r\kappa(s) \geq 1 - \epsilon \|\kappa\|_{L^\infty} \geq \frac{1}{2},$$

when  $N_0 \gg 1$  is chosen large enough.  $\square$

Now, we are at the position to prove the main result Theorem 3.1.9. As Theorem 3.1.7, we will combine Corollary 3.4.1 and the trace estimate (3.4.21).

*Proof of Theorem 3.1.9.* By comparing the consequence (3.4.1) of virial identity and the target inequality (3.1.29), it suffices to control the integral  $I$  defined in (3.4.1) (see also its reformulation (3.4.18)) by the right-hand side of (3.1.29). Since the arguments below are independent of time, we shall omit the time dependence for simplicity.

Recall that in the condition (3) of Definition 3.1.2, we have assumed that the total mass  $M$  is identically zero. Thus, from the reformulation (3.4.18) and the definition (3.1.16) of  $\psi$ , we can write the integral  $I$  as

$$I = \sum_{\pm} \pm \underline{\rho}^{\pm} \int_0^L \beta \alpha_s \phi^{\pm}|_{\Sigma} ds = \sum_{\pm} \pm \underline{\rho}^{\pm} \int_0^L \beta \alpha_s (\phi^{\pm}|_{\Sigma} - m^{\pm}) ds,$$

where  $m^{\pm} \in \mathbb{R}$  is the mean value of  $\phi^{\pm}$  on  $\Sigma$ ,

$$m^{\pm} = \frac{1}{L} \int_0^L \phi^{\pm}(\gamma(s)) ds.$$

Clearly,  $\phi^{\pm} - m^{\pm}$  has zero mean on the interface  $\Sigma$ , allowing us to apply the trace estimate (3.4.5).

$$\begin{aligned} I &\leq \sum_{\pm} \underline{\rho}^{\pm} \|\beta \alpha_s\|_{L^2(\Sigma)} \|\phi^{\pm}|_{\Sigma} - m^{\pm}\|_{L^2(\Sigma)} \\ &\lesssim \sum_{\pm} \underline{\rho}^{\pm} \|\beta \alpha_s\|_{L^2(\Sigma)} (H^+ + H^-) \epsilon^{-3/2} \|\nabla_{x,y} \phi^{\pm}\|_{L^2(\Omega^{\pm})} \\ &\lesssim (H^+ + H^-) \epsilon^{-3/2} \|\beta \alpha_s\|_{L^2(\Sigma)} \sqrt{E_k}, \end{aligned}$$

where the last inequality follows from the definition (3.1.13) and the fact  $\underline{\rho}^{\pm} = \rho^{\pm}/(\rho^+ + \rho^-) \in [0, 1]$ . On the right-hand side, the  $L^2(\Sigma)$ -norm of  $\beta \alpha_s$  can be estimated by observing  $|\beta| \leq L$  and  $|\alpha_s| \leq |\gamma_s| = 1$ ,

$$\|\beta \alpha_s\|_{L^2(\Sigma)}^2 = \int_0^L |\beta|^2 |\alpha_s|^2 ds \leq L^2 \int_0^L ds = L^3,$$

while  $E_k$  can be estimated through the expression (3.4.18) of  $E_p$ ,

$$E_k = E - E_p \leq E + \frac{|A|g}{2} \int_0^L \beta^2 \alpha_s ds \leq E + |A|gL^3.$$

In conclusion, we have shown that the integral  $I$  can be bounded by

$$I \lesssim (H^+ + H^-) \epsilon^{-3/2} L^{3/2} \sqrt{E + |A|gL^3},$$

which, thanks to Lemma 3.4.6, implies

$$I \leq C \epsilon^{-3} \sqrt{|E| + |A|g\epsilon^{-3}},$$

for some constant  $C > 0$  depending on  $H^{\pm}$ . The proof of (3.1.29) is completed.  $\square$

### 3.A . Normal geodesic coordinate

The purpose of this section is the following proposition, stating that the mapping  $\Phi$ , known as *normal geodesic coordinate*, defined in (3.4.22), realizes a diffeomorphism to its image.

**Proposition 3.A.1.** *Consider  $L > 0$ ,  $0 < c_0 \leq 1$ , and  $N_0 > 0$ . Let  $\gamma : \mathbb{R}/L \rightarrow \mathbb{T} \times \mathbb{R}$  be the arc-length parametrization of a  $C^2$  curve  $\Sigma$  with length  $L$ , which satisfies the chord-arc condition : for all  $s_1, s_2 \in \mathbb{R}/L$ ,*

$$(3.A.1) \quad c_0 |s_1 - s_2| \leq |\gamma(s_1) - \gamma(s_2)| \leq |s_1 - s_2|.$$

We define  $\epsilon \in ]0, +\infty]$  as

$$(3.A.2) \quad \epsilon := \frac{c_0}{N_0 \|\kappa\|_{L^\infty}},$$

where  $\kappa$  is the curvature of the curve  $\Sigma$ . Then, if  $N_0$  is large enough, the mapping

$$(3.A.3) \quad \begin{aligned} \Phi : \mathbb{R}/L \times ]-\epsilon, \epsilon[ &\rightarrow \mathbb{T} \times \mathbb{R} \\ (s, r) &\mapsto \gamma(s) + rn(s) \end{aligned}$$

is a diffeomorphism to its image, where  $n(s)$  is the unit normal vector.

**Remark 3.A.2.** *Note that the choice of  $\epsilon$  above is not the same as (3.4.19). The extra  $d_0$  in (3.4.19) is to make sure that the image of  $\Phi$  lies in the fluid domain  $\mathbb{T} \times ]-H^+, H^-]$ , especially in finite depth case. Actually, once Proposition 3.A.1 is proved, the injectivity of  $\Phi$  can be obtained for any choice of  $\epsilon$  smaller than (3.A.2) above.*

If  $\|\kappa\|_{L^\infty} = 0$ , the curve  $\Sigma$  is straight. In this case we have  $\epsilon = +\infty$  and that  $\Phi$  is trivially a bijection. For non-trivial case  $\|\kappa\|_{L^\infty} > 0$  (thus  $\epsilon \in ]0, +\infty[$ ), one can compute the determinant  $j$  of the Jacobian of  $\Phi$ ,

$$j(s, r) = 1 + r\kappa(s) \in [1/2, 3/2],$$

when  $N_0$  is large enough. Thus,  $\Phi$  is at least a local diffeomorphism with the desired correspondence. In the sequel, we shall check that this property is global by showing the injectivity of  $\Phi$ . If  $\Phi$  is not injective, then

$$(3.A.4) \quad \exists \text{ distinct points } (s, r), (s', r') \in [0, L] \times [-\epsilon, \epsilon], \text{ such that } \gamma(s) + rn(s) = \gamma(s') + r'n(s').$$

Without loss of generality, we may assume  $s < s'$ , since  $s = s'$  implies  $r = r'$  by definition of  $\Phi$ . We will first prove that (3.A.4) holds only when  $s$  and  $s'$  are close enough (see Lemma 3.A.3 below) and deduce Proposition 3.A.1 by showing that  $\Phi$  must be locally injective.

In this section, the identification that  $[0, 2\pi] \times \mathbb{R}$  is a subdomain of  $\mathbb{C}$  will also be used, which enables us to define a real function  $\theta$  such that

$$(3.A.5) \quad \gamma_s(s) = (\alpha_s(s), \beta_s(s)) = e^{i\theta(s)}.$$

Then, by definition, the curvature reads

$$(3.A.6) \quad \kappa = -\tau_s \cdot n = -\gamma_{ss} \cdot n = -\text{Im}(\gamma_{ss} i \overline{\gamma_s}) = -\theta'.$$

### 3.A.1 . Reduction to local injectivity.

**Lemma 3.A.3.** *Let  $N_0 \geq 2$  and  $\|\kappa\|_{L^\infty} > 0$ . Then, under the assumption of Proposition 3.A.1, (3.A.4) implies*

$$(3.A.7) \quad |s - s'| \leq \frac{4}{N_0 \|\kappa\|_{L^\infty}}.$$

*Proof.* In fact, the assumption (3.A.4) can be written as

$$\gamma(s') - \gamma(s) = rn(s) - r'n(s') = (r - r')n(s) + r' \int_{s'}^s \kappa(s_1) \tau(s_1) ds_1,$$

where  $\tau$  is the unit tangent vector of  $\Sigma$ . The length of left-hand side can be bounded from below by applying chord-arc condition (3.A.1),

$$c_0 |s - s'| \leq |\gamma(s') - \gamma(s)|,$$

while the length of right-hand side is controlled by

$$|r - r'| + \left| r' \int_{s'}^s \kappa(s_1) \tau(s_1) ds_1 \right| \leq |r - r'| + |s - s'| \epsilon \|\kappa\|_{L^\infty} \leq |r - r'| + \frac{c_0}{N_0} |s - s'|.$$

By combining the estimates above, we have

$$c_0 |s - s'| \leq |r - r'| + \frac{c_0}{N_0} |s - s'|.$$

Then, we can conclude by using the condition  $N_0 \geq 2$ ,

$$|s - s'| \leq \frac{2}{c_0} |r - r'| \leq \frac{4}{c_0} \epsilon \leq \frac{4}{N_0 \|\kappa\|_{L^\infty}},$$

which gives the desired inequality (3.A.7).  $\square$

### 3.A.2 . Local injectivity.

We shall check below that, when  $s$  and  $s'$  are close in the sense of (3.A.7), (3.A.4) can not hold, which is a contradiction and Proposition 3.A.1 follows.

**Lemma 3.A.4.** *Let  $N_0 \gg 1$  and  $\|\kappa\|_{L^\infty} > 0$ . Then, under the assumption of Proposition 3.A.1, the inequality (3.A.7) yields*

$$\tau(s) \cdot \tau(s') \geq \frac{1}{2}.$$

*Proof.* By identifying the unit tangent vector  $\tau$  as complex number (see (3.A.5)), we have

$$\tau(s) \cdot \tau(s') = \cos(\theta(s) - \theta(s')).$$

Then the expression (3.A.6) of the curvature  $\kappa$  gives

$$|\theta(s) - \theta(s')| \leq |s - s'| \max_{[s, s']} |\theta'| = |s - s'| \max_{[s, s']} |\kappa| \leq \frac{4}{N_0 \|\kappa\|_{L^\infty}} \|\kappa\|_{L^\infty} = \frac{4}{N_0}.$$

By choosing  $N_0$  large enough, we can assume  $\cos(4/N_0) \geq 1/2$ , which proves the desired result.  $\square$

*Proof of Proposition 3.A.1.* Now we are able to deduce a contradiction to (3.A.4), based on Lemma 3.A.4. Let us consider the real-valued function :

$$\begin{aligned} f : [-\epsilon, \epsilon] &\rightarrow \mathbb{R} \\ h &\mapsto (\gamma(s') + hn(s') - \gamma(s)) \cdot \tau(s) \end{aligned} .$$

Once (3.A.4) holds true,  $f$  will admit a zero  $h = r'$ , while

$$\begin{aligned} f(h) &= (\gamma(s') + hn(s') - \gamma(s) - hn(s)) \cdot \tau(s) \\ &= \left( \int_s^{s'} \tau(s_1) ds_1 + h \int_s^{s'} n_s(s_1) ds_1 \right) \cdot \tau(s) \\ &= \left( \int_s^{s'} \tau(s_1) ds_1 + h \int_s^{s'} \kappa(s_1) \tau(s_1) ds_1 \right) \cdot \tau(s) \\ &= \int_s^{s'} (1 + h\kappa(s_1)) \tau(s_1) \cdot \tau(s) ds_1. \end{aligned}$$

From Lemma 3.A.4,  $\tau(s_1) \cdot \tau(s) \geq 1/2$ , and the definition of  $\epsilon$  ensures that

$$1 + h\kappa(s_1) \geq 1 - \epsilon \|\kappa\|_{L^\infty} \geq 1 - \frac{c_0}{N_0} \geq 1/2,$$

for large enough  $N_0$ . Therefore,  $f$  has a lower bound  $|s - s'|/4 > 0$ , which is a contradiction.  $\square$

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## 4 - Cauchy problem for cylinder-like capillary jets

The motion of liquid jets plays an important role in physics and engineering, and needs rigorous mathematical investigations. Recently, Huang-Karakhanyan proved the first local well-posedness in Sobolev spaces for axisymmetric jets. In this chapter, we will extend this result to general jets, namely without any axisymmetry condition.

### 4.1 . Introduction

#### 4.1.1 . History of the problem

The study of liquid jets has a long history in physics. It is until the beginning 19th century that researchers discovered that the singularity (namely, break-up) of jets is regardless of any exterior forces (gravity, for example), and should be due to surface tension, which is a nature of the fluid. In 1873, Plateau [Pla73] firstly observed that this instability is related to the area of jets. Soon in 1879, Rayleigh [Ray79] developed a linear stability method and explained this phenomenon from a theoretical point of view. Meanwhile, other physical nature of jets were also widely studied. We refer to [EV08] for more histories and results on physics.

In mathematics, the study of jets is concentrated on the steady case (system independent of time), especially the flow in nozzles. For the research on fixed infinitely long nozzles, we refer to [BL58; XX07; XX09; DXY11]. In the case of semi-infinite nozzle (with half free boundary), the early study using hodograph transformation and conformal mappings can be found in [BZ90; Gil60] and a stronger method via variational formulation is developed by Alt, Caffarelli, and Friedman [CA81; ACF82; ACF83]. Recent progress on this topic can be found in [CDW18; WX19; Li24] and the references therein. Except for nozzle problem, there are also some other results on stationary jets [GN18; GMN20].

The purpose of this chapter is to demonstrate the local well-posedness of the time-dependent system, which is the first step to the mathematical justification of *Rayleigh-Plateau instability* [Pla73; Ray79]. For the system depending on time, some formal construction of special solutions is established [Joh52; AR92]. The first study of the Cauchy problem has recently been given by Huang-Karakhanyan [HK23], where the authors prove that the axisymmetric jets are locally well-posed in Sobolev spaces. In [HK24], the same authors studied in detail the Dirichlet-to-Neumann operator in the problem of Taylor-cone ari-

sing from break-up of jets. In what follows, we will extend the well-posedness obtained in [HK23] to the case without the axisymmetry condition.

### 4.1.2 . Setting of the problem

#### Cylinder-like jets

In this chapter, we are interested in the 3D jets that are not necessarily invariant by rotation. The shape of such jets can be characterized by

$$(4.1.1) \quad \Omega(t) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : x = 0 \text{ or } |x| < \eta(t, x/|x|, z)\},$$

and its free boundary is given by

$$(4.1.2) \quad \Sigma(t) = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : |x| = \eta(t, x/|x|, z)\},$$

where  $\eta$  is a strictly positive function defined on  $\mathbb{R} \times \mathbb{T} \times \mathbb{R}$ . From physical background of the problem, we focus on two cases : steady interface with little perturbation and periodic interfaces. Rigorously, the following hypotheses on initial data  $\eta_0 = \eta|_{t=0}$  will be made in the whole chapter :

- (H0)  $\exists C_0, c_0 > 0$ , such that  $c_0 < \eta_0 < C_0$ ;
- (H1) (Perturbative case)  $\exists R > 0$ , such that  $\eta_0 - R \in H^s(\mathbb{T} \times \mathbb{R})$ , for some proper  $s$ ;
- (H1') (Periodic case)  $\exists L > 0$ , such that  $\eta_0$  is periodic in  $z$  of period  $L$ .

In perturbative case, (H1) indicates that, when  $s > 1$  and  $z \rightarrow \pm\infty$ ,  $\eta_0(\omega, z)$  tends to  $R$  uniformly in  $\omega$ , which can be formally preserved for all time due to the governing equation (4.1.7), while (H1) and (H0) will hold for short time from our main result Theorem 4.1.2. In periodic case, one may check that (H1') is preserved for all time by using the fact that equations (4.1.3), (4.1.7), and (4.1.8) are invariant by translations in  $z$ -direction. Therefore,  $\eta$  should be viewed as a function on  $\mathbb{R} \times \mathbb{T}^2$  (for simplicity, we may take  $L = 2\pi$ ). In what follows, we focus on the perturbative case, while all the results hold true in periodic case with normalizations eliminated.

We also assume that the fluid is inviscid, incompressible, and irrotational. Consequently, the velocity field  $u$  satisfies *Euler equation* with divergence-free and rotation-free conditions

$$(4.1.3) \quad \begin{cases} \partial_t u + u \cdot \nabla_{x,z} u + \nabla_{x,y} P = 0, & \text{in } \Omega(t), \\ \operatorname{div}_{x,z} u = 0, \quad \operatorname{curl}_{x,z} u = 0, & \text{in } \Omega(t), \end{cases}$$

where  $P$  is the pressure driven only by surface tension.

### Boundary conditions

In order to fully determine the motion of fluid and free surface, we introduce two boundary conditions on  $\Sigma(t)$ . The first one is the *kinematic boundary condition*,

$$(4.1.4) \quad n \cdot \gamma_t = n \cdot u, \quad \text{on } \Sigma(t),$$

where, for all time  $t$ ,  $n = n(t)$  is the outward unit normal direction to  $\Sigma(t)$  in  $\mathbb{R}^3$  and  $\gamma = \gamma(t)$  is the parametrization of  $\Sigma(t)$ ,

$$\begin{aligned} \gamma(t) : \mathbb{T} \times \mathbb{R} &\rightarrow \mathbb{R}^3 \\ (\omega, z) &\mapsto (\eta(t, \omega, z) \cos \omega, \eta(t, \omega, z) \sin \omega, z). \end{aligned}$$

In coordinate  $(\omega, z)$ ,  $n$  equals

$$(4.1.5) \quad n = \frac{1}{l} \left( e_r - \frac{\eta_\omega}{\eta} e_\omega - \eta_z e_z \right),$$

with

$$e_r = (\cos \omega, \sin \omega, 0), \quad e_\omega = (-\sin \omega, \cos \omega, 0), \quad e_z = (0, 0, 1),$$

and

$$(4.1.6) \quad l = \sqrt{1 + \left( \frac{\eta_\omega}{\eta} \right)^2 + \eta_z^2}.$$

As a result, (4.1.4) can be rewritten as

$$(4.1.7) \quad \eta_t = \left( e_r - \frac{\eta_\omega}{\eta} e_\omega - \eta_z e_z \right) \cdot u|_\Sigma.$$

The second boundary condition in need is the balance of forces on the free boundary  $\Sigma$ :

$$(4.1.8) \quad P|_\Sigma = \sigma \left( H - \frac{1}{2R} \right),$$

where  $\sigma > 0$  is a constant and  $H$  is the mean curvature. The normalization  $1/(2R)$  is necessary in perturbative case, since the mean curvature

$$(4.1.9) \quad H = \frac{1}{2} \left[ \frac{1}{\eta l} - \frac{\partial_\omega}{\eta} \left( \frac{\eta_\omega}{\eta l} \right) - \partial_z \left( \frac{\eta_z}{l} \right) \right]$$

tends to  $1/(2R)$  as  $z \rightarrow \pm\infty$  under the hypothesis (H1) with  $s > 3$ . In periodic case, this normalization can be omitted.

**Remark 4.1.1.** *Although gravity is typically an essential factor in water-wave problems, in this chapter we focus exclusively on the model without gravity. This choice is motivated by our ultimate goal : the study of the Rayleigh–Plateau instability, where surface tension serves as the only source of instability. It is worth noting that the zero-gravity model can be viewed as an approximation of systems with large Froude number—a regime that characterizes thin and elongated liquid jets (see, for example, the experimental study in [RJ70]).*

### 4.1.3 . Craig-Sulem-Zakharov formulation

Unlike standard 3D Euler equation, (4.1.3) is defined in a domain varying in time. To overcome this difficulty, we follow the idea from Zakharov [Zak68] and Craig-Sulem [CS93], which can reduce the problem to equations on a fixed domain  $\mathbb{T} \times \mathbb{R}$  (or  $\mathbb{T}^2$  in periodic case).

To begin with, we observe that the irrotational and incompressible conditions guarantee the existence of harmonic scalar potential  $\phi$

$$(4.1.10) \quad \Delta_{x,z}\phi = 0, \quad \nabla_{x,z}\phi = u, \quad \text{in } \Omega(t).$$

In terms of  $\phi$ , (4.1.3) can be written as

$$\nabla_{x,z} \left( \partial_t \phi + \frac{|\nabla_{x,z}\phi|^2}{2} + P \right) = 0,$$

which yields the *Bernoulli's equation*

$$(4.1.11) \quad \partial_t \phi + \frac{|\nabla_{x,z}\phi|^2}{2} + P = 0, \quad \text{in } \Omega(t).$$

Note that the right hand side should be a constant depending only on time, while, by absorbing this constant in  $P$ , we may take it to be zero for simplicity.

Now, let us consider the trace  $\psi$  of  $\phi$  at the free surface  $\Sigma(t)$ , namely

$$(4.1.12) \quad \psi(t, \omega, z) = \phi(\eta(t, \omega, z) \cos \omega, \eta(t, \omega, z) \sin \omega, z).$$

By definition,  $\phi$  can be uniquely determined by  $\psi$  (and  $\eta$  implicitly) via the following linear elliptic equation

$$(4.1.13) \quad \begin{cases} \Delta_{x,z}\phi = 0, & \text{in } \Omega(t), \\ \phi|_{\Sigma(t)} = \psi. \end{cases}$$

Consequently, the right hand side of (4.1.7) can be regarded as a function depending linearly on  $\psi$  and implicitly on  $\eta$ , which gives rise to the (formal) definition of *Dirichlet-to-Neumann operator*,

$$(4.1.14) \quad G(\eta)\psi := \left( e_r - \frac{\eta_\omega}{\eta} e_\omega - \eta_z e_z \right) \cdot \nabla_{x,z}\phi|_{r=\eta(\omega,z)}$$

It is easy to check that  $G(\eta)$  is linear, positive, and that  $\eta G(\eta)$  is self-adjoint. A rigorous study of this operator will be given in Section 4.2 and 4.3. In terms of the quantities below

$$(4.1.15) \quad N = BV \cdot \left( \frac{\eta_\omega}{\eta} e_\omega + \eta_z e_z \right) + \frac{|V|^2 - B^2}{2},$$

$$(4.1.16) \quad B = e_r \cdot \nabla_{x,z}\phi|_{\Sigma},$$

$$(4.1.17) \quad V = e_\omega (e_\omega \cdot \nabla_{x,z}\phi)|_{\Sigma} + e_z (e_z \cdot \nabla_{x,z}\phi)|_{\Sigma}.$$

we have

$$\eta_t = G(\eta)\psi = B - V \cdot \left( \frac{\eta_\omega}{\eta} e_\omega + \eta_z e_z \right),$$

and, from (4.1.11) and (4.1.8),

$$\begin{aligned} \psi_t &= \partial_t (\phi(t, \eta \cos \omega, \eta \sin \omega, z)) = \phi_t|_\Sigma + \eta_t e_r \cdot \nabla_{x,z} \phi|_\Sigma \\ &= - \left( \frac{|\nabla_{x,z} \phi|^2}{2} + P \right) \Big|_\Sigma + \left( B - V \cdot \left( \frac{\eta_\omega}{\eta} e_\omega + \eta_z e_z \right) \right) B \\ &= - \frac{|V|^2 + B^2}{2} - \sigma \left( H - \frac{1}{2R} \right) + \left( B - V \cdot \left( \frac{\eta_\omega}{\eta} e_\omega + \eta_z e_z \right) \right) B \\ &= - \sigma \left( H - \frac{1}{2R} \right) - N. \end{aligned}$$

In conclusion, the system (4.1.3), (4.1.7), and (4.1.8) is reformulated as the following equations on  $\mathbb{T} \times \mathbb{R}$  (or  $\mathbb{T}^2$  in periodic case)

$$(4.1.18) \quad \begin{cases} \eta_t = G(\eta)\psi, \\ \psi_t + \sigma \left( H - \frac{1}{2R} \right) + N = 0. \end{cases}$$

#### 4.1.4. Main results

The main result of this chapter is that the system (4.1.18) is locally well-posed in Sobolev spaces. More precisely,

**Theorem 4.1.2.** *Let  $(\eta_0 - R, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R})$  with  $s > 3$ . Assume that  $\eta_0$  satisfies the hypotheses (H0) and (H1), or (H0) and (H1') for periodic case. Then there exists  $T > 0$ , such that the system (4.1.18) with initial data  $(\eta_0, \psi_0)$  admits a unique solution*

$$(\eta - R, \psi) \in C \left( [0, T[; H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R}) \right).$$

Moreover, the hypotheses (H0), (H1) or (H1') are preserved for all  $t \in [0, T[$ .

Moreover, we shall prove that the flow map is continuous in the sense of following theorem :

**Theorem 4.1.3.** *Under the hypotheses of Theorem 4.1.2, for all  $0 < r \ll 1$ , we denote by  $B_s(\eta_0, \psi_0; r)$  the collection of all  $(\tilde{\zeta}_0, \tilde{\psi}_0)$  with*

$$\|(\eta_0 - R) - \tilde{\zeta}_0\|_{H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} + \|\psi_0 - \tilde{\psi}_0\|_{H^s(\mathbb{T} \times \mathbb{R})} < r,$$

which is a subset of  $H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R})$ . Then there exists  $T_r > 0$  (depending on  $(\eta_0, \psi_0)$ ), such that the system (4.1.18) with initial data in  $B_s(\eta_0, \psi_0; r) + (R, 0)$  admits a unique solution  $(\tilde{\eta}, \tilde{\psi})$  on  $[0, T_r]$  with  $(\tilde{\eta} - R, \tilde{\psi}) \in C \left( [0, T_r[; H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R}) \right)$ , and the hypotheses (H0), (H1) or (H1') are preserved for all  $t \in [0, T_r[$ .

Moreover, the flow map

$$(4.1.19) \quad \begin{aligned} \mathfrak{F} : B_s(\eta_0, \psi_0; r) &\rightarrow L^\infty([0, T_r[; H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R})) \\ (\tilde{\zeta}_0, \tilde{\psi}_0) &\mapsto \left( \tilde{\eta}(t) - R, \tilde{\psi}(t) \right). \end{aligned}$$

is continuous.

These well-posedness results have attracted lots of attention for planar water-wave [Wu97; Wu99; Lan05; AL08; GMS12; IP15; AD15; AIT23; AIT22] in recent years. And the same problem with different geometric setting such as water drops [BG98; CL00; Lin05; SZ08b; SZ08a; Sha23; BJM24] has also been widely studied. [HK23] provides the first result on cylinder, and the theorems above extend it by deleting the axisymmetric condition.

#### 4.1.5 . Idea of the proof

Formally speaking, we attempt to investigate the hyperbolic nature of water-wave equation (4.1.3) with (4.1.7), (4.1.8), and reformulate the equivalent system (4.1.18) as

$$(4.1.20) \quad \partial_t Y + A(Y)Y = F(Y),$$

where  $Y$  is a new variable defined as an elliptic operator acting on  $(\eta, \psi)$  and  $F(Y)$  is a (relatively) regular source term relying implicitly on  $Y$ . The high order linear operator  $A(Y)$  also depends implicitly on  $Y$  with

$$A(Y)^* = -A(Y).$$

Then classical methods for hyperbolic equations allow us to construct a series of approximate solutions and prove their convergence to a solution to (4.1.20) via energy estimates. With a similar argument, one may also show the uniqueness of the solution, which completes the well-posedness stated in Theorem 4.1.2.

In the context of water-wave problem (4.1.18), there are several difficulties :

1. How to rewrite the nonlinear terms as  $A(Y)Y$  up to acceptable remainders?
2. What is the dependence of  $A(Y)$  in  $Y$ ? Or equivalently, how do  $G(\eta)$ ,  $N$ , and  $H$  depend on  $(\eta, \psi)$ ?
3. Does the unique solution depend continuously on time and initial data?

To solve these problems, we follow the idea of [AM09; ABZ11] and apply the techniques of paradifferential calculus. More precisely, we shall write all the nonlinear terms as paradifferential operators acting on  $(\eta, \psi)$ , where the implicit dependence in  $(\eta, \psi)$  will be reflected in the symbols to be calculated explicitly.

**Paralinearization of  $G(\eta)$ .**

In order to write  $G(\eta)$  as a paradifferential operator with symbol depending explicitly on  $\eta$ , from definition (4.1.14), one can see that the core of this problem is to express normal derivative of potential  $\phi$  as tangential ones, where the implicit dependence on  $\eta$  is mainly hidden in Poisson's equation (4.1.13). As in [AM09], we will reformulate (4.1.13) as an elliptic equation on a fixed domain (see (4.2.22)), where the Laplacian  $\Delta_g$  relies on metric  $g$ , or equivalently  $\eta$ . Then, by decomposing the Laplacian in normal and tangential part, we can see that the difference of normal derivative of  $\phi$  and a tangential operator acting on  $\phi$  is governed by a hyperbolic equation, implying the desired paralinearization (4.3.52).

A major difficulty in this step is that the coefficients are smooth in  $\eta$ , which has limited regularity. As a consequence, if one applies directly the calculus above, most of the remainders will have the same regularity as the principal terms. To deal with this lack of regularity, we replace  $\psi$  by *Alinhac's good unknown*  $U = \psi - T_B\eta$  defined by (4.3.19), which is inspired by paracomposition (see Appendix 4.C.4 and [AM09]). Note that it is possible to recover  $\psi$  from  $U$  and  $\eta$ .

**Paralinearization of  $N, H$ .**

The treatment of  $N$  is easy since it can be expressed as a smooth function of  $G(\eta)\psi, \eta, \psi$ , and their derivatives. The desired paralinearization follows easily from classical formula (4.C.19). The same formula also works for mean curvature  $H$  which is a smooth function of  $\eta$  and its derivatives.

**Symmetrization.**

Given the paralinearizations above, we can rewrite the main equation (4.1.18) as (4.3.68), which takes the form of (4.1.20) with

$$A(Y) = T_V \cdot \bar{\nabla} + \mathcal{L}.$$

The first part  $T_V \cdot \bar{\nabla}$  can be roughly viewed as transport  $V \cdot \nabla$  with a Lipschitzian vector  $V$ , and the classical arguments can be applied. The second part  $\mathcal{L}$  is a matrix of paradifferential operators and the symbol of each entry is homogeneous in Fourier variable  $\xi$ . After simple linear transform, the matrix of symbols becomes anti-Hermitian (see Section 4.4). To do the same symmetrization at operator level, it suffices to apply symbolic calculus as in Section 4 of [ABZ11].

**Cauchy problem.**

As mentioned above, we will first construct a series of approximate solutions governed by

$$(4.1.21) \quad \partial_t Y_\epsilon + A_\epsilon(Y_\epsilon)Y_\epsilon = F(Y_\epsilon),$$

where  $A_\epsilon(Y)$  is a mollification of  $A(Y)$ . Note that, in order to maintain the property  $A_\epsilon(Y)^* = -A_\epsilon(Y)$ , this mollifier cannot be taken as a simple localization in Fourier variable. Instead,  $A_\epsilon(Y)$  will be defined by inserting a well-chosen paradifferential operator depending on  $\eta$  in  $A(Y)$ .

Another difficulty arising from the convergence of approximate solutions  $Y_\epsilon$  is that, to compare  $Y_\epsilon$  with another  $Y_{\epsilon'}$ , it is necessary to calculate the Lipschitz norm of  $F$ . We will see in Section 4.3.5 that the core of this problem is to study the derivative-in- $\eta$  of  $G(\eta)\psi$ , which is known as *shape derivative*. A standard proof of this can be found in Section 3.3 of [Lan13], which does not work here due to some difficulties from the geometric nature of the cylinder. Inspired by [BG98], we first study the Hamiltonian formulation (Proposition 4.2.5) of the system (4.1.18), which then implies the desired shape derivative formula (4.2.42).

### Continuity of the solution.

It remains to check that the unique solution constructed as the limit of approximate solutions is continuous in time and initial data. It is possible to use the same strategy as in Section 6 of [ABZ11], while, in this chapter, we will use another method. In a recent paper [Ala+24], the authors prove a nonlinear interpolation theorem (see Theorem 4.5.19), which can be briefly stated as : if the flow map satisfies the contraction (4.5.50) and tame estimate (4.5.51), the solution will be continuous in time and initial data. The former one follows easily from the energy estimate, while the tame estimate holds due to the paradifferential calculus. In fact, all the estimates involving paradifferential operators require the index of regularity to belong to an open interval. Then it is harmless to replace these index by slightly smaller one, which gives the desired tame estimate.

#### 4.1.6 . Plan of the chapter

In Section 4.2, we will construct a proper change of variable to pull the Poisson's equation (4.1.13) back to a fixed domain and prove the elliptic regularity with details left to Appendix 4.B. Moreover, the Hamiltonian formulation will be given and, as a result, we will deduce the shape derivative formula.

In Section 4.3, we first introduce the homogeneous symbols to be studied in this chapter and clarify the basic properties of paradifferential operator of these symbols (the general theory will be reviewed in Appendix 4.C). Then the parilinearization of  $G(\eta)$ ,  $N$ , and  $H$  will be given as well as the parilinearization of the system (4.1.18). In the end of this section, we will justify that all the remainders arising from the parilinearization are Lipschitzian in  $(\eta, \psi)$ .

With the symmetrization proved in Section 4.4, we are able to define the approximate system in Section 4.5, where the energy estimate will also be proved. As a consequence, we deduce the convergence of approximate solutions and the uniqueness of the solution,

whose continuity in time and initial data is left to the end of Section 4.5 via nonlinear interpolation.

The Appendix 4.A is devoted to a review of Sobolev spaces defined on domains with smooth boundary which is used in the pull-back of the Poisson's equation 4.1.13. And Appendix 4.D contains a direct (but complicated) variational calculus, which proves the most important identity in the Hamiltonian formulation of (4.1.18).

### 4.1.7 . Notations and conventions

In what follows, we list the notations and conventions frequently used in the whole chapter. And a list of symbols is available at the end of this chapter.

- Let  $W \in C_b^\infty$  be a fixed function (usually it is a constant). We denote by  $H_W^s$  the set of functions  $f$  such that

$$\|f\|_{H_W^s} := \|f - W\|_{H^s} < +\infty,$$

where  $H^s$  is the Sobolev norm. For example, all  $\eta$  satisfying (H1) with  $R > 0$  belong to  $H_R^s(\mathbb{T} \times \mathbb{R})$ . Furthermore, an operator  $L : H_{W_0}^{s_0} \rightarrow H_{W_1}^{s_1}$  is said to be linear when  $(f - W_0) \mapsto (Lf - W_1)$  is linear from  $H^{s_0}$  to  $H^{s_1}$ .

- We denote by  $\mathbb{D}$  the open unit disk in 2D, namely  $\{y \in \mathbb{R}^2 : |y| < 1\}$ . Its boundary is denoted as  $\mathbb{S}$ , which will be identify as the 1D torus  $\mathbb{T}$ .

- We say  $\chi$  is a smooth truncation near  $K \subset \mathbb{R}^d$ , if  $\chi \in C_b^\infty(\mathbb{R}^d)$  is supported in a (usually small) neighborhood of  $K$  and is equal to 1 in a smaller neighborhood of  $K$ .

- All the large constants will be denoted as  $C$  or  $C_\alpha$  if this constant depends on some parameter  $\alpha$ . Moreover, we write  $C(Q)$  for some quantity  $Q > 0$  when  $C > 0$  is a smooth increasing function of  $Q$ .

- We use the Einstein summation convention : if an index appears twice, a summation in this index should be added automatically. For simplicity, theses summations will not be precised in the formulas.

- We use double integral  $\iint$  for the integrals in domain  $\Omega(t)$  (or its pull-back by diffeomorphisms), and single integral  $\int$  for those on interface  $\Sigma(t)$  without precising the region of integration.

- We say a linear operator  $T$  is of order  $m \in \mathbb{R}$ , if it is bounded from  $H^s$  to  $H^{s-m}$  for all  $s \in \mathbb{R}$ . When an operator is of order  $m$  for all  $m \in \mathbb{R}$ , it is said to be a smoothing operator.

- For real number  $s$ , we write  $s+$  for  $s + \epsilon$ , where  $\epsilon > 0$  is a small number, when the exact value of  $\epsilon$  is not important. Similarly,  $s-$  stands for  $s - \epsilon$  with  $0 < \epsilon \ll 1$ .

## 4.2 . Preliminaries

In Section 4.1.3, we have reformulated our problem as (4.1.18) on fixed domain  $\mathbb{T} \times \mathbb{R}$ , while the implicit dependence on the shape of fluid, or equivalently  $\eta$ , is hidden in the definition of Dirichlet-to-Neumann operator  $G(\eta)$  and the non-linear term  $N$ . In order to clarify this dependence, it is essential to rigorously investigate the relation between  $\psi$

and  $\phi$ , or equivalently the Poisson's equation (4.1.13), where the main difficulty is that the domain  $\Omega(t)$  varies in time. To reduce the problem to a fixed domain, in axis-symmetric case [HK23], the authors use an explicit change of variable :

$$[0, 1] \times \mathbb{T} \times \mathbb{R} \rightarrow \Omega(t), \quad (\rho, \theta, z) \mapsto (\rho\eta(t, z) \cos(\theta), \rho\eta(t, z) \sin(\theta), z),$$

where the dependence on angular variable  $\theta$  can be omitted due to the symmetricity. In this way, one may rewrite the Poisson's equation (4.1.13) as an elliptic equation on  $[0, 1] \times \mathbb{R}$ ,

$$(4.2.1) \quad \begin{cases} \nabla_{\rho, z} \cdot Q(\eta, \eta_z) \nabla_{\rho, z} \varphi = 0, & \text{in } [0, 1] \times \mathbb{R}, \\ \varphi|_{\rho=1} = \psi, \end{cases}$$

where  $Q$  is a smooth function of  $\eta, \eta_z$  that can be calculated explicitly. At  $\rho = 0$ , there is no boundary condition directly from (4.1.13), while one may add the compatibility condition  $\partial_\rho \varphi|_{\rho=0} = 0$  to avoid singularity at the axis  $\{\rho = 0\} \subset \Omega(t)$ .

Clearly, this boundary condition is not reasonable in general case (with dependence on angular variable). To overcome this difficulty, instead of looking for a generalized boundary condition, we attempt to flatten  $\Omega(t)$  in an alternative way so that no extra boundary will be generated. In the study of water-drop, where  $\Omega(t)$  is a perturbation of unit ball, Beyer-Günther [BG98] introduce a diffeomorphism from unit ball to the water-drop which equals identity near zero, up to multiple of positive constants. Inspired by this work, we shall extend  $\eta$  defined on  $\mathbb{T} \times \mathbb{R}$  to a positive function  $\zeta = \zeta(y, z)$  on  $\mathbb{D} \times \mathbb{R}$  (see (4.2.5)), such that in polar coordinate  $(y, z) = (\rho\theta, z)$ ,  $\zeta$  equals  $\eta$  at  $\rho = 1$  and behaves like  $\rho$  near  $\rho = 0$ . In this way, the domain  $\Omega(t)$  can be characterized by  $x = \zeta(\rho\theta, z)\theta, z = z$ , and there is no singularity at  $\rho = 0$  since  $x = \zeta(\rho\theta, z)\theta$  behaves like  $\rho\theta = y$  near  $y = 0$  (see Proposition 4.2.2).

Via this change of coordinate, (4.1.13) can be reduced to an elliptic equation on  $\mathbb{D} \times \mathbb{R}$ , whose coefficients depend smoothly on  $\eta$ . The desired elliptic regularity (Proposition 4.2.3) then follows from some classical arguments. As a corollary, we may obtain the high-order estimates for Dirichlet-to-Neumann operator  $G(\eta)$  (Corollary 4.2.4).

Furthermore, this change of variable gives a rigorous meaning of the variational calculus, which is required for Hamiltonian formulation (Proposition 4.2.5). In fact, from the definition (4.2.29) of energy Hamiltonian  $\mathcal{H}$ , it is clear that its variation in  $\eta$  can be reduced to derivative-in- $\eta$  of metric, which is a smooth function of  $\eta$  and its derivatives, due to our construction of coordinate  $(x, z) = (\zeta(y, z)y, z)$ . Hence, in consideration of the elliptic regularity (which guarantees all the integrands are integrable), all the derivatives-in- $\eta$  of potential and kinetic energy make sense.

As a result of Hamiltonian formulation (Proposition 4.2.5), we shall also deduce the shape derivative formula (4.2.42), which provides a preliminary explanation of how Dirichlet-to-Neumann operator  $G(\eta)$  depends on  $\eta$  (while more delicate results will be given in Section 4.3 via paradifferential calculus).

Note that, in planar water-wave case (where free surface is homotopic to hyperplane), it is easier to start by proving shape derivative formula (see Chapter 3 of [Lan13] for instance), which implies the Hamiltonian formulation. To use this method, it is essential to

find explicitly a harmonic extension of  $B\delta\eta$  appearing in the right hand side of (4.2.42) so that one could calculate  $G(\eta)(B\delta)$ . However, this harmonic extension is not evident in our context, preventing us to repeat the argument for planar water-wave. That is why we begin with the Hamiltonian formulation and then prove the shape derivative formula. As a by-product, during the proof of Proposition 4.2.8, we shall see that the shape derivative formula is somehow equivalent to the variation of kinetic energy in  $\eta$ , which is the core of Hamiltonian formulation.

It is also worth mentioning the recent work by Baldi-Julin-Manna [BJM24] on the study of liquid water drop. The authors develop an alternative method based on differential geometry to deal with the singularity at the origin. This method allows us to prove directly the shape derivative formula with little interior information involved (i.e. one can work solely on the boundary). And it could be expected to work well for jets and the planar water-wave with rough bottom (for example, the case studied in [ABZ11]). Nevertheless, these geometrical calculations require further justifications in low-regular context, which may end up with more technical details. In the mean time, the equivalence between shape derivative formula and Hamiltonian formulation is not revealed during the proof.

In this section, the hypotheses (H0) and (H1) (or (H1')) play an important role, while we will not precise them in each statement for simplicity.

### 4.2.1 . Change of coordinate

As explained above, we aim to construct a diffeomorphism of the form

$$(4.2.2) \quad \begin{aligned} \iota(t) : \mathbb{D} \times \mathbb{R} &\rightarrow \Omega(t) \subset \mathbb{R}^3 \\ (y, z) &\mapsto (y\zeta(t, y, z), z) \end{aligned}$$

with boundary correspondence, where  $\zeta$  is an explicit extension of  $\eta$ . To begin with, we fix a general extension  $\tilde{\eta}$  of  $\eta$  to  $\mathbb{D} \times \mathbb{R}$ , and construct  $\zeta$  in a proper way so that the mapping above is a bijective and thus a diffeomorphism. For simplicity, time variable  $t$  will be omitted in this part since each step below is independent of time. The construction of  $\zeta$  has been given in [BG98] for water droplets (free surface homotopic to  $\mathbb{S}^d$ ). We will see in the sequel that it also works for jets.

#### Extension of $\eta$ .

Let  $\eta$  be a function in  $H_R^s(\mathbb{T} \times \mathbb{R})$  with  $s > 2$ , satisfying (H0) and (H1) (or (H1')). Then we may define  $\tilde{\eta}$ , in polar coordinate  $(y, z) = (\rho\theta, z)$ , as

$$(4.2.3) \quad \tilde{\eta}(\rho\theta, z) := \chi_1(\rho)\hat{\chi}_0((\rho - 1)\langle D_{\theta, z} \rangle) [\eta(\theta, z) - R] + R,$$

where  $\chi_0, \chi_1$  are smooth truncation near 0 and 1 respectively,  $\chi_0$  is even and  $\int \chi_0 = 1$ . It is easy to check that this extension satisfies the following properties :

**Lemma 4.2.1.** *Let  $s > 2$ . The scalar function  $\tilde{\eta}$  defined by (4.2.3) belongs to  $H_R^{s+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  with*

(1)  $\tilde{\eta}|_{\rho=1} = \eta$ ;

(2)  $\partial_\rho \tilde{\eta}$  is uniformly continuous in  $\mathbb{D} \times \mathbb{R}$  with  $\partial_\rho \tilde{\eta}|_{\rho=1} = 0$ ;

(3) The linear application  $L_0 : (\eta - R) \mapsto (\tilde{\eta} - R)$  is bounded from  $H^s(\mathbb{T} \times \mathbb{R})$  to  $H^{s+\frac{1}{2}}(\mathbb{R}^3)$ , and from  $C^{s-1}(\mathbb{T} \times \mathbb{R})$  to  $C^{s-1}(\mathbb{R}^3)$ ;

(4) There exists  $0 < \delta \ll 1$ , such that, for all  $\rho \in [1 - \delta, 1]$ ,  $\tilde{\eta} > \frac{c_0}{2}$  and  $|\partial_\rho \tilde{\eta}| < \frac{c_0}{4}$ . Moreover,  $\delta$  depends only on the  $H^s$  norm of  $\eta$ .

Recall that  $c_0$  is the constant appearing in (H0).

*Proof.* These claims are trivial except for the boundedness in Hölder space (from  $C^{s-1}(\mathbb{T} \times \mathbb{R})$  to  $C^{s-1}(\mathbb{R}^3)$ ). To prove this, we first show that the multiplier  $\chi_1(\rho)\hat{\chi}_0((\rho - 1)\langle D_{\theta,z} \rangle)$  is bounded from  $C^0(\mathbb{T} \times \mathbb{R})$  to  $C^0(\mathbb{R}^3)$ . The kernel of this multiplier is given by

$$K(w, w'; \rho) = \frac{1}{(2\pi)^2} \int e^{i(w-w') \cdot \xi} \chi_1(\rho) \hat{\chi}_0((\rho - 1)\langle \xi \rangle) d\xi,$$

where  $w = (\theta, z)$ ,  $w' = (\theta', z')$ , and  $\xi$  is the Fourier variable associated to  $(\theta, z)$ . A simple calculus gives that

$$|K(w, w'; \rho)| \lesssim \chi_1(\rho) |1 - \rho|^{-2},$$

while

$$\begin{aligned} |(w - w')K(w, w'; \rho)| &= \left| \frac{1}{(2\pi)^2} \int e^{i(w-w') \cdot \xi} \chi_1(\rho) \partial_\xi (\hat{\chi}_0((\rho - 1)\langle \xi \rangle)) d\xi \right| \\ &\lesssim \chi_1(\rho) |\rho - 1| \left| \int \hat{\chi}_0'((\rho - 1)\langle \xi \rangle) \frac{\xi}{\langle \xi \rangle} d\xi \right| \lesssim \chi_1(\rho) |1 - \rho|^{-1}. \end{aligned}$$

Similarly, one gains  $|1 - \rho|$  from each product with  $(w - w')$ , and thus for all  $N \gg 1$ ,

$$(4.2.4) \quad |K(w, w'; \rho)| \lesssim \chi_1(\rho) |1 - \rho|^{-2} \left\langle \frac{w - w'}{|1 - \rho|} \right\rangle^{-N},$$

which proves the boundedness of  $L_0$  from  $L^\infty(\mathbb{T} \times \mathbb{R})$  to  $L^\infty(\mathbb{R}^3)$ . Provided that  $\eta \in C^0(\mathbb{T} \times \mathbb{R})$ , then continuity of  $\tilde{\eta} = L_0(\eta - R) + R$  on  $\{\rho \neq 1\}$  is obvious due to the continuity of  $K$  in  $(w, w'; \rho)$  and the estimate (4.2.4) of  $K$ . To obtain the continuity at  $\{\rho = 1\}$ , we notice that

$$\int K(w, w'; \rho) dw' = \chi_1(\rho),$$

which, together with (4.2.4), guarantees that  $\tilde{\eta}(\rho) \rightarrow \eta$  as  $\rho \rightarrow 1$ , locally uniformly in  $(\theta, z)$  (see Section 1.2.4 of [Gra08]).

For high order regularity, we observe that the derivatives in  $w = (\theta, z)$  is commutative with  $L_0$ , i.e.

$$\|\partial_w^\alpha \tilde{\eta}\|_{L^\infty(\mathbb{R}^3)} = \|L_0(\partial_w^\alpha \eta)\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\partial_w^\alpha \eta\|_{L^\infty(\mathbb{T} \times \mathbb{R})}, \quad \forall \alpha \in \mathbb{N}^2.$$

As for the derivatives in  $\rho$ , one can see from the definition (4.2.3) that each derivative in  $\rho$  leads to a  $\langle D_w \rangle$ , reducing the problem to derivatives in  $w$ . In conclusion, we have proved that  $L_0$  is bounded from  $C^k(\mathbb{T} \times \mathbb{R})$  to  $C^k(\mathbb{R}^3)$  for all  $k \in \mathbb{N}$  and the case of non-integer  $s$  follows from the interpolation between Hölder spaces (see Section 1.1.1 of [Lun18]).  $\square$

Here, we have managed to define  $\tilde{\eta}$  on  $\mathbb{R}^3$ , while we only consider its restriction in  $\mathbb{D} \times \mathbb{R}$ . Note that this construction is not unique and the next step is valid for any  $\tilde{\eta}$  verifying Lemma 4.2.1.

### Construction of $\zeta$ .

Let  $\chi \in C_c^\infty(\mathbb{R})$  be an even function decreasing on  $[0, +\infty[$ , supported in  $] -1, 1[$ , and equal to 1 on  $[-1 + \delta, 1 - \delta]$  with  $0 < \delta \ll 1$ . Then we define

$$(4.2.5) \quad \zeta(r\theta, z) := (1 - \chi)(\rho)\tilde{\eta}(\rho\theta, z) + \epsilon\chi(\rho)M\eta,$$

where  $0 < \epsilon \ll 1$  is a constant depending on  $c_0, C_0$  in (H0) and  $M\eta \in \mathbb{R}$  is the mean of  $\eta$ . In perturbative case, when  $s > 2$ , (H1) implies that

$$M\eta := \lim_{l \rightarrow +\infty} \frac{1}{4\pi l} \int_{-l}^l \int_0^{2\pi} \eta(\theta, z) d\theta dz = R.$$

In the periodic case,  $M\eta$  is the usual average of  $\eta$  on  $\mathbb{T}^2$ . Note that (H0) guarantees  $c_0 < M\eta < C_0$ .

**Proposition 4.2.2.** *Let  $s > 2$ ,  $\delta$  be chosen as in (4) of Lemma 4.2.1, and  $\epsilon = \frac{c_0}{4C_0}$ . Then the mapping  $\iota$  defined by (4.2.2) is bijective with boundary correspondence  $\iota(\mathbb{S} \times \mathbb{R}) = \Sigma$ . The Jacobian of  $\iota$  is bounded from below by*

$$(4.2.6) \quad \det J := \det(\partial_\alpha \iota^\beta) \geq \frac{c_0^3}{4C_0},$$

where  $c_0, C_0$  are constants in hypothesis (H0). Moreover, the linear application

$$L : (\eta - R) \mapsto (\zeta - R_\epsilon(\rho)),$$

is bounded from  $H^s(\mathbb{T} \times \mathbb{R})$  to  $H^{s+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ , and from  $C^{s-1}(\mathbb{T} \times \mathbb{R})$  to  $C^{s-1}(\mathbb{D} \times \mathbb{R})$ , with  $R_\epsilon$  defined by

$$(4.2.7) \quad R_\epsilon(\rho) := R - (1 - \epsilon)\chi(\rho)R.$$

*Proof.* The boundedness of  $L$  is a direct consequence of Lemma 4.2.1. To prove the bijectivity of  $\iota$ , from the definition (4.2.2), it suffices to check that, in polar coordinate  $(y, z) = (\rho\theta, z)$ ,  $\partial_\rho(\rho\zeta) > 0$  for all  $0 \leq \rho \leq 1$ . By definition (4.2.5) of  $\zeta$ , we have

$$\partial_\rho(\rho\zeta) = (1 - \chi)(\rho) (\tilde{\eta} + \rho\partial_\rho\tilde{\eta}) - \rho\chi'(\rho)(\tilde{\eta} - \epsilon M\eta) + \epsilon\chi(\rho)M\eta.$$

The first term is positive since it is supported for  $1 - \delta \leq r \leq 1$ , where  $\tilde{\eta} > c_0/2$  and  $|\rho \partial_\rho \tilde{\eta}| < c_0/4$  due to (4) of Lemma 4.2.1. The positivity of the second term follows from the fact that  $\epsilon M \eta \leq \epsilon C_0 = c_0/4$  and  $\tilde{\eta} > c_0/2$  on  $\text{Supp } \chi' \subset [1 - \delta, 1]$ . The last term is trivially bounded from below by  $\epsilon \chi(\rho) c_0$ . To sum up, for all  $\rho \geq 0$ ,

$$(4.2.8) \quad \partial_\rho(\rho\zeta) \geq \frac{c_0}{4}(1 - \chi)(\rho) - \frac{c_0}{4}\rho\chi'(\rho) + \epsilon c_0\chi(\rho) \geq \frac{c_0^2}{4C_0}.$$

Thus, by definition (4.2.2) of  $\iota$ , the Jacobian of  $\iota$  admits the lower bound  $\det J = \zeta \partial_\rho(\rho\zeta) \geq c_0^3/(4C_0)$ .  $\square$

For simplicity, in the sequel, we shall ignore the normalizations and say that  $\zeta = L\eta$  is linear as in periodic case. As the extension  $\tilde{\eta}$  defined by (4.2.3),  $\zeta$  is well-defined in  $H^{s+\frac{1}{2}}(\mathbb{R}^3)$ , up to  $C_b^\infty(\mathbb{R}^3)$  normalization, whose restriction in  $\mathbb{D} \times \mathbb{R}$  will be studied. Before entering the next section, we introduce some conventions and notations to be used frequently in the following sections.

- We use Latin letters for the indices involving  $y \in \mathbb{R}^2$  and Greek letters for those related to  $(y, z) \in \mathbb{R}^3$ .

- We use  $u \cdot v$  for the scalar product of vectors in Euclidean spaces, while  $uv^T$  should be understood as the matrix product. Besides, all the vectors are columns if there is no further specifications.

- Let  $J = (a_{\alpha\beta})$  be the Jacobian matrix of  $\iota$  defined by (4.2.2). It is easy to see that

$$(4.2.9) \quad J = \begin{pmatrix} J_0 & J_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \zeta + y \nabla_y^T \zeta & y \zeta_z \\ 0 & 1 \end{pmatrix},$$

whose inverse is denoted by

$$(4.2.10) \quad J^{-1} = (a^{\alpha\beta}) = \begin{pmatrix} J_0^{-1} & -J_0^{-1}J_1 \\ 0 & 1 \end{pmatrix}.$$

- Let  $(g_{\alpha\beta}) := J^T J$  be the metric tensor whose inverse is denoted by  $(g^{\alpha\beta}) = J^{-1} J^{-T}$ . We use  $g$  to represent the determinant of  $(g_{\alpha\beta})$ , which is equal to

$$(4.2.11) \quad g := \det(g_{\alpha\beta}) = (\det J)^2 = (\det J_0)^2 = [\zeta(\zeta + y \cdot \nabla_y \zeta)]^2 = \zeta^2 (\partial_\rho(\rho\zeta))^2 > 0.$$

Note that its restriction at  $\rho = 1$  is  $\eta^4$  since  $\partial_\rho \zeta|_{\rho=1} = 0$  due to our construction.

- We denote by  $\nabla_g$  the pull-back of  $\nabla_{x,z}$  by  $\iota$ , namely

$$(4.2.12) \quad \nabla_g = J^{-T} \nabla_{y,z}.$$

Moreover the pull-back of Laplacian  $\Delta_{x,z}$  reads

$$(4.2.13) \quad \Delta_g = \frac{1}{\sqrt{g}} \partial_\alpha \left( \sqrt{g} g^{\alpha\beta} \partial_\beta \right),$$

which is clearly uniform elliptic on  $\mathbb{D} \times \mathbb{R}$ .

- Let  $n_g$  be the pull-back of the conormal vector  $n$  defined by (4.1.5). One may check that

$$(4.2.14) \quad n_g = J^{-T}Y,$$

where  $Y = (y_1, y_2, 0)^T$ .

- In polar coordinate  $(y, z) = (\rho\theta, z)$ , the restriction of  $\nabla_g$  at  $\{\rho = 1\}$  is

$$(4.2.15) \quad \nabla_g = e_\rho \frac{1}{\eta} \partial_\rho + e_\theta \left( \frac{1}{\eta} \partial_\theta - \frac{\eta_\theta}{\eta^2} \partial_\rho \right) + e_z \left( \partial_z - \frac{\eta_z}{\eta} \partial_\rho \right),$$

where  $(e_\rho, e_\theta, e_z)$  is the orthonormal basis associated to polar variables  $(\rho, \theta, z)$ . Moreover,  $n_g$  can be expressed as

$$(4.2.16) \quad n_g = \frac{1}{\eta} \left( e_\rho - \frac{\eta_\theta}{\eta} e_\theta - \eta_z e_z \right),$$

while  $B, V, N$  defined in (4.1.16), (4.1.17), (4.1.15) read

$$(4.2.17) \quad B = B(\psi) = \frac{1}{\eta} \varphi_\rho|_{\rho=1},$$

$$(4.2.18) \quad V = V(\psi) = \left[ \frac{1}{\eta} \left( \varphi_\theta - \frac{\eta_\theta}{\eta} \varphi_\rho \right) e_\theta + \left( \varphi_z - \frac{\eta_z}{\eta} \varphi_\rho \right) e_z \right] \Big|_{\rho=1},$$

$$(4.2.19) \quad N = N(\psi) = BV \cdot \left( \frac{\eta_\theta}{\eta} e_\theta + \eta_z e_z \right) + \frac{|V|^2 - B^2}{2},$$

where  $\varphi = \phi \circ \iota$ . As a result, the Dirichlet-to-Neumann operator defined by (4.1.14) can be expressed as

$$(4.2.20) \quad G(\eta)\psi = \eta n_g \cdot \nabla_g \varphi|_{\rho=1} = \eta(J^{-T}Y) \cdot \nabla_g \varphi|_{r=1} = B - V \cdot \left( \frac{\eta_\theta}{\eta}, \eta_z \right).$$

In addition, a simple calculation gives

$$(4.2.21) \quad \left( e_\theta \frac{\partial_\theta}{\eta} + e_z \partial_z \right) \psi = V + B \left( e_\theta \frac{\eta_\theta}{\eta} + e_z \eta_z \right).$$

### 4.2.2 . Elliptic regularity

In this section, we shall solve (4.1.13) via variational formulation and illustrate the regularity of  $\phi$  from those of  $(\eta, \psi)$  with the help of diffeomorphism constructed in (4.2.2). As a consequence, a rigorous definition of the Dirichlet-to-Neumann operator  $G(\eta)$ , which is defined formally by (4.1.14), and some high order estimates will also be established.

In the rest of this chapter, we denote by  $\varphi$  the pull-back of  $\phi$  by change of variable  $\iota$ . By definition (4.1.12) of  $\psi$ , it is clear that in polar coordinate  $(y, z) = (\rho\theta, z)$ ,

$$\psi(\theta, z) = \varphi|_{\rho=1} = \varphi(\theta, z).$$

From (4.1.13),  $\varphi$  is the solution to

$$(4.2.22) \quad \begin{cases} \Delta_g \varphi = 0, \\ \varphi|_{\rho=1} = \psi \in H^{s_0}(\mathbb{T} \times \mathbb{R}). \end{cases}$$

Let  $\Psi \in H^{s_0 + \frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  be an extension of  $\psi$  supported near  $\rho = 1$ . One may repeat the procedure in previous sections (construction of  $\tilde{\eta}$  from  $\eta$ ) to obtain the desired  $\Psi$ . Then, by writing  $\varphi = \Phi' + \Psi$ , the equation above can be restated as the following variational problem

$$(4.2.23) \quad \iint \nabla_g \Phi' \cdot \nabla_g H \sqrt{g} dy dz = - \iint \nabla_g \Psi \cdot \nabla_g H \sqrt{g} dy dz, \quad \forall H \in H_0^1(\mathbb{D} \times \mathbb{R}).$$

Provided that  $\eta \in H_R^{s + \frac{1}{2} -}(\mathbb{T} \times \mathbb{R})$  with  $s > \frac{3}{2}$ , we have  $\zeta \in H_{R_\epsilon}^{s+1-}(\mathbb{D} \times \mathbb{R})$  from Proposition 4.2.2 and thus  $g_{\alpha\beta}, g^{\alpha\beta}, g \in L^\infty(\mathbb{D} \times \mathbb{R})$ . A classical argument by Lions–Lax–Milgram theorem (see for example [Bre11]) guarantees a unique solution  $\Phi' \in H_0^1(\mathbb{D} \times \mathbb{R})$  to the variational problem (4.2.23), which yields  $\varphi \in H^1(\mathbb{D} \times \mathbb{R})$ . Moreover, by choosing  $H = \Phi'$ , we obtain the estimate

$$c_1 \inf_{\mathbb{D} \times \mathbb{R}} \sqrt{g} \|\Phi'\|_{H_0^1(\mathbb{D} \times \mathbb{R})}^2 \leq C_1 \sup_{\mathbb{D} \times \mathbb{R}} \sqrt{g} \|\Phi'\|_{H_0^1(\mathbb{D} \times \mathbb{R})} \|\Psi\|_{H^1(\mathbb{D} \times \mathbb{R})},$$

where  $0 < c_1 < C_1$  satisfies  $c_1 < (g^{\alpha\beta}) < C_1$ . Under the hypothesis (H0), by using the definition  $(g^{\alpha\beta}) = J^{-1} J^{-T}$  with  $J$  defined by (4.2.9) and lower bound (4.2.6), we have

$$\|\Phi'\|_{H_0^1(\mathbb{D} \times \mathbb{R})} \leq C'_1 \|\Psi\|_{H^1(\mathbb{D} \times \mathbb{R})},$$

for some  $C'_1$  depending only on  $c_0, C_0$  appearing in the hypothesis (H0).

Consequently, we are able to define  $G(\eta)$  by duality :

$$(4.2.24) \quad \int G(\eta) \psi h d\theta dz := \iint \nabla_g \varphi \cdot \nabla_g H \sqrt{g} dy dz, \quad \forall H \in H^1(\mathbb{D} \times \mathbb{R}), \quad h = H|_{\rho=1}.$$

An application of the trace theorem and duality of Sobolev spaces implies the following estimate

$$(4.2.25) \quad \|G(\eta) \psi\|_{H^{-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{\frac{1}{2}}(\mathbb{T} \times \mathbb{R})}, \quad \forall s > \frac{3}{2}.$$

In order to obtain high order estimates, we need to prove the regularity of the elliptic equation (4.2.22) or its variational form (4.2.23).

**Proposition 4.2.3.** *Let  $\eta \in H_R^{s + \frac{1}{2} -}(\mathbb{T} \times \mathbb{R})$  and  $\psi \in H^{s_0}(\mathbb{T} \times \mathbb{R})$  with  $s > \frac{3}{2}$  and  $\frac{1}{2} \leq s_0 \leq s$ . Then the unique solution  $\varphi = \Phi' + \Psi$  to (4.2.23) verifies*

$$(4.2.26) \quad \|\varphi\|_{H^{s_0 + \frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})},$$

where  $C > 0$  is a smooth increasing function.

The proof of this proposition is classic and we leave it to Appendix 4.B. As a corollary, we are able to generalize the estimate (4.2.25) for high order Sobolev norms,

**Corollary 4.2.4.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$   $s > \frac{3}{2}$ . Then for all  $s_0 \in [\frac{1}{2}, s]$ , we have*

$$(4.2.27) \quad \|G(\eta)\|_{\mathcal{L}(H^{s_0}(\mathbb{T} \times \mathbb{R}); H^{s_0-1}(\mathbb{T} \times \mathbb{R}))} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right),$$

where  $C > 0$  is a smooth increasing function.

### 4.2.3 . Hamiltonian structure

In this section, we investigate the Hamiltonian structure of (4.1.18). Recall that for standard water-wave equation where the interface is given by  $\{(x, y) \in \mathbb{R}^{d+1} : y = \eta(x)\}$ , the Hamiltonian  $\mathcal{H}$  is taken as the total energy, which is preserved in time and can be written in terms of  $\eta$  and  $\psi$  (restriction of scalar potential), and the Hamiltonian formulation reads

$$\begin{cases} \eta_t = \frac{\delta \mathcal{H}}{\delta \psi}, \\ \psi_t = -\frac{\delta \mathcal{H}}{\delta \eta}. \end{cases}$$

This formulation has been firstly discovered by Zakharov [Zak68], while the case of two-phase fluid is studied by Benjamin-Bridges [BB97]. For fluids with nonzero vorticity, Castro-Lannes [CL15] formally prove the Hamiltonian equation for the triple  $(\eta, \psi, \omega)$ , where  $\psi$  can be defined by projection and  $\omega$  is the vorticity.

In the case of sphere-like interface, Beyer-Günther [BG98] pointed out that one should modify  $\psi$  by  $p = \eta^d \psi$ , where  $d$  is the dimension of spherical interface (perturbation of unit sphere  $\mathbb{S}^d$ ). Inspired by this, we turn to the new variable  $(\eta, p) := (\eta, \eta^d \psi)$  satisfying

$$(4.2.28) \quad \begin{cases} \eta_t = G(\eta)\psi, \\ -p_t = -\psi G(\eta)\psi + \eta \left( \sigma \left( H - \frac{1}{2R} \right) + N \right). \end{cases}$$

due to equation (4.1.18).

The total energy of the system (4.1.18) (or (4.2.28)) is

$$(4.2.29) \quad \mathcal{H} := E_k + E_p.$$

Here  $E_k$  is kinetic energy

$$(4.2.30) \quad E_k := \frac{1}{2} \iint_{\Omega(t)} |\nabla_{x,z} \phi|^2 dx dz = \frac{1}{2} \iint_{D \times \mathbb{R}} |\nabla_g \varphi|^2 \sqrt{g} dy dz = \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}} \psi (\eta G(\eta)) \psi d\theta dz,$$

where the last equality follows from divergence theorem and  $\Delta_g \varphi = 0$  (see (4.2.24)).  $E_p$  is potential energy

$$(4.2.31) \quad E_p := \frac{\sigma}{2} A = \frac{\sigma}{2} \int_{\mathbb{T} \times \mathbb{R}} \left[ \eta \left( \sqrt{1 + \left( \frac{\eta_\theta}{\eta} \right)^2 + \eta_z^2} - 1 \right) - \frac{|\eta - R|^2}{2R} \right] d\theta dz,$$

where  $A$  is the (normalized) area of interface  $\Sigma(t)$  and  $R$  is the constant from (H1). In the periodic case, these normalization can be omitted.

By regarding  $\mathcal{H}$  as a function on  $\eta$  and  $p = \eta\psi$ , we have the following Hamiltonian formulation.

**Proposition 4.2.5.** *Let  $1 \leq s_0 \leq s$  and  $s > \frac{3}{2}$ . For all  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,  $\delta\eta \in H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  and  $p, \delta p \in H^{s_0}(\mathbb{T} \times \mathbb{R})$ , we have*

$$(4.2.32) \quad \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{H}(p + \epsilon\delta p, \eta) = \int \eta_t \delta p d\theta dz,$$

$$(4.2.33) \quad \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{H}(p, \eta + \epsilon\delta\eta) = \int (-p_t) \delta\eta d\theta dz,$$

where  $\eta_t$  and  $-p_t$  should be understood as the right hand side of (4.2.28).

Note that the results in previous sections imply that  $g, (g_{\alpha\beta}), (g^{\alpha\beta}) \in C^{s-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})$  and  $\varphi \in H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ , so as their variations in  $p$  or  $\eta$ . These ensure that all the integrands below are integrable.

### Variation in $p$ .

We first prove (4.2.32). In this paragraph, we fix  $\eta$  and use the notation

$$\delta Q := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} Q(p + \epsilon\delta p)$$

for any quantity depending on  $p$ . Since the potential energy (4.2.31) depends only on  $\eta$ , the variation can be applied merely for kinetic energy (4.2.30),

$$\begin{aligned} \delta\mathcal{H} &= \delta E_k = \iint \nabla_g^T \varphi \nabla_g \delta\varphi \sqrt{g} dy dz \\ &= \iint (\sqrt{g} J^{-1} \nabla_g \varphi) \cdot \nabla_{y,z} \delta\varphi dy dz \\ &= \int Y \cdot (\sqrt{g} J^{-1} \nabla_g \varphi) \delta\psi d\theta dz, \end{aligned}$$

where  $Y = (y_1, y_2, 0)^T$  and the last equality follows from divergence theorem with  $\delta\psi$  the restriction of  $\delta\varphi$ . By (4.2.11) and (4.2.20), we have

$$Y \cdot (\sqrt{g} J^{-1} \nabla_g \varphi) \delta\psi = (J^{-T} Y) \cdot \nabla_g \varphi \eta^2 \delta\psi = G(\eta) \psi \eta \delta\psi = G(\eta) \psi \delta p,$$

Note that the last equality follows from  $\psi = p/\eta$  and the fact that  $\eta$  is fixed. This completes the proof of (4.2.32).

**Variation in  $\eta$ .**

Let us fix  $p$  and denote

$$\delta Q := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} Q(\eta + \epsilon\delta\eta)$$

for any quantity depending on  $\eta$ . The variation of potential part  $E_p$  can be calculated directly,

$$(4.2.34) \quad \delta E_p = \frac{\sigma}{2} \delta A = \sigma \int \eta \left( H - \frac{1}{2R} \right) \delta\eta d\theta dz,$$

where  $H$  is the mean curvature defined in (4.1.9). For the kinetic part, we have

**Lemma 4.2.6.** *Let  $1 \leq s_0 \leq s$  and  $s > \frac{3}{2}$ . For all  $\eta, \delta\eta \in C^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  and  $p, \delta p \in H^{s_0}(\mathbb{T} \times \mathbb{R})$ , we have*

$$(4.2.35) \quad \delta E_k := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} E_k(p, \eta + \epsilon\delta\eta) = \int (-\psi G(\eta)\psi + \eta N) \delta\eta d\theta dz.$$

Note that, by Sobolev embedding, (4.2.35) and (4.2.34) implies (4.2.33). Here we may temporarily ignore the hypothesis (H1) since all the integrals below make sense whenever  $\eta, \delta\eta$  (and  $\zeta, \delta\zeta$ ) have Lipschitz regularity, which is ensured by the condition  $\eta, \delta\eta \in C^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,  $s > \frac{3}{2}$ .

*Proof.* By definition (4.2.30) of kinetic energy  $E_k$ , we have

$$(4.2.36) \quad \delta E_k = I_1 + I_2,$$

$$(4.2.37) \quad I_1 = \frac{1}{2} \iint \partial_\alpha \varphi \delta(\sqrt{g} g^{\alpha\beta}) \partial_\beta \varphi dy dz,$$

$$(4.2.38) \quad I_2 = \iint \nabla_g \varphi \cdot \nabla_g \delta\varphi \sqrt{g} dy dz.$$

As before, by divergence theorem,  $I_2$  equals

$$(4.2.39) \quad I_2 = \int (G(\eta)\psi) \eta \delta\psi d\theta dz = - \int \psi (G(\eta)\psi) \delta\eta d\theta dz,$$

where the last equality follows from  $\delta\psi = \delta(p/\eta) = -\delta\eta p/(\eta^2) = -\psi\delta\eta/\eta$ .

**Remark 4.2.7.** *Our final goal is to establish the Hamiltonian formulation w.r.t. variable  $(\eta, p)$ . Therefore, during the calculation of variation in  $\eta$  above, the kinetic energy  $E_k$ , which is a function of  $(\eta, \psi)$  due to definition (4.2.30), should be regarded as a function of  $(\eta, p)$ , namely*

$$E_k = E_k \left( \eta, \frac{p}{\eta} \right) = \frac{1}{2} \int p G(\eta) \left( \frac{p}{\eta} \right) dx.$$

*And the variation in  $\eta$  should be understood for fixed  $p$ ,*

$$\delta E_k = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \frac{1}{2} \int p G(\eta + \epsilon\delta\eta) \left( \frac{p}{\eta + \epsilon\delta\eta} \right) dx$$

However, in most parts of this chapter, we are interested in variable  $(\eta, \psi)$  instead of  $(p, \psi)$ . Then, the derivative in  $\eta$  should be defined with fixed  $\psi$  instead of fixed  $p$  (in such case,  $p = \eta\psi$  may vary in  $\eta$ ). The main difference is that, when  $\psi$  is fixed,  $I_2$  is automatically zero due to the first equality of (4.2.39). This is the case of shape derivative to be studied in Section 4.2.4.

As for  $I_1$ , a direct calculus gives that

$$(4.2.40) \quad \begin{aligned} \frac{1}{2}\delta(\sqrt{g}g^{\alpha\beta})\partial_\alpha\varphi\partial_\beta\varphi &= \frac{1}{2}\nabla_{y,z}^T \left( J^{-1}Y |\nabla_g\varphi|^2 \sqrt{g}\delta\zeta \right) - \nabla_{y,z}^T (\sqrt{g}J^{-1}\nabla_g\varphi Y^T \nabla_g\varphi\delta\zeta) \\ &\quad + \sqrt{g}\Delta_g\varphi Y \cdot \nabla_g\varphi\delta\zeta, \end{aligned}$$

the proof of this is left to Lemma 4.D.1. The technical identity above implies, via divergence theorem and  $\Delta_g\varphi = 0$ , that

$$I_1 = \frac{1}{2} \int Y^T J^{-1}Y |\nabla_g\varphi|^2 \eta^2 \delta\eta d\theta dz - \int Y^T J^{-1}\nabla_g\varphi Y^T \nabla_g\varphi \eta^2 \delta\eta d\theta dz.$$

Note that, at boundary  $\rho = 1$ , we have  $\sqrt{g} = \eta^2$ , as it has been noticed after (4.2.11). By formula (4.2.10) of  $J^{-1}$ , definition (4.2.9) of  $J_0$ , and  $Y = (y, 0)^T$ , we have,

$$Y^T J^{-1}Y = \frac{\zeta}{\sqrt{g}}|y|^2.$$

The expression (4.2.15) yields that, as  $\rho = 1$ ,

$$|\nabla_g\varphi|^2 = B^2 + |V|^2, \quad Y^T \nabla_g\varphi = B.$$

In addition, we have seen in (4.2.20) that the trace of  $Y^T J^{-1}\nabla_g\varphi$  is  $G(\eta)\psi/\eta$ . Consequently,

$$(4.2.41) \quad I_1 = \frac{1}{2} \int \eta(B^2 + |V|^2)\delta\eta d\theta dz - \int \eta G(\eta)\psi B \delta\eta d\theta dz = \int \eta N \delta\eta d\theta dz,$$

thanks to (4.2.17), (4.2.18), (4.2.19), and (4.2.20).  $\square$

In conclusion, the variational identity (4.2.33) follows from (4.2.34) and (4.2.35). Formally, we may write (4.2.32) and (4.2.33) in classical form

$$\begin{cases} \eta_t = \frac{\delta\mathcal{H}}{\delta p}, \\ p_t = -\frac{\delta\mathcal{H}}{\delta\eta}. \end{cases}$$

As a consequence of this Hamiltonian formulation, the total energy  $\mathcal{H}$  is preserved in time. Note that for Cauchy problem, there is no difference between  $(\eta, \psi)$  and  $(\eta, p)$ , since (H0) together with  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,  $s > 3/2$  guarantees that  $\psi \in H^s(\mathbb{T} \times \mathbb{R})$  if and only if  $p \in H^s(\mathbb{T} \times \mathbb{R})$ . In the rest of this chapter, we shall still work on  $(\eta, \psi)$  for the simplicity of notation.

#### 4.2.4 . Shape derivative

We have proved in Corollary 4.2.4 that  $G(\eta)$  is a 1-order operator for any fixed  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  with  $s > \frac{3}{2}$ . In this section, we shall show that, for fixed  $\psi$ ,  $G(\eta)\psi$  is derivable w.r.t.  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  and this derivative can be calculated explicitly, which is known as *shape derivative*.

In the case of planar water-wave ( $\Sigma$  as a perturbation of  $\mathbb{R}^d$ ), we refer to [Lan13], Section 3.3 for a detailed study of shape derivative. The axis-symmetric version (system independent of  $\theta$ ) of shape derivative (4.2.42) is also proved in [HK23]. In these references, the authors calculate directly the derivative in  $\eta$  of (4.2.20). The main difficulty of this method is to represent  $G(\eta)(B\delta\eta)$  in terms of  $\eta$ ,  $\varphi$ , and  $\delta\varphi$ . To do so, one needs to apply variational calculus for the elliptic equation (4.2.22) and then determine the harmonic extension of  $B\delta\eta$ , which cannot be obtained simply due to the (relatively complicated) change of coordinate (4.2.2). Nevertheless, one will see in the proof of Proposition 4.2.8 that the variational calculus (4.2.35) of kinetic energy is equivalent to (4.2.42), where the former one has already been proved in previous section without using shape derivative.

**Proposition 4.2.8.** *Let  $s > \frac{3}{2}$  and  $1 \leq s_0 \leq s$ . Then for all fixed  $\psi \in H^{s_0}$ , the map*

$$\begin{aligned} H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) &\rightarrow H^{s_0-1}(\mathbb{T} \times \mathbb{R}) \\ \eta &\mapsto G(\eta)\psi \end{aligned}$$

is  $C^1$  in the sense that, for all  $\delta\eta \in H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  or  $\delta\eta \in C^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,

$$(4.2.42) \quad \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} G(\eta + \epsilon\delta\eta)\psi = -G(\eta)(B\delta\eta) - \nabla_{\theta,z} \cdot \left( \left( \frac{V^\theta}{\eta} e_\theta + V^z e_z \right) \delta\eta \right) - \frac{B\delta\eta}{\eta},$$

where  $V^\theta, V^z$  are defined by orthogonal decomposition  $V = (V^\theta, V^z) = V^\theta e_\theta + V^z e_z$ . Recall that the second term on the right hand side should be understood as

$$\nabla_{\theta,z} \cdot \left( \left( \frac{V^\theta}{\eta} e_\theta + V^z e_z \right) \delta\eta \right) = \partial_\theta \left( \frac{V^\theta}{\eta} \delta\eta \right) + \partial_z (V^z \delta\eta).$$

*Proof.* As in previous section,  $\delta$  denotes the derivative in  $\eta$ . To begin with, we write (4.2.42) into integrated form, namely, for all  $f \in C_c^\infty(\mathbb{T} \times \mathbb{R})$ ,

$$\begin{aligned} &\int \eta \delta (G(\eta)\psi) f d\theta dz \\ &= - \int \eta G(\eta)(B\delta\eta) f d\theta dz - \int \eta \nabla_{\theta,z} \cdot \left( \left( \frac{V^\theta}{\eta} e_\theta + V^z e_z \right) \delta\eta \right) f d\theta dz - \int B\delta\eta f d\theta dz \\ &= - \int B\eta G(\eta) f \delta\eta d\theta dz - \int B f \delta\eta d\theta dz + \int \left( V^\theta e_\theta + V^z e_z \right) \cdot \left( e_\theta \frac{\partial_\theta}{\eta} + e_z \partial_z \right) (\eta f) \delta\eta d\theta dz \\ &= - \int B G(\eta) f \eta \delta\eta d\theta dz - \int G(\eta)\psi f \delta\eta d\theta dz + \int \left( V^\theta \frac{f_\theta}{\eta} + V^z f_z \right) \eta \delta\eta d\theta dz. \end{aligned}$$

Let  $F$  be the harmonic extension of  $f$ , namely  $\Delta_g F = 0$  and  $F|_{\rho=1} = f$ . Since  $f$  is regular,  $F$  is well-defined as well as  $B(f)$  and  $V(f)$ . By applying (4.2.20) and (4.2.21), we may write the equality above as

$$\begin{aligned} & \int \eta \delta (G(\eta) \psi) f d\theta dz + \int G(\eta) \psi f \delta \eta d\theta dz \\ &= \int \left( B(\psi) V(f) \cdot \left( \frac{\eta_\theta}{\eta}, \eta_z \right) + B(f) V(\psi) \left( \frac{\eta_\theta}{\eta}, \eta_z \right) - B(\psi) B(f) + V(\psi) \cdot V(f) \right) \eta \delta \eta d\theta dz, \end{aligned}$$

from which, one can see that the desired result (4.2.42) is equivalent to : for all  $f \in C_c^\infty(\mathbb{T} \times \mathbb{R})$ ,

(4.2.43)

$$\begin{aligned} & \delta \left( \int \eta G(\eta) \psi f d\theta dz \right) \\ &= \int \left( B(\psi) V(f) \cdot \left( \frac{\eta_\theta}{\eta}, \eta_z \right) + B(f) V(\psi) \left( \frac{\eta_\theta}{\eta}, \eta_z \right) - B(\psi) B(f) + V(\psi) \cdot V(f) \right) \eta \delta \eta d\theta dz. \end{aligned}$$

When  $\psi$  belongs to the same class  $C_c^\infty(\mathbb{T} \times \mathbb{R})$  as  $f$ , due to the fact that both sides are symmetric quadratic form of  $(\psi, f)$ , it suffices to check the case  $\psi = f$ . According to (4.2.30), the left hand side becomes

$$\delta \left( \int \eta G(\eta) \psi \psi d\theta dz \right) = 2\delta E_k.$$

Note that, unlike in the proof of Proposition 4.2.5,  $\psi$  is required to be fixed, instead of  $p$ . Thus the terms involving  $\delta\psi$  are automatically zero. By applying the same calculation as in Lemma 4.2.6, we can conclude that

$$2\delta E_k = 2(I_1 + I_2) = 2 \int \eta N \delta \eta d\theta dz,$$

where  $I_1, I_2$  are defined in (4.2.37) and (4.2.38) with  $I_2 = 0$  as explained above. From (4.2.19), when  $\psi = f$ , the right hand side of (4.2.43) equals

$$2 \int \left( BV \cdot \left( \frac{\eta_\theta}{\eta}, \eta_z \right) + \frac{|V|^2 - B^2}{2} \right) \eta \delta \eta d\theta dz = 2 \int \eta N \delta \eta d\theta dz.$$

Till now, we have proved (4.2.42) for  $\psi \in C_c^\infty(\mathbb{T} \times \mathbb{R})$ . Since  $C_c^\infty(\mathbb{T} \times \mathbb{R})$  is dense in  $H^{s_0}(\mathbb{T} \times \mathbb{R})$ , it remains to check that the left hand side of (4.2.42) satisfies

$$(4.2.44) \quad \left\| \frac{d}{d\epsilon} \Big|_{\epsilon=0} G(\eta + \epsilon \delta \eta) \psi \right\|_{H^{s_0-1}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\delta \eta\|_{C^{s-\frac{1}{2}}} \right) \|\psi\|_{H^{s_0-1}}.$$

To prove this, we apply the variation in  $\eta$  to the equation (4.2.22),

$$\begin{cases} \Delta_g \delta \varphi = -\frac{1}{\sqrt{g}} \partial_\alpha (\delta (\sqrt{g} g^{\alpha\beta}) \partial_\beta \varphi), \\ \delta \varphi|_{\rho=1} = 0. \end{cases}$$

Recall that  $\delta$  should be understood as the derivative in  $\epsilon$  at  $\epsilon = 0$ . It follows immediately from (4.2.9)-(4.2.11) that  $\delta(\sqrt{g}g^{\alpha\beta})$  is a smooth function of  $\zeta$  and  $\nabla_{y,z}\zeta$ , depending linearly in  $\delta\eta$  and  $\nabla_{y,z}\delta\eta$ . Consequently,  $\varphi \in H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  implies that

$$\left\| \frac{1}{\sqrt{g}} \partial_\alpha \left( \delta(\sqrt{g}g^{\alpha\beta}) \partial_\beta \varphi \right) \right\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})}, \|\delta\eta\|_{C^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\varphi\|_{H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})}.$$

In fact,  $\delta(\sqrt{g}g^{\alpha\beta})$  can be written in terms of  $F_1(\zeta, \nabla_{\rho,\theta,z}\zeta)\delta\zeta$  and  $F_2(\zeta, \nabla_{\rho,\theta,z}\zeta)\nabla_{\rho,\theta,z}\delta\zeta$ , where  $F_1$  and  $F_2$  are smooth functions with  $F_1(0) = F_2(0) = 0$ . Since  $\zeta \in H_{R_\epsilon}^{s+1}(\mathbb{D} \times \mathbb{R})$  with  $R_\epsilon$  defined by (4.2.7), due to Proposition 4.2.2 and the assumption  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , the coefficients before  $\delta\zeta$  and  $\nabla_{\rho,\theta,z}\delta\zeta$  belong to  $H^s(\mathbb{D} \times \mathbb{R})$ , up to some  $C_b^\infty$  normalizations (see Proposition 4.C.22). In the mean time, when  $\delta\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,  $\delta\zeta \in H_W^{s+1}(\mathbb{D} \times \mathbb{R})$  and  $\nabla_{\rho,\theta,z}\delta\zeta \in H_{W'}^s(\mathbb{D} \times \mathbb{R})$  for some  $W, W' \in C_b^\infty(\mathbb{D} \times \mathbb{R})$  due to Proposition 4.2.2, which also guarantees that, as  $\delta\eta \in C^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ ,  $\delta\zeta \in C^{s+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  and  $\nabla_{\rho,\theta,z}\delta\zeta \in C^{s-\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ . Therefore, the desired inequality follows from the estimate of products (Proposition 4.C.8). As a consequence, an application of elliptic regularity (4.B.2) gives that

$$(4.2.45) \quad \|\delta\varphi\|_{H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})}, \|\delta\eta\|_{C^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})}.$$

By expression (4.2.20), the variation of

$$G(\eta)\psi = \zeta Y^\alpha g^{\alpha\beta} \partial_\beta \varphi|_{\rho=1}$$

is composed by two parts, the variation in coefficients (depending only on  $\zeta$ ) and in  $\varphi$ , namely,

$$\begin{aligned} \delta(G(\eta)\psi) &= \delta \left( \zeta Y^\alpha g^{\alpha\beta} \partial_\beta \varphi \right) \Big|_{\rho=1} \\ &= \left( \delta(\zeta Y^\alpha g^{\alpha\beta}) \partial_\beta \varphi \right) \Big|_{\rho=1} + \left( \zeta Y^\alpha g^{\alpha\beta} \partial_\beta \delta\varphi \right) \Big|_{\rho=1} \end{aligned}$$

We can deduce from (4.2.45) that  $\partial_\beta \delta\varphi|_{\rho=1} \in H^{s_0-1}(\mathbb{T} \times \mathbb{R})$ , while, up to  $C_b^\infty(\mathbb{D} \times \mathbb{R})$  normalizations,  $\zeta Y^\alpha g^{\alpha\beta}$  has  $H^s(\mathbb{D} \times \mathbb{R})$  regularity, the trace at  $\rho = 1$  of which belongs to  $H^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ . By applying Corollary 4.C.9, we are able to conclude that the second part (variation in  $\varphi$ ) lies in  $H^{s_0-1}(\mathbb{T} \times \mathbb{R})$ . As for the first part, we observe that  $\zeta Y^\alpha g^{\alpha\beta}$  can be written as a smooth function of  $\zeta$ ,  $\nabla_{y,z}\zeta$ , and  $Y$ , i.e.

$$\zeta Y^\alpha g^{\alpha\beta} = G(\zeta, \nabla_{y,z}\zeta, Y),$$

for some smooth function  $G$ . The the variation in  $\eta$  reads

$$\delta \left( \zeta Y^\alpha g^{\alpha\beta} \right) = \partial_1 G(\zeta, \nabla_{y,z}\zeta, Y) \delta\zeta + \partial_2 G(\zeta, \nabla_{y,z}\zeta, Y) \nabla_{y,z}\delta\zeta,$$

whose trace at  $\rho = 0$  belongs to  $H^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , thanks to the same argument as above. Then, by Corollary 4.C.9, its product with  $\partial_\beta \varphi|_{\rho=1} \in H^{s_0-1}(\mathbb{T} \times \mathbb{R})$  remains in  $H^{s_0-1}(\mathbb{T} \times \mathbb{R})$ , which completes the proof.  $\square$

In the formula (4.2.42), the left hand side belongs to  $H^{s_0-1}(\mathbb{T} \times \mathbb{R})$  while the first two terms on the right hand side are only in  $H^{s_0-2}(\mathbb{T} \times \mathbb{R})$ . This indicates that there exists an implicit cancellation between these terms. In fact, by choosing  $\delta\eta = 1 \in C^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , we can deduce from  $B/\eta \in H^{s_0-1}(\mathbb{T} \times \mathbb{R})$ , which is a consequence of Corollary 4.C.9 and 4.C.22, that

**Proposition 4.2.9.** *Let  $s > \frac{3}{2}$  and  $1 \leq s_0 < s$ . Then for all  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$  and  $\psi \in H^{s_0}(\mathbb{T} \times \mathbb{R})$ , we have*

$$(4.2.46) \quad \left\| G(\eta)B + \left( \frac{e_\theta \partial_\theta}{\eta} + e_z \partial_z \right) \cdot V \right\|_{H^{s_0-1}(\mathbb{T} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})},$$

where  $C > 0$  is an increasing smooth function.

*Proof.* It remains to check that

$$\left\| \left( \frac{e_\theta \partial_\theta}{\eta} + e_z \partial_z \right) \cdot V - \nabla_{\theta,z} \cdot \left( \frac{V^\theta}{\eta} e_\theta + V^z e_z \right) \right\|_{H^{s_0-1}(\mathbb{T} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})},$$

which is equivalent to

$$\left\| \partial_\theta \eta^{-1} V^\theta \right\|_{H^{s_0-1}(\mathbb{T} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})}.$$

We have seen that  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , which implies  $\partial_\theta \eta^{-1} \in H_R^{s-\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , thanks to Proposition 4.C.22. Since  $V^\theta \in H^{s_0-1}$ , the desired result follows from Corollary 4.C.9.  $\square$

### 4.3 . Paralinearization of the system

This section is devoted to an explicit formulation of Dirichlet-to-Neumann operator  $G(\eta)$  as paradifferential operator, up to some remainder of lower order. And, as a result, we shall write the system (4.1.18) in paralinear form. Namely, all the nonlinear terms will be replaced by paradifferential operators acting on  $(\eta, \psi)$ , up to proper regular remainders. And the symbol of these paradifferential operators is a smooth function of  $\eta$  and its derivatives and homogeneous in Fourier variable  $\xi$ . We shall follow the strategy used in [AM09] and [ABZ11] (see also [ABZ14]).

To achieve this, we first observe that, by (4.2.20),

$$(4.3.1) \quad \begin{aligned} G(\eta)\psi &= \frac{1}{\eta} \left( 1 + \left( \frac{\eta_\theta}{\eta} \right)^2 + \eta_z^2 \right) \partial_\rho \varphi|_{\rho=1} - \frac{\eta_\theta}{\eta} \frac{\varphi_\theta|_{\rho=1}}{\eta} - \eta_z \varphi_z|_{\rho=1} \\ &= \frac{1}{\eta} \left( 1 + \left( \frac{\eta_\theta}{\eta} \right)^2 + \eta_z^2 \right) \partial_\rho \varphi|_{\rho=1} - \frac{\eta_\theta}{\eta} \frac{\psi_\theta}{\eta} - \eta_z \psi_z, \end{aligned}$$

where the only implicit term is  $\partial_\rho \varphi|_{\rho=1}$ , which is linear in  $\psi$  and depends implicitly on  $\eta$ . In order to represent normal derivative  $\partial_\rho|_{\rho=1}$  as tangential ones, we follow the idea in

[AM09] and return to the Poisson's equation (4.2.22). Formally speaking, we shall rewrite Laplacian  $\Delta_g$  as

$$\Delta_g = A_0(\partial_\rho + A'_1)(\partial_\rho - A_1) + \text{remainders},$$

where  $A_0$ ,  $A_1$ , and  $A'_1$  are elliptic operators involving only tangential derivatives. This decomposition will be proved in Lemma 4.3.14, while the tangential operators should be understood as paradifferential operator since the coefficients are smooth functions of  $\eta$  and its derivatives and thus have limited regularity. As a result, the Poisson's equation (4.2.22) ensures that  $v := (\partial_\rho - A_1)\varphi$  solves the parabolic equation

$$(\partial_\rho + A'_1)v = \text{regular terms},$$

whose smoothing effect leads to a higher regularity of  $v|_{\rho=1}$  than  $\varphi$  (or  $\psi$ ). Consequently, one may replace  $\partial_\rho|_{\rho=1}$  by  $A_1|_{\rho=1}$  and then deduce the parilinearization of (4.3.1) (see Proposition 4.3.17).

With parilinear formula (4.3.52) of  $G(\eta)\psi$ , we are able to write the nonlinear terms in (4.1.18) in a similar form. In fact, these terms (quadratic term  $N$  and mean curvature  $H$ ) can be represented as a function of  $\eta$ ,  $\psi$ , and  $G(\eta)\psi$ , together with their derivatives. Hence, one may simply apply the parilinear formula (4.C.19) and the parilinearization of  $G(\eta)\psi$ , which will be proved in Section 4.3.4.

In the end of this section, we shall check that all the remainders appearing in the normalization of the system are Lipschitzian w.r.t.  $(\eta, \psi)$  in proper functional spaces. This is essential to prove convergence of approximate solution and uniqueness of the solution (which will be studied in Section 4.5.3).

Inside this section, we always assume that  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ , which is the regularity required in the main theorem 4.1.2. One should pay attention to the fact that, thanks to the paradifferential calculus (reviewed in Appendix 4.C), all the estimates to be proved are *tame*, in the sense that they can be written as

$$Q \leq K \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right),$$

where  $Q$  stands for quantities to be studied, and the constant  $K = K \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right)$  depends only on  $\|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}}$  instead of  $\|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s}$ . In Section 4.5, this observation will lead to the *tame estimate* (4.5.51), implying the continuity of the solution to (4.1.18) in time and initial data via a nonlinear interpolation (Theorem 4.5.19).

To simplify the computations, instead of the change of variable  $r = \rho\zeta(\rho\theta, z), \omega = \theta, z = z$  defined in Section 4.2, where  $(x, z) = (r\omega, z)$  is the polar coordinate, we shall use the same change of variable as in [HK23],

$$(4.3.2) \quad r = \bar{\rho}\eta(\bar{\theta}, \bar{z}), \quad \omega = \bar{\theta}, \quad z = \bar{z}.$$

Recall that this change of variable is easy to calculate and behaves well near the interface  $r = \eta$ , while singularities exists at  $r = 0$ , which is out of concern in this section. By definition, the relation between  $(\bar{\rho}, \bar{\theta}, \bar{z})$  and the variable  $(\rho, \theta, z)$  studied in previous section is given by

$$(4.3.3) \quad (\bar{\rho}, \bar{\theta}, \bar{z}) = \bar{v}(\rho, \theta, z) := \left( \frac{\rho \zeta(\rho \theta, z)}{\eta(\theta, z)}, \theta, z \right).$$

We denote by  $\bar{\varphi}$  the potential  $\phi$  in coordinate  $(\bar{\rho}, \bar{\theta}, \bar{z})$  (equivalently,  $\bar{\varphi} \circ \bar{v} = \varphi$ ). It can be checked that  $\bar{\varphi}$  has the same regularity as  $\varphi$  when  $\bar{\rho}$  is close to 1. The proof of lemma below is left to the end of Appendix 4.B.4.

**Lemma 4.3.1.** *Let  $s > 3$  and  $\delta > 0$  be small enough. If  $\psi \in H^{s_0}(\mathbb{T} \times \mathbb{R})$  with  $\frac{3}{2} < s_0 \leq s$  and  $\eta \in H_R^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ , the potential  $\bar{\varphi}$  in new coordinate  $(\bar{\rho}, \bar{\theta}, \bar{z})$  satisfies*

$$(4.3.4) \quad \|\partial_{\bar{\rho}}^l \bar{\varphi}\|_{C^0([1-\delta, 1]; H^{s_0-l}(\mathbb{T} \times \mathbb{R}))} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \quad l = 0, 1, 2, 3.$$

From now on, we will no more use the coordinate introduced in Section 4.2 and, for simplicity of notation, we omit the bar over  $(\bar{\rho}, \bar{\theta}, \bar{z})$  and  $\bar{\varphi}$ . Moreover, all the functions to be studied should be regarded as functions defined on  $\mathbb{T} \times \mathbb{R}$  with parameter  $\rho \in [1-\delta, 1]$ . By definition (4.3.2), the potential  $\varphi$  satisfies

$$(4.3.5) \quad \begin{cases} L\varphi := \left( \alpha \partial_\rho^2 + \beta \cdot \nabla_{\theta, z} \partial_\rho + \gamma \partial_\rho + \frac{1}{\rho^2 \eta^2} \partial_\theta^2 + \partial_z^2 \right) \varphi = 0, & \forall 1-\delta \leq \rho \leq 1, \\ \varphi|_{\rho=1} = \psi. \end{cases}$$

where

$$(4.3.6) \quad \alpha = \frac{1}{\eta^2} \left( 1 + \left( \frac{\eta_\theta}{\eta} \right)^2 + \rho^2 \eta_z^2 \right),$$

$$(4.3.7) \quad \beta = \left( -\frac{2\eta_\theta}{\rho \eta^3}, -\frac{2\rho \eta_z}{\eta} \right),$$

$$(4.3.8) \quad \gamma = -\frac{1}{\rho \eta} \left( \frac{\eta_\theta}{\eta^2} \right)_\theta - \rho \eta \left( \frac{\eta_z}{\eta^2} \right)_z + \frac{1}{\rho \eta^2}.$$

Note that by construction (4.2.5) of  $\zeta$ , we have  $\partial_\rho \varphi|_{\rho=1} = \partial_{\bar{\rho}} \bar{\varphi}|_{\bar{\rho}=1}$ . Thus, the formula (4.3.1) for Dirichlet-to-Neumann operator as well as formulation (4.2.17), (4.2.18), (4.2.19) of  $B, V, N$  remain unchanged. Before entering the next part, we introduce the following estimates for  $\alpha, \beta$ , and  $\gamma$ , which will be frequently used in this section.

**Lemma 4.3.2.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}$  with  $s > 3$ . Then we have*

$$(4.3.9) \quad \|\alpha\|_{C_\rho^0 H_R^{s-\frac{1}{2}-}} + \|\beta\|_{C_\rho^0 H^{s-\frac{1}{2}-}} + \|\gamma\|_{C_\rho^0 H_{\rho^{-1}R^{-2}}^{s-\frac{1}{2}-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta\|_{H_R^{s+\frac{1}{2}-}}.$$

The proof of this Lemma is no more than an application of Proposition 4.C.22.

### 4.3.1 . Preliminaries in paradifferential calculus

During the whole chapter, the involved paradifferential operators have symbol homogeneous in Fourier variable  $\xi$  and depending smoothly in  $\eta$  and its derivatives (see Definition 4.3.3 below). In this case, the general theory of paradifferential calculus reviewed in Appendix 4.C can be refined with simpler formulas, which are collected in this section. Note that most of these technical results have been presented in Section 4.1 of [ABZ11], where one may find detailed proofs.

**Definition 4.3.3.** *Given  $m \in \mathbb{R}$ , we denote by  $\Sigma^m$  the collection of symbols on  $\mathbb{T} \times \mathbb{R}$  that take the form*

$$(4.3.10) \quad a(w, \xi) = a^{(m)}(w, \xi) + a^{(m-1)}(w, \xi),$$

where  $a^{(m)}$  and  $a^{(m-1)}$  takes the form

$$\begin{aligned} a^{(m)}(w, \xi) &= F(\eta, \nabla_{\theta, z} \eta; \xi), \\ a^{(m-1)}(w, \xi) &= \sum_{|\alpha| \leq 2} G_\alpha(\eta, \nabla_{\theta, z} \eta; \xi) \partial_{\theta, z}^\alpha \eta \end{aligned}$$

Here  $F, G_\alpha$  are smooth functions and homogeneous of degree  $m, m - 1$  in  $\xi$ , respectively.

Under the assumption that  $\eta \in H_R^{s+\frac{1}{2}}$  with  $s > 3$  (true in the rest of this chapter), we have  $a^{(m)} \in \Gamma_{3/2+}^m$  and  $a^{(m-1)} \in \Gamma_{1/2+}^{m-1}$  (see Definition 4.C.14). In the sequel, we shall regard  $a \in \Sigma^m$  as an element in  $\Gamma_{3/2+}^m + \Gamma_{1/2+}^{m-1}$ .

As a convention, we shall write all the equations in polar coordinate  $(y, z) = (\rho\theta, z)$  and denote the Fourier variable associated to  $w = (\theta, z) \in \mathbb{T} \times \mathbb{R}$  as  $\xi = (\xi_\theta, \xi_z) \in \mathbb{R}^2$  (see Section 4.C.1 for Fourier variables on torus). Furthermore, all the estimates in this section are uniform in time, the dependence on which will be omitted for simplicity.

For symbols  $a, b \in \Sigma^m$ , we write

$$T_a \approx T_b$$

if  $T_a - T_b$  is of order  $m - \frac{3}{2}$ , i.e. maps  $H^s$  to  $H^{s-m+\frac{3}{2}}$  for all  $s \in \mathbb{R}$ , with operator norm controlled by  $C \left( \|\eta\|_{H_R^{\frac{7}{2}+}} \right)$ . Then an application of Proposition 4.C.18 gives that

**Proposition 4.3.4.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$  with  $s > 3$ . For symbols  $a = a^{(m)} + a^{(m-1)} \in \Sigma^m$  and  $b = b^{(m')} + b^{(m'-1)} \in \Sigma^{m'}$  with  $m, m' \in \mathbb{R}$ , we have*

$$(4.3.11) \quad T_a T_b \approx T_{a\sharp b},$$

$$(4.3.12) \quad T_a^* \approx T_{a^*},$$

where

$$(4.3.13) \quad a\sharp b = a^{(m)} b^{(m')} + \left( \partial_\xi a^{(m)} \cdot D_w b^{(m')} + a^{(m)} b^{(m'-1)} + a^{(m-1)} b^{(m')} \right) \in \Sigma^{m+m'},$$

$$(4.3.14) \quad a^* = \overline{a^{(m)}} + \left( D_w \cdot \partial_\xi \overline{a^{(m)}} + \overline{a^{(m-1)}} \right) \in \Sigma^m.$$

Recall that we have regarded  $\Sigma^m$  as a subclass of  $\Gamma_{3/2+}^m + \Gamma_{1/2+}^{m-1}$ . Thus the parameter  $\rho$  in Proposition 4.C.18 should be taken as  $3/2+$  or  $1/2+$ . Moreover, all the involved operator norms can actually be controlled by a positive smooth increasing function of  $\|\eta\|_{H_R^{s+\frac{1}{2}-}}$ . For simplicity, we shall not precise these in the sequel.

As a consequence, we are able to calculate the symbol of commutator,

**Corollary 4.3.5.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}$  with  $s > 3$ . For symbols  $a = a^{(m)} + a^{(m-1)} \in \Sigma^m$  and  $b = b^{(m')} + b^{(m'-1)} \in \Sigma^{m'}$  with  $m, m' \in \mathbb{R}$ , their commutator  $[T_a, T_b]$  is of order at most  $m + m' - 1$ . More precisely,  $[T_a, T_b] \approx T_{-i\{a^{(m)}, b^{(m')}\}}$ , where  $\{\cdot, \cdot\}$  is Poisson bracket,*

$$\{a, b\} := \partial_\xi a^{(m)} \cdot \partial_w b^{(m')} - \partial_w a^{(m)} \cdot \partial_\xi b^{(m')} \in \Gamma_{1/2+}^{m+m'-1}.$$

**Definition 4.3.6.** *Let symbol  $a = a^{(m)} + a^{(m-1)} \in \Sigma^m$  with  $m \in \mathbb{R}$ . We say that  $a$  is elliptic, if there exists  $0 < c < C$  such that*

$$c|\xi|^m \leq \operatorname{Re} a^{(m)}(w, \xi) \leq C|\xi|^m, \quad \forall w \in \mathbb{T} \times \mathbb{R}, \xi \neq 0,$$

**Proposition 4.3.7.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}$  with  $s > 3$ . For any elliptic symbol  $a \in \Sigma^m$  with  $m \in \mathbb{R}$ , we can construct another elliptic symbol  $\tilde{a} \in \Sigma^{-m}$  such that*

$$(4.3.15) \quad T_a T_{\tilde{a}} \approx T_{\tilde{a}} T_a \approx \operatorname{id}.$$

*Proof.* It suffices to take  $\tilde{a}^{(-m)} a^{(m)} = 1$  and  $\tilde{a}^{(-m-1)}$  as the solution to

$$\partial_\xi a^{(m)} \cdot D_w \frac{1}{a^{(m)}} + a^{(m)} \tilde{a}^{(-m-1)} + a^{(m-1)} \tilde{a}^{(-m)} = 0.$$

The ellipticity of  $\tilde{a}$  is obvious from this definition. The desired estimate (4.3.15) can be checked directly from Proposition 4.3.4. More precisely, we have  $T_a T_{\tilde{a}} \sim T_{a\sharp\tilde{a}}$  where

$$\begin{aligned} a\sharp\tilde{a} &= a^{(m)} \tilde{a}^{(m)} + \left( \partial_\xi a^{(m)} \cdot D_w \tilde{a}^{(-m)} + a^{(m)} \tilde{a}^{(-m-1)} + a^{(m-1)} \tilde{a}^{(-m)} \right) \\ &= 1 + \left( \partial_\xi a^{(m)} \cdot D_w \frac{1}{a^{(m)}} + a^{(m)} \tilde{a}^{(-m-1)} + a^{(m-1)} \tilde{a}^{(-m)} \right) = 1. \end{aligned}$$

Therefore,  $T_a T_{\tilde{a}} \approx T_1$ , which equals identity up to a smoothing operator. The proof of  $T_{\tilde{a}} T_a \approx \operatorname{id}$  is similar. It suffices to observe that

$$\partial_\xi \tilde{a}^{(-m)} \cdot D_w a^{(m)} = \partial_\xi \frac{1}{a^{(m)}} \cdot D_w a^{(m)} = \frac{i \partial_\xi a^{(m)} \cdot \partial_w a^{(m)}}{(a^{(m)})^2} = \partial_\xi a^{(m)} \cdot D_w \frac{1}{a^{(m)}} = \partial_\xi a^{(m)} \cdot D_w \tilde{a}^{(-m)},$$

which implies  $\tilde{a}\sharp a = a\sharp\tilde{a} = 1$ . □

**Proposition 4.3.8.** *Let  $\eta \in H_R^{2+}$ . For any elliptic symbol  $a \in \Sigma^m$  with  $m \in \mathbb{R}$ , we have the following estimate,*

$$(4.3.16) \quad \|u\|_{H^r} \leq C \left( \|\eta\|_{H_R^{2+}} \right) (\|T_a u\|_{H^{r-m}} + \|u\|_{L^2}), \quad \forall r \in \mathbb{R},$$

where  $C > 0$  is a smooth increasing function and the  $L^2$ -norm of  $u$  can be replaced with any other Sobolev norms.

If  $\eta \in H_R^{s+\frac{1}{2}-}$  with  $s > 3$ , (4.3.16) is a consequence of Proposition 4.3.7. For the refined version  $\eta \in H_R^{2+}$ , we refer to Proposition 4.6 of [ABZ11].

**Proposition 4.3.9.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}$  with  $s > 3$ . For any  $a = a^{(m)} + a^{(m-1)} \in \Sigma^m$  with  $m \in \mathbb{R}$  and  $\text{Im } a^{(m)} = 0$ , we have the following equivalence,*

$$(4.3.17) \quad T_a \approx T_a^* \Leftrightarrow (T_a - T_a^*) \text{ is of order } (m-1-) \Leftrightarrow \text{Im } a^{(m-1)} = -\frac{1}{2} \partial_w \cdot \partial_\xi a^{(m)}.$$

*Proof.* The first assertion implying the second is easy. By Proposition 4.3.4, we have  $T_a^* \approx T_{a^*}$ , ensuring that  $T_a - T_a^*$  is of order  $(m-3/2-)$  and thus of order  $(m-1-)$ .

To show that the second assertion implies the third one, we observe that, since  $T_a^* \approx T_{a^*}$ ,  $T_{a-a^*}$  is also of order  $(m-1-)$ . Moreover,  $a, a^*$  belong to the class  $\Sigma^m$ , meaning that  $a - a^*$  is the sum of a symbol homogeneous in  $\xi$  of degree  $m$  and another one of degree  $m-1$ . Once the order of  $T_{a-a^*}$  is strictly smaller than  $m-1$ , both components vanish and consequently  $a = a^*$ , while, due to the expression (4.3.14) of  $a^*$ ,

$$a - a^* = a^{(m)} - \overline{a^{(m)}} - D_w \cdot \partial_\xi \overline{a^{(m)}} + a^{(m-1)} - \overline{a^{(m-1)}} = i \partial_w \cdot \partial_\xi a^{(m)} + 2i \text{Im } a^{(m-1)},$$

which gives the third assertion.

If the third assertion holds true, the calculation above ensures that  $a = a^*$  and the first assertion follows from  $T_a^* \approx T_{a^*} = T_a$ .  $\square$

### 4.3.2 . Paralinearization of normal derivative and good unknown

In this part, we shall separate Poisson's equation (4.2.22) in normal and tangential directions near boundary  $\rho = 1$  and deduce that  $\partial_\rho \varphi|_{\rho=1}$  equals some one order tangential operator acting on  $\psi$ , up to some remainders. To do so, we use the strategy in [AM09] (see also [ABZ11]) where the authors apply paradifferential calculus in tangential direction and reduce the problem to a paralinear parabolic equation which yields the regularity of remainders. The main result of this section is

**Proposition 4.3.10.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Then there exists  $\tau \in \Sigma^1$  such that*

$$(4.3.18) \quad \|\partial_\rho \varphi|_{\rho=1} - T_\tau U\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\psi\|_{H^{s_0}} + \|\eta\|_{H_R^{s+\frac{1}{2}}} \|\psi\|_{H^{s_0-}} \right),$$

where

$$(4.3.19) \quad U := \psi - T_B \eta \in H^{s_0}(\mathbb{T} \times \mathbb{R}).$$

Moreover,  $\tau = A|_{\rho=0}$ , where  $A(\rho) = A^{(1)}(\rho) + A^{(0)}(\rho)$  is defined by (4.3.33) and (4.3.35) below.

Note that, under the assumption of Proposition 4.3.18, due to Proposition 4.C.8 and Lemma 4.3.16, the new variable  $U$  satisfies

$$(4.3.20) \quad \|U\|_{H^{s_0}} \leq \|\psi\|_{H^{s_0}} + \|T_B\eta\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \quad \forall s > 3, \frac{3}{2} < s_0 \leq s.$$

As indicated in [AM09], the use of  $U$  instead of  $\psi$  is essential, which will be explained briefly below. Let us consider the parilinearized version of Laplacian operator  $L$  appearing in (4.3.5),

$$(4.3.21) \quad P := T_\alpha \partial_\rho^2 + T_{i\beta \cdot \xi + \gamma} \partial_\rho - T_{\rho^{-2} \eta^{-2} \xi_\theta^2 + \xi_z^2}.$$

One may expect that,  $P\varphi - L\varphi$  is much more regular than  $\psi$ , which is true if the coefficients  $\alpha, \beta, \gamma, \eta^{-2}$  are smooth. However, under the assumption of Proposition 4.3.10, these coefficients have only Sobolev regularity and then the paraproducts  $T_{\partial_\rho^2 \varphi} \alpha, T_{\nabla_{\theta, z} \partial_\rho \varphi} \cdot \beta$  appearing in the remainders lie in  $C_\rho^0 H^{s_0 - \frac{1}{2}}$ , which is not enough to conclude (4.3.18) (we need  $C_\rho^0 H^{s_0 - \frac{1}{2}+}$ ). This lack of regularity is actually a consequence of paracomposition, the general theory of which is sketched in Appendix 4.C.4. Let  $\chi$  represent the diffeomorphism from  $(\rho, \theta, z) \in [1 - \delta, 1] \times \mathbb{T} \times \mathbb{R}$  to  $(x, z) \in \Omega(t)$  and  $X^*$  be the paracomposition operator associated to  $\chi$ . A formal application of Proposition 4.C.24 gives that

$$0 = X^* \Delta_{x, z} \phi = T_a X^* \phi + R,$$

where the symbol  $a$  is the pull-back of symbol of Laplacian operator and remainder  $R$  has the desired regularity (see Lemma 4.3.11 below). Note that  $\Delta_{x, z}$  should be understood as the parilinearized Laplacian operator differing from  $\Delta_{x, z}$  only in low frequency. Therefore, this difference only generates smooth remainders and can be omitted. Formally, one may identify  $T_a$  as  $P$  defined by (4.3.21), since Laplacian  $\Delta_{x, z}$  is a differential operator. To sum up, we have  $PX^* \phi \in C_\rho^0 H^{s_0 - \frac{1}{2}+}$ . Meanwhile, by Proposition 4.C.23, up to remainders, the term  $X^* \phi$  equal

$$X^* \phi = \varphi - T_{\phi' \circ \chi} \chi = \varphi - T_{\eta^{-1} \partial_\rho \varphi} \rho \eta,$$

which suggests us to turn to *Alinhac's good unknown*

$$(4.3.22) \quad \Phi := \varphi - T_{\eta^{-1} \partial_\rho \varphi} \rho \eta,$$

whose trace at  $\rho = 1$  is equal to  $U$  defined by (4.3.19).

**Lemma 4.3.11.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Then*

$$(4.3.23) \quad P\Phi = r_1.$$

where  $\Phi$  is defined by (4.3.22) and  $r_1$  satisfies

$$(4.3.24) \quad \|r_1\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

During the proof of Lemma 4.3.11, we shall use the equivalence : for  $u, v$  defined on  $[1 - \delta] \times \mathbb{T} \times \mathbb{R}$ ,

$$(4.3.25) \quad u \sim v \Leftrightarrow \|u - v\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}},$$

with  $0 < \epsilon \ll 1$  to be determined later.

To begin with, we check that

**Lemma 4.3.12.** *Under the hypothesis of Lemma 4.3.11, we have,*

$$(4.3.26) \quad P\Phi \sim -T_{\partial_\rho^2 \varphi} \alpha - T_{\nabla_w \partial_\rho \varphi} \cdot \beta - T_{\partial_\rho \varphi} \gamma - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta,$$

in the sense of (4.3.25). Recall that  $w$  stands for the couple of variable  $(\theta, z)$ .

*Proof.* By definition (4.3.22) of the good unknown  $\Phi$ , we have

$$\begin{aligned} P\Phi &= P\varphi - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta \\ &= T_\alpha \partial_\rho^2 \varphi + T_{i\beta \cdot \xi + \gamma} \partial_\rho \varphi - T_{\rho^{-2} \eta^{-2} \xi_\theta^2 + \xi_z^2} \varphi - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta \\ &= T_\alpha \partial_\rho^2 \varphi + T_\beta \cdot \nabla_w \partial_\rho \varphi + T_\gamma \partial_\rho \varphi + T_{\rho^{-2} \eta^{-2}} \partial_\theta^2 \varphi + \partial_z^2 \varphi - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta + (-T_{\xi_z^2} \varphi - \partial_z^2 \varphi) \\ &= L\varphi - T_{\partial_\rho^2 \varphi} \alpha - T_{\nabla_w \partial_\rho \varphi} \cdot \beta - T_{\partial_\rho \varphi} \gamma - T_{\partial_\theta^2 \varphi} \rho^{-2} \eta^{-2} - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta \\ &\quad - R(\partial_\rho^2 \varphi, \alpha) - R(\nabla_w \partial_\rho \varphi, \beta) - R(\partial_\rho \varphi, \gamma) - R(\partial_\theta^2 \varphi, \rho^{-2} \eta^{-2}) + (-T_{\xi_z^2} \varphi - \partial_z^2 \varphi) \\ &\sim L\varphi - T_{\partial_\rho^2 \varphi} \alpha - T_{\nabla_w \partial_\rho \varphi} \cdot \beta - T_{\partial_\rho \varphi} \gamma - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta \\ &= -T_{\partial_\rho^2 \varphi} \alpha - T_{\nabla_w \partial_\rho \varphi} \cdot \beta - T_{\partial_\rho \varphi} \gamma - PT_{\eta^{-1} \partial_\rho \varphi} \rho \eta. \end{aligned}$$

Recall that  $L$  is the Laplacian in coordinate  $(\rho, \theta, z)$  defined in (4.3.5). To check this equivalence, we first observe that, thanks to Lemma 4.3.1,  $\nabla_{\rho, w}^2 \varphi \in C_\rho^0 H^{s_0 - 2 -}$  and  $\partial_\rho \varphi \in C_\rho^0 H^{s_0 - 1 -}$ . By Proposition 4.C.8, Remark 4.C.6, and estimate (4.3.9), we have the following estimates :

$$\begin{aligned} &\|T_{\partial_\theta^2 \varphi} \rho^{-2} \eta^{-2}\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} + \|R(\partial_\theta^2 \varphi, \rho^{-2} \eta^{-2})\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \\ &\leq \|T_{\partial_\theta^2 \varphi} (\eta^{-2} - R^{-2})\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} + \|R(\partial_\theta^2 \varphi, \eta^{-2})\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \\ &\lesssim \left( \|\eta^{-2}\|_{H_{R^{-2}}^{\max(\frac{5}{2}, s_0 - \frac{1}{2}) + \epsilon +}} + 1 \right) \|\partial_\theta^2 \varphi\|_{C_\rho^0 H^{s_0 - 2 -}} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}}; \\ &\|R(\partial_\rho^2 \varphi, \alpha) + R(\nabla_w \partial_\rho \varphi, \beta)\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \\ &\lesssim \left( \|\alpha\|_{C_\rho^0 H_{R^{-2}}^{\frac{5}{2} + \epsilon +}} + 1 \right) \|\partial_\rho^2 \varphi\|_{C_\rho^0 H^{s_0 - 2 -}} + \|\beta\|_{C_\rho^0 H^{\frac{5}{2} + \epsilon +}} \|\nabla_w \partial_\rho \varphi\|_{C_\rho^0 H^{s_0 - 2 -}} \\ &\leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}}; \\ &\|R(\partial_\rho \varphi, \gamma)\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \lesssim \left( \|\gamma\|_{C_\rho^0 H_{\rho^{-1} R^{-2}}^{\frac{3}{2} + \epsilon +}} + 1 \right) \|\partial_\rho \varphi\|_{C_\rho^0 H^{s_0 - 1 -}} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}}. \end{aligned}$$

As for the remaining term  $-T_{\xi_z^2}\varphi - \partial_z^2\varphi$ , we observe that, since the symbol of  $\partial_z^2$  equals  $-\xi_z^2$ , the difference  $-T_{\xi_z^2} - \partial_z^2$  is a smoothing operator. Namely, for all  $M \gg 1$ , we have

$$\| -T_{\xi_z^2}\varphi - \partial_z^2\varphi \|_{C^0H^{s_0-\frac{1}{2}+\epsilon}} \lesssim \|\varphi\|_{C_\rho^0H^{-M}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

□

Let us proceed with the proof of Lemma 4.3.11. We observe that  $r_1 = P\Phi$  is linear in  $\psi$ . Therefore, an interpolation argument allows us to reduce to the endpoints  $s_0 = \frac{3}{2} + \delta$  and  $s_0 = s$ . If  $s_0 = \frac{3}{2} + \delta$  with  $\delta > 0$  fixed and small enough, the regularity of  $\varphi$  reads  $\nabla_{\rho,w}^2\varphi \in C_\rho^0H^{-\frac{1}{2}+\delta}$  and  $\partial_\rho\varphi \in C_\rho^0H^{\frac{1}{2}+\delta}$  (see Lemma 4.3.1). As a result, by Proposition 4.C.8 and (4.3.9), we have

$$\begin{aligned} & \|T_{\partial_\rho^2}\alpha + T_{\nabla_w\partial_\rho\varphi} \cdot \beta + T_{\partial_\rho\varphi}\gamma\|_{C_\rho^0H^{1+\delta+\epsilon}} \\ & \lesssim \|\partial_\rho^2\varphi\|_{C_\rho^0H^{-\frac{1}{2}+\delta}} \|\alpha\|_{C_\rho^0H^{\frac{5}{2}+\epsilon}} + \|\nabla_w\partial_\rho\varphi\|_{C_\rho^0H^{-\frac{1}{2}+\delta}} \|\beta\|_{C_\rho^0H^{\frac{5}{2}+\epsilon}} + \|\partial_\rho\varphi\|_{C_\rho^0H^{\frac{1}{2}+\delta}} \|\gamma\|_{C_\rho^0H^{\frac{3}{2}+\epsilon}} \\ & \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{\frac{3}{2}+\delta}}. \end{aligned}$$

Meanwhile, Proposition 4.C.8 and Corollary 4.C.9 give

$$\|T_{\eta^{-1}\partial_\rho\varphi}\rho\eta\|_{C_\rho^0H^{3+\delta+\epsilon}} \lesssim \|\eta^{-1}\partial_\rho\varphi\|_{C_\rho^0H^{\frac{1}{2}+\delta}} \|\rho\eta\|_{C_\rho^0H^{\frac{7}{2}+\epsilon}} \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{\frac{3}{2}+\delta}}.$$

Since  $P$  is an operator of order 2, we can conclude

$$P\Phi \sim -T_{\partial_\rho^2}\alpha - T_{\nabla_w\partial_\rho\varphi} \cdot \beta - T_{\partial_\rho\varphi}\gamma - PT_{\eta^{-1}\partial_\rho\varphi}\rho\eta \sim 0.$$

Note that the extra term  $T_{\eta^{-1}\partial_\rho\varphi}\rho\eta$  from *Alinhac's good unknown* is useless in this case  $s_0 = 3/2 + \delta$ , while it will be crucial in the case  $s_0 = s$ . In fact, when  $s_0 = s$ , the estimates above are no more correct. For example, we can only prove  $T_{\partial_\rho^2}\alpha \in C_\rho^0H^{s-\frac{1}{2}} = C_\rho^0H^{s_0-\frac{1}{2}}$  instead of  $C_\rho^0H^{s_0-\frac{1}{2}+\epsilon}$ .

From now on, we take  $s_0 = s$  and check that the extra term  $PT_{\eta^{-1}\partial_\rho\varphi}\rho\eta$  is able to cancel the low regular components of  $T_{\partial_\rho^2}\alpha$ ,  $T_{\nabla_w\partial_\rho\varphi} \cdot \beta$ , and  $T_{\partial_\rho\varphi}\gamma$ . To begin with, by definition (4.3.21) of  $P$ ,

$$PT_{\eta^{-1}\partial_\rho\varphi}\rho\eta = T_\alpha\partial_\rho^2T_{\eta^{-1}\partial_\rho\varphi}\rho\eta + T_{i\beta \cdot \xi}\partial_\rho T_{\eta^{-1}\partial_\rho\varphi}\rho\eta + T_\gamma\partial_\rho T_{\eta^{-1}\partial_\rho\varphi}\rho\eta - T_{\rho^{-2}\eta^{-2}\xi_\theta^2 + \xi_z^2}T_{\eta^{-1}\partial_\rho\varphi}\rho\eta,$$

and we will separately study each terms on the right hand side. As a corollary of the following estimates, we are able to deduce that

(4.3.27)

$$\begin{aligned} P\Phi \sim & -T_{\partial_\rho^2}\alpha - T_{\nabla_w\partial_\rho\varphi} \cdot \beta - T_{\partial_\rho\varphi}\gamma - T_\beta \cdot T_{\eta^{-1}\partial_\rho^2\varphi}\rho\nabla_w\eta - T_\beta \cdot T_{\eta^{-1}\partial_\rho\varphi}\nabla_w\eta \\ & - 2\rho^{-1}T_{\eta^{-2}}T_{\partial_\theta(\eta^{-1}\partial_\rho\varphi)}\eta\theta - \rho^{-1}T_{\eta^{-2}}T_{\eta^{-1}\partial_\rho\varphi}\eta\theta\theta - 2\rho T_{\partial_z(\eta^{-1}\partial_\rho\varphi)}\eta z - \rho T_{\eta^{-1}\partial_\rho\varphi}\eta z z, \end{aligned}$$

when  $s_0 = s$ .

**Lemma 4.3.13.** *Under the hypotheses of Lemma 4.3.11, we have the following equivalences :*

$$(4.3.28) \quad T_\alpha \partial_\rho^2 T_{\eta^{-1} \partial_\rho \varphi} \rho \eta \sim 0,$$

$$(4.3.29) \quad T_{i\beta \cdot \xi} \partial_\rho T_{\eta^{-1} \partial_\rho \varphi} \rho \eta \sim T_\beta \cdot T_{\eta^{-1} \partial_\rho^2 \varphi} \rho \nabla_w \eta + T_\beta \cdot T_{\eta^{-1} \partial_\rho \varphi} \nabla_w \eta,$$

$$(4.3.30) \quad T_\gamma \partial_\rho T_{\eta^{-1} \partial_\rho \varphi} \rho \eta \sim 0,$$

$$(4.3.31) \quad -T_{\rho^{-2} \eta^{-2} \xi_\theta^2 + \xi_z^2} T_{\eta^{-1} \partial_\rho \varphi} \rho \eta \sim 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1} \partial_\rho \varphi)} \eta \theta + \rho^{-1} T_{\eta^{-2}} T_{\eta^{-1} \partial_\rho \varphi} \eta \theta \theta \\ + 2\rho T_{\partial_z(\eta^{-1} \partial_\rho \varphi)} \eta z + \rho T_{\eta^{-1} \partial_\rho \varphi} \eta z z,$$

in the sense of (4.3.25) with  $s_0 = s$ .

*Proof.* As before, we observe that  $\eta^{-1} \partial_\rho \varphi \in C_\rho^0 H^{s-1}$  and thus  $T_{\eta^{-1} \partial_\rho \varphi} \rho \eta \in C_\rho^0 H^{s+\frac{1}{2}}$ . To prove the first equivalence (4.3.28), we apply Proposition 4.C.8, Corollary 4.C.9, together with (4.3.9) and obtain

$$\begin{aligned} & \|T_\alpha \partial_\rho^2 T_{\eta^{-1} \partial_\rho \varphi} \rho \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} = \|T_\alpha T_{\eta^{-1} \partial_\rho^3 \varphi} \rho \eta + 2T_\alpha T_{\eta^{-1} \partial_\rho^2 \varphi} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \\ & \lesssim \left( \|\alpha\|_{C_\rho^0 H^{1+}} + 1 \right) \left( \|T_{\eta^{-1} \partial_\rho^3 \varphi} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} + \|T_{\eta^{-1} \partial_\rho^2 \varphi} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \right) \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\eta^{-1} \partial_\rho^3 \varphi\|_{C_\rho^0 H^{s-3-}} \|\eta\|_{H_R^{\max(\frac{7}{2}, s-\frac{1}{2})+\epsilon+}} + \|\eta^{-1} \partial_\rho^2 \varphi\|_{C_\rho^0 H^{s-2-}} \|\eta\|_{H_R^{\max(\frac{5}{2}, s-\frac{1}{2})+\epsilon+}} \right) \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\eta^{-1} \partial_\rho^3 \varphi\|_{C_\rho^0 H^{s-3-}} + \|\eta^{-1} \partial_\rho^2 \varphi\|_{C_\rho^0 H^{s-2-}} \right) \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\eta^{-1}\|_{H_{R-1}^{\max(1+, s-2)}} + 1 \right) \left( \|\partial_\rho^3 \varphi\|_{C_\rho^0 H^{s-3-}} + \|\partial_\rho^2 \varphi\|_{C_\rho^0 H^{s-2-}} \right) \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}. \end{aligned}$$

Recall that  $\partial_\rho^3 \varphi \in C_\rho^0 H^{s-3}$  thanks to (4.3.4). The term  $T_{i\beta \cdot \xi} \partial_\rho T_{\eta^{-1} \partial_\rho \varphi} \rho \eta$  in the second equivalence (4.3.29) can be written as

$$T_\beta \cdot T_{\nabla_w(\eta^{-1} \partial_\rho^2 \varphi)} \rho \eta + T_\beta \cdot T_{\eta^{-1} \partial_\rho^2 \varphi} \rho \nabla_w \eta + T_\beta \cdot T_{\nabla_w(\eta^{-1} \partial_\rho \varphi)} \eta + T_\beta \cdot T_{\eta^{-1} \partial_\rho \varphi} \nabla_w \eta,$$

with

$$\begin{aligned} & \|T_\beta \cdot T_{\nabla_w(\eta^{-1} \partial_\rho^2 \varphi)} \rho \eta + T_\beta \cdot T_{\nabla_w(\eta^{-1} \partial_\rho \varphi)} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \\ & \lesssim \|\beta\|_{C_\rho^0 H^{1+}} \|T_{\nabla_w(\eta^{-1} \partial_\rho^2 \varphi)} \rho \eta + T_{\nabla_w(\eta^{-1} \partial_\rho \varphi)} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\nabla_w(\eta^{-1} \partial_\rho^2 \varphi)\|_{C_\rho^0 H^{s-3-}} + \|\nabla_w(\eta^{-1} \partial_\rho \varphi)\|_{C_\rho^0 H^{s-2-}} \right) \|\eta\|_{C_\rho^0 H^{\max(\frac{7}{2}, s-\frac{1}{2})+\epsilon+}} \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}, \end{aligned}$$

which proves (4.3.29). The third equivalence (4.3.30) is due to the estimate

$$\|T_\gamma \partial_\rho T_{\eta^{-1} \partial_\rho \varphi} \rho \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} = \|T_\gamma T_{\eta^{-1} \partial_\rho^2 \varphi} \rho \eta + T_\gamma T_{\eta^{-1} \partial_\rho \varphi} \eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}}$$

$$\lesssim \left( \|\gamma\|_{C_\rho^0 H_{\rho^{-1}R^{-2}}^{1+}} + 1 \right) \|T_{\eta^{-1}\partial_\rho^2\varphi}\rho\eta + T_{\eta^{-1}\partial_\rho\varphi}\eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s},$$

where one may apply Proposition 4.C.22 to deal with  $\eta^{-1}$ . The last equivalence can be treated as follow,

$$\begin{aligned} -T_{\rho^{-2}\eta^{-2}\xi_z^2}T_{\eta^{-1}\partial_\rho\varphi}\rho\eta &= \rho^{-1}T_{\eta^{-2}}T_{\partial_\theta^2(\eta^{-1}\partial_\rho\varphi)}\eta + 2\rho^{-1}T_{\eta^{-2}}T_{\partial_\theta(\eta^{-1}\partial_\rho\varphi)}\eta\theta + \rho^{-1}T_{\eta^{-2}}T_{\eta^{-1}\partial_\rho\varphi}\eta\theta\theta, \\ -T_{\xi_z^2}T_{\eta^{-1}\partial_\rho\varphi}\rho\eta &= \rho T_{\partial_z^2(\eta^{-1}\partial_\rho\varphi)}\eta + 2\rho T_{\partial_z(\eta^{-1}\partial_\rho\varphi)}\eta z + \rho T_{\eta^{-1}\partial_\rho\varphi}\eta z z \\ &\quad + (-T_{\xi_z^2} - \partial_z^2) T_{\eta^{-1}\partial_\rho\varphi}\rho\eta, \end{aligned}$$

where similar estimates as above can be derived :

$$\begin{aligned} \|\rho^{-1}T_{\eta^{-2}}T_{\partial_\theta^2(\eta^{-1}\partial_\rho\varphi)}\eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} &\lesssim \left( \|\eta^{-2}\|_{H_R^{1+}} + 1 \right) \|\partial_\theta^2(\eta^{-1}\partial_\rho\varphi)\|_{C_\rho^0 H^{s-3-}} \|\eta\|_{H_R^{\max(\frac{7}{2}, s-\frac{1}{2})+\epsilon+}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta^{-1}\partial_\rho\varphi\|_{C_\rho^0 H^{s-1-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}, \end{aligned}$$

and

$$\begin{aligned} \|\rho T_{\partial_z^2(\eta^{-1}\partial_\rho\varphi)}\eta\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} &\lesssim \|\partial_z^2(\eta^{-1}\partial_\rho\varphi)\|_{C_\rho^0 H^{s-3-}} \|\eta\|_{H_R^{\max(\frac{7}{2}, s-\frac{1}{2})+\epsilon+}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta^{-1}\partial_\rho\varphi\|_{C_\rho^0 H^{s-1-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}. \end{aligned}$$

The desired result (4.3.31) follows from  $(-T_{\xi_z^2} - \partial_z^2) T_{\eta^{-1}\partial_\rho\varphi}\rho\eta \sim 0$  since  $(-T_{\xi_z^2} - \partial_z^2)$  is a smoothing operator.  $\square$

*Proof of Lemma 4.3.11.* Till now, we have managed to prove that, in the sense of (4.3.25),

$$P\Phi \sim 0$$

when  $s_0 = \frac{3}{2} + \delta$  with  $0 < \delta \ll 1$ , and moreover, when  $s_0 = s$ , (4.3.27) holds. It remains to check that the right hand side of (4.3.27) is equivalent to zero, namely

$$\begin{aligned} &-T_{\partial_\rho^2\varphi}\alpha - T_{\nabla_w\partial_\rho\varphi} \cdot \beta - T_{\partial_\rho\varphi}\gamma - T_\beta \cdot T_{\eta^{-1}\partial_\rho^2\varphi}\rho\nabla_w\eta - T_\beta \cdot T_{\eta^{-1}\partial_\rho\varphi}\nabla_w\eta \\ &- 2\rho^{-1}T_{\eta^{-2}}T_{\partial_\theta(\eta^{-1}\partial_\rho\varphi)}\eta\theta - \rho^{-1}T_{\eta^{-2}}T_{\eta^{-1}\partial_\rho\varphi}\eta\theta\theta - 2\rho T_{\partial_z(\eta^{-1}\partial_\rho\varphi)}\eta z - \rho T_{\eta^{-1}\partial_\rho\varphi}\eta z z \sim 0. \end{aligned}$$

Firstly, we claim that

$$-T_{\partial_\rho^2\varphi}\alpha - T_\beta \cdot T_{\eta^{-1}\partial_\rho^2\varphi}\rho\nabla_w\eta \sim 0.$$

In fact, by symbolic calculus (Proposition 4.3.4) and commutator estimate (Corollary 4.C.19), we have

$$\begin{aligned} T_\beta \cdot T_{\eta^{-1}\partial_\rho^2\varphi}\rho\nabla_w\eta &\sim T_{\eta^{-1}\partial_\rho^2\varphi}T_\beta \cdot \rho\nabla_w\eta \sim T_{\partial_\rho^2\varphi}T_{\eta^{-1}}T_\beta \cdot \rho\nabla_w\eta \\ &\sim T_{\partial_\rho^2\varphi}T_{\eta^{-1}\beta} \cdot \rho\nabla_w\eta, \end{aligned}$$

where we use the fact that  $\beta, \eta^{-1} \in C_\rho^0 H^{s-\frac{1}{2}} \subset \Gamma_{0+}^0$  and  $\partial_\rho^2 \varphi \in C_\rho^0 H^{s-2} \subset \Gamma_{0+}^0$ . As a consequence, in each step above, the error can be written as a  $(0-)$ -order operator acting on  $\nabla_w \eta$ , which lies in  $C_\rho^0 H^{s-\frac{1}{2}+}$ . Meanwhile, by definition (4.3.6) of  $\alpha$ ,

$$T_{\partial_\rho^2 \varphi} \alpha \sim T_{\partial_\rho^2 \varphi} (\eta^{-4} \eta_\theta^2 + \rho^2 \eta^{-2} \eta_z^2) \sim T_{\partial_\rho^2 \varphi} (T_{2\eta^{-4} \eta_\theta} \eta_\theta + T_{2\rho^2 \eta^{-2} \eta_z} \eta_z) = T_{\partial_\rho^2 \varphi} T_{-\rho \eta^{-1} \beta} \cdot \nabla_w \eta,$$

where the first equivalence is due to

$$\|T_{\partial_\rho^2 \varphi} \eta^{-2}\|_{C_\rho^0 H^{s-\frac{1}{2}+\epsilon}} \lesssim \|\partial_\rho^2 \varphi\|_{C_\rho^0 H^{s-2-}} \|\eta^{-2}\|_{H_{R^{-2}}^{\max(\frac{5}{2}, s-\frac{1}{2})+\epsilon+}} \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s},$$

and the second equivalence is a consequence of Corollary 4.C.21.

Now our problem is reduced to

$$\begin{aligned} P\Phi &\sim -T_{\nabla_w \partial_\rho \varphi} \cdot \beta - T_{\partial_\rho \varphi} \gamma - T_\beta \cdot T_{\eta^{-1} \partial_\rho \varphi} \nabla_w \eta \\ &\quad - 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1} \partial_\rho \varphi)} \eta_\theta - \rho^{-1} T_{\eta^{-2}} T_{\eta^{-1} \partial_\rho \varphi} \eta_{\theta\theta} - 2\rho T_{\partial_z(\eta^{-1} \partial_\rho \varphi)} \eta_z - \rho T_{\eta^{-1} \partial_\rho \varphi} \eta_{zz}. \end{aligned}$$

We observe that

$$\begin{aligned} 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1} \partial_\rho \varphi)} \eta_\theta &= 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1})} \partial_\rho \varphi \eta_\theta + 2\rho^{-1} T_{\eta^{-2}} T_{\eta^{-1} \partial_\theta \partial_\rho \varphi} \eta_\theta \\ &\sim 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\rho \varphi} T_{\partial_\theta(\eta^{-1})} \eta_\theta + 2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta \partial_\rho \varphi} T_{\eta^{-1}} \eta_\theta \\ &\sim 2\rho^{-1} T_{\partial_\rho \varphi} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1})} \eta_\theta + 2\rho^{-1} T_{\partial_\theta \partial_\rho \varphi} T_{\eta^{-2}} T_{\eta^{-1}} \eta_\theta \\ &\sim 2\rho^{-1} T_{\partial_\rho \varphi} T_{\eta^{-2} \partial_\theta(\eta^{-1})} \eta_\theta + 2\rho^{-1} T_{\partial_\theta \partial_\rho \varphi} T_{\eta^{-3}} \eta_\theta \end{aligned}$$

These equivalences are consequences of symbolic calculus (Proposition 4.C.18 and Corollary 4.C.19) and commutator estimate (Corollary 4.C.19), as well as the fact that all the involved symbols belong to  $C_\rho^0 H^{s-2-} \subset \Gamma_{1/2+}^0$ , which guarantees that the error in each step takes the form of  $T\eta_\theta$  with  $T$  linear and of order  $-\frac{1}{2}-$ . Moreover, an application of Proposition 4.C.8 gives

$$\begin{aligned} \|T_{\eta^{-3}} \eta_\theta - \eta^{-3} \eta_\theta\|_{H^{s-\frac{1}{2}+}} &\leq \|T_{\eta_\theta} \eta^{-3} + R(\eta^{-3}, \eta_\theta)\|_{H^{s-\frac{1}{2}+}} \\ &\lesssim \|\eta_\theta\|_{H^{1+}} \|\eta^{-3}\|_{H_{R^{-3}}^{s-\frac{1}{2}+\epsilon}} \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right), \end{aligned}$$

and thus

$$2\rho^{-1} T_{\eta^{-2}} T_{\partial_\theta(\eta^{-1} \partial_\rho \varphi)} \eta_\theta \sim 2\rho^{-1} T_{\partial_\rho \varphi} T_{\eta^{-2} \partial_\theta(\eta^{-1})} \eta_\theta + 2\rho^{-1} T_{\partial_\theta \partial_\rho \varphi} (\eta^{-3} \eta_\theta).$$

Similarly, we have

$$2\rho T_{\partial_z(\eta^{-1} \partial_\rho \varphi)} \eta_z \sim 2\rho T_{\partial_\rho \varphi} T_{\partial_z(\eta^{-1})} \eta_z + 2\rho T_{\partial_z \partial_\rho \varphi} (\eta^{-1} \eta_z).$$

By observing that

$$T_{\nabla_w \partial_\rho \varphi} \cdot \beta = -2\rho^{-1} T_{\partial_\theta \partial_\rho \varphi} \eta^{-3} \eta_\theta - 2\rho T_{\partial_z \partial_\rho \varphi} \eta^{-1} \eta_z,$$

we can further reduce our problem to

$$\begin{aligned}
P\Phi &\sim -T_{\partial_\rho\varphi}\gamma - T_\beta \cdot T_{\eta^{-1}\partial_\rho\varphi}\nabla_w\eta \\
&\quad - 2\rho^{-1}T_{\partial_\rho\varphi}T_{\eta^{-2}\partial_\theta(\eta^{-1})}\eta_\theta - \rho^{-1}T_{\eta^{-2}}T_{\eta^{-1}\partial_\rho\varphi}\eta_{\theta\theta} - 2\rho T_{\partial_\rho\varphi}T_{\partial_z(\eta^{-1})}\eta_z - \rho T_{\eta^{-1}\partial_\rho\varphi}\eta_{zz} \\
&\sim -T_{\partial_\rho\varphi}\gamma - T_{\partial_\rho\varphi}T_{\eta^{-1}\beta} \cdot \nabla_w\eta \\
&\quad - 2\rho^{-1}T_{\partial_\rho\varphi}T_{\eta^{-2}\partial_\theta(\eta^{-1})}\eta_\theta - \rho^{-1}T_{\partial_\rho\varphi}T_{\eta^{-3}}\eta_{\theta\theta} - 2\rho T_{\partial_\rho\varphi}T_{\partial_z(\eta^{-1})}\eta_z - \rho T_{\partial_\rho\varphi}T_{\eta^{-1}}\eta_{zz},
\end{aligned}$$

due to symbolic calculus (Proposition 4.C.18 and Corollary 4.C.19). From definition (4.3.7) and (4.3.8) of  $\beta$  and  $\gamma$ , respectively, it is easy to calculate that

$$\begin{aligned}
-\gamma &= -\eta^{-1}\beta \cdot \nabla_w\eta + \rho^{-1}\eta^{-3}\eta_{\theta\theta} + \rho\eta^{-1}\eta_{zz} + \rho^{-1}\eta^{-2}, \\
\eta^{-1}\beta &= (2\rho^{-1}\eta^{-2}\partial_\theta(\eta^{-1}), 2\rho\partial_z(\eta^{-1})).
\end{aligned}$$

As a result,

$$\begin{aligned}
P\Phi &\sim -\rho^{-1}T_{\partial_\rho\varphi}\eta^{-2} + T_{\partial_\rho\varphi}(\eta^{-1}\beta \cdot \nabla_w\eta - 2T_{\eta^{-1}\beta} \cdot \nabla_w\eta) \\
&\quad + \rho^{-1}T_{\partial_\rho\varphi}(\eta^{-3}\eta_{\theta\theta} - T_{\eta^{-3}}\eta_{\theta\theta}) + \rho T_{\partial_\rho\varphi}(\eta^{-1}\eta_{zz} - T_{\eta^{-1}}\eta_{zz}),
\end{aligned}$$

the right hand side of which is equivalent to zero in the sense of (4.3.25), due to the estimates

$$\begin{aligned}
\|T_{\partial_\rho\varphi}\eta^{-2}\|_{H^{s-\frac{1}{2}+\epsilon}} &\lesssim \|\partial_\rho\varphi\|_{H^1} + \|\eta^{-2}\|_{H^{s-\frac{1}{2}+\epsilon}} \leq C \left( \|\eta\|_{H^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}, \\
\|T_{\partial_\rho\varphi}(\eta^{-3}\eta_{\theta\theta} - T_{\eta^{-3}}\eta_{\theta\theta})\|_{H^{s-\frac{1}{2}+\epsilon}} &\leq \|T_{\partial_\rho\varphi}T_{\eta_{\theta\theta}}\eta^{-3}\|_{H^{s-\frac{1}{2}+\epsilon}} + \|T_{\partial_\rho\varphi}R(\eta^{-3}, \eta_{\theta\theta})\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\lesssim \|\partial_\rho\varphi\|_{H^1} + \|\eta_{\theta\theta}\|_{H^1} + \|\eta^{-3}\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\leq C \left( \|\eta\|_{H^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s}, \\
\|T_{\partial_\rho\varphi}(\eta^{-1}\eta_{zz} - T_{\eta^{-1}}\eta_{zz})\|_{H^{s-\frac{1}{2}+\epsilon}} &\leq \|T_{\partial_\rho\varphi}T_{\eta_{zz}}\eta^{-1}\|_{H^{s-\frac{1}{2}+\epsilon}} + \|T_{\partial_\rho\varphi}R(\eta^{-1}, \eta_{zz})\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\lesssim \|\partial_\rho\varphi\|_{H^1} + \|\eta_{zz}\|_{H^1} + \|\eta^{-1}\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\leq C \left( \|\eta\|_{H^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s},
\end{aligned}$$

as well as

$$\begin{aligned}
&\|T_{\partial_\rho\varphi}(\eta^{-1}\beta \cdot \nabla_w\eta - 2T_{\eta^{-1}\beta} \cdot \nabla_w\eta)\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\lesssim \|\partial_\rho\varphi\|_{H^1} + \|\eta^{-1}\beta \cdot \nabla_w\eta - 2T_{\eta^{-1}\beta} \cdot \nabla_w\eta\|_{H^{s-\frac{1}{2}+\epsilon}} \\
&\leq C \left( \|\eta\|_{H^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s} \left( \|\eta^{-4}\eta_\theta^2 - 2T_{\eta^{-4}\eta_\theta}\eta_\theta\|_{H^{s-\frac{1}{2}+\epsilon}} + \|\eta^{-2}\eta_z^2 - 2T_{\eta^{-2}\eta_z}\eta_z\|_{H^{s-\frac{1}{2}+\epsilon}} \right) \\
&\leq C \left( \|\eta\|_{H^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^s},
\end{aligned}$$

which is a consequence of Corollary 4.C.21. □

Now we are in a position to separate the normal and tangential derivatives.

**Lemma 4.3.14.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$  with  $s > 3$ . There exists elliptic symbols  $a, A \in \Sigma^1$ , such that*

$$(4.3.32) \quad P = T_\alpha(\partial_\rho + T_a)(\partial_\rho - T_A) + R_0 + R_1\partial_\rho,$$

where  $R_0$  is of order  $\frac{1}{2}-$ , and  $R_1$  is of order  $-\frac{1}{2}-$ , satisfying

$$\|R_0\|_{\mathcal{L}(H^t; H^{t-\frac{1}{2}+})} + \|R_1\|_{\mathcal{L}(H^t; H^{t+\frac{1}{2}+})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right), \quad \forall t \in \mathbb{R}.$$

Moreover,  $a = a^{(1)} + a^{(0)}$  and  $A = A^{(1)} + A^{(0)}$  can be computed explicitly,

$$(4.3.33) \quad A^{(1)} = \frac{1}{2\alpha} \sqrt{4\alpha \left( \frac{\xi_\theta^2}{\rho^2 \eta^2} + \xi_z^2 \right) - (\beta \cdot \xi)^2 - \frac{i\beta \cdot \xi}{2\alpha}},$$

$$(4.3.34) \quad a^{(1)} = \frac{1}{2\alpha} \sqrt{4\alpha \left( \frac{\xi_\theta^2}{\rho^2 \eta^2} + \xi_z^2 \right) - (\beta \cdot \xi)^2 + \frac{i\beta \cdot \xi}{2\alpha}},$$

and

$$(4.3.35) \quad A^{(0)} = -\frac{1}{A^{(1)} + a^{(1)}} \left( \frac{A^{(1)}\gamma}{\alpha} + \partial_\rho A^{(1)} + \partial_\xi a^{(1)} \cdot D_w A^{(1)} \right),$$

$$(4.3.36) \quad a^{(0)} = -\frac{1}{A^{(1)} + a^{(1)}} \left( -\frac{a^{(1)}\gamma}{\alpha} + \partial_\rho A^{(1)} + \partial_\xi a^{(1)} \cdot D_w A^{(1)} \right).$$

Note that all the operators are defined on  $\mathbb{T} \times \mathbb{R}$  with  $\rho \in [1 - \delta, 1]$  regarded as a parameter.

*Proof.* From (4.3.32), we shall see what is the conditions required for  $a$  and  $A$ . In fact, the right hand side of (4.3.32) equals

$$\begin{aligned} & T_\alpha(\partial_\rho + T_a)(\partial_\rho - T_A) + R_0 + R_1\partial_\rho \\ &= T_\alpha \left( \partial_\rho^2 + T_{a-A}\partial_\rho - T_a T_A - T_{\partial_\rho A} \right) + R_0 + R_1\partial_\rho \\ &= T_\alpha \partial_\rho^2 + (T_\alpha T_{a-A} + R_1) \partial_\rho - T_\alpha (T_a T_A + T_{\partial_\rho A}) + R_0. \end{aligned}$$

By comparing it with definition (4.3.21) of  $P$ , one could see that (4.3.32) holds if the construction of  $a, A$  satisfies

$$\begin{cases} T_\alpha T_{a-A} + R_1 = T_{i\beta \cdot \xi + \gamma}, \\ T_\alpha (T_a T_A + T_{\partial_\rho A}) - R_0 = T_{\rho^{-2} \eta^{-2} \xi_\theta^2 + \xi_z^2}. \end{cases}$$

Recall that by (4.3.6) and definition 4.3.3,  $\alpha \in \Sigma^0$ . Since  $\eta \in H_R^{s+\frac{1}{2}}$ , the end of definition 4.3.3 shows in particular that  $\alpha \in \Gamma_{3/2+}^0 + \Gamma_{1/2+}^{-1}$ . Moreover,  $\alpha$  is elliptic and independent of  $\xi$ . For all  $a, A \in \Sigma^1$ , symbolic calculus (Proposition 4.3.4) gives that

$$T_\alpha T_{a-A} \approx T_{\alpha \sharp (a-A)} = T_{\alpha(a-A)},$$

$$T_\alpha (T_a T_A + T_{\partial_\rho A}) \approx T_\alpha T_{a\sharp A + \partial_\rho A} \approx T_{\alpha\sharp(a\sharp A + \partial_\rho A)} = T_{\alpha(a\sharp A + \partial_\rho A)} \approx T_{\alpha(a\sharp A + \partial_\rho A^{(1)})},$$

where the difference in the first line is of order  $0 + 1 - \frac{3}{2} - = -\frac{1}{2} -$  and those in the second line is of order  $0 + 1 + 1 - \frac{3}{2} - = \frac{1}{2} -$ , allowing us to define

$$\begin{aligned} R_1 &:= T_{\alpha(a-A)} - T_\alpha T_{a-A}, \\ R_0 &:= T_{\alpha(a\sharp A + \partial_\rho A)} - T_\alpha (T_a T_A + T_{\partial_\rho A^{(1)}}). \end{aligned}$$

Therefore, it suffices to construct  $A = A^{(1)} + A^{(0)}$  and  $a = a^{(1)} + a^{(0)}$  verifying

$$\begin{cases} \alpha(a - A) = i\beta \cdot \xi + \gamma, \\ \alpha(a\sharp A + \partial_\rho A^{(1)}) = \rho^{-2}\eta^{-2}\xi_\theta^2 + \xi_z^2, \end{cases}$$

From (4.3.13) and correspondence of order (in  $\xi$ ), we only need to solve

$$\begin{cases} \alpha(a^{(1)} - A^{(1)}) = i\beta \cdot \xi, \\ \alpha(a^{(0)} - A^{(0)}) = \gamma, \\ \alpha a^{(1)} A^{(1)} = \rho^{-2}\eta^{-2}\xi_\theta^2 + \xi_z^2, \\ \partial_\xi a^{(1)} \cdot D_w A^{(1)} + a^{(0)} A^{(1)} + a^{(1)} A^{(0)} + \partial_\rho A^{(1)} = 0, \end{cases}$$

which allows us to obtain (4.3.33)-(4.3.36).  $\square$

By combining Lemma 4.3.11 and 4.3.14, we have

$$(4.3.37) \quad T_\alpha(\partial_\rho + T_a)(\partial_\rho - T_A)\Phi = -R_0\Phi - R_1\partial_\rho\Phi + r_1.$$

Under the hypothesis of Proposition 4.3.10, we have

$$(4.3.38) \quad \|-R_0\Phi - R_1\partial_\rho\Phi + r_1\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}}.$$

In fact, the estimate for  $r_1$  is a consequence of Lemma 4.3.11, and the definition (4.3.22) of  $\Phi$ , together with Lemma 4.3.1 and Proposition 4.C.8, implies that

$$(4.3.39) \quad \|\Phi\|_{C_\rho^0 H^{s_0}} + \|\partial_\rho\Phi\|_{C_\rho^0 H^{s_0 - 1}} + \|\partial_\rho^2\Phi\|_{C_\rho^0 H^{s_0 - 2}} \leq C \left( \|\eta\|_{H_R^{s + \frac{1}{2} -}} \right) \|\psi\|_{H^{s_0}},$$

which implies (4.3.38).

By definition (4.3.6) of  $\alpha$ , under the hypothesis of Proposition 4.3.10,  $\alpha \in H_{R^{-2}}^{s - \frac{1}{2} -}$ , which is a symbol in  $\Gamma_{s-3/2-}^0 \subset \Gamma_{3/2+}^0$ . Since  $\alpha$  is strictly positive due to hypothesis (H0),  $1/\alpha$  belongs to the same class of symbol. Consequently, Proposition 4.C.8 guarantees that

$$T_{\frac{1}{\alpha}} T_\alpha = id + R_2,$$

where  $R_2$  is of order  $\frac{3}{2} +$ . Now we may apply  $T_{1/\alpha}$  on both hands of (4.3.37) and obtain that

$$(\partial_\rho + T_a)(\partial_\rho - T_A)\Phi = T_{\frac{1}{\alpha}} (-R_0\Phi - R_1\partial_\rho\Phi + r_1) - R_2(\partial_\rho + T_a)(\partial_\rho - T_A)\Phi.$$

We claim that the  $C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}$  norm of the right hand side is bounded by  $C(\|\eta\|_{H_R^{s+\frac{1}{2}-}})\|\psi\|_{H^{s_0}}$ .

In fact, the estimate of the first term follows from (4.3.38) and the fact that  $T_{1/\alpha}$  is of order zero. As for the second term, since  $T_a$  and  $T_A$  are of order 1, we have

$$\|R_2(\partial_\rho + T_a)(\partial_\rho - T_A)\Phi\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \lesssim \|(\partial_\rho + T_a)(\partial_\rho - T_A)\Phi\|_{C_\rho^0 H^{s_0 - 2}} \lesssim C\left(\|\eta\|_{H_R^{s+\frac{1}{2}-}}\right)\|\psi\|_{H^{s_0}},$$

where the last inequality is a consequence of (4.3.39).

To sum up, under the hypothesis of Proposition 4.3.10, we have seen that  $\Phi$  defined by (4.3.22) satisfies

$$(\partial_\rho + T_{a(1)})(\partial_\rho - T_A)\Phi = T_{-a(0)}(\partial_\rho - T_A)\Phi + r_2,$$

with

$$\|r_2\|_{C_\rho^0 H^{s_0 - \frac{1}{2} + \epsilon}} \leq C\left(\|\eta\|_{H_R^{s+\frac{1}{2}-}}\right)\|\psi\|_{H^{s_0}}.$$

**Lemma 4.3.15** (Proposition 3.19 of [ABZ11]). *Let  $b_1 \in \Gamma_1^1$  and  $b_0 \in \Gamma_0^0$  with*

$$\operatorname{Re} b_1(w, \xi) \geq c|\xi|$$

*for some  $c > 0$ . If  $v \in C_\rho^1 H^M$  for some  $M \in \mathbb{R}$  that solves the equation*

$$\partial_\rho v + T_{b_1} v = T_{b_0} v + f,$$

*with  $f \in C_\rho^0 H^r$  for some  $r \in \mathbb{R}$ , then, for all  $\epsilon' > 0$ , we have*

$$v|_{\rho=1} \in H^{r+1-\epsilon'}.$$

One may apply this lemma for  $b_1 = a^{(1)} \in \Gamma_{3/2+}^1 \subset \Gamma_1^1$ ,  $b_0 = -a^{(0)} \in \Gamma_{1/2+}^0 \subset \Gamma_0^0$ , and  $v = (\partial_\rho - T_A)\Phi$  with  $M = s_0 - 1$ ,  $r = s_0 - \frac{1}{2} + \epsilon$ , and  $\epsilon' = \frac{\epsilon}{2}$  to conclude that

$$\left\|((\partial_\rho - T_A)\Phi)|_{\rho=1}\right\|_{H^{s_0 + \frac{1}{2}}} \leq C\left(\|\eta\|_{H_R^{s+\frac{1}{2}-}}\right)\|\psi\|_{H^{s_0}}.$$

In order to recover  $\partial_\rho \varphi$  (appearing on the left hand side of (4.3.18)) from  $\partial_\rho \Phi$ , we use the definition (4.3.22) of  $\Phi$ ,

$$((\partial_\rho - T_A)\Phi)|_{\rho=1} = \partial_\rho \varphi|_{\rho=1} - T_\tau U - T_{\eta^{-1} \partial_\rho^2 \varphi} \eta|_{\rho=1} - T_{\eta^{-1} \partial_\rho \varphi} \eta|_{\rho=1},$$

where  $U$  is defined by (4.3.19) with  $B$  given in (4.2.17),  $\tau = A|_{\rho=1}$ , and the remaining terms verify

$$\|T_{\eta^{-1} \partial_\rho^2 \varphi} \eta\|_{C_\rho^0 H^{s_0 + \frac{1}{2}}} \lesssim \|\eta^{-1} \partial_\rho^2 \varphi\|_{C_\rho^0 H^{s_0 - 2}} \|\eta\|_{H_R^{s+\frac{1}{2}}} \leq C\left(\|\eta\|_{H_R^{s+\frac{1}{2}-}}\right)\|\psi\|_{H^{s_0}} \|\eta\|_{H_R^{s+\frac{1}{2}}},$$

$$\|T_{\eta^{-1} \partial_\rho \varphi} \eta\|_{C_\rho^0 H^{s_0 + \frac{1}{2}}} \lesssim \|\eta^{-1} \partial_\rho \varphi\|_{C_\rho^0 H^{s_0 - 1}} \|\eta\|_{H_R^{s+\frac{1}{2}}} \leq C\left(\|\eta\|_{H_R^{s+\frac{1}{2}-}}\right)\|\psi\|_{H^{s_0}} \|\eta\|_{H_R^{s+\frac{1}{2}}},$$

thanks to Proposition 4.C.8 and Corollary 4.C.9 together with Lemma 4.3.1. Note that, when  $s_0 - 2 > 1$ , the estimate follows from  $s_0 \leq s$ , while, when  $s_0 - 2 \leq \frac{d}{2}$ , we use the fact that  $(s_0 -) - 2 + s + \frac{1}{2} - 1 > s_0 + \frac{1}{2}$  since  $s > 3$ .

### 4.3.3. Parilinearization of Dirichlet-to-Neumann operator

Thanks to Proposition 4.3.10, we are now able to write the Dirichlet-to-Neumann operator (4.3.1) in terms of tangential derivatives. We introduce the modified gradient (in  $w$ )

$$(4.3.40) \quad \bar{\nabla} := \frac{e_\theta \partial_\theta}{\eta} + e_z \partial_z,$$

which allows us to do the following calculation from (4.2.17)-(4.2.21) (attention that we abuse the notation  $(\rho, \theta, z)$  which has different meaning in Section 4.2 and here, whose relation is given by (4.3.3)),

$$(4.3.41) \quad B = \frac{\partial_\rho \varphi|_{\rho=1}}{\eta},$$

$$(4.3.42) \quad V = \bar{\nabla} \varphi|_{\rho=1} - \frac{\bar{\nabla} \eta}{\eta} \partial_\rho \varphi|_{\rho=1},$$

$$(4.3.43) \quad N = BV \cdot \bar{\nabla} \eta + \frac{|V|^2 - B^2}{2},$$

$$(4.3.44) \quad G(\eta)\psi = B - V \cdot \bar{\nabla} \eta,$$

$$(4.3.45) \quad \bar{\nabla} \psi = V + B \bar{\nabla} \eta.$$

Note that these formulas also imply

$$(4.3.46) \quad B = \frac{G(\eta)\psi + \bar{\nabla} \psi \cdot \bar{\nabla} \eta}{1 + |\bar{\nabla} \eta|^2},$$

$$(4.3.47) \quad V = \bar{\nabla} \psi - B \bar{\nabla} \eta,$$

$$(4.3.48) \quad N = \frac{1}{2} \frac{(G(\eta)\psi + \bar{\nabla} \psi \cdot \bar{\nabla} \eta)^2}{1 + |\bar{\nabla} \eta|^2}.$$

The following estimate for  $(B, V)$  is no more than a consequence of Lemma 4.3.1 and formula (4.3.41), together with Corollary 4.C.9 and Proposition 4.C.22.

**Lemma 4.3.16.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Then, by expressing  $B, V$  as linear operators acting on  $\psi$ ,*

$$(4.3.49) \quad B = \mathcal{B}(\eta)\psi, \quad V = \mathcal{V}(\eta)\psi,$$

we have

$$(4.3.50) \quad \|\mathcal{B}(\eta)\|_{\mathcal{L}(H^{s_0}; H^{s_0-1})} + \|\mathcal{V}(\eta)\|_{\mathcal{L}(H^{s_0}; H^{s_0-1})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right),$$

and, in particular,

$$(4.3.51) \quad \|B\|_{H^{s_0-1}} + \|V\|_{H^{s_0-1}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

We can now state the main result of this section as

**Proposition 4.3.17.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Let  $U = \psi - T_B \eta$  be the good unknown as defined in (4.3.19). Then there exists elliptic symbol  $\lambda \in \Sigma^1$ , such that*

$$(4.3.52) \quad \begin{aligned} G(\eta)\psi &= T_\lambda U - T_V \cdot \bar{\nabla} \eta + f_1 \\ &= T_\lambda (\psi - T_{\mathcal{B}(\eta)} \eta) - T_{\mathcal{V}(\eta)} \psi \cdot \bar{\nabla} \eta + f_1, \end{aligned}$$

where  $\mathcal{B}(\eta), \mathcal{V}(\eta)$  are defined by (4.3.49) above and  $f_1 = f_1(\eta, \psi)$  is linear in  $\psi$  with

$$(4.3.53) \quad \|f_1\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \left( \|\psi\|_{H^{s_0}} + \|\eta\|_{H_R^{s+\frac{1}{2}}} \|\psi\|_{H^{s_0-}} \right).$$

Moreover,  $\lambda = \lambda^{(1)} + \lambda^{(0)} \in \Sigma^1$  can be calculated explicitly,

$$(4.3.54) \quad \lambda^{(1)} = \sqrt{\left( \frac{\xi_\theta^2}{\eta^2} + \xi_z^2 \right) + \left( \frac{\xi_\theta}{\eta} \eta_z - \xi_z \frac{\eta_\theta}{\eta} \right)^2}, \quad \lambda^{(0)} = \frac{l^2}{\eta} A^{(0)}|_{\rho=1},$$

where  $A^{(0)}$  is defined in (4.3.35), and  $l$  is as defined in (4.1.6),

$$l = \sqrt{1 + \left( \frac{\eta_\theta}{\eta} \right)^2 + \eta_z^2}.$$

*Proof.* During the proof of Proposition 4.3.17, we shall use the equivalence : for  $u, v$  defined on  $\mathbb{T} \times \mathbb{R}$ ,

$$u \sim v \Leftrightarrow \|u - v\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

Recall that, from (4.3.1),  $G(\eta)\psi$  can be written as

$$\begin{aligned} G(\eta)\psi &= \frac{l^2}{\eta} \partial_\rho \varphi|_{\rho=1} - \frac{\eta_\theta}{\eta} \frac{\psi_\theta}{\eta} - \eta_z \psi_z \\ &= T_{\eta^{-1}l^2} \partial_\rho \varphi|_{\rho=1} + T_{\partial_\rho \varphi|_{\rho=1}} \eta^{-1} l^2 + R(\eta^{-1} l^2, \partial_\rho \varphi|_{\rho=1}) - T_{\eta^{-1} \eta_\theta} \eta^{-1} \psi_\theta - T_{\eta^{-1} \psi_\theta} \eta^{-1} \eta_\theta \\ &\quad - R(\eta^{-1} \eta_\theta, \eta^{-1} \psi_\theta) - T_{\eta_z} \psi_z - T_{\psi_z} \eta_z - R(\eta_z, \psi_z) \\ &\sim T_{\eta^{-1} l^2} \partial_\rho \varphi|_{\rho=1} + T_{\partial_\rho \varphi|_{\rho=1}} \eta^{-1} l^2 - T_{\eta^{-1} \eta_\theta} \eta^{-1} \psi_\theta - T_{\eta^{-1} \psi_\theta} \eta^{-1} \eta_\theta - T_{\eta_z} \psi_z - T_{\psi_z} \eta_z \\ &= T_{\eta^{-1} l^2} \partial_\rho \varphi|_{\rho=1} + T_{\partial_\rho \varphi|_{\rho=1}} \eta^{-1} l^2 - T_{\bar{\nabla} \eta} \cdot \bar{\nabla} \psi - T_{\bar{\nabla} \psi} \cdot \bar{\nabla} \eta \\ &= T_{\eta^{-1} l^2} \partial_\rho \varphi|_{\rho=1} + T_{\partial_\rho \varphi|_{\rho=1}} \eta^{-1} l^2 - T_{\bar{\nabla} \eta} \cdot \bar{\nabla} \psi - T_{B \bar{\nabla} \eta} \cdot \bar{\nabla} \eta - T_V \cdot \bar{\nabla} \eta. \end{aligned}$$

where we use the formula (4.3.45). To prove the equivalence, we observe that  $\partial_\rho \varphi|_{\rho=1}$ ,  $\eta^{-1} \psi_\theta$  and  $\psi_z$  belong to  $H^{s_0-1}$  with norm bounded by  $C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}$  using (4.3.4). Thus, an application of Proposition 4.C.22 and Corollary 4.C.9 gives

$$\begin{aligned} &\|R(\eta^{-1} l^2, \partial_\rho \varphi|_{\rho=1}) + R(\eta^{-1} \eta_\theta, \eta^{-1} \psi_\theta) + R(\eta_z, \psi_z)\|_{H^{s_0+\frac{1}{2}}} \\ &\leq \left( \|\eta^{-1} l^2\|_{H_R^{\frac{5}{2}+}} + \|\eta^{-1} \eta_\theta\|_{H^{\frac{5}{2}+}} + \|\eta_z\|_{H^{\frac{5}{2}+}} \right) C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}} \end{aligned}$$

$$\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

Thanks to definition (4.3.19) of good unknown  $U$ , formula (4.3.41), and Proposition 4.3.10, we have

$$\begin{aligned} G(\eta)\psi &\sim T_{\eta^{-1}l^2}\partial_\rho\varphi|_{\rho=1} + T_{\partial_\rho\varphi|_{\rho=1}}\eta^{-1}l^2 - T_{\bar{\nabla}\eta} \cdot \bar{\nabla}\psi - T_{B\bar{\nabla}\eta} \cdot \bar{\nabla}\eta - T_V \cdot \bar{\nabla}\eta \\ &\sim T_{\eta^{-1}l^2}T_\tau U + T_{\partial_\rho\varphi|_{\rho=1}}\eta^{-1}l^2 - T_{\bar{\nabla}\eta} \cdot \bar{\nabla}(U + T_B\eta) - T_{B\bar{\nabla}\eta} \cdot \bar{\nabla}\eta - T_V \cdot \bar{\nabla}\eta \\ &= T_{\eta^{-1}l^2}T_\tau U - T_{\bar{\nabla}\eta} \cdot \bar{\nabla}U - T_V \cdot \bar{\nabla}\eta + T_{\eta B}\eta^{-1}l^2 - T_{\bar{\nabla}\eta} \cdot \bar{\nabla}T_B\eta - T_{B\bar{\nabla}\eta} \cdot \bar{\nabla}\eta, \end{aligned}$$

where the equivalence is due to estimate (4.3.18) and  $\eta^{-1}l^2 \in H_{R^{-1}}^{s-\frac{1}{2}-}$ , ensuring that  $T_{\eta^{-1}l^2}$  is of order 0.

We claim that

$$(4.3.55) \quad T_{\eta^{-1}l^2}T_\tau U \sim T_{\eta^{-1}l^2\tau}U,$$

$$(4.3.56) \quad T_{\bar{\nabla}\eta} \cdot \bar{\nabla}U \sim -\frac{1}{2}T_{i\eta\xi \cdot \beta|_{\rho=1}}U,$$

$$(4.3.57) \quad T_{\eta B}\eta^{-1}l^2 - T_{\bar{\nabla}\eta} \cdot \bar{\nabla}T_B\eta - T_{B\bar{\nabla}\eta} \cdot \bar{\nabla}\eta \sim 0.$$

With these equivalences (whose proof will be given later), we may conclude that

$$G(\eta)\psi \sim T_\lambda U - T_V \cdot \bar{\nabla}\eta, \quad \lambda = \eta^{-1}l^2\tau + \frac{i\eta\xi \cdot \beta|_{\rho=1}}{2},$$

which coincides with (4.3.54).  $\square$

*Proof of (4.3.55).* In Proposition 4.3.10 and Lemma 4.3.14, we have seen that  $\tau = A|_{\rho=1} \in \Sigma^1$ . By applying Proposition 4.3.4 and using the fact that  $\eta^{-1}l^2 \in H_{R^{-1}}^{s-\frac{1}{2}-} \subset \Sigma^0$  (since  $s - \frac{1}{2}- > \frac{3}{2} + 1$  due to  $s > 3$ ), we have

$$\|T_{\eta^{-1}l^2}T_\tau U - T_{(\eta^{-1}l^2)\sharp\tau}U\|_{H^{s_0+\frac{1}{2}}} \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|U\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}.$$

Note that, since  $\eta^{-1}l^2 \in \Sigma^0$  is independent of  $\xi$  and has no  $\Gamma_{1/2+}^{-1}$  components, the formula (4.3.13) yields

$$(\eta^{-1}l^2)\sharp\tau = \eta^{-1}l^2\tau. \quad \square$$

*Proof of (4.3.56).* By definition (4.3.7), it is easy to see that

$$-\frac{i\eta\xi \cdot \beta|_{\rho=1}}{2} = i\xi_\theta \frac{\eta_\theta}{\eta^2} + i\xi_z \eta_z.$$

We may decompose the left hand side of (4.3.56) as

$$\begin{aligned} T_{\bar{\nabla}\eta} \cdot \bar{\nabla}U &= T_{\eta^{-1}\eta_\theta} (\eta^{-1}\partial_\theta U) + T_{\eta_z} \partial_z U \\ &= T_{\eta^{-1}\eta_\theta} T_{\eta^{-1}} \partial_\theta U + T_{\eta^{-1}\eta_\theta} T_{\partial_\theta U} \eta^{-1} + T_{\eta^{-1}\eta_\theta} R(\eta^{-1}, \partial_\theta U) + T_{i\xi_z \eta_z} U. \end{aligned}$$

Since  $\eta^{-1}\eta_\theta, \eta^{-1} \in \Gamma_{3/2+}^0$ , an application of Proposition 4.C.18 and (4.3.20) gives that

$$\begin{aligned} \|(T_{(\eta^{-1}\eta_\theta)\sharp(\eta^{-1})} - T_{\eta^{-1}\eta_\theta}T_{\eta^{-1}})\partial_\theta U\|_{H^{s_0+\frac{1}{2}}} &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\partial_\theta U\|_{H^{s_0-1}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \end{aligned}$$

with  $(\eta^{-1}\eta_\theta)\sharp(\eta^{-1}) = \eta^{-2}\eta_\theta$ , from which one may deduce that

$$T_{\eta^{-1}\eta_\theta}T_{\eta^{-1}}\partial_\theta U \sim T_{\eta^{-2}\eta_\theta}\partial_\theta U = T_{i\xi_\theta\eta^{-2}\eta_\theta}U,$$

and the desired estimate (4.3.56) can be reduced to

$$T_{\eta^{-1}\eta_\theta}T_{\partial_\theta U}\eta^{-1} + T_{\eta^{-1}\eta_\theta}R(\eta^{-1}, \partial_\theta U) \sim 0.$$

In fact, thanks to Proposition 4.C.8 and Corollary 4.C.9, we have

$$\begin{aligned} \|T_{\eta^{-1}\eta_\theta}T_{\partial_\theta U}\eta^{-1}\|_{H^{s_0+\frac{1}{2}}} &\lesssim \|\eta^{-1}\eta_\theta\|_{H^1} + \|T_{\partial_\theta U}\eta^{-1}\|_{H^{s_0+\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\partial_\theta U\|_{H^{s_0-1}} \|\eta^{-1}\|_{H^{\max(\frac{5}{2}+, s_0+\frac{1}{2})}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \\ \|T_{\eta^{-1}\eta_\theta}R(\eta^{-1}, \partial_\theta U)\|_{H^{s_0+\frac{1}{2}}} &\lesssim \|\eta^{-1}\eta_\theta\|_{H^1} + \|R(\eta^{-1}, \partial_\theta U)\|_{H^{s_0+\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta^{-1}\|_{H^{\frac{5}{2}+}} \|\partial_\theta U\|_{H^{s_0-1}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}. \end{aligned}$$

which completes the proof.  $\square$

*Proof of (4.3.57).* To prove (4.3.57), as in the proof of Lemma 4.3.11, we only need to consider the case  $s_0 = \frac{3}{2} + \delta$  with  $0 < \delta \ll 1$  and the case  $s_0 = s$ . In the first case, each term on the left hand side of (4.3.57) is equivalent to zero. In fact, Proposition 4.C.8 ensures that

$$\begin{aligned} \|T_{\eta B}\eta^{-1}l^2\|_{H^{2+\delta}} &\leq \|\eta B\|_{H^{\frac{1}{2}+\delta}} \|\eta^{-1}l^2\|_{H^{\frac{5}{2}-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \\ \|T_{\bar{\nabla}\eta} \cdot \bar{\nabla}T_B\eta\|_{H^{2+\delta}} &\leq \|\bar{\nabla}\eta\|_{H^1} + \|\bar{\nabla}T_B\eta\|_{H^{2+\delta}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|T_B\eta\|_{H^{3+\delta}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|B\|_{H^{\frac{1}{2}+\delta}} \|\eta\|_{H^{\frac{7}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \\ \|T_{B\bar{\nabla}\eta} \cdot \bar{\nabla}\eta\|_{H^{2+\delta}} &\lesssim \|B\bar{\nabla}\eta\|_{H^{\frac{1}{2}+\delta}} \|\bar{\nabla}\eta\|_{H^{\frac{5}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|B\|_{H^{\frac{1}{2}+\delta}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}. \end{aligned}$$

Note that the estimate for  $B$  has been given by (4.3.51).

If  $s_0 = s > 3$ , we are able to apply symbolic calculus (Proposition 4.C.18), since all the involved symbols has at least Hölder regularity in  $w$ . Recall that  $\eta \in H_R^{s+\frac{1}{2}-}$  and  $B \in H^{s-1-}$ . Then we have

$$\begin{aligned}
& T_{\eta B} \eta^{-1} l^2 - T_{\bar{\nabla} \eta} \cdot \bar{\nabla} T_B \eta - T_B \bar{\nabla} \eta \cdot \bar{\nabla} \eta \\
& = T_{\eta B} \eta^{-1} + T_{\eta B} (\eta^{-1} |\bar{\nabla} \eta|^2) - T_{\eta^{-1} \eta_\theta} (\eta^{-1} \partial_\theta (T_B \eta)) - T_{\eta_z} \partial_z (T_B \eta) \\
& \quad - T_{\eta^{-1} \eta_\theta B} (\eta^{-1} \eta_\theta) - T_{\eta_z B} \eta_z \\
& \sim T_{\eta B} (\eta^{-1} |\bar{\nabla} \eta|^2) - T_{\eta^{-1} \eta_\theta} (\eta^{-1} T_B \eta_\theta) - T_{\eta_z} T_B \eta_z - T_{\eta^{-1} \eta_\theta B} (\eta^{-1} \eta_\theta) - T_{\eta_z B} \eta_z \\
& \sim T_B T_\eta (\eta^{-1} |\bar{\nabla} \eta|^2) - T_{\eta^{-1} \eta_\theta} T_{\eta^{-1} T_B \eta_\theta} - 2 T_B T_{\eta_z} \eta_z - T_B T_{\eta^{-1} \eta_\theta} (\eta^{-1} \eta_\theta) \\
& \sim T_B |\bar{\nabla} \eta|^2 - T_B T_{\eta^{-1} \eta_\theta} T_{\eta^{-1} \eta_\theta} - T_B T_{\eta^{-1} \eta_\theta} (\eta^{-1} \eta_\theta) - T_B \eta_z^2 \\
& \sim T_B |\bar{\nabla} \eta|^2 - 2 T_B T_{\eta^{-1} \eta_\theta} (\eta^{-1} \eta_\theta) - T_B \eta_z^2 \\
& \sim T_B |\bar{\nabla} \eta|^2 - T_B (\eta^{-1} \eta_\theta)^2 - T_B \eta_z^2 = 0.
\end{aligned}$$

□

As a result of Proposition 4.3.17, we can replace the first equation in (4.1.18) by

$$(4.3.58) \quad \eta_t + T_V \cdot \bar{\nabla} \eta - T_\lambda U = f_1.$$

Before entering to the parilinearization of nonlinear terms, we provide a calculation concerning the subprincipal part  $\lambda^{(0)}$  of symbol  $\lambda$  appearing in (4.3.52). We claim that

**Lemma 4.3.18.** *The symbol  $\lambda = \lambda^{(1)} + \lambda^{(0)}$  constructed in Proposition 4.3.17 satisfies*

$$(4.3.59) \quad \text{Im } \lambda^{(0)} = -\frac{1}{2} \partial_w \cdot \partial_\xi \lambda^{(1)} - \frac{1}{2} \frac{\partial_w \eta}{\eta} \cdot \partial_\xi \lambda^{(1)}.$$

*Proof.* Set  $a = \eta \lambda$ . Since by (4.3.54),  $\text{Im } \lambda^{(1)} = 0$ , Proposition 4.3.9 shows that (4.3.59) will hold if we prove that the symbol  $\eta \lambda$  verifies the second condition of (4.3.17), i.e. that  $T_{\eta \lambda} - T_{\eta \lambda}^*$  is of order  $0-$ .

To prove this, we assume that  $\eta \in H_R^{+\infty} := \cap_{s \in \mathbb{R}} H_R^s$  and  $\psi \in H^s$  for some fixed  $s \gg 1$ . Under this regularity assumption, from (4.3.52) and (4.3.19), we have

$$\begin{aligned}
\eta G(\eta) \psi & = \eta T_\lambda \psi - \eta T_\lambda T_B \eta - \eta T_V \cdot \bar{\nabla} \eta + f_1 \\
& = T_\eta T_\lambda \psi + T_{T_\lambda \psi} \eta + R(\eta, T_\lambda \psi) - \eta T_\lambda T_B \eta - \eta T_V \cdot \bar{\nabla} \eta + f_1 \\
& = T_{\eta \lambda} \psi + (T_\eta T_\lambda - T_{\eta \lambda}) \psi + T_{T_\lambda \psi} \eta + R(\eta, T_\lambda \psi) - \eta T_\lambda T_B \eta - \eta T_V \cdot \bar{\nabla} \eta + f_1.
\end{aligned}$$

Since  $\eta \in H_R^{+\infty}$ , thanks to Proposition 4.C.18, 4.C.8, and Corollary 4.C.9, it is clear that the following terms are smoothing operator w.r.t.  $\psi$ ,

$$(T_\eta T_\lambda - T_{\eta \lambda}) \psi, T_{T_\lambda \psi} \eta, R(\eta, T_\lambda \psi), \eta T_\lambda T_B \eta, \eta T_V \cdot \bar{\nabla} \eta,$$

while  $f_1$  is of order  $-\frac{1}{2}$  due to (4.3.53). Consequently, we have that

$$T_{\eta \lambda} - \eta G(\eta) \text{ is of order } -\frac{1}{2},$$

which, combined with the fact that  $\eta G(\eta)$  is self-adjoint, yields that

$$T_{\eta\lambda} - T_{\eta\lambda}^* = (T_{\eta\lambda} - \eta G(\eta)) - (T_{\eta\lambda} - \eta G(\eta))^* \text{ is of order } -\frac{1}{2},$$

which concludes the desired result (4.3.59) for regular enough  $(\eta, \psi)$ .

We emphasize that the regularity assumption on  $\eta$  and  $\psi$  has no impact in the identity (4.3.59) since it can also be obtained by algebraic calculus from (4.3.54) (which is far more complicated).  $\square$

#### 4.3.4 . Paralinearization of the nonlinear terms

In this section, we aim to rewrite the second equation of (4.1.18) in the similar form as (4.3.58). The main difficulty is the paralinearization of the nonlinear terms, i.e.

**Proposition 4.3.19.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ . Then there exists  $\mu \in \Sigma^2$  and  $r_2, r_3 \in H^{s+}$  such that*

$$(4.3.60) \quad N = T_V \cdot \bar{\nabla} \psi - T_B G(\eta) \psi - T_B T_V \cdot \bar{\nabla} \eta + r_2,$$

$$(4.3.61) \quad H - \frac{1}{2R} = T_\mu \eta + r_3.$$

where  $r_2 = r_2(\eta; \psi, \psi)$  is quadratic in  $\psi$  and  $r_3 = r_3(\eta)$  is independent of  $\psi$  with, for all  $\frac{3}{2} < s_0 \leq s$  and  $\psi_1, \psi_2 \in H^{s_0}$ ,

$$(4.3.62) \quad \|r_2(\eta; \psi_1, \psi_2)\|_{H^{s_0}} + \|r_2(\eta; \psi_2, \psi_1)\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi_1\|_{H^{s_0}} \|\psi_2\|_{H^{s_0}},$$

$$(4.3.63) \quad \|r_3(\eta)\|_{H^s} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta\|_{H_R^{s+\frac{1}{2}}}.$$

Moreover,  $\mu = \mu^{(2)} + \mu^{(1)} \in \Sigma^2$  is elliptic with

$$(4.3.64) \quad \mu^{(2)} = \frac{1}{2l^3} \left[ \left( \frac{\xi_\theta}{\eta} \right)^2 + \xi_z^2 + \left( \frac{\xi_\theta}{\eta} \eta_z - \xi_z \frac{\eta_\theta}{\eta} \right)^2 \right].$$

Once this proposition is proved, the second equation in (4.1.18) can be replaced by

$$(4.3.65) \quad \psi_t + T_V \cdot \bar{\nabla} \psi + \sigma T_\mu \eta - T_B G(\eta) \psi - T_B T_V \cdot \bar{\nabla} \eta = -r_2 - \sigma r_3,$$

which can be further written as

$$(\partial_t + T_V \cdot \bar{\nabla}) \psi - T_B (\partial_t + T_V \cdot \bar{\nabla}) \eta + \sigma T_\mu \eta = -r_2 - \sigma r_3.$$

By combining this equation with (4.3.58), we obtain a reformulation of (4.1.18),

$$(4.3.66) \quad \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} (\partial_t + T_V \cdot \bar{\nabla}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} + \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

where

$$(4.3.67) \quad f_2 := f_2(\eta; \psi, \psi) = -r_2(\eta; \psi, \psi) - \sigma r_3(\eta).$$

Note that

$$\begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix}.$$

This identity allows us to recover from (4.3.66) an evolution equation for  $(\eta, \psi)$ ,

$$(4.3.68) \quad (\partial_t + T_V \cdot \bar{\nabla}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} + \mathcal{L} \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f,$$

where

$$(4.3.69) \quad \mathcal{L} := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}$$

and

$$(4.3.70) \quad f := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ T_B f_1 + f_2 \end{pmatrix}.$$

Since  $\lambda$  and  $\mu$  are elliptic, the operator  $\mathcal{L}$  can be symmetrized, which is the purpose of Section 4.4.

### Paralinearization of $N$

In this part, we shall prove (4.3.60) and (4.3.62). As before, during this proof, we shall use the equivalence : for  $u, v$  defined on  $\mathbb{T} \times \mathbb{R}$  and bilinear in  $(\psi_1, \psi_2)$ ,

(4.3.71)

$$u \sim v \Leftrightarrow \|(u-v)(\eta; \psi_1, \psi_2)\|_{H^{s_0}} + \|(u-v)(\eta; \psi_2, \psi_1)\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi_1\|_{H^{s-}} \|\psi_2\|_{H^{s_0}}.$$

To begin with, from formula (4.3.43) for  $N$ , it is clear that  $N$  is quadratic in  $\psi$  as well as the three first terms on the right hand side of (4.3.60). As a result,  $r_2$  is quadratic in  $\psi$  (not necessarily symmetric).

Due to (4.3.43) and (4.3.44), one may rewrite  $N$  as follow,

(4.3.72)

$$\begin{aligned} N &= \frac{|V|^2 + B^2}{2} - BG(\eta)\psi \\ &= T_V \cdot V + T_B B - T_B G(\eta)\psi - T_{B-V} \cdot \bar{\nabla}_\eta B + R(V, V) + R(B, B) + R(B, G(\eta)\psi) \\ &\sim T_V \cdot V + T_B B - T_B G(\eta)\psi - T_{B-V} \cdot \bar{\nabla}_\eta B = T_V \cdot V - T_B G(\eta)\psi + T_V \cdot \bar{\nabla}_\eta B \end{aligned}$$

To prove the equivalence, it suffices to apply Proposition 4.C.8 and use the fact that  $B, V, G(\eta)\psi$  are linear in  $\psi$  with

$$(4.3.73) \quad \|B\|_{H^{s'-1}} + \|V\|_{H^{s'-1}} + \|G(\eta)\psi\|_{H^{s'-1}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s'}}, \quad \forall s' \in \left] \frac{3}{2}, s \right].$$

More precisely, we have

$$\begin{aligned} \|R(B(\psi_1), B(\psi_2))\|_{H^{s_0}} &\lesssim \|B(\psi_1)\|_{H^{2+}} \|B(\psi_2)\|_{H^{s_0-1-}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi_1\|_{H^{s-}} \|\psi_2\|_{H^{s_0}}, \end{aligned}$$

and the similar estimate holds for  $R(V, V)$  (note that  $R(\cdot, \cdot)$  is symmetric), while

$$\begin{aligned} &\|R(B(\psi_1), G(\eta)\psi_2)\|_{H^{s_0}} + \|R(B(\psi_2), G(\eta)\psi_1)\|_{H^{s_0}} \\ &\lesssim \|B(\psi_1)\|_{H^{2+}} \|G(\eta)\psi_2\|_{H^{s_0-1-}} + \|G(\eta)\psi_1\|_{H^{2+}} \|B(\psi_2)\|_{H^{s_0-1-}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi_1\|_{H^{s-}} \|\psi_2\|_{H^{s_0}}. \end{aligned}$$

Inserting (4.3.45) in the right hand side of (4.3.60), we can write it, up to  $r_2$ , as

$$T_V \cdot \bar{\nabla} \psi - T_B G(\eta) \psi - T_B T_V \cdot \bar{\nabla} \eta = T_V V + T_V \cdot (B \bar{\nabla} \eta) - T_B G(\eta) \psi - T_B T_V \cdot \bar{\nabla} \eta.$$

According to (4.3.72), this will be equivalent to  $N$  if and only if

$$T_{V \cdot \bar{\nabla} \eta} B \sim T_V \cdot (B \bar{\nabla} \eta) - T_B T_V \cdot \bar{\nabla} \eta.$$

The difference of the two sides can be expressed as

$$\begin{aligned} &T_V \cdot (B \bar{\nabla} \eta) - T_B T_V \cdot \bar{\nabla} \eta - T_{V \cdot \bar{\nabla} \eta} B \\ &= T_V T_B \bar{\nabla} \eta + T_V \cdot T_{\bar{\nabla} \eta} B + T_V R(B, \bar{\nabla} \eta) - T_B T_V \cdot \bar{\nabla} \eta - T_{V \cdot \bar{\nabla} \eta} B \\ &= [T_V, T_B] \bar{\nabla} \eta + (T_V \cdot T_{\bar{\nabla} \eta} - T_{V \cdot \bar{\nabla} \eta}) B + T_V R(B, \bar{\nabla} \eta). \end{aligned}$$

Therefore, it suffices to check the following equivalences :

$$(4.3.74) \quad [T_V, T_B] \bar{\nabla} \eta \sim 0,$$

$$(4.3.75) \quad (T_V \cdot T_{\bar{\nabla} \eta} - T_{V \cdot \bar{\nabla} \eta}) B \sim 0,$$

$$(4.3.76) \quad T_V R(B, \bar{\nabla} \eta) \sim 0,$$

in the sense of (4.3.71). As before, the interpolation argument allows us to focus on the case  $s_0 = \frac{3}{2} + \delta$  with  $0 < \delta \ll 1$  and the case  $s_0 = s$ . In the latter one, by observing that  $B, V \in H^{s-1-} \subset \Gamma_{1+}^0$ ,  $\bar{\nabla} \eta \in H^{s-\frac{1}{2}-} \subset \Gamma_{1+}^0$ , the equivalences (4.3.74), (4.3.75), and (4.3.76) follows from Corollary 4.C.19, Proposition 4.C.18, and Proposition 4.C.8, respectively, together with (4.3.73).

In the case  $s_0 = \frac{3}{2} + \delta$ , we shall check that all the terms concerned are equivalent to zero. In fact, we have

$$\begin{aligned} &\|T_V(\psi_1) T_B(\psi_2) \bar{\nabla} \eta\|_{H^{\frac{3}{2}+\delta}} + \|T_V(\psi_2) T_B(\psi_1) \bar{\nabla} \eta\|_{H^{\frac{3}{2}+\delta}} \\ &\lesssim \|V(\psi_1)\|_{H^{1+}} \|B(\psi_2)\|_{H^{\frac{1}{2}+\delta}} \|\bar{\nabla} \eta\|_{H^2} + \|V(\psi_2)\|_{H^{\frac{1}{2}+\delta}} \|B(\psi_1)\|_{H^{1+}} \|\bar{\nabla} \eta\|_{H^2} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi_1\|_{H^{s-}} \|\psi_2\|_{H^{s_0}}, \end{aligned}$$

which implies  $T_V T_B \bar{\nabla} \eta \sim 0$  and  $T_B T_V \bar{\nabla} \eta \sim 0$  in the same way, yielding (4.3.74). For (4.3.75), we apply Proposition 4.C.8,

$$\begin{aligned} & \left\| \left( T_V(\psi_1) \cdot T_{\bar{\nabla} \eta} - T_V(\psi_1) \cdot \bar{\nabla} \eta \right) B(\psi_2) \right\|_{H^{\frac{3}{2}+\delta}} + \left\| T_V(\psi_2) \cdot T_{\bar{\nabla} \eta} B(\psi_1) \right\|_{H^{\frac{3}{2}+\delta}} \\ & + \left\| T_V(\psi_2) \cdot \bar{\nabla} \eta B(\psi_1) \right\|_{H^{\frac{3}{2}+\delta}} \\ & \leq \left\| V(\psi_1) \right\|_{H^{2+}} \left\| \bar{\nabla} \eta \right\|_{H^{2+}} \left\| B(\psi_2) \right\|_{H^{\frac{1}{2}+\delta}} + \left\| V(\psi_2) \right\|_{H^{\frac{1}{2}+\delta}} \left\| \bar{\nabla} \eta \right\|_{H^{1+}} \left\| B(\psi_2) \right\|_{H^2} \\ & \leq C \left( \left\| \eta \right\|_{H_R^{s+\frac{1}{2}-}} \right) \left\| \psi_1 \right\|_{H^{s-}} \left\| \psi_2 \right\|_{H^{s_0}}. \end{aligned}$$

The estimate corresponding to (4.3.76) is due to Proposition 4.C.8.

$$\begin{aligned} & \left\| T_V(\psi_1) R(B(\psi_2), \bar{\nabla} \eta) \right\|_{H^{\frac{3}{2}+\delta}} + \left\| T_V(\psi_2) R(B(\psi_1), \bar{\nabla} \eta) \right\|_{H^{\frac{3}{2}+\delta}} \\ & \leq \left\| V(\psi_1) \right\|_{H^{1+}} \left\| B(\psi_2) \right\|_{H^{\frac{1}{2}+\delta}} \left\| \bar{\nabla} \eta \right\|_{H^2} + \left\| V(\psi_2) \right\|_{H^{\frac{1}{2}+\delta}} \left\| \bar{\nabla} \eta \right\|_{H^1} \left\| B(\psi_2) \right\|_{H^2} \\ & \leq C \left( \left\| \eta \right\|_{H_R^{s+\frac{1}{2}-}} \right) \left\| \psi_1 \right\|_{H^{s-}} \left\| \psi_2 \right\|_{H^{s_0}}. \end{aligned}$$

### Paralinearization of $H$

In this part we prove (4.3.61) and (4.3.63). As before, we shall use the equivalence : for  $u, v$  functions of  $\eta$ ,

$$(4.3.77) \quad u \sim v \Leftrightarrow \|u(\eta) - v(\eta)\|_{H^s} \leq C \left( \left\| \eta \right\|_{H_R^{s+\frac{1}{2}-}} \right) \left\| \eta \right\|_{H_R^{s+\frac{1}{2}}}.$$

By definition (4.1.9) of  $H$  and expression (4.1.6) of  $l$ , it is easy to calculate that the left hand side of (4.3.61) reads

$$\begin{aligned} H - \frac{1}{2R} &= \frac{1}{2} \left[ \frac{1}{\eta l} - \frac{\partial_\theta}{\eta} \left( \frac{\eta_\theta}{\eta l} \right) - \partial_z \left( \frac{\eta_z}{l} \right) \right] - \frac{1}{2R} \\ &= \frac{1}{2\eta l} - \frac{1}{2R} + \frac{1}{2} \left[ \frac{1}{\eta} \frac{\eta_\theta}{l} \frac{\eta_\theta}{\eta^2} + \frac{1}{\eta} \frac{\eta_\theta}{\eta} \frac{l_\theta}{l^2} - \frac{1}{\eta^2 l} \eta_{\theta\theta} \right] + \frac{1}{2} \left[ \eta_z \frac{l_z}{l^2} - \frac{1}{l} \eta_{zz} \right] \\ &= \frac{1}{2\eta l} - \frac{1}{2R} + \frac{1}{2} \left[ \frac{\eta_\theta^2}{\eta^3 l} + \frac{\eta_\theta}{\eta^2 l^2} \frac{\eta_\theta}{\eta} \frac{\partial_\theta}{\eta} \left( \frac{\eta_\theta}{\eta} \right) + \eta_z \eta_{\theta z} - \frac{1}{\eta^2 l} \eta_{\theta\theta} \right] \\ & \quad + \frac{1}{2} \left[ \frac{\eta_z}{l^2} \frac{\eta_\theta}{\eta} \partial_z \left( \frac{\eta_\theta}{\eta} \right) + \eta_z \eta_{zz} - \frac{1}{l} \eta_{zz} \right] \\ &= \frac{1}{2\eta l} - \frac{1}{2R} + \frac{1}{2} \left[ \frac{\eta_\theta^2}{\eta^3 l} - \frac{\eta_\theta^4}{\eta^5 l^3} + \frac{\eta_\theta \eta_z}{\eta^2 l^3} \eta_{\theta z} - \left( \frac{1}{\eta^2 l} - \frac{\eta_\theta^2}{\eta^4 l^3} \right) \eta_{\theta\theta} \right] \\ & \quad + \frac{1}{2} \left[ -\frac{\eta_\theta^2 \eta_z^2}{\eta^3 l^3} + \frac{\eta_\theta \eta_z}{\eta^2 l^3} \eta_{\theta z} - \left( \frac{1}{l} - \frac{\eta_z^2}{l^3} \right) \eta_{zz} \right] \\ &= \frac{1}{2\eta l} - \frac{1}{2R} + \frac{\eta_\theta^2}{2\eta^3 l^3} - \frac{1}{2l^3} \left[ (1 + \eta_z^2) \frac{\eta_{\theta\theta}}{\eta^2} - 2 \frac{\eta_\theta \eta_z}{\eta} \frac{\eta_{\theta z}}{\eta} + \left( 1 + \left( \frac{\eta_\theta}{\eta} \right)^2 \right) \eta_{zz} \right] \end{aligned}$$

$$= F(\eta, \eta_\theta, \eta_z) + \sum_{j,k \in \{\theta, z\}} G^{jk}(\eta, \eta_\theta, \eta_z) \eta_{jk},$$

where  $F$  and  $G^{jk}$  are smooth functions defined by

$$(4.3.78) \quad \begin{aligned} F(x, u, v) &:= \frac{1}{2xh} - \frac{1}{2R} + \frac{u^2}{2x^2h^3}, \\ G^{\theta\theta}(x, u, v) &:= -\frac{1}{2h^3} \frac{1+v^2}{x^2}, \quad G^{zz}(x, u, v) := -\frac{1}{2h^3} \left(1 + \left(\frac{u}{x}\right)^2\right), \\ G^{\theta z}(x, u, v) = G^{z\theta}(x, u, v) &:= \frac{1}{2h^3} \frac{uv}{x^2}, \end{aligned}$$

with  $h(x, u, v) := \sqrt{1 + (u/x)^2 + v^2}$ . Since  $F(R, 0) = G^{\theta z}(R, 0) = G^{z\theta}(R, 0) = 0$ ,  $G^{\theta\theta}(R, 0) = -(2R^2)^{-1}$ , and  $G^{zz}(R, 0) = -1/2$ , one may apply Proposition 4.C.22 and obtain

$$(4.3.79) \quad \begin{aligned} H - \frac{1}{2R} &= T_{F_x(\eta, \nabla_w \eta)} \eta + T_{(\nabla_{u,v} F)(\eta, \nabla_w \eta)} \cdot \nabla_w \eta + T_{G^{jk}(\eta, \nabla_w \eta)} \eta_{jk} \\ &\quad + T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta + T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \cdot \nabla_w \eta + R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) + r_4, \end{aligned}$$

where Einstein summation convention is applied for simplicity and the remainder  $r_4$  is equal to

$$(4.3.80) \quad \begin{aligned} r_4 &:= F(\eta, \nabla_w \eta) - T_{\nabla_{x,u,v} F(\eta, \nabla_w \eta)} \cdot (\eta, \nabla_w \eta) \\ &\quad + T_{\eta_{jk}} \left( G^{jk}(\eta, \nabla_w \eta) - T_{\nabla_{x,u,v} G^{jk}(\eta, \nabla_w \eta)} \cdot (\eta, \nabla_w \eta) \right), \end{aligned}$$

which lies in  $H^{2(s-\frac{1}{2})-1} \subset H^s$  due to (4.C.19) and Proposition 4.C.8, together with the fact that  $\eta \in H_R^{s+\frac{1}{2}-}$ . We claim that

**Lemma 4.3.20.** *Under the hypotheses of Proposition 4.3.19, we have the following equivalences in the sense of (4.3.77),*

$$(4.3.81) \quad T_{F_x(\eta, \nabla_w \eta)} \eta \sim 0,$$

$$(4.3.82) \quad T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta, R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) \sim 0,$$

$$(4.3.83) \quad \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \eta \sim 0.$$

*Proof.* From (4.C.20), we know that the composition of smooth functions with  $(\eta, \nabla_w \eta)$  belong to  $H^{s-\frac{1}{2}}$ . Thus, from Proposition 4.C.8, we have

$$\begin{aligned} \|T_{F_x(\eta, \nabla_w \eta)} \eta\|_{H^s} &\lesssim \|F_x(\eta, \nabla_w \eta)\|_{H^{1+}} \|\eta\|_{H_R^s} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta\|_{H_R^{s+\frac{1}{2}}}, \\ \|T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta\|_{H^s} + \|R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right)\|_{H^s} \\ &\lesssim \|\eta_{jk}\|_{H^{1+}} \|G_x^{jk}(\eta, \nabla_w \eta)\|_{H^{1+}} \|\eta\|_{H^s} + \|\eta_{jk}\|_{H^{\frac{3}{2}+}} \|G^{jk}(\eta, \nabla_w \eta)\|_{H^{s-\frac{1}{2}-}} \end{aligned}$$

$$\begin{aligned}
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \\
&\quad \left\| \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \eta \right\|_{H^s} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\nabla_w \eta\|_{H^{s-\frac{3}{2}-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\eta\|_{H_R^{s+\frac{1}{2}-}},
\end{aligned}$$

where in the last inequality we apply Proposition 4.C.18 with the fact that  $\eta_{jk}$  and  $(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)$  are both in  $H^{s-\frac{3}{2}-} \subset \Gamma_{3/2+}^0$ .  $\square$

To sum up, we have proved that

$$\begin{aligned}
(4.3.84) \quad H - \frac{1}{2R} &\sim T_{(\nabla_{u,v} F)(\eta, \nabla_w \eta)} \cdot \nabla_w \eta + T_{G^{jk}(\eta, \nabla_w \eta)} \eta_{jk} + T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \cdot \nabla_w \eta \\
&= T_{(\nabla_{u,v} F)(\eta, \nabla_w \eta)} \cdot i\xi \eta - T_{G^{jk}(\eta, \nabla_w \eta)} \xi_j \xi_k \eta + T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \cdot i\xi \eta,
\end{aligned}$$

and we can define  $\mu^{(2)}, \mu^{(1)}$  as

$$\begin{aligned}
(4.3.85) \quad &-G^{jk}(\eta, \nabla_w \eta) \xi_j \xi_k = \mu^{(2)}, \\
&(\nabla_{u,v} F)(\eta, \nabla_w \eta) \cdot i\xi + \eta_{jk} (\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta) \cdot i\xi = \mu^{(1)}.
\end{aligned}$$

One can check by direct computation that  $\mu^{(2)}$  is given by (4.3.64). As for  $\mu^{(1)}$ , we are only interested in its imaginary part, which will be useful in Section 4.4. We claim that

**Lemma 4.3.21.** *The symbol  $\mu = \mu^{(2)} + \mu^{(1)}$  constructed in (4.3.61) satisfies*

$$(4.3.86) \quad \operatorname{Im} \mu^{(1)} = -\frac{1}{2} \partial_w \cdot \partial_\xi \mu^{(2)} - \frac{1}{2} \frac{\partial_w \eta}{\eta} \cdot \partial_\xi \mu^{(2)}.$$

*Proof.* The idea of proof is similar to that of Lemma 4.3.18. We may assume  $\eta \in H_R^{+\infty} := \cap_{s \in \mathbb{R}} H_R^s$  and reduce the problem to show that  $T_{\eta\mu}^* - T_{\eta\mu}$  is of order 1- (we shall prove that it is a smoothing operator).

To begin with, we recall that we can represent  $H$  as

$$H - \frac{1}{2R} = F(\eta, \eta_\theta, \eta_z) + \sum_{j,k \in \{\theta, z\}} G^{jk}(\eta, \eta_\theta, \eta_z) \eta_{jk},$$

where  $F(x, u, v)$  and  $G^{jk}(x, u, v)$  are smooth functions defined in (4.3.78). Then by regarding  $H - \frac{1}{2R}$  as a nonlinear operator of  $\eta$  (not  $(\eta, \eta_\theta, \eta_z)$ ), one may calculate the derivative w.r.t.  $\eta$  of  $\eta(H - \frac{1}{2R})$ . Namely, for all  $\delta\eta \in H^s$  with  $s \gg 1$ , we have

$$\begin{aligned}
&\frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[ (\eta + \epsilon\delta\eta) \left( H(\eta + \epsilon\delta\eta) - \frac{1}{2R} \right) \right] \\
&= \frac{d}{d\epsilon} \Big|_{\epsilon=0} [(\eta + \epsilon\delta\eta) F(\eta + \epsilon\delta\eta, \partial_\theta(\eta + \epsilon\delta\eta), \partial_z(\eta + \epsilon\delta\eta))]
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{j,k \in \{\theta, z\}} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[ (\eta + \epsilon\delta\eta) G^{jk}(\eta + \epsilon\delta\eta, \partial_\theta(\eta + \epsilon\delta\eta), \partial_z(\eta + \epsilon\delta\eta)) \partial_j \partial_k (\eta + \epsilon\delta\eta) \right] \\
 & = F(\eta, \nabla_w \eta) \delta\eta + \eta F_x(\eta, \nabla_w \eta) \delta\eta + \eta (\nabla_{u,v} F)(\eta, \nabla_w \eta) \cdot \nabla_w \delta\eta + \eta_{jk} G^{jk}(\eta, \nabla_w \eta) \delta\eta \\
 & \quad + \eta \eta_{jk} G_x^{jk}(\eta, \nabla_w \eta) \delta\eta + \eta \eta_{jk} \left( \nabla_{u,v} G^{jk} \right) (\eta, \nabla_w \eta) \cdot \nabla_w \delta\eta + \eta G^{jk}(\eta, \nabla_w \eta) \partial_j \partial_k \delta\eta,
 \end{aligned}$$

where the right hand side is a differential operator acting on  $\delta\eta$ . By comparing it with (4.3.85), we can see that the symbol of this differential operator equals  $\eta\mu + F_0(\eta, \nabla_w \eta, \nabla_w^2 \eta)$ , where  $F_0$  is a smooth real-valued function.

Now, for any  $\delta\eta_1, \delta\eta_2 \in H_R^s$  with  $s \gg 1$ , we define

$$h(\epsilon_1, \epsilon_2) := A(\eta + \epsilon_1 \delta\eta_1 + \epsilon_2 \delta\eta_2),$$

where  $A$  is the normalized area of interface  $\Sigma(t)$ , which is defined in (4.2.31). By (4.2.34), we have

$$\partial_{\epsilon_1} h|_{\epsilon_1=0} = \int (\eta + \epsilon_2 \delta\eta_2) \left( H(\eta + \epsilon_2 \delta\eta_2) - \frac{1}{2R} \right) \delta\eta_1 dw,$$

which, together with the calculus above, yields

$$\partial_{\epsilon_2} \partial_{\epsilon_1} h|_{\epsilon_1=\epsilon_2=0} = \int \text{Op}(\eta\mu + F_0(\eta, \nabla_w \eta, \nabla_w^2 \eta)) \delta\eta_2 \delta\eta_1 dw.$$

Due to the regularity of  $\eta$ ,  $h$  is a smooth function of  $(\epsilon_1, \epsilon_2)$  near  $(0, 0)$ . Therefore, the order of derivative is not important, from which one can deduce that  $\text{Op}(\eta\mu + F_0(\eta, \nabla_w \eta, \nabla_w^2 \eta))$  is self-adjoint. Since  $F_0$  is real-valued, the differential operator  $\text{Op}(\eta\mu)$  is self-adjoint. Form the fact that the difference between differential operator and paradifferential operators with the same symbol is a smoothing operator, we conclude that  $T_{\eta\mu} - T_{\eta\mu}^*$  is a smoothing operator, which completes the proof.  $\square$

Note that, as a by-product, we have re-expressed  $H - (2R)^{-1}$  as

$$\begin{aligned}
 (4.3.87) \quad H - \frac{1}{2R} & = T_\mu \eta + T_{F_x(\eta, \nabla_w \eta)} \eta + T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta + R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) \\
 & \quad + \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \eta + r_4
 \end{aligned}$$

which can be seen by inserting (4.3.85) in (4.3.79). That is to say, the error  $r_3$  appearing in (4.3.61) admits an explicit formulation :

$$\begin{aligned}
 (4.3.88) \quad r_3 & = T_{F_x(\eta, \nabla_w \eta)} \eta + T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta + R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) \\
 & \quad + \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \eta + r_4.
 \end{aligned}$$

Recall that  $F, G$  are defined in (4.3.78) and  $r_4$  is given by (4.3.80).

### 4.3.5 . Continuity of source term

In previous sections, we manage to rewire (4.1.18) in parilinear form (4.3.68). Moreover, by combining estimates in Proposition 4.3.17 and Proposition 4.3.19, we are able to decompose the source term  $f$  (see (4.3.70) and (4.3.67)) as

$$(4.3.89) \quad f = f^{(0)} + f^{(1)} + f^{(2)},$$

where

$$(4.3.90) \quad f^{(0)} = \begin{pmatrix} 0 \\ -\sigma r_3(\eta) \end{pmatrix}, \quad f^{(1)} = \begin{pmatrix} f_1(\eta, \psi) \\ 0 \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} 0 \\ T_{B(\eta, \psi)} f_1(\eta, \psi) - r_2(\eta; \psi, \psi) \end{pmatrix},$$

are respectively independent, linear, and quadratic in  $\psi$ , with estimates (4.3.53), (4.3.62), and (4.3.63). In these estimates, by taking  $s_0 = s$ , we have, for all  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ ,  $B \in H^{s-1}$  (see (4.3.51)) where  $s - 1 > 1$ , which, thanks to Proposition 4.C.8, yields

$$(4.3.91) \quad \|f(\eta, \psi)\|_{H^{s+\frac{1}{2}} \times H^s} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} + \|\psi\|_{H^s} \right).$$

Meanwhile, with  $\frac{3}{2} < s_0 < s - \frac{3}{2}$ , we can prove the local Lipschitz regularity of  $f(\eta, \psi)$  in  $\psi$ , i.e. for all  $\eta \in H_R^{s+\frac{1}{2}}$  and  $\psi_1, \psi_2 \in H^s$  with  $s > 3$ ,

$$(4.3.92) \quad \|f(\eta, \psi_1) - f(\eta, \psi_2)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi_1\|_{H^{s_0}}, \|\psi_2\|_{H^{s_0}} \right) \|\psi_1 - \psi_2\|_{H^{s_0}}.$$

The goal of this section is to show that  $f(\eta, \psi)$  is also locally Lipschitzian in  $\eta$ , which is important to prove the convergence of approximate solutions to the parilinear system (4.3.68) and the uniqueness (see Section 4.5.3).

**Proposition 4.3.22.** *Let  $(\eta_1, \psi_1), (\eta_2, \psi_2) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ , such that*

$$\|(\eta_1, \psi_1)\|_{H_R^{s+\frac{1}{2}} \times H^s} + \|(\eta_2, \psi_2)\|_{H_R^{s+\frac{1}{2}} \times H^s} \leq M$$

for some  $M > 0$ . Then we have, for all  $\frac{3}{2} < s_0 < s - \frac{3}{2}$

$$(4.3.93) \quad \|f(\eta_1, \psi_1) - f(\eta_2, \psi_2)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C(M) \|(\eta_1 - \eta_2, \psi_1 - \psi_2)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}}.$$

The estimate (4.3.92) allows us to focus on the continuity in  $\eta$  by taking  $\psi := \psi_1 = \psi_2 \in H^s$ , and the desired result can be reduced to

$$(4.3.94) \quad \|f(\eta_1, \psi) - f(\eta_2, \psi)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\eta_1 - \eta_2\|_{H^{s_0+\frac{1}{2}}},$$

which can be further reduced to the boundedness of derivation in  $\eta$ .

**Lemma 4.3.23.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$ ,  $\delta\eta \in H^{s+\frac{1}{2}}$ , and  $\psi \in H^s$  with  $s > 3$ . Then the derivative of  $f$  in  $\eta$ , defined by*

$$(4.3.95) \quad \delta f := \frac{d}{d\eta} f(\eta, \psi) \cdot \delta\eta = \frac{d}{d\epsilon} \Big|_{\epsilon=0} f(\eta + \epsilon\delta\eta, \psi),$$

*satisfies, for all  $\frac{3}{2} < s_0 < s - \frac{3}{2}$ ,*

$$(4.3.96) \quad \|\delta f(\eta, \psi)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

Since  $f$  can be decomposed as (4.3.89) with  $f^{(0)}$ ,  $f^{(1)}$ , and  $f^{(2)}$  given by (4.3.90), we need to study  $\delta f_1$  and  $\delta f_2$  (more precisely  $\delta r_2$  and  $\delta r_3$ , see definition (4.3.67) of  $f_2$ ). The study of  $f_1$  is the most difficult one, since the only formula for  $f_1$  is (4.3.52), meaning that it is unavoidable to calculate the derivative-in- $\eta$  of  $G(\eta)\psi$  (depending implicitly in  $\eta$ ), which has been done in Proposition 4.2.8. As for the study of  $r_2$ , we shall rewrite it as a function of  $B$ ,  $V$ , and  $G(\eta)\psi$  and reduce the problem again to shape derivative (Proposition 4.2.8). The last term  $r_3$ , according to its definition (4.3.61), depends explicitly on  $\eta$  and the corresponding variational calculus is direct.

### Derivative of $f_1$

In this part, we shall prove (4.3.96) for  $f_1$  (defined in Proposition 4.3.17), namely

$$(4.3.97) \quad \|\delta f_1(\eta, \psi)\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

Before starting the proof, we introduce several technical lemmas which will be frequently used :

**Lemma 4.3.24.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$  and  $\delta\eta \in H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . All the functions  $u$  taking the form  $u = F(\eta, \nabla_w \eta, \dots, \nabla_w^k \eta)$  ( $k = 0, 1, 2$ ) belong to  $H_{F(R,0)}^{s+\frac{1}{2}-k}$  with*

$$(4.3.98) \quad \|\delta u\|_{H^{s_0+\frac{1}{2}-k}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

*Proof.* The fact that  $u \in H_{F(R,0)}^{s+\frac{1}{2}-k}$  is no more than a consequence of Proposition 4.C.22. For the proof of (4.3.98), we focus on the case  $k = 2$ , while the other cases can be proved in the same way. By definition, the derivative of  $u$  in  $\eta$  reads

$$\delta u = \sum_{j=0}^k \partial_j F(\eta, \nabla_w \eta, \dots, \nabla_w^k \eta) \nabla_w^j \delta\eta,$$

where, again by Proposition 4.C.22,  $\partial_j F(\eta, \nabla_w \eta, \dots, \nabla_w^k \eta) \in H^{s+\frac{1}{2}-k}$ , up to constant normalization, and  $\nabla_w^j \delta\eta \in H^{s_0+\frac{1}{2}-j} \subset H^{s_0+\frac{1}{2}-k}$ . Thus, Corollary 4.C.9 gives the desired

estimate,

$$\begin{aligned} \|\delta u\|_{H^{s_0+\frac{1}{2}-k}} &\leq \sum_{j=0}^k \|\partial_j F(\eta, \nabla_w \eta, \dots, \nabla_w^k \eta) \nabla_j^k \delta \eta\|_{H^{s_0+\frac{1}{2}-k}} \\ &\lesssim \sum_{j=0}^k \|\partial_j F(\eta, \nabla_w \eta, \dots, \nabla_w^k \eta)\|_{H^{s+\frac{1}{2}-k}} \|\nabla_j^k \delta \eta\|_{H^{s_0+\frac{1}{2}-k}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta \eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

□

**Lemma 4.3.25.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$ ,  $\psi \in H^s$ , and  $\delta \eta \in H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Then the following estimate holds,*

$$(4.3.99) \quad \|\delta B\|_{H^{s_0-1}} + \|\delta V\|_{H^{s_0-1}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta \eta\|_{H^{s_0+\frac{1}{2}}}.$$

*Proof.* Recall that, since  $\delta \eta \in H^{s_0+\frac{1}{2}}$ , the derivative in  $\eta$  of  $G(\eta)\psi$  lies in  $H^{s_0-1}$  due to Proposition 4.2.8. The formula (4.3.46) allows us to estimate  $\delta B$ . In fact, from (4.3.46), we have

$$\delta B = \frac{\delta(G(\eta)\psi)}{1+|\bar{\nabla}\eta|^2} + \frac{\delta(\bar{\nabla}\psi \cdot \bar{\nabla}\eta)}{1+|\bar{\nabla}\eta|^2} + (G(\eta)\psi + \bar{\nabla}\psi \cdot \bar{\nabla}\eta) \delta \left( (1+|\bar{\nabla}\eta|^2)^{-1} \right).$$

From  $\delta(G(\eta)\psi) \in H^{s_0-1}$ , the first term on the right hand side belongs to  $H^{s_0-1}$ , while the other terms belong to the same space  $H^{s_0-1}$  due to Lemma 4.3.24 and Corollary 4.C.9, which completes the proof for  $\delta B$ . As for  $\delta V$ , the formula (4.3.47) implies

$$\delta V = \delta(\bar{\nabla}\psi) + \delta B \bar{\nabla}\eta + B \delta(\bar{\nabla}\eta).$$

The desired estimate then follows from that of  $\delta B$  and Lemma 4.3.24. □

**Lemma 4.3.26.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$  and  $\delta \eta \in H^{s_0+\frac{1}{2}}$  with  $s > 3$  and  $\frac{3}{2} < s_0 \leq s$ . Let  $a = a^{(m)} + a^{(m-1)}$  be a symbol with  $m \in \mathbb{R}$  and*

$$a^{(m)} = F(\eta, \nabla_{\theta,z}\eta; \xi), \quad a^{(m-1)} = \sum_{|\alpha| \leq 2} G_\alpha(\eta, \nabla_{\theta,z}\eta; \xi) \partial_{\theta,z}^\alpha \eta,$$

where  $F(x, u, v; \xi)$  and  $G_\alpha(x, u, v; \xi)$ 's are smooth function of order  $m$  and  $m-1$  in  $\xi$ , respectively. Namely, for all  $\beta \in \mathbb{N}^3$  and  $\gamma \in \mathbb{N}^2$

$$\sup_{x,u,v} \left| \partial_{x,u,v}^\beta \partial_\xi^\gamma F(x, u, v; \xi) \right| \lesssim \langle \xi \rangle^{m-|\gamma|}, \quad \sup_{x,u,v} \left| \partial_{x,u,v}^\beta \partial_\xi^\gamma G_\alpha(x, u, v; \xi) \right| \lesssim \langle \xi \rangle^{m-1-|\gamma|}.$$

Then  $T_a$  and  $T_{\delta a}$  are of order  $m$ .

*Proof.* Clearly  $a \in \Gamma_{3/2+}^m + \Gamma_{1/2+}^{m-1}$  and thus  $T_a$  is of order  $m$ . As for  $T_{\delta a}$ , a direct calculus gives that

$$\delta a = \delta(F(\eta, \nabla_w \eta; \xi)) + \sum_{|\alpha| \leq 2} \delta(G_\alpha(\eta, \nabla_w \eta; \xi)) \partial_w^\alpha \eta + \sum_{|\alpha| \leq 2} G_\alpha(\eta, \nabla_w \eta; \xi) \partial_w^\alpha \delta \eta.$$

By Lemma 4.3.24 and  $s > 3$ , all the terms on the right hand side belong to  $\Gamma_{0+}^m$  or  $\Gamma_{0+}^{m-1}$  except for those  $G_\alpha(\eta, \nabla_w \eta; \xi) \partial_{\theta, z}^\alpha \delta \eta$  with  $|\alpha| = 2$ . To deal with these terms, we observe that  $G_\alpha(\eta, \nabla_w \eta; \xi)$  has  $H^{s-\frac{1}{2}}$  regularity in  $w$  variable while the low regularity comes from  $\partial_{\theta, z}^\alpha \delta \eta \in H^{s_0-\frac{3}{2}}$ . By applying Corollary 4.C.9, it is easy to check that, for all  $|\xi| > \frac{1}{2}$  and  $\gamma \in \mathbb{N}^2$ ,

$$\|\partial_\xi^\gamma G_\alpha(\eta, \nabla_w \eta; \xi) \partial_{\theta, z}^\alpha \delta \eta\|_{H^{s_0-\frac{3}{2}}} \lesssim \|G_\alpha(\eta, \nabla_w \eta; \xi)\|_{H^{s-\frac{1}{2}}} \|\delta \eta\|_{H^{s_0+\frac{1}{2}}} \langle \xi \rangle^{m-1-|\gamma|}.$$

Then by Proposition 4.C.16, the associated paradifferential operator is of order  $m - 1$  when  $s_0 - \frac{3}{2} > 1$ , or

$$m - 1 + 1 - \left(s_0 - \frac{3}{2}\right) = m - \left(s_0 - \frac{3}{2}\right) < m,$$

when  $s_0 - \frac{3}{2} < 1$  (in the critical case  $s_0 - \frac{3}{2} = 1$ , one may replace  $s_0 - \frac{3}{2}$  by  $s_0 - \frac{3}{2} - \epsilon$ ), which completes the proof.  $\square$

Now we are ready to prove (4.3.97). As before, during this proof, we denote

$$(4.3.100) \quad u \sim v \Leftrightarrow \|u - v\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta \eta\|_{H^{s_0+\frac{1}{2}}}.$$

The main idea of the proof is to use (4.3.52) together with shape derivative formula (4.2.42) to obtain an explicit expression of  $\delta f_1$ , where most terms can be treated separately and the remaining ones (see (4.3.104)) will be proved to admit some cancellation thanks to an application of parilinearization (4.3.52) with  $\psi$  replaced by  $B\delta\eta$  and  $B$ .

From (4.3.52), the derivative of  $f_1$  in  $\eta$  reads

$$\begin{aligned} \delta f_1 &= \delta(G(\eta)\psi - T_\lambda U + T_V \cdot \bar{\nabla} \eta) \\ &= \delta G(\eta)\psi - T_{\delta\lambda} U - T_\lambda \delta U + T_{\delta V} \cdot \bar{\nabla} \eta + T_V \cdot \delta(\bar{\nabla} \eta) \\ &= \delta G(\eta)\psi - T_{\delta\lambda} U - T_\lambda \delta U + T_{\delta V} \cdot \bar{\nabla} \eta + T_V \cdot \left( -\frac{\eta_\theta}{\eta^2} \delta \eta, 0 \right) + T_V \cdot \bar{\nabla} \delta \eta. \end{aligned}$$

By (4.2.42) and definition (4.3.19) of  $U$ , the right hand side can be further written as

$$\begin{aligned} \delta f_1 &= -G(\eta)(B\delta\eta) - \nabla_w \cdot \left( \frac{V^\theta}{\eta} \delta \eta, V^z \delta \eta \right) - \frac{B\delta\eta}{\eta} - T_{\delta\lambda} U + T_\lambda \delta(T_B \eta) \\ &\quad + T_{\delta V} \cdot \bar{\nabla} \eta + T_V \cdot \left( -\frac{\eta_\theta}{\eta^2} \delta \eta, 0 \right) + T_V \cdot \bar{\nabla} \delta \eta \\ &= -G(\eta)(B\delta\eta) - \bar{\nabla} \cdot (V \delta \eta) + \frac{\eta_\theta}{\eta^2} V^\theta \delta \eta - \frac{B\delta\eta}{\eta} - T_{\delta\lambda} U + T_\lambda T_{\delta B} \eta + T_\lambda T_B \delta \eta \\ &\quad + T_{\delta V} \cdot \bar{\nabla} \eta + T_V \cdot \left( -\frac{\eta_\theta}{\eta^2} \delta \eta, 0 \right) + T_V \cdot \bar{\nabla} \delta \eta. \end{aligned}$$

**Lemma 4.3.27.** *Under the hypotheses of Lemma 4.3.23, we have*

$$(4.3.101) \quad \frac{\eta_\theta}{\eta^2} V^\theta \delta\eta, \frac{B\delta\eta}{\eta}, T_V \cdot \left( -\frac{\eta_\theta}{\eta^2} \delta\eta, 0 \right) \sim 0,$$

$$(4.3.102) \quad T_\lambda T_{\delta B} \eta, T_{\delta V} \cdot \bar{\nabla} \eta \sim 0,$$

$$(4.3.103) \quad T_{\delta\lambda} U \sim 0,$$

in the sense of (4.3.100)

*Proof of Lemma 4.3.27.* The proof of (4.3.101) is direct. Since  $B, V \in H^{s-1}$ , we are able to deduce the following estimates from Proposition 4.C.8, Corollary 4.C.9, and Proposition 4.C.22,

$$\begin{aligned} \left\| \frac{\eta_\theta}{\eta^2} V^\theta \delta\eta \right\|_{H^{s_0+\frac{1}{2}}} &\lesssim \left\| \frac{\eta_\theta}{\eta^2} V^\theta \right\|_{H^{s-1}} \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ \left\| \frac{B\delta\eta}{\eta} \right\|_{H^{s_0+\frac{1}{2}}} &\lesssim \left\| \frac{B}{\eta} \right\|_{H^{s-1}} \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ \left\| T_V \cdot \left( -\frac{\eta_\theta}{\eta^2} \delta\eta, 0 \right) \right\|_{H^{s_0+\frac{1}{2}}} &\lesssim \|V\|_{H^{1+}} \left\| \frac{\eta_\theta}{\eta^2} \delta\eta \right\|_{H^{s_0+\frac{1}{2}}} \lesssim \|V\|_{H^{1+}} \left\| \frac{\eta_\theta}{\eta^2} \right\|_{H^{s_0+\frac{1}{2}}} \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

To prove (4.3.102), we recall that, due to Lemma 4.3.25,  $\delta B, \delta V \in H^{s_0-1}$ . As a result,

$$\begin{aligned} \|T_\lambda T_{\delta B} \eta\|_{H^{s_0+\frac{1}{2}}} &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|T_{\delta B} \eta\|_{H^{s_0+\frac{3}{2}}} \lesssim \|\delta B\|_{H^{s_0-1}} \|\eta\|_{H^{\max(\frac{7}{2}, s_0+\frac{3}{2})}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ \|T_{\delta V} \cdot \bar{\nabla} \eta\|_{H^{s_0+\frac{1}{2}}} &\lesssim \|\delta V\|_{H^{s_0-1}} \|\bar{\nabla} \eta\|_{H^{\max(\frac{5}{2}, s_0+\frac{1}{2})+}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

As for the last equivalence (4.3.103), it is no more than a consequence of  $U \in H^s \subset H^{s_0+\frac{3}{2}}$  and  $\lambda \in \Sigma^1$  with Lemma 4.3.26.  $\square$

It remains to prove (4.3.97), namely that

$$(4.3.104) \quad \delta f_1 \sim -G(\eta) (B\delta\eta) - \bar{\nabla} \cdot (V\delta\eta) + T_\lambda T_B \delta\eta + T_V \cdot \bar{\nabla} \delta\eta$$

is equivalent to zero in the sense of (4.3.100). Note that  $\delta\eta$  is assumed to be in  $H^{s+\frac{1}{2}}$ , permitting us to apply the parilinearization of Dirichlet-to-Neumann operator (Proposition 4.3.17). More precisely, we have

$$\begin{aligned} &\left\| G(\eta) (B\delta\eta) - T_\lambda (B\delta\eta - T_{\mathcal{B}(\eta)(B\delta\eta)} \eta) + T_{\mathcal{V}(\eta)(B\delta\eta)} \cdot \bar{\nabla} \eta \right\|_{H^{s_0+\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|B\delta\eta\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

Recall that linear operators  $\mathcal{B}(\eta)$  and  $\mathcal{V}(\eta)$  are defined in (4.3.49) with estimate (4.3.50). Then an application of Lemma 4.3.16, together with  $B\delta\eta \in H^{s_0+\frac{1}{2}}$ , gives the following estimates

$$\begin{aligned} \|T_{\mathcal{V}(\eta)(B\delta\eta)} \cdot \bar{\nabla}\eta\|_{H^{s_0+\frac{1}{2}}} &\leq \| \mathcal{V}(\eta)(B\delta\eta) \|_{H^{1+}} \| \bar{\nabla}\eta \|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ \|T_{\mathcal{B}(\eta)(B\delta\eta)}\eta\|_{H^{s_0+\frac{3}{2}}} &\leq \| \mathcal{B}(\eta)(B\delta\eta) \|_{H^{s_0-\frac{1}{2}}} \|\eta\|_{H^{s_0+\frac{3}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

Therefore, the following paradifferential calculus holds :

$$\begin{aligned} \delta f_1 &\sim -T_\lambda(B\delta\eta) + T_\lambda T_B \delta\eta - (\bar{\nabla} \cdot V)\delta\eta - V \cdot \bar{\nabla}\delta\eta + T_V \cdot \bar{\nabla}\delta\eta \\ &= -T_\lambda(T_{\delta\eta}B + R(B, \delta\eta)) - (\bar{\nabla} \cdot V)\delta\eta - (T_{\bar{\nabla}\delta\eta} \cdot V + R(V, \bar{\nabla}\delta\eta)) \\ &\sim -T_\lambda T_{\delta\eta}B + (G(\eta)B)\delta\eta, \end{aligned}$$

where the last equivalence is from Proposition 4.2.9 and the following estimate (recall that we assume  $s > s_0 + 3/2$ )

$$\begin{aligned} &\|R(B, \delta\eta)\|_{H^{s_0+\frac{3}{2}}} + \|T_{\bar{\nabla}\delta\eta}V\|_{H^{s_0+\frac{1}{2}}} + \|R(V, \bar{\nabla}\delta\eta)\|_{H^{s_0+\frac{1}{2}}} \\ &\lesssim \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \|B\|_{H^{s-1}} + \|\bar{\nabla}\delta\eta\|_{H^{s_0-\frac{1}{2}}} \|V\|_{H^{s-1}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

Our problem is now reduced to

$$(G(\eta)B)\delta\eta - T_\lambda T_{\delta\eta}B \sim 0.$$

To prove this, we apply again Proposition 4.3.17 with  $\psi$  replaced by  $B \in H^{s-1}$ , namely

$$G(\eta)B \sim T_\lambda (B - T_{\mathcal{B}(\eta)B}\eta) - T_{\mathcal{V}(\eta)B} \cdot \bar{\nabla}\eta.$$

Recall that linear operators  $\mathcal{B}(\eta)$  and  $\mathcal{V}(\eta)$  are defined in (4.3.49) and the estimate (4.3.50) ensures that  $\mathcal{B}(\eta)B, \mathcal{V}(\eta)B \in H^{s-2}$ . Since  $\eta \in H_R^{s+\frac{1}{2}}$  and  $\bar{\nabla}\eta \in H^{s-\frac{1}{2}}$  due to Corollary 4.C.9, one can apply Proposition 4.C.8 to show that  $T_\lambda T_{\mathcal{B}(\eta)B}\eta$  and  $T_{\mathcal{V}(\eta)B} \cdot \bar{\nabla}\eta$  belong to  $H^{s-\frac{1}{2}}$ , which implies that

$$G(\eta)B \sim T_\lambda B,$$

since  $s - 1/2 > s_0 + 1/2$  (recall that in our assumption  $s > s_0 + 3/2$ ). Consequently,

$$(G(\eta)B)\delta\eta \sim (T_\lambda B)\delta\eta = T_{\delta\eta}T_\lambda B + T_{T_\lambda B}\delta\eta + R(T_\lambda B, \delta\eta) \sim T_{\delta\eta}T_\lambda B,$$

where the last equivalence is due to

$$\begin{aligned} \|T_{T_\lambda B}\delta\eta\|_{H^{s_0+\frac{1}{2}}} + \|R(T_\lambda B, \delta\eta)\|_{H^{s_0+\frac{1}{2}}} &\lesssim \|T_\lambda B\|_{H^{s-2}} \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|B\|_{H^{s-1}} \|\delta\eta\|_{H^{s_0+\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

Thus,

$$G(\eta)B\delta\eta - T_\lambda T_{\delta\eta}B \sim T_{\delta\eta}T_\lambda B - T_\lambda T_{\delta\eta}B = [T_{\delta\eta}, T_{\lambda^{(1)}}]B + [T_{\delta\eta}, T_{\lambda^{(0)}}]B.$$

Since  $\delta\eta \in H^{s_0+\frac{1}{2}} \subset \Gamma_{1+}^0$ ,  $\lambda^{(1)} \in \Gamma_{3/2+}^1$ , and  $\lambda^{(0)} \in \Gamma_{1/2+}^0$ , the commutator estimate (Corollary 4.C.19) implies that  $[T_{\delta\eta}, T_{\lambda^{(1)}}]$  and  $[T_{\delta\eta}, T_{\lambda^{(0)}}]$  are both of order less than 0, and the proof of (4.3.97) is completed by observing that  $B \in H^{s-1} \subset H^{s_0+\frac{1}{2}}$ .

### Derivative of $f_2$

The purpose of this part is to show that the derivative in  $\eta$  of  $f_2$  (defined by (4.3.67)) is bounded, namely

$$(4.3.105) \quad \|\delta f_2(\eta, \psi)\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

We first claim that this estimate implies the desired result (4.3.96). In fact, by (4.3.97) and the definition (4.3.70) of  $f$ , it suffices to show

$$(4.3.106) \quad \|\delta(T_B f_1 + f_2)\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}},$$

which can be reduced to

$$\|\delta(T_B f_1)\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}},$$

provided (4.3.105) is correct. In (4.3.97) and Lemma 4.3.25, we have seen  $\delta B \in H^{s_0-1}$  and  $\delta f_1 \in H^{s_0+\frac{1}{2}}$ , while  $B \in H^{s-1}$  and  $f_1 \in H^{s+\frac{1}{2}}$ . These estimates give

$$\begin{aligned} \|\delta(T_B f_1)\|_{H^{s_0}} &\leq \|T_{\delta B} f_1\|_{H^{s_0}} + \|T_B \delta f_1\|_{H^{s_0}} \\ &\lesssim \|\delta B\|_{H^{s_0-1}} \|f_1\|_{H^{\max(s_0, 2+)}} + \|B\|_{H^{1+}} \|\delta f_1\|_{H^{s_0}} \\ &\lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \end{aligned}$$

which completes the proof of (4.3.106).

As before, during the proof of (4.3.105), we denote

$$(4.3.107) \quad u \sim v \Leftrightarrow \|u - v\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

From definition (4.3.67), it suffices to prove that  $r_2$  (defined in (4.3.60)) and  $r_3$  (defined in (4.3.61)) are equivalent to zero. We shall begin with the estimate of  $r_2$ .

**Study of  $r_2$ .**

By (4.3.60),  $r_2$  reads

$$r_2 = N - T_V \cdot \bar{\nabla} \psi + T_B G(\eta) \psi + T_B T_V \cdot \bar{\nabla} \eta.$$

As in Section 4.3.4, we shall decompose it as a function of  $B$ ,  $V$ , and  $G(\eta)\psi$  and apply Proposition 4.2.8. In this part, the shape derivative formula (4.2.42) is not necessary, we only need the regularity of  $\delta(G(\eta)\psi)$  (see also Lemma 4.3.25).

Via the same calculus as in (4.3.72), we can reformulate  $r_2$  as

$$\begin{aligned} r_2 &= \frac{|V|^2 + B^2}{2} - BG(\eta)\psi - T_V \cdot (V + B\bar{\nabla}\eta) + T_B G(\eta)\psi + T_B T_V \cdot \bar{\nabla}\eta \\ &= T_B B - T_{G(\eta)\psi} B - T_V \cdot (B\bar{\nabla}\eta) + T_B T_V \cdot \bar{\nabla}\eta + R(V, V) + R(B, B) + R(B, G(\eta)\psi) \\ &= T_{V \cdot \bar{\nabla}\eta} B - T_V \cdot (B\bar{\nabla}\eta) + T_B T_V \cdot \bar{\nabla}\eta + R(V, V) + R(B, B) + R(B, G(\eta)\psi) \\ &= (T_{V \cdot \bar{\nabla}\eta} - T_V \cdot T_{\bar{\nabla}\eta}) B + [T_B, T_V] \cdot \bar{\nabla}\eta - T_V \cdot R(B, \bar{\nabla}\eta) \\ &\quad + R(V, V) + R(B, B) + R(B, G(\eta)\psi). \end{aligned}$$

And the desired result  $r_2 \sim 0$  is no more than a consequence of the following lemma :

**Lemma 4.3.28.** *Under the hypotheses of Lemma 4.3.23, we have the following equivalences in the sense of (4.3.107),*

$$(4.3.108) \quad \delta \left( (T_{V \cdot \bar{\nabla}\eta} - T_V \cdot T_{\bar{\nabla}\eta}) B \right) \sim 0,$$

$$(4.3.109) \quad \delta \left( [T_B, T_V] \cdot \bar{\nabla}\eta \right) \sim 0,$$

$$(4.3.110) \quad \delta \left( T_V \cdot R(B, \bar{\nabla}\eta) \right) \sim 0,$$

$$(4.3.111) \quad \delta \left( R(V, V) + R(B, B) + R(B, G(\eta)\psi) \right) \sim 0.$$

*Proof.* The left hand side of (4.3.108) is equal to

$$\begin{aligned} \delta \left( (T_{V \cdot \bar{\nabla}\eta} - T_V \cdot T_{\bar{\nabla}\eta}) B \right) &= T_{\delta V \cdot \bar{\nabla}\eta} B - T_{\delta V} \cdot T_{\bar{\nabla}\eta} B + T_{V \cdot \delta(\bar{\nabla}\eta)} B - T_V \cdot T_{\delta(\bar{\nabla}\eta)} B \\ &\quad + (T_{V \cdot \bar{\nabla}\eta} - T_V \cdot T_{\bar{\nabla}\eta}) \delta B, \end{aligned}$$

where each term on the right hand side is equivalent to zero. In fact, since  $\delta B, \delta V \in H^{s_0-1}$  (see Lemma 4.3.25) and  $\delta(\bar{\nabla}\eta) \in H^{s_0-\frac{1}{2}}$  (see Lemma 4.3.24), one may obtain from Proposition 4.C.18, 4.C.8, and Corollary 4.C.9 that

$$\begin{aligned} &\|T_{\delta V \cdot \bar{\nabla}\eta} B\|_{H^{s_0}} + \|T_{\delta V} \cdot T_{\bar{\nabla}\eta} B\|_{H^{s_0}} \\ &\lesssim \|\delta V \cdot \bar{\nabla}\eta\|_{H^{s_0-1}} \|B\|_{H^{\max(s_0, 2+)}} + \|\delta V\|_{H^{s_0-1}} \|\bar{\nabla}\eta\|_{H^{1+}} \|B\|_{H^{\max(s_0, 2+)}} \\ &\lesssim \|\delta V\|_{H^{s_0-1}} \|\bar{\nabla}\eta\|_{H^{\max(1+, s_0-1)}} \|B\|_{H^{\max(s_0, 2+)}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ &\|T_{V \cdot \delta(\bar{\nabla}\eta)} B\|_{H^{s_0}} + \|T_V \cdot T_{\delta(\bar{\nabla}\eta)} B\|_{H^{s_0}} \\ &\lesssim \|V \cdot \delta(\bar{\nabla}\eta)\|_{H^{1+}} \|B\|_{H^{s_0}} + \|V\|_{H^{1+}} \|\delta(\bar{\nabla}\eta)\|_{H^{1+}} \|B\|_{H^{s_0}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|V\|_{H^{1+}} \|\delta(\bar{\nabla}\eta)\|_{H^{1+}} \|B\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\
&\| (T_V \cdot \bar{\nabla}\eta - T_V \cdot T_{\bar{\nabla}\eta}) \delta B \|_{H^{s_0}} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta B\|_{H^{s_0-1}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.
\end{aligned}$$

Recall that the regularity of  $B$  and  $V$  has been studied in Lemma 4.3.16.

The proof of (4.3.109) is similar. We first write its left hand side as

$$\delta([T_B, T_V] \cdot \bar{\nabla}\eta) = [T_{\delta B}, T_V] \cdot \bar{\nabla}\eta + [T_B, T_{\delta V}] \cdot \bar{\nabla}\eta + [T_B, T_V] \cdot \delta(\bar{\nabla}\eta),$$

where we shall apply commutator estimate (Corollary 4.C.19) only for the last term on the right hand side.

$$\begin{aligned}
&\|[T_{\delta B}, T_V] \cdot \bar{\nabla}\eta\|_{H^{s_0}} + \|[T_B, T_{\delta V}] \cdot \bar{\nabla}\eta\|_{H^{s_0}} \\
&\leq \|T_{\delta B} T_V \cdot \bar{\nabla}\eta\|_{H^{s_0}} + \|T_V \cdot T_{\delta B} \bar{\nabla}\eta\|_{H^{s_0}} + \|T_B T_{\delta V} \cdot \bar{\nabla}\eta\|_{H^{s_0}} + \|T_{\delta V} \cdot T_B \bar{\nabla}\eta\|_{H^{s_0}} \\
&\lesssim (\|B\|_{H^{1+}} + \|V\|_{H^{1+}}) (\|\delta B\|_{H^{s_0-1}} + \|\delta V\|_{H^{s_0-1}}) \|\bar{\nabla}\eta\|_{H^{\max(s_0, 2+)}} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\
&\|[T_B, T_V] \cdot \delta(\bar{\nabla}\eta)\|_{H^{s_0}} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta(\bar{\nabla}\eta)\|_{H^{s_0-\frac{1}{2}}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.
\end{aligned}$$

As for the estimate (4.3.110), we shall apply Proposition 4.C.8.

$$\begin{aligned}
&\|\delta(T_V \cdot R(B, \bar{\nabla}\eta))\|_{H^{s_0}} \\
&\leq \|T_{\delta V} \cdot R(B, \bar{\nabla}\eta)\|_{H^{s_0}} + \|T_V \cdot R(\delta B, \bar{\nabla}\eta)\|_{H^{s_0}} + \|T_V \cdot R(B, \delta(\bar{\nabla}\eta))\|_{H^{s_0}} \\
&\lesssim \|\delta V\|_{H^{s_0-1}} \|R(B, \bar{\nabla}\eta)\|_{H^{\max(s_0, 2+)}} + \|V\|_{H^{1+}} \|R(\delta B, \bar{\nabla}\eta)\|_{H^{s_0}} \\
&\quad + \|V\|_{H^{1+}} \|R(B, \delta(\bar{\nabla}\eta))\|_{H^{s_0}} \\
&\lesssim \|\delta V\|_{H^{s_0-1}} \|B\|_{H^{s-1}} \|\bar{\nabla}\eta\|_{H^{s-\frac{1}{2}}} + \|V\|_{H^{1+}} \|\delta B\|_{H^{s_0-1}} \|\bar{\nabla}\eta\|_{H^{2+}} \\
&\quad + \|V\|_{H^{1+}} \|B\|_{H^{\frac{3}{2}+}} \|\delta(\bar{\nabla}\eta)\|_{H^{s_0-\frac{1}{2}}} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.
\end{aligned}$$

The last equivalence (4.3.111) is merely a consequence of Proposition 4.C.8.

$$\begin{aligned}
&\|\delta(R(V, V) + R(B, B) + R(B, G(\eta)\psi))\|_{H^{s_0}} \\
&\leq \|R(\delta V, V)\|_{H^{s_0}} + \|R(\delta B, B)\|_{H^{s_0}} + \|R(\delta B, G(\eta)\psi)\|_{H^{s_0}} + \|R(B, \delta(G(\eta)\psi))\|_{H^{s_0}} \\
&\lesssim \|\delta V\|_{H^{s_0-1}} \|V\|_{H^{2+}} + \|\delta B\|_{H^{s_0-1}} \|B\|_{H^{2+}} + \|\delta B\|_{H^{s_0-1}} \|G(\eta)\psi\|_{H^{2+}} \\
&\quad + \|B\|_{H^{2+}} \|\delta(G(\eta)\psi)\|_{H^{s_0-1}} \\
&\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}}, \|\psi\|_{H^s} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}},
\end{aligned}$$

where we use the fact that  $G(\eta)\psi \in H^{s-1}$  and  $\delta(G(\eta)\psi) \in H^{s_0-1}$  due to Proposition 4.2.8.  $\square$

**Study of  $r_3$ .**

In the end of Section 4.3.4, we have obtained an explicit formula (4.3.88) for  $r_3$ . Then it suffices to show that the derivative-in- $\eta$  of the right hand side of (4.3.88) is equivalent to zero in the sense of (4.3.107), which can be reduced to the following lemma.

**Lemma 4.3.29.** *Under the hypotheses of Lemma 4.3.23, we have the following equivalences in the sense of (4.3.107),*

$$(4.3.112) \quad \delta \left( F(\eta, \nabla_w \eta) - T_{\nabla_{x,u,v} F(\eta, \nabla_w \eta)} \cdot (\eta, \nabla_w \eta) \right) \sim 0,$$

$$(4.3.113) \quad \delta \left( T_{\eta_{jk}} \left( G^{jk}(\eta, \nabla_w \eta) - T_{\nabla_{x,u,v} G^{jk}(\eta, \nabla_w \eta)} \cdot (\eta, \nabla_w \eta) \right) \right) \sim 0,$$

$$(4.3.114) \quad \delta \left( T_{F_x(\eta, \nabla_w \eta)} \eta \right) \sim 0,$$

$$(4.3.115) \quad \delta \left( T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta \right), \delta \left( R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) \right) \sim 0,$$

$$(4.3.116) \quad \delta \left( \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \eta \right) \sim 0.$$

Recall that  $F, G^{jk}$  are smooth functions defined in (4.3.78).

*Proof of (4.3.114)-(4.3.116).* The proof of these equivalences are similar to the study of  $r_2$ . Due to Lemma 4.3.24, the composition of smooth functions with  $(\eta, \nabla_w \eta)$  belongs to  $H^{s-\frac{1}{2}}$  and their derivative in  $\eta$  lie in  $H^{s_0-\frac{1}{2}}$ . Then (4.3.114) follows from

$$\begin{aligned} \|\delta \left( T_{F_x(\eta, \nabla_w \eta)} \eta \right)\|_{H^{s_0}} &\leq \|T_{\delta(F_x(\eta, \nabla_w \eta))} \eta\|_{H^{s_0}} + \|T_{F_x(\eta, \nabla_w \eta)} \delta \eta\|_{H^{s_0}} \\ &\lesssim \|\delta \left( F_x(\eta, \nabla_w \eta) \right)\|_{H^{1+}} \|\eta\|_{H^{s_0}} + \|F_x(\eta, \nabla_w \eta)\|_{H^{1+}} \|\delta \eta\|_{H^{s_0}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta \eta\|_{H^{s_0+\frac{1}{2}}}, \end{aligned}$$

while (4.3.115) is a consequence of

$$\begin{aligned} &\|\delta \left( T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta \right)\|_{H^{s_0}} + \|\delta \left( R \left( \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right) \right)\|_{H^{s_0}} \\ &\leq \|T_{\delta \eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \eta\|_{H^{s_0}} + \|T_{\eta_{jk}} T_{\delta(G_x^{jk}(\eta, \nabla_w \eta))} \eta\|_{H^{s_0}} + \|T_{\eta_{jk}} T_{G_x^{jk}(\eta, \nabla_w \eta)} \delta \eta\|_{H^{s_0}} \\ &\quad + \|R \left( \delta \eta_{jk}, G^{jk}(\eta, \nabla_w \eta) \right)\|_{H^{s_0}} + \|R \left( \eta_{jk}, \delta \left( G^{jk}(\eta, \nabla_w \eta) \right) \right)\|_{H^{s_0}} \\ &\lesssim \|\delta \eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \|G_x^{jk}(\eta, \nabla_w \eta)\|_{H^{1+}} \|\eta\|_{H^{\max(\frac{5}{2}, s_0)}} + \|\eta_{jk}\|_{H^{1+}} \|\delta \left( G_x^{jk}(\eta, \nabla_w \eta) \right)\|_{H^{1+}} \|\eta\|_{H^{s_0}} \\ &\quad + \|\eta_{jk}\|_{H^{1+}} \|G_x^{jk}(\eta, \nabla_w \eta)\|_{H^{1+}} \|\delta \eta\|_{H^{s_0}} + \|\delta \eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \|G_x^{jk}(\eta, \nabla_w \eta)\|_{H^{\frac{5}{2}+}} \\ &\quad + \|\eta_{jk}\|_{H^{\frac{3}{2}+}} \|\delta \left( G_x^{jk}(\eta, \nabla_w \eta) \right)\|_{H^{s_0-\frac{1}{2}}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta \eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

As for (4.3.116), when the derivative  $\delta$  acts on  $\nabla_w \eta$ , by Proposition 4.C.18, we have

$$\left\| \left( T_{\eta_{jk}} T_{(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} - T_{\eta_{jk}(\nabla_{u,v} G^{jk})(\eta, \nabla_w \eta)} \right) \cdot \nabla_w \delta \eta \right\|_{H^{s_0}}$$

$$\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{s_0-\frac{1}{2}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}},$$

since the symbols  $\eta_{jk}$  and  $(\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta)$  belong to  $H^{s-\frac{3}{2}} \subset \Gamma_{3/2+}^0$ . When  $\delta$  acts on the symbols, each terms are equivalent to zero,

$$\begin{aligned} \|T_{\delta\eta_{jk}}T_{(\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta)} \cdot \nabla_w\eta\|_{H^{s_0}} &\lesssim \|\delta\eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \left\| (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right\|_{H^{1+}} \|\nabla_w\eta\|_{H^{\max(\frac{5}{2}+, s_0)}}, \\ \|T_{\eta_{jk}}T_{\delta((\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta))} \cdot \nabla_w\eta\|_{H^{s_0}} &\lesssim \|\eta_{jk}\|_{H^{1+}} \|\delta \left( (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right)\|_{H^{s_0-\frac{1}{2}}} \|\nabla_w\eta\|_{H^{s_0}}, \\ \|T_{\delta\eta_{jk}}T_{(\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta)} \cdot \nabla_w\eta\|_{H^{s_0}} &\lesssim \|\delta\eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \left\| (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right\|_{H^{s_0-\frac{3}{2}}} \|\nabla_w\eta\|_{H^{\max(\frac{5}{2}+, s_0)}} \\ &\leq \|\delta\eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \left\| (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right\|_{H^{s-1}} \|\nabla_w\eta\|_{H^{\max(\frac{5}{2}+, s_0)}}, \\ \|T_{\eta_{jk}\delta((\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta))} \cdot \nabla_w\eta\|_{H^{s_0}} &\lesssim \|\eta_{jk}\delta \left( (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right)\|_{H^{s_0-\frac{1}{2}}} \|\nabla_w\eta\|_{H^{s_0}} \\ &\leq \|\eta_{jk}\|_{H^{s-\frac{3}{2}}} \|\delta \left( (\nabla_{u,v}G^{jk})(\eta, \nabla_w\eta) \right)\|_{H^{s_0-\frac{1}{2}}} \|\nabla_w\eta\|_{H^{s_0}}, \end{aligned}$$

the right hand side of which are all bounded by  $C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}$ .  $\square$

To prove (4.3.112) and (4.3.113), we need the following lemma,

**Lemma 4.3.30.** *Let  $\eta \in H_R^{s+\frac{1}{2}}$  and  $\delta\eta \in H^{s_0}$  with  $s > 3$  and  $\frac{3}{2} < s_0 < s - \frac{3}{2}$ . For all smooth function  $F = F(x, u, v)$ , we have*

$$(4.3.117) \quad \|\delta(F(\eta, \nabla_w\eta) - T_{(\nabla_{x,u,v}F)(\eta, \nabla_w\eta)} \cdot (\eta, \nabla_w\eta))\|_{H^{s_0}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}.$$

Recall that  $\delta$  stands for the derivation in  $\eta$ .

*Proof.* The function on the left hand side of (4.3.117) reads

$$(\nabla_{x,u,v}F)(\eta, \nabla_w\eta) \cdot (\delta\eta, \nabla_w\delta\eta) - T_{\delta((\nabla_{x,u,v}F)(\eta, \nabla_w\eta))} \cdot (\eta, \nabla_w\eta) - T_{(\nabla_{x,u,v}F)(\eta, \nabla_w\eta)} \cdot (\delta\eta, \nabla_w\delta\eta).$$

Recall that Lemma 4.3.24 ensures that  $\delta((\nabla_{x,u,v}F)(\eta, \nabla_w\eta)) \in H^{s_0-\frac{1}{2}}$ . Then Proposition 4.C.8 implies

$$\begin{aligned} \|T_{\delta((\nabla_{x,u,v}F)(\eta, \nabla_w\eta))} \cdot (\eta, \nabla_w\eta)\|_{H^{s_0}} &\lesssim \|\delta((\nabla_{x,u,v}F)(\eta, \nabla_w\eta))\|_{H^{1+}} \|(\eta, \nabla_w\eta)\|_{H^{s_0}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

The remaining term

$$\begin{aligned} &(\nabla_{x,u,v}F)(\eta, \nabla_w\eta) \cdot (\delta\eta, \nabla_w\delta\eta) - T_{(\nabla_{x,u,v}F)(\eta, \nabla_w\eta)} \cdot (\delta\eta, \nabla_w\delta\eta) \\ &= T_{(\delta\eta, \nabla_w\delta\eta)} \cdot (\nabla_{x,u,v}F)(\eta, \nabla_w\eta) + R((\nabla_{x,u,v}F)(\eta, \nabla_w\eta), (\delta\eta, \nabla_w\delta\eta)), \end{aligned}$$

which, due to Proposition 4.C.8, can be bounded by

$$\|T_{(\delta\eta, \nabla_w\delta\eta)} \cdot (\nabla_{x,u,v}F)(\eta, \nabla_w\eta)\|_{H^{s_0}} \lesssim \|(\delta\eta, \nabla_w\delta\eta)\|_{H^{s_0-\frac{1}{2}}} \|(\nabla_{x,u,v}F)(\eta, \nabla_w\eta)\|_{H^{s_0}}$$

$$\begin{aligned} &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}, \\ \|R((\nabla_{x,u,v}F)(\eta, \nabla_w\eta), (\delta\eta, \nabla_w\delta\eta))\|_{H^{s_0}} &\lesssim \|(\delta\eta, \nabla_w\delta\eta)\|_{H^{s_0-\frac{1}{2}}} \|(\nabla_{x,u,v}F)(\eta, \nabla_w\eta)\|_{H^{\frac{3}{2}+}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

□

*Proof of (4.3.112) and (4.3.113).* (4.3.112) is a direct consequence of Lemma 4.3.30. For (4.3.113), one may rewrite its left hand side as

$$\begin{aligned} &T_{\delta\eta_{jk}} \left( G^{jk}(\eta, \nabla_w\eta) - T_{\nabla_{x,u,v}} G^{jk}(\eta, \nabla_w\eta) \cdot (\eta, \nabla_w\eta) \right) \\ &+ T_{\eta_{jk}} \delta \left( \left( G^{jk}(\eta, \nabla_w\eta) - T_{\nabla_{x,u,v}} G^{jk}(\eta, \nabla_w\eta) \cdot (\eta, \nabla_w\eta) \right) \right), \end{aligned}$$

where the second term is equivalent to zero due to Lemma 4.3.30 and Proposition 4.C.8, while the first term can be estimated via (4.C.9) and (4.C.19),

$$\begin{aligned} &\|T_{\delta\eta_{jk}} \left( G^{jk}(\eta, \nabla_w\eta) - T_{\nabla_{x,u,v}} G^{jk}(\eta, \nabla_w\eta) \cdot (\eta, \nabla_w\eta) \right)\|_{H^{s_0}} \\ &\lesssim \|\delta\eta_{jk}\|_{H^{s_0-\frac{3}{2}}} \|G^{jk}(\eta, \nabla_w\eta) - T_{\nabla_{x,u,v}} G^{jk}(\eta, \nabla_w\eta) \cdot (\eta, \nabla_w\eta)\|_{H^{\max(\frac{5}{2}, s_0)}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}}} \right) \|\delta\eta\|_{H^{s_0+\frac{1}{2}}}. \end{aligned}$$

□

## 4.4 . Symmetrization of the system

In previous section, the main system (4.1.18) has been reformulated into parilinear form (4.3.68), which we recall here

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} + (T_V \cdot \bar{\nabla} + \mathcal{L}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f,$$

where

$$\mathcal{L} = \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}.$$

The goal of this section is to check that this system is symmetrizable. Namely, the operator  $T_V \cdot \bar{\nabla} + \mathcal{L}$  will be shown to be anti-self-adjoint, up to remainders with proper regularity. Under the hypotheses of Theorem 4.1.2,  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  and thus the vector  $V$  and  $\eta$  has Lipschitz regularity, which guarantees that  $T_V \cdot \bar{\nabla}$  is anti-self-adjoint up to a remainder of order 0 (see Lemma 4.5.14). The main difficulty is to prove that  $\mathcal{L}$  is symmetrizable. In fact, the matrix of principal symbols associated to  $\mathcal{L}$  reads

$$\begin{pmatrix} 0 & -\lambda^{(1)} \\ \sigma\mu^{(2)} & 0 \end{pmatrix},$$

where  $\lambda^{(1)}, \mu^{(2)}$  are real and elliptic with  $\sigma > 0$ . When  $\eta$  (the coefficients of  $\lambda, \mu$ ) is regular enough, a simple calculation of matrices gives the desired symmetrization

$$\begin{pmatrix} \sqrt{\sigma\mu^{(2)}/\lambda^{(1)}} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\sqrt{\sigma\lambda^{(1)}\mu^{(2)}} \\ \sqrt{\sigma\lambda^{(1)}\mu^{(2)}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma\mu^{(2)}/\lambda^{(1)}} & 0 \\ 0 & 1 \end{pmatrix}.$$

This heuristic calculation indicates us to try to construct symmetrizer  $S$  such that

$$S \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} S^{-1}$$

is anti-self-adjoint up to remainders. Meanwhile, one could also see the proper order of symbols, which is stated in detail in Proposition 4.4.1. To rigorously construct the symmetrizer  $S$ , we shall repeat the method in [ABZ11] with the symbolic calculus introduced in Section 4.3.1 and Appendix 4.C.3.

As in previous section, for  $a, b \in \Sigma^m$ , we say that  $T_a$  and  $T_b$  are equivalent and write  $T_a \approx T_b$  if their difference is of order  $m-3/2-$  with operator norm bounded by  $C(\|\eta\|_{H^{\frac{7}{2}+}})$ . Besides, for any symbol  $a \in \Sigma^m \subset \Gamma_{3/2+}^m + \Gamma_{1/2+}^{m-1}$ , we shall also use the notation  $a^{(m)} \in \Gamma_{3/2+}^m$  and  $a^{(m-1)} \in \Gamma_{1/2+}^{m-1}$  to present its principal and subprincipal component respectively (note that this decomposition is unique if we further assume that  $a^{(m)}$  is homogeneous in  $\xi$  of order  $m$ ).

**Proposition 4.4.1.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ . There exist elliptic symbols  $p \in \Sigma^{\frac{1}{2}}$  and  $q \in \Sigma^0$  such that*

$$(4.4.1) \quad S := \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}$$

*satisfies, for some elliptic  $\gamma \in \Sigma^{\frac{3}{2}}$ ,*

$$(4.4.2) \quad S \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & -T_p T_\lambda \\ \sigma T_q T_\mu & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -T_\gamma T_q \\ T_\gamma^* T_p & 0 \end{pmatrix} = \begin{pmatrix} 0 & -T_\gamma \\ T_\gamma^* & 0 \end{pmatrix} S$$

*in the sense that  $T_p T_\lambda \approx T_\gamma T_q$  and  $\sigma T_q T_\mu \approx T_\gamma^* T_p$ .*

*Furthermore, we have,*

$$(4.4.3) \quad T_p T_\lambda \approx T_\gamma T_q, \quad T_q T_\mu \approx T_\gamma T_p, \quad T_\gamma^* \approx T_\gamma,$$

*which is stronger than (4.4.2)*

The construction of  $p, q, \gamma$  from  $\lambda, \mu$  is exactly the same as in Section 4.2 of [ABZ11], which we recall here. To begin with, we investigate the conditions on  $\gamma$  such that  $T_\gamma^* \sim T_\gamma$ . By choosing

$$(4.4.4) \quad \text{Im } \gamma^{(3/2)} = 0,$$

one may apply Proposition 4.3.9 to deduce that

$$(4.4.5) \quad \operatorname{Im} \gamma^{(1/2)} = -\frac{1}{2} \partial_w \cdot \partial_\xi \gamma^{(3/2)}.$$

We shall next determine  $\operatorname{Re} \gamma^{(3/2)}$ ,  $\operatorname{Re} \gamma^{(1/2)}$  and  $q$  such that

$$(4.4.6) \quad T_q T_{\sigma\mu} T_\lambda \approx T_\gamma T_\gamma T_q, \quad \text{and} \quad T_p T_\lambda \approx T_\gamma T_q.$$

This will provide a solution to the first two equivalences in (4.4.3), plugging the last formula of (4.4.6) inside the first one and using that  $\lambda$  has a parametrix, since  $\lambda$  is elliptic (see Proposition 4.3.7). By applying symbolic calculus (4.3.11), the first equivalence of (4.4.6) follows from

$$q^\# (\sigma\mu^\# \lambda) = (\gamma^\# \gamma)^\# q.$$

By (4.3.13), we can write this equality as

$$(4.4.7) \quad \begin{aligned} & q^{(0)} (\sigma\mu^\# \lambda) + \partial_\xi q^{(0)} \cdot D_w \left( \sigma\mu^{(2)} \lambda^{(1)} \right) + q^{(-1)} (\sigma\mu^\# \lambda)^{(2)} \\ &= (\gamma^\# \gamma) q^{(0)} + \partial_\xi \left( \gamma^{(3/2)} \gamma^{(3/2)} \right) \cdot D_w q^{(0)} + (\gamma^\# \gamma)^{(2)} q^{(-1)}. \end{aligned}$$

By comparing the principal part of both sides (terms of degree 3 in  $\xi$ ), we are able to determine

$$(4.4.8) \quad \gamma^{(3/2)} = \sqrt{\sigma\mu^{(2)} \lambda^{(1)}}.$$

To deal with the subprincipal terms of (4.4.7), one may set

$$(4.4.9) \quad q^{(-1)} = 0$$

and (4.4.7) becomes

$$(4.4.10) \quad \begin{aligned} q^{(0)} (\sigma\mu^\# \lambda - \gamma^\# \gamma) &= \partial_\xi \left( \gamma^{(3/2)} \gamma^{(3/2)} \right) \cdot D_w q^{(0)} - \partial_\xi q^{(0)} \cdot D_w \left( \sigma\mu^{(2)} \lambda^{(1)} \right) \\ &= \partial_\xi \left( \sigma\mu^{(2)} \lambda^{(1)} \right) \cdot D_w q^{(0)} - \partial_\xi q^{(0)} \cdot D_w \left( \sigma\mu^{(2)} \lambda^{(1)} \right) \\ &= \frac{1}{i} \{ \sigma\mu^{(2)} \lambda^{(1)}, q^{(0)} \}, \end{aligned}$$

where  $\{ \cdot, \cdot \}$  is Poisson bracket. By comparing real and imaginary part of both sides of (4.4.10) and setting

$$(4.4.11) \quad \operatorname{Im} q^{(0)} = 0,$$

we have

$$(4.4.12) \quad \operatorname{Re} (\sigma\mu^\# \lambda - \gamma^\# \gamma) = 0, \quad q^{(0)} \operatorname{Im} (\sigma\mu^\# \lambda - \gamma^\# \gamma) = -\{ \sigma\mu^{(2)} \lambda^{(1)}, q^{(0)} \}.$$

From the first equation in (4.4.12), we are able to solve  $\operatorname{Re} \gamma^{(1/2)}$ ,

$$(4.4.13) \quad 2\gamma^{(3/2)} \operatorname{Re} \gamma^{(1/2)} = \sigma \left( \mu^{(2)} \operatorname{Re} \lambda^{(0)} + \operatorname{Re} \mu^{(1)} \lambda^{(1)} \right) = \sigma\mu^{(2)} \operatorname{Re} \lambda^{(0)},$$

where the first equality holds since  $\gamma^{(3/2)}$ ,  $\lambda^{(1)}$ , and  $\mu^{(2)}$  are real with (4.4.8), and the second follows from  $\operatorname{Re} \mu^{(1)} = 0$ , which can be seen from the second formula in (4.3.85). Then (4.4.13), together with (4.4.8) and (4.4.5), completes the definition of  $\gamma$ . Meanwhile, to solve the second equation in (4.4.12), we first calculate the coefficient before  $q^{(0)}$  by applying (4.3.59), (4.3.86), and (4.4.5),

$$\begin{aligned}
& \operatorname{Im}(\sigma\mu\sharp\lambda - \gamma\sharp\gamma) \\
&= -\sigma\partial_\xi\mu^{(2)} \cdot \partial_w\lambda^{(1)} + \sigma\operatorname{Im}\mu^{(1)}\lambda^{(1)} + \sigma\mu^{(2)}\operatorname{Im}\lambda^{(0)} \\
&\quad + \partial_\xi\gamma^{(3/2)} \cdot \partial_w\gamma^{(3/2)} - 2\gamma^{(3/2)}\operatorname{Im}\gamma^{(1/2)} \\
&= -\sigma\partial_\xi\mu^{(2)} \cdot \partial_w\lambda^{(1)} - \frac{\sigma}{2}\partial_w \cdot \partial_\xi\mu^{(2)}\lambda^{(1)} - \frac{\sigma}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi\mu^{(2)}\lambda^{(1)} - \frac{\sigma}{2}\mu^{(2)}\partial_w \cdot \partial_\xi\lambda^{(1)} \\
&\quad - \frac{\sigma}{2}\mu^{(2)}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi\lambda^{(1)} + \partial_\xi\gamma^{(3/2)} \cdot \partial_w\gamma^{(3/2)} + \gamma^{(3/2)}\partial_w \cdot \partial_\xi\gamma^{(3/2)} \\
&= -\frac{\sigma}{2}\partial_\xi\mu^{(2)} \cdot \partial_w\lambda^{(1)} - \left(\frac{\sigma}{2}\partial_\xi\mu^{(2)} \cdot \partial_w\lambda^{(1)} + \frac{\sigma}{2}\partial_w \cdot \partial_\xi\mu^{(2)}\lambda^{(1)} + \frac{\sigma}{2}\mu^{(2)}\partial_w \cdot \partial_\xi\lambda^{(1)}\right) \\
&\quad - \left(\frac{\sigma}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi\mu^{(2)}\lambda^{(1)} + \frac{\sigma}{2}\mu^{(2)}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi\lambda^{(1)}\right) + \left(\partial_\xi\gamma^{(3/2)} \cdot \partial_w\gamma^{(3/2)} + \gamma^{(3/2)}\partial_w \cdot \partial_\xi\gamma^{(3/2)}\right) \\
&= -\frac{\sigma}{2}\partial_\xi\mu^{(2)} \cdot \partial_w\lambda^{(1)} + \frac{\sigma}{2}\partial_w\mu^{(2)} \cdot \partial_\xi\lambda^{(1)} - \frac{\sigma}{2}\partial_w \cdot \partial_\xi(\mu^{(2)}\lambda^{(1)}) - \frac{\sigma}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi(\mu^{(2)}\lambda^{(1)}) \\
&\quad + \frac{1}{2}\partial_w \cdot \partial_\xi(\gamma^{(3/2)}\gamma^{(3/2)}) \\
&= \frac{\sigma}{2}\{\lambda^{(1)}, \mu^{(2)}\} - \frac{\sigma}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi(\mu^{(2)}\lambda^{(1)}) + \frac{1}{2}\partial_w \cdot \partial_\xi(\gamma^{(3/2)}\gamma^{(3/2)}) - \sigma\mu^{(2)}\lambda^{(1)} \\
&= \frac{\sigma}{2}\{\lambda^{(1)}, \mu^{(2)}\} - \frac{\sigma}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi(\mu^{(2)}\lambda^{(1)}),
\end{aligned}$$

where we apply (4.4.8) to obtain the last equality. The equation for  $q^{(0)}$  becomes

$$\frac{1}{2}q^{(0)} \left[ \{\lambda^{(1)}, \mu^{(2)}\} - \frac{\partial_w\eta}{\eta} \cdot \partial_\xi(\mu^{(2)}\lambda^{(1)}) \right] = -\{\mu^{(2)}\lambda^{(1)}, q^{(0)}\}.$$

From (4.3.54) and (4.3.64), we have  $\mu^{(2)} = a^2(\lambda^{(1)})^2$  with

$$(4.4.14) \quad a := \frac{1}{\sqrt{2}}(1 + |\bar{\nabla}\eta|^2)^{-\frac{3}{4}},$$

which is positive and independent of  $\xi$ . Furthermore, we look for  $q^{(0)}$  also independent of  $\xi$ . Then the equation for  $q^{(0)}$  can be written as

$$\frac{\partial_w q^{(0)}}{q^{(0)}} \cdot \partial_\xi\lambda^{(1)} = \frac{1}{2}\frac{\partial_w\eta}{\eta} \cdot \partial_\xi\lambda^{(1)} - \frac{1}{3}\frac{\partial_w a}{a} \cdot \partial_\xi\lambda^{(1)},$$

an obvious solution of which reads

$$(4.4.15) \quad q^{(0)} = \eta^{\frac{1}{2}}a^{-\frac{1}{3}} = 2^{\frac{1}{6}}\sqrt{\eta}(1 + |\bar{\nabla}\eta|^2)^{\frac{1}{4}}.$$

Clearly,  $q = q^{(0)}$  is elliptic.

Till now, we have constructed  $\gamma \in \Sigma^2$  and  $q \in \Sigma^0$  satisfying (4.4.6), according to our discussion before, it remains to construct  $p \in \Sigma^{\frac{1}{2}}$  with either  $T_p T_\lambda \approx T_\gamma T_q$  or  $T_q T_{\sigma\mu} \approx T_\gamma T_p$ . Here we focus on the former one. In fact, since  $\lambda \in \Sigma^1$  is elliptic, by Proposition 4.3.7, there exists  $\tilde{\lambda} \in \Sigma^{-1}$  such that  $T_\lambda T_{\tilde{\lambda}} \approx T_{\tilde{\lambda}} T_\lambda \approx id$ . Thus, by symbolic calculus (Proposition 4.3.4), one has

$$T_p T_\lambda \approx T_\gamma T_q \Leftrightarrow T_p T_\lambda T_{\tilde{\lambda}} \approx T_\gamma T_q T_{\tilde{\lambda}} \Leftrightarrow T_p \approx T_{(\gamma \# q) \# \tilde{\lambda}},$$

which gives a possible definition of  $p$  :

$$(4.4.16) \quad p := (\gamma \# q) \# \tilde{\lambda} \in \Sigma^{\frac{1}{2}},$$

whose principal symbol equals

$$(4.4.17) \quad p^{(1/2)} = \frac{\gamma^{(3/2)} q^{(0)}}{\lambda^{(1)}}.$$

Hence,  $p$  is elliptic.

From Proposition 4.3.7, the ellipticity of  $p, q$  implies that  $S$  has a parametrix  $\tilde{S}$ , in the sense that

$$(4.4.18) \quad S \tilde{S} \approx \tilde{S} S \approx id,$$

where

$$(4.4.19) \quad \tilde{S} := \begin{pmatrix} T_{\tilde{p}} & 0 \\ 0 & T_{\tilde{q}} \end{pmatrix}$$

with  $\tilde{p} \in \Sigma^{-1/2}$  and  $\tilde{q} \in \Sigma^0$  the elliptic symbols constructed by Proposition 4.3.7.

To end this section, we introduce some commutator estimates involving  $S$  (see also Section 2.4 of [ABZ14] for the case of planar water-wave).

**Proposition 4.4.2.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ . For all symbol  $a \in \Sigma^m$  with  $m \in \mathbb{R}$ , the commutator  $[T_a, \partial_t]$  is of order  $m$ , with*

$$(4.4.20) \quad \|[T_a, \partial_t]\|_{\mathcal{L}(H^r; H^{r-m})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right), \quad \forall r \in \mathbb{R}.$$

In particular, the commutator between  $S$  and  $\partial_t$  is equal to

$$[S, \partial_t] = - \begin{pmatrix} T_{\partial_t p} & 0 \\ 0 & T_{\partial_t q} \end{pmatrix},$$

where  $T_{\partial_t p}$  is of order  $\frac{1}{2}$  and  $T_{\partial_t q}$  is of order 0.

*Proof.* We intend to apply Lemma 4.3.26, which gives the invariance of order of  $T_a$  under derivative in  $\eta$  in a weaker space  $H^{s_0+\frac{1}{2}}$  with  $\frac{3}{2} < s_0 \leq s$ . By Leibniz rule, it suffices to check that  $\eta_t$  belongs to this space for some  $s_0 \in ]\frac{3}{2}, s]$ , which is a consequence of the first equation of (4.1.18) together with Corollary 4.2.4 (with  $s_0 = s$ ), namely

$$\eta_t = G(\eta)\psi \in H^{s-1}.$$

□

**Proposition 4.4.3.** *Let  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ . Let  $a = a^{(m)} + a^{(m-1)}$  be a symbol in  $\Gamma_{\frac{3}{2}+}^m + \Gamma_{\frac{1}{2}+}^{m-1}$  with  $m \in \mathbb{R}$  and*

$$\|\partial_\xi^\alpha a^{(m)}(\xi)\|_{C^{\frac{3}{2}+}} \langle \xi \rangle^{-m+|\alpha|} + \|\partial_\xi^\alpha a^{(m-1)}(\xi)\|_{C^{\frac{1}{2}+}} \langle \xi \rangle^{-m-1+|\alpha|} \lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right),$$

for all  $\alpha \in \mathbb{N}^2$  and  $\xi \in \mathbb{R}^2$ . Then we have, for all  $r \leq \min(s + m + \frac{1}{2}, s + \frac{3}{2}-)$ ,

$$(4.4.21) \quad \|[T_a, T_V \cdot \bar{\nabla}]\|_{\mathcal{L}(H^r; H^{r-m})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right).$$

In particular, we have, for all  $r \leq s + \frac{1}{2}$ ,

$$(4.4.22) \quad \|[T_p, T_V \cdot \bar{\nabla}]\|_{\mathcal{L}(H^r; H^{r-\frac{1}{2}})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right),$$

$$(4.4.23) \quad \|[T_q, T_V \cdot \bar{\nabla}]\|_{\mathcal{L}(H^r; H^r)} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right).$$

*Proof.* Let  $u$  be any function in  $H^r$  with  $r \leq s + m + \frac{1}{2}$ . One may write

$$\begin{aligned} [T_a, T_V \cdot \bar{\nabla}] u &= [T_a, T_{V^z i \xi_z}] u + T_a T_{V^\theta} (T_{\eta^{-1}} \partial_\theta u + T_{\partial_\theta u} \eta^{-1} + R(\eta^{-1}, \partial_\theta u)) \\ &\quad - T_{V^\theta} (T_{\eta^{-1}} \partial_\theta (T_a u) + T_{\partial_\theta (T_a u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (T_a u))) \\ &= [T_a, T_{V^z i \xi_z}] u + [T_a, T_{V^\theta} T_{\eta^{-1}}] \partial_\theta u + T_a T_{V^\theta} (T_{\partial_\theta u} \eta^{-1} + R(\eta^{-1}, \partial_\theta u)) \\ &\quad - T_{V^\theta} (T_{\eta^{-1}} T_{\partial_\theta u} + T_{\partial_\theta (T_a u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (T_a u))). \end{aligned}$$

We shall check that each term on the right hand side lies in  $H^{r-m}$ . The first term  $[T_a, T_{V^z i \xi_z}] u \in H^{r-m}$  is no more than a consequence of commutator estimate (Corollary 4.C.19). For the second term  $[T_a, T_{V^\theta} T_{\eta^{-1}}] \partial_\theta u$ , one may replace  $T_{V^\theta} T_{\eta^{-1}}$  by  $T_{\eta^{-1} V^\theta}$ , since their difference is of order  $-1-$  leading to an error term in  $H^{r-m+}$ , and apply again Corollary 4.C.19. The third term can be estimated via Proposition 4.C.8,

$$\begin{aligned} &\|T_a T_{V^\theta} (T_{\partial_\theta u} \eta^{-1} + R(\eta^{-1}, \partial_\theta u))\|_{H^{r-m}} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \|T_{\partial_\theta u} \eta^{-1} + R(\eta^{-1}, \partial_\theta u)\|_{H^r} \\ &\leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \|u\|_{H^r} \|\eta^{-1}\|_{H_R^{s+\frac{1}{2}-}} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \|u\|_{H^r}, \end{aligned}$$

which is bounded due to Proposition 4.C.22. To deal with the last term, we observe that  $T_a u \in H^{r-m}$ , and that  $T_{\partial_\theta a}$  has the same order as  $T_a$  due to Proposition 4.C.16 and the fact that  $\eta_\theta \in H^{s-\frac{1}{2}}$ . Then, again by Proposition 4.C.8 and 4.C.22, we have

$$\begin{aligned} & \|T_{V^\theta} (T_{\eta^{-1}} T_{\partial_\theta a} u + T_{\partial_\theta(T_a u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta(T_a u)))\|_{H^{r-m}} \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \|T_{\eta^{-1}} T_{\partial_\theta a} u + T_{\partial_\theta(T_a u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta(T_a u))\|_{H^{r-m}} \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \left( \|u\|_{H^r} + \|\partial_\theta(T_a u)\|_{H^{r-m-1}} \|\eta^{-1}\|_{H_R^{s+\frac{1}{2}-}} \right) \\ & \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}}, \|\psi\|_{H^{s-}} \right) \|u\|_{H^r}. \end{aligned}$$

Note that in the second inequality, we need the assumption  $r \leq \min(s + m + \frac{1}{2}, s + \frac{3}{2}-)$  to apply Proposition 4.C.8.  $\square$

## 4.5 . Cauchy problem

In this section, we shall prove Theorem 4.1.2 and 4.1.3. Recall that in Section 4.3, we have managed to reformulate the equation (4.1.18) as (4.3.68),

$$(\partial_t + T_V \cdot \bar{\nabla}) \begin{pmatrix} \eta \\ \psi \end{pmatrix} + \mathcal{L} \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f,$$

where the linear operator  $\mathcal{L}$  admits a symmetrizer  $S$  defined by (4.4.1) (see Proposition 4.4.1). Moreover, provided that  $(\eta, \psi) \in H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ ,  $f = f(\eta, \psi)$  belongs to  $H^{s+\frac{1}{2}} \times H^s$  and has local Lipschitz regularity from  $H_R^{s_0+\frac{1}{2}} \times H^{s_0}$  to  $H^{s_0+\frac{1}{2}} \times H^{s_0}$ , for all  $\frac{3}{2} < s_0 < s - \frac{3}{2}$  (see Proposition 4.3.22).

Based on the definition (4.3.69) of  $\mathcal{L}$ , we shall introduced a mollified version  $\mathcal{L}^\epsilon$  by inserting proper smoothing operators (see (4.5.9)). The resulting approximate system (4.5.10), which is nothing else than an ODE, has a unique solution  $(\eta^\epsilon, \psi^\epsilon)$  on the time interval  $[0, T_\epsilon]$  due to Cauchy-Lipschitz Theorem. Energy estimates (Proposition 4.5.5) guarantee that the lifespan  $T_\epsilon$  admits a uniform lower bound  $T$  (see Corollary 4.5.6), and the approximate solution  $(\eta^\epsilon, \psi^\epsilon)$  is bounded in  $H_R^{s+\frac{1}{2}} \times H^s$  with  $s > 3$ , and converges on  $[0, T]$  in a weaker norm  $H_R^{s_0+\frac{1}{2}} \times H^{s_0}$  where  $\frac{3}{2} < s_0 < s - \frac{3}{2}$ . Consequently,  $(\eta^\epsilon, \psi^\epsilon)$  tends to a solution  $(\eta, \psi)$  to (4.1.18) and the uniqueness follows from a similar argument used in the convergence of approximate solution. Note that, by classical interpolation argument, the limit of approximate solution has only  $L^\infty$  regularity in time, which can be optimized to  $C^0$  via nonlinear interpolation introduced in [Ala+24], which completes the proof of Theorem 4.1.2. As a by-product, we can also deduce the continuity of flow map, i.e. Theorem 4.1.3.

Before starting the proof, we introduce some notations and conventions used in this section,

**Definition 4.5.1.** For  $s \in \mathbb{R}$ ,  $R > 0$ , and  $T > 0$ ,  $\mathcal{X}_R^s$  denotes the collection of all  $(\eta, \psi)$  such that  $\psi \in H^s$  and  $\eta - R \in H^{s+\frac{1}{2}}$ , where we further assume that  $\eta$  satisfy hypotheses (H0) and (H1) (or (H1')). For all  $(\eta_1, \psi_1), (\eta_2, \psi_2) \in \mathcal{X}^s$ , we denote

$$(4.5.1) \quad d_{\mathcal{X}_R^s}((\eta_1, \psi_1), (\eta_2, \psi_2)) := \|\eta_1 - \eta_2\|_{H^{s+\frac{1}{2}}} + \|\psi_1 - \psi_2\|_{H^s}.$$

It is clear that  $(\mathcal{X}_R^s, d_{\mathcal{X}_R^s})$  is a complete metric space. Note that in periodic case (i.e. with hypothesis (H1')), the normalization  $R$  should be omitted and  $(\mathcal{X}^s, d_{\mathcal{X}^s})$  becomes a Banach space. Moreover, if  $(\eta, \psi)$  depends on time  $t \in [0, T[$ , we denote by  $L_T^\infty \mathcal{X}_R^s$  the space of  $L^\infty$ -functions with value in  $\mathcal{X}_R^s$  such that

$$\sup_{t \in [0, T[} \|(\eta(t) - R, \psi(t))\|_{H^{s+\frac{1}{2}} \times H^s} < +\infty.$$

Furthermore, we denote by  $C_T \mathcal{X}^s$  the subspace of  $L_T^\infty \mathcal{X}^s$  where  $(\eta(t), \psi(t))$  is continuous in time w.r.t. the distance (4.5.1).

In the sequel, we focus on the perturbative case (with hypothesis (H1)) and the proof for periodic case can be obtained simply by deleting all the normalization.

As in previous sections, for linear operators  $A, B$  of order  $m \in \mathbb{R}$ , we write

$$(4.5.2) \quad A \approx B \Leftrightarrow \begin{array}{l} A - B \text{ is of order } m - \frac{3}{2}, \\ \text{with operator norm bounded by } C \left( \|(\eta, \psi)\|_{H_R^{s+\frac{1}{2}} \times H^s} \right). \end{array}$$

### 4.5.1 . Construction of approximate solutions

Recall that  $\mathcal{L}$  is defined by (4.3.69),

$$\mathcal{L} := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}.$$

As in [ABZ11], we introduce the mollifier  $J_\epsilon := T_{j_\epsilon}$  ( $0 \leq \epsilon \ll 1$ ), where the symbol  $j_\epsilon$  is given by

$$(4.5.3) \quad j_\epsilon = j_\epsilon^{(0)} + j_\epsilon^{(-1)}, \quad j_\epsilon^{(0)} := \exp\left(-\epsilon \gamma^{(3/2)}\right), \quad j_\epsilon^{(-1)} := -\frac{i}{2} \partial_w \cdot \partial_\xi j_\epsilon^{(0)}.$$

Recall that  $\gamma^{(3/2)}$  is constructed in Proposition 4.4.1. As a consequence of Corollary 4.3.5 and 4.3.9, all the symbols constructed as above satisfy the following property.

**Lemma 4.5.2.** Given  $m \in \mathbb{R}$ , we denote by  $\mathcal{G}^m$  the collection of real functions  $F = F(\epsilon, \rho)$  which are smooth on  $\{\rho > 0\}$  with parameter  $\epsilon \in ]0, 1]$ , satisfying that, for all  $k \in \mathbb{N}$  and  $\rho > 0$ ,

$$(4.5.4) \quad \sup_{\epsilon \in ]0, 1]} \left| \partial_\rho^k F(\epsilon, \rho) \right| \lesssim \rho^{m-k}.$$

Let  $\eta \in L_T^\infty H_R^{s+\frac{1}{2}-}$  with  $s > 3$  and  $T > 0$ . For all  $F \in \mathcal{G}^m$ , we define symbol  $a_{F,\epsilon}$  as

$$(4.5.5) \quad a_{F,\epsilon} := a_{F,\epsilon}^{(3m/2)} + a_{F,\epsilon}^{(3m/2-1)}, \quad a_{F,\epsilon}^{(3m/2)} = F(\epsilon, \gamma^{(3/2)}), \quad a_{F,\epsilon}^{(3m/2-1)} = -\frac{i}{2} \partial_w \cdot \partial_\xi a_{F,\epsilon}^{(3m/2)}.$$

Then  $a_{F,\epsilon}$  belongs to  $\Gamma_{3/2+}^{3m/2} + \Gamma_{1/2+}^{3m/2-1}$  uniformly in  $\epsilon$  and  $t \in [0, T[$  with

$$(4.5.6) \quad T_{a_{F,\epsilon}} \approx T_{a_{F,\epsilon}}^*.$$

Moreover, for any  $F, G \in \mathcal{G}^m$ , we have

$$(4.5.7) \quad T_{a_{F,\epsilon}} T_{a_{G,\epsilon}} \approx T_{a_{G,\epsilon}} T_{a_{F,\epsilon}}.$$

*Proof.* The uniform-in- $(\epsilon, t)$  boundedness of  $a_{F,\epsilon}^{(3m/2)}$  and  $a_{F,\epsilon}^{(3m/2-1)}$  in  $\Gamma_{3/2+}^{3m/2}$  and  $\Gamma_{1/2+}^{3m/2-1}$ , respectively, can be directly checked from the definition (4.4.8) of  $\gamma^{(3/2)}$  and the condition (4.5.4) of  $F$ . Since  $F$  is real-valued,  $a_{F,\epsilon}^{(3m/2)}$  is also real. Then an application of Proposition 4.3.9 gives (4.5.6). To prove the last equivalence (4.5.7), we first reduce it to  $a_{F,\epsilon} \sharp a_{G,\epsilon} = a_{G,\epsilon} \sharp a_{F,\epsilon}$ , thanks to (4.3.11). In fact, by formula (4.3.13) and definition (4.5.5), we have

$$\begin{aligned} & a_{F,\epsilon} \sharp a_{G,\epsilon} - a_{G,\epsilon} \sharp a_{F,\epsilon} \\ &= \partial_\xi a_{F,\epsilon}^{(3m/2)} \cdot D_w a_{G,\epsilon}^{(3m/2)} - \partial_\xi a_{G,\epsilon}^{(3m/2)} \cdot D_w a_{F,\epsilon}^{(3m/2)} \\ &= F'(\epsilon, \gamma^{(3/2)}) G'(\epsilon, \gamma^{(3/2)}) \partial_\xi \gamma^{(3/2)} \cdot D_w \gamma^{(3/2)} - G'(\epsilon, \gamma^{(3/2)}) F'(\epsilon, \gamma^{(3/2)}) \partial_\xi \gamma^{(3/2)} \cdot D_w \gamma^{(3/2)} \\ &= 0. \end{aligned}$$

□

In particular, by choosing  $F(\epsilon, \rho) = \exp(-\epsilon\rho)$  and  $G(\epsilon, \rho) = \rho$ , we have the following property on  $j_\epsilon$  constructed in (4.5.3),

**Lemma 4.5.3.** *Let  $\eta \in L_T^\infty H_R^{s+\frac{1}{2}-}$  with  $s > 3$  and  $T > 0$ . Then the symbol  $j_\epsilon$  defined above is elliptic and belong to  $\Gamma_{3/2+}^0 + \Gamma_{1/2+}^{-1}$  uniformly in  $\epsilon \geq 0$  and  $t \in [0, T[$ . Moreover, we have*

$$(4.5.8) \quad J_\epsilon T_\gamma \approx T_\gamma J_\epsilon, \quad J_\epsilon^* \approx J_\epsilon,$$

uniformly in  $\epsilon, t$ .

Note that when  $\epsilon > 0$ ,  $j_\epsilon \in \Gamma_{3/2+}^m + \Gamma_{1/2+}^{m-1}$  for all  $m < 0$ , and as a consequence,  $J_\epsilon$  serves as a smoothing operator. Nevertheless, the estimates of  $j_\epsilon$  are uniform in  $\epsilon \geq 0$  only for  $m \geq 0$ . In the sequel, we shall regard  $j_\epsilon$  as an element in  $\Gamma_{3/2+}^0 + \Gamma_{1/2+}^{-1}$  in symbolic calculus.

Now, we define the mollification of  $\mathcal{L}$  as

$$(4.5.9) \quad \mathcal{L}^\epsilon := \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \tilde{S} J_\epsilon S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix},$$

where  $S$  is the symmetrizer of  $\mathcal{L}$  defined by (4.4.1) and  $\tilde{S}$  is an inverse of  $S$  (in the sense of (4.4.18)), which can be taken as (4.4.19). The resulting approximate system reads

$$(4.5.10) \quad \begin{cases} (\partial_t + T_V \cdot \bar{\nabla} J_\epsilon + \mathcal{L}^\epsilon) \begin{pmatrix} \eta \\ \psi \end{pmatrix} = f(J_\epsilon \eta, J_\epsilon \psi), \\ (\eta, \psi)|_{t=0} = (\eta_0, \psi_0) \in H_R^{s+\frac{1}{2}} \times H^s, \end{cases}$$

where  $s > 3$ .

**Proposition 4.5.4.** *Let  $s > 3$  and  $\epsilon > 0$ . For all  $(\eta_0, \psi_0) \in \mathcal{X}_R^s$  there exists  $T'_\epsilon > 0$  and a unique solution  $(\eta^\epsilon, \psi^\epsilon) \in C_{T'_\epsilon} \mathcal{X}_R^s$  to the Cauchy problem (4.5.10). Moreover, if  $T_\epsilon$  denotes the supremum of all such  $T'_\epsilon$ , we have,*

$$(4.5.11) \quad T_\epsilon = +\infty, \text{ or } T_\epsilon < +\infty \text{ with } \limsup_{t \rightarrow T_\epsilon^-} \left( \|\eta^\epsilon(t) - R\|_{H^{s+\frac{1}{2}}} + \|\psi^\epsilon(t)\|_{H^s} \right) = +\infty.$$

In the following, we still denote the approximate solution by  $(\eta, \psi)$  if there is no confusion.

*Proof.* To prove Proposition 4.5.4, we attempt to apply Cauchy-Lipschitz theorem, requiring the following quantities to be locally Lipschitzian on  $\mathcal{X}_R^s$ :

$$T_V \cdot \bar{\nabla} J_\epsilon \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad \mathcal{L}^\epsilon \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad f(J_\epsilon \eta, J_\epsilon \psi).$$

As in Section 4.3.5, it suffices to check that the derivative-in- $(\eta, \psi)$  of these quantities are bounded in  $\mathcal{X}_R^s$  (in this case, we can assume  $\delta\eta \in H^{s+\frac{1}{2}}$ ). This boundedness is no more than a consequence of Lemma 4.3.25, 4.3.26, and Proposition 4.3.22, 4.C.8. The only term that needs to be treated carefully is the  $\eta^{-1}$  arising from the definition (4.3.40) of  $\bar{\nabla}$ , namely

$$T_{V^\theta} \left( \frac{1}{\eta} \partial_\theta J_\epsilon \begin{pmatrix} \eta \\ \psi \end{pmatrix} \right).$$

After applying derivatives in  $\eta$  or  $\psi$ ,  $\epsilon > 0$  ensures that  $\partial_\theta J_\epsilon \eta$ ,  $\partial_\theta J_\epsilon \psi$ , and their derivatives in  $(\eta, \psi)$  lie in  $H^{+\infty} := \cap_{k \in \mathbb{Z}} H^k$ . Therefore, their product with  $\eta^{-1}$  or  $\delta(\eta^{-1})$  has regularity  $H^{s+\frac{1}{2}}$  due to Corollary 4.C.9. Since  $T_{V^\theta}$  is of order 0, the contribution of these terms is still in  $\mathcal{X}_R^s$ .  $\square$

## 4.5.2 . Energy estimates

In this part, we shall make energy estimates for approximate solutions constructed in Proposition 4.5.4, which also hold for the original system (4.3.68), namely in the case  $\epsilon = 0$ . One should keep in mind that all the estimates in this section are uniform for  $\epsilon \in [0, 1]$ .

**Proposition 4.5.5.** *Let  $(\eta_0, \psi_0) \in \mathcal{X}_R^s$  with  $s > 3$ . The approximate solution  $(\eta^\epsilon, \psi^\epsilon)$  obtained in Proposition 4.5.4 satisfies, for some  $T_0$  depending only on  $H_R^{s+\frac{1}{2}} \times H^s$ -norm of initial data  $(\eta_0, \psi_0)$ ,*

$$(4.5.12) \quad M_{T,\epsilon} \leq C(M_{0,\epsilon})M_{0,\epsilon}, \quad \forall T \in [0, \min(T_0, T_\epsilon)[, \quad \forall 0 < \epsilon \leq 1,$$

where  $C$  is a positive increasing function independent of  $\epsilon$ ,  $T_\epsilon > 0$  is the lifespan of the approximate solution, and

$$(4.5.13) \quad M_{T,\epsilon} := \sup_{t \in [0, T[} \left( \|\eta^\epsilon(t) - R\|_{H^{s+\frac{1}{2}}} + \|\psi^\epsilon(t)\|_{H^s} \right).$$

As a corollary, the blow-up criteria (4.5.11) implies

**Corollary 4.5.6.** *Under the hypotheses of Proposition 4.5.4, there exists  $T_0 > 0$  depending only on  $H_R^{s+\frac{1}{2}} \times H^s$ -norm of initial data  $(\eta_0, \psi_0)$ , such that the lifespan  $T_\epsilon$  satisfies,*

$$(4.5.14) \quad T_\epsilon \geq T_0, \quad \forall \epsilon > 0.$$

As a result, the energy estimate (4.5.12) becomes

$$(4.5.15) \quad M_{T,\epsilon} \leq C(M_{0,\epsilon})M_{0,\epsilon}, \quad \forall T \in [0, T_0[, \quad \forall 0 < \epsilon \leq 1.$$

**Remark 4.5.7.** *Note that, once we manage to prove the convergence of approximate solutions  $(\eta^\epsilon, \psi^\epsilon)$  in  $L_T^\infty \mathcal{X}_R^s$  or weaker space  $L_T^\infty \mathcal{X}_R^{s_0}$  ( $s_0 < s$ ), the limit  $(\eta, \psi)$  will inherit this uniform bound, namely,*

$$\|(\eta, \psi)\|_{L_T^\infty(H_R^{s+\frac{1}{2}} \times H^s)} \leq C \left( \|(\eta_0, \psi_0)\|_{H_R^{s+\frac{1}{2}} \times H^s} \right) \|(\eta_0, \psi_0)\|_{H_R^{s+\frac{1}{2}} \times H^s}, \quad \forall T \in [0, T_0[.$$

Instead of proving Proposition 4.5.5, we shall consider a more general case.

**Proposition 4.5.8.** *Let  $(\eta'_0, \psi'_0) \in H_{R'}^{s+\frac{1}{2}} \times H^s$  and  $(\eta, \psi) \in L_T^\infty \mathcal{X}_R^s$  with  $R' \in \mathbb{R}$ ,  $R > 0$ ,  $T > 0$ , and  $s > 3$ . Given  $s_0 \in [\frac{3}{2}, s]$ , we assume that*

(1) *The system*

$$(4.5.16) \quad \begin{cases} (\partial_t + T_V \cdot \bar{\nabla} J_\epsilon + \mathcal{L}^\epsilon) \begin{pmatrix} \eta' \\ \psi' \end{pmatrix} = f, \\ (\eta', \psi')|_{t=0} = (\eta'_0, \psi'_0), \end{cases}$$

*admits a solution  $(\eta', \psi') \in L_T^\infty(H_{R'}^{s+\frac{1}{2}} \times H^s)$ , where the operators  $T_V \cdot \bar{\nabla} J_\epsilon$  and  $\mathcal{L}^\epsilon$  are associated to  $(\eta, \psi)$ ;*

(2) *The source term  $f$  verifies, for all  $t \in [0, T[$ ,*

$$(4.5.17) \quad \|f\|_{L_t^\infty(H^{s_1+\frac{1}{2}} \times H^{s_1})} \leq C(N_t^{s^-})(M_t^{s_1} + c_f),$$

where  $c_f \in [0, 1]$  is a constant,

$$(4.5.18) \quad M_T^r := \sup_{t \in [0, T[} \left( \|\eta'(t) - R'\|_{H^{r+\frac{1}{2}}} + \|\psi'(t)\|_{H^r} \right),$$

and  $N_T^r$  is defined as

$$(4.5.19) \quad N_T^r := \sup_{t \in [0, T[} \left( \|\eta(t) - R\|_{H^{r+\frac{1}{2}}} + \|\psi(t)\|_{H^r} \right) + M_T^r,$$

(3) There exists  $\frac{3}{2} < s_1 < \min(s - \frac{3}{2}, s_0)$  such that  $(\eta, \psi)$  satisfies for all  $t \in [0, T[$ ,

$$(4.5.20) \quad \|\eta_t\|_{H^{s_1+\frac{1}{2}}} + \|\psi_t\|_{H^{s_1}} \leq C(N_t^{s^-})N_t^{s_1+\frac{3}{2}}.$$

Then there exists  $0 < T_0 \leq T$  depending only on  $\|(\eta'_0, \psi'_0)\|_{H^{s+\frac{1}{2}} \times H^s}$  and  $c_f \in [0, 1]$ , such that for all  $\epsilon \in [0, 1]$  and  $t \in [0, T_0]$ ,

$$(4.5.21) \quad (M_t^{s_0})^2 \leq C(N_0^{s^-})(M_0^{s_0})^2 + tC(N_t^{s^-})(N_t^{s_0} + 1)((N_t^{s_0})^2 + c_f^2),$$

where the smooth increasing function  $C > 0$  is independent of  $\epsilon \in [0, 1]$ .

In particular, if  $s_0 < s$ , the estimate above can be written as

$$(4.5.22) \quad (M_t^{s_0})^2 \leq C(N_0^{s^-})(M_0^{s_0})^2 + tC(N_t^{s^-})((M_t^{s_0})^2 + c_f^2), \quad \forall t \in [0, T_0].$$

Meanwhile, if  $s_0 \geq s^-$ ,  $c_f = 0$ ,  $R' = R$ , and  $(\eta', \psi') = (\eta, \psi) \in C_T \mathcal{X}_R^s$ , we have  $N_t^r = 2M_t^r$  as well as

$$(4.5.23) \quad N_t^{s_0} \leq C(N_0^{s^-})N_0^{s_0}, \quad \forall t \in [0, T_0].$$

**Remark 4.5.9.** In application of this proposition, we always have that  $(\eta, \psi)$  solves (4.3.68) or the approximate system (4.5.10). Therefore, condition (3) is trivial. Besides,  $(\eta', \psi')$  is equal to either  $(\eta, \psi)$  or the difference of two solutions to (4.3.68) or (4.5.10). In both cases, we can easily derive the equation (4.5.16) in (1).

When Proposition 4.5.8 is proved, Proposition 4.5.5 follows from (4.3.91) and (4.5.23) with  $s_0 = s$ ,  $c_f = 0$ , and  $(\eta', \psi') = (\eta, \psi)$ . In order to prove Proposition 4.5.8, we begin with the following two lemmas.

**Lemma 4.5.10.** Let  $(\eta, \psi) \in L_T^\infty \mathcal{X}_R^s$  with  $s > 3$  and  $T > 0$ . Then the following estimate holds uniformly in  $\epsilon \in [0, 1]$  and  $t \in [0, T[$ ,

$$(4.5.24) \quad S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \mathcal{L}^\epsilon \approx \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}.$$

The meaning of this equivalence between matrix of parilinear operators is that the corresponding entries are equivalent in the sense of (4.5.2).

*Proof.* By definition (4.5.9) of  $\mathcal{L}^\epsilon$ , we have

$$S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \mathcal{L}^\epsilon = S \begin{pmatrix} 0 & -T_\lambda \\ \sigma T_\mu & 0 \end{pmatrix} \tilde{S} J_\epsilon S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}.$$

Thanks to Proposition 4.4.1, the right hand side is equivalent to

$$\begin{pmatrix} 0 & -T_\gamma \\ T_\gamma^* & 0 \end{pmatrix} S \tilde{S} J_\epsilon S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \approx \begin{pmatrix} 0 & -T_\gamma \\ T_\gamma^* & 0 \end{pmatrix} J_\epsilon S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix},$$

where we use  $S \tilde{S} \approx id$  (see (4.4.18)). It remains to check that  $T_\gamma^* J_\epsilon \approx (T_\gamma J_\epsilon)^*$ . In fact, by (4.4.3) and Lemma 4.5.3,

$$(T_\gamma J_\epsilon)^* = J_\epsilon^* T_\gamma^* \approx J_\epsilon T_\gamma^* \approx J_\epsilon T_\gamma \approx T_\gamma J_\epsilon \approx T_\gamma^* J_\epsilon.$$

□

**Lemma 4.5.11.** *Let  $(\eta, \psi) \in L_T^\infty \mathcal{X}_R^s$  with  $s > 3$  and  $T > 0$ . Then the following estimate holds uniformly in  $\epsilon \in [0, 1]$  and  $t \in [0, T]$*

$$(4.5.25) \quad \|[T_B, \partial_t]\|_{\mathcal{L}(H^r; H^{r-\frac{1}{2}+})} + \|[T_B, T_V \cdot \bar{\nabla}]\|_{\mathcal{L}(H^r; H^{r-\frac{1}{2}+})} \leq C(N_T^{s-}), \quad \forall r \leq s + \frac{1}{2}.$$

*Proof.* The commutator  $T_B$  and  $\partial_t$  reads

$$[T_B, \partial_t] = T_{\partial_t B}.$$

By Proposition 4.C.16, it suffices to prove that  $\partial_t B \in H^{\frac{1}{2}+}$ . Thanks to formula (4.3.46) of  $B$ , one may regard  $B$  as a smooth function of  $G(\eta)\psi$ ,  $(\eta, \nabla_w \eta)$ , and  $(\psi, \nabla_w \psi)$ . By Proposition 4.C.22 and Corollary 4.C.9, it suffices to check that the derivative in time of these quantities belongs to  $H^{\frac{1}{2}+}$ . From equation (4.1.18), we have

$$\partial_t \eta = G(\eta)\psi \in H^{s-1}, \quad \partial_t \psi = -\sigma \left( H - \frac{1}{2R} \right) - N \in H^{s-\frac{3}{2}},$$

which is a consequence of Corollary 4.2.4, 4.C.9, and Proposition 4.C.22 (recall that  $B, V \in H^{s-1}$  due to Lemma 4.3.16). This observation implies that  $\partial_t(\eta, \nabla_w \eta)$  and  $\partial_t(\psi, \nabla_w \psi)$  have regularity  $H^{\frac{1}{2}+}$  since  $s > 3$ . As for the time derivative of  $G(\eta)\psi$ , we have

$$\partial_t (G(\eta)\psi) = \frac{d}{d\eta} G(\eta)\psi \cdot \partial_t \eta + G(\eta)\partial_t \psi \in H^{s-\frac{5}{2}} \subset H^{\frac{1}{2}+},$$

thanks to Proposition 4.2.8 and 4.2.4.

It remains to study the commutator  $[T_B, T_V \cdot \bar{\nabla}]$ , whose estimate follows from Proposition 4.4.3 with  $a = B \in \Gamma_{2+}^0$  (see also (4.3.51)). □

Now, we are ready to prove Proposition 4.5.8. Recall that  $(\eta', \psi')$  solves (4.5.16), namely

$$(\partial_t + T_V \cdot \bar{\nabla} J_\epsilon + \mathcal{L}^\epsilon) \begin{pmatrix} \eta' \\ \psi' \end{pmatrix} = f \in L_T^\infty(H^{s+\frac{1}{2}} \times H^s).$$

By applying operator

$$(4.5.26) \quad S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}$$

on both sides (recall that  $S$  is defined in (4.4.1)), one has

$$(4.5.27) \quad \begin{aligned} & S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} (\partial_t + T_V \cdot \bar{\nabla} J_\epsilon + \mathcal{L}^\epsilon) \\ &= (\partial_t + T_V \cdot \bar{\nabla} J_\epsilon) S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} + \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} S \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} + R_1. \end{aligned}$$

We claim that the remainder  $R_1$  satisfies the following property.

**Lemma 4.5.12.** *Under the assumption of Proposition 4.5.8,  $R_1$  defined by (4.5.27) satisfies, for all  $r \in [0, s]$ ,*

$$(4.5.28) \quad \|R_1\|_{\mathcal{L}(H_{R'}^{r+\frac{1}{2}} \times H^r; H^r \times H^r)} \leq C(N_t^{s-}).$$

*Proof.* We observe from the definition (4.4.1) of  $S$ , that

$$\begin{aligned} R_1 &= \left[ S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix}, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon \right] \\ &\quad + S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \mathcal{L}^\epsilon - \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} S \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \\ &= \left[ \begin{pmatrix} T_p & 0 \\ -T_q T_B & T_q \end{pmatrix}, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon \right] + R_2'' = R_2' + R_2'', \end{aligned}$$

where

$$\begin{aligned} R_2' &= \begin{pmatrix} [T_p, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] & 0 \\ [-T_q T_B, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] & [T_q, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] \end{pmatrix}, \\ R_2'' &= S \begin{pmatrix} I & 0 \\ -T_B & I \end{pmatrix} \mathcal{L}^\epsilon - \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} S \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix}. \end{aligned}$$

From Lemma 4.5.10,  $R_2''$  satisfies the desired estimate, so as the commutator between  $T_p$  (and  $T_q$ ) and  $\partial_t + T_V \cdot \bar{\nabla} J_\epsilon$  (see Proposition 4.4.2 and 4.4.3). Thus, it remains to show that the commutator  $[T_q T_B, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon]$  is bounded from  $H_{R'}^{r+\frac{1}{2}}$  to  $H^r$ , with operator norm bounded by  $C(N_t^{s-})$ . In fact, we have

$$\begin{aligned} [T_q T_B, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] &= T_q [T_B, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] + [T_q, \partial_t + T_V \cdot \bar{\nabla} J_\epsilon] T_B \\ &= T_q [T_B, \partial_t] + T_q [T_B, T_V \cdot \bar{\nabla} J_\epsilon] + T_q T_V \cdot \bar{\nabla} [T_B, J_\epsilon] \\ &\quad + [T_q, \partial_t] T_B + [T_q, T_V \cdot \bar{\nabla} J_\epsilon] T_B + T_V \cdot \bar{\nabla} [T_q, J_\epsilon] T_B. \end{aligned}$$

Since  $T_q$ ,  $T_B$ , and  $J_\epsilon$  are of order zero, thanks to Proposition 4.4.2, 4.4.3, and Lemma 4.5.11, it suffices to study the term  $T_q T_V \cdot \bar{\nabla} [T_B, J_\epsilon]$  and  $T_V \cdot \bar{\nabla} [T_q, J_\epsilon] T_B$ , which are bounded from  $H_{R'}^{r+\frac{1}{2}}$  to  $H^{r+\frac{1}{2}}$  since the commutators  $[T_B, J_\epsilon]$ ,  $[T_q, J_\epsilon]$  are of order  $(-1)$  due to Corollary 4.C.19 and 4.3.5.  $\square$

As a consequence of Lemma 4.5.12, the quantity

$$(4.5.29) \quad Y := S \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} \begin{pmatrix} \eta' \\ \psi' \end{pmatrix} \in C_T(H^s \times H^s),$$

verifying the direct estimate

$$(4.5.30) \quad \|Y(t)\|_{H^{s_0} \times H^{s_0}} \leq C(N_t^{s-}) M_t^{s_0}, \quad \forall s_0 \in [0, s], \quad t \in [0, T[,$$

solves the equation

$$(4.5.31) \quad \partial_t Y + T_V \cdot \bar{\nabla} J_\epsilon Y + \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} Y = F_1,$$

with

$$(4.5.32) \quad F_1 = S \begin{pmatrix} I & 0 \\ T_B & I \end{pmatrix} f - R_1 \begin{pmatrix} \eta' \\ \psi' \end{pmatrix}.$$

Hence, the new source term  $F_1$  verifies, for all  $0 \leq s_0 \leq s$  and  $t \in [0, T[$ ,

$$(4.5.33) \quad \|F_1(t)\|_{C_T(H^{s_0} \times H^{s_0})} \leq C(N_t^{s-}) \left( \|f(t)\|_{H^{s_0 + \frac{1}{2}} \times H^{s_0}} + M_t^{s_0} \right).$$

To obtain the  $H_{R'}^{s_0 + \frac{1}{2}} \times H^{s_0}$  estimate of  $(\eta', \psi')$  (namely (4.5.21)), we introduce a new symbol  $\beta := \beta^{(s_0)} + \beta^{(s_0-1)} \in \Sigma^s$ , defined by

$$(4.5.34) \quad \beta^{(s_0)} := \left( \gamma^{(3/2)} \right)^{\frac{2s_0}{3}}, \quad \beta^{(s_0-1)} = -\frac{i}{2} \partial_w \cdot \partial_\xi \beta^{(s_0)},$$

where  $\gamma^{(3/2)}$  is constructed in Proposition 4.4.1. By applying Lemma 4.5.2 with  $F(\epsilon, \rho) = \rho^{2s_0/3}$ ,  $G_1(\epsilon, \rho) = \exp(-\epsilon\rho)$ , and  $G_2(\epsilon, \rho) = \rho$ , we obtain the following properties of  $\beta$ .

**Lemma 4.5.13.** *Let  $\eta \in L_T^\infty H_R^{s + \frac{1}{2}}$  with  $s > 3$  and  $T > 0$ . Then the symbol  $\beta$  defined above is elliptic and belong to  $\Gamma_{3/2+}^{s_0} + \Gamma_{1/2+}^{s_0-1}$  uniformly in  $t \in [0, T[$ . Moreover, we have*

$$(4.5.35) \quad T_\beta T_\gamma \approx T_\gamma T_\beta, \quad T_\beta J_\epsilon \approx J_\epsilon T_\beta,$$

uniformly in  $\epsilon, t$ , in the sense of (4.5.2).

As a result of Lemma 4.5.13 together with (4.4.3) and (4.5.8), we have

$$T_\beta \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} \approx \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} T_\beta.$$

Moreover, thanks to Proposition 4.4.2, 4.4.3, and Corollary 4.C.19, the commutator between  $T_\beta$  and  $(\partial_t + T_V \cdot \bar{\nabla} J_\epsilon)$  is bounded from  $H^r$  to  $H^{r-s_0}$  for all  $r \in [0, s]$ , whose operator norm is bounded by  $C(N_t^{s-})$ . Now, we apply  $T_\beta$  to (4.5.31) and write the resulting equation as

$$(4.5.36) \quad \partial_t T_\beta Y + T_V \cdot \bar{\nabla} J_\epsilon T_\beta Y + \begin{pmatrix} 0 & -T_\gamma J_\epsilon \\ (T_\gamma J_\epsilon)^* & 0 \end{pmatrix} T_\beta Y = F_2,$$

with

$$(4.5.37) \quad F_2 = T_\beta F_1 + R_2 Y,$$

where the remainder  $R_2$  is bounded from  $H^r \times H^r$  to  $H^{r-s_0} \times H^{r-s_0}$  for all  $r \in [0, s]$ , whose operator norm is bounded by  $C(N_t^{s-})$ , which, combined with (4.5.30) and (4.5.33), yields the following estimate for  $F_2$ ,

$$(4.5.38) \quad \|F_2(t)\|_{C_T(L^2 \times L^2)} \leq C(N_t^{s-}) \left( \|f(t)\|_{H^{s_0 + \frac{1}{2}} \times H^{s_0}} + M_t^{s_0} \right), \quad \forall t \in [0, T[.$$

The  $L^2$ -scalar product of (4.5.36) with  $T_\beta Y$  gives

$$\frac{1}{2} \frac{d}{dt} \|T_\beta Y\|_{L^2 \times L^2}^2 = -\operatorname{Re} \langle T_V \cdot \bar{\nabla} J_\epsilon T_\beta Y, T_\beta Y \rangle + \operatorname{Re} \langle F_2, T_\beta Y \rangle.$$

We claim that

**Lemma 4.5.14.** *Let  $(\eta, \psi) \in L_T^\infty \mathcal{X}_R^s$  with  $s > 3$  and  $T > 0$ . Then we have*

$$(4.5.39) \quad |\operatorname{Re} \langle T_V \cdot \bar{\nabla} J_\epsilon T_\beta Y, T_\beta Y \rangle| \leq C(N_t^{s-}) \|T_\beta Y\|_{L^2 \times L^2}^2, \quad \forall t \in [0, T[,$$

where  $C > 0$  is an  $\epsilon$ -independent smooth increasing function.

Once Lemma 4.5.14 is proved (the proof is deferred to the end of this section), we can conclude using (4.5.30) and (4.5.38) that, for all  $t \in [0, T[$ ,

$$\begin{aligned} & \|T_{\beta(t)} Y(t)\|_{L^2 \times L^2}^2 \\ & \leq \|T_\beta Y(0)\|_{L^2 \times L^2}^2 + \int_0^t (C(N_{t'}^{s-}) \|T_\beta Y(t')\|_{L^2 \times L^2}^2 + \|T_\beta Y(t')\|_{L^2 \times L^2} \|F_2(t')\|_{L^2 \times L^2}) dt' \\ & \leq C(N_0^{s-}) \|Y(0)\|_{H^{s_0} \times H^{s_0}}^2 + \int_0^t C(N_{t'}^{s-}) \|Y(t')\|_{H^{s_0} \times H^{s_0}} (\|Y(t')\|_{H^{s_0} \times H^{s_0}} + \|F_2(t')\|_{L^2 \times L^2}) dt' \\ & \leq C(N_0^{s-}) (M_0^{s_0})^2 + \int_0^t C(N_{t'}^{s-}) M_{t'}^{s_0} \left( M_{t'}^{s_0} + \|f(t')\|_{H^{s_0 + \frac{1}{2}} \times H^{s_0}} \right) dt' \\ & \leq C(N_0^{s-}) (M_0^{s_0})^2 + t C(N_t^{s-}) M_t^{s_0} \left( M_t^{s_0} + \|f\|_{L_t^\infty(H^{s_0 + \frac{1}{2}} \times H^{s_0})} \right). \end{aligned}$$

An application of (4.5.17) leads to the following estimate of  $T_\beta Y$ ,

$$(4.5.40) \quad \|T_{\beta(t)} Y(t)\|_{L^2 \times L^2}^2 \leq C(N_0^{s-}) (M_0^{s_0})^2 + t C(N_t^{s-}) M_t^{s_0} (M_t^{s_0} + c_f).$$

*Proof of (4.5.21).* Compared with the desired estimate (4.5.12), we need to pass from the estimate of  $T_\beta Y$  to that of  $(\eta', \psi')$ . Notice that  $\beta$ ,  $p$ , and  $q$  are elliptic symbols. Then, by Proposition 4.3.8, we have

$$\|(\eta', \psi')(t)\|_{H_{R'}^{s_0 + \frac{1}{2}} \times H^{s_0}} \leq C(N_t^{\frac{3}{2}+}) \left( \|T_{\beta(t)} Y(t)\|_{L^2 \times L^2} + \|(\eta', \psi')(t)\|_{H_{R'}^{s_0 - 1} \times H^{s_0 - \frac{3}{2}}} \right).$$

Recall that,  $\mathcal{L}^\varepsilon$  (defined in (4.5.9)) is bounded from  $H_{R'}^{r+\frac{1}{2}} \times H^r$  to  $H_{R'}^{r-1} \times H^{r-\frac{3}{2}}$  for all  $r \in \mathbb{R}$  with operator norm controlled by  $C(N_t^{s-})$ . Then, by (4.5.16), one can easily see that

$$\|\eta'_t\|_{H^{s'-1}} + \|\psi'_t\|_{H^{s'-\frac{3}{2}}} \leq C(N_t^{s-})M_t^{s'} + \|f(t)\|_{H^{s'-1} \times H^{s'-\frac{3}{2}}}, \quad \forall t \in [0, T[, s' \in \left[\frac{3}{2}, s - \frac{3}{2}\right],$$

which yields, for all  $t \in [0, T[$ ,

$$\begin{aligned} & \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + \|(\eta', \psi')(t)\|_{H_{R'}^{s_0-1} \times H^{s_0-\frac{3}{2}}}^2 \\ & \leq \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + \|(\eta'_0, \psi'_0)\|_{H_{R'}^{s_0-1} \times H^{s_0-\frac{3}{2}}}^2 \\ & \quad + 2 \int_0^t \|(\eta', \psi')(t')\|_{H_{R'}^{s_0-1} \times H^{s_0-\frac{3}{2}}} \left( \|\eta'_t\|_{H^{s_0-1}} + \|\psi'_t\|_{H^{s_0-\frac{3}{2}}} \right) dt' \\ & \leq \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + \|(\eta'_0, \psi'_0)\|_{H_{R'}^{s_0+\frac{1}{2}} \times H^{s_0}}^2 \\ & \quad + 2 \int_0^t \|(\eta', \psi')(t')\|_{H_{R'}^{s_0-1} \times H^{s_0-\frac{3}{2}}} \left( C(N_{t'}^{s-})M_{t'}^{s_0} + \|f(t')\|_{H^{s_0-1} \times H^{s_0-\frac{3}{2}}} \right) dt' \\ & \leq \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + (M_0^{s_0})^2 + 2 \int_0^t M_{t'}^{s_0} \left( C(N_{t'}^{s-})M_{t'}^{s_0} + \|f(t')\|_{H^{s_0-1} \times H^{s_0-\frac{3}{2}}} \right) dt' \\ & \leq \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + (M_0^{s_0})^2 + tC(N_t^{s-})M_t^{s_0} \left( M_t^{s_0} + \|f\|_{L_t^\infty(H^{s_0-1} \times H^{s_0-\frac{3}{2}})} \right) \\ & \leq \|T_\beta Y(t)\|_{L^2 \times L^2}^2 + (M_0^{s_0})^2 + tC(N_t^{s-})M_t^{s_0} (M_t^{s_0} + c_f) \\ & \leq C(N_0^{s-})(M_0^{s_0})^2 + tC(N_t^{s-})M_t^{s_0} (M_t^{s_0} + c_f), \end{aligned}$$

where we used again the assumption (4.5.17) and estimate (4.5.40) of  $T_\beta Y$ .

To sum up, we have proved that, for all  $t \in [0, T[$ ,

(4.5.41)

$$\|(\eta, \psi)(t)\|_{H_{R'}^{s_0+\frac{1}{2}} \times H^{s_0}}^2 \leq C \left( N_t^{\frac{3}{2}+} \right) \left( C(N_0^{s-})(M_0^{s_0})^2 + tC(N_t^{s-})M_t^{s_0} (M_t^{s_0} + c_f) \right).$$

Notice that, it is harmless to replace  $C \left( N_t^{\frac{3}{2}+} \right)$  by  $C \left( (N_t^{s_1})^2 \right)$  (recall that  $\frac{3}{2} < s_1 < \min(s - \frac{3}{2}, s_0)$ ), where  $C > 0$  is smooth increasing function. From the definition (4.5.19) of  $N_t^{s_1}$ , equation (4.5.16), and assumption (4.5.20), it is clear that, for all  $t \in [0, T[$ ,

$$\begin{aligned} C \left( (N_t^{s_1})^2 \right) & \leq C \left( (N_0^{s_1})^2 \right) + \int_0^t C' \left( (N_{t'}^{s_1})^2 \right) \|(\eta, \psi)(t')\|_{H_{R'}^{s_1+\frac{1}{2}} \times H^{s_1}} \|(\eta_t, \psi_t)(t')\|_{H^{s_1+\frac{1}{2}} \times H^{s_1}} dt' \\ & \quad + \int_0^t C' \left( (N_{t'}^{s_1})^2 \right) \|(\eta', \psi')(t')\|_{H_{R'}^{s_1+\frac{1}{2}} \times H^{s_1}} \|(\eta'_t, \psi'_t)(t')\|_{H^{s_1+\frac{1}{2}} \times H^{s_1}} dt' \\ & \leq C(N_0^{s_1}) + \int_0^t C(N_{t'}^{s_1}) N_{t'}^{s_1} C(N_{t'}^{s-}) N_{t'}^{s_1+\frac{3}{2}} dt' \\ & \leq C(N_0^{s-}) + \int_0^t C(N_{t'}^{s-}) N_{t'}^{s_0} dt' \\ & \leq C(N_0^{s-}) + tC(N_t^{s-}) N_t^{s_0}, \end{aligned}$$

and the right hand side of (4.5.41) can be bounded by

$$\begin{aligned} \|(\eta, \psi)(t)\|_{H_R^{s_0+\frac{1}{2}} \times H^{s_0}}^2 &\leq (C(N_0^{s^-}) + tC(N_t^{s^-})N_t^{s_0}) \\ &\quad \times (C(N_0^{s^-})(M_0^{s_0})^2 + tC(N_t^{s^-})M_t^{s_0}(M_t^{s_0} + c_f)) \\ &\leq C(N_0^{s^-})(M_0^{s_0})^2 + tC(N_t^{s^-})N_t^{s_0}(M_t^{s_0})^2 + tC(N_t^{s^-})((M_t^{s_0})^2 + c_f^2) \\ &\quad + t^2C(N_t^{s^-})N_t^{s_0}((M_t^{s_0})^2 + c_f^2) \\ &\leq C(N_0^{s^-})(M_0^{s_0})^2 + (t + t^2)C(N_t^{s^-})(N_t^{s_0} + 1)((M_t^{s_0})^2 + c_f^2), \end{aligned}$$

which completes the proof of (4.5.21) by taking  $T_0 < 1$ .  $\square$

*Proof of (4.5.22) and (4.5.23).* When  $s_0 < s$ , (4.5.22) can be obtained by the trivial inequality  $N_t^{s_0} \leq N_t^{s^-}$ . Now, we assume  $s_0 \geq s^-$ ,  $c_f = 0$ , and prove (4.5.23) via a bootstrap argument. Let

$$B_\nu := \sqrt{2C(N_0^{s^-})(N_0^{s_0})^2 + \nu^2} \geq \nu > 0,$$

where  $\nu > 0$  is a small parameter. We fix  $T_{0,\nu} \in ]0, T[$  such that

$$C(N_0^{s^-})(N_0^{s_0})^2 + T_{0,\nu}C(B_\nu)(B_\nu + 1)B_\nu^2 < B_\nu^2,$$

or equivalently,

$$T_{0,\nu}C(B_\nu)(B_\nu + 1)B_\nu^2 < \frac{B_\nu^2 + \nu^2}{2}.$$

One may take  $T_{0,\nu}$  as

$$T_{0,\nu} = \frac{B_\nu^2 + \nu^2}{4C(B_\nu)(B_\nu + 1)B_\nu^2},$$

which depends only on initial data  $(\eta, \psi)(0)$  and parameter  $0 < \nu \ll 1$ . We claim that  $N_{T_{0,\nu}}^{s_0} \leq B_\nu$ . Otherwise, there exists  $t_0 \in ]0, T_{0,\nu}[$ , such that

$$t_0 = \sup\{t \in ]0, T_{0,\nu}[ : N_t^{s_0} \leq B_\nu\}.$$

The continuity-in-time of  $(\eta, \psi)$  guarantees that  $N_{t_0}^{s_0} = B_\nu$ . As a result, (4.5.21) gives

$$B_\nu^2 = (N_{t_0}^{s_0})^2 \leq C(N_0^{s^-})(N_0^{s_0})^2 + t_0C(B_\nu)(B_\nu + 1)B_\nu^2 < B_\nu^2,$$

which is a contradiction. Now, by passing to the limit  $\nu \rightarrow 0 + 0$ , one obtains (4.5.23). Note that  $T_{0,\nu}$  tends to  $T_0$ , which is a positive smooth function of  $N_0^{s_0}$  and  $N_0^{s^-}$ .  $\square$

*Proof of Lemma 4.5.14.* We shall prove a general result covering the inequality (4.5.39) : for all  $u \in L^2$  and  $t \in [0, T[$ ,

$$(4.5.42) \quad |\operatorname{Re}\langle T_V \cdot \bar{\nabla} J_\epsilon u, u \rangle| \lesssim C(N_t^{s^-})\|u\|_{L^2}^2.$$

To begin with, a simple calculus gives

$$T_V \cdot \bar{\nabla} J_\epsilon u = T_{Vz} \bar{z}_i \xi_z T_{j_\epsilon} u + T_{V\theta} T_{\eta^{-1}} \partial_\theta (J_\epsilon u) + T_{V\theta} (T_{\partial_\theta (J_\epsilon u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (J_\epsilon u)))$$

$$\begin{aligned}
 &= T_{V^z i \xi_z} T_{j_\epsilon} u + T_{V^\theta} T_{\eta^{-1}} T_{\partial_\theta j_\epsilon} u + T_{V^\theta} T_{\eta^{-1}} T_{j_\epsilon i \xi_\theta} u \\
 &\quad + T_{V^\theta} (T_{\partial_\theta (J_\epsilon u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (J_\epsilon u))),
 \end{aligned}$$

where the last term can be omitted since, due to Proposition 4.C.8,

$$\begin{aligned}
 \|T_{V^\theta} (T_{\partial_\theta (J_\epsilon u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (J_\epsilon u)))\|_{L^2} &\lesssim \|T_{\partial_\theta (J_\epsilon u)} \eta^{-1} + R(\eta^{-1}, \partial_\theta (J_\epsilon u))\|_{L^2} \\
 &\lesssim \|\partial_\theta (J_\epsilon u)\|_{H^{-1}} \|\eta^{-1}\|_{H_{R^{-1}}^{2+}} \lesssim C(N_t^{s-}) \|u\|_{L^2}.
 \end{aligned}$$

Moreover, by Lemma 4.3.26 and the fact that  $\eta_\theta \in H^{s-\frac{1}{2}}$ ,  $T_{\partial_\theta j_\epsilon}$  is of order  $\leq 0$ , and thus

$$\|T_{V^\theta} T_{\eta^{-1}} T_{\partial_\theta j_\epsilon} u\|_{L^2} \lesssim C(N_t^{s-}) \|u\|_{L^2}.$$

This estimate is independent of  $\epsilon$  since  $j_\epsilon \in \Sigma^0$  uniformly in  $\epsilon \geq 0$  (that is, all the involved estimates are uniform in  $\epsilon$ ). Furthermore, by symbolic calculus (Proposition 4.C.18),

$$\begin{aligned}
 \|T_{V^z i \xi_z} T_{j_\epsilon} u - T_{V^z j_\epsilon i \xi_z} u\|_{L^2} &\lesssim C(N_t^{s-}) \|u\|_{L^2}, \\
 \|T_{V^\theta} T_{\eta^{-1}} T_{j_\epsilon i \xi_\theta} u - T_{\eta^{-1} V^\theta j_\epsilon i \xi_\theta} u\|_{L^2} &\lesssim C(N_t^{s-}) \|u\|_{L^2},
 \end{aligned}$$

which reduces to

$$|\operatorname{Re}\langle T_{h_\epsilon} u, u \rangle| \lesssim C(N_t^{s-}) \|u\|_{L^2}^2, \quad h_\epsilon = j_\epsilon (V^z i \xi_z + \eta^{-1} V^\theta i \xi_\theta) \in \Gamma_{1+}^1.$$

By Proposition 4.C.18,  $T_{h_\epsilon}^* = T_{h_\epsilon^*}$ , up to some operators of order  $0-$ , with

$$h_\epsilon^* = \bar{h}_\epsilon + \partial_\xi \cdot D_w \bar{h}_\epsilon = -h_\epsilon + \partial_\xi \cdot D_w \bar{h}_\epsilon.$$

Consequently,

$$\operatorname{Re}\langle T_{h_\epsilon} u, u \rangle = \operatorname{Re} \left\langle T_{\frac{h_\epsilon + h_\epsilon^*}{2}} u, u \right\rangle + \frac{1}{2} \operatorname{Re} \langle u, (T_{h_\epsilon}^* - T_{h_\epsilon^*}) \rangle,$$

where both terms on the right hand side are bounded by  $C(N_t^{s-}) \|u\|_{L^2}^2$ .  $\square$

### 4.5.3. Convergence of approximate solutions and uniqueness

In previous sections, we have constructed a sequence of approximate solution  $(\eta^\epsilon, \psi^\epsilon)$  solving the approximate system (4.5.10), and that, for all  $\epsilon > 0$ ,  $(\eta^\epsilon, \psi^\epsilon)$  is well-defined on time interval  $[0, T_0]$  and uniformly (in  $\epsilon > 0$ ) bounded. The goal of this section is to check that these approximate solutions form a Cauchy sequence in a weaker topology  $L_{T_1}^\infty \mathcal{X}_R^{s_0}$ , where  $s_0 < s - \frac{3}{2}$  and  $0 < T_1 \leq T_0$ , which proves the existence in Theorem 4.1.2. As a by-product, one will see that the same argument allows us to compare two different solutions to the system (4.3.68) and deduce the uniqueness part of Theorem 4.1.2.

**Proposition 4.5.15.** *Under the hypotheses of Proposition 4.5.4, there exists  $0 < T_1 \leq T_0$ , where  $T_0$  is defined in Corollary 4.5.6, such that the sequence  $\{(\eta^\epsilon, \psi^\epsilon)\}_{\epsilon \in ]0,1[}$  is Cauchy in  $C_{T_1} \mathcal{X}_R^{s_0}$  with  $3/2 < s_0 < s - 3/2$ . More precisely, we have*

$$(4.5.43) \quad \lim_{\epsilon_2 \rightarrow 0^+} \sup_{0 < \epsilon_1 < \epsilon_2 < 1} \sup_{t \in [0, T_1[} \left( \|\eta^{\epsilon_1}(t) - \eta^{\epsilon_2}(t)\|_{H^{s_0 + \frac{1}{2}}} + \|\psi^{\epsilon_1}(t) - \psi^{\epsilon_2}(t)\|_{H^{s_0}} \right) = 0.$$

During the proof of Proposition (4.5.15), we emphasize that the appearance of mollifier  $J_\epsilon$  has no impact in the estimates and one may delete all the  $J_\epsilon$ 's to obtain the proof of uniqueness in Theorem 4.1.2.

Recall that, with  $0 < \epsilon_1 < \epsilon_2 < 1$ , the approximate solutions  $(\eta^{\epsilon_j}, \psi^{\epsilon_j})$  ( $j = 1, 2$ ) satisfy (4.5.10), namely

$$\begin{cases} (\partial_t + T_{V^{\epsilon_j}} \cdot \bar{\nabla}^{\epsilon_j} J_{\epsilon_j} + \mathcal{L}^{\epsilon_j}) \begin{pmatrix} \eta^{\epsilon_j} \\ \psi^{\epsilon_j} \end{pmatrix} = f(J_{\epsilon_j} \eta^{\epsilon_j}, J_{\epsilon_j} \psi^{\epsilon_j}), \\ (\eta^{\epsilon_j}, \psi^{\epsilon_j})|_{t=0} = (\eta_0, \psi_0) \in H_R^{s+\frac{1}{2}} \times H^s, \end{cases}$$

where  $V^{\epsilon_j}$  and  $\bar{\nabla}^{\epsilon_j}$  stand for the  $V$  and  $\bar{\nabla}$  associated to  $(\eta^{\epsilon_j}, \psi^{\epsilon_j})$ , respectively. The difference between approximate solutions

$$(4.5.44) \quad (\delta\eta, \delta\psi) := (\eta^{\epsilon_2} - \eta^{\epsilon_1}, \psi^{\epsilon_2} - \psi^{\epsilon_1}) \in C_{T_0}(H^{s+\frac{1}{2}} \times H^s)$$

satisfies

$$(4.5.45) \quad \begin{cases} (\partial_t + T_{V^{\epsilon_2}} \cdot \bar{\nabla}^{\epsilon_2} J_{\epsilon_2} + \mathcal{L}^{\epsilon_2}) \begin{pmatrix} \delta\eta \\ \delta\psi \end{pmatrix} = h, \\ (\delta\eta, \delta\psi)|_{t=0} = 0, \end{cases}$$

where  $h$  equals

$$\begin{aligned} h &= h_1 + h_2 - h_3 - h_4 - h_5, \\ h_1 &= f(J_{\epsilon_2} \eta^{\epsilon_2}, J_{\epsilon_2} \psi^{\epsilon_2}) - f(J_{\epsilon_2} \eta^{\epsilon_1}, J_{\epsilon_2} \psi^{\epsilon_1}), \\ h_2 &= f(J_{\epsilon_2} \eta^{\epsilon_1}, J_{\epsilon_2} \psi^{\epsilon_1}) - f(J_{\epsilon_1} \eta^{\epsilon_1}, J_{\epsilon_1} \psi^{\epsilon_1}), \\ h_3 &= (T_{V^{\epsilon_2}} \cdot \bar{\nabla}^{\epsilon_2} - T_{V^{\epsilon_1}} \cdot \bar{\nabla}^{\epsilon_1}) J_{\epsilon_2} \begin{pmatrix} \eta^{\epsilon_1} \\ \psi^{\epsilon_1} \end{pmatrix}, \\ h_4 &= (T_{V^{\epsilon_1}} \cdot \bar{\nabla}^{\epsilon_1}) (J_{\epsilon_2} - J_{\epsilon_1}) \begin{pmatrix} \eta^{\epsilon_1} \\ \psi^{\epsilon_1} \end{pmatrix}, \\ h_5 &= (\mathcal{L}^{\epsilon_2} - \mathcal{L}^{\epsilon_1}) \begin{pmatrix} \eta^{\epsilon_1} \\ \psi^{\epsilon_1} \end{pmatrix}. \end{aligned}$$

Let us denote by  $M$  the upper bound of  $L_{T_0}^\infty(H_R^{s+\frac{1}{2}} \times H^s)$ -norm of approximate solutions  $(\eta^\epsilon, \psi^\epsilon)$ , namely

$$(4.5.46) \quad M := \sup_{\epsilon \in ]0,1]} \|(\eta^\epsilon, \psi^\epsilon)\|_{L_{T_0}^\infty(H_R^{s+\frac{1}{2}} \times H^s)},$$

which is finite due to Corollary 4.5.6. We claim that

**Lemma 4.5.16.** *Under the assumption of Proposition 4.5.15, we have the following estimates,*

$$(4.5.47) \quad \|h_k(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C(M) \|(\delta\eta, \delta\psi)(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}}, \quad \forall t \in [0, T_0[, \quad k = 1, 3, 5,$$

(4.5.48)

$$\|h_k(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} \leq C(M) \left( \|(\delta\eta, \delta\psi)(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} + \epsilon_2^\nu \right), \quad \forall t \in [0, T_0[, \quad k = 2, 4,$$

where  $0 < \nu \ll 1$  is a constant and  $C > 0$  is an increasing smooth function that does not depend on time  $t$  and  $\epsilon_j$ 's.

Once Lemma 4.5.16 is true, one may apply energy estimate (4.5.22), with  $\frac{3}{2} < s_0 < s - \frac{3}{2}$  and  $c_f = \epsilon_2^\nu \in [0, 1]$ , to conclude that, for all  $t \in [0, T_0[$ ,

$$\begin{aligned} \|(\delta\eta, \delta\psi)(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}}^2 &\leq C \left( \|(\eta_0, \psi_0)\|_{H_R^{s+\frac{1}{2}} \times H^s} \right) \|(\delta\eta, \delta\psi)(0)\|_{H^{s+\frac{1}{2}} \times H^s}^2 \\ &\quad + tC(M) \left( \|(\delta\eta, \delta\psi)\|_{L_t^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})}^2 + \epsilon_2^{2\nu} \right) \\ &= tC(M) \left( \|(\delta\eta, \delta\psi)\|_{L_t^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})}^2 + \epsilon_2^{2\nu} \right), \end{aligned}$$

where we used the fact that  $(\delta\eta, \delta\psi)|_{t=0} = 0$ . Then, by choosing  $T_1 > 0$  such that  $T_1 C(M) < \frac{1}{2}$ , we have

$$\|(\delta\eta, \delta\psi)\|_{L_{T_1}^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})} \leq \epsilon_2^\nu \rightarrow 0, \quad \text{as } \epsilon_2 \rightarrow 0,$$

which completes the proof of (4.5.43).

**Remark 4.5.17.** In Lemma 4.5.16,  $\epsilon_2^\nu$  appears solely in the estimates concerning  $J_{\epsilon_2} - J_{\epsilon_1}$ . Thus, in the proof of uniqueness, there is no  $\epsilon_2^\nu$  and the above inequality becomes  $(\delta\eta, \delta\psi) = 0$  for all  $t \in [0, T_1[$ , which proves the uniqueness (see Proposition 4.5.21).

By Proposition 4.5.15, the approximate solutions converges to  $(\eta, \psi) \in L_{T_1}^\infty \mathcal{X}_R^{s_0}$  which is the unique solution to (4.1.18). Besides, from the uniform bound of approximate solutions in  $L_{T_0}^\infty \mathcal{X}_R^s$ , one may apply an interpolation argument to conclude that this solution  $(\eta, \psi)$  has regularity  $L_{T_1}^\infty \mathcal{X}_R^s$  and  $C_{T_1} \mathcal{X}_R^{s-}$ . The continuity in time and the continuous dependence in initial data  $(\eta_0, \psi_0)$  (w.r.t. the topology of  $\mathcal{X}_R^s$ ) will be given in the next section.

*Proof of (4.5.47).* In the term  $h_1, h_3$ , and  $h_5$ , the difference comes from that of  $(\eta^{\epsilon_j}, \psi^{\epsilon_j})$ . Thus, this estimate is equivalent to boundedness of derivatives in  $\eta, \psi$  of

$$f(J_\epsilon \eta, J_\epsilon \psi), \quad T_{V^\epsilon} \cdot \bar{\nabla}^\epsilon J_\epsilon \begin{pmatrix} \eta^\epsilon \\ \psi^\epsilon \end{pmatrix}, \quad \mathcal{L}^\epsilon \begin{pmatrix} \eta^\epsilon \\ \psi^\epsilon \end{pmatrix}$$

in  $H^{s_0+\frac{1}{2}} \times H^{s_0}$ . For the derivative of the first term, if  $\epsilon = 0$ , the desired result has been given in Proposition 4.3.22. Otherwise, the following extra terms will appear

$$\left( \frac{\delta}{\delta\eta} f \right) (J_\epsilon \eta, J_\epsilon \psi) T_{\delta_{j_\epsilon}} \eta, \quad \left( \frac{\delta}{\delta\psi} f \right) (J_\epsilon \eta, J_\epsilon \psi) T_{\delta_{j_\epsilon}} \psi,$$

where the extra factors  $T_{\delta_{j_\epsilon}} \eta$  and  $T_{\delta_{j_\epsilon}} \psi$  do not change the desired estimate thanks to Lemma 4.3.23. As for the other two terms, we observe that they are written in terms of finitely many paradifferential operators (with symbol in  $\Sigma^m$  for some  $m \in \mathbb{R}$  or equal to

$B, V$ ) acting on  $\eta^\epsilon$  or  $\psi^\epsilon$ . Thus the desired estimates are no more than a consequence of Lemma 4.3.26 and 4.3.25. The only case uncovered is when the derivative in  $\eta$  acts on  $(\eta^\epsilon)^{-1}$  from  $\bar{\nabla}^\epsilon$ , whose contribution reads

$$-T_{(V^\epsilon)^\theta} \left[ \frac{\delta\eta}{(\eta^\epsilon)^2} \partial_\theta \left( J_\epsilon \begin{pmatrix} \eta^\epsilon \\ \psi^\epsilon \end{pmatrix} \right) \right].$$

The estimate of this term simply follows from Proposition 4.C.8 and Corollary 4.C.9.  $\square$

By Proposition 4.3.22, 4.C.8 and Corollary 4.C.9, (4.5.48) can be reduced to the following lemma

**Lemma 4.5.18.** *Under the hypotheses of Proposition 4.5.15, for all  $r \in \mathbb{R}$  and  $0 < \nu \ll 1$ ,*

$$(4.5.49) \quad \|J_{\epsilon_2} - J_{\epsilon_1}\|_{\mathcal{L}(H^r; H^{r-\frac{3}{2}\nu})} \leq C(M) \left( \|(\delta\eta, \delta\psi)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} + \epsilon_2^\nu \right),$$

where  $C > 0$  is an increasing smooth function that does not depend on time  $t$  and  $\epsilon_j$ 's.

*Proof.* By definition (4.5.3) of  $j_\epsilon$ , the principal symbol of  $J_{\epsilon_2} - J_{\epsilon_1}$  reads (the subprincipal part can be treated in the same way)

$$\exp(-\epsilon_2 \gamma_2^{(3/2)}) - \exp(-\epsilon_1 \gamma_1^{(3/2)}),$$

where  $\gamma_j^{(3/2)}$  stands for the  $\gamma$  (defined in Proposition 4.4.1) associated to  $\eta^{\epsilon_j}$ . One may decompose this symbol as

$$\left( \exp(-\epsilon_2 \gamma_2^{(3/2)}) - \exp(-\epsilon_2 \gamma_1^{(3/2)}) \right) + \exp(-\epsilon_1 \gamma_1^{(3/2)}) \left( \exp(-(\epsilon_2 - \epsilon_1) \gamma_1^{(3/2)}) - 1 \right).$$

The contribution of the first part is of order 0 due to Lemma 4.3.26. For the second part, since  $\gamma_1^{(3/2)} \in \Gamma_{3/2+}^{\frac{3}{2}}$ , we have,

$$\exp(-\epsilon_1 \gamma_1^{(3/2)}) \frac{\exp(-(\epsilon_2 - \epsilon_1) \gamma_1^{(3/2)}) - 1}{(\epsilon_2 - \epsilon_1)^\nu |\xi|^{\frac{3}{2}\nu}} \in \Gamma_{3/2+}^0,$$

and consequently,

$$(\epsilon_2 - \epsilon_1)^{-\nu} \exp(-\epsilon_1 \gamma_1^{(3/2)}) \left( \exp(-(\epsilon_2 - \epsilon_1) \gamma_1^{(3/2)}) - 1 \right) \in \Gamma_{3/2+}^{\frac{3}{2}\nu},$$

which implies the desired estimate (4.5.49).  $\square$

#### 4.5.4. Continuity in time and initial data

In this section, we finish the proof of Theorem 4.1.2 and 4.1.3 by showing that the unique solution constructed in previous section is continuous in time  $t$  and initial data

$(\eta_0, \psi_0)$ . To achieve this, we shall apply the nonlinear interpolation theorem recently proved in [Ala+24]. Recall that, in the statement of Theorem 4.1.3, we have defined a small ball in  $H^{s+\frac{1}{2}} \times H^s$ , which we recall here

$$B_s(\eta_0, \psi_0; r) := \{v_0 = (\tilde{\zeta}_0, \tilde{\psi}_0) \in H^{s+\frac{1}{2}} \times H^s : \|(\eta_0 - R) - \tilde{\zeta}_0\|_{H^{s+\frac{1}{2}}} + \|\psi_0 - \tilde{\psi}_0\|_{H^s} < r\}.$$

And in previous section, we have proved that the following flow map (already defined in (4.1.19)) is well-defined for all  $s > 3$ ,

$$\begin{aligned} \mathfrak{F} : B_s(\eta_0, \psi_0; r) &\rightarrow L_T^\infty(H^{s+\frac{1}{2}}(\mathbb{T} \times \mathbb{R}) \times H^s(\mathbb{T} \times \mathbb{R})) \\ (\tilde{\zeta}_0, \tilde{\psi}_0) &\mapsto (\tilde{\eta}(t) - R, \tilde{\psi}(t)), \end{aligned}$$

where  $(\tilde{\eta}(t), \tilde{\psi}(t))$  is the unique solution to (4.1.18) with initial data  $(\tilde{\zeta}_0 + R, \tilde{\psi}_0)$ . Note that, when  $r > 0$  is small enough, the life span of these solutions admits a minimum which is denoted by  $T$  here. To prove this, it suffices to combine the energy estimate (4.5.12) and the same argument in Corollary 4.5.6.

Then Theorem 1 of [Ala+24] can be stated as,

**Theorem 4.5.19.** *Let  $s_0, s, s_1, r \in \mathbb{R}$  with  $s_0 < s < s_1$ ,  $r > 0$ , and  $0 < T \ll 1$ . Assume that the flow map  $\mathfrak{F}$  satisfies*

$$(1)(\text{contraction}) \text{ for all } v_0, v'_0 \in B_s(\eta_0, \psi_0; r) \cap H^{+\infty} \times H^{+\infty}, \quad (4.5.50)$$

$$\|\mathfrak{F}(v_0) - \mathfrak{F}(v'_0)\|_{L_T^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})} \leq C \left( \|v_0\|_{H^{s+\frac{1}{2}} \times H^s} + \|v'_0\|_{H^{s+\frac{1}{2}} \times H^s} \right) \|v_0 - v'_0\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}};$$

$$(2)(\text{tame estimate}) \text{ for all } v_0 \in B_s(\eta_0, \psi_0; r) \cap H^{+\infty} \times H^{+\infty},$$

$$(4.5.51) \quad \|\mathfrak{F}(v_0)\|_{L_T^\infty(H^{s_1+\frac{1}{2}} \times H^{s_1})} \leq C \left( \|v_0\|_{H^{s+\frac{1}{2}} \times H^s} \right) \|v_0\|_{H^{s_1+\frac{1}{2}} \times H^{s_1}}.$$

Then  $\mathfrak{F}$  is continuous on the ball  $B_s(\eta_0, \psi_0; r)$  and  $\mathfrak{F}(v_0) \in C_T(H^{s+\frac{1}{2}} \times H^s)$  for all  $v_0 \in B_s(\eta_0, \psi_0; r)$ .

**Remark 4.5.20.** *This theorem can be generalized to more general case, for instant, Besov spaces. One may refer to Theorem 18 of [Ala+24] for the setting.*

*Proof of tame estimate.* The tame estimate (4.5.51) is a direct result of energy estimate (4.5.21) (replacing  $(s_0, s_-)$  by  $(s_1, s)$ ). In fact, let us consider any initial data  $(\eta_0, \psi_0)$  such that  $v_0 = (\eta_0 - R, \psi_0) \in B_{s_1}(\eta_0, \psi_0; r)$ . The energy of the resulting solution  $(\eta, \psi)$  is still denoted as

$$N_T^{s_1} = \sup_{t \in [0, T[} \left( \|\eta(t) - R\|_{H^{s_1+\frac{1}{2}}} + \|\psi(t)\|_{H^{s_1}} \right) = \|\mathfrak{F}(v_0)\|_{L_T^\infty(H^{s_1+\frac{1}{2}} \times H^{s_1})}.$$

We recall that (4.3.91) gives

$$\|f\|_{L_t^\infty(H^{s_1+\frac{1}{2}} \times H^{s_1})} \leq C \left( \|(\eta, \psi)\|_{L_t^\infty(H_R^{s+\frac{1}{2}} \times H^s)} \right) \|(\eta, \psi)\|_{L_t^\infty(H_R^{s_1+\frac{1}{2}} \times H^{s_1})},$$

which, combined with energy estimates (4.5.23) yields the desired estimate.  $\square$

The contraction condition (4.5.50) can be obtained directly from the following proposition,

**Proposition 4.5.21.** *Given  $s > 3$ , we take arbitrary  $(\eta_0, \psi_0), (\eta'_0, \psi'_0) \in \mathcal{X}_R^s$ . Let  $(\eta, \psi), (\eta', \psi') \in L_T^\infty \mathcal{X}_R^s$  be the solutions to (4.1.18) with initial data  $(\eta_0, \psi_0), (\eta'_0, \psi'_0)$ , respectively. Then, when  $T \ll 1$ , the following estimate holds :*

$$(4.5.52) \quad \|(\eta - \eta', \psi - \psi')\|_{L_T^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})} \leq C(M) \|(\eta_0 - \eta'_0, \psi_0 - \psi'_0)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}},$$

where  $\frac{3}{2} < s_0 < s - \frac{3}{2}$ , and  $M > 0$  is the maximum of  $\|(\eta_0, \psi_0)\|_{L_T^\infty(H_R^{s+\frac{1}{2}} \times H^s)}$  and  $\|(\eta'_0, \psi'_0)\|_{L_T^\infty(H_R^{s+\frac{1}{2}} \times H^s)}$ .

*Proof.* We shall only give the sketch of the proof, since the argument is the same as Proposition 4.5.15. By calculating the difference between the equations for  $(\eta, \psi)$  and  $(\eta', \psi')$ , we are able to write the equation for

$$(\delta\eta, \delta\psi) := (\eta - \eta', \psi - \psi') \in L_T^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0}).$$

More precisely,

$$\begin{cases} (\partial_t + T_V \cdot \bar{\nabla} + \mathcal{L}) \begin{pmatrix} \delta\eta \\ \delta\psi \end{pmatrix} = g, \\ (\delta\eta, \delta\psi)|_{t=0} = (\eta_0 - \eta'_0, \psi_0 - \psi'_0) \in H^{s+\frac{1}{2}} \times H^s, \end{cases}$$

where the source term  $g = g_1 - g_2 - g_3$  is equal to

$$g_1 = f(\eta, \psi) - f(\eta', \psi'), \quad g_2 = (T_V \cdot \bar{\nabla} - T_{V'} \cdot \bar{\nabla}') \begin{pmatrix} \eta' \\ \psi' \end{pmatrix}, \quad g_3 = (\mathcal{L} - \mathcal{L}') \begin{pmatrix} \eta' \\ \psi' \end{pmatrix},$$

and  $V', \bar{\nabla}', \mathcal{L}'$  are associated to  $(\eta', \psi')$ . By using the same proof as for (4.5.47), one can show that,

$$\begin{aligned} \|g\|_{L_t^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})} &\leq C \left( \|(\eta, \psi)\|_{L_t^\infty(H_R^{s+\frac{1}{2}} \times H^s)} + \|(\eta', \psi')\|_{L_t^\infty(H_R^{s+\frac{1}{2}} \times H^s)} \right) \\ &\quad \times \|(\delta\eta, \delta\psi)(t)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}}. \end{aligned}$$

Then one may apply energy estimate (4.5.21) with  $s_0$  and obtain

$$\|(\delta\eta, \delta\psi)\|_{L_T^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})} \leq C(M) \|(\delta\eta, \delta\psi)(0)\|_{H^{s_0+\frac{1}{2}} \times H^{s_0}} + TC(M) \|(\delta\eta, \delta\psi)\|_{L_T^\infty(H^{s_0+\frac{1}{2}} \times H^{s_0})}.$$

Then the desired result follows by choosing  $TC(M) < \frac{1}{2}$ .  $\square$

## 4.A . Fractional Sobolev spaces in regular domains

In this appendix, we give a brief review of the fractional Sobolev spaces defined on regular domain  $\Omega$ . We focus on four types of domain : (1) Euclidean spaces, torus, and their products, namely  $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$  with  $d_1, d_2 \in \mathbb{N}$ ; (2) half spaces  $\mathbb{R}_+^d := \{(x', d_d) \in \mathbb{R}^{d-1} \times$

$\mathbb{R}, x_d > 0$ }; (3) bounded subdomains of  $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$  with smooth boundary; (4) cylindrical domains  $\Omega_0 \times \mathbb{R}$ , where  $\Omega_0$  is a bounded subdomain of  $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$  with smooth boundary. For simplicity, the properties below will be stated only for  $\mathbb{R}^d$  and its subdomains, while the same results holds for  $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$  and its subdomains. We mainly refer to [LM12; Dem12; AF03] for the detailed demonstrations.

**Definition 4.A.1.** Let  $s \in \mathbb{R}$ . When  $s \in \mathbb{N}$ ,  $H^s(\Omega)$  is the collection of distributions  $u$  with

$$\|u\|_{H^s(\Omega)}^2 := \sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2(\Omega)}^2 < +\infty.$$

When  $s = n + \sigma$  with  $n \in \mathbb{N}$  and  $\sigma \in ]0, 1[$ , the space  $H^s(\Omega)$  is defined by interpolation (see Section 2.1 of [LM12] for the definition and [Lio60; LP61] for general theory of interpolation spaces)

$$H^s(\Omega) = [H^n(\Omega), H^{n+1}(\Omega)]_\sigma.$$

It is clear that the spaces defined above are Hilbert spaces, which allows us to define  $H^s(\Omega)$  with  $s < 0$  by duality

$$H^s(\Omega) = \left( H_0^{s|\Omega}(\Omega) \right)',$$

where  $H_0^{s|\Omega}(\Omega)$  is the closure of  $C_c^\infty(\Omega)$  w.r.t. the  $H^{s|\Omega}(\Omega)$ -norm.

In the case of full space  $\mathbb{R}^d$ , the Sobolev space can be characterized by Fourier transform.

**Proposition 4.A.2.** If  $\Omega = \mathbb{R}^d$ , for all  $s \in \mathbb{R}$ , we have the equivalence

$$\|u\|_{H^s(\Omega)}^2 \sim \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi.$$

From Definition 4.A.1, one can deduce the boundedness of restriction operator.

**Proposition 4.A.3.** For all  $s \in \mathbb{R}$ , the restriction operator

$$\begin{aligned} \mathcal{R} : H^s(\mathbb{R}^d) &\rightarrow H^s(\Omega) \\ u &\mapsto u|_\Omega \end{aligned}$$

is bounded.

In the mean time, any function in  $H^s(\Omega)$  can be continuously extended to  $\mathbb{R}^d$ .

**Proposition 4.A.4.** There exists an extension operator

$$\begin{aligned} \mathcal{E} : H^s(\Omega) &\rightarrow H^s(\mathbb{R}^d) \\ u &\mapsto \tilde{u}, \end{aligned}$$

such that  $\tilde{u}|_\Omega = u$  and  $\mathcal{E} \in \mathcal{L}(H^s(\Omega); H^s(\mathbb{R}^d))$  for all  $s \in \mathbb{R}$ .

The construction of operator  $\mathcal{E}$  is not unique and we refer to Section 5.1 of [Agr15] for one possible construction (see also Chapter 5 of [Ste70] for non-negative index  $s$ ). Note that, in these references, the extension operator is constructed for bounded domains and half-space. As for the case of cylindrical domains  $\Omega = \Omega_0 \times \mathbb{R} \subset \mathbb{R}^d$  (where  $\Omega_0$  is bounded domain), it suffices to fix a finite cover  $\{U_j\}$  of  $\Omega_0$  such that  $U_j \cap \Omega_0$  is diffeomorphic to some subset of half-space  $\mathbb{R}_+^{d-1}$ . Thus  $U_j \times \mathbb{R} \cap \Omega$  is diffeomorphic to half-space  $\mathbb{R}_+^d$  and the problem is reduced to the case of half-space. The boundedness of restriction and extension operator guarantees the following equivalence

**Proposition 4.A.5.** *For all  $s \in \mathbb{R}$ , we have the equivalence*

$$\|u\|_{H^s(\Omega)} \sim \inf_{\tilde{u}|_{\Omega}=u} \|\tilde{u}\|_{H^s(\mathbb{R}^d)}.$$

As a corollary, Sobolev space  $H^s(\Omega)$  is invariant by multiplication with  $C_b^\infty(\Omega)$  functions.

**Corollary 4.A.6.** *Let  $W \in C_b^\infty(\Omega)$ . Then for all  $s \in \mathbb{R}$ , we have the following estimate,*

$$(4.A.1) \quad \|Wu\|_{H^s(\Omega)} \leq C \|W\|_{C^M(\Omega)} \|u\|_{H^s(\Omega)},$$

where  $C > 0$  and  $M \in \mathbb{N}$  are constants depending on  $s$  and dimension  $d$  and

$$\|W\|_{C^M(\Omega)} = \sup_{|\alpha| \leq M} \|\partial^\alpha W\|_{L^\infty(\Omega)}.$$

**Proposition 4.A.7.** *Let  $s \in \mathbb{R}$  and  $\Omega, \Omega' \subset \mathbb{R}^d$  be two subdomains.  $\chi : \Omega' \rightarrow \Omega$  is a smooth diffeomorphism such that  $\chi$  and  $\chi^{-1}$  are bounded as well as their derivatives. Then for all  $u \in H^s(\Omega)$ ,*

$$\|u\|_{H^s(\Omega)} \sim \|u \circ \chi\|_{H^s(\Omega')}.$$

We refer to Section 12.9 of [LM12] for a proof of this invariance by change of coordinate (see also Chapter 3 of [AF03]). More precisely, we have the following refined version.

**Proposition 4.A.8.** *Let  $s \in \mathbb{R}$  with  $s > \frac{d}{2} + 1$ .  $\chi : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a diffeomorphism such that  $(\chi - id)$  or  $(\chi^{-1} - id) \in H^s$ . Then for all  $u \in H^{s_0}(\mathbb{R}^d)$ ,  $0 \leq s_0 \leq s$ ,*

$$\|u \circ \chi\|_{H^{s_0}(\mathbb{R}^d)} \sim \|u\|_{H^{s_0}(\mathbb{R}^d)}.$$

We refer to Theorem 1.1 of [IKT13] for the proof of this proposition (see also [BV17]). Local result can also be obtained via paracomposition (see Proposition 4.C.23).

## 4.B . Proof of elliptic regularity

This appendix serves as a proof of Proposition 4.2.3 and Lemma 4.3.1. The former one is no more than a special case of the following proposition,

**Proposition 4.B.1.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  and  $F \in H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})$  with  $s > \frac{3}{2}$  and  $\frac{1}{2} \leq s_0 \leq s$ . Then the equation*

$$(4.B.1) \quad \begin{cases} \Delta_g \varphi = F \in H^{s_0-\frac{3}{2}}, \\ \varphi|_{\rho=1} = 0, \end{cases}$$

*admits a unique solution  $\varphi$  in  $H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  with*

$$(4.B.2) \quad \|\varphi\|_{H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|F\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})}.$$

*where  $C > 0$  is an increasing smooth function.*

The idea of the proof of Proposition 4.B.1 comes from [Lan13], Chapter 2, where the author focus on the water-wave with a bottom, the regularity of which is the same as interface. In this case, the boundary condition at bottom should also be considered, which does not exist in our problem. For axis-symmetric jets, one may find a similar proof in [HK23].

Note that the  $s_0 = \frac{1}{2}$  case can be proved by classical arguments via Lions–Lax–Milgram theorem (see Section 4.2.2), which gives a unique solution. Therefore, for general case  $s_0 > \frac{1}{2}$ , it remains to show that this solution is regular in the sense of (4.B.2). We shall apply an iteration in  $s_0$ . More precisely, we claim that

$$(4.B.3) \quad \text{If (4.B.2) is true for all } s_0 \in \left[ \frac{1}{2}, s_1 \right] \text{ with } s_1 \leq s - \delta, \text{ it also holds for } s_0 = s_1 + \delta.$$

Here  $\delta > 0$  is a small constant depending only on  $s$ , the choice of which will be precised during the proof. From now on, we assume the assumption in (4.B.3) to be correct, and turn to prove (4.B.2) with  $s_0 = s_1 + \delta$ .

To do so, we first show that it is possible to add a cut-off before  $\varphi$  via a commutator estimate, which allows us to study the equation (4.B.1) in interior part  $\{\rho < \frac{1}{2}\}$  and boundary part  $\{\frac{1}{2} < \rho < 1\}$ .

### 4.B.1 . Localization

Let  $\chi \in C_b^\infty(\mathbb{R}^3)$  be any smooth truncation. Then (4.B.1) can be localized as

$$(4.B.4) \quad \begin{cases} \Delta_g(\chi\varphi) = F_1 := \chi F + [\Delta_g, \chi]\varphi, \\ (\chi\varphi)|_{\rho=1} = 0. \end{cases}$$

**Lemma 4.B.2.** *If  $\chi \in C_b^\infty(\mathbb{R}^3)$  and  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  with  $s > \frac{3}{2}$ , for all  $0 \leq \sigma \leq s$ , we have*

$$(4.B.5) \quad \|[\Delta_g, \chi]w\|_{H^{\sigma-1}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|w\|_{H^\sigma(\mathbb{D} \times \mathbb{R})}, \quad \forall w \in H^\sigma(\mathbb{D} \times \mathbb{R}).$$

*Proof.* A simple calculus gives that

$$[\Delta_g, \chi]w = \partial_\alpha \chi g^{\alpha\beta} \partial_\beta w + \frac{1}{\sqrt{g}} \partial_\alpha \left( g^{\alpha\beta} \sqrt{g} \partial_\beta \chi w \right).$$

Recall that, from the construction of  $(g^{\alpha\beta}) = J^{-1}J^{-T}$  and  $g$  by (4.2.10) and (4.2.11), respectively, we have  $g^{\alpha\beta}, g \in H^{s-}(\mathbb{D} \times \mathbb{R})$  up to  $C_b^\infty(\mathbb{D} \times \mathbb{R})$  normalizations, provided that  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  (see Proposition 4.2.2). As a result,  $\partial_\alpha \chi g^{\alpha\beta}, \frac{1}{\sqrt{g}}, g^{\alpha\beta} \sqrt{g} \partial_\beta \chi$  belong to  $H^{s-}(\mathbb{D} \times \mathbb{R})$  up to  $C_b^\infty(\mathbb{D} \times \mathbb{R})$  normalizations, while  $w \in H^\sigma(\mathbb{D} \times \mathbb{R})$  and  $w \in H^\sigma(\mathbb{D} \times \mathbb{R})$ . Since  $s > 3/2$  and  $\sigma \leq s$ , by Corollary 4.A.6 and 4.C.9, each term on the right hand side is linear in  $w$  and belongs to  $H^{\sigma-1}(\mathbb{D} \times \mathbb{R})$ . Note that the estimate for product, Corollary 4.C.9 is stated for Euclidean spaces, which also holds for domains such as  $\mathbb{D} \times \mathbb{R}$  via Proposition 4.A.5.  $\square$

As a result, under the hypotheses of Proposition 4.B.1, the new source term  $F_1$  verifies (4.B.6)

$$\begin{aligned} \|F_1\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} &\lesssim \|F\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + \|\varphi\|_{H^{s_0-\frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \\ &\lesssim \|F\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + \|\varphi\|_{H^{s_1}(\mathbb{D} \times \mathbb{R})} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|F\|_{H^{s_0-\frac{3}{2}}(\mathbb{T} \times \mathbb{R})}, \end{aligned}$$

once  $\delta > 0$  is chosen to be strictly smaller than  $1/2$ , i.e.  $s_0 - 1/2 = s_1 + \delta - 1/2 \leq s_1$ .

In the following, we shall take  $\chi$  to be smooth truncations near  $\{\rho = 0\}$  and  $\{\rho = 1\}$ , respectively, and show (4.B.2) with  $s_0 = s_1 + \delta$  and  $\varphi$  replaced by  $\chi\varphi$ .

#### 4.B.2 . Interior regularity

Let us fix  $\chi \in C_b^\infty(\mathbb{R}^3)$  a smooth truncation near  $\{\rho < 1/2\}$ . The goal of this paragraph is to prove the regularity of  $\chi\varphi$  and deduce (4.B.2) with  $s_0 = s_1 + \delta \in ]1/2, s]$  and  $\varphi$  replaced by  $\chi\varphi$ . With such localization, it is harmless to extend the boundary value problem (4.B.4) to  $\mathbb{R}^3$  by zero extension away from  $\mathbb{D} \times \mathbb{R}$ , since  $\chi\varphi$  vanishes in a neighborhood of  $\{\rho < 1/2\}$ .

Let  $\Lambda = \langle D_{y,z} \rangle$  be the Fourier multiplier of symbol  $\langle \tilde{\xi} \rangle$ , where  $\tilde{\xi} \in \mathbb{R}^3$  is the Fourier variable associated to  $(y, z) \in \mathbb{R}^3$ . Now, we apply  $\Lambda^\delta$  to both sides of (4.B.4) and obtain

$$(4.B.7) \quad \begin{cases} \Delta_g \Lambda^\delta(\chi\varphi) = F_2 := \Lambda^\delta F_1 - [\Lambda^\delta, \Delta_g](\chi\varphi), \\ \Lambda^\delta(\chi\varphi)|_{\rho=1} = 0. \end{cases}$$

**Lemma 4.B.3.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  and  $\varphi \in H^{s_1+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  with  $\frac{1}{2} \leq s_1 \leq s - \delta$  and  $s > \frac{3}{2}$ . Then, when  $\delta > 0$  is small enough, we have*

$$(4.B.8) \quad \|[\Lambda^\delta, \Delta_g](\chi\varphi)\|_{H^{s_1-\frac{3}{2}}(\mathbb{R}^3)} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|\varphi\|_{H^{s_1+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})}.$$

Note that  $\chi \in C_c^\infty(\mathbb{R}^3)$  is supported near  $\{\rho < 1/2\}$  and thus  $\chi\varphi$  can be extended to  $\mathbb{R}^3$  by taking zero value outside  $\mathbb{D} \times \mathbb{R}$ .

Once this lemma holds true, by using the assumption of iteration ((4.B.2) holds for  $s_0 \in [1/2, s_1]$  and  $F \in H^{s_0 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})$  with  $s_0 = s_1 + \delta$ ), we have

$$\begin{aligned} \|F_2\|_{H^{s_1 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})} &\leq \|\Lambda^\delta F_1\|_{H^{s_1 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + \|[\Lambda^\delta, \Delta_g](\chi\varphi)\|_{H^{s_1 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})} \\ &\lesssim \|F_1\|_{H^{s_0 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + C \left( \|\eta\|_{H_R^{s_1 + \frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|\varphi\|_{H^{s_1 + \frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \\ &\lesssim C \left( \|\eta\|_{H_R^{s_1 + \frac{1}{2}}(\mathbb{T} \times \mathbb{R})} \right) \|F\|_{H^{s_0 - \frac{3}{2}}(\mathbb{D} \times \mathbb{R})}, \end{aligned}$$

where the last inequality can be seen from estimate (4.B.6) as well as (4.B.2) with  $s_0$  replaced by  $s_1$ . Then, for equation (4.B.7), the assumption of iteration ensures that  $\Lambda^\delta(\chi\varphi) \in H^{s_1 + \frac{3}{2}}(\mathbb{D} \times \mathbb{R})$ . Furthermore, the desired result  $\chi\varphi \in H^{s_0 + \frac{3}{2}}(\mathbb{D} \times \mathbb{R})$  follows from the elliptic regularity of  $\Lambda^\delta$  in  $\mathbb{R}^3$  and Proposition 4.A.5.

*Proof of Lemma 4.B.3.* By definition,  $\Lambda^\delta$  is commutable with derivatives, then

$$[\Lambda^\delta, \Delta_g](\chi\varphi) = \left[ \Lambda^\delta, \frac{1}{\sqrt{g}} \right] \partial_\alpha \left( g^{\alpha\beta} \sqrt{g} \partial_\beta(\chi\varphi) \right) + \frac{1}{\sqrt{g}} \partial_\alpha \left( \left[ \Lambda^\delta, g^{\alpha\beta} \sqrt{g} \right] \partial_\beta(\chi\varphi) \right).$$

From  $\varphi \in H^{s_1 + \frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ , where  $s_1 \leq s - \delta$ , and  $g^{\alpha\beta}, \sqrt{g} \in H^{s-}(\mathbb{D} \times \mathbb{R})$  up to  $C_b^\infty$  normalization, it is easy to see that  $\partial_\alpha (g^{\alpha\beta} \sqrt{g} \partial_\beta(\chi\varphi)) \in H^{s_1 - \frac{3}{2}}(\mathbb{R}^3)$  with  $s_1 - 3/2 \leq (s-) - \delta$  due to Corollary 4.C.9. Thus by Corollary 4.C.20 and the fact that  $1/\sqrt{g} \in H^{s-}(\mathbb{D} \times \mathbb{R})$ , we have

$$\left[ \Lambda^\delta, \frac{1}{\sqrt{g}} \right] \partial_\alpha \left( g^{\alpha\beta} \sqrt{g} \partial_\beta(\chi\varphi) \right) \in H^{s_1 - \frac{3}{2}}(\mathbb{R}^3),$$

for some small  $\delta > 0$ . Similarly, by noticing that  $\partial_\beta(\chi\varphi) \in H^{s_1 - \frac{1}{2}}$  where  $s_1 - \frac{1}{2} \leq s - 1 - \delta < (s-) - \delta$  and that  $g^{\alpha\beta} \sqrt{g} \in H^{s-}(\mathbb{D} \times \mathbb{R})$  up to  $C_b^\infty$  normalization, we apply again Corollary 4.C.20 and obtain that

$$\left[ \Lambda^\delta, g^{\alpha\beta} \sqrt{g} \right] \partial_\beta(\chi\varphi) \in H^{s_1 - \frac{1}{2}}(\mathbb{R}^3)$$

and the desired estimate follows due to Corollary 4.C.9.  $\square$

### 4.B.3 . Boundary regularity

The proof of the case where  $\chi$  is a smooth truncation near  $\{\rho = 1\}$  is similar to the interior case. The main difference is that, to maintain the boundary condition, it is impossible to apply  $\Lambda^\delta$ , which involves normal derivatives at boundary. Instead, we introduce tangential multiplier  $\Lambda_0^\delta$ . Let  $\langle D_{\theta,z} \rangle$  be a Fourier multiplier on  $\mathbb{T} \times \mathbb{R}$  with symbol  $\langle \xi \rangle$ , where  $\xi$  is the Fourier variable associated to  $(\theta, z) \in \mathbb{T} \times \mathbb{R}$ . Then, with polar coordinate  $\kappa : (\rho, \theta, z) \mapsto (y, z)$ , we can define  $\Lambda_0^\delta$  as

$$\Lambda_0^\delta f := \left( \langle D_{\theta,z} \rangle^\delta f \circ \kappa \right) \circ \kappa^{-1},$$

for functions  $f$  defined on  $\mathbb{D} \times \mathbb{R}$ . Note that, for any  $0 < \delta \ll 1$ ,  $\varphi|_{\rho=1} = 0$  implies  $\Lambda_0^\delta \varphi|_{\rho=1} = 0$ , and thus (4.B.7) becomes

$$(4.B.9) \quad \begin{cases} \Delta_g \Lambda_0^\delta(\chi\varphi) = F_3 := \Lambda_0^\delta F_1 - [\Lambda_0^\delta, \Delta_g](\chi\varphi), \\ \Lambda_0^\delta(\chi\varphi)|_{\rho=1} = 0, \end{cases}$$

where the source term  $F_3$  can be estimated as  $F_2$  in interior case,

$$\begin{aligned} \|F_3\|_{H^{s_1-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} &\leq \|\Lambda_0^\delta F_1\|_{H^{s_1-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + \|[\Lambda_0^\delta, \Delta_g](\chi\varphi)\|_{H^{s_1-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} \\ &\lesssim \|F_1\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})} + C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|\varphi\|_{H^{s_1+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})} \\ &\lesssim C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})} \right) \|F\|_{H^{s_0-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})}, \end{aligned}$$

where the second inequality follows from Lemma 4.B.3 with  $\Lambda$  replaced by  $\Lambda_0$ , which is possible since Corollary 4.C.20, the main step of Lemma 4.B.3, is also stated for multipliers independent of the normal variable. As a result, the assumption of iteration ensures that  $\Lambda_0^\delta(\chi\varphi) \in H^{s_1+\frac{1}{2}}$ .

Now we prove the regularity in normal direction. Recall that the coefficients of  $\Delta_g$  lie in  $H^{s-}(\mathbb{D} \times \mathbb{R})$  (2-order terms) or  $H^{s-1-}(\mathbb{D} \times \mathbb{R})$  (1-order terms), up to  $C_b^\infty(\mathbb{D} \times \mathbb{R})$  normalizations. Then, in polar coordinate,  $\Delta_g$  can be written as

$$\alpha \partial_\rho^2 + \beta \cdot \nabla_{\theta,z} \partial_\rho + \gamma \partial_\rho + R_2 + R_1,$$

where  $R_j$  is a  $j$ -order differential operator in  $\theta, z$  with coefficients in  $H^{s+j-2-}(\mathbb{D} \times \mathbb{R})$ ,  $\alpha, \beta \in H^{s-}(\mathbb{D} \times \mathbb{R})$ , and  $\gamma \in H^{s-1-}(\mathbb{D} \times \mathbb{R})$ , up to  $C_b^\infty(\mathbb{D} \times \mathbb{R})$  normalizations. Moreover,  $\alpha$  is strictly positive and its lower bound depends only on  $c_0, C_0$  from (H0) and  $\|\eta\|_{H_R^{s+\frac{1}{2}-}}$ .

Therefore, Corollary 4.C.9 and Proposition 4.C.22 gives that

$$\partial_\rho^2(\chi\varphi) = -\frac{\beta}{\alpha} \cdot \nabla_{\theta,z} \partial_\rho(\chi\varphi) - \frac{\gamma}{\alpha} \partial_\rho(\chi\varphi) - \frac{1}{\alpha} (R_2 + R_1)(\chi\varphi) \in H^{s_1+\delta-\frac{3}{2}}(\mathbb{D} \times \mathbb{R}),$$

where we use the following regularities

$$\nabla_{\theta,z} \partial_\rho(\chi\varphi), \nabla_{\theta,z}^2(\chi\varphi) \in H^{s_1+\delta-\frac{3}{2}}(\mathbb{D} \times \mathbb{R}), \quad \nabla_{\rho,\theta,z}(\chi\varphi) \in H^{s_1+\delta-\frac{1}{2}}(\mathbb{D} \times \mathbb{R}).$$

Till now, we have showed  $\nabla_{\rho,\theta,z} \partial_\rho(\chi\varphi) \in H^{s_1+\delta-\frac{3}{2}}(\mathbb{D} \times \mathbb{R})$ , i.e.  $\partial_\rho(\chi\varphi) \in H^{s_1+\delta-\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ , which gives  $\chi\varphi \in H^{s_1+\delta+\frac{1}{2}}(\mathbb{D} \times \mathbb{R}) = H^{s_0+\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$  due to  $\nabla_{\theta,z}(\chi\varphi) \in H^{s_1+\delta-\frac{1}{2}}(\mathbb{D} \times \mathbb{R})$ . And the proof of Proposition 4.B.1 is finished.

#### 4.B.4 . Proof of Lemma 4.3.1

Recall that the goal of Lemma 4.3.1 is to show that, after a change of variable

$$(\bar{\rho}, \bar{\theta}, \bar{z}) = \bar{t}(\rho, \theta, z) := \left( \frac{\rho \zeta(\rho\theta, z)}{\eta(\theta, z)}, \theta, z \right),$$

the new scalar potential  $\bar{\varphi} := \varphi \circ \bar{\iota}^{-1}$  satisfies (4.3.4). The proof is divided into four steps. Firstly, we check that  $\bar{\iota} - id$  has  $H^{s+\frac{1}{2}-}$ -regularity away from  $\rho = 0$  (so as  $\bar{\iota}^{-1} - id$ ). Secondly, we adopt Proposition 4.A.8 to prove that  $\bar{\varphi}$  belongs to  $H^{s_0+\frac{1}{2}}([1-\bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})$  for some  $0 < \bar{\delta} \ll 1$  where  $s_0$  can be taken as any value in  $]3/2, s[$ . In the third step, we complete the case of  $s_0 = s$  by reducing the problem to Proposition 4.B.1. The last step is the application of trace theorem so that one can obtain the continuity in  $\bar{\rho}$ . During the proof, we identify  $(y, z) \in \mathbb{D} \times \mathbb{R}$  with polar coordinate  $(\rho, \theta, z) \in [0, 1] \times \mathbb{T} \times \mathbb{R}$ , if the information near the axis  $\{\rho = 0\}$  is not important.

### Step 1 : Regularity of $\bar{\iota}$

**Lemma 4.B.4.** *Let  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  with  $s > \frac{3}{2}$ . Then  $\bar{\iota}$  defined in (4.3.3) is a diffeomorphism from  $[0, 1] \times \mathbb{T} \times \mathbb{R}$  to  $[0, 1] \times \mathbb{T} \times \mathbb{R}$ . Moreover, there exists  $0 < \delta, \bar{\delta} \ll 1$  such that  $\bar{\iota} - id \in H^{s+\frac{1}{2}-}([1-\delta, 1] \times \mathbb{T} \times \mathbb{R})$  and  $\bar{\iota}^{-1} - id \in H^{s+\frac{1}{2}-}([1-\bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})$ .*

*Proof.* By definition (4.3.3) of  $\bar{\iota}$ ,

$$\bar{\iota}(\rho, \theta, z) - id = \left( \rho \frac{\zeta(\rho\theta, z) - \eta(\theta, z)}{\eta(\theta, z)}, 0, 0 \right),$$

which reduces our problem to showing that  $(\zeta(\rho\theta, z) - \eta(\theta, z))/\eta(\theta, z)$  belongs to  $H^{s+\frac{1}{2}-}$  near  $\rho = 1$ . In Proposition 4.2.2, we have seen that  $\zeta \in H_{R_\epsilon}^{s+1}(\mathbb{D} \times \mathbb{R})$  with  $R_\epsilon$  defined in (4.2.7), which indicates that  $R_\epsilon$  equals  $R$  when  $\rho$  is close to 1. Thus, by choosing  $\delta > 0$  small enough, we have  $\zeta - R \in H^{s+1-}([1-\delta, 1] \times \mathbb{T} \times \mathbb{R})$ . And  $\eta \in H_R^{s+\frac{1}{2}-}(\mathbb{T} \times \mathbb{R})$  implies  $\zeta - \eta \in H^{s+\frac{1}{2}-}([1-\delta, 1] \times \mathbb{T} \times \mathbb{R})$ , from which one may conclude  $\bar{\iota} - id \in H^{s+\frac{1}{2}-}([1-\delta, 1] \times \mathbb{T} \times \mathbb{R})$  by applying Proposition 4.C.22 and Corollary 4.C.9.

To obtain the regularity of  $\bar{\iota}^{-1}$  near  $\{\bar{\rho} = 1\}$ , due to [BV17], Section 2, it suffices to extend  $\bar{\iota}$  as a diffeomorphism on  $\mathbb{R} \times \mathbb{T} \times \mathbb{R}$  and check that its Jacobian has strictly positive lower bound. One possible extension is as follow,

$$\bar{\iota}_{ext}(\rho, \theta, z) := (f(\rho, \theta, z), \theta, z), \quad f(\rho, \theta, z) = \chi_-(\rho)f_-(\rho) + \chi_1(\rho) \frac{\rho\zeta(\rho\theta, z)}{\eta(\theta, z)} + \chi_+(\rho)f_+(\rho),$$

where  $\chi_1$  is a smooth truncation near  $\{\rho = 1\}$  increasing on  $\{\rho < 1\}$  and decreasing on  $\{\rho > 1\}$ ,  $\chi_-$  ( $\chi_+$  resp.) is a smooth function supported in  $\{\rho < 1\}$  ( $\{\rho > 1\}$  resp.) with  $\chi_- + \chi_1 + \chi_+ = 1$  for all  $\rho \in \mathbb{R}$ , and  $f_\pm$  is a smooth increasing function such that  $f_\pm(\rho) = \rho$  when  $\rho$  is away from 1 and  $f_\pm(\rho) = l_\pm \rho$  for  $\rho \in \text{Supp } \chi_1$  with constants  $l_\pm > 0$  to be determined later.

Clearly  $\bar{\iota}_{ext}$  coincides with the original one (4.3.3) when  $\rho$  is close enough to 1 and equals identity when  $\rho$  is away from 1. Moreover, since the extended parts are smooth and equal to identity when  $\rho$  is away from 1, we have  $\bar{\iota}_{ext} - id \in H^{s-\frac{1}{2}-}(\mathbb{R} \times \mathbb{T} \times \mathbb{R})$ . The Jacobian of  $\bar{\iota}_{ext}$  reads

$$\partial_\rho f = \chi_-(\rho)f'_-(\rho) - \chi'_-(\rho) \left( \frac{\rho\zeta}{\eta} - f_-(\rho) \right) + \chi_1(\rho) \frac{\partial_\rho(\rho\zeta)}{\eta}$$

$$\begin{aligned}
& + \chi'_+(\rho) \left( f_+(\rho) - \frac{\rho\zeta}{\eta} \right) + \chi_+(\rho) f'_+(\rho) \\
& = \chi_-(\rho) f'_-(\rho) - \chi'_-(\rho) \rho \left( \frac{\zeta}{\eta} - l_- \right) + \chi_1(\rho) \frac{\partial_\rho(\rho\zeta)}{\eta} \\
& + \chi'_+(\rho) \rho \left( l_+ - \frac{\zeta}{\eta} \right) + \chi_+(\rho) f'_+(\rho).
\end{aligned}$$

Due to our construction,  $f'_{\pm}$ ,  $-\chi'_-$ , and  $\chi'_+$  are non-negative. Besides, due to Proposition 4.2.2 (and estimate (4.2.8)),  $\zeta/\eta$  and  $\partial_\rho(\rho\zeta)/\eta$  have upper and lower bounds depending only on constants  $c_0, C_0$  appearing in hypothesis (H0). Thus, by choosing  $0 < l_- \ll 1 \ll l_+$ , the Jacobian of  $\bar{t}_{ext}$  can be bounded from below by some positive constant.  $\square$

**Step 2 : The case  $s_0 < s$**  Recall that  $\bar{\varphi} = \varphi \circ \bar{t}^{-1}$  with  $\varphi$  verifying (4.2.26). Note that, in polar coordinate  $(\rho, \theta, z) \in [0, 1] \times \mathbb{T} \times \mathbb{R}$ , we can only deduce that  $\varphi \in H^{s_0 + \frac{1}{2}}([1 - \delta, 1] \times \mathbb{T} \times \mathbb{R})$  for  $\delta > 0$ . Then the condition  $s_0 < s$  guarantees  $s_0 + 1/2 \leq s + 1/2 -$ , which allows us to apply Proposition 4.A.8 and conclude that

$$(4.B.10) \quad \|\bar{\varphi}\|_{H^{s_0 + \frac{1}{2}}([1 - \bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})} \leq C \left( \|\eta\|_{H^{s + \frac{1}{2} -}(\mathbb{T} \times \mathbb{R})} \right) \|\psi\|_{H^{s_0}(\mathbb{T} \times \mathbb{R})},$$

for some  $0 < \bar{\delta} \ll 1$  such that  $[1 - \bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R}$  lies in  $\bar{t}^{-1}([1 - \delta, 1] \times \mathbb{T} \times \mathbb{R})$ .

**Step 3 : The case  $s_0 = s$**  Let  $\epsilon > 0$  be small enough (to be determined later). When  $\psi \in H^s(\mathbb{T} \times \mathbb{R})$ , from Step 2, we have seen that  $\bar{\varphi} \in H^{s + \frac{1}{2} - \epsilon}([1 - \bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})$ . To obtain the  $H^{s + \frac{1}{2}}$ -regularity, we notice that  $\bar{\varphi}$  is the solution to equation (4.3.5), which we recall here

$$\begin{cases} L\bar{\varphi} := \left( \alpha \partial_{\bar{\rho}}^2 + \beta \cdot \nabla_{\bar{\theta}, \bar{z}} \partial_{\bar{\rho}} + \gamma \partial_{\bar{\rho}} + \frac{1}{\bar{\rho}^2 \eta^2} \partial_{\bar{\theta}}^2 + \partial_{\bar{z}}^2 \right) \bar{\varphi} = 0, & \forall 1 - \bar{\delta} \leq \bar{\rho} \leq 1, \\ \varphi|_{\bar{\rho}=1} = \psi, \end{cases}$$

where  $\alpha, \beta, \gamma$  are defined in (4.3.6)-(4.3.8). Let  $\chi_1 = \chi_1(\rho)$  be a smooth truncation near  $\{\bar{\rho} = 1\}$  supported in  $[1 - \bar{\delta}, +\infty[$ . By denoting  $\Lambda_1 := \langle D_{\bar{\theta}, \bar{z}} \rangle$ , we have

$$\begin{cases} L\Lambda_1^\epsilon(\chi_1 \bar{\varphi}) = \bar{f} := -\Lambda_1^\epsilon[\chi_1, L]\bar{\varphi} - [\Lambda_1^\epsilon, L](\chi_1 \bar{\varphi}), & \forall \bar{\rho} \in [0, 1], \\ \varphi|_{\bar{\rho}=1} = \Lambda_1^\epsilon \psi. \end{cases}$$

Note that the first equation is valid for all  $\bar{\rho} \in [0, 1]$  since both sides vanish when  $\bar{\rho} < 1 - \bar{\delta}$ , thanks to the truncation  $\chi_1$ . It is clear that  $\bar{f} \in H^{s - \frac{3}{2} - \epsilon}([1 - \bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})$  (it suffices to apply Corollary 4.C.20 to deal with  $[\Lambda_1^\epsilon, L]$ ). Then, via change of variable  $\bar{t}$ , this equation can be rewritten as

$$\begin{cases} \Delta_g \varphi_1 = \bar{f} \circ \bar{t}, & \text{on } \mathbb{D} \times \mathbb{R}, \\ \varphi_1|_{\rho=1} = \Lambda_1^\epsilon \psi, \end{cases}$$

where  $\varphi_1 := (\Lambda_1^\epsilon(\chi_1 \bar{\varphi})) \circ \bar{t}$ . Recall that we identify  $(y, z) \in \mathbb{D} \times \mathbb{R}$  with polar coordinate  $(\rho, \theta, z) \in [0, 1] \times \mathbb{T} \times \mathbb{R}$ , since all the involved functions are supported away from the

axis  $\{\rho = 0\}$ . By Proposition 4.A.8 and Lemma 4.B.4, the source term  $\bar{f} \circ \bar{v}$  belongs to  $H^{s-\frac{3}{2}-\epsilon}(\mathbb{D} \times \mathbb{R})$ , while the boundary data  $\Lambda_1^\epsilon \psi \in H^{s-\epsilon}(\mathbb{T} \times \mathbb{R})$ . Due to Proposition 4.B.1, we have  $\varphi_1 \in H^{s+\frac{1}{2}-\epsilon}(\mathbb{D} \times \mathbb{R})$ . Finally, we apply again Proposition 4.A.8 and Lemma 4.B.4 to deduce  $\Lambda_1^\epsilon(\chi_1 \bar{\varphi}) \in H^{s+\frac{1}{2}-\epsilon}([1-\bar{\delta}, 1] \times \mathbb{T} \times \mathbb{R})$  and thus  $\Lambda_1^\epsilon \bar{\varphi} \in H^{s+\frac{1}{2}-\epsilon}([1-\bar{\delta}', 1] \times \mathbb{T} \times \mathbb{R})$  for some  $0 < \bar{\delta}' < \bar{\delta}$ . One may recover the regularity in  $\bar{\rho}$  from elliptic operator  $L$  as in the proof of Proposition 4.B.1 and conclude that  $\bar{\varphi} \in H^{s+\frac{1}{2}}([1-\bar{\delta}', 1] \times \mathbb{T} \times \mathbb{R})$ , i.e. estimate (4.B.10) also holds for  $s_0 = s$ . Here the difference between  $\bar{\delta}$  and  $\bar{\delta}'$  can be ignored since we are solely interested in the behavior near  $\{\bar{\rho} = 1\}$ .

**Step 4 : Trace estimate** To complete the proof of Lemma 4.3.1, it remains to deduce the estimate below from (4.B.10).

$$\|\partial_{\bar{\rho}}^l \bar{\varphi}\|_{C^0([1-\bar{\delta}, 1]; H^{s_0-l}(\mathbb{T} \times \mathbb{R}))} \leq C \left( \|\eta\|_{H_R^{s+\frac{1}{2}-}} \right) \|\psi\|_{H^{s_0}}, \quad l = 0, 1, 2, 3.$$

Recall that in the assumption of Lemma 4.3.1, we have  $s > 3$  and  $3/2 < s_0 \leq s$ . Then the case  $l = 0, 1$  follows directly from trace estimate. For  $l = 2$ , we observe that

$$\partial_{\bar{\rho}}^2 \bar{\varphi} = -\frac{1}{\alpha} \left( \beta \cdot \nabla_{\bar{\theta}, \bar{z}} \partial_{\bar{\rho}} + \gamma \partial_{\bar{\rho}} + \frac{1}{\bar{\rho}^2 \eta^2} \partial_{\bar{\theta}}^2 + \partial_{\bar{z}}^2 \right) \bar{\varphi}.$$

where  $\alpha, \beta, \gamma$  are estimated in Lemma 4.3.2 with  $\alpha > c$  for some constant  $c > 0$  (depending on  $c_0, C_0$  appearing in hypothesis (H0)). An application of Lemma 4.3.2, Proposition 4.C.22, and Corollary 4.C.9 gives  $\partial_{\bar{\rho}}^2 \bar{\varphi} \in C^0([1-\bar{\delta}, 1]; H^{s_0-2}(\mathbb{T} \times \mathbb{R}))$ . The last case  $l = 3$  can be obtained by applying an extra  $\partial_{\bar{\rho}}$  and repeating this argument.

## 4.C . Paradifferential calculus

In this section, we shall review the definition and properties of paradifferential operators on  $\mathbb{R}^d, \mathbb{T}^d$ , or  $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$  with  $d_1 + d_2 = d$ .

### 4.C.1 . Pseudo-differential operator

**Definition 4.C.1.** Let  $a$  be a tempered distribution on  $\mathbb{R}^d \times \mathbb{R}^d$ . For all Schwartz function  $u \in \mathcal{S}(\mathbb{R}^d)$ , we define

$$(4.C.1) \quad \langle \text{Op}(a) u, v \rangle_{\mathcal{S}' \times \mathcal{S}} := \frac{1}{(2\pi)^d} \iint e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \overline{\hat{v}(x)} d\xi dx.$$

It is easy to check that  $\text{Op}(a)$  is a continuous application from Schwartz functions  $\mathcal{S}(\mathbb{R}^d)$  to tempered distributions  $\mathcal{S}'(\mathbb{R}^d)$ . We say that  $\text{Op}(a)$  is a pseudo-differential operator with symbol  $a$ .

**Definition 4.C.2.** Let  $\rho, \delta \in [0, 1]$  and  $m \in \mathbb{R}$ . The class of symbols  $S_{\rho, \delta}^m = S_{\rho, \delta}^m(\mathbb{R}^d)$  is defined as the collection of all symbols in  $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ , such that, for all  $\alpha, \beta \in \mathbb{N}^d$ ,

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \langle \xi \rangle^{m + \delta|\alpha| - \rho|\beta|}.$$

The following results are classic, whose proof can be founded in [Hör07; CM78].

**Proposition 4.C.3.** Let  $0 \leq \delta < \rho \leq 1$  or  $0 \leq \rho = \delta < 1$ . Then the pseudo-differential operator  $\text{Op}(a) \in \mathcal{L}(H^s; H^{s-m})$  for all  $a \in S_{\rho, \delta}^m$  and  $s, m \in \mathbb{R}$ .

**Proposition 4.C.4.** Let  $0 \leq \delta < \rho \leq 1$  and  $m, m' \in \mathbb{R}$ . For all symbols  $a \in S_{\delta, \rho}^m$  and  $b \in S_{\delta, \rho'}^{m'}$ , the composition  $\text{Op}(a) \text{Op}(b)$  and the adjoint  $\text{Op}(a)^*$  are also pseudo-differential operators, with symbol  $a \# b \in S_{\delta, \rho}^{m+m'}$ ,  $a^* \in S_{\delta, \rho'}^m$ , respectively. Moreover, for all  $N \in \mathbb{N}$ , we have the following symbolic calculus,

$$(4.C.2) \quad a \# b - \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \in S_{\rho, \delta}^{m+m'-N(\rho-\delta)},$$

$$(4.C.3) \quad a^* - \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a} \in S_{\rho, \delta}^{m-N(\rho-\delta)},$$

where  $D_x = -i\partial_x$ .

In periodic case, there are two ways to define  $\text{Op}(a)$ . One is to regard the Fourier variable  $\xi$  as an element in  $\mathbb{T}^d$ , and symbols as distributions on  $\mathbb{T}^d \times \mathbb{N}^d$ . The only difference with  $\mathbb{R}^d$  case is that the derivative in  $\xi$  should be understood as finite difference :

$$\partial_\xi a(x, \xi) := a(x, \xi + 1) - a(x, \xi).$$

Another method is to regard functions on  $\mathbb{T}^d$  as periodic functions on  $\mathbb{R}^d$ . Thus, for symbols  $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ ,  $\text{Op}(a)u$  is a well-defined periodic distribution, which can be viewed as a distribution on  $\mathbb{T}^d$ . Then, for all  $a \in S_{\rho, \delta}^m$ , one may define  $\text{Op}(a)$  by density arguments. Via this method, the derivatives in  $\xi$  is just the same as those on Euclidean space. These two methods are equivalent in the sense that the difference of pseudo-differential operators defined by two methods is of lower order than the itself. That is to say, if  $a \in S_{\delta, \rho}^m$ , the difference of two definition should be an operator of order  $m - \rho$ , i.e. belonging to  $\mathcal{L}(H^s; H^{s-(m-\rho)})$  for all  $s \in \mathbb{R}$ . A rigorous study of these definitions can be found in [RT10; Sai23]. For simplicity, we shall focus on  $\mathbb{R}^d$  case in the sequel.

One observes that all the results above requires the symbols to be smooth in  $x$  and  $\xi$ . However, in applications, they are usually rough in  $x$ . For example, the  $\Delta_g$  defined by (4.2.13) has only Sobolev regularity in  $(y, z)$  but smooth in Fourier variables. To overcome this difficulty, we turn to a refined version known as *paradifferential calculus* developed by Bony [Bon81] in 1980s.

### 4.C.2 . Paraproduct

To begin with, we consider the simplest case, multiplication operators. We fix a dyadic decomposition

$$1 = \chi(\xi) + \sum_{j=0}^{\infty} \varphi\left(\frac{\xi}{2^j}\right),$$

where  $\chi, \varphi$  are radial positive smooth truncations near  $\{|\xi| \leq 1\}$  and  $\{1 \leq |\xi| \leq 2\}$ , respectively, such that

$$\chi\left(\frac{\xi}{2^j}\right) = \chi(\xi) + \sum_{k=0}^j \varphi\left(\frac{\xi}{2^k}\right), \quad \forall \xi \in \mathbb{R}^d, j \in \mathbb{N}.$$

Then we can define the following multipliers :

$$\Delta_j := \varphi\left(\frac{D_x}{2^j}\right), \quad S_j := \chi\left(\frac{D_x}{2^j}\right).$$

**Definition 4.C.5** (Bony's decomposition). *For all functions  $a, b \in \mathcal{S}(\mathbb{R}^d)$ ,*

$$T_a b := \sum_{j=2}^{\infty} S_{j-2} a \Delta_j b,$$

$$R(a, b) := ab - T_a b - T_b a.$$

*The linear operator  $T_a$  is known as paraproduct.*

**Remark 4.C.6.** *In the definition of  $T_a b$ , the low frequency part of  $b$  is eliminated. Thus, it makes no difference to replace  $b$  by  $b$  plus any function whose Fourier transform is supported near zero. In particular, we have*

$$T_a b = T_a(b - R), \quad \forall R \in \mathcal{R}.$$

Now, we review the boundedness of  $T_a b$  and  $R(a, b)$  in Sobolev spaces,

**Proposition 4.C.7.** *Let  $a \in H^s, b \in H^{s'}$  with  $s, s' \in \mathbb{R}$ . Then we have,*

$$(4.C.4) \quad \|T_a b\|_{H^{\min(s', s+s'-\frac{d}{2})}} \lesssim \|a\|_{H^s} \|b\|_{H^{s'}}, \quad \text{if } s \neq \frac{d}{2};$$

$$(4.C.5) \quad \|R(a, b)\|_{H^{s+s'-\frac{d}{2}}} \lesssim \|a\|_{H^s} \|b\|_{H^{s'}}, \quad \text{if } s + s' > 0.$$

*Moreover, when  $a \in C_b^\infty$  and  $b \in H^{s'}$  with  $s' \in \mathbb{R}$ , we have*

$$(4.C.6) \quad \|T_a b\|_{H^{s'}} \lesssim \|a\|_{L^\infty} \|b\|_{H^{s'}},$$

$$(4.C.7) \quad \|R(a, b)\|_{H^{s'}} \lesssim \|a\|_{C^M} \|b\|_{H^{s'}},$$

*for some  $M \in \mathbb{N}$  depending on  $s'$  and dimension  $d$ . Recall that  $\|a\|_{C^M} := \sup_{|\alpha| \leq M} \|\partial^\alpha a\|_{L^\infty}$ . And when  $a \in H^s$  and  $b \in C_b^\infty$ , the estimate (4.C.6) above becomes*

$$(4.C.8) \quad \|T_a b\|_{H^s} \lesssim \|a\|_{H^s} \|b\|_{C^N},$$

*where  $N \in \mathbb{N}$  depends on  $s$  and dimension  $d$ .*

These results are classical, one may find a proof in [Bon81] or Chapter 2 of [BCD11], where the results are generalized to Besov spaces. By combining Proposition 4.C.7 and Remark 4.C.6, we have the following result to be used frequently in this chapter.

**Proposition 4.C.8.** *Let  $a \in H_W^s$  and  $b \in H_R^{s'}$  for some  $s, s' \in \mathbb{R}$ , with  $W \in C_b^\infty$  and  $R \in \mathbb{R}$ . Then, we have, for all  $r \in \mathbb{R}$ ,*

$$(4.C.9) \quad \|T_a b\|_{H^r} \leq C(\|a\|_{H_W^s} + \|W\|_{L^\infty})\|b\|_{H_R^{\max(r, r + \frac{d}{2} - s)_+}}, \quad \text{if } \max(r, r + \frac{d}{2} - s) < s';$$

$$(4.C.10) \quad \|R(a, b)\|_{H^r} \leq C(\|a\|_{H_W^s} + \|W\|_{C^M})\|b\|_{H_R^{r + \frac{d}{2} - s}}, \quad \text{if } r > -\frac{d}{2},$$

where constants  $C > 0$  and  $M \in \mathbb{N}$  rely solely on  $r, s, s'$  and dimension  $d$ .

If the normalization of  $b$  is not constant but a  $C_b^\infty$  function, the following corollary follows from Proposition 4.C.8 with estimates (4.C.8) and (4.C.7).

**Corollary 4.C.9.** *Given  $r > -\frac{d}{2}$ , we assume that  $a \in H_{R_1}^s$ ,  $b \in H_{R_2}^{s'}$  with  $s, s' > r$  and  $R_1, R_2 \in C_b^\infty$ . If  $s' + s > r + \frac{d}{2}$ , there exists constants  $C > 0$  and  $M \in \mathbb{N}$  depending on  $r, s, s'$  and dimension  $d$ , such that*

$$(4.C.11) \quad \|ab\|_{H_{R_1 R_2}^r} \leq C(\|a\|_{H_{R_1}^s} + \|R_1\|_{C^M})(\|b\|_{H_{R_2}^{s'}} + \|R_2\|_{C^M}).$$

### 4.C.3 . Paradifferential operators

The paraproduct operator  $T_a$  defined in previous section can be regarded as a refinement of multiplication operator, which is equal to the pseudo-differential operator  $\text{Op}(a)$ . This inspires us to study  $\text{Op}(a)$ , where  $a = a(x, \xi)$  is a symbol with limited regularity in  $x$ , by turning to the *paradifferential operator*  $T_a$  defined below, which is no more than a generalization of Definition 4.C.5.

**Definition 4.C.10.** *Let  $a = a(x, \xi)$  be a symbol smooth in  $\xi \neq 0$  with Sobolev or Hölder regularity in  $x$ . Then the paradifferential operator of symbol  $a$  is defined as*

$$(4.C.12) \quad T_a u(x) := \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} \sum_{j=2}^{\infty} S_{j-2} a(x, \xi) \varphi\left(\frac{\xi}{2^j}\right) \hat{u}(\xi) d\xi,$$

where  $S_{j-2}$  acts on  $x$  variable.

**Remark 4.C.11.** *By construction, the low-frequency information  $|\xi| \ll 1$  of  $u$  is eliminated, which means that paradifferential operators are never bijective. Nevertheless, for elliptic symbols (see Definition 4.3.6), it is possible to construct left and right inverse, up to some reasonable remainders, which is known as a parametrix.*

**Remark 4.C.12.** *The definition above is a special case of the general one introduced by Bony [Bon81],*

$$T_a := \text{Op}(\tilde{a}), \quad \tilde{a}(x, \xi) = \tilde{\chi}(D_x, \xi)a(\cdot, \xi),$$

where  $\tilde{\chi} = \tilde{\chi}(\eta, \xi)$  is a smooth truncation near  $\{|\eta| < \epsilon(1 + |\xi|)\}$ , such that  $\tilde{\chi} = 0$  on  $\{|\eta| > \epsilon'(1 + |\xi|)\}$  with  $0 < \epsilon < \epsilon' < 1$  and for all  $\alpha, \beta \in \mathbb{N}^d$ ,

$$|\partial_\eta^\alpha \partial_\xi^\beta \tilde{\chi}(\eta, \xi)| \lesssim_{\alpha, \beta} \langle \xi \rangle^{-|\alpha| - |\beta|}.$$

In fact, with different choice of  $\tilde{\chi}$ , the resulting paradifferential operators are equivalent in the sense that their difference is a smoothing operator (see [Mét08] for more details).

**Example 4.C.13.** *If  $a = a(x)$ , the definition 4.C.5 and 4.C.10 coincide. For  $a = a(\xi)$ ,*

$$T_a = a(D_x) - \chi\left(\frac{D_x}{4}\right)a(D_x),$$

i.e.  $T_a$  equals the Fourier multiplier  $a(D_x)$  up to a smoothing operator. Furthermore, for general symbol  $a = a(x, \xi)$  and multiplier  $b = b(\xi)$ , we have

$$T_a \circ b(D_x) = T_{ab}.$$

**Definition 4.C.14.** *Let  $\rho \geq 0$  and  $m \in \mathbb{R}$ . The symbol class  $\Gamma_\rho^m$  is defined as the collection of symbols  $a = a(x, \xi)$  Hölder in  $x$  and smooth in  $\xi$ , such that for all  $\alpha \in \mathbb{N}^d$ ,*

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{C^\rho} \leq C_\alpha \langle \xi \rangle^{m - |\alpha|}, \quad \forall |\xi| > \frac{1}{2},$$

where  $C^\rho$  is the class of Hölder function for non-integer  $\rho$  and the usual Sobolev space  $W^{\rho, \infty}$  for  $\rho \in \mathbb{N}$ .

**Remark 4.C.15.** *If  $a = a(x, \xi)$  is homogeneous in  $\xi$  of degree  $m$ , then it belongs to  $\Gamma_\rho^m$  if and only if*

$$\sup_{|\xi|=1} \|\partial_\xi^\alpha a(\cdot, \xi)\|_{C^\rho} \leq C_\alpha, \quad \forall \alpha \in \mathbb{N}^d.$$

In this chapter, we are interested in symbols with Sobolev regularity in  $x$ . An application of Bernstein's Lemma (see, for example, Lemma 2.1 of [BCD11]) implies that

**Proposition 4.C.16.** *For all  $m \in \mathbb{R}$  and  $a \in \Gamma_r^m$  with  $r < 0$ ,  $T_a$  is of order  $m - r$ . In particular, let  $s < \frac{d}{2}$  be a real number. We assume that  $a = a(x, \xi)$  is smooth in  $\xi$  with Sobolev regularity in  $x$ , namely*

$$\|\partial_\xi^\alpha a(\cdot, \xi)\|_{H^s} \leq C_\alpha \langle \xi \rangle^{m - |\alpha|}, \quad \forall |\xi| > \frac{1}{2}.$$

Then  $T_a$  is of order  $m - s + \frac{d}{2}$ .

Now, we are able to generalize Proposition 4.C.3 and 4.C.4. The proof of following results can be found in Chapter 5 of [Mét08].

**Proposition 4.C.17.** *Given  $m \in \mathbb{R}$ , for all  $a \in \Gamma_0^m$ , the paradifferential operator  $T_a$  is of order  $m$ , namely*

$$(4.C.13) \quad T_a \in \mathcal{L}(H^s; H^{s-m}), \quad \forall s \in \mathbb{R}.$$

**Proposition 4.C.18.** *Let  $a \in \Gamma_\rho^m$  and  $b \in \Gamma_\rho^{m'}$  with  $m, m' \in \mathbb{R}$  and  $\rho > 0$ . Then the composition  $T_a T_b$  and adjoint  $T_a^*$  are both paradifferential operators, such that  $T_a T_b - T_{a\#_\rho b}$  is of order  $m + m' - \rho$  and  $T_a^* - T_{a^*}$  is of order  $m - \rho$ , where*

$$(4.C.14) \quad a\#_\rho b = \sum_{|\alpha| < \rho} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b,$$

$$(4.C.15) \quad a^* = \sum_{|\alpha| < \rho} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \bar{a}.$$

As a corollary, we have the following commutator estimate :

**Corollary 4.C.19.** *If  $a \in \Gamma_\rho^m$ ,  $b \in \Gamma_\rho^{m'}$  with  $m, m' \in \mathbb{R}$  and  $\rho > 0$ , the commutator  $[T_a, T_b]$  is of order  $m + m' - \min(\rho, 1)$ .*

The following estimate concerning commutator will also be used,

**Corollary 4.C.20.** *Let  $a \in H_W^s$  with  $s > \frac{d}{2}$  and  $W \in C_b^\infty$ . There exists  $0 < \delta_0 \ll 1$ , such that, for all symbol  $\lambda = \lambda(\xi)$  in the class  $S_{1,0}^\delta(\mathbb{R}^d)$  (see Definition 4.C.2) with  $\delta \leq \delta_0$ , we have*

$$(4.C.16) \quad \|[\text{Op}(\lambda), a]\|_{\mathcal{L}(H^r)} \leq C, \quad \forall r \leq s - \delta,$$

where  $C > 0$  and  $\delta_0$  depends only on  $s, r$ .

Moreover, the same result holds true if the symbol  $\lambda = \lambda(\xi')$  depends only on  $\xi'$  (we write  $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ ) and belongs to the class  $S_{1,0}^\delta(\mathbb{R}^{d-1})$ .

*Proof.* For all  $u \in H^r$ , we decompose,

$$[\text{Op}(\lambda), a]u = [\text{Op}(\lambda), T_a]u + \text{Op}(\lambda)(T_u a + R(a, u)) - (T_{\text{Op}(\lambda)u} a + R(a, \text{Op}(\lambda)u)).$$

For the last two terms, we will use estimates in Proposition 4.C.8 and estimate (4.C.8). In fact,  $u \in H^r$  implies

$$T_u a \in H^{\min(s, s+r-\frac{d}{2})-} \subset H^{r+\delta}, \quad R(a, u) \in H^{s+r-\frac{d}{2}} \subset H^{r+\delta},$$

whenever  $s \geq r + \delta_0$  and  $s - d/2 > \delta_0$ . Consequently,  $\text{Op}(\lambda)(T_u a + R(a, u))$  lies in  $H^r$ . With this choice of  $\delta_0$ , from  $\text{Op}(\lambda)u \in H^{r-\delta}$  one may also deduce that

$$T_{\text{Op}(\lambda)u} a \in H^{\min(s, s+r-\delta-\frac{d}{2})-} \subset H^r, \quad R(a, \text{Op}(\lambda)u) \in H^{s+r-\delta-\frac{d}{2}} \subset H^r.$$

It remains to deal with the principal part  $[\text{Op}(\lambda), T_a]u$ . If  $\lambda \in S_{1,0}^0(\mathbb{R}^d)$ , it is harmless to replace  $\text{Op}(\lambda)$  by  $T_\lambda$  since their difference is a smoothing operator. Then by Corollary

4.C.19, it is easy to see that  $[T_\lambda, T_a]u$  belongs to  $H^r$  since  $\lambda \in \Gamma_\delta^\delta$  and  $a \in \Gamma_\delta^0$  by choosing  $s - d/2 > \delta_0$ . The proof for the case of full space ( $\lambda \in S_{1,0}^0(\mathbb{R}^d)$ ) is completed.

If  $\lambda$  depends only on  $\xi'$  and belongs to  $S_{1,0}^0(\mathbb{R}^{d-1})$ , we observe from Definition 4.C.5 that

$$[\text{Op}(\lambda), T_a]u = \sum_{j \geq 2} [\text{Op}(\lambda), S_j a] \Delta_j u,$$

It is easy to see that, for each  $j \geq 2$ , the Fourier transform of  $[\text{Op}(\lambda), S_j a] \Delta_j u$  is supported in  $2^j \mathcal{C}$ , where  $\mathcal{C}$  is an annulus. Thus, by almost orthogonality, it suffices to check that  $[\text{Op}(\lambda), S_j a]$  is bounded from  $L^2(\mathbb{R}^d)$  to  $L^2(\mathbb{R}^d)$  uniformly in  $j$ , and the problem can be reduced to the following commutator estimate

$$(4.C.17) \quad \|[\text{Op}(\lambda), b]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim \|b\|_{H^s}.$$

For any  $u \in L^2(\mathbb{R}^d)$ , we fix arbitrary  $x_d \in \mathbb{R}$  and apply the result proved above (full space case) on  $\mathbb{R}^{d-1}$  (recall that we write  $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ ).

$$\|[\text{Op}(\lambda), b(\cdot, x_d)]u(\cdot, x_d)\|_{L_{x'}^2} \lesssim \|b\|_{L_{x_d}^\infty H_{x'}^{s-\frac{1}{2}}} \|u(\cdot, x_d)\|_{L_{x'}^2} \lesssim \|b\|_{H^s(\mathbb{R}^d)} \|u(\cdot, x_d)\|_{L_{x'}^2},$$

where the second inequality is due to the classical trace theorem. The desired estimate (4.C.17) follows by taking  $L_{x_d}^2$  norm on both sides.  $\square$

Another useful corollary is

**Corollary 4.C.21.** *Let  $a \in H_R^s$  and  $u \in H^{s'}$  with  $\frac{d}{2} < s' \leq s$  and  $R \in \mathbb{R}$  constant. Then there exists  $C > 0$  such that*

$$(4.C.18) \quad \|au^2 - 2T_{au}u\|_{H^{\min(s, 2s' - \frac{d}{2})}} \leq C (\|a\|_{H_R^s}) \|u\|_{H^{s'}}^2.$$

*Proof.* We may write  $au^2$  as

$$\begin{aligned} au^2 &= T_{au}u + T_u au + R(au, u) = T_{au}u + T_u(T_a u + T_u a + R(a, u)) + R(au, u) \\ &= 2T_{au}u + (T_u T_a - T_{au})u + T_u^2 a + T_u R(a, u) + R(au, u). \end{aligned}$$

Since  $a, u$  can be regarded as symbols in  $\Gamma_{s'-d/2}^0$ , respectively, the operator  $T_u T_a - T_{au}$  is of order  $-(s' - d/2)$  and thus  $(T_u T_a - T_{au})u \in H^{s+s'-\frac{d}{2}}$  (note that when  $a, u$  are independent of  $\xi$ ,  $a \#_\rho u = au$  for all  $\rho > 0$ ). By applying Proposition 4.C.8 and Corollary 4.C.9, we have  $T_u^2 a \in H^s$ ,  $T_u R(a, u) \in H^{s+s'-\frac{d}{2}}$ , and  $R(au, u) \in H^{2s'-\frac{d}{2}}$ , which completes the proof.  $\square$

#### 4.C.4 . Composition

To end this section, we review the paradifferential calculus concerning composition.

**Proposition 4.C.22.** *Let  $F$  be a smooth function and  $u \in H^s$  with  $s > \frac{d}{2}$ . Then  $F(u) \in H^s_{F(0)}$  with*

$$(4.C.19) \quad \|F(u) - F(0) - T_{F'(u)}u\|_{H^{2s-\frac{d}{2}}} \leq C(\|u\|_{H^s})\|u\|_{H^s}.$$

*Consequently, when  $u \in H^s_R$  with  $s > \frac{d}{2}$  and  $R \in C_b^\infty$ , we have*

$$(4.C.20) \quad \|F(u)\|_{H^s_{F(R)}} \leq C(\|u\|_{H^s_R})\|u\|_{H^s_R},$$

*where  $C > 0$  is a smooth increasing function depending on  $F$  and  $R$ .*

The proof of this proposition can be found in [Bon81] or [BCD11], Chapter 2.

Now we focus on the study of  $u \circ \chi$ , where  $u$  and  $\chi$  both have limited regularity. In this case, the singularity concentrates on two terms,  $T_{u' \circ \chi}\chi$  and the *paracomposition*  $X^*u$ , which is firstly studied by Alinhac in [Ali86]. In the sequel, we assume that  $\chi : \Omega' \rightarrow \Omega$  is a diffeomorphism, where  $\Omega, \Omega'$  are bounded domains in  $\mathbb{R}^d$  with smooth boundary, and  $u$  is a function on  $\Omega$  with Sobolev regularity. The following results can be found in [Ali86] (see also [Ali85]), while the case of  $\mathbb{R}^d$  is studied in [Tay00] (see also [Sai23]).

**Proposition 4.C.23.** *Let  $\chi \in H^{1+\frac{d}{2}+\sigma}_{loc}$  and  $\sigma \in \mathbb{R}_+ \setminus \mathbb{N}$ . Then there exists a paracomposition operator  $X^*$ , such that, for all  $u \in H^s_{loc}$ ,  $s > \frac{d}{2} + 1$ ,*

$$(4.C.21) \quad u \circ \chi = T_{u' \circ \chi}\chi + X^*u + Ru,$$

*where the remainder  $R \in H^{1+\sigma+\min(1+\sigma+\frac{d}{2}, s-1)}_{loc}$ .*

An important property of paracomposition operator  $\chi^*$  is the following conjugation formula,

**Proposition 4.C.24.** *Let  $\chi \in H^{1+\frac{d}{2}+\sigma}_{loc}$  be as in Proposition 4.C.23 and  $a \in \Gamma_r^m$  with  $m \in \mathbb{R}$  and  $r \geq 0$ . The  $X^*$  defined in Proposition 4.C.23 satisfies*

$$(4.C.22) \quad X^*T_a = T_{\chi^*a}X^* + R,$$

*where  $\chi^*$  is the pull-back by  $\chi$  and the remainder  $R$  is also a paradifferential operator with symbol in  $\Gamma_0^{m-r}$ .*

**Remark 4.C.25.** *The rigorous definition of the pull-back  $\chi^*a$  depends on a delicate study of pseudo-differential operators on domains (or smooth manifolds), which will not be precised here (refer to [Hör07], Theorem 18.1.17). In the case where  $a$  is a differential operator  $a(s, \xi) = \sum_\alpha a_\alpha(x)\xi^\alpha$  (Laplacian operator, for example),  $\chi^*a$  equals the symbol of the pull-back of  $\text{Op}(a)$ , which is clearly a differential operator and can be calculated simply by usual change of variable.*

**Remark 4.C.26.** *An explicit definition of paracomposition operator  $X^*$  is*

$$X^*u := \sum_j \Delta_j (\Delta_j u \circ \chi),$$

*which is hard to use in applications. Usually, we may define  $X^*u$  via (4.C.21) when the remainder is not important.*

#### 4.D . Variation in metric

This section is devoted to the proof of identity (4.2.40), which can be obtained via a direct but complicated calculus.

**Lemma 4.D.1.** *Under the hypotheses of Proposition 4.2.5, we have*

$$(4.D.1) \quad \begin{aligned} \frac{1}{2} \delta(\sqrt{g} g^{\alpha\beta}) \partial_\alpha \varphi \partial_\beta \varphi &= \frac{1}{2} \nabla_{y,z}^T \left( J^{-1} Y |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta \right) - \nabla_{y,z}^T \left( \sqrt{g} J^{-1} \nabla_g \varphi Y^T \nabla_g \varphi \delta \zeta \right) \\ &\quad + \sqrt{g} \Delta_g \varphi Y \cdot \nabla_g \varphi \delta \zeta. \end{aligned}$$

Recall that, in the coordinate  $(y_1, y_2, z)$ , Greek letters are indices for  $\{1, 2, z\}$ , while Latin letters correspond to  $\{1, 2\}$ .

*Proof.* To begin with, by using  $(g^{\alpha\beta}) = J^{-1} J^{-T}$ , one could right the left hand side as

$$\begin{aligned} \frac{1}{2} \delta(\sqrt{g} g^{\alpha\beta}) \partial_\alpha \varphi \partial_\beta \varphi &= \frac{1}{2} \delta(\sqrt{g}) g^{\alpha\beta} \varphi_\alpha \varphi_\beta + \frac{1}{2} \nabla_{y,z}^T \varphi \delta (J^{-1} J^{-T}) \nabla_{y,z} \varphi \sqrt{g} \\ &= \frac{1}{2} \delta(\det J) |\nabla_g \varphi|^2 - \nabla_{y,z}^T \varphi J^{-1} \delta J J^{-1} J^{-T} \nabla_{y,z} \varphi \sqrt{g} \\ &= \frac{1}{2} \operatorname{tr} (J^{-1} \delta J) |\nabla_g \varphi|^2 \sqrt{g} - \nabla_g^T \varphi \delta J J^{-1} \nabla_g \varphi \sqrt{g}, \end{aligned}$$

where we use formulas (4.2.11), (4.2.12), and

$$\delta(\det J) = \operatorname{tr} (J^{-1} \delta J) \det J \quad \text{and} \quad \delta(J^{-1}) = -J^{-1} \delta J J^{-1}.$$

From the definition (4.2.9), one may compute

$$J^{-1} \delta J = \begin{pmatrix} J_0^{-1} \delta \zeta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J_0^{-1} y \nabla_y^T \delta \zeta & J_0^{-1} y \delta \zeta_z \\ 0 & 0 \end{pmatrix},$$

with  $J_0 = (\zeta \delta_{ij} + y_i \zeta_j)_{ij}$ . As a result,

$$\begin{aligned} &\frac{1}{2} \operatorname{tr} (J^{-1} \delta J) |\nabla_g \varphi|^2 \sqrt{g} \\ &= \frac{1}{2} \operatorname{tr} (J_0^{-1}) |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta + \frac{1}{2} \operatorname{tr} (J_0^{-1} y \nabla_y^T \delta \zeta) |\nabla_g \varphi|^2 \sqrt{g} \\ &= \frac{1}{2} \operatorname{tr} (J_0^{-1}) |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta + \frac{1}{2} J_0^{-1} y \cdot \nabla_y \delta \zeta |\nabla_g \varphi|^2 \sqrt{g} \\ &= \frac{1}{2} \operatorname{tr} (J_0^{-1}) |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta + \frac{1}{2} \nabla_y \cdot (J_0^{-1} y \delta \zeta |\nabla_g \varphi|^2 \sqrt{g}) - \frac{1}{2} \nabla_y \cdot (J_0^{-1} y |\nabla_g \varphi|^2 \sqrt{g}) \delta \zeta \\ &= \frac{1}{2} \operatorname{tr} (J_0^{-1}) |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta + \frac{1}{2} \nabla_y \cdot (J_0^{-1} y \delta \zeta |\nabla_g \varphi|^2 \sqrt{g}) - \frac{1}{2} \nabla_y \cdot (J_0^{-1} y |\nabla_g \varphi|^2) \sqrt{g} \delta \zeta \\ &\quad - \frac{1}{2} |\nabla_g \varphi|^2 J_0^{-1} y \cdot (\nabla_y \sqrt{g}) \delta \zeta \\ &= \frac{1}{2} \operatorname{tr} (J_0^{-1}) |\nabla_g \varphi|^2 \sqrt{g} \delta \zeta + \frac{1}{2} \nabla_y \cdot (J_0^{-1} y \delta \zeta |\nabla_g \varphi|^2 \sqrt{g}) - \frac{1}{2} \nabla_y \cdot (J_0^{-1} y |\nabla_g \varphi|^2) \sqrt{g} \delta \zeta \end{aligned}$$

$$-\frac{1}{2}|\nabla_g\varphi|^2 J_0^{-1}y \cdot \text{tr}(J_0^{-1}\nabla_y J_0) \sqrt{g}\delta\zeta.$$

Note that the last equality is a consequence of

$$\nabla_y\sqrt{g} = \nabla_y \det J_0 = \text{tr}(J_0^{-1}\nabla_y J_0) \det J_0 = \text{tr}(J_0^{-1}\nabla_y J_0) \sqrt{g}.$$

Therefore, we obtain the identity

$$\frac{1}{2} \text{tr}(J^{-1}\delta J) |\nabla_g\varphi|^2 \sqrt{g} = r_1 \sqrt{g}\delta\zeta + \frac{1}{2} \nabla_{y,z}^T (J^{-1}Y\delta\zeta |\nabla_g\varphi|^2 \sqrt{g}),$$

where

$$(4.D.2) \quad r_1 = \frac{1}{2} \text{tr}(J_0^{-1}) |\nabla_g\varphi|^2 - \frac{1}{2} \nabla_y \cdot (J_0^{-1}y |\nabla_g\varphi|^2) - \frac{1}{2} |\nabla_g\varphi|^2 J_0^{-1}y \cdot \text{tr}(J_0^{-1}\nabla_y J_0).$$

In the mean time, it is clear that

$$\delta J = \begin{pmatrix} \delta\zeta + y\nabla_y^T\delta\zeta & y\delta\zeta_z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta\zeta & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} y\nabla_y^T\delta\zeta & y\delta\zeta_z \\ 0 & 0 \end{pmatrix},$$

which yields

$$\begin{aligned} & \nabla_g^T\varphi\delta J J^{-1}\nabla_g\varphi\sqrt{g} \\ &= \nabla_g^i\varphi (J^{-1}\nabla_g\varphi)^i \sqrt{g}\delta\zeta + y_i\nabla_g^i\varphi\delta\zeta_\alpha (J^{-1}\nabla_g\varphi)^\alpha \sqrt{g} \\ &= \nabla_g^i\varphi (J^{-1}\nabla_g\varphi)^i \sqrt{g}\delta\zeta + \nabla_{y,z} \cdot (J^{-1}\nabla_g\varphi y_i\nabla_g^i\varphi\delta\zeta\sqrt{g}) - \nabla_{y,z} \cdot (\sqrt{g}J^{-1}\nabla_g\varphi y_i\nabla_g^i\varphi) \delta\zeta \\ &= \nabla_g^i\varphi (J^{-1}\nabla_g\varphi)^i \sqrt{g}\delta\zeta + \nabla_{y,z}^T (J^{-1}\nabla_g\varphi Y^T\nabla_g\varphi\delta\zeta\sqrt{g}) - \sqrt{g}J^{-1}\nabla_g\varphi \cdot \nabla_{y,z} (y_i\nabla_g^i\varphi) \delta\zeta \\ & \quad - \sqrt{g}\Delta_g\varphi Y^T\nabla_g\varphi\delta\zeta, \end{aligned}$$

where the last equality is due to (4.2.12) and (4.2.13). Then the following identity holds true,

$$\nabla_g^T\varphi\delta J J^{-1}\nabla_g\varphi\sqrt{g} = r_2\sqrt{g}\delta\zeta + \nabla_{y,z}^T (J^{-1}\nabla_g\varphi Y^T\nabla_g\varphi\delta\zeta) - \sqrt{g}\Delta_g\varphi Y^T\nabla_g\varphi\delta\zeta,$$

with

$$(4.D.3) \quad r_2 = \nabla_g^i\varphi (J^{-1}\nabla_g\varphi)^i - (J^{-1}\nabla_g\varphi \cdot \nabla_{y,z}) (y_i\nabla_g^i\varphi).$$

It follows that the desired result (4.D.1) is equivalent to

$$r_1 - r_2 = 0.$$

We notice that

$$r_1 = -((J_0^{-1}y \cdot \nabla_y) \nabla_g^\alpha\varphi) \nabla_g^\alpha\varphi - \frac{1}{2}\partial_i a^{ij} y_j |\nabla_g\varphi|^2 - \frac{1}{2} J_0^{-1}y \cdot \text{tr}(J_0^{-1}\nabla_y J_0) |\nabla_g\varphi|^2$$

(recall that  $(a^{ij})$  is defined in (4.2.10) with  $i, j$  taken in  $\{y_1, y_2\}$ ), and

$$r_2 = -y_i (J^{-1}\nabla_g\varphi \cdot \nabla_{y,z}) \nabla_g^i\varphi.$$

Then the problem is reduced the following identities

$$(4.D.4) \quad -\partial_i a^{ij} y_j = J_0^{-1} y \cdot \text{tr} (J_0^{-1} \nabla_y J_0),$$

$$(4.D.5) \quad ((J_0^{-1} y \cdot \nabla_y) \nabla_g^\alpha \varphi) \nabla_g^\alpha \varphi = y_i (J^{-1} \nabla_g \varphi \cdot \nabla_{y,z}) \nabla_g^i \varphi.$$

To prove these, we shall use the following formulas which can be checked easily from (4.2.9),

$$(4.D.6) \quad \partial_\alpha a^{\beta\gamma} = -a^{\beta\beta'} \partial_\alpha a_{\beta'\gamma'} a^{\gamma'\gamma},$$

$$(4.D.7) \quad \partial_\alpha a_{z\beta} = 0,$$

$$(4.D.8) \quad \partial_\alpha a_{i\beta} = \delta_{i\beta} \zeta_\alpha + \delta_{\alpha i} \zeta_\beta + y_i \zeta_{\alpha\beta}.$$

Note that we write the version for  $J$ , while the same formulas hold for  $J_0$ , which means one could replace the Greek letters by Latin ones.

We first check (4.D.4). By (4.D.8), the left hand side is equal to

$$\begin{aligned} -\partial_i a^{ij} y_j &= a^{ik} \partial_i a_{kl} a^{lj} y_j \\ &= a^{ik} (\delta_{kl} \zeta_i + \delta_{ik} \zeta_l + y_k \zeta_{il}) a^{lj} y_j \\ &= \zeta_i a^{ik} a^{kj} y_j + \zeta_l a^{lj} y_j a^{ii} + a^{ik} y_k a^{lj} y_j \zeta_{il}, \end{aligned}$$

while the right hand side reads

$$\begin{aligned} J_0^{-1} y \cdot \text{tr} (J_0^{-1} \nabla_y J_0) &= a^{ij} y_j a^{kl} \partial_i a_{lk} \\ &= a^{ij} y_j a^{kl} (\delta_{lk} \zeta_i + \delta_{il} \zeta_k + y_l \zeta_{ik}) \\ &= \zeta_i a^{ij} y_j a^{kk} + \zeta_k a^{ki} a^{ij} y_j + a^{ij} y_j a^{kl} y_l \zeta_{ik}, \end{aligned}$$

which proves (4.D.4).

As for (4.D.5), the left hand side equals

$$\begin{aligned} &a^{ij} y_j \partial_i (a^{\beta\alpha} \varphi_\beta) \nabla_g^\alpha \varphi \\ &= a^{ij} y_j \partial_i a^{\beta\alpha} \varphi_\beta \nabla_g^\alpha \varphi + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi \\ &= -a^{ij} y_j a^{\beta\lambda} \partial_i a_{\lambda\mu} a^{\mu\alpha} \varphi_\beta \nabla_g^\alpha \varphi + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi \\ &= -a^{ij} y_j a^{\beta k} \partial_i a_{k\mu} a^{\mu\alpha} \varphi_\beta \nabla_g^\alpha \varphi + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi \\ &= -a^{ij} y_j a^{\beta k} (\delta_{k\mu} \zeta_i + \delta_{ik} \zeta_\mu + y_k \zeta_{i\mu}) a^{\mu\alpha} \varphi_\beta \nabla_g^\alpha \varphi + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi \\ &= -\zeta_i a^{ij} y_j \varphi_\beta a^{\beta k} a^{k\alpha} \nabla_g^\alpha \varphi - \varphi_\beta a^{\beta i} a^{ij} y_j \zeta_\mu a^{\mu\alpha} \nabla_g^\alpha \varphi - \varphi_\beta a^{\beta k} y_k a^{ij} y_j \zeta_{i\mu} a^{\mu\alpha} \nabla_g^\alpha \varphi \\ &\quad + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi \\ &= -\zeta_i a^{ij} y_j \varphi_l a^{lk} a^{k\alpha} \nabla_g^\alpha \varphi - \varphi_l a^{li} a^{ij} y_j \zeta_\mu a^{\mu\alpha} \nabla_g^\alpha \varphi - \varphi_l a^{lk} y_k a^{ij} y_j \zeta_{i\mu} a^{\mu\alpha} \nabla_g^\alpha \varphi \\ &\quad + a^{ij} y_j a^{\beta\alpha} \varphi_{i\beta} \nabla_g^\alpha \varphi, \end{aligned}$$

while the right hand side can be written as

$$y_i a^{\beta\alpha} \nabla_g^\alpha \varphi \partial_\beta (a^{\gamma i} \varphi_\gamma)$$

$$\begin{aligned}
&= y_i a^{\beta\alpha} \nabla_g^\alpha \varphi \partial_\beta (a^{ji} \varphi_j) \\
&= y_i a^{\beta\alpha} \partial_\beta a^{ji} \varphi_j \nabla_g^\alpha \varphi + a^{ji} y_i a^{\beta\alpha} \varphi_{j\beta} \nabla_g^\alpha \varphi \\
&= - y_i a^{\beta\alpha} a^{jk} \partial_\beta a_{kl} a^{li} \varphi_j \nabla_g^\alpha \varphi + a^{ji} y_i a^{\beta\alpha} \varphi_{j\beta} \nabla_g^\alpha \varphi \\
&= - y_i a^{\beta\alpha} a^{jk} (\delta_{kl} \zeta_\beta + \delta_{k\beta} \zeta_l + y_k \zeta_{\beta l}) a^{li} \varphi_j \nabla_g^\alpha \varphi + a^{ji} y_i a^{\beta\alpha} \varphi_{j\beta} \nabla_g^\alpha \varphi \\
&= - \varphi_j a^{jk} a^{ki} y_i \zeta_\beta a^{\beta\alpha} \nabla_g^\alpha \varphi - \zeta_l a^{li} y_i \varphi_j a^{jk} a^{k\alpha} \nabla_g^\alpha \varphi - \varphi_j a^{jk} y_k a^{li} y_i \zeta_{\beta l} a^{\beta\alpha} \nabla_g^\alpha \varphi \\
&\quad + a^{ji} y_i a^{\beta\alpha} \varphi_{j\beta} \nabla_g^\alpha \varphi,
\end{aligned}$$

which gives (4.D.5). □

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## Liste des Symboles

- $a^*$  Adjoint of symbol, 246  
 $B$  Radial component of velocity fluid at free boundary, 155  
 $\mathcal{B}(\eta)$  Trace of radial derivative of harmonic extension, 190  
 $C_b^\infty$  Bounded smooth functions with every derivative bounded, 159  
 $C_T \mathcal{X}_R^s$  Solution space (continuous in time), 220  
 $a \# b$  Composition of symbols, 246  
 $c_0, C_0$  Lower and upper bounds of  $\eta$ , 152  
 $\mathbb{D}$  2D Unit disk, 159  
 $\Delta_g$  Pull-back of  $\Delta_{x,z}$  by  $\iota$ , 164  
 $\frac{\delta}{\delta \eta}$  Variation in  $\eta$ , 167  
 $E_k$  Kinetic energy, 167  
 $E_p$  Potential energy, 167  
 $\approx$  Equivalence between parilinear operators, 177  
 $\eta$  Radius of free boundary, 152  
 $\sim$  Equivalence between functions, 181  
 $\mathfrak{F}$  Flow map, 156  
 $f_1$  Error in parilinearization of Dirichlet-to-Neumann operator, 191  
 $G(\eta)$  Dirichlet-to-Neumann operator, 154  
 $\Gamma_\rho^m$  Symbol class for parilinear operators, 249  
 $g$  Determinant of  $(g_{\alpha\beta})$ , 164  
 $g^{\alpha\beta}$  Inverse of  $(g_{\alpha\beta})$ , 164  
 $g_{\alpha\beta}$  Metric tensor of  $\iota$ , 164  
 $H$  Mean curvature of free boundary, 153  
 $H^s$  Sobolev space, 159  
 $H_W^s$  Sobolev space with normalization  $W$ , 159  
 $\mathcal{H}$  Hamiltonian, 167  
 $\iota$  Diffeomorphism to straight cylinder, 161  
 $J$  Jacobian matrix of  $\iota$ , 163  
 $J_\epsilon$  Smoothing operator, 221  
 $j_\epsilon$  Symbol of  $J_\epsilon$ , 221  
 $L_T^\infty \mathcal{X}_R^s$  Solution space (bounded in time), 220  
 $\mathcal{L}$  Principal operator in parilinearization, 196  
 $\mathcal{L}^\epsilon$  Approximate principal operator, 222  
 $\lambda$  Symbol of Dirichlet-to-Neumann operator, 191  
 $M_T^r$  Sobolev energy, 224  
 $\mu$  Symbol of mean curvature, 195  
 $N$  Nonlinear term, 155  
 $N_T^r$  Auxiliary Sobolev energy, 224  
 $\bar{\nabla}$  Modified gradient, 190  
 $\nabla_g$  Pull-back of  $\nabla_{x,z}$  by  $\iota$ , 164  
 $n$  Conormal vector of free boundary, 153  
 $n_g$  Pull-back of the conormal vector  $n$ , 165

- $\Omega(t)$  Domain of fluid, 152  
 $\text{Op}(a)$  Pseudo-differential operator, 245  
 $P$  Paralinearized Laplacian, 180  
 $\Phi$  Alinhac's good unknown, 180  
 $\phi$  Scalar potential, 154  
 $\psi$  Trace of scalar potential, 154  
 $R(a, b)$  Remainder in Bony's decomposition, 247  
 $r_2$  Error in paralinearization of nonlinear term, 195  
 $r_3$  Error in paralinearization of mean curvature, 195  
 $S$  Symmetrizer of the system, 214  
 $S_{\rho, \delta}^m$  Symbol class, 246  
 $\Sigma(t)$  Free boundary, 152  
 $\Sigma^m$  Homogeneous symbols depending on  $\eta$ , 177  
 $\mathbb{S}$  1D circle, 159  
 $T_a$  Paralinear operator (paraproduct), 247  
 $\mathbb{T}$  1D Torus, 159  
 $V$  Angular component of velocity fluid at free boundary, 155  
 $\mathcal{V}(\eta)$  Trace of angular derivative of harmonic extension, 190  
 $X^*$  Paracomposition operator, 252  
 $\mathcal{X}_R^s$  Data space, 220  
 $\zeta$  Extension of  $\eta$ , 163