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# THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

ÉCOLE DOCTORALE N° 601

*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,  
Électronique*

Spécialité : *Mathématiques et leurs interactions*

Par

**Louis DAILLY**

## Caractérisation numérique des quotients d'espaces projectifs complexes

Thèse présentée et soutenue à Rennes, le 8 juillet 2025

Unité de recherche : UMR CNRS 6625 - Institut de recherche mathématique de Rennes (IRMAR)

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Co-dir. de thèse :	Henri GUENANCIA	Directeur de recherche, CNRS et Université de Bordeaux



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---

Le premier théorème que j'aimerais énoncer dans cette thèse est le suivant :

**Théorème 1.** Soit  $M$  le présent manuscrit et  $E$  l'entourage de son auteur. Alors  $M$  est obtenu comme un barycentre des points de  $E$ .

Si je ne suis pas en mesure d'en donner une preuve complète, j'aimerais tout de même ici en donner une esquisse.

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# INTRODUCTION

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## Introduction en français

### Caractérisation des espaces projectifs par l'uniformisation

#### Uniformisation de variétés et courbure sectionnelle constante

A la fin du XIXe siècle, l'étude d'équations différentielles de la variable complexe amène à repenser la notion de fonctions : si on considère par exemple l'équation différentielle complexe sur  $\mathbb{C}^\times$  donnée par  $zy' = 1$ , on ne dispose pas de solution globalement définie. Le prix à payer pour disposer d'un objet global est de travailler avec des fonctions multivaluées, au sens où l'image d'un point est désormais un ensemble discret. L'étude de telles fonctions conduit Poincaré à l'introduction de la notion de revêtement et de groupe fondamental d'une variété complexe lisse  $X$ . Une correspondance existe entre ces deux objets et la notion de revêtement universel, qui englobe tous les revêtements possibles, émerge. La question de la description du revêtement universel d'une variété  $X$  se pose naturellement : c'est le problème de l'uniformisation. Celui-ci est de prime abord de nature purement topologique, mais on peut montrer qu'un revêtement d'une variété complexe peut être muni de manière unique d'une structure de variété complexe qui rend l'application du revêtement holomorphe. Lorsque des métriques riemanniennes ou hermitiennes apparaissent sur ces variétés, on demandera par ailleurs que l'application de revêtement soit une isométrie locale.

L'histoire de l'uniformisation des surfaces de Riemann (c'est-à-dire les variétés complexes de dimension complexe 1) est largement retracée dans [Sai10]. En 1907, Poincaré et Koebe classifient les surfaces de Riemann simplement connexes.

**Théorème 0.0.1.** [Sai10, Thm. p.31] *Toute surface de Riemann simplement connexe est biholomorphe à la sphère de Riemann  $\mathbb{P}^1$ , au plan complexe  $\mathbb{C}$  ou au disque unité  $\mathbb{D}$ .*

Si ce résultat clôt la classification des surfaces de Riemann simplement connexes, la résolution du problème de l'uniformisation des surfaces de Riemann compactes connexes était connue plus tôt grâce aux contributions de Clebsch en 1865 et Poincaré en 1884 :

**Théorème 0.0.2.** [Sai10, Thm. II.2.15, II.2.16, VIII.5.3] *Soit  $X$  une surface de Riemann compacte connexe de genre  $g$ . Alors, le revêtement universel de  $X$  est :*

- $\mathbb{P}^1$  si  $g = 0$ ,
- $\mathbb{C}$  si  $g = 1$ ,
- $\mathbb{D}$  si  $g \geq 2$ .

Qu'en est-il en dimension supérieure ? Avant d'explicitier une généralisation du théorème précédent, il est nécessaire d'interpréter ce résultat en termes de géométrie riemannienne. On a le théorème suivant dû à Killing et Hopf :

**Théorème 0.0.3.** *Soit  $(M, g)$  une variété riemannienne compacte de dimension réelle  $n$ . On suppose que la courbure sectionnelle de  $M$  est constante égale à  $\lambda \in \mathbb{R}$ . Alors, le revêtement universel  $(\tilde{M}, \tilde{g})$  de  $(M, g)$  est isométrique à l'un des trois modèles suivants :*

- la sphère munie de la métrique sphérique  $(\mathbb{S}^n, \frac{1}{\lambda}g_{\text{sph}})$  si  $\lambda > 0$ ,
- l'espace euclidien munie de la métrique euclidienne  $(\mathbb{R}^n, g_{\text{euc}})$  si  $\lambda = 0$ ,
- l'espace hyperbolique réel muni de la métrique hyperbolique  $(\mathbb{H}^n, -\frac{1}{\lambda}g_{\text{hyp}})$  si  $\lambda < 0$ .

Pour  $n = 2$ , on retrouve les trois variétés lisses précédentes, compte tenu du fait que  $\mathbb{P}^1 \simeq \mathbb{S}^2$ ,  $\mathbb{R}^2 \simeq \mathbb{C}$  et  $\mathbb{H}^2 \simeq \mathbb{D}$ . On peut être un peu plus précis car en dimension réelle 2, il y a un lien exceptionnel entre les structures riemanniennes et complexes qu'on peut mettre sur une surface. En effet, il se trouve que les transformations conformes coïncident avec les biholomorphismes. Ainsi, une surface orientée munie d'une métrique riemannienne détermine une unique structure de surface de Riemann. Il en découle qu'une isométrie entre surfaces munies de métriques riemanniennes induit un biholomorphisme entre les structures complexes sous-jacentes.

Revenons au cas où l'on dispose d'une structure complexe sur notre variété  $X$ . Dans ce contexte, on peut également munir la variété lisse  $M$  sous-jacente d'une métrique riemannienne  $\omega$ . On dit que la métrique est kählérienne<sup>1</sup> si elle est compatible avec la structure complexe de  $X$ . On a alors un avatar complexe du théorème de Killing et Hopf. Hawley et Igusa établissent indépendamment une classification semblable des revêtements universels des variétés kählériennes en 1953-1954. Celle-ci a été reprise par Kobayashi et Nomizu [KN69, Thm. IX.7.8-9] et s'énonce de la manière suivante.

---

1. Plus précisément, cette métrique induit une connexion  $\nabla_{\omega, \text{End}(T_M)}$  sur le fibré  $\text{End}(T_M)$ . On dit que la métrique est kählérienne si la structure presque complexe  $J \in \mathcal{C}^\infty(M, \text{End}(T_M))$  est parallèle par rapport à cette connexion :  $\nabla_{\omega, \text{End}(T_M)} J = 0$ .

**Théorème 0.0.4.** [Haw53; Igu54] *Soit  $(X, \omega)$  une variété kählérienne complète de dimension complexe  $n$ . On suppose que  $\omega$  est à courbure sectionnelle holomorphe<sup>2</sup> constante égale à  $\lambda \in \mathbb{R}$ .*

*Alors, le revêtement universel  $(\widetilde{X}, \widetilde{\omega})$  est isométriquement biholomorphe à :*

- *l'espace projectif muni de la métrique de Fubini–Study  $(\mathbb{P}^n, \frac{1}{\lambda}\omega_{FS})$  si  $\lambda > 0$ ,*
- *l'espace euclidien muni de la métrique euclidienne  $(\mathbb{C}^n, \omega_{\text{euc}})$  si  $\lambda = 0$ ,*
- *la boule unité dans  $\mathbb{C}^n$  muni de la métrique de Bergman  $(\mathbb{B}^n, -\frac{1}{\lambda}\omega_{\text{Ber}})$  si  $\lambda < 0$ .*

### De la constance de la courbure sectionnelle à l'égalité de Miyaoka–Yau

Les trois différents modèles fournissent en particulier des métriques de Kähler–Einstein, c'est-à-dire qu'elles vérifient  $\text{Ric } \widetilde{\omega} = \lambda \widetilde{\omega}$ . Ceci nous permet d'en conclure qu'une variété kählérienne  $(X, \omega)$  à courbure sectionnelle holomorphe constante est en particulier une variété Kähler–Einstein. On peut se demander si ces deux conditions sont équivalentes. En 1974, Chen et Ogiue montrent que la constance de la courbure sectionnelle holomorphe est équivalente à une égalité de classes de Chern, si on suppose que la métrique est Kähler–Einstein :

**Théorème 0.0.5.** [CO75, Thm. 2] *Soit  $(X, \omega)$  une variété kählérienne compacte de dimension  $n$ . On suppose que  $\omega$  est une métrique Kähler–Einstein. Alors, on a l'inégalité suivante :*

$$\left(2(n+1)c_2(X) - nc_1(X)^2\right) \cdot [\omega]^{n-2} \geq 0.$$

*De plus, l'égalité est satisfaite dans l'inégalité précédente si et seulement si la courbure sectionnelle holomorphe de  $\omega$  est constante.*

Cette inégalité est connue sous le nom de *inégalité de Miyaoka–Yau*, Miyaoka ayant montré la formule  $c_1(X)^2 \leq 3c_2(X)$  lorsque  $X$  est une surface de type général (voir [Miy77, Thm. 4]) . On en déduit le théorème suivant :

**Théorème 0.0.6.** *Soit  $(X, \omega)$  une variété kählérienne compacte de dimension  $n$  telle que les deux conditions suivantes sont satisfaites :*

(KE $_{\lambda}$ ) *La métrique  $\omega$  est Kähler–Einstein :  $\text{Ric } \omega = \lambda \omega$  pour  $\lambda \in \{-1, 0, 1\}$ ,*

(MY) *L'égalité suivante est satisfaite :*

$$\left(2(n+1)c_2(X) - nc_1(X)^2\right) \cdot [\omega]^{n-2} = 0.$$

---

2. Il s'agit de la courbure sectionnelle restreinte aux plans tangents holomorphes

Alors, le revêtement universel de  $(X, \omega)$  est isométriquement isomorphe à :

- $(\mathbb{P}^n, \omega_{FS})$  si  $\lambda = 1$ ,
- $(\mathbb{C}^n, \omega_{\text{euc}})$  si  $\lambda = 0$ ,
- $(\mathbb{B}^n, \omega_{\text{Ber}})$  si  $\lambda = -1$ .

Dans le cas où  $\lambda = 1$  dans le théorème précédent, on dispose d'un isomorphisme  $(X, \omega) \simeq (\mathbb{P}^n, \omega_{FS})$ . En d'autres termes, ceci fournit une caractérisation de  $(\mathbb{P}^n, \omega_{FS})$ . En effet, la variété  $X$  est biholomorphe à un quotient de  $\mathbb{P}^n$  par un groupe  $G$  agissant proprement et librement. La propriété de l'action implique que  $G$  doit être fini, et la liberté impose que  $G$  doit être trivial. Ainsi, l'application quotient  $\mathbb{P}^n \rightarrow X$  est un biholomorphisme.

### Quelques variations sur les hypothèses du théorème 0.0.6

Compte tenu de la résolution du problème d'existence de métrique de Kähler–Einstein sur une variété compacte  $X$  satisfaisant  $c_1(X) < 0$  par Aubin et Yau, une variété  $X$  dont le fibré canonique est ample est un quotient de la boule si et seulement si l'égalité

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot c_1(X)^{n-2} = 0$$

est satisfaite. D'autre part, une variété kählérienne  $X$  satisfaisant  $c_1(X) = 0$  peut être munie d'une métrique Ricci plate  $\omega$  par la résolution du problème de Calabi due à Yau [Yau78]. Si on suppose de plus que  $c_2(X) = 0$ , on obtient que la variété  $X$  est un quotient fini d'un tore complexe comme conséquence d'un théorème de Bieberbach sur les groupes cristallographiques (voir [Kob87, Cor. 4.4.15]).

Enfin, dans le cas où  $c_1(X) > 0$ , Tian montre en 1992 [Tia92, Thm. 0.1] que l'hypothèse  $(KE_{>0})$  implique la condition suivante :

$$(ECP) \text{ L'extension canonique } E_X \text{ est polystable par rapport à } c_1(X).$$

On rappelle d'abord la définition d'extension canonique. Soit  $S$  et  $Q$  deux fibrés vectoriels sur  $X$ . Alors, on a une correspondance entre les classes d'isomorphisme d'extension de  $Q$  par  $S$  et le groupe  $H^1(X, \text{Hom}(Q, S))$ . Lorsque  $Q = T_X$  et  $S = \mathcal{O}_X$ , on a  $H^1(X, \text{Hom}(Q, S)) = H^1(X, \Omega_X^1)$  qui contient la classe  $c_1(X)$ . Ainsi, la classe d'extension correspondante est celle de l'extension canonique  $E_X$ . C'est donc un fibré vectoriel qui est une extension du fibré tangent  $T_X$  par le fibré trivial  $\mathcal{O}_X$ . En particulier, les classes

de Chern de l'extension canonique sont exactement les classes de Chern de la variété par la formule de Whitney pour les extensions de fibrés :  $c_i(E_X) = c_i(X)$  pour tout  $i$ .

On peut également considérer l'assertion (ECS) en remplaçant dans la condition précédente la polystabilité par la semistabilité de l'extension canonique. Ceci nous permet, dans une autre direction, de donner la caractérisation de  $\mathbb{P}^n$  suivante.

**Théorème 0.0.7.** *Soit  $X$  une variété complexe compacte telle que  $c_1(X) > 0$  et satisfaisant les deux hypothèses suivantes :*

(ECS) *l'extension canonique  $E_X$  est semi-stable par rapport à  $c_1(X)$ ,*

(MY) *l'égalité suivante est satisfaite :  $(2(n+1)c_2(X) - nc_1(X)^2) \cdot c_1(X)^{n-2} = 0$ .*

*Alors,  $X$  est biholomorphe à  $\mathbb{P}^n$ .*

Donnons une idée de la preuve de ce théorème. On montre d'abord que l'extension canonique  $E_X$  est projectivement plate. Ceci est une conséquence de la correspondance de Simpson appliquée à  $F = \text{End}(E_X)$ , puisque  $F$  est encore semi-stable par rapport à  $c_1(X)$  et on a :

$$\text{ch}_1(F) \cdot c_1(X)^{n-1} = \text{ch}_2(F) \cdot c_1(X)^{n-2} = 0,$$

si bien que le fibré des endomorphismes  $F$  est plat. Ainsi, il en découle que  $E_X$  est projectivement plat. Or, la variété  $X$  étant Fano, elle est simplement connexe [Bes08, Thm. 11.26], donc on en déduit que le fibré projectif  $\mathbb{P}(E_X)$  associé à  $E_X$  est trivial. Il existe ainsi un fibré en droites ample  $L$  tel que  $E_X = L^{\oplus n+1}$ . L'égalité suivante :

$$c_1(E_X) = c_1(X) = (n+1)c_1(L)$$

couplée avec le théorème de Kobayashi–Ochiai [KO73, Cor. of Thm.1] permet de conclure que  $X \simeq \mathbb{P}^n$ .

## Le problème de l'uniformisation dans le cas singulier

### Singularités et ramification de l'application quotient

Donnons d'abord une autre approche du problème d'uniformisation pour des variétés qui rendra naturelle la généralisation au cadre singulier. On considère une variété lisse  $Y$  ainsi qu'un sous-groupe discret  $G$  du groupe des automorphismes de  $Y$ . On peut considérer l'espace quotient  $X := Y/G$ . C'est a priori simplement un espace topologique. Toutefois, sous l'hypothèse que l'action de  $G$  sur  $Y$  est propre et libre, l'espace quotient peut être

muni d'une unique structure complexe telle que  $X$  soit lisse et l'application quotient  $\pi_G : Y \rightarrow X$  soit holomorphe. C'est d'ailleurs une application étale, c'est-à-dire un biholomorphisme local qui fournit donc un revêtement galoisien de  $X$ , de groupe de Galois  $G$ . Le problème de l'uniformisation peut se reformuler de la façon suivante : étant donné  $X$ , peut-on retrouver tous les couples  $(Y, G)$  tels que  $X$  est obtenu comme quotient de  $Y$  par une action de  $G$  propre et libre sur  $Y$  ? Bien sûr, on dispose d'un couple universel  $(\widetilde{X}, \pi_1(X))$  où  $\widetilde{X}$  est le revêtement universel de  $X$  et  $\pi_1(X)$  est le groupe fondamental de  $X$ . On peut déduire tous les couples  $(Y, G)$  possibles par la théorie des revêtements et la correspondance de Galois géométrique.

Discutons des hypothèses lorsque  $G$  n'agit plus de manière propre et libre sur  $Y$ . Afin de conserver la séparation de l'espace des orbites, il est nécessaire de conserver la propriété de l'action de  $G$  sur  $Y$ . Si on supprime l'hypothèse de liberté, deux phénomènes apparaissent.

D'une part, l'espace  $X$  peut avoir des singularités. Il a été montré par Cartan en 1957 que l'espace des orbites  $X$  est muni dans ce cas d'une structure d'espace analytique complexe normal, et cette structure peut être décrite explicitement [Car57]. L'espace  $X$  est une *orbifold* : c'est un espace topologique dont les modèles locaux sont des quotients de  $\mathbb{C}^n$  par des groupes finis agissant linéairement.

D'autre part, l'application quotient  $\pi_G : Y \rightarrow X$  n'est plus une application de revêtement topologique : on dit que c'est un revêtement ramifié. Une première question est donc d'adapter les revêtements universels et groupes fondamentaux dans ce contexte. Une hypothèse simplificatrice est d'abord de considérer que le lieu de branchement de  $\pi_G$  est constitué de composantes de codimension au moins 2. Alors, par le théorème de pureté du lieu de branchement de Nagata–Zariski, ceci indique que  $\pi_G$  est étale au-dessus de  $X_{\text{reg}}$ , et donc le lieu de ramification est inclus dans un ensemble analytique de codimension au moins 2 : on dit que  $\pi_G$  est un *revêtement quasi-étale*. Ainsi, l'ensemble des points qui ont un stabilisateur non-trivial est un ensemble analytique qui doit être de codimension au moins 2 : on dit que  $G$  agit sans point fixe en codimension 1. Dans cette situation, l'application  $\pi_G$  induit un revêtement au-dessus de  $X_{\text{reg}}$ . Ainsi, le groupe fondamental de  $X_{\text{reg}}$  donne des informations sur les revêtements quasi-étales de  $X$ .

## État de l'art de l'uniformisation dans le cas singulier

Dans cette optique, on peut se demander si les caractérisations précédentes s'appliquent dans le cadre singulier. De nombreux résultats ont été établis ces dernières années

généralisant les théorèmes précédents, dans des contextes où les singularités sont supposées *Kawamata log terminales* (ou *klt*) : c'est une classe de singularités naturelle en géométrie algébrique plus large que la classe des singularités orbifoldes. Dans ce cadre, les résultats d'uniformisation qu'on souhaite avoir requièrent de définir les classes de Chern orbifoldes dans un contexte singulier, ainsi que la notion d'extension canonique associée à  $X$ .

Initialement, les premiers résultats d'uniformisation dans le cadre singulier concernaient le cas plat, c'est-à-dire le cas où on suppose  $c_1(X) = 0$ . Les articles suivants énoncent des résultats avec de nombreuses variantes : d'abord sur la codimension du lieu singulier, puis sur le caractère projectif ou kählérien de la variété  $X$  qu'on veut uniformiser, on se reportera par exemple à [GKP16a ; LT18 ; GK20 ; GKP22 ; CGG22].

Dans l'article [Gre+19b], Greb, Kebekus, Peternell et Taji traitent le cas où  $c_1(X) < 0$ , et ils donnent une caractérisation des quotients singuliers de la boule pour des variétés klt.

Enfin, dans l'article [GKP22], Greb, Kebekus et Peternell donnent notamment une caractérisation des quotients de  $\mathbb{P}^n$ , via une généralisation du théorème 0.0.7 :

**Théorème 0.0.8.** [GKP22, Thm. 1.3] *Soit  $X$  une variété projective Fano à singularités klt. Alors, il existe un groupe fini  $G$  agissant sans point fixe en codimension 1 sur  $\mathbb{P}^n$  tel que  $X \simeq \mathbb{P}^n/G$  si et seulement s'il existe un diviseur ample  $H$  sur  $X$  tel que les deux assertions suivantes sont satisfaites :*

(ECS) *L'extension canonique  $\mathcal{E}_X$  est semi-stable par rapport à  $H$ ,*

(MY) *L'égalité suivante est satisfaite :*

$$\left(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1(X)^2\right) \cdot H^{n-2} = 0.$$

Dans le théorème précédent, les auteurs introduisent la notion d'extension canonique  $\mathcal{E}_X$  de  $X$  : c'est un faisceau réflexif sur  $X$  qui est obtenu comme extension du faisceau tangent  $\mathcal{T}_X$  par le faisceau trivial  $\mathcal{O}_X$ . L'égalité de Miyaoka–Yau fait intervenir des classes de Chern orbifoldes introduites par Mumford en 1983 [Mum83, §2-3].

Donnons une idée de la preuve de ce théorème. Dans le cas où  $X$  est une variété projective normale à singularités klt, on dispose d'un revêtement maximale-ment quasi-étale  $\pi : Y \rightarrow X$ . C'est une application finie galoisienne quasi-étale, tel que  $Y$  est une variété klt et l'inclusion  $i : Y_{\text{reg}} \hookrightarrow Y$  induit un isomorphisme de groupes :  $\widehat{\pi}_1(Y_{\text{reg}}) \simeq \widehat{\pi}_1(Y)$ , où  $\widehat{\pi}_1$  est la complétion profinie du groupe fondamental. C'est ce revêtement qui fera

office d'application uniformisante, si bien qu'on va montrer que  $Y \simeq \mathbb{P}^n$ . Les propriétés Fano, (ECS) et (MY) se transférant à  $Y$ , il suffit de montrer que  $Y$  est lisse, et la preuve précédente dans le cas lisse permet de conclure.

Pour montrer que  $Y$  est lisse, on utilise la résolution de la conjecture de Zariski–Lipman pour les variétés klt (voir par exemple [Gre+11 ; Dru14 ; Gra15]). Ainsi, il s'agit de montrer que le faisceau tangent  $\mathcal{T}_Y$  de  $Y$  est localement libre. Pour cela, on utilise une version klt de la correspondance de Simpson [LT18, Thm. 1.4]. En effet, les hypothèses du théorème permettent d'établir que le faisceau  $\mathcal{F} := \text{End}(\mathcal{E}_Y)$  est semi-stable par rapport à  $H$  et satisfait l'égalité de classes de Chern orbifoldes :

$$\widehat{\text{ch}}_1(\mathcal{F}) \cdot H^{n-1} = \widehat{\text{ch}}_2(\mathcal{F}) \cdot H^{n-2} = 0,$$

La correspondance de Simpson dans ce cadre singulier donne la platitude de  $\mathcal{F}$  sur  $Y_{\text{reg}}$ . Puisque  $Y$  est maximale quasi-étale, il existe une extension plate, donc localement libre de  $\mathcal{F}$  à  $Y$  tout entier, qui coïncide avec  $\mathcal{F}$  puisque les deux faisceaux sont réflexifs [GKP16a, Thm. 1.14]. Alors, on obtient que  $\mathcal{F}$  est localement libre. Ainsi, autour d'un point  $y \in Y$ , on dispose d'un ouvert affine  $U$  dans lequel on a un scindage :

$$\mathcal{F}|_U = \mathcal{O}_U^{\oplus(n+1)^2} = \mathcal{O}_U \oplus \mathcal{T}_U \oplus \Omega_U^{[1]} \oplus \text{End}(\mathcal{T}_U).$$

Une conséquence du lemme de Nakayama [Kob87, Lemma. 5.5.3], donne que  $\mathcal{T}_U \simeq \mathcal{O}_U^{\oplus n}$ . Ainsi,  $\mathcal{T}_Y$  est localement libre.

Enfin, pour résoudre en toute généralité le problème d'uniformisation, il faut considérer des revêtements ramifiés qui peuvent avoir des phénomènes de branchement en codimension 1. Ceci soulève le problème suivant : ces comportements divisoriels ne sont plus totalement encodés dans la géométrie de  $X$ . En effet, lorsque l'on considèrerait des revêtements quasi-étales, on disposait du fait que le branchement de l'application quotient devait être concentré sur le lieu singulier de  $X$ . Pour résoudre ceci, on doit spécifier un diviseur supplémentaire auquel on doit penser comme un lieu de singularités en codimension 1. Dans cette optique, les bons objets géométriques à considérer sont les *paires* : ce sont des couples  $(X, \Delta)$  constitués d'une variété normale  $X$  et d'un  $\mathbb{Q}$ -diviseur de Weil à coefficients standards, c'est-à-dire de la forme  $\Delta := \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$  où  $m_i \in \mathbb{N}_{\geq 2}$ .

Dans ce cadre, Claudon, Graf et Guenancia caractérisent les paires  $(X, \Delta)$  qui sont

obtenues comme quotients par des groupes agissant proprement sur  $\mathbb{C}^n$  et  $\mathbb{B}^n$  en termes d'égalité de Miyaoka–Yau et de positivité de  $K_X + \Delta$  dans [CGG24, Thm. A-B].

## Présentation des résultats

L'objectif de la thèse est d'étudier une caractérisation des paires  $(X, \Delta)$  obtenues comme quotients de  $\mathbb{P}^n$  via l'égalité de Miyaoka–Yau, avec pour motivation une généralisation du théorème 0.0.8. Plus précisément, on montre le résultat suivant.

**Théorème 0.0.9.** *Soit  $(X, \Delta)$  une paire log Fano à coefficients standards de dimension  $n$ .*

*On suppose que les deux assertions suivantes sont satisfaites :*

(ECS) *L'extension canonique  $\mathcal{E}_{(X, \Delta)}$  est semi-stable par rapport à  $c_1(X, \Delta)$ ,*

(MY) *On a l'égalité suivante :*

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

*Alors, il existe un groupe fini  $G$  tel que  $(X, \Delta) \simeq (\mathbb{P}^n/G, \Delta_G)$ .*

Pour établir le résultat dans le cadre des paires, plusieurs difficultés adviennent du fait de la présence du bord  $\Delta$ .

### Définition de l'extension canonique $\mathcal{E}_{(X, \Delta)}$

Tout d'abord, il faut donner un sens à l'extension canonique  $\mathcal{E}_{(X, \Delta)}$ . Cet objet n'existe pas sur la variété  $X$  en elle-même et, à l'instar de faisceaux orbifoldes comme le faisceau  $\Omega_{(X, \Delta)}^{[1]}$  ou  $\mathcal{T}_{(X, \Delta)}$ , ces objets font sens sur des revêtements adaptés à la paire  $(X, \Delta)$ .

Notamment, lorsque l'on dispose d'un revêtement  $f: Y \rightarrow X$ , Miyaoka introduit en 2008 la notion de faisceau des différentielles adaptées  $\Omega_{X, \Delta, f}^{[1]}$  [Miy08, p. 412], reprise dans [CKT16, Sect. I.3]. C'est un faisceau réflexif sur  $Y$  qui encode le comportement de ramification de  $f$  et la présence du diviseur  $\Delta$ . Dans la même veine, on introduit la notion d'extension canonique adaptée  $\mathcal{E}_{X, \Delta, f}$ . C'est un faisceau réflexif sur  $Y$  qui est une extension du faisceau tangent adapté  $\mathcal{T}_{X, \Delta, f}$  par le faisceau trivial  $\mathcal{O}_Y$ . Ainsi, la condition (ECS) se réécrit :

(ECS) *Il existe un revêtement  $f: Y \rightarrow X$  strictement adapté à la paire  $(X, \Delta)$  tel que  $\mathcal{E}_{X, \Delta, f}$  est semistable par rapport à  $f^*(-(K_X + \Delta))$ .*

Dans une autre direction, on peut définir une notion d'orbifibré vectoriel sur une orbifold. Il s'agit d'une collection de fibrés vectoriels sur les uniformisations locales satisfaisant des conditions de compatibilité. Cette notion de  $\mathbb{Q}$ -fibré sur une  $\mathbb{Q}$ -structure est notamment développée dans [Mum83, §3]. On pourra définir ainsi de la même façon l'extension canonique  $\mathcal{E}_{(X,\Delta)}$  associée à une orbifold en mimant la construction dans le cas lisse.

On donne une interprétation de ces faisceaux qui rend leur construction naturelle du point de vue des orbifaisceaux : ces faisceaux adaptés  $\Omega_{X,\Delta,f}^{[1]}$  ou  $\mathcal{E}_{X,\Delta,f}$  peuvent être vus comme des images réciproques des orbifibrés  $\Omega_{(X,\Delta)}^{[1]}$  et  $\mathcal{E}_{(X,\Delta)}$  par  $f$ .

**Théorème 0.0.10.** *Soit  $(X, \Delta)$  une paire klt et  $f: Y \rightarrow X$  un morphisme strictement  $\Delta$ -adapté. On note  $X_{\text{orb}} \subseteq X$  l'ouvert maximal pour l'inclusion tel que la paire  $(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})$  peut être munie d'une structure d'orbifold. Alors, on a les isomorphismes d'orbifaisceaux suivants au-dessus de  $f^{-1}(X_{\text{orb}})$  :*

$$f^* \Omega_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}^1 \simeq \left( \Omega_{X,\Delta,f}^{[1]} \right)^{[\text{orb}]} \quad \text{et} \quad f^* \mathcal{E}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})} \simeq \left( \mathcal{E}_{X,\Delta,f} \right)^{[\text{orb}]}.$$

L'intérêt d'une telle interprétation est de permettre de calculer les classes de Chern orbifoldes associées à ces faisceaux.

## Le revêtement universel d'une paire

Ensuite, il est possible de définir la notion de revêtement universel  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  associé à une paire  $(X, \Delta)$  suivant [Nam87; CGG24]. Toutefois, l'objet géométrique résultant est encore une paire, si bien qu'un résultat semblable au théorème de Zariski–Lipman dans ce cadre doit être développé. Ainsi, un tel résultat assure que la variété  $\widetilde{X}_\Delta$  est lisse, mais aussi qu'elle est dépourvue de singularités en codimension 1, c'est-à-dire  $\widetilde{\Delta} = 0$ . On montre un tel résultat pour les paires  $(X, \Delta)$  orbifoldes.

**Proposition 0.0.11** (Critère de développabilité d'une orbifold). *Soit  $(X, \Delta)$  une orbifold et  $\mathcal{F}$  un orbifibré plat. On suppose qu'il existe un morphisme surjectif d'orbifibrés :*

$$\mathcal{F} \twoheadrightarrow \mathcal{T}_{(X,\Delta)}.$$

*Alors, le revêtement universel  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  de  $(X, \Delta)$  satisfait  $\widetilde{\Delta} = 0$  et  $\widetilde{X}_\Delta$  lisse.*

## Esquisse de la démonstration

La preuve du théorème 0.0.9 se découpe en deux parties. La première consiste à établir que la paire  $(X, \Delta)$  est une orbifolde sous les hypothèses du théorème. La seconde montre que le revêtement universel doit être lisse en utilisant le critère de développabilité précédent.

Pour montrer que  $(X, \Delta)$  est une orbifolde, on construit autour de chaque point de  $(X, \Delta)$  une carte uniformisante locale. Cette carte est obtenue à partir d'un revêtement  $f: Y \rightarrow X$  strictement adapté à  $\Delta$ , et on prouve qu'elle est lisse au-dessus de  $x$ , en utilisant le même genre d'argument que le théorème 0.0.8 : l'égalité de Miyaoka–Yau impose la platitude de  $\text{End}(\mathcal{E}_{X,\Delta,f})$ , et on en déduit similairement que  $\mathcal{T}_{X,\Delta,f}$  doit être localement libre au-dessus de  $x$ . Si on choisit correctement  $f$ , on peut avoir un isomorphisme de faisceaux autour d'une préimage  $y$  de  $x$  :  $\mathcal{T}_{X,\Delta,f,y} \simeq \mathcal{T}_{Y,y}$ . Il s'ensuit en appliquant la résolution de la conjecture de Zariski–Lipman dans le cas klt, que la variété  $Y$  est lisse au-dessus de  $x$ .

Dans un second temps, on montre que le revêtement universel orbifold est la paire  $(\mathbb{P}^n, 0)$  : il suffit pour cela de montrer que  $\tilde{\Delta} = 0$  et  $\tilde{X}_\Delta$  est lisse, et on conclut par 0.0.7. Comme conséquence du critère de développabilité précédent, il suffit d'établir que  $\text{End}(\mathcal{E}_{(X,\Delta)})$  est plat. Ceci est vérifié si l'on parvient à montrer que le faisceau  $\text{End}(\mathcal{E}_{X,\Delta,f})$  est plat pour un revêtement strictement  $\Delta$ -adapté  $f: Y \rightarrow X$ , et la connexion est invariante par le groupe de Galois de  $f$ . Le premier point est une conséquence de [LT18, Thm. 1.4] appliqué au faisceau  $\text{End}(\mathcal{E}_{X,\Delta,f})$  dont on peut calculer les classes de Chern orbifoldes. En effet, ce faisceau est semistable par rapport à  $f^*c_1(X, \Delta)$  et l'égalité de Miyaoka–Yau impose :

$$\widehat{\text{ch}}_1(\text{End}(\mathcal{E}_{X,\Delta,f})) \cdot f^*c_1(X, \Delta)^{n-1} = \widehat{\text{ch}}_2(\text{End}(\mathcal{E}_{X,\Delta,f})) \cdot f^*c_1(X, \Delta)^{n-2} = 0,$$

si bien que  $\text{End}(\mathcal{E}_{X,\Delta,f})$  est plat sur  $Y_{\text{reg}}$ . D'autre part, cette connexion est invariante par  $\text{Gal}(f)$  par functorialité de la correspondance de Simpson klt [Gre+19a, Cor. 3.8]. Ceci induit que la connexion "redescend" sur chacune des uniformisations locales, si bien que  $\text{End}(\mathcal{E}_{(X,\Delta)})$  est muni d'une orbiconnexion plate, ce qui conclut la preuve.

## Introduction in English

### Characterization of projective spaces via uniformization

#### Uniformization of manifolds and constant sectional curvature

At the end of the 19th century, the study of complex differential equations led to a rethinking of the notion of functions. For example, consider the complex differential equation on  $\mathbb{C}^\times$  given by  $zy' = 1$ . This equation does not admit a globally defined solution. To obtain a globally defined object, one must work with multivalued functions, in the sense that the image of a point is now a discrete set. The study of such functions led Poincaré to introduce the notion of a covering space and the fundamental group of a complex manifold  $X$ . There is a correspondence between these two notions, and the concept of the universal covering space, which encapsulates all possible coverings, emerges naturally. Thus arises the question of describing the universal cover of a manifold  $X$ : this is the uniformization problem.

Initially, this problem is of a purely topological nature, but one can show that a covering of a complex manifold can be endowed uniquely with a complex manifold structure making the covering map holomorphic. When Riemannian or Hermitian metrics appear on these varieties, one also requires that the covering map be a local isometry.

The history of the uniformization of Riemann surfaces (i.e., complex manifolds of complex dimension 1) is thoroughly recounted in [Sai10]. In 1907, Poincaré and Koebe classified simply connected Riemann surfaces:

**Theorem 0.0.12.** [Sai10, Thm. p.31] *Every simply connected Riemann surface is biholomorphic to the Riemann sphere  $\mathbb{P}^1$ , the complex plane  $\mathbb{C}$ , or the unit disk  $\mathbb{D}$ .*

While this result completes the classification of simply connected Riemann surfaces, the problem of uniformizing compact connected Riemann surfaces had been known earlier, thanks to contributions by Clebsch in 1865 and Poincaré in 1884:

**Theorem 0.0.13.** [Sai10, Thm. II.2.15, II.2.16, VIII.5.3] *Let  $X$  be a compact connected Riemann surface of genus  $g$ . Then the universal cover of  $X$  is:*

- $\mathbb{P}^1$  if  $g = 0$ ,
- $\mathbb{C}$  if  $g = 1$ ,
- $\mathbb{D}$  if  $g \geq 2$ .

What about higher dimensions? Before presenting a generalization of the previous theorem, it is necessary to interpret the above result in terms of Riemannian geometry. The following theorem, due to Killing and Hopf, provides the right framework.

**Theorem 0.0.14.** *Let  $(M, g)$  be a compact Riemannian manifold of real dimension  $n$ . Suppose the sectional curvature of  $M$  is constant and equal to  $\lambda \in \mathbb{R}$ . Then the universal cover  $(\widetilde{M}, \widetilde{g})$  of  $(M, g)$  is isometric to one of the following three models:*

- the sphere with the spherical metric  $(\mathbb{S}^n, \frac{1}{\lambda}g_{\text{sph}})$  if  $\lambda > 0$ ,
- Euclidean space with the Euclidean metric  $(\mathbb{R}^n, g_{\text{euc}})$  if  $\lambda = 0$ ,
- real hyperbolic space with the hyperbolic metric  $(\mathbb{H}^n, -\frac{1}{\lambda}g_{\text{hyp}})$  if  $\lambda < 0$ .

For  $n = 2$ , we recover the three smooth manifolds mentioned earlier, since  $\mathbb{P}^1 \simeq \mathbb{S}^2$ ,  $\mathbb{R}^2 \simeq \mathbb{C}$ , and  $\mathbb{H}^2 \simeq \mathbb{D}$ . In real dimension 2, there is a remarkable relationship between Riemannian and complex structures: conformal transformations coincide with biholomorphisms. As a result, an oriented surface endowed with a Riemannian metric determines a unique Riemann surface structure. It follows that an isometry between Riemannian surfaces induces a biholomorphism between the underlying complex structures.

Let us now return to the case where our manifold  $X$  is equipped with a complex structure. In this context, we can also equip the underlying smooth manifold  $M$  with a Riemannian metric  $\omega$ . We say that the metric is Kähler<sup>3</sup> if it is compatible with the complex structure of  $X$ .

In this setting, we also have a complex analogue of the Killing–Hopf theorem. Hawley and Igusa independently established a similar classification of the universal coverings of Kähler manifolds in 1953–1954. This result was later reproduced by Kobayashi and Nomizu [KN69, Thm. IX.7.8-9], and can be stated as follows:

**Theorem 0.0.15.** [Haw53; Igu54] *Let  $(X, \omega)$  be a complete Kähler manifold of complex dimension  $n$ . Suppose that  $\omega$  has constant holomorphic sectional curvature<sup>4</sup> equal to  $\lambda \in \mathbb{R}$ .*

*Then the universal cover  $(\widetilde{X}, \widetilde{\omega})$  is holomorphically isometric to:*

- projective space with the Fubini–Study metric  $(\mathbb{P}^n, \frac{1}{\lambda}\omega_{FS})$  if  $\lambda > 0$ ,
- Euclidean space with the Euclidean metric  $(\mathbb{C}^n, \omega_{\text{euc}})$  if  $\lambda = 0$ ,

3. More precisely, this metric induces a connection  $\nabla_{\omega, \text{End}(T_M)}$  on the bundle  $\text{End}(T_M)$ . The metric is Kähler if the almost complex structure  $J \in \mathcal{C}^\infty(M, \text{End}(T_M))$  is parallel with respect to this connection:  $\nabla_{\omega, \text{End}(T_M)} J = 0$ .

4. That is, the sectional curvature of the Riemannian metric restricted to planes that are invariant by the almost complex structure.

— the unit ball in  $\mathbb{C}^n$  with the Bergman metric  $(\mathbb{B}^n, -\frac{1}{\lambda}\omega_{\text{Ber}})$  if  $\lambda < 0$ .

### From constant sectional curvature to the Miyaoka–Yau equality

The three classical models actually provide Kähler–Einstein metrics, meaning they satisfy  $\text{Ric } \tilde{\omega} = \lambda \tilde{\omega}$ . In 1974, Chen and Ogiue showed that constant holomorphic sectional curvature is equivalent to a Chern class equality, assuming the metric is Kähler–Einstein:

**Theorem 0.0.16.** [CO75, Thm. 2] *Let  $(X, \omega)$  be a compact Kähler manifold of dimension  $n$ . Assume that  $\omega$  is Kähler–Einstein. Then the following inequality holds:*

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot [\omega]^{n-2} \geq 0.$$

Moreover, equality holds in the above inequality if and only if the holomorphic sectional curvature of  $\omega$  is constant.

This inequality is known as the *Miyaoka–Yau inequality*, as Miyaoka proved the formula  $c_1(X)^2 \leq 3c_2(X)$  when  $X$  is a surface of general type [Miy77, Thm. 4]. From this, we deduce the following theorem:

**Theorem 0.0.17.** *Let  $(X, \omega)$  be a compact Kähler manifold of dimension  $n$  such that the following two conditions are satisfied:*

(KE $_{\lambda}$ ) *The metric  $\omega$  is Kähler–Einstein:  $\text{Ric } \omega = \lambda \omega$  for  $\lambda \in \{-1, 0, 1\}$ ,*

(MY) *The following equality holds:*

$$(2(n+1)c_2(X) - nc_1(X)^2) \cdot [\omega]^{n-2} = 0.$$

Then the universal cover of  $(X, \omega)$  is isometrically isomorphic to:

- $(\mathbb{P}^n, \omega_{FS})$  if  $\lambda = 1$ ,
- $(\mathbb{C}^n, \omega_{\text{euc}})$  if  $\lambda = 0$ ,
- $(\mathbb{B}^n, \omega_{\text{Ber}})$  if  $\lambda = -1$ .

In the case  $\lambda = 1$ , the previous theorem provides a characterization of  $(\mathbb{P}^n, \omega_{FS})$ . Indeed, the manifold  $X$  is biholomorphic to a quotient of  $\mathbb{P}^n$  by a group  $G$  acting properly and freely. The properness of the action implies that  $G$  must be finite, and freeness forces  $G$  to be trivial. Therefore, the quotient map  $\mathbb{P}^n \rightarrow X$  is a biholomorphism.

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**Some variations on the assumptions of Theorem 0.0.17**

Given the resolution of the existence problem for Kähler–Einstein metrics on compact manifolds  $X$  with  $c_1(X) < 0$  by Aubin and Yau, such a manifold  $X$  is a quotient of the complex unit ball if and only if the equality

$$\left(2(n+1)c_2(X) - nc_1(X)^2\right) \cdot c_1(X)^{n-2} = 0$$

is satisfied. On the other hand, a Kähler manifold  $X$  satisfying  $c_1(X) = 0$  can be endowed by a Ricci-flat metric  $\omega$  by the resolution of Calabi’s problem due to Yau [Yau78]. Moreover, if we assume that  $c_2(X) = 0$ , we infer that  $X$  is a finite quotient of a complex torus, as a consequence of a theorem of Bieberbach on crystallographic groups (see [Kob87, Cor. 4.4.15]).

Finally, in the case where  $c_1(X) > 0$ , Tian showed in 1992 [Tia92, Thm. 0.1] that the assumption  $(KE_{>0})$  implies the following condition:

(ECP) *The canonical extension  $E_X$  is polystable with respect to  $c_1(X)$ .*

We first recall the definition of the canonical extension. Let  $S$  and  $Q$  be two vector bundles on  $X$ . Then, there is a correspondence between isomorphism classes of extensions of  $Q$  by  $S$  and the group  $H^1(X, \text{Hom}(Q, S))$ . When  $Q = T_X$  and  $S = \mathcal{O}_X$ , we get  $H^1(X, \text{Hom}(Q, S)) = H^1(X, \Omega_X^1)$ , which contains the class  $c_1(X)$ . Thus, the corresponding extension class defines the canonical extension  $E_X$ . This is a vector bundle which is an extension of the tangent bundle  $T_X$  by the trivial bundle  $\mathcal{O}_X$ . In particular, the Chern classes of the canonical extension are exactly the Chern classes of the variety, by the Whitney formula for extensions of vector bundles:  $c_i(E_X) = c_i(X)$  for all  $i$ .

One may also consider the condition (ECS) by replacing polystability with semistability of the canonical extension in the previous definition. In another direction, we may state the following characterization of  $\mathbb{P}^n$ :

**Theorem 0.0.18.** *Let  $X$  be a compact complex manifold with  $c_1(X) > 0$  satisfying the following two assumptions:*

(ECS) *the canonical extension  $E_X$  is semistable with respect to  $c_1(X)$ ,*

(MY) *The following equality is satisfied:*

$$\left(2(n+1)c_2(X) - nc_1(X)^2\right) \cdot c_1(X)^{n-2} = 0.$$

*Then  $X$  is biholomorphic to  $\mathbb{P}^n$ .*

Let us sketch the proof of this theorem. One first shows that the canonical extension  $E_X$  is projectively flat. This follows from Simpson’s correspondence applied to  $F = \text{End}(E_X)$ , since  $F$  is still semistable with respect to  $c_1(X)$  and we have:

$$\text{ch}_1(F) \cdot c_1(X)^{n-1} = \text{ch}_2(F) \cdot c_1(X)^{n-2} = 0,$$

so the endomorphism bundle  $F$  is flat. It follows that  $E_X$  is projectively flat. Now, since  $X$  is Fano, it is simply connected [Bes08, Thm. 11.26], and so the projectivization  $\mathbb{P}(E_X)$  is trivial. Therefore, there exists an ample line bundle  $L$  such that  $E_X = L^{\oplus n+1}$ . The following equality:

$$c_1(E_X) = c_1(X) = (n+1)c_1(L)$$

combined with the characterization of projective spaces due to Kobayashi–Ochiai [KO73, Cor. of Thm.1] allows one to conclude that  $X \simeq \mathbb{P}^n$ .

## The uniformization problem in the singular case

### Singularities and ramification of the quotient map

Let us first present another approach to the uniformization problem for manifolds, which will naturally lead to its generalization to the singular setting.

We consider a manifold  $Y$  and a discrete subgroup  $G$  of the group of automorphisms of  $Y$ . We can then consider the quotient space  $X := Y/G$ . A priori, this is simply a topological space. However, under the assumption that the action of  $G$  on  $Y$  is proper and free, the quotient space can be endowed with a unique complex structure such that  $X$  is smooth and the quotient map  $\pi_G : Y \rightarrow X$  is holomorphic. In fact, it is an étale map—that is, a local biholomorphism—thus providing a Galois cover of  $X$  with Galois group  $G$ . The uniformization problem can then be reformulated as follows: given a manifold  $X$ , can we recover all pairs  $(Y, G)$  such that  $X$  is the quotient of  $Y$  by a proper and free action of  $G$ ? Of course, there is a universal pair  $(\widetilde{X}, \pi_1(X))$ , where  $\widetilde{X}$  is the universal cover of  $X$  and  $\pi_1(X)$  is its fundamental group. All possible pairs  $(Y, G)$  can then be deduced via the theory of covering spaces and the geometric Galois correspondence.

Let us now examine what happens when the group  $G$  no longer acts properly and freely on  $Y$ . To preserve the separability of the orbit space, it is necessary to retain the properness of the action of  $G$  on  $Y$ . However, if we drop the freeness assumption, two phenomena arise.

On the one hand, the space  $X$  may acquire singularities. It was shown by Cartan in 1957 that, in such cases, the orbit space  $X$  is endowed with the structure of a normal complex analytic space, and this structure can be explicitly described [Car57]. The space  $X$  is then an *orbifold*: a topological space whose local models are quotients of  $\mathbb{C}^n$  by finite groups acting linearly.

On the other hand, the quotient map  $\pi_G : Y \rightarrow X$  is no longer a topological covering map: it is said to be a branched covering. The first question that arises is how to adapt the notions of universal coverings and fundamental groups in this context. A simplifying assumption is to suppose that the branch locus of  $\pi_G$  consists of components of codimension at least 2. Then, by the purity of the branch locus theorem due to Nagata–Zariski, this implies that  $\pi_G$  is étale over  $X_{\text{reg}}$ , and thus the ramification locus is contained in an analytic subset of codimension at least 2. We say in this case that  $\pi_G$  is a *quasi-étale cover*. Consequently, the set of points with nontrivial stabilizer is an analytic subset that must have codimension at least 2: we say that  $G$  acts without fixed points in codimension 1. In this setting, the map  $\pi_G$  induces a covering over  $X_{\text{reg}}$ . Hence, the fundamental group of  $X_{\text{reg}}$  gives information about the quasi-étale covers of  $X$ .

### State of the Art of uniformization in the singular case

In this context, one might wonder if the previous characterizations apply in the singular setting. Several results have been established in recent years, generalizing the previous theorems, in contexts where singularities are assumed to be *Kawamata log terminal* (or *klt*): this is a natural class of singularities in algebraic geometry, broader than the class of quotient singularities. In this framework, the uniformization results we seek require the definition of orbifold Chern classes in a singular context, as well as the notion of canonical extension associated with  $X$ .

Initially, the first uniformization results in the singular setting concerned the flat case, where we assume  $c_1(X) = 0$ . The papers [GKP16a; LT18; GK20; GKP22; CGG22] state results with many variants: first on the codimension of the singular locus, then on the projective or Kähler nature of the variety  $X$  to be uniformized, etc.

In the paper [Gre+19b], Greb, Kebekus, Peternell, and Taji address the case where  $c_1(X) < 0$ , and provide a characterization of singular quotients of the ball for klt varieties.

Finally, in the paper [GKP22], Greb, Kebekus, and Peternell give a characterization of quotients of  $\mathbb{P}^n$ , via a generalization of Theorem 0.0.18:

**Theorem 0.0.19.** [GKP22, Thm. 1.3] *Let  $X$  be a projective Fano klt variety. Then,*

there exists a finite group  $G$  acting without fixed points in codimension 1 on  $\mathbb{P}^n$  such that  $X \simeq \mathbb{P}^n/G$  if and only if there exists an ample divisor  $H$  on  $X$  such that the following two conditions are satisfied:

(ECS) The canonical extension  $\mathcal{E}_X$  is semi-stable with respect to  $H$ ,

(MY) The following equality is satisfied:

$$\left(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1(X)^2\right) \cdot H^{n-2} = 0.$$

In the previous theorem, the authors introduce the notion of the canonical extension  $\mathcal{E}_X$  of  $X$ : it is a reflexive sheaf on  $X$  obtained as an extension of the tangent sheaf  $\mathcal{T}_X$  by the trivial sheaf  $\mathcal{O}_X$ . The Miyaoka—Yau equality involves orbifold Chern classes introduced by Mumford in 1983 [Mum83, §2-3].

Let us give an idea of the proof of this theorem. In the case where  $X$  is a projective variety with normal klt singularities, we have a maximally quasi-étale covering  $\pi: Y \rightarrow X$ . This is a finite Galois quasi-étale map, where  $Y$  is a klt variety such that the inclusion map  $i: Y_{\text{reg}} \hookrightarrow Y$  induces a group isomorphism  $\widehat{\pi}_1(Y_{\text{reg}}) \simeq \widehat{\pi}_1(Y)$ , where  $\widehat{\pi}_1$  denotes the profinite completion of the fundamental group. This covering serves as the uniformizing map, and we will show that  $Y \simeq \mathbb{P}^n$ . The properties Fano, (ECS) and (MY) are transferred to  $Y$ , so it suffices to show that  $Y$  is smooth, and the previous proof in the smooth case allows us to conclude.

To show that  $Y$  is smooth, we use the solution of the Zariski–Lipman conjecture for klt varieties [Gre+11; Dru14; Gra15]. Thus, the task is to show that the tangent sheaf  $\mathcal{T}_Y$  of  $Y$  is locally free. For this, we use a klt version of Simpson’s correspondence [LT18, Thm. 1.4]. Indeed, the assumptions of the theorem allow us to establish that the sheaf  $\mathcal{F} := \text{End}(\mathcal{E}_Y)$  is semistable with respect to  $H$  and satisfies the orbifold Chern class equality:

$$\widehat{\text{ch}}_1(\mathcal{F}) \cdot H^{n-1} = \widehat{\text{ch}}_2(\mathcal{F}) \cdot H^{n-2} = 0,$$

Simpson’s correspondence in this singular setting implies the flatness of  $\mathcal{F}$  over  $Y_{\text{reg}}$ . Since  $Y$  is maximally quasi-étale, there exists a flat, hence locally free extension of  $\mathcal{F}$  to all of  $Y$ , which coincides with  $\mathcal{F}$  since the two sheaves are reflexive [GKP16a, Thm. 1.14]. Hence, we obtain that  $\mathcal{F}$  is locally free. Therefore, around a point  $y \in Y$ , there exists an affine

open set  $U$  in which we have a splitting:

$$\mathcal{F}|_U = \mathcal{O}_U^{\oplus(n+1)^2} = \mathcal{O}_U \oplus \mathcal{T}_U \oplus \Omega_U^{[1]} \oplus \text{End}(\mathcal{T}_U).$$

A consequence of Nakayama's lemma [Kob87, Lemma. 5.5.3] gives that  $\mathcal{T}_U \simeq \mathcal{O}_U^{\oplus n}$ . Thus,  $\mathcal{T}_Y$  is locally free.

Finally, to solve the uniformization problem in full generality, we must consider branched coverings that may have branching phenomena in codimension 1. This raises the following problem: these divisor behaviors are no longer fully encoded in the geometry of  $X$ . Indeed, when considering quasi-étale coverings, we had the fact that the branch locus of the quotient map must be concentrated on the singular locus of  $X$ . To resolve this, we need to specify an additional divisor, which should be thought of as the locus of singularities in codimension 1. In this context, the correct geometric objects to consider are *pairs*: these are couples  $(X, \Delta)$  consisting of a normal variety  $X$  and a Weil  $\mathbb{Q}$ -divisor with standard coefficients, i.e., of the form  $\Delta := \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$ , where  $m_i \in \mathbb{N}_{\geq 2}$ .

In this framework, Claudon, Graf, and Guenancia characterize the pairs  $(X, \Delta)$  that arise as quotients by groups acting properly on  $\mathbb{C}^n$  and  $\mathbb{B}^n$  in terms of the Miyaoka–Yau equality and positivity of  $K_X + \Delta$  in [CGG24, Thm. A-B].

## Presentation of the results

The aim of this thesis is to characterize pairs  $(X, \Delta)$  that are obtained as quotients of  $\mathbb{P}^n$  by the equality case in the Miyaoka–Yau inequality. Motivated by a generalization of Theorem 0.0.19, we show the following result.

**Theorem 0.0.20.** *Let  $(X, \Delta)$  be a log Fano pair with standard coefficients of dimension  $n$ . Assume that the following two assertions hold:*

- (ECS) *The canonical extension  $\mathcal{E}_{(X, \Delta)}$  is semistable with respect to  $c_1(X, \Delta)$ ,*
- (MY) *The Miyaoka–Yau equality holds:*

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

*Then, there exists a finite subgroup  $G \subseteq \text{PGL}(n+1, \mathbb{C})$  such that  $(X, \Delta) \simeq (\mathbb{P}^n/G, \Delta_G)$ .*

Several difficulties arise from the presence of the boundary  $\Delta$  when we deal with pairs.

**Definition of the canonical extension  $\mathcal{E}_{(X,\Delta)}$**

First, we need to make sense of the canonical extension  $\mathcal{E}_{(X,\Delta)}$ . This object does not exist on the variety  $X$ , and similarly to orbifold sheaves  $\Omega_{(X,\Delta)}^{[1]}$  or  $\mathcal{T}_{(X,\Delta)}$ , it is well-defined on adapted covers above  $(X, \Delta)$ .

In particular, when a covering map  $f: Y \rightarrow X$  is given, Miyaoka introduced in 2008 the notion of an adapted differential sheaf  $\Omega_{X,\Delta,f}^{[1]}$  [Miy08, p. 412], which is also presented in [CKT16, Sect. I.3]. It is a reflexive sheaf on  $Y$  that encodes the branching behavior of  $f$  and the presence of the divisor  $\Delta$ . In the same vein, we introduce the notion of the adapted canonical extension  $\mathcal{E}_{X,\Delta,f}$ . This is a reflexive sheaf on  $Y$  that is an extension of the adapted tangent sheaf  $\mathcal{T}_{X,\Delta,f}$  by the trivial sheaf  $\mathcal{O}_Y$ . Thus, the condition (ECS) can be rewritten as:

(ECS) *There exists a covering map  $f: Y \rightarrow X$  strictly adapted to the pair  $(X, \Delta)$  such that  $\mathcal{E}_{X,\Delta,f}$  is semistable with respect to  $f^*(-(K_X + \Delta))$ .*

In another direction, one can define the notion of an orbibundle over an orbifold. This is a collection of vector bundles over the local uniformizations satisfying compatibility conditions. This notion of a  $\mathbb{Q}$ -bundle over a  $\mathbb{Q}$ -structure is developed in [Mum83, §3]. Similarly, the canonical extension  $\mathcal{E}_{(X,\Delta)}$  associated with an orbifold can be defined by mimicking the construction in the smooth case.

We give an interpretation of these sheaves that makes their construction natural from the point of view of orbisheaves: these adapted sheaves  $\Omega_{X,\Delta,f}^{[1]}$  or  $\mathcal{E}_{X,\Delta,f}$  can be viewed as pullbacks of the orbibundles  $\Omega_{(X,\Delta)}^{[1]}$  and  $\mathcal{E}_{(X,\Delta)}$  by  $f$ .

**Theorem 0.0.21.** *Let  $(X, \Delta)$  be a klt pair and  $f: Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism. Consider  $X_{\text{orb}}$  the maximal open subset of  $X$  such that  $(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})$  can be endowed with an orbifold structure. Then we have the following orbisheaf isomorphisms over  $f^{-1}(X_{\text{orb}})$ :*

$$f^*\Omega_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}^{[1]} \simeq \left(\Omega_{X,\Delta,f}^{[1]}\right)^{[\text{orb}]} \quad \text{and} \quad f^*\mathcal{E}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})} \simeq \left(\mathcal{E}_{X,\Delta,f}\right)^{[\text{orb}]}.$$

The interest of such an interpretation is to allow the computation of the orbifold Chern classes associated with these sheaves.

## The universal covering of a klt pair

Next, it is possible to define the notion of a universal covering  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  associated with a pair  $(X, \Delta)$  following [Nam87; CGG24]. However, the resulting geometric object is still a pair, so a result similar to the Zariski–Lipman theorem in this context needs to be developed. Thus, such a result ensures that the variety  $\widetilde{X}_\Delta$  is smooth, but also that it has no singularities in codimension 1, i.e.,  $\widetilde{\Delta} = 0$ . A similar result is shown for orbifolds  $(X, \Delta)$ .

**Proposition 0.0.22** (Criterion for the developability of an orbifold). *Let  $(X, \Delta)$  be an orbifold and  $\mathcal{F}$  a flat orbibundle. Suppose there exists a surjective morphism of orbibundles:  $\mathcal{F} \twoheadrightarrow \mathcal{T}_{(X, \Delta)}$ .*

*Then, the universal covering  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  of  $(X, \Delta)$  satisfies  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is smooth.*

### Sketch of proof

The proof of Theorem 0.0.20 is divided into two parts. The first part establishes that the pair  $(X, \Delta)$  is an orbifold under the assumptions of the theorem. The second part shows that the universal covering must be smooth using the previous developability criterion.

To show that  $(X, \Delta)$  is an orbifold, one constructs a local uniformizing chart around each point  $x$  of  $(X, \Delta)$ . This chart is obtained from a strictly  $\Delta$ -adapted covering map  $f: Y \rightarrow X$ , and it is shown to be smooth over  $x$ , using an argument similar to Theorem 0.0.19: the Miyaoka–Yau equality imposes the flatness of  $\text{End}(\mathcal{E}_{X, \Delta, f})$ , and similarly, it follows that  $\mathcal{T}_{X, \Delta, f}$  must be locally free over  $x$ . By appropriately choosing  $f$ , one can obtain a sheaf isomorphism around a preimage  $y$  of  $x$ :  $\mathcal{T}_{X, \Delta, f, y} \simeq \mathcal{T}_{Y, y}$ . It follows from applying the solution of the Zariski–Lipman conjecture in the klt case that the variety  $Y$  is smooth over  $x$ .

In the second part, we show that the orbifold universal covering is the pair  $(\mathbb{P}^n, 0)$ : to do this, it suffices to show that  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is smooth, and we conclude by applying Theorem 0.0.18. As a consequence of the previous developability criterion, it is sufficient to establish that  $\text{End}(\mathcal{E}_{(X, \Delta)})$  is flat. This is verified if we manage to show that the sheaf  $\text{End}(\mathcal{E}_{X, \Delta, f})$  is flat for a strictly  $\Delta$ -adapted covering map  $f: Y \rightarrow X$ , and the connection is invariant under the Galois group of  $f$ . The first point follows from [LT18, Thm. 1.4] applied to the sheaf  $\text{End}(\mathcal{E}_{X, \Delta, f})$  whose orbifold Chern classes can be computed. Indeed,

this sheaf is semistable with respect to  $f^*c_1(X, \Delta)$ , and the Miyaoka–Yau equality imposes:

$$\widehat{\text{ch}}_1(\text{End}(\mathcal{E}_{X,\Delta,f})) \cdot f^*c_1(X, \Delta)^{n-1} = \widehat{\text{ch}}_2(\text{End}(\mathcal{E}_{X,\Delta,f})) \cdot f^*c_1(X, \Delta)^{n-2} = 0,$$

so that  $\text{End}(\mathcal{E}_{X,\Delta,f})$  is flat over  $Y_{\text{reg}}$ . Moreover, this connection is invariant under  $\text{Gal}(f)$  by the functoriality of the klt Simpson correspondence [Gre+19a, Cor. 3.8]. This implies that the connection "descends" on each of the local uniformizations, so that  $\text{End}(\mathcal{E}_{(X,\Delta)})$  is equipped with a flat orbiconnection, which concludes the proof.

## Conventions and notations

Throughout, we denote by  $X$  a *complex analytic space*, that is, a topological space which is Hausdorff and has a countable basis of open sets, equipped with a sheaf of local rings  $\mathcal{O}_X$ , such that around each point  $x \in X$ , we have:

- a neighborhood  $U$  of  $x$ ,
- an integer  $n$  and a neighborhood  $V$  of  $0 \in \mathbb{C}^n$  (we denote by  $\mathcal{O}_V$  the sheaf of holomorphic functions on  $V$ ),
- a collection of holomorphic functions  $(f_i)_{i \in I}$  defined on  $V$ ,
- an isomorphism of locally ringed spaces  $(U, \mathcal{O}_{X|U}) \simeq (V(\mathcal{I}), \mathcal{O}_V/\mathcal{I})$ , where  $\mathcal{I}$  is the ideal sheaf generated by the  $(f_i)$ , and

$$V(\mathcal{I}) = \{x \in V \mid f_i(x) = 0 \text{ for all } i\}.$$

A point  $x \in X$  is called *smooth* if in the above definition one can choose an empty collection of holomorphic functions. The *regular locus* of  $X$ , denoted  $X_{\text{reg}}$ , is the open subset of smooth points of  $X$ . The set of *singular points* of  $X$  is denoted  $X_{\text{sing}} := X \setminus X_{\text{reg}}$ .

We assume that  $X$  is *irreducible*, meaning that  $X_{\text{reg}}$  is connected (in particular  $X$  is connected). We assume that  $X$  is *normal*, meaning that for every point  $x \in X$ , the local ring  $\mathcal{O}_{X,x}$  is an integrally closed domain. This implies in particular that  $X_{\text{sing}}$  is a closed analytic subspace of  $X$  of codimension at least 2 and  $X$  is locally irreducible.

The topological fundamental group of  $X$  is denoted by  $\pi_1(X)$ .

On such a space  $X$ , *Weil divisors* (respectively,  $\mathbb{Q}$ -*Weil divisors*) are defined as formal sums with integer (respectively, rational) coefficients of irreducible reduced hypersurfaces in  $X$ . The *support* of a divisor  $D := \sum_i a_i \Delta_i$ , denoted  $|D|$ , is the divisor  $|D| := \sum_i \Delta_i$ . A

divisor is said to be *reduced* if  $D = |D|$ . A *Cartier divisor* is a Weil divisor that is locally principal. A  *$\mathbb{Q}$ -Cartier divisor* is a Weil  $\mathbb{Q}$ -divisor  $D$  such that there exists an integer  $m \in \mathbb{N}_{\geq 1}$  with  $mD$  being a Cartier divisor.

A morphism  $f: Y \rightarrow X$  between normal analytic spaces is said to be *finite* if it is proper and has discrete (hence finite) fibers.



PART I

# Singular pairs

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# INTRODUCTION

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## Why do we consider pairs ?

Let  $G$  be a group acting properly on a smooth variety  $Y$ . Following Appendix A, one can establish that the quotient  $Y/G$  is a normal complex analytic space. A more precise description of the neighborhoods of points can be given analytically.

First, Cartan's linearization process [Car57, Lemma 1] allows us to show that the action of the stabilizer  $G_x$  fixing a point  $x \in Y$  is linearizable: one can find coordinates in which the action of  $G_x$  is linear. In other words, there exists an injective group morphism  $\rho : G_x \hookrightarrow \mathrm{GL}(n, \mathbb{C})$  and a biholomorphism between neighborhoods of 0,  $\sigma : U \rightarrow V$ , where  $U$  is an open set invariant under the action of  $G_x$  and  $V$  is invariant under the action of  $\rho(G_x)$ , such that  $\sigma$  is equivariant with respect to both actions:  $\sigma(g \cdot x) = \rho(g) \cdot \sigma(x)$ .

Thus, we now suppose that we are given a finite group  $H$  acting linearly on  $\mathbb{C}^n$ . This action can be studied algebraically: the linear action of  $H$  on  $\mathbb{C}^n$  induces a  $\mathbb{C}$ -algebra action of  $H$  on the polynomial ring  $\mathbb{C}[X_1, \dots, X_n]$ . As a consequence of Noether's lemma [Noe15], the ring of invariant polynomials is a finitely generated  $\mathbb{C}$ -algebra, i.e., we have an isomorphism:

$$\mathbb{C}[X_1, \dots, X_n]^H \simeq \mathbb{C}[T_1, \dots, T_p]/I,$$

where  $I = (P_1, \dots, P_q)$  is an ideal. In practice, we obtain generators that are  $H$ -invariant polynomials  $f_1, \dots, f_p \in \mathbb{C}[X_1, \dots, X_n]^H$  satisfying the relations  $P_i(f_1, \dots, f_p) = 0$ .

Hence, we get an isomorphism of analytic spaces between the analytifications of the affine schemes associated to these two  $\mathbb{C}$ -algebras. This leads to the following theorem:

**Theorem 1.0.1.** [Car57, Thm. 4] *Let  $H$  be a finite group acting linearly on  $\mathbb{C}^n$ . Let  $f_1, \dots, f_p$  be generators of the ring of invariant polynomials, and let  $I$  be the ideal of relations among the  $f_i$ . Then, the map*

$$x \in \mathbb{C}^n \longmapsto (f_1(x), \dots, f_p(x)) \in V(I) \subseteq \mathbb{C}^p$$

---

induces, via the quotient by  $H$ , an isomorphism of normal analytic spaces between  $\mathbb{C}^n/H$  and the analytification of  $(V(I), \mathcal{O}_{\text{Spec}(\mathbb{C}[X_1, \dots, X_n]/I)})$ .

Chevalley showed [Spr77, Thm. 7.2.5] conditions under which the invariant algebra is smooth. A linear group  $H \subseteq \text{GL}(n, \mathbb{C})$  is called a *complex reflection group* if it is generated by its elements that fix a hyperplane. Then we have:

**Theorem 1.0.2.** *Let  $H$  be a finite linear group acting on  $\mathbb{C}^n$ . Then the following statements are equivalent:*

- (i)  $H$  is a complex reflection group,
- (ii) the invariant ring  $\mathbb{C}[X_1, \dots, X_n]^H$  is generated by  $n$  algebraically independent homogeneous polynomials.

According to [Pri67], the subgroup  $K \subseteq H$  generated by

$$\{g \in H \mid \dim \ker(g - \text{id}) = n - 1\}$$

is a normal subgroup. It is a complex reflection group, so that the quotient  $\mathbb{C}^n/K$  is smooth.

Now, the quotient group  $H/K$  acts on  $\mathbb{C}^n/K \simeq \mathbb{C}^n$ . This action is free in codimension 1 (i.e., the fixed-point locus  $\text{Fix } H = \bigcup_{g \in H \setminus \{\text{id}\}} \text{Fix } g$  is a union of subspaces of codimension at least 2). We then have an isomorphism:

$$\mathbb{C}^n/H \simeq (\mathbb{C}^n/K)/(H/K).$$

Thus, we decompose the action of  $H$  into two linear actions on  $\mathbb{C}^n$ . The first, by the reflection subgroup  $K$ , is an action with pure codimension-1 branching; the quotient map  $\pi_K : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is ramified and branched in codimension 1. The second is an action of a smaller group  $H/K$  on  $\mathbb{C}^n$ , whose structure is reflected in the geometry of the quotient  $\mathbb{C}^n/(H/K)$ : notably, we have  $\pi_1((\mathbb{C}^n/(H/K))_{\text{reg}}) = H/K$ .

We now give two examples illustrating these two phenomena.

**Example 1.0.3.** Consider the diagonal action of the group  $\{\pm 1\}$  on  $\mathbb{C}^2$ :  $(-1) \cdot (x, y) = (-x, -y)$ . Then, by Cartan's theorem, we can determine the quotient by studying the invariant polynomials. Let  $P$  be a polynomial invariant under the action of  $\pm 1$ , then  $P(x, y) = P(-x, -y)$ . Writing  $P = \sum_{i,j} a_{i,j} x^i y^j$ , we obtain the condition  $(-1)^{i+j} a_{i,j} = a_{i,j}$ ,

so if  $i+j$  is odd,  $a_{i,j} = 0$ . It follows that we can write  $P(x, y) = Q(x^2, y^2) + xy R(x^2, y^2)$ , so the invariant polynomials are generated by  $U(x, y) = x^2$ ,  $V(x, y) = y^2$ , and  $W(x, y) = xy$ . These satisfy the relation  $W^2 = UV$ . Thus, the quotient  $\mathbb{C}^2/\pm 1$  is analytically isomorphic to  $V(w^2 - uv) \subseteq \mathbb{C}^3$ , which is singular at 0 (it is a cone over a smooth projective conic), and the quotient map is:

$$\pi_{\pm 1} : (x, y) \in \mathbb{C}^2 \longmapsto (x^2, y^2, xy) \in V(w^2 - uv).$$

**Example 1.0.4.** Consider the action of the group of  $m$ -th roots of unity  $\mu_m$  on  $\mathbb{C}^n$  given by:

$$\zeta \cdot (x_1, \underline{x}) = (\zeta x_1, \underline{x}),$$

where  $\underline{x} = (x_2, \dots, x_n)$  denotes the last  $n-1$  coordinates. To determine the quotient, study the invariant polynomials. Let  $P$  be invariant under  $\mu_m$ , then  $P(x_1, \underline{x}) = P(\zeta x_1, \underline{x})$ . Writing  $P(x_1, \dots, x_n) = \sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ , the condition becomes  $\zeta^{i_1} a_I = a_I$ . For  $\zeta = e^{\frac{2\pi i}{m}}$ , this means that  $a_I = 0$  unless  $i_1$  is divisible by  $m$ . Hence,  $P(x) = Q(x_1^m, \underline{x})$ , and the invariant polynomials are generated by  $x_1^m, x_2, \dots, x_n$ , which are algebraically independent. Therefore, the quotient  $\mathbb{C}^n/\mu_m$  is analytically isomorphic to  $\mathbb{C}^n$ , which is smooth, and the quotient map is:

$$\pi_{\mu_m} : (x_1, \underline{x}) \in \mathbb{C}^n \longrightarrow (x_1^m, \underline{x}) \in \mathbb{C}^n.$$

## Organization of Part I

The aim of this part is to introduce the general notions involved in singular uniformization theory. We first introduce the notion of pairs and adapted morphisms which have a good behaviour with respect to a pair. Namely, they have ramification indices above the pair that are given by the coefficients appearing in  $\Delta$ .

The first examples of singular pairs were given by orbifolds. Inspired by the original definition due to Satake, we introduce the notion of orbistruktures that is analogous to the notion of orbifold atlas: an orbistrukture is a collection of local uniformizing charts, that are adapted morphisms on open subsets of a pair  $(X, \Delta)$ . Because we do not require any smoothness for the uniformizing spaces, an orbifold is a special type of orbistrukture, that consists in adapted morphisms with smooth uniformizing spaces. This framework is well-suited to define orbifold versions of sheaves. Several classical objects coming from geometry can be defined and several properties are transferred to this orbifold setting.

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For instance, we introduce the notion of Chern classes.

To conclude this part, we deal with a broader type of singularities than those obtained from orbifolds but that are natural from the point of view of algebraic geometry and pairs setting: these are klt singularities.

# PAIRS AND ORBIFOLD OBJECTS

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## 2.1 Morphisms adapted to pairs

### 2.1.1 The category of pairs

**Definition 2.1.1.** A pair  $(X, \Delta)$  consists of a normal analytic space  $X$  and a Weil  $\mathbb{Q}$ -divisor  $\Delta$  with standard coefficients, i.e written in the form:

$$\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i,$$

where  $\Delta_i$  is an irreducible reduced hypersurface and  $m_i \in \mathbb{N}_{\geq 2}$ .

We denote by  $X^* := X_{\text{reg}} \setminus |\Delta|$  the regular locus of the pair  $(X, \Delta)$ .

**Definition 2.1.2.** Let  $(X, \Delta)$  and  $(Y, \Delta')$  be two pairs, written as:

$$\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i \quad \text{and} \quad \Delta' = \sum_j \left(1 - \frac{1}{m'_j}\right) \Delta'_j.$$

A morphism between pairs  $f: (Y, \Delta') \rightarrow (X, \Delta)$  is a finite morphism of analytic spaces  $f: Y \rightarrow X$  such that, the following condition holds: if we write

$$f^* \Delta_i = \sum_j t_{ij} \Delta'_j + \sum_k t_{ik} \Delta''_{ik},$$

where  $\Delta''_{ik}$  are irreducible reduced components of  $f^* \Delta_i$  that are not included in  $|\Delta'|$ , then, for all  $i, j, k$ , the multiplicity  $m_i$  divides  $t_{ij} m'_j$  and  $t_{ik}$ .

One can check that the previous definitions define the *category of pairs*. Indeed, let  $(X, \Delta), (Y, \Delta'), (Z, \Delta'')$  be three pairs, and consider two morphisms  $f: (Z, \Delta'') \rightarrow (Y, \Delta')$  and  $g: (Y, \Delta') \rightarrow (X, \Delta)$ . The composition  $g \circ f$  defines a morphism of pairs from  $(Z, \Delta'')$  to  $(X, \Delta)$ . Moreover, the morphism  $\text{id}_X$  is a morphism of pairs  $(X, \Delta) \rightarrow (X, \Delta)$  that

is an identity morphism for the composition. Following this definition, an isomorphism of pairs  $\varphi: (Y, \Delta') \rightarrow (X, \Delta)$  is an isomorphism  $\varphi: Y \rightarrow X$  such that  $\Delta' = \varphi^* \Delta$ .

If  $U$  is a connected open subset of a pair  $(X, \Delta)$ , we can define the pair  $(U, \Delta|_U)$  by setting

$$\Delta|_U = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_{i|U}.$$

The canonical inclusion  $i_U: U \hookrightarrow X$  of an open subset  $U$  of  $X$  does not induce a morphism of pairs as defined above, since it is not finite.

### 2.1.2 The quotient pair $(Y/G, \Delta_G)$

Let  $Y$  be a normal analytic space and  $G$  a group acting properly on  $Y$ . Following Appendix A, the quotient space  $X := Y/G$  can be endowed with a natural analytic structure. One can construct a *pair structure on  $X$* . Let  $\pi_G: Y \rightarrow X$  be the quotient map, and define the ramification locus:

$$\text{Ram}(\pi_G) := \{y \in Y \mid \mathcal{O}_{Y/G, \pi_G(y)} \rightarrow \mathcal{O}_{Y, y} \text{ is not an isomorphism}\},$$

and the branch locus  $\text{Br}(\pi_G) := \pi_G(\text{Ram}(\pi_G))$ , which is the image of the ramification locus. The branch locus is an analytic subset of  $X$ . If we denote by  $(\Delta_i)_i$  the irreducible components of codimension 1 of the branch locus, we can associate to each of these components a ramification index  $m_i$  as follows: choose a general point of  $\Delta_i$  (i.e., a point of  $\Delta_i$  that is a smooth point of both  $X$  and  $\Delta_i$ , and does not belong to any other component of the branching locus). Around this point, we can choose a coordinate chart  $\varphi: V \rightarrow \mathbb{D}^n$  adapted to  $\Delta_i$ , meaning that  $\Delta_i$  is given by  $(z_1 = 0)$ , where  $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ . Choose a connected component  $U$  of  $\pi_G^{-1}(V)$ , and by Lemma 11.3.9, we can even assume that the map  $\pi_G|_U$  is finite. We know that  $\pi_G$  is étale on  $U \setminus \pi_G^{-1}(\Delta_i)$  and finite, so it is a covering of  $\mathbb{D}^* \times \mathbb{D}^{n-1}$ . By the geometric Galois correspondence, we have a biholomorphism  $U \setminus \pi_G^{-1}(\Delta_i) \simeq \mathbb{D}^* \times \mathbb{D}^{n-1}$ , and  $\pi_G$  is given in these coordinates by the map  $(z_1, z_2, \dots, z_n) \mapsto (z_1^{m_i}, z_2, \dots, z_n)$ .

The integer  $m_i$  is independent of the choice of the connected component  $U$ . Indeed, if  $U'$  is another connected component, there exists an element  $g \in G$  such that  $U' = g(U)$ , so  $g$  defines an isomorphism of coverings  $U' \setminus \pi_G^{-1}(\Delta_i) \simeq U \setminus \pi_G^{-1}(\Delta_i)$ . Furthermore, the integer  $m_i$  is independent of the choice of the general point  $x$ : the set in which  $x$  was chosen is an open Zariski set of  $\Delta_{i, \text{reg}}$  (the complement of an analytic subset of  $X$ ), which

is connected because  $\Delta_i$  is an irreducible component. Now, for each element of this open set, we have constructed an integer  $m_i(x)$  defined as above, and the function  $m_i$  is locally constant, hence constant. We set  $\Delta := \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$ . This defines a pair  $(Y/G, \Delta_G)$ .

**Remark 2.1.3.** Consider  $\pi_G : Y \rightarrow Y/G$ . Let  $x \in Y/G$ . As we saw earlier, there exists a neighborhood  $V$  of  $x$  and an open set  $U$  around a preimage  $y$  of  $x$  such that  $\pi_G|_U$  is the quotient map of the action of the stabilizer  $G_y$  on  $U$ . Given the definition of  $\Delta_G$ , we observe that  $(V, \Delta_G|_V) \simeq (U/G_y, \Delta_{G_y})$ .

**Remark 2.1.4.** Consider the special case where  $G$  is a finite group acting faithfully and linearly on  $\mathbb{C}^n$ . We can give another definition of  $m_i$  in terms of the elements of  $G$  that fix a hyperplane. First, consider  $\text{Fix } G = \bigcup_{g \in G \setminus \{1\}} \text{Fix } g$ . It is a finite union of proper vector subspaces of  $\mathbb{C}^n$ .

If  $H$  is a hyperplane in  $\text{Fix } G$ , there exists an element  $g \in G \setminus \{1\}$  such that  $\text{Fix } g = H$ .<sup>1</sup> Hence, we can consider  $G(H) := \bigcap_{x \in H} G_x$ . It is a non-trivial group since it contains  $g$ . We can show that  $G(H)$  is cyclic. Indeed, the group  $G(H)$  fixes the hyperplane  $H$ , and that representation can be split so that  $G(H)$  acts on a line (for instance, we can pick a Hermitian product on  $\mathbb{C}^n$  such that  $G$  acts unitarily, and  $H^\perp$  is a  $G$ -stable line). Since  $G$  acts faithfully, we conclude that  $G(H)$  acts faithfully on the line, so  $G(H)$  is a finite subgroup of  $\mathbb{C}^\times$ , and hence it is cyclic. We can show that for a general  $x \in H$ , we have  $G_x = G(H)$ .<sup>2</sup> Let  $m(H)$  denote its order.

That integer  $m(H)$  coincides with the geometric integer  $m_i$  associated with a meridian loop around general points of the branched divisor. Indeed, we have  $\text{Fix } G = \text{Ram}(\pi_G)$ , in particular their codimension 1 locus are the same. If we pick a general point  $p$  of an irreducible component  $\Delta_i$  of the branch divisor  $\Delta_G$ , we have a neighborhood of this point of the form  $U_x/G_x$ , where  $x \in \mathbb{C}^n$  is in the preimage of  $p$ , which must lie on a fixed hyperplane  $H_i$  by the characterization of smooth points of  $\mathbb{C}^n/G$ . As we pick a general point, we have  $G_x$  which is cyclic of order  $m(H_i)$ , and we can find coordinates around  $x$  such that the action of  $G_x$  is given by the multiplication by an  $m(H_i)$ -th root of unity

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1. This is a consequence of the fact that  $H = \bigcup_{g \in G \setminus \{1\}} \text{Fix } g \cap H$ , and as it is a finite union of subspaces, one of them is equal to the whole of  $H$ .

2. Indeed, if  $g_0$  is a generator of  $G(H)$ , we have  $H = \text{Fix } g_0$ . The direct inclusion is clear by definition of  $G(H)$ , and the second inclusion is due to the fact that  $\text{Fix } g_0$  contains a hyperplane but it is not the whole of  $\mathbb{C}^n$ . To conclude, it suffices to show that for any  $x \in H \setminus \bigcup_{g \notin G(H)} \text{Fix } g \cap H$ , we have  $G_x = G(H)$ . First, the removed subset of  $H$  is a finite union of proper subspaces, because, for  $g \in G$  that is not in  $G(H)$ , we have  $H \cap \text{Fix } g \subset H$ . Moreover, for such a  $x$ , we have  $G(H) \subseteq G_x$  by definition of  $G(H)$ , but it is equal: if not we pick an element  $g \in G_x \setminus G(H)$ . It follows that  $x$  belongs to  $\text{Fix } g \cap H$  that is not allowed.

on the first coordinate. Hence, the quotient map  $\pi_G$  restricted to  $U_x$  is isomorphic to the quotient by  $G_x$ , which in normal form is  $(z_1, \underline{z}) \mapsto (z_1^{m(H_i)}, \underline{z})$ . Therefore,  $m(H_i) = m_i$ .

**Definition 2.1.5.** *Let  $X, Y$  be two analytic spaces. We say that a morphism  $f: Y \rightarrow X$  is Galois if there exists a finite group  $G \subseteq \text{Aut}(Y)$  and an isomorphism of analytic spaces  $\varphi: Y/G \rightarrow X$  such that  $\varphi \circ \pi_G = f$ .*

*We say that a finite surjective morphism  $f$  is quasi-étale if there exists an analytic subset  $Z \subseteq Y$  of codimension at least 2 such that  $f$  is étale over  $Y \setminus Z$ .*

By Nagata–Zariski’s branching locus purity theorem, the morphism  $f$  is quasi-étale if and only if  $f$  is étale over  $X_{\text{reg}}$ . Indeed, if  $f$  is étale over  $X_{\text{reg}}$ , then it suffices to consider the analytic subset  $Z = f^{-1}(X_{\text{sing}})$  whose codimension is at least 2. If  $f$  is not étale over  $X_{\text{reg}}$ , then by the purity theorem, the branching locus has pure codimension 1 and is non-empty, so the ramification locus contains an irreducible component of codimension 1.

### 2.1.3 General facts about adapted morphisms

**Definition 2.1.6.** *Let  $(X, \Delta)$  be a pair with  $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$ , and let  $Y$  be a normal analytic space. We say that a finite Galois morphism  $f: Y \rightarrow X$  is:*

- $\Delta$ -adapted if for every  $i$ , there exists a reduced Weil divisor  $\Delta'_i$  and an integer  $a_i \in \mathbb{N}_{\geq 1}$  such that:  $f^* \Delta_i = a_i m_i \Delta'_i$ ,
- strictly  $\Delta$ -adapted if  $f$  is  $\Delta$ -adapted and additionally  $a_i = 1$  for all  $i$ ,
- orbi-étale if  $f$  is strictly  $\Delta$ -adapted and étale over  $X^* := X_{\text{reg}} \setminus |\Delta|$ .

These definitions have an interpretation in terms of classical morphisms of pairs as explained in [Cam11, Def. 2.4]: since  $f$  is Galois, we have a group  $G$  and an isomorphism  $\varphi: Y/G \rightarrow X$ . Saying that  $f$  is  $\Delta$ -adapted means that  $\varphi$  induces a morphism of pairs  $\varphi: (Y/G, \Delta_G) \rightarrow (X, \Delta)$  as in the above mentioned article. Saying that  $f$  is orbi-étale means that  $\varphi$  induces an isomorphism between the pairs.

The property of being orbi-étale is invariant under  $\Delta$ -adapted base change :

**Proposition 2.1.7.** *Let  $(X, \Delta)$  be a pair, and let  $f: Y \rightarrow X$  be an orbi-étale morphism for  $\Delta$  and  $g: Z \rightarrow X$  a  $\Delta$ -adapted morphism. Let us denote by  $N := \text{norm}(Y \times_X Z)$  the normalization of the fiber product and  $F: N \rightarrow Z$  the induced projection. Then the morphism  $F$  is quasi-étale.*

*Proof.* We consider the open subset  $U \subseteq X$  which is the complement of the analytic subset  $X_{\text{sing}} \cup |\Delta|_{\text{sing}} \cup f(Y_{\text{sing}}) \cup \text{Br}(f)_{\text{sing}} \cup g(Z_{\text{sing}}) \cup \text{Br}(g)_{\text{sing}}$ , which has codimension at least 2. We show that the morphism  $F$  is étale over  $g^{-1}(U)$ . This is an open subset of  $Z$  whose complement is contained in an analytic subset of codimension at least 2 by finiteness of  $g$ . We then study the normal forms of the various morphisms. If  $x \in U$ , we consider two points  $y \in Y$  and  $z \in Z$  as preimages of  $x$  by  $f$  and  $g$ , respectively. Then, several cases occur depending on the position of  $x$ .

If  $x$  is not on the divisor  $\Delta$ , then  $f$  is étale over  $x$ , so  $f$  establishes a biholomorphism between neighborhoods of  $y$  and  $x$ . This leads us to conclude that the fiber product around the point  $(y, z)$  is a normal variety, and the morphism  $F$  is likewise a biholomorphism between neighborhoods of  $(y, z)$  and  $z$ .

If  $x$  is on the divisor  $\Delta$ , then there is a unique irreducible component on which  $x$  lies, and let  $m$  denote the multiplicity associated with this component. We can find coordinates around the points  $x$ ,  $y$ , and  $z$  such that the morphisms  $f$  and  $g$  are written in these coordinates as  $f(z_1, \underline{z}) = (z_1^m, \underline{z})$  and  $g(w_1, \underline{w}) = (w_1^{ma}, \underline{w})$ . Then, the fiber product is:

$$\{(z, w) \mid z_1^m = w_1^{ma}, \underline{z} = \underline{w}\} = \bigcup_{0 \leq k \leq m-1} \{(z, w) \mid z_1 = \zeta^k w_1^a, \underline{z} = \underline{w}\},$$

where  $\zeta = e^{\frac{2\sqrt{-1}\pi}{m}}$ . The normalization is isomorphic to  $\mathbb{D}^n \times \mu_m$ . In fact, we have a morphism:

$$(w, \zeta^k) \in \mathbb{D}^n \times \mu_m \mapsto ((\zeta^k w_1^a, \underline{w}), w) \in \{(z, w) \mid z_1^m = w_1^{ma}, \underline{z} = \underline{w}\}.$$

This induces a morphism between normal varieties from  $\mathbb{D}^n \times \mu_m$  to the normalization of the fiber product. This is a morphism of degree 1 away from the locus  $(z_1 = w_1 = 0)$ , so it is an isomorphism by Zariski's main theorem [Rem94, Prop. 14.7].

Finally, the morphism  $F$  is written in normal form  $F(w, \zeta^k) = w$ , which is étale.  $\square$

## 2.2 Orbistruktures and orbisheaves

### 2.2.1 Orbistruktures

We can now define the notion of an orbistrukture, which is necessary to define what an orbifold is:

**Definition 2.2.1.** Let  $(X, \Delta)$  be a pair. An orbistrukture on  $(X, \Delta)$  is a collection  $\mathcal{C}$  of triples  $(U_\alpha, V_\alpha, \eta_\alpha)_\alpha$ , called local uniformizing charts, such that:

- the  $U_\alpha$  are euclidean open sets that cover  $X$ :  $X = \bigcup_\alpha U_\alpha$ ,
- the  $V_\alpha$  are normal analytic spaces,
- the  $\eta_\alpha: V_\alpha \rightarrow U_\alpha$  are morphisms adapted to the pair  $(U_\alpha, \Delta|_{U_\alpha})$  satisfying the following condition: if we denote by  $V_{\alpha\beta}$  the normalization of the fiber product  $V_\alpha \times_X V_\beta$ , and by  $g_{\alpha\beta}: V_{\alpha\beta} \rightarrow V_\alpha$  the natural projection morphisms, then these are quasi-étale.

We say that the structure is strictly  $\Delta$ -adapted (resp. orbi-étale) if the morphisms  $\eta_\alpha$  are. We say that the orbistrukture is smooth if the spaces  $V_\alpha$  are smooth.

We often denote by  $G_\alpha := \text{Gal}(\eta_\alpha)$  the finite group of automorphisms acting on  $V_\alpha$ . Note that  $V_{\alpha\alpha} = V_\alpha \times G_\alpha$ . Indeed, we have a morphism

$$(v, g) \in V_\alpha \times G_\alpha \mapsto (v, g \cdot v) \in V_\alpha \times V_\alpha.$$

Since  $\eta_\alpha(x) = \eta_\alpha(g \cdot x)$ , this morphism factors through the fiber product  $V_\alpha \times_X V_\alpha$  by the universal property, and then through  $V_{\alpha\alpha}$  because  $V_\alpha \times G_\alpha$  is normal. The morphism  $V_\alpha \times G_\alpha \rightarrow V_{\alpha\alpha}$  is finite, and is an isomorphism at least over  $V_\alpha^* := \eta_\alpha^{-1}(X^*)$ . By Zariski's Main Theorem [Rem94, Prop. 14.7], it is an isomorphism. Furthermore, the projections are given by  $g_{\alpha\alpha}(v, g) = v$  and  $g'_{\alpha\alpha}(v, g) = g \cdot v$ .

The orbistrukture associated with a pair generalizes the notion of an atlas: if  $X$  is a manifold and  $\mathcal{U}$  is an open cover, we can define the associated orbistrukture  $\mathcal{C}_\mathcal{U} := \{(U, U, \text{id}_U)_{U \in \mathcal{U}}\}$  on the pair  $(X, 0)$ . In this example, the normalization of the fiber product of two open sets  $U$  and  $V$  is simply the intersection  $U \cap V$ , and the projections  $g_{\alpha\beta}$  are the canonical inclusions  $U \cap V \hookrightarrow U$ .

This analogy motivates the following terminology: we say that  $V_{\alpha\beta}$  is the intersection of  $V_\alpha$  and  $V_\beta$ . Let us now study multiple intersections. Suppose we have three uniformizing charts  $(U_\alpha, V_\alpha, \eta_\alpha)$ ,  $(U_\beta, V_\beta, \eta_\beta)$ , and  $(U_\gamma, V_\gamma, \eta_\gamma)$ . Consider the normalization of the limit of the following diagram, denoted by  $V_{\alpha\beta\gamma}$ :

$$\begin{array}{ccc} V_\alpha & & V_\beta & & V_\gamma \\ & \searrow & \downarrow & \swarrow & \\ & & X & & \end{array}$$

Then, by the universal properties of normalization and fiber product, there are unique

isomorphisms:

$$V_{\alpha\beta\gamma} \simeq \text{norm}(V_{\alpha\beta} \times_X V_\gamma) \simeq \text{norm}(V_{\beta\gamma} \times_X V_\alpha) \simeq \text{norm}(V_{\gamma\alpha} \times_X V_\beta),$$

where  $\text{norm}(V)$  denotes the normalization of  $V$ .

We denote by  $h_{\alpha\beta,\gamma} : V_{\alpha\beta\gamma} \rightarrow V_{\alpha\beta}$  the natural projection, and we have

$$h_{\alpha\beta,\gamma} \circ g_{\alpha\beta} = h_{\gamma\alpha,\beta} \circ g_{\alpha\gamma}.$$

The morphisms  $h_{\alpha\beta,\gamma}$  corresponding to the orbistruature  $\mathcal{C}_{\mathcal{U}}$  associated with an open cover are simply the inclusions between triple and double intersections, of the form  $U \cap V \cap W \hookrightarrow U \cap V$ .

The objects we have introduced rely on analytic notions, but other definitions exist in the algebraic setting: if  $(X, \Delta)$  is a pair where  $X$  is a normal projective variety, one can replace the first condition in the above definition by requiring that the  $(U_\alpha)$  form a covering for the étale topology.

**Definition 2.2.2.** *Let  $(X, \Delta)$  and  $(Y, \Delta')$  be two pairs equipped with orbistruatures  $\mathcal{C}$  and  $\mathcal{D}$  respectively. An orbifold morphism from  $(Y, \Delta')$  to  $(X, \Delta)$  is a morphism of analytic spaces  $f : Y \rightarrow X$  such that, for each point  $y \in Y$  and for each local uniformizing chart  $(U, V, \eta)$  around  $f(y)$ , there exists a local uniformizing chart  $(T, W, \mu)$  around  $y$ , together with a morphism of analytic spaces  $f_{\eta\mu} : W \rightarrow V$  such that*

$$\eta \circ f_{\eta\mu} = f \circ \mu.$$

Moreover, if  $(X, \Delta)$ ,  $(Y, \Delta')$ , and  $(Z, \Delta'')$  are three pairs equipped with orbistruatures  $\mathcal{C}$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  respectively, and we have two morphisms  $f : (Z, \Delta'') \rightarrow (Y, \Delta')$  and  $g : (Y, \Delta') \rightarrow (X, \Delta)$ , then their composition  $g \circ f$  defines an orbifold morphism. The identity morphism  $\text{id}_X$  induces an identity morphism, which is a left and right identity for the composition previously defined.

Thus, we define a category whose objects are pairs equipped with orbistruatures and whose morphisms are orbifold morphisms.

When we have an orbifold morphism  $f : (Y, \Delta') \rightarrow (X, \Delta)$  and two local uniformizing charts  $(U_\alpha, V_\alpha, \eta_\alpha) \in \mathcal{C}$  and  $(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'}) \in \mathcal{D}$ , we denote by  $f_{\alpha\alpha'} := f_{\eta_\alpha \mu_{\alpha'}}$  to simplify notation.

**Remark 2.2.3.** Here we follow the definition of morphisms from [Sat56]. However, several

definitions of orbifold morphisms exist (see [MP97] for the notion of strong morphisms).

**Remark 2.2.4.** Consider an orbifold morphism  $f: (Y, \Delta') \rightarrow (X, \Delta)$ , and let  $\mathcal{C} := \{(U_\alpha, V_\alpha, \eta_\alpha)\}$  with transition morphisms, and  $\mathcal{D} := \{(U'_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})\}$  be the orbistruktures on  $(X, \Delta)$  and  $(Y, \Delta')$ , respectively. Let  $g_{\alpha\beta}$  and  $l_{\alpha'\beta'}$  be the projection morphisms of  $\mathcal{C}$  and  $\mathcal{D}$  respectively.

By universal properties of the fiber product and normalization, we obtain a morphism  $f_{\alpha\beta, \alpha'\beta'}: W_{\alpha'\beta'} \rightarrow V_{\alpha\beta}$  that sits in the following diagram.

$$\begin{array}{ccccc}
 & & \exists f_{\alpha\beta, \alpha'\beta'} & & \\
 & & \dashrightarrow & & \\
 W_{\alpha'\beta'} & & & & V_{\alpha\beta} \\
 \downarrow l_{\alpha'\beta'} & \searrow l_{\beta'\alpha'} & & & \downarrow \\
 & W_{\beta'} & \xrightarrow{f_{\beta\beta'}} & & V_{\beta} \\
 & \downarrow \mu_{\beta'} & & & \downarrow \\
 W_{\alpha'} & \xrightarrow{f_{\alpha\alpha'}} & & & V_{\alpha} \\
 \downarrow \mu_{\alpha'} & & & & \downarrow \\
 Y & \xrightarrow{f} & & & X
 \end{array}$$

## 2.2.2 Orbisheaves

**Definition 2.2.5.** Let  $(X, \Delta)$  be a pair equipped with an orbistrukture  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)_\alpha\}$ . We use the same notations as in Definition 2.2.1. An orbisheaf  $\mathcal{F}$  on<sup>3</sup>  $\mathcal{C}$  is the data of a collection of sheaves  $\mathcal{F}_\alpha$  on  $V_\alpha$  such that:

- the  $\mathcal{F}_\alpha$  are coherent  $\mathcal{O}_{V_\alpha}$ -modules,
- for all  $\alpha, \beta$ , there exists an isomorphism  $\varphi_{\alpha\beta}: g_{\beta\alpha}^* \mathcal{F}_\beta \rightarrow g_{\alpha\beta}^* \mathcal{F}_\alpha$ , satisfying compatibility conditions on triple intersections, that is, over  $V_{\alpha\beta\gamma}$  we have:

$$h_{\alpha\beta, \gamma}^* \varphi_{\alpha\beta} \circ h_{\beta\gamma, \alpha}^* \varphi_{\beta\gamma} = h_{\gamma\alpha, \beta}^* \varphi_{\alpha\gamma}.$$

We say that  $\mathcal{F}$  is torsion-free (resp. reflexive) if, for each  $\alpha$ , the sheaf  $\mathcal{F}_\alpha$  is torsion-free (resp. reflexive). We say that  $\mathcal{F}$  is an orbibundle if the  $\mathcal{F}_\alpha$  are locally free.

In the same way, we define a weak orbisheaf as a collection of coherent  $\mathcal{O}_{V_\alpha}$ -modules such that there are transition morphisms  $\varphi_{\alpha\beta}: g_{\beta\alpha}^{[*]} \mathcal{F}_\beta \rightarrow g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha$ , and a compatibility

3. As an abuse of notation, we will say that  $\mathcal{F}$  is an orbisheaf or an orbisheaf on  $(X, \Delta)$  without specifying the orbistrukture  $\mathcal{C}$  when there is no ambiguity.

condition:

$$h_{\alpha\beta,\gamma}^{[*]} \varphi_{\alpha\beta} \circ h_{\beta\gamma,\alpha}^{[*]} \varphi_{\beta\gamma} = h_{\gamma\alpha,\beta}^{[*]} \varphi_{\alpha\gamma}.$$

Here, we use the notation  $[*]$  to denote the reflexive hull of the pullback, meaning for example:  $g_{\beta\alpha}^{[*]} \mathcal{F}_\beta := (g_{\beta\alpha}^* \mathcal{F}_\beta)^{\vee\vee}$ .

**Remark 2.2.6.** Let  $(X, \Delta)$  be a pair endowed with an orbistructure  $\mathcal{C}$ . One can observe that an orbibundle  $\mathcal{F}$  defines a vector bundle  $\mathcal{F}|_{X^*}$  on  $X^*$ . First, the collection  $\mathcal{C}$  induces a manifold structure on  $X^*$  by restriction. More precisely, there is an orbi-étale orbistructure  $\mathcal{C}^* = \{(U_\alpha^*, V_\alpha^*, \eta_\alpha^*)\}$  where  $U_\alpha^* = U_\alpha \cap X^*$ ,  $V_\alpha^* = \eta_\alpha^{-1}(U_\alpha^*)$  and  $\eta_\alpha^*$  is the restriction of  $\eta_\alpha$  to  $V_\alpha^*$ . Up to shrinking  $U_\alpha^*$  and choosing a connected component of  $V_\alpha^*$ , one can assume that  $\eta_\alpha^*$  is a biholomorphism. Hence, we get that  $\mathcal{C}^*$  defines an atlas of  $X^*$  which is compatible with the natural manifold structure on  $X^*$ . Hence, by considering the restriction of locally free sheaves  $\mathcal{F}_\alpha$  and transition morphisms to these charts of  $X^*$  and their intersections, we get a locally free sheaf, because these transition morphisms satisfy cocycle conditions as they are restrictions of the initial ones to appropriate open subsets. We get that these locally free sheaves glue together on  $X^*$ , so that it defines a locally free sheaf denoted  $\mathcal{F}|_{X^*}$ .

We investigate what happens in the definition of orbisheaves when  $\alpha = \beta$ . The following lemma ensures that the local sheaves  $\mathcal{F}_\alpha$  are naturally  $G_\alpha$ -sheaves.

**Lemma 2.2.7.** *Let  $(X, \Delta)$  be a pair equipped with an orbistructure  $\mathcal{C}$ . Let  $\mathcal{F}$  be a weak orbisheaf. Then, the local sheaves  $\mathcal{F}_\alpha$  are  $G_\alpha$ -sheaves.*

*Proof.* We have established that  $V_{\alpha\alpha} = V_\alpha \times G_\alpha$ , equipped with two projections  $g_{\alpha\alpha}$  and  $g'_{\alpha\alpha}$ . These two morphisms are étale, hence flat, so they preserve the reflexivity of sheaves [Har80, Prop. 1.8]. In other words, we have:  $g_{\alpha\alpha}^* \mathcal{F}_\alpha = g_{\alpha\alpha}^{[*]} \mathcal{F}_\alpha$ .

Consider a transition morphism  $\varphi_{\alpha\alpha} : g_{\alpha\alpha}^* \mathcal{F}_\alpha \rightarrow g'_{\alpha\alpha}{}^* \mathcal{F}_\alpha$ . On the stalk over a point  $(v, g) \in V_\alpha \times G_\alpha$ , this yields an isomorphism:

$$\varphi_{\alpha\alpha,(v,g)} : \mathcal{F}_{\alpha,v} \otimes_{\mathcal{O}_{V_\alpha,v}} \mathcal{O}_{V_{\alpha\alpha,(v,g)}} \longrightarrow \mathcal{F}_{\alpha,g \cdot v} \otimes_{\mathcal{O}_{V_\alpha,g \cdot v}} \mathcal{O}_{V_{\alpha\alpha,(v,g)}}.$$

Since  $g_{\alpha\alpha}$  and  $g'_{\alpha\alpha}$  are étale, their stalks provide isomorphisms

$$g_{\alpha\alpha,(v,g)} : \mathcal{O}_{V_{\alpha\alpha,(v,g)}} \longrightarrow \mathcal{O}_{V_\alpha,v}, \quad g'_{\alpha\alpha,(v,g)} : \mathcal{O}_{V_{\alpha\alpha,(v,g)}} \longrightarrow \mathcal{O}_{V_\alpha,g \cdot v},$$

which induce isomorphisms on the tensor products. Ultimately, tensoring with  $\mathcal{F}_{\alpha,v}$  yields an isomorphism

$$\mathcal{F}_{\alpha,v} \simeq \mathcal{F}_{\alpha,v} \otimes_{\mathcal{O}_{V_\alpha,v}} \mathcal{O}_{V_{\alpha\alpha},(v,g)}.$$

Composing these maps produces, for each  $g \in G_\alpha$ , an isomorphism of sheaves

$$\mathcal{F}_\alpha \xrightarrow{\sim} g^* \mathcal{F}_\alpha,$$

given by the composition:

$$\mathcal{F}_{\alpha,v} \xrightarrow{\sim} \mathcal{F}_{\alpha,v} \otimes_{\mathcal{O}_{V_\alpha,v}} \mathcal{O}_{V_{\alpha\alpha},(v,g)} \longrightarrow \mathcal{F}_{\alpha,g \cdot v} \otimes_{\mathcal{O}_{V_\alpha,g \cdot v}} \mathcal{O}_{V_{\alpha\alpha},(v,g)} \xrightarrow{\sim} \mathcal{F}_{\alpha,g \cdot v}.$$

This construction defines a  $G_\alpha$ -sheaf structure. However, we observe a reversal compared to the conventions established in Definition 11.3.11. This results from the fact that the transformation induced by  $g$  above is given by precomposition with  $g^{-1}$ : if  $s_v$  is a germ of a section at  $v$ , then  $s \circ g^{-1}$  yields a germ at  $g \cdot v$ .

This definition has the advantage of producing a left action of  $G$  on functions. The definition we gave for orbifold sheaves was based on a left action on the underlying space, which induces a right action on the sections by precomposition.

Therefore, to align with our conventions, it suffices to consider the construction above at the point  $(g \cdot v, g^{-1})$ . This gives a morphism

$$g_{\mathcal{F}_\alpha} : g^* \mathcal{F}_\alpha \longrightarrow \mathcal{F}_\alpha.$$

Compatibility is preserved when composing two such morphisms  $g_{\mathcal{F}}$  and  $h_{\mathcal{F}}$ , by virtue of the triple intersection compatibility condition. Indeed, we have  $V_{\alpha\alpha\alpha} = V_\alpha \times G_\alpha \times G_\alpha$ , and the three projections to  $V_{\alpha\alpha}$  are given by:

$$h_\alpha : (x, g, h) \longmapsto (x, g), \quad h'_\alpha : (x, g, h) \longmapsto (g \cdot x, h), \quad h''_\alpha : (x, g, h) \longmapsto (x, hg).$$

Then, from the compatibility relation, we obtain:

$$h'_\alpha{}^* \varphi_{\alpha\alpha} \circ h_\alpha{}^* \varphi_{\alpha\alpha} = h''_\alpha{}^* \varphi_{\alpha\alpha},$$

applied at  $((hg) \cdot x, h^{-1}, g^{-1})$ , which gives the desired compatibility condition for a  $G_\alpha$ -sheaf.  $\square$

**Remark 2.2.8.** If  $\mathcal{F}$  is a torsion-free orbisheaf, then it naturally carries the structure of a weak orbisheaf: if  $\varphi_{\alpha\beta}$  is a transition morphism, then the morphism  $\varphi_{\alpha\beta}^{\vee\vee}$  (the double dual) is a transition morphism between the reflexive hulls, satisfying the compatibility conditions.

Indeed, applying the biduality functor to the compatibility identities and identifying the morphisms<sup>4</sup> — namely,  $(h_{\alpha\beta,\gamma}^* \varphi_{\alpha\beta})^{\vee\vee} = h_{\alpha\beta,\gamma}^{[*]} \varphi_{\alpha\beta}^{\vee\vee}$  — we obtain the desired identity.

**Definition 2.2.9.** Let  $(X, \Delta)$  be a pair equipped with an orbistructure  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)_\alpha\}$ , and let  $\mathcal{F}$  and  $\mathcal{G}$  be two orbisheaves, whose transition morphisms are denoted  $\varphi_{\alpha\beta}$  and  $\psi_{\alpha\beta}$  respectively. We keep the notations from 2.2.1.

A morphism of orbisheaves  $u : \mathcal{F} \rightarrow \mathcal{G}$  is a collection of morphisms of  $\mathcal{O}_{V_\alpha}$ -modules  $(u_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{G}_\alpha)_\alpha$  such that the following diagram commutes:

$$\begin{array}{ccc} g_{\beta\alpha}^* \mathcal{F}_\beta & \xrightarrow{\varphi_{\alpha\beta}} & g_{\alpha\beta}^* \mathcal{F}_\alpha \\ g_{\beta\alpha}^* u_\beta \downarrow & & \downarrow g_{\alpha\beta}^* u_\alpha \\ g_{\beta\alpha}^* \mathcal{G}_\beta & \xrightarrow{\psi_{\alpha\beta}} & g_{\alpha\beta}^* \mathcal{G}_\alpha \end{array}$$

Similarly, we can define morphisms between weak orbisheaves.

**Definition 2.2.10.** Let  $(X, \Delta)$  be a pair equipped with an orbistructure  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)_\alpha\}$ , and let  $\mathcal{F}$  and  $\mathcal{G}$  be two weak orbisheaves, whose transition morphisms are denoted  $\varphi_{\alpha\beta}$  and  $\psi_{\alpha\beta}$  respectively.

A morphism of weak orbisheaves  $u : \mathcal{F} \rightarrow \mathcal{G}$  is a collection of morphisms of  $\mathcal{O}_{V_\alpha}$ -modules  $(u_\alpha : \mathcal{F}_\alpha \rightarrow \mathcal{G}_\alpha)_\alpha$  such that the following diagram commutes:

$$\begin{array}{ccc} g_{\beta\alpha}^{[*]} \mathcal{F}_\beta & \xrightarrow{\varphi_{\alpha\beta}} & g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha \\ g_{\beta\alpha}^{[*]} u_\beta \downarrow & & \downarrow g_{\alpha\beta}^{[*]} u_\alpha \\ g_{\beta\alpha}^{[*]} \mathcal{G}_\beta & \xrightarrow{\psi_{\alpha\beta}} & g_{\alpha\beta}^{[*]} \mathcal{G}_\alpha \end{array}$$

If  $u : \mathcal{F} \rightarrow \mathcal{G}$  and  $v : \mathcal{G} \rightarrow \mathcal{H}$  are two morphisms of orbisheaves (resp. weak orbisheaves), there exists a composition  $v \circ u : \mathcal{F} \rightarrow \mathcal{H}$  given by the collection  $(v_\alpha \circ u_\alpha)_\alpha$ .

---

4. We can identify these morphisms because we have assumed that  $\mathcal{F}$  is torsion-free. In that case, the  $\mathcal{F}_\alpha$  are locally free on a Zariski open subset whose complement has codimension at least 2, so the two previous sections coincide on this open set (since in this case we have  $\varphi_{\alpha\beta}^{\vee\vee} = \varphi_{\alpha\beta}$  and  $h_{\alpha\beta,\gamma}^* = h_{\alpha\beta,\gamma}^{[*]}$ ). We conclude that the two sections are equal by the reflexivity of the sheaf  $\mathcal{H}om(h_{\alpha\beta,\gamma}^{[*]} g_{\beta\alpha}^{[*]} \mathcal{F}_\beta, h_{\alpha\beta,\gamma}^{[*]} g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)$ .

Moreover, there is an identity morphism  $\text{id}_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F}$  given by the collection  $(\text{id}_{\mathcal{F}_\alpha})_\alpha$ , which acts as an identity for the above composition.

In this way, we define the category of orbisheaves defined on an orbisttructure  $\mathcal{C}$ , denoted by  $\text{OrbiSh}_{\mathcal{C}}$  (resp. weak orbisheaves denoted by  $\text{WOrbiSh}_{\mathcal{C}}$ ).

**Example 2.2.11.** Let  $(X, \Delta)$  be a pair endowed with an orbisttructure  $\mathcal{C}$  and  $\mathcal{F}$  a sheaf on  $X$ . Hence, one can define *the orbisheaf induced by  $\mathcal{F}$* , denoted  $\mathcal{F}^{\text{orb}}$ . It is defined by the collection of sheaves  $\mathcal{F}_\alpha := \eta_\alpha^* \mathcal{F}$  and the natural transition morphisms coming from the identity  $\eta_\alpha \circ g_{\alpha\beta} = \eta_\beta \circ g_{\beta\alpha}$ :

$$\text{nat}_{\alpha\beta} : g_{\beta\alpha}^* \eta_\beta^* \mathcal{F} \xrightarrow{\sim} g_{\alpha\beta}^* \eta_\alpha^* \mathcal{F}.$$

The classical constructions associated with ordinary sheaves carry over to the framework of orbisheaves.

**Definition 2.2.12.** Let  $(X, \Delta)$  be a pair equipped with an orbisttructure  $\mathcal{C}$ , and let  $\mathcal{F}, \mathcal{G}$  be two orbisheaves (resp. weak orbisheaves), given respectively by transition morphisms  $\varphi_{\alpha\beta}$  and  $\psi_{\alpha\beta}$ .

We define the direct sum  $\mathcal{F} \oplus \mathcal{G}$  as the orbisheaf (resp. weak orbisheaf) given by the collection  $(\mathcal{F}_\alpha \oplus \mathcal{G}_\alpha)$ , and the transition morphisms  $\theta_{\alpha\beta}$ :

$$\begin{aligned} g_{\beta\alpha}^*(\mathcal{F}_\beta \oplus \mathcal{G}_\beta) &\longrightarrow g_{\beta\alpha}^* \mathcal{F}_\beta \oplus g_{\beta\alpha}^* \mathcal{G}_\beta \xrightarrow{\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}} g_{\alpha\beta}^* \mathcal{F}_\alpha \oplus g_{\alpha\beta}^* \mathcal{G}_\alpha \longrightarrow g_{\alpha\beta}^*(\mathcal{F}_\alpha \oplus \mathcal{G}_\alpha). \\ \left( \text{resp. } g_{\beta\alpha}^{[*]}(\mathcal{F}_\beta \oplus \mathcal{G}_\beta) &\longrightarrow g_{\beta\alpha}^{[*]} \mathcal{F}_\beta \oplus g_{\beta\alpha}^{[*]} \mathcal{G}_\beta \xrightarrow{\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}} g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha \oplus g_{\alpha\beta}^{[*]} \mathcal{G}_\alpha \longrightarrow g_{\alpha\beta}^{[*]}(\mathcal{F}_\alpha \oplus \mathcal{G}_\alpha) \right) \end{aligned}$$

We define the tensor product  $\mathcal{F} \otimes \mathcal{G}$  as the orbisheaf (resp. weak orbisheaf) given by  $(\mathcal{F}_\alpha \otimes \mathcal{G}_\alpha)$  and the transition morphisms:

$$\begin{aligned} g_{\beta\alpha}^*(\mathcal{F}_\beta \otimes \mathcal{G}_\beta) &\longrightarrow g_{\beta\alpha}^* \mathcal{F}_\beta \otimes g_{\beta\alpha}^* \mathcal{G}_\beta \xrightarrow{\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}} g_{\alpha\beta}^* \mathcal{F}_\alpha \otimes g_{\alpha\beta}^* \mathcal{G}_\alpha \longrightarrow g_{\alpha\beta}^*(\mathcal{F}_\alpha \otimes \mathcal{G}_\alpha). \\ \left( \text{resp. } g_{\beta\alpha}^{[*]}(\mathcal{F}_\beta \otimes \mathcal{G}_\beta) &\longrightarrow g_{\beta\alpha}^{[*]} \mathcal{F}_\beta \otimes g_{\beta\alpha}^{[*]} \mathcal{G}_\beta \xrightarrow{\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}} g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha \otimes g_{\alpha\beta}^{[*]} \mathcal{G}_\alpha \longrightarrow g_{\alpha\beta}^{[*]}(\mathcal{F}_\alpha \otimes \mathcal{G}_\alpha) \right) \end{aligned}$$

To show that these are indeed orbisheaves, it remains to verify compatibility on triple intersections. This follows from the fact that

$$h_{\alpha\beta,\gamma}^*(\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}) = h_{\alpha\beta,\gamma}^* \varphi_{\alpha\beta} \oplus h_{\alpha\beta,\gamma}^* \psi_{\alpha\beta},$$

from the formula  $g_{\alpha\beta} \circ h_{\alpha\beta,\gamma} = g_{\alpha\gamma} \circ h_{\gamma\alpha,\beta}$ , and from the compatibility conditions satisfied by the transition morphisms  $(\varphi_{\alpha\beta})$  and  $(\psi_{\alpha\beta})$ .

Likewise, for the tensor product, this also follows from

$$h_{\alpha\beta,\gamma}^*(\varphi_{\alpha\beta} \otimes \psi_{\alpha\beta}) = h_{\alpha\beta,\gamma}^*\varphi_{\alpha\beta} \otimes h_{\alpha\beta,\gamma}^*\psi_{\alpha\beta}.$$

The two previous definitions illustrate the fact that if one has orbisheaves and an operation that commutes with pullbacks, then this operation defines a new orbisheaf. In the same way, one can define the  $p$ -th exterior power  $\wedge^p \mathcal{F}$ .

A difficulty arises when trying to define the dual of an orbisheaf because there is a priori no isomorphism between the pullback of the dual and the dual of the pullback. When a morphism is finite, it is flat in codimension 1,<sup>5</sup> so one has an isomorphism  $g_{\alpha\beta}^* \mathcal{F}_\alpha^\vee \simeq (g_{\alpha\beta}^* \mathcal{F}_\alpha)^\vee$  on a Zariski open subset of codimension 2, using in particular [Har80, Proof of Prop. 1.8].

It follows that the reflexive hulls are isomorphic on a big open subset, and hence isomorphic everywhere by reflexivity. Thus, we obtain an isomorphism:

$$I_{\alpha,\beta} : g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha^\vee \simeq (g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)^\vee.$$

Therefore, we can define transition morphisms for the dual by  $\theta_{\alpha\beta} := I_{\alpha,\beta}^{-1} \circ \varphi_{\beta\alpha}^\vee \circ I_{\beta,\alpha}$ . If we consider a big open subset where  $h_{\alpha\beta,\gamma}$  is flat, then under this identification, we obtain the commutativity of the following diagram<sup>6</sup>:

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5. This is a consequence of the normal form of finite morphisms over general points of the branch locus in codimension 1.

6. At the level of stalks, it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(M, R) \otimes_R Q & \xrightarrow{\mathrm{Hom}(u, R) \otimes_R Q} & \mathrm{Hom}_R(N, R) \otimes_R Q \\ \sim \downarrow & & \downarrow \sim \\ \mathrm{Hom}_Q(M \otimes_R Q, Q) & \xrightarrow{\mathrm{Hom}_Q(u \otimes Q, Q)} & \mathrm{Hom}_Q(N \otimes_R Q, Q) \end{array}$$

where  $Q$  is a flat  $R$ -algebra, and  $u : N \rightarrow M$  is a morphism of finitely generated  $R$ -modules. This is effective by explicitly writing out each of the four morphisms in the diagram above. In the proof of [Har80, Proof of Prop. 1.8], the isomorphism between the vertical arrows is shown by invoking the coincidence of two functors when  $M$  or  $N$  are  $R$ -free modules. The argument just given shows that the isomorphism in question is a natural transformation between these two functors.

$$\begin{array}{ccccc}
 h_{\alpha\beta,\gamma}^* g_{\beta\alpha}^{[*]} \mathcal{F}_\beta^\vee & \xrightarrow{h_{\alpha\beta,\gamma}^* I_{\beta,\alpha}} & h_{\alpha\beta,\gamma}^* (g_{\beta\alpha}^{[*]} \mathcal{F}_\beta)^\vee & \xrightarrow{h_{\alpha\beta,\gamma}^* \varphi_{\beta\alpha}^\vee} & h_{\alpha\beta,\gamma}^* (g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)^\vee & \xrightarrow{h_{\alpha\beta,\gamma}^* I_{\alpha,\beta}^{-1}} & h_{\alpha\beta,\gamma}^* g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha^\vee \\
 & & \downarrow & & \downarrow & & \\
 & & (h_{\alpha\beta,\gamma}^* g_{\beta\alpha}^{[*]} \mathcal{F}_\beta)^\vee & \xrightarrow{(h_{\alpha\beta,\gamma}^* \varphi_{\beta\alpha}^\vee)} & (h_{\alpha\beta,\gamma}^* g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)^\vee & & 
 \end{array}$$

The morphisms appearing in the diagram can be viewed as sections on a dense open subset of sheaves  $\mathcal{H}om(\mathcal{R}_1, \mathcal{R}_2)$  where the  $\mathcal{R}_i$  are explicit reflexive sheaves. Thus, they can be uniquely extended over the whole of  $V_{\alpha\beta\gamma}$ . It then follows, by dualizing the compatibility identity and using the commutativity of the previous diagram, that the following equality holds:

$$h_{\alpha\beta,\gamma}^{[*]} \theta_{\alpha\beta} \circ h_{\beta\gamma,\alpha}^{[*]} \theta_{\gamma\beta} = h_{\gamma\alpha,\beta}^{[*]} \theta_{\alpha\gamma}.$$

We thus conclude that one can define the dual of a weak orbisheaf.

**Definition 2.2.13.** *Let  $(X, \Delta)$  be a pair equipped with an orbistrukture  $\mathcal{C}$  such that the projection morphisms  $g_{\alpha\beta}$  and  $h_{\alpha\beta,\gamma}$  are flat, and let  $\mathcal{F}$  be a weak orbisheaf whose transition morphisms are denoted  $\varphi_{\alpha\beta}$ . Then, the dual orbisheaf  $\mathcal{F}^\vee$  is the orbisheaf defined by the collection  $(\mathcal{F}_\alpha^\vee)$  and the transition morphisms given by the maps  $\theta_{\alpha\beta}$ :*

$$g_{\beta\alpha}^{[*]}(\mathcal{F}_\beta^\vee) \xrightarrow{I_{\beta,\alpha}} (g_{\beta\alpha}^{[*]} \mathcal{F}_\beta)^\vee \xrightarrow{\varphi_{\beta\alpha}^\vee} (g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)^\vee \xrightarrow{I_{\alpha,\beta}^{-1}} g_{\alpha\beta}^{[*]}(\mathcal{F}_\alpha^\vee).$$

If we assume that the morphisms  $g_{\alpha\beta}$  and  $h_{\alpha\beta,\gamma}$  are flat, then pullbacks commute with duality, so that the dual of an orbisheaf is an orbisheaf. Similarly, if we assume that  $\mathcal{F}$  is an orbibundle, the previous construction yields an orbisheaf (it is even an orbibundle).

The flatness of the  $g_{\alpha\beta}$  also ensures the correct definition of the kernel of an orbisheaf morphism.

**Proposition 2.2.14.** *Let  $(X, \Delta)$  be a pair equipped with an orbistrukture  $\mathcal{C}$  such that the projections  $g_{\alpha\beta}$  and  $h_{\alpha\beta,\gamma}$  are flat morphisms. Then, the category of orbisheaves on  $\mathcal{C}$  is an abelian category.*

*Proof.* The category of orbisheaves is additive: if  $\mathcal{F}$  and  $\mathcal{G}$  are two orbisheaves, the set of morphisms  $u : \mathcal{F} \rightarrow \mathcal{G}$  is an abelian group (if  $u, v$  are two morphisms, then  $(u_\alpha + v_\alpha)_\alpha$  is also a morphism, and the neutral element is  $(0_\alpha)_\alpha$ ). Composition is bilinear because for each  $\alpha$ , composition is bilinear for the sheaves of  $\mathcal{O}_{V_\alpha}$ -modules. The orbisheaf  $(0)_\alpha$  is both

an initial and terminal object in this category, and the category admits finite products and coproducts.

Now, this category is abelian: if  $u : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism, we have a kernel  $\ker u : \text{Ker } u \rightarrow \mathcal{F}$  and a cokernel  $\text{coker } u : \mathcal{G} \rightarrow \text{Coker } u$  obtained by considering the collections  $(\text{Ker } u_\alpha)_\alpha$  and  $(\text{Coker } (u_\alpha))$ . These collections define orbisheaves. Indeed, for  $(\text{Ker } u)$ , if we denote  $\varphi_{\alpha\beta}$  the transition morphisms of  $\mathcal{F}$ , we have a natural isomorphism  $\text{Ker } g_{\alpha\beta}^* u_\alpha \simeq g_{\alpha\beta}^* \text{Ker } u_\alpha$  because  $g_{\alpha\beta}$  is flat, so the pullback by  $g_{\alpha\beta}$  is a left-exact functor. Additionally, the universal property of the kernel applied to  $\varphi_{\alpha\beta} \circ \ker g_{\beta\alpha}^* u_\beta$  shows that we have a unique morphism  $k_{\alpha\beta} : \text{Ker } g_{\beta\alpha}^* u_\beta \rightarrow \text{Ker } g_{\alpha\beta}^* u_\alpha$  (this is in some sense the restriction of the transition morphism  $\varphi_{\alpha\beta}$  to the kernel of  $g_{\beta\alpha}^* u_\beta$ . Since  $\varphi_{\alpha\beta}$  is an isomorphism,  $k_{\alpha\beta}$  is also an isomorphism (the inverse is obtained by applying the universal property of the kernel of  $g_{\beta\alpha}^* u_\beta$  to the morphism  $\varphi_{\alpha\beta}^{-1} \circ \ker g_{\alpha\beta}^* u_\alpha$ ). Moreover, since the morphisms  $h_{\alpha\beta,\gamma}$  are flat, they preserve kernels, and by the uniqueness of the morphisms arising from the universal properties, we obtain the cocycle relation:

$$h_{\alpha\beta,\gamma}^* k_{\alpha\beta} \circ h_{\beta\gamma,\alpha}^* k_{\beta\gamma} = h_{\gamma\alpha,\beta}^* k_{\alpha\gamma}.$$

Thus, we have defined the orbisheaf  $\text{Ker } u$ . Given the definition of the morphisms  $k_{\alpha\beta}$ , we have the compatibility relation:

$$g_{\alpha\beta}^* \ker u_\alpha \circ k_{\alpha\beta} = \varphi_{\alpha\beta} \circ g_{\beta\alpha}^* \ker u_\beta,$$

so we have a morphism  $\ker u : \text{Ker } u \rightarrow \mathcal{F}$ . This morphism satisfies  $u \circ \ker u = 0$ , and if  $v : \mathcal{H} \rightarrow \mathcal{F}$  satisfies  $u \circ v = 0$ , then, for each  $\alpha$ , we have a morphism  $\widetilde{v}_\alpha : \mathcal{H}_\alpha \rightarrow \text{Ker } u_\alpha$  by the definition of the kernel of a sheaf morphism. These morphisms satisfy  $v_\alpha = \ker u_\alpha \circ \widetilde{v}_\alpha$ . Now, the morphisms  $\widetilde{v}_\alpha$  satisfy compatibility conditions on the intersections. If we denote  $\theta_{\alpha\beta}$  the transition morphisms of  $\mathcal{H}$ , we have to show that  $g_{\alpha\beta}^* \widetilde{v}_\alpha \circ \theta_{\alpha\beta} = k_{\alpha\beta} \circ g_{\beta\alpha}^* \widetilde{v}_\beta$ . This follows from the compatibility conditions satisfied by  $v$  and  $\ker u$  with transition morphisms and the injectivity of the morphisms  $g_{\alpha\beta}^* \ker u_\alpha$ .

Similarly, we show that we can construct a morphism for the cokernel. Finally, we have an image, i.e., we have a natural isomorphism  $\text{Ker } \text{coker } u \simeq \text{Coker } \ker u$ . This follows from the fact that we have this isomorphism for each  $u_\alpha$ , and given the exactness of  $g_{\alpha\beta}^*$  and the universal properties of kernels and cokernels, these morphisms are compatible on the intersections.  $\square$

# ORBIFOLDS

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Orbifolds were introduced by I. Satake in [Sat56] under the term "V-manifold". These are topological spaces whose local models are given by quotients of  $\mathbb{C}^n$  by finite groups acting linearly.

## 3.1 General facts about orbifolds

**Definition 3.1.1.** *An orbifold is a pair  $(X, \Delta)$  equipped with a smooth orbi-étale orbistructure.*

Just like smooth varieties determined by an atlas of charts, to which a unique maximal atlas is associated, one can uniquely determine a maximal smooth orbi-étale orbistructure associated to an orbifold.

**Remark 3.1.2.** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two orbistructures on a pair  $(X, \Delta)$ . We say that they are *compatible* if  $\mathcal{C}_1 \cup \mathcal{C}_2$  is an orbistructure. Compatibility is an equivalence relation on orbistructures, and each equivalence class has a natural order relation given by the inclusion. A *maximal* orbistructure is a maximal element in a class.

Now, we can show that the orbi-étale property is invariant under  $\Delta$ -adapted base change by Proposition 2.1.7. This implies that two orbi-étale orbistructures are always compatible. Then, there is a unique maximal structure by taking the union of all orbi-étale orbistructures on  $(X, \Delta)$ . To show that a pair  $(X, \Delta)$  is an orbifold, it suffices to show the existence of a smooth orbi-étale orbistructure.

**Remark 3.1.3.** When  $(X, \Delta)$  is endowed with a smooth orbi-étale orbistructure, the branch locus of the maps  $g_{\alpha\beta}$  is of pure codimension 1 by purity of the branch locus. By quasi-étaleness of these morphisms, these loci are of codimension at least 2, hence they are empty. We obtain that  $g_{\alpha\beta}$  are étale maps.

Several remarks follow from the previous point. First of all, one can define orbibundles relative to the geometry of the orbifold.

**Definition 3.1.4.** Let  $(X, \Delta)$  be an orbifold equipped with a maximal smooth orbi-étale orbistructure  $\mathcal{C} := \{(U_\alpha, V_\alpha, \eta_\alpha)\}$ . The orbifold cotangent sheaf  $\Omega_{(X, \Delta)}^1$  is the orbibundle defined by the collection  $\Omega_{V_\alpha}^1$  and the transition morphisms:

$$\Xi_{\alpha\beta} : g_{\alpha\beta}^* \Omega_{V_\alpha}^1 \xrightarrow{\sim} \Omega_{V_{\alpha\beta}}^1 \xrightarrow{\sim} g_{\beta\alpha}^* \Omega_{V_\beta}^1.$$

We define the orbifold tangent sheaf  $\mathcal{T}_{(X, \Delta)} := (\Omega_{(X, \Delta)}^1)^\vee$  and the orbifold  $p$ -forms sheaf  $\Omega_{(X, \Delta)}^p = \bigwedge^p \Omega_{(X, \Delta)}^1$ .

Secondly, it follows from Proposition 2.2.14 that the category of orbisheaves on an orbifold is abelian. In this way, one can define short exact sequences of orbisheaves, and even extensions, and we have an analogue of the characterization of extension classes of  $\mathcal{Q}$  by  $\mathcal{S}$ , up to isomorphism, by  $H^1(X, \mathcal{H}om(\mathcal{Q}, \mathcal{S}))$ .

**Definition 3.1.5.** Let  $(X, \Delta)$  be an orbifold, and let  $\mathcal{S}, \mathcal{Q}$  be two orbisheaves. We call an extension of  $\mathcal{Q}$  by  $\mathcal{S}$  any orbisheaf  $\mathcal{E}$  that can be placed in a short exact sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

**Proposition 3.1.6.** Let  $(X, \Delta)$  be an orbifold, and let  $\mathcal{S}, \mathcal{Q}$  be two orbisheaves with transition morphisms denoted by  $\varphi_{\alpha\beta}$  and  $\psi_{\alpha\beta}$ , respectively.

(1) Let  $(\omega_{\alpha\beta} : g_{\beta\alpha}^* \mathcal{Q}_\beta \longrightarrow g_{\alpha\beta}^* \mathcal{S}_\alpha)$  be a collection of morphisms of  $\mathcal{O}_{V_{\alpha\beta}}$ -modules satisfying the following cocycle condition on  $V_{\alpha\beta\gamma}$ :

$$h_{\alpha\beta, \gamma}^* \varphi_{\alpha\beta} \circ h_{\beta\gamma, \alpha}^* \omega_{\beta\gamma} + h_{\alpha\beta, \gamma}^* \omega_{\alpha\beta} \circ h_{\beta\gamma, \alpha}^* \psi_{\beta\gamma} = h_{\gamma\alpha, \beta}^* \omega_{\alpha\gamma}.$$

Then, this collection induces an extension  $\mathcal{E}$  of  $\mathcal{Q}$  by  $\mathcal{S}$ .

(2) Moreover, if there exists a morphism of orbisheaves  $s : \mathcal{Q} \longrightarrow \mathcal{S}$  such that:

$$\omega_{\alpha\beta} = g_{\alpha\beta}^* s_\alpha \circ \psi_{\alpha\beta} - \varphi_{\alpha\beta} \circ g_{\beta\alpha}^* s_\beta.$$

Then, the extension  $\mathcal{F}$  is isomorphic to  $\mathcal{S} \oplus \mathcal{Q}$ .

**Remark 3.1.7.** If  $X$  is smooth and  $S, Q$  are two vector bundles on  $X$ , it is well-known that extension classes of  $Q$  by  $S$  are parameterized by  $H^1(X, \mathcal{H}om(Q, S))$ . The previous proposition is an adaptation of this fact in the orbifold setting: the collection  $(\omega_{\alpha\beta})_{\alpha, \beta}$  replaces a Čech cocycle which represents a class in  $H^1(X, \mathcal{H}om(Q, S))$ . If this collection

is exact, there is a collection  $(s_\alpha)_\alpha$  such that  $s_\alpha - s_\beta = \omega_{\alpha\beta}$ , and in this case the extension splits.

*Proof. (1):* Let  $\theta_{\alpha\beta} : g_{\beta\alpha}^*(\mathcal{S}_\beta \oplus \mathcal{Q}_\beta) \longrightarrow g_{\alpha\beta}^*(\mathcal{S}_\alpha \oplus \mathcal{Q}_\alpha)$  be the  $\mathcal{O}_{V_{\alpha\beta}}$ -modules morphism defined by:

$$\begin{bmatrix} \varphi_{\alpha\beta} & \omega_{\alpha\beta} \\ 0 & \psi_{\alpha\beta} \end{bmatrix}$$

The cocycle conditions satisfied by  $(\varphi_{\alpha\beta})_{\alpha\beta}$ ,  $(\psi_{\alpha\beta})_{\alpha\beta}$ ,  $(\omega_{\alpha\beta})_{\alpha\beta}$  induce those on  $\theta_{\alpha\beta}$ . These cocycles determine an orbisheaf  $\mathcal{F}$ , given by the collection  $(\mathcal{S}_\alpha \oplus \mathcal{Q}_\alpha)_\alpha$  and the transition morphisms  $(\theta_{\alpha\beta})_{\alpha\beta}$ .

(2): Assume that we have a morphism  $s : \mathcal{Q} \longrightarrow \mathcal{S}$  such that:

$$\omega_{\alpha\beta} = g_{\alpha\beta}^* s_\alpha \circ \psi_{\alpha\beta} - \varphi_{\alpha\beta} \circ g_{\beta\alpha}^* s_\beta.$$

Denote by  $u_\alpha : \mathcal{S}_\alpha \oplus \mathcal{Q}_\alpha \longrightarrow \mathcal{F}_\alpha = \mathcal{S}_\alpha \oplus \mathcal{Q}_\alpha$  the  $\mathcal{O}_{V_\alpha}$ -modules isomorphism:

$$\begin{bmatrix} \text{id}_{\mathcal{S}_\alpha} & s_\alpha \\ 0 & \text{id}_{\mathcal{Q}_\alpha} \end{bmatrix}$$

This isomorphism is compatible with the transition morphisms because of the following computation:

$$\begin{aligned} \theta_{\alpha\beta} \circ g_{\beta\alpha}^* u_\beta &= \begin{bmatrix} \varphi_{\alpha\beta} & \omega_{\alpha\beta} \\ 0 & \psi_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \text{id}_{g_{\beta\alpha}^* \mathcal{S}_\alpha} & g_{\beta\alpha}^* s_\beta \\ 0 & \text{id}_{g_{\beta\alpha}^* \mathcal{Q}_\alpha} \end{bmatrix} = \begin{bmatrix} \varphi_{\alpha\beta} & \omega_{\alpha\beta} + \varphi_{\alpha\beta} \circ g_{\beta\alpha}^* s_\beta \\ 0 & \psi_{\alpha\beta} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{\alpha\beta} & g_{\alpha\beta}^* s_\alpha \circ \psi_{\alpha\beta} \\ 0 & \psi_{\alpha\beta} \end{bmatrix} = g_{\alpha\beta}^* u_\alpha \circ (\varphi_{\alpha\beta} \oplus \psi_{\alpha\beta}) \end{aligned}$$

So,  $u : \mathcal{S} \oplus \mathcal{Q} \longrightarrow \mathcal{F}$  is an isomorphism, and the extension splits.  $\square$

**Example 3.1.8.** We apply the previous proposition when  $\mathcal{Q}$  is trivial and  $\mathcal{S}$  is an orbibundle. A collection  $\Omega$  is given by morphisms  $\omega_{\alpha\beta} : \mathcal{O}_{V_{\alpha\beta}} \rightarrow g_{\alpha\beta}^* \mathcal{S}_\alpha$ . These morphisms are totally determined by the image of 1, we denote it by  $\Omega_{\alpha\beta}$ . The cocycle relation is:

$$h_{\alpha\beta,\gamma}^* \Omega_{\alpha\beta} + h_{\alpha\beta,\gamma}^* \varphi_{\alpha\beta} (h_{\beta\gamma,\alpha}^* \Omega_{\beta\gamma}) = h_{\gamma\alpha,\beta}^* \Omega_{\alpha\gamma}.$$

In the smooth setting, it is the condition to get a closed Čech cocycle.

When  $\mathcal{Q} = \mathcal{T}_{(X,\Delta)}$  and  $\mathcal{S} = \mathcal{O}_X$ , we can define the canonical extension  $\mathcal{E}_{(X,\Delta)}$ .

Let  $\Omega = (\omega_{\alpha\beta} : g_{\beta\alpha}^* \mathcal{T}_{V_\beta} \rightarrow \mathcal{O}_{V_{\alpha\beta}})$  a collection as previously. Under the identification  $\mathcal{T}_{V_{\alpha\beta}}^\vee = \Omega_{V_{\alpha\beta}}^1$ , the collection  $\Omega$  can be interpreted as a collection of 1-forms verifying a closed Čech cocycle relation:  $h_{\alpha\beta,\gamma}^* \omega_{\alpha\beta} + h_{\beta\gamma,\alpha}^* \omega_{\beta\gamma} = h_{\gamma\alpha,\beta}^* \omega_{\alpha\gamma}$ .

Now, we assume that the orbifold structure is such that the  $V_\alpha$  are trivializing open sets for the rank 1 orbibundle  $\Omega_{(X,\Delta)}^n$ . Let  $\varphi_{\alpha\beta}$  be the transition morphisms of this orbibundle. These are automorphisms of  $\mathcal{O}_{V_{\alpha\beta}}$ . The image of 1 gives a section  $\Phi_{\alpha\beta} \in \mathcal{O}_{V_{\alpha\beta}}(V_{\alpha\beta})^\times$ . We have the cocycle condition:  $h_{\alpha\beta,\gamma}^* \Phi_{\alpha\beta} h_{\beta\gamma,\alpha}^* \Phi_{\beta\gamma} = h_{\gamma\alpha,\beta}^* \Phi_{\alpha\gamma}$ . By differentiating its logarithm, we get:

$$h_{\alpha\beta,\gamma}^* \left( \frac{d\Phi_{\alpha\beta}}{\Phi_{\alpha\beta}} \right) + h_{\beta\gamma,\alpha}^* \left( \frac{d\Phi_{\beta\gamma}}{\Phi_{\beta\gamma}} \right) = h_{\gamma\alpha,\beta}^* \left( \frac{d\Phi_{\alpha\gamma}}{\Phi_{\alpha\gamma}} \right).$$

We obtain a collection of 1-forms  $\Omega$  as previously, that induces an extension  $\mathcal{E}_{(X,\Delta)}$  of  $\mathcal{O}_X$  by  $\mathcal{T}_{(X,\Delta)}$ , called the *canonical extension* of  $(X, \Delta)$ . In the smooth setting, this is the extension given by the class  $-c_1(K_X) \in H^1(X, \Omega_X^1)$ .

## 3.2 Orbifold Chern classes

**Definition 3.2.1.** Let  $(X, \Delta)$  be an orbifold and let  $\mathcal{F}$  be an orbibundle. A metric  $h$  on  $\mathcal{F}$  is a hermitian metric  $h^*$  on the vector bundle  $\mathcal{F}^* := \mathcal{F}|_{X^*}$  on  $X^*$  such that for each  $\alpha$ ,  $\eta_\alpha^* h^*$  can be extended to a metric  $h_\alpha$  on  $\mathcal{F}_\alpha$ .

**Proposition 3.2.2.** Let  $(X, \Delta)$  be an orbifold and let  $\mathcal{F}$  be an orbibundle. There exists a metric  $h$  on  $\mathcal{F}$ .

*Proof.* For each  $\alpha$ , there exists a metric  $h_\alpha$  on  $\mathcal{F}_\alpha$ . The metric  $h_\alpha^{G_\alpha} := \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} g^* h_\alpha$  is invariant under the action of  $G_\alpha$ . So it induces a metric  $\widetilde{h}_\alpha$  on  $U_\alpha$ . Now, by [Sat56, p. 6.], there exists a smooth locally finite partition of unity  $(\rho_\alpha)_\alpha$  adapted to the open cover  $\{U_\alpha\}$ . The metric  $h := \sum_\alpha \rho_\alpha \widetilde{h}_\alpha$  is well-defined and is a metric for the orbibundle  $\mathcal{F}$ .  $\square$

**Definition 3.2.3.** Let  $(X, \Delta)$  be an orbifold. An orbiconnection  $\nabla$  on an orbibundle  $\mathcal{F}$  is a collection of connections  $(\nabla_\alpha)_\alpha$  on  $\mathcal{F}_\alpha$  such that the transition morphisms  $\varphi_{\alpha\beta}$  of  $\mathcal{F}$  satisfy compatibility<sup>1</sup> with connections  $\nabla_\alpha$  and  $\nabla_\beta$ :

$$\varphi_{\alpha\beta} : g_{\beta\alpha}^*(\mathcal{F}_\beta, \nabla_\beta) \xrightarrow{\sim} g_{\alpha\beta}^*(\mathcal{F}_\alpha, \nabla_\alpha)$$

An orbiconnection induces a connection  $\nabla^*$  on  $\mathcal{F}^*$ . We say that  $\nabla$  is flat if  $\nabla^*$  is flat (or equivalently, each  $\nabla_\alpha$  is flat).

If  $\mathcal{F}$  is endowed with a metric  $h$ , we can define the Chern orbiconnection  $\nabla_h$  associated to  $h$  by taking the Chern connections  $\nabla_{h_\alpha}$  on each  $\alpha$ , and we say that the metric  $h$  is flat if  $\nabla_h$  is flat.

**Remark 3.2.4.** Following [Bla96, Sect. 2], if  $\mathcal{F}$  is an orbibundle on  $(X, \Delta)$  and  $h$  a metric on  $\mathcal{F}$ , we can define Chern-Weil forms associated to  $h$ . Indeed, the restriction  $h^*$  is a smooth metric on  $\mathcal{F}^*$ , and its curvature is a  $(1, 1)$ -form

$$\Theta(\mathcal{F}^*, h^*) \in \mathcal{C}^\infty(X^*, \bigwedge^{1,1} T_X \otimes \text{End}(\mathcal{F}|_{X^*})).$$

For  $0 \leq p \leq d$ , let  $P_p : \bigwedge^{1,1} T_X \times \text{End}(\mathcal{F}^*) \longrightarrow \bigwedge^{1,1} T_X$  be the  $p$ -th invariant polynomial by the conjugacy action of  $\text{GL}(\mathcal{F}^*)$ . The  $2p$ -form

$$c_p^{\text{orb}}(\mathcal{F}, h) := P_p \left( \frac{\sqrt{-1}}{2\pi} \Theta(\mathcal{F}^*, h^*) \right)$$

gives a closed orbifold form. Indeed, on a local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$ , the metric  $\eta_\alpha^* h^*$  is extended to  $h_\alpha$  on  $V_\alpha$ , its curvature is  $\Theta(\mathcal{F}_\alpha, h_\alpha)$ . This form extends  $\eta_\alpha^* \Theta(\mathcal{F}^*, h^*)$ . So the curvature form is an orbifold section. Likewise the  $2p$ -forms  $c_p(\mathcal{F}, h)$  are orbifold forms.

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1. More precisely, connections are differential objects, that means that we need to move to  $\mathcal{C}^\infty$ -category. One can adapt the definition of holomorphic orbibundles we gave by replacing locally free  $\mathcal{O}_{V_\alpha}$ -modules by locally free  $\mathcal{C}_{V_\alpha}^\infty$ -modules and hence define differential orbibundles. We can even define a notion of orbisheaves of  $\mathbb{C}$ -vector spaces by replacing  $\mathcal{O}_{V_\alpha}$ -modules by  $\underline{\mathbb{C}}$ -modules. Since we have the inclusions

$$\underline{\mathbb{C}} \hookrightarrow \mathcal{O}_{V_\alpha} \hookrightarrow \mathcal{C}_{V_\alpha}^\infty,$$

a  $\underline{\mathbb{C}}$ -orbisheaf is in particular a holomorphic orbibundle, and the latter induces a differential orbibundle. Because  $g_{\alpha\beta}$  are  $\mathcal{C}^\infty$ -étale, we can also define the orbibundle of orbifold 1-forms  $\mathcal{A}_{(X, \Delta)}^1$ , similarly to the orbifold cotangent sheaf  $\Omega_{(X, \Delta)}^1$ . Hence, we obtain that an orbiconnection on a differential orbibundle  $\mathcal{F}$  is a morphism of  $\underline{\mathbb{C}}$ -orbisheaves  $\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{A}_{(X, \Delta)}^1$ , such that each  $\nabla_\alpha$  is a connection, that means they satisfy the Leibniz rule.

**Definition 3.2.5.** Let  $(X, \Delta)$  be an orbifold, and  $\mathcal{F}$  an orbibundle. We define the orbifold  $p$ -th Chern class of  $\mathcal{F}$  by:

$$c_p^{\text{orb}}(\mathcal{F}) := \left[ c_p^{\text{orb}}(X, \Delta, \mathcal{F}, h) \right] \in H_{\text{dR, orb}}^{2p}(X, \mathbb{C}),$$

where  $h$  is any metric on  $\mathcal{F}$ . We denote  $c_p^{\text{orb}}(X, \Delta) := c_p^{\text{orb}}(\mathcal{T}_{(X, \Delta)})$ .

Because we have an extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}_{(X, \Delta)} \longrightarrow \mathcal{T}_{(X, \Delta)} \longrightarrow 0,$$

the Whitney formula for orbifold classes gives  $c_p^{\text{orb}}(\mathcal{E}_{(X, \Delta)}) = c_p^{\text{orb}}(X, \Delta)$  for all  $p$ .

# KAWAMATA LOG TERMINAL SINGULARITIES

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## 4.1 First definitions

In the previous section, we studied singular pairs whose singularities can be obtained as quotients of smooth varieties by finite groups acting linearly. These are called quotient singularities. We now define a broader and more natural class of singularities from the perspective of algebraic geometry: the so-called klt singularities.

**Definition 4.1.1.** *Let  $(X, \Delta)$  be a pair, and let  $x$  be a point on  $X$ . We say that  $x$  is a quotient singularity of  $(X, \Delta)$  if there exists a neighborhood  $U$  of  $x$  and a morphism  $\eta: V \rightarrow U$  that is orbi-étale for  $(U, \Delta|_U)$ , with  $V$  smooth.*

**Definition 4.1.2.** *Let  $(X, \Delta)$  be a pair and let  $x$  be a point in  $X$ . We say that  $(X, \Delta)$  is klt at  $x$  (for Kawamata log terminal) if:*

- $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, i.e. there exists an integer  $m \in \mathbb{N}^*$  such that  $m(K_X + \Delta)$  is a Weil divisor satisfying the following condition: there exists a neighborhood  $U$  of  $x$ , and a nowhere vanishing section  $\tau \in H^0(U_{\text{reg}}, m(K_{U_{\text{reg}}} + \Delta|_{U_{\text{reg}}}))$ .
- The integral  $\int_{U_{\text{reg}} \setminus |\Delta|} (\tau \wedge \bar{\tau})^{1/m}$  is finite.

We say that  $(X, \Delta)$  is a klt pair if it is klt at any point  $x \in X$ .

On  $U_{\text{reg}}$ , we define  $(\tau \wedge \bar{\tau})^{1/m}$  as follows: let  $p$  be a point of  $|\Delta|_{\text{reg}}$ , and  $\Delta_i$  the irreducible component on which  $p$  lies. We consider local coordinates  $z_1, \dots, z_n$  adapted to the divisor  $\Delta_i$ , meaning that in these coordinates,  $\Delta_i$  corresponds to the subvariety  $(z_1 = 0)$ . Around a general point of an irreducible component  $\Delta_i$  of  $\Delta$ , we have a local trivialization given by

$$\tau = f \frac{1}{z_1^{m - \frac{m}{m_i}}} (dz_1 \wedge \dots \wedge dz_n)^{\otimes m}.$$

Then we define:

$$(\tau \wedge \bar{\tau})^{1/m} := \frac{|f|^{2/m}}{|z_1|^{2(1-\frac{1}{m_i})}} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

This definition is independent of the choice of coordinates. Indeed, if  $w_1, \dots, w_n$  are other coordinates adapted to  $\Delta_i$ , let  $w = \varphi(z)$  be the coordinate change. For simplicity, write  $dw := dw_1 \wedge \cdots \wedge dw_n$ , and similarly for  $dz$ . Then we have:

$$(dw)^{\otimes m} = (\det \varphi)^m (dz)^{\otimes m}.$$

Writing  $\tau$  in the new coordinates:  $\tau = g(w) \frac{1}{w_1^{m-\frac{m}{m_i}}} (dw)^{\otimes m}$ , we obtain:

$$f(z) = \frac{z_1^{m-\frac{m}{m_i}} \cdot g(\varphi(z)) \cdot (\det \varphi)^m}{\varphi_1(z)},$$

where  $\varphi_1(z) = w_1$ , by definition of  $\varphi$ . Thus,

$$\frac{|f|^{2/m}}{|z_1|^{2(1-\frac{1}{m_i})}} dz \wedge d\bar{z} = \frac{|g|^{2/m} |z_1|^{2(1-\frac{1}{m_i})} |\det \varphi|^2}{|z_1|^{2(1-\frac{1}{m_i})} |w_1|^{2(1-\frac{1}{m_i})}} dz \wedge d\bar{z} = \frac{|g|^{2/m}}{|w_1|^{2(1-\frac{1}{m_i})}} dw \wedge d\bar{w}.$$

This shows that the definition of  $(\tau \wedge \bar{\tau})^{1/m}$  is indeed independent of the choice of coordinates. For points  $p \in U_{\text{reg}} \setminus |\Delta|$ , similar computations show that the definition of  $(\tau \wedge \bar{\tau})^{1/m}$  is independent of coordinates.

Moreover, the convergence of the integral is also independent of the chosen trivialization. Indeed, if  $\tau'$  is another trivialization of  $m(K_X + \Delta)$  on a neighborhood  $U'$  of  $x$ , there exists a holomorphic function  $\psi : (U \cap U')_{\text{reg}} \rightarrow \mathbb{C}^\times$  such that  $\tau' = \psi\tau$ . Since  $X$  is normal,  $\psi$  extends as a holomorphic function over  $U \cap U'$  that does not vanish on the singular locus. Therefore:

$$\int_{(U \cap U')_{\text{reg}} \setminus |\Delta|} (\tau' \wedge \bar{\tau}')^{1/m} = \int_{(U \cap U')_{\text{reg}} \setminus |\Delta|} |\psi|^{2/m} (\tau \wedge \bar{\tau})^{1/m} < \infty.$$

We thus obtain the following result, which establishes that klt singularities form a larger class of singularities than quotient singularities.

**Proposition 4.1.3.** *Let  $(X, \Delta)$  be a pair and let  $x \in X$ . Then:*

$$x \text{ is a quotient singularity of } (X, \Delta) \implies (X, \Delta) \text{ is klt at } x.$$

*Proof.* By Cartan's linearization process [Car57, Lemma 1], it suffices to show that for any finite group  $G \subseteq \mathrm{GL}(n, \mathbb{C})$  acting linearly on  $\mathbb{C}^n$ , the pair  $(X, \Delta) := (\mathbb{C}^n/G, \Delta_G)$  is klt. For this, observe that the canonical projection  $\pi_G : \mathbb{C}^n \rightarrow \mathbb{C}^n/G$  is étale over the complement of  $\bigcup_{g \in G \setminus \{1\}} \mathrm{Fix}(g)$ , and thus over  $X^* := (\mathbb{C}^n/G)_{\mathrm{reg}} \setminus |\Delta_G|$ .

Let  $dz := dz_1 \wedge \cdots \wedge dz_n$ , and consider  $dz^{\otimes |G|} \in H^0(\mathbb{C}^n, K_{\mathbb{C}^n}^{\otimes |G|})$ .

The action of  $G$  on  $\mathbb{C}^n$  induces a left action on  $n$ -forms via  $g \cdot \omega := g^{-1*} \omega$ , and then an action on tensor powers of sections. It turns out that  $dz^{\otimes |G|}$  is invariant under the  $G$ -action. Indeed:

$$g \cdot dz^{\otimes |G|} = (g^{-1*} dz)^{\otimes |G|} = \det(g)^{-|G|} dz^{\otimes |G|} = dz^{\otimes |G|}.$$

Thus, this form descends to a section  $\tau \in H^0(X^*, K_{X^*}^{\otimes |G|})$ , which extends meromorphically along  $|\Delta|$  on  $X_{\mathrm{reg}}$ .

Let us consider a general point  $p$  of an irreducible component  $\Delta_i$  of  $\Delta$ . Then,  $dz^{\otimes |G|} = \pi_G^* \tau$  can be written in normal form as  $f'(dz')^{\otimes |G|}$ , where  $z'_1, \dots, z'_n$  are coordinates adapted to a hyperplane fixed by  $G$ , and  $f'$  is a non-vanishing holomorphic function invariant under  $G$ , hence  $f' = \pi_G^* \tilde{f}$  for some holomorphic function  $\tilde{f}$  on the quotient.

The isotropy group  $G_p = \{g \in G \mid g \cdot p = p\}$  of such a point is cyclic, with order given by the multiplicity  $m_i$  (cf. Remark 2.1.4). In these coordinates, the map  $\pi_G$  is given by  $\pi_G(z'_1, \underline{z}') = (z'_1^{m_i}, \underline{z}')$ , and if  $w_1, \underline{w}'$  are coordinates on the quotient, then writing  $\tau = f(dw)^{\otimes |G|}$  gives:

$$dz^{\otimes |G|} = \pi_G^* f \cdot z'_1^{|G|(m_i-1)} (dz')^{\otimes |G|}.$$

Identifying both sides, we get  $f' = \pi_G^* f \cdot z'_1^{|G|(m_i-1)}$ . Since

$$|G|(m_i - 1) = m_i \left( |G| - \frac{|G|}{m_i} \right),$$

we conclude:

$$z'_1^{|G|(m_i-1)} = \pi_G^* (w_1^{|G| - \frac{|G|}{m_i}}), \quad \text{thus} \quad f = \frac{\tilde{f}}{w_1^{|G|(1 - \frac{1}{m_i})}}.$$

Hence,  $\tau$  extends meromorphically along  $\Delta$ , and  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, with  $m = |G|$ .

Finally, since  $\pi$  is a finite étale morphism of degree  $|G|$  over  $X^*$ , we compute:

$$\begin{aligned} \int_{X^*} (\tau \wedge \bar{\tau})^{1/|G|} &= \frac{1}{\deg \pi_G} \int_{(\mathbb{C}^n \setminus \text{Fix}G, 0)} (\pi_G^* \tau \wedge \pi_G^* \bar{\tau})^{1/|G|} \\ &= \frac{1}{|G|} \int_{(\mathbb{C}^n \setminus \text{Fix}G, 0)} (dz \wedge \bar{d}z) < \infty, \end{aligned}$$

since the integral is taken over a germ against the volume form on  $\mathbb{C}^n$ , and one may choose a relatively compact open set with finite Lebesgue measure. Hence, quotient singularities are klt.  $\square$

We also have the following alternative characterization of klt singularities:

**Proposition 4.1.4.** *Let  $(X, \Delta)$  be a pair such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. Consider a log resolution of singularities  $\mu: \hat{X} \rightarrow X$  of  $(X, \Delta)$ . Assume that  $\mu$  is an isomorphism above the snc locus<sup>1</sup> of the pair  $(X, \Delta)$ , and denote by  $E_i$  the components of the exceptional divisor. We have:*

$$m(K_{\hat{X}} + \mu_*^{-1}\Delta) = m\mu^*(K_X + \Delta) + \sum_i b_i E_i,$$

where  $\mu_*^{-1}\Delta$  is the strict transform of  $\Delta$  and  $b_i \in \mathbb{Z}$ . Define  $a_i := \frac{b_i}{m}$ . Then, the following statements are equivalent:

- (i)  $(X, \Delta)$  is a klt pair,
- (ii) For all  $i$ ,  $a_i > -1$ .

*Proof.* Let  $U$  be an open neighborhood of a point  $x \in X$ . Since  $\mu$  is an isomorphism above the snc locus of  $(X, \Delta)$ , we have:

$$\int_{U_{\text{reg}} \setminus |\Delta|} (\tau \wedge \bar{\tau})^{1/m} = \int_{\mu^{-1}(U) \setminus (|\mu_*^{-1}\Delta| \cup E)} (\mu^* \tau \wedge \mu^* \bar{\tau})^{1/m},$$

where  $E = \bigcup_i E_i$  is a normal crossings divisor. Since  $E$  is compact, we can cover it by finitely many open charts locally homeomorphic to affine open sets. Using partitions of unity, the convergence of the integral reduces to convergence in each chart.

Let  $U'$  be a chart intersecting  $E$ , and  $V' := \prod \mathcal{D}(0, \rho_i)$  a polydisk, image of  $U'$  in  $\mathbb{C}^n$  under local coordinates  $z$ , such that the divisor  $\sum_i b_i E_i$  is given by  $z_1^{b_1} \cdots z_k^{b_k} = 0$ . Then,

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1. It is the largest open subset  $U \subseteq X$  such that  $(U, \Delta|_U)$  is a log smooth pair.

the local integral becomes:

$$\begin{aligned}
& \int_{V'} \left( |z_1|^{2b_1} \cdots |z_k|^{2b_k} \left( \left( \frac{\sqrt{-1}}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \right)^{\otimes m} \right)^{1/m} \\
&= \int_{V'} (|z_1|^{2a_1} \cdots |z_k|^{2a_k}) \left( \left( \frac{\sqrt{-1}}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \right) \\
&= \int_{V'} \prod_{i=1}^k r_i^{2a_i+1} \prod_{i=k+1}^n r_i dr_1 \cdots dr_n d\theta_1 \cdots d\theta_n \\
&= (2\pi)^n \prod_{i=k+1}^n \frac{\rho_i^2}{2} \prod_{i=1}^k \int_0^{\rho_i} r_i^{2a_i+1} dr_i.
\end{aligned}$$

This integral is finite if and only if  $2a_i + 1 > -1$ , which is equivalent to  $a_i > -1$ . Since this condition holds in each chart covering  $E$ , we conclude the equivalence.  $\square$

Finally, we have the following lemma stating that the klt property behaves well under finite morphisms:

**Lemma 4.1.5.** [KM08, Prop. 5.20] *Let  $(X, \Delta)$  be a pair, and let  $\pi: Y \rightarrow X$  be a finite morphism. Define the Weil  $\mathbb{Q}$ -divisor  $\Delta' := \pi^*(K_X + \Delta) - K_Y$ . Then, the pair  $(Y, \Delta')$  is klt if and only if  $(X, \Delta)$  is klt.*

## 4.2 Orbifold Chern classes on klt pairs

In this section, we give some constructions of Chern classes on klt pairs. The first definitions have been initiated by Mumford [Mum83, §2]. In this article, the author gives the definitions of  $\mathbb{Q}$ -varieties and  $\mathbb{Q}$ -sheaves that are algebraic versions of orbistruures and orbisheaves on a pair  $(X, 0)$ , where  $X$  is a quasi-projective variety. For instance, the orbistruures  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)\}$  above  $\mathbb{Q}$ -varieties consist of triples where  $U_\alpha$  are étale open sets on  $X$ ,  $V_\alpha$  are smooth quasi-projective varieties, and  $\eta_\alpha: V_\alpha \rightarrow U_\alpha$  are finite Galois morphisms.

The construction of Chern classes requires to define a global cover  $\widetilde{X} \rightarrow X$ . It is a finite Galois ramified cover such that for each  $\alpha$ , there exists a factorization  $\widetilde{X} \rightarrow V_\alpha$ . This global cover is obtained by constructing a finite Galois extension of  $\mathbb{C}(X)$  that contains copies of the field extensions  $\mathbb{C}(V_\alpha)$ . Hence, a  $\mathbb{Q}$ -sheaf  $\mathcal{F}$  can be lifted to a coherent sheaf  $\widetilde{\mathcal{F}}$  (we lift each  $\mathcal{F}_\alpha$  on  $\widetilde{X}$ , and the compatibility conditions of  $\mathbb{Q}$ -sheaf provide compatibility condition between the lifted  $\mathcal{F}_\alpha$ ).

In general on manifolds, we can define Chern classes for coherent sheaves by computing those of locally free sheaves that appear in a projective resolution [Kob87]. In order to define Chern classes of  $\mathbb{Q}$ -sheaves, Mumford uses the fact that when  $\widetilde{X}$  is Cohen–Macaulay (that means that the local rings  $\mathcal{O}_{\widetilde{X},x}$  are Cohen–Macaulay), any  $\mathbb{Q}$ -sheaf  $\mathcal{F}$  lifts to a coherent sheaf  $\widetilde{\mathcal{F}}$  that admits a projective resolution.

We give a definition based on [GK20, Sect. 5] for klt spaces and developed in [CGG24, Sect. 3] that uses orbifold Chern classes on the orbifold locus of a klt pair.

**Lemma 4.2.1.** [CGG24, Lemma 14] *Let  $(X, \Delta)$  be a klt pair. Then, there is an analytic subset  $Z \subseteq X$  of codimension at least 3 and if we denote by  $X^\circ := X \setminus Z$ , the pair  $(X^\circ, \Delta|_{X^\circ})$  is an orbifold.*

We denote by  $X_{\text{orb}}$  the biggest open subset such that  $(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})$  is an orbifold. The previous lemma states that  $X \setminus X_{\text{orb}}$  is contained in an analytic subset of codimension at least 3. Now, we have two isomorphisms

$$H^{2n-2}(X, \mathbb{C}) \simeq H_c^{2n-2}(X_{\text{orb}}, \mathbb{C}) \text{ and } H^{2n-4}(X, \mathbb{C}) \simeq H_c^{2n-4}(X_{\text{orb}}, \mathbb{C})$$

induced by long exact sequence associated to the inclusion  $X_{\text{orb}} \hookrightarrow X$ , see for instance [Ive86, p. III.7.6]. Now, the orbifold de Rham complex yields isomorphisms

$$H_{\text{dR},c}^k(X_{\text{orb}}, \mathbb{C}) \simeq H_c^k(X_{\text{orb}}, \mathbb{C}).$$

We denote by

$$\varphi : H^{2n-2}(X, \mathbb{C}) \longrightarrow H_{\text{dR},c}^{2n-2}(X_{\text{orb}}, \mathbb{C}) \text{ and } \psi : H^{2n-4}(X, \mathbb{C}) \longrightarrow H_{\text{dR},c}^{2n-4}(X_{\text{orb}}, \mathbb{C})$$

the composed isomorphisms. If  $\alpha \in H^{2n-2}(X, \mathbb{C})$  and  $(\mathcal{F}, h)$  is an orbibundle of  $X_{\text{orb}}$  endowed with a metric  $h$ , we set:

$$\widehat{c}_1(\mathcal{F}) \cdot \alpha := \int_{X_{\text{orb}}} c_1^{\text{orb}}(\mathcal{F}, h) \wedge A$$

where  $A$  is a compactly supported orbifold  $(2n-2)$ -form representing  $\varphi(\alpha)$ . We set likewise for  $\beta \in H^{2n-4}(X, \mathbb{C})$ :

$$\widehat{c}_2(\mathcal{F}) \cdot \beta := \int_{X_{\text{orb}}} c_2^{\text{orb}}(\mathcal{F}, h) \wedge B, \text{ and}$$

$$\widehat{c}_1^2(\mathcal{F}) \cdot \beta := \int_{X_{\text{orb}}} c_1^{\text{orb}}(\mathcal{F}, h)^2 \wedge B.$$

where  $B$  is a compactly supported orbifold  $(2n - 4)$ -form representing  $\psi(\beta)$ .

Finally, we set:

$$c_1(X, \Delta) := \widehat{c}_1(\mathcal{T}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}) \in H^{2n-2}(X, \mathbb{C})^\vee,$$

$$c_2(X, \Delta) := \widehat{c}_2(\mathcal{T}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}) \in H^{2n-4}(X, \mathbb{C})^\vee, \text{ and}$$

$$c_1^2(X, \Delta) := \widehat{c}_1^2(\mathcal{T}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}) \in H^{2n-4}(X, \mathbb{C})^\vee.$$

Later, we will need to compute Chern classes of orbibundles  $\mathcal{F}$  defined on an open subset  $X^\circ \subseteq X^{\text{orb}}$ , such that the complement is contained in an analytic subset of codimension at least 3. The same construction as previously yields klt Chern classes  $\widehat{c}_i^{X^\circ}(\mathcal{F})$  for  $i = 1, 2$ , defined as follows. Let  $\alpha \in H^{2n-2i}(X, \mathbb{C}) \simeq H_{\text{dR}, c}^{2n-2i}(X^\circ, \mathbb{C})$  be a class represented by a  $(2n - 2i)$ -orbifold form  $A$  with compact support. Then, we define :

$$\widehat{c}_i^{X^\circ}(\mathcal{F}) \cdot \alpha := \int_{X^\circ} c_i^{\text{orb}}(\mathcal{F}, h) \wedge A$$

where  $h$  is a metric on  $\mathcal{F}$ .

The following lemma establishes that klt Chern classes are compatible with inclusions:

**Lemma 4.2.2.** *Let  $(X, \Delta)$  be a klt pair, and let  $X^1 \subseteq X^0 \subseteq X^{\text{orb}}$  be open subsets such that  $X \setminus X^1$  is contained in an analytic subset of codimension at least 3, and  $\mathcal{F}$  an orbibundle on  $X^0$ . We denote by  $\mathcal{F}|_{X^1}$  the restricted orbisheaf<sup>2</sup>. Then:*

$$\widehat{c}_i^{X^1}(\mathcal{F}|_{X^1}) \cdot \alpha = \widehat{c}_i^{X^0}(\mathcal{F}) \cdot \alpha.$$

*Proof.* The different inclusions give morphisms:

$$H_{\text{dR}, c}^{2n-2i}(X^1, \mathbb{C}) \longrightarrow H_{\text{dR}, c}^{2n-2i}(X^0, \mathbb{C}) \longrightarrow H^{2n-2i}(X, \mathbb{C}),$$

where the composition and the second morphism are isomorphisms. Therefore, so is the first morphism. Then, if  $A$  is a  $(2n - 2i)$ -form with compact support in  $X^1$  which represents

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2. When  $U \subseteq X$  is an open subset, we can pullback an orbistructure  $\mathcal{C} = (U_\alpha, V_\alpha, \eta_\alpha)$  on  $U$ . It suffices to consider the collection  $(U \cap U_\alpha, V_\alpha^U := \eta_\alpha^{-1}(U), \eta_\alpha|_{V_\alpha^U})$ . With these orbistructures, the inclusion  $i : (U, \Delta|_U) \longrightarrow (X, \Delta)$  is an orbifold morphism.

a class  $\alpha \in H^{2n-2i}(X, \mathbb{C})$ , it is a compact supported form in  $X^0$  too. We get :

$$\widehat{c}_i^{X^1}(\mathcal{F}) \cdot \alpha = \int_{X^1} c_i^{\text{orb}}(\mathcal{F}, h)|_{X^1} \wedge A = \int_{X^0} c_i^{\text{orb}}(\mathcal{F}, h) \wedge A = \widehat{c}_i^{X^0}(\mathcal{F}) \cdot \alpha. \quad \square$$

To simplify the notations, we will denote by  $\widehat{c}_i(\mathcal{F})$  the orbifold Chern classes without specifying the open subset on which it is defined.

To construct Chern classes, we use that  $X \setminus X_{\text{orb}} \subseteq Z$  with  $Z$  analytic with  $\text{codim } Z \geq 3$ . In general, if  $(X, \Delta)$  is orbifold in codimension  $d$  (it means that  $\text{codim } Z \geq d + 1$ ), we can define as previously  $\widehat{c}_i(\mathcal{F}) \in H_{\text{dR}}^{2n-2i}(X_{\text{orb}}, \mathbb{C})$  for  $i = 1, \dots, d$ . When  $(X, \Delta)$  is an orbifold, we have  $d = n$ , and we can consider  $\widehat{c}_i$  for any  $i$ . Hence, these Chern classes coincide with the intersection with the classical Chern classes we defined for an orbifold, under the Poincaré duality identification.

**Proposition 4.2.3.** *Let  $(X, \Delta)$  be a compact orbifold and  $\mathcal{F}$  an orbibundle endowed with a metric  $h$ . Then, for each  $1 \leq i \leq n$  and each class  $\alpha \in H^{2n-2i}(X, \mathbb{C})$  such that the corresponding orbifold Chern class is represented by a  $(2n - 2i)$ -form  $\omega$ , we have:*

$$\widehat{c}_i(\mathcal{F}) \cdot \alpha = c_i^{\text{orb}}(\mathcal{F}) \cdot \alpha = \int_{X^*} c_i^{\text{orb}}(\mathcal{F}, h)|_{X^*} \wedge \omega|_{X^*}$$

**Remark 4.2.4.** Another way to define  $c_1(X, \Delta)$  is to set:

$$c_1(X, \Delta) := [-(K_X + \Delta)] \in H^2(X, \mathbb{Q}).$$

This definition is compatible with the previous one. More precisely, there are two possible arrows "first Chern class" when we have  $\mathbb{Q}$ -Cartier divisors. Here, we denote by  $\text{Div}(X)$  the group of Cartier divisors, and  $\text{Div}(X)_{\mathbb{Q}} := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  the vector space of  $\mathbb{Q}$ -Cartier divisors. The map  $\text{Div}(X) \longrightarrow \text{Pic}(X)$  induces a morphism  $\text{Div}(X)_{\mathbb{Q}} \longrightarrow \text{Pic}(X)_{\mathbb{Q}}$ , where  $\text{Pic}(X)_{\mathbb{Q}} := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

On the one hand, we can construct the arrow  $c_1^{\text{orb}} : \text{Pic}(X) \longrightarrow H_{\text{dR}}^2(X_{\text{orb}}, \mathbb{C})$  by associating to a line bundle  $\mathcal{L}$  the de Rham class induced by the curvature of an orbifold metric on the bundle  $\mathcal{L}|_{X_{\text{orb}}}$ . This morphism yields a  $\mathbb{Q}$ -linear map  $\text{Pic}(X)_{\mathbb{Q}} \longrightarrow H_{\text{dR}}^2(X_{\text{orb}}, \mathbb{C})$  by extension of scalars. On the other hand, we have a morphism:

$$c_1^{\text{exp}} : \text{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{C})$$

coming from the connection morphism of the cohomology long exact sequence associated

to the exponential exact sequence. Under the previous identification, we have the following commutative diagram:

$$\begin{array}{ccc}
 & \text{Pic}(X)_{\mathbb{Q}} & \\
 c_1^{\text{orb}} \swarrow & & \searrow c_1^{\text{exp}} \\
 H_{\text{dR}}^2(X_{\text{orb}}, \mathbb{C}) & \xrightarrow{\sim} & H^2(X, \mathbb{C})
 \end{array}$$

Now, we have a Poincaré duality  $H_{\text{dR}}^2(X_{\text{orb}}, \mathbb{C}) \simeq H_{\text{dR},c}^{2n-2}(X_{\text{orb}}, \mathbb{C})^{\vee}$  given by the pairing:

$$\int_{X_{\text{orb}}} : (\alpha, \beta) \in H_{\text{dR}}^k(X_{\text{orb}}, \mathbb{C}) \times H_{\text{dR},c}^{2n-k}(X_{\text{orb}}, \mathbb{C}) \mapsto \int_{X_{\text{orb}}} A \wedge B \in \mathbb{C}$$

where  $A$  (resp.  $B$ ) is a representative of  $\alpha$  (resp.  $\beta$ ). Hence, the linear form in  $H_{\text{dR},c}^{2n-2}(X_{\text{orb}}, \mathbb{C})$  which is Poincaré dual to  $c_1^{\text{orb}}(-K_X + \Delta)$  is  $\hat{c}_1(\mathcal{T}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})})$ .



PART II

# Orbifold fundamental group

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# INTRODUCTION

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## Context

In order to take into account singularities of  $X$  in uniformization theory, Greb, Kebekus and Peternell introduce the notion of quasi-étale covers in [GKP16a, Thm. 1.5].

**Definition 5.0.1.** *Let  $X$  be a normal projective variety, and let  $Y$  be a normal variety. A quasi-étale cover  $\pi : Y \rightarrow X$  is a finite surjective morphism such that there exists an analytic subset  $Z \subseteq Y$  of codimension at least 2 such that  $\pi$  is étale on  $Y \setminus Z$ .*

Due to Nagata's purity of the branch locus, one can show that the morphism  $\pi$  is quasi-étale if and only if  $\pi$  is étale over  $X_{\text{reg}}$ . Hence, the notion of quasi-étale covers is related to the fundamental group of  $\pi_1(X_{\text{reg}})$ .

**Definition 5.0.2.** *Let  $X$  be a normal klt variety. We say that  $X$  is maximally quasi-étale if one of the following equivalent assertions is satisfied:*

- (i) *The inclusion  $i : X_{\text{reg}} \hookrightarrow X$  induces an isomorphism of groups:  $\widehat{\pi}_1(X_{\text{reg}}) \simeq \widehat{\pi}_1(X)$ ,*
- (ii) *Any quasi-étale morphism  $\pi : Y \rightarrow X$  is étale.*

**Theorem 5.0.3.** [GKP16a, Thm. 1.5] *Let  $X$  be a normal klt variety. There exists a quasi-étale cover  $\pi : Y \rightarrow X$  such that  $Y$  is maximally quasi-étale. Such a morphism is called a maximally quasi-étale cover.*

Let us recall roughly how the proof of Theorem 0.0.8 goes. The authors first establish that the assumptions of semistability of the canonical extension  $\mathcal{E}_X$  and the Miyaoka–Yau equality:

$$\left(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1(X)^2\right) \cdot H^{n-2} = 0$$

are satisfied on the maximally quasi-étale cover  $\pi : Y \rightarrow X$ . Hence, they show that this maximally quasi-étale space  $Y$  is smooth. In order to do this, they use the fact that  $Y$  is still a klt space and it suffices to show that the tangent sheaf  $\mathcal{T}_Y$  is locally free, by the

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resolution of Zariski–Lipman conjecture [Gre+11; Dru14]. To show that the tangent sheaf is locally free, we use the fact that conditions of semistability of  $\mathcal{E}_Y$  and Miyaoka–Yau equality imply that  $\text{End}(\mathcal{E}_Y)$  is flat on  $Y_{\text{reg}}$ . The maximally quasi-étale property of  $Y$  ensures that this sheaf can be extended over  $Y$  by a flat sheaf. In particular, we get that  $\text{End}(\mathcal{E}_Y)$  is locally free, and we conclude that  $\mathcal{T}_Y$  is locally free, as it is locally a direct summand.

When we want to deal with pairs, implementing the preceding arguments requires to overpass some issues that occur because of the presence of a boundary. First, a suitable theory of uniformization is needed. This has been developed in [CGG24, Sect. 2.3]. In this article, the authors introduce the notions of orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$  of a pair  $(X, \Delta)$  and covers branched at most along  $\Delta$ . They are branched covers over  $X$  that have ramification phenomena controlled by the boundary divisor  $\Delta$ . Then, they show that a geometric Galois correspondence holds in this setting. More precisely, there is a one-to-one correspondence between subgroups of  $\pi_1^{\text{orb}}(X, \Delta)$  and isomorphism classes of branched covers at most along  $\Delta$ . From this correspondence we extract the definition of universal cover  $(X, \Delta)$ . This is a pair  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  that has same type of singularities than  $(X, \Delta)$  (namely, it is klt (resp. orbifold) if  $(X, \Delta)$  is klt (resp. orbifold)).

A consequence of Braun’s finiteness of orbifold fundamental group of log Fano pairs is the fact that, in our case, we do not need to consider generalization of maximally quasi-étale covers, since the universal covering map  $\pi : (\widetilde{X}_\Delta, \widetilde{\Delta}) \rightarrow (X, \Delta)$  is a finite morphism. Despite this, we introduce in this part the notion of maximally orbi-étale pairs, that is a slight generalization to pairs of maximally quasi-étale spaces, and we prove that maximally orbi-étale covers exist for klt pairs:

**Proposition 5.0.4.** *Let  $(X, \Delta)$  be a quasi-projective klt pair. There exists a finite cover  $\pi : (Y, \Delta') \rightarrow (X, \Delta)$  branched at most along  $\Delta$  such that  $(Y, \Delta')$  is maximally orbi-étale.*

If we want to generalize previous ideas, a further argument is needed to ensure that  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is smooth. We develop in this part a developability criterion that is analogous to Zariski–Lipman’s result for orbifold pairs:

**Proposition 5.0.5.** *Let  $(X, \Delta)$  be a quasi-projective orbifold and let  $\mathcal{F}$  be a flat orbibundle on  $(X, \Delta)$ . Let us assume that there exists a surjective morphism of orbibundles:  $\mathcal{F} \twoheadrightarrow \mathcal{T}_{(X, \Delta)}$ . Then, the universal cover  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  satisfies  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is smooth.*

## Organization of Part II

The first chapter is devoted to the introduction of the main tools of the uniformization of pairs. Namely, we define orbifold fundamental groups, covers branched at most along  $\Delta$  and we show the geometric Galois correspondence for klt pairs. Secondly, we give a construction of maximally orbi-étale covers and we show the existence of such covers for klt pairs.

The second chapter of this part is devoted to the relationship between smoothing phenomena of the universal cover and flatness of orbibundles that are related to the geometry of  $(X, \Delta)$ . We first introduce orbi-local freeness that is a well-suited version of local freeness in order to generalize to pairs Zariski–Lipman’s result. At the same time, we explore how previous constructions related to functors  $(\cdot)^{\text{orb}}$  and  $(\cdot)^{\text{inv}}$  are involved to detect if an orbisheaf is orbi-locally free. Then, we give a proof of Riemann–Hilbert correspondence that links representations of orbifold fundamental groups and isomorphism classes of flat orbibundles on an orbifold  $(X, \Delta)$ . That is an important tool in order to show the developability criterion.

# ORBIFOLD FUNDAMENTAL GROUP AND COVERS BRANCHED ALONG A DIVISOR

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In order to define the universal cover of a pair  $(X, \Delta)$ , one important tool is the fundamental group that encodes intermediate covers with the geometric Galois correspondence. In this section, we introduce the notion of orbifold fundamental group and we give this correspondence in this setting. A main tool in the proof of uniformization result in [GKP22, Thm. 1.3] is the maximally quasi-étale cover, whose existence has been established by the same authors in [GKP16a]. We give an analogous construction of "maximally orbi-étale" covers in the pair setting.

## 6.1 General facts

**Definition 6.1.1.** *Let  $(X, \Delta)$  be a pair. Locally around a general point of an irreducible component  $\Delta_i$ , the model of  $X^*$  is  $\mathbb{D}^* \times \mathbb{D}^{n-1}$  (here  $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ ) and its fundamental group is  $\mathbb{Z}$ . Let us denote by  $\gamma_i$  a loop that generates this group and by  $\langle\langle \gamma_i^{m_i} \rangle\rangle$  the normal subgroup of  $\pi_1(X^*)$  generated by the homotopy classes of the loops  $\gamma_i^{m_i}$ . The orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$  is the group defined by:*

$$\pi_1^{\text{orb}}(X, \Delta) := \pi_1(X^*) / \langle\langle \gamma_i^{m_i} \rangle\rangle.$$

**Remark 6.1.2.** The construction of the fundamental group depends usually on the choice of a basepoint  $x_0 \in X$ . To be more precise, once we have chosen a general point of a component  $\Delta_i$ , we can consider a loop  $l_i$  around the divisor based on a point  $y_i$  in a neighborhood of the divisor  $\Delta_i$  as above, and we choose a path  $c_i$  from  $x_0$  to  $y_i$ . Finally, we define  $\gamma_i$  as the composition  $c_i l_i c_i^{-1}$ . We can show that the definition is independent of the choice of the paths  $c_i$ : if  $c'_i$  are other paths, and  $\gamma'_i$  are the new loop based on  $x_0$ , then we have  $\gamma'_i = c'_i l_i c_i^{-1} = c'_i c_i^{-1} \gamma_i (c'_i c_i^{-1})^{-1}$ , so that both paths are conjugated and they

generate the same normal subgroups  $\langle\langle\gamma_i^{m_i}\rangle\rangle = \langle\langle\gamma_i^{m_i}\rangle\rangle$ . Moreover, similarly to the usual case, the orbifold fundamental group is independent of the base point  $x_0 \in X$ .

**Example 6.1.3.** As an example, let  $G$  be a finite group acting linearly, faithfully on  $\mathbb{C}^n$ . Then, we have  $\pi_1^{\text{orb}}(\mathbb{C}^n/G, \Delta_G) = G$ . Let us denote by  $(X, \Delta)$  this pair.

First, we observe that the projection map  $\pi_G : \mathbb{C}^n \rightarrow \mathbb{C}^n/G$  is étale over  $X^*$ , and the preimage of  $X^*$  by  $\pi_G$  is  $\mathbb{C}^n \setminus \bigcup_{g \in G \setminus \{1\}} \text{Fix } g$ . The fundamental group of this space is generated by meridian loops  $\gamma_1, \dots, \gamma_p$  around hyperplanes. Now, we can associate to each hyperplane an integer  $m_i$ .

We have a short exact sequence of groups because  $\mathbb{C}^n \setminus \text{Fix } G \rightarrow X^*$  is a covering map, that is Galois of group  $G$ :

$$1 \longrightarrow \pi_1(\mathbb{C}^n \setminus \text{Fix } G) \longrightarrow \pi_1(X^*) \longrightarrow G \longrightarrow 1.$$

Remark 2.1.4 establishes that the image of the loop  $\gamma_i$  by the first inclusion is  $\gamma_i^{m_i}$ , so that  $G \simeq \pi_1(X^*) / \langle\langle\gamma_i^{m_i}\rangle\rangle = \pi_1^{\text{orb}}(X, \Delta)$ .

The previous example shows that for an orbifold  $(X, \Delta)$ , if  $(U_\alpha, V_\alpha, \eta_\alpha)$  is a local uniformizing chart, we have  $\pi_1^{\text{orb}}(U_\alpha, \Delta|_{U_\alpha}) = G_\alpha$ . Especially, around each point  $x \in X$ , we can pick a local uniformizing chart such that the orbifold fundamental group is  $G_x$ .

More generally, the finiteness of the local fundamental group holds for klt pairs.

**Theorem 6.1.4.** [Bra21, Thm. 1] *Let  $(X, \Delta)$  be a pair and  $x$  a klt singularity of  $(X, \Delta)$ . Then, there exists a neighborhood  $U$  of  $x$  such that  $\pi_1^{\text{orb}}(U, \Delta|_U)$  is finite.*

In the same article, the author shows the following theorem which will be an important tool for us in order to characterize log Fano pairs.

**Theorem 6.1.5.** [Bra21, Thm. 2] *Let  $(X, \Delta)$  be a log Fano pair. Then, the orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$  is finite.*

## 6.2 Orbifold Galois correspondence

**Definition 6.2.1.** *A surjective morphism  $\pi : Y \rightarrow X$  is said to be a cover branched at most along  $\Delta$  if:*

- $Y$  is a connected, normal complex analytic space,
- $\pi$  is étale over  $\pi^{-1}(X^*)$ , where  $X^* := X_{\text{reg}} \setminus |\Delta|$ ,

- for every  $i \in I$  and every irreducible divisor  $\Delta'_{ij} \subseteq \pi^{-1}(\Delta_i)$ , there exists an integer  $r_{ij} \in \mathbb{N}_{\geq 1}$  such that  $r_{ij}$  divides  $m_i$  and

$$\pi^* \Delta_i = \sum_j r_{ij} \Delta'_{ij}.$$

**Remark 6.2.2.** If  $\pi$  is a finite Galois cover and  $r_{ij} = m_i$  for all  $i, j$ , then the morphism is orbi-étale with respect to the pair  $(X, \Delta)$ .

**Remark 6.2.3.** If  $\pi : Y \rightarrow X$  is a cover branched at most along  $\Delta$ , one can define a divisor

$$\Delta' := \sum_{i,j} \left(1 - \frac{r_{ij}}{m_i}\right) \Delta'_{ij},$$

so that the pair  $(Y, \Delta')$  satisfies the relation:  $K_Y + \Delta' = \pi^*(K_X + \Delta)$ . Moreover, the pair  $(Y, \Delta')$  is klt if  $(X, \Delta)$  is, by Lemma 4.1.5. Finally, note that the coefficients of  $\Delta'$  are less than or equal to those of  $\Delta$ . In particular, when  $\Delta = 0$ , a cover branched at most along  $\Delta$  is a quasi-étale cover.

**Theorem 6.2.4.** [CGG24, Thm. 2.16] *Let  $(X, \Delta)$  be a klt pair.*

*There exists a natural one-to-one correspondence between morphisms  $\pi : Y \rightarrow X$  branched at most along  $\Delta$  and subgroups  $G \subseteq \pi_1^{\text{orb}}(X, \Delta)$ . Furthermore:*

- $G$  is of finite index if and only if  $\pi$  is finite.
- $G$  is a normal subgroup if and only if  $\pi$  is Galois.
- Let  $Y_{1,2} \rightarrow X$  be two covers branched at most along  $\Delta$ , with corresponding subgroups  $G_{1,2} \subseteq \pi_1^{\text{orb}}(X, \Delta)$ . Then there is a factorization:

$$\begin{array}{ccc} & & Y_2 \\ & \exists & \downarrow \\ Y_1 & \longrightarrow & X \end{array}$$

*if and only if  $G_1 \subseteq G_2$ .*

*Proof. Step 1 : From branched covers to subgroups of  $\pi_1^{\text{orb}}(X, \Delta)$*

First, suppose that  $(Y, \Delta')$  is a pair such that  $\pi : Y \rightarrow X$  is a cover branched at most along  $\Delta$ , and define  $\Delta' := \pi^*(K_X + \Delta) - K_Y$ . Then we may consider the orbifold fundamental group  $\pi_1^{\text{orb}}(Y, \Delta')$ . We claim there is an injection

$$\pi_1^{\text{orb}}(Y, \Delta') \hookrightarrow \pi_1^{\text{orb}}(X, \Delta).$$

Indeed, since  $\pi$  is étale over  $X^* := X_{\text{reg}} \setminus |\Delta|$ , there is a natural injection  $\pi_1(Y^*) \hookrightarrow \pi_1(X^*)$ . Let  $\gamma_i$  be a meridian loop around a general point of  $\Delta_i$ , and let  $\delta_{ij}$  denote a meridian loop around  $\Delta'_{ij}$ . By definition of  $\Delta'_{ij}$ , the image of the class of  $\delta_{ij}$  under  $\pi$  is the class of  $\gamma_i^{r_{ij}}$ . Therefore,  $\pi_* \left( \delta_{ij}^{\frac{m_i}{r_{ij}}} \right)$  maps to  $\gamma_i^{m_i}$ , so

$$\pi_* \left\langle \left\langle \delta_{ij}^{\frac{m_i}{r_{ij}}} \right\rangle \right\rangle \subseteq \langle \langle \gamma_i^{m_i} \rangle \rangle.$$

Now, suppose  $\delta$  is a loop in  $Y^*$  such that  $\pi_*\delta \in \langle \langle \gamma_i^{m_i} \rangle \rangle$ . Then we can write:

$$\pi_*\delta = g_1 \gamma_{i_1}^{m_{i_1}} g_1^{-1} \cdots g_n \gamma_{i_n}^{m_{i_n}} g_n^{-1}$$

for some loops  $g_k$  in  $X^*$ . By uniqueness of path lifting, we can write:

$$\delta = c_1 \delta_{i_1 j_1}^{\frac{m_{i_1}}{r_{i_1 j_1}}} c_1^{-1} \cdots c_n \delta_{i_n j_n}^{\frac{m_{i_n}}{r_{i_n j_n}}} c_n^{-1}$$

for some paths  $c_k$  in  $Y^*$ . Hence,  $\delta$  lies in the normal subgroup  $\left\langle \left\langle \delta_{ij}^{\frac{m_i}{r_{ij}}} \right\rangle \right\rangle$  by Remark 6.1.2.

Therefore, the kernel of the morphism  $\pi_1(Y^*) \rightarrow \pi_1^{\text{orb}}(X, \Delta)$  is precisely  $\left\langle \left\langle \delta_{ij}^{\frac{m_i}{r_{ij}}} \right\rangle \right\rangle$ , and so we obtain an injection:

$$\pi_1^{\text{orb}}(Y, \Delta') \hookrightarrow \pi_1^{\text{orb}}(X, \Delta).$$

### Step 2 : From subgroups of $\pi_1^{\text{orb}}(X, \Delta)$ to branched covers

Assume first that  $(X, \Delta)$  is an orbifold. Let  $G \subseteq \pi_1^{\text{orb}}(X, \Delta)$  be a subgroup. Define  $G^* \subseteq \pi_1(X^*)$  to be a subgroup containing the normal subgroup  $\langle \langle \gamma_i^{m_i} \rangle \rangle$ , such that:

$$G = G^* / \langle \langle \gamma_i^{m_i} \rangle \rangle.$$

By the usual Galois correspondence on  $X^*$ , there exists an étale cover  $\pi^* : Y^* \rightarrow X^*$  corresponding to the subgroup  $G^*$ . We can then complete this cover to obtain a morphism  $\pi : Y \rightarrow X$  branched at most along  $\Delta$ , endowing  $(Y, \Delta')$  with a smooth orbi-étale structure.

Let  $x \in X_{\text{sing}} \cup |\Delta|$ . There exists a neighborhood  $U$  and a finite orbi-étale morphism  $\eta : V \rightarrow U$ , where  $V$  is smooth. Then  $\pi_1^{\text{orb}}(U, \Delta|_U) = \text{Gal}(\eta)$  is finite. Let  $U^* := U_{\text{reg}} \setminus |\Delta|$ , and let  $W^*$  be a connected component of  $\pi^{*-1}(U^*)$ . It corresponds to a normal subgroup

$\pi_1(W^*)$  of  $\pi_1(U^*)$  under the Galois correspondence. Suppose that  $\Delta_i$  is an irreducible component of  $\Delta$  that intersects  $U$ . Then, by construction of  $Y^*$  and the fact that  $\gamma_i^{m_i} \in G^*$ , the lift of  $\gamma_i^{m_i}$  to  $W^*$  is a loop<sup>1</sup>. Therefore,  $\gamma_i^{m_i}$  lies in  $\pi_1(W^*)$ , so  $\langle\langle\gamma_i^{m_i}\rangle\rangle \subseteq \pi_1(W^*)$ . Hence, the index  $[\pi_1(U^*) : \pi_1(W^*)]$  is finite, and the cover  $W^* \rightarrow U^*$  is finite.

By Theorem 11.3.12, this extends to a finite morphism  $W \rightarrow U$  with  $W$  a normal analytic space. Gluing together  $Y^*$  with these extensions yields a connected normal analytic space  $Y$  and a morphism  $\pi : Y \rightarrow X$  étale over  $X^*$ .

Now let  $\delta_{ij}$  be a generator of a loop around a component  $\Delta'_{ij}$  of  $\pi^*\Delta_i$ . The lift of  $\gamma_i^{m_i}$  is a loop around  $\Delta'_{ij}$ , so it can be written as  $\delta_{ij}^k$ . Let  $r_{ij}$  be the ramification index along  $\Delta'_{ij}$ . Then:

$$\pi_*(\delta_{ij}^k) = \gamma_i^{kr_{ij}} = \gamma_i^{m_i},$$

so  $r_{ij}$  divides  $m_i$ . Therefore,  $\pi$  is a cover branched at most along  $\Delta$ .

Finally, we assumed that  $(X, \Delta)$  is an orbifold, and we obtain that  $(Y, \Delta')$  is also endowed with an explicit smooth orbi-étale orbistruature. More generally, if  $(X, \Delta)$  is assumed klt, we can similarly consider  $\pi^* : Y^* \rightarrow X^*$ , and we can show similarly that around each point  $x \in X_{\text{sing}} \cup |\Delta|$ , there exists a neighborhood  $U$  such that the restriction of  $\pi^*$  to any connected component  $W^*$  of  $\pi^{*-1}(U^*)$  is finite, thanks to local finiteness of orbifold fundamental group by Theorem 6.1.4. This is a crucial assumption to apply the finite morphism completion theorem.

### Step 3 : Constructions are mutually inverse

We observe that the orbifold fundamental group of the cover constructed in Step 2 is:

$$\pi_1^{\text{orb}}(Y, \Delta') = G^* / \langle\langle\gamma_i^{m_i}\rangle\rangle = G.$$

Moreover, if there is another cover  $(Y_1, \Delta'_1)$  branched at most along  $\Delta$  with the same orbifold fundamental group  $G$ , then:

- $Y_1^* \simeq Y^*$  as covers of  $X^*$  by Galois theory,
- This isomorphism extends to an isomorphism of analytic branched covers by [Gro03, Prop. XII.5.3].

Hence, the assignments  $G \mapsto (Y, \Delta')$  and  $(Y, \Delta') \mapsto G$  are mutual inverses to each other.

The additional statements follow from the properties of the regular part of the pair, and from the fact that the completion preserves the finiteness and Galois nature of mor-

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1. To be more precise, we should consider loops  $l_i$  as defined in Remark 6.1.2

phisms.<sup>2</sup> □

From this correspondence, the definition of universal cover follows.

**Definition 6.2.5.** *Let  $(X, \Delta)$  be an orbifold. The universal cover of  $(X, \Delta)$  is the orbifold  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  corresponding to the subgroup  $\{1\} \subseteq \pi_1^{\text{orb}}(X, \Delta)$  under the correspondence of Theorem 6.2.4.*

We say that  $(X, \Delta)$  is simply connected if  $\pi_1^{\text{orb}}(X, \Delta) = \{1\}$

**Remark 6.2.6.** When  $(X, \Delta)$  is an orbifold, so is  $(\widetilde{X}_\Delta, \widetilde{\Delta})$ , and we can describe explicitly the smooth orbi-étale orbistruure induced by that on  $(X, \Delta)$ . In the above construction, one can observe that we showed the finiteness of the morphism on neighborhoods by showing that there is a factorization  $V^* \rightarrow W^* \rightarrow U^*$  where  $(U, V, \eta)$  is a local uniformizing chart, given by a group  $\pi_1(V^*) \subseteq H \subseteq \pi_1(U^*)$ , where  $\pi_1(V^*)$  is normal in  $\pi_1(U^*)$  and the quotient is isomorphic to  $\text{Gal}(\eta)$ . Hence, we can use Theorem 11.3.12 applied to  $V^* \rightarrow W$ . By composition by  $W \rightarrow U$  and by unicity of the completion, the completed space is  $V$  that is smooth. Moreover, the morphism  $V \rightarrow W$  is Galois of group  $H/\pi_1(V^*) \subseteq \text{Gal}(\eta)$ .

To sum up, the smooth orbi-étale orbistruure on  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  is given by a collection  $(\widetilde{U}_\alpha, V_\alpha, \widetilde{\eta}_\alpha)$ , where  $V_\alpha$  are the same spaces that define the orbistruure on  $(X, \Delta)$ , and morphisms  $\widetilde{\eta}_\alpha$  are quotient maps associated to a subgroup  $H_\alpha$  of  $G_\alpha$ .

**Example 6.2.7.** We consider the log smooth pair  $(\mathbb{P}^2, \frac{1}{2}H_0 + \frac{1}{2}H_1)$  where  $H_i = (z_i = 0)$  is a coordinate hyperplane. Let us compute its fundamental group and its universal cover. First, we have a homeomorphism  $\mathbb{P}^2 \setminus |\Delta| \simeq \mathbb{C}^\times \times \mathbb{C}$ . We get  $\pi_1(\mathbb{P}^2 \setminus |\Delta|) = \mathbb{Z}$ , so that  $\pi_1^{\text{orb}}(\mathbb{P}^2, \Delta)$  is either  $\mathbb{Z}$  or cyclic.

The group  $\pi_1^{\text{orb}}(\mathbb{P}^2, \Delta)$  is generated by two loops:

$$\gamma_0(t) = [e^{2\sqrt{-1}\pi t} : 1 : 1] \quad \text{and} \quad \gamma_1(t) = [1 : e^{2\sqrt{-1}\pi t} : 1].$$

We can observe that in the chart  $U_0 = (z_0 \neq 0)$ , we have  $\gamma_0(t) = (e^{-2\sqrt{-1}\pi t}, e^{-2\sqrt{-1}\pi t})$  and  $\gamma_1(t) = (e^{2\sqrt{-1}\pi t}, 1)$ , so that  $\gamma_0$  is homotopic to  $\gamma_1^{-1}$ . Hence, we have:

$$\pi_1^{\text{orb}}(X, \Delta) = \langle \gamma_0, \gamma_1 \mid \gamma_0^2, \gamma_1^2, \gamma_0\gamma_1 \rangle = \mathbb{Z}/(2).$$

---

2. For finiteness: if  $G$  is of finite index, so is  $G^*$ , hence the cover  $Y^* \rightarrow X^*$  is finite. The finite morphism completion lemma yields a finite morphism  $Y \rightarrow X$ .

For the Galois property: if  $G \triangleleft \pi_1^{\text{orb}}(X, \Delta)$ , then  $G^* \triangleleft \pi_1(X^*)$ , and the cover  $Y^* \rightarrow X^*$  is Galois with group  $K := \pi_1^{\text{orb}}(X, \Delta)/G$ . The uniqueness in the completion of finite morphisms yields compatible extensions with the action of the group  $K$  on the whole of  $Y$ , thanks to Corollary 11.3.13.

Following the construction of Theorem 6.2.4, one can see that the universal covering map for  $(X, \Delta)$  above the regular part  $\pi^* : Y^* \rightarrow X^*$  of  $X^* = \mathbb{P}^2 \setminus |\Delta|$  corresponding to  $\langle \langle \gamma_0^2, \gamma_1^2 \rangle \rangle$  is such that  $Y^*$  is biholomorphic to  $\mathbb{P}^2 \setminus |\Delta|$ , and the local uniformizations above general points of the divisor are given by ramified coverings of the form  $(z_1, z_2) \mapsto (z_1^2, z_2)$ .

It remains to study the situation at the point  $[0 : 0 : 1]$ . Let us consider the coordinate chart  $U_2 = (z_2 \neq 0)$  and the local uniformizing chart:

$$\eta : (z_0, z_1) \in \mathbb{C}^2 =: V \mapsto (z_0^2, z_1^2) \in \mathbb{C}^2 =: U_2.$$

We set:  $V^* = \mathbb{C}^2 \setminus (z_0 z_1 = 0)$ . We know that  $W^* := \pi^{*-1}(U_2^*)$  is connected, and there exists a map  $V^* \rightarrow W^*$  of degree 2 that is an intermediate cover of the Galois cover  $V^* \rightarrow U_2^*$  of Galois group  $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$ . We have three possibilities for the subgroup of  $\mathbb{Z}/(2)$  that induces the cover  $W^*$ . Here, the lifts of paths  $\gamma_0$  and  $\gamma_1$  on  $W^*$  are not loops, so that the only possible subgroup is the one generated by the automorphism of  $V^* : (z_0, z_1) \mapsto (-z_0, -z_1)$ . Hence, the quotient map can be completed to the map  $V = \mathbb{C}^2 \rightarrow W = \mathbb{C}^2 / \pm 1$ . It follows that the universal cover  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  satisfies  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is a normal surface which has only one singular point (an  $A_1$ -singularity).

### 6.3 Maximally orbi-étale cover

Following the definition of [GKP16a], we define maximally orbi-étale covers.

**Definition 6.3.1.** *Let  $(X, \Delta)$  be a pair. We say that  $(X, \Delta)$  is maximally orbi-étale if every finite Galois cover  $\pi : Y \rightarrow X$  branched at most along  $\Delta$  is étale.*

We observe that  $X$  is maximally quasi-étale if and only if the pair  $(X, 0)$  is maximally orbi-étale. One can show that there is a description of maximally orbi-étaleness in terms of fundamental groups.

Let  $G$  be a group. We denote by

$$\widehat{G} := \varprojlim_{\substack{H \triangleleft G \\ [G:H] < \infty}} G/H$$

the profinite completion of  $G$ . Given a surjective group morphism  $\alpha : G \rightarrow G'$ , the finite index normal subgroups of  $G'$  are exactly the images of the finite index normal subgroups

of  $G$ , so this induces a natural surjective morphism

$$\widehat{\alpha} : \widehat{G} \rightarrow \widehat{G}'.$$

In our setting, we have a surjective morphism of groups  $\pi_1(X^*) \rightarrow \pi_1(X)$ , which factors through the normal subgroup  $\langle\langle \gamma_i^{m_i} \rangle\rangle$ , thereby inducing a natural surjective morphism

$$\text{nat} : \pi_1^{\text{orb}}(X, \Delta) \rightarrow \pi_1(X).$$

As explained above, we thus obtain a morphism

$$\widehat{\text{nat}} : \widehat{\pi}_1^{\text{orb}}(X, \Delta) \rightarrow \widehat{\pi}_1(X).$$

More precisely, we have the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(X^*) & \twoheadrightarrow & \pi_1(X_{\text{reg}}) & \twoheadrightarrow & \pi_1(X) \\ & \searrow & \uparrow & \nearrow & \\ & & \pi_1^{\text{orb}}(X, \Delta) & & \end{array} \quad \begin{array}{c} \\ \\ \text{nat} \\ \\ \end{array}$$

**Proposition 6.3.2.** *Let  $(X, \Delta)$  be a pair. Assume the natural morphism*

$$\widehat{\text{nat}} : \widehat{\pi}_1^{\text{orb}}(X, \Delta) \longrightarrow \widehat{\pi}_1(X)$$

*is an isomorphism. Then, the pair  $(X, \Delta)$  is maximally orbi-étale.*

*Proof.* Assume  $\widehat{\text{nat}}$  is an isomorphism. Let  $\pi : Y \rightarrow X$  be a cover branched at most along  $\Delta$ . Then, by Theorem 6.2.4, there exists a finite quotient  $G$  of  $\pi_1^{\text{orb}}(X, \Delta)$  such that  $\pi$  is isomorphic, as a  $\Delta$ -branched cover over  $X$ , to the quotient  $Y \rightarrow Y/G$ . The isomorphism  $\widehat{\text{nat}}$  implies that  $G$  is also a quotient of  $\pi_1(X)$  (see [GKP16a, Sect. 8.1]).

By classical Galois theory, there exists a finite étale Galois cover  $Z \rightarrow X$  with Galois group  $G$ . The two covers  $Y$  and  $Z$  are isomorphic over  $X^*$ , and by uniqueness of the extension of finite covers, we conclude that  $\pi$  is étale.  $\square$

**Proposition 6.3.3.** *Let  $(X, \Delta)$  be a klt pair with  $X$  compact. Then there exists a finite cover  $\pi : Y \rightarrow X$  branched at most along  $\Delta$  such that the pair  $(Y, \Delta')$  is maximally orbi-étale.*

*Proof.* Assume for contradiction that there exists a klt pair  $(Y_0, \Delta_0)$  such that for every

finite Galois cover  $\pi : Y \rightarrow Y_0$  branched at most along  $\Delta_0$ , the pair  $(Y, \Delta)$  is not maximally orbi-étale.

In particular,  $(Y_0, \Delta_0)$  itself is not maximally orbi-étale, otherwise the identity map would yield such a cover. Hence, there exists a finite non-étale cover  $\pi_0 : Y_1 \rightarrow Y_0$  branched at most along  $\Delta_0$ . Equipping  $Y_1$  with the divisor  $\Delta_1 := \pi_0^*(K_{Y_0} + \Delta_0) - K_{Y_1}$ , we again find that  $(Y_1, \Delta_1)$  is not maximally orbi-étale. Repeating this process, we get a finite non-étale cover  $\pi_1 : Y_2 \rightarrow Y_1$  branched at most along  $\Delta_1$ , and so on. The composition  $\pi_0 \circ \pi_1$  is then a finite cover branched at most along  $\Delta_0$ , and the corresponding divisor is  $\Delta_2$ .<sup>3</sup>

We thus construct by induction a sequence of klt pairs  $(Y_n, \Delta_n)$  and morphisms  $\pi_n : Y_{n+1} \rightarrow Y_n$  that are finite non-étale and branched at most along  $\Delta_n$ , with compositions  $\pi_0 \circ \cdots \circ \pi_n : Y_{n+1} \rightarrow Y_0$  branched at most along  $\Delta_0$ , and such that

$$K_{Y_{n+1}} + \Delta_{n+1} = \pi_n^*(K_{Y_n} + \Delta_n).$$

By Theorem 6.2.4, we obtain a descending chain of subgroups:

$$\cdots \subseteq G_n \subseteq \cdots \subseteq G_1 \subseteq G_0 \subseteq \pi_1^{\text{orb}}(Y_0, \Delta_0),$$

where  $G_n$  corresponds to the cover  $\pi_0 \circ \cdots \circ \pi_n$ .

By Theorem 6.1.4 and compactness of  $Y_0$ , we can choose a finite open cover  $\{U_1, \dots, U_p\}$  of  $Y_0$  such that each local orbifold fundamental group  $\pi_1^{\text{orb}}(U_i, \Delta_{0|U_i})$  is finite.

Restricting the morphisms to preimages of the  $U_i$ , we extract a connected component  $V_{n+1} \subset (\pi_0 \circ \cdots \circ \pi_n)^{-1}(U_i)$  such that  $\pi_n(V_{n+1}) = V_n$  (with  $V_0 := U_i$ ). These induce a descending sequence of subgroups

$$\cdots \subseteq H_n^i \subseteq \cdots \subseteq H_1^i \subseteq H_0^i \subseteq \pi_1^{\text{orb}}(U_i, \Delta_{0|U_i}).$$

This sequence is independent of the choice of connected components: another sequence  $\widetilde{V}_n$  would differ by automorphisms  $g_n \in \text{Gal}(\pi_{n-1})$ , and these induce the same subgroup by Theorem 6.2.4.

By finiteness, the sequence stabilizes at some rank  $M_i$ . By Galois theory, the morphisms

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3. Indeed, since  $\pi_0^{-1}(X^*) \subset Y^*$  and  $\pi_1$  is étale over  $Y^*$ , their composition is étale over  $X^*$ . Moreover, we have  $\Delta_2 = \pi_1^*(K_{Y_1} + \Delta_1) - K_{Y_2} = \pi_1^* \pi_0^*(K_X + \Delta) - K_{Y_2}$ , which shows it corresponds to the divisor of the composition. If the composition were étale at a point  $y_2 \in Y_2$ , then the induced ring maps  $\mathcal{O}_{Y_2, y_2} \rightarrow \mathcal{O}_{Y_1, \pi_1(y_2)} \rightarrow \mathcal{O}_{X, \pi_0(\pi_1(y_2))}$  would be isomorphisms, implying that both  $\pi_0$  and  $\pi_1$  are étale individually, contradicting the assumption.

$\pi_n$  become isomorphisms over  $V_n$  for  $n \geq M_i$ . Changing components if needed, we conclude that the  $\pi_n$  are étale over  $V_n$  and thus over every connected component of the preimage of  $U_i$ . Hence, for  $n \geq \max_i M_i$ , the morphisms  $\pi_n$  are étale, contradicting their construction.  $\square$

# DEVELOPABILITY CRITERION

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## 7.1 Orbi-local freeness

### 7.1.1 The functor orb

First, we recall some constructions and interpret them in more categorical terms. Let us denote by  $(X, \Delta)$  a pair endowed with an orbistructure  $\mathcal{C}$ . We recall that we denote by  $\mathcal{OrbiSh}_{\mathcal{C}}$  (resp.  $\mathcal{WOrbiSh}_{\mathcal{C}}$ ) the category of orbisheaves (resp. weak orbisheaves) defined on  $\mathcal{C}$ .

First, if we have an orbisheaf  $\mathcal{F}$ , it defines a weak orbisheaf, and if  $u : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of orbisheaves it defines especially a morphism of weak orbisheaves by taking double duals in the compatibility conditions. It defines a forgetful functor

$$\text{For} : \mathcal{OrbiSh}_{\mathcal{C}} \rightarrow \mathcal{WOrbiSh}_{\mathcal{C}}.$$

We defined a construction that associates to each coherent sheaf  $\mathcal{F}$  on  $X$  an orbisheaf  $\mathcal{F}^{\text{orb}}$ . Moreover, if  $u : \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of coherent sheaves on  $X$ , we can define  $u^{\text{orb}} : \mathcal{F}^{\text{orb}} \rightarrow \mathcal{G}^{\text{orb}}$  that is defined by the collection of  $\eta_{\alpha}^* u : \mathcal{F}_{\alpha} \rightarrow \mathcal{G}_{\alpha}$ . This collection satisfies compatibility condition, as we have  $g_{\alpha\beta}^* \eta_{\alpha}^* u = g_{\beta\alpha}^* \eta_{\beta}^* u$ . This construction is compatible with identity morphisms and composition so that it defines a functor

$$(\cdot)^{\text{orb}} : \mathcal{Coh}_X \rightarrow \mathcal{OrbiSh}_{\mathcal{C}}.$$

We defined the dual of a weak orbisheaf  $\mathcal{F}$ . We can show as previously that for a morphism of weak orbisheaves  $u : \mathcal{F} \rightarrow \mathcal{G}$ , one can define a morphism of weak orbisheaves  $u^{\vee} : \mathcal{G}^{\vee} \rightarrow \mathcal{F}^{\vee}$  by considering the collection  $u_{\alpha}^{\vee} : \mathcal{G}_{\alpha}^{\vee} \rightarrow \mathcal{F}_{\alpha}^{\vee}$ . The fact that it defines a morphism is a consequence of the commutativity of the following diagram, that comes

from the compatibility between  $(g_{\alpha\beta}^* u_\alpha)^\vee$  and  $g_{\alpha\beta}^{[*]} u_\alpha^\vee$ :

$$\begin{array}{ccc}
 g_{\beta\alpha}^{[*]} \mathcal{G}_\beta^\vee & \xrightarrow{\psi_{\alpha\beta}} & g_{\alpha\beta}^{[*]} \mathcal{G}_\alpha^\vee \\
 \downarrow g_{\beta\alpha}^{[*]} u_\beta & \searrow \sim & \swarrow \sim \\
 & (g_{\beta\alpha}^{[*]} \mathcal{G}_\beta)^\vee \longrightarrow (g_{\alpha\beta}^{[*]} \mathcal{G}_\alpha)^\vee & \\
 & \downarrow & \downarrow \\
 & (g_{\beta\alpha}^{[*]} \mathcal{F}_\beta)^\vee \longrightarrow (g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha)^\vee & \\
 & \swarrow & \searrow \\
 g_{\beta\alpha}^{[*]} \mathcal{F}_\beta^\vee & \xrightarrow{\varphi_{\alpha\beta}} & g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha^\vee \\
 & & \downarrow g_{\alpha\beta}^{[*]} u_\alpha
 \end{array}$$

It is compatible with identity morphisms and composition that defines a contravariant functor

$$(\cdot)^\vee : \mathcal{WOrbiSh}_{\mathcal{C}} \longrightarrow \mathcal{WOrbiSh}_{\mathcal{C}}.$$

Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . One can define the reflexive hull of the induced orbisheaf that is the following functor:

$$(\cdot)^{[\text{orb}]} := (\cdot)^{\vee\vee} \circ \text{For} \circ (\cdot)^{\text{orb}} : \mathcal{Coh}_X \longrightarrow \mathcal{WOrbiSh}_{\mathcal{C}}.$$

This functor yields more precisely a reflexive weak orbisheaf, so that the image can be chosen in the full subcategory of reflexive weak orbisheaves  $\mathcal{RWOrbiSh}_{\mathcal{C}}$ .

**Lemma 7.1.1.** *Let  $(X, \Delta)$  be a pair and  $\mathcal{C}$  an orbistructure. Let  $\mathcal{F}$  be a weak orbisheaf and  $\mathcal{G}$  an orbibundle. Assume that  $u : \mathcal{F} \longrightarrow \mathcal{G}$  is an isomorphism of weak orbisheaves.*

*Then,  $\mathcal{F}$  is an orbibundle, and  $u$  is an isomorphism of orbisheaves.*

*Proof.* The isomorphisms  $u_\alpha : \mathcal{F}_\alpha \longrightarrow \mathcal{G}_\alpha$  show that  $\mathcal{F}_\alpha$  are locally free sheaves. Hence, we have that  $g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha$  is naturally isomorphic to  $g_{\alpha\beta}^* \mathcal{F}_\alpha$ , and  $g_{\alpha\beta}^{[*]} u_\alpha$  is compatible with  $g_{\alpha\beta}^* u_\alpha$  under this natural isomorphism, so that  $u$  induces an isomorphism of orbisheaves.  $\square$

### 7.1.2 The functor $\text{inv}$

We define a natural manner to construct a coherent reflexive sheaf on the underlying space  $X$  from a reflexive weak orbisheaf on  $(X, \Delta)$ .

Let  $(X, \Delta)$  be a pair endowed with an orbistructure, and  $\mathcal{F}$  a reflexive weak orbisheaf. We can define a coherent sheaf on  $X$  in the following way. For each local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$ , there is a sheaf  $\mathcal{F}_\alpha^{\text{inv}}$  on  $U_\alpha$  defined by:

$$\mathcal{F}_\alpha^{\text{inv}} : U \longmapsto H^0(\eta_\alpha^{-1}(U), \mathcal{F}_\alpha)^{G_\alpha},$$

that associates to each open subset  $U$  of  $U_\alpha$  the set of  $G_\alpha$ -invariant sections of  $\mathcal{F}_\alpha$  defined on  $\eta_\alpha^{-1}(U)$ . It is coherent and reflexive by [Gre+11, Lemmas. A.3-4].

There are morphisms  $\theta_{\alpha\beta} : \mathcal{F}_\beta^{\text{inv}}|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}_\alpha^{\text{inv}}|_{U_\alpha \cap U_\beta}$  induced by transition maps of the orbisheaf  $\mathcal{F}$  (here we use reflexivity of  $\mathcal{F}_\alpha$  to extend a section of  $\mathcal{F}_\alpha$  obtained by descent of a  $G_\alpha$ -invariant section on  $g_{\alpha\beta}^{[*]} \mathcal{F}_\alpha$  away from the non-étale locus of  $g_{\alpha\beta}$ ). They satisfy cocycle conditions, and it follows that the sheaves  $\mathcal{F}_\alpha^{\text{inv}}$  glue together to define a reflexive sheaf  $\mathcal{F}^{\text{inv}}$  on  $X$ .

Moreover, if  $u : \mathcal{F} \longrightarrow \mathcal{G}$  is a morphism of reflexive orbisheaves, we can construct a morphism  $u^{\text{inv}} : \mathcal{F}^{\text{inv}} \longrightarrow \mathcal{G}^{\text{inv}}$  as follows. For each  $\alpha$ , there is a morphism  $u_\alpha^{\text{inv}} : \mathcal{F}_\alpha^{\text{inv}} \longrightarrow \mathcal{G}_\alpha^{\text{inv}}$  induced by  $u_\alpha$ . Indeed, the morphism  $u_\alpha$  sends  $G_\alpha$ -invariant sections of  $\mathcal{F}_\alpha$  on invariant sections of  $\mathcal{G}_\alpha$  because it is  $G_\alpha$ -equivariant. These morphisms are compatible with cocycles of  $\mathcal{F}^{\text{inv}}$  and  $\mathcal{G}^{\text{inv}}$ , because the morphisms  $u_\alpha$  are compatible with transition morphisms of the orbisheaves  $\mathcal{F}$  and  $\mathcal{G}$ . They glue together to compose a global morphism  $u^{\text{inv}} : \mathcal{F}^{\text{inv}} \longrightarrow \mathcal{G}^{\text{inv}}$ . This construction is compatible with composition of orbisheaves and preserves identity morphisms. Hence, it defines a functor

$$(\cdot)^{\text{inv}} : \mathcal{WOrbiSh}_{\mathcal{C}} \longrightarrow \mathcal{Coh}_X.$$

**Definition 7.1.2.** *Let  $(X, \Delta)$  be a pair endowed with an orbistructure  $\mathcal{C}$ , and  $\mathcal{F}$  a reflexive weak orbisheaf. The reflexive sheaf  $\mathcal{F}^{\text{inv}}$  is called the sheaf of invariant sections of  $\mathcal{F}$ .*

**Lemma 7.1.3.** *Let  $(X, \Delta)$  be a pair endowed with an orbistructure such that the category of orbisheaves is abelian. Then, the functor  $(\cdot)^{\text{inv}}$  is exact.*

*Proof.* It is a direct application of [Gre+11, Lemma. A.3]. □

We sum up the previous constructions in the following diagram. Here, the notation  $\mathcal{Ref}_X$  denotes the full subcategory of  $\mathcal{Coh}_X$  composed of reflexive sheaves. The dashed

arrows are canonical inclusions.

$$\begin{array}{ccccc}
 & & \text{orb} & & \\
 & & \curvearrowright & & \\
 \text{OrbiSh}_C & \xrightarrow{\text{For}} & \mathcal{W}\text{OrbiSh}_C & & \text{Coh}_X \\
 & & \uparrow \quad \downarrow & & \uparrow \\
 & & \mathcal{R}\mathcal{W}\text{OrbiSh}_C & \xrightarrow{(\cdot)^{\text{inv}}} & \mathcal{R}ef_X \\
 & & \downarrow & & \downarrow \\
 & & & & \mathcal{O}_{U_0}^{\oplus r}
 \end{array}$$

### 7.1.3 Orbi-local freeness

In the previous section, we have defined two functors  $(\cdot)^{[\text{orb}]}$  and  $(\cdot)^{\text{inv}}$ . The goal of this section is to study to what extent these functors produce an equivalence of category.

This question is related to the detection of orbibundles obtained as pullbacks of bundles on  $X$ . In order to investigate the latter question, we have the following proposition.

**Proposition 7.1.4.** *Let  $(X, \Delta)$  be a pair endowed with an orbistructure, and let  $\mathcal{F}$  be an orbibundle of rank  $r$  defined on  $(X, \Delta)$ . Let  $x$  be a point of  $X$ . The following assertions are equivalent:*

- (i) *There exists a neighborhood  $U_0$  around  $x$  and an isomorphism of orbisheaves:  $\mathcal{F}|_{U_0} \simeq \mathcal{O}_{U_0}^{\oplus r}$ .*
- (ii) *There exists a neighborhood  $U_0$  of  $x$  such that for all local uniformizing chart  $(U, V, \eta)$  around  $x$ ,  $\mathcal{F}_{\eta|_{\eta^{-1}(U_0)}}$  is a free  $\mathcal{O}_{\eta^{-1}(U_0)}$ -module of rank  $r$ , and there exists a local frame consisting of  $G_\eta$ -invariant sections.*
- (iii) *There exists a neighborhood  $U_0$  and a local uniformizing chart  $(U, V, \eta)$  around  $x$  such that  $\mathcal{F}_{\eta|_{\eta^{-1}(U_0)}}$  is a free  $\mathcal{O}_{\eta^{-1}(U_0)}$ -module of rank  $r$ , and there exists a local frame consisting of  $G_\eta$ -invariant sections.*

This proposition establishes criteria to know if an orbisheaf is locally free at  $x$  in an orbifold sense, meaning that there is an isomorphism of orbisheaves between  $\mathcal{F}|_{U_0}$  and the trivial sheaf  $\mathcal{O}_{U_0}^{\oplus r}$  on a neighborhood  $U_0$  of  $x$ .

*Proof.* (i)  $\implies$  (ii): By definition, there exists for each chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  an isomorphism of bundles  $u_\alpha : \mathcal{F}_{\eta_\alpha|_{\eta_\alpha^{-1}(U_0)}} \longrightarrow \mathcal{O}_{\eta_\alpha^{-1}(U_0)}^{\oplus r}$  where  $r$  is the rank of  $\mathcal{F}$ . By compatibility with transition maps, they are  $G_\alpha$ -equivariant. Because the canonical local frame  $(e_i)_{1 \leq i \leq r}$  on  $\mathcal{O}_{\eta_\alpha^{-1}(U_0)}(\eta_\alpha^{-1}(U_0))^{\oplus r}$  consists of invariant sections under the action of  $G_\alpha$ , so are  $s_i := u_\alpha^{-1}(e_i)$  which define a local frame of  $\mathcal{F}_{\eta_\alpha|_{\eta_\alpha^{-1}(U_0)}}$ .

(ii)  $\implies$  (iii): It suffices to pick any chart  $(U, V, \eta)$  around  $x$ .

(iii)  $\implies$  (i): By assumption, let  $\widetilde{U}_0$  be a neighborhood of  $x$  and  $(U_\alpha, V_\alpha, \eta_\alpha)$  a local uniformizing chart around  $x$  such that  $\mathcal{F}_\alpha|_{\eta_\alpha^{-1}(\widetilde{U}_0)}$  has a local frame of  $G_\alpha$ -invariant sections. In other words, there exists an isomorphism of  $G_\alpha$ -sheaves  $u_\alpha : \mathcal{F}_\alpha|_{\eta_\alpha^{-1}(\widetilde{U}_0)} \longrightarrow \mathcal{O}_{\eta_\alpha^{-1}(\widetilde{U}_0)}^{\oplus r}$ . We set  $U_0 := \widetilde{U}_0 \cap U_\alpha$ . It suffices to define for each local uniformizing chart  $(U_\beta, V_\beta, \eta_\beta)$  around  $x$  a morphism  $u_\beta : \mathcal{F}_\beta|_{\eta_\beta^{-1}(U_0)} \longrightarrow \mathcal{O}_{\eta_\beta^{-1}(U_0)}^{\oplus r}$  compatible with transition morphisms. Let us denote  $\mathcal{V}_{\alpha\beta} := g_{\alpha\beta}^{-1}(\eta_\alpha^{-1}(U_0))$ . We set:

$$\psi_{\alpha\beta} = g_{\alpha\beta}^* u_\alpha \circ \varphi_{\alpha\beta}|_{\mathcal{V}_{\alpha\beta}} : (g_{\beta\alpha}^* \mathcal{F}_\beta)|_{\mathcal{V}_{\alpha\beta}} \longrightarrow \mathcal{O}_{\mathcal{V}_{\alpha\beta}}^{\oplus r}.$$

Now, both groups  $G_\alpha$  and  $G_\beta$  acts on  $\mathcal{V}_{\alpha\beta}$ , and the sheaves  $g_{\alpha\beta}^* \mathcal{F}_\alpha$  are naturally endowed with an action of these groups. The morphisms  $g_{\alpha\beta}^* u_\alpha$  are equivariant with respect to both actions. Indeed, the initial  $G_\alpha$ -equivariance of  $u_\alpha$  induces that on  $g_{\alpha\beta}^* u_\alpha$ , and pulling back by  $g_{\alpha\beta}$  which is Galois with group  $G_\beta$  implies naturally a  $G_\beta$ -equivariance on the pullback  $g_{\alpha\beta}^* u_\alpha$ . Moreover, the compatibility conditions on transition maps imply that morphisms  $\varphi_{\alpha\beta}$  are also equivariant with respect to both actions, so that  $\psi_{\alpha\beta}$  is equivariant with respect to these actions.

Hence, the morphism  $\psi_{\alpha\beta}$  can be realized as a section on  $\mathcal{V}_{\alpha\beta}$  of the locally free sheaf  $\mathcal{H}om(g_{\beta\alpha}^* \mathcal{F}_\beta, \mathcal{O}_{\mathcal{V}_{\alpha\beta}}^{\oplus r}) = g_{\beta\alpha}^* \mathcal{H}om(\mathcal{F}_\beta, \mathcal{O}_{V_\beta}^{\oplus r})$ . Both previous actions induce an action on this bundle<sup>1</sup>. Because the morphism  $\psi_{\alpha\beta}$  is equivariant with respect to these actions, the associated section in  $\mathcal{H}om(g_{\beta\alpha}^* \mathcal{F}_\beta, \mathcal{O}_{\mathcal{V}_{\alpha\beta}}^{\oplus r})$  is invariant under the induced action on the sheaf of morphisms. It follows that there exists an isomorphism  $u_\beta \in H^0(\eta_\beta^{-1}(U_0), \mathcal{H}om(\mathcal{F}_\beta, \mathcal{O}_{V_\beta}^{\oplus r}))$  such that  $\psi_{\alpha\beta} = g_{\beta\alpha}^* u_\beta$ .

Now, it remains to show that the collection above constructed satisfies compatibility condition, which means that for local uniformizing charts  $(U_\beta, V_\beta, \eta_\beta)$  and  $(U_\gamma, V_\gamma, \eta_\gamma)$  we have on  $\mathcal{V}_{\beta\gamma}$  the equality  $g_{\gamma\beta}^* u_\gamma \circ \varphi_{\gamma\beta} = g_{\beta\gamma}^* u_\beta$ . As a consequence of the surjectivity of  $\eta_\alpha$  onto  $U_0$ , the morphism  $h_{\beta\gamma, \alpha}$  is surjective. Hence, it suffices to show that the following

---

1. If  $\mathcal{E}$  and  $\mathcal{F}$  are two  $G_\alpha$ -sheaves, then so is  $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ , and the action of  $G_\alpha$  is defined as following: if  $g \in G_\alpha$ ,  $x \in V_\alpha$  and  $s \in \mathcal{H}om(\mathcal{F}, \mathcal{G})_x$ , we define:

$$(g \cdot s)_{g \cdot x} : e_{g \cdot x} \mapsto g_{\mathcal{F}} \cdot s(g_{\mathcal{E}}^{-1} e_{g \cdot x}),$$

where  $g_{\mathcal{E}}$  and  $g_{\mathcal{F}}$  denote the action of  $g$  on  $\mathcal{E}$  and  $\mathcal{F}$  respectively. If  $\mathcal{E}$  and  $\mathcal{F}$  are orbisheaves, one can show that it is the action induced by the cocycles associated to the orbisheaf  $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ .

equality holds:

$$h_{\beta\gamma,\alpha}^* g_{\gamma\beta}^* u_\gamma \circ h_{\beta\gamma,\alpha}^* \varphi_{\gamma\beta} = h_{\beta\gamma,\alpha}^* g_{\beta\gamma}^* u_\beta.$$

We have cocycle conditions:  $h_{\gamma\alpha,\beta}^* \varphi_{\alpha\gamma} \circ h_{\beta\gamma,\alpha}^* \varphi_{\gamma\beta} = h_{\alpha\beta,\gamma}^* \varphi_{\alpha\beta}$ . By composing by  $h_{\alpha\beta,\gamma}^* g_{\alpha\beta}^* u_\alpha = h_{\gamma\alpha,\beta}^* g_{\alpha\gamma}^* u_\alpha$ , and using the compatibility:  $g_{\alpha\beta}^* u_\alpha \circ \varphi_{\alpha\beta} = g_{\beta\alpha}^* u_\beta$  by definition of  $\psi_{\alpha\beta}$ , we obtain the sought equality.  $\square$

**Definition 7.1.5.** *With notation as above, we say that  $\mathcal{F}$  is orbi-locally free around  $x$  if one of the assertions of Proposition 7.1.4 is satisfied. If  $\mathcal{F}$  is orbi-locally free around each point, we say that  $\mathcal{F}$  is orbi-locally free.*

**Proposition 7.1.6.** *Let  $(X, \Delta)$  be a pair endowed with an orbistructure. We have the following statements.*

A. *If  $\mathcal{A}$  is a reflexive sheaf on  $X$ , then we have a natural isomorphism:*

$$\mathcal{A} \simeq (\mathcal{A}^{[\text{orb}]})^{\text{inv}}.$$

B. *If  $\Delta = 0$  and  $\mathcal{F}$  is a reflexive weak orbisheaf, then:*

$$\mathcal{F} \simeq (\mathcal{F}^{\text{inv}})^{[\text{orb}]}$$

C. *The orbibundle  $\mathcal{F}$  is orbi-locally free if and only if  $\mathcal{F}^{\text{inv}}$  is locally free and there is an isomorphism of orbisheaves:  $\mathcal{F} \simeq (\mathcal{F}^{\text{inv}})^{\text{orb}}$ .*

*Proof.* For the point A, let  $U \hookrightarrow U_\alpha$  be an open subset where  $(U_\alpha, V_\alpha, \eta_\alpha)$  is a local uniformizing chart. The isomorphism is locally given by:

$$s \in \mathcal{A}|_{U_\alpha}(U) \longmapsto s \circ \eta_\alpha \in H^0(\eta_\alpha^{-1}(U), \eta_\alpha^* \mathcal{A})^{G_\alpha}.$$

First, we show that it is an isomorphism when  $\mathcal{A}$  is locally free.

It is injective as  $\eta_\alpha$  is onto  $U_\alpha$ . Let us show that this morphism is surjective. It is true if  $\mathcal{A} = \mathcal{O}_X$  by definition of functions on a quotient space. Now, if  $\mathcal{A}|_U$  is isomorphic to  $\mathcal{O}_U^{\oplus r}$  via a trivialization  $s$ , the precomposition  $\tilde{s} := s \circ \eta_\alpha$  provides a trivialization for  $\eta_\alpha^* \mathcal{A}|_U$  by invariant sections. It follows by the case of the trivial bundle that a section  $t := \sum f_i \tilde{s}_i$  which is  $G_\alpha$ -invariant yields functions  $g_1, \dots, g_r$  in  $\mathcal{O}_U$  such that  $g_i = f_i \circ \eta_\alpha$ . Hence, the section  $t$  is obtained by the precomposition by  $\eta_\alpha$  of the section  $\sum g_i s_i$ . It implies that the morphism is onto.

Now, we assume that  $\mathcal{A}$  is reflexive. Because  $\mathcal{A}|_{U_\alpha}$  and  $\mathcal{A}_\alpha^{\text{inv}}$  are reflexive and are isomorphic in codimension 1 (since  $\mathcal{A}$  is locally free in codimension 1), the isomorphism extends to  $U_\alpha$ . Then, the isomorphism  $g_{\alpha\beta}^{[*]}\eta_\alpha^{[*]}\mathcal{A} = g_{\beta\alpha}^{[*]}\eta_\beta^{[*]}\mathcal{A}$  is given by the identity of  $\mathcal{A}$ , and it follows that this isomorphism commutes with the transition morphism  $\theta_{\alpha\beta}$  defined in Definition 7.1.2 on intersections  $U_\alpha \cap U_\beta$ . Finally,  $\mathcal{A}$  is naturally isomorphic to the sheaf of invariant sections of its induced orbisheaf.

For the point B, if  $(U_\alpha, V_\alpha, \eta_\alpha)$  is a local uniformizing chart, we observe by using the compatibility condition between  $\mathcal{F}_\alpha$  and a trivializing open subset of  $\mathcal{F}^{\text{inv}}$  on  $X_{\text{reg}}$  that  $\mathcal{F}_\alpha|_{\eta_\alpha^{-1}(X_{\text{reg}})} \simeq \eta_\alpha^*\mathcal{F}^{\text{inv}}|_{X_{\text{reg}}}$  (see Lemma 7.2.1 below). We have an isomorphism  $\mathcal{F}_\alpha \simeq \eta_\alpha^{[*]}\mathcal{F}^{\text{inv}}$  because both sheaves are reflexive and coincide on  $\eta_\alpha^{-1}(X_{\text{reg}})$  which is an open subset whose complement is included in a analytic subset of codimension at least 2. The compatibility conditions with transition maps follow from that on triple intersections with trivializing open subsets of  $\mathcal{F}^{\text{inv}}$  on  $X_{\text{reg}}$ .

For the point C, if  $\mathcal{F}$  is orbi-locally free, then the sheaves  $\mathcal{F}_\alpha^{\text{inv}}$  are locally free by (i) in Proposition 7.1.4. Hence, the sheaf  $\mathcal{F}^{\text{inv}}$  is locally free. Moreover, we have  $\eta_\alpha^*\mathcal{F}_\alpha^{\text{inv}} \simeq \mathcal{F}_\alpha$  (as a consequence of [HL10, Thm. 4.2.15], there exists a locally free sheaf  $\mathcal{G}_\alpha$  such that  $\mathcal{F}_\alpha \simeq \eta_\alpha^*\mathcal{G}_\alpha$ , and the argument of point A shows that  $\mathcal{F}_\alpha^{\text{inv}} \simeq (\eta_\alpha^*\mathcal{G}_\alpha)^{\text{inv}} = \mathcal{G}_\alpha$ ). This isomorphism is compatible with the transition maps of  $\mathcal{F}$  and that of  $(\mathcal{F}^{\text{inv}})^{\text{orb}}$  (these are  $g_{\alpha\beta}^*\eta_\alpha^*\theta_{\alpha\beta}$ , see the discussion before Definition 7.1.2 for notation), by taking local invariant frames. For the converse, as  $\mathcal{F}^{\text{inv}}$  has local trivializations, we get that the orbisheaf  $\mathcal{F}$  is locally isomorphic to the trivial  $\mathcal{O}_X^{\oplus r}$ , that exactly means that  $\mathcal{F}$  is orbi-locally free.  $\square$

**Example 7.1.7.** We give here an example of an orbisheaf  $\mathcal{F}$  on a pair  $(X, \Delta)$  such that  $\mathcal{F}$  is not isomorphic to  $(\mathcal{F}^{\text{inv}})^{\text{orb}}$ . Consider the pair  $(X, \Delta) = (\mathbb{C}, (1 - \frac{1}{m})\{0\})$ . It can be endowed with an orbifold structure, since there is a local uniformizing chart  $\eta: V \rightarrow X$  where  $V = \mathbb{C}$  and  $\eta(z) = z^m$ . The morphism  $\eta$  is Galois of group  $\mathbb{Z}/(m)$ . Set  $\mathcal{F} = \Omega_{(X, \Delta)}^1$ , that means that it is given by the sheaf  $\Omega_V^1$  on the chart. First, there is an isomorphism of sheaves  $\eta_*(\Omega_V^1)^G \simeq \Omega_X^1$  on  $X$  given by:

$$\omega \in \Omega_X^1(U) \mapsto \eta^*\omega \in \Omega_V^1(\eta^{-1}(U))^G$$

for any open subset  $U \subseteq X$ . It is an isomorphism because the induced morphisms on stalks are isomorphisms.

We get that  $\mathcal{F}^{\text{inv}} = \Omega_X^1$ , which is in particular locally free. Let us assume by contradiction that  $\mathcal{F} = (\mathcal{F}^{\text{inv}})^{\text{orb}}$ . Hence, we conclude by the point C of Proposition 7.1.6 that  $\mathcal{F}$  should be orbi-locally free. It means by point (ii) of Proposition 7.1.4 that there exists a  $G$ -invariant local section  $S$  around  $0 \in V$  that is a trivialization of the bundle  $\Omega_V^1$ . This implies that the section  $dz$  around 0 is of the form  $dz = fS$  where  $f$  is a holomorphic function defined around 0. We observe that  $f$  does not vanish since both are trivializations of the cotangent bundle.

Now, the action of a generator  $[1] \in G = \mathbb{Z}/(m)$  on a form  $\omega$  is given by  $[1] \cdot \omega = \zeta^{-1}\omega$ , where  $\zeta := e^{\sqrt{-1}\frac{2\pi}{m}}$  so that we have:

$$\zeta^{-1}fS = \zeta^{-1}dz = [1] \cdot dz = [1] \cdot fS = f(\zeta^{-1}\cdot)S.$$

By evaluating at 0, we get  $f(0) = 0$  that contradicts that  $f$  does not vanish.

One can give a more detailed description of  $(\mathcal{F}^{\text{inv}})^{\text{orb}}$ . On  $V$ , it is given by  $\eta^*\Omega_X^1$ , which is the submodule of  $\Omega_V^1$  generated locally by the  $G$ -invariant form  $z^{m-1}dz$ .

## 7.2 Flatness and developability

### 7.2.1 Riemann–Hilbert correspondence

In this section, we give a correspondence between representations of the orbifold fundamental group and the orbundles endowed with a flat connection on an orbifold  $(X, \Delta)$ . Such a correspondence has been proved by Shen and Yu (see [SY22, Thm. 2.35]), in a slightly different setting. In this section we translate this result in the complex orbifold setting we introduced.

First, let  $(X, \Delta)$  be an orbifold. We want to show that there is a one-to-one correspondence between isomorphism classes of orbundle endowed with a flat orbiconnection  $(\mathcal{F}, \nabla)$  and representations of the orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$ . Before we construct the two inverse arrows of this correspondence, we state two important lemmas.

**Lemma 7.2.1.** *Let  $(X, \Delta)$  be an orbifold, and  $\mathcal{F}$  an orbundle on it. For a local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$ , we denote  $V_\alpha^* := \eta_\alpha^{-1}(X^*)$ .*

*Then, there is a natural isomorphism of vector bundles  $\psi_\alpha^* : \eta_\alpha^*\mathcal{F}^* \simeq \mathcal{F}_{\alpha|V_\alpha^*}$  above  $V_\alpha^*$ ,*

such that the following diagram commutes:

$$\begin{array}{ccc}
 g_{\beta\alpha}^* \eta_{\beta}^* \mathcal{F}^* & \xrightarrow{\text{nat}} & g_{\alpha\beta}^* \eta_{\alpha}^* \mathcal{F}^* \\
 g_{\beta\alpha}^* \psi_{\beta}^* \downarrow & & \downarrow g_{\alpha\beta}^* \psi_{\alpha}^* \\
 g_{\beta\alpha}^* \mathcal{F}_{\beta|V_{\beta}^*} & \xrightarrow{\varphi_{\alpha\beta}} & g_{\alpha\beta}^* \mathcal{F}_{\alpha|V_{\alpha}^*}
 \end{array}$$

Moreover, if  $\mathcal{F}$  is endowed with an orbiconnection  $\nabla$ , the previous morphisms preserve all the connections. Namely, the morphism  $\psi_{\alpha}^*$  is an isomorphism between bundles endowed with a connection :  $\eta_{\alpha}^*(\mathcal{F}^*, \nabla^*) \simeq (\mathcal{F}_{\alpha}, \nabla_{\alpha})|_{V_{\alpha}^*}$ , and the following compatibility holds:

$$\begin{array}{ccc}
 g_{\beta\alpha}^* \eta_{\beta}^*(\mathcal{F}^*, \nabla^*) & \xrightarrow{\text{nat}} & g_{\alpha\beta}^* \eta_{\alpha}^*(\mathcal{F}^*, \nabla^*) \\
 g_{\beta\alpha}^* \psi_{\beta}^* \downarrow & & \downarrow g_{\alpha\beta}^* \psi_{\alpha}^* \\
 g_{\beta\alpha}^*(\mathcal{F}_{\beta}, \nabla_{\beta})|_{V_{\beta}^*} & \xrightarrow{\varphi_{\alpha\beta}} & g_{\alpha\beta}^*(\mathcal{F}_{\alpha}, \nabla_{\alpha})|_{V_{\alpha}^*}
 \end{array}$$

*Proof.* This isomorphism comes from the compatibility condition of orbisheaf and orbiconnection. More precisely, a local uniformizing chart  $(U_{\delta}, V_{\delta}, \eta_{\delta})$  above  $X^*$  is a diffeomorphism  $\eta_{\delta} : V_{\delta} \rightarrow U_{\delta}$ . We consider in the following that  $\eta_{\delta} : V_{\delta} \hookrightarrow X^*$  is an inclusion map. If we have an open subset  $V_{\delta} \subseteq U_{\alpha}^*$ , we get that the normalization of the fiber product  $V_{\alpha\delta}$  is the subset  $\eta_{\alpha}^{-1}(V_{\delta})$ , and the morphism  $g_{\alpha\delta} : V_{\alpha\delta} \hookrightarrow V_{\alpha}$  is an inclusion map too, and  $g_{\delta\alpha} : V_{\alpha\delta} \rightarrow V_{\delta}$  is the morphism  $\eta_{\alpha}$  restricted to  $\eta_{\alpha}^{-1}(V_{\delta})$ .

Hence, the isomorphism  $g_{\alpha\delta}^* \mathcal{F}_{\alpha} \rightarrow g_{\delta\alpha}^* \mathcal{F}_{\delta}$  yields a local isomorphism on  $V_{\alpha}$  given by  $\varphi_{\alpha\delta}$ :

$$\mathcal{F}_{\alpha}|_{V_{\alpha\delta}} \simeq \eta_{\alpha}^* \mathcal{F}^*|_{V_{\alpha\delta}}.$$

These local morphisms glue together because of compatibility condition of transition maps. Indeed, if we pick another trivialization  $\eta_{\gamma} : V_{\gamma} \hookrightarrow X^*$  of  $\mathcal{F}^*$ , then we have  $V_{\delta\gamma} = V_{\delta} \cap V_{\gamma}$ ,  $V_{\alpha\delta\gamma} = \eta_{\alpha}^{-1}(V_{\delta\gamma})$  and we have that  $h_{\alpha\delta,\gamma}$  and  $h_{\gamma\alpha,\delta}$  are inclusions when the morphism  $h_{\delta\gamma,\alpha} : V_{\alpha\delta\gamma} \rightarrow V_{\delta\gamma}$  coincides with the morphism  $\eta_{\alpha}$  restricted to  $V_{\alpha\delta\gamma}$ . Hence, the compatibility condition gives the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{F}_{\alpha}|_{V_{\alpha\delta\gamma}} & \xrightarrow{h_{\alpha\delta,\gamma}^* \varphi_{\alpha\delta}} & \eta_{\alpha}^* \mathcal{F}^*|_{V_{\alpha\delta\gamma}} \\
 \text{id}_{\mathcal{F}_{\alpha}} \downarrow & & \downarrow h_{\delta\gamma,\alpha}^* \varphi_{\delta\gamma} \\
 \mathcal{F}_{\alpha}|_{V_{\alpha\delta\gamma}} & \xrightarrow{h_{\gamma\alpha,\delta}^* \varphi_{\gamma\alpha}} & \eta_{\alpha}^* \mathcal{F}^*|_{V_{\alpha\delta\gamma}}
 \end{array}$$

We deduce that there is an isomorphism:  $\eta_\alpha^* \mathcal{F}^* \simeq \mathcal{F}_{\alpha|V_\alpha^*}$ .

This isomorphism satisfies the stated compatibility condition because of compatibility conditions of the orbisheaf  $\mathcal{F}$  applied to  $\varphi_{\alpha\beta}$ ,  $\varphi_{\beta\delta}$  and  $\varphi_{\alpha\delta}$ .

For the second part of the proposition, we can argue similarly, since every structure morphisms preserve connections by definition of an orbiconnection.  $\square$

**Lemma 7.2.2.** *Let  $X$  be a manifold and let  $(F, \nabla_F)$  and  $(G, \nabla_G)$  be two flat vector bundles on  $X$ . Let  $X^\circ$  be an analytic Zariski open subset of  $X$ , and assume that there is an isomorphism of vector bundles endowed with connections  $\phi^\circ : (F, \nabla_F)|_{X^\circ} \simeq (G, \nabla_G)|_{X^\circ}$ .*

*Then, this isomorphism extends to an isomorphism of vector bundles endowed with flat connections  $\phi : (F, \nabla_F) \simeq (G, \nabla_G)$ .*

*Proof.* We consider an open cover of  $X$  by trivializing open subset  $U_\alpha$ . Let us assume that there exists on each trivialization  $U_\alpha$  a trivialization of  $F$  (resp. of  $G$ ) by flat sections  $s_\alpha = (s_{\alpha,1}, \dots, s_{\alpha,r})$  (resp.  $t_\alpha = (t_{\alpha,1}, \dots, t_{\alpha,r})$ ). We write  $\varphi_{\alpha\beta} : U_{\alpha\beta} \rightarrow \mathrm{GL}(r, \mathbb{C})$  (resp.  $\psi_{\alpha\beta}$ ) the transition matrices of these local frames. We have  $s_\beta = s_\alpha \varphi_{\alpha\beta}$  and  $t_\beta = t_\alpha \psi_{\alpha\beta}$ . If  $U_\delta$  is included in  $X^\circ$ , the isomorphism  $\Phi^\circ$  provides us with local isomorphisms  $A_\delta : U_\delta \rightarrow \mathrm{GL}(r, \mathbb{C})$  such that  $s_\delta = t_\delta A_\delta$ . Hence, if  $U_\gamma$  is another trivializing open subset included in  $X^\circ$ , we have the compatibility condition  $A_\gamma \varphi_{\gamma\delta} = \psi_{\gamma\delta} A_\delta$ . Because all the considered frames are flat sections, the matrices are all locally constant.

Now, we want to extend the isomorphism through  $X$ . The goal is to construct local isomorphisms  $A_\alpha : U_\alpha \rightarrow \mathrm{GL}(r, \mathbb{C})$  for trivializations  $U_\alpha$  that are not included in  $X^\circ$ , that satisfy compatibility conditions. We consider such an open subset  $U_\alpha$ , and a local trivialization  $U_\delta$  included in  $U_\alpha^\circ := U_\alpha \cap X^\circ$ . Hence, we set  $A_{\alpha,\delta} : U_\delta \rightarrow \mathrm{GL}(r, \mathbb{C})$  defined by  $A_{\alpha,\delta} = \psi_{\alpha\delta} A_\delta \varphi_{\delta\alpha}$ . This is a locally constant matrix, and we observe that for another trivialization  $U_\gamma$  included in  $U_\alpha^\circ$ , we have  $A_{\alpha,\gamma} = A_{\alpha,\delta}$ . Hence, these morphisms glue together, and they produce a map  $A_\alpha : U_\alpha^\circ \rightarrow \mathrm{GL}(r, \mathbb{C})$  that is locally constant. Hence, the local constancy allows us to extend this map through the whole of  $U_\alpha$ . We obtain a map  $A_\alpha : U_\alpha \rightarrow \mathrm{GL}(r, \mathbb{C})$ .

It remains to show that these morphisms  $A_\alpha$  satisfy compatibility conditions. If we have two trivializations  $U_\alpha$  and  $U_\beta$ , we can pick a trivialization  $U_\delta$  included in  $U_\alpha \cap U_\beta \cap X^\circ$ , and we have:

$$A_{\alpha,\delta} = \psi_{\alpha\delta} A_\delta \varphi_{\delta\alpha} = \psi_{\alpha\beta} \psi_{\beta\delta} A_\delta \varphi_{\delta\beta} \varphi_{\beta\alpha} = \psi_{\alpha\beta} A_{\beta,\delta} \varphi_{\beta\alpha}.$$

By local constancy of the morphisms, this shows that  $A_\alpha = \psi_{\alpha\beta} A_\beta \varphi_{\beta\alpha}$ . Hence, this produce an isomorphism of vector bundles  $\Phi : F \simeq G$ . Moreover, it preserves flat sections

because the local matrices are locally constant. It is an isomorphism of flat vector bundles  $\Phi : (F, \nabla_F) \simeq (G, \nabla_G)$ .  $\square$

We consider  $(\mathcal{F}, \nabla)$  a flat orbifold on  $(X, \Delta)$ , with transition maps denoted  $\varphi_{\alpha\beta}$ . Hence, the restriction to  $X^*$  is a genuine flat vector bundle  $(\mathcal{F}^*, \nabla^*)$  above the manifold  $X^*$ , and the classical Riemann–Hilbert correspondence yields an associated representation  $\rho^* : \pi_1(X^*) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ . It suffices to show that this representation factors through  $\langle\langle \gamma_i^{m_i} \rangle\rangle$  to construct a representation of  $\pi_1^{\mathrm{orb}}(X, \Delta)$ . Above a general point of a component  $\Delta_i$  of the divisor  $\Delta$ , we can consider a local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  such that  $V_\alpha = \mathbb{D}^n$ , and the morphism  $\eta_\alpha$  can be written in normal form :  $\eta_\alpha(z_1, \underline{z}) = (z_1^{m_i}, \underline{z})$ . We get that  $V_\alpha^* := \eta_\alpha^{-1}(X^*) = \mathbb{D}^* \times \mathbb{D}^{n-1}$  and its fundamental group is generated by a loop  $\delta_i$  around the component  $(z_1 = 0)$ . Its image by  $\eta_\alpha$  in  $\pi_1(X^*)$  is homotopic to  $\gamma_i^{m_i}$ .

It suffices to show that the image of  $\delta_i$  by the composition is trivial in  $\mathrm{GL}(r, \mathbb{C})$ . By Lemma 7.2.1, one can observe that there is a natural isomorphism of flat bundles  $\eta_\alpha^*(\mathcal{F}^*, \nabla^*) \simeq (\mathcal{F}_\alpha, \nabla_\alpha)|_{V_\alpha^*}$ . Hence, the composition  $\pi_1(V_\alpha^*) \longrightarrow \pi_1(X^*) \longrightarrow \mathrm{GL}(r, \mathbb{C})$  is a representation of the fundamental group of  $V_\alpha^*$  that is associated to  $\eta_\alpha^*(\mathcal{F}^*, \nabla^*)$ . But the previous isomorphism shows that this flat vector bundle on  $V_\alpha^*$  is obtained as the restriction of  $(\mathcal{F}_\alpha, \nabla_\alpha)$ , so that the previous composition is isomorphic as a representation of  $\pi_1(V_\alpha^*)$  to the composition  $\pi_1(V_\alpha^*) \longrightarrow \pi_1(V_\alpha) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ . We have that the image of  $\delta_i$  is trivial, because  $V_\alpha$  is simply connected. Hence, we obtained by factorization a representation  $\rho : \pi_1^{\mathrm{orb}}(X, \Delta) \longrightarrow \mathrm{GL}(r, \mathbb{C})$ .

$$\begin{array}{ccccc} \pi_1(V_\alpha^*) & \longrightarrow & \pi_1(X^*) & \longrightarrow & \mathrm{GL}(r, \mathbb{C}) \\ & \searrow & & \nearrow & \\ & & \pi_1(V_\alpha) & & \end{array}$$

**Definition 7.2.3.** *Let  $(X, \Delta)$  be an orbifold, and  $(\mathcal{F}, \nabla)$  a flat orbifold on it. Then the representation associated to the flat orbifold, denoted  $\rho_{(\mathcal{F}, \nabla)}$  is the representation constructed above:*

$$\rho_{(\mathcal{F}, \nabla)} : \pi_1^{\mathrm{orb}}(X, \Delta) \longrightarrow \mathrm{GL}(r, \mathbb{C}).$$

The inverse arrow is obtained by the following construction. Let us consider a representation  $\rho : \pi_1^{\mathrm{orb}}(X, \Delta) \longrightarrow \mathrm{GL}(r, \mathbb{C})$  and let  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  be the universal cover of  $(X, \Delta)$ . We denote by  $\pi : \widetilde{X}_\Delta \longrightarrow X$  the covering map. First, we recall the link between the orbifold structures on  $(\widetilde{X}_\Delta, \widetilde{\Delta})$ . If we pick a local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  on  $(X, \Delta)$ , then there are local uniformizing charts on  $\widetilde{X}_\Delta$  denoted  $(W_\alpha, V_\alpha, \mu_\alpha)$  obtained as completions

of connected components  $W_\alpha^*$  of  $\pi^{-1}(U_\alpha^*)$ . If we denote by  $G_\alpha$  the Galois group of  $\eta_\alpha$ , then Galois groups of these charts are subgroups  $H_\alpha$  of  $G_\alpha$ .

Now, we consider the trivial orbifold of rank  $r$  on  $\widetilde{X}_\Delta$ . It is the flat bundle  $(\mathcal{O}_{\widetilde{X}_\Delta}^{\oplus r}, d)$  given by the collection  $(\mathcal{O}_{V_\alpha}^{\oplus r}, d)$  and trivial transition maps. We recall that the compatibility conditions imply that there are local actions of  $H_\alpha$  on  $V_\alpha \times \mathbb{C}^r$ . There exists an action of  $\pi_1^{\text{orb}}(X, \Delta)$  on  $\widetilde{X}_\Delta \times \mathbb{C}^r$  in the following way:

$$g \cdot (x, v) = (g \cdot x, \rho(g)v).$$

This action is proper, and we can consider the quotient space  $F_\rho := \widetilde{X}_\Delta \times \mathbb{C}^r / \pi_1^{\text{orb}}(X, \Delta)$ . On  $\widetilde{X}_\Delta^*$ , the previous action is free, and the action is compatible with the action on  $\widetilde{X}_\Delta$  so that the quotient  $F_\rho^*$  is a vector bundle above  $X^*$ , that can be endowed with a flat connection  $\nabla^*$ . Now, the goal is to construct an orbifold structure on  $F_\rho$  in order to get a flat orbifold  $\mathcal{F}_\rho$  on  $X$ .

First, by restriction of  $\pi$  and  $\mu_\alpha$ , the bundle  $\eta_\alpha^*(F_\rho^*, \nabla^*)$  is isomorphic to the restriction of  $(\mathcal{O}_{V_\alpha}^{\oplus r}, d)$  to  $V_\alpha^*$ . The usual Riemann–Hilbert correspondence applied to  $U_\alpha^*$  yields a representation  $\rho : \pi_1(U_\alpha^*) \rightarrow \text{GL}(r, \mathbb{C})$ . Because the pullback on  $V_\alpha^*$  is trivial, this representation is trivial on the normal subgroup  $\pi_1(V_\alpha^*) \triangleleft \pi_1(U_\alpha^*)$ , and there is a representation  $\rho_\alpha : G_\alpha \rightarrow \text{GL}(r, \mathbb{C})$ . To sum up, there is an action of  $G_\alpha$  on  $V_\alpha^* \times \mathbb{C}^r$  given by  $g \cdot (x, v) = (g \cdot x, \rho_\alpha(g)v)$ . This action extends uniquely to  $V_\alpha \times \mathbb{C}^r$  by Corollary 11.3.13 applied to  $V_\alpha^* \times \mathbb{C}^r \rightarrow F_\rho^*$ . Hence, we get a collection of trivial flat bundles  $(\mathcal{O}_{V_\alpha}^{\oplus r}, d)$  with actions of  $G_\alpha$ . Now, we construct the transition maps  $\varphi_{\alpha\beta}$ . Let us consider a connected component  $C$  of  $V_{\alpha\beta}^* := g_{\alpha\beta}^{-1}(\eta_\alpha^{-1}(X^*))$ . The composition  $\eta_\alpha \circ g_{\alpha\beta} = \eta_\beta \circ g_{\beta\alpha}$  is a covering map above  $U_{\alpha\beta}^*$ . Given the fact that  $\eta_\alpha^*(F_\rho^*, \nabla^*)$  is trivial, we deduce that  $g_{\alpha\beta}^* \eta_\alpha^*(F_\rho^*, \nabla^*)$  is trivial too. Hence, we have a natural isomorphism:

$$\begin{array}{ccc} g_{\beta\alpha}^* \eta_\beta^*(F_\rho^*, \nabla^*) & \xrightarrow{\text{nat}} & g_{\alpha\beta}^* \eta_\alpha^*(F_\rho^*, \nabla^*) \\ \downarrow \sim & & \sim \downarrow \\ (\mathcal{O}_{V_{\alpha\beta}}^{\oplus r}, d)|_C & \dashrightarrow & (\mathcal{O}_{V_{\alpha\beta}}^{\oplus r}, d)|_C \end{array}$$

We set  $\varphi_{\alpha\beta|_C} : C \rightarrow \text{GL}(r, \mathbb{C})$  the dashed arrow on the diagram above. This morphism is constant and extends to the closure of  $C$  in  $V_{\alpha\beta}$ , that is a connected component of  $V_{\alpha\beta}$  because of the smoothness of  $V_{\alpha\beta}$ . Hence, we get a morphism  $\varphi_{\alpha\beta} : V_{\alpha\beta} \rightarrow \text{GL}(r, \mathbb{C})$  locally constant.

These transition matrices satisfy compatibility conditions by pulling back the diagram above on a connected component of  $V_{\alpha\beta\gamma}^*$  and identifying the morphisms by the commutation conditions satisfied by  $h_{\alpha\beta,\gamma}$ ,  $g_{\alpha\beta}$  and  $\eta_\alpha$ .

Finally, we have constructed an orbibundle  $\mathcal{F}_\rho$  on  $(X, \Delta)$ . This bundle is endowed with an orbiconnection: the trivial connections  $d_\alpha$  on each  $V_\alpha$  are compatible with the transition matrices  $\varphi_{\alpha\beta}$  because these matrices are locally constant. Hence, we have constructed a flat orbibundle  $(\mathcal{F}_\rho, \nabla_\rho)$ .

This bundle is constructed up to isomorphism, because we did several choices that produce isomorphic orbibundles. For instance, we state that  $F_\rho^*$  is a vector bundle on  $X^*$ , but there are different structures of vector bundle on  $F_\rho^*$  obtained by choosing trivializations using the universal cover. These different choices produce isomorphic vector bundles.

**Definition 7.2.4.** *Let  $(X, \Delta)$  be an orbifold and let  $\rho : \pi_1^{\text{orb}}(X, \Delta) \longrightarrow \text{GL}(r, \mathbb{C})$  be a representation. The orbibundle associated to  $\rho$  is denoted  $(\mathcal{F}_\rho, \nabla_\rho)$ .*

**Proposition 7.2.5.** *Let  $(X, \Delta)$  be an orbifold.*

*Then, for any representation  $\rho : \pi_1^{\text{orb}}(X, \Delta) \longrightarrow \text{GL}(r, \mathbb{C})$ , we have:*

$$\rho_{(\mathcal{F}_\rho, \nabla_\rho)} \simeq \rho$$

*and for any flat orbibundle  $(\mathcal{F}, \nabla)$ , we have:*

$$(\mathcal{F}_{\rho_{(\mathcal{F}, \nabla)}}, \nabla_{\rho_{(\mathcal{F}, \nabla)}}) \simeq (\mathcal{F}, \nabla).$$

*Proof.* First, we show that  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)} \simeq \rho$ . We first set  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)}^*$  the composition:

$$\pi_1(X^*) \longrightarrow \pi_1^{\text{orb}}(X, \Delta) \xrightarrow{\rho_{(\mathcal{F}_\rho, \nabla_\rho)}} \text{GL}(r, \mathbb{C})$$

and  $\rho^*$  the composition:

$$\pi_1(X^*) \longrightarrow \pi_1^{\text{orb}}(X, \Delta) \xrightarrow{\rho} \text{GL}(r, \mathbb{C}).$$

Let us reinterpret this in geometric terms. We set  $\widetilde{X}^*$  the universal cover of  $X^*$ . We can pull back on  $\widetilde{X}^*$  the trivial bundle on  $\widetilde{X}_\Delta^*$  and the action of  $\pi_1^{\text{orb}}(X, \Delta)$ . This yields an action of  $\rho^*$  on  $\widetilde{X}^* \times \mathbb{C}^r$ . The usual Riemann–Hilbert correspondence associates to  $(\mathcal{F}_\rho^*, \nabla_\rho^*)$  the representation  $\rho^*$ . Hence, by definition of  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)}$ , we get that  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)}^* \simeq \rho^*$ .

This isomorphism means that there exists  $P \in \text{GL}(r, \mathbb{C})$  such that  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)}^* = P\rho^*P^{-1}$ . It remains true for the induced representations on  $\pi_1^{\text{orb}}(X, \Delta)$ :  $\rho_{(\mathcal{F}_\rho, \nabla_\rho)} \simeq \rho$ .

Now, let us prove that:  $(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}, \nabla_{\rho(\mathcal{F}, \nabla)}) \simeq (\mathcal{F}, \nabla)$ . First, the flat vector bundle  $(\mathcal{F}^*, \nabla^*)$  is obtained as a quotient of  $\widetilde{X}^* \times \mathbb{C}^r$  by the action induced by  $\rho_{(\mathcal{F}, \nabla)}^*$ . But we showed before Definition 7.2.3 that  $\rho_{(\mathcal{F}, \nabla)}^*$  is trivial on  $\langle \langle \gamma_i^{m_i} \rangle \rangle$ . Hence, this action of  $\rho_{(\mathcal{F}, \nabla)}^*$  on  $\widetilde{X}^* \times \mathbb{C}^r$  factors through an action of  $\pi_1^{\text{orb}}(X, \Delta)$  via  $\rho_{(\mathcal{F}, \nabla)}$  on  $\widetilde{X}_\Delta^* \times \mathbb{C}^r$ . But the quotient is a flat vector bundle isomorphic to  $(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}^*, \nabla_{\rho(\mathcal{F}, \nabla)}^*)$ .

Let us denote by  $\varphi_{\alpha\beta}$  (resp.  $\psi_{\alpha\beta}$ ) the transition matrices of  $(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}, \nabla_{\rho(\mathcal{F}, \nabla)})$  (resp. the transition matrices of  $(\mathcal{F}, \nabla)$ ). By pulling back on each local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  and using Lemma 7.2.1, we have an isomorphism of flat vector bundles on  $V_\alpha^*$ :

$$(\mathcal{O}_{V_\alpha^*}^{\oplus r}, d) \simeq \eta_\alpha^*(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}^*, \nabla_{\rho(\mathcal{F}, \nabla)}^*) \simeq \eta_\alpha^*(\mathcal{F}^*, \nabla^*) \simeq (\mathcal{F}_\alpha^*, \nabla_\alpha^*).$$

This isomorphism extends to an isomorphism of flat vector bundles on the whole of  $V_\alpha$  by Lemma 7.2.2. The compatibility condition being satisfied by the natural isomorphism of this lemma on  $V_{\alpha\beta}^*$ , we have the following diagram where each square commutes:

$$\begin{array}{ccccccc} (\mathcal{O}_{V_{\alpha\beta}^*}^{\oplus r}, d) & \longrightarrow & g_{\beta\alpha}^* \eta_\beta^*(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}^*, \nabla_{\rho(\mathcal{F}, \nabla)}^*) & \longrightarrow & g_{\beta\alpha}^* \eta_\beta^*(\mathcal{F}^*, \nabla^*) & \longrightarrow & g_{\beta\alpha}^*(\mathcal{F}_\beta^*, \nabla_\beta^*) \\ \varphi_{\alpha\beta} \downarrow & & \downarrow & & \downarrow & & \downarrow \psi_{\alpha\beta} \\ (\mathcal{O}_{V_{\alpha\beta}^*}^{\oplus r}, d) & \longrightarrow & g_{\alpha\beta}^* \eta_\alpha^*(\mathcal{F}_{\rho(\mathcal{F}, \nabla)}^*, \nabla_{\rho(\mathcal{F}, \nabla)}^*) & \longrightarrow & g_{\alpha\beta}^* \eta_\alpha^*(\mathcal{F}^*, \nabla^*) & \longrightarrow & g_{\alpha\beta}^*(\mathcal{F}_\alpha^*, \nabla_\alpha^*) \end{array}$$

Finally, all these isomorphisms extend through  $V_{\alpha\beta}$  by Lemma 7.2.2, and it gives the desired isomorphism of flat orbibundles.  $\square$

**Corollary 7.2.6.** *Let  $(X, \Delta)$  be a simply connected orbifold. Hence, any flat orbibundle  $(\mathcal{F}, \nabla)$  of rank  $r$  is isomorphic to  $(\mathcal{O}_X^{\oplus r}, d)$ .*

## 7.2.2 The criterion

In order to give a criterion of developability of a pair, we give the following two results that describe orbifolds whose orbifold tangent sheaf is orbi-locally free:

**Lemma 7.2.7.** *Let  $(X, \Delta)$  be an orbifold. Assume that  $\mathcal{T}_{(X, \Delta)}$  is orbi-locally free. Then,  $\Delta = 0$  and  $X$  is smooth.*

*Proof.* The orbi-locally freeness of  $\mathcal{T}_{(X, \Delta)}$  is equivalent to that of  $\Omega_{(X, \Delta)}^1$ , because they are dual to each other. Hence, we can assume that  $\Omega_{(X, \Delta)}^1$  is orbi-locally free. We denote:  $\Delta =$

$\sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$ . At a general point  $x \in \Delta_i$ , we have an adapted local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  above  $x$  of degree  $m_i$  (it means that its normal form is  $\eta_\alpha(z_1, \underline{z}) = (z_1^{m_i}, \underline{z})$ ). We consider  $(s_1, \dots, s_n)$  a system of generators of  $\Omega_{V_\alpha, y}^1$  at a point  $y \in \eta_\alpha^{-1}(x)$ , invariant under the action of  $\mathbb{Z}/(m_i)$ . Let  $dz_1 := \sum_k f_k s_k$ , with  $f_{k_0}(0) \neq 0$  for some  $1 \leq k_0 \leq n$ .

The action of the class  $[1] \in \mathbb{Z}/(m_i)$  on  $dz_1$  is:  $[1] \cdot dz_1 = \zeta^{-1} dz_1$ , where  $\zeta = e^{\sqrt{-1} \frac{2\pi}{m_i}}$ . On the other hand, it is:  $[1] \cdot dz_1 = \sum_k f_k(\zeta^{-1} z_1, \underline{z}) s_k$ . By evaluating at 0, we have  $\zeta f_{k_0}(0) = f_{k_0}(0)$ , then  $\zeta = 1$ , hence  $m_i = 1$ . It is true for each  $i$ , so  $\Delta = 0$ .

Now, we have  $\mathcal{T}_{(X,0)}$  is orbi-locally free because so is  $\Omega_{(X,0)}^1$ . By the point C of Proposition 7.1.6, we get that  $\mathcal{T}_{(X,0)}^{\text{inv}}$  is locally free, and it coincides on  $X_{\text{reg}}$  with  $\mathcal{T}_X$ . Hence, they are isomorphic because both are reflexive, and by [Dru14, Thm. 1.1],  $X$  is smooth.  $\square$

**Corollary 7.2.8.** *Let  $(X, \Delta)$  be an orbifold. Assume that there exists a surjective morphism of orbibundles:  $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathcal{T}_{(X, \Delta)}$ . Then,  $\Delta = 0$  and  $X$  is smooth.*

*Proof.* By Lemma 7.2.7, it suffices to show that  $\mathcal{T}_{(X, \Delta)}$  is orbi-locally free. By Proposition 7.1.4, it suffices to construct for any local uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  and  $y \in V_\alpha$  a local frame of  $G_\alpha$ -invariant sections for  $\mathcal{T}_{V_\alpha}$  on a  $G_\alpha$ -invariant open neighborhood  $V$  of  $y$ . By assumption, we have a morphism of  $G_\alpha$ -sheaves:  $\mathcal{O}_{V_\alpha}^{\oplus r} \twoheadrightarrow \mathcal{T}_{V_\alpha}$ . We denote by  $t_1, \dots, t_r$  the image of the canonical local frame  $e_1, \dots, e_r$  by this morphism. Hence, these sections are  $G_\alpha$ -invariant, and they generate the tangent bundle. It follows that there exists a neighborhood of  $y$  denoted  $V^\circ \hookrightarrow V_\alpha$  such that we can extract  $n$  sections (for example  $t_1, \dots, t_n$ ) that form a local frame of  $\mathcal{T}_{V_\alpha}$ . Moreover, if  $g \in G_\alpha$ , then the family  $(g(t_1), \dots, g(t_n))$  forms a trivialization of  $\mathcal{T}_{V_\alpha}$  around  $g(y)$ . Because the sections  $t_i$  are  $G_\alpha$ -invariant, we infer that  $(t_1, \dots, t_n)$  provides a local frame on a  $G_\alpha$ -invariant open neighborhood  $V = \bigcup g \cdot V^\circ$ , providing the sought trivialization of  $\mathcal{T}_{V_\alpha|_V}$  by  $G_\alpha$ -invariant sections.  $\square$

**Example 7.2.9.** We give the example of a cone over a smooth conic: the group  $\{\pm 1\}$  acts on the affine plane  $\mathbb{C}^2$ , the quotient space  $X$  is isomorphic to  $(w^2 - uv = 0) \subseteq \mathbb{C}^3$  (we denote by  $(u, v, w)$  the coordinates in the target space) and the quotient map is given by  $\pi: (x, y) \in \mathbb{C}^2 \mapsto (x^2, y^2, xy) \in X$ . This map is quasi-étale, and the local uniformizing chart  $(X, \mathbb{C}^2, \pi)$  provides an orbi-étale orbistrukture for the pair  $(X, 0)$ . We have  $\Delta = 0$  and  $X$  is singular at 0. One can show that there does not exist a  $G$ -equivariant surjective map  $\mathcal{O}_{\mathbb{C}^2}^{\oplus r} \twoheadrightarrow \mathcal{T}_{\mathbb{C}^2}$ . Indeed, if such a map exists, then the image of the canonical local frame provides vector fields that generate  $\mathcal{T}_{\mathbb{C}^2}$ , and they are invariant by the action of

$\{\pm 1\}$ . Hence, such vector fields have their germs at 0 of the form:

$$f(x^2, y^2)x \frac{\partial}{\partial x} + g(x^2, y^2)y \frac{\partial}{\partial x} + h(x^2, y^2)x \frac{\partial}{\partial y} + k(x^2, y^2)y \frac{\partial}{\partial y},$$

where  $f, g, h, k \in \mathcal{O}_{\mathbb{C}^2, 0}$ . It belongs to  $\mathfrak{m}\mathcal{T}_{\mathbb{C}^2, 0}$ , where  $\mathfrak{m}$  is the maximal ideal of  $\mathcal{O}_{\mathbb{C}^2, 0}$ . Hence, the image of  $(\mathcal{O}_{\mathbb{C}^2}^{\oplus r} \longrightarrow \mathcal{T}_{\mathbb{C}^2})$  can not generate  $\mathcal{T}_{\mathbb{C}^2}$  at 0.

This example illustrates that, in the case  $\Delta = 0$ , the functors  $(\cdot)^{[\text{orb}]}$  and  $(\cdot)^{\text{inv}}$  are essentially surjective by points A and B of Proposition 7.1.6, but they are not equivalences of categories. Indeed, let us show that the functor  $(\cdot)^{[\text{orb}]}$  is not exact, because it does not preserve surjective morphisms. Let us consider a surjective morphism  $\mathcal{O}_X^{\oplus r} \twoheadrightarrow \mathcal{T}_X$ . Its reflexive pullback by  $\pi$  provides a family of vector fields of  $\mathbb{C}^2$  which are invariant by the action of  $\{\pm 1\}$ . The resulting morphism is no more surjective, as its image is included in  $\mathfrak{m}\mathcal{T}_{\mathbb{C}^2, 0}$ .

We have the following criterion of developability of an orbifold.

**Proposition 7.2.10** (Developability criterion for an orbifold). *Let  $(X, \Delta)$  be an orbifold and let  $\mathcal{F}$  be a flat orbibundle on  $(X, \Delta)$ . Let us assume that there exists a surjective morphism of orbibundles:  $\mathcal{F} \twoheadrightarrow \mathcal{T}_{(X, \Delta)}$ . Then, the universal cover  $(\tilde{X}_\Delta, \tilde{\Delta})$  satisfies  $\tilde{\Delta} = 0$  and  $\tilde{X}_\Delta$  is smooth.*

*Proof.* Let us consider the universal cover map:  $\pi: (\tilde{X}_\Delta, \tilde{\Delta}) \longrightarrow (X, \Delta)$ . The pullback<sup>2</sup>  $\pi^*\mathcal{F}$  remains flat on  $(\tilde{X}_\Delta, \tilde{\Delta})$ . By Corollary 7.2.6, we get that it is a trivial bundle  $\mathcal{O}_{\tilde{X}_\Delta}^{\oplus r}$ . Moreover, we have  $\pi^*\mathcal{T}_{(X, \Delta)} = \mathcal{T}_{(\tilde{X}_\Delta, \tilde{\Delta})}$ . Hence, we have a surjective morphism  $\mathcal{O}_{\tilde{X}_\Delta}^{\oplus r} \twoheadrightarrow \mathcal{T}_{(\tilde{X}_\Delta, \tilde{\Delta})}$ . We are in position to apply Corollary 7.2.8, and we get  $\tilde{\Delta} = 0$  and  $\tilde{X}_\Delta$  is smooth.  $\square$

**Corollary 7.2.11.** *Let  $(X, \Delta)$  be an orbifold such that the orbibundle  $\text{End}(\mathcal{E}_{(X, \Delta)})$  is flat. Then,  $\tilde{X}_\Delta$  is smooth and  $\tilde{\Delta} = 0$ .*

*Proof.* It suffices to construct a surjective morphism of orbibundles:

$$\text{End}(\mathcal{E}_{(X, \Delta)}) \twoheadrightarrow \mathcal{T}_{(X, \Delta)}$$

and to apply Proposition 7.2.10. The orbibundle  $\text{End}(\mathcal{E}_{(X, \Delta)})$  is defined by the collection:

$$\text{End}(\mathcal{E}_{(X, \Delta)})_\alpha = \mathcal{O}_{V_\alpha} \oplus \mathcal{T}_{V_\alpha} \oplus \Omega_{V_\alpha}^1 \oplus \text{End}(\mathcal{T}_{V_\alpha})$$

---

2. For the definition of pullbacks of orbibundles, see Definition 9.3.5

and the transition maps in matrix notation following this splitting:

$$\Phi_{\alpha\beta} := \begin{bmatrix} 1 & \omega_{\alpha\beta} & 0 & 0 \\ 0 & \varphi_{\mathcal{T}_{(X,\Delta)},\alpha\beta} & 0 & 0 \\ \omega_{\alpha\beta}^\top & \omega_{\alpha\beta}^\top \otimes \omega_{\alpha\beta} & \varphi_{\Omega_{(X,\Delta)}^1,\alpha\beta} & \omega_{\alpha\beta} \otimes \varphi_{\Omega_{(X,\Delta)}^1,\alpha\beta} \\ 0 & \omega_{\alpha\beta}^\top \otimes \varphi_{\mathcal{T}_{(X,\Delta)},\alpha\beta} & 0 & \varphi_{\mathcal{T}_{(X,\Delta)},\alpha\alpha} \otimes \varphi_{\Omega_{(X,\Delta)}^1,\alpha\beta} \end{bmatrix},$$

where  $\varphi_{\mathcal{T}_{(X,\Delta)},\alpha\beta}$  (resp.  $\varphi_{\Omega_{(X,\Delta)}^1,\alpha\beta}$ ) are the transition morphisms associated to the orbibundle  $\mathcal{T}_{(X,\Delta)}$  (resp.  $\Omega_{(X,\Delta)}^1$ ), and  $\omega_{\alpha\beta}$  are the 1-forms that define the canonical extension, see Example 3.1.8. It suffices to consider the projection morphisms:

$$p_\alpha : \mathcal{O}_{V_\alpha} \oplus \mathcal{T}_{V_\alpha} \oplus \Omega_{V_\alpha}^1 \oplus \text{End}(\mathcal{T}_{V_\alpha}) \longrightarrow \mathcal{T}_{V_\alpha}.$$

These morphisms satisfy compatibility conditions with the matrices  $\Phi_{\alpha\beta}$ , so that it yields a surjective morphism  $p : \text{End}(\mathcal{E}_{(X,\Delta)}) \longrightarrow \mathcal{T}_{(X,\Delta)}$ . Indeed, we observe that the projection can be written in matrix notation  $p_\alpha = \begin{bmatrix} 0 & \text{id}_{\mathcal{T}_{V_\alpha}} & 0 & 0 \end{bmatrix}$ , so that we have:

$$\varphi_{\mathcal{T}_{(X,\Delta)},\alpha\beta} \circ g_{\beta\alpha}^* p_\beta = \begin{bmatrix} 0 & \varphi_{\mathcal{T}_{(X,\Delta)},\alpha\beta} & 0 & 0 \end{bmatrix} = g_{\alpha\beta}^* p_\alpha \circ \Phi_{\alpha\beta}.$$

□



PART III

**A characterization of quotients of  
projective spaces via Chern numbers**

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# INTRODUCTION

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## Uniformization of klt Fano varieties

The following result gives a characterization of singular quotients of  $\mathbb{P}^n$  by finite groups acting freely in codimension 1.

**Theorem 8.0.1.** [GKP22, Thm. 1.3] *Let  $X$  be a klt Fano projective variety. Then the following two assertions are equivalent:*

- (a) *There exists a finite group  $G \subseteq \mathrm{PGL}(n+1, \mathbb{C})$  acting freely in codimension 1 such that  $X \simeq \mathbb{P}^n/G$ .*
- (b) *There exists an ample divisor  $H$  such that the canonical extension  $\mathcal{E}_X$  is semistable with respect to  $H$  and the following equality holds:*

$$\left(2(n+1)\widehat{c}_2(X) - n\widehat{c}_1(X)^2\right) \cdot H^{n-2} = 0.$$

In that article, the authors construct the canonical extension  $\mathcal{E}_X$  of a klt variety  $X$ , which is a reflexive sheaf on  $X$ . It is an extension of  $\mathcal{T}_X$  by  $\mathcal{O}_X$ .

One considers, by [GKP16a, Thm. 1.5], a maximally quasi-étale cover  $\pi : Y \rightarrow X$ . We aim to show that  $Y \simeq \mathbb{P}^n$ . The two assumptions of the theorem are still satisfied on  $Y$ . Indeed, the variety  $Y$  is again klt because  $K_Y = \pi^*K_X$  and by Lemma 4.1.5. It is also Fano by [Laz04, Prop. 1.2.13]. Moreover, we can observe from [Gre+19b, Lemma 3.16] that:

$$\begin{aligned} \widehat{c}_2(Y) \cdot \pi^*H^{n-2} &= \deg \pi \widehat{c}_2(X) \cdot H^{n-2}, \\ \widehat{c}_1(Y)^2 \cdot \pi^*H^{n-2} &= \deg \pi \widehat{c}_1(X)^2 \cdot H^{n-2}. \end{aligned}$$

The canonical extension  $\mathcal{E}_Y$  remains  $c_1(Y)$ -semistable since  $\mathcal{E}_Y = \pi^{[*]}\mathcal{E}_X$ , and if  $\mathcal{E}_Y$  were not semistable, there would exist a maximal destabilizing subsheaf  $\mathcal{F}$ . By uniqueness, this subsheaf is stabilized by the Galois group of  $\pi$ , and the action of each stabilizer is trivial on  $\mathcal{F}$ , so it descends to a subsheaf  $\mathcal{G}$  of  $\mathcal{E}_X$  (i.e., we would have  $\mathcal{F} = \pi^*\mathcal{G}$ ). This subsheaf  $\mathcal{G}$

would also destabilize  $\mathcal{E}_X$ , by the same Chern class computation on a cover as above. This is excluded, since  $\mathcal{E}_X$  is assumed to be  $c_1(X)$ -semistable. The equality in the Miyaoka–Yau inequality is also still satisfied.

Hence, the reflexive sheaf  $\mathcal{F} := \text{End}(\mathcal{E}_Y)$  is  $c_1(Y)$ -semistable and satisfies the equalities:

$$\widehat{\text{ch}}_1(\mathcal{F}) \cdot c_1(X)^{n-1} = \widehat{\text{ch}}_2(\mathcal{F}) \cdot c_1(X)^{n-2} = 0.$$

A version of Simpson’s correspondence in the klt setting was established in [LT18, Thm. 1.4], so that the sheaf  $\mathcal{F}$  is locally free and flat on  $Y_{\text{reg}}$ . By the extension lemma for flat bundles over maximally quasi-étale covers from [GKP16a, Thm. 1.14], we conclude that  $\mathcal{F}$  is locally free and flat.

In particular, in a neighborhood of a point  $y \in Y$ , there exists an affine open neighborhood  $U$ , so that we have a splitting:

$$\mathcal{O}_U^{\oplus(n+1)^2} \simeq \text{End}(\mathcal{E}_Y)|_U = \mathcal{O}_U \oplus \mathcal{T}_{Y|U} \oplus \Omega_{Y|U}^1 \oplus \text{End}(\mathcal{T}_Y)|_U.$$

Thus,  $\mathcal{T}_{Y|U}$  is locally free as a direct summand of a free module (see [Kob87, Lemma. 5.3.3]). Then, by [Dru14, Thm. 1], the variety  $Y$  is smooth.

To conclude that  $Y \simeq \mathbb{P}^n$ , we observe that  $\text{End}(\mathcal{E}_Y)$  is flat, that implies the projective flatness of the canonical extension  $\mathcal{E}_Y$  by [GKP22, Prop. 3.7]. Because  $Y$  is simply connected as a Fano manifold, we get that the projectivized bundle  $\mathbb{P}(\mathcal{E}_Y) \simeq \mathbb{P}(\mathcal{O}_Y^{\oplus n+1})$ , so that there exists a line bundle  $\mathcal{L}$  such that  $\mathcal{E}_Y \simeq \mathcal{L}^{\oplus n+1}$ . We conclude by computing first Chern classes:

$$c_1(Y) = c_1(\mathcal{E}_Y) = (n+1)c_1(\mathcal{L}).$$

Hence, the line bundle  $\mathcal{L}$  is ample and we conclude that  $Y \simeq \mathbb{P}^n$  by [KO73, Cor. of Thm.1].

## Presentation of the results

If we want to generalize the previous to the pair setting, one main issue is a good definition of the canonical extension of a pair  $\mathcal{E}_{(X,\Delta)}$ . In the case where  $(X, \Delta)$  is orbifold, we gave a definition, see Definition 3.1.8. A first observation is that such orbisheaves do not exist as objects on  $X$ , but they can be embodied through strictly adapted morphisms. For instance, for a strictly  $\Delta$ -adapted morphism  $f : Y \rightarrow X$ , one can define the adapted cotangent sheaf  $\Omega_{X,\Delta,f}^{[1]}$ . That is a reflexive sheaf on  $Y$  that takes into account

the ramification of  $f$  and the presence of the boundary  $\Delta$ . In the same spirit, we define an adapted canonical extension  $\mathcal{E}_{X,\Delta,f}$ . This is a reflexive sheaf on  $Y$ , that is an extension of the adapted tangent sheaf  $\mathcal{T}_{X,\Delta,f}$  by the trivial sheaf  $\mathcal{O}_Y$ . The semistability condition is expressed in this case by the following:

(SCE+) *There exists a strictly  $\Delta$ -adapted morphism  $f_0 : Y_0 \rightarrow X$  such that the adapted canonical extension  $\mathcal{E}_{X,\Delta,f_0}$  is  $-f_0^*(K_X + \Delta)$ -semistable.*

In order to compute orbifold Chern classes of these sheaves, we give them an interpretation in terms of pullbacks of orbibundles. Indeed, we introduce the pullback of an orbistruature. It is an orbistruature on  $Y$  obtained by considering normalizations of fiber products of  $f$  with local uniformizing charts  $\eta_\alpha$  above  $X$ . On this special orbistruature, we can define the pullback orbisheaf  $f^*\mathcal{F}$  of an orbibundle  $\mathcal{F}$  above  $(X, \Delta)$ : it is an orbisheaf on  $Y$ . By taking invariant sections of the pullback orbisheaf, we define a reflexive sheaf  $\mathcal{F}_{X,\Delta,f}$  on  $Y$ . This procedure produces reflexive sheaves that are similar to classical adapted sheaves  $\Omega_{X,\Delta,f}^{[1]}$  or  $\mathcal{E}_{X,\Delta,f}$ . In fact, these two constructions give isomorphic sheaves.

**Proposition 8.0.2.** *Let  $(X, \Delta)$  be a klt pair and  $f : Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism. Then, we have an isomorphism of orbisheaves above  $f^{-1}(X_{\text{orb}})$ :*

$$f^*\Omega_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}^1 \simeq \left(\Omega_{X,\Delta,f}^{[1]}\right)^{[\text{orb}]} \quad \text{and} \quad f^*\mathcal{E}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})} \simeq \left(\mathcal{E}_{X,\Delta,f}\right)^{[\text{orb}]}.$$

When  $(Y, 0)$  is klt, we can compute their orbifold Chern classes. Because they are obtained as pullbacks, they satisfy the following equality:  $\widehat{c}_i(\mathcal{F}_{X,\Delta,f}) \cdot f^*\kappa = (\deg f)\widehat{c}_i(\mathcal{F}) \cdot \kappa$ .

Such computations are crucial in order to show that a log Fano pair  $(X, \Delta)$  satisfying (SCE+) and the Miyaoka–Yau equality is orbifold. Indeed, we can first consider a family of strictly  $\Delta$ -adapted morphism, whose additional branch locus  $H$  moves in a very ample linear system. If we pick a morphism  $f : Y \rightarrow X$ , one can show by implementing same computations as the case  $\Delta = 0$  that the reflexive sheaf  $\text{End}(\mathcal{E}_{X,\Delta,f})$  is flat and locally free, so much that  $\mathcal{T}_{X,\Delta,f}$  is locally free. Because it is isomorphic to  $\mathcal{T}_Y$  away from the additional ramification locus  $f^{-1}(H)$ . Because  $Y$  is a klt space, we conclude by Zariski–Lipman’s theorem.

It remains to show the result if we assume that  $(X, \Delta)$  is a log Fano pair that is orbifold. The previous flatness of  $\text{End}(\mathcal{E}_{X,\Delta,f})$  can be described in terms of holomorphic flat connection, equivariant with respect to  $\text{Gal}(f)$ . We deduce that it yields a flat orbiconnection on  $\text{End}(\mathcal{E}_{(X,\Delta)})$  by descent theorems about flat connections, and we conclude that  $\text{End}(\mathcal{E}_{(X,\Delta)})$  is flat. Hence, we conclude that the universal cover  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  satisfies  $\widetilde{\Delta} = 0$

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and  $\widetilde{X}_\Delta$  is smooth as a consequence of the developability criterion stated in the previous part.

## Organization of Part III

The aim of this part is to prove uniformization results of log Fano pairs. Based on the strategy of [CGG24], we recall some useful propositions of this article, related to the construction of orbi-étale orbistruktures on a klt pair  $(X, \Delta)$ . This gives us the setup in which we will implement the proof. Next, we recall and develop the construction of classical adapted sheaves, that are involved in the geometry of klt pairs (e.g the sheaf of adapted differentials  $\Omega_{X, \Delta, f}^{[1]}$ , the adapted tangent sheaf  $\mathcal{T}_{X, \Delta, f}$ , and the adapted canonical extension  $\mathcal{E}_{X, \Delta, f}$ ). We make sense of pullbacks of orbistruktures and orbibundles and we interpret the above-mentioned sheaves in terms of pullbacks of orbibundles. Then, this allows us to compute orbifold Chern classes of these sheaves.

Finally, we show uniformization results. We first assume that the pair  $(X, \Delta)$  is a log Fano orbifold, and we show by using tools developed in the previous part that its universal cover is  $(\mathbb{P}^n, 0)$  under the assumptions of semistability of the canonical extension and Miyaoka–Yau equality. Then, we show that if  $(X, \Delta)$  is a log Fano pair satisfying these assumptions, it is actually an orbifold by constructing a smooth orbi-étale orbistrukture as explained above.

We conclude this part by giving some explicit computations and examples. For instance, we give some insights of uniformization of orbifold curves, and give some examples where adapted canonical extensions are not semistable. Next, we give the example of weighted projective spaces that are quotients of projective spaces, and we characterize log smooth surfaces that are obtained as quotients of  $\mathbb{P}^2$  by a finite group.

# STRICTLY ADAPTED MORPHISMS ABOVE A KLT PAIR

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## 9.1 Orbistruktures on a klt pair

### 9.1.1 Construction of branched covers

A difficulty that arises when working with pairs equipped with orbifold structures and orbifold objects is their virtual nature: these objects only exist in local uniformizations. One way to overcome this issue is to realize these objects in strictly  $\Delta$ -adapted covers  $f : Y \rightarrow X$ , so that the study of orbifold objects on  $X$  reduces to concrete objects on the space  $Y$ , which account for the special ramification that occurs along  $\Delta$ .

For instance, on an orbifold pair  $(X, \Delta)$ , the orbifold tangent sheaf  $\mathcal{T}_{(X, \Delta)}$  is defined on a smooth orbi-étale orbistrukture. Providing a strictly adapted cover leads to the definition of the adapted tangent sheaf  $\mathcal{T}_{X, \Delta, f}$ .

We begin with the following well-known lemma, which allows us to construct such adapted covers:

**Lemma 9.1.1.** *Let  $X$  be a smooth variety and let  $D = \sum_i n_i D_i$  be an effective divisor such that there exists a line bundle  $L$  on  $X$  satisfying  $\mathcal{O}(D) \simeq L^{\otimes N}$ . Then, there exists a normal variety  $Y$  and a finite morphism  $f : Y \rightarrow X$  of degree  $N$  such that  $\text{Br}(f) \subseteq |D|$ . Moreover, the ramification index above an irreducible component  $D_i$  of  $D$  is  $\frac{N}{\gcd(N, n_i)}$ .*

*Proof.* Consider a section  $\sigma \in H^0(X, L^{\otimes N})$  such that  $\text{div}(\sigma) = D$ . Define  $Y$  as the normalization of:

$$Z := \{l \in \mathbb{L} \mid l^{\otimes N} = \sigma(p(l))\},$$

where  $\mathbb{L}$  is the total space of the line bundle  $L$ , and  $p : \mathbb{L} \rightarrow X$  is the projection. Let  $f : Y \rightarrow X$  be the composition of the normalization map with  $p$ .

Let  $x^0 \in X$ , and choose a trivialization of  $L$  and coordinates on a neighborhood  $U$  of  $x^0$ , which we assume to be a polydisc. If  $S_U$  is a local frame of  $L$ , we can write  $\sigma|_U = fS_U^{\otimes N}$  for some  $f \in \mathcal{O}(U)$ . Then an element  $l \in L$  over  $U$  is of the form  $l = zS_U$  with  $z \in \mathbb{C}$ , and the space becomes:

$$Z \simeq \{(x, z) \in U \times \mathbb{C} \mid z^N = f(x)\}.$$

This space is smooth at any point  $(x, z)$  where  $f(x) \neq 0$ , or where  $f(x) = 0$  and  $f$  is a submersion at  $x$  (i.e.,  $x$  is a smooth point of  $D$ ).

Indeed, choose coordinates  $(x_1, \dots, x_n) : U \rightarrow V \subseteq \mathbb{C}^n$  around  $x^0$ , and define  $g(x, z) = z^n - f(x)$ . Then:

$$dg_{(x,z)} = -\sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i + nz^{n-1} dz.$$

If  $f(x) \neq 0$ , then  $z \neq 0$ , so  $dg_{(x,z)} \neq 0$ . If  $f(x) = 0$  but  $df_x \neq 0$ , then  $z = 0$  and  $dg_{(x,z)} = df_x \neq 0$ . In both cases, the function  $g$  is a submersion at  $(x, z)$ . Hence, the hypersurface is smooth at  $(x, z)$ .

Now suppose  $x$  is a smooth point of  $D$ , say on an irreducible component  $D_i$ , and assume  $D$  is locally defined by  $x_1 = 0$ . Then, locally,  $Y$  is the normalization of the analytic variety:

$$S := \{(x, z) \in U \times \mathbb{C} \mid z^N = x_1^{n_i}\}.$$

Let  $d_i := \gcd(N, n_i)$ ,  $N' := N/d_i$ , and  $n'_i := n_i/d_i$ . Then the normalization of  $S$  decomposes into a union of irreducible hypersurfaces:

$$S_\zeta := \{(x, z) \in U \times \mathbb{C} \mid z^{N'} = \zeta x_1^{n'_i}\}, \quad \text{for } \zeta \in \mu_{d_i},$$

so  $S = \bigcup_{\zeta \in \mu_{d_i}} S_\zeta$ .

The normalization map is then given by:

$$(x, \zeta) \mapsto \left( x_1^{N'}, x_2, \dots, x_n, \zeta^{1/N'} x_1^{n'_i} \right),$$

where  $\zeta^{\pm 1/N'}$  is the unique  $N'$ -th root of  $\zeta$  with argument in  $[0, \pm 2\pi/N')$ . Since  $U \times \mu_{d_i}$  is normal, it suffices to show that the map is an isomorphism over the open subset  $(z \neq 0)$  by [Rem94, Prop. 14.7].

By Bézout's identity, there exist integers  $u, v$  such that  $uN' + vn'_i = 1$ . For  $(x, z) \in S_\zeta$ ,

define:

$$t := x_1^u \left( \zeta^{-1/N'} z \right)^v.$$

Then:

$$\begin{aligned} t^{N'} &= \zeta^{-v} x_1^{N'u} z^{N'v} = \zeta^{-v} x_1^{N'u} \zeta^v x_1^{vn'_i} = x_1, \\ \zeta^{1/N'} t^{n'_i} &= \zeta^{1/N'} x_1^{n'_i u} \zeta^{-vn'_i/N'} z^{vn'_i} = \zeta^{\frac{1-vn'_i-uN'}{N'}} z^{uN'+v'_i} = z. \end{aligned}$$

Hence, the point  $(t, x_2, \dots, x_n, \zeta)$  defines a preimage, which is unique. Indeed, if  $x, y \in \mathbb{C}^\times$  satisfy  $x^{N'} = y^{N'}$  and  $\zeta^{1/N'} x^{n'_i} = \zeta^{1/N'} y^{n'_i}$ , then  $x = y$ . This fact comes from the fact that there exists two integers  $p, q$  such that:

$$x = \exp(2\sqrt{-1}\pi p/N')y = \exp(2\sqrt{-1}\pi q/n'_i)y.$$

We get that  $\frac{p}{N'} - \frac{q}{n'_i}$  is an integer. By multiplying by  $N'$ , the integer  $n'_i$  divides  $Nq$ . Since  $\gcd(N', n'_i) = 1$ , we get that  $n'_i$  divides  $q$  and  $x = y$ .

The morphism  $S \rightarrow X$  is projection on the first  $n$  coordinates, so the composition with the normalization gives:

$$f : (x_1, \dots, x_n, \zeta) \mapsto (x_1^{N'}, \dots, x_n).$$

From this normal form of  $f$ , we observe two things:

- The ramification index above  $D_i$  is  $N' = \frac{N}{\gcd(N, n_i)}$  as required.
- The map  $f$  is étale outside of  $D$ , so the branch locus  $\text{Br}(f)$  is contained in  $D$  and contains a Zariski open subset of  $D$ . Since both are closed, we conclude  $\text{Br}(f) = |D|$ .  $\square$

Similarly, we have variations on this type of constructions if  $X$  is a possibly singular normal variety and  $D$  is a principal divisor, or if  $L$  is a torsion-free sheaf of rank 1 [KM08, Def. 2.52].

**Lemma 9.1.2.** *Let  $X$  be a normal complex space, and let  $D$  be a principal divisor on  $X$ . Then, for any  $N$ , there exists a finite cyclic map  $f : Y \rightarrow X$  of degree  $N$ , such that  $\text{Br}(f) \subseteq |D|$ , and the ramification index above an irreducible divisor  $Z \subseteq X$  is  $\frac{N}{\gcd(N, \text{ord}_Z(f))}$ .*

*Proof.* Let  $f \in \mathcal{M}_X^\times(X)$  be a meromorphic function such that  $D = \text{div}(f)$ . We get a map  $f : U \rightarrow \mathbb{P}^1$  that is well defined on a Zariski open subset  $U \subseteq X$  whose complement has

codimension at least 2. We construct  $Y^\circ$  as the normalization of the subspace:

$$\{(x, z) \in U \times \mathbb{P}^1 \mid [z_0^N : z_1^N] = f(x)\},$$

and there is a natural map  $\pi^\circ : Y^\circ \rightarrow U$  that is finite of degree  $N$ . By Theorem 11.3.12, there exists a normal variety  $Y$  and  $f : Y \rightarrow X$  that extends  $f^\circ$ . To compute the ramification indices, we study the normal forms of this map above a general point of an irreducible component of  $D$ , as we did in the previous lemma.  $\square$

**Remark 9.1.3.** Let  $X$  be a normal complex space and  $D$  a principal divisor. We set  $D = \sum_i d_i D_i$ . If we choose  $N$  such that  $d_i$  divides  $N$  and  $d_i \neq N$  for any  $i$ , then the morphism produced by the previous lemma satisfies  $\text{Br}(f) = |D|$ .

**Remark 9.1.4.** It can happen that the previous procedures produce reducible analytic spaces. For instance, let us consider  $D = 0$  on a compact connected simply connected manifold  $X$ . The map of degree  $N$  given by the previous lemma  $f : Y \rightarrow X$  is étale over  $X$ . Now, we can show that  $Y$  consists of  $N$  connected components  $C_1, \dots, C_N$ , that are isomorphic to  $X$  via  $f$ , because of the simple connectedness of  $X$ .

## 9.1.2 Orbistruktures associated to a pair

We consider a klt pair  $(X, \Delta)$ . First, we construct some orbistruktures on this pair, and we recall the construction of adapted sheaves (the sheaf of adapted differentials, or the adapted tangent sheaf) to a strictly  $\Delta$ -adapted morphism  $f : Y \rightarrow X$ .

**Lemma 9.1.5.** [CGG24, Prop. 13] *Let  $(X, \Delta)$  be a projective klt pair. Then, there exists an ample line bundle  $L$  over  $X$  such that for general  $H \in |L|$ , we have a morphism  $f : Y \rightarrow X$  of degree  $N$  that is orbi-étale for the pair  $(X, \Delta + (1 - \frac{1}{N})H)$ . Moreover, if  $(X, \Delta)$  is klt, so is  $(Y, 0)$ .*

*Proof.* We consider  $K$  a canonical Weil divisor and  $A$  a general very ample divisor on  $X$ . Then, for  $N \gg 1$ , we have  $N(A - (K + \Delta))$  that is Cartier, very ample, we denote by  $L$  the line bundle induced by this divisor. For a general  $H \in |L|$ , we get that the divisor:

$$D := H - N(A - (K + \Delta))$$

is principal. By Lemma 9.1.2, we consider a Galois morphism  $f : Y \rightarrow X$  that branches along  $D$ . Let us replace  $Y$  by one of its irreducible components. We get that the restriction

of  $f$  to this component is still a surjective Galois morphism. In the following, we denote  $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) \Delta_i$  and  $K = \sum_i k_i \Delta_i + \widetilde{K}$  where the irreducible components of  $\widetilde{K}$  are all different from the  $\Delta_i$ 's. Moreover, we can choose  $H, A$  irreducible and different from all the irreducible components appearing in  $K + \Delta$ . It follows that we have:

$$D := H + \sum_i d'_i \Delta'_i + \sum_i d_i \Delta_i$$

where  $\Delta'_i$  are irreducible components of  $D$  different from  $H$  and the  $\Delta_i$ 's. First, we have  $d'_i$  that is a multiple of  $N$ , so that the ramification indices above these components are equal to 1, and  $f$  is étale above these components. Secondly, the coefficient associated to the component  $\Delta_i$  is  $d_i := k_i N + N - \frac{N}{m_i}$  where  $k_i$  is the coefficient associated to  $\Delta_i$  in  $K$ . If we set  $N = p_i m_i$ , we have  $d_i = p_i((k_i + 1)m_i - 1)$ . It follows that the ramification indices above  $\Delta_i$  are  $\frac{N}{\gcd(N, d_i)} = \frac{N}{p_i} = m_i$ . The ones above  $H$  are equal to  $N$ . We get that the morphism is orbi-étale for the pair  $(X, \Delta + \left(1 - \frac{1}{N}\right) H)$ .

Following Proposition 4.1.5, we have the formula

$$K_Y = f^* \left( K_X + \Delta + \left(1 - \frac{1}{N}\right) H \right),$$

so that it suffices to show that for general  $H$ , the pair  $(X, \Delta + \left(1 - \frac{1}{N}\right) H)$  is klt. Let us consider a log resolution  $\mu : \widehat{X} \rightarrow X$  of the pair  $(X, \Delta)$ . First, by basepoint-freeness of  $|L|$ , we have  $\mu^* H = \mu_*^{-1} H$  for general  $H \in |L|$ . Now, by basepoint-freeness of  $|\mu^* L|$ , the support of the divisor  $\text{Exc } \mu + \mu_*^{-1} \Delta + \widehat{H}$  has simple normal crossings for general element  $\widehat{H} \in |\mu^* L|$ . It follows that for a general  $H \in |L|$  such a resolution provides one for the pair  $(X, \Delta + \left(1 - \frac{1}{N}\right) H)$  because the strict transform of  $H$  in  $Y$  can be chosen in a way that  $\text{Exc } \mu + \mu_*^{-1} \Delta + \mu_*^{-1} H$  has simple normal crossings, because of basepoint-freeness of  $|L|$  and of  $|\mu^* L|$ . Moreover, we have:

$$K_{\widehat{X}} + \mu_*^{-1} \Delta = \mu^*(K_X + \Delta) + \sum_i a_i E_i,$$

where all the coefficients  $a_i$  are greater than -1. Hence, we have:

$$K_{\widehat{X}} + \mu_*^{-1} \left( \Delta + \left(1 - \frac{1}{N}\right) H \right) = \mu^* \left( K_X + \Delta + \left(1 - \frac{1}{N}\right) H \right) + \sum_i a_i E_i.$$

It follows that the discrepancies are greater than -1. □

**Remark 9.1.6.** We can moreover assume that  $Y$  is maximally quasi-étale, since we can compose by a maximally quasi-étale cover [GKP22, Facts 5.1.1-3] and take the Galois closure of the morphism. Indeed, we can replace  $Y \rightarrow X$  by the composition  $Y' \rightarrow Y \rightarrow X$  where  $Y' \rightarrow Y$  is the maximally quasi-étale cover of  $Y$ . It follows that the composition is branched exactly along  $\Delta + \left(1 - \frac{1}{N}\right)H$ . We can take the Galois closure  $Y'' \rightarrow Y' \rightarrow X$  by [CGG24, Cor. 27]. This map is orbi-étale for this divisor, hence strictly  $\Delta$ -adapted. Moreover, the definition of the maximally quasi-étale cover and Galois closure ensures that  $(Y'', 0)$  is still klt.

The previous proposition yields an orbistructure orbi-étale by considering the collection of l.u.c.  $\{X \setminus H, Y \setminus f^{-1}(H), f_{|Y \setminus f^{-1}(H)}\}_H$  general in  $|L|$ .

**Remark 9.1.7.** One can construct another orbi-étale orbistructure above  $(X, \Delta)$  in the following way. The  $\mathbb{Q}$ -divisor  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, hence, for an integer  $m \gg 1$ , we can cover  $X$  by open sets  $U_\alpha$  such that  $m(K_{U_\alpha} + \Delta_{|U_\alpha})$  is principal. The cyclic covering map of Lemma 9.1.2 of degree  $m$  branched along this principal divisor provides an orbi-étale map  $\eta_\alpha: V_\alpha \rightarrow U_\alpha$ . One can show that the morphisms are compatible on intersections, by studying local forms of maps above general points of  $\Delta$ . The collection  $(U_\alpha, V_\alpha, \eta_\alpha)$  provides us with an orbi-étale orbistructure.

## 9.2 Classical adapted sheaves

In this section, we recall the constructions of adapted differentials and adapted tangent sheaf, and we give a first definition of the adapted canonical extension associated to a strictly  $\Delta$ -adapted morphism  $f: Y \rightarrow X$ .

### 9.2.1 Sheaf of adapted differentials, adapted tangent sheaf

The following construction initiated by Miyaoka [Miy08, p. 412] follows the exposition of [CKT16, Sect. I.3].

We consider  $(X, \Delta)$  a pair and  $f: Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism. Assume that  $f$  is good in the sense of [CKT16, Def. 3.5]. Let us remind the definition of the adapted cotangent sheaf in this case. Let us denote by  $\Omega_X^1$  the cotangent sheaf, and  $\Omega_X^1(\log[\Delta])$  is the sheaf of forms generated as a  $\mathcal{O}_X$ -module by  $\frac{dz_1}{z_1}, dz_2, \dots, dz_n$ , where  $(z_1 = 0)$  is a local equation of  $\Delta$ .

Hence, we have a short exact sequence induced by the residue map:

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log[\Delta]) \xrightarrow{\text{res}_\Delta} \bigoplus_i \mathcal{O}_{\Delta_i} \longrightarrow 0,$$

where the residue map is defined on stalks at  $x \in X$  by:

$$\text{res}_\Delta : f_1(z) \frac{dz_1}{z_1} + \sum_{2 \leq k \leq n} f_k(z) dz_k \in \Omega_X^1(\log[\Delta])_x \longmapsto f_1(0, z_2, \dots, z_n) \in \mathcal{O}_{\Delta, x}.$$

We recall that the strictly  $\Delta$ -adapted condition ensures that for each irreducible component  $\Delta_i$  of  $\Delta$  there exists a reduced divisor  $\Delta'_i$  on  $Y$  such that  $f^*\Delta_i = m_i\Delta'_i = m_i \sum_j \Delta'_{ij}$  where  $\Delta'_{ij}$  are irreducible components of  $\Delta'_i$ . The pullback of sheaves by  $f$  yields a surjective morphism:

$$f^*\Omega_X^1(\log[\Delta]) \xrightarrow{f^*\text{res}} \bigoplus_i \mathcal{O}_{m_i\Delta'_i} \twoheadrightarrow \bigoplus_{i,j} \mathcal{O}_{\Delta'_{ij}}.$$

We set  $\Omega_{X,\Delta,f}^1$  the kernel of this morphism.

One can give an equivalent definition of this sheaf that has advantages for effective computations. The pullback of forms by  $f$  induces a morphism:

$$df_\Delta : f^*\Omega_X^1(\log[\Delta]) \longrightarrow \Omega_Y^1(\log[\Delta']),$$

where  $\Delta'$  is the reduced divisor associated to  $f^*\Delta$ . By studying normal forms of the sheaves above general points of  $\Delta$  and the additional ramification, there is an isomorphism:

$$\Omega_{X,\Delta,f}^1 = \text{im } df_\Delta \cap \Omega_Y^1 = \left[ \text{im} \left( df_0 : f^*\Omega_X^1 \longrightarrow \Omega_Y^1 \right) \otimes \mathcal{O}(f^*\Delta) \right] \cap \Omega_Y^1.$$

It is in particular a locally free sheaf, and the following inclusions hold:

$$\text{im} \left( f^*\Omega_X^1 \longrightarrow \Omega_Y^1 \right) \subseteq \Omega_{X,\Delta,f}^1 \subseteq \Omega_Y^1.$$

In the following we assume that  $X$  is projective. Under this assumption, the manifold  $Y$  is projective by [Laz04, Prop. 1.2.13], and the sheaf  $\Omega_{X,\Delta,f}^1$  is the analytification of the algebraic coherent sheaf defined in the same way in the algebraic setting.

Now, let us consider  $X_0 \subseteq X$ , and  $\iota : Y_0 \hookrightarrow Y$  the maximal subsets where  $f|_{Y_0}^{X_0} : Y_0 \longrightarrow X_0$  is good for  $\Delta_0 := \Delta|_{X_0}$ . These are Zariski open subsets whose complements have

codimension at least 2. We define the sheaf of adapted differentials as  $\Omega_{X,\Delta,f}^{[1]} := \iota_* \Omega_{X_0,\Delta_0,f_0}^1$ . It is a coherent reflexive sheaf on  $Y$  by [Ser66, Thm. 4]. We sum up the previous discussion in the following definition.

**Definition 9.2.1.** [CGG24, Def. 17] *Let  $(X, \Delta)$  be a projective pair and  $f: Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism. Let  $X_0 \subseteq X$  and  $\iota: Y_0 \hookrightarrow Y$  be the maximal open subsets where  $f$  is good for  $\Delta_0 := \Delta|_{X_0}$  in the sense of [CKT16, Def. 3.5]. The sheaf of adapted differentials, denoted by  $\Omega_{X,\Delta,f}^{[1]}$ , is the following sheaf:*

$$\Omega_{X,\Delta,f}^{[1]} = \iota_* \left[ \left( \text{im}(f^* \Omega_{X_0}^1 \rightarrow \Omega_{Y_0}^1) \otimes \mathcal{O}(f^* \Delta) \right) \cap \Omega_{Y_0}^1 \right],$$

where  $\Omega_{X_0}^1$  (resp.  $\Omega_{Y_0}^1$ ) denotes the cotangent sheaf on  $X_0$  (resp. on  $Y_0$ ). The adapted tangent sheaf, denoted by  $\mathcal{T}_{X,\Delta,f}$ , is its dual  $\mathcal{T}_{X,\Delta,f} := \Omega_{X,\Delta,f}^{[1]\vee}$ .

We have the following properties for the sheaf of adapted differentials:

**Lemma 9.2.2.** [CGG24, Lemma. 18] *Let  $(X, \Delta)$  be a pair and  $f: Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism.*

- (i) *If  $f$  is orbi-étale we have  $\Omega_{X,\Delta,f}^{[1]} = \Omega_Y^{[1]}$ ,*
- (ii) *If  $\gamma: Z \rightarrow Y$  is a quasi-étale map, we have  $\gamma^{[*]} \Omega_{X,\Delta,f}^{[1]} = \Omega_{X,\Delta,f \circ \gamma}^{[1]}$*

Let  $\mathcal{C}$  be a strictly adapted orbistructure on  $(X, \Delta)$ . One can construct a reflexive weak orbisheaf  $\Omega_{\mathcal{C}}^{[1]}$  by considering the collection  $\Omega_{X,\Delta,\eta_\alpha}^{[1]}$  on each  $V_\alpha$ , and we have the compatibility condition:

$$g_{\beta\alpha}^{[*]} \Omega_{X,\Delta,\eta_\beta}^{[1]} \simeq \Omega_{X,\Delta,\eta_\beta \circ g_{\beta\alpha}}^{[1]} = \Omega_{X,\Delta,\eta_\alpha \circ g_{\alpha\beta}}^{[1]} \simeq g_{\alpha\beta}^{[*]} \Omega_{X,\Delta,\eta_\alpha}^{[1]}.$$

## 9.2.2 Adapted canonical extension

Now, we construct the adapted canonical extension  $\mathcal{E}_{X,\Delta,f}$ . It is a reflexive sheaf obtained as an extension of  $\mathcal{T}_{X,\Delta,f}$  by  $\mathcal{O}_Y$  as follows.

Let  $m \gg 1$  be such that  $-m(K_X + \Delta)$  is Cartier. It provides a Čech closed cocycle  $(h_{\alpha\beta})_{\alpha\beta}$  on  $X$ . The pullback of the derivative of its logarithm  $\left( f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}} \right)$  is a closed cocycle with values in:  $\text{im}(f^* \Omega_X^1 \rightarrow \Omega_Y^1)$ . We have a natural morphism from this subsheaf of  $\Omega_Y^1$  into  $\Omega_{X,\Delta,f}^{[1]}$ . Hence, it gives rise to a class  $H^1(Y, \Omega_{X,\Delta,f}^{[1]})$ , we denote it by  $mf^*c_1(X, \Delta)$ . We consider the extension induced by  $f^*c_1(X, \Delta)$ :

$$0 \rightarrow \Omega_{X,\Delta,f}^{[1]} \rightarrow \mathcal{W}_{X,\Delta,f} \rightarrow \mathcal{O}_Y \rightarrow 0.$$

We use the duality functor  $\mathcal{H}om(-, \mathcal{O}_Y)$ . This produces a short exact sequence because  $\mathcal{E}xt^1(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ :

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{E}_{X,\Delta,f} \longrightarrow \mathcal{T}_{X,\Delta,f} \longrightarrow 0.$$

The reflexive sheaf  $\mathcal{E}_{X,\Delta,f}$  is the desired sheaf called the *adapted canonical extension*. We have the following properties for this sheaf.

**Lemma 9.2.3.** *With notation as above:*

(i) *If  $f$  is orbi-étale, then we have  $\mathcal{E}_{X,\Delta,f} = \mathcal{E}_Y$  (see [GKP22, Sect. 4] to see how  $\mathcal{E}_Y$  is constructed when  $Y$  is klt).*

(ii) *Let  $\gamma: Y' \rightarrow Y$  be a quasi-étale morphism with  $Y'$  normal.*

*Then, we have  $\mathcal{E}_{X,\Delta,\gamma \circ f} = \gamma^{[*]} \mathcal{E}_{X,\Delta,f}$ .*

*Proof.* (i) When  $f$  is orbi-étale, we have  $\Omega_Y^{[1]} \simeq \Omega_{X,\Delta,f}^{[1]}$  by the point (i) of Lemma 9.2.2. Moreover, we have  $K_Y = f^*(K_X + \Delta)$  and the cocycles  $\left(f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}}\right)$  defined previously correspond to those (up to some exact cocycles) defined in [GKP22, Sect. 4], as they are cocycles associated to the divisor  $-mK_Y$ . Hence, they produce the same class in  $H^1(Y, \Omega_{X,\Delta,f}^{[1]})$ . It follows that the extensions  $\mathcal{E}_{X,\Delta,f}$  and  $\mathcal{E}_Y$  are isomorphic.

(ii) We observe that the pullback of the previous cocycles by  $\gamma$  are given by  $\left(\gamma^* f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}}\right)$ . They provide cocycles that define  $\mathcal{E}_{X,\Delta,\gamma \circ f}$ .

Let  $U \subseteq Y'_{\text{reg}}$  be an open subset such that  $\gamma$  is étale on  $U$  and  $(f \circ \gamma)^{-1}(\Delta + H)$  (here  $H$  denote the additional branching locus of  $f$ ) is a smooth hypersurface on  $U$ . We can choose  $U$  in a way that its complement has codimension at least 2. The explicit description of  $\Omega_{X,\Delta,f}^{[1]}$  at general points (see [CKT16, Rem. 3.8]) and the étale property imply that we have an isomorphism  $\gamma^* \Omega_{X,\Delta,f}^{[1]} \xrightarrow{\sim} \Omega_{X,\Delta,f \circ \gamma}^{[1]}$ , which maps the pullback of  $\left(f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}}\right)$  on  $\left(\gamma^* f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}}\right)$ . Hence, both sheaves  $\gamma^* \mathcal{E}_{X,\Delta,f}|_U$  and  $\mathcal{E}_{X,\Delta,\gamma \circ f}|_U$  coincide. It follows that we have the isomorphism  $\gamma^{[*]} \mathcal{E}_{X,\Delta,f} \simeq \mathcal{E}_{X,\Delta,\gamma \circ f}$  because both sheaves are reflexive.  $\square$

As previously, we can construct a canonical extension  $\mathcal{E}_{\mathcal{C}}$  associated to a strictly adapted orbistructure  $\mathcal{C}$  on  $(X, \Delta)$ . It is a reflexive weak orbisheaf given by the collection  $\mathcal{E}_{X,\Delta,\eta_\alpha}$  and the compatibility condition:

$$g_{\beta\alpha}^{[*]} \mathcal{E}_{X,\Delta,\eta_\beta} \simeq \mathcal{E}_{X,\Delta,\eta_\beta \circ g_{\beta\alpha}} = \mathcal{E}_{X,\Delta,\eta_\alpha \circ g_{\alpha\beta}} \simeq g_{\alpha\beta}^{[*]} \mathcal{E}_{X,\Delta,\eta_\alpha}.$$

## 9.3 Pullbacks

### 9.3.1 Pullback of an orbistructure

There is a way to pull back orbistructures by  $\Delta$ -adapted morphisms.

**Definition 9.3.1.** *Let  $(X, \Delta)$  be a pair with an orbi-étale orbistructure  $\mathcal{C}$ , and we consider a  $\Delta$ -adapted morphism  $f: Y \rightarrow X$ . The pull-back orbistructure, denoted by  $\mathcal{C}_f$  is an orbistructure on  $(Y, 0)$  given by the triples  $(f^{-1}(U_\alpha), W_\alpha, \mu_\alpha)$ , where  $W_\alpha$  is the normalization of  $Y \times_X V_\alpha$ , and  $\mu_\alpha$  is the morphism  $W_\alpha \rightarrow Y$ .*

One can show that the collection defining  $\mathcal{C}_f$  is an orbistructure because it satisfies the compatibility condition on intersections. Indeed, let  $l_{\alpha\beta}: W_{\alpha\beta} \rightarrow W_\alpha$  be the maps from the normalization of  $W_\alpha \times_Y W_\beta$  to  $W_\alpha$ . In normal forms above general points of  $\Delta$  and additional branching locus, we see that the morphisms  $l_{\alpha\beta}$  are étale. It follows that  $l_{\alpha\beta}$  are quasi-étale.

Moreover, the morphism  $f$  is naturally an orbifold morphism with respect to orbistructures  $\mathcal{C}_f$  and  $\mathcal{C}$ .

### 9.3.2 Pullback of an orbisheaf

Let us begin by the following naive definition of pullback of orbisheaves. Let us consider two pairs  $(X, \Delta)$  and  $(Y, \Delta')$ . Let  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)\}_\alpha$  and  $\mathcal{D} = \{(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})\}_{\alpha'}$  be two orbistructures on  $(X, \Delta)$  and  $(Y, \Delta')$  respectively. Let  $\mathcal{F}$  be an orbisheaf on  $(X, \Delta)$  and let  $f: (Y, \Delta') \rightarrow (X, \Delta)$  be an orbifold morphism. *The pullback of  $\mathcal{F}$  by  $f$ , denoted  $f^*\mathcal{F}$ , is the orbisheaf defined by the collection  $(f_{\alpha\alpha'}^*\mathcal{F}_\alpha)$ . If  $(\varphi_{\alpha\beta})$  are the transition maps of  $\mathcal{F}$ , the transition maps of  $f^*\mathcal{F}$  are  $(f_{\alpha\beta, \alpha'\beta'}^{[*]}\varphi_{\alpha\beta})$  where  $f_{\alpha\beta, \alpha'\beta'}^*$  has been defined in Remark 2.2.4, and they satisfy cocycle conditions on triple intersections.*

One issue that occurs with this definition is the ambiguity of the definition in the following situation. We consider a local uniformizing chart  $(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})$  on  $(Y, \Delta')$  such that there are two local uniformizing charts  $(U_\alpha, V_\alpha, \eta_\alpha)$  and  $(U_\beta, V_\beta, \eta_\beta)$  on  $(X, \Delta)$  and two morphisms  $f_{\alpha\alpha'}: W_{\alpha'} \rightarrow V_\alpha$  and  $f_{\beta\alpha'}: W_{\alpha'} \rightarrow V_\beta$ . Hence, the previous procedures produce two different sheaves on  $W_{\alpha'}: f_{\alpha\alpha'}^*\mathcal{F}_\alpha$  and  $f_{\beta\alpha'}^*\mathcal{F}_\beta$ . One can show that these two sheaves are isomorphic.

In order to avoid such ambiguities, we introduce the notion of orbistructure  $(f, \mathcal{C})$ -admissible.

**Definition 9.3.2.** Let  $(X, \Delta), (Y, \Delta')$  be two pairs endowed respectively with orbistructures  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)\}_{\alpha \in A}$  and  $\mathcal{D} = \{(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})\}_{\alpha' \in A'}$ , and let  $f : Y \rightarrow X$  be an orbifold morphism. We say that  $\mathcal{D}$  is  $(f, \mathcal{C})$ -admissible if there exists an application  $u : A' \rightarrow A$  such that:

$$\{(\alpha', \alpha) \in A' \times A \mid \text{there exists a morphism } f_{\alpha\alpha'} : W_{\alpha'} \rightarrow V_\alpha\} \subseteq \text{Graph } u.$$

In other words, the orbistructure  $\mathcal{D}$  is  $(f, \mathcal{C})$ -admissible if for any l.u.c.  $(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})$ , there is at most one chart  $(U_\alpha, V_\alpha, \eta_\alpha)$  and one morphism  $f_{\alpha\alpha'} : W_{\alpha'} \rightarrow V_\alpha$  such that  $\eta_\alpha \circ f_{\alpha\alpha'} = f \circ \mu_{\alpha'}$ .

**Example 9.3.3.** For instance, if  $(X, \Delta)$  is a pair endowed with an orbistructure  $\mathcal{C}$  and  $f : Y \rightarrow X$  is a  $\Delta$ -adapted morphism, the orbistructure  $\mathcal{C}_f$  is such that  $f$  is an orbifold morphism, and  $\mathcal{C}_f$  is  $(f, \mathcal{C})$ -admissible. Following notation of Definition 9.3.1, we denote by  $f_\alpha : W_\alpha \rightarrow V_\alpha$  the morphisms between two local uniformizing charts for more simplicity.

**Example 9.3.4.** Let  $(X, \Delta)$  be an orbifold given by the orbistructure  $\mathcal{C}$ , and let  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  be its universal cover. Let  $\pi : \widetilde{X}_\Delta \rightarrow X$  be the covering map. We gave a description of the orbistructure constructed on the universal cover in Remark 6.2.6. With this description, one can show that  $\pi$  is an orbifold morphism, and the orbistructure constructed on  $(\widetilde{X}_\Delta, \widetilde{\Delta})$  is  $(\pi, \mathcal{C})$ -admissible.

Now, let us give a definition of pullback of an orbisheaf.

**Definition 9.3.5.** Let  $(X, \Delta), (Y, \Delta')$  be two pairs endowed with orbistructures  $\mathcal{C} = \{(U_\alpha, V_\alpha, \eta_\alpha)\}_{\alpha \in A}$  and  $\mathcal{D} = \{(T_{\alpha'}, W_{\alpha'}, \mu_{\alpha'})\}_{\alpha' \in A'}$  respectively. Let  $f : (Y, \Delta') \rightarrow (X, \Delta)$  be an orbifold morphism, such that  $\mathcal{D}$  is an  $(f, \mathcal{C})$ -admissible orbistructure (in this case, there exists a map  $u : A' \rightarrow A$ ). Let  $\mathcal{F}$  be an orbisheaf on  $(X, \Delta)$ , whose transition matrices are denoted  $\varphi_{\alpha\beta}$ .

In the following, the indices  $\alpha, \beta \in A$  will denote images of indices  $\alpha', \beta' \in A'$  by the map  $u$ , in order to simplify notation. The pullback of  $\mathcal{F}$  by  $f$ , denoted  $f^*\mathcal{F}$ , is the orbisheaf on  $(Y, \Delta')$  defined by the collection  $(f_{\alpha\alpha'}^*\mathcal{F}_\alpha)$  and with transition maps  $\psi_{\alpha'\beta'}$ , where  $\psi_{\alpha'\beta'}$  is the composition:

$$l_{\beta'\alpha'}^* f_{\beta\beta'}^* \mathcal{F}_\beta \xrightarrow{\sim \text{nat}} f_{\alpha\beta, \alpha'\beta'}^* g_{\beta\alpha}^* \mathcal{F}_\beta \xrightarrow{f_{\alpha\beta, \alpha'\beta'}^* \varphi_{\alpha\beta}} f_{\alpha\beta, \alpha'\beta'}^* g_{\alpha\beta}^* \mathcal{F}_\alpha \xrightarrow{\sim \text{nat}} l_{\alpha'\beta'}^* f_{\alpha\alpha'}^* \mathcal{F}_\alpha$$

Here,  $f_{\alpha\beta, \alpha'\beta'}$  has been defined in Remark 2.2.4. One can show that they satisfy cocycle

conditions on triple intersections.

## 9.4 Adapted sheaves and pullbacks

### 9.4.1 Comparison with classical adapted sheaves

We show that classical adapted sheaves can be interpreted in terms of pullbacks of orbisheaves. Let  $(X, \Delta)$  be an orbifold, and  $f : Y \rightarrow X$  be a strictly  $\Delta$ -adapted morphism. We denote by  $\mathcal{C} := \{(U_\alpha, V_\alpha, \eta_\alpha)\}_{\alpha \in A}$  the smooth orbi-étale orbistruature on  $(X, \Delta)$  and  $\mathcal{C}_f := (S_\alpha, R_\alpha, \nu_\alpha)_{\alpha \in A}$  the pullback orbistruature. It is an orbi-étale orbistruature on  $(Y, 0)$ , that is  $(f, \mathcal{C})$ -admissible. We denote by  $g_{\alpha\beta}$  (resp.  $l_{\alpha\beta}$ ) the morphisms of  $\mathcal{C}$  (resp.  $\mathcal{C}_f$ ) induced by intersections.

We recall that we have defined classical adapted sheaves  $\mathcal{E}_{X,\Delta,f}$  and  $\Omega_{X,\Delta,f}^{[1]}$  on  $Y$ . On the one hand, they induce weak orbisheaves on  $\mathcal{C}_f$ ,  $\mathcal{E}_{X,\Delta,f}^{[\text{orb}]}$  and  $(\Omega_{X,\Delta,f}^{[1]})^{[\text{orb}]}$ . On the other hand, we can consider the pullbacks  $f^*\mathcal{E}_{(X,\Delta)}$  and  $f^*\Omega_{(X,\Delta)}^1$ . The aim of the following proposition is to show that these two procedures produce isomorphic orbisheaves on  $Y$ .

**Proposition 9.4.1.** *Let  $(X, \Delta)$  be an orbifold, and  $f : Y \rightarrow X$  be a strictly  $\Delta$ -adapted morphism. With notation as above, we have the isomorphisms of orbisheaves above  $Y$ :*

$$f^*\Omega_{(X,\Delta)}^1 \simeq (\Omega_{X,\Delta,f}^{[1]})^{[\text{orb}]} \quad \text{and} \quad f^*\mathcal{E}_{(X,\Delta)} \simeq (\mathcal{E}_{X,\Delta,f})^{[\text{orb}]}.$$

*Proof.*  $f^*\Omega_{(X,\Delta)}^1 \simeq (\Omega_{X,\Delta,f}^{[1]})^{[\text{orb}]}$ : To construct such an isomorphism of orbisheaves, we first observe that  $(\Omega_{X,\Delta,f}^{[1]})^{[\text{orb}]}$  is a weak orbisheaf. Hence, we have to construct an isomorphism of weak orbisheaves. If we have such an isomorphism, we conclude by Lemma 7.1.1 that  $(\Omega_{X,\Delta,f}^{[1]})^{[\text{orb}]}$  is an orbibundle, and the induced isomorphism is an isomorphism of orbisheaves.

First, we construct a collection of isomorphisms  $\theta_\alpha : f_\alpha^*\Omega_{V_\alpha}^1 \xrightarrow{\sim} \nu_\alpha^{[*]}\Omega_{X,\Delta,f}^{[1]}$ . Because this is an isomorphism of reflexive sheaves on  $R_\alpha$ , it suffices to construct the isomorphism on an open subset whose complement is included in an analytic subset of codimension 2. For instance, it suffices to show that they are isomorphic on smooth points, and above general points of  $\Delta, H$ .

The strategy to construct such isomorphisms is to identify them as subsheaves of  $\Omega_{R_\alpha}^1$ , and verify that these isomorphisms are compatible with transition maps.

First, we describe such a local isomorphism for general points above  $H$ . Because  $f$  is an orbifold morphism and the pullback orbistruature is admissible, there are morphisms  $f_\alpha : R_\alpha \rightarrow V_\alpha$ , and  $f_{\alpha\beta} : R_{\alpha\beta} \rightarrow V_{\alpha\beta}$  that satisfy compatibility conditions described in Definition 2.2.2 and Remark 2.2.4. We can write in normal forms locally above a general point of  $H$ :

$$\begin{array}{ccc}
 \{(z_1, w_1, \underline{z}) \mid z_1^N = w_1\} \subseteq R_\alpha & \xrightarrow{f_\alpha} & w \in V_\alpha \\
 \downarrow \nu_\alpha \sim & & \parallel \eta_\alpha \\
 z = (z_1, \underline{z}) \in Y & \xrightarrow{f} & (z_1^N, \underline{z}) = w \in H
 \end{array}$$

We have two inclusions  $f_\alpha^* \Omega_{V_\alpha}^1 \hookrightarrow \Omega_{R_\alpha}^1$  and  $\nu_\alpha^* \Omega_{X,\Delta,f}^{[1]} \hookrightarrow \Omega_{R_\alpha}^1$ . These inclusions have the same image: it is a locally free sheaf generated by  $(z_1^{N-1} dz_1, dz_2, \dots, dz_n)$ .

Secondly, above a general point of  $\Delta$ , the morphisms  $f_\alpha$  and  $\nu_\alpha$  are étale, and we have naturally an isomorphism  $f_\alpha^* \Omega_{V_\alpha}^1 \simeq \Omega_{R_\alpha}^1 \simeq \nu_\alpha^* \Omega_{X,\Delta,f}^{[1]}$ . Now, on more complicated local uniformizing charts  $V_\alpha$  and  $R_\alpha$ , the previous cases correspond to normal forms of the morphisms in codimension one. Hence, we have an isomorphism on a big open subset of  $R_\alpha$ , that extends to an isomorphism  $\theta_\alpha : f_\alpha^* \Omega_{V_\alpha}^1 \rightarrow \nu_\alpha^* \Omega_{X,\Delta,f}^{[1]}$  because both are reflexive on  $R_\alpha$ . Hence, we have constructed a collection of isomorphisms  $\theta_\alpha$ .

Now, let us check the compatibility on intersections  $R_{\alpha\beta}$ . As previously, we can show the compatibility by identifying subsheaves of  $\Omega_{R_{\alpha\beta}}^1$  on big open subsets of  $R_{\alpha\beta}$ . Let us denote by  $l_{\alpha\beta} : R_{\alpha\beta} \rightarrow R_\alpha$  the morphisms of the pullback orbistruature  $\mathcal{C}_f$  on  $Y$ . We

have the following diagram of sheaves<sup>1</sup> on  $R_{\alpha\beta}$ :

$$\begin{array}{ccccc}
 & & f_{\alpha\beta}^* \Xi_{\beta\alpha} & & \\
 & & \sim & & \\
 & \swarrow & & \searrow & \\
 l_{\alpha\beta}^* f_{\alpha}^* \Omega_{V_{\alpha}}^1 & \xrightarrow{\quad} & f_{\alpha\beta}^* \Omega_{V_{\alpha\beta}}^1 & \xrightarrow{\quad} & l_{\beta\alpha}^* f_{\beta}^* \Omega_{V_{\beta}}^1 \\
 \downarrow l_{\alpha\beta}^* \theta_{\alpha} & \searrow & \downarrow & \swarrow & \downarrow l_{\beta\alpha}^* \theta_{\beta} \\
 & & l_{\alpha\beta}^* \Omega_{R_{\alpha}}^1 = \Omega_{R_{\alpha\beta}}^1 = l_{\beta\alpha}^* \Omega_{R_{\beta}}^1 & & \\
 l_{\alpha\beta}^* \nu_{\alpha}^{[*]} \Omega_{X,\Delta,f}^{[1]} & \xrightarrow{\text{nat}_{\beta\alpha}} & & \xrightarrow{\quad} & l_{\beta\alpha}^* \nu_{\beta}^{[*]} \Omega_{X,\Delta,f}^{[1]} \\
 & \sim & & & 
 \end{array}$$

By compatibility of the pullback of forms with composition, the morphisms in the previous diagram commute, and we get the compatibility condition.

$f^* \mathcal{E}_{(X,\Delta)_{\text{orb}}} \simeq (\mathcal{E}_{X,\Delta,f})^{[\text{orb}]}$ : We want to show that there exists a collection of isomorphisms:

$$\Theta_{\alpha} : \mathcal{O}_{R_{\alpha}} \oplus f_{\alpha}^* \Omega_{V_{\alpha}}^1 \xrightarrow{\sim} \mathcal{O}_{R_{\alpha}} \oplus \nu_{\alpha}^{[*]} \Omega_{X,\Delta,f}^{[1]}$$

defined on the refined orbisstructure above  $Y^{\circ}$ , that are compatible with transition maps.

We can define :

$$\Theta_{\alpha} := \left[ \begin{array}{c|c} 1 & 0 \\ \hline 0 & \theta_{\alpha} \end{array} \right].$$

It remains to show that these  $\Theta_{\alpha}$  are compatible with transition maps. Both orbisheaves have their transition maps that are defined by collections of 1-forms. For  $\mathcal{E}_{X,\Delta,f}^{[\text{orb}]}$ , this is the collection  $(\omega_{\alpha\beta})$  coming from the one that defines  $\mathcal{E}_{X,\Delta,f}$ :  $(f^* \frac{dh_{\alpha\beta}}{h_{\alpha\beta}})$ . For  $f^* \mathcal{E}_{(X,\Delta)}$ , this is the collection  $(\xi_{\alpha\beta})$  obtained from the trivialization of the orbundle  $\Omega_{(X,\Delta)}^n \vee$ . Up to an exact cocycle, they are equal as they can be obtained as pullbacks on  $R_{\alpha\beta}$  of the same collection  $(\frac{dh_{\alpha\beta}}{h_{\alpha\beta}})$ , where  $(h_{\alpha\beta})$  are the transition maps defining the line bundle  $-m(K_X + \Delta)$  on  $(X, \Delta)$  for  $m \gg 1$ .  $\square$

From this proposition, and Fact 4.2.1, we deduce the following corollary applied to the orbifold locus  $X_{\text{orb}}$  of  $(X, \Delta)$ .

1. We recall that  $\Xi_{\alpha\beta}$  are transition morphisms of  $\Omega_{(X,\Delta)}^1$  following Definition 3.1.4

**Corollary 9.4.2.** *Let  $(X, \Delta)$  be a klt pair and  $f : Y \rightarrow X$  a strictly  $\Delta$ -adapted morphism. Then, we have an isomorphism of orbisheaves above  $f^{-1}(X_{\text{orb}})$ :*

$$f^* \Omega_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})}^1 \simeq \left( \Omega_{X, \Delta, f}^{[1]} \right)^{[\text{orb}]} \quad \text{and} \quad f^* \mathcal{E}_{(X_{\text{orb}}, \Delta|_{X_{\text{orb}}})} \simeq \left( \mathcal{E}_{X, \Delta, f} \right)^{[\text{orb}]}.$$

## 9.4.2 A general construction of adapted sheaves coming from pullbacks

Let  $(X, \Delta)$  be a klt pair. With Remark 9.1.7, we can endow  $(X, \Delta)$  with an orbifold structure, and refine it to have a smooth one  $\mathcal{C}$  above  $(X, \Delta)_{\text{orb}}$ , see [CGG24, Lemma. 14].

**Definition 9.4.3.** *Let  $\mathcal{F}$  be an orbifold on  $(X, \Delta)_{\text{orb}}$ . Let  $f : Y \rightarrow X$  be a strictly  $\Delta$ -adapted morphism. The morphism  $f$  is orbifold between  $(f^{-1}((X, \Delta)_{\text{orb}}), 0)$  endowed with  $\mathcal{C}_f$  and  $(X, \Delta)_{\text{orb}}$ , and  $\mathcal{C}_f$  is  $(f, \mathcal{C})$ -admissible by Example 9.3.3. Hence, we can consider the pullback orbifold  $f^* \mathcal{F}$ . Let us denote by  $i : f^{-1}((X, \Delta)_{\text{orb}}) \hookrightarrow Y$  the canonical inclusion. The adapted sheaf associated to  $\mathcal{F}$ , denoted by  $\mathcal{F}_{X, \Delta, f}$  is a sheaf on  $Y$  defined by:  $\mathcal{F}_{X, \Delta, f} := i_*((f^* \mathcal{F})^{\text{inv}})$ .*

A consequence of point B of Proposition 7.1.6 is that we have an isomorphism:

$$\left( \mathcal{F}_{X, \Delta, f} |_{f^{-1}((X, \Delta)_{\text{orb}})} \right)^{[\text{orb}]} \simeq f^* \mathcal{F}.$$

We have the following lemma that establishes that the construction of adapted sheaf behaves well with duals, tensor products, direct sums.

**Lemma 9.4.4.** *Let  $\mathcal{F}, \mathcal{G}$  be two orbifolds on  $(X, \Delta)_{\text{orb}}$ . Then, we have the following isomorphisms:*

$$\begin{aligned} \mathcal{F}_{X, \Delta, f}^{\vee} &= \mathcal{F}_{X, \Delta, f}^{\vee} \\ (\mathcal{F} \oplus \mathcal{G})_{X, \Delta, f} &= \mathcal{F}_{X, \Delta, f} \oplus \mathcal{G}_{X, \Delta, f} \\ (\mathcal{F} \otimes \mathcal{G})_{X, \Delta, f} &= \mathcal{F}_{X, \Delta, f} \otimes \mathcal{G}_{X, \Delta, f}. \end{aligned}$$

*Proof.* For the first isomorphism, we observe that  $f^*(\mathcal{F}^{\vee}) \simeq (f^* \mathcal{F})^{\vee}$ . Indeed, pullbacks commute with duals for locally free sheaves, so that we have collection of isomorphisms  $f_{\alpha}^*(\mathcal{F}_{\alpha}^{\vee}) \simeq (f_{\alpha}^* \mathcal{F}_{\alpha})^{\vee}$ . Moreover, the two transition matrices of  $f^*(\mathcal{F}^{\vee})$  and  $(f^* \mathcal{F})^{\vee}$  are compatible because of compatibility of pullbacks of morphisms and the definition of the

dual. Once we have this compatibility, it remains to show that for an orbisheaf  $\mathcal{H}$  above  $(f^{-1}(X_{\text{orb}}), 0)$ , we have  $\mathcal{H}^{\vee \text{inv}} \simeq \mathcal{H}^{\text{inv} \vee}$ . First, we observe that the dual commutes with the functor  $(\cdot)^{[\text{orb}]}$ . Indeed, it suffices to observe that for a coherent sheaf  $\mathcal{S}$  we have natural isomorphisms  $\eta_{\alpha}^{[*]}(\mathcal{S}^{\vee}) \simeq (\eta_{\alpha}^{[*]}\mathcal{S})^{\vee}$  that are compatible on intersections, so that  $(\mathcal{S}^{\vee})^{[\text{orb}]} \simeq (\mathcal{S}^{[\text{orb}]})^{\vee}$ . Now, we deduce from the point B of Proposition 7.1.6:

$$\mathcal{H}^{\text{inv} \vee [\text{orb}]} \simeq \mathcal{H}^{\text{inv} [\text{orb}] \vee} \simeq \mathcal{H}^{\vee}.$$

Hence, when we apply the functor  $(\cdot)^{\text{inv}}$  and point A of the above-mentioned proposition, we get :  $\mathcal{H}^{\vee \text{inv}} \simeq \mathcal{H}^{\text{inv} \vee}$ . Applied to  $\mathcal{H} = f^*\mathcal{F}$ , and the compatibility with pullbacks, we have:

$$\mathcal{F}^{\vee}_{X,\Delta,f} = f^*(\mathcal{F}^{\vee})^{\text{inv}} \simeq ((f^*\mathcal{F})^{\vee})^{\text{inv}} \simeq (f^*\mathcal{F}^{\text{inv}})^{\vee} = \mathcal{F}_{X,\Delta,f}^{\vee}.$$

For the other identifications, we prove them similarly. It suffices to show that:

- these constructions satisfy compatibilities with pullbacks when they are applied to orbibundles,
- they commute with functor  $(\cdot)^{[\text{orb}]}$ . □

**Remark 9.4.5.** The previous section establishes that in the case where  $\mathcal{F} = \Omega^1_{(X,\Delta)_{\text{orb}}}$  (resp.  $\mathcal{F} = \mathcal{E}_{(X,\Delta)_{\text{orb}}}$ ), we have  $\mathcal{F}_{X,\Delta,f} = \Omega^1_{X,\Delta,f}$  (resp.  $\mathcal{F}_{X,\Delta,f} = \mathcal{E}_{X,\Delta,f}$ ) by application of Proposition 9.4.2 and point A of Proposition 7.1.6. The previous lemma indicates that we can extend it to other orbibundles  $\mathcal{F}$ . For instance, if  $\mathcal{F} = \mathcal{T}_{(X,\Delta)_{\text{orb}}}, \text{End}(\mathcal{E}_{(X,\Delta)_{\text{orb}}})$ , we have  $\mathcal{F}_{X,\Delta,f} = \mathcal{T}_{X,\Delta,f}, \text{End}(\mathcal{E}_{X,\Delta,f})$ .

**Remark 9.4.6.** We use notation of § 9.4.3. On  $Y^{\circ}$ , if  $\mathcal{F} = \Omega^1_{(X,\Delta)_{\text{orb}}}$  (resp.  $\mathcal{F} = \mathcal{E}_{(X,\Delta)_{\text{orb}}}$ ), then we have the isomorphism for the adapted sheaf  $\mathcal{F}_{X,\Delta,f} = \Omega^1_{X,\Delta,f}$  (resp.  $\mathcal{F}_{X,\Delta,f} = \mathcal{E}_{X,\Delta,f}$ ) as an application of the previous lemma, and using functoriality of  $(\cdot)^{\text{inv}}$  detailed before Definition 7.1.2 and the point A of Proposition 7.1.6. As these classical adapted sheaves are algebraic coherent on  $Y$ , we infer by [Ser66, Thm. 2] that  $\mathcal{F}_{X,\Delta,f}$  are coherent, reflexive sheaves on  $Y$ , and isomorphic to  $\Omega^1_{X,\Delta,f}$  (resp.  $\mathcal{E}_{X,\Delta,f}$ ) on the whole of  $Y$ .

We have stated for classical adapted sheaves some results about pullbacks by a quasi-étale morphism in Proposition 9.2.2 and Proposition 9.2.3. The description of these sheaves in terms of pullbacks gives a slight improvement, the results (ii) of these propositions hold if we assume that we pull back the sheaves by morphisms such that the strictly-adapted property is preserved.

**Lemma 9.4.7.** *Let  $\gamma : Z \rightarrow Y$  be a finite morphism such that the composition morphism  $f \circ \gamma : Z \rightarrow X$  is a strictly  $\Delta$ -adapted morphism. Let  $\mathcal{F}$  be an orbifold on  $(X, \Delta)_{\text{orb}}$ .*

*Then, we have the isomorphism:*

$$\gamma^{[*]} \mathcal{F}_{X, \Delta, f} \simeq \mathcal{F}_{X, \Delta, f \circ \gamma}.$$

*Proof.* First, we observe that the orbifold structures  $(\mathcal{C}_f)_\gamma$  and  $\mathcal{C}_{f \circ \gamma}$  coincide. Hence, we have a natural isomorphism on this orbifold structure  $\gamma^* f^* \mathcal{F} = (f \circ \gamma)^* \mathcal{F}$ .

Now, let us justify the isomorphism:  $\mathcal{F}_{X, \Delta, f \circ \gamma} \simeq \gamma^{[*]} \mathcal{F}_{X, \Delta, f}$ . One can observe that  $\gamma^*(\mathcal{F}_{X, \Delta, f}^{\text{orb}}) \simeq (\gamma^* \mathcal{F}_{X, \Delta, f})^{\text{orb}}$  by pulling back sheaves on l.u.c. above  $f^{-1}(X_{\text{orb}})$  in two different ways. Because finite morphisms are flat in codimension one, so are  $\gamma$  and morphisms of the local uniformizing charts. Flat morphisms preserve reflexive sheaves, so that these local sheaves are moreover reflexive on big open subsets. It follows that the reflexive hulls  $(\gamma^{[*]} \mathcal{F}_{X, \Delta, f})^{[\text{orb}]}$  and  $\gamma^{[*]}(\mathcal{F}_{X, \Delta, f}^{[\text{orb}]})$  are isomorphic in codimension 1, hence everywhere. Moreover, these isomorphisms are compatible with transition morphisms on a big open subset, hence everywhere by reflexivity. To sum up, we get an isomorphism  $\gamma^{[*]}(\mathcal{F}_{X, \Delta, f}^{[\text{orb}]}) \simeq (\gamma^{[*]} \mathcal{F}_{X, \Delta, f})^{[\text{orb}]}$ . It implies by taking invariant sections and using the fact that:  $\gamma^{[*]}(\mathcal{F}_{X, \Delta, f}^{[\text{orb}]}) \simeq \gamma^* f^* \mathcal{F}$  that  $\mathcal{F}_{X, \Delta, f \circ \gamma} \simeq \gamma^{[*]} \mathcal{F}_{X, \Delta, f}$ .  $\square$

Now, we can show that the semistability property of one adapted sheaf transmits to any strictly  $\Delta$ -adapted morphism.

**Proposition 9.4.8.** *Let  $(X, \Delta)$  be a klt pair. Assume that  $X$  is projective, and let  $A$  be an ample divisor on  $X$  and let  $\mathcal{F}$  be an orbifold on  $(X, \Delta)_{\text{orb}}$ . Assume that there exists a strictly  $\Delta$ -adapted morphism  $f_0 : Y_0 \rightarrow X$  such that  $\mathcal{F}_{X, \Delta, f_0}$  is semistable with respect to  $f_0^* A$ .*

*Then, for any strictly  $\Delta$ -adapted morphism  $f : Y \rightarrow X$ , the adapted sheaf  $\mathcal{F}_{X, \Delta, f}$  is semistable with respect to  $f^* A$ .*

*Proof.* Let  $f_0 : Y_0 \rightarrow X$  be the strictly  $\Delta$ -adapted morphism of the proposition, and let  $f : Y \rightarrow X$  be another strictly  $\Delta$ -adapted morphism. We denote by  $Z$  the normalization of  $Y_0 \times_X Y$ . Let  $F : Z \rightarrow Y$ ,  $F_0 : Z \rightarrow Y_0$  be the natural projections so that  $f \circ F = f_0 \circ F_0$ . First, we have an isomorphism of sheaves on  $Z$ :  $F^{[*]} \mathcal{F}_{X, \Delta, f} \simeq F_0^{[*]} \mathcal{F}_{X, \Delta, f_0}$ . Indeed, one can observe with Lemma 9.4.7:

$$F^{[*]} \mathcal{F}_{X, \Delta, f} \simeq \mathcal{F}_{X, \Delta, f \circ F} \simeq \mathcal{F}_{X, \Delta, f_0 \circ F_0} \simeq F_0^{[*]} \mathcal{F}_{X, \Delta, f_0}.$$

Now, we argue by contradiction, following the proof of [GT22, Prop. 3.6]. Assume that  $\mathcal{F}_{X,\Delta,f}$  is not  $f^*A$ -semistable. Hence, using arguments of [GKP22, Fact 5.1.3], the sheaf  $F^{[*]}\mathcal{F}_{X,\Delta,f}$  is not  $F^*f^*A$ -semistable. Hence, the previous isomorphism implies that  $F_0^{[*]}\mathcal{F}_{X,\Delta,f_0}$  is not  $F_0^*f_0^*A$ -semistable. The polarization is invariant under the action of  $\text{Gal}(F_0)$ , hence the maximal destabilizing subsheaf, denoted  $\mathcal{G}$ , is  $\text{Gal}(F_0)$ -invariant. By [HL10, Thm. 4.2.15], it follows that there exists a subsheaf  $\mathcal{H}$  of  $\mathcal{F}_{X,\Delta,f_0}$  such that  $\mathcal{G} = f_0^*\mathcal{H}$  that destabilizes  $\mathcal{F}_{X,\Delta,f_0}$ , contradicting its semistability.  $\square$

### 9.4.3 Computation of orbifold Chern classes of $\mathcal{F}_{X,\Delta,f}$

In order to describe the pullback of orbibundles, we refine the pullback orbistructure above a big open subset  $Y$  in order to get a smooth one, which is well-suited to compute orbifold Chern classes with Chern–Weil theory.

*Setup.* Let  $(X, \Delta)$  be a klt pair, and let  $f: Y \rightarrow X$  be a strictly  $\Delta$ -adapted morphism with  $(Y, 0)$  klt. Let us endow  $(X, \Delta)_{\text{orb}}$  with a smooth orbi-étale orbistructure. We denote by  $\{(S_\alpha, R_\alpha, \nu_\alpha)\}_\alpha$  the pullback orbistructure on  $(f^{-1}(X_{\text{orb}}), 0)$ . Let us justify that the space  $R_\alpha$  is klt. Indeed, the morphism  $\nu_\alpha: R_\alpha \rightarrow Y$  is finite and satisfies  $K_{R_\alpha} = \nu_\alpha^*K_Y$  by quasi-étaleness. We get that  $(R_\alpha, 0)$  is klt as a consequence of Lemma 4.1.5.

By [Gre+11, Prop. 9.3], there is an open subset  $R_{\alpha \text{ orb}}$  which has only quotient singularities, and its complement has codimension at least 3. Its image by  $\nu_\alpha$  is open in  $Y$  by the open mapping theorem. Hence, the set  $Y^\circ = \bigcup_\alpha \nu_\alpha(R_{\alpha \text{ orb}})$  is open, and its complement is included in an analytic set of codimension at least 3. By taking smooth orbi-étale orbistructures on each  $(R_{\alpha \text{ orb}}, 0)$  and taking Galois closures (see [CGG24, Cor. 27]) of the composition of a l.u.c. by  $\nu_\alpha$ , we get that there are smooth spaces  $W_{\alpha\alpha'}$  above  $R_{\alpha \text{ orb}}$  such that  $\mu_{\alpha\alpha'}: W_{\alpha\alpha'} \rightarrow Y^\circ$  is a Galois quasi-étale morphism.

This construction gives an orbistructure on  $(Y^\circ, 0)$  in which  $f$  becomes an orbifold morphism between  $Y^\circ$  and  $(X, \Delta)_{\text{orb}}$ . Moreover, we get that this orbistructure is  $(f, \mathcal{C})$ -compatible, so that we can consider the pullback of an orbibundle above  $(X, \Delta)_{\text{orb}}$  on this orbistructure.

To simplify notation, we will denote by  $\mu_{\alpha'}: W_{\alpha'} \rightarrow \mu_{\alpha'}(W_{\alpha'}) =: T_{\alpha'} \subseteq Y^\circ$  a morphism of this refined orbistructure,  $W_{\alpha'\beta'}$  the normalization of the fiber product and  $l_{\alpha'\beta'}: W_{\alpha'\beta'} \rightarrow W_{\alpha'}$  the corresponding étale morphisms:

$$\begin{array}{ccccc}
 W_{\alpha\alpha'} \text{ smooth} & & & & \\
 \downarrow & \nearrow f_{\alpha\alpha'} & & & \\
 R_{\alpha} \text{ orb} & \hookrightarrow & R_{\alpha} & \xrightarrow{f_{\alpha}} & V_{\alpha} \text{ smooth} \\
 \downarrow \nu_{\alpha} & & \downarrow \nu_{\alpha} & & \downarrow \eta_{\alpha} \\
 Y^{\circ} & \hookrightarrow & f^{-1}((X, \Delta)_{\text{orb}}) & \xrightarrow{f} & (X, \Delta)_{\text{orb}}
 \end{array}$$

**Remark 9.4.9.** Let  $\mathcal{F}$  be an orbibundle defined on  $(X, \Delta)_{\text{orb}}$ . The adapted sheaf  $\mathcal{F}_{X, \Delta, f}$  can be constructed in the following way: we can pullback  $\mathcal{F}$  by  $f$  on this refined orbistructure, and taking the sheaf of invariants. It provides a sheaf on  $Y^{\circ}$  that coincides with  $\mathcal{F}_{X, \Delta, f}$  on  $Y^{\circ}$ . This is a convenient orbistructure, because we deal with genuine orbibundles above orbifolds.

We can compute orbifold Chern classes of  $\mathcal{F}_{X, \Delta, f}$ . Let us first recall what are Chern classes for klt pairs.

**Definition 9.4.10.** Let  $(X, \Delta)$  be a klt pair, and  $\mathcal{F}$  an orbibundle on an open subset  $U \subseteq (X, \Delta)_{\text{orb}}$ , whose complement is included in an analytic subset of codimension at least 3.

Then, for  $i = 1, 2$ , we have an isomorphism  $\iota: H_{dR, \text{orb}, c}^{2n-2i}(U, \mathbb{C}) \simeq H^{2n-2i}(X, \mathbb{C})$  (see [GK20, Section. 5]) that provides a linear form  $\widehat{c}_i(\mathcal{F}) \in H^{2n-2i}(X, \mathbb{C})^{\vee}$  defined by:

$$\widehat{c}_i(\mathcal{F}) \cdot \kappa := \int_U \Omega \wedge A,$$

where  $\kappa \in H^{2n-2i}(X, \mathbb{C})$ ,  $A$  is an orbifold  $(2n-2i)$ -form with compact support in  $U$  which represents the class  $\iota^{-1}(\kappa)$ , and  $\Omega$  is an orbifold  $(2i)$ -form representing the orbifold Chern class  $c_i^{\text{orb}}(\mathcal{F})$ .<sup>2</sup>

The intersection  $c_1^{\text{orb}}(\mathcal{F})^2$  yields similarly a class  $\widehat{c}_1^2(\mathcal{F}) \in H^{2n-4}(X, \mathbb{C})^{\vee}$ . We denote  $c_i(X, \Delta) = \widehat{c}_i(\mathcal{T}_{(X, \Delta)_{\text{orb}}})$ .

**Remark 9.4.11.** In the case where  $\Delta = 0$ , and  $\mathcal{G}$  is a reflexive sheaf on  $X$ , one can show that there exists an open subset  $X^0 \subseteq X_{\text{orb}}$  and an orbistructure orbi-étale on  $X^0$  such that  $\mathcal{G}^{[\text{orb}]}$  is an orbibundle. In this case, Proposition [GT22, Prop. 3.5] states that Chern classes are independent of the smooth orbi-étale orbistructure on  $X^0$ . Hence, we can define  $\widehat{c}_i(\mathcal{G}) = \widehat{c}_i(\mathcal{G}^{[\text{orb}]})$ .

---

2. The definition is independent of the choices of  $\Omega$  and  $A$  thanks to Stokes formula for orbifolds.

As  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier, we can compute the class:

$$c_1(-(K_X + \Delta)) := \frac{1}{m}c_1(-m(K_X + \Delta)) \in H^2(X, \mathbb{C}),$$

for  $m \gg 1$ . It is independent of  $m$ . This class yields by intersection pairing an element of  $H^{2n-2}(X, \mathbb{C})^\vee$  which coincides with  $c_1(X, \Delta)$ . This definition allows us to consider  $c_1(X, \Delta)^2 \in H^{2n-4}(X, \mathbb{C})^\vee$ , as the intersection with  $c_1(-(K_X + \Delta))^2$ .

**Proposition 9.4.12.** *Let  $(X, \Delta)$  be a klt pair and let  $f : Y \rightarrow X$  be a strictly  $\Delta$ -adapted morphism such that  $(Y, 0)$  is klt. Let  $\mathcal{F}$  be an orbifold over  $(X, \Delta)_{\text{orb}}$ . We have:*

$$\widehat{c}_1(\mathcal{F}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) \widehat{c}_1(\mathcal{F}) \cdot \kappa,$$

$$\widehat{c}_1^2(\mathcal{F}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) \widehat{c}_1^2(\mathcal{F}) \cdot \kappa,$$

$$\widehat{c}_2(\mathcal{F}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) \widehat{c}_2(\mathcal{F}) \cdot \kappa,$$

*Proof.* First, we observe that adapted sheaves satisfy  $\mathcal{F}_{X,\Delta,f}^{[\text{orb}]} \simeq f^* \mathcal{F}$ , so that they are orbifolds on the smooth orbi-étale orbifold structure on  $(Y^\circ, 0)$ . We can hence compute first and second Chern classes. We show the equality for  $\widehat{c}_2$ , the other ones are obtained similarly.

Let  $h$  be a hermitian metric on  $\mathcal{F}$ . There is a pullback orbifold hermitian metric  $f^* h = (f_{u(\alpha')\alpha'}^* h_{u(\alpha')})$  on  $f^* \mathcal{F} \simeq \mathcal{F}_{X,\Delta,f}^{[\text{orb}]}$ . Let  $A$  be a  $(2n - 4)$ -orbifold form which represents  $\kappa$  with compact support in  $f(Y^\circ)$ . The previous constructions show that the intersection  $\widehat{c}_2(\mathcal{F}_{X,\Delta,f}) \cdot f^* \kappa$  is well-defined on  $Y^\circ$ . In other terms, the integral  $\int_{Y^\circ} c_2^{\text{orb}}(\mathcal{F}_{X,\Delta,f}^{[\text{orb}]}, f^* h) \wedge f^* A$  is finite. By removing subsets of measure zero, we get:

$$\begin{aligned} \widehat{c}_2(\mathcal{F}_{X,\Delta,f}) \cdot f^* \kappa &= \int_{Y^{**}} c_2^{\text{orb}}(\mathcal{F}_{X,\Delta,f}, f^* h)|_{Y^{**}} \wedge f^* A \\ &= \int_{Y^{**}} c_2^{\text{orb}}(f^* \mathcal{F}, f^* h)|_{Y^{**}} \wedge f^* A \\ &= (\deg f) \int_{X^{**}} c_2^{\text{orb}}(\mathcal{F}, h)|_{X^{**}} \wedge A \\ &= (\deg f) \widehat{c}_2(\mathcal{F}) \cdot \kappa. \end{aligned}$$

Here  $X^{**} := X_{\text{reg}} \setminus \text{Br}(f)$ , where  $\text{Br}(f)$  is the branching locus of  $f$ , and  $Y^{**} = f^{-1}(X^{**})$ . We used the fact the morphism  $f|_{Y^{**}}$  is étale, finite of degree  $\deg f$ .  $\square$

A consequence of this computation is:

$$\widehat{c}_1(\mathcal{E}_{X,\Delta,f}) \cdot f^* \kappa = \widehat{c}_1(\mathcal{T}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) c_1(X, \Delta) \cdot \kappa,$$

$$\widehat{c}_1^2(\mathcal{E}_{X,\Delta,f}) \cdot f^* \kappa = \widehat{c}_1^2(\mathcal{T}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) c_1^2(X, \Delta) \cdot \kappa,$$

$$\widehat{c}_2(\mathcal{E}_{X,\Delta,f}) \cdot f^* \kappa = \widehat{c}_2(\mathcal{T}_{X,\Delta,f}) \cdot f^* \kappa = (\deg f) c_2(X, \Delta) \cdot \kappa.$$

# UNIFORMIZATION OF LOG FANO PAIRS

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In this chapter, we prove Theorem 0.0.20. Let  $(X, \Delta)$  be a log Fano pair with standard coefficients and we denote the following assertions:

(Quo): *There exists a finite subgroup  $G$  of  $\mathrm{PGL}(n+1, \mathbb{C})$  such that  $(X, \Delta) \simeq (\mathbb{P}^n/G, \Delta_G)$ ,*

(Unif): *The pair  $(X, \Delta)$  satisfies assumptions (SCE+) and (MY).*

Here, the assumptions (SCE+) and (MY) are:

(SCE+) : There exists a strictly  $\Delta$ -adapted morphism  $f_0 : Y_0 \rightarrow X$  such that the adapted canonical extension  $\mathcal{E}_{X, \Delta, f_0}$  is semistable with respect to  $f_0^*c_1(X, \Delta)$ .

(MY) : The equality case in the Miyaoka–Yau inequality holds:

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

## 10.1 (Quo) $\implies$ (Unif)

We first show the implication : (Quo)  $\implies$  (Unif).

Let  $G$  be a finite subgroup of  $\mathrm{PGL}(n+1, \mathbb{C})$  such that  $(X, \Delta)$  is isomorphic to the orbifold  $\mathcal{P}_G := (\mathbb{P}^n/G, \Delta_G)$ . We get that the quotient map  $\pi_G : \mathbb{P}^n \rightarrow \mathbb{P}^n/G$  is orbi-étale, and we have:

$$\pi_G^* \Omega_{\mathcal{P}_G}^1 = \Omega_{\mathbb{P}^n}^1 \quad \text{and} \quad \pi_G^* \mathcal{E}_{\mathcal{P}_G} = \mathcal{E}_{\mathbb{P}^n},$$

as a consequence of Proposition 9.4.2, and an application of Lemmas 9.2.2 and 9.2.3. In particular, we have  $\mathcal{E}_{\mathbb{P}^n/G, \Delta_G, \pi_G} = \mathcal{E}_{\mathbb{P}^n}$  which is semistable with respect to  $\pi_G^*c_1(\mathcal{P}_G) = c_1(\mathbb{P}^n)$  (see [Tia92, Thm. 0.1], with the fact that the Fubini–Study metric  $\omega_{FS}$  is Kähler–Einstein). Hence, we get that assumption (SCE+) is satisfied.

Moreover, we have:

$$\begin{aligned} & (2(n+1)c_2^{\mathrm{orb}}(X, \Delta) - nc_1^{\mathrm{orb}}(X, \Delta)^2) \cdot c_1^{\mathrm{orb}}(X, \Delta)^{n-2} \\ &= \frac{1}{|G|} \left(2(n+1)c_2^{\mathrm{orb}}(\mathbb{P}^n) - nc_1^{\mathrm{orb}}(\mathbb{P}^n)^2\right) \cdot c_1^{\mathrm{orb}}(\mathbb{P}^n)^{n-2} = 0. \end{aligned}$$

Hence, the pair satisfies Miyaoka–Yau equality (MY).  $\square$

**Remark 10.1.1.** The group  $G$  can be recovered from  $(X, \Delta)$ , as  $G = \pi_1^{\text{orb}}(X, \Delta)$ . If  $\Delta_G = 0$ , it is a consequence of [GKP22, Lemma. 7.8]. If  $\Delta_G \neq 0$ , one can show this isomorphism by arguing in the same way as Example 6.1.3.

The sequel is devoted to the proof of the implication (Unif)  $\implies$  (Quo).

## 10.2 (Unif) $\implies$ (Quo) : orbifold case

### 10.2.1 When the canonical extension is polystable

First, we define the different notions of stability for orbibundles, and metric notion of Hermite–Einstein.

**Definition 10.2.1.** Let  $(X, \Delta)$  be an orbifold. A Kähler metric on  $(X, \Delta)$  is an orbifold  $(1, 1)$ -form such that on each uniformizing chart  $(U_\alpha, V_\alpha, \eta_\alpha)$ , the form  $\omega_\alpha$  is a Kähler form. A class  $\alpha \in H_{\text{dR}, \text{orb}}^2(X, \mathbb{R})$  is said Kähler if there exists a Kähler form  $\omega$  such that  $\omega \in \alpha$ .

**Definition 10.2.2.** Let  $(X, \Delta)$  be a Kähler orbifold. Let  $\mathcal{F}$  be an orbibundle on  $(X, \Delta)$  and let  $\alpha \in H_{\text{dR}, \text{orb}}^2(X, \mathbb{R})$  be a Kähler class. The  $\alpha$ -slope of  $\mathcal{F}$  is the quantity:

$$\mu_\alpha(\mathcal{F}) := \frac{\int_X c_1^{\text{orb}}(\mathcal{F}, h) \wedge \omega^{n-1}}{\text{rk } \mathcal{F}},$$

where  $h$  is any hermitian metric on  $\mathcal{F}$ , and  $\omega$  is any orbifold 2-form representing  $\alpha$ .

**Remark 10.2.3.** This definition of  $\mu_\alpha(\mathcal{F})$  can be extended when  $\mathcal{F}$  is a torsion-free orbisheaf. For instance, one can consider the orbibundle  $\det(\mathcal{F})$  which consists of locally free sheaves  $\det(\mathcal{F}_\alpha) := (\wedge^r \mathcal{F}_\alpha)^{\vee\vee}$  where  $r$  is the rank of  $\mathcal{F}_\alpha$ . The transition maps are  $(\wedge^r \varphi_{\alpha\beta})^{\vee\vee}$ . Hence, we denote  $c_1^{\text{orb}}(\mathcal{F}) := c_1^{\text{orb}}(\det(\mathcal{F}))$ .

**Definition 10.2.4.** Let  $(X, \Delta)$  be an orbifold. Let  $\mathcal{F}$  be a torsion-free orbisheaf, and let  $\alpha \in H_{\text{dR}, \text{orb}}^2(X, \mathbb{R})$  be a Kähler class. We say that  $\mathcal{F}$  is:

- $\alpha$ -stable if for every proper sub-orbisheaf  $\mathcal{G}$ , we have  $\mu_\alpha(\mathcal{G}) < \mu_\alpha(\mathcal{F})$ ,
- $\alpha$ -semistable if for every proper sub-orbisheaf  $\mathcal{G}$ , we have  $\mu_\alpha(\mathcal{G}) \leq \mu_\alpha(\mathcal{F})$ ,
- $\alpha$ -polystable if  $\mathcal{F} = \bigoplus \mathcal{F}_i$  where  $\mathcal{F}_i$  are  $\alpha$ -stable and have same  $\alpha$ -slopes.

**Definition 10.2.5.** Let  $(X, \Delta, \omega)$  be a Kähler orbifold. Let  $\mathcal{F}$  be an orbibundle. A hermitian metric  $h$  on  $\mathcal{F}$  is said Hermite–Einstein if there exists a constant  $\lambda \in \mathbb{R}$  such that the following equality of orbifold  $(n, n)$ -forms with value in  $\text{End}(\mathcal{F})$  holds:

$$\sqrt{-1}\Theta(\mathcal{F}, h) \wedge \omega^{n-1} = \lambda \text{id}_{\mathcal{F}} \omega^n.$$

**Remark 10.2.6.** An orbifold version of Donaldson–Uhlenbeck–Yau theorem holds in this setting due to [ES18, Theorem 4.2].

**Proposition 10.2.7.** Let  $(X, \Delta)$  be an orbifold,  $\mathcal{F}$  an orbibundle of rank  $r$  and let  $\alpha \in H_{\text{dR}}^2(X, \mathbb{C})$  be a Kähler class. We assume that  $\mathcal{F}$  is polystable with respect to  $\alpha$ .

Then, we have the following inequality:

$$\left(2rc_2^{\text{orb}}(\mathcal{F}) - (r-1)c_1^{\text{orb}}(\mathcal{F})^2\right) \cdot \alpha^{n-2} \geq 0.$$

Moreover, if equality holds, there exists a metric  $h$  on  $\mathcal{F}$  such that  $(\mathcal{F}, h)$  is projectively flat.

*Proof.* Let us pick a Kähler form  $\omega$  representing  $\alpha$ . The orbibundle  $\mathcal{F}$  is polystable with respect to  $\alpha$ . By Remark 10.2.6, there is an orbifold Hermite–Einstein metric  $h$  on  $\mathcal{F}$  with respect to  $\omega$ . On  $X^*$ , we have a pointwise equality as explained in the proof of [Kob87, Thm. 4.4.7]:

$$(2rc_2(\mathcal{F}^*, h^*) - (r-1)c_1(\mathcal{F}^*, h^*)^2) \cdot \omega^{n-2} = C|\mathring{\Theta}(\mathcal{F}^*, h^*)|_{\omega}^2 \omega^n,$$

where  $\mathring{\Theta}(\mathcal{F}^*, h^*)$  is the traceless curvature tensor, and  $C > 0$  is a quantity depending only on the dimension of  $X$ . By integrating over  $X^*$ , we get the inequality.

If the equality holds, we get that the tensor  $\mathring{\Theta}(\mathcal{F}^*, h^*)$  is identically zero, and  $\Theta(\mathcal{F}^*, h^*) = \frac{1}{r}\theta \otimes \text{id}_{\mathcal{F}^*}$  where  $\theta$  is a 2-form. On local uniformizations, the curvature tensor extends to  $\Theta(\mathcal{F}_\alpha, h_\alpha)$  which is a diagonal matrix of 2-forms. Hence, the form  $\theta$  extends to an orbifold 2-form such that  $\Theta(\mathcal{F}, h) = \frac{1}{r}\theta \otimes \text{id}_{\mathcal{F}}$ . It follows that  $\mathcal{F}$  is projectively flat.  $\square$

We get the following proposition as an application of previous ones:

**Lemma 10.2.8.** Let  $(X, \Delta)$  be an orbifold. Assume that  $\mathcal{E}_{(X, \Delta)}$  is polystable with respect to a Kähler class  $\alpha \in H_{\text{dR}}^2(X, \mathbb{C})$ , and that  $(X, \Delta)$  satisfies the Miyaoka–Yau equality:

$$\left(2(n+1)c_2^{\text{orb}}(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot \alpha^{n-2} = 0.$$

Then,  $\tilde{\Delta} = 0$  and  $\tilde{X}_\Delta$  smooth.

*Proof.* As a consequence of Proposition 10.2.7, the orbibundle  $\mathcal{E}_{(X,\Delta)}$  is projectively flat. It suffices to adapt the local computation of [Kob87, Prop. 1.2.9] to conclude that the orbibundle  $\text{End}(\mathcal{E}_{(X,\Delta)})$  is flat. By Corollary 7.2.11, we conclude that its universal cover  $(\tilde{X}_\Delta, \tilde{\Delta})$  satisfies  $\tilde{\Delta} = 0$  and  $\tilde{X}_\Delta$  smooth.  $\square$

**Theorem 10.2.9.** *Let  $(X, \Delta)$  be a Fano orbifold such that:*

(PCE) *The canonical extension  $\mathcal{E}_{(X,\Delta)}$  is polystable with respect to  $c_1(X, \Delta)$ ,*

(MY) *The equality case in the Miyaoka–Yau inequality holds:*

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

*Then, the universal cover satisfies  $(\tilde{X}_\Delta, \tilde{\Delta}) = (\mathbb{P}^n, 0)$ .*

*Proof.* We observe by Theorem 6.1.5, that the morphism  $\pi : \tilde{X}_\Delta \rightarrow X$  is finite. By applying Lemma 10.2.8 to  $\alpha = c_1(X, \Delta)$ , we have that  $\tilde{\Delta} = 0$  and  $\tilde{X}_\Delta$  is smooth. Hence, we get that  $\pi$  is an orbi-étale map, and  $\pi^*\text{End}(\mathcal{E}_{(X,\Delta)}) = \text{End}(\mathcal{E}_{\tilde{X}_\Delta})$  by Lemma 9.2.3. Hence,  $\text{End}(\mathcal{E}_{\tilde{X}_\Delta})$  is flat on  $\tilde{X}_\Delta$ . We conclude by [GKP22, Prop. 3.7] that  $\mathcal{E}_{\tilde{X}_\Delta}$  is projectively flat on  $\tilde{X}_\Delta$  that is a Fano manifold<sup>1</sup>. We conclude that there exists a line bundle  $\mathcal{L}$  such that  $\mathcal{E}_{\tilde{X}_\Delta} = \mathcal{L}^{\oplus n+1}$ . Finally, we have:

$$c_1(\tilde{X}_\Delta) = c_1(\mathcal{E}_{\tilde{X}_\Delta}) = (n+1)c_1(L),$$

and we conclude by the characterization of projective spaces [KO73, Cor. of Thm. 1.1].  $\square$

## 10.2.2 When the canonical extension is semistable

We show the following theorem.

**Theorem 10.2.10.** *Let  $(X, \Delta)$  be a log Fano orbifold such that:*

(SCE+) *There exists a strictly  $\Delta$ -adapted morphism  $f_0 : Y_0 \rightarrow X$  such that the adapted canonical extension  $\mathcal{E}_{X,\Delta,f_0}$  is semistable with respect to  $f_0^*c_1(X, \Delta)$ ,*

(MY) *The equality case in the Miyaoka–Yau inequality holds:*

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

---

1. It is the case, because we have  $-mK_{\tilde{X}_\Delta} = -\pi^*m(K_X + \Delta)$ . Indeed, we have that for  $m \gg 1$ , the  $\mathbb{Q}$ -divisor  $-m(K_X + \Delta)$  is Cartier, ample. Since  $\pi$  is a finite morphism, we conclude by using finiteness of  $\pi$  and [Laz04, Prop. 1.2.13].

Then, the universal cover satisfies  $(\widetilde{X}_\Delta, \widetilde{\Delta}) = (\mathbb{P}^n, 0)$ .

**Remark 10.2.11.** The assertions (SCE+) implies the semistability of the canonical extension  $\mathcal{E}_{(X,\Delta)}$  in the sense of Definition 10.2.4. Indeed, if  $\mathcal{F}$  is a proper sub-orbisheaf of  $\mathcal{E}_{(X,\Delta)}$ , hence it yields a proper reflexive sheaf  $\mathcal{F}_{X,\Delta,f_0}$  such that there is an inclusion  $\mathcal{F}_{X,\Delta,f_0|U} \subseteq \mathcal{E}_{X,\Delta,f_0|U}$ , where  $U$  is an open subset whose complement is included in an analytic subset of codimension at least 2.<sup>2</sup> Then, we can compute the  $f_0^*c_1(X, \Delta)$ -slopes of these two torsion-free sheaves thanks to Proposition 9.4.12:<sup>3</sup>

$$\begin{aligned} \mu_{f_0^*c_1(X,\Delta)}(\mathcal{F}_{X,\Delta,f_0}) &= \frac{c_1(\mathcal{F}_{X,\Delta,f_0}) \cdot f_0^*c_1(X, \Delta)^{n-1}}{\text{rk } \mathcal{F}_{X,\Delta,f_0}} \\ &= \text{deg}(f_0) \frac{c_1(\mathcal{F}) \cdot c_1(X, \Delta)^{n-1}}{\text{rk } \mathcal{F}} \\ &= \text{deg}(f_0) \mu_{c_1(X,\Delta)}(\mathcal{F}). \end{aligned}$$

The same computation holds for  $\mathcal{E}_{X,\Delta,f_0}$ .

Hence, we get by (SCE+) that  $\mu_{f_0^*c_1(X,\Delta)}(\mathcal{F}_{X,\Delta,f_0}) \leq \mu_{f_0^*c_1(X,\Delta)}(\mathcal{E}_{X,\Delta,f_0})$  that implies that  $\mu_{c_1(X,\Delta)}(\mathcal{F}) \leq \mu_{c_1(X,\Delta)}(\mathcal{E}_{(X,\Delta)})$ .

*Proof.* By Theorem 6.1.5, the log Fano pair  $(X, \Delta)$  has a finite orbifold fundamental group. So, the universal covering map  $\pi: (\widetilde{X}_\Delta, \widetilde{\Delta}) \longrightarrow (X, \Delta)$  is finite.

Let us denote  $\alpha := -(K_X + \Delta)$  and we consider the orbibundle  $\mathcal{F} := \text{End}(\mathcal{E}_{(X,\Delta)})$ . Let  $f: Y \longrightarrow X$  be a morphism constructed in Lemma 9.1.5. In particular, we assume by Remark 9.1.6 that  $Y$  is maximally quasi-étale.

One can observe that the corresponding reflexive sheaf  $\mathcal{F}_{X,\Delta,f}$  constructed in Definition 9.4.3 is  $\text{End}(\mathcal{E}_{X,\Delta,f})$  by Lemma 9.4.4. Now, the assumption (SCE+) implies that  $\mathcal{E}_{X,\Delta,f}$  is semistable with respect to  $f^*\alpha$  by Proposition 9.4.8. As a consequence of [GKP16a, Prop. 4.4], we get that  $\mathcal{F}_{X,\Delta,f}$  is semistable with respect to  $f^*\alpha$ . Moreover, it satisfies by

2. Since  $f^{[*]}\mathcal{F}$  is a reflexive weak orbisheaf on the pullback orbistructure above  $Y$ , we can define  $\mathcal{F}_{X,\Delta,f} := (f^{[*]}\mathcal{F})^{\text{inv}}$ . Hence, because the morphism  $f$  is flat in codimension 1, so are the morphisms  $f_\alpha$  defined in Example 9.3.3 since flatness is preserved under base change. Hence, we have the inclusion of orbisheaves  $f^*\mathcal{F} \hookrightarrow f^*\mathcal{E}$  above  $f^{-1}(U)$ , and exactness of functor  $(\cdot)^{\text{inv}}$  gives the sought inclusion.

3. For the second inequality, we need to observe that we have a natural isomorphism  $(\det \mathcal{F})_{X,\Delta,f} \simeq \det \mathcal{F}_{X,\Delta,f}$ . One can show that there is a natural isomorphism  $(\bigwedge^r f^{[*]}\mathcal{F})^{\vee\vee} \simeq f^*((\bigwedge^r \mathcal{F})^{\vee\vee})$ , where  $\mathcal{F}$  is a torsion-free orbisheaf of rank  $r$ . Indeed, such an isomorphism holds on an open subset where  $\mathcal{F}$  is locally free, and it extends because both previous orbisheaves are reflexive. This argument allows us to show that for a reflexive sheaf  $\mathcal{A}$  of rank  $r$  on  $Y$ , we have  $(\bigwedge^r \mathcal{A}^{\text{orb}})^{\vee\vee} \simeq ((\bigwedge^r \mathcal{A})^{\vee\vee})^{\text{orb}}$ . We obtain the result by arguing as Lemma 9.4.4.

the assumption (MY):

$$\widehat{\text{ch}}_1(\mathcal{F}_{X,\Delta,f}) \cdot f^* \alpha^{n-1} = \widehat{\text{ch}}_2(\mathcal{F}_{X,\Delta,f}) \cdot f^* \alpha^{n-2} = 0.$$

We deduce from the non-abelian Hodge correspondence for klt projective spaces that  $\mathcal{F}_{X,\Delta,f}$  can be endowed with a flat holomorphic connection  $\nabla$  on  $Y_{\text{reg}}$ , that is  $\text{Gal}(f)$ -equivariant by functoriality of the correspondence (see Theorems 3.4 and Corollary 3.8 of [Gre+19a]). Hence, with notation of § 9.4.3, it yields an orbiconnection  $(\nu_\alpha^* \nabla)$  on the restriction of the pullback orbistructure to  $Y_{\text{reg}}$  (namely, the connections  $\nu_\alpha^* \nabla$  are defined above  $S_{\text{areg}}$  on the bundles  $\nu_\alpha^* \mathcal{F}_{X,\Delta,f} \simeq f_\alpha^* \mathcal{F}_\alpha$ ).

By descent theorems for connections [Gre+19b, Prop. 2.7], the orbibundle  $\mathcal{F}_\alpha$  is endowed with a flat connection  $\nabla_\alpha$  on an open subset of  $V_\alpha$  whose complement has codimension at least 2. The collection  $(\nabla_\alpha)$  satisfies moreover compatibility conditions on  $V_{\alpha\beta}$ , as it suffices to consider the conditions on  $R_{\alpha\beta}$  between  $f_\alpha^* \nabla_\alpha$  and  $f_\beta^* \nabla_\beta$ , and to take the functor of  $G$ -invariant sections, in order to descend them on a big open subset of  $V_{\alpha\beta}$ . Because removing an analytic subset of codimension 2 in a smooth variety does not change the fundamental group and the flat bundles are in correspondence with representations of the fundamental group, we obtain that the connection  $\nabla_\alpha$  extends through the whole of  $V_\alpha$ , and the compatibility conditions too. Finally, the orbisheaf  $\mathcal{F}$  is endowed with a flat orbiconnection. Hence, by Corollary 7.2.11, we get that  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  is smooth.

To sum up, the pair  $(X, \Delta)$  is a Fano orbifold such that  $\text{End}(\mathcal{E}_{(X,\Delta)})$  is flat, and we have  $\widetilde{\Delta} = 0$  and  $\widetilde{X}_\Delta$  smooth. We conclude that  $\widetilde{X}_\Delta = \mathbb{P}^n$  by the same arguments appearing in the proof of Theorem 10.2.9.  $\square$

### 10.3 (Unif) $\implies$ (Quo) : Klt case

In this section we prove the following theorem.

**Theorem 10.3.1.** *Let  $(X, \Delta)$  be a klt log Fano pair such that:*

(SCE+) *There exists a strictly  $\Delta$ -adapted morphism  $f_0 : Y_0 \rightarrow X$  such that the adapted canonical extension  $\mathcal{E}_{X,\Delta,f_0}$  is semistable with respect to  $f_0^* c_1(X, \Delta)$ ,*

(MY) *The equality case in the Miyaoka–Yau inequality holds:*

$$(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2) \cdot c_1(X, \Delta)^{n-2} = 0.$$

Then, the universal cover satisfies  $(\widetilde{X}_\Delta, \widetilde{\Delta}) = (\mathbb{P}^n, 0)$ .

It suffices to show that  $(X, \Delta)$  can be endowed with a smooth orbi-étale orbistructure, and we conclude by the theorem proved in the previous section. In order to construct such an orbistructure, we use Proposition 9.1.5. It states that there exists a very ample line bundle  $L$  which produces many orbi-étale morphisms above an open set of  $X$ . Let  $x \in X$ , we pick a divisor  $H \in |L|$  such that  $x \notin H$  (it is possible since  $L$  is basepoint-free). A consequence of this proposition is the existence of a strictly  $\Delta$ -adapted morphism  $f: Y \rightarrow X$  that is orbi-étale for the pair  $(X, \Delta + (1 - \frac{1}{N})H)$  for some integer  $N \gg 1$ . We show that  $Y \setminus f^{-1}(H)$  is smooth.

By Proposition 9.4.12, and assumption (MY), we have:

$$\begin{aligned} & \left(2(n+1)\widehat{c}_2(\mathcal{E}_{X,\Delta,f}) - n\widehat{c}_1(\mathcal{E}_{X,\Delta,f})^2\right) \cdot [f^*(-(K_X + \Delta))]^{n-2} \\ &= (\deg f)(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2) \cdot c_1(X, \Delta)^{n-2} \\ &= 0. \end{aligned}$$

The morphism  $f$  being strictly  $\Delta$ -adapted, the extension  $\mathcal{E}_{X,\Delta,f}$  is semistable with respect to  $f^*(-(K_X + \Delta))$  as a consequence of Proposition 9.4.8. Its rank is  $n+1$  and it satisfies the Bogomolov–Gieseker equality. By [GKP22, Prop. 1.6], the extension  $\mathcal{E}_{X,\Delta,f}|_{Y_{\text{reg}}}$  is locally free and projectively flat. So,  $\text{End}(\mathcal{E}_{X,\Delta,f})$  is locally free and flat [GKP22, Cor. 3.6]. Now, the extension  $\mathcal{E}_{X,\Delta,f}$  splits locally, we have on sufficiently small open subsets  $U \subseteq Y \setminus f^{-1}(H)$ :

$$\text{End}(\mathcal{E}_{X,\Delta,f})|_U = \mathcal{O}_U \oplus \mathcal{T}_{X,\Delta,f}|_U \oplus \Omega_{X,\Delta,f|U}^{[1]} \oplus \text{End}(\mathcal{T}_{X,\Delta,f})|_U$$

and  $\mathcal{T}_{X,\Delta,f}$  is locally free as a direct summand of a locally free sheaf. On  $Y \setminus f^{-1}(H)$ , there is an isomorphism:  $\mathcal{T}_{X,\Delta,f} \simeq \mathcal{T}_Y$ . By [Dru14, Thm 1.1], the space  $Y \setminus f^{-1}(H)$  is smooth. Moreover, the morphism  $f: Y \setminus f^{-1}(H) \rightarrow X \setminus H$  is orbi-étale, providing a suitable l.u.c. around  $x$  to make a smooth orbi-étale orbistructure on  $(X, \Delta)$ .

We conclude that  $(\widetilde{X}_\Delta, \widetilde{\Delta}) = (\mathbb{P}^n, 0)$  by Theorem 10.2.10.  $\square$

# EXAMPLES

## 11.1 One-dimensional orbifold uniformization

In dimension 1, we have an orbifold version of the Poincaré–Koebe theorem.

**Theorem 11.1.1.** *Let  $(C, \Delta)$  be an orbifold curve. Then the orbifold universal cover  $(\widetilde{C}_\Delta, \widetilde{\Delta})$  of  $(C, \Delta)$  is:*

- $(\mathbb{P}^1, 0)$ ,  $(\mathbb{P}^1, (1 - \frac{1}{m})\{pt\})$ ,  $(\mathbb{P}^1, (1 - \frac{1}{m_1})\{pt\} + (1 - \frac{1}{m_2})\{pt\})$  with  $\gcd(m_1, m_2) = 1$ , if  $\deg(K_C + \Delta) < 0$ ,
- $\mathbb{C}$  if  $\deg(K_C + \Delta) = 0$ ,
- $\mathbb{D}$  (the unit disc) if  $\deg(K_C + \Delta) > 0$ .

The results of Bundgaard–Nielsen [BN51] and Fox [Fox52] establish the existence of an orbi-étale cover of  $(C, \Delta)$  in the following cases:

- the genus of  $C$  is greater than or equal to 1,
- $C \simeq \mathbb{P}^1$  and the support of  $\Delta$  consists of at least three points,
- $C \simeq \mathbb{P}^1$  and there exists  $m \geq 2$  such that  $\Delta = (1 - \frac{1}{m})\{0\} + (1 - \frac{1}{m})\{\infty\}$ .

We now describe an orbi-étale morphism in a special case:

**Example 11.1.2.** Consider  $E := \mathbb{C}/\mathbb{Z}[j]$ . The subgroup  $\text{Aut}_0(E)$  of automorphisms fixing 0 is cyclic of order 6, generated by complex multiplication by  $e^{\frac{\sqrt{-1}\pi}{3}}$ . Consider multiplication by  $j := e^{\frac{2\sqrt{-1}\pi}{3}}$ , and the action of  $\langle j \rangle$  on  $E$ . This action has three fixed points.

Indeed, write  $x = [a + jb]$  with  $0 \leq a, b < 1$ . Then  $x = j \cdot x$  implies  $[a + jb] = [j(a + jb)] = [-b + j(a - b)]$ , so we get  $a + b, a - 2b \in \mathbb{Z}$ . In particular,  $3b \in \mathbb{Z}$ , so  $b \in \{0, \frac{1}{3}, \frac{2}{3}\}$ . From  $a + b \in \mathbb{Z}$  we deduce:

- if  $b = 0$ , then  $a = 0$ ,
- if  $b = \frac{1}{3}$ , then  $a = \frac{2}{3}$ ,
- if  $b = \frac{2}{3}$ , then  $a = \frac{1}{3}$ .

Thus the three fixed points are  $[0]$ ,  $[\frac{1}{3}(1 + 2j)]$  and  $[\frac{1}{3}(2 + j)]$ .

These three points are the ramification points of the quotient map  $\pi : E \rightarrow E/\langle j \rangle$ , and each has order 3. The quotient is a normal (hence smooth) curve  $C$ . By the Riemann–Hurwitz formula, we compute:

$$2 + 2 + 2 = 2g(E) - 2 + 3(2 - 2g(C)) \Rightarrow g(C) = 0.$$

Thus  $C \simeq \mathbb{P}^1$ , and the branch points have order 3. We have constructed an orbi-étale morphism  $E \rightarrow \mathbb{P}^1$  for the pair  $(\mathbb{P}^1, \frac{2}{3}\{0\} + \frac{2}{3}\{1\} + \frac{2}{3}\{\infty\})$ .

**Example 11.1.3.** Let  $X$  be a curve of genus  $g(X) \geq 2$ . Its automorphism group  $G$  is bounded:  $|G| \leq 84(g(X) - 1)$ . If equality holds,  $X$  is called a Hurwitz curve.

Let  $(C, \Delta)$  be an orbifold curve. Then there exists a Hurwitz curve  $X$  such that  $(X/G, \Delta_G) \simeq (C, \Delta)$  if and only if  $(C, \Delta) \simeq (\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{6}{7}\{\infty\})$ .

Indeed, consider  $(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{6}{7}\{\infty\})$ . By Bundgaard–Nielsen’s result, there exists an orbi-étale cover  $f : X \rightarrow C$ . The map  $f$  corresponds to the quotient by a finite group  $H$  of automorphisms. By the Riemann–Hurwitz formula:

$$2(g(X) - 1) = |H| \left( -2 + \frac{1}{2} + \frac{2}{3} + \frac{6}{7} \right) = \frac{-84 + 21 + 28 + 36}{42} = \frac{1}{42}.$$

So  $2(g(X) - 1) > 0$ , hence  $g(X) \geq 2$ , and  $|H| = 84(g(X) - 1)$ , thus  $X$  is a Hurwitz curve, and  $f$  is the quotient by the automorphisms group  $G$ .

Conversely, if  $X$  is a Hurwitz curve, by Riemann–Hurwitz:

$$2(g(X) - 1) = 84(g(X) - 1) \left( 2g(C) - 2 + \sum_{i=1}^s \left( 1 - \frac{1}{m_i} \right) \right).$$

Define  $\chi(C, \Delta) := 2g(C) - 2 + \sum_{i=1}^s \left( 1 - \frac{1}{m_i} \right)$ . Then  $\chi(C, \Delta) = \frac{1}{42}$ .

We must have  $g(C) = 0$ : if  $g(C) \geq 2$  or  $g(C) = 1$  with  $s \geq 1$ , then  $\chi(C, \Delta) \geq \frac{1}{2}$ . If  $g(C) = 1$  and  $s = 0$ , then  $\chi(C, \Delta) = 0$ .

Now, since  $1 - \frac{1}{m_i} \in [1/2, 1)$ , we have:

$$-2 + \frac{s}{2} \leq \chi(C, \Delta) < -2 + s,$$

which implies  $s = 3$  or  $s = 4$ .

If  $s = 4$ , let  $m_1 \leq m_2 \leq m_3 \leq m_4$  be the multiplicities. Then  $m_3 = 2$ , because if

$m_3 \geq 3$ , then:

$$\chi(C, \Delta) \geq -2 + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} = \frac{1}{3}.$$

So  $m_1 = m_2 = m_3 = 2$ , and  $m_4 \geq 3$  implies  $\chi(C, \Delta) \geq \frac{1}{6}$ , which is too big. But  $m_4 = 2$  gives  $\chi(C, \Delta) = 0$  — contradiction.

So  $s = 3$ , with  $m_1 \leq m_2 \leq m_3$ . First,  $m_1 = 2$ : if  $m_1 \geq 3$ , then for  $m_2 \geq 4$  and  $m_3 \geq 4$ , we have  $\chi(C, \Delta) \geq \frac{1}{6}$ . Hence  $m_2 = 3$ , and  $m_3 = 3$  gives  $\chi = 0$  — contradiction.

Similarly,  $m_2 = 3$ . If  $m_2 \geq 4$  and  $m_3 \geq 5$ , then  $\chi \geq \frac{1}{20}$ . If  $m_3 = 4$ , then  $\chi = 0$ . Finally, we find:

$$\chi(C, \Delta) = \frac{1}{42} \iff m_1 = 2, \quad m_2 = 3, \quad m_3 = 7.$$

## 11.2 Semistability of the canonical extension

Let  $(X, \Delta)$  be an orbifold curve such that  $-(K_X + \Delta)$  is ample. On curves, the Miyaoka–Yau inequality does not make sense, but we can still study the semistability of the canonical extension  $\mathcal{E}_{(X, \Delta)}$  for curves that are not uniformized by  $\mathbb{P}^1$ . In the classification of Fano orbifold curves, the simply connected curves are:

$$(\mathbb{P}^1, 0), \quad \left( \mathbb{P}^1, \left(1 - \frac{1}{m}\right) \{0\} \right),$$

$$\left( \mathbb{P}^1, \left(1 - \frac{1}{m_1}\right) \{0\} + \left(1 - \frac{1}{m_2}\right) \{\infty\} \right) \text{ with } \gcd(m_1, m_2) = 1.$$

**Example 11.2.1.** For instance, we take  $(X, \Delta) = \left( \mathbb{P}^1, \left(1 - \frac{1}{m}\right) \{0\} \right)$  with  $m$  even. Let us consider the covering

$$f : [z_0 : z_1] \in \mathbb{P}^1 \longrightarrow [z_0^m : z_1^m] \in \mathbb{P}^1.$$

This is a strictly adapted cover for the divisor  $\left(1 - \frac{1}{m}\right) \{0\}$ , but not orbi-étale.

The sheaf  $\mathcal{T}_{X, \Delta, f}$  is reflexive on a smooth curve, hence a line bundle on  $\mathbb{P}^1$ . Its degree is:

$$\deg(\mathcal{T}_{X, \Delta, f}) = \deg(f^*(-(K_X + \Delta))) = -m(-2 + 1 - \frac{1}{m}) = m + 1.$$

Thus, the adapted extension  $\mathcal{E}_{X, \Delta, f}$  is a rank-2 vector bundle on  $\mathbb{P}^1$ , an extension of  $\mathcal{O}_{\mathbb{P}^1}(m + 1)$  by  $\mathcal{O}_{\mathbb{P}^1}$ . Therefore, it is isomorphic to a direct sum  $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ , with determinant  $\mathcal{O}_{\mathbb{P}^1}(m + 1)$ , hence  $a_1 + a_2 = m + 1$ .

Now, suppose for contradiction that the bundle is semistable. Then  $a_1 = a_2 = \frac{m+1}{2}$ , which is not an integer since  $m$  is even — contradiction.

**Example 11.2.2.** For instance, we take now  $(X, \Delta) = (\mathbb{P}^1, \frac{1}{2}\{0\} + (1 - \frac{1}{m})\{\infty\})$  with  $m$  odd. The dihedral group  $D_{2m}$  of order  $2m$  acts faithfully on  $\mathbb{P}^1$ . The action is given by the representation

$$D_{2m} \longrightarrow \mathrm{GL}(2, \mathbb{C}) \longrightarrow \mathrm{PGL}(2, \mathbb{C}),$$

where the reflection and the rotation are respectively given by

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho = \begin{bmatrix} \cos \frac{2\pi}{m} & -\sin \frac{2\pi}{m} \\ \sin \frac{2\pi}{m} & \cos \frac{2\pi}{m} \end{bmatrix}.$$

Consider the quotient map  $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1/D_{2m} = \mathbb{P}^1$ . The fixed points of the action fall into three orbits:

- For  $0 \leq k \leq m-1$ , the point  $[\cos \frac{2k\pi}{m} : \sin \frac{2k\pi}{m}]$  is fixed by the reflection  $\rho^{2k}\sigma$ .
- The point  $[-\sin \frac{2k\pi}{m} : \cos \frac{2k\pi}{m}]$  is also fixed by the reflection  $\rho^{2k}\sigma$ .
- The points  $[1 : i]$  and  $[1 : -i]$  are stabilized by  $\rho$ , which has order  $m$ .

The resulting orbifold pair is

$$\left( \mathbb{P}^1, \frac{1}{2}\{0\} + \frac{1}{2}\{\lambda\} + \left(1 - \frac{1}{m}\right)\{\infty\} \right),$$

so the quotient map is strictly  $\Delta$ -adapted. The sheaf  $\mathcal{T}_{X,\Delta,f}$  is again reflexive and hence a line bundle on  $\mathbb{P}^1$ . Its degree is:

$$\deg(\mathcal{T}_{X,\Delta,f}) = \deg(f^*(-(K_X + \Delta))) = -2m(-2 + \frac{1}{2} + 1 - \frac{1}{m}) = m + 2.$$

Therefore, the adapted extension  $\mathcal{E}_{X,\Delta,f}$  is a rank-2 vector bundle over  $\mathbb{P}^1$ , an extension of  $\mathcal{O}_{\mathbb{P}^1}(m+2)$  by  $\mathcal{O}_{\mathbb{P}^1}$ , and is thus isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ . With determinant  $\mathcal{O}_{\mathbb{P}^1}(m+2)$ , we must have  $a_1 + a_2 = m+2$ . Assuming, for contradiction, that this bundle is semistable, we would have  $a_1 = a_2 = \frac{m+2}{2}$ , which is not an integer because  $m$  is odd — contradiction.

## 11.3 Quotients of projective spaces

### 11.3.1 Weighted projective spaces

Here, we study the examples of weighted projective spaces. Let  $\mathbf{a} = (a_0, \dots, a_n) \in \mathbb{Z}_{\geq 1}^{n+1}$  be  $n + 1$  positive integers. There is an action of  $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$  on  $\mathbb{C}^{n+1} \setminus \{0\}$  given by:

$$\lambda \cdot (x_i)_{0 \leq i \leq n} = (\lambda^{a_i} x_i)_{0 \leq i \leq n}$$

The weighted projective space  $\mathbb{P}(\mathbf{a})$  is the orbit space of this action. We denote by  $[x_i]_{\mathbf{a}}$  the class of  $(x_i)$  in  $\mathbb{P}(\mathbf{a})$ .

There is an order on  $\mathbb{Z}_{\geq 1}^{n+1}$  defined by:  $\mathbf{a} | \mathbf{b}$  iff  $\forall 0 \leq i \leq n, a_i | b_i$ . If  $\mathbf{a} | \mathbf{b}$ , there is a map:

$$\pi_{\mathbf{a} | \mathbf{b}} : [x_i]_{\mathbf{a}} \in \mathbb{P}(\mathbf{a}) \longmapsto \left[ x_i^{\frac{b_i}{a_i}} \right]_{\mathbf{b}} \in \mathbb{P}(\mathbf{b}),$$

which is the quotient map of the action of  $\prod_{0 \leq i \leq n} \mu_{b_i/a_i}$  where  $\mu_m$  is the group of  $m$ -th root of unity in  $\mathbb{C}$ . The action is given by the diagonal action:

$$(\zeta_i)_{0 \leq i \leq n} \cdot [x_i]_{\mathbf{a}} := [\zeta_i x_i]_{\mathbf{a}}.$$

The kernel of this action is  $\bigcap_{0 \leq i \leq n} \mu_{\frac{b_i}{a_i}} = \mu_{\gcd(\frac{b_i}{a_i})}$ . Hence, the action is effective if and only if  $\gcd\left(\frac{b_i}{a_i}\right) = 1$ .

If  $\mathbf{a} = (1, \dots, 1)$ , we have  $\mathbf{a} | \mathbf{b}$  for any  $\mathbf{b}$ , and  $\mathbb{P}(\mathbf{a}) = \mathbb{P}^n$ . It follows that the spaces  $\mathbb{P}(\mathbf{b})$  are quotient of  $\mathbb{P}^n$  by the diagonal action of roots of the unity. Hence, each  $\mathbb{P}(\mathbf{b})$  is obtained as a finite quotient of  $\mathbb{P}^n$ .

**Example 11.3.1.** If  $\mathbf{b} = (m, \dots, m)$ , we have  $\mathbb{P}(\mathbf{b}) = \mathbb{P}^n$ , and the map  $\pi_{\mathbf{a} | \mathbf{b}}$  is the quotient by the diagonal action of  $(\mu_m)^{n+1}$ . This action is not effective, and it can be reduced to the action of  $G := (\mu_m)^n$  on  $\mathbb{P}^n$  given by:

$$(\zeta_1, \dots, \zeta_n) \cdot [x_i]_{0 \leq i \leq n} = [x_0 : \zeta_1 x_1 : \dots : \zeta_n x_n],$$

and the quotient map is given by  $[x_i] \in \mathbb{P}^n \longmapsto [x_i^m] \in \mathbb{P}^n$ .

The set of fixed points is:

$$\text{Fix } G := \bigcup_{g \in G \setminus \{\text{id}\}} \{[x] \in \mathbb{P}^n \mid g \cdot [x] = [x]\} = \bigcup_{0 \leq k \leq n} (x_k = 0).$$

Hence, it follows that  $(\mathbb{P}^n/G, \Delta_G) = \left( \mathbb{P}^n, \sum_{0 \leq k \leq n} \left(1 - \frac{1}{m}\right) (x_k = 0) \right)$

**Example 11.3.2.** If  $\mathbf{b} = (1, 1, 2)$ , the group  $G := \mu_2$  acts on  $\mathbb{P}^2$  as follows:

$$(-1) \cdot [x_0 : x_1 : x_2] = [x_0 : x_1 : -x_2]$$

The set of fixed points under this action is  $\text{Fix}(-1) = (x_2 = 0) \cup \{[0 : 0 : 1]\}$ . One can show that the following morphism

$$[x_0 : x_1 : x_2] \in \mathbb{P}^2 \longmapsto [x_0^2 : x_0x_1 : x_1^2 : x_2^2] =: [u : v : w : t] \in \mathbb{P}^3$$

factors through  $G$ .

Moreover, it induces an isomorphism between  $\mathbb{P}^2/G$  and the cone  $(v^2 = uw)$  in  $\mathbb{P}^3$ . It is a singular variety with a conical singularity in  $[0 : 0 : 0 : 1]$ .

The branching divisor  $\Delta_G$  is the image of  $(x_2 = 0)$  by the morphism above, it corresponds to  $\Delta_1 := (v^2 = uw) \cap (t = 0)$ , which is a smooth conic in the projective plane  $(t = 0)$ . Finally, we get  $(\mathbb{P}^2/G, \Delta_G) = \left( (v^2 = uw), \frac{1}{2}\Delta_1 \right)$ .

### 11.3.2 Log smooth surfaces which are quotients of the projective plane

In this section we establish the following lemma.

**Lemma 11.3.3.** *Let  $(X, \Delta)$  be a log smooth surface which is a quotient of the form  $(\mathbb{P}^n/G, \Delta_G)$ . Then, we have  $X = \mathbb{P}^2$ , and there exists an integer  $m$  such that*

$$\Delta = \left(1 - \frac{1}{m}\right) (H_0 + H_1 + H_2)$$

where  $H_i$  are hyperplanes in general position. In particular,  $G = \mu_m \times \mu_m$ , where the action is that described in Example 11.3.1, conjugated by an automorphism of  $\mathbb{P}^2$ .

*Proof.* First, by [Bar+04, Example. 1.4], we have  $X = \mathbb{P}^2$ . Let us denote by

$$\Delta = \sum_{1 \leq i \leq k} \left(1 - \frac{1}{m_i}\right) \Delta_i$$

where  $\Delta_i$  is an irreducible curve of degree  $d_i$ , and  $m_i \geq 2$ . For simplicity, let us set  $\lambda_i := \left(1 - \frac{1}{m_i}\right)$ .

We compute the discriminant of Miyaoka–Yau  $\Delta_{MY} := 3c_2(X, \Delta) - c_1(X, \Delta)^2$ . Because  $(X, \Delta)$  is a quotient of  $\mathbb{P}^2$ , we have  $\Delta_{MY} = 0$ . We will show that this happens only for  $\Delta$  as in the statement.

The pair  $(X, \Delta)$  is log smooth, so by [GT22, Example 3.9], and using that  $c_1(\mathbb{P}^2) = -3[\omega_{FS}]$  and  $c_2(\Omega_{\mathbb{P}^2}^1) = 3[\omega_{FS}]^2$ :

$$\begin{aligned} c_1(X, \Delta)^2 &= \left(-3 + \sum_{1 \leq i \leq k} \lambda_i d_i\right)^2, \\ c_2(X, \Delta) &= 3 - 3 \sum_i \lambda_i d_i + \sum_i \lambda_i d_i^2 + \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j d_i d_j. \end{aligned}$$

We get:

$$\begin{aligned} \Delta_{MY} &= -3 \sum_i \lambda_i d_i + 3 \sum_i \lambda_i d_i^2 - \sum_{1 \leq i \leq k} \lambda_i^2 d_i^2 + \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j d_i d_j \\ &= \sum_i \lambda_i d_i (-3 + 3d_i - \lambda_i d_i) + \sum_{1 \leq i < j \leq k} \lambda_i \lambda_j d_i d_j. \end{aligned}$$

First, we can show that at least one of the  $d_i$ 's equals one. If not, we have the following inequality:  $-3 + 3d_i - \lambda_i d_i \geq -3 + 3d_i - d_i = 2d_i - 3 > 0$ , and it follows that  $\Delta_{MY} > 0$  that contradicts  $\Delta_{MY} = 0$ .

In fact, each  $d_i$  equals one. If not, let us assume that  $d_1 = \dots = d_t = 1 < d_{t+1} \leq \dots \leq d_k$ . We have by the lower bound  $d_i \geq 1$ :

$$\begin{aligned} \Delta_{MY} &\geq \sum_{1 \leq i \leq t} -\lambda_i^2 + \sum_{1 \leq i \leq k-1} \lambda_i \lambda_k d_k \\ &\quad + \sum_{t+1 \leq i \leq k} \lambda_i d_i (-3 + 3d_i - \lambda_i d_i) + \sum_{1 \leq i < j \leq k-1} \lambda_i \lambda_j d_i d_j. \end{aligned}$$

As  $\lambda_i^2 < \lambda_i \leq \lambda_i \lambda_k d_k$ , the sum of the first two sums is positive, and it follows from the previous remarks that  $\Delta_{MY} > 0$ .

Now we study the discriminant according to the number  $k$  of hyperplanes in  $\Delta$ . In the following, we assume that  $\lambda_1 \leq \dots \leq \lambda_k$  up to changing indices.

If  $k = 1$ , we have  $\Delta_{MY} = -\lambda_1^2 < 0$ .

If  $k = 2$ , we have  $\Delta_{MY} = -\lambda_1^2 - \lambda_2^2 + \lambda_1 \lambda_2 = -(\lambda_1 - \lambda_2)^2 - \lambda_1 \lambda_2 < 0$ .

If  $k \geq 4$ , one can observe that we have for  $1 \leq i \leq k - 1$  the following inequality:  $\lambda_i^2 \leq \lambda_i \lambda_{i+1}$ . Hence, we have:

$$\begin{aligned} \Delta_{MY} &> - \sum_{1 \leq i \leq k} \lambda_i^2 + \sum_{1 \leq i \leq k-1} \lambda_i \lambda_{i+1} + \lambda_k \lambda_1 + \lambda_k \lambda_2 \\ &\geq -\lambda_k^2 + \lambda_k \lambda_1 + \lambda_k \lambda_2 = \lambda_k (\lambda_1 + \lambda_2 - \lambda_k). \end{aligned}$$

Because  $\lambda_i \in \left[\frac{1}{2}, 1\right)$ , we have  $\lambda_1 + \lambda_2 - \lambda_k > 0$ . Hence,  $\Delta_{MY} > 0$ .

If  $k = 3$ , we have:

$$\begin{aligned} \Delta_{MY} &= -\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\ &= -\frac{1}{2} \left( (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2 \right). \end{aligned}$$

We have  $\Delta_{MY} = 0$  iff  $\lambda_1 = \lambda_2 = \lambda_3$ , which is equivalent to  $m_1 = m_2 = m_3$ . Hence,  $\Delta = \left(1 - \frac{1}{m}\right) (H_0 + H_1 + H_2)$ . As  $\Delta$  has simple normal crossings, the three hyperplanes  $H_i$  are in general position. Hence, there exists  $A \in \text{PGL}(3, \mathbb{C})$  such that  $A(z_i = 0) = H_i$ . Hence, the quotient map induced by the action of  $G = (\mu_m)^2$  described in the previous section conjugated by  $A$  branches exactly along  $H_i$  with multiplicity  $m_i$ . Hence, it follows that  $(X, \Delta)$  is the quotient of  $\mathbb{P}^2$  by this action conjugated by  $A$ .  $\square$



# CONCLUSION

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We finally showed that there is a characterization of projective spaces for pairs, providing a generalization of Theorem 0.0.7. In the smooth case, we explained that the assumptions of this theorem come from a geometric description of pairs in terms of metrics and constancy of holomorphic sectional curvature. One can ask if such a result still holds in the singular setting. We expect that we have the following characterization of quotients of  $\mathbb{P}^n$ :

**Conjecture 11.3.4.** *Let  $(X, \Delta)$  be a log Fano pair such that the following two assumptions are satisfied:*

(KE) *The pair can be endowed with a singular Kähler–Einstein metric  $\omega_{(X, \Delta)}$ . That is a metric such that we have the following equality of currents:*

$$\text{Ric } \omega_{(X, \Delta)} = \omega_{(X, \Delta)} + [\Delta].$$

(MY) *The equality case in Miyaoka–Yau equality holds:*

$$\left(2(n+1)c_2(X, \Delta) - nc_1(X, \Delta)^2\right) \cdot c_1(X, \Delta)^{n-2} = 0.$$

*Then, there exists a finite subgroup  $G \subseteq \text{PGL}(n+1, \mathbb{C})$  such that:  $(X, \Delta) \simeq (\mathbb{P}^n/G, \Delta_G)$ .*

In the smooth case, it is possible to give a proof in purely differential geometric terms, but one can argue by using [Tia92, Thm. 0.1], that ensures the implication: (KE)  $\implies$  (SCE), and by proving the theorem by replacing (KE) assumption by (SCE). That is how the proof runs in the singular setting with  $\Delta = 0$ . Indeed, it has been shown by Druel–Guenancia–Păun a result analogous to Tian’s one in the klt framework.

**Theorem 11.3.5.** [DGP24, Thm. B] *Let  $X$  be a klt Fano variety admitting a singular Kähler–Einstein metric. Then, the canonical extension  $\mathcal{E}_X$  of  $X$  is  $c_1(X)$ -polystable (hence  $c_1(X)$ -semistable).*

Combined with Theorem 0.0.8, we obtain that the previous conjecture holds when  $\Delta = 0$ .

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In order to prove the conjecture for any  $\Delta$ , it remains to show a Tian's type result that takes into account the boundary.

**Conjecture 11.3.6.** *Let  $(X, \Delta)$  be a log Fano pair endowed with a singular Kähler-Einstein metric. Then, for any strictly  $\Delta$ -adapted morphism  $f: Y \rightarrow X$ , the adapted canonical extension  $\mathcal{E}_{X, \Delta, f}$  is  $f^*(-(K_X + \Delta))$ -polystable.*

# APPENDICES

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## Appendix A: Proper action of a discrete group on an analytic space: topological and analytical side

The goal being to classify quotients of projective spaces, we first define the structures associated with quotients of analytic spaces by groups. Since we require that the quotient space  $X$  carries a Hausdorff topology, it is natural to study discrete groups acting properly on an analytic space  $Y$ .

**Definition 11.3.7.** *Let  $Y$  be a locally compact topological space and let  $G$  be a discrete group acting continuously on  $Y$ . We say that the action is proper if for every compact set  $K \subseteq Y$ , the set*

$$\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$$

*is finite.*

Let  $G$  be a group acting properly on  $Y$ . Then for a singleton  $K = \{y\}$ , this implies that the stabilizer  $G_y$  is finite.

**Definition 11.3.8.** *Let  $Y$  be a topological space and  $G$  be a group acting continuously on  $Y$ . We define the quotient map*

$$\pi_G : Y \longrightarrow Y/G,$$

*where  $Y/G$  is the space of orbits. It is endowed with the quotient topology, which makes  $\pi_G$  continuous and open.*

The open subsets of  $Y/G$  can be described more precisely in terms of the stabilizers of points in their preimages.

**Lemma 11.3.9.** *Let  $G$  be a group acting properly on an analytic space  $Y$ . Then for every  $y \in Y$ , there exist an open neighborhood  $U_y$  of  $y$  stable under the action of  $G_y$ , and an open neighborhood  $V_y$  of  $\pi_G(y)$  in  $Y/G$ , and a homeomorphism*

$$\varphi_y : U_y/G_y \longrightarrow V_y.$$

---

*Proof.* Since  $Y$  is locally compact, there exists a compact neighborhood  $\widetilde{U}_y$  of  $y$ . By properness of the action, the set

$$H := \{g \in G \mid g \cdot \widetilde{U}_y \cap \widetilde{U}_y \neq \emptyset\}$$

is finite; write  $H = \{h_1, \dots, h_p\}$ .

As  $Y$  is Hausdorff, for each left coset  $hG_y \in H/G_y$ , we can find open neighborhoods  $W_{h \cdot y}$  of  $h \cdot y$  such that for  $hG_y \neq h'G_y$ , we have  $W_{h \cdot y} \cap W_{h' \cdot y} = \emptyset$ . Let

$$U := \bigcap_{i=1}^p h_i^{-1}(W_{h_i \cdot y}),$$

an open neighborhood of  $y$ . Replacing  $U$  with

$$U_y := \bigcap_{g \in G_y} g(U \cap \widetilde{U}_y),$$

we obtain an open neighborhood of  $y$  stable under  $G_y$ , included in  $\widetilde{U}_y$ . The restriction  $\pi_G|_{U_y}$  factors through  $U_y/G_y$ , giving rise to a map

$$\varphi_y : U_y/G_y \longrightarrow V_y := \pi_G(U_y).$$

The map  $\varphi_y$  is surjective: any point in  $V_y$  has a representative in  $U_y$ , hence arises from a point in  $U_y/G_y$ .

The map  $\varphi_y$  is injective: if  $z, z' \in U_y$  satisfy  $\pi_G(z) = \pi_G(z')$ , then  $z' = g \cdot z$  for some  $g \in G$ . Since  $z' \in g\widetilde{U}_y \cap \widetilde{U}_y$ , we have  $g \in H$ . Moreover, since  $z' \in W_{g \cdot y} \cap W_y \neq \emptyset$ , we must have  $g \in G_y$ .

Finally,  $\varphi_y$  is continuous and open by construction: for any open  $O \subseteq U_y/G_y$ ,

$$\pi_G^{-1}(\varphi_y(O)) = \bigcup_{g \in G} g(\pi_G^{-1}(O)),$$

which is open by the definition of the quotient topology on  $U_y/G_y$ . □

From the analytic point of view, we want to define the appropriate notion of analytic functions on open subsets of the quotient space  $Y/G$ . To do this, the action and the analytic structure need to be compatible.

**Definition 11.3.10.** *Let  $Y$  be a normal analytic space, and let  $G$  be a discrete group acting*

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faithfully on  $Y$ . We say that the action is analytic if there is an injective homomorphism  $G \hookrightarrow \text{Aut}(Y, \mathcal{O}_Y)$ .

In other terms, an analytic action of  $G$  on  $Y$  means that for each  $g \in G$ , we have a morphism of locally ringed spaces: a homeomorphism  $g : Y \rightarrow Y$ , along with a morphism of sheaves

$$g^\# : g^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_Y,$$

which induces an isomorphism of stalks  $\mathcal{O}_{Y,gy} \rightarrow \mathcal{O}_{Y,y}$  for each  $y \in Y$ . Moreover, we require compatibility: for all  $g, h \in G$ ,

$$g^\# \circ g^{-1}h^\# = (hg)^\#.$$

**Definition 11.3.11.** *Let  $\mathcal{F}$  be an  $\mathcal{O}_Y$ -module. We say that  $\mathcal{F}$  is a  $G$ -sheaf if for each  $g \in G$ , there exists an  $\mathcal{O}_Y$ -module morphism  $g_{\mathcal{F}} : g^*\mathcal{F} \rightarrow \mathcal{F}$ , satisfying the compatibility condition:*

$$(hg)_{\mathcal{F}} = g_{\mathcal{F}} \circ g^*h_{\mathcal{F}} \quad \text{for all } g, h \in G.$$

In practice, if  $g \in G$ ,  $y \in Y$ ,  $f$  is an analytic function defined near  $g \cdot y$ , and  $s$  is a germ of a section of  $\mathcal{F}$  near  $g \cdot y$ , then

$$g(fs) = g^\#(f) \cdot g_{\mathcal{F}}(s),$$

or more simply:

$$g(fs) = g(f) \cdot g(s).$$

We can now define the analytic structure on the quotient space. Motivated by the smooth case and the universal property of the quotient, we define an analytic function on an open set  $U \subseteq Y/G$  as an analytic function on  $\pi_G^{-1}(U)$  that is invariant under the action of  $G$ .

Thus, the structure sheaf of  $Y/G$  is defined by:

$$\mathcal{O}_{Y/G}(U) := \mathcal{O}_Y(\pi_G^{-1}(U))^G.$$

It follows that the ringed space  $(Y/G, \mathcal{O}_{Y/G})$  is a normal analytic space (see [Pop77, Thm. 3.4]).

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## Appendix B: Completion of finite morphisms

To construct branched coverings over a variety  $X$ , we often use classical constructions over general points, which constitute a large open  $X^\circ \hookrightarrow X$ . The following lemma is crucial for extending these constructions to singular loci. We follow the proof of [Gro03, Thm. XII.5.4].

**Theorem 11.3.12.** *Let  $X, Y^\circ$  be normal complex analytic spaces, and  $Z$  a nowhere dense analytic subspace of  $X$ , and we denote  $X^\circ := X \setminus Z$ . Let  $f^\circ : Y^\circ \rightarrow X^\circ$  be a finite morphism.*

*Then, there exists a normal analytic space  $Y$ , a dense image embedding  $\iota : Y^\circ \hookrightarrow Y$ , and a finite morphism  $f : Y \rightarrow X$  such that  $f^\circ = f \circ \iota$ . Furthermore, this completion is unique up to unique isomorphism: if there is another completion  $Y', \iota'$ , and  $f'$ , then there exists a unique isomorphism  $\phi : Y \rightarrow Y'$  such that  $f' \circ \phi = f$  and  $\phi \circ \iota = \iota'$ .*

*Proof. Step 1: Reduction to the case where  $Z$  is a snc divisor in a smooth  $X$*

First, by possibly removing the branch locus of  $f^\circ$ , we may assume that  $f^\circ$  is étale. By Hironaka's resolution of singularities theorem, there exists a resolution  $\mu : \widehat{X} \rightarrow X$  such that  $E := \mu^{-1}(Z)$  is a divisor with simple normal crossings, denoted  $\sum_i E_i$ . Hence, since  $X^\circ \simeq \widehat{X} \setminus E$ , we have a finite morphism  $Y^\circ \rightarrow \widehat{X} \setminus E$ . We complete this morphism over  $\widehat{X}$ . If we consider a point  $x \in E$ , we can choose a chart around  $x$  such that  $E \simeq (z_1 \cdots z_k = 0)$ . In other words, we have a neighborhood  $\widehat{U} \simeq \mathbb{D}^n$  such that  $\widehat{U} \setminus E \simeq (\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$ .

We consider a connected component  $W$  of  $f^{\circ-1}(\widehat{U} \setminus E)$ . The restriction of  $f^\circ$  to  $W$  is finite and étale over  $(\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$ . This is a finite covering, so the loops  $\gamma_i$  generating  $\pi_1(\widehat{U} \setminus E) = \mathbb{Z}^k$  have a certain minimal power  $m_i$  that belongs to  $\pi_1(W)$  since it is a finite index subgroup. Thus, by the geometric Galois correspondence, we have a composition of coverings:

$$(\mathbb{D}^*)^k \times \mathbb{D}^{n-k} \rightarrow W \rightarrow (\mathbb{D}^*)^k \times \mathbb{D}^{n-k},$$

given by  $(z_1, \dots, z_k, z_{k+1}, \dots, z_n) \mapsto (z_1^{m_1}, \dots, z_k^{m_k}, z_{k+1}, \dots, z_n)$ . Thus,  $W$  is obtained as a quotient by a subgroup  $H$  of the automorphism group  $T := \mathbb{Z}/(m_1) \oplus \cdots \oplus \mathbb{Z}/(m_k)$  of this covering<sup>1</sup>. Thus, this action extends to  $\mathbb{D}^n$ . Consequently, the action of  $H$  also extends, and we define  $\widehat{W} := \mathbb{D}^n/H$ . By the universal property of the quotient, we obtain a natural extension  $\widehat{W} \rightarrow \mathbb{D}^n/T = \mathbb{D}^n$ .

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1. The group  $T$  acts on  $(\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$  by diagonal multiplication on the coordinates by  $n$ -th roots of unity.

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We consider the gluing  $\widehat{Y}$  of  $Y^\circ$  with the  $\widehat{W}$ 's, so that we obtain an extension of  $f^\circ$  by a finite morphism  $\widehat{f} : \widehat{Y} \rightarrow \widehat{X}$  of the morphism  $f^\circ$ .

**Step 2: From the completion over a resolution  $\widehat{X}$  to the completion over  $X$**   
 To obtain a completion over  $X$ , we observe that  $\mathcal{A} := (\mu \circ \widehat{f})_* \mathcal{O}_{\widehat{Y}}$  is a coherent  $\mathcal{O}_X$ -algebra. Indeed, both  $\mu$  and  $\widehat{f}$  are proper morphisms, so the direct image of  $\mathcal{O}_{\widehat{Y}}$  is coherent by [GR84, Thm. 10.4.6]. We can associate to such a coherent algebra its analytic spectrum [PR94, Thm. 3.11]: it is an analytic space  $(Y, \mathcal{O}_Y)$  and a finite morphism  $f : Y \rightarrow X$  such that  $f_* \mathcal{O}_Y = \mathcal{A}$ . By possibly composing with the normalization of  $Y$ , we can assume  $Y$  is normal.

Let us show that this morphism extends  $f$ . First, the fact that  $\mu$  is a biholomorphism over  $X^\circ$  and that  $\widehat{f}$  coincides with  $f^\circ$  over  $X^\circ$  indicates that we have an isomorphism of coherent algebras:  $\mathcal{A}|_{Y^\circ} \simeq f^\circ_* \mathcal{O}_{Y^\circ}$ . By the equivalence of categories induced by the analytic spectrum functor, we obtain that the analytic spectrum associated with  $\mathcal{A}|_{Y^\circ}$  is  $Y^\circ$  by [PR94, Thm. 3.14.4]. Hence, we obtain an embedding  $\iota : Y^\circ \hookrightarrow Y$ , which must be dense by the finiteness of  $f$ . But it is also the restriction of  $f$  over  $X^\circ$  by applying [PR94, Prop. 3.12.3] to  $i^* \mathcal{A}$  for  $i : X^\circ \hookrightarrow X$ .

The uniqueness is a consequence of the fully faithful property of the restriction functor for branched coverings over a dense open subset [Gro03, Prop. XII.5.3]. Indeed, if we have two completions  $(Y, \iota, f)$  and  $(Y', \iota', f')$ , then the restriction map:

$$\mathrm{Hom}(Y, Y') \longrightarrow \mathrm{Hom}(\iota(Y^\circ), \iota'(Y^\circ)),$$

is bijective. In particular, there exists a unique morphism of coverings  $\varphi : Y \rightarrow Y'$ , which extends the isomorphism of coverings between  $\iota(Y^\circ)$  and  $\iota'(Y^\circ)$ . This morphism establishes an isomorphism between  $Y$  and  $Y'$ . Indeed, it suffices to consider in the same way the extension  $\psi : Y' \rightarrow Y$  of the inverse isomorphism between  $\iota'(Y^\circ)$  and  $\iota(Y^\circ)$ . Since the composition  $\varphi \circ \psi$  coincides with  $\mathrm{id}_{Y'}$  over  $X^\circ$ , we obtain by injectivity on the morphisms of the restriction that  $\varphi \circ \psi = \mathrm{id}_{Y'}$ . Finally, in the same way, we have  $\psi \circ \varphi = \mathrm{id}_Y$ , so  $\varphi$  is uniquely determined as an isomorphism.  $\square$

**Corollary 11.3.13.** *Let  $X, Y$  be two normal analytic spaces, and  $f : Y \rightarrow X$  a finite morphism. Let  $X^\circ$  be a Zariski open subset of  $X$  and  $\iota : Y^\circ := f^{-1}(X^\circ) \rightarrow Y$ . We denote by  $\mathrm{Aut}(f)$  (resp.  $\mathrm{Aut}(f^\circ)$ ) the group of automorphisms  $\Phi : Y \rightarrow Y$  such that  $f \circ \Phi = f$  (resp. automorphisms  $\Phi^\circ : Y^\circ \rightarrow Y^\circ$  such that  $f^\circ \circ \Phi^\circ = f^\circ$ ).*

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Then, the restriction and corestriction of maps:

$$\Phi \in \text{Aut}(f) \longmapsto \Phi|_{Y^\circ} \in \text{Aut}(f^\circ)$$

induces an isomorphism of groups.

*Proof.* First, the map is well-defined. Indeed, if we pick  $\Phi \in \text{Aut}(f)$ , then the image of an element  $y \in Y^\circ$  by  $\Phi$  satisfies  $f(\Phi(y)) = f(y) \in X^\circ$  so that  $\Phi(y) \in Y^\circ$ . It is clearly a group morphism. Let us construct the inverse map. We consider  $\Phi^\circ \in \text{Aut}(f^\circ)$ . We consider the two completions  $(Y, \iota, f)$  and  $(Y, \iota \circ \Phi^\circ, f)$  of the morphism  $f^\circ$ . By Theorem 11.3.12, we obtain a unique automorphism  $\Phi \in \text{Aut}(Y)$  such that  $\iota \circ \Phi^\circ = \Phi \circ \iota$  and  $f \circ \Phi = f$ . In other terms, there is a unique automorphism  $\Phi \in \text{Aut}(f)$  such that  $\Phi|_{Y^\circ} = \Phi^\circ$ .  $\square$

## Appendix C: Slope stability

**Definition 11.3.14.** Let  $X$  be a compact Kähler manifold of dimension  $n$ , and  $E$  a holomorphic vector bundle of rank  $r$  over  $X$ . Let  $\alpha \in H_{\text{dR}}^2(X, \mathbb{R})$  be a Kähler class. We define the  $\alpha$ -slope of  $E$  as the quantity:

$$\mu_\alpha(E) := \frac{\int_X c_1(E, h) \wedge \omega^{n-1}}{\text{rk } E},$$

where  $\omega$  is a Kähler metric representing  $\alpha$  and  $h$  is any metric on  $E$ .

**Remark 11.3.15.** More generally, we can define Chern classes of a coherent sheaf when  $X$  is projective in the following way. By the syzygies theorem, there exists a finite resolution of  $\mathcal{E}$  by locally free sheaves :

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow 0,$$

and we can define the total Chern class of  $\mathcal{E}$ , denoted by  $c(\mathcal{E})$  as :

$$c(\mathcal{E}) := \prod_i c(\mathcal{E}_i)^{(-1)^i} \in H^{2*}(X, \mathbb{Z}).$$

One can show that it is well defined<sup>2</sup> and independent of the chosen resolution. Finally,

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2. For instance,  $c(\mathcal{E}_i)^{-1}$  makes sense as it is of the form  $1 + n$  where  $n$  is a nilpotent element in the ring  $H^{2*}(X, \mathbb{Z})$

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one can define the rank of a coherent sheaf  $\mathcal{E}$  as following : there exists a Zariski open subset  $U$  of  $X$  such that  $\mathcal{E}|_U$  is a locally free sheaf, the rank of  $\mathcal{E}$  is the rank of  $\mathcal{E}|_U$ .

**Definition 11.3.16.** *Let  $X$  be a compact Kähler manifold and let  $E$  be a vector bundle. We denote by  $\mathcal{E}$  its sheaf of holomorphic sections. Let  $\alpha \in H_{\text{dR}}^2(X, \mathbb{R})$  be a Kähler class. We say that  $E$  is:*

- $\alpha$ -stable if for any proper coherent subsheaf  $\mathcal{F} \subseteq \mathcal{E}$ , we have  $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ ,
- $\alpha$ -semistable if for any proper coherent subsheaf  $\mathcal{F} \subseteq \mathcal{E}$ , we have  $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ ,
- $\alpha$ -polystable if  $\mathcal{E} = \sum_i \mathcal{E}_i$  where  $\mathcal{E}_i$  are  $\alpha$ -stable and have same  $\alpha$ -slopes.

Let us consider a short exact sequence of  $\mathcal{O}_X$ -modules:

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

Then, we have:

$$\text{rk } E \mu_\alpha(E) = c_1(E) \cdot \alpha^{n-1} = c_1(S) \cdot \alpha^{n-1} + c_1(Q) \cdot \alpha^{n-1} = \text{rk } S \mu_\alpha(S) + \text{rk } Q \mu_\alpha(Q).$$

Hence, we have  $\text{rk } E = \text{rk } S + \text{rk } Q$ . If we set  $t = \frac{\text{rk } S}{\text{rk } E}$ , we get  $\mu_\alpha(E) = t\mu_\alpha(S) + (1-t)\mu_\alpha(Q)$ . As a consequence of this computation, we get that a torsion-free sheaf  $\mathcal{E}$  is  $\alpha$ -semistable if and only if for any quotient  $\mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$ , we have  $\mu_\alpha(\mathcal{Q}) \geq \mu_\alpha(\mathcal{E})$ .

**Lemma 11.3.17.** *Let  $X$  be a compact Kähler manifold and let  $E$  be a vector bundle. Let  $\alpha \in H_{\text{dR}}^2(X, \mathbb{R})$  be a Kähler class. Then, we have:*

$$E \text{ is } \alpha\text{-stable} \implies E \text{ is } \alpha\text{-polystable} \implies E \text{ is } \alpha\text{-semistable}$$

*Proof.* It is clear that stability implies polystability. We infer the second implication from the proof of the following lemma. □

**Lemma 11.3.18.** *Let  $X$  be a compact Kähler manifold and let  $E$  and  $F$  be two vector bundles. Let  $\alpha \in H_{\text{dR}}^2(X, \mathbb{R})$  be a Kähler class. Then, we have:*

$$E \oplus F \text{ is } \alpha\text{-semistable of slope } \mu \iff E, F \text{ are } \alpha\text{-semistable and } \mu_\alpha(E) = \mu_\alpha(F) = \mu.$$

*Proof.* For the direct implication, we observe first that the sheaf of sections  $\mathcal{E}$  is a subsheaf and a quotient of  $\mathcal{E} \oplus \mathcal{F}$ . Hence, its slope is equal to  $\mu$ , and we have the same for  $\mu_\alpha(F)$ .

---

Secondly, a subsheaf  $\mathcal{G}$  of  $\mathcal{E}$  is in particular a subsheaf of  $\mathcal{E} \oplus \mathcal{F}$ , so that  $\mu_\alpha(\mathcal{G}) \leq \mu_\alpha(E \oplus F) = \mu$ . Hence,  $E$  is  $\alpha$ -semistable. We argue similarly for  $F$ .

For the converse, we first observe that if  $\mu_\alpha(E) = \mu_\alpha(F) =: \mu$ , we have  $\mu_\alpha(E \oplus F) = \mu$  by the formula of the slope of a short exact sequence explained above. We consider the projection of vector bundles  $p : E \oplus F \rightarrow F$ . Hence, if  $\mathcal{G}$  is a proper subsheaf of  $\mathcal{E} \oplus \mathcal{F}$ , the morphism  $p$  induces a morphism of sheaves by restriction to  $\mathcal{G}$ :

$$0 \longrightarrow \text{Ker}(p|_{\mathcal{G}}) \longrightarrow \mathcal{G} \longrightarrow \text{Im}(p|_{\mathcal{G}}) \longrightarrow 0.$$

We have  $\text{Ker}(p|_{\mathcal{G}}) \subseteq \mathcal{E}$  and  $\text{Im}(p|_{\mathcal{G}}) \subseteq \mathcal{F}$ . If  $\text{Ker}(p|_{\mathcal{G}}) = 0$  or  $\text{Im}(p|_{\mathcal{G}}) = 0$ , we get that  $\mathcal{G}$  is isomorphic to a subsheaf of  $\mathcal{F}$  or  $\mathcal{E}$ , so that  $\mu_\alpha(\mathcal{G}) \leq \mu$ . In the other cases, we get that  $\mu_\alpha(\mathcal{G}) = t\mu_\alpha(\text{Ker}(p|_{\mathcal{G}})) + (1-t)\mu_\alpha(\text{Im}(p|_{\mathcal{G}})) \leq t\mu + (1-t)\mu = \mu$ , for some  $t \in (0, 1)$ .  $\square$

In the sequel, we assume that  $X$  is projective, and we focus on torsion-free sheaves. These assumptions guarantee the boundedness of the set of slopes of subsheaves. For proofs and details, see the arXiv version of [GKP16b, Sect. 2.5-6]. We have the following important lemmas.

**Lemma 11.3.19** (Existence of the maximal destabilizing subsheaf). *Let  $X$  be a smooth projective variety (we denote by  $A$  an ample divisor on  $X$ ), and let  $\mathcal{E}$  be a torsion-free sheaf on  $X$  that is not  $A$ -stable. Then there exists a proper subsheaf  $\mathcal{F} \subseteq \mathcal{E}$  such that:*

$$\mu_A(\mathcal{F}) = \sup \{ \mu_A(\mathcal{G}) \mid \mathcal{G} \subseteq \mathcal{E} \text{ proper coherent subsheaf} \}.$$

Moreover, this sheaf  $\mathcal{F}$  is maximal in the sense that if  $\mu_A(\mathcal{F}') = \mu_A(\mathcal{F})$ , we have  $\mathcal{F}' \subseteq \mathcal{F}$ .

**Definition 11.3.20.** *This sheaf  $\mathcal{F}$  is called the maximal destabilizing subsheaf of  $\mathcal{E}$ . It is a  $A$ -semistable torsion-free sheaf.*

**Lemma 11.3.21** (Jordan-Hölder filtration). *With notation as above, if  $\mathcal{E}$  is a  $A$ -semistable sheaf, then there exists a filtration:*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subseteq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = \mathcal{E},$$

where  $\mathcal{F}_i$  are torsion-free sheaves such that the successive quotients  $\mathcal{Q}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$  are  $A$ -stable torsion-free sheaves, and they have slopes  $\mu_A(\mathcal{Q}_i) = \mu_A(\mathcal{E})$ .

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We conclude this survey on stability by the algebro-geometric correspondence between polystability and the presence of Hermite–Einstein metrics, due to Donaldson–Uhlenbeck–Yau.

**Definition 11.3.22.** *Let  $(X, \omega)$  be a Kähler compact manifold and  $(E, h)$  a hermitian vector bundle. We say that the metric  $h$  is Hermite–Einstein if there exists a constant  $\lambda \in \mathbb{R}$  such that the following equality in  $\mathcal{C}^\infty(X, \wedge^{n,n} T_X \otimes \text{End}(E))$  holds:*

$$\sqrt{-1}\Theta(E, h) \wedge \omega^{n-1} = \lambda \text{id}_E \omega^n.$$

If we apply the trace with respect to the endomorphism part, we get an equality of  $(n, n)$ -forms, and we have the following after integration:

$$\frac{c_1(E) \cdot [\omega]^{n-1}}{\text{rk } E} = \lambda \text{Vol}_\omega(X),$$

where  $\text{Vol}_\omega(X) := \int_X \omega^n$ . Hence, we can express the coefficient  $\lambda$  in terms of slopes:  $\lambda = \frac{\mu_{[\omega]}(E)}{\text{Vol}_\omega(X)}$ . We have the following theorem that is an algebro-geometric correspondence due to Donaldson–Uhlenbeck–Yau.

**Theorem 11.3.23.** *Let  $(X, \omega)$  be a compact Kähler manifold and  $E$  a vector bundle over  $X$ . Then, the following assertions are equivalent:*

- (i) *There exists a Hermite–Einstein metric  $h$  on  $E$  with respect to  $\omega$ ,*
- (ii) *The vector bundle  $E$  is polystable with respect to  $[\omega]$ .*

The implication (i)  $\implies$  (ii) has been established by Kobayashi and Lübke [Lüb83, Prop. 4], and Donaldson showed the converse (ii)  $\implies$  (i) in the case of curves, and the result has been extended by Uhlenbeck and Yau [UY86, Thm. 4.1].



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**Titre :** Caractérisation numérique des quotients d'espaces projectifs complexes

**Mot clés :** géométrie complexe, paires orbifoldes, uniformisation, égalité de Miyaoka–Yau, variétés de Fano

**Résumé :** Cette thèse est consacrée à l'étude du problème de l'uniformisation dans le cadre singulier. Plus précisément, on donne une caractérisation des paires singulières obtenues comme quotients d'espaces projectifs complexes. Ce résultat généralise aux travaux de Greb, Kebekus et Peternell qui établissent une telle caractérisation pour les variétés singulières.

Après avoir rappelé le formalisme des paires et la théorie de l'uniformisation dans ce cadre, nous donnons un critère qui permet d'assurer que le revêtement universel d'une paire est effectivement lisse. Celui-ci re-

pose sur un résultat analogue au théorème de Zariski–Lipman pour les paires orbifoldes, et des considérations sur la platitude dans ce contexte.

Ensuite, on étudie la bonne définition d'extension canonique d'une paire singulière. On introduit la notion d'extension canonique adaptée, qui mime la construction du faisceau des différentielles adaptées introduit par Miyaoka. On montre que ces faisceaux peuvent être interprétés comme des tirés en arrière d'orbifaisceaux, permettant de calculer effectivement les classes de Chern orbifoldes de ces objets.

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**Title:** Numerical characterization of quotients of complex projective spaces

**Keywords:** Complex Geometry, Orbifold pairs, Uniformization, Miyaoka–Yau equality, Fano varieties

**Abstract:** This thesis is devoted to the study of the problem of uniformization in the singular framework. We provide a characterization of singular pairs that are obtained as quotients of complex projective spaces. This generalizes recent results due to Greb, Kebekus, and Peternell, who established such a characterization for singular varieties.

Subsequent to a review of the formalism of pairs and the theory of uniformization within this framework, we state a criterion that guarantees the smoothness of the universal covering of a pair. This criterion is founded on a

result analogous to the Zariski–Lipman theorem for orbifold pairs, and on considerations of flatness in this context.

Next, we study the correct definition of the canonical extension of a singular pair. We introduce the adapted canonical extension, whose construction is similar to the adapted differentials one that Miyaoka introduced. We show that these bundles can be understood as pullbacks of orbibundles, so that this provides us with effective computations of their orbifold Chern classes.