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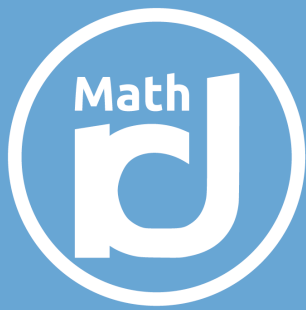
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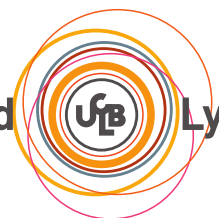
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Étude des groupes et corps dans certaines théories modérées



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Thèse de doctorat

Université Claude Bernard



Lyon 1

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Étude des groupes et corps dans certaines théories modérées

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Résumé en français

Cette thèse est dédiée à l'étude des structures algébriques dans une certaine classe de la classification modèle-théorique des structures, celle des théories NSOP₁.

On commence par présenter comment s'est développée la classification en général et l'étude de NSOP₁ en particulier, ainsi que les différents outils modèle-théoriques que l'on utilise dans ce contexte, c'est-à-dire les relations d'indépendance, la Shelah-indépendance et surtout la Kim-indépendance.

On aborde ensuite la construction d'exemples de théories NSOP₁. Dans le Chapitre 2 on considère des extensions générique de certaines théories. On démontre que sous certaines hypothèses ces extensions génériques sont NSOP₁ et on caractérise la Kim-indépendance et la Shelah-indépendance dans celles-ci. Ceci nous fournit entre autres un exemple de groupe définissable dans une théorie NSOP₁.

Un autre exemple de théorie NSOP₁ est étudié dans le Chapitre 3. Il s'agit de la théorie d'une forme bilinéaire symétrique générique sur un espace vectoriel de dimension infinie sur un corps K . On démontre que si le corps K est NSOP₁ alors cette théorie l'est aussi et on caractérise la Kim-indépendance sur des modèles dans celle-ci. On démontre aussi des résultats plus précis sur cette théorie en ajoutant des conditions sur la théorie du corps K . On fournit un contre-exemple à une conjecture portant sur un lien entre Kim-indépendance et Shelah-indépendance dans le cas où K est un corps algébriquement clos.

Dans le Chapitre 4 on s'intéresse aux outils modèle-théoriques pour étudier les groupes définissables, et on cherche à les généraliser au contexte de NSOP₁. On définit le Kim-stabilisateur et on démontre que dans plusieurs exemples de théories NSOP₁ on obtient un sous-groupe type-définissable. On définit aussi certaines hypothèses sur un corps K sous lesquelles en utilisant le Kim-stabilisateur on peut établir un lien entre un groupe définissable dans K et les points K -rationnels d'un groupe algébrique définissable sur K .

Dans l'appendice on s'intéresse à une généralisation de la conjecture de la déviation stable au cadre des théories NSOP₁ et de la Kim-déviation.

Résumé en anglais

This thesis is dedicated to the study of algebraic structures within a certain class of the model-theoretic classification of structures, namely that of NSOP_1 theories.

We begin by presenting how the classification developed in general and the study of NSOP_1 in particular, as well as the various model-theoretic tools used in this context, namely independence relations, forking-independence, and especially Kim-independence.

We then address the construction of NSOP_1 theories. In Chapter 2, we consider generic extensions of certain theories. We show that under certain assumptions, these generic extensions are NSOP_1 , and in that case we characterize Kim-independence and forking-independence in them. This provides us, among other things, with an example of a definable group in an NSOP_1 theory.

An other example of NSOP_1 theory is studied in Chapter 3. It is the theory of a generic symmetric bilinear form over a field K . We show that if the field K is NSOP_1 , then this theory is as well, and we characterize Kim-independence over models in it. We also demonstrate more precise results about this theory by adding conditions on the theory of the field K . We provide a counterexample to a conjecture concerning a link between Kim-independence and forking-independence in the case where K is an algebraically closed field.

In Chapter 4, we focus on model-theoretic tools for studying definable groups, and we aim to generalize them to the NSOP_1 context. We define the Kim-stabilizer and show that in several examples of NSOP_1 theories, we obtain a type-definable subgroup. We also define certain assumptions on a field K under which, using the Kim-stabilizer, we can establish a connection between a definable group in K and the K -rational points of an algebraic group definable over K .

In the appendix, we focus on a generalization of the stable forking conjecture to the setting of NSOP_1 theories and Kim-forking.

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Introduction (version française)

On commence par introduire chronologiquement les théories NSOP₁ et la classification de Shelah. On présente ensuite les différentes notions qui sont abordées au cours du manuscrit en les plaçant dans leur contexte.

Stabilité et Classification

L'un des problèmes qui intéressait les théoriciens des modèles en 1970 était de classer les modèles d'une théorie, c'est à dire de compter les modèles de cardinal κ à isomorphisme près. L'une des notions importantes pour étudier ce problème est celle de théorie κ -stable. On considère ici que tout les langages sont dénombrables. Une théorie T est dite κ -catégorique pour un cardinal κ si elle admet un unique modèle de cardinal κ à isomorphisme près.

Definition 0.1. Une théorie complète T est κ -stable pour un cardinal infini κ si pour tout modèle M de T de cardinal κ , l'espace des types $S_n(M)$ a pour cardinal κ .

Un résultat marquant dans cette voie fut le théorème de Morley :

Theorem 0.2. *Si une théorie dans un langage dénombrable est κ -catégorique pour un certain cardinal κ indénombrable alors elle l'est pour tout les cardinaux indénombrables. De plus une théorie \aleph_1 -catégorique est ω -stable.*

Dans sa preuve Morley développe de nombreux outils permettant d'analyser la complexité des ensembles définissables. Il définit les notions suivantes :

Definition 0.3. *Ensemble fortement minimal :* Soit M une L -structure et $D \subseteq M$ un sous-ensemble définissable infini, on dit que D est minimal dans M si tout sous-ensemble définissable $Y \subseteq D$ est fini ou cofini. On dit que D est fortement minimal si il est minimal dans toute extension élémentaire de M . Une formule est fortement minimale si elle définit un ensemble fortement minimal.

Definition 0.4. *Dimension sur les ensembles fortement minimaux :* Soit M une L -structure et $D \subseteq M$ un ensemble définissable fortement minimal. Un ensemble $A \subseteq D$ est dit indépendant si $a \notin \text{acl}(A \setminus \{a\})$ pour tout $a \in A$. A est une base de D si de plus $D \subseteq \text{acl}(A)$, et alors la dimension de D est la cardinalité de A , notée $\dim(D)$ (qui ne dépend pas de l'ensemble A).

Cette notion de dimension coïncide avec plusieurs notions bien connues : Le degré de transcendance dans le cadre de la théorie d'un corps algébriquement clos dans le langage des anneaux, ou encore la dimension linéaire dans la théorie d'un espace vectoriel E sur un corps K dans le langage à une sorte $L_K = \{0, +\} \cup \{\lambda_k : k \in K\}$ où λ_k est la multiplication par le scalaire k pour tout $k \in K$.

Une théorie est fortement minimale si la formule $x = x$ est fortement minimale. Morley démontre qu'un modèle d'une telle théorie T est caractérisé à isomorphisme près par sa dimension. Dans

ce cas il est facile de déduire de ce résultat que T est \aleph_1 -catégorique : en effet la cardinalité de la clôture algébrique d'un ensemble infini X est $\max\{|X|, \aleph_0\}$, donc la dimension d'un ensemble indénombrable est égal à sa cardinalité. Morley démontre aussi une réciproque de cette propriété.

Proposition 0.5. *Soit T une théorie ω -stable, alors si $M \models T$ il y a un ensemble définissable fortement minimal dans M .*

L'étude des ensembles fortement minimaux a connu deux développements parallèles. D'un côté la théorie de la stabilité géométrique, qui analyse plus précisément les ensembles définissables avec pour principal outil la notion de dimension (le rang et degré de Morley dans les théories ω -stables par exemple). De l'autre la théorie de la stabilité au sens de Shelah, qui est basée sur les propriétés combinatoire des formules. C'est à cette deuxième notion que l'on va s'intéresser à présent.

Definition 0.6. Une formule $\varphi(x, y)$ dans une théorie T est dite *stable* si dans tout modèle M de T il n'existe pas de paramètres $(a_i)_{i < \omega}, (b_i)_{i < \omega} \in M$ tels que $M \models \varphi(a_i, b_j)$ si et seulement si $i \leq j$, c'est à dire si cette formule n'ordonne pas d'ensemble infini dans un modèle de T . Une théorie est dite stable si toutes ses formules sont stables.

Il existe de nombreuses caractérisations de cette propriété, en termes de taille de l'espace de types : en effet une théorie est stable si et seulement si elle est κ -stable pour un cardinal κ infini, mais aussi de complexité topologique de celui-ci avec le rang de Cantor-Bendixon, ou encore de définissabilité des types.

Toutes ces caractérisations et les théorèmes qui en découlent font de la classe des théories stables un cadre riche et de la propriété de stabilité la première 'ligne de division' définie par Shelah dans son programme de classification des théories du premier ordre. Une ligne de division est une propriété telle qu'elle-même ou sa négation ait des conséquences sur la structures des modèles d'une théorie, ladite propriété impliquant des résultats positifs (par exemple l'existence de certaines relations, d'une notion de rang sur les ensembles définissables), et la négation témoignant d'une structure 'chaotique'.

C'est un domaine actif de la recherche en théorie des modèles de classifier les théories selon ces lignes de division, d'en définir de nouvelles et de prouver des résultats structurels dans celles-ci. Ces équivalences entre propriétés combinatoires et structurelles sont très intéressantes et permettent de démontrer de nombreux résultats.

Cette thèse est dédiée à l'une de ces classes de théories : les théories NSOP₁. Il est impossible de faire ici l'inventaire complet des lignes de division tellement elles sont nombreuses, on se contentera de donner une carte de l'état actuel de la classification :

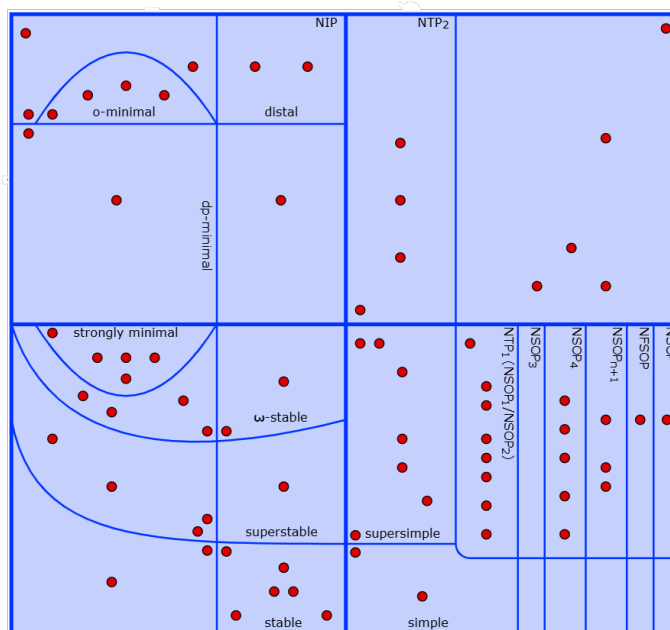


Figure 0.1: La carte de l'Univers - forkinganddividing.com

Comme vous pouvez le voir, ce tableau est tout à fait indéchiffrable. Heureusement dans ce manuscrit nous ne mentionnerons que quelques classes de théories :

1. les théories stables, dont les propriétés et les preuves de celles-ci constituent une source d'inspiration pour chercher des résultats équivalents dans les classes plus générales.
2. les théories simples auxquelles est dédiée la section suivante, qui généralisent les théories stables,
3. et enfin les théories NSOP₁, qui généralisent les théories simples, auxquelles est dédiée cette thèse, et dont nous reparlerons avec beaucoup de détail.

Simplicité

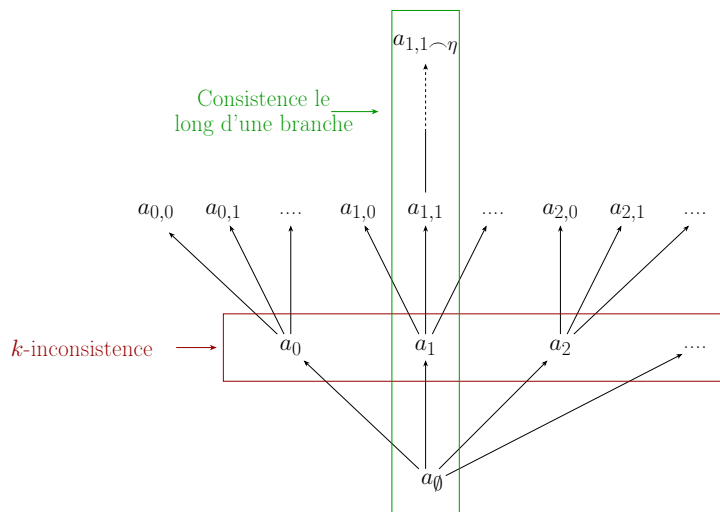
Comme mentionné précédemment la simplicité est une généralisation de la stabilité. Une théorie est dite simple si dans tout modèle M de T et pour tout entier k il n'existe pas de formule $\varphi(x, y)$ et d'arbre de paramètres $(a_\nu)_{\nu \in \omega^{<\omega}} \in M$ tels que :

1. Consistance le long d'une branche : $\{\varphi(x, a_{\nu|n}) : n < \omega\}$ est consistant pour tout $\nu \in \omega^\omega$.
2. k -inconsistance des successeurs : $\{\varphi(x, a_{\nu \smallfrown n}) : n < \omega\}$ est k -inconsistant pour tout $\nu \in \omega^{<\omega}$.

Les théories simples ont beaucoup de caractérisations en termes de propriétés des types, et plus particulièrement d'extensions de types. Pour exprimer celles-ci nous allons avoir besoin d'un nouvel outil, la notion d'indépendance de Shelah, ou de façon plus concise la Shelah-indépendance. On note $a \perp_A^f B$ la propriété que a est Shelah-indépendant de B sur A . Nous définirons en détail ces notions dans la section préliminaire, pour l'instant on se contentera de savoir que \perp^f est une relation ternaire entre les parties d'un modèle d'une théorie T qui est définie pour toute théorie T .

Certaines propriétés de \perp^f caractérisent la simplicité, par exemple :

Theorem 0.7. [Wag00, Théorème 2.4.7] Une théorie T est simple si et seulement si la relation 'être Shelah-indépendant sur E ' est symétrique pour tout ensemble E .

Figure 0.2: La propriété du k -arbre

Reciproquement, le théorème de Kim-Pillay énonce que l'existence d'une relation ternaire entre les parties d'un modèle monstre de T satisfaisant certaine de ces propriétés caractérise la Shelah-indépendance dans une théorie simple. De façon légèrement plus formelle :

Theorem 0.8. [Wag00, Théorème 2.6.1] (Théorème de Kim-Pillay) Soit T une théorie complète et \mathbb{M} un modèle monstre de M . Alors T est simple si et seulement si il existe une relation ternaire \downarrow^* entre les parties de \mathbb{M} satisfaisant un certain ensemble de propriétés, et dans ce cas \downarrow^f coïncide avec \downarrow^* .

Le théorème de Kim-Pillay est très intéressant car il établit un lien entre cette relation de Shelah-indépendance théorique qui peut éventuellement être complexe à caractériser, et ses propriétés que l'on peut vérifier pour d'autres relations d'indépendance apparaissant naturellement dans une théorie, et que l'on peut calculer facilement. Avec cet outil on peut démontrer par exemple que la relation de forking indépendance dans la théorie des graphes aléatoires est donnée par $A \downarrow_E^f B$ si et seulement si $A \cap B \subseteq E$ en démontrant des propriétés basique de cette relation.

Du point de vue des lignes de divisions ce résultat motive l'étude des relations d'indépendance, qui nous donnent un nouvel outil pour l'analyse des ensembles définissables dans une théorie.

Relations d'indépendance en théorie des modèles

Une relation d'indépendance dans un modèle M d'une théorie T est une relation ternaire entre les sous-ensembles de M , elles sont notées habituellement \downarrow . Les relations d'indépendance sont l'un des outils principaux de la théorie des modèles moderne et nous en ferons usage tout au long du manuscrit.

Les relations d'indépendance sont nombreuses et variées, elles sont de deux types principaux. Elles peuvent être issues d'une définition purement modèle-théorique, c'est le cas de l'indépendance de Shelah \downarrow^f mentionnée précédemment ainsi que de la Kim-indépendance \downarrow^K que nous introduirons dans la prochaine section. Elles peuvent aussi être issues de propriétés algébriques de la théorie, c'est le cas de l'indépendance linéaire dans un espace vectoriel, ou encore l'indépendance algébrique \downarrow^a , qui est définie dans toute théorie comme $A \downarrow_E^a B$ si et seulement si $\text{acl}(EA) \cap \text{acl}(EB) = \text{acl}(E)$.

L'un des enjeux dans l'étude d'une théorie est de démontrer les propriétés des relations \downarrow^f et \downarrow^K , pour pouvoir ensuite exploiter celles-ci. Ceci peut se faire grâce à l'utilisation d'autres

relations d'indépendance, plus naturelles et plus simple à calculer. Un exemple de telle propriété est celle d'existence : une théorie satisfait l'axiome d'existence si pour tout $a, A \in M$ modèle de T on a $a \downarrow_A^f A$. Une stratégie que l'on emploie dans le chapitre 2 pour démontrer cette propriété dans une théorie T est de définir une nouvelle relation \downarrow^Γ , que l'on peut caractériser facilement, et de montrer que $a \downarrow_A^\Gamma A$ pour tout $a, A \in M$ ainsi que \downarrow^Γ implique \downarrow^f .

Les relations d'indépendance peuvent aussi être vues comme des propriétés des types. Dans ce cas elles servent à isoler certaines extensions d'un type que l'on reconnaît comme particulièrement libres, ce qu'on peut appeler alors ses extensions 'génériques'. On considère ici un exemple de ce fait avec le cas de la théorie T du graphe aléatoire dans le langage $L = \{R\}$ ($R(a, b)$ signifie que a et b sont connectés dans le graphe). T est axiomatisée par le schéma suivant : pour tout entier n et toutes paire d'ensembles disjoints de n sommets A et B il existe un sommet x connecté à tout les éléments de A et à aucun de ceux de B . En appliquant le critère du va-et-vient on peut montrer aisément que T a l'élimination des quantificateurs dans L .

Soit \mathbb{M} un modèle monstre de T et \downarrow^a la relation d'indépendance algébrique définie précédemment. Dans [TZ12, Corollary 7.3.14] Tent et Ziegler démontrent en utilisant le théorème de Kim-Pillay que T est simple et que $\downarrow^a = \downarrow^f$. Pour ce faire Tent et Ziegler démontrent des propriétés calculatoires de \downarrow^a dans \mathbb{M} , parmi celles-ci il y a le théorème d'indépendance, ce résultat permet d'étendre l'union de deux types indépendants sur E de façon indépendante.

Proposition 0.9. *Soit $E, A_i, B_i \subseteq \mathbb{M}$ tels que $A_0 \equiv_E A_1$, $A_i \downarrow_E^a B_i$ pour $i = 0, 1$ et $B_0 \downarrow_E^a B_1$. Alors il existe $A \subseteq \mathbb{M}$ tel que $A \downarrow_E^a B_0 B_1$ et $A \equiv_{B_i} A_i$ pour $i = 0, 1$.*

Ce résultat exprime le fait que \downarrow^a témoigne d'une certaine indépendance, en effet $\text{tp}(A_0/B_0)$ et $\text{tp}(A_1/B_1)$ ne se contredisent pas mutuellement du fait que leurs paramètres sont indépendants au sens de \downarrow^a .

On peut définir une autre relation plus forte \downarrow^Γ par $A \downarrow_E^\Gamma B$ si et seulement si $A \downarrow_E^a B$ et pour tout $a \in A$ et $b \in B$, $R(a, b)$ entraîne $a \in E$ ou $b \in E$. Cette relation \downarrow^Γ est une relation d'indépendance dite stationnaire, ce qui signifie qu'elle définit une unique extension : formellement, si $A' \equiv_E A$ et $A' \downarrow_E^\Gamma B$ alors $A' \equiv_B A$. Cette relation exprime que A et B sont 'les plus indépendants possible sur E ', on peut voir ça du point de vue de la catégorie des graphes : en effet si $A \downarrow_E^\Gamma B$ alors le sous-graphe de \mathbb{M} induit sur $E \cup A \cup B$ correspond à la somme directe des graphes induits sur $E \cup A$ et $E \cup B$ au dessus de E .

NSOP₁ et la Kim-indépendance

Comme mentionné précédemment les théories NSOP₁ sont une généralisation des théories simples. Elles sont définies par une propriété d'arbre similaire à celle des théories simples. Le premier exemple de théorie NSOP₁ non-simple est celui des relations d'équivalences paramétrées étudié par Shelah et Usvyatsov dans [SU08].

Cette classe a été étudiée intensément au cours des dernières années en suivant le modèle des résultats dans les théories simples. Le principal moteur de cette étude fut un analogue de l'indépendance de Shelah, la Kim-indépendance, qui est vue comme une indépendance de Shelah générique et est définie dans un premier temps avec des modèles pour base, on parlera dans ce cas de Kim-indépendance sur les modèles. Chernikov, Kaplan, Kim, Ramsey et Shelah démontrent les propriétés de la Kim-indépendance dans les théories NSOP₁ dans [KR17], [KRS19] et [KR21]. Cette relation coïncide avec l'indépendance de Shelah dans une théorie simple, et de nombreuses propriétés de l'indépendance de Shelah dans les théories simples ont un analogue dans le cadre NSOP₁. Par exemple :

Theorem 0.10. [KR17, Théorème 5.9] *Une théorie complète T est NSOP₁ si et seulement si la relation de Kim-indépendance est symétrique sur tous les modèles, c'est à dire pour tout $M \models T$, $M, a, b \in \mathbb{M}$ dans un modèle monstre de T on a $a \perp_M^K b$ ssi $b \perp_M^K a$.*

Kaplan et Ramsey démontrent aussi un critère semblable au théorème de Kim-Pillay pour la Kim-indépendance dans les théories NSOP₁ :

Theorem 0.11. [KR20, Theorem 8.1] *(Théorème de Kim-Pillay pour la Kim-indépendance) Soit T une théorie complète et \mathbb{M} un modèle monstre de M . Alors T est NSOP₁ si et seulement si il existe une relation ternaire \perp^* entre les parties de \mathbb{M} satisfaisant un certain ensemble de propriétés, et dans ce cas \perp^K coïncide avec \perp^* .*

Ce résultat permet à Chernikov et Ramsey de démontrer que certaines théories, dans lesquelles l'existence de relations d'indépendance satisfaisant presque toutes les conditions du théorème de Kim-Pillay pour les théories simples était déjà connue, sont NSOP₁. C'est le cas de la théorie des formes bilinéaires génériques sur un espace vectoriel de dimension infinie sur un corps algébriquement clos que nous aborderons dans une section suivante, ainsi que de la théorie des corps PAC e -libres pour e un entier, et des corps PAC ω -libres, sur lesquels nous reviendrons aussi.

Une seconde évolution de l'étude des propriétés de la Kim-indépendance a lieu lorsque Dobrowolski, Kim et Ramsey la définissent sur une base quelconque au lieu d'un modèle dans [DKR22], sous l'hypothèse que la théorie ambiante satisfait l'existence. On parle alors de Kim-indépendance sur des ensembles quelconques. Les propriétés calculatoires de la Kim-indépendance dans les théories NSOP₁ sont généralisées aux bases quelconques dans [DKR22] et [CKR23], et plus tard aux hyperimaginaires dans [DK21] et [Bos22].

Un critère de Kim-Pillay pour la Kim-indépendance dans les théories avec existence est démontré par Chernikov, Kim et Ramsey [CKR23, Théorème 5.1.] et caractérise la Kim-indépendance sur les ensembles quelconques dans les théories NSOP₁ avec existence. On applique ce résultat dans le Chapitre 2 pour caractériser la Kim-indépendance sur des bases quelconques.

Dans la partie préliminaire on définira formellement les notions que nous avons mentionné jusque-là, les relations d'indépendance, la propriété NSOP₁, et on énoncera les propriétés de la Kim-indépendance dans les théories NSOP₁.

Les corps PAC

On introduit à présent un exemple de théorie NSOP₁ particulièrement intéressant pour deux raisons, premièrement historiquement car ces théories et certaines propriétés liées à NSOP₁ étaient étudiées avant que cette propriété soit reconnue, deuxièmement car c'est un exemple que nous considérerons dans le Chapitre 4 de ce manuscrit pour y étudier les groupes définissables.

Un corps pseudo-algébriquement clos est un corps F tel que toute variété absolument irréductible définie sur F admet un point F -rationnel. Ils apparaissent pour la première fois dans le travail de Ax sur les corps pseudo-finis ([Ax68]), et ont depuis été étudié notamment par Fried et Jarden dans [FJ⁺08], Cherlin, Van Den Dries, Macintyre dans [CvdDM80] et Chatzidakis dans [Cha10], [Cha02] et [CP98].

Pour F un corps, son groupe de Galois absolu $\mathcal{G}(F)$ est défini comme la limite profinie des groupes de Galois de ses extensions finies, ce que l'on note $\mathcal{G}(F) \cong \lim Gal(L/F)$ avec L parcourant les extensions galoisiennes finies, et l'application de restriction de $Gal(L'/F)$ vers $Gal(L/F)$ lorsque $L \subseteq L'$ sont deux extensions galoisiennes finies. $\mathcal{G}(F)$ ainsi défini est un groupe topologique ayant pour base d'ouverts les images réciproques des éléments des groupes du système, ce qui correspond

aux ensembles d'automorphismes de F^{alg} coïncidant sur une extension galoisienne finie spécifique. Un corps est dit borné si pour tout entier n il n'admet qu'un nombre fini d'extensions séparables de degré n .

De tels systèmes de groupes finis associés à un groupe topologique, alors appelé groupe profini, furent étudiés dans [CvdDM80] par Cherlin, Van Den Dries et Macintyre à l'aide d'un langage L_ω avec une sorte pour chaque entier n , correspondant aux classes d'équivalences modulo les sous-groupes ouverts normaux d'indice $\leq n$ du groupe profini. Il appliquent ces resultat a l'étude des corps PAC dans [CvdDM80](appelés alors régulièrement clos) ou ils démontrent que certaines propriétés de ces corps peuvent se déduire de propriétés du groupe de Galois absolu. Plus spécifiquement que la théorie d'un corps PAC F est entièrement déterminée par trois invariants : le type d'isomorphisme des éléments de F algébriques sur le sous-corps premier (càd de la clôture algébrique de l'ensemble vide), le degré d'imperfection de F , et la théorie du système projectif des groupes de Galois au sens de L_ω .

Cette étude est reprise plus tard par Chatzidakis dans [Cha98] dans le cas de groupes profinis avec la propriété d'Iwasawa. Ce cas inclu les groupes profinis libres sur e générateurs, notés \hat{F}_e , avec e un entier ou $e = \omega$. \hat{F}_e est défini comme la limite des quotients finis du groupe libre à e générateurs : en notant F_e le groupe libre à e générateurs, \hat{F}_e est la limite projective du système des F/N avec N parcourant les sous-groupes normaux d'indice fini de F . Un corps PAC est dit e -libre si son groupe de Galois absolu est \hat{F}_e .

Sur le plan de la classification Chatzidakis démontre dans [Cha10, Théorème 3.9] qu'un corps PAC parfait est simple si et seulement si il est borné (en particulier les corps PAC e -libres parfaits sont simple), et que les corps PAC ω -libres sont NSOP et non simples (plus précisément qu'ils sont NSOP₃ dans [Cha19, Theorem 2.10]). Elle déduit ces résultats de plusieurs théorèmes d'amalgamation qu'elle démontre, et qui sont plus tard appliqués dans [KR17, Section 9.3] par Kaplan et Ramsey pour prouver, grâce au critère de Kim-Pillay pour le Kim-forking, que les corps PAC ω -libres sont NSOP₁. Dans le Chapitre 4 on démontre un résultat sur les groupes définissables dans la théorie des corps PAC ω -libres parfaits de caractéristique 0.

Classe de Fraïssé avec amalgamation libre

Dans le Chapitre 2 on s'intéresse à la construction d'exemples de théories NSOP₁. La méthode principale employée lors de la construction de tels exemples est celle d'extension générique. On commence avec une théorie stable ou simple T dans un langage L . On étend le langage en L' et la théorie en une théorie incomplète T' sur L' . On s'intéresse ensuite, lorsqu'elle existe, à un modèle-complétion de T' .

Les exemples de telles extensions nombreux, on peut citer les expansions génériques de corps, étudiées par D'Elbée dans sa thèse [d'E19].

Dans le Chapitre 2 on construit donc la modèle-complétion de certaines théories. A l'origine de ce travail il y a une construction apparaissant dans [Bau02], la paramétrisation d'une théorie. Etant donné une théorie T dans un langage L , on considère le langage L_P où l'on ajoute une nouvelle sorte P , appelée sorte paramètre, et une arité dans P à tout les symboles de L , par exemple une constante c dans L de la sorte V devient une fonction unaire $c(x) : P \rightarrow V$. On définit ensuite la théorie T_P exprimant que pour tout paramètre $p \in P$ la L -structure induite par p est un modèle de T . Cette théorie est incomplète et on s'intéresse à un modèle-companion, c'est à dire à une théorie qui, si elle existe, décrit une collection de modèles de T sur un même ensemble de base interagissant de façon aléatoire.

Dans [CR16, Section 6.3] Chernikov et Ramsey donnent un exemple de théorie T et étudient un modèle-companion de T_P . Cette théorie T ainsi que le modèle-companion de T_P sont construites comme les limites d'une classe de Fraïssé. Ramsey et Chernikov considèrent un langage relationnel L et une classe K de L -structures avec la propriété d'amalgamation forte. La classe paramétrée K_P est alors définie comme la classe des L_P structures A telles que pour tout $p \in P(A)$ la L -structure induite par p sur A est dans K . Ils montrent que la limite de Fraïssé de cette classe est NSOP_1 , et donnent des conditions sur K pour qu'elle ne soit pas simple.

Dans le Chapitre 2 on donne un contexte pour faire une construction similaire sans supposer que le langage de base soit relationnel. Ceci se fait au coût de fortes hypothèses sur la structure engendrée, on donne un ensemble d'hypothèses (H) renforçant les propriétés de Fraïssé usuelles. On démontre des résultats similaires à ceux de Chernikov et Ramsey : Si une classe K satisfait (H) , alors on peut définir la théorie limite, cette théorie est NSOP_1 , satisfait l'existence, et on peut caractériser la Kim-indépendance et la Shelah-indépendance dans T . Ceci se fait grâce à l'utilisation d'une relation d'indépendance provenant de l'hypothèse que la classe K est close par amalgamation libre au sens catégorique. Cette relation d'indépendance était déjà mentionnée dans [Bau14], et ne coïncide pas avec la notion d'amalgamation libre de [Con17], qui est aussi une relation stationnaire mais requiert une condition supplémentaire sur la clôture algébrique, qui est que si A et B sont indépendants et algébriquement clos alors leur union est algébriquement close —ce qui ne sera pas vérifié dans notre cas à moins que la clôture algébrique soit triviale.

On démontre aussi que la condition (H) est préservée par certaines modifications de la classe K : la paramétrisation que nous avons présenté précédemment et l'ajout de structure générique à K . On donne une condition sur K sous laquelle la théorie limite de la classe paramétrée K_P est non-simple. Ce contexte nous permet de construire un modèle-companion de certaines théories exotiques, par exemple la théorie du groupe abélien avec une fonction unaire f telle que $f(0) = 0$, qui nous servira d'exemple de théorie de groupe NSOP_1 dans le Chapitre 4.

Théorie des modèles des formes bilinéaires

La théorie à deux sortes des formes bilinéaires a été étudiée en premier lieu par Granger dans sa thèse [Gra99]. Granger démontre que la théorie des formes bilinéaires symétriques non dégénérées sur un espace vectoriel de dimension infinie sur un corps algébriquement clos, que l'on notera sT_∞^{ACF} , a l'élimination des quantificateurs une fois que l'on ajoute au langage les fonctions donnant les coefficients des relations de dépendance linéaires. Il définit aussi une relation d'indépendance forte \perp^Γ et démontre certaines de ses propriétés. Cet exemple est repris plus tard par Kaplan et Ramsey dans [CR16, Section 6.1] qui démontrent que cette théorie est NSOP_1 et qui y calculent la relation de Kim-indépendance sur les modèles.

Il y a quelques problèmes avec certaines preuves de [Gra99], en particulier avec celle de l'élimination des quantificateurs. On peut trouver une preuve correcte et des remarques à ce sujet par Abd Aldaim, Connant et Terry dans la Section 5 de [AACT25] ou encore par Chernikov et Valentin-Hempel dans [CH24]. Dans le Chapitre 3 nous avons une hypothèse supplémentaire portant sur la forme bilinéaire, notée (\dagger) avant Lemma 3.1, ce qui nous permet de faire une preuve plus élémentaire de l'élimination des quantificateurs que dans ces deux articles.

Dobrovolski donne une étude plus profonde des ensembles définissables de cette théorie dans [Dob23]. Il définit une notion de dimension pour les ensembles définissables qui lui permet de démontrer que les groupes définissables dans cette théorie sont (algébrique-par-abélien)-par-algébrique. Il s'intéresse aussi au groupe additif de l'espace vectoriel et démontre qu'il ne satisfait pas certaines propriétés des groupes définissables dans une théorie simple.

Dans le Chapitre 3 on généralise certains des résultats sur sT_∞^{ACF} en considérant d'autres théories de corps que ACF . On montre que si K est la théorie d'un corps $NSOP_1$ alors la théorie sT_∞^K des formes bilinéaires non-dégénérées sur un espace vectoriel de dimension infinie sur un modèle de K est aussi $NSOP_1$ et on calcule la Kim-indépendance sur des modèles dans ce cas. Cette preuve se fait en appliquant le théorème de Kim-Pillay pour la Kim-indépendance sur les modèles. On montre aussi que lorsque la théorie des corps K est simple alors la théorie sT_∞^K satisfait l'existence et dans ce cas on calcule la Kim-indépendance sur des ensembles quelconques.

On se sert aussi de l'exemple de $sT_\infty^{ACF,eq}$ pour aborder une conjecture à propos d'un lien entre \downarrow^f et \downarrow^K dans les théories $NSOP_1$. Pour ceci on se sert d'un résultat d'élimination des imaginaires de Dobrowolski pour caractériser $sT_\infty^{ACF,eq}$.

On peut aussi citer à ce sujet un article plus récent de Chernikov et Valentin-Hempel, [CH24], qui considère le cas plus général des formes n -linéaires et d'autres propriétés combinatoires : NIP et la n -dépendance. Ils démontrent, entre autres, que si la théorie du corps K est $NSOP_1$ alors la théorie des formes n -linéaires alternées sur un espace vectoriel de dimension infinie sur un modèle de K est aussi $NSOP_1$.

Stabilisateur dans un groupe

Lorsque l'on étudie une théorie T on souhaite comprendre ses ensembles définissables. En particulier on veut analyser les groupes définissables, c'est-à-dire des groupes dont l'ensemble de base G est définissable et la loi de groupe, vue comme un sous-ensemble de G^3 , l'est aussi. On a mentionné précédemment à ce sujet les résultats de Dobrowolski dans [Dob23] sur les formes bilinéaires et de Chatzidakis et Ramsey dans [CR23] sur les groupes définissables dans les corps PAC e -libres avec e fini.

Un résultat efficace pour ceci est le théorème de configuration de groupe, que Hrushovski démontra dans sa thèse [Hru86]. Ce résultat permet d'obtenir un groupe type-définissable à partir d'une certaine configuration d'éléments indépendants dans une théorie stable. Comme application typique de ce résultat on peut citer :

Theorem 0.12. [HP94, Théorème C] *Soit G un groupe définissable dans un corps pseudo-fini F . Alors G est virtuellement définissablement isogène à l'ensemble des points F -rationnels d'un groupe algébrique sur F .*

Il y a deux notions majeures dans l'étude des groupes en théorie des modèles que l'on va présenter maintenant de façon assez informelle : la notion de formule et type générique dans un groupe, et celle de stabilisateur d'un type dans un groupe. Soit G un groupe définissable dans un modèle \mathbb{M} d'une théorie T .

Un sous-ensemble définissable X de G est dit générique à gauche si un nombre fini de translatés à gauche de X recouvre G , similairement pour générique à droite. Un type est dit générique (à gauche/droite) si il n'implique que des formules génériques (à gauche/droite). Dans une théorie stable il existe des types complets génériques sur tout ensemble de paramètres, mais ce n'est pas vrai en général. Il existe une notion similaire adaptée aux théories simples. Étant donné un type complet $p \in S_G(A)$ le stabilisateur de p est un sous-groupe $Stab(p) \leq G$ engendré par des éléments 'stabilisant' le type, au sens suivant : $g \in Stab(p)$ ssi $p \cup g \cdot p$ est consistant, ou $g \cdot p$ est le type p translaté par g . Le lien entre ces deux notions est qu'un type générique doit avoir un stabilisateur 'grand' - d'indice borné par exemple.

La stratégie typique pour étudier un groupe définissable G dans un modèle F d'une théorie de corps est la suivante :

1. On prend a, b, x des éléments génériques (c'est à dire dont le type est générique) indépendants dans le groupe G , on note $c := a \cdot b$.
2. Le théorème de configuration de groupe nous fournit des éléments a', b', c' dans un groupe H définissable dans la cloture algébrique de F^{alg} de F (c'est à dire un groupe algébrique) interalgébriques avec a, b, c respectivement sur un ensemble de paramètre E .
3. On considère alors le stabilisateur de $\text{tp}(a, a'/E)$ dans $G \times H$, et on montre qu'il définit une isogénie entre un G et H .

C'est cette méthode que l'on veut appliquer pour étudier les groupes définissables dans les corps PAC ω -libres de caractéristique 0. Pour ceci on cherche à généraliser les notions de généricité et de stabilisateur à NSOP₁. Pour la notion de générique en prenant la généralisation 'naïve' du générique dans le cadre simple on a une réponse négative (ce qui était déjà remarqué par Dobrovolski : [Dob23, Proposition 8.14]). Pour la notion de stabilisateur on donne une définition généralisant celle du cas simple, qui définit en général un groupe invariant. On montre que dans certains exemples le sous-groupe obtenu, que l'on appellera le Kim-stabilisateur, est type-définissable. On généralise certains des résultats sur les groupes définissables dans une théorie simple à cette notion.

On fait aussi intervenir une autre notion de stabilisateur, due à Hrushovski dans [Hru12]. Un idéal de formules μ est un ensemble de formules clos par disjonction et implication logique (si $\theta \in \mu$ et $\theta \vdash \theta'$ alors $\theta' \in \mu$). Un idéal μ est dit *S1* sur un ensemble de paramètres A si il est A -invariant et si $\varphi(x, a_i) \wedge \varphi(x, a_j) \in \mu$ pour $i \neq j < \omega$ entraîne $\varphi(x, a_i) \in \mu$ pour toute suite A -indiscernable $(a_i)_{i < \omega}$ et toute formule $\varphi(x, y)$. Sous l'hypothèse de l'existence d'un idéal de formules *S1* dans un groupe il définit une notion de stabilisateur et montre qu'il s'agit d'un sous-groupe type-définissable.

En utilisant les deux notions de stabilisateurs, celle de Hrushovski et le Kim-stabilisateur on démontre sous certaines hypothèses sur un corps NSOP₁ F et un groupe G définissable dans F qu'il existe un sous groupe type-définissable de G d'indice borné et un plongement de noyau fini de celui-ci dans les points F -rationnels d'un groupe algébrique définissable sur F . On montre ensuite que les groupes définissables dans les corps PAC ω -libres de caractéristique 0 satisfont ces hypothèses.

Appendice : Kim-forking stable

La conjecture de la déviation stable est la conjecture selon laquelle dans une théorie simple, la déviation est témoinnée par une formule stable. C'est une conjecture de longue date, on peut à son sujet se référer à l'article [KP01] qui donne une exposition complète de ce problème. Dans cet appendice on donne un énoncé analogue dans le cadre des théories NSOP₁ et de la Kim-déviation, la Kim-déviation faiblement stable. On connecte cette notion à certains résultats sur la déviation stable que l'on peut trouver dans [CP18]. On peut aussi citer à ce sujet [BFM24] un récent article de Baldwin, Freitag et Mutchnik dans lequel ils démontrent entre autre que la Kim-déviation dans une théorie NSOP₁ est toujours témoinnée par une formule simple, c'est-à-dire n'ayant pas la propriété de l'arbre.

Introduction (English version)

We begin by chronologically introducing NSOP₁ theories and classification. We then present the different concepts that are addressed throughout the manuscript by placing them in their context.

Stability and Classification

One of the problems that interested model theorists in 1970 was to classify the models of a theory, that is, to count the models of cardinality κ up to isomorphism. One of the important notions for studying this problem is that of κ -stable theory. Here, we consider that all languages are countable. A theory T is said to be κ -categorical for a cardinal κ if it admits a unique model of cardinality κ up to isomorphism.

Definition 0.13. A complete theory T is κ -stable for an infinite cardinal κ if for all model M of T of cardinality κ the space of types $S_n(M)$ has cardinality κ .

An important result in this direction was Morley's theorem:

Theorem 0.14. *If a theory in a countable language is κ -categorical for some uncountable cardinal κ , then it is categorical for all uncountable cardinals. Moreover, an \aleph_1 -categorical theory is ω -stable.*

In his proof Morley develops numerous tools for analyzing the complexity of definable sets. He defines the following notions:

Definition 0.15. *Strongly minimal set:* Let M be an L -structure and $D \subseteq M$ be an infinite definable subset, we say that D is minimal in M if for every definable subset $Y \subseteq D$, Y is finite or cofinite. We say that D is strongly minimal if it is minimal in every elementary extension of M . A formula is strongly minimal if it defines a strongly minimal set.

Definition 0.16. *Dimension on strongly minimal sets:* Let M be an L -structure and $D \subseteq M$ a definable strongly minimal set. A set $A \subseteq D$ is said to be independent if $a \notin \text{acl}(A \setminus \{a\})$ for all $a \in A$. A is a basis of D if moreover $D \subseteq \text{acl}(A)$, and then the dimension of D is the cardinality of A , denoted $\dim(D)$ (which does not depend on the set A).

This notion of dimension coincides with several well-known notions: The transcendence degree in the theory of an algebraically closed field with the language of rings, or the linear dimension in the theory of a vector space E over a field K in the one-sorted language $L_K = \{0, +\} \cup \{\lambda_k: k \in K\}$ where λ_k is the scalar multiplication k for any $k \in K$. A theory is strongly minimal if the formula $x = x$ is strongly minimal. Morley demonstrates that a model of such a theory T is characterized up to isomorphism by its dimension. In this case it is easy to deduce from this result that T is \aleph_1 -categorical: In fact, the cardinality of the algebraic closure of an infinite set X is $\max\{|X|, \aleph_0\}$, so the dimension of an uncountable set is equal to its cardinality. Morley also demonstrates a reciprocal of this property.

Proposition 0.17. *Let T be an ω -stable theory, then there is a definable set which is strongly minimal in any model M of T .*

The study of strongly minimal sets has undergone two parallel developments. On the one hand, the theory of geometric stability, which analyzes more precisely definable sets with the notion of dimension as its main tool (Morley's rank and degree in ω -stable theories for example). On the other hand, the theory of stability in the sense of Shelah, which is based on the combinatorial properties of formulas. It is this second notion that we present now.

Definition 0.18. A formula $\varphi(x, y)$ in a theory T is called *stable* if in every model M of T there are no parameters $(a_i)_{i < \omega}, (b_i)_{i < \omega} \in M$ such that $M \models \varphi(a_i, b_j)$ if and only if $i \leq j$, meaning if this formula does not order an infinite set in a model of T . A theory is stable if all of its formulas are stable.

There are many characterizations of this property, in terms of the size of the type space: indeed a theory is stable if and only if it is κ -stable for an infinite cardinal κ , but also of the topological complexity of the latter with the Cantor-Bendixon rank, or the definability of the types. All these characterizations and the theorems derived from them make the class of stable theories a rich framework and the stability property the first 'dividing line' defined by Shelah in his project to classify first-order theories. A dividing line is a property such that it or its negation has consequences on the structure of the models of a theory, the said property implying positive results (e.g. the existence of certain relations, of a notion of rank on definable sets), and the negation implying that the structure is 'chaotic'.

It is an active area of research in model theory to classify theories along these dividing lines, to define such properties, and to prove structural results in them. These equivalences between combinatorial and structural properties are very interesting and allow us to demonstrate many results. This thesis is dedicated to one of these classes of theories: the NSOP₁ theories. It is impossible to make a complete inventory of the dividing lines here because they are too many, we will content ourselves with giving a map of the current state of the classification:

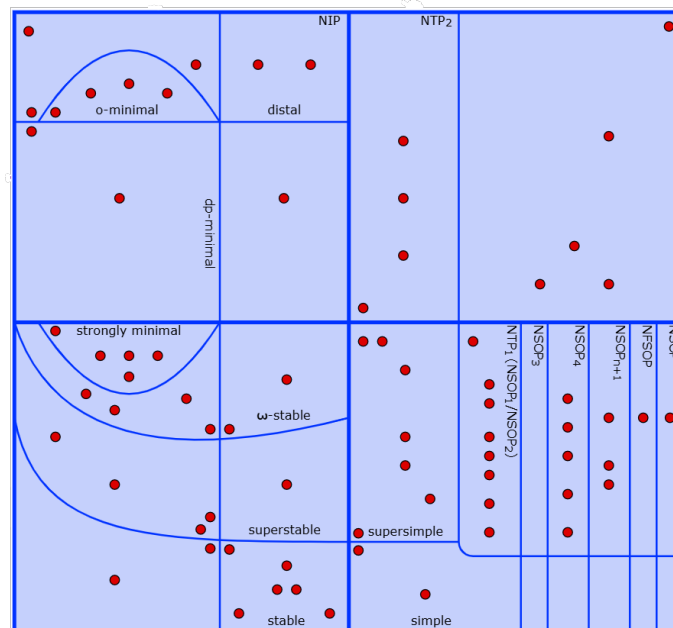


Figure 0.3: The map of the Universe - forkinganddividing.com

As you can see, this picture is completely indecipherable. Fortunately, in this manuscript, we will only mention a few classes of theories:

1. stable theories, whose properties and proofs of them constitute a source of inspiration for seeking equivalent results in the more general classes.
2. the simple theories, to which the next section is dedicated, which generalize stable theories,
3. and finally the NSOP₁ theories, which generalize simple theories, to which this thesis is dedicated, and which we will discuss in great detail.

Simplicity

As mentioned earlier, simplicity is a generalization of stability. A theory is said to be simple if in any model M of T and for any integer k there is no formula $\varphi(x, y)$ and no tree of parameter $(a_\nu)_{\nu \in \omega^{<\omega}} \in M$ such that:

1. Consistency along a branch: $\{\varphi(x, a_{\nu|n}) : n < \omega\}$ is consistent for all $\nu \in \omega^\omega$.
2. k -inconsistency of successors: $\{\varphi(x, a_{\nu \smallfrown n}) : n < \omega\}$ est k -inconsistent for all $\nu \in \omega^{<\omega}$.

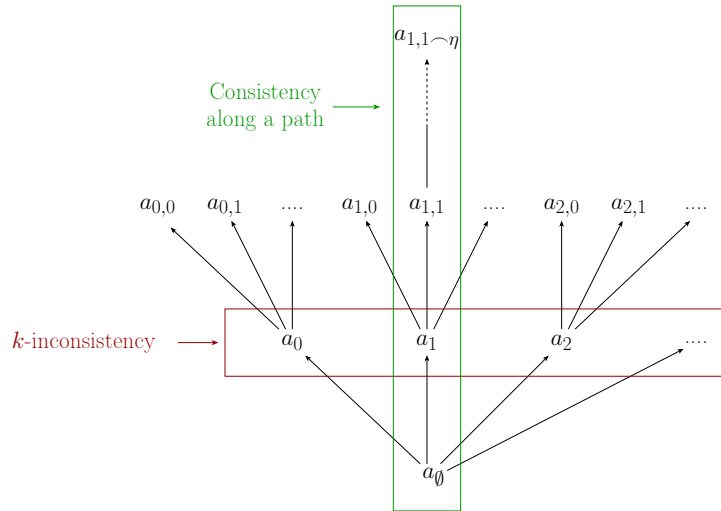


Figure 0.4: The k -tree property

Simple theories have many characterizations in terms of the properties of types, and more particularly the extension of types. To express these we will need a new tool, the notion of Shelah-independence. We denote $a \perp_A^f B$ the property that a is Shelah-independent of B over A . We will define these notions in detail in the preliminary section, for now we will just consider \perp^f as a ternary relation between the parts of a model of a theory T that is defined for any theory T .

Some properties of \perp^f characterize simplicity, for example:

Theorem 0.19. [Wag00, Theorem 2.4.7] *A theory T is simple if and only if $a \perp_E^f b$ is equivalent to $b \perp_E^f a$ for every E, a, b .*

Conversely, the Kim-Pillay theorem states that the existence of a ternary relation between the parts of a monster model of T satisfying some of these properties characterizes Shelah-independence in a simple theory. In a slightly more formal way:

Theorem 0.20. [Wag00, Theorem 2.6.1] (Kim-Pillay theorem) *Let T be a complete theory and \mathbb{M} be a monster model of T . T is simple if and only if there exists a ternary relation \perp^* between the subsets of \mathbb{M} satisfying a particular set of properties, and in that case \perp^f coincides with \perp^* .*

The Kim-Pillay theorem is very interesting because it establishes a link between this theoretical Shelah-independence relation, which can possibly be complex to characterize, and its properties that can be verified for other independence relations that appear naturally in a theory, and that can be easily calculated. With this tool we can demonstrate for example that the Shelah-independence relation in the theory of the random graph is given by $A \downarrow_E^f B$ if and only if $A \cap B \subseteq E$, and we do this by proving basic properties of this relation.

From the point of view of dividing lines, this result motivates the study of independence relations, which give us a new tool for the analysis of definable sets in a theory.

Independence relations in Model Theory

An independence relation in a model M of a theory T is a ternary relation between subsets of M , they are usually denoted \downarrow . Independence relations are one of the main tools of modern model theory and we will use them throughout the manuscript.

Relations of independence are many and varied, they are of two main types. They can be derived from a purely model-theoretical definition, as is the case of the Shelah independence \downarrow^f mentioned above as well as the Kim-independence \downarrow^K that we will introduce in the next section. They can also be derived from algebraic properties of the theory, such as linear independence in a vector space, or the algebraic independence \downarrow^a , which is defined in any theory as $A \downarrow_E^a B$ if and only if $\text{acl}(EA) \cap \text{acl}(EB) = \text{acl}(E)$.

One of the challenges in the study of a theory is to demonstrate the properties of the relations \downarrow^f and \downarrow^K , in order to then be able to exploit them. This can be done through the use of other, more natural and simpler to calculate relations of independence. An example of such a property is that of existence: a theory satisfies the axiom of existence if for any $a, A \in M$ model of T we have $a \downarrow_A^f A$. A strategy used in Chapter 2 to prove this property in a T theory is to define a new relation \downarrow^Γ , which can be easily characterized, and to show that $a \downarrow_A^\Gamma A$ for any $a, A \in M$ and that \downarrow^Γ implies \downarrow^f .

Independence relations can also be seen as properties of types. In this case, they are used to isolate certain extensions of a type that we recognize as particularly free, which we can then call its ‘generic’ extensions. An example of this fact is the case of the T theory of the random graph in the language $L = \{R\}$ ($R(a, b)$ means that a and b are connected in the graph). T is axiomatized by the following scheme: for any integer n and all disjoint pairs A and B , each consisting of n vertices there exists a vertex connecting x to any elements of A no vertex connecting x to an element of B . By applying the back-and-forth criterion, it can easily be shown that T has quantifier elimination in L .

Let \mathbb{M} be a monster model of T and \downarrow^a be the algebraic independence relation previously defined. In [TZ12, Corollary 7.3.14] Tent and Ziegler show by using the Kim-Pillay theorem that T is simple and that $\downarrow^a = \downarrow^f$. To do this, they prove computational properties of \downarrow^a in \mathbb{M} , among these there is the independence theorem, which allows to extend the union of two types independent on E in an independent way.

Proposition 0.21. *Let $E, A_i, B_i \subseteq \mathbb{M}$ be such that $A_0 \equiv_E A_1$, $A_i \downarrow_E^a B_i$ for $i = 0, 1$ and $B_0 \downarrow_E^a B_1$. Then there exists $A \subseteq \mathbb{M}$ such that $A \downarrow_E^a B_0 B_1$ and $A \equiv_{B_i} A_i$ for $i = 0, 1$.*

This result expresses the fact that \downarrow^a implies a certain independence, in fact $\text{tp}(A_0/B_0)$ and $\text{tp}(A_1/B_1)$ do not contradict each other because their parameters are independent in the sense of \downarrow^a .

We can define another stronger relation \downarrow^Γ by $A \downarrow_E^\Gamma B$ if and only if $A \downarrow_E^a B$ and for any $a \in A$ and $b \in B$, $R(a, b)$ implies that in $a \in E$ or $b \in E$. This relation \downarrow^Γ is a so-called stationary relation of independence, which means that it defines a single extension: formally, if $A' \equiv_E A$ and $A' \downarrow_E^\Gamma B$ then $A' \equiv_B A$. This relation expresses that A and B are ‘as independent as possible on E ’, we can see this from the point of view of the category of graphs: indeed if $A \downarrow_E^\Gamma B$ then the subgraph of $\mathit{mathbb{M}}$ induced on $E \cup A \cup B$ corresponds to the direct sum of the graphs induced on $E \cup A$ and $E \cup B$ above E .

NSOP₁ and Kim-independence

As mentioned earlier, NSOP₁ theories are a generalization of simple theories. They are defined by a tree property similar to the one for simple theories. The first example of non-simple NSOP₁ theories is the theory of parameterized equivalence relations studied by Shelah and Usvyatsov in [SU08].

This class of theories has been studied intensively over the past few years, following the case of the class of simple theories. The principal tool in this study is Kim-independence, an analogue of Shelah’s independence. Kim-independence is seen as a generic forking-independence and is initially defined with models as a basis, in this case we will speak of Kim-independence on models. Chernikov, Kaplan, Kim, Ramsey and Shelah demonstrate the properties of Kim-independence in the NSOP₁ theories in [KR17], [KRS19], and [KR21]. This relation coincides with Shelah-independence in a simple theory, and many properties of Shelah-independence in simple theories have an analogue in the NSOP₁ framework. For example:

Theorem 0.22. [KR17, Theorem 5.9] *A complete theory T is NSOP₁ if and only if the relation of Kim-independence is symmetric above all models, meaning that for all $M \models T$, $M, a, b \in \mathbb{M}$ in a monster model of T we have $a \downarrow_M^K b$ if and only if $b \downarrow_M^K a$.*

Kaplan and Ramsey also prove a criterion similar to the Kim-Pillay theorem for Kim-independence in NSOP₁ theories:

Theorem 0.23. [KR20, Theorem 8.1] *(Kim-Pillay Theorem for Kim-independence) Let T be a complete theory and \mathbb{M} be a monster model of M . Then T is NSOP₁ if and only if there exist a ternary relation \downarrow^* between the parts of \mathbb{M} satisfying a certain set of properties, and in that case \downarrow^K coincides with \downarrow^* .*

This result allows Kaplan and Ramsey to prove that some theories, in which the existence of independence relations satisfying almost all the conditions of the Kim-Pillay theorem for simple theories was already known, are NSOP₁. This is the case of the theory of generic bilinear forms on an infinite-dimensional vector space on an algebraically closed field, which we will discuss in a next section, as well as the theory of e -free PAC fields for e an integer, and of ω -free PAC fields, which we will also come back to.

A second evolution in the study of the properties of Kim-independence takes place when Dobrowolski, Kim and Ramsey define it on an arbitrary basis instead of a model in [DKR22], under the assumption that the ambient theory satisfies existence. We then speak of Kim-independence on arbitrary sets. The computational properties of Kim-independence in NSOP₁ theories are generalized to any bases in [DKR22] and [CKR23], and later to hyperimaginaries in [DK21] and [Bos22].

A Kim-Pillay criterion for Kim-independence in theories with existence is proved by Chernikov, Kim, and Ramsey in [CKR23, Theorem 5.1.] and characterizes the Kim-independence over arbitrary sets in NSOP₁ theories with existence. This result is applied in Chapter 2 to characterize Kim-independence on arbitrary bases.

In the preliminary section, we will formally define the notions we have mentioned so far, the relations of independence, the NSOP₁ property, and we will state the properties of Kim-independence in NSOP₁ theories.

PAC fields

We now introduce an example of NSOP₁ theory that is particularly interesting for two reasons, firstly historically because these theories and certain properties related to NSOP₁ were studied before the NSOP₁ property was recognized, and secondly because it is an example that we will consider in Chapter 4 of this manuscript to study definable groups.

A pseudo-algebraically closed field F is a field such that every absolutely irreducible variety defined on F has an F -rational point. These fields appear for the first time in the work of Ax on pseudo-finite fields ([Ax68]), and since then they have been studied by Fried and Jarden in [FJ⁺08], Cherlin, Van Den Dries, Macintyre in [CvdDM80] and Chatzidakis in [Cha10], [Cha02] and [CP98].

For a field F its absolute Galois group $\mathcal{G}(F)$ is defined as the profinite limit of the Galois groups of its finite extensions, which is denoted $\mathcal{G}(F) \cong \lim Gal(L/F)$ where L is taken among the finite Galois extensions, and the restriction map from $Gal(L'/F)$ to $Gal(L/F)$ when $L \subseteq L'$ are two finite Galois extensions. $\mathcal{G}(F)$ thus defined is a topological group having as a basis of open sets the reciprocal images of the elements of the groups of the system, which corresponds to the sets of automorphisms of F^{alg} that coincide on a specific finite Galois extension. A field is said to be bounded if for any integer n it admits only a finite number of separable extensions of degree n .

Such systems of finite groups associated with a topological group, which is then said to be profinite, were studied in [CvdDM80] by Cherlin, Macintyre, and Van Den Dries using a language L_ω with a sort for each integer n , corresponding to the cosets of open normal subgroups of index $\leq n$ of the profinite group. They applied these results to the study of PAC fields in [CvdDM80] (which are called regularly closed in this paper) where they demonstrated that certain properties of these fields can be deduced from properties of the absolute Galois group. More specifically, the theory of a PAC field F is entirely determined by three invariants: the type of isomorphism of the elements of F algebraic on the prime subfield (i.e. of the algebraic closure of the empty set), the degree of imperfection of F , and the theory of the projective system of the Galois groups in the sense of L_ω .

This study was later taken up by Chatzidakis in [Cha98] in the case of profinite groups with the Iwasawa property. This case includes free profinite groups on e generators, which are denoted \hat{F}_e , with e an integer or $e = \omega$. \hat{F}_e is defined as the limit of the system of finite quotients of the free group with e generators: We denote F_e the free group at e generators, \hat{F}_e is the projective limit of the system of the F_e/N with N taken among the normal subgroups of finite index of F_e . A PAC field is said to be e -free if its absolute Galois group is \hat{F}_e .

In terms of classification, Chatzidakis demonstrates in [Cha10, Theorem 3.9] that a perfect PAC field is simple if and only if it is bounded (in particular perfect e -free PAC fields are simple), and that ω -free PAC fields are NSOP and not simple (more precisely she showed that they are NSOP₃ in [Cha19, Theorem 2.10]). She deduced these results from several amalgamation theorems she proved, and which were later applied in [KR17, Section 9.3] by Kaplan and Ramsey to prove, using the Kim-Pillay criterion for Kim-forking, that ω -free PAC fields are NSOP₁. In Chapter 4 we demonstrate a result on the groups definable in the theory of ω -free PAC field of characteristic 0.

Fraïssé classes with Free Amalgamation

In Chapter 2 we are interested in the construction of examples of NSOP₁ theories. The main method used when constructing such examples is that of generic extension. We start with a stable or simple theory T in a language L . We extend the language to L' and the theory to an incomplete theory T' over L' . We then consider, when it exists, a model-completion of T' .

To summarize this process, we start with a theory that we know well and to which we add language on which we place few or no restrictions. Examples of this case are numerous, we can cite the generic expansions of fields, studied by D'Elbée in his thesis [d'E19].

In Chapter 2 we construct the model-completion of certain theories. At the origin of this work there is a construction appearing in [Bau02], the parameterization of a theory. Given a theory T in a language L , we consider the language L_P where we add a new sort P , called parameter sort, and an arity for P to each symbol in L , for example a constant c in L of the sort V becomes a unary function $c(x) : P \rightarrow V$. We then define the theory T_P expressing that for any parameter $p \in P$ the L -structure induced by p is a model of T . This theory is incomplete and we are interested in a model companion, that is to say a theory which, if it exists, describes a collection of models of T on the same basic set interacting randomly.

In [CR16, Section 6.3] Ramsey and Chernikov give an example of a theory T and study a model companion of T_P . This theory T and the companion model of T_P are constructed as the limits of a Fraïssé class. Chernikov and Ramsey consider a relational language L and a class K of L -structures with the strong amalgamation property. The parameterized class K_P is then defined as the class of L_P structures A such that for all $p \in P(A)$ the L -structure induced by p on A is in K . They show that the Fraïssé limit of this class is NSOP₁, and give conditions on K so that it is not simple.

In Chapter 2 we provide a context for making a similar construction without assuming that the base language is relational. This is done at the cost of strong assumptions about the generated structure, we give a set of assumptions (H) reinforcing the usual Fraïssé properties. We prove results similar to those of Chernikov and Ramsey: If a class K satisfies (H) , then we can define its limit theory T , this theory is NSOP₁, satisfies existence, and we can characterize Kim-independence and Shelah-independence in T . This is done through the use of an independence relation coming from the assumption that the class K is closed under free amalgamation in the categorical sense. This independence relation was already mentioned in [Bau14], and does not coincide with the notion of free amalgamation from [Con17], which is also a stationary relation but requires an additional condition on algebraic closure, which is that if A and B are independent and algebraically closed then their union is algebraically closed—which will not be verified in our case unless the algebraic closure is trivial.

We also show that the condition (H) is preserved by certain modifications of the class K : The parameterization we presented previously and the addition of generic structure to K . We give a condition on K under which the limit theory of the parameterized class K_P is non-simple. This framework allows us to construct the model companion of certain exotic theories, for example the theory of the abelian group with a unary function f such that $f(0) = 0$, which will serve as an example of NSOP₁ group in Chapter 4.

Model theory of bilinear forms

The two-sorted theory of bilinear forms was first studied by Granger in his thesis. Granger demonstrates that the theory of nondegenerate symmetric bilinear forms on an infinite-dimensional vector space over an algebraically closed field, denoted by sT_∞^{ACF} , has quantifier elimination once

some functions giving the coefficients of linear dependence relations are added to the language. He also defines a strong independence relation \downarrow^Γ and demonstrates some of its properties. This example is later taken up by Chernikov and Ramsey in [CR16, Section 6.1] who demonstrate that this theory is NSOP_1 and compute the Kim-independence relation on models.

There are some problems with certain proofs in [Gra99], in particular with the one concerning quantifier elimination. A correct proof and related remarks can be found by Abd Aldaim, Connant, and Terry in Section 5 of [AACT25], or by Chernikov and Valentin-Hempel in [CH24]. In Chapter 3, we have an additional assumption on the bilinear form, denoted by (\dagger) before Lemma 3.1, which allows us to give a more elementary proof of quantifier elimination than in these two articles.

Dobrovolski gives a more in-depth study of definable sets of this theory in [Dob23]. He defines a notion of dimension for definable sets that allows him to demonstrate that the definable groups in this theory are (algebraic-by-abelian)-by-algebraic. He also considers the additive group of the vector space and demonstrates that it does not satisfy certain properties of definable groups in simple theories.

In Chapter 3 we generalize some of the results on sT_∞^{ACF} by considering other theories of fields than ACF . We show that if K is the theory of a NSOP_1 field then the theory sT_∞^K of non-degenerate bilinear forms on an infinite-dimensional vector space over a model of K is also NSOP_1 and we compute Kim-independence on models in this case. This proof is done by applying the Kim-Pillay theorem for Kim-independence on models. We also show that when the theory of the field K is simple then the theory sT_∞^K satisfies existence and in this case we compute Kim-independence on arbitrary sets.

We also use $sT_\infty^{ACF,eq}$ to discuss a conjecture about a link between \downarrow^f and \downarrow^K in NSOP_1 theories. For this matter we use a result of elimination of imaginaries to characterize $sT_\infty^{ACF,eq}$.

We can also cite on this subject a more recent article by Chernikov and Valentin-Hempel, [CH24], which considers the more general case of n -linear forms and other combinatorial properties: NIP and n -dependence. They demonstrate, among other things, that if the theory of the field K is NSOP_1 then the theory of alternating n -linear forms on an infinite-dimensional vector space on a model of K is also NSOP_1 .

Stabilizer in a group

When we study a theory T we want to understand its definable sets. In particular we want to analyze definable groups, i.e. groups whose base set G is definable and whose group law, seen as a subset of G^3 , is also definable. We have previously mentioned in this regard the results of Dobrowolski in [Dob23] on bilinear forms and of Chatzidakis and Ramsey in [CR23] on groups definable in e -free PAC fields when e is finite.

An effective result for this is the group configuration theorem, which Hrushovski proved in his thesis [Hru86]. This result allows one to obtain a type-definable group from a certain configuration of independent elements in a stable theory. Typical applications of this result include:

Theorem 0.24. [HP94, Theorem C] *Let G be a group definable in a pseudo-finite field F . Then G is virtually definably isogenous to the set of F -rational points of an algebraic group over F .*

There are two major notions in the study of groups in model theory that we will now present in a rather informal way: the notion of formula and generic type in a group, and that of stabilizer of a type in a group. Let G be a group definable in a model \mathbb{M} of a theory T .

A definable subset X of G is said to be left-generic if a finite number of left translates of X cover G , similarly for right-generic. A type is said to be (left/right) generic if it implies only (left/right) generic formulas. In a stable theory there exist generic complete types on any set of parameters, but this is not true in general. There is a similar notion in the context of simple theories. Given a complete type $p \in S_G(A)$ the stabilizer of p is a subgroup $Stab(p) \leq G$ generated by elements ‘stabilizing’ the type, in the following sense: $g \in Stab(p)$ iff $p \cup g \cdot p$ is consistent, where $g \cdot p$ is the type p translated by g . The link between these two notions is that a generic type must have a ‘large’ stabilizer - of bounded index for example.

The typical strategy for studying a definable group G in a model F of a theory of field is as follows:

1. Take a, b, x generic elements (meaning that their type is generic) independent in the group G , write $c := a \cdot b$.
2. The group configuration theorem provides us with elements a', b', c' in a group H definable in the algebraic closure of F^{alg} of F (i.e., an algebraic group) interalgebraic with a, b, c respectively on a set of parameters E .
3. We then consider the stabilizer of $tp(a, a'/E)$ in $G \times H$, and we show that it defines an isogeny between G and H .

This is the method that we want to apply to study groups definable in ω -free PAC fields of characteristic 0. To achieve this we seek to generalize the notions of genericity and stabilizer to NSOP₁. For the notion of generic, by taking the ‘naive’ generalization of the generic in the simple setting we have a negative answer (which was already noted by Dobrowolski: [Dob23, Proposition 8.14]). For the notion of stabilizer we give a definition generalizing that of the simple case, which defines in general an invariant group. We show that in certain examples the subgroup obtained, which we will call the Kim-stabilizer, is type-definable. We generalize some of the results on groups definable in a simple theory to this notion.

We also introduce another notion of stabilizer, due to Hrushovski in [Hru12]. An ideal of formula is a set of formula closed under disjunction and logical implication (if $\theta \in \mu$ and $\theta \vdash \theta'$ then $\theta' \in \mu$). An ideal μ is said to be $S1$ on a set of parameters A if it is A -invariant and if $\varphi(x, a_i) \wedge \varphi(x, a_j) \in \mu$ for $i \neq j < \omega$ implies $\varphi(x, a_i) \in \mu$ for any A -indiscernible sequence $(a_i)_i$ and any formula $\varphi(x, y)$. Under the assumption of the existence of an $S1$ ideal of formulas in a group, he defines a notion of stabilizer and shows that it is a type-definable subgroup.

Using the two notions of stabilizers, the Hrushovski one and the Kim-stabilizer one we prove under certain assumptions on a NSOP₁ field F and a group G definable in F that there exists a type-definable subgroup of G of bounded index and a finite kernel embedding of this subgroup into the F -rational points of an algebraic group definable over F . We also show that the groups definable in ω -free PAC fields of characteristic 0 satisfy these assumptions.

Appendix: Stable Kim-forking

The stable forking conjecture is the conjecture according to which in a simple theory, forking is always witnessed by a stable formula. This is a long-standing conjecture; we refer to the article [KP01] which gives a complete exposition of this problem. In this appendix we give an analogous statement in the framework of NSOP₁ theories and Kim-forking, weakly stable Kim-forking. We connect this notion to certain results on stable forking that can be found in [CP18]. We can also cite on this subject [BFM24] a recent article by Baldwin, Freitag and Mutchnik in which they demonstrate, among other things, that Kim-forking in an NSOP₁ theory is always witnessed by a simple formula, that is, one that does not have the tree property.

Chapter 1

Preliminaries

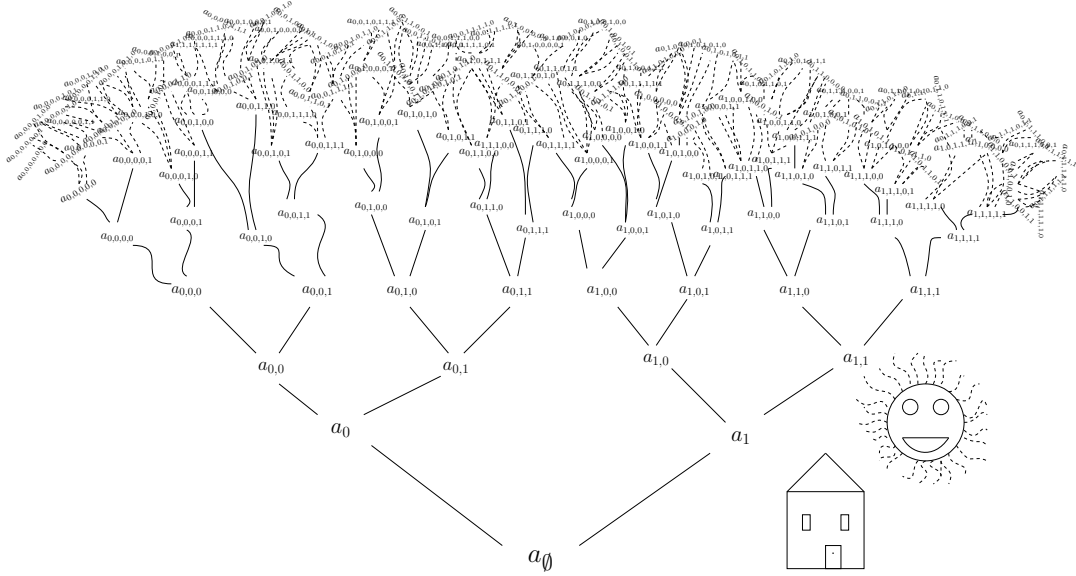


Figure 1.1: A weak tree Morley landscape

1.1 Generalities

We assume the knowledge of the basic notions of model theory, by this we mean the notions of types, compactness and saturation. A type will in general mean a complete type. We will denote by $x, y, z, x^i, \bar{x}, \tilde{x}$ tuples of variables, and write x_j for the j -th coordinate of the tuple x . We use capital letters E, A, B, C for sets of parameters, and the letters N, M for models. For a tuple or a set A , $|A|$ is the cardinality of A and $|T| = \max\{\aleph_0, |L|\}$ is the size of the language for a complete theory T . If A and B are two sets, we use the juxtaposition AB to denote their union. We will use $\langle A \rangle$ for the structure generated by the set A , or eventually for the linear span of A when working in a vector space.

When working with a complete theory T in some language L we will most often use a monster model of T . This is a model \mathbb{M} which is strongly κ -homogeneous and κ -saturated for some cardinal κ large enough. In this context, the prefix small (as in a small model, a small cardinal or a small set of parameters) will mean of size smaller than κ . Every object we will consider will be small if not stated otherwise. We use the standard notations tp , acl , dcl with T or L in index, as in tp_L if we have to specify the language.

Definition 1.1. Let A be a set of parameters. A sequence $(a_i)_{i < \omega}$ is A -indiscernible if $a_{i_0}..a_{i_{n-1}} \equiv_A a_{j_0}..a_{j_{n-1}}$ for all $n < \omega$ and for all pair of strictly increasing sequences of n integers $i_0 < .. < i_{n-1}$ and $j_0 < .. < j_{n-1}$.

An interesting way to view this notion, which leads to the notion of generalized indiscernible, is to see the set of indices, the integers, equipped with the language $L' = \{<\}$, and to read the definition as: if two sets of indices have the same quantifier free L' -types, then the associated tuples in the sequence have the same L -types.

Proposition 1.2. [Cas11, Proposition 1.6] (Erdős–Rado Theorem) If $\kappa \leq |T| + |A|$, $\lambda = \beth_{(2^\kappa)^+}$ and $(a_i)_{i < \kappa}$ is a sequence of tuples of the same length $\leq \kappa$, then there is an A -indiscernible sequence $(b_i)_{i < \omega}$ finitely based on $(a_i)_{i < \lambda}$ over A , which means that for all $n < \omega$ there is $i_0 < .. < i_n < \lambda$ such that $b_0..b_n \equiv_A a_{i_0}..a_{i_n}$.

An imaginary is an equivalence class of some definable equivalence relation. We use the exponent eq to specify that some model theoretic notions are considered in the theory of imaginaries T^{eq} , which is the extension of T where we add a new sort E for any equivalence relation $E(x, y)$ definable over \emptyset .

An hyperimaginary is an equivalence class of some type-definable equivalence relation. If $E(x, y)$ is a type-definable equivalence relation and a is an x -tuple we will write a_E for the E -class of a and refer to such an object as an hyperimaginary element of sort E . An hyperimaginary a of sort E is countable if the tuple of variable x on which E is defined is countable.

Definition 1.3. Let A be a set of hyperimaginaries.

1. The hyperimaginary definable closure of A , written $\text{dcl}^{heq}(A)$, is the set of all hyperimaginaries that are fixed under all A -automorphisms.
2. The hyperimaginary bounded closure of A , written $\text{bdd}(A)$, is the set of all hyperimaginaries with a bounded orbit under the action of the group of A -automorphisms.

We will write \mathbb{M}^{heq} for \mathbb{M} together with the collection of all countable hyperimaginaries modulo type-definable equivalence relations over \emptyset . The main difference between hyperimaginaries and imaginaries is that there is no notion of formulas for hyperimaginaries, but instead we have a partial type over some representative. A consequence of this is that there is no negation in this

context, as the negation of a partial type $\pi(x)$ is not a partial type unless $\pi(x)$ is equivalent to a formula. We give now some illustration of this fact.

If $E(x, y)$ is a definable equivalence relation and $a = \bar{a}_E$ is the E -class of \bar{a} , for some formula $\varphi(x)$ we can express the satisfaction of $\varphi(a)$ by: $\exists x(E(\bar{a}, x) \wedge \varphi(x))$. This is a formula that can be written using any representative of a as parameters, and its negation is $\forall x(E(\bar{a}, x) \implies \neg \varphi(x))$. Now if F is a type-definable equivalence relation and $b = \bar{b}_F$ is an hyperimaginary of sort F , $\exists x(F(\bar{b}, x) \wedge \varphi(x))$ is a partial type over the representative \bar{b} , and the ‘negation’ $\forall x(F(\bar{a}, x) \implies \neg \varphi(x))$ can not be written as a partial type.

For two tuples a, b and some set of parameters e we will write $a \sim_e b$ to mean that a and b are interdefinable over e , i.e. that $a \in dcl(eb)$ and $b \in dcl(ea)$. The definitions and basic properties of hyperimaginaries can be found in [Cas11].

1.2 Forking and dividing

Let $\mathbb{M} \models T$ be a monster model of some complete theory. We introduce the notions of dividing and forking and their basic properties.

Definition 1.4. Let b_0, e be tuples of \mathbb{M} and $\varphi(x, b_0)$ be a formula.

1. $\varphi(x, b_0)$ *divides over* e if there is an e -indiscernible sequence $I = (b_i : i < \omega)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent. In that case we say that $\varphi(x, b_0)$ divides over e with respect to I .
2. A partial type $p(x, b)$ *divides over* e if it implies a formula which divides over e . A partial type $p(x, b)$ *forks over* e if it implies a finite disjunction of formulas each of which divides over e .
3. Let $a \in \mathbb{M}$. We write $a \downarrow_e^f b$ to denote the assertion that $\text{tp}(a/eb_0)$ does not fork over e and $a \downarrow_e^d b_0$ to denote the assertion that $\text{tp}(a/eb_0)$ does not divide over e .
4. A sequence of tuples $(a_i : i < \omega)$ is called *e -Morley* if it is e -indiscernible and if $a_i \downarrow_e^f a_{<i}$ for all $i < \omega$.

Definition 1.5. A set $e \subseteq \mathbb{M}$ is an *extension base* if $a \downarrow_e^f e$ for all $a \in \mathbb{M}$. A theory T has *existence* if every set is an extension base. This is equivalent to saying that for all $e \in \mathbb{M}$ and all $p(x) \in S(e)$ there is an e -Morley sequence in p .

Definition 1.6. Let $A \subseteq B$ be sets of parameters and a be a tuple. A partial type $\pi(x/AB)$ is finitely satisfiable in A if for all formula $\varphi(x, b)$ implied by π for $b \in B$ there is $a \in A$ such that $\models \varphi(a, b)$. We write $c \downarrow_A^u B$ to mean that $\text{tp}(c/AB)$ is finitely satisfiable in A .

Lemma 1.7. [Wag00, Lemma 2.2.6] *Let $A \subseteq B$ be sets of parameters and a be a tuple. If a partial type $\pi(x/AB)$ is finitely satisfiable in A there is a completion of π that is also finitely satisfiable. Also if $c \downarrow_A^u B$ then $c \downarrow_A^f B$.*

Remark 1.8. Let M be a model and $B \supseteq M$ be a set of parameters. Then any partial type $q(x, B)$ is finitely satisfiable in M . From this and the previous lemma we can deduce that every model is an extension base.

Lemma 1.9. [Kim14, Lemma 2.1.6] *(Characterization of dividing) For any $a, B, e \in \mathbb{M}$ the following are equivalent:*

- (1) $\text{tp}(a/eB)$ does not divide over e .

- (2) For all infinite e -indiscernible sequence I such that $B \in I$, there is some $a' \equiv_{eB} a$ such that I is ea' -indiscernible.
- (3) For all infinite e -indiscernible sequence I such that $B \in I$, there is some $J \equiv_{eB} I$ such that J is ea -indiscernible.

Proposition 1.10. [Wag00, Lemma 2.2.2] (Left transitivity of dividing) Let $a, b, C, e \in \mathbb{M}$. If $\text{tp}(b/Ce)$ does not divide over e and $\text{tp}(a/bCe)$ does not divide over b then $\text{tp}(ab/Ce)$ does not divide over e .

Proposition 1.11. [Wag00, Lemma 2.2.3] (Extension) Let $e \subseteq A$ be some sets of parameters.

1. Let $\pi(x)$ be a partial type over A . If $\pi(x)$ does not fork over $e \in \mathbb{M}$ there is $p \in S(A)$ a completion of π that does not fork over e .
2. Let $a, b, c \in \mathbb{M}$ such that $a \downarrow_b^f c$. Then, for any $d \in \mathbb{M}$ there is $a' \equiv_{bc} a$ such that $a' \downarrow_b^f cd$.

Proposition 1.12. [Cas11, 12.7 and 12.10] (Left transitivity of forking) Let $a, b, c, e \in \mathbb{M}$. If $a \downarrow_e^f c$ and $b \downarrow_{ea}^f c$ then $ab \downarrow_e^f c$.

Proposition 1.13. Let $a, b, e \in \mathbb{M}$. Then $a \downarrow_e^f b$ if and only if $a \downarrow_e^f \text{acl}(b)$ iff $a \downarrow_{\text{acl}(e)}^f b$.

Remark 1.14. A sequence is e -Morley if and only if it is $\text{acl}(e)$ -Morley.

Remark 1.15. Forking and dividing generalize to hyperimaginaries, and in this context the properties we wrote here hold when we replace the algebraic closure of some set of parameters A by the bounded closure of some sets of hyperimaginaries. The one difference is that there is no notion of formula over an hyperimaginary, but only partial types over some representative so we have to define dividing for partial types directly. The definitions and properties of forking and dividing for hyperimaginaries can be found in [Wag00, Section 3.2].

1.3 Independence relations

Let $\mathbb{M} \models T$ be a monster model of some complete theory. We refer to [Adl05] and [d'E23] on the topic of the axiomatic approach to independence relations.

An independence relation is a ternary relation between subsets of the monster model \mathbb{M} . For some sets A, B, e we will write $A \downarrow_e B$ to mean that A and B are independent over e in the sense of \downarrow . We present the different properties that independence relation can satisfy. We say that \downarrow satisfies:

1. *Invariance* if it is invariant under automorphisms of the monster model: $A \downarrow_e B$ if and only if $\sigma(A) \downarrow_{\sigma(e)} \sigma(B)$ for all $\sigma \in \text{Aut}(\mathbb{M})$ and $e, A, B \in \mathbb{M}$.
2. *Finite character* if $A \downarrow_e B$ if and only if $a \downarrow_e b$ for all finite tuples $a \in A, b \in B$ and $e, A, B \in \mathbb{M}$.
3. *Existence* if $A \downarrow_e e$ for all $e, A \in \mathbb{M}$.
4. *Symmetry* if $A \downarrow_e B$ if and only if $B \downarrow_e A$ for all $e, A, B \in \mathbb{M}$.
5. *Extension* if whenever $A \downarrow_e B$ and $e \subseteq B \subseteq C$ there is $A' \equiv_B A$ such that $A' \downarrow_e C$ for all $A, e \subseteq B \subseteq C \in \mathbb{M}$.
6. *Local character* if for all $A \in \mathbb{M}$ there is a cardinal $\kappa = \kappa(A)$ such that for all $B \in \mathbb{M}$ there is $B_0 \subseteq B$ of cardinality $\leq \kappa$ such that $A \downarrow_{B_0} B$.

7. *Right normality* if $A \downarrow_e B$ implies $A \downarrow_e eB$ for all $e, A, B \in \mathbb{M}$.
8. *Right monotonicity* if $A \downarrow_e B$ implies $A \downarrow_e B_0$ for all $e, A, B \in \mathbb{M}$ and $B_0 \subseteq B$.
9. *Right transitivity* if $A \downarrow_e B$ and $A \downarrow_B C$ implies $A \downarrow_e C$ for all $e, A, B \in \mathbb{M}$ such that $e \subseteq B \subseteq C$.
10. *Right base monotonicity* if $A \downarrow_e B$ implies $A \downarrow_{eB_0} B$ for all $e, A, B \in \mathbb{M}$ and $B_0 \subseteq B$.
11. *Stationarity* if whenever $A \downarrow_e B$, $e \subseteq B$, $A' \equiv_e A$ and $A' \downarrow_e B$, then $A' \equiv_B A$.
12. *Closure* if whenever $A \downarrow_e B$ then $\text{acl}(A) \downarrow_{\text{acl}(e)} \text{acl}(B)$ for all $e, A, B \in \mathbb{M}$.

The notion of left normality / monotonicity / transitivity and base monotonicity are defined similarly. We say that \downarrow satisfies normality if it satisfies both left and right normality, similarly for transitivity ect..

In the previous section we stated that \downarrow^f satisfies extension, right base monotonicity, left transitivity, normality and monotonicity. It is easy to show that it satisfies invariance: Let $\sigma \in \text{Aut}(\mathbb{M})$ and let $\varphi(x, b_0)$ divide over e with respect to the e -indiscernible sequence $(b_i)_{i < \omega}$. Then $\varphi(x, \sigma(b_0))$ divides over $\sigma(e)$ with respect to the sequence $\sigma(b_i)_{i < \omega}$. This implies that if a formula $\psi(x, b)$ forks over e then the formula $\psi(x, \sigma(b))$ forks over $\sigma(e)$. So if $a \not\downarrow_e^f B$ then a satisfies some formula $\varphi(x, b)$ that forks over e so $\sigma(a)$ satisfies $\varphi(x, \sigma(b))$ and $\sigma(a) \not\downarrow_{\sigma(e)}^f \sigma(B)$.

The fact that \downarrow^f is defined from some properties of formulas also implies that \downarrow^f satisfies finite character on both sides: If $A \not\downarrow_e^f B$ then there is a formula $\varphi(x, b) \in \text{tp}(A/eB)$ that forks over e . If a and b are respectively finite subtuples of A and B corresponding to the elements appearing in the formula φ then $\varphi(x, b) \in \text{tp}(a/eb)$ forks over e so $a \not\downarrow_e^f b$.

As we mentioned in the introduction forking independence satisfies additional properties in simple theories:

Proposition 1.16. [Wag00, Theorem 2.3.13] *In a simple theory the relation \downarrow^f satisfies existence, symmetry and transitivity.*

Definition 1.17. We say that some sets A, B are algebraically independent over e , written $A \downarrow_e^a B$ if $\text{acl}(eA) \cap \text{acl}(eB) = \text{acl}(e)$.

Clearly the relation \downarrow^a always satisfies invariance, finite character, existence, monotonicity, symmetry, normality, transitivity and closure. In general it does not satisfy base monotonicity. It also satisfies extension in any theory, as proven in [Hod93, Theorem 6.4.5].

From an independence relation we can define other relations by ‘forcing’ some properties:

Definition 1.18. Forcing base monotonicity: Let \downarrow^i be an independence relation defined on sets e, A, B . We define \downarrow^{iM} as the weakest independence relation that implies \downarrow^i and satisfies base monotonicity, algebraic closure and normality. So $A \downarrow_e^{iM} B$ iff $\text{acl}(AB') \downarrow_{\text{acl}(eB')} B$ for all $B' \subseteq \text{acl}(eB)$.

Definition 1.19. Forcing extension: Let \downarrow^i be an independence relation defined on algebraically closed sets A, B, C such that $C \subseteq A, B$. We define \downarrow^{i*} as the weakest independence relation that implies \downarrow^i and satisfies extension, meaning: $A \downarrow_C^{i*} B$ iff for all $B' \supseteq B$ there exist $A' \equiv_B A$ such that $A \downarrow_C^i B'$.

Definition 1.20. We define M -independence as the weakest relation implying \downarrow^a and satisfying base monotonicity and closure, i. e. $A \downarrow_e^M B$ iff $A \downarrow_e^{aM} B$ iff $\text{acl}(AB') \downarrow_{B'}^a \text{acl}(Be)$ for all $e \subseteq B' \subseteq \text{acl}(Be)$.

This relation was defined by Onshuus in the context of rosy theories (see [Ons03] for reference). Onshuus also defines thorn-forking as \perp^{M^*} in [Ons03, Definition 2.1] (see [Adl09b] for an introduction to thorn forking in terms of independence relations) and studies in the theories in which \perp^{M^*} satisfies some local character property.

1.4 NSOP₁ and Kim-Forking

We mentioned in the introduction the tree property defining simple theories. NSOP₁ theories are defined similarly.

Definition 1.21. A formula $\varphi(x, y)$ in \mathcal{L} has the strong order property of the first kind (SOP₁) in T if there is a tree of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ in a model of T such that:

$$\begin{cases} \forall \eta \in 2^\omega, \{\varphi(x, a_{\eta \upharpoonright \alpha}) : \alpha < \omega\} \text{ is consistent} \\ \forall \nu, \eta \in 2^{<\omega}, \text{ if } \nu \supseteq \eta \frown 0 : \{\varphi(x, a_\nu), \varphi(x, a_{\eta \frown 1})\} \text{ is inconsistent} \end{cases}$$

The theory T has SOP₁ if one of its formulas does. T is NSOP₁ otherwise.

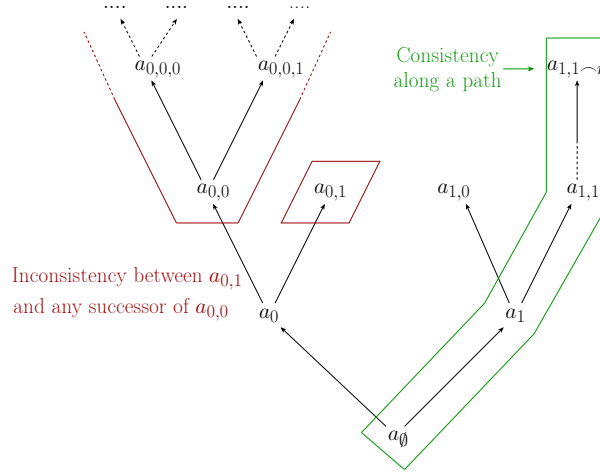


Figure 1.2: The strong order property of the first kind

In [KR20] Kaplan and Ramsey prove a more easily exploitable combinatorial property characterizing NSOP₁:

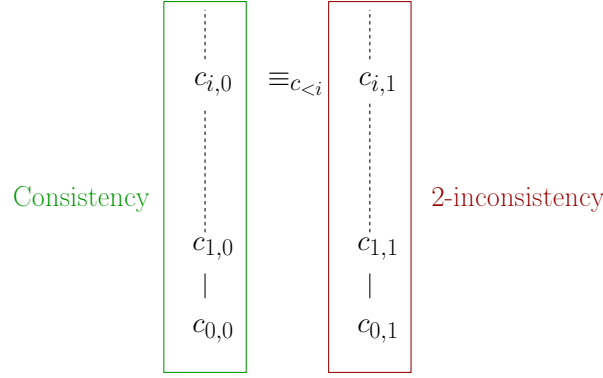
Proposition 1.22. [KR20, Proposition 2.4]: Let T be a complete theory. The following are equivalent:

- (1) T has SOP₁.
- (2) There is a formula $\varphi(x, y)$ and a sequence $(c_{i,j} : i < \omega, j < 2)$ such that:

$$\begin{cases} (a) \forall i < \omega, c_{i,0} \equiv_{c_{<i}} c_{i,1} \\ (b) \{\varphi(x, c_{i,0}) : i < \omega\} \text{ is consistent} \\ (c) \{\varphi(x, c_{i,1}) : i < \omega\} \text{ is 2-inconsistent} \end{cases}$$

- (3) There is a formula $\varphi(x, y)$, an integer k and a sequence $(c_{i,j} : i < \omega, j < 2)$ such that:

$$\begin{cases} (a) \forall i < \omega, c_{i,0} \equiv_{c_{<i}} c_{i,1} \\ (b) \{\varphi(x, c_{i,0}) : i < \omega\} \text{ is consistent} \\ (c) \{\varphi(x, c_{i,1}) : i < \omega\} \text{ is } k\text{-inconsistent} \end{cases}$$

Figure 1.3: A more practical characterization of SOP₁

We will now define the relation of Kim-independence and give some of the basic properties. Kim-dividing is defined as dividing along Morley sequences, more formally:

Definition 1.23. Let $b_0, e \in \mathbb{M}$ and $p(x, b_0)$ be a partial type over eb_0 .

1. $p(x, b_0)$ *Kim-divides over e* if there is an e -Morley sequence $(b_i)_{i < \omega}$ such that $\{p(x, b_i) : i < \omega\}$ is inconsistent. We will say that $p(x, b_0)$ *Kim-divides over e with respect to $(b_i)_{i < \omega}$* .
2. $p(x, b_0)$ *Kim-forks over e* if it implies a finite disjunction of formulas each of which *Kim-divides over e* .
3. Let $a \in \mathbb{M}$. We write $a \downarrow_e^K b_0$ to denote the assertion that $\text{tp}(a/eb_0)$ does not *Kim-fork over e* and $a \downarrow_e^{Kd} b_0$ to denote the assertion that $\text{tp}(a/eb_0)$ does not *Kim-divide over e* .
4. $I = (a_i)_{i < \omega}$ is *Kim-Morley over e* if it is e -indiscernible and if $a_i \downarrow_e^K a_{<i}$ for all $i < \omega$.

Remark 1.24. $a \downarrow_e^f b$ implies $a \downarrow_e^K b$ for all $a, b, e \in \mathbb{M}$ since *Kim-dividing implies dividing*.

The following result and its proof are analogous to the case of usual forking we gave previously in Lemma 1.9.

Lemma 1.25. [KR20, Lemma 3.18] (*Characterization of Kim-dividing*) Let $a, e, b \in \mathbb{M}$. The following are equivalent:

- (1) $\text{tp}(a/eb)$ does not *Kim-divide over e* .
- (2) For all infinite e -Morley sequence I such that $b \in I$, there is some $a' \equiv_{eb} a$ such that I is ea' -indiscernible.
- (3) For all infinite e -Morley sequence I such that $b \in I$, there is some $J \equiv_{eb} I$ such that J is ea -indiscernible.

Remark 1.26. The same way as forking and dividing generalize to hyperimaginaries, *Kim-forking and Kim-dividing* do. The proofs of the properties of *Kim-forking* in hyperimaginaries can be found in [Bos22]. We also mention [DK21] in which Dobrowolski and Kamsma generalize *Kim-independence* and NSOP₁ in the context of positive logic and deduce from this some results on hyperimaginaries in [DK21, Section 10.3].

We define an other tree property:

Definition 1.27. A formula $\varphi(x, y)$ in \mathcal{L} has the tree property of the second kind (TP₂) in T if there is an array of parameters $(a_{i,j})_{i,j < \omega}$ in a model of T such that:

$$\begin{cases} \forall \sigma \in \omega^\omega, \{\varphi(x, a_{n,\sigma(n)}) : n < \omega\} \text{ is consistent} \\ \forall n < \omega \text{ and } i < j < \omega, \{\varphi(x, a_{n,i}), \varphi(x, a_{n,j})\} \text{ is inconsistent} \end{cases}$$

The theory T has TP_2 if one of its formulas does, T is NTP_2 otherwise.

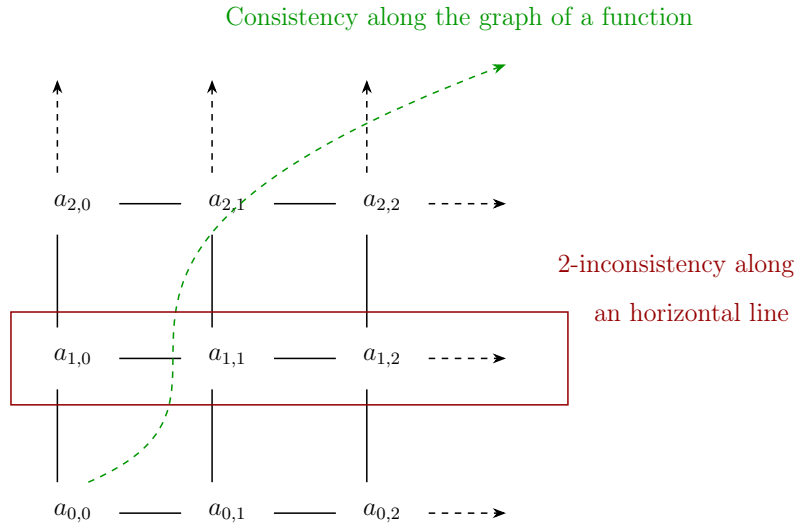


Figure 1.4: The tree property of the second kind

Proposition 1.28. [KR17, Corollary 8.5]: *A complete theory T is simple if and only if it is NTP_2 and $NSOP_1$.*

Since we work mostly with $NSOP_1$ theories the way we will use the previous proposition is the following: In a non-simple $NSOP_1$ theory (we will refer to such theories as ‘strictly $NSOP_1$ ’) there is a formula witnessing TP_2 .

1.5 Properties of Kim-forking in $NSOP_1$

We now present the properties of Kim-forking in $NSOP_1$ theories. It is interesting to compare these with the properties of forking in simple theories for multiple reasons:

1. $NSOP_1$ is a generalization of simplicity (a simple theory is $NSOP_1$), and in a simple theory Kim-forking is equal to forking.
2. Most of the proofs and a lot of ideas used in the context of $NSOP_1$ are inspired and a lot of times built from analogous proofs in the simple context.

One of the differences with simple theory is that any simple theory satisfies existence, this is not the case for $NSOP_1$ theories: It was a conjecture that any $NSOP_1$ theory satisfies existence before Mutchnik recently gave a counter-example in [Mut24].

To know that a formula Kim-divides over some set of parameters A we need to have some A -Morley sequence, i.e. we need A to be an extension base. Without the existence of such a sequence the following can happen:

Let $\varphi(x, b_0)$ be a formula over some parameters b_0 such that there is no e -Morley sequence in $\text{tp}(b_0/e)$. Then $\varphi(x, b_0)$ does not Kim-divide. Let $\psi(x, c_0)$ be a formula that is implied by $\varphi(x, b_0)$ over some parameters c_0 such that there is a Morley sequence in $\text{tp}(c_0/e)$. It can happen that $\psi(x, c_0)$ Kim-divides. We would like Kim-dividing to not depend on the parameters of the formulas, so we want to avoid this case. For this there are two options. The first one is to consider Kim-dividing over models, since models are extension bases this situation can not happen. The second one is to assume that the theory we are working with satisfies existence.

We take this second option since we will work mostly with theories that satisfy existence. The only times we will not have this assumption are when we are proving existence in some theory, and in the appendix. We mentioned previously that Kim-forking generalizes to hyperimaginaries, for this it is good to know the following result that ensures that we can use Kim-forking over arbitrary bases also in the hyperimaginaries:

Theorem 1.29. [DKR21, Theorem 10.20]: *The theory T satisfies the existence axiom for forking if and only if T^{eq} satisfies the existence axiom for forking.*

1.5.1 Kim's Lemma for Kim-forking

Let T be an NSOP₁ theory with existence and \mathbb{M} be a monster model of T . The first property of Kim-forking in NSOP₁ is an analogue of the Kim's Lemma for forking in simple theory that states that if a formula forks, then it forks with respect to any Morley sequence.

Theorem 1.30. [DKR22, Theorem 3.5]: *T satisfies Kim's lemma for Kim-dividing over an arbitrary set e : a formula $\varphi(x, a)$ Kim-divides over e with respect to some Morley sequence in $\text{tp}(a/e)$ if and only if it Kim-divides over e with respect to any such sequence.*

From this and some basic properties of Morley sequences we can deduce the following results:

Proposition 1.31. [DKR22, Proposition 4.1] (Kim-forking = Kim-dividing) *For any $e \in \mathbb{M}$, $\varphi(x, y)$ \mathcal{L} -formula and b tuple if $\varphi(x, b)$ Kim-forks over e then $\varphi(x, b)$ Kim-divides over e .*

Proof. Suppose $\varphi(x, b) \vdash \bigvee_{j < n} \psi_j(x, c^j)$, where each $\psi_j(x, c^j)$ Kim-divides over e . Let $(b_i c_i^0 \dots c_i^{n-1})_{i < \omega}$ be a Morley sequence in $\text{tp}(b_i c^0 \dots c^{n-1}/e)$. Since $(b_i)_{i < \omega}$ is a Morley sequence in $\text{tp}(b/e)$, to get that $\varphi(x, b)$ Kim-divides over e it is enough to show that $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent.

If not, then there is some $a \models \{\varphi(x, b_i) : i < \omega\}$. Then, by the pigeonhole principle, we get that, for some $j < n$, a realizes $\psi_j(x, c_i^j)$, for infinitely many i 's. As $(c_i^j)_{i < \omega}$ is a Morley sequence in $\text{tp}(c^j/e)$, it follows from Kim's Lemma for Kim-forking that $\psi_j(x, c^j)$ does not Kim-divide over e , a contradiction. \square

Corollary 1.32. [DKR22, Proposition 4.5] *Let $I = (a_i)_{i < \omega}$ be e -Morley and ea' -indiscernible, then $a' \downarrow_e^K a_0$.*

Corollary 1.33. *If $I = (a_i)_{i < \omega}$ is e -Morley and ea' -indiscernible then $a' \downarrow_e^K I$.*

Proof. It is enough to show that $a' \downarrow_e a_{[0, n]}$ for all $n < \omega$, but this is a direct consequence of Corollary 1.32 and of the fact that $(a_{[n \cdot i, n \cdot (i+1)]})_{i < \omega}$ is also an e -Morley sequence. \square

Remark 1.34. Since a sequence is a -indiscernible if and only if it is $\text{acl}(a)$ -indiscernible for any a and since a sequence is e -Morley if and only if it is $\text{acl}(e)$ -Morley we have $a \downarrow_e^K b$ if and only if $a \downarrow_{\text{acl}(e)}^K b$ if and only if $\text{acl}(a) \downarrow_e^K b$.

Proposition 1.35. *Let $a, b, e \in \mathbb{M}$. If $a \downarrow_e^K b$ and $I = (b_i)_{i < \omega}$ is a Morley sequence in $\text{tp}(b/e)$ with $b_0 = b$, then there is some $a' \equiv_{eb} a$ such that $a' \downarrow_e^K I$ and I is ea' -indiscernible*

Proof. $a \downarrow_e^K b$, by Lemma 1.25 there is $a' \equiv_{eb} a$ such that I is ea' -indiscernible, then by Corollary 1.33 $a' \downarrow_e^K I$ holds. \square

1.5.2 Symmetry of Kim-Forking

Theorem 1.36. [DKR22, Corollary 4.9] *Kim-forking satisfies symmetry: $a \downarrow_e^K b$ if and only if $b \downarrow_e^K a$ for any $a, b, e \in \mathbb{M}$.*

Definition 1.37. A sequence $I = (a_i)_{i < \omega}$ is called strong Kim-Morley over $e \in \mathbb{M}$ if for all $n < \omega$ the sequence of the $\bar{a}_i := (a_{i \cdot n} \cdot a_{i \cdot n + n - 1})$ is Kim-Morley over e .

Definition 1.38. A sequence $I = (a_i)_{i < \omega}$ is called total Kim-Morley over $e \in \mathbb{M}$ if $a_{\geq n} \downarrow_e^K a_{< n}$ for all $n < \omega$.

Remark 1.39. Using base monotonicity and left transitivity of forking we can show that a Morley sequence is strong and total in the above sense. Since these sequence are also Kim-Morley they are strong Kim-Morley and total Kim-Morley.

Proposition 1.40. *$I = (a_i)_{i < \omega}$ is strong Kim-Morley over e if and only if it is total Kim-Morley over e .*

Proof. [\Leftarrow]: Immediate by finite character. [\Rightarrow]: Let $i < \omega$, we show that $a_{\geq i} \downarrow_e^K a_{< i}$ for all $i < \omega$. By finite character, it is enough to show that for all $i < j < \omega$, $a_{[i,j]} \downarrow_e^K a_{< i}$. For this let $k > j$, by assumption we have that $a_{[k,2k-1]} \downarrow_e^K a_{[0,k-1]}$, so by indiscernibility and invariance, $a_{[i,j]} \downarrow_e^K a_{< i}$. \square

There is another strengthening of the notion of Kim-Morley sequence: weak tree Morley sequences. We will not define since we will not use it and the definition is really technical (it can be found in [KR17, Definition 5.7]). The proof of most properties of Kim-forking relies on this notion, and once we have all of these properties the following can be proven:

Proposition 1.41. [KR20, Proposition 7.9 and Lemma 5.9] *Let $e \in \mathbb{M}$, then $(a_i)_{i < \omega}$ is a total Kim-Morley sequence over e if and only if it is a weak tree Morley sequence over e .*

The previous result is not in the chronological order, we quote it here to close the matter of weak tree Morley sequences so that we can state some of the following results in terms of total Kim-Morley sequences instead of weak tree Morley sequences.

Lemma 1.42. [CKR23, Lemma 1.18] *Let $e \in \mathbb{M}$ and $p \in S(e)$ a complete type over e and H be a sort.*

- (1) *The set of (a, b) such that $b \models p$, a of sort H and $a \downarrow_e^K b$ is type-definable over e .*
- (2) *The set of Kim-Morley sequences in p is type-definable over e .*

Proof. (1): Let (x, y) be a tuple of variables such that a is a x -tuple and b is a y -tuple. Then the partial type $p(y) \cup \{\neg\varphi(x, y) : \varphi(x, b) \text{ Kim-divides over } e\}$ expresses the right conditions. (2): The conditions defining (2) in the tuple of variables $(x_i)_{i < \omega}$ are given by the partial type expressing that:

- $\models p(x_i)$ for all $i < \omega$.
- The sequence $(x_i)_{i < \omega}$ is e -indiscernible.
- $x_i \downarrow_e^K x_{< i}$ for all $i < \omega$.

The last condition is type-definable over e since by symmetry $x_i \downarrow_e^K x_{< i}$ if and only if $x_{< i} \downarrow_e^K x_i$ and we can apply (1). \square

From the proof of Lemma 1.42 we can see that (1) always holds, and that for (2) we need the symmetry of Kim-forking.

1.5.3 The independence theorem for Kim-forking

Theorem 1.43. [KR20, Theorem 5.6] (The independence theorem for Kim-forking) Let $a, a', b, c, e \in \mathbb{M}$. Let $b \downarrow_e^K c$, $a \equiv_e^L a'$, $a \downarrow_e^K b$ and $a' \downarrow_e^K c$. Then there is a'' such that $a'' \downarrow_e^K bc$ and $a'' \equiv_{eb}^L a$, $a'' \equiv_{ec}^L a'$.

As in simple theories for the amalgamation theorem to hold we need to amalgamate over a Lascar strong type. This might seem like a problem but in some situations by passing to the bounded closure in the hyperimaginaries we can sometimes assume that a type is Lascar strong.

Remark 1.44. By [Bos22, Remark 2.2.13] and symmetry of \downarrow^K in an NSOP₁ theory with existence $a \downarrow_e^K b$ if and only if $bdd(ea) \downarrow_{bdd(e)}^K bdd(eb)$ for any $e, a, b \in \mathbb{M}^{heq}$. We also know from [DKR22, Corollary 5.9] that NSOP₁ theories with existence are G-compact (this is not known for NSOP₁ theories in general), which means that $a \equiv_e^L a'$ if and only if $a \equiv_{bdd(e)} a'$ for any $a, a', e \in \mathbb{M}^{heq}$.

We now give corollaries of the independence theorem for Kim-forking. As for the case of simple theories (with [Cas11, Proposition 10.10] for example), we have the following:

Corollary 1.45. Let $e \in \mathbb{M}$, $I = (a_i)_{i < \omega}$ a Kim-Morley over e and p_i a partial type over a_i that does not Kim-fork over e . Assume that for any realizations $b_i \models p_i$, $b_j \models p_j$ we have $b_i \equiv_e^L b_j$. Then $\{p_i(x) : i < \omega\}$ does not Kim-fork over e .

Proof. We show by induction on $n < \omega$ that $\{p_i(x) : i \leq n\}$ does not Kim-divide over e . For $n = 0$ it is clear, if it is true for $n \geq 0$ we have a realization $b \models \{p_i(x) : i \leq n\}$ such that $b \downarrow_e^K a_{\leq n}$. By hypothesis there is $b' \models p_{n+1}(x)$ such that $b' \downarrow_e^K a_{n+1}$. $a_{n+1} \downarrow_e^K a_{\leq n}$ holds since I is Kim-Morley over e and by assumption $b \equiv_e^L b'$. By Theorem 1.43 there is $b'' \models \{p_i(x) : i \leq n+1\}$ such that $b' \downarrow_e^K a_{\leq n+1}$, and so $\{p_i(x) : i \leq n+1\}$ does not Kim-divide over e , which finishes our induction, and also the proof by the finite character of \downarrow^K . \square

For the following result we also follow the same track as in simple theories, see 10.11 of [Cas11]). This result is a generalization of [KR20, Corollary 6.6]:

Proposition 1.46. [Bos22, Corollary 2.5.12] Let $b, b', e \in \mathbb{M}$, if $b \downarrow_e^K b'$ and $b \equiv_e^L b'$ then there is a total Kim-Morley sequence $(b_i)_{i < \omega}$ over e , with $b_0 = b$ and $b_1 = b'$.

Corollary 1.47. (Chain condition for Kim-Morley sequences) Let $e \in \mathbb{M}$, $I = (a_i)_{i < \omega}$ be a Kim-Morley sequence over e and $b \in \mathbb{M}$ such that $b \downarrow_e^K a_0$, then there is $b' \equiv_{ea_0}^L b$ such that I is eb' -indiscernible and $b' \downarrow_e^K I$.

Proof. We can extend the sequence I into a long enough sequence $I' = (a_i)_{i < \kappa}$ also Kim-Morley over e . Let $p(x, a_0) := \text{tp}(b/a_0)$. By Corollary 1.45 $\{p(x, a_i) : i < \kappa\}$ does not Kim-fork over e , so there is $b'' \models \{p(x, a_i) : i < \kappa\}$ such that $b'' \downarrow_e^K (a_i)_{i < \kappa}$. We can extract a sequence $\tilde{I} = (\tilde{a}_i)_{i < \omega}$ using Erdős–Rado which is eb'' -indiscernible and finitely based on I' over eb'' . This implies that $\tilde{I} \equiv_e I$, so there is some b' such that $\tilde{I} \equiv_e Ib'$. This implies that I is eb' -indiscernible, and also $b' \equiv_{ea_0} b$ since $b' \models \{p(x, a_i) : i < \omega\}$. So b' is the tuple we want. \square

1.5.4 Transitivity of Kim-Forking

Lemma 1.48. [CKR23, Lemma 2.1] Let $a_0, b, e \in \mathbb{M}$. If $a_0 \downarrow_e^K b$, $e \in \text{dcl}(b)$ then there is a total Kim-Morley sequence $(a_i)_{i < \omega}$ over b such that $a_i \downarrow_e^K ba_{<i}$ for all $i < \omega$.

Proposition 1.49. [CKR23, Proposition 2.3] Let $a_0, b, e \in \mathbb{M}$. If $a_0 \downarrow_e^K b$, then there is an eb -indiscernible sequence $I = (a_i)_{i < \omega}$ which is a total Kim-Morley sequence over e and a Kim-Morley sequence over eb .

Theorem 1.50. [CKR23, Theorem 2.4] (Transitivity of Kim-independence) Let $a, b, c, e \in \mathbb{M}$. If $e \in \text{dcl}(b)$, $a \downarrow_e^K b$ and $a \downarrow_b^K c$, then $a \downarrow_e^K bc$.

Proof. Since $a \downarrow_e^K b$ by Proposition 1.49 there is a sequence $I = (a_i)_{i < \omega}$ with $a_0 = a$ such that I is a Kim-Morley sequence over b and a weak tree Morley sequence over e . By symmetry $c \downarrow_b^K a$, so by the chain condition Proposition 1.35 there is $I' \equiv_{ba} I$ such that $I' = (a'_i)_{i < \omega}$ is bc -indiscernible. I' is then also a total Kim-Morley sequence over e such that $a'_0 = a$. Then by Kim's lemma $bc \downarrow_e^K a$, and we conclude by symmetry. \square

1.5.5 Witnessing and local character

Definition 1.51. Let $e \in \mathbb{M}$ and $(a_i)_{i < \omega}$ be an e -indiscernible sequence.

- (1) Say $(a_i)_{i < \omega}$ is a witness for Kim-dividing over e if, whenever a partial type $p(x, a_0)$ over ea_0 Kim-divides over e , $\{p(x, a_i) : i < \omega\}$ is inconsistent.
- (2) Say $(a_i)_{i < \omega}$ is a strong witness for Kim-dividing over e if for all $n < \omega$, the sequence $(a_{[n \cdot i, n \cdot (i+1)]})_{i < \omega}$ is a witness for Kim-dividing over e .

Proposition 1.52. [Kim21, Lemma 2.3] Let $e \in \mathbb{M}$ and $I = (a_i)_{i < \omega}$ be a Kim-Morley sequence over e . For all $b \in \mathbb{M}$ there is a Kim-Morley sequence $(a_i b_i)_{i < \omega}$ over e such that $b \equiv_{ea_0} b_0$.

We now explain the local character of Kim-forking. By local character we mean two different type of properties, either that for a finitary type there is some bound on the length of a chain of Kim-forking extensions, or that there is some cardinal κ such that whenever p is a finitary type over some set of parameters A there is a subset $A_0 \subseteq A$ of size $\leq \kappa$ such that p does not Kim-fork over A_0 . By base monotonicity of \downarrow^f these two properties are equivalent for \downarrow^f . On this matter we recall a result that characterizes simple theories in terms of local character for forking independence:

Proposition 1.53. [Wag00, Corollary 2.8.9] and [Cas11, Proposition 4.13] Let $\kappa \geq |T|$ be a cardinal. The following are equivalent.

- (1) T is simple.
- (2) There is no increasing sequence of parameters $(e_i : i < \kappa^+)$ and finite tuple d such that $|e_i| \leq \kappa$ and $d \not\downarrow_{e_i}^f e_{i+1}$ for all $i < \kappa^+$.
- (3) For any finitary type p over A there is $A_0 \subseteq A$ such that $|A_0| \leq \kappa$ and p does not fork over A_0 .

In the following proposition by an increasing continuous sequence of parameters $(e_i : i < \kappa^+)$ we mean that if $\alpha < \beta < \kappa^+$ then $e_\alpha \in dcl(e_\beta)$ and if $\lambda < \kappa^+$ is a limit ordinal, then $e_\lambda \sim (e_i : i < \lambda)$. By ‘union e ’ in (3) we mean that $e_\lambda \sim (e_i : i < \kappa^+)$. The following proposition is the version of local character that holds for Kim-forking in NSOP₁ theories.

Proposition 1.54. [CKR23, Proposition 2.6] Let $\kappa \geq |T|$ be a cardinal. The following are equivalent.

- (1) T is NSOP₁.
- (2) There is no increasing continuous sequence $(e_i : i < \kappa^+)$ and finite tuple d such that $|e_i| \leq \kappa$ and $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < \kappa^+$.
- (3) There is no $e \in \mathbb{M}$ such that $|e| = \kappa^+$ and $p(x) \in S(e)$ with x a finite tuple of variables such that for some increasing and continuous sequence $(e_i : i < \kappa^+)$ with union e , we have $|e_i| \leq \kappa$ and p Kim-divides over e_i for all $i < \kappa^+$.

It is interesting to know that the conditions of Proposition 1.54 are maximal: In fact Casanovas and Kim show the following:

Proposition 1.55. [CK20, Proposition 3.6] Assume that T has TP₂. Let κ be an infinite cardinal.

1. There is an increasing non-continuous sequence $(e_i : i < \kappa^+)$ and finite tuple d such that $|e_i| \leq \kappa$ and $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < \kappa^+$.
2. There is an increasing continuous sequence $(e_i : i < |T|)$ and a finite tuple d such that $|e_i| \leq |T|$ and $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < |T|$.

Remark 1.56. The previous results imply that in an NSOP₁ theory T , \downarrow^K satisfies base monotonicity if and only if T is simple, and in that case $\downarrow^K = \downarrow^f$.

Proof. Assume that \downarrow^K satisfies base monotonicity in some NSOP₁ theory T that has TP₂.

By Proposition 1.55 there is an increasing non-continuous sequence $(e_i : i < \kappa^+)$ and finite tuple d such that $|e_i| \leq \kappa$ and $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < \kappa^+$. If we let e'_i be the sequence defined by $e'_i = e_{<i}$ then $(e'_i)_{i < \kappa^+}$ is a continuous increasing sequence which has the same union as $(e_i)_{i < \kappa^+}$. By 3 of Proposition 1.54 there is some $i < \kappa^+$ such that $d \downarrow_{e'_i}^K e_{<\kappa^+}$. By base monotonicity this implies $d \downarrow_{e_i}^K e_{<\kappa^+}$ which contradicts $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < \kappa^+$. So T does not have TP₂, which implies that T is simple by Proposition 1.28. \square

Kim-Morley sequences are witnesses to Kim-dividing:

Theorem 1.57. [CKR23, Theorem 2.8] (*Witnessing*) Let $e \in \mathbb{M}$ and $I = (a_i)_{i < \omega}$ be a Kim-Morley sequence over e . If $\varphi(x, a_0)$ Kim-divides over e then $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent, i.e. Kim-Morley sequences over e are witness to Kim-dividing over e . In particular strong Kim-Morley sequences over e are strong witnesses for Kim-dividing over e .

Proof. Suppose towards contradiction that $\varphi(x, a_0)$ Kim-divides over e and $I = (a_i)_{i < \omega}$ is an e -Morley sequence such that $\{\varphi(x, a_i) : i < \omega\}$ is consistent. We may stretch I such that $I = (a_i)_{i < \kappa^+}$ with $|T| + |e| \leq \kappa$. Let $e_i \sim ea_{<i}$. Then $(e_i)_{i < \kappa^+}$ is increasing and continuous and $|e_i| \leq \kappa$.

Let $d \models \{\varphi(x, a_i) : i < \kappa^+\}$. We claim that $d \not\downarrow_{e_i}^K e_{i+1}$ for all $i < \kappa^+$. If not, then for some $i < \kappa^+$, we have $d \downarrow_{e_i}^K e_{i+1}$, i.e. $d \downarrow_{ea_{<i}}^K a_i$. Since I is a Kim-Morley sequence over e we also have $a_i \downarrow_e^K a_{<i}$, hence $da_{<i} \downarrow_e^K a_i$ by transitivity. This entails, in particular, that $d \downarrow_e^K a_i$, which is a contradiction, since $\varphi(x, a_i)$ Kim-divides over e . This completes the proof. \square

From Proposition 1.52 and Theorem 1.57 we deduce:

Corollary 1.58. Let $a_0, e \in \mathbb{M}$, $I = (a_i)_{i < \omega}$ a Kim-Morley sequence over e and $p(x, a_0)$ a partial type over a_0 such that $\{p(x, a_i) : i < \omega\}$ is consistent. Then $p(x, a_0)$ does not Kim-fork over e .

Corollary 1.59. Let $e \in \mathbb{M}$, $I = (a_i)_{i < \omega}$ be a Kim-Morley sequence over e and $b \in \mathbb{M}$ such that all the a_i have the same type over eb . Then $b \downarrow_e^K a_0$.

Corollary 1.60. Let $e \in \mathbb{M}$, $I = (a_i)_{i < \omega}$ a total Kim-Morley sequence over e and $b \in \mathbb{M}$ such that I is eb -indiscernible. Then $b \downarrow_e^K I$.

Proof. By finite character to show that $b \downarrow_e^K I$ it is enough to show that $b \downarrow_e^K a_{<n}$ for all $n < \omega$. We have that $(a_{n \cdot i} \dots a_{n \cdot i + n - 1})_{i < \omega}$ is Kim-Morley over e by hypothesis and also $\hat{e}b$ -indiscernible. Then, by Corollary 1.59 we have $b \downarrow_e^K a_{<n}$. \square

1.5.6 The Kim-Pillay criterion for Kim-forking

We begin by recalling the Kim-Pillay theorem that characterizes simple theories in terms of independence relation:

Theorem 1.61. [KP97, Theorem 4.2] *Assume that there is an invariant independence relation \perp satisfying the following properties for arbitrary tuples from \mathbb{M} :*

1. *Finite character: $a \perp_E B$ if and only if $a \perp_E b$ for all finite tuples $a, b \in B$.*
2. *Existence and extension: For any a, E, B there is $a' \equiv_E a$ such that $a' \perp_E B$.*
3. *Local character: For any finite a and any set of parameters B there is $B_0 \subseteq B$ such that $|B_0| \leq |T|$ and $a \perp_{B_0} B$.*
4. *Symmetry: $a \perp_E b$ implies $b \perp_E a$.*
5. *Transitivity: $a \perp_E BC$ if $a \perp_E B$ and $a \perp_{EB} C$.*
6. *Base monotonicity: $a \perp_E BC$ implies $a \perp_{EB} C$.*
7. *The independence theorem over a model: If $a_0 \equiv_M a_1$, $a_i \perp_M b_i$ for $i = 0, 1$ and $b_0 \perp_M b_1$ there is an $a \in \mathbb{M}$ such that $a \equiv_{eb_i} a_i$ for $i = 0, 1$ and $a \perp_M b_0 b_1$.*

Then T is simple and \perp coincide with \perp^f , i.e. $a \perp_E b$ if and only if $a \perp_E^f b$ for all $E, a, b \in \mathbb{M}$.

There is a similar result characterizing Kim-forking over models in NSOP₁ theories:

Theorem 1.62. [CR16, Proposition 5.8] and [KR17, Theorem 9.1] *Assume that there is an invariant independence relation \perp satisfying the following properties for an arbitrary $M \models T$ as the basis and arbitrary tuples from \mathbb{M} :*

1. *Strong finite character: If $a \not\perp_M b$ then there is a formula $\varphi(x, b, m) \in \text{tp}(a/Mb)$ such that $a' \not\perp_M b$ whenever $\models \varphi(a', b, m)$.*
2. *Existence over models: $M \models T$ implies $a \perp_M M$ for all $a \in \mathbb{M}$.*
3. *Monotonicity over models: $aa' \perp_M bb'$ implies $a \perp_M b$.*
4. *Symmetry over models: $a \perp_M b$ implies $b \perp_M a$.*
5. *The independence theorem: If $a_0 \equiv_M a_1$, $a_i \perp_M b_i$ for $i = 0, 1$ and $b_0 \perp_M b_1$ there is an $a \in \mathbb{M}$ such that $a \equiv_{eb_i} a_i$ for $i = 0, 1$ and $a \perp_M b_0 b_1$.*

Then T is NSOP₁ and \perp strengthens \perp^K , i.e. if $a \perp_M b$ then $a \perp_M^K b$. If \perp also satisfies:

6. *Witnessing: If $M \models T$ and $(b_i)_{i < \omega}$ is an M -indiscernible sequence satisfying $b_i \perp_M b_{<i}$ then whenever $a \not\perp_M b_0$ there is a formula $\varphi(x, m, b_0) \in \text{tp}(a/Mb_0)$ such that $\{\varphi(x, m, b_i) : i < \omega\}$ is inconsistent.*

Then \perp and \perp^K coincide over models: $a \perp_M b$ if and only if $a \perp_M^K b$ for any $M \models T$ and $a, b \in \mathbb{M}$.

There is a similar results for theories with existence that characterizes Kim-independence over arbitrary sets in NSOP₁ theories with existence:

Theorem 1.63. [CKR23, Theorem 5.1] Assume that T satisfies existence. The theory T is NSOP₁ if and only if there is an invariant independence relation \perp on small subsets of the monster model which satisfies the following properties, for an arbitrary set of parameters e and arbitrary tuples from \mathbb{M} .

1. *Strong finite character:* If $a \not\perp_e b$ then there is a formula $\varphi(x, b, e) \in \text{tp}(a/eb)$ such that $a' \not\perp_e b$ whenever $\models \varphi(a', b, e)$.
2. *Existence and extension:* For any a, e, B there is $a' \equiv_e a$ such that $a' \perp_e B$.
3. *Monotonicity:* $aa' \perp_e bb'$ implies $a \perp_e b$.
4. *Symmetry:* $a \perp_e b$ implies $b \perp_e a$.
5. *The independence theorem:* If $a_0 \equiv_e^L a_1$, $a_i \perp_e b_i$ for $i = 0, 1$ and $b_0 \perp_e b_1$ there is an $a \in \mathbb{M}$ such that $a \equiv_{eb_i} a_i$ for $i = 0, 1$ and $a \perp_e b_0 b_1$.

Then T is NSOP₁ and \perp strengthens \perp^K , i.e. if $a \perp_e b$ then $a \perp_e^K b$. If \perp also satisfies:

6. *Transitivity:* If $a \perp_e b$ and $a \perp_{eb} c$ then $a \perp_e bc$.
7. *Local character:* If $\kappa \geq |T|^+$ is a regular cardinal, $(A_i)_{i < \kappa}$ is an increasing continuous sequence of sets of size $< \kappa$, $A_\kappa = \bigcup_{i < \kappa} A_i$ and $|A_\kappa| = \kappa$, then for any finite d , there is some $\alpha < \kappa$ such that $d \perp_{A_\alpha} A_\kappa$.

Then \perp and \perp^K coincide: $a \perp_e b$ if and only if $a \perp_e^K b$ for any $e, a, b \in \mathbb{M}$.

Notice that the conditions 1, 2 and 4 of Theorem 1.62 imply extension over models, so that the conditions of the first part of Theorem 1.63 are just those of Theorem 1.62 where we replace models by arbitrary sets: Let M be a model, $M \subseteq B$ and a be some tuple. Consider an extension of $\text{tp}(a/M)$ to B that is finitely satisfiable in M and let a' realize this extension. We show that $a' \perp_M B$. Else by 1 there is a finite tuple $b \in B$ and a formula $\varphi(x, b, m) \in \text{tp}(a'/Mb)$ such that $c \not\perp_m b$ whenever $\models \varphi(c, b, m)$. Since $\text{tp}(a'/Mb)$ is finitely satisfiable in M we can find $n \in N$ realizing $\varphi(x, b, m)$. Then $n \not\perp_M b$, so by symmetry $b \not\perp_M n$ which yields $b \not\perp_M M$ by 3, which contradicts 2.

The difference between the conditions of Theorem 1.61 and the conditions of the first part of Theorem 1.63 is that in the case of NSOP₁ we do not require \perp to satisfy base monotonicity. The properties of Kim-independence in NSOP₁ theories with existence can be summed-up by this informal rule: it is the same as the properties of forking independence in simple theories once you remove every instance of base monotonicity.

Chapter 2

Fraissé Classes with Free Amalgamation



In this chapter we give a way to construct some examples of NSOP₁ theories as limits of some Fraïssé class satisfying strong conditions. These limits will satisfy existence, that Kim-independence coincides with algebraic independence, and that forking independence is obtained by forcing base monotonicity on Kim-forking. These theories also have a stationary independence relation. This study is based on results of Baudisch, Chernikov, Kruckman and Ramsey.

2.1 Constructing the limit

In this section we define a set of hypotheses (H) on a class K of structures. We begin by showing that if K satisfies (H) then we can construct the limit theory T of K . We show that this limit T is an NSOP₁ theory, has existence, that $\downarrow^K = \downarrow^a$, that $\downarrow^M = \downarrow^f = \downarrow^d$ over arbitrary sets and that it has weak elimination of imaginaries. We give some basic examples of such classes K . The bases of this study are [CR16, Section 6.3] and [Bau02]. The work of Baudisch takes place in a much more general context, but we focus here on the study of independence relations, which appears in his paper in [Bau02, Theorem 4.1] under some stronger hypotheses.

In this section we will consider the notion of morphism of L -structures for a given language L . To avoid any confusion with the notion of L -embedding we will remind the definition of these two notions. Let M and N be two L -structures and let $\varphi : M \rightarrow N$ be a map.

1. φ is a morphism of L -structures if for any atomic formula $\psi(x)$ in L and any x -tuple $a \in M$, $M \models \psi(a)$ implies $N \models \psi(\varphi(a))$.
2. φ is an L -embedding if for any atomic formula $\psi(x)$ in L and any x -tuple $a \in M$, $M \models \psi(a)$ holds if and only if $N \models \psi(\varphi(a))$ does.

Another way to say this is that a morphism of L -structures preserves the satisfaction of positive quantifier free formulas and that an L -embedding preserves the satisfaction of all quantifier free formulas.

2.1.1 Hypothesis on the class and basic examples

Definition 2.1. We consider a countable language L and a class K of finite and countable L -structures with the following properties, which we will refer to as (H):

1. **Universal Axiomatisability:** If an L -structure A is not in K there is a quantifier free formula φ and a finite tuple $a \in A$ such that $A \models \varphi(a)$ and $A' \not\models \varphi(a')$ for every $a' \in A' \in K$.
2. **Quantifier Elimination:** For every quantifier free formula $\varphi(x, y)$ satisfied by (a, b) in some structure A of K there is a quantifier free formula $\psi(x)$ such that $A \models \psi(a)$ and if $a' \in A' \in K$ and $A' \models \psi(a')$ then there is $A' \subseteq B' \in K$ an extension and $b' \in B'$ such that $B' \models \varphi(a', b')$.
3. **Free Amalgamation:** If $C \subseteq A, B$ are structures in K (where we allow $C = \emptyset$ when the language L does not have any constants) there is $D \in K$ and some embeddings i_A, i_B of A and B respectively to D satisfying $i_A|_C = i_B|_C$, $i_A(A) \cap i_B(B) = i_A(C)$, D is generated by the images of A and B and such that given any morphisms of L -structures $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ to $D' \in K$ such that $\varphi_A|_C = \varphi_B|_C$ there is a unique morphism $\varphi : D \rightarrow D'$ such that $\varphi_A = \varphi \circ i_A$ and $\varphi_B = \varphi \circ i_B$. We will write $A \oplus_C B$ for this structure D . We also assume that the elements of K agree on the structure generated by the constants, which is equivalent to the joint embedding property when Free Amalgamation holds.
4. **Generic Element:** If V is a sort there is a structure $\langle \{x\} \rangle \in K$ generated by a single element $x \in V$ such that for every structure $A \in K$ and element $a \in V(A)$ there is a morphism $\varphi : \langle \{x\} \rangle \rightarrow A \in K$ sending x to a .

5. **Algebraically Independent 3-Amalgamation:** Given a commutative diagram in K as on the left below, where the arrows are embeddings, $\varphi_{B_i}(B_i) \cap \varphi_{A_i}(A) = \varphi_{A_i} \circ i_A(E) = \varphi_{B_i} \circ i_{B_i}(E)$ for $i = 0, 1$ and $\psi_{B_0}(B_0) \cap \psi_{B_1}(B_1) = \psi_{B_0} \circ i_{B_0}(E) = \psi_{B_1} \circ i_{B_1}(E)$, we can complete it inside of K to a commutative diagram as on the right (where the arrows are embeddings) such that $\varphi_{D_i}(D_i) \cap \varphi_B(B) = \varphi_{D_i} \circ \varphi_{B_i}(B_i) = \varphi_B \circ \psi_{B_i}(B_i)$ for $i = 0, 1$ and $\varphi_{D_0}(D_0) \cap \varphi_{D_1}(D_1) = \varphi_{D_0} \circ \varphi_{A_0}(A) = \varphi_{D_1} \circ \varphi_{A_1}(A)$ (the blank arrows of the right diagram are those of the left diagram; their names have been omitted for clarity).

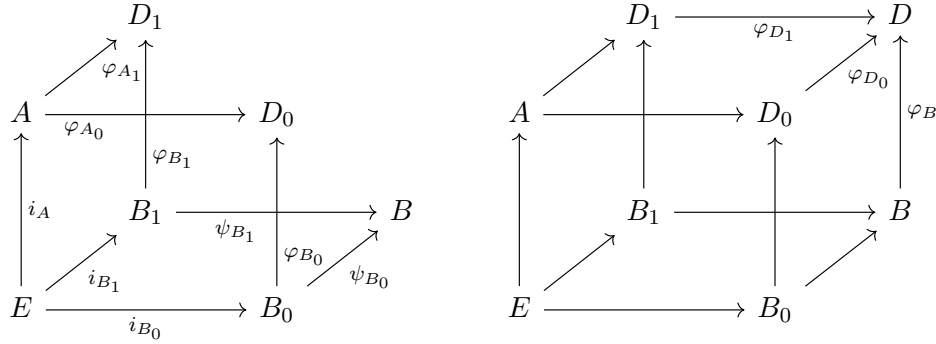


Figure 2.1: Algebraically Independent 3-Amalgamation

Notice that we do not assume that there are countably many isomorphism classes (i.e. that the class K is ‘essentially countable’), nor that the structures in K are finitely generated. If K satisfies Condition 1 then K has the hereditary property: If $A \subseteq B \in K$ is a substructure then $A \in K$. To check Condition 2 for a formula $\varphi(x, y)$ realized by a tuple $(a, b) \in A$ it is enough to check it for a strengthening of this formula that is also realized by (a, b) in A .

Applying Condition 4 to find $\langle\{x\}\rangle$ and then Condition 3 to some $A \in K$ and $\langle\{x\}\rangle$ above $\langle\emptyset\rangle$ gives us a structure $A \oplus_{\langle\emptyset\rangle} \langle\{x\}\rangle$ satisfying Condition 4 above A , meaning that for every morphism φ from A to $B \in K$ and element $b \in V(B)$ there is a morphism $\varphi_b : A \oplus_{\langle\emptyset\rangle} \langle\{x\}\rangle \rightarrow B \in K$ sending x to b and extending φ .

We can define by induction $\langle\{(x_i)_{i < n}\}\rangle$ for every $n < \omega$ the following way: $\langle\{(x_i)_{i < n+1}\}\rangle$ is defined as $\langle\{(x_i)_{i < n}\}\rangle \oplus \langle\{x_n\}\rangle$. From this definition it is easy to show that $\langle\{(x_i)_{i < n}\}\rangle$ satisfies the genericity property 4 for n -tuples of elements of the sort V . Condition 1 implies that K is closed under unions of countable chains. With this and the previous remark we can define:

$$\langle\{(x_i)_{i < \omega}\}\rangle := \bigcup_{n < \omega} \langle\{(x_i)_{i < n}\}\rangle.$$

This structure lies in K , since morphisms of L -structures are closed under unions of countable chains it is easy to show that it satisfies the genericity property 4 for countable tuples of elements of the sort V .

The notation $\langle\{x\}\rangle$ is here to insist on the fact that the set in between the braces is considered only as a set of generators. This notation $A \oplus_{\langle\emptyset\rangle} \langle\{D\}\rangle$ where $A, D \in K$ means the Generic Extension of A with generators over A indexed on D and should not be confused with $A \oplus_{\langle\emptyset\rangle} D$ which is the Free Amalgam of the structures $A, D \in K$ over the structure generated by the constants.

Condition 5 is a strengthening of 3-amalgamation for \perp^a ; the fact that the images of A and B_1 are disjoint over the image of E in D_1 should be read as ‘ A and B are algebraically independent

inside of D_1 (which is written $A \downarrow_E^{a, D_1} B_1$ with the notations of [AHKK23], see this reference for some use of independence relation in categories). In the usual 3-amalgamation we would ask that D_i or B are generated by the image of the two embeddings from A and B_i or from B_0 and B_1 respectively. Here however we allow ourselves to take an extension of the generated structure which will be useful when adding structure in Section 4.2.

Some examples of classes K satisfying (H) are one based stable theories that can be presented as Fraïssé limits: The class of countable vector spaces over a finite (or countable) field (which is in the language), the class of countable abelian groups, the class of countable sets with no structure. In all those cases the conditions 1 – 4 are easy to check, and the Condition 5 is a consequence of the following proposition. An other example is the class of countable graph, which is not stable.

Lemma 2.2. *Suppose that some class K satisfies 1 – 4 and that free independence and algebraic independence coincide, i. e. that for every $E \subseteq A, B \in K$ such that $A, B \subseteq D \in K$, the canonical map $\varphi : A \oplus_E B \rightarrow D$ is an embedding if and only if $A \cap B = E$. Then Condition 5 holds in K . The classes associated to one based stable theories that we quoted previously satisfy this condition.*

Proof. These hypotheses imply in particular that algebraic independence $\downarrow^{a, D}$ satisfies base monotonicity in the structures of K (it is shown later in Proposition 2.11 that free independence always satisfies base monotonicity, and it is clear that $\downarrow^{a, D}$ satisfies transitivity). Given E, A, B_0, B_1, D_0, D_1 and a diagram satisfying the assumptions of 6 consider the following diagram, where $D := [D_0 \oplus_{B_0} B] \oplus_{A \oplus_E B} [D_1 \oplus_{B_1} B]$.

We want to show that $D_0 \cap D_1 = A$, $D_0 \cap B = B_0$ and $D_1 \cap B = B_1$ inside of D (i. e. $D_0 \downarrow_A^{a, D} D_1$, $D_0 \downarrow_{B_0}^{a, D} B$ and $D_1 \downarrow_{B_1}^{a, D} B$). We abuse notation and write $\langle AB \rangle$ for the substructure of D generated by the images of A and B , and for substructures F, G, H of D such that $F \cap G = H$, we will indifferently write $\langle FG \rangle$ and $F \oplus_H G$ for the generated structure.

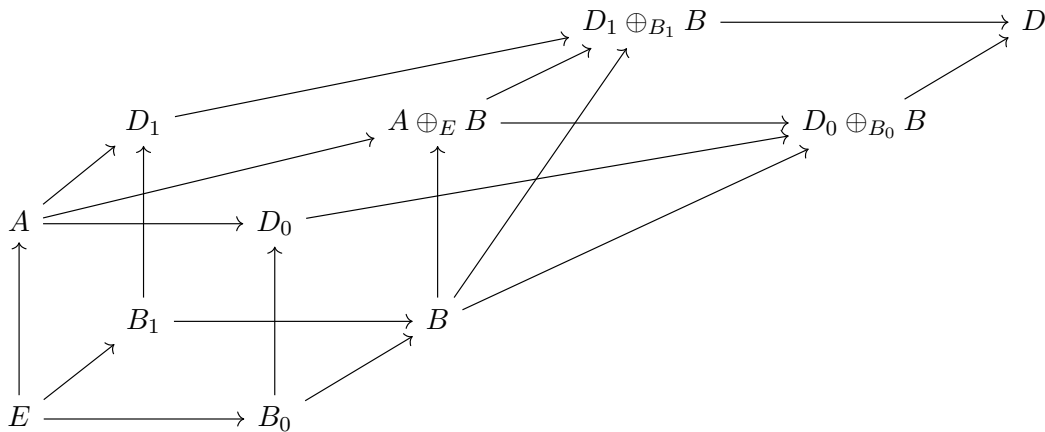


Figure 2.2: Proving Algebraically Independent 3-Amalgamation

The maps from $A \oplus_E B$ to $D_i \oplus_{B_i} B$ come from the fact that A and B are free over E in $D_i \oplus_{B_i} B$. In fact if some element of $D_i \oplus_{B_i} B$ is in the image of both A and B , then it is also in the image of D_i , so it is in the image of B_i , and by assumption B_i and A are free over E in D_i . This diagram is commutative and all arrows are embeddings. If some element of D is in the image of D_i and B then it is in the image of B_i , so $D_i \downarrow_{B_i}^{a, D} B$ for $i \in \{0, 1\}$. If some element of D is in the image of D_0 and D_1 then it is in the image of $A \oplus_E B$ which is just the substructure generated by the images of A and B .

Since $D_i \downarrow_{B_i}^{a, D} B$ by base monotonicity $D_i \downarrow_{\langle AB_i \rangle}^{a, D} \langle AB \rangle$, so it is enough to show that $\langle AB_0 \rangle \downarrow_A^{a, D} \langle AB_1 \rangle$. Now $A \downarrow_E^{a, D} B$ since $D_0 \downarrow_{B_0}^{a, D} B$ and $A \downarrow_E^{a, D} B_0$. By base monotonicity $\langle AB_0 \rangle \downarrow_{B_0}^{a, D} B$.

$\langle B_0 B_1 \rangle$, since $B_0 \downarrow_E^{a,D} B_1$ transitivity yields $\langle AB_0 \rangle \downarrow_E^{a,D} B_1$, so $\langle AB_0 \rangle \downarrow_A^{a,D} \langle AB_1 \rangle$ by base monotonicity and Condition 5 holds. \square

2.1.2 Properties of the limit theory

We begin by constructing the limit theory T of K and then we prove some of its properties. We shall assume the continuum hypothesis as we need an uncountable regular cardinal κ such that $2^{<\kappa} = \kappa$, and it is convenient to take \aleph_1 as such.

Definition 2.3. Given an L -structure M and a cardinal κ the κ -age of M is the class of the substructures of M of cardinality $< \kappa$.

Definition 2.4. Let \mathcal{K} be a class of countable structures in some countable language \mathcal{L} . An \mathcal{L} -structure \mathcal{M} is \mathcal{K} - \aleph_1 -saturated if its \aleph_1 -age is \mathcal{K} and if for every $\mathcal{U}, \mathcal{B} \in \mathcal{K}$ countable and any embeddings $f_0 : \mathcal{U} \rightarrow \mathcal{M}$, $f_1 : \mathcal{U} \rightarrow \mathcal{B}$ there is some embedding $g : \mathcal{B} \rightarrow \mathcal{M}$ such that $g \circ f_1 = f_0$.

Proposition 2.5. *There exists an L -structure \mathcal{M} which is K - \aleph_1 -saturated and has cardinal \aleph_1 . Every structure of \aleph_1 -age included in K and of cardinality $\leq \aleph_1$ embeds into \mathcal{M} , and \mathcal{M} is unique up to isomorphism.*

Proof. Let $(\mathcal{B}_i)_{i < \aleph_1}$ enumerate the elements of K up to isomorphism. We construct by induction an chain $(\mathcal{U}_i)_{i < \aleph_1}$ of L -structures in K . Let $\mathcal{U}_0 = \mathcal{B}_0$. If \mathcal{U}_i has been defined, for every $k < \aleph_1$ let $(f_j^{\mathcal{U}_i, k})_{j < \aleph_1}$ enumerate the embeddings of \mathcal{U}_i into \mathcal{B}_k . We begin by showing that we can extend a countable number of partial embeddings in K .

Claim: If $\mathcal{U} \in K$, $B_i \in K$ for $i < \omega$ and $f_i : \mathcal{U} \rightarrow B_i$ are partial embeddings, then there is an extension $\mathcal{U} \subseteq \mathcal{U}' \in K$ and embeddings $g_i : B_i \rightarrow \mathcal{U}'$ such that $g_i \circ f_i$ is the identity on its domain for all $i < \omega$.

Proof of the claim: We build an increasing sequence $(\mathcal{U}'_i)_{i < \omega}$ of extensions of \mathcal{U} . Let $E_i \subseteq \mathcal{U}$ be the domain of f_i for $i < \omega$. Consider the extensions $E_0 \subseteq \mathcal{U}$ and $f_0 : E_0 \rightarrow B_0$. By Free Amalgamation we can define $\mathcal{U}'_0 = \mathcal{U} \oplus_{E_0} B_0 \in K$, and we set g_0 to be the natural embedding $B_0 \rightarrow \mathcal{U}'_0$. Assume that \mathcal{U}'_{i-1} and g_{i-1} are defined for $i-1 < \omega$. Consider the extensions $E_i \subseteq \mathcal{U}'_{i-1}$ and $f_i : E_i \rightarrow B_i$. We can define $\mathcal{U}'_i = \mathcal{U}'_{i-1} \oplus_{E_i} B_i$, and we set g_i to be the natural embedding $B_i \rightarrow \mathcal{U}'_i$. Now let $\mathcal{U}' := \bigcup_{i < \omega} \mathcal{U}'_i$. Since K is closed under unions of countable chains this is a structure in K and it clearly satisfies the condition. \square

We can now construct the chain. Assume that \mathcal{U}_j has been defined for $j < i < \aleph_1$. Let $\mathcal{U}' := \bigcup_{j < i} \mathcal{U}_j$.

Consider the sequence of embeddings $(f_l^{\mathcal{U}_j, k})_{j, k, l < i}$ seen as partial embeddings on $\mathcal{U}' \supseteq \mathcal{U}_j$. By the claim there is an extension $\mathcal{U}_i \supseteq \mathcal{U}'$ and a sequence of embeddings $(g_l^{\mathcal{U}_j, k})_{j, k, l < i} : \mathcal{B}_k \rightarrow \mathcal{U}_i$ such that $g_l^{\mathcal{U}_j, k} \circ f_l^{\mathcal{U}_j, k}$ is the identity on its domain \mathcal{U}_j for every $j, k, l < i$. We define $\mathcal{M} := \bigcup_{i < \aleph_1} \mathcal{U}_i$.

Now we show that \mathcal{M} is K - \aleph_1 -saturated. Let \mathcal{U} be a countable substructure of \mathcal{M} . By isomorphic correction we can assume that the map f_0 of the saturation property is the inclusion. Then $\mathcal{U} \subseteq \mathcal{U}_j$ for some $j < \aleph_1$ so $\mathcal{U} \in K$ by Heredity thus \aleph_1 -age(\mathcal{U}) $\subseteq K$.

Consider an embedding $f : \mathcal{U} \rightarrow \mathcal{B} \in K$. \mathcal{B} is isomorphic to \mathcal{B}_k for some $k < \aleph_1$ via $\varphi : \mathcal{B} \rightarrow \mathcal{B}_k$. By Free Amalgamation there is some $k' < \aleph_1$ and some embeddings $g : \mathcal{B}_k \rightarrow \mathcal{B}_{k'}$ and $f' : \mathcal{U}_j \rightarrow \mathcal{B}_{k'}$ such that $g \circ \varphi \circ f = f'$ (here k' is such that $\mathcal{B}_{k'}$ is isomorphic to $\mathcal{U}_i \oplus \mathcal{B}_k$). Now $f' : \mathcal{U}_j \rightarrow \mathcal{B}_{k'}$ is equal to $f_l^{\mathcal{U}_j, k'}$ for some $l < \aleph_1$. Set $i = \max\{j, k, l\}$, by construction, there is a $h = g_l^{\mathcal{U}_j, k'} : \mathcal{B}_{k'} \rightarrow \mathcal{U}_{i+1}$ such that $h \circ f'$ is the inclusion $\mathcal{U}_j \subseteq \mathcal{U}_{i+1}$. Then $h \circ g \circ \varphi$ is the embedding we want.

$$\begin{array}{ccccc}
\mathcal{U}_i & \xrightarrow{\subseteq} & \mathcal{U}_{i+1} & \subseteq & \mathcal{M} \\
\uparrow \subseteq & & & & \uparrow h \\
\mathcal{U}_j & \xrightarrow{f'} & \mathcal{B}_{k'} & & \\
\uparrow \subseteq & & \uparrow g & & \\
\mathcal{U} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{\varphi} & \mathcal{B}_k
\end{array}$$

Figure 2.3: The embedding we needed

For the second point we proceed as in Fraïssé theory and only use the property of \aleph_1 -saturation and the fact that we can amalgamate over the structure generated by the constants: if \mathcal{M}' is a structure of age inside of K and of cardinality $\leq \aleph_1$ we fix an enumeration $(m_i)_{i < \aleph_1}$ of \mathcal{M}' and define by induction an increasing sequence of embeddings $f_i : \langle m_{<i} \rangle \rightarrow \mathcal{M}$, then the union $f = \bigcup_{i < \aleph_1} f_i : \mathcal{M} \rightarrow \mathcal{M}'$ is our embedding.

Let f_0 be the embedding of the constants from \mathcal{M}' to \mathcal{M} (possibly f_0 is the empty set). If f_j is defined for $j < i < \aleph_1$, let $f' := \bigcup_{j < i} f_j : \mathcal{B}' \subseteq \mathcal{M}' \rightarrow \mathcal{M}$.

Let \mathcal{B} be the image of \mathcal{M} and $g' : \mathcal{B} \rightarrow \mathcal{B}'$ be the inverse map of f' , we can compose g' with the inclusion $\mathcal{B}' \subseteq \langle m_{<i} \rangle$. Since $\langle m_{<i} \rangle \in K$ we can use the \aleph_1 -saturation of \mathcal{M} to find a map $f_i : \langle m_{<i} \rangle \rightarrow \mathcal{M}$ such that $f_i \circ g' = id_{\mathcal{B}}$, which implies that $f_i|_{\mathcal{B}'} = f$ by pre-composing with f , i. e. that f_i extends f_j for $j < i$. In a similar fashion, to show uniqueness we construct an isomorphism by back and forth. \square

Let T be the theory of \mathcal{M} constructed in Proposition 2.5, we refer to T as the *limit theory* of K . It is a complete theory so we can consider a monster model \mathbb{M} of T .

Lemma 2.6. *T has quantifier elimination in L .*

Proof. This is clear from the Condition 2 : If we consider any \aleph_1 -saturated model it is K - \aleph_1 -saturated, so partial isomorphisms between finitely generated structures have the back and forth property. \square

Remark 2.7. A consequence of the Free Amalgamation property is that $dcl(A) = acl(A) = \langle A \rangle$ for every set $A \subseteq \mathbb{M}$. In fact if some element a is not in the countable structure $A = \langle A \rangle$ then we can form $\langle A, a \rangle \oplus_A \langle A, a \rangle$ and get two copies of a over A , repeating the process we get that $qftp(a/A)$ is not algebraic.

Lemma 2.8. *The relation of algebraic independence \downarrow^a on subsets of \mathbb{M} satisfies the independence theorem: If $E, A_i, B_i \subseteq \mathbb{M}$ for $i \in \{0, 1\}$, $A_0 \equiv_E A_1$, $A_i \downarrow_E^a B_i$ for every $i \in \{0, 1\}$ and $B_0 \downarrow_E^a B_1$ there is an A such that $A \equiv_{B_i} A_i$ for $i \in \{0, 1\}$ and $A \downarrow_E^a B_0 B_1$.*

Proof. Consider E, A_i, B_i for $i = 0, 1$ as in the statement, \bar{x}_A a tuple of variables indexed by A_0 (and so also by A_1 through the isomorphism $\varphi : A_0 \rightarrow A_1$ induced by $A_0 \equiv_E A_1$). We write x_a the variable in \bar{x}_A associated to $a \in A_0$ and $\varphi(a) \in A_1$. We extend \bar{x}_A into $\bar{x}_A \bar{y}_0$, a tuple of variable indexed on $\langle A_0 B_0 \rangle$ and into $\bar{x}_A \bar{y}_1$, a tuple of variable indexed on $\langle A B_1 \rangle$ disjoint with \bar{y}_0 .

Let $q_i(\bar{x}_A \bar{y}_i) := qftp(\langle A_i B_i \rangle / B_i)$ for $i = 0, 1$. We show that:

$$q_0(\bar{x}_A \bar{y}_0) \cup q_1(\bar{x}_A \bar{y}_1) \cup \{t(\bar{x}_{\bar{a}}) \notin \langle B_0 B_1 \rangle : t \text{ terms and } \bar{a} \in A_0 \text{ tuples such that } t(\bar{a}) \notin E\}$$

is a consistent type. By compactness is it enough to consider finitely generated structures E, A_i, B_i , and in that case we get consistency by just applying Condition 5. \square

Definition 2.9. We define the Γ -independence relation on small subsets of \mathbb{M} , the monster model of T . For $E \subseteq A, B$ say that $A \perp_E^\Gamma B$ if the structure $\langle AB \rangle$ and the inclusions $i_A : A \rightarrow \langle AB \rangle$, $i_B : B \rightarrow \langle AB \rangle$ satisfy the universal property that every common extension of A and B in \mathbb{M} factorizes uniquely through $\langle AB \rangle$. In the case where $E \subseteq A, B$ are countable $A \perp_E^\Gamma B$ if and only if $A \oplus_E B$ is isomorphic to $\langle AB \rangle$ via the canonical map of Condition 3.

Lemma 2.10. *Full existence: For every $E \subseteq A, B \subseteq \mathbb{M}$ there is $A' \equiv_E A$ such that $A' \perp_E^\Gamma B$.*

Proof. We want to construct an L -structure D extending B and an E -embedding $i_A : A \rightarrow D$ such that for $A' := i_A(A)$, $D = \langle A'B \rangle$ and $A' \perp_E^\Gamma B$. For countable substructures $E' \subseteq E$, $A' \subseteq A$, $B' \subseteq B$ we consider $A' \oplus_{E'} B' \in K$. The structure D that we build is the inductive limit of these structures. We begin by showing that they indeed form an inductive system.

Let $E'' \subseteq E' \subseteq E$, $A'' \subseteq A' \subseteq A$, $B'' \subseteq B' \subseteq B$ be countable substructure. Then there is a canonical map from $A'' \oplus_{E''} B''$ to $A' \oplus_{E'} B'$ where $E'' \subseteq E'$, $A'' \subseteq A'$ and $B'' \subseteq B'$. We describe it in the following diagram. This map assures us that the system of the $A' \oplus_{E'} B'$ we are considering is directed and that we can define the L -structure D as its direct limit. Also $\text{Diag}(D/B)$ is consistent with T because any formula that appears in it is satisfied in some of the $A' \oplus_{E'} B' \in K$.

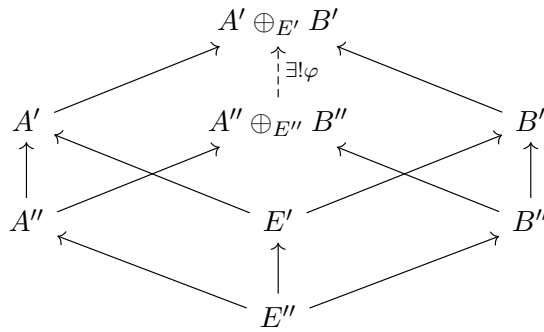


Figure 2.4: The inductive system of $A' \oplus_{E'} B'$

Now we show that D satisfies the right property: Let $D' \subseteq \mathbb{M}$, and $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ form a commutative square over E . For every $E'' \subseteq E' \subseteq E$, $A'' \subseteq A' \subseteq A$, $B'' \subseteq B' \subseteq B$ countable substructures the restricted morphism $\varphi_{E|E''} \subseteq \varphi_{E|E'}$, $\varphi_{A|A''} \subseteq \varphi_{A|A'}$, $\varphi_{B|B''} \subseteq \varphi_{B|B'}$ factorizes uniquely via $\varphi_{A'' \oplus_{E''} B''}$ and $\varphi_{A' \oplus_{E'} B'}$ respectively through $A'' \oplus_{E''} B''$ and $A' \oplus_{E'} B'$ respectively. By uniqueness if φ is the canonical map from $A'' \oplus_{E''} B''$ to $A' \oplus_{E'} B'$ then $\varphi_{A' \oplus_{E'} B'} \circ \varphi = \varphi_{A'' \oplus_{E''} B''}$, so the system of the $\varphi_{A' \oplus_{E'} B'}$ is also directed and we can consider its limit $\varphi_{A \oplus_E B}$, which is the map we want. \square

The difference with the previous free amalgam notion is that it was defined for the elements of K only, which corresponds to the countable substructures of \mathbb{M} , whereas this notion is defined for small substructures. Also since our class K is axiomatisable every countable substructure of \mathbb{M} is inside of K so there is no problem in considering the structures $A' \oplus_{E'} B'$ for any countable substructures E', A', B' of \mathbb{M} .

Proposition 2.11. *The relation $A \perp_E^\Gamma B$ defined for structures $E \subseteq A, B$ is invariant, symmetric, satisfies existence, extension, monotonicity, finite character, stationarity, base monotonicity and transitivity.*

Proof. By existence here we mean that $A \perp_E^\Gamma E$ for every extension $E \subseteq A$. The properties of symmetry, existence and the fact that it implies \perp^a are immediate. Stationarity is a consequence

of quantifier elimination: Up to isomorphism over B there is a unique structure $D = A \oplus_E B$ that satisfies $A \downarrow_E^\Gamma B$, which means a unique quantifier free type over B .

For monotonicity assume that $A \downarrow_E^\Gamma B$ and consider $E \subseteq C \subseteq B$ a substructure (this is enough by symmetry). We consider the free amalgam D of A and C and the associated embeddings i_A, i_C . Now let us consider the free amalgam D' of D and B over C and its associated embeddings j_D and j_B . There is a unique morphism $\varphi_D : D \rightarrow \langle AB \rangle$ such that $\varphi_D \circ i_A = id_A$ and $\varphi_D \circ i_C = id_C$ and $\varphi_{D'} \circ j_B = id_B$. Since $\langle AB \rangle = A \oplus_E B$ there is a unique morphism $\psi_{D'} : D' \rightarrow \langle AB \rangle \rightarrow D'$ such that $\psi_{D'}$ restricted to A and B respectively is $j_D \circ i_A$ and j_B respectively. From this we get that $\varphi_{D'} \circ \psi_{D'} = id_{D'}$ and $\psi_{D'} \circ \varphi_{D'} = id_D$, so $\varphi_{D'}$ is an isomorphism.

There is a unique morphism $\varphi_{D'} : D' \rightarrow \langle AB \rangle$ such that $\varphi_{D'} \circ j_D = \varphi_D$. So $\varphi_{D'} \circ j_D \circ i_A = id_A$, and we get that D' is generated by the images of A and B and that $\varphi_{D'}$ is an isomorphism. We know that D is generated by the images of A and C , and that it is sent to $\langle AC \rangle \subseteq \langle AB \rangle$ by j_D , so $\varphi_{D'} \circ j_D : D \rightarrow \langle AC \rangle$ is an isomorphism by a similar argument, so $A \downarrow_E^\Gamma C$.

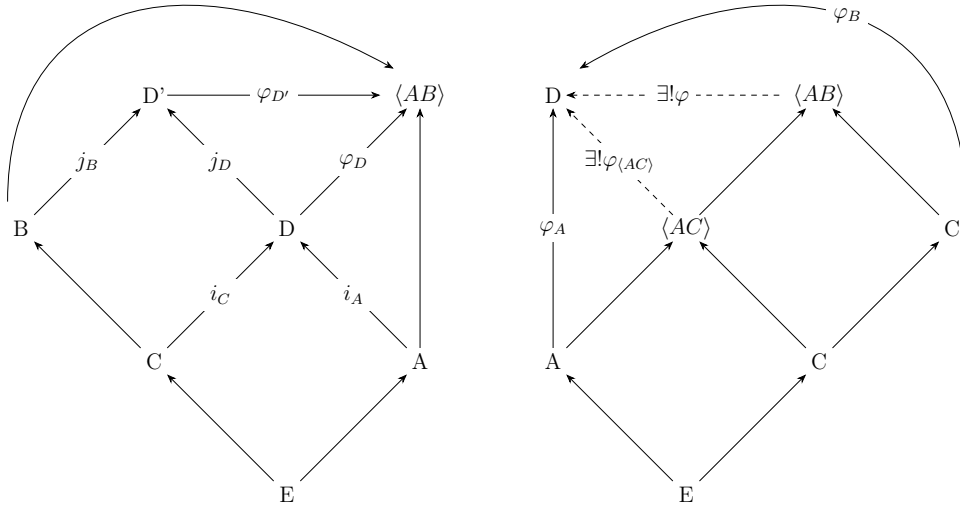


Figure 2.5: Monotonicity (on the left) and transitivity (on the right) of \downarrow^Γ

For transitivity, given structures $E \subseteq A, C, C \subseteq B$ such that $A \downarrow_E^\Gamma C$ and $\langle AC \rangle \downarrow_C^\Gamma B$ we want to show that $A \downarrow_E^\Gamma B$.

Consider a structure D and two embeddings φ_A, φ_B that coincide in E from A and B respectively to D . Then the embeddings φ_A and $\varphi_B|_C$ coincide over E , and by assumption there is a unique map $\varphi_{\langle AC \rangle} : \langle AC \rangle \rightarrow D$ such that $\varphi_B|_C = \varphi_{\langle AC \rangle}|_C$ and $\varphi_A = \varphi_{\langle AC \rangle}|_A$.

Now we consider the two maps $\varphi_{\langle AC \rangle}$ and φ_B which coincide over C . Since $\langle AC \rangle \downarrow_C^\Gamma B$ there is a unique map $\varphi : \langle \langle AC \rangle B \rangle = \langle AB \rangle \rightarrow D$ such that $\varphi|_B = \varphi_B$ and $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$, so in particular $\varphi|_A = \varphi_A$. This proof consist just in saying that the square formed by two adjacent pushouts forms a pushout.

For extension and stationarity, given structures $E \subseteq A, C, C \subseteq B$ such that $A \downarrow_E^\Gamma C$ we can form the free amalgam of $\langle AB \rangle$ and C over B inside of \mathbb{M} thanks to Lemma 2.10. This allows us to find some A' such that $\langle A'C \rangle \cong \langle AC \rangle$ (since the free amalgam defines a unique isomorphism type), so by Q. E. $A' \equiv_B A$ and $\langle A'B \rangle \downarrow_B^\Gamma C$, so $A' \downarrow_E^\Gamma C$ by transitivity, and this gives us a unique type over C for the same reason.

Finite character is also a direct consequence of the universal property: assume that $A \not\downarrow_E^K B$, so there is a structure D and two embeddings φ_A, φ_B from A and B respectively to D that coincide

in E such that there is no map $\varphi : \langle AB \rangle \rightarrow D$ satisfying $\varphi_A = \varphi|_A$ and $\varphi_B = \varphi|_B$. This is equivalent to saying that the function that to an element $t(\bar{a}, \bar{b}) \in \langle AB \rangle$ associates $t(\varphi_A(\bar{a}), \varphi_B(\bar{b}))$ is not well defined (since this function is the only candidate) or that it is not a morphism. In either of those cases we can find some finite tuples $\bar{a} \in A, \bar{b} \in B$ such that the same thing is happening with the function that to an element $t(\bar{a}', \bar{b}') \in \langle E\bar{a}\bar{b} \rangle$ associates $t(\varphi_A(\bar{a}'), \varphi_B(\bar{b}'))$, so $\bar{a} \not\downarrow_E^K \bar{b}$.

For base monotonicity, given structures $E \subseteq A, C$ and $C \subseteq B$ such that $A \downarrow_E^\Gamma B$ we want to show that $\langle AC \rangle \downarrow_C^\Gamma B$. We take the same notations as for transitivity. As previously $A \downarrow_E^\Gamma C$ (by monotonicity this time). Consider a structure D and two embeddings $\varphi_{\langle AC \rangle}, \varphi_B$ from $\langle AC \rangle$ and B respectively to D that coincide on C . Then the embeddings $\varphi_{\langle AC \rangle}|_A$ and φ_B coincide over E . By assumption there is a unique map $\varphi : \langle AB \rangle = \langle \langle AC \rangle B \rangle \rightarrow D$ such that $\varphi|_A = \varphi_{\langle AC \rangle}|_A$ and $\varphi|_B = \varphi_B$. Then we also have that $\varphi|_C = \varphi_B|_C = \varphi_{\langle AC \rangle}|_C$, so $\varphi|_{\langle AC \rangle} = \varphi_{\langle AC \rangle}$ since $A \downarrow_E^\Gamma C$. This proof consists in noticing that if the square formed by two adjacent squares is a pushout and the left one also is then the right one is also a pushout. \square

Corollary 2.12. *For every $E \subseteq A, B$ if $A \downarrow_E^\Gamma B$ then $A \downarrow_E^f B$, so T has existence. Also T is NSOP₁ and for every substructures $E \subseteq A, B$ we have $A \downarrow_E^K B$ if and only if $A \downarrow_E^a B$.*

Proof. We show that $A \downarrow_E^\Gamma B$ implies $A \downarrow_E^d B$. Assume that $A \downarrow_E^\Gamma B$ and let $(B_i)_{i < \omega}$ be an E -indiscernible sequence with $B_0 = B$. By extension we can find $A' \equiv_{B_0} A$ such that $A' \downarrow_{B_0}^\Gamma B_{<\omega}$. By transitivity and invariance $A' \downarrow_E^\Gamma B_{<\omega}$. By indiscernibility $B_i \equiv_E B_0$ for every $i < \omega$. Since $B_0 \downarrow_E^\Gamma A'$ and $B_i \downarrow_E^\Gamma A'$ it follows from stationarity that $A'B_i \equiv_E A'B_0 \equiv_E AB_0$ for every $i < \omega$. So $A \downarrow_E^d B$. Since \downarrow^Γ satisfies extension we get that if $A \downarrow_E^\Gamma B$ then $A \downarrow_E^f B$, and existence for \downarrow^f follows from existence for \downarrow^Γ .

Now that we know that T has existence we can consider Kim-independence over arbitrary sets. We use the Kim-Pillay theorem for Kim-forking Theorem 1.63. By [KR18, Lemma 2.7] \downarrow^a satisfies extension, existence, monotonicity, symmetry, strong finite character, and witnessing. By Lemma 2.8 it satisfies the amalgamation theorem, and it is clear that it satisfies transitivity.

We now show that it satisfies the local character property (7 in Theorem 1.63). Let \bar{a} be a finite tuple, κ an uncountable regular cardinal, $(A_i)_{i < \kappa}$ a continuous increasing sequence of structures of cardinality smaller than κ and $A_\kappa = \bigcup_{i < \kappa} A_i$. We want to show that there is an $i < \kappa$ such that $\langle A_i \bar{a} \rangle \downarrow_{A_i}^a A_\kappa$. We choose some $i_0 < \kappa$. If $\langle A_{i_0} \bar{a} \rangle \not\downarrow_{A_{i_0}}^a A_\kappa$ the set $\langle A_{i_0} \bar{a} \rangle \cap A_\kappa$ is non empty and has cardinal smaller than κ , so it is contained in A_{i_1} for some $i_1 > i_0$. We iterate this process to define a sequence $(i_j)_{j < \omega}$ and set $i = \bigcup_{j < \omega} i_j$ which is smaller than κ by regularity. Now if $a \in \langle A_i \bar{a} \rangle \cap A_\kappa$ it is in $\langle A_{i_j} \bar{a} \rangle \cap A_\kappa$ for some $j < \omega$ so by construction it is inside of $A_{i_{j+1}} \subseteq A_i$. \square

Remark 2.13. In the case where the limit T of the class K is a stable theory $\downarrow^K = \downarrow^f = \downarrow^a$ and $\downarrow^a = \downarrow^\Gamma$ since both of these relations are stationary and satisfy existence. Also the Kim-Pillay criterion for simple theories Theorem 1.63 applied to the stationary relation \downarrow^Γ tells us that \downarrow^Γ satisfies local character if and only if T is stable.

Remark 2.14. Notice that in order to get Corollary 2.12 we only need 1 – 5 and Lemma 2.8 to hold, and not the full generality of Algebraically Independent 3-Amalgamation. In our setting when applying the amalgamation theorem to E, A_0, A_1, B_0, B_1 by Lemma 2.8 and Algebraically Independent 3-Amalgamation we have $AB_i \downarrow_{B_i}^K B_0 B_1$ for $i = 0, 1$ and $AB_0 \downarrow_A^K AB_1$. We give a proof later in Proposition 4.12 that this holds in any NSOP₁ theory with existence.

Remark 2.15. The proof of Corollary 2.12 can be generalized to show that if T' is a theory that admits a stationary independence relation \downarrow satisfying the list of properties of Proposition 2.11 and such that \downarrow^a satisfies the independence theorem over arbitrary sets, then \downarrow implies \downarrow^f , and so T' is NSOP₁, has existence and $\downarrow^a = \downarrow^K$ over arbitrary sets.

The following proposition is stated in [KR18] for the model companion of the empty theory in a language L but it works more generally for any theory in which \perp^a satisfies the independence theorem. This fact is part of some more general results about deducing weak elimination of imaginaries from the properties of some independence relations (see [CK19, Section 4.4] for example).

Proposition 2.16. [KR18, Proposition 3.22] *If in a monster model of a complete theory T' the relation \perp^a satisfies the independence theorem then T' has weak elimination of imaginaries.*

Corollary 2.17. *T has weak elimination of imaginaries.*

Proposition 2.18. *For any $E \subseteq A, B$ we have $A \perp_E^f B$ iff $A \perp_E^d B$ iff $A \perp_E^M B$, where $A \perp_E^M B$ if and only if $\langle AC \rangle \perp_C^a B$ for every substructure $E \subseteq C \subseteq B$, as defined in Definition 1.20.*

Proof. We begin by showing that in \mathbb{M} the relation \perp^M satisfies extension. Consider $E \subseteq A, B \subseteq \mathbb{M}$ such that $A \perp_E^M B$ and $B \subseteq B'$. We can find A' such that $A' \equiv_B A$ and $\langle A'B \rangle \perp_B^{\Gamma} B'$, let us write $D' = \langle A'B' \rangle$. We show that $A' \perp_E^M B'$. Consider $C' \subseteq B'$ and define $C = C' \cap B$, we want to show that $\langle A'C' \rangle \cap B' = C'$. By Condition 5, since $\langle A'C \rangle \cap C' = C$, $\langle A'C \rangle \cap B = C$ (because $A \perp_E^M B$) and $B \cap C' = C$, the following diagram with the solid arrows, where $\langle A'C' \rangle$ is the structure generated by A' and C' in D' , can be completed with the dashed arrows and a structure $D_{C'}$ such that $\varphi_{B'}(B') \cap \varphi_{\langle A'C' \rangle}(\langle A'C' \rangle) = \varphi_{B'}(C') = \varphi_{\langle A'C' \rangle}(C')$.

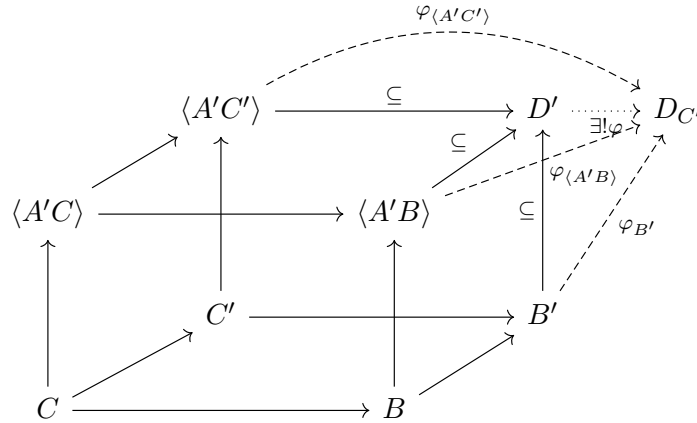


Figure 2.6: Characterizing \perp^f

There is a unique map φ such that $\varphi|_{\langle A'B \rangle} = \varphi_{\langle A'B \rangle}$ and $\varphi|_{B'} = \varphi_{B'}$. The two maps $\varphi_{\langle A'C' \rangle}$ and $\varphi|_{\langle A'C' \rangle}$ coincide on the sets C' and $\langle A'C \rangle$, and these sets generate $\langle A'C' \rangle$, so $\varphi|_{\langle A'C' \rangle} = \varphi_{\langle A'C' \rangle}$. If $t(a', c') = t'(b')$ for two terms $t, t', a' \in A', c' \in C'$ and $b' \in B'$ inside of D' then by applying φ we get that $\varphi_{\langle A'C' \rangle}(t(a', c')) = \varphi_{B'}(t'(b'))$, so by Condition 5 there is a $c \in C'$ such that $\varphi_{\langle A'C' \rangle}(c) = \varphi_{\langle A'C' \rangle}(t(a', c'))$ and $\varphi_{B'}(c) = \varphi_{B'}(t'(b'))$. Since the maps $\varphi_{B'}$ and $\varphi_{\langle A'C' \rangle}$ are embeddings we get that $t(a', c') = t'(b') \in B'$, so $\langle A'C' \rangle \cup B' = C'$. So \perp^M satisfies extension.

It is always true that $\perp^d \implies \perp^M$. To show that $\perp^M \implies \perp^f$ it is sufficient to show that $\perp^M \implies \perp^d$ since \perp^M satisfies extension. For this assume that $A \perp_E^M B_0$ and consider an indiscernible sequence $I = (B_i)_{i < \omega}$. Let $p(x, B_0) := \text{tp}(A/B_0)$, we want to show that $\bigcup_{i < \omega} p(x, B_i)$ is consistent.

We begin defining some set $E' \supseteq E$ and an E' indiscernible sequence $(B'_i)_{i < \omega}$ such that $B_i \subseteq B'_i$ for all $i < \omega$, $B'_i \perp_{E'}^a B'_{<i}$ for every $i < \omega$, and some set $A' \supseteq A$. Let $q(x', B'_0) := \text{tp}(A'/B'_0)$ then it is sufficient to show that $\bigcup_{i < \omega} q(x', B'_i)$ is consistent.

Begin by adding a copy of ω at the beginning of the sequence $I := (B_i)_{i < \omega}$, i.e. we find $\tilde{I} = (\tilde{B}_i)_{i < \omega}$ such that $\tilde{I} \frown I$ is E -indiscernible. By extension we can assume that $A \downarrow_E^M \langle \tilde{I} \cup I \rangle$. Now set $E' = \langle \tilde{I} \rangle$ to be the structure generated by \tilde{I} . Consider the structures $B'_i := \langle E' B_i \rangle$ for $i < \omega$. They form an E' -indiscernible sequence. Define $A' := \langle E' A \rangle$, by base monotonicity $A' \downarrow_{E'}^M \langle (B'_i)_{i < \omega} \rangle$. Now, if $b_i \in B'_i$ is also in $\langle B'_{<i} \rangle$ then it has to be in E' by indiscernibility of $(\tilde{B}_i)_{i < \omega} \frown (B_i)_{i < \omega}$ (this is the point of adding $(\tilde{B}_i)_{i < \omega}$: algebraic relations happening in the sequence are already happening inside of this part of the sequence). So we got the sequence and the sets we want.

Since $B'_0 \equiv_{E'} B'_i$ there is some \tilde{A}_i such that $A' B'_0 \equiv_{E'} \tilde{A}_i B'_i$. We define by induction on $i < \omega$ a sequence of structures A'_i such that $A'_i \downarrow_{E'}^a B'_{\leq i}$ and for all $j \leq i$ $A' B'_0 \equiv_{E'} A'_i B'_j$. Let $A'_0 = \tilde{A}_0 = A'$. If A'_i has been defined then $A'_i \downarrow_{E'}^a B'_{\leq i}$, $\tilde{A}_{i+1} \downarrow_{E'}^a B'_{i+1}$, $B'_{i+1} \downarrow_{E'}^a B'_{\leq i}$ and $\tilde{A}_i \equiv_{E'} A'_i$, so by Lemma 2.8 we can find a suitable A'_{i+1} . By compactness there is some A'' such that $A'' B'_i \equiv_{E'} A' B'_0$ for all $i < \omega$ and we get the consistency that we want.

To conclude if $A \downarrow_E^M B$ then $A \downarrow_E^d B$, by extension $A \downarrow_E^f B$ also holds, so these three notions are equivalent. \square

Remark 2.19. We get from Proposition 2.18 that in T thorn-forking coincides with forking, T is rosy (which is defined as thorn-forking satisfying local character - see [Ons03] about rosy theories) if and only if T is simple.

In the previous proposition we insist on the fact that we are considering two algebraically closed sets A, B containing the basis E ; the reason for this is that in NSOP₁ theories it is not always true that $\downarrow^f = \downarrow^d$ over arbitrary sets. A counterexample is given in [CK23, Section 3.2]: if T_f^\emptyset is the model completion of the empty theory in a language containing only a binary function and T_f^{eq} its imaginary expansion, then in this paper a real tuple a , small model M and some unordered pair d such that $a \downarrow_M^d Md$ and $a \not\downarrow_M^d acl^{eq}(Md)$ are given.

2.2 The parameterized class

In this section we consider a Fraïssé class K_0 in a language L_0 and a second Fraïssé class K_P in a language L_P which is one sorted. We show that we can ‘parameterize’ the class K_0 by K_P to get a new class K of structures in a new language that satisfies (H) whenever K_0 and K_P also does, and that in some of these cases the limit theory of K is not simple, and so is strictly NSOP₁ by Corollary 2.12.

2.2.1 The definition of the parameterized theory

Definition 2.20. Let L be a new language built from L_0 by adding a new sort P equipped with an L_P -structure. We will refer to P as the parameter sort, we will refer to the sorts of L_0 as the object sorts, denoted O . We replace every symbol of constant c in L_0 of sort O by a function symbol $\bar{c} : P \rightarrow O$, every function symbol $f : \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ by a function symbol $\bar{f} : P \times \prod_{i < n} O_i \rightarrow \prod_{i < m} O'_i$ and every relation symbol $R \subseteq \prod_{i < n} O_i$ by a relation symbol $\bar{R} \subseteq P \times \prod_{i < n} O_i$.

Let us fix some notations: For a symbol of function $f \in L$ we will sometimes write $f_p(\bar{x})$ instead of $\bar{f}(p, \bar{x})$, and similarly for relation symbols. For an L_0 -formula φ and a parameter p we will write φ_p for the L -formula parameterized by p . For an L -structure A and $p \in P(A)$ we write A_p to denote the L_0 -structure induced by p on $O(A)$. For a set of objects A and a tuple of parameters \bar{p} we will write $\langle A \rangle_{\bar{p}}$ for the structure generated by (A, \bar{p}) , we will also write $\langle A \rangle$ for the structure generated by a set A (possibly containing parameters and objects), so $\langle A \rangle = \langle O(A) \rangle_{\langle P(A) \rangle}$.

We begin by citing some results of Baudisch on the parametrization of a theory showing that this construction works in a more general context than the one we are considering here. These results are proven essentially by decomposing quantifier-free L_P -formulas into a conjunction of L -formulas evaluated in different parameters.

Theorem 2.21. [Bau02, Theorem 2.9]: *Let T_0 be a model complete theory which eliminates the quantifier \exists^∞ . Let L_P be the empty language and L be the parametrized language defined above. Let T' be the L theory stating that for every parameter $p \in P$ the L_0 -structure induced by p is a model of T_0 . Then the theory T' has a model companion T .*

Corollary 2.22. [Bau02, Corollary 2.10] and [Bau02, Theorem 3.1] *If T_0 is model complete and eliminates \exists^∞ then T is complete and eliminates \exists^∞ .*

2.2.2 Basic properties

After this small parenthesis we now return to the context of classes of L -structures and the set of assumptions (H).

Definition 2.23. Let K be the class of countable L -structures A such that $A_p \in K_0$ for every $p \in P(A)$ and $P(A) \in K_P$.

Remark 2.24. We will most often consider the case where L_P is the empty language and K_P is the class of countable sets. It might be easier to read the proofs with the case of L_P being the empty language in mind.

Lemma 2.25. *If K_0 and K_P satisfies Condition 1 then K also does.*

Proof. If an L -structure A is not in K then either there is some parameter $p \in P(A)$ such that $A_p \notin K_0$ or $P(A) \notin K_P$. In the last case by 1 in K_0 there is an L_0 -formula φ such that $A_p \models \varphi$ and such that no structures in K_0 satisfies φ . So $A \models \varphi_p$ and no structure in K satisfies this formula. In the second case $P(A) \notin K_P$ is expressed by an L_P formula by Condition 1 in K_P . \square

Lemma 2.26. *If K_0 and K_P satisfies Conditions 1, 3 and 4 then K also does.*

Proof. Condition 1 follows from Lemma 2.25. This condition is required here because we need K_0 to be closed under unions of countable chains. For Condition 3 consider some structures $E \subseteq A, B \in K$. We construct an L -structure in K satisfying the Free Amalgamation property. Let us write $E = (O(E), P(E))$, $A = (O_A, P_A)$ and $B = (O_B, P_B)$. We begin by forming the free amalgam $P(D) := P(A) \oplus_{P(E)} P(B) \in K_P$ which will be the parameter sort of the free amalgam in K .

Let $(p_i)_{i < \omega}$ enumerate $P(D)$. We define a sequence $((D_{p_i}^k)_{i < \omega})_{k < \omega}$ of tuples of countable L_0 -structures indexed by $P(D)$ by induction such that the sequence $(D_p^k)_{k < \omega}$ is a chain of L_0 -structures for any $p \in P(D)$ and that for any $k < \omega$ the sequence $(D_{p_i}^k)_{i < \omega}$ is an increasing sequence of sets. Let $D_{p_{-1}}^0 := A \dot{\cup} (B \setminus A)$ and let E_0 be the structure generated by the constants inside of K_0 .

Using Conditions 3 and 4 in K_0 we define:

$$\begin{aligned} D_{p_i}^0 &:= (A_{p_i} \oplus_{E_{p_i}} B_{p_i}) \oplus_{\langle \emptyset \rangle} \langle \{D_{p_{i-1}}^0 \setminus (D_{-1}^0)\} \rangle \text{ if } p_i \in P(E), \\ D_{p_i}^0 &:= A_{p_i} \oplus_{\langle \emptyset \rangle} \langle \{D_{p_{i-1}}^0 \setminus A\} \rangle \text{ if } p_i \in P(A) \setminus P(E), \\ D_{p_i}^0 &:= B_{p_i} \oplus_{\langle \emptyset \rangle} \langle \{D_{p_{i-1}}^0 \setminus B\} \rangle \text{ if } p_i \in P(B) \setminus P(E), \\ D_{p_i}^0 &:= E_0 \oplus_{\langle \emptyset \rangle} \langle \{D_{p_{i-1}}^0\} \rangle \text{ if } p_i \in P(D) \setminus (P(A) \cup P(B)). \end{aligned}$$

Set $D^0 = \bigcup_{i < \omega} D_{p_i}^0$. If D^k is defined for some $1 \leq k < \omega$ we set:

$$D_{p_0}^{k+1} = D_{p_0}^k \oplus_{\langle \emptyset \rangle} \{D^k \setminus D_{p_0}^k\}$$

Similarly if $D_{p_i}^k$ is defined for some $1 \leq i < \omega$ we set:

$$D_{p_{i+1}}^{k+1} = D_{p_{i+1}}^k \oplus_{\langle \emptyset \rangle} \langle \{D_i^{k+1} \setminus D_{p_i}^k\} \rangle.$$

Every D_p^k is an L_0 -structure for the parameter p . We set $O(D) = \bigcup_{k < \omega} D^k$, this set is countable and has an L_0 -structure for the parameter $p \in P(D)$ given by $D_p = \bigcup_{k < \omega} D_p^k \in K_0$, so $D \in K$.

This defines an L -structure $D = (O(D), P(D)) \in K$ which extends A and B , and such that $A \cap B = E$ inside of D . Also if A and B are generated by $(\bar{a}, P(A))$ and $(\bar{b}, P(B))$ respectively $(\overline{ab}, P(A), P(B))$ generates D .

Now that we constructed the structure D we show that it satisfies the amalgamation property. Consider $D' \in K$, $\varphi_A : A \rightarrow D'$ and $\varphi_B : B \rightarrow D'$ as in the statement, and write φ_E their restriction to E . The uniqueness of φ is clear from the fact that D is generated by the images of A and B . For existence we define φ on the parameters: By free amalgamation in K_P there is a unique morphism φ_P of L_P -structures from $P(D)$ to $P(D')$ such that the diagram commutes. We then construct an increasing sequence of tuples of morphism of L_0 -structures indexed on $P(D)$:

$$((\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_P(p_i)})_{i < \omega})_{k < \omega}.$$

For $k = 0$, if $p_i \in P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p)}$ and $\varphi_B(p, -) : B_p \rightarrow D'_{\varphi_P(p)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$.

If $p_i \in P(A) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_A(p, -) : A_p \rightarrow D'_{\varphi_P(p)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus A$ to $\varphi_{p_{i-1}}^0(d)$.

Similarly if $p_i \in P(B) \setminus P(E)$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that extends $\varphi_B(p, -) : B_{p_i} \rightarrow D'_{\varphi_P(p_i)}$ and sends any element $d \in D_{p_{i-1}}^0 \setminus B$ to $\varphi_{p_{i-1}}^0(d)$, and finally if $p_i \in P(D) \setminus (P(A) \cup P(B))$ we take the unique morphism $\varphi_{p_i}^0 : D_{p_i}^0 \rightarrow D'_{\varphi_P(p_i)}$ that sends any element $d \in D_{p_{i-1}}^0$ to $\varphi_{p_{i-1}}^0(d)$.

Assume that for every $i < \omega$ the functions $\varphi_{p_i}^k : D_{p_i}^k \rightarrow D'_{\varphi_P(p_i)}$ have been defined. Define $\varphi_{p_0}^{k+1} : D_{p_0}^{k+1} \rightarrow D'_{\varphi_P(p_0)}$ as the unique morphism extending $\varphi_{p_0}^k : D_{p_0}^k \rightarrow D'_{\varphi_P(p_0)}$ by sending $x \in D_{p_i}^k$ to $\varphi_{p_i}^k(x)$ for every $i < \omega$. We just repeat the same steps for the other parameters. Once all these functions are constructed we consider $\varphi = \bigcup_{k < \omega} \varphi_p^k$ for every $p \in P(D)$. This function defines a

map in K from D to D' such that its restrictions to A and B are φ_A and φ_B respectively, and it is a morphism of L -structures since $\varphi = \bigcup_{k < \omega} \varphi_p^k : D_p \rightarrow D'_{\varphi_P(p)}$ is a morphism of L_0 -structures for every parameter $p \in P(D)$ and $\varphi_P : P(D) \rightarrow P(D')$ is a morphism of L_P -structures.

Checking Condition 4 for some object sort in K_P is immediate, since the structure formed by a single element in this sort will suffice. For the case of the parameter sort we consider the structure $\langle \{p\} \rangle \in K_P$ obtained by applying Condition 4 in K_P . To the elements $q \in \langle \{p\} \rangle$ we associate disjoint copies of the L_0 -structure generated by the constants and we amalgamate these copies freely, as in the previous proof, to get the L -structure $\langle \{p\} \rangle \in K$. \square

Remark 2.27. In the proof of Lemma 2.26 we have shown that K seen as a category has pushouts for pairs of monomorphisms. This proof also implies that D does not depend (up to isomorphism)

on the choice of the enumeration of the parameters: given the same construction for a different enumeration $D^* = \bigcup_{i < \omega} D_{p_i}^{*,k}$, by the same proof D^* also has the universal property so D and D^* are isomorphic over A and B .

Remark 2.28. A substructure of a finitely generated structure is not necessarily finitely generated. If we consider the examples of parameterized infinite dimensional vector spaces over a finite field with L_P being the empty language, the structure $E = \langle v, p_1 p_2 \rangle$ freely generated by two parameters and one object is infinite however the substructure $(O(E), p_1)$ where we removed one parameter is not finitely generated.

Lemma 2.29. *If K_0 and K_P satisfies Conditions 1, 2, 3 and 4 then K also does. For every quantifier-free formula $\varphi(\bar{x}, \bar{a})$ satisfied in some structure of K there is a quantifier-free formula $\psi(\bar{x})$ such that if $\bar{a} \in A \in K$ and $A \models \psi(\bar{a})$ then there is $A \subseteq B \in K$ an extension and $\bar{b} \in B$ such that $B \models \varphi(\bar{a}, \bar{b})$.*

Proof. Conditions 1, 3 and 4 follow from Lemma 2.26, we need them here because we want to amalgamate structures in K_0 and as previously we need K_0 to be closed under countable unions. For Condition 2 consider a quantifier free formula $\varphi(\bar{x}\bar{v}, \bar{y}\bar{w})$ satisfied by some $\bar{a}\bar{p}\bar{b}\bar{q} \in A \in K$ with \bar{x}, \bar{y} in the object sorts and \bar{v}, \bar{w} in the parameters.

We can assume that $\varphi(\bar{x}\bar{v}, \bar{y}\bar{w})$ is of the form:

$$\bigwedge_{i,j} E_i(f_i, g_i) \wedge \bigwedge_i \bar{R}_i(v_i, n_i, m_i) \wedge \bigwedge_i \bar{R}'_i(w_i, n'_i, m'_i) \wedge \bigwedge_i \bar{R}''_i(w'_i, n''_i, m''_i),$$

where R_i, R'_i are relations symbol or negated relations symbols from L_0 , n_i, n'_i, m_i, m'_i are terms in $\bar{x}\bar{v}, \bar{y}\bar{w}$, E_i are relations symbols or negated relation symbols form L_P and w'_i, f_i, g_i are terms in \bar{v}, \bar{w} .

We consider a strengthening of φ satisfied by $\bar{a}\bar{p}\bar{b}\bar{q}$. Let $(h_i)_{i < m}$ enumerate the interpretation of all the sub-terms of the variable terms appearing in φ when we fix $\bar{v}\bar{w}$ to be $\bar{p}\bar{q}$. Let $(c_i)_{i < n}$ enumerate the interpretation of all the sub-terms of the object terms appearing in φ when we fix $\bar{x}\bar{y}\bar{v}\bar{w}$ to be $\bar{a}\bar{p}\bar{b}\bar{q}$. Let $\bar{k} = (k_i)_{i < m}$ and $\bar{z} = (z_i)_{i < n}$ be tuples of variable of the same length. Enlarging $\bar{a}\bar{p}$ we assume that it contains every term c_i and h_i such that $c_i \in \langle \bar{a}\bar{p} \rangle$ and $h_i \in \langle \bar{p} \rangle$, let $\bar{x}\bar{v}$ be the corresponding tuples of variables.

We replace the tuple $\bar{b}\bar{q}$ by $\bar{b}\bar{c}\bar{q}\bar{h}$ and write $\varphi = \varphi(\bar{x}\bar{v}, \bar{y}\bar{z}\bar{w}\bar{k})$. We add to φ the formulas that expresses the fact that k_i is a sub-term, meaning that if $h_i = t_i(\bar{p}, \bar{q}, h_{<i})$ for t_i an L_P -term we add $k_i = t_i(\bar{x}, \bar{y}, k_{<i})$ to φ and replace the occurrences of $t_i(\bar{x}, \bar{y}, k_{<i})$ by occurrences of k_i .

Similarly we add to φ the formulas that expresses the fact that z_i is a sub-term, meaning that if $c_i = \bar{t}_i(\rho, \bar{a}, \bar{b}, c_{<i})$ for \bar{t}_i an L_0 -term and $\rho \in \bar{p}\bar{b}\bar{h}$ we add $z_i = \bar{t}_i(\nu, \bar{x}, \bar{y}, z_{<i})$ to φ where $\nu \in \bar{v}\bar{w}\bar{k}$ is the variable corresponding to ρ and replace the occurrences of $\bar{t}_i(\nu, \bar{x}, \bar{y}, z_{<i})$ by occurrences of z_i . After this the object terms that appears in φ only use one parameter.

For every parameter variable $\nu \in \bar{v}\bar{w}\bar{k}$ consider the L_0 -formula $\varphi_\nu(\bar{x}, \bar{y}\bar{z})$ that specifies every equality $z_j = \bar{t}_j(\nu, \bar{x}, \bar{y}, z_{<j})$ (we consider the previous equalities and only mention those where the parameter ν is used), the relations $R_{j,\nu}$ that hold between the elements of $\bar{a}\bar{b}\bar{c}$ (so here we look at the L_0 -structure induced by ν), and we include the instances of equality and inequality among these relations.

We consider the formula φ_P that specifies every equality $k_i = t_i(\bar{x}, \bar{y}, k_{<i})$ (we consider the previous equalities and only mention those in the parameter sort), the relations E_j that holds between the elements of $\bar{p}\bar{h}$, and we include the instances of equality and inequality among these relations. We can write the formula φ as:

$$\varphi(\overline{xv}, \overline{yzw\bar{k}}) = \varphi_P(\overline{v}, \overline{w\bar{k}}) \wedge \bigwedge_{\nu \in \overline{v\bar{w}\bar{k}}} \varphi_\nu(\overline{x}, \overline{yz}).$$

We can consider ψ_P which is the L_P -formula associated to φ_P by 2 in K_P and ψ_ν the L_0 -formula associated to φ_ν by Condition 2 in K_0 for every $\nu \in \overline{v\bar{w}\bar{k}}$. Let $\psi(\overline{x}, \overline{v}) := \psi_P(\overline{v}) \wedge \bigwedge_{\nu \in \overline{v\bar{w}\bar{k}}} \varphi_\nu(\overline{x})$. It satisfies the condition, and we conclude as in Lemma 2.29: If $\psi(\overline{x}, \overline{v})$ is satisfied by $\overline{a'p'} \in A' \in K$ we can extend $P(A)$ in K_P thanks to ψ_P , choose an extension for every parameter variable $\overline{v\bar{w}\bar{k}}$, and amalgamate these extension in an L -structure extending A as in Lemma 2.26. \square

2.2.3 Amalgamation and Combinatorial Properties

In this subsection we assume that K_0 and K_P both satisfy (H). By the previous section we can construct the limit theory T_0 of K_0 and \mathbb{M}_0 a monster model of it. T_0 has quantifier elimination and Kim-independence and algebraic independence coincide inside of it, similarly we can construct the limit theory T_P of K_P and a monster model \mathbb{M}_P .

By Remark 2.7 the algebraic closure and the generated structure coincide in \mathbb{M}_0 we can represent non-Kim-forking extensions as commutative squares as in the following figure:

$$\begin{array}{ccc} A & \xrightarrow{f} & D \\ \uparrow j & & \uparrow g \\ E & \xrightarrow{i} & B \end{array}$$

Figure 2.7: A commutative square

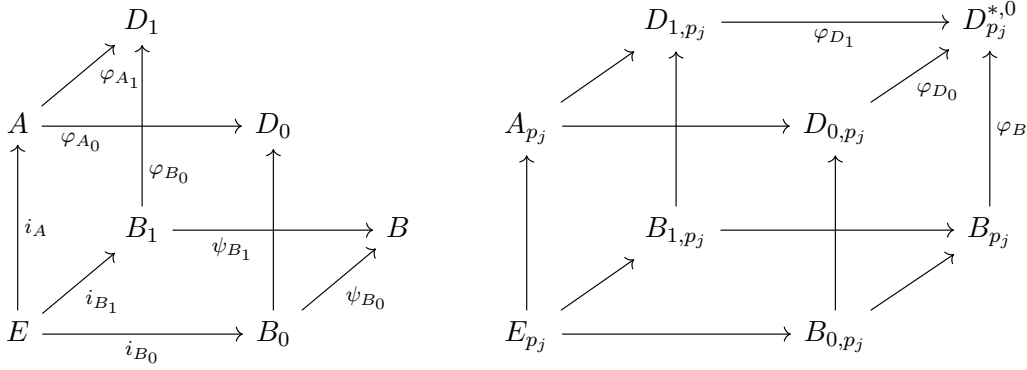
The arrows are embeddings, the structure D is generated by the images $f(A)$ and $g(B)$, and $f(A) \cap g(B) = f \circ j(E) = g \circ i(E)$. Then the isomorphism type of $f(A)$ over $g(B)$ gives us a type over $g(B)$ which corresponds to a non-Kim-forking extension of $\text{tp}(A/E)$. We will refer to such squares as strong amalgams. However for the (Algebraically Independent 3-Amalgamation) the squares we are considering consist in some ‘extensions’ of such squares, meaning that we compose with an embedding from D to some $D' \in K_0$.

Proposition 2.30. *K satisfies Algebraically Independent 3-Amalgamation.*

Proof. Consider the following diagram in K , which is as in the definition of Algebraically Independent 3-Amalgamation and where for simplicity we take the embeddings to be inclusions. We want to complete it.

Firstly by Condition 5 in K_P we can complete the parameter part of this diagram with an L_P -structure $P(D) \in K_P$ which is going to be the parameter sort of our structure D . After this we begin by choosing a completion for every parameter. Fix an enumeration $(p_i)_{i < \omega}$ of $P(D)$. Looking at the L_0 -structure indexed by any of the parameters $p_j \in P(E)$ we can find an L_0 -structure $D_{p_j}^{*,0}$ as in the right diagram. We begin by filling the cube for the L_0 -structures induced by each of the parameters.

These sets $D_{p_j}^{*,0}$ for $p_j \in P(E)$ are also equipped with the L -structure on the sets $\varphi_B(B)$ and $\varphi_{D_i}(D_i)$ for $i = 0, 1$ induced by the embeddings. We construct the L_0 -structure $D_{p_j}^{*,0}$ for

Figure 2.8: Proving Algebraically Independent 3-Amalgamation in T_P

$j \in P(D) \setminus P(E)$ and consider the natural mappings from D_0 , D_1 and B to $D_{p_j}^{*,0}$.

- If $p_j \in P(A) \setminus P(E)$ set $D_{p_j}^{*,0} = [D_{0,p_j} \oplus_{A_{p_j}} D_{1,p_j}] \oplus_{\langle \emptyset \rangle} \langle \{B \setminus (B_0 \cup B_1)\} \rangle$
- If $p_j \in P(B_i) \setminus P(E)$ set $D_{p_j}^{*,0} = [D_0 \oplus_{B_0} B] \oplus_{\langle \emptyset \rangle} \langle \{D_1 \setminus (A \cup B_1)\} \rangle$
- If $p_j \in P(D_0) \setminus (P(A) \cup P(B_0))$ set $D_{p_j}^{*,0} = D_{0,p_j} \oplus_{\langle \emptyset \rangle} \langle \{(B \setminus B_0) \cup (D_1 \setminus A)\} \rangle$
- If $p_j \in P(B) \setminus (P(B_0) \cup P(B_1))$ set $D_{p_j}^{*,0} = B_{p_j} \oplus_{\langle \emptyset \rangle} \langle \{(D_0 \setminus B_0) \cup (D_1 \setminus B_1)\} \rangle$
- If $p_j \in P(D) \setminus (P(D_0) \cup P(D_1) \cup P(B))$ set $D_{p_j}^{*,0} = E_0 \oplus_{\langle \emptyset \rangle} \langle \{(D_0 \cup D_1 \cup B)\} \rangle$

In this last union we assimilate elements that have the same predecessors in the diagram. Now that we have all of our L_0 -structures we will just put them together in a canonical way: we define an increasing sequence of tuples of L_0 -structures $((D_{p_j}^k)_{j < \omega})_{k < \omega}$.

Let $D_{p_0}^0 = D_{p_0}^{*,0}$, if $D_{p_j}^0$ has been defined for some $j < \omega$ set:

$$D_{p_{j+1}}^0 := D_{p_{j+1}}^{*,0} \oplus_{\langle \emptyset \rangle} \langle \{D_{p_j}^0 \setminus (D_0 \cup D_1 \cup B)\} \rangle,$$

where we assimilate D_0 , D_1 and B with their image in $D_{p_j}^0$. Once $D_{p_j}^0$ has been defined for every $j < \omega$ we set $D^0 := \bigcup_{i < \omega} D_{p_i}^0$. If D^k is defined we set:

$$D_{p_0}^{k+1} := D_{p_0}^k \oplus_{\langle \emptyset \rangle} \langle \{D^k \setminus D_{p_0}^k\} \rangle.$$

Now assume that $D_{p_j}^{k+1}$ is defined for some $k < \omega, j < \omega$, we set:

$$D_{p_{j+1}}^{k+1} := D_{p_{j+1}}^k \oplus_{\langle \emptyset \rangle} \langle \{D_{p_j}^{k+1} \setminus D_{p_{j+1}}^k\} \rangle.$$

Then take $D := \bigcup_{k < \omega} D_{p_j}^k$ with the L_0 -structure induced for every $j < \omega$. This defines an L -structure. The natural mappings from D_0 , D_1 and B to D are embeddings for the L -structure we put on D and they satisfy the conditions. \square

Finally we can state that the class K satisfies the conditions of (H) by Proposition 2.30, Lemma 2.25 and Lemma 2.29. We can define the limit theory T of K and take \mathbb{M} a monster model of T . By Corollary 2.12 we know that T is an NSOP₁ theory with existence such that $\downarrow^K = \downarrow^a$ over arbitrary sets.

Proposition 2.31. T implies the following sets of formulas for any $n < \omega$:

1. If \bar{a} is an m -tuple of objects, $\pi_i(\bar{x})$ is an m -type in T_0 for every $i \leq n$ such that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ and $q(\bar{v})$ in a type in T_P , then there exists an infinite number of tuples of parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$ and $\models q((p_i)_{i \leq n})$.
2. If $\bar{p} = p_0, \dots, p_n$ is a n -tuple of parameters and $\pi_i(\bar{x}, \bar{b}_{p_i})$ is a type in \mathcal{M}_{p_i} over a n -tuple of objects \bar{b} such that the induced type in any of the x_i is non-algebraic and that $\pi_i \vdash x_j \neq x_{j'}$ for $j \neq j'$ then there is a tuple of objects \bar{a} such that $\models \pi_i(\bar{a}_{p_i}, \bar{b}_{p_i})$ for every $i \leq n$.

Proof. Consider the limit \mathcal{M} of K built in Proposition 2.5, a finite tuple of objects $\bar{a} \in \mathcal{M}$ and a finite set of types $\pi_i(\bar{x})$ as in the first point of the statement. We can build a structure $A \in K$ containing \bar{a} and with parameters $(p_i)_{i \leq n}$ such that $\models \pi_i(\bar{a}_{p_i})$ for every $i \leq n$ and $\models q((p_i)_{i \leq n})$. Then by saturation we can embed this structure A inside of \mathcal{M} over \bar{a} and find the parameters.

The second point is similar: Consider the structure B generated by \bar{b}, \bar{p} and types $\pi_i(\bar{x}, \bar{b}_{p_i})$ as in the statement. We can extend B into $B' \in K$ that contains a tuple of objects \bar{a}' such that $\bar{a}'\bar{b}$ has the right type for the parameter p_i for every i and then embed B' into \mathcal{M} over B to find the right objects. \square

Definition 2.32. A formula $\varphi(x, y)$ has the *independence property*, I. P. for short, if there is $(a_i)_{i < \omega}$ and $(b_I)_{I \subseteq \omega}$ two sequences of tuples such that $\models \varphi(a_i, b_I)$ if and only if $i \in I$. A theory T has I. P. if some formula in it has I. P..

Corollary 2.33. *The theory T has I. P. if there are two distinct non-algebraic types in T_0 . T is non-simple if T_0 is non-trivial, in the sense that there are $a, b \in \mathbb{M}_0$ such that $\langle a, b \rangle \setminus (\langle a \rangle \cup \langle b \rangle) \neq \emptyset$.*

Proof. For I.P. consider two distinct T_0 -types π_i in \bar{x} for $i = 0, 1$ and a formula $\psi(\bar{x})$ such that $\pi_0(\bar{x}) \vdash \psi(\bar{x})$ and $\pi_1(\bar{x}) \vdash \neg\psi(\bar{x})$. If we consider a sequence $(p_i)_{i < \omega}$ of distinct parameters in \mathbb{M} , then by 2.31 for every $I \subseteq \omega$ we can find \bar{a}_I such that $\models \pi_{0, p_i}(\bar{a}_I)$ if $i \in I$ and $\models \pi_{1, p_i}(\bar{a}_I)$ if $i \notin I$. Then the formula $\psi_y(\bar{x})$ has I.P..

Now for simplicity let $t(x, y)$ be a L_0 -term such that $t(a, b) \notin \langle a \rangle \cup \langle b \rangle$ and consider a tree of parameters all distinct $(p_\mu)_{\mu \in \omega < \omega}$. Lets now consider a tree of distinct objects $(a_\mu)_{\mu \in \omega < \omega}$ and the formulas $\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) := t_{p_\mu}(x, y) = a_{\mu \smallfrown i}$ along this tree. Clearly $\{\varphi(x, y, a_{\mu \smallfrown i}, p_\mu) : i < \omega\}$ is 2-inconsistent for every $\mu \in \omega < \omega$, and by 2.31 every path is consistent. \square

Remark 2.34. Even when the relation \downarrow^Γ satisfies local character in \mathbb{M}_P and \mathbb{M}_0 it does not need to do so in \mathbb{M} (otherwise it would satisfy all conditions of the Kim-Pillay theorem): Take some distinct objects $\bar{o} = o_1, o_2, o_3$ and a set $P = (p_i)_{i < \kappa}$ of parameters for $\kappa = |T_P|^+$ such that $o_3 \in \langle o_1 o_2 \rangle_{p_i}$ for every $i < \kappa$ and $o_j \notin \langle P \rangle$ for every $j \in \{0, 1, 2\}$. Then there is no subset P_0 of cardinality $|T|$ of P such that $\bar{o} \downarrow_{P_0}^\Gamma P$.

Remark 2.35. In the case where we can find in K_0 some $c = t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ the relation \downarrow^a in \mathbb{M} does not admit 4-amalgamation: Take 3 distinct parameters p_0, p_1, p_2 and distinct objects b_0, b_1, b_2 such that $b_i, b_j \notin \langle p_k \rangle$ for every i, j, k such that $\{i, j, k\} = \{0, 1, 2\}$ and such that $t_{p_0}(b_1, b_2) = t_{p_1}(b_0, b_2)$, $t_{p_1}(b_0, b_2) = t_{p_2}(b_0, b_1)$ and $t_{p_0}(b_1, b_2) \neq t_{p_2}(b_1, b_2)$. Then the type $\text{tp}(p_0/b_1 b_2) \cup \text{tp}(p_1/b_0 b_2) \cup \text{tp}(p_2/b_0 b_1)$ is inconsistent.

2.3 Examples of generic extensions preserving (H)

Let K be a class of countable L -structures satisfying (H). We describe now how adding generic structure to the class K might affect the conditions (H) and some properties of the theory.

2.3.1 Adding a generic predicate

Definition 2.36. Consider a new symbol U for a unary predicate in some sort S , we define $L_U = L \cup \{U\}$ and the class K_U of L_U -structures consisting of the structures A in K equipped with an arbitrary subset $U(A) \subseteq S(A)$.

Proposition 2.37. *The class K_U satisfies (H).*

Proof. The Condition 1 is satisfied in K_U . For Condition 3 we take

$$U(A \oplus_E B) := U(A) \cup U(B),$$

where $A \oplus_E B$ is the free amalgam in K . Proving Condition 4 is straightforward, we take $\langle \{x\} \rangle \in K$ and set U to be empty. For Condition 5 we find an amalgam D using Condition 5 in K and set $U(D)$ to be $U(B_0) \cup U(B_1) \cup U(B)$.

For Condition 2 consider a finitely generated structure $\langle a, b \rangle \in K_U$ and a quantifier free L_U formula φ such that $\models \varphi(a, b)$. Without loss of generality we can assume that:

$$\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} U(t_i(x, y)) \wedge \bigwedge_{i < n} \neg U(t'_i(x, y)),$$

where $\varphi'(x, y)$ is an L -formula and the t_i and t'_i are L -terms. By taking a strengthening of φ we can assume that $\varphi'(x, y) \vdash \bigwedge_{j < n} t'_i(x, y) \neq t_j(x, y)$ for every $i < n$.

Now we consider the formula $\psi'(x)$ associated to the L -formula $\varphi'(x, y) \wedge \bigwedge_{i, j < n} (t'_i(x, y) \neq t_j(x, y))$

by Condition 2 in K . If $A' \models \psi(a')$ for some $a' \in A' \in K_R$ then by assumption we can find some $b' \in B'$ an extension of A' in K such that:

$$B' \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle t_j(a', b') \rangle_{j < n}).$$

Define the subset U on B' to be $U(A') \cup \{t_j(a', b')\}_{j < n}$, we get that $B' \models \neg U(t'_i(a', b'))$ for all $i < n$, so $B' \models \varphi(a', b')$, and B' is an extension of A' in K_U . \square

Remark 2.38. If there are some $a, b \in A \in K$ and a term $t(x, y)$ such that $t(a, b) \in \langle ab \rangle \setminus (\langle a \rangle \cup \langle b \rangle)$ the limit theory T_U of K_U is non stable, in fact the formula $U(t(x, y))$ has the order property.

Remark 2.39. The proof of Proposition 2.37 work more generally for a relation of any arity: If we were working with an n -ary relation we would have some tuples of terms \bar{t}_i and \bar{t}'_i in the formula φ from the beginning of the proof and the formula φ' should express that $\bar{t}_i \neq \bar{t}'_j$ for any i, j .

Definition 2.40. Let $L_R = L \cup \{R\}$ be a new unary predicate symbol. We define the class K_R of L_R -structures consisting of a structure A in K equipped with an arbitrary substructure $R(A)$.

Proposition 2.41. *If the structures of K are uniformly locally finite the class K_R satisfies Condition 1 – 4. However if the generated structure in K is non-trivial (i.e. in $\langle A \rangle \neq A$ for some set A) then Condition 5 needs not to hold.*

Proof. The Conditions 1, 3 and 4 are still true by the same arguments as before, we will need the local finiteness assumption on K to get Condition 2. We define $P(A \oplus_E B) := \langle P(A)P(B) \rangle$ and $P(\langle \{x\} \rangle) := P(\langle \emptyset \rangle)$. For Condition 2 consider a finitely generated structure $\langle a, b \rangle \in K_R$ and a quantifier free L_R formula φ such that $\langle a, b \rangle \models \varphi(a, b)$. Without loss of generality we can assume that:

$$\varphi(x, y) = \varphi'(x, y) \wedge \bigwedge_{i < n} R(t_i(x, y)) \wedge \bigwedge_{i < n} \neg R(t'_i(x, y)),$$

where $\varphi'(x, y)$ is an L -formula, for all $c \in R(\langle ab \rangle)$ there is some $i < n$ such that $c = t_i(a, b)$ and for all $c \in \langle ab \rangle \setminus R(\langle ab \rangle)$ there is some $i < n$ such that $c = t'_i(a, b)$.

Since our theory is uniformly locally finite the Condition $t'_i(x, y) \notin \langle (t_j(x, y))_{j < n} \rangle$ is definable. Now we consider the formula $\psi'(x)$ associated to the L -formula $\varphi'(x, y) \wedge \bigwedge_{i < n} (t'_i(x, y) \notin \langle (t_j)_{j < n} \rangle)$ by Condition 2 in K , and then the formula:

$$\psi(x) := \psi'(x) \wedge \bigwedge_{k < m} R(t_{i_k}(x)) \wedge \bigwedge_{k < m} \neg R(t'_{i_k}(x)),$$

where $t_{i_k}(x)$ enumerates the terms t such that $R(t(a))$ and $t'_{i_k}(x)$ enumerates the terms t' such that $\neg R(t'(a))$. If $A' \models \psi(a')$ for some $a' \in A' \in K_R$ then by assumption we can find some b' such that $\langle a'b' \rangle \models \varphi'(a', b') \wedge \bigwedge_{i < n} (t'_i(a', b') \notin \langle (t_j(a', b'))_{j < n} \rangle)$. Define the substructure R on $\langle a'b' \rangle$ to be $\langle t_j(a', b') \rangle_{j < n}$. We obtain that $\langle a'b' \rangle \models \neg R(t'_i(a', b'))$ for all $i < n$, so $\langle a'b' \rangle \models \varphi(a', b')$, and this property still holds in $B' := A' \oplus_{\langle a' \rangle} \langle a'b' \rangle$, which is an extension of A as an L_R -structure.

Now about Condition 5 : By Condition 5 in the class K we can find some structure $D \in K$ and some embeddings as in the statement. We want to define $R(D)$. We know that in any case $\langle R(D_0) \cup R(D_1) \cup R(B) \rangle \subseteq R(D)$, however it might happen that $b \in \langle R(D_0) \cup R(D_1) \cup R(B) \rangle$ for some $b \in B \setminus R(B)$, and in that case the L -embedding φ_B can not be an L_R -embedding. \square

To give a simple example of this last fact, let us look at some vector space over a finite field with a predicate for a subspace. Take some independent vectors a_0, a_1, b_0, b_1 as in the statement of the independence theorem for \perp^a such that $\neg R(a_0 + b_0)$, $R(a_1 - b_1)$ and $R(b_0 + b_1)$ are satisfied. Then $R(a + b_0)$ is satisfied if a satisfies $\text{tp}(a_i/b_i)$ for $i = 0, 1$, which is a contradiction. In that case the limit of K_R exists, and it might be NSOP_1 , however $\perp^K \neq \perp^a$ in the limit theory.

Remark 2.42. If we assume that the class K is uniformly locally finite we can restrict it to the class of finite structures in K and have a Fraïssé class in the usual sense, then its limit is ω -categorical.

As we have seen in the previous section Condition 5 puts a strong restriction on what we can add to the structure, and some classes with an NSOP_1 limit can satisfy the Conditions 1 – 4 and not satisfy Condition 5. For example the class of finite equivalence relations, with limit the (stable) theory of the equivalence relation with an infinite number of classes all infinite (which we will write T_∞), or its parameterized version (see [KR20] or [Bos23]) which is strictly NSOP_1 , satisfies existence and that $\perp^{K^M} = \perp^f = \perp^d$ over arbitrary sets.

Definition 2.43. Consider a new symbol E for a binary predicate in some sort S , we define the language $L_E = L \cup \{V, E, p\}$ where V is a new sort, E a binary relation on S , and p a unary map from S to V . Define the class K_E of L_E -structures consisting of any structure A in K equipped with an equivalence relation $E^A \subseteq S(A)^2$, a new sort $V(A)$ and a map $p : S(A) \rightarrow V(A)$ such that $p(a) = p(b)$ if and only if $E(a, b)$ for all $a, b \in S(A)$.

Proposition 2.44. *The class K_E satisfies (H).*

Proof. If we do not add the quotient to the structure Conditions 1 – 4 are satisfied but Condition 5 is not, also if we ask for the map p to be surjective the Condition 4 will not hold. It is easy to see that Conditions 1 – 4 are satisfied in K_E : The only structure on the quotient is equality and we can do ‘in parallel’ free amalgamation in K and in the quotient. For 5 we proceed similarly by forming the amalgam in K on one end and the amalgam of the quotient on the other end. \square

Notice that this construction coincides with adding a new sort V with no structure other than equality and then a generic function from S to V .

2.3.2 Adding generic functions

Definition 2.45. Consider a new symbol f for a binary function, we define $L_f = L \cup \{f\}$ and the class K_f of L_f -structures consisting of a structure A in K equipped with a binary function $f_A : A \times A \rightarrow A$ such that the structure generated by the constants is some specified L_f -structure (in order to have the joint embedding property and uniqueness of the limit the structures need to agree on the constants).

Proposition 2.46. *The class K_f satisfies (H).*

Proof. Condition 1 is clearly satisfied in K_f . For Condition 3 we construct some L_f -structure D using 3 and 4 in K :

Consider $E \subseteq A, B$ two extensions of L_f -structures. We define an increasing sequence of L -structures $(D_i)_{i < \omega}$ equipped with a partial binary function $f_{i+1} : D_i^2 \rightarrow D_{i+1}$. Let $D_0 := A \oplus_E B$ and $f_0 = f_E$. Now set:

$$D_1 := D_0 \oplus_{\langle \emptyset \rangle} \langle \{D_0^2 \setminus (A^2 \cup B^2)\} \rangle,$$

and consider $f_1 : D_0^2 \rightarrow D_1$ defined the following way:

1. $f_1(x, y) := f_A(x, y)$ if $(x, y) \in A^2$,
2. $f_1(x, y) := f_B(x, y)$ if $(x, y) \in B^2$,
3. (x, y) is sent to the element of D_1 with the corresponding label otherwise.

We iterate this process: if D_i, f_{i+1} are defined for $i > 0$ we let:

$$D_{i+1} := D_i \oplus_{\langle \emptyset \rangle} \langle \{D_i^2 \setminus (D_{i-1}^2)\} \rangle,$$

and f_{i+1} extends f_i by sending $(x, y) \in D_i^2 \setminus (D_{i-1}^2)$ to the element of D_{i+1} with the corresponding label. We then set $D := \bigcup_{i < \omega} D_i$ and $f := \bigcup_{i < \omega} f_i : D^2 \rightarrow D$. This structure satisfies the universal property: it is generated by the images of A and B as an L_f -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphisms of L -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_f -structures. The construction of $\langle \{x\} \rangle$ in K_f follows the same method.

Condition 5 is easy to check: Consider E, A, B_i, D_i for $i = 0, 1$ and D as in the statement. We apply 5 inside of K_0 to find embeddings of L -structures into some $D \in K$, and define the function f_D in D so that these maps become embeddings of L_f -structures: Take $f_D|_{D_i^2}$ to be the image of f_{D_i} by φ_{D_i} , similarly for B , and we can set f to be anything outside of D_i^2 and B^2 .

For Condition 2 consider a structure $\langle ab \rangle \in K_f$ and some quantifier-free L_f -formula φ such that $\langle ab \rangle \models \varphi(a, b)$. By replacing every instance of f in φ by a new variable and separating the instances of f applied only to elements of $\langle a \rangle$ from the other ones we can reshape $\varphi(x, y)$ into a formula of the following form:

$$\varphi'(x_{\leq m}, y_{\leq n}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x_{< i}), s'_i(x_{< i}))) \wedge \bigwedge_{i \leq n} (y_i = f(t_i(x_{\leq m}, y_{< i}), t'_i(x_{\leq m}, y_{< i})))$$

where φ' is an L quantifier-free formula and the t_i, s_i, t'_i, s'_i are L -terms. We define by induction $x_0(a) = a$, $x_k(a) = f(s_k(x_{< k}(a)), s'_k(x_{< k}(a)))$ if $k > 0$, $y_0(a, b) = b$ and if $k > 0$

$$y_k(a, b) := f(t_k(x_{\leq m}(a), y_{< k}(a, b)), t'_k(x_{\leq m}(a), y_{< k}(a, b))).$$

If $y_k(a, b) \neq y_{k'}(a, b)$, we strengthen the formula φ to specify that:

$$(t_k(x_{\leq m}, y_{< k}), t'_k(x_{\leq m}, y_{< k})) \neq (t_{k'}(x_{\leq m}, y_{< k'}), t'_{k'}(x_{\leq m}, y_{< k'})),$$

and similarly for $(s_k(x_{\leq k}), s'_k(x_{\leq k}))$ and $(s_{k'}(x_{\leq k'}), s'_{k'}(x_{\leq k'}))$.

Now consider the L -formula $\psi'(x_{\leq m})$ associated by Condition 2 in K to $\varphi'(x_{\leq m}, y_{\leq n})$ and then the L_f -formula:

$$\psi(x) := \psi'(x_{\leq m}) \wedge \bigwedge_{i \leq m} (x_i = f(s_i(x_{< i}), s'_i(x_{< i}))).$$

Assume that $A' \models \psi(a')$ for some $a' \in A'$. By assumption there is an L -extension $A' \subseteq B'$ and $b'_{\leq n} \in B'$ such that:

$$B' \models \varphi'(a'_{\leq m}, b'_{\leq n}) \wedge \bigwedge_{i \leq m} (a'_i = f(s_i(a'_{< i}), s'_i(a'_{< i}))).$$

We just need to define a binary function $f_{B'}$ on B' extending $f_{A'}$ such that

$$b'_k = f(t_i(a'_{\leq m}, b'_{< i}), t'_i(a'_{\leq m}, b'_{< i}))$$

for every $k \leq n$. We begin by setting these values, this is possible thanks to the strengthening we had, and we can let $f_{B'}$ take any values elsewhere. We write B'_f for the L_f -structure defined on B' , then $B'_f \models \varphi(a', b')$. \square

Remark 2.47. Adding a generic binary function assures us that the theory of any limit is strictly NSOP₁, as proven in [KR18, Section 3]. Also we get that the generated structure is non-trivial. With the same construction we can add generic functions of any arity.

Remark 2.48. The proof of Proposition 2.46 work more generally for a function of any arity: Adding variables to the generic function just makes the L' -formulas ψ' longer to write down but the proof works similarly. This remark added with Remark 2.39 gives us that if some class of structures K satisfy (H) then the expansion of K by some generic language also does.

Definition 2.49. We define $L_\pi = L \cup \{\pi, \pi^{-1}\}$ new symbols of unary functions and the class K_π of L_π -structures consisting of a structure A in K equipped with a bijection π_A of inverse function π_A^{-1} such that the structure generated by the constants is some specified L_π -structure.

Proposition 2.50. *The class K_π satisfies (H).*

Proof. Condition 1 is clearly satisfied in K_π , for the Conditions 2, 4 and 5 we can proceed as previously. Now we describe how to build the free amalgam for Condition 3. Let $E \subseteq A, B$ be two extensions in K_π .

In the same fashion as previously we construct an increasing sequence of L -structures $(D_i)_{i < \omega}$, and an increasing sequence of subsets $H_i \subseteq D_i$ and two increasing sequences of functions $\pi_i, \pi_i^{-1} : H_i \rightarrow D_i$ such that $D_i \subseteq H_{i+1}$ and $\pi_{i+1} \circ \pi_i^{-1} = \pi_{i+1}^{-1} \circ \pi_i = id_{H_i}$ for every $i < \omega$. Let $D_0 := A \oplus_E B$, $H_0 = A \cup B$, and $\pi_0 = \pi_A \cup \pi_B$, $\pi_0^{-1} = \pi_A^{-1} \cup \pi_B^{-1}$.

$$D_1 := D_0 \oplus_{\{\emptyset\}} \langle \{[D_0 \setminus (A \cup B)]_k : k \in \mathbb{Z} \setminus \{0\}\} \rangle,$$

I.e., we consider the L -structure freely generated over D_0 by copies of $D_0 \setminus (A \cup B)$ indexed on $\mathbb{Z} \setminus \{0\}$, we see the element of $D_0 \setminus (A \cup B)$ as those corresponding to the copy of index 0. Set

$$H_1 = D_0 \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} [D_0 \setminus (A \cup B)]_k.$$

We define π_1 as the extension of π_0 that sends an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k+1$, i. e. it sends $t(a, b)_k$ to $t(a, b)_{k+1}$, similarly for π_0^{-1} which sends an element of $[D_0 \setminus (A \cup B)]_k$ to its copy of index $k-1$. Assume that D_{i+1}, H_i and π_i, π_i^{-1} have been defined for $i > 0$. We set

$$D_{i+1} := D_i \oplus_{\langle \emptyset \rangle} \langle \{[D_i \setminus D_{i-1}]_k : k \in \mathbb{Z} \setminus \{0\}\} \rangle,$$

$$H_{i+1} := D_i \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} [D_i \setminus D_{i-1}]_k,$$

and take π_{i+1} to be the function extending π_i that sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k+1$, similarly for π_0^{-1} which sends an element of $[D_i \setminus D_{i-1}]_k$ to its copy of index $k-1$. Once this is done we take $D := \bigcup_{i < \omega} D_i$, $\pi_D := \bigcup_{i < \omega} \pi_i$ and $\pi_D^{-1} := \bigcup_{i < \omega} \pi_i^{-1}$.

This defines an L_π -structure of bijection π_D , and D satisfies the universal property: it is generated by the images of A and B as an L_π -structure, given D' and embeddings $\varphi_A : A \rightarrow D'$, $\varphi_B : B \rightarrow D'$ as in the statement we can define an increasing sequence of morphism of L -structures $\varphi_i : D_i \rightarrow D'$ in a canonical way and show that its union is a morphism of L_π -structures. We take $\varphi_i : D_i \rightarrow D'$ to be the only morphism of L -structures such that $\varphi_i|_A = \varphi_A$ and $\varphi_i|_B = \varphi_B$. If $\varphi_i : D_i \rightarrow D'$ is defined we take $\varphi_{i+1} : D_{i+1} \rightarrow D'$ to be its unique morphism extending φ_i that sends $\pi_D^k(t)$ to $\pi_{D'}^k(\varphi(t))$ for every $t \in D_i$.

We define $\varphi := \bigcup_{i < \omega} \varphi_i$, it is a morphism of L -structures and by construction it commutes with π , so it is a morphism of L_π -structures. The construction of $\langle \{x\} \rangle$ in K_π follows the same method. \square

Definition 2.51. We define $L_\sigma = L \cup \{\sigma, \sigma^{-1}\}$ new symbols of unary functions and the class K_σ of L_σ -structures consisting of a structure A in K equipped with an automorphism σ_A and its inverse function σ_A^{-1} .

In K_σ the Conditions 1 – 4 are easy to check: we use the Free Amalgam property to show that if $E \subseteq A, B \in K_\sigma$ then there is an (unique) automorphism $\sigma_{A \oplus_E B}$ of $A \oplus_E B$ extending σ_A and σ_B and $(A \oplus_E B, \sigma_{A \oplus_E B})$ satisfies the universal property.

For Condition 4 we consider the L -structure $\langle \{(x_i)_{i \in \mathbb{Z}}\} \rangle \in K$ equipped with the unique automorphism that extends σ_A and sends x_k to x_{k+1} for every $k \in \mathbb{Z}$. As for the generic substructure we need the structures of K to be uniformly locally finite for Condition 2 to hold, in this case the proof follows the same method. Also Condition 5 need not to hold even in this case for similar reasons as in K_R . The limit of K_σ exists, and it might be NSOP₁, however in that case $\downarrow^K \neq \downarrow^a$ in the limit theory.

Remark 2.52. To sum up the last two sections, the theories we are building are model companions of theories which are made from very basic one-based stable theories, like the theory of the \mathbb{F}_2 -vector space of infinite dimension, to which we add some basic structure (equivalence relation, bijection,..) that does not interact - typically some generic functions and relations as in Remark 2.48. This is slightly more general than the model companion of the empty theory, and it allows us to give some minimalist examples of strictly NSOP₁ theory: The model companion of the theory of an abelian group with a unary function f such that $f(0) = 0$.

2.4 Some perspectives

The theory T of the limit of some K satisfying (H) is not a free amalgamation theory in the sense of [Con17]. The stationary independence relation \downarrow^Γ does not in general satisfies the closure condition (which is defined as : if $C \subseteq A, B$ are algebraically closed and $A \downarrow_C B$ then $A \cup B$ is algebraically closed), however the relation \downarrow^Γ is interesting.

There are other known examples of NSOP₁ theories admitting a strong independence relation satisfying Proposition 2.11, which could motivate the study of such theories :

1. The theory of ω -free PAC fields of characteristic 0 (see [Cha08] and [Cha02]) with the relation \downarrow^{III} defined as: for $C \subseteq A, B$, $A \downarrow_C^{III} B$ if $A \downarrow_C^{lin} B$ and $acl(AB) = AB$. This relation is named strong independence in [Cha10, Definition 4.5], written as here in [Cha08, Section 3.6], but this definition is not the same as the strong independence of [Cha02, Definition 1.2].
2. The theory of vector spaces of infinite dimension over an ACF with a non degenerate bilinear form with the relation \downarrow^Γ (see [Adl09a], [Dob23] or [Bos23] for the definition of \downarrow^Γ). This is not a free amalgamation theory because of the closure condition.
3. The theory ACFG of an ACF with a generic multiplicative subgroup, see strong independence in [d'E21, Section 2.2] and the remark following [d'E21, Section 4.4].

An other context that might be interesting to study is the one of 'one-based' NSOP₁ theory. By this we mean an NSOP₁ theory with existence and weak elimination of imaginaries such that Kim-independence coincides with algebraic independence over arbitrary sets.

1. If T is a one-based NSOP₁ theory does forking independence coincides with M -independence?
2. If T is an NSOP₁ theory such that Kim-independence coincides with algebraic independence over models does T satisfies existence? If that is the case does \downarrow^K coincides with \downarrow^a over arbitrary sets?

Chapter 3

Model theory of bilinear forms



This chapter is dedicated to the study the two sorted theory of bilinear forms. Let T_K be the theory of a field in a language \mathcal{L}_K eventually extending the language of rings, we write sT_∞^K the theory of a vector space of infinite dimension over a model of T_K with a non-degenerate symmetric bilinear form. We will study these theories and show different results depending on T_K . This study is following some works of Chernikov, Dobrowolski, Granger and Ramsey.

We define \mathcal{L}_0 as the language \mathcal{L}_K on the field sort K , the sum and the vector 0_V on the vector sort V and the scalar product $\cdot : K \times V \rightarrow V$. We define the theta functions $\theta_n : V^{n+1} \rightarrow K^n$ for any $n < \omega$ which is defined the following way: It sends any tuple of vectors $\bar{v} = (v_i)_{i \leq n}$ with $v_{<n}$ linearly independent and v_n in the linear span of $v_{<n}$ to the unique tuple of scalars $(\lambda_i)_{i < n}$ such that $v_n = \sum_{i < n} \lambda_i \cdot v_i$, and any other tuple to $(0_K, \dots, 0_K)$. We will work with the language $\mathcal{L} = \mathcal{L}_0 \cup \{\theta_n : n < \omega\}$.

The theta functions are definable with quantifiers in \mathcal{L}_0 , the point of adding them into the language is to have quantifier elimination in \mathcal{L} . For two substructures A and B of some model M of sT_∞^K we will write $\langle A \rangle$ for the linear span of $V(A)$, and use the concatenation AB to denote the structure generated by A and B .

3.1 Generic bilinear form on an NSOP₁ field

We generalize the characterization of Kim-independence over models in sT_∞^{ACF} by Chernikov and Ramsey in [CR16, Section 6.1] to the case of NSOP₁ fields. To have the most generality as possible on the theory of the field we add an additional axiom to sT_∞^K so that this theory is NSOP₁ if and only if the theory T_K of the field is NSOP₁, and compute what Kim-forking is over models, and over algebraically closed sets in the case when we assume that sT_∞^K satisfies existence. We now begin by giving a description of the generated structures that we will use throughout the chapter.

Let $M \models sT_\infty^K$ and let $A \subseteq M$ be a substructure. We will describe the extensions of A generated by a single element. First of all we can choose a basis $(a_i)_{i < \kappa}$ of $V(A)$ as a $K(A)$ -vector space, since A is closed by the theta functions then $(a_i)_{i < \kappa}$ is also a basis of $\langle A \rangle$ as a $K(M)$ -vector space.

Consider $\alpha \in K(M)$. Let $A' = A\alpha$ be the substructure of M generated by $K(A)\alpha'$. Then $K(A')$ is the substructure of $K(M)$ generated by $K(A)\alpha$ and $V(A') = \text{Lin}_{K(A')} (V(A))$.

Consider now $a \in V(M)$. If $a \in \langle A \rangle$ then a can be written $a = \sum_{i < \kappa} \lambda_i \cdot a_i$, and it is clear that the structure A' generated by Aa is equal to the structure generated by $A(\lambda_i)_{i < \kappa}$, which by the previous case $K(A')$ is the substructure of $K(M)$ generated by $K(A)(\lambda_i)_{i < \kappa}$ and $V(A') = \text{Lin}_{K(A')} (A)$. If $a \notin \langle A \rangle$, then the A' generated by Aa has field sort $K(A')$ equal to the the substructure of $K(M)$ generated by $K(A)([a_i, a])_{i < \kappa}$, and vector sort $V(A') = \text{Lin}_{K(A')} (V(A)a)$.

3.1.1 NSOP₁ and Kim-independence

Let us consider a complete NSOP₁ theory of fields T_K with quantifier elimination in the language \mathcal{L}_K and let sT_∞^K be the two sorted theory of infinite dimensional vector spaces over a model of T_K with a non degenerate symmetric bilinear form (written $[,]$) that also satisfy the condition (†) below. For a set of parameters A we will write A^\perp to mean $\{u : [u, x] = 0 \text{ for all } x \in V(A)\}$. We will add the following scheme of axioms to sT_∞^K :

(†): For any finite tuple \bar{b} in the vector sort there is $u \in \bar{b}^\perp \setminus \{0\}$ such that $[u, u] = 0$.

The condition (†) is a sufficient condition for this theory to have quantifier elimination, (†) is in particular satisfied when every element of the field is a square (see the last part of the proof of Lemma 3.1), which Granger assume in [Gra99]. Let $M = (V, K)$ be a model of sT_∞^K .

Lemma 3.1. *Let $(\lambda_i)_{i < n}$ be a tuple of scalars, $(b_i)_{i < n}$ be linearly independent vectors in M , $\alpha \in K$ and \bar{e} be a finite tuple of vectors. Then there is $x \in V$ such that $[x, x] = \alpha$, $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$.*

Proof. We begin by showing that there is $x \in V$ such that $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$, for this we do not need the assumption (†). What we want to show is that given a linearly independent tuple $\bar{b} = b_0, \dots, b_n$ the linear function $\psi_{\bar{b}}: x \rightarrow ([x, b_i])_{i \leq n}$ from V to K^{n+1} is surjective, the other condition follows then from the fact that its kernel has infinite dimension.

Now let us prove surjectivity by induction on $n < \omega$. For $n = 0$ it follows trivially from the fact that the bilinear form is non-degenerate. Now assume that it holds for $n \geq 0$. We want to find $b'_0 \in (b_1, \dots, b_n)^\perp \setminus b_0^\perp$. By assumption there is b'_i for $1 \leq i \leq n$ such that $[b'_i, b_j] = \delta_{i,j}$ for all $1 \leq i, j \leq n$. With this every $u \in V$ can be written:

$$u = \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i + (u - \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i).$$

So $V = \bigoplus_{1 \leq i \leq n} K \cdot b'_i + (b_1, \dots, b_n)^\perp$, we will write $u_0 = u - \sum_{1 \leq i \leq n} [u, b_i] \cdot b'_i$.

Now if such a b'_0 does not exist we would have that $(b_1, \dots, b_n)^\perp \subseteq b_0^\perp$, using the previous decomposition, for all $u \in V$ we would have:

$$[u, b_0] = \sum_{1 \leq i \leq n} [b'_i, b_0] \cdot [u, b_i] + [u_0, b_0] = \sum_{1 \leq i \leq n} [b'_i, b_0] \cdot [u, b_i],$$

which by non-degeneracy contradicts the fact that $b_0 \notin \langle b_1, \dots, b_n \rangle$.

We now use our assumption (†) to choose the value of the quadratic form. Choose $x \in V$ such that $[x, b_i] = \lambda_i$ for all $i < n$ and $x \notin \langle \bar{e} \rangle$. By what we have just shown and (†) there are x', x'' linearly independent such that $x', x'' \notin \langle \bar{e}, x \rangle$, $x', x'' \in (\bar{b}, x)^\perp$, $[x', x'] = 0$ and $[x', x''] = 1$. Then the element $x + (\alpha - [x, x] - [x'', x'']) \cdot x' + x''$ corresponds to what we are looking for.

We now show that when every element of the field is a square the condition (†) is satisfied. Let \bar{b} be a finite tuple of vectors and $x \in \bar{b}^\perp \setminus \langle \bar{b} \rangle$. By the previous point we can find $y \in \bar{b}^\perp \setminus \langle \bar{b}, x \rangle$ such that $[x, y] = 0$. Then, if $[x, x] = 0$ or $[y, y] = 0$ we have the element we are looking for, and else if $\lambda^2 = [x, x] \neq 0$ and $\mu^2 = -[y, y] \neq 0$ then the vector $\mu \cdot x + \lambda \cdot y$ satisfies our conditions. \square

Corollary 3.2. *The theory sT_∞^K has Q.E. in the language $\mathcal{L} = \mathcal{L}_K \cup \{[\ , \], \cdot, +\} \cup \{\theta_n : n < \omega\}$. This implies in particular that the field is stably embedded and that sT_∞^K is complete.*

Proof. For this we show that finite partial isomorphism between to finitely generated substructure of two \aleph_0 -saturated models $M = (V, K)$ and $M' = (V', K')$ of sT_∞^K have the back and forth property. Consider two finitely generated substructures A and B of M and M' respectively and $\varphi: A \rightarrow B$ an isomorphism. We show that we can extend it to any element $a \in M$.

Let a_1, \dots, a_{n-1} be a basis of $V(A)$ as a $K(A)$ vector space and $b_i := \varphi(a_i)$ for $i < n$. Then b_1, \dots, b_{n-1} is a basis of $V(B)$ as a $K(B)$ vector space. Let $\varphi_K: K(A) \rightarrow K(B)$ be the induced field isomorphism. By quantifier elimination in the field sort we can extend φ_K to any element of K , so if $a \in K$ we can extend φ to Aa . If $a \in \langle A \rangle$ and $a = \sum_i \lambda_i \cdot a_i$ it is clear that extending φ to the structure generated by Aa is equivalent to extending φ to $A(\lambda_i)_{i < n}$, which we can do by quantifier elimination in the field sort.

Now if $a \notin \langle A \rangle$ let us consider an extension φ'_K of φ_K to the field sort of the structure generated by Aa (which is just the structure generated by $K(A)([a, a], [a, a_i])_{i < n}$ in \mathcal{L}_K). Now using Lemma 3.1 we can find $b \notin \langle B \rangle$ such that $[b, b] = \varphi'_K([a, a])$ and $[b, b_i] = \varphi'_K([a, a_i])$ for all $i < n$. It is clear that sending a to b extends the isomorphism φ . \square

Without the assumption (\dagger) quantifier elimination in the theory of infinite dimensional vector spaces over a model of T_K with a non degenerate symmetric bilinear form in the language \mathcal{L} depends on the theory of the field (having squares for example) and also on the bilinear form (being positively defined in a RCF for example), the (\dagger) assumption allows us to keep full generality on the theory of the field. Now that we have proven that sT_∞^K is complete we can work in a monster model $\mathbb{M} = (\mathbb{V}, \mathbb{K})$ of sT_∞^K .

Definition 3.3. For $E \subseteq A, B$ algebraically closed subsets of \mathbb{M} and $E \models sT_\infty^K$ we define the relation $A \downarrow_E^\infty B$ by: $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \downarrow_{K(E)}^K K(B)$ in the theory of the field. If $E \not\subseteq A, B$ then $A \downarrow_E^\infty B$ is defined as $\langle AE \rangle \downarrow_E^\infty \langle BE \rangle$.

Theorem 3.4. sT_∞^K is NSOP₁ and $\downarrow^\infty = \downarrow^K$ over models.

Proof. We will now show that this relation satisfies the conditions of the first part of Theorem 1.62.

Clearly it satisfies symmetry, monotonicity and existence over models. For strong finite character if $A \not\downarrow_E^\infty B$ and $K(A) \not\downarrow_{K(E)}^K K(B)$ then we can take the formula in $\text{tp}(A/B)$ that Kim-forks over E . Otherwise if $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$ let $(e_i)_{i < \alpha}$ be a basis of $V(E)$ (it is then also a basis of $\langle E \rangle$), and $(b_i)_{i < \beta}$ completing it into a basis of $V(B)$. Then we have finite some tuples \bar{e} and \bar{b} in these bases and a finite tuple $\bar{a} \in V(A)$ which is a free family of vectors such that there exists a vector $a \in \langle \bar{a} \rangle$ satisfying $a \in \langle \bar{e}\bar{b} \rangle \setminus \langle \bar{e} \rangle$. Say that \bar{a} is an n -tuple, \bar{e} an m -tuple, \bar{b} an l -tuple and let \bar{x} be an n -tuple of variables.

Then the formula:

$$\psi(\bar{x}, \bar{e}, \bar{b}) := (\bar{x} \text{ is a free family}) \wedge \exists x ((\theta_n(\bar{x}, x) = 0) \wedge (\theta_{m+l}(\bar{e}\bar{b}, x) = 0) \wedge (\theta_m(\bar{e}, x) \neq 0))$$

expresses what we want. The quantified x in this formula can not in fact be in $\langle E \rangle$ (which is without context not a definable condition), because otherwise by the transfer principle some element of the tuple \bar{b} would then be in $\langle E \rangle$, which is absurd.

We show that \downarrow^∞ satisfies the independence theorem over models. Let $E \subseteq A_0, A_1, B_0, B_1$ with E a model, $A_0 \equiv_E A_1$, $A_j \downarrow_E^\infty B_j$ for $j = 0, 1$ and $B_0 \downarrow_E^\infty B_1$. Let $(e_i)_{i < \alpha}$ be a basis of $V(E)$, complete it by $(b_i^j)_{i < \beta_j}$ into a basis of $V(B_j)$ for $j = 0, 1$, so $(e_i)_{i < \alpha} (b_i^0)_{i < \beta_0} (b_i^1)_{i < \beta_1}$ is a basis of $B_0 B_1$ since B_0 and B_1 are linearly independent on E .

Let $(a_i^0)_{i < \gamma}$ complete $(e_i)_{i < \alpha}$ into a basis of A_0 . Let $a_0 = k_0 v_0$ be a tuple enumerating A_0 , $A_0 \equiv_E A_1$ so we can find a tuple $a_1 = k_1 v_1$ enumerating A_1 such that $\varphi : a_0 \rightarrow a_1$ is an E -elementary embedding. Define $(a_i^1)_{i < \gamma} := \varphi((e_i)_{i < \alpha})$. Since φ is elementary over E we have $\varphi([e_i, a_j^0]) = [e_i, a_j^1]$ and $\varphi([a_i^0, a_j^0]) = [a_i^1, a_j^1]$ for all i, j .

We have that $\text{tp}(k_0/K(E)) = \text{tp}(k_1/K(E))$, $k_j \downarrow_{K(E)}^K K(B_j)$ for $j = 0, 1$ and $K(B_0) \downarrow_{K(E)}^K K(B_1)$.

By applying the independence theorem for Kim-forking in T_K we find $k \models \text{tp}(k_0/K(B_0)) \cup \text{tp}(k_1/K(B_1))$ such that $k \downarrow_{K(E)}^K K(B_0)K(B_1)$. Using extension we can assume that

$$k \downarrow_{K(E)}^K K(B_0 B_1).$$

We have two partial embeddings $\varphi_0 : k_0 \rightarrow k$ and $\varphi_1 : k_1 \rightarrow k$ which are elementary over $K(B_0)$ and $K(B_1)$ respectively. By saturation and homogeneity we can extend them into automorphisms of \mathbb{K} , which we still write φ_0 and φ_1 . By construction we have that $\varphi_0|_{k_0} = \varphi_1|_{k_1} \circ \varphi$. We will now define our structure A . Let $(a_i)_{i < \gamma}$ be a tuple of linearly independent vectors not in $\langle B_0 B_1 \rangle$ such that:

$$\begin{aligned} [e_i, a_j] &= \varphi_0([e_i, a_j^0]) = \varphi_1([e_i, a_j^1]) \text{ for all } i < \alpha, j < \gamma, \\ [a_i, a_j] &= \varphi_0([a_i^0, a_j^0]) = \varphi_1([a_i^1, a_j^1]) \text{ for all } i, j < \gamma, \\ [a_i, b_j^0] &= \varphi_0([a_i^0, b_j^0]) \text{ for all } i < \gamma, j < \beta_0, \\ [a_i, b_j^1] &= \varphi_1([a_i^1, b_j^1]) \text{ for all } i < \gamma, j < \beta_1. \end{aligned}$$

This is possible by Lemma 3.1 and compactness. Then the structure $A := (\text{Lin}_k((e_i)_{i < \alpha} (a_i)_{i < \gamma}), k)$ satisfies $A \downarrow_E^\infty B_0 B_1$ and $A \equiv_{B_j} A_j$ for $j = 0, 1$ by quantifier elimination.

By Theorem 1.62 we have that sT_∞^K is NSOP₁ and that $\downarrow^\infty \implies \downarrow^K$ over models. For the other implication, it is clear that if $A \downarrow_E^K B$ then $K(A) \downarrow_{K(E)}^K K(B)$:

In fact if $A \downarrow_E^K B$ there is a sequence $(A_i)_{i < \omega}$ which is coheir Morley over E and B indiscernible. Then the sequence $(K(A_i))_i$ is coheir Morley over $K(E)$ and $K(B)$ indiscernible, which by Kim's Lemma for Kim-forking in T_K implies that $K(A) \downarrow_{K(E)}^K K(B)$.

Now if $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$, let $a \in \langle \bar{a} \rangle$ such that $a \in \langle B \rangle \setminus \langle E \rangle$ with \bar{a} a finite tuple in A . Let $(B_i)_{i < \omega}$ be a coheir Morley sequence over E with $B_0 = B$. We have $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$, so if $q(x, y) := \text{tp}(\bar{a}, B)$ then $\bigcup_{i < \omega} q(x, B_i)$ is inconsistent and $A \not\downarrow_E^K B$. We will give a more general argument when generalizing this to algebraically closed sets in Proposition 3.8. \square

In particular if T_K is a theory of fields, the theory sT_∞^K is NSOP₁ if and only if the theory T_K is, and sT_∞^K is also always non-simple, as we can construct infinite decreasing sequences of type definable groups of unbounded index using orthogonality.

3.1.2 More properties of independence relations

We now assume that the theory of the field T_K has existence. \downarrow^∞ can now be defined on arbitrary sets. If $E \subseteq A, B$ then $A \downarrow_E^\infty B$ if $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \downarrow_{K(E)}^K K(B)$ in the theory of the field. We show that in this case some properties passes from \downarrow^K in T_K to \downarrow^∞ .

Proposition 3.5. *If Kim-forking in T_K satisfies strong local character, i.e. if for all finite tuple $\bar{a} \in \mathbb{K}$ and for all $B \subseteq \mathbb{K}$ small set there is a finite B_0 such that $a \downarrow_{B_0}^K B$, then \downarrow^∞ also does. This is the case when T_K is supersimple for example.*

Proof. Let A be a finitely generated structure and let \bar{a} be a basis of $V(A)$ (the tuple \bar{a} is then finite). Let $B \subseteq \mathbb{M}$ be a substructure, with $(b_i)_{i < \beta}$ a $K(B)$ -basis of $V(B)$. Since \bar{a} is a finite tuple we can find a finite $\bar{b} \subseteq \{b_i : i < \beta\}$ such that $\langle \bar{a} \rangle \cap \langle (b_i)_{i < \beta} \rangle \subseteq \langle \bar{b} \rangle$. We then have that $\langle \bar{a} \bar{b} \rangle \cap \langle (b_i)_{i < \beta} \rangle = \langle \bar{b} \rangle$.

We now consider the structure A_0 generated by $A\bar{b}$. We have $\langle A_0 \rangle = \langle \bar{a} \bar{b} \rangle$, and the field sort is finitely generated in \mathcal{L}_K , since $K(A)$ is finitely generated \mathcal{L}_K by assumption and by adding \bar{b} we only have to add a finite number of images by the bilinear form and coefficients from linearly dependent tuples to the field sort.

By assumption there is a finitely generated $E_0 \subseteq K(B)$ such that $K(A_0) \downarrow_{E_0}^K B$, and by setting B_0 to be the substructure of B generated by (\bar{b}, E_0) we get that $A \downarrow_{B_0}^\infty B$. \square

Lemma 3.6. $A \downarrow_E^d B$ implies that $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ for all $E \subseteq A, B$.

Proof. We can assume that A is a finite tuple. As previously let $(e_i)_{i < \alpha}$ be a basis of $V(E)$ and let $(e_i)_{i < \alpha}(b_i)_{i < \beta}$ a basis of $V(B)$.

We consider a tuple $(b_i^j)_{i < \beta, j < \omega}$ linearly independent over E such that:

1. $[b_i^j, e_l] = [b_i, e_l]$ for all $i < \beta, j < \omega$,
2. $[b_i^j, b_l^j] = [b_i, b_l]$ for all $i, l < \beta, j < \omega$,
3. $[b_i^j, b_l^{j'}] = 0$ for all $i, l < \beta$ and $j \neq j' < \omega$.

We can find such a sequence by Lemma 3.1 and compactness. By quantifier elimination it is an indiscernible sequence over E . This implies that $\langle A \rangle \cap \langle B \rangle \subseteq \langle E \rangle$. In fact, otherwise the type of A over B_0 would divide along that sequence since $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$ for all $i < \omega$, as we will show now.

If $\langle A \rangle \cap \langle B \rangle \not\subseteq \langle E \rangle$ then as in the proof of strong finite character in Theorem 3.4 we can find some finite some tuples $\bar{e} \in (e_i)_{i < \alpha}$, $\bar{b}_0 \in (b_i^0)_{i < \beta}$ and a finite tuple $\bar{a} \in V(A)$ which is a free family of vectors such that there exists a vector $a \in \langle \bar{a} \rangle$ satisfying $a \in \langle \bar{e}\bar{b} \rangle \setminus \langle \bar{e} \rangle$. Say that \bar{a} is an n -tuple, \bar{e} an m -tuple, \bar{b} an l -tuple and let \bar{x} be an n -tuple of variables. We write $\bar{b}_i \in (b_j^0)_{j < \beta}$ the tuple corresponding to \bar{b}_0 , so that the sequence $(\bar{b}_i)_{i < \omega}$ is E -indiscernible.

Then the formula:

$$\psi(\bar{x}, \bar{e}, \bar{b}_0) := (\bar{x} \text{ is a free family}) \wedge \exists x ((\theta_n(\bar{x}, x) = 0) \wedge (\theta_{m+l}(\bar{e}\bar{b}_0, x) = 0) \wedge (\theta_m(\bar{e}, x) \neq 0))$$

divides along the sequence $(\bar{b}_i)_{i < \omega}$. In fact assume that a tuple \bar{a}' satisfies the formula $\psi(\bar{x}, \bar{e}, \bar{b}_i)$ for every $i < \omega$. Then for every $i < \omega$ there exists $a'_i \in \langle \bar{a}' \rangle$ such that $a'_i \in \langle \bar{e}\bar{b}_i \rangle \setminus \langle E \rangle$ (by the same argument as in Theorem 3.4). Since $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$ for every $i < \omega$ the family $(a'_i)_{i < \omega}$ is free, which contradicts the fact that $\langle \bar{a}' \rangle$ has finite dimension. \square

Proposition 3.7. Let A be a structure such that \downarrow^K in T_K satisfies the independence theorem over $K(A)$. If D^0, D^1 and B^0, B^1 are such that $B^0 \downarrow_A^\infty B^1$, $D^0 \equiv_A D^1$ and $D^j \downarrow_A^\infty B^j$ for $j = 0, 1$ then there is D' such that $D' \equiv_{AB^j} D^j$ for $j = 0, 1$ and $D' \downarrow_A^\infty B^0 B^1$.

Proof. The proof is strictly similar to the one of amalgamation over models. \square

We will now additionally assume that the theory sT_∞^K has existence, which was only proven for ACF . This assumption allows us to use Kim-forking independence over arbitrary sets and its properties, here we use Kim's lemma for Kim-forking Theorem 1.30.

Proposition 3.8. If $E \subseteq A, B$ are algebraically closed, then $A \downarrow_E^K B$ if and only if $A \downarrow_E^\infty B$.

Proof. $[\Rightarrow]$: If $\langle A \rangle \cap \langle B \rangle \neq \langle E \rangle$, let B_i be a Morley sequence with $B_0 = B$. By Lemma 3.6 we have that $\langle B_i \rangle \cap \langle B_{< i} \rangle = \langle E \rangle$ for all $i < \omega$, so the type of A over B is inconsistent along that sequence, and so it Kim-divides over E . Now if $K(A) \not\downarrow_{K(E)}^K K(B)$, the induced structure in the field sort is the one of T_K since the field is stably embedded, so $A \not\downarrow_E^K B$.

$[\Leftarrow]$: Let $A \downarrow_E^\infty B$, we show that $A \downarrow_E^K B$ using Kim's Lemma for Kim-forking.

Let $(e_i)_{i < \kappa_E}$ be a basis of $V(E)$, complete it by $(a_i)_{i < \kappa_A}$ into a basis of $V(A)$ and by $(b_i^0)_{i < \kappa_B}$ into a basis of $V(B)$. Let $(B_i)_{i < \omega}$ be an E -Morley sequence with $B_0 = B$. Let $(b_i^j)_{i < \kappa_B}$ be the basis of B_i over $V(E)$ corresponding to $(b_i^0)_{i < \kappa_B}$ for all $j < \omega$.

Then the sequence $(K(B_j))_{j < \omega}$ is Morley over $K(E)$ and $K(A) \downarrow_{K(E)}^K K(B_0)$. Let k_a be a tuple enumerating $K(A)$ and let $q(x, B_0) := \text{tp}(k_a K(B_0))$. By Kim's Lemma for Kim-forking we can find $k' \models \bigcup_{j < \omega} q(x, K(B_j))$.

As for amalgamation over models there is a partial elementary embedding $\varphi: k_a \rightarrow k'$ over E such that for all $j < \omega$ this embedding extends into $\varphi_j: k_a K(B) \rightarrow k' K(B_j)$, which is also elementary over $K(E)$. As previously in Theorem 3.4 we can extend each φ_j into an automorphism of \mathbb{K} which we also write φ_j .

Let us now define a structure A' . Using Lemma 3.1 we can find a tuple $(a'_i)_{i < \gamma}$ of linearly independent vectors outside of $\langle B_{< \omega} \rangle$ such that:

$$[e_i, a'_j] = \varphi_0([e_i, a_j^0]) = \varphi_l([e_i, a_j]) \text{ for all } i < \kappa_E, j < \kappa_A \text{ and } l < \gamma,$$

$$[a'_i, a'_j] = \varphi_0([a_i, a_j]) = \varphi_l([a_i, a_j]) \text{ for all } i, j < \kappa_A \text{ and } l < \gamma,$$

$$[a'_i, b_j^l] = \varphi_l([a_i, b_j]) \text{ for all } i < \kappa_A, j < \kappa_B \text{ and } l < \gamma.$$

By quantifier elimination the structure $A' = (\text{Lin}_{k'}((e_i)_{i < \alpha}, (a'_i)_{i < \gamma}), k')$ is such that $A' B_j \equiv_E A B$ for all $j < \omega$, so $A \downarrow_E^K B$ by Kim's Lemma for Kim-independence. \square

3.2 Proving existence in the case of simple fields

We will now assume that T_K is a simple theory of fields (see [Wag00] for properties of simple fields). We will generalize the notion of Gamma-forking that Granger developed in [Gra99] for the case of ACF and also give complete proofs. We use the same terminology as previously, and to lighten the notations we will write A for the structure generated by the set A . As previously we work in a monster model of sT_∞^K . We begin by recalling the properties of f -generics in groups definable in simple theories.

Definition 3.9. Let (G, \cdot) be a group definable over E in a model M of some complete theory. For $E \subseteq B$ a set of parameters we say that an element $a \in G$ is f -generic in G over B if $a \cdot b \downarrow_B^f b$ for every $b \in G$ such that $a \downarrow_B^f b$.

Proposition 3.10. [Wag00, Proposition 4.1.7] Let (G, \cdot) be a group definable over E in a model M of some simple theory and $B \supseteq E$. Then there is an f -generic for G over B .

Lemma 3.11. [Wag00, Lemma 4.1.2] Let g be f -generic for G over A and let $A \subseteq B$. If $g \downarrow_A^f B$ then g is f -generic over B .

We say that a tuple $(x_i)_{i < \kappa} \in \mathbb{K}$ is f -generic over $K \subseteq \mathbb{K}$ if x_i is f -generic over \emptyset for all $i < \kappa$ and $x_i \downarrow_{\emptyset}^f K x_{\neq i}$. By Lemma 3.11 this implies that x_i is f -generic over $K x_{\neq i}$.

We recall that in a simple field the notion of genericity for the additive group and the multiplicative group coincide. These notions coincide with genericity for affine transformations: if x is f -generic over $K \subseteq \mathbb{K}$, $g \in \mathbb{K}^*$ and $h \in \mathbb{K}$ such that $x \downarrow_K^f g, h$ then $g \cdot x + h \downarrow_K^f g, h$.

Lemma 3.12. f -generic tuples in \mathbb{K} satisfy the following properties:

1. *Extension:* If a tuple $(x_i)_{i < \kappa}$ is f -generic over K and if $(x_i)_{i < \kappa} \downarrow_K^f K'$ for $K \subseteq K'$ then $(x_i)_{i < \kappa}$ is f -generic over K' .
2. *Genericity:* If a finite tuple $(x_i)_{i < n}$ is f -generic over K , $M \in GL_n(K)$ is an invertible matrix and $(k_i)_{i < n} \in K^n$ then the tuple $M \times (x_i)_{i < n} + (k_i)_{i < n}$ is also f -generic over K .

3. *Transitivity:* If \bar{x} and \bar{y} are finite tuples, \bar{x} is f -generic over K and \bar{y} is f -generic over $K(\bar{x})$ if and only if \overline{xy} is f -generic over K .

Proof. The first and last points are just transitivity of \downarrow^f and Lemma 3.11. For the second point we know that in a simple field genericity for the additive group and the multiplicative group are the same.

This tells us that if $(\alpha_i) \in K^*$ and $(k_i)_{i < n} \in K$ the tuple $(\alpha_i \cdot x_i + k_i)_{i < n}$ is f -generic over K . The tuple $(x_1 + x_2, x_2, \dots, x_{n-1})_{i < n}$ is f -generic over K since x_1 is f -generic over $Kx_{\neq 1}$ and x_2 is f -generic over $Kx_{\neq 2}$, and it is clear that with these two transformations and the permutations of the tuple $(x_i)_{i < n}$ we can generate the transformations $(x_i)_{i < n} \rightarrow M \times (x_i)_{i < n} + (k_i)_{i < n}$ described in the statement. \square

Definition 3.13. Let $E \subseteq A, B \subseteq \mathbb{M}$. We say that A is Gamma-independent of B over E , written $A \downarrow_E^\Gamma B$ if the following conditions holds:

1. $K(A) \downarrow_{K(E)}^f K(B)$.
2. $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$.
3. For all $b_1, \dots, b_n \in V(B)$ linearly independent over E and $a_1, \dots, a_m \in V(A)$ linearly independent over E the tuple $([a_i, b_j])_{i \leq m, j \leq n}$ is f -generic over $K(A)K(B)$.

Remark 3.14. Using Lemma 3.12 it is easy to show that when A and B have finite dimension over E it is enough to check 3 for some basis $b_1, \dots, b_n, a_1, \dots, a_m$ of B and A over E . In fact changing the basis of A over E consists of applying a transformation of the form $(x_i)_{i < n} \rightarrow M \times (x_i)_{i < n} + (k_i)_{i < n}$ with $M \in GL_n(K(A))$ and $k_i \in \text{Lin}_{K(A)}(V(E))$.

A similar result holds when the dimension is infinite: If for some basis $(a_i)_{i < \kappa_A}$ of A over E and $(b_i)_{i < \kappa_B}$ of B over E the tuple $([a_i, b_j])_{i < \kappa_A, j < \kappa_B}$ is f -generic over $K(A)K(B)$ then for any basis $(a'_i)_{i < \kappa_A}$ of A over E and $(b'_i)_{i < \kappa_B}$ of B over E the tuple $([a'_i, b'_j])_{i < \kappa_A, j < \kappa_B}$ is f -generic over $K(A)K(B)$.

Proof. Let $(a_i)_{i < \kappa_A}, (a'_i)_{i < \kappa_A}, (b_i)_{i < \kappa_B}$ be as in the statement (it is enough to show that we can change one of the two basis, once this is done we can change the other one). We begin by showing that any finite subtuple of $([a'_i, b_j])_{i < \kappa_A, j < \kappa_B}$ is f -generic over $K(A)K(B)$.

Consider $(a'_{j'_k})_{k < n'}$ for $n' < \omega$, $(j'_k)_{k < n'} \in \kappa_A$ a finite part of $(a'_i)_{i \leq \kappa_A}$. We can find some $n < \omega$ and $(j_k)_{k < n} \in \kappa_A$ such that:

$$\langle E(a'_{j'_k})_{k < n'}(a_{j_k})_{k < n} \rangle = \langle E(a_{j_k})_{k < n} \rangle,$$

and that $(a'_{j'_k})_{k < n'}(a_{j_k})_{n' \leq k < n}$ is a basis of $\langle E(a_{j_k})_{k < n} \rangle$ over E .

By the finite dimensional case we know that the tuple $([a'_{j'_k}, b_{l_k}])_{k' < n', k < m}$ is f -generic over $K(A)K(B)$ for any $m < \omega$ and $(l_k)_{k < m} \in \kappa_B$. Since this holds for any finite parts of $(a'_i)_{i < \kappa_A}$ and $(b_i)_{i < \kappa_B}$ by finite character of \downarrow^f we get that the tuple $([a'_i, b_j])_{i < \kappa_A, j < \kappa_B}$ is f -generic over $K(A)K(B)$. \square

Proposition 3.15. *The relation of Gamma-independence is invariant, transitive, symmetric, satisfies existence, monotonicity, finite character and extension.*

Proof. It is clear that \downarrow^Γ satisfies invariance, monotonicity, existence and symmetry. We begin by showing that \downarrow^Γ satisfies finite character.

Assume that for every finitely generated structures over E $A_0 \subseteq A$ and $B_0 \subseteq B$ we have $A_0 \downarrow_E^\Gamma B_0$. Then clearly $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$. $K(A) \downarrow_{K(E)}^f K(B)$ follows from the finite character of \downarrow^f . The other condition of \downarrow^Γ follows from the same fact:

If $b_1, \dots, b_n \in V(B)$ are linearly independent over E and $a_1, \dots, a_m \in V(A)$ are linearly independent over E then by assumption the tuple $([a_i, b_j])_{i \leq m, j \leq n}$ is f -generic over $K(A_0)K(B_0)$ for any $A_0 \subseteq A$ and $B_0 \subseteq B$ finitely generated structures over E containing $(a_i)_{i \leq m}$ and $(b_j)_{j \leq n}$ respectively. In particular for any finitely generated subfield $K \subseteq K(A)K(B)$ the tuple $([a_i, b_j])_{i \leq m, j \leq n}$ is f -generic over K , so it is f -generic over $K(A)K(B)$ and $A \downarrow_E^\Gamma B$.

We now prove the transitivity of \downarrow^Γ . Let $E \subseteq A, B$ and $B \subseteq C$. By finite character we can assume that A is finitely generated over E and that C is finitely generated over B . Let $(e_i)_{i < \kappa_E}$ be a basis of $V(E)$, we complete it by $(a_i)_{i < \kappa_A}$ and $(b_i)_{i < \kappa_B}$ into some basis of $V(A)$ and $V(B)$ respectively, and complete $(e_i)_{i < \kappa_E}(b_i)_{i < \kappa_B}$ by $(c_i)_{i < \kappa_C}$ into a basis of $V(C)$. We assume that $A \downarrow_E^\Gamma C$ and show that $AB \downarrow_B^\Gamma C$. Clearly $\langle AB \rangle \cap \langle C \rangle = \langle B \rangle$.

By assumption $K(C) \downarrow_{K(E)}^f K(A)K(B)$, so by base monotonicity $K(C) \downarrow_{K(B)}^f K(A)K(B)$. $K(AB)$ is generated in \mathcal{L}_K by $K(A)K(B)[a_i, b_j]_{i < \kappa_A, j < \kappa_B}$, and by assumption

$$[a_i, b_{j_k}]_{i < \kappa_A, k < m} \downarrow_{\emptyset}^K K(A)K(C)$$

for any $m < \omega$ and $(j_k)_{k < m} \in \kappa_B$ so, by finite character,

$$[a_i, b_j]_{i < \kappa_A, j < \kappa_B} \downarrow_{\emptyset}^K K(A)K(C).$$

Then using base monotonicity we get:

$$K(A)K(B)[a_i, b_j]_{i < \kappa_A, j < \kappa_B} \downarrow_{K(A)K(B)}^K K(A)K(C),$$

and using transitivity we get that $K(AB) \downarrow_{K(B)}^f K(C)$.

By assumption the tuples $[a_i, c_j]_{i < \kappa_A, j < \kappa_C}$ for any $m < \omega$ and $(j_k)_{k < m} \in \kappa_C$ are f -generic over $K(A)K(C)$. By 3. of Lemma 3.12 this implies that the tuple $[a_i, c_j]_{i < \kappa_A, j < \kappa_C}$ is f -generic over $K(A)K(C)[a_i, b_{j_k}]_{i < \kappa_A, k < m'}$, and finite character entails that $[a_i, c_j]_{i < \kappa_A, j < \kappa_C}$ is f -generic over $K(AB)K(C)$, so $AB \downarrow_B^\Gamma C$.

We now assume that $A \downarrow_E^\Gamma B$ and $AB \downarrow_B^\Gamma C$ and we show that $A \downarrow_E^\Gamma C$. Clearly $\langle A \rangle \cap \langle C \rangle = \langle E \rangle$.

By assumption $K(AB) \downarrow_{K(B)}^f K(C)$ and $K(A) \downarrow_{K(E)}^f K(B)$, so $K(A) \downarrow_{K(E)}^f K(C)$. By assumption also the tuple $[a_i, c_j]_{i < \kappa_A, j < \kappa_C}$ is f -generic over $K(AB)K(C)$ which is generated by $K(A)K(B)[a_i, b_j]_{i < \kappa_A, j < \kappa_B}K(C)$ in \mathcal{L}_K , and $[a_i, b_j]_{i < \kappa_A, j < \kappa_B}$ is f -generic over $K(A)K(B)$, so by Lemma 3.12 the tuple

$$[a_i, b_j]_{i < \kappa_A, j < \kappa_B} [a_i, c_j]_{i < \kappa_A, j < \kappa_C}$$

is f -generic over $K(A)K(C)$ and $A \downarrow_E^\Gamma C$.

We now show that \downarrow^Γ satisfies extension. Let $E \subseteq A, B$, let $(e_i)_{i < \kappa_E}$ be a basis of $V(E)$, we complete it by $(a_i)_{i < \kappa_A}$ and $(b_i)_{i < \kappa_B}$ into some basis of $V(A)$ and $V(B)$ respectively. Let \bar{a}_k be a tuple enumerating $K(A)$. Let $\bar{a}'_k \equiv_{K(E)} \bar{a}_k$ be such that $\bar{a}'_k \downarrow_{K(E)}^f K(B)$. We write $\alpha_{i,j}$ for the element of \bar{a}'_k corresponding to $[a_i, a_j] \in \bar{a}_k$ via this isomorphism, similarly for $\gamma_{i,j}$ and $[a_i, e_j] \in \bar{a}_k$.

Let $(\beta_{i,j})_{i < \kappa_A, j < \kappa_B}$ be an f -generic tuple over $\bar{a}'_k K(B)$. At this point we have collected all of the elements of the field sort that we want, and we just need to find the vector sort. By Lemma 3.1 we can find some vectors $(a'_i)_{i < \kappa_A}$ linearly independent over $\langle B \rangle$ such that:

- $[a'_i, e_j] = \gamma_{i,j}$ for all $i < \kappa_A, j < \kappa_E$.
- $[a'_i, a'_j] = \alpha_{i,j}$ for all $i, j < \kappa_A$.
- $[a'_i, b_j] = \beta_{i,j}$ for all $i < \kappa_A, j < \kappa_B$.

Let A' be the structure defined the following way: $K(A') = \bar{a}'_k$ and $(a'_i)_{i < \kappa_A}$ is a basis of $V(A')$ over $V(E)$. Then $A' \equiv_E A$ by quantifier elimination (sending a_i to a'_i and $K(A)$ to $K(A')$ via the isomorphism $\bar{a}_k \rightarrow \bar{a}'_k$ in the field sort gives us the isomorphism we want). $K(A') \downarrow_{K(E)} K_B$ and $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$. The last condition of \downarrow^Γ follows from Remark 3.14. \square

Proposition 3.16. *Let E be a structure and $p \in S(K(E))$ be a type in the field sort that is an amalgamation base in \mathbb{K} . If A^0, A^1 and B^0, B^1 are structures containing E such that $\models p(K(A^0))$, $A^0 \equiv_E A^1$, $A^i \downarrow_E^\Gamma B^i$ for $i = 0, 1$ and $B^0 \downarrow_A^\infty B^1$. Then there is A such that $A \equiv_{B^j} A^j$ for $j = 0, 1$ and $A \downarrow_E^\Gamma B^0 B^1$. So the relation of Gamma-independence satisfies type amalgamation over models.*

Proof. Let E and A^j, B^j for $j = 0, 1$ be as in the statement. Let $(e_i)_{i < \kappa_E}$ be a basis of $V(E)$, we complete it by $(a_i^0)_{i < \kappa_A}$ and $(b_i^j)_{i < \kappa_{B^j}}$ into some basis of $V(A^0)$ and $V(B^j)$ for $j = 0, 1$ respectively. Let $(a_i^1)_{i < \kappa_A}$ be the basis of A^1 over E corresponding to $(a_i^0)_{i < \kappa_A}$ through the isomorphism $A^0 \equiv_E A^1$. Let k_A^0 enumerate $K(A^0)$ and let k_A^1 be the corresponding enumeration of $K(A^1)$. Since the isomorphism is an isomorphism of \mathcal{L} -structures it sends $[a_j^0, a_j^0]$ to $[a_j^1, a_j^1]$ for any $i, j < \kappa_A$ and similarly for $[a_j^0, e_j]$ and $[a_j^1, e_j]$.

Then $k_A^0 \equiv_{K(E)} k_A^1$, $k_A^j \downarrow_{K(E)}^f K(B^j)$ for $j = 0, 1$ and $K(B^0) \downarrow_{K(E)}^f K(B^1)$. By type amalgamation in T_K there is $k \models \text{tp}(k_A^0/K(B^0)) \cup \text{tp}(k_A^1/K(B^1))$ such that $k \downarrow_{K(E)}^f K(B^0)K(B^1)$. By extension we can assume that $k \downarrow_{K(E)}^f K(B^0 B^1)$.

By assumption the tuple $([a_i^0, b_l^0])_{i < \kappa_A, j < \kappa_{B^0}}$ is f -generic over $K(A^0)K(B^0)$. By extension there is $(\beta_{i,j}^0)_{i < \kappa_A, j < \kappa_{B^0}}$ such that $k(\beta_{i,j}^0)_{i < \kappa_A, j < \kappa_{B^0}} \equiv_{K(B^0)} K(A^0)([a_i^0, b_l^0])_{i < \kappa_A, j < \kappa_{B^0}}$ and $(\beta_{i,j}^0)_{i < \kappa_A, j < \kappa_{B^0}}$ is f -generic over $kK(B^0 B^1)$.

By the same argument there is a tuple $(\beta_{i,j}^1)_{i < \kappa_A, j < \kappa_{B^1}}$ such that:

$$k(\beta_{i,j}^1)_{i < \kappa_A, j < \kappa_{B^1}} \equiv_{K(B^1)} K(A^1)([a_i^1, b_l^1])_{i < \kappa_A, j < \kappa_{B^1}},$$

and $(\beta_{i,j}^1)_{i < \kappa_A, j < \kappa_{B^1}}$ is f -generic over $k'K(B^0 B^1)(\beta_{i,j}^0)_{i < \kappa_A, j < \kappa_{B^0}}$. Then by Lemma 3.12 the tuple $(\beta_{i,j}^0)_{i < \kappa_A, j < \kappa_{B^0}} (\beta_{i,j}^1)_{i < \kappa_A, j < \kappa_{B^1}}$ is f -generic over $kK(B^0 B^1)$. Let $\gamma_{i,j}$ be the element of k corresponding to $[a_i^0, e_j]$ for every $i < \kappa_A, j < \kappa_E$ and $\alpha_{i,j}$ the one corresponding to $[a_i^0, a_j^0]$ for every $i < \kappa_A, j < \kappa_A$ through the isomorphism $k \equiv_{K(B^0)} k_A^0$ (notice that using A^1 instead gives the same elements of k).

By Lemma 3.1 we can find some vectors $(a_i)_{i < \kappa_A}$ linearly independent over $\langle B^0 B^1 \rangle$ such that:

- $[a_i, e_j] = \gamma_{i,j}$ for all $i < \kappa_A, j < \kappa_E$.
- $[a_i, a_j] = \alpha_{i,j}$ for all $i, j < \kappa_A$.
- $[a_i, b_j^0] = \beta_{i,j}^0$ for all $i < \kappa_A, j < \kappa_{B^0}$.
- $[a_i, b_j^1] = \beta_{i,j}^1$ for all $i < \kappa_A, j < \kappa_{B^1}$.

Let A be the structure defined by $K(A) = k$ and $(a_i)_{i < \kappa_A}$ is a basis of $V(A)$ over $V(E)$. Then $A \equiv_{B^i} A^i$ for $i = 0, 1$ by quantifier elimination (sending a_j to a_j^i and $K(AB^i)$ to $K(A^i B^i)$ via the isomorphism $k(\beta_{j,k}^i)_{j < \kappa_A, k < \kappa_{B^i}} \equiv_{K(B)} k_A^i([a_j^i, b_l^i])_{j < \kappa_A, l < \kappa_{B^i}}$ in the field sort gives us the isomorphism we want). $K(A) \downarrow_{K(E)}^f K(B)$ and $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$. The fact that $A \downarrow_E^\Gamma B^0 B^1$ follows from Remark 3.14. \square

Proposition 3.17. $A \downarrow_E^\Gamma B$ implies $A \downarrow_E^f B$ for all $E \subseteq A, B$. So the theory sT_∞^K satisfies existence.

Proof. By finite character it is enough to prove this for $V(A)$ and $V(B)$ of finite dimension over $V(E)$. Since \downarrow^Γ satisfies extension it is enough to show that $A \downarrow_E^\Gamma B$ implies $A \downarrow_E^d B$. Let $(e_i)_{i < \kappa_E}$ be a basis of $V(E)$, we complete it by $(a_i)_{i < n}$ into a basis of $V(A)$. Let $(B_i)_{i < \omega}$ be an E -indiscernible sequence with $B_0 = B$. We want to find an A' such that $A' B_i \equiv_E A B_0$ for all $i < \omega$.

We find a negative part and extend the sequence to a sequence $(B_i)_{i \in \mathbb{Z}}$ indexed on the relative integers. By extension of \downarrow^Γ we can assume that $A \downarrow_E^\Gamma B_\mathbb{Z}$, where $B_\mathbb{Z}$ is the structure generated by $\{B_i : i \in \mathbb{Z}\}$. We will write $B_{[i,j]}$ for the structure generated by $B_i \dots B_j$ for every $i < j \in \mathbb{Z}$. $A \downarrow_{EB_{<0}}^\Gamma B_\mathbb{Z}$ by base monotonicity of \downarrow^Γ . We define $E' := B_{<0}$, by indiscernibility the B_i for $i \in \omega$ are linearly disjoint over E' and they form an E' indiscernible sequence. In fact if some vectors $b_i \in B_i$ is in $\langle B_{<i} \rangle$ then by indiscernibility it is in $\langle B_{<0} \rangle$.

Thanks to this we can assume without loss of generality (eventually replacing E by E') that the sequence $(B_i)_{i < \omega}$ is linearly disjoint over E . What happens here is that the elements of $(B_i)_{i < \omega}$ might overlap outside of the original $\langle E \rangle$ but since the property of being linearly disjoint has local character, by indiscernibility, all the overlapping that might happen is already happening inside of the negative part of the sequence. Let $(b_i^0)_{i < n}$ be a basis of $V(B^0)$ over $V(E)$, and let $(b_i^i)_{j < n}$ be the corresponding basis of $V(B^i)$.

Let k be a tuple enumerating $K(A)$ and $p(x, y, z) = \text{tp}(k, ([a_i, b_j^0])_{i < n, j < m}, K(B_0)/K(E))$. By assumption $k([a, b_j^0])_{j < n} \downarrow_{K(E)}^f K(B_0)$, so there is a tuple $k', (\beta_{i,j})_{i < n, j < m}$ such that

$$k', (\beta_{i,j})_{i < n, j < m} \models \bigcup_{i < \omega} p(x, y, K(B_i)).$$

Write $\gamma_{i,j}$ the element of k' corresponding to $[a_i, e_j] \in k$ for all $i < n, j < \kappa_E$ and $\alpha_{i,j}$ the element of k' corresponding to $[a_i, a_j] \in k$ for all $i, j < n$. By Lemma 3.1 we can find some vectors $(a'_i)_{i < n}$ linearly independent over $\langle B_{<\omega} \rangle$ such that:

- $[a'_i, e_j] = \gamma_{i,j}$ for all $i < \kappa_A, j < \kappa_E$.
- $[a'_i, a'_j] = \alpha_{i,j}$ for all $i, j < \kappa_A$.
- $[a'_i, b_l^j] = \beta_{i,j}$ for all $i < n, j < \omega$ and $l < m$.

Let A' be the structure defined by $K(A') = k'$ and $V(A') = \text{Lin}_{k'}((a'_i)_{i < n})$. By quantifier elimination $A' B_i \equiv_E A B_0$ for all $i < \omega$, which concludes our proof. \square

3.3 Forcing Base Monotonicity of Kim-Independence

In Chapter 2 we gave some example of NSOP₁ theories such that $\downarrow^{K^M} = \downarrow^f$. It was conjectured that this holds in any NSOP₁ theory. A counter-example was given in [Bos23] in sT_∞^{ACF} .

This counter-example is partial in the following sense: sT_∞^{ACF} does not eliminate imaginaries, and in the imaginary extension $sT_\infty^{ACF,eq}$ this example is not a counter-example anymore.

We begin by presenting the said counter example and then we fill this gap by showing that $\downarrow^{K^M} \neq \downarrow^f$ in $sT_\infty^{ACF,eq}$. For this we use a result of weak elimination of imaginaries of Dobrowolski to characterize $sT_\infty^{ACF,eq}$. We will write \downarrow^{ACF} for forking independence in ACF .

3.3.1 A counter-example for the home sorts

Consider $E \models sT_\infty^{ACF}$ a small model and let $(e_i)_{i < \omega}$ be a $K(E)$ -basis of $V(E)$. Let a, b_0, b_1 be vectors linearly independent over $\langle E \rangle$ such that $a, b_0, b_1 \in E^\perp$, $[a, a] = [b_0, b_0] = [b_1, b_1] = 0$ and $[a, b_0] = [a, b_1] = [b_0, b_1] = 1$.

Let $A = (\text{Lin}_{K(E)}(V(E), a), K(E))$ and $B = (\text{Lin}_{K(E)}(V(E), b_0, b_1), K(E))$. Then B is a model of sT_∞^{ACF} and $K(AB) = K(E)$ where $K(AB)$ is the field sort of the structure generated by A and B . Both A and B are algebraically closed and contain E , and by definition we have $A \downarrow_E^K B$. If B_0 is such that $E \subseteq B_0 \subseteq B$ then B_0 can be E and B , and in these two cases $AB_0 \downarrow_{B_0}^K B$ holds trivially.

Otherwise $B_0 = Eb$ is generated over E by a single vector $b \in B$. In that case this vector b can be written as $b = \lambda_0 \cdot b_0 + \lambda_1 \cdot b_1$ for $\lambda_0, \lambda_1 \in K(E)$. Then $AB_0 = (\text{Lin}_{K(E)}(V(E), a, b), K(E))$ and $AB_0 \downarrow_{B_0}^K B$. In all of these cases $AB_0 \downarrow_{B_0}^K B$, so $A \downarrow_E^{K^M} B$.

Now let us consider some $\alpha \in \mathbb{K}$ transcendental over $K(E)$ and $B' := (\text{Lin}_{K(E)[\alpha]}(V(E), b_0, b_1), K(E)[\alpha])$. We show that there is no $A' \equiv_B A$ such that $A' \downarrow_E^{K^M} B'$, which implies that \downarrow^{K^M} does not satisfy extension.

If there were such an A' , setting $B'_0 = (\text{Lin}_{K(E)}(V(E), \alpha \cdot b_0, \alpha^{-1} \cdot b_1), K(E))$ we would have $A'B'_0 \downarrow_{B'_0}^K B'$, so $[a', \alpha \cdot b_0] \downarrow_{K(E)}^f \alpha$ and $\alpha \downarrow_{K(E)}^f \alpha$ so $\alpha \in K(E)$, a contradiction. This shows us that in an NSOP₁ theory with existence \downarrow^{K^M} is not necessary equal to \downarrow^f over models.

As we mentioned sT_∞^{ACF} does not weakly eliminate imaginaries: In fact the equivalence relation between n -tuples of vector defined by $E(\bar{x}, \bar{y})$ if and only if \bar{x} and \bar{y} generate the same vector space is not eliminated. We will write $\langle \bar{u} \rangle$ for the E -class of \bar{u} . If we consider Kim-independence in the imaginary extension then $A \downarrow_E^{K^M} B$ does not hold for the previous E, A, B :

Let $\langle b_0 \rangle$ be the line generated by b_0 seen as an imaginary. Then $A \langle b_0 \rangle \not\downarrow_{E \langle b_0 \rangle}^K B$ since $b_0 \in \text{acl}(A, \langle b_0 \rangle)$ and $b_0 \notin \text{acl}(E, \langle b_0 \rangle)$: In fact b_0 is the only point x of the line $\langle b_0 \rangle$ that satisfies $[a, x] = 1$.

3.3.2 A counter-example for $sT_\infty^{ACF,eq}$

We consider the imaginary expansion $sT_\infty^{ACF,eq}$ of sT_∞^{ACF} . As we mentioned in the previous subsection sT_∞^{ACF} does not weakly eliminate the equivalence classes of the relations ‘generating the same vector subspace’. We add additional sorts for the equivalence classes of these relations and write $sT_{\infty, Gr}^{ACF}$ for this theory in the extended language.

The following result is due to Dobrowolski.

Proposition 3.18. $sT_{\infty, Gr}^{ACF}$ has weak elimination of imaginaries.

Proof. The proof was kindly communicated to me by Dobrowolski (personal communication). \square

We now show that \downarrow^{K^M} does not satisfy extension in $sT_\infty^{ACF,eq}$, which yields that $\downarrow^{K^M} \neq \downarrow^f$.

Let $a, b_0 \in \mathbb{V}$ and $\alpha \in \mathbb{K}$ be such that $[b_0, b_0] = 1$, $[a, a] = 0$, $[a, b_0] = \alpha$ and α is transcendental over the prime field.

It is easy to see that $\alpha, a \downarrow_{\emptyset}^{K^M} b_0$, in fact by weak elimination of imaginaries all of the elements in $acl^{eq}(b_0) \setminus acl^{eq}(\emptyset)$ are interalgebraic with b_0 or $\langle b_0 \rangle$, and these two elements are interalgebraic since b_0 is one of the two elements $x \in \langle b_0 \rangle$ such that $[x, x] = 1$.

Let b_1 be such that $b_1 \notin \langle b_0 \rangle$, $[b_1, b_1] = 0$ and $[b_0, b_1] = 0$. We show that there is no $\alpha', a' \equiv_{b_0} \alpha, a$ such that $\alpha', a' \downarrow_{\emptyset}^{K^M} b_0 b_1$. Otherwise let α', a' be such a tuple.

If $[a', b_1] = 0$, then $\alpha' a' \not\downarrow_{\langle b_0 b_1 \rangle}^K b_0 b_1$: The set $b_0 + \langle b_1 \rangle$ can be seen as the class of (b_0, b_1) for the relation $(x, y)E(x', y') := (\langle y \rangle = \langle y' \rangle) \wedge (x - x' \in \langle y \rangle)$. This set is definable over $a' \alpha' \langle b_0 b_1 \rangle$ as $\{x : x \in \langle b_0 b_1 \rangle \text{ and } [x, a'] = \alpha'\}$, so as an imaginary it is in the algebraic closure of $a' \alpha' \langle b_0 b_1 \rangle$. However $b_0 + \langle b_1 \rangle$ is not in the algebraic closure of $\langle b_0 b_1 \rangle$ since $b_0 + \langle b_1 \rangle \equiv_{\langle b_0 b_1 \rangle} (b_0 + \lambda \cdot b_1) + \langle b_1 \rangle$ for all $\lambda \in \mathbb{K}$.

Otherwise assume that $[a', b_1] = \beta' \neq 0$. We show that $\alpha', a' \not\downarrow_{\langle b_0 b_1 \rangle, b_1}^K b_0 b_1$. Then $a'^{\perp} \cap \langle b_0 b_1 \rangle = \langle \beta' \cdot b_0 - \alpha' \cdot b_1 \rangle$. We can define the set of the points x in this line that satisfy $[x, x] = 1$. This set is finite and contains $b_0 + \frac{\alpha'}{\beta'} \cdot b_1$. Since $\alpha', \beta', b_1 \in acl(\alpha' a' \langle b_0 b_1 \rangle b_1)$ we deduce that $b_0 \in acl(\alpha' a' \langle b_0 b_1 \rangle b_1)$. However b_0 is not in the algebraic closure of $\langle b_0 b_1 \rangle b_1$. In fact $b_0 \equiv_{\langle b_0 b_1 \rangle b_1} b_0 + \lambda \cdot b_1$ for all $\lambda \in \mathbb{K}$.

Remark 3.19. This theory is to my knowledge the only example of NSOP₁ theory with existence that does not satisfy that $\downarrow^f = \downarrow^{K^M}$. I should also mention that there is not yet a characterization of forking independence in this theory.

Chapter 4

Groups and fields

Definition 0.1. c ————— b ————— a ————— x ————— y ————— z

that
of a set of tuples
we can arrange
consists
as below
group configuration
and
line are forking independent
inter-algebraic
so that two points in
over
A
the same
the third one.

In this chapter we study some examples of groups definable and type-definable in NSOP₁ theories. We exhibit some behaviors of these groups that differ from the ones definable in simple theories. We take interest in the notions of generics and stabilizers, and define the Kim-stabilizer.

We apply the notion of Kim-stabilizer and the stabilizer from [Hru12] to the context of a group G definable in an NSOP₁ field F satisfying some assumptions to show that there is a finite to one embedding of a type-definable subgroup of G of bounded index into an algebraic group over F . We then show that definable groups in ω -free PAC fields satisfy these conditions.

4.1 NSOP₁ groups

4.1.1 Kim-generics

In this section we consider the notion of generic from stability and simplicity theory and try to see if we can find an analogy in the context of NSOP₁. The answer to this question seems to be negative: we give here some examples of ‘bad’ behavior of NSOP₁ groups. The main example we consider in this section is the model completion of the theory of abelian groups with a generic unary function f such that $f(0) = 0$. We denote this theory by $T_{f,ab}$.

This theory is constructed in the Chapter 2 of this manuscript (see Remark 2.48). It is an NSOP₁ theory with existence and has quantifier elimination in the language $L = \{+, 0, f\}$. Let \mathbb{M} be a monster model of $T_{f,ab}$, we recall some of the properties of \mathbb{M} :

1. Remark 2.7: The algebraic closure of any set $A \subseteq \mathbb{M}$ is equal to the structure generated by A .
2. Corollary 2.12: Kim-forking independence coincides with algebraic independence \perp^a .
3. Proposition 2.11: There is an independence relation \perp^Γ that is invariant, stationary, satisfies existence, extension, transitivity, base monotonicity and symmetry.
4. (5) in Definition 2.1: For any subset $A \subseteq \mathbb{M}$ there is a type $p_A \in S(A)$ that satisfies the following property: Let $a \models p$ and let $\psi : A \rightarrow B$ be morphism of L -structures from A to $B \subseteq \mathbb{M}$ and let $b \in B$ be any element. Then there is a unique extension of ψ into a morphism from $\langle A, a \rangle$ to B sending a to b . p_A is called the Γ -generic type over A .

Remark 4.1. For any subsets $A \subseteq B \subseteq \mathbb{M}$ the type p_B is the unique non- Γ forking extension of p_A to B , in particular p_A is the non- Γ forking extension of p_\emptyset to A .

We now give the ‘naive’ definition of a Kim-generic element of a group.

Definition 4.2. Let (G, \cdot) be a group definable over E in a model M of an NSOP₁ theory with existence. For $E \subseteq B$ a set of parameters we say that an element $a \in G$ is *Kim-generic in G over B* if $a \cdot b \perp_B^K b$ for every $b \in G$ such that $a \perp_B^K b$.

Remark 4.3. The group $(\mathbb{M}, +)$ does not have Kim-generics: let $a \in \mathbb{M}$, and consider b such that $a \perp_{\langle 0 \rangle}^a b$ with the function f being equal to 0 everywhere in $\langle a, b \rangle \setminus \langle a \rangle$ except that $f(a + b) = b$. Then $a \perp_{\langle 0 \rangle}^a b$ and $a + b \not\perp_{\langle 0 \rangle}^a b$, so a is not Kim-generic. This behavior was already observed in the theory sT_∞^{ACF} in [Dob23, Proposition 8.14]. A similar counter-example can be constructed with a model as the basis (i.e. instead of $\langle 0 \rangle$ here).

Proposition 4.4. *There is a global type p in \mathbb{M} which is invariant under translation and does not Γ -fork over $\langle 0 \rangle$, meaning that if $g \models p|_A$ for a set A and if $a \in A$ then $g + a \models p|_A$ and $g + a \perp_{\langle 0 \rangle}^\Gamma A$.*

Proof. Consider first the Γ -generic type $p_{\langle 0 \rangle}$ over $\langle 0 \rangle$. We define p as its global non- Γ -forking extension $p = p_{\mathbb{M}}$. By definition this type does not Γ -fork over $\langle 0 \rangle$. We now show that it is invariant under translations. Let A be a set of parameters, $a \in A$ and $g \models p|_A$.

By base monotonicity $g + a \downarrow_a^\Gamma A$, so by transitivity and stationarity of \downarrow^Γ it is enough to show that $g + a \downarrow_{\langle 0 \rangle}^\Gamma a$ and that $g + a \equiv_{\langle 0 \rangle} g$. For the second point by quantifier elimination it is enough to show that there is a morphism $\varphi : \langle g + a \rangle \rightarrow \langle g \rangle$ sending $g + a$ to g : In fact then the morphism sending g to $g + a$ (which exists by the Γ -genericity of g) is its inverse so these two structures are isomorphic.

We know that $g \downarrow_{\langle 0 \rangle}^\Gamma a$. By genericity of g there is a morphism sending g to $g + a \in \langle g, a \rangle$. Then \downarrow^Γ -independence implies that there is a morphism $\psi : \langle a, g + a \rangle \rightarrow \langle a, g \rangle$ that sends a to a and $g + a$ to g . We can consider the restriction of ψ to $\langle g + a \rangle$ which is the morphism we want.

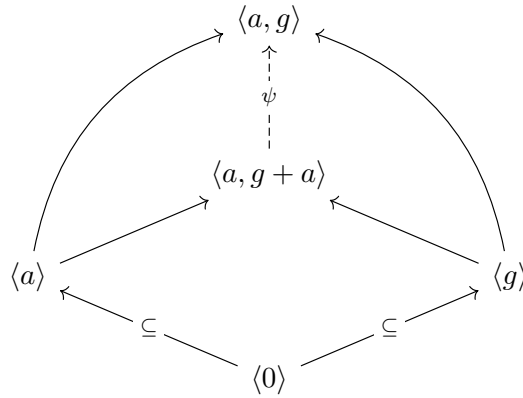


Figure 4.1: The \downarrow^Γ -generic

Now we show that $g + a \downarrow_{\langle 0 \rangle}^\Gamma a$. For this we consider two morphisms $\psi_1 : \langle g + a \rangle \rightarrow D$ and $\psi_2 : \langle a \rangle \rightarrow D$ and we show that they factorize through $\langle g + a, a \rangle = \langle g, a \rangle$. Consider the morphism $\psi_0 : \langle g \rangle \rightarrow D$ sending g to $\psi_1(g + a) - \psi_2(-a)$, since $g \downarrow_{\langle 0 \rangle}^\Gamma a$ there is a morphism $\psi : D \rightarrow \langle g, a \rangle$ factorizing ψ_0 and ψ_1 through D , and it is easy to see that it also factorizes ψ_2 . \square

Lemma 4.5. *Let T be any theory with a definable group G . Consider a formula $\varphi(x, b)$ that forks over a set of parameters A . Let $g \downarrow_A^f b$, then the formula $\varphi(g \cdot x, b)$ forks over A .*

Proof. If $\varphi(g \cdot x, b)$ does not fork over A there is some $a \models \varphi(g \cdot x, b)$ such that $a \downarrow_A^f g, b$. Then $g, a \downarrow_{A, g}^f g, b$, and since $g \downarrow_A^f b$ by transitivity we have $g \cdot a \downarrow_A^f b$, contradicting that $\varphi(x, b)$ forks over A . \square

Remark 4.6. This does not hold for Kim-forking, even under stronger conditions: Let $b \in \mathbb{M}$, consider the formula $\varphi(x, b) := f(x) = b$. This formula Kim-divides over $\langle 0 \rangle$. Let $g \models p|_b$, with p the type from Proposition 4.4. The formula $\varphi(g + x, b)$ Kim-divides over g , however it does not Kim-divide over $\langle 0 \rangle$.

To make things simpler assume that f is the identity on $\langle b \rangle$. Consider a such that $a \notin \langle b, g \rangle$, $a + a = 0$, and f is the identity on the group generated by $a, \langle b, g \rangle$ with the exception of $f(g + a) = b$ (this defines in fact a structure in K). Then $a \downarrow_{\langle 0 \rangle}^K g, b$ and $\models \varphi(g + a, b)$ (also $a \downarrow_b^K g, g \downarrow_{\langle 0 \rangle}^\Gamma b$ and $a \not\downarrow_g^K b$).

To sum up in an NSOP₁ group, translating a Kim-forking or non-Kim-forking formula by some independent parameter can either give a Kim-forking or a non-Kim-forking formula.

4.1.2 The Kim-Stabilizer

4.1.2.1 Strengthenings of the Independence Theorem in NSOP₁

We know that NSOP₁ theories with existence satisfy 3-amalgamation for Lascar strong types. We will write $a \equiv_E^L b$ to mean that a and b have the same Lascar strong type over E .

Theorem 4.7. [DKR22, Theorem 5.6] *Let $b \downarrow_E^K c$ and $a \equiv_E^L a'$ with $a \downarrow_E^K b$ and $a' \downarrow_E c$. Then $\text{tp}(a/Eb) \cup \text{tp}(a'/Ec)$ does not Kim-fork over E .*

In this subsection we prove some strengthenings of this result that we will use in the following subsection. For this we begin by recalling some facts about Kim-independence in NSOP₁ theories. It is shown in [Bos22, Theorem 2.5.10] that Theorem 4.7 holds in hyperimaginaries, this with the fact that $a \downarrow_E^K b$ if and only if $\text{bdd}(Ea) \downarrow_{\text{bdd}(E)}^K \text{bdd}(Eb)$ for all tuples $a, b, e \in \mathbb{M}^{\text{heq}}$ ([Bos22, Remark 2.2.13] and symmetry of \downarrow^K) implies that we can assume that the set of parameter E we consider in the amalgamation problem is boundedly closed.

Definition 4.8. Let $E \subseteq \mathbb{M}$ be a set of parameters and $p(x)$ be a type over E . We call p a *Kim-amalgamation base* if whenever p_1 and p_2 are non-Kim-forking extensions of p over some sets of parameters A and B respectively, with $A, B \subseteq \mathbb{M}^{\text{eq}}$ such that $A \downarrow_E^K B$, then the type $p_1 \cup p_2$ does not Kim-fork over E . Any complete type over a model or more generally a boundedly closed set is a Kim-amalgamation base: [Bos22, Theorem 2.5.8].

Proposition 4.9. [Kim21, Proposition 2.2] *Let $a_0 \downarrow_E^K B$ with $E \subseteq B$ and let $I = (a_i)_{i < \omega}$ be a Kim-Morley sequence over E . Then there is $I' \equiv_{Aa_0} I$ such that I' is Kim-Morley over B .*

Theorem 4.10. [CKR23, Theorem 2.8] *Let a_0 be a tuple and let $E \subseteq B$ be sets of parameters. Then $a_0 \downarrow_E^K B$ if and only if there exist $I = (a_i)_{i < \omega}$ a Kim-Morley sequence over E which is B -indiscernible.*

Using these results Kim shows the following version of the independence theorem:

Theorem 4.11. [Kim21, Theorem 2.7] *Let $b \downarrow_E^K c$ and $a \equiv_E^L a'$ with $a \downarrow_E^K b$ and $a' \downarrow_E c$. Then there is $\alpha \models \text{tp}(a/Eb) \cup \text{tp}(a'/Ec)$ such that $\alpha \downarrow_E^K bc$, $\alpha \downarrow_{Eb}^K bc$ and $\alpha \downarrow_{Ec}^K bc$.*

We begin with a strengthening of this result, which was already known, and we give a proof for the sake of completeness.

Proposition 4.12. *An NSOP₁ theory with existence satisfies the following version of the independence theorem: Whenever there is $E, b, c, \beta, \gamma \in \mathbb{M}$ such that $\beta \downarrow_E^K b$, $\gamma \downarrow_E^K c$, $\beta \equiv_E^L \gamma$ and $b \downarrow_E^K c$ there exists an α such that $\alpha \equiv_{Eb} \beta$, $\alpha \equiv_{Ec} \gamma$, $\alpha \downarrow_{Eb}^K bc$, $\alpha \downarrow_{Ec}^K bc$, $b \downarrow_{E\alpha}^K c$ and $\alpha \downarrow_E^K bc$.*

Proof. We begin by showing that we can replace b by a sequence $\tilde{B} = (\tilde{b}_i)_{i < \kappa_0}$ which is E -Morley and Kim-Morley over $E\beta$ and Ec . Consider an E -Morley sequence $B' = (b'_i)_{i < \kappa}$ such that $b'_0 = b$ for κ a large enough cardinal.

Since $\alpha \downarrow_E^K b'_0$ by Proposition 4.9 we can assume that $\beta \downarrow_E^K B'$ and that B' is Kim-Morley over $E\beta$. Similarly since $c \downarrow_E^K b'_0$ there is $B'' \equiv_{Ec} B'$ such that $B'' \downarrow_E^K c$ and that B'' is Kim-Morley over Ec . By applying the independence theorem to $\text{tp}(B'/E\beta)$ and $\text{tp}(B''/Ec)$ over E we get a new sequence $\tilde{B} = (\tilde{b}_i)_{i < \kappa}$ which is E -Morley and Kim-Morley both over $E\beta$ and Ec . Since $\tilde{B} \equiv_{Ec} B$ by moving β we can assume that $\tilde{b}_0 = b$, which concludes the first step.

Now we apply Theorem 4.11 to $\text{tp}(\beta/E\tilde{B})$ and $\text{tp}(\gamma/Ec)$. We find some $\alpha' \models \text{tp}(\beta/E\tilde{B}) \cup \text{tp}(\gamma/Ec)$ such that $\alpha' \downarrow_E^K \tilde{B}c$, $\alpha' \downarrow_{E\tilde{B}}^K \tilde{B}c$ and $\alpha' \downarrow_{Ec}^K \tilde{B}c$. By Proposition 1.2 we can find a sequence

$\tilde{B}' = (\tilde{b}'_i)_{i < \omega}$ which is $E\alpha'c$ -indiscernible and finitely based on \tilde{B} over $E\alpha'c$. Since \tilde{B} is Ec -Kim-Morley and $E\alpha'$ -Kim-Morley this implies that $\tilde{B}' \equiv_{Ec} \tilde{B}$ and $\tilde{B}' \equiv_{E\alpha'} \tilde{B}$. So we can find a sequence $B = (b_i)_{i < \omega}$ with $b_0 = b$ and α such that $\alpha B \equiv_{Ec} \alpha' \tilde{B}'$.

B is then an E -Morley sequence which is also Kim-Morley over $E\alpha$ and Ec . In particular the sequence $(b_i)_{1 \leq i < \omega}$ is Eb_0 -Morley and Eb_0c -indiscernible, by Theorem 4.10 this implies that $B \downarrow_{Eb_0}^K c$. Since $\alpha \downarrow_{EB}^K c$ transitivity yields that $\alpha \downarrow_{Eb}^K c$. We also have $\alpha \downarrow_{E\alpha}^K cb$. Since B is Kim-Morley over $E\alpha$ and $E\alpha c$ -indiscernible by Theorem 4.10 $b \downarrow_{E\alpha}^K c$, so α is the tuple we want. \square

Definition 4.13. We say that *Kim-amalgamation holds* between two sets of parameters B_0 and B_1 over a common subset E if for all a^0, a^1 such that $a^0 \equiv_E^L a^1$, $a^0 \downarrow_E^K B_0$ and $a^1 \downarrow_E^K B_1$, then there is an a such that $a \equiv_{B_i} a^i$ for $i = 0, 1$ and $a \downarrow_E^K B_0 B_1$.

We know that Kim-amalgamation over E holds between two sets whenever they are Kim-independent over E , that is just the usual independence theorem for Kim-forking. However in some NSOP₁ theories the condition that Kim-amalgamation between some sets of parameters B_0 and B_1 holds over E is weaker than the condition $B_0 \downarrow_E^K B_1$. That is the case in ω -free PAC fields for example with Lemma 4.24. The point of the following lemma is to show that under this weaker assumption we can still get the stronger conclusion of Theorem 4.11.

Lemma 4.14. *Assume that B_0 and B_1 are two sets containing E such that Kim-amalgamation holds between B_0 and B_1 over E . Then, for all a^0, a^1 such that $a^0 \equiv_E^L a^1$, $a^0 \downarrow_E^K B_0$ and $a^1 \downarrow_E^K B_1$, then there is an a such that $a \equiv_{B_i} a^i$ for $i = 0, 1$ and $a \downarrow_E^K B_0 B_1$ and additionally $a \downarrow_{EB_0}^K B_0 B_1$ and $a \downarrow_{EB_1}^K B_0 B_1$.*

Proof. This proof is just a restatement of the one of [Kim21, Theorem 2.7] where we make sure that we only use the independence theorem over the parameters B_0 and B_1 . Consider an E -Morley sequence $I_0 = (a_i^0)_{i < \kappa}$ with $a_0^0 = a^0$ and κ a large enough cardinal. Since $a^0 \downarrow_E B_0$ by Proposition 4.9 we can assume that $I_0 \downarrow_E^K B_0$ and that I_0 is Kim-Morley over B_0 .

Similarly we can find an E -Morley sequence $I_1 = (a_i^1)_{i < \kappa}$ such that $a_1^1 = a^1$, $I_1 \equiv_E I_0$, $I_1 \downarrow_E^K B_1$ and I_1 is Kim-Morley over B_1 . By assumption there is a sequence $I' = (a_i)_{i < \kappa}$ such that $I' \equiv_{B_i} I_i$ for $i = 0, 1$ and $I' \downarrow_E^K B_0 B_1$. By Proposition 1.2 we can find a sequence $I = (a_i)_{i < \omega}$ which is indiscernible over $B_0 B_1$ and finitely based on I' over $B_0 B_1$. Let $a := a_0$.

Then $I \downarrow_E^K B_0 B_1$, I is E -Morley, Kim-Morley over B_0 and over B_1 , and $a \equiv_{B_i} a^i$ for $i = 0, 1$. By Theorem 4.10 $a \downarrow_{B_0}^K B_1$ and $a \downarrow_{B_1}^K B_0$, so a is the tuple we want. \square

4.1.2.2 Defining the Kim-stabilizer

In this subsection we consider the notion of stabilizer: We give a definition extending the one of f -stabilizers in simple theories (see [Wag00, Chapter 4] on this topic) to the NSOP₁ context. We begin by recalling the definition of the f -stabilizer.

Definition 4.15. [Wag00, Definition 4.1.17] Let (G, \cdot) be a group definable over E in a model M of a theory with existence and let $p \in S(E)$ be a type concentrating in G . We define the following set:

$$S^f(p) := \{g : \exists a \models p, g \cdot a \models p, a \downarrow_E^f g \text{ and } g \cdot a \downarrow_E^f g\}.$$

For $a \in G$ we will write $S^f(a/E)$ to mean $S^f(\text{tp}(a/E))$. We denote by $\text{Stab}^f(p)$ the subgroup of G generated by $S^f(p)$, which we call the f -stabilizer of p .

Proposition 4.16. [Wag00, Proposition 4.1.21] Let (G, \cdot) be a type-definable group in a simple theory and p be a type concentrating in G . If p is Lascar-strong then $\text{Stab}^f(p) = S^f(p) \cdot S^f(p)$ is a type-definable subgroup of G .

Definition 4.17. Let (G, \cdot) be a group definable over E in a model M of an NSOP₁ theory with existence and let $p \in S(E)$ be a type concentrating in G . We define the following set:

$$S^K(p) := \{g : \exists a \models p, g \cdot a \models p, a \downarrow_E^K g \text{ and } g \cdot a \downarrow_E^K g\}.$$

For some $a \in G$ we will write $S^K(a/E)$ to mean $S^K(\text{tp}(a/E))$. We denote by $\text{Stab}^K(p)$ the subgroup of G generated by $S^K(p)$, which we call the *Kim-stabilizer* of p .

If $q \in S(E)$ is a non-Kim-forking extension of p then $S^K(q) \subseteq S^K(p)$. The set $S^K(p)$ is type-definable for a given complete type p , and it coincides with $S^f(p)$ in simple theories.

In fact if $a \models p$ then $S^K(p)$ is defined by the partial type:

$$\begin{aligned} \exists y(p(y) \cup p(x \cdot y) \cup \{\neg\varphi(x, y) : \varphi(x, a) \text{ Kim-forks over } E\} \\ \cup \{\neg\varphi(x, x \cdot y) : \varphi(x, a) \text{ Kim-forks over } E\}) \end{aligned}$$

Lemma 4.18. Let (G, \cdot) be a group type-definable over E in a model M of an NSOP₁ theory with existence and let $p \in S(E)$ be a Kim-amalgamation base concentrating in G . Consider $g, g' \in S^K(p)$ such that Kim-amalgamation holds between g and g' over E . Then $g \cdot g' \in S^K(p)$.

Proof. Let $g, g' \in S^K(p)$ as in the statement. By definition there are $a, a' \models p$ such that $g \cdot a, g' \cdot a' \models p$, $g \downarrow_E^K a$, $g' \downarrow_E^K a'$ and $g \cdot a \downarrow_E^K a$, $g' \cdot a' \downarrow_E^K a'$. By assumption we can apply Lemma 4.14 to $\text{tp}(a/g)$ and $\text{tp}(g' \cdot a'/g')$ to find some $a'' \models \text{tp}(a/g) \cup \text{tp}(g' \cdot a'/g')$ such that $a'' \downarrow_{Eg}^K g'$ and $a'' \downarrow_{Eg'}^K g$.

Then $g'^{-1} \cdot a''$ is the witness we want. In fact:

$$g'^{-1} \cdot a'' \models p \text{ and } g \cdot g' \cdot (g'^{-1} \cdot a'') \models p. \text{ Also } a'' \downarrow_{Eg}^K g, g' \text{ which implies } g \cdot g' \cdot a'' \downarrow_{Eg}^K g, g'.$$

Since $g \cdot a \downarrow_E^K g$ transitivity yields $g \cdot g' \cdot (g'^{-1} \cdot a'') \downarrow_E^K g \cdot g'$. Similarly $a'' \downarrow_{Eg'}^K g, g'$ implies $g'^{-1} \cdot a'' \downarrow_{Eg'}^K g, g'$. Since $a' \downarrow_E^K g'$ by transitivity $g'^{-1} \cdot a'' \downarrow_E^K g \cdot g'$ holds, so $g \cdot g' \in S^K(p)$. \square

Remark 4.19. In particular $S^K(p)$ is stable under Kim-independent product: If $g, g' \in S^K(p)$ and $g \downarrow_E^K g'$ then $g \cdot g' \in S^K(p)$. In simple theory we can show that if a type-definable subset X of a group is stable by \downarrow^f -independent product then the subgroup generated by X is $X \cdot X$. It is not clear that we can generalize this result to NSOP₁ since the proof revolves around the notion of stratified Δ -rank, which does not generalize well to NSOP₁.

Remark 4.20. $S^K(p)$ is also stable by inversion: If $g \in S^K(p)$ and $a \models p$ is such that $g \cdot a \models p$, $g \downarrow_E^K a$ and $g \cdot a \downarrow_E^K a$. Let $a' := g \cdot a$, then $a', g^{-1} \cdot a' \models p$, $g^{-1} \downarrow_E^K a'$ and $g^{-1} \cdot a' \downarrow_E^K a'$, so $g^{-1} \in S^K(p)$.

4.1.2.3 Examples

We give some examples of groups G definable in strictly NSOP₁ theories and types p such that $\text{Stab}^K(p) := S^K(p) \cdot S^K(p)$.

1 - The group $G = (\mathbb{M}, +)$ in $T_{f,ab}$, where $T_{f,ab}$ is the model completion of the theory of abelian groups with a generic unary function f such that $f(0) = 0$.

Lemma 4.21. For any unary type $p \in S(E)$ over some set E of parameters $\text{Stab}^K(p)$ is a type-definable subgroup of G and is equal to $S^K(p) + S^K(p)$.

Proof. We define the following independence relation: say that two structures A, B extending E are 0-independent, written $A \downarrow_E^0 B$ if $A \downarrow_E^a B$ and $f(a+b) = 0$ for all elements $a \in A \setminus E$, $b \in B \setminus E$. It is easy to show that \downarrow^0 satisfies existence, extension, invariance, symmetry, stationarity, transitivity.

Let $E \subseteq \mathbb{M}$ be a substructure and $p \in S(E)$ be a type. Consider $g_1, g_2, g_3 \in S^K(p)$. Consider $g \equiv_E g_1$ such that $g \downarrow_E^0 g_1 g_2 g_3$. Then any term in $g+g_2$ over E is equal to $n \cdot (g+g_2) + e$ for some $n \in \mathbb{Z}$ and $e \in E$. From this it is clear that $g+g_2 \downarrow_E^K g_3$ since $g \downarrow_E^K g_2, g_3$. Now $g_1 - g \in S^K(p)$ and $g+g_2+g_3 \in S^K(p)$ by Lemma 4.18. So $g_1+g_2+g_3 = (g_1-g) + (g+g_2+g_3) \in S^K(p) + S^K(p)$ and the group generated by $S^K(p)$ is the type-definable group $S^K(p) + S^K(p)$. \square

2 - Let T_F be the theory of an NSOP₁ field \mathbb{F} with existence such that $S^K(q) + S^K(q) = \text{Stab}^K(q)$ for every type $q \in (F^\kappa, +)$ where κ is a small cardinal. By Proposition 4.16 this is true when T_F is a simple theory. We consider the additive group $(\mathbb{V}, +)$ of a monster model (\mathbb{V}, \mathbb{F}) of the theory $sT_\infty^{T_F}$ of vector space of infinite dimension over a model of T_F with a non-degenerate symmetric bilinear form.

We will also assume that T satisfies existence, which holds when T_F is a simple theory, as we have shown in Proposition 3.17.

For all $E \subseteq A, B$ algebraically closed $A \downarrow_E^K B$ if and only if $V(A)$ and $V(B)$ are linearly independent over $V(E)$ and $F(A) \downarrow_{F(E)}^F F(B)$, where \downarrow^F is Kim-forking independence in the sense of the theory of the field T_F .

Lemma 4.22. *Consider a unary type $p(x) \in S(E)$ in the additive group $(\mathbb{V}, +)$, where E is an algebraically closed set of parameters. Then the equality $\text{Stab}^K(p) := S^K(p) \cdot S^K(p)$ holds.*

Proof. Fix $(e_i)_{i < \kappa}$ a $K(E)$ basis of $V(E)$. If p specifies that $x \in \langle E \rangle$, let $v \models p$. Then $v = \sum_{i < \kappa} \lambda_i \cdot e_i$. Consider the type $q((y_i)_{i < \kappa}) := \text{tp}((\lambda_i)_{i < \kappa}/K(E))$. It is easy to see from the characterization of Kim-forking that $g \in S^K(p)$ if and only if $g = \sum_{i < \kappa} \lambda'_i \cdot e_i$ and $(\lambda'_i)_{i < \kappa} \in S^K(q)$. From this we deduce that $g \in S^K(p)$ if and only if $g = \sum_{i < \kappa} \lambda'_i \cdot e_i$ and $(\lambda'_i)_{i < \kappa} \in S^K(q)$, so $\text{Stab}^K(p) = S^K(p) \cdot S^K(p)$ since $\text{Stab}^K(q) = S^K(q) \cdot S^K(q)$ by assumption.

In the other case p specifies that $x \notin \langle E \rangle$. Let $v \models p$ and consider the type

$$q((y_i)_{i < \kappa}) := \text{tp}([v, e_i]_{i < \kappa}/K(E)).$$

Claim: $u \in S^K(p)$ if and only if $([u, e_i]_{i < \kappa}) \in S^K(q)$ for every $u \in \mathbb{V}$.

Proof: The inclusion \subseteq is clear, for the other direction consider $u \in \mathbb{V}$ such that $([u, e_i]_{i < \kappa}) \in S^K(q)$. There is $(\lambda_i)_{i < \kappa} \models q$ such that $(\lambda_i + [u, e_i]_{i < \kappa}) \equiv_{K(E)} (\lambda_i)_{i < \kappa}$, $(\lambda_i)_{i < \kappa} \downarrow_E^F ([u, e_i]_{i < \kappa})$ and $(\lambda_i + [u, e_i]_{i < \kappa}) \downarrow_E^F (\lambda_i)_{i < \kappa}$. By extension we can assume that $(\lambda_i)_{i < \kappa} \downarrow_{E([u, e_i]_{i < \kappa})}^F K(\text{acl}(E, u))$, so $(\lambda_i)_{i < \kappa} \downarrow_E^F K(\text{acl}(E, u))$ by transitivity.

Let $w' \models p$. Then $([w', e_i]_{i < \kappa}) \equiv_{K(E)} (\lambda_i)_{i < \kappa}$, so there is $\alpha, \beta \in \mathbb{K}$ such that:

$$([w', e_i]_{i < \kappa} [w', w']) \equiv_{K(E)} (\lambda_i)_{i < \kappa} \alpha \equiv_{K(E)} (\lambda_i + [u, e_i]_{i < \kappa}) \beta.$$

Similarly since $(\lambda_i)_{i < \kappa} \downarrow_E^F K(\text{acl}(E, u))$ and $K(\text{acl}(E, u)) \downarrow_E^F (\lambda_i + [u, e_i]_{i < \kappa})$ by extension we can assume that $(\lambda_i)_{i < \kappa} \alpha \downarrow_E^F K(\text{acl}(E, u))$ and $K(\text{acl}(E, u)) \downarrow_E^F (\lambda_i + [u, e_i]_{i < \kappa}) \beta$.

By Lemma 3.1 we can find $w \in \mathbb{V}$ such that:

1. $w \notin \langle Eu \rangle$,
2. $[w, e_i] = \lambda_i$ for every $i < \kappa$,
3. $[w, w] = \alpha$,
4. $[w, u] = \frac{1}{2} \cdot ([u, u] - \alpha - \beta)$.

Then $[w + u, w + u] = \beta$. By quantifier elimination and the characterization of Kim-forking it is clear that $u \downarrow_E^K w$, $u \downarrow_E^K w + u$ and $w + u \equiv_E w \equiv_E w'$, so $u \in S^K(p)$. \square

As in the case of $p \in \langle E \rangle$ we deduce by the assumption on $Stab^K(q)$ in the field that $Stab^K(p) = S^K(p) \cdot S^K(p)$. \square

A similar proof of the **Claim** would work in the context of the additive group $(\mathbb{V}^n, +)$ and the semi-direct product $\mathbb{V}^n \rtimes GL_n(\mathbb{K})$.

3 - Type-definable groups of finite arity in ω -free PAC fields of characteristic 0. We will need the following results:

Theorem 4.23. *Let $E = acl(E)$, $E \subseteq A, B$ for $i = 0, 1$ be algebraically closed subsets of an ω -free PAC field F of characteristic 0. Then $A \downarrow_E^K B$ if and only if A and B are linearly disjoint over E and $A^{acl}B^{acl} \cap F = AB$.*

Proof. Kaplan and Ramsey prove in [KR20, Theorem 9.33] that if E is a model Kim-independence over E is equivalent to weak independence over E , a notion defined in [Cha02, Definition 1.2] by Chatzidakis. Chatzidakis proves in [Cha02, Lemma 2.8] that in an ω -free PAC-field F if $E \subseteq A, B$ are algebraically closed subfields of F then A and B are weakly independent over E if and only if A and B are linearly disjoint over E and $A^{acl}B^{acl} \cap F = AB$. We can extend the result of Kaplan and Ramsey to E algebraically closed by using the Kim-Pillay characterization of Kim-forking over arbitrary sets [CKR23, Theorem 5.1]. \square

Lemma 4.24. [Cha02, Lemma 3.2] *Let $E = acl(E)$, $E \subseteq A_i, B_i$ for $i = 0, 1$ be algebraically closed subsets of an ω -free PAC field F such that $A_i \downarrow_E^K B_i$ for $i = 0, 1$, $A_0 \equiv_E A_1$ and $B_0 \downarrow_E^{ACF} B_1$. Then there is $A \equiv_{B_i} A_i$ for $i = 0, 1$ such that $A \downarrow_E^K B_0B_1$.*

Remark 4.25. Lemma 4.24 together with Lemma 4.14 implies that if p is a type in a type-definable group of finite arity then $S^K(p)$ is stable by \downarrow^{ACF} independent product. The following proof uses this and the fact that \downarrow^{ACF} has generic elements in G , namely the elements of maximal transcendence degree over E .

Lemma 4.26. *Consider a Lascar strong type $p \in S(E)$ in a group (G, \cdot) which is finitary and type-definable over E in a ω -free PAC field $F \supseteq E$. Then $Stab^K(p) := S^K(p) \cdot S^K(p)$.*

Proof. Notice that if the transcendence degree of the type p over E is $n < \omega$ then the elements of $S^K(p)$ have transcendence degree $\leq n$ over E . Consider $g_i \in S^K(p)$ for $i = 0, 1, 2$. Consider $g \in S^K(p)$ of maximal transcendence degree over E such that $g \downarrow_E^K g_0, g_1, g_2$. Since $g \downarrow_E^K g_0$ we know that $g_0 \cdot g^{-1} \in S^K(p)$ by Lemma 4.18 ($g^{-1} \in S^K(p)$ since $S^K(p)$ is closed by inversion - Remark 4.20). We want to show that $g \cdot g_1 \cdot g_2 \in S^K(p)$.

By Lemma 4.18 we know that $g \cdot g_1 \in S^K(p)$. We show that $g \cdot g_1 \downarrow_E^{ACF} g_2$, where \downarrow_E^{ACF} is forking independence in the sense of the algebraic closure. This follows from the fact that $g \cdot g_1$ and g are interalgebraic over Eg_1g_2 and that $g \downarrow_E^{ACF} g_1g_2$: $\text{trdeg}(g \cdot g_1/E) \geq \text{trdeg}(g \cdot g_1/Eg_1g_2) = \text{trdeg}(g/Eg_1g_2) = \text{trdeg}(g/E)$ is maximal in $S^K(p)$, so $\text{trdeg}(g \cdot g_1/E) = \text{trdeg}(g \cdot g_1/Eg_2)$ and $g \cdot g_1 \downarrow_E^{ACF} g_2$.

Then Remark 4.25 yields that $g \cdot g_1 \cdot g_2 \in S^K(p)$, so $g_0 \cdot g_1 \cdot g_2 \in S^K(p) \cdot S^K(p)$ and $Stab^K(p) = S^K(p) \cdot S^K(p)$. \square

Remark 4.27. The previous proof relies on the finiteness of the transcendence degree over E to show that $g \cdot g_1 \downarrow_E^{ACF} g_2$. In the more general case of a group of infinite arity this argument would not work.

4.1.2.4 A Counter-Example

We now show that the Kim-stabilizer does not satisfy certain properties of the f -stabilizer in simple theories. For this we consider the example of the additive group $(\mathbb{V}, +)$ in sT_∞^{ACF} .

Let $(e_i)_{i < \omega}$ be a family of linearly independent vectors. For $n < \omega$ let $p_n \in S(e_{<n})$ be the type defined by $x \notin \langle e_{<n} \rangle$, $[x, x] = 0$ and $[x, e_i] = 0$ for all $i < n$. p_m is a non-Kim-forking extension of p_n for all $n < m$. From the proof of Lemma 4.22 it is easy to see that:

$$Stab^K(p_n) = \{x : [x, e_i] = 0 \text{ for all } i < n\}.$$

We can then notice two things happening in this case:

- 1 - $Stab^K(p_n)$ is not a subgroup of bounded index of $Stab^K(p_m)$ when $n < m$.
- 2 - There is $v \in Stab^K(p_m)$ such that $v \downarrow_{e_{<n}}^K e_{<m}$ and such that $v + Stab^K(p_m)$ seen as an hyperimaginary is not in the bounded closure of $e_{<m}$. In fact it is easy to see that $v + Stab^K(p_m) \in bdd(A)$ for some set A if and only if $[v, e_i] \in bdd(A)$ for all $i < m$.

These two things can not happen in a simple group. After seeing this we are left with the following questions. Let G be a group type-definable over E in an NSOP₁ theory with existence:

- 1 - If $p \in S_G(A)$ and $E \subseteq A \subseteq B$ is there a non-Kim-forking extension $q \in S_G(B)$ of p such that $Stab^K(q)$ has bounded index in $Stab^K(p)$?
- 2 - Is there a type $p \in S_G(E)$ such that $Stab^K(p)$ has bounded index in G ?

1 holds in the group $(\mathbb{V}, +)$ in sT_∞^{ACF} : Given any type $p \in S_{\mathbb{V}}(A)$ if q is the \downarrow^Γ -independent extension of p to B then $Stab^K(p) = Stab^K(q)$.

4.2 Stabilizer and expansion of the language

In this subsection we define a set of conditions on a field F and on a group G type-definable in F . We show that under these conditions there is a definable finite to one homomorphism from a type-definable subgroup of G of bounded index to an algebraic group H .

4.2.1 S1 ideals of formulas

We begin by recalling the notions of S1 ideal and wide types from [Hru12].

Definition 4.28. Let X be a definable set over some set of parameters A . Let μ be an A -invariant ideal of definable subsets of X . μ is said to be S1 if $\varphi(x, a_0) \wedge \varphi(x, a_1) \in \mu$ if and only if $\varphi(x, a_0) \in \mu$ for every formula $\varphi(x, y)$ and A -indiscernible sequence $(a_i)_{i < \omega}$.

Definition 4.29. Let μ be an ideal. A definable set X is medium if μ is S1 when restricted to X .

Definition 4.30. Let μ be an ideal. A partial type $\pi(x)$ is said to be μ -wide (or just wide if there is no ambiguity) if it does not imply a formula in μ . A type is medium if it concentrates on a medium set.

Lemma 4.31. [Hru12, Lemma 2.9] *Let μ be an A -invariant ideal which is $S1$ on some A -definable set X . Then any wide type p concentrating in X does not fork over A .*

Remark 4.32. If G is a type-definable group over A , μ is an ideal $S1$ on G , A invariant and invariant under translations. Then any wide subgroup $H \leq G$ type-definable over A has bounded index in G .

Proof. If $H \leq G$ does not have bounded index there is an A -indiscernible sequence $(a_i)_{i < \omega}$ such that all of the a_i are in different cosets of H . Let $\pi(x)$ be a partial type over A defining H . Then $\pi(a_i^{-1} \cdot x) \wedge \pi(a_j^{-1} \cdot x)$ is inconsistent for every $i \neq j < \omega$.

By indiscernibility and compactness there is a formula $\varphi(x) \in \pi(x)$ such that $\varphi(a_i^{-1} \cdot x) \wedge \varphi(a_j^{-1} \cdot x)$ is inconsistent. So $\varphi(a_0^{-1} \cdot x) \wedge \varphi(a_1^{-1} \cdot x) \in \mu$, and $\varphi(a_0^{-1} \cdot x) \in \mu$ since μ is $S1$. Since μ is invariant under translation $\varphi(x) \in \mu$ which contradicts the fact that H is wide. \square

In this context Hrushovski defines the following notion of stabilizer:

Definition 4.33. Let G be a group type-definable over A . Let μ be an A -invariant ideal in G which is invariant under translations by elements of G . If $p \in G$ is a wide type we define:

$$S^\mu(p) := \{g : p \cdot g \cap p \text{ is wide}\}.$$

We define $Stab^\mu(p)$ as the subgroup of G generated by $S^\mu(p)$.

Definition 4.34. Let p and q be two types over some model M . We define:

$$p \times_{nf} q := \{(a, b) : a \models p, b \models q, b \perp_M^f a\}$$

We will use the two following results:

Theorem 4.35. [MOS20, Theorem 2.15] *Let μ and λ be M invariant ideals of G that are invariant under left and right translation and such that μ is $S1$ in any $X \in \lambda$. Assume that we are given a wide and medium type p in G and that the following conditions are satisfied:*

- (A) *for any types q, r , if for some $(c, d) \models q \times_{nf} r$, $\text{tp}(cd/M)$ or $\text{tp}(dc/M)$ is λ -medium then q is λ -medium;*
- (B) *for any $(a, b) \models p \times_{nf} p$, $\text{tp}(a^{-1}b/M)$ is λ -wide.*
- (F) *there are $(a, b) \models p \times_{nf} p$ such that $\text{tp}(a/Mb)$ does not fork over M .*

Then $Stab^\mu(p) = S^\mu(p)^2 = (pp^{-1})^2$ is a type-definable, wide and λ -medium group. Also $Stab^\mu(p) \setminus S^\mu(p)$ is contained in a union of non-wide M -definable sets.

Lemma 4.36. [Hru12][Lemma 2.16] *Let μ be an A -invariant ideal. Then there is a model $M \supseteq A$ and a global M -invariant type p finitely satisfiable in M such that if $a \models p|_M$ and $b \models p|_{M_a}$ then $\text{tp}(a/Mb)$ is wide.*

4.2.2 The assumptions on F

Let \mathbb{F} be a monster model of some theory T in a countable language L extending the theory of fields and let (G, \cdot) be a group of finite arity type-definable over a set of parameters $A \subseteq \mathbb{F}$. We write \mathbb{F}^{alg} the algebraic closure of \mathbb{F} in the sense of fields and similarly $B^{alg} \subseteq F^{alg}$ the algebraic closure of some $B \subseteq \mathbb{F}$. We will write dcl^{alg} for the definable closure in the sense of \mathbb{F}^{alg} . We will write \perp^{ACF} for forking-independence in the sense of $\mathbb{F}^{alg} \models ACF$.

We assume the following:

- (H1) T is NSOP₁, satisfies existence and $Stab^K(p) = S^K(p) \cdot S^K(p)$ for every type $p \in S_G(B)$ and parameters $B \supseteq A$.
- (H2) The group law of G is algebraic in the sense of \mathbb{F}^{alg} : $a^{-1} \in (A, a)^{alg}$ and $a \cdot b \in (A, a, b)^{alg}$ for every $a, b \in G$.
- (H3) We can extend the language L to L' and \mathbb{F} in an L' -structure which we write \mathbb{F}' such that there is an invariant $S1$ ideal of formulas μ_0 which is invariant by translation in \mathbb{F}' and for which G is wide.

If $N' \prec \mathbb{F}'$ we will write N for N' seen as an L -structure, in this context we have that $N \prec \mathbb{F}$. We recall the following result about Kim-independence in fields:

Proposition 4.37. [KR20, Proposition 9.28] *Suppose F is an arbitrary field. We denote by A^s the separable closure a field A . Let $M \prec F$ and $a, b \in F$ such that $a \perp_M^K b$. Then the fields $A = acl(Ma)$ and $B = acl(Mb)$ satisfy the following conditions:*

1. A and B are linearly disjoint over M .
2. F is a separable extension of AB .
3. $acl(AB) \cap A^s B^s = AB$.

With this we can show the following in NSOP₁ fields:

Lemma 4.38. *Let F be an NSOP₁ field with existence. Let $E \subseteq A, B \subseteq F$ be algebraically closed sets such that $A \perp_E^K B$. Then $A \perp_E^{ACF} B$.*

Proof. Let $(A_i)_{i < \omega + \omega + 1}$ be an E -Morley sequence with $A_0 = A$ which is B -indiscernible and satisfies

$$(A_i)_{i < \omega + \omega + 1} \perp_E^K B.$$

Let $I = (A_i)_{i < \omega}$ and $J = (A_i)_{\omega \leq i < \omega + \omega}$. Since $tp(A_{\omega + \omega} / EIJB)$ is finitely satisfiable in J we have that $A_{\omega + \omega} \perp_J^{ACF} EIJB$. By stability and quantifier elimination in ACF the sequence $(A_i)_{i < \omega + \omega + 1}$ is totally indiscernible in the theory ACF , so $A_{\omega + \omega} \perp_I^{ACF} EIJB$. Let $\bar{a} := cb(A_{\omega + \omega} / EIJB)$, then $\bar{a} \in I^{alg} \cap J^{alg}$ by elimination of imaginaries in ACF .

We show that $\bar{a} \in E^{alg}$, which implies that $A_{\omega + \omega} \perp_E^{ACF} IJB$, so $A \perp_E^{ACF} B$ by indiscernibility. We know that $\bar{a} \in I^{alg} \cap J^{alg}$.

Let $P(X)$ be the minimal polynomial of an element $a \in \bar{a}$ over the field generated by I . Then P has coefficients in the field generated by $A_{<k}$ for some $k < \omega$. Let Q be the minimal polynomial of a over the field generated by $A_{[k, 2k[}$. Since $A_{<k}$ and $A_{[k, 2k[}$ begin a $JA_{\omega + \omega}$ -indiscernible sequence in F^{alg} we know that $A_{<k} \equiv_{J^{alg} A_{\omega + \omega}} A_{[k, 2k[}$.

In particular P and Q have the same degree. If $P \neq Q$ then a is a root of $P - Q$ which has degree strictly smaller than P and coefficients in the field generated by I , which contradicts the fact that P is the minimal polynomial of a over I . So $P = Q$ and P has coefficients in the intersection of the fields generated by $A_{<k}$ and $A_{[k, 2k[}$, which is E since $(A_i)_{i < \omega}$ is a total E -Morley sequence. From this we have that $a \in E^{alg}$ for any $a \in \bar{a}$, and so that $\bar{a} \in E^{alg}$. \square

Lemma 4.39. *If a type $p' = tp_{L'}(a/N'B)$ is wide for an elementary substructure $N' \prec \mathbb{F}'$, then $p := tp_L(a/NB)$ its restriction to L does not Kim-fork over N (here N is the restriction of N' to the language L).*

Proof. Let $(B_i)_{i < \omega}$ a coheir Morley sequence with $B_0 = B$ over N' . Then since $\text{tp}(a'/N'B_0)$ does not fork in the sense of L' there is an $a'' \in F$ such that $a''NB_i \equiv_N a'NB_0$ in the sense of L' for all $i < \omega$, so in particular $a''B_i \equiv_N aB_0$ for all $i < \omega$ in the sense of L . Since $(B_i)_{i < \omega}$ is coheir Morley over N in the sense of L , by Kim's Lemma for Kim-forking, we get that $a' \downarrow_N^K B$. \square

Lemma 4.40. *There exist a model M' and a global M' -invariant type $p' \in S_G(\mathbb{F})$ in the language L' finitely satisfiable in M' such that $A \subseteq M' \prec \mathbb{F}'$, if $b \models p'_{|M'}$ and $a \models p'_{|M'b}$ then $\text{tp}(b/M'a)$ is wide and also that M' has infinite transcendence degree over A . We write M for M' seen as an L -structure (so $M \prec \mathbb{F}$) and $p \in S_G(\mathbb{F})$ for the restriction of p' to L .*

Proof. Let $A \subseteq B$ be such that B contains a sequence $(g_i)_{i < \omega} \in G$ of elements of maximal transcendence degree over A such that $g_i \downarrow_A^{ACF} g_{<i}$ for all $i < \omega$. We apply Lemma 4.36 to B and find a model $A \subseteq B \subseteq M' \prec \mathbb{F}'$ and a global M' -invariant type p' finitely satisfiable in M' such that if $a \models p'_{|M'}$ and $b \models p'_{|M'a}$ then $\text{tp}(a/M'b)$ is wide. Then M' has infinite transcendence degree over A . \square

In this section we are going to work on three different levels:

1. The stable theory of \mathbb{F}^{alg} , i.e. the quantifier free types, in which we will build a group configuration in the following subsection.
2. The original L -theory of the field \mathbb{F} which is NSOP_1 and has a notion of stabilizer.
3. The L' -theory of \mathbb{F}' which has an $S1$ ideal, so we can apply the results of [MOS20].

4.2.3 Definable measures and (H3)

Definition 4.41. A measure μ is said to be definable over a set of parameters B if for every formula $\varphi(x, a)$ with parameters a , whenever $\mu(\varphi(x, a)) \in]\alpha, \beta[$ there is a formula $\theta(y) \in \text{tp}(a/B)$ such that $\mu(\varphi(x, a)) \in]\alpha, \beta[$ for all $\models \theta(a')$.

A measure definable over B is B -invariant (i.e. invariant under automorphisms fixing B). If $\varphi(x, y)$ is a formula and $] \alpha, \beta [$ is an open interval then what the definition states is that the set $\{b : \mu(\varphi(x, b)) \in] \alpha, \beta [\}$ is \vee -definable over B .

By expanding the language it is always possible to extend a measure μ on the definable subsets of some definable set X into a definable measure on the definable subsets of X in this expansion.

For this all we have to do is to add to the language the relation symbols $R_{\varphi, q}$ for $\varphi(x, y)$ a formula such that $\varphi(x, b) \subseteq X$ for all b and $q \in \mathbb{Q}$ which are interpreted by

$$\{b : \mu(\varphi(x, b)) < q\}.$$

Then extend the measure to the new definable sets, which is possible by applying [Sim12, Lemma 7.3], and iterate the process. Notice that if the measure μ is defined on the definable subsets of a definable group G and is invariant by translation there is no reason for the expanded measure that we get in this process to also be invariant by translation.

In the context of the assumption (H3) this is a problem: We want to get an ideal which is invariant by translation and also by automorphisms, and if we have a measure μ on the definable subsets of G in \mathbb{F} we would like this ideal to correspond to the ideal of formulas of measure 0 in the expanded language.

With the following two lemmas, we show that in a very specific case, it is indeed possible to expand the language and extend a measure which is invariant by translation into a definable measure that remains invariant by translation, so the ideal of formulas of measure 0 in the expansion will be invariant by translation and also $S1$, and with it we can satisfy (H3).

Lemma 4.42. *Consider a definable measure μ on definable subsets of some B -definable set X for some set of parameters B such that for every formula $\varphi(x, y)$ the measure $\mu(\varphi(x, b))$ can only take a finite number of values for b ranging over a model. Then for every formula $\varphi(x, y)$ and real number α there is a formula $\theta_\alpha^\varphi(y)$ such that $\models \theta_{\varphi, \alpha}(a)$ if and only if $\mu(\varphi(x, a)) = \alpha$ for all y -tuples a .*

Proof. Let $\varphi(x, y)$ be a formula. The set $\{b : \mu(\varphi(x, b)) \in]\alpha, \beta[\}$ is \forall -definable over A for any open interval. In the case when the measure can only take a finite number of values this set coincides with the pre-image of a closed interval, so it is both open and closed in $S_y(A)$, i.e. it is a definable set. \square

Lemma 4.43. *Let L be a countable language. Let $(M_e)_{e < \omega}$ be a family of L -structures and $G_e \subseteq M_e$ be a uniformly definable family of groups. Assume that μ_e is a measure on definable subsets of G_e which is definable over a set of parameters $A_e \subseteq M_e$, can only take a finite number of values for a given formula $\varphi(x, y)$ and is invariant by translation. Assume also that $\mu_e(X_e) = 1$ for all $e < \omega$. Let \mathcal{U} be an ultrafilter on ω . Let T be the L -theory of the ultraproduct $M := \prod_i M_e/\mathcal{U}$.*

Then there is an expansion of L into a countable language L' and an L' -structure on M such that there exist a measure μ' on $G := \prod_e G_e/\mathcal{U} \subseteq M$ which extends the ultraproduct measure $\prod_e \mu_e/\mathcal{U}$, is definable on $\prod_e A_e/\mathcal{U}$ in the sense of L' and is invariant by translation.

Proof. We expand the language L into a language L' which formulas are interpreted in M as ultraproduct of L -formulas in the coordinates M_e .

This is done through an inductive construction. Let $L_0 = L$. Assume that L_i has been defined. For every L_i -formula $\varphi(x, y) = \prod_e \varphi^e(x, y)/\mathcal{U}$ which implies $x \in X$, integers n and $1 \leq j \leq n$ we add a new relation symbol $\theta_{\varphi, j, n}(y)$ that is interpreted in M as the ultraproduct of the formulas $\theta_{\varphi^e, j, n}^e(y)$, where $\theta_{\varphi^e, j, n}^e(y)$ expresses that $\frac{j-1}{n} < \mu_e(\varphi^e(x, y)) < \frac{j+1}{n}$ in the coordinates M_e (such a formula exists by Lemma 4.42).

These new formulas are measurable in a canonical way, if $\theta(x) = \prod_{e < \omega} \theta^e(x)/\mathcal{U}$ is an L_{i+1} formula that implies that $x \in G$, then $\theta^e(x)$ is an L -formula which implies that $x \in G_e$ for almost all e and we can take the measure $\mu'(\theta(x))$ to be the standard part of $\prod_{e < \omega} \mu_e(\theta^e(x))/\mathcal{U}$. Since all of the measures μ_e are invariant by translation we have that the standard part of $\prod_{e < \omega} \mu_e(\theta^e(x))/\mathcal{U}$ is equal to the standard part of $\prod_{e < \omega} \mu_e(\theta^e(g_e \cdot x))/\mathcal{U}$ for every $g = [g_e]_{e < \omega} \in G$, so $\mu'(\theta(x)) = \mu'(\theta^e(g \cdot x))$ and the measure we get is invariant by translation.

By construction if $\varphi(x, y)$ is a formula in L_i , there is $\theta_{\varphi, i, n}(y) \in L_{i+1}$ such that $M \models \theta_{\varphi, i, n}(a)$ if and only if $\mu(\varphi(x, a)) \in]\frac{j-1}{n}, \frac{j+1}{n}[$ for all y -tuple a in M . So the measure μ' is definable in L' . Since the language L is countable L_i is also countable for every $i < \omega$, so $L' = \bigcup_{i < \omega} L_i$ also is. \square

In the context of the previous lemma we have a definable group which is definably amenable by some measure μ in the language L , and an expansion L' of the language and μ' of the measure in which the group is definably amenable with a definable measure in the sense of L' . As we saw before it is not true in general that we can get a definable measure which is invariant by translation from a measure on definable subsets which is invariant by translation by extending the language, and it is an interesting question whether we can write some sufficient conditions for this to happen.

In technical terms the proof of Lemma 4.43 consists of adding some ultraproducts of the definitions of the measures to the language, take the generated algebra, extend the measure to all of these sets (since the ultraproduct measure extends to the algebra of internal sets of the ultraproduct) and iterate this process.

4.2.4 The group configuration

In this subsection we construct a group configuration over M' in the sense of \mathbb{F}^{alg} with points in \mathbb{F} . This consists of a set of tuples (a, b, c, x, y, z) that we can arrange as in Figure 4.2 so that:

1. Two points in the same line are \downarrow^{ACF} -independent over M' .
2. Two points in the same line are inter-algebraic over the third one and M' .
3. Any other pair or triplets are \downarrow^{ACF} -independent.

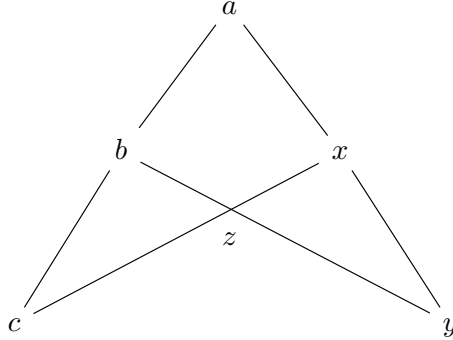


Figure 4.2: The group configuration

Let M' and p' be as in Lemma 4.40. By Lemma 4.31 p' does not \downarrow^f -fork over M' in the sense of L' and by Lemma 4.39 p does not Kim-fork over M in the sense of L . By Lemma 4.38 p does not \downarrow^{ACF} -fork over M' . Let $b \models p'_{M'}$, $x \models p'_{M'b}$ and $a \models p'_{M'bx}$. Then $\text{tp}_{L'}(a/M'b)$ is wide and both $\text{tp}_{L'}(a/M'b)$ and $\text{tp}_{L'}(b/M'a)$ do not fork over M' .

Let $c := b \cdot a$, $y := a \cdot x$ and $z = b \cdot a \cdot x$. Then $\text{tp}(a/M')$ is wide and since $a \equiv_{M'} b \equiv_{M'} x$ these three tuples have the same transcendence degree n over M' . The tuples a, b, c, x, y, z form a group configuration over M' in \mathbb{F}^{alg} :

1. Since p' is finitely satisfiable in M' we have that $a \downarrow_{M'}^{ACF} b, x$ and $x \downarrow_{M'}^{ACF} b$, so $a \downarrow_b^{ACF}$ and $a \downarrow_{M'}^{ACF} x$. Since $a \downarrow_{M'b}^{ACF} x$ by base monotonicity we have $c \downarrow_{M'b}^{ACF} x$ and by transitivity we deduce $c \downarrow_{M'}^{ACF} x$. Similarly we can show that $b \downarrow_{M'}^{ACF} y$. The types $\text{tp}(b/M'a)$ and $\text{tp}(x/M'a)$ are wide. Since the ideal μ_0 is invariant by translation the types $\text{tp}(c/M'a)$ and $\text{tp}(y/M'a)$ are also wide, so by Lemma 4.38 $c \downarrow_{M'}^{ACF} a$ and $y \downarrow_{M'}^{ACF} a$. The remaining relations can be deduced from these by the properties of \downarrow^{ACF} .
2. Two points in the same line are inter-algebraic over the third one and M' since the third point can be obtained by using the group law which is definable over M' .
3. As written previously $a \downarrow_{M'}^{ACF} b, x$ and $x \downarrow_{M'}^{ACF} b$. From the previous relations, the fact that the group law is algebraic in the sense of F^{alg} and the properties of \downarrow^{ACF} we can deduce the remaining independence relations for 3.

4.2.4.1 Germs and the Hrushovski-Weil theorem

We recall the definition of the germ of a function and the ‘Group Chunk Theorem’ which we will apply in \mathbb{F}^{alg} . A presentation of this result can be found in [BNP89]. We will here refer here to the course of Bays on geometric stability theory [Bay18] which suits us better. To keep things simple we state these notion in the context of $\mathbb{F}^{alg} \models ACF$.

Definition 4.44. Let $p_0, q_0 \in S(A)$ be stationary types. In our context by stability and elimination of imaginaries this just means types over an algebraically closed subfield of \mathbb{F}^{alg} . Let $p, q \in S(\mathbb{F}^{alg})$ be their respective global non-forking extensions.

Let f be a definable partial function and let $\varphi(x, y, m)$ be a formula defining f over the parameters m . We say that f is defined at p if $p(x) \models \exists! y(\varphi(x, y, m))$, which we can also write $p(x) \models x \in \text{dom}(f)$.

Two definable partial functions f_1, f_2 defined at p are said to have the same germ at p if $p(x) \models f_1(x) = f_2(x)$. This is a definable equivalence relation between definable partial functions, the class of f is denoted \tilde{f} and we call it the *germ of f at p* .

To a germ \tilde{f} corresponds a unique function on p : If f_1, f_2 are two representatives of \tilde{f} defined over parameters m_1 and m_2 respectively, then $f_1(a) = f_2(a)$ whenever $a \models p|_{Am_1m_2}$. We write $\tilde{f} : p \rightarrow q$ if when f is a representative of \tilde{f} defined over parameters b then $f(a) \models q|_{Ab}$ whenever $a \models p|_{Ab}$.

$\text{Aut}(\mathbb{F})$ acts on germs: Let $\sigma \in \text{Aut}(\mathbb{M})$ and $\tilde{f} : p \rightarrow q$ a germ on p . If f_1, f_2 are two representatives of \tilde{f} then f_1^σ, f_2^σ have the same germ at p^σ , which we write $\tilde{f}^\sigma : p^\sigma \rightarrow q^\sigma$.

We say that a tuple \bar{b} in \mathbb{F}^{alg} is a code for \tilde{f} if:

$$\bar{b} = \bar{b}^\sigma \text{ if and only if } \tilde{f} = \tilde{f}^\sigma \text{ for all } \sigma \in \text{Aut}(\mathbb{F}^{alg}).$$

We then define $\ulcorner \tilde{f} \urcorner := \text{dcl}^{\text{eq}}(\bar{b})$, this is in fact well-defined since two codes are fixed by the same automorphisms, so they are interdefinable. Since ACF is stable and has elimination of imaginaries such a code always exists: The type p is definable over A . Let $\varphi(x, y, n)$ be a formula defining a representative f of \tilde{f} . Then $f = f_n$ where $(f_z)_z$ is a definable family of partial functions defined by $\varphi(x, y, z)$. We consider the relation:

$$E_f(m, m') := d_p x.(f_m(x) = f_{m'}(x))$$

Then $E_f(m, m')$ if and only if f_m and $f_{m'}$ have the same germ at p . If $\sigma \in \text{Aut}(\mathbb{M})$ then $f_m^\sigma = f_{m^\sigma}$. From this it is easy to see that the E_f -class of n is a code for \tilde{f} , i.e. $\ulcorner \tilde{f} \urcorner = \text{dcl}^{\text{eq}}(n/E_f)$.

If $\tilde{f} : p \rightarrow q$ and $\tilde{g} : q \rightarrow s$ are two germs we can define their composition $\tilde{g} \circ \tilde{f} : p \rightarrow s$, if f and g are representatives defined over parameters n and m respectively then $g \circ f$ is a representative of $\tilde{g} \circ \tilde{f}$.

If a germ $\tilde{f} : p \rightarrow q$ defines an injective function, i.e. if for any representative f defined over parameters n there is $\theta \in p$ such that $\models ((\theta(x) \wedge \theta(x')) \implies ((f(x) = f(x')) \implies (x = x')))$, then we can define the inverse germ \tilde{f}^{-1} of \tilde{f} which is the unique germ $\tilde{f}^{-1} : q \rightarrow p$ such that $\tilde{f}^{-1} \circ \tilde{f} = \text{id}$.

Definition 4.45. If $p, q, s \in S(N)$ are stationary, a *family \tilde{f}_s of germs $p \rightarrow q$* is a family $\tilde{f}_s := (\tilde{f}_b)_{b \models s}$ of germs at p of an N -definable family f_b of partial functions, which is such that $\tilde{f}_b : p \rightarrow q$ whenever $b \models s$.

- The family is *canonical* if b is a code for \tilde{f}_b , for all $b \models s$.
- The family is *generically transitive* if $\tilde{f}_b(x) \downarrow_N^f x$ for some (any) b and x such that $b \models s$ and $x \models p|_{Nb}$.

Lemma 4.46. [Bay18, Lemma 5.1] Let (b, z, y) be a triplet of tuples and B be a tuple of parameters satisfying $b \downarrow_B^{ACF} z$, $b \downarrow_B^{ACF} y$ and $y \in \text{dcl}(Bbz)$. Define $s := \text{tp}(b/B)$, $q := \text{tp}(y/B)$ and $p := \text{tp}(z/B)$. Let $f_b(z) = y$ be a formula witnessing $y \in \text{dcl}(bz)$. We can consider the family of germs $\tilde{f}_s : p \rightarrow q$.

- (i) \tilde{f}_s can be taken invertible if and only if $z \in \text{dcl}(Bby)$;
- (ii) \tilde{f}_s is generically transitive if and only if $z \downarrow_B^{ACF} y$;
- (iii) $\ulcorner \tilde{f}_b \urcorner = \text{cb}(zy/Bb)$. So \tilde{f}_s is canonical if and only if $\text{cb}(zy/Bb) = \text{dcl}(b)$.

Definition 4.47. An homogeneous space consists of a group G together with a set S on which G acts transitively. We say that (G, S) is an A type-definable homogeneous space for a set of parameters A whenever G is an A type-definable group, S is an A type-definable set and the action of G on S is also type-definable over A . In that case we say that S is connected if and only if there exist a global type $s \in S$ such that $g \cdot b \models s|_{A_g}$ whenever $g \in G$ and $b \models s|_{A_g}$. This type is then called the generic type of S .

Proposition 4.48. [Bay18, Lemma 5.2](Hrushovski-Weil Group chunk Theorem) Let $p, s \in S(N)$ be stationary, and suppose that \tilde{f}_s is a generically transitive canonical family of invertible germs $p \rightarrow p$. Suppose that \tilde{f}_s is closed under inverse and generic composition, meaning that for $b \models s$ there exists $b' \models s$ such that $\tilde{f}_b^{-1} = \tilde{f}_{b'}$, and for $b_1, b_2 \models s$, $b_1 \downarrow_{\emptyset}^f b_2$ there exists $b_3 \models s$ such that:

$$\tilde{f}_{b_1} \circ \tilde{f}_{b_2} = \tilde{f}_{b_3}, \text{ and } b_i \downarrow_{\emptyset}^f b_3 \text{ for } i = 1, 2.$$

Then there exists a connected faithful \emptyset type-definable homogeneous space (G, S) , a definable embedding of s into G as its unique generic type, and a definable embedding of p into S as its unique generic type, such that the generic action of s on p agrees with the action $*$ of G on S , i.e. $\tilde{f}_b(a) = b * a$ for $b \models s$ and $a \models p$ such that $a \downarrow_{\emptyset}^f b$.

4.2.4.2 Obtaining an algebraic group

In the next proof we follow similar steps as in [Bay18, Theorem 6.1]. We begin by constructing a new group configuration (a', b', c', x', y', z') that satisfies the conditions of Proposition 4.48. This new configuration is obtained by applying some modifications to the original configuration (a, b, c, x, y, z) from the previous section. These modifications consist of replacing some tuple by another one that is interalgebraic over M' in the sense of \mathbb{F}^{alg} .

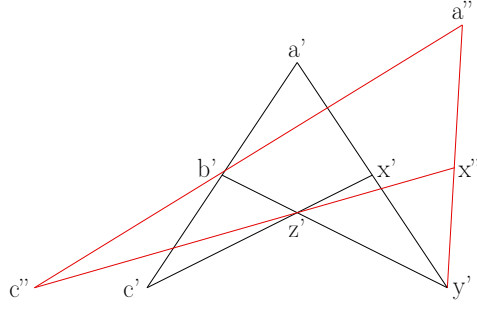
To keep track of the modifications we did we will also consider a set of parameters $A \subseteq B \subseteq M$, finitely generated over A . (a, b, c, x, y, z) is a group configuration over M' whose interalgebraicity relations are over A (they all come from the group law).

If $A \subseteq A' \subseteq M'$ is such that $(a, b, c, x, y, z) \downarrow_{A'}^{ACF} M$ then (a, b, c, x, y, z) is a group configuration over A' . Let A' be such a set, A' can be taken finitely generated over A (we just add to A the coefficients of the algebraic relations between (a, b, c, x, y, z) over M'), and let $B := A'$. We construct a sequence of sets B and a sequence of group configurations over B .

Proposition 4.49. There is an M' -definable algebraic group H and dimension generic elements $a', b', c' \in H(\mathbb{F})$ such that $a' \cdot b' = c'$, $(M'a')^{alg} = (M'a)^{alg}$, $(M'b')^{alg} = (M'b)^{alg}$ and $(M'c')^{alg} = (M'c)^{alg}$. In particular a and a' , b and b' , c and c' are interalgebraic over M' in the model theoretic sense.

Proof. Let $(a', b', c', x', y', z') := (a, b, c, x, y, z)$. We begin by adding some parameters to B from M' in order to make b' and y' interdefinable over Bz' in the sense of \mathbb{F}^{alg} .

Let $a'' \in G(M')$ of maximal transcendence degree such that $a'' \downarrow_B^{ACF} a'b'c'x'y'z'$ (such an element exists since M' has infinite transcendence degree over B). Define $c'' := b' \cdot a'$ and $x'' := a''^{-1} \cdot y'$. Let \bar{z}' be the tuple of the coefficients of the minimal polynomial of z over $Ay'b'x''c''$, then $\bar{z}' \in \mathbb{F}$. We replace the set of parameters B by $B = Aa''$ and replace: y' by $y'x''$, b' by $b'c''$, z' by \bar{z}' , so that now $z' \in \text{dcl}^{alg}(By'b')$.

Figure 4.3: A copy of the configuration over the line $(b'z'y')$

Let $b'' \in G(M')$ of maximal transcendence degree such that $b'' \downarrow_B^{ACF} a'b'c'x'y'z'$. Define $c'' := b'' \cdot a'$ and $z'' := c'' \cdot x'$. Let $\bar{x} \in \mathbb{F}$ be the tuple of the coefficients of the minimal polynomial of x' over $Ba'y'c''z''$. We replace the set of parameters B by Bb'' and replace: a' by $a'c''$, y' by $y'z''$, x' by \bar{x} , so that now $x' \in dcl^{alg}(Ba'y')$.

Let $c'' \in G(M')$ of maximal transcendence degree such that $c'' \downarrow_B^{ACF} a'b'c'x'y'z'$. Define $a'' := c'' \cdot b'^{-1}$ and $x'' := c''^{-1} \cdot z'$. Let $\bar{y} \in \mathbb{F}$ be the tuple of the coefficients of the minimal polynomial of y' over $Bb'a''x''$. We replace the set of parameters B by Bc'' and replace: b' by $b'a''$, z' by $z'x''$, y' by \bar{y} , so that now $y' \in dcl^{alg}(Bb'z')$ and $z' \in dcl^{alg}(Bb'y')$. $cb(y'z'/Bb') \in \mathbb{F}^{alg}$, and is interalgebraic with b' over B in \mathbb{F}^{alg} , it consists of the coefficients of the algebraic relations of $y'z'$ on Bb' , which are included in \mathbb{F} .

At this step we have a group configuration in \mathbb{F}^{alg} with points inside of \mathbb{F} such that $y' \in dcl^{alg}(Bz'b')$, $z' \in dcl^{alg}(Bb'y')$, b' is interdefinable with $cb(z'y'/Bb')$ over B . This group configuration is interalgebraic with the original group configuration over M' . Notice that whenever an element of \mathbb{F}^{alg} is definable over parameters in \mathbb{F} then it lies in \mathbb{F} (\mathbb{F} is definably closed in \mathbb{F}^{alg}), in particular if an imaginary in $\mathbb{F}^{alg,eq}$ is definable over parameters in \mathbb{F} by elimination of imaginaries in \mathbb{F}^{alg} it has a representative in \mathbb{F} .

Let $p^{alg} = tp(y'/B^{alg})$, $r^{alg} = tp(b'/B^{alg})$ and $q^{alg} = tp(z'/B^{alg})$ be types in the sense of the stable theory of \mathbb{F}^{alg} . Let f_r be the generic family of invertible germs from p^{alg} to q^{alg} corresponding to (b', z', y') . To apply Proposition 4.48 to (b', y', z') we need to check whenever $b'' \models r_{Bb'}^{alg}$ and $dcl(d) := [\tilde{f}_{b''}^{-1} \cdot \tilde{f}_b]$ then $d \downarrow_B^{ACF} b'$ and $d \downarrow_B^{ACF} b''$. This follows from the same verification as in (II) of [Bay18, Theorem 6.1].

We can apply the group chunk theorem Proposition 4.48 to get a connected faithful homogeneous space (H, S) type-definable in \mathbb{F}^{alg} over B and a definable embedding of p^{alg} which is the generic type of S into H as its unique generic type s_G in the sense of \mathbb{F}^{alg} .

We do one more base change to get a group configuration of (H, S) . Let $b'' \in G(M')$ of maximal transcendence degree such that $b'' \downarrow_B^{ACF} a'b'c'x'y'z'$. Define $y'' = z' \cdot b''^{-1}$. By the previous remark about imaginaries we can find $g \in M'$ such that $dcl^{eq}(g) = [\tilde{f}_{b''}^{-1} \cdot \tilde{f}_b]$. Then $g \models s_G$, $y', y'' \models p^{alg}$ and $y'' = g \cdot y'$.

We add b'' to the parameters and interalgebraically replace b' by g and z' by $g \cdot y' \models p^{alg}$. Let $c'' \in G(M')$ of maximal transcendence degree such that $c'' \downarrow_B^{ACF} a'b'c'x'y'z'$. Define $b'' = c'' \cdot a'^{-1}$ and $z'' := c'' \cdot x'^{-1}$. We add c'' to the parameters and interalgebraically replace a' by $b'' \models s_G$ and x' by z'' . Let $h := b' \cdot a'^{-1}$. Then $x' = a' \cdot y'$, $z' = b' \cdot y'$ so $z' = h \cdot x'$. By Lemma 4.46 $cb(x'z'/Ba'b') = h$. Since x' and c' are interalgebraic over B we get that $cb(x'z'/Ba'b')^{alg} = cb(x'z'/Bc')^{alg} = acl^{eq}(Bc')$, and we can interalgebraically replace c' with h . Then (a', b', c', x', y', z') is a group configuration of (H, S) .

In we look at the edge (a', b', c') of the configuration we have three points of a type-definable group H over B in \mathbb{F}^{alg} . Such a group H is definable in ACF and is definably isomorphic to an algebraic group, so by adding to B the parameters under which this bijection is defined, which we can find in M' , and composing by this bijection we can assume that (a', b', c') lies in an algebraic group H definable over B . Then each tuple a, b and c are interalgebraic in the sense of \mathbb{F}^{alg} over M' with their corresponding primed tuple. \square

Theorem 4.50. *There exists an algebraic group H definable over M , an M type-definable (in the sense of L) subgroup G_0 of bounded index in G and a definable finite to one homomorphism from G_0 to $H(\mathbb{F})$.*

Proof. Consider the algebraic group H and the interalgebraic elements a and a' from Proposition 4.49.

We define the ideal μ on $G \times H$ as set of the definable subsets of $G \times H$ whose projection to G is in μ_0 . We define the ideal λ as the set of the definable subsets of $G \times H$ whose projections to G and H each have finite fibers. Let $\tilde{p}' = \text{tp}_{L'}(a, a'/M')$ and $p' = \text{tp}_{L'}(a/M')$. a and a' are interalgebraic over M' so \tilde{p} is wide (in the sense of μ since $\text{tp}(a/M')$ is wide in the sense of μ_0) and medium (since its coordinates are interalgebraic), and satisfies the conditions of Theorem 4.35 (the conditions A and B follows from the same verifications as in the proof of [MOS20, Theorem 2.19], and F holds by the choice of M').

We can apply Theorem 4.35 to get that $K' := \text{Stab}^\mu(\tilde{p}') = S^\mu(\tilde{p}) \cdot S^\mu(\tilde{p}) \leq G \times H$ is a connected wide subgroup type-definable in L' . Since K' is wide, $\pi_1(K')$ is μ_0 -wide, so by Remark 4.32 $\pi_1(K')$ is a type-definable subgroup of bounded index in G .

Consider the original language L on \mathbb{F} . Let \tilde{p} be the restriction of \tilde{p}' to L and p the restriction of p' to L . Define $K \leq G \times H$ as $K = \text{Stab}^K(\tilde{p}) = S^K(\tilde{p}) \cdot S^K(\tilde{p})$. K is a group type-definable over M by (H1).

Claim 1: $K' \leq K$.

Proof of Claim 1. We show that $S^\mu(\tilde{p}') \subseteq S^K(\tilde{p})$. If $g \in S^\mu(\tilde{p}')$ there is $\alpha \models \tilde{p}'$ such that $g \cdot \alpha \models \tilde{p}'$ and $\text{tp}(\alpha/M'g)$ is μ -wide. So $\alpha \models \tilde{p}$ and $g \cdot \alpha \models \tilde{p}$. Since μ is invariant by translation $\text{tp}(g \cdot \alpha/M'g)$ is μ -wide. By Lemma 4.39 we have that $g \cdot \alpha \downarrow_M^K g$ and $\alpha \downarrow_M^K g$, so $g \in S^K(\tilde{p})$. \square

We consider the two projections $\pi_1 : K \rightarrow G$ and $\pi_2 : K \rightarrow H$. $K' \leq K \leq G \times H$, so $\pi_1(K)$ has bounded index in G . Let $K_1 := \text{Ker}(\pi_1)$.

Claim 2: Both projections from K to G and H have finite fibers.

Proof of Claim 2. We begin by showing that whenever $(g, \tilde{g}) \in S^K(\tilde{p})$ then g and \tilde{g} are interalgebraic over M in the sense of \mathbb{F}^{alg} . Let $(g, \tilde{g}) \in S^K(\tilde{p})$, there is a realization $(a, \tilde{a}) \models \tilde{p}$ such that $g, \tilde{g} \downarrow_M^K a, \tilde{a}$ and $g \cdot a, \tilde{g} \cdot \tilde{a} \models \tilde{p}$, so a and \tilde{a} are interalgebraic over M' and $g \cdot a$ and $\tilde{g} \cdot \tilde{a} \models p$ are also interalgebraic over M' , both in the sense of \mathbb{F}^{alg} .

From this we deduce that g and \tilde{g} are interalgebraic over Ma . Since $g, \tilde{g} \downarrow_M^K a, \tilde{a}$ by base monotonicity of \downarrow^{ACF} we get that $\tilde{g} \downarrow_{Mg}^{ACF} a$, so $\text{trdeg}(\tilde{g}/M, g) = \text{trdeg}(\tilde{g}/M, g, a) = 0$, and similarly $\text{trdeg}(g/M, \tilde{g}) = 0$, so g and \tilde{g} are interalgebraic over M in the sense of \mathbb{F}^{alg} .

Assume that $(1_G, h) \in K$ for some $h \in H$. We show that in that case $h \in M^{alg} \cap \mathbb{F} = M$. There is $(g, \tilde{g}) \in S^K(\tilde{p})$ such that $(g^{-1}, \tilde{g}^{-1} \cdot h) \in S^K(\tilde{p})$. Consider some $(g', \tilde{g}') \in S^K(\tilde{p})$ such that $(g', \tilde{g}') \downarrow_M^K g, \tilde{g}, h$. Then by Lemma 4.18 $(g \cdot g', \tilde{g} \cdot \tilde{g}') \in S^K(\tilde{p})$ and $(g'^{-1} \cdot g^{-1}, \tilde{g}'^{-1} \cdot \tilde{g}^{-1} \cdot h) \in S^K(\tilde{p})$.

We know from the previous point and Lemma 4.18 that $g \cdot g'$ and $\tilde{g} \cdot \tilde{g}'$ and $g'^{-1} \cdot g^{-1}$ and $\tilde{g}'^{-1} \cdot \tilde{g}^{-1} \cdot h$ are interalgebraic over M , so in particular $h \in \text{acl}(M, g' \cdot g)$. Also since $g' \downarrow_M^{ACF} g, h$

we have $g \cdot g' \downarrow_{M,g}^{ACF} h$ and by maximality of the transcendence degree $g' \cdot g \downarrow_M^{ACF} h$, which implies that $h \in M^{alg} \cap \mathbb{F} = M$. $Ker(\pi_1)$ is a type-definable subgroup included in $acl(M)$, so by compactness the partial type defining this set implies an algebraic formula and so $Ker(\pi_1)$ is finite. The proof for π_2 is similar. \square

By **Claim 2** K_1 is finite. Note that K_1 is normal in K . K_1 is central in K since K is connected over M . Let $C \leq H$ be the centralizer of $\pi_2(K_1)$ inside H . C is an algebraic subgroup of H and is definable in L over M' . We can replace H by $C/\pi_2(K_1)$ which is also an algebraic group definable over M' , so we can assume that K_1 is trivial, so K defines a finite to one homomorphism from $\pi_1(K)$ to H and $G_0 = \pi_1(K)$ is the subgroup we want. \square

4.3 The case of ω -free PAC fields

In this section apply the result of the previous section to the case of a group definable in an ω -free PAC field.

4.3.1 Some facts about e -free PAC fields

We begin by recalling some results of Chatzidakis and Ramsey about measures and definable groups in e -free PAC fields when e is finite, these results are from [CR23]. On this topic we can also quote a more recent article of Chatzidakis and Ramsey, [CR25], which generalizes these results to bounded perfect PAC fields.

Proposition 4.51. [CP98, Corollary of 4.5] *Let F be a PAC field of characteristic 0. If $E \subseteq F$, then $acl(E)$ is obtained by taking the relative field theoretic algebraic closure in F .*

Let k be an e -free PAC field of characteristic 0. In [CR23] Chatzidakis and Ramsey define a measure on the definable subsets of an absolutely irreducible variety. We begin by recalling the definition of this measure. Let V be an absolutely irreducible variety defined over k .

Definition 4.52. Let E be a subfield of k . A test formula $\theta(x)$ over E is a boolean combination of formulas of the form $\exists y(P(x, y) = 0)$ with y a single variable and $P \in E[x, y]$.

Proposition 4.53. [CR23, Corollary 2.4] *Let k be a perfect e -free PAC field and $E \subseteq k$ be a subfield of k . Every formula over a field E is equivalent in k to a test formula over E .*

Definition 4.54. [CR23, Definition 3.1] Let V be an absolutely irreducible variety defined over an e -free PAC field k and let a be a generic of V over k . We define the measure μ_V on definable subsets of V in k . Let $\varphi(x, b)$ be a formula over k . We can assume that φ is a test formula. Let L be the splitting field over $k(a)$ of all the polynomials appearing inside of $\varphi(x, b)$. Whenever some field K contains $k(a)$, the satisfaction of the formula $\varphi(a, b)$ inside of K only depends on the $k(a)$ isomorphism type of $K \cap L$.

The measure consists of counting the intermediate fields K up to isomorphism above $k(a)$ such that $k(a) \subseteq K \subseteq L$ and that are realized as $\mathcal{K} \cap L$ for $k \prec \mathcal{K}$. We will call such fields *permissible*. A field K is permissible if and only if K/k is a regular extension and $\text{Gal}(L/K)$ is an image of $\mathcal{G}(k)$ (in our case this is equivalent to saying that $\text{Gal}(L/K)$ is e -generated). Then $\mu_V(\varphi(x, b))$ is the proportion of the permissible fields (up to $k(a)$ isomorphism) in which $\varphi(a, b)$ is satisfied.

Since the field k is PAC notice that such a generic a can be found inside of an elementary extension of k .

Definition 4.55. Let G and H be two definable groups in some structure. We say that G and H are *definably isogenous* if there exist a definable subgroup S of $G \times H$ such that

- The projection of S to G , denoted by G_S , has a finite index in G ,
- The projection of S to H , denoted by H_S , has a finite index in H ,
- The kernel $\ker(S) = \{g \in G : (g, 1) \in S\}$ and the co-kernel $\text{coker}(S) = \{h \in H : (1, h) \in S\}$ are finite.

This subgroup S is then an isogeny between G and H . If S is an isogeny between G and H then there is an isomorphism between $G_S/\ker(S)$ and $H_S/\text{coker}(S)$.

Theorem 4.56. [HP94, Theorem C] *Let G be a group definable in a perfect e -free PAC-field. Then there is an algebraic group H defined over F and a definable (in F) isogeny between G and $H(F)$.*

Remark 4.57. The above theorem is stated for pseudo-finite fields in [HP94], Hrushovski and Pillay then state that it also holds for perfect PAC fields whose absolute Galois group is finitely generated as a profinite group.

Chatzidakis and Ramsey apply Theorem 4.56 and the isogeny to move the measure μ_H they defined on H into G and prove the following:

Theorem 4.58. *Let k be a perfect e -free PAC field, and let G be a group definable in k . Then G is definably amenable.*

The remainder of this subsection is dedicated to showing that the measure μ_V is definable over models. For this we begin by a quick proof to get a good understanding of this measure.

Proposition 4.59. *Let k be an e -free PAC field and V be an absolutely irreducible variety defined in k . The measure μ_V is invariant under elementary extension: Let $k \prec k'$ be an elementary extension, $b \in k$ and $\varphi(x, y)$ be a test formula. Then if we write μ'_V for the measure on the definable subsets of V computed in k' we have $\mu_V(\varphi(x, b)) = \mu'_V(\varphi(x, b))$.*

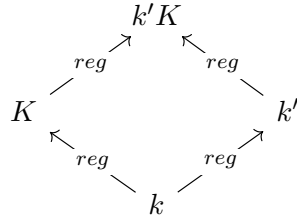
Proof. Let $k \prec k'$ be an elementary extension, $b \in k$ and $\varphi(x, y)$ be a test formula. We write μ'_V for the measure on the definable subsets of V in the sense of k' . Let a be a generic of V over k' . Since V is defined over k we have that $a \downarrow_k^{ACF} k'$. Let L be the splitting field of the polynomials appearing in $\varphi(a, b)$ over $k(a)$.

Define $L' := Lk'$. $L'/k'(a)$ is a finite Galois extension that contains all of the roots of the polynomials appearing in $\varphi(a, b)$. $L \downarrow_k^{ACF} k'$ and k'/k is a regular extension, so L and k' are linearly disjoint over k . This implies that the Galois groups $\text{Gal}(L/k(a))$ and $\text{Gal}(L'/k'(a))$ are isomorphic via the restriction map, and that any intermediate extension $k'(a) \subseteq K' \subseteq L'$ can be written $K' = k'(K' \cap L)$. We show that an intermediate extension $k(a) \leq K \leq L$ is permissible if and only if $k'(a) \leq k'K \leq L'$ is permissible.

Let $k(a) \leq K \leq L$ be a permissible field. Define $K' := k'K$. K/k is a regular extension, so K'/k' and K'/K both are regular extensions. Since $\text{Gal}(L/K)$ is e -generated and since there is a surjective map from this group to $\text{Gal}(L'/K')$ the group $\text{Gal}(L'/K')$ is e -generated, and $k'(a) \leq K' \leq L'$ is permissible.

Conversely assume that K' is a permissible intermediate extension $k'(a) \subseteq K' \subseteq L'$. Let $K := K' \cap L$, then $K' = k'K$. K'/k' is regular, so K/k also is. Since $L \downarrow_k^{ACF} k'$ and K'/K is regular L and K' are linearly disjoint over K . This implies that the restriction map $\text{Gal}(L'/K') \rightarrow \text{Gal}(L/K)$ is surjective, so $k(a) \leq K \leq L$ is permissible.

The extension K'/K is regular, and K contains the parameters of the formula $\varphi(a, b)$. This formula is satisfied in K' if and only if it is satisfied in K (since φ is a test formula). From this we see that there is the same number of admissible fields in L and L' , and that an admissible

Figure 4.4: $K \downarrow_k^{ACF} k'$, K/k and k'/k are regular

field $k(a) \leq K \leq L$ satisfies the formula φ if and only if $K' := k'K$ satisfies it, so $\mu_V(\varphi(x, b)) = \mu_{V'}(\varphi(x, b))$. \square

We now recall and prove a few facts about field extensions which we will need for the proof of Proposition 4.63.

Remark 4.60. Let k be an e -free PAC field of characteristic 0, V an absolutely irreducible variety defined over some subfield $v \subseteq k$ and $a = a_1..a_k$ be a generic of V over k . Consider $P(X) \in k[X]$ a polynomial in a tuple over k in a tuple of variable X , with a an X -tuple. Then $P(a) = 0$ if and only if $P(x) = 0$ for all $x \in V(k)$. This means that the equality inside of $k(a)$ between two polynomials $P(a), Q(a) \in k(a)$ is definable by a formula over v in their coefficients.

Corollary 4.61. *Let k be an e -free PAC field of characteristic 0, V an absolutely irreducible variety defined over some subfield $v \subseteq k$ and a be a generic of V over k . Let x, y be two tuples of variables such that a is an x -tuple. Given a sequence of rational fractions $(f_i(x, y))_{i \leq n} \in \mathbb{Q}(x, y)$ there is a formula $\theta(y)$ over v such that $k \models \theta(b)$ if and only if $\sum_{i \leq n} f_i(a, b) \cdot X^i \in k(a)[X]$ is irreducible in $k(a)[X]$ for all y -tuples $b \in k$.*

Proof. To state that a polynomial is irreducible we need to quantify over the polynomials of smaller degree. The problem here is that this case the coefficients of such polynomials are in $k(a)$. However we can find a bound on the degree of the coefficients in a :

Let $P = \sum_{i \leq m} g_i(a) \cdot X^i \in k(a)[X]$ where $g_i = \frac{\sum_{j \leq n_i} \lambda_{i,j} \cdot \prod_l a_l^{n_{l,i,j}}}{\sum_{j \leq m_i} \mu_{i,j} \cdot \prod_l a_l^{m_{l,i,j}}}$ with $\lambda_{i,j}, \mu_{i,j} \in k$ and the polynomials $\sum_{j \leq n_i} \lambda_{i,j} \cdot \prod_l X_l^{n_{l,i,j}}$ and $\sum_{j \leq m_i} \mu_{i,j} \cdot \prod_l X_l^{m_{l,i,j}}$ are coprime in $k[\bar{Y}]$ where $\bar{Y} = Y_1..Y_k$.

We define the a -degree of P as:

$$\deg_a(P) = \max\{\sum_l(n_{l,i,j}) + \sum_l(m_{l,i,j}) : i \leq m, j \leq m_i\}.$$

By a straightforward induction of the degree of P we can show that if P is a unitary polynomial in $k(a)[X]$ and if Q is also unitary and divides P then $\deg_a(Q) < l$ for some integer l depending only on $\deg_a(P)$ and $\deg(P)$. With this in mind we can state that $\sum_{i \leq n} f_i(a, b) \cdot X^i \in k(a)[X]$ is irreducible in $k(a)[X]$ by stating:

$$Q \times S \neq \sum_{i \leq n} f_i(a, b) \cdot X^i \text{ for all unitary } Q, S \in k(a)[X] \text{ such that } \deg(Q), \deg(S) < \deg(P) \\ \text{and } \deg_a(Q), \deg_a(S) < l.$$

This is indeed a formula over v since $Q \times S \neq \sum_{i \leq n} f_i(a, b) \cdot X^i$ is a disjunction of inequalities in $k(a)$ which are expressible by formulas over v by Remark 4.60. \square

Remark 4.62. Let F be a field, $F(\bar{t})$ be a purely transcendental extension of F with $\bar{t} = t_1..t_m$ a transcendence basis and let $L = F(\bar{t})(\alpha)$ be an algebraic extension of degree n of $F(\bar{t})$. Suppose that $P(X) = X^n + \sum_{i < n} \lambda_i(\bar{t}) \cdot X^i$ is the minimal polynomial of α over $F(\bar{t})$. Then there exists some integer M depending only on $n = \deg(P)$ and on $\deg_{\bar{t}}(P)$ such that:

$$F^{alg} \cap L \subseteq \text{Lin}_F\left\{\left(\prod_i t_i^{k_i}\right) \cdot \alpha^j : k_i < M \text{ for all } i \leq m \text{ and } j < n\right\}.$$

Proof. The extension $F(\bar{t})/F$ is purely transcendental, so it is regular. The extension $L \cap F^{alg}/F$ is algebraic, so $F(\bar{t})$ and $L \cap F^{alg}$ are linearly disjoint over F . From this we deduce that the extension $L \cap F^{alg}/F$ has degree $\leq n$: Indeed any family of elements of $L \cap F^{alg}$ linearly independent over F is linearly independent over $F(\bar{t})$, so it should have size $\leq n = [F(\bar{t})(\alpha) : F(\bar{t})]$.

So if an element of L is algebraic over F it is witnessed by a polynomial of degree $\leq n$ in $F[X]$. Also:

$$\mathcal{B} := \left\{ \left(\prod_i t_i^{k_i} \right) \cdot \alpha^j : \bar{k} \in \omega^m \text{ and } j < n \right\}$$

is a basis of L as a F -vector space. From these two facts it is easy to see that if a t_i with too high exponent appears inside of some $x \in L$ when writing x as a linear combination of the elements of \mathcal{B} with coefficients in F , then $P(x) \neq 0$ for every polynomial $Q \in F[X]$ of degree $\leq n$. \square

The proof of the following proposition was hinted to me by Ramsey:

Proposition 4.63. *Let F be a monster e -free PAC field and V be an absolutely irreducible variety defined in F . Let $k \prec F$ be an elementary substructure over which V is defined. Then the measure μ_V is definable over k .*

Proof. Let $(P_i(X, x, y))_{1 \leq i \leq m}$ be the polynomials appearing in the test formula $\varphi(x, y)$. Let a be a generic of V over F . Let b be a y -tuple. We write $k_b := acl(k(b)) = F \cap (k(b))^{alg}$. Let L_b be the splitting field of the polynomials $P_i(X, a, b) \in k(a, b)[X]$ over $k_b(a)$. Since $k_b(a) \downarrow_{k_b}^{ACF} F$ and F/k_b is regular we have $G := \text{Gal}(FL_b/F(a)) \cong \text{Gal}(L/k_b(a))$.

Claim 1: There is a formula $\theta_1(y) \in \text{tp}(b/k)$ such that $\text{Gal}(FL_{b'}/F(a)) \cong G$ whenever $b' \models \theta_1(y)$.

Proof of Claim 1: We can write $L_b = k_b(a)(\alpha_b)$. Let $P_0(X) \in k_b(a)[X]$ be the minimal polynomial of α_b over $F(a)$, let n_0 be the degree of P_0 . There exist some polynomials $p_{i,j}(Z) \in k_b(a)[Z]$ of degree $\leq n_0$ with $p_{0,0}(Z) = Z$ and some polynomial $q((Z_{i,j})_{i,j}) \in k_b(a)[(Z_{i,j})_{i,j}]$ such that:

1. $P_0(X)$ is irreducible in $F(a)[X]$
2. $P_i(X, a, b) \equiv \prod_{j < n_i} (X - p_{i,j}(Z)) \pmod{P_0(Z)}$ for every $i \leq m$,
3. $q((p_{i,j}(Z))_{i,j}) \equiv Z \pmod{P_0(Z)}$

This can be expressed by a formula $\theta(y)$ with parameters in k : The coefficients are in $k_b(a)$ so we can take the algebraic formulas that they satisfy over k, b , by Corollary 4.61 we can state that the polynomial we get is irreducible since it has the same deg and deg_a as the P_0 of b . The congruence relations are witnessed by some equalities inside of $k(a)$, and so inside of k by Remark 4.60. Then the condition $\text{Gal}(P_0(X), F(a)) \cong G$ as permutation groups can be expressed using these polynomials.

Let $z_j := p_{0,j}(\alpha_b)$ for $1 \leq j \leq l$. If $\sigma \in \text{Gal}(P_0(X)/F(a))$ then there is a unique $\sigma' \in S_{n_0}$ such that $\sigma(z_i) = z_{\sigma'(i)}$ so we can see $\text{Gal}(P_0(X)/F(a))$ as a subgroup of S_{n_0} . Then $\text{Gal}(P_0(X), F(a)) \cong G$ is equivalent to:

$$\bigwedge_{\sigma \in G} \left(\bigwedge_{j=1}^l (p_{0,\sigma(j)}(z_1) = p_{0,j}(z_{\sigma(j)})) \right) \wedge \bigwedge_{\sigma \in S_{n_0} \setminus G} \left(\bigvee_{j=1}^l (p_{0,\sigma(j)}(z_1) \neq p_{0,j}(z_{\sigma(j)})) \right)$$

These equalities and inequalities in $F(a)[X]/(P_0(X))$ correspond to systems of equalities and inequalities in $F(a)$, so by Remark 4.60 they correspond to formulas over k . \square

Let $F_{L_b} := FL_b \cap F^{alg}$ and $H := \text{Gal}(F_{L_b}/F)$.

Claim 2: There is a formula $\theta_2(y) \in \text{tp}(b/k)$ strengthening θ_1 such that $\text{Gal}(F_{L_{b'}}/F) \cong H$ whenever $b' \models \theta_2(y)$.

Proof of Claim 2: Let $\beta_b = q(\alpha_b)$ be a generator of FL_b over F , with $q \in F(a)[X]$. Let $Q(X) \in F[X]$ be the minimal polynomial of $q(\alpha_b)$ over F and let $l := \deg(Q)$. Let P_0 be the polynomial from Claim 1. There exist some polynomials $q_i(Z) \in F[Z]$ for $i < l$ of degree $\leq l$ with $q_0(Z) = Z$ such that:

1. $Q(X) \in F[X]$ is irreducible in $F[X]$,
2. $Q(X) \equiv \prod_{j < n_i} (X - q_i(q(Z))) \pmod{P_0(Z)}$,
3. For every $x = g(\alpha_b) \in FL_b \cap F^{alg}$ with $g \in F(a)[X]$ there is $f \in F[X]$ of degree $\leq l$ such that $g(Z) \equiv f(q(Z)) \pmod{P_0(Z)}$.

These conditions are expressible by some formulas, for the first two ones it is clear. For the third one, in order to express that $FL_b \cap F^{alg} \subseteq F(\beta_b)$ it is enough to let g range over the polynomials of degree $\leq n$ that also have \deg_a smaller than M , where M is the bound from Remark 4.62 - so there is only a finite number of coefficients in F over which we have to quantify. As previously we can express that $\text{Gal}(Q, F) \cong H$ as a permutation group on the roots $q_i(Z) \in F[Z]$. \square

We have a formula expressing both $\text{Gal}(FL_{b'}/F(a))$ and $\text{Gal}(FL_{b'}/F)$ as permutation groups. An intermediate field $F(a) \subseteq F' \subseteq FL_{b'}$ is regular over F if and only if the restriction $\text{Gal}(FL_{b'}/F') \rightarrow \text{Gal}(FL_{b'}/F)$ is surjective. The conditions defining the measure $\mu_V(\varphi(x, b))$ are expressible by formulas over k :

- A subgroup $G' \subseteq G$ corresponds to a permissible field $F' = \text{Fix}(G')$ if and only if the restriction $G' \rightarrow H$ is surjective and if G' is e -generated.
- Given a subgroup $G' \subseteq G$ such that $\text{Fix}(G')$ is permissible we can express that $\text{Fix}(G') \models \varphi(a, b')$ by saying which roots of the P_i are fixed by the elements of G' and which are not. \square

4.3.2 ω -free PAC fields

Let F be an ω -free PAC field. By [CR23, Lemma 4.2] any ω -free PAC is elementary equivalent to an ultraproduct of e -free PAC fields. Replacing F by an elementarily equivalent field, we may assume that $F = \mathbb{F} := \prod_{e < \omega} \mathbb{F}_e / \mathcal{U}$, where the \mathbb{F}_e are κ -saturated e -free PAC field of characteristic 0 for $e < \omega$, and \mathcal{U} is a non-principal ultrafilter.

Let G be a group definable in \mathbb{F} over a finite set A . The group G can be written as $G = \prod_{e < \omega} G_e / \mathcal{U}$ where G_e is a group definable in \mathbb{F}_e over some finite set of parameters A_e for \mathcal{U} -almost all $e < \omega$.

Using the measures from the previous subsection Chatzidakis and Ramsey prove the following:

Proposition 4.64. [CR23, Proposition 4.13] *Let G be a group definable in an ω -free PAC field of characteristic 0. Then G is definably amenable.*

In this context every group G_e in M_e is definably amenable with a measure μ_e which is invariant under translations, is such that $\mu(G_e) = 1$ and is definable over an e -free PAC field $A_e \subseteq M_e \prec \mathbb{F}_e$ by Proposition 4.63. G is also definably amenable with the ultraproduct measure $\mu = \prod_{e < \omega} \mu_e / \mathcal{U}$, which is also invariant under translation. Let $M := \prod_{e < \omega} M_e / \mathcal{U} \prec \mathbb{F}$.

In the previous subsection we saw that the definable amenability of definable groups in e -free PAC fields with e finite is proven using the fact that such a group is definably isogenous to the rational points of an algebraic group. This raises an insightful question as to whether the same applies to ω -free PAC field. With the previous results from this chapter we get a partial answer:

Theorem 4.65. *There exist an algebraic group definable over M , a type-definable subgroup G_0 of bounded index in G and a definable finite to one group homomorphism from G_0 to $H(\mathbb{F})$.*

Proof. We apply Theorem 4.50. By Lemma 4.26 the assumption (H1) is satisfied. (H2) follows from Proposition 4.51. We want to show that (H3) holds, for this we construct an expansion of the language.

By Lemma 4.43 and Lemma 4.43 there is a countable extension L' of the language L_{ring} and a canonical L' -structure on \mathbb{F} such that the group G is definably amenable in L' by a measure μ' extending μ which is definable over M and invariant under translations. We will write \mathbb{F}' for \mathbb{F} seen as an L' -structure.

We have a definably amenable group in L' , so the ideal of formulas of measure 0 is $S1$ on G , M -invariant and invariant under translation, so (H3) holds. By Theorem 4.50 there is a type-definable subgroup G_0 of G of bounded index, an algebraic group H and a definable finite to one group homomorphism φ from G_0 to $H(\mathbb{F})$. \square

Remark 4.66. We can ask ourselves if we can get a stronger conclusion: In fact the subgroup G_0 is wide and the group homomorphism φ from G_0 to $H(\mathbb{F})$ is defined on a domain X_φ , which is a definable set included in G containing G_0 . So $\mu(X_\varphi) > \frac{1}{n}$ for some $n < \omega$. This implies that any group containing X_φ has finite index, and so finding an extension of φ to such a group would strengthen the conclusion to G_0 being of finite index.

Remark 4.67. Theorem 4.65 does not hold for interpretable groups: In [HP94] (p. 207) Hrushovski and Pillay point out that if F is a PAC field of characteristic 0 such that the subgroup of n -th powers have infinite index in (F^*, \cdot) , then the group $G = F^*/(F^*)^n$ can not be abstractly isomorphic to $H(F)$ for H any algebraic group. More generally no type-definable subgroup of G of bounded index can be sent by a morphism of finite kernel into $H(F)$.

4.4 Some questions

The main difficulty we face when trying to prove some result on the Kim-stabilizer is that without additional assumptions it is difficult to show that the set $S^K(p)$ is not trivial, or that it is large in some sense for a certain type. In the proof of Theorem 4.50 we use the existence of the ideal μ to know that the type defining $S^K(p)$ is μ -wide.

We conclude this chapter with some questions that we collected from the different parts of this work:

1. Is there an interesting notion of genericity in groups definable in NSOP₁ theories?
2. Is $Stab^K(p)$ type-definable for every type p ? Is it always equal to $S^K(p) \cdot S^K(p)$?
3. More accurately, if X is a definable or type-definable subset of a group G definable in an NSOP₁ theory that is stable by Kim-independent product (over the parameters defining the group), is the subgroup generated by X definable or type-definable?
4. When does $S^K(p)$ contain a translate of p ?
5. Consider a measure μ on definable subsets of a definable group G which is invariant by translation in a L -theory T . Are there some conditions under which there exist an extension T' of T in some language L' and an extension of μ into a measure μ' on L' -definable subsets of G that is definable and also invariant by translation?

Chapter 5

Appendix: Weakly stable Kim-forking



Figure 5.1: Le Chiengue

In this section we consider Kim-forking at the level of formulas. The stable forking conjecture is the conjecture that in a simple theory, if a complete type forks over a set of parameter, then there is a stable formula inside of this type which forks. In short, that forking is witnessed by stable formulas.

A generalization of the notion of stable forking to the context of NSOP₁ theories, namely weak stable Kim-forking, is given here. We connect this notion to some results known about stable forking which can be found in [CP18]. It is known that if a theory T has stable forking then the imaginary extension T^{eq} also does. However I could not prove the equivalent result here for some reasons discussed in Section 5.

We recall some basic notions around local types. Let $\varphi(x, y)$ be a formula.

Definition 5.1. A φ -formula over A is a boolean combination of formulas of the form $\varphi(x, a)$ for $a \in A$. A *generalized φ -formula over A* is a formula over A that is equivalent to a boolean combination of φ -formulas, possibly with parameters outside of A .

A *complete φ -type over A* (resp. *complete generalized φ -type over A*) p is a consistent set of φ -formulas (resp. generalized φ -formulas) over A such that given any $\psi(x)$ φ -formula (resp. generalized φ -formula) over A either $\psi(x) \in p$ or $\neg\psi(x) \in p$.

The set of complete φ -types (resp. generalized φ -types) over A is written $S_\varphi(A)$ (resp. $S_{\varphi^*}(A)$) and the φ -type (resp. generalized φ -type) of a over A is written $\text{tp}_\varphi(a/A)$ (resp. $\text{tp}_{\varphi^*}(a/A)$).

Definition 5.2. A φ -type p is said to be definable over some set of parameters A if there is a formula $d_px\varphi(x, y)$ with parameters in A such that $\varphi(x, b) \in p$ if and only if $\models d_px\varphi(x, b)$.

Fact 5.3. [KP97, Chapter 2] If $\varphi(x, y, e)$ is a stable formula and M is a model such that $e \in M$, then every $\varphi(x, y, e)$ -type p over M is definable over M and has a unique non-forking extension, which has the same definition. We write $d_px\varphi(x, y, e)$ its definition and $Cb(d_px\varphi(x, y, e))$ the canonical parameter of this definition.

The previous result holds more generally for types over algebraically closed sets in T^{eq} .

Fact 5.4. [Cas11, Proposition 6.13]: Let $\varphi(x, y)$ be a stable formula and $E = \text{acl}^{eq}(E) \subseteq B$. Then any generalized φ -type $p_\varphi \in S_{\varphi^*}(E)$ is definable over E , and it has a unique extension $q \in S_{\varphi^*}(B)$ which is also definable over E . The formula $d_px\varphi(x, y)$ defining p then admits a canonical parameter in E , which we write $Cb(d_px\varphi(x, y))$.

5.1 Weak stable Kim-forking over models

In this section we work in any complete theory T in a language \mathcal{L} , all sets of parameters and elements are assumed to be in a monster model \mathbb{M} of T . We show some basic results about what can and can not happen when Kim-dividing is witnessed by stable formulas, and isolate the notion of ‘weak stable Kim-forking’ as the correct generalization of stable forking in simple theories to the NSOP₁ context. In this section we only consider types over models and Kim-forking over models.

Definition 5.5. A theory T has *stable Kim-forking over models* (resp. *stable forking over models*) if whenever $a \not\downarrow_M^K B$ for $M \subseteq B$ and M a model (resp. $a \not\downarrow_M^f B$) there is a stable formula $\varphi(x, y) \in \mathcal{L}$ and $b \in B$ such that $a \models \varphi(x, b)$ and $\varphi(x, b)$ Kim-forks (resp. forks) over M .

Definition 5.6. A theory T has *weak stable Kim-forking over models* (resp. *weak stable forking over models*) if whenever $a \not\downarrow_M^K B$ (resp. $a \not\downarrow_M^f B$) for $M \subseteq B$ and M a model there is a stable formula $\varphi(x, y, m) \in \mathcal{L}_m$ (i.e. the formula with fixed $m \in M$ is stable) and $b \in B$ such that $\models \varphi(a, b, m)$ and $\varphi(x, b, m)$ Kim-forks (resp. forks) over M .

It is known that a theory with stable forking is simple [CP18, Proposition 1.3]. We show that a theory with stable Kim-forking over models is simple and that a theory with weak stable forking over models is simple. We also show that a theory with weak stable Kim-forking over models is NSOP_1 and not necessarily simple.

Lemma 5.7. *Let $M \subseteq N$ be models of a theory T with weak stable forking over models. For any type $p = \text{tp}(a/N)$, $a \downarrow_M^f N$ if and only if for every stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $\text{Cb}(d_p x \varphi(x, y, m))$ is in M .*

Proof. [\implies] By Fact 5.3 we know that the $\varphi(x, y, m)$ -type of a over M is definable over M and admits a unique non-forking extension to N , which has the same definition over M , so in particular the same canonical parameter $\text{Cb}(d_p x \varphi(x, y, m)) \in M$.

[\impliedby] Conversely assume that p forks over M and that M contains the canonical parameters. Then there is a stable formula $\varphi(x, y, m)$ with $m \in M$ and $n_0 \in N$ such that $\varphi(x, n_0, m) \in p$ and $\varphi(x, n_0, m)$ forks over M .

So there is an M -indiscernible sequence $(n_i)_{i < \omega}$ such that $\{\varphi(x, n_i, m) : i < \omega\}$ is inconsistent. Now since $\text{Cb}(d_p x \varphi(x, y, m)) \in \text{acl}^{\text{eq}}(M)$ by indiscernibility $\models d_p x \varphi(x, n_i, m)$ for all $i < \omega$, so $\{\varphi(x, n_i, m) : i < \omega\}$ is satisfied by any realization of a non-forking extension of p to $N(n_i)_{i < \omega}$, a contradiction. \square

Proposition 5.8. *Let T be a theory with weak stable forking over models, then T is simple.*

Proof. We show that \downarrow^f satisfies local character. Let a be a finite tuple, M a model, $p = \text{tp}(a/M)$. We consider the set $C_0 = \{\text{Cb}(d_p x \varphi(x, y)) : \varphi(x, y) \in \mathcal{L} \text{ a stable formula}\}$. We take a model $M_0 \subseteq M$ such that $|M_0| = |T|$ and M_0 contains a representative of every element of C_0 .

We iterate the process taking $C_{i+1} = \{\text{Cb}(d_p x \varphi(x, y, e)) : \varphi(x, y, e) \in \mathcal{L}_e \text{ a stable formula with } e \in M_i\}$ and $M_i \subseteq M_{i+1} \subseteq M$ such that $|M_{i+1}| = |T|$ and M_{i+1} contains a representative of every element of C_{i+1} . Let $M' := \bigcup_{i < \omega} M_i$. By Lemma 5.7 $a \downarrow_{M'}^f M$ and M' has size $|T|$. \square

Lemma 5.9. *Let \mathbb{M} be a monster model of a theory T with weak stable Kim-forking over models. Let $M \subseteq B \subseteq \mathbb{M}$ with M a model and $p := \text{tp}(a/B)$ a type. Then $a \downarrow_M^K B$ if for every stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $\text{Cb}(d_p x \varphi(x, y, m))$ is in M .*

Proof. Assume that p Kim-forks over M and that M contains all of the canonical parameters. There is a stable formula $\varphi(x, y, m)$ with $m \in M$ and $n_0 \in N$ such that $\varphi(x, n_0, m) \in p$ and $\varphi(x, n_0, m)$ Kim-forks over M . So there is an M -Morley sequence $(n_i)_{i < \omega}$ such that $\{\varphi(x, n_i, m) : i < \omega\}$ is inconsistent.

By assumption $\text{Cb}(d_p x \varphi(x, y, m)) \in M$, so by indiscernibility $\models d_p x \varphi(x, n_i, m)$ for all $i < \omega$, and $\{\varphi(x, n_i, m) : i < \omega\}$ is satisfied by any realization of a non-forking extension of p to $N(n_i)_{i < \omega}$, which contradicts the fact that $\varphi(x, n_0, m)$ Kim-forks over M . \square

Proposition 5.10. *Let T be a theory with weak stable Kim-forking over models, then T is NSOP_1 .*

Proof. We show that \downarrow^K satisfies the local character criterion of Proposition 1.54 to show that T is NSOP_1 . Assume that T has weak stable Kim-forking and that there exists a continuous increasing sequence of models of size $|T|$ $(M_i)_{i < \kappa}$ for $\kappa = |T|^+$ and a finite tuple a such that $a \not\downarrow_{M_i}^K M_{i+1}$ for all $i < \kappa$. Let $M := \bigcup_{i < \kappa} M_i$ and $p := \text{tp}(a/M)$.

For every $i < \kappa$ the number of stable formulas with parameters in M_i is at most $|T|$, since $\kappa > |T|$ is regular there is a smallest index $\alpha_i \geq i$ such that for every stable formula $\varphi(x, y, e)$ with $e \in M_i$, $\text{Cb}(d_p x \varphi(x, y, e)) \in M_{\alpha_i}$. Now let us consider $(\beta_i)_{i < \omega}$ defined by induction by $\beta_0 = 0$

and $\beta_{i+1} = \alpha_{\beta_i}$. Take $\beta = \bigcup_{i < \omega} \beta_i$, by continuity $\alpha_\beta = \beta$, so $a \downarrow_{M_\beta}^K M$ by Lemma 5.9, which contradicts $a \not\downarrow_{M_\beta}^K M_{\beta+1}$. \square

Corollary 5.11. *Let T be a theory with stable Kim-forking over models, then T is simple.*

Proof. By Proposition 5.10 we know that T is NSOP₁. If T is non-simple we can build an arbitrary long sequence of Kim-forking extensions with an instance of TP₂ (as in [CK20, Proposition 3.6]). Now we can run the same proof as in [CP18, Proposition 1.3]: There is a stable formula with a dividing chain of infinite length and hence with the tree property, which is not possible. \square

Lemma 5.12. *Let T be any theory. If $p = \text{tp}(a/M)$ is a complete type over a model M , $M \subseteq N$ is a model, and $\varphi(x, y, m)$ is a stable formula for some fixed $m \in M$, then there is a unique complete φ -type over N consistent with p that does not Kim-fork over M . The unique global $\varphi(x, y, e)$ -type q_φ such that $p \cup q_\varphi$ does not Kim-fork over M is definable over M , we write $d_p^K x\varphi(x, y, e)$ its definition and $\text{Cb}(d_p^K x\varphi(x, y, e))$ the canonical parameter of this definition.*

Proof. Assume that there are $a', a'' \models p$ such that $a' \downarrow_M^K N$, $a'' \downarrow_M^K N$ and that for some $b \in N$, $\models \varphi(a', b, m)$ and $\models \neg\varphi(a'', b, m)$. Consider an M -coheir Morley sequence $(N_i)_{i < \omega}$ such that $N = N_0$. Let $(b_i)_{i < \omega}$ be the corresponding sequence with $b_0 = b$, and let a'_i, a''_i be such that $a'_i N_i \equiv_M a' N_0$ and $a''_i N_i \equiv_M a'' N_0$, so $\models \varphi(a'_i, b_i, m)$ and $\models \neg\varphi(a''_i, b_i, m)$ for every $i < \omega$.

By applying the amalgamation theorem for Kim-forking over M along the sequence N_i we can construct a sequence $(a_i)_{i < \omega}$ such that $\models \varphi(a_i, b_j, m)$ iff $i < j$, contradicting the stability of $\varphi(x, y, m)$. So $\varphi(x, b, e) \in q_\varphi$ iff $p \cup \{\varphi(x, b, e)\}$ does not Kim-fork over M , and the complement of this set is $\{b : p \cup \{\varphi(x, b, e)\} \text{ Kim forks over } M\}$. Both of these sets are M -type definable, so they are definable over M . \square

Remark 5.13. By Fact 5.3 we know that there is a unique non-forking extension of $p \in S_\varphi(M)$ to any set of parameters, and by the above result we know that there is a unique non Kim-forking extension. So these two extension coincide, and we get that the Shelah definition $d_p^K x\varphi(x, y, e)$ coincides with the definition $d_p^K x\varphi(x, y, e)$ from Lemma 5.12.

Lemma 5.14. *Let $M \subseteq N$ be models in a theory T with weak stable Kim-forking over models. Let $q = \text{tp}(a/N)$ and $p = \text{tp}(a/M)$. Then $a \downarrow_M^K N$ if and only if for every stable formula $\varphi(x, y, m)$ with $m \in M$ the canonical parameter $\text{Cb}(d_q^K x\varphi(x, y, m)) \in N$ is equal to $\text{Cb}(d_q^K x\varphi(x, y, m)) \in M$. These two canonical parameters are equal if $\text{Cb}(d_q^K x\varphi(x, y, m)) \in M$.*

Proof. [\implies] Assume that $a \downarrow_M^K N$ and that $\text{Cb}(d_p^K x\varphi(x, y, e)) \notin M$ for some stable formula $\varphi(x, y, m)$ with $m \in M$. Let $p_\varphi := \text{tp}_\varphi(a/M)$. Lemma 5.12, which we can apply by Proposition 5.10, yields that there is a unique complete $\varphi(x, y, m)$ -type q_φ over N such that $p \cup q_\varphi$ does not Kim-fork over M . In particular since a non-forking extension always exists we get that $p \cup q_\varphi$ does not fork over M , so q_φ coincides with the unique non-forking extension of p_φ to N , so q_φ and p_φ have the same definition and the same canonical parameter.

[\impliedby] Is just Lemma 5.9. \square

5.2 Over algebraically closed sets

In this section we work in a monster model \mathbb{M} of a theory T with existence. We can define weak stable Kim-forking over algebraically closed sets in a similar fashion. We will make a small use of hyperimaginaries. As previously we will consider stable formulas $\varphi(x, y, e)$ for some fixed parameters, for convenience we will sometimes shorten to φ and write $\text{tp}_\varphi(a/A)$ instead of $\text{tp}_{\varphi(x, y, e)}(a/A)$ for example.

Definition 5.15. A theory T has *weak stable Kim-forking* if whenever $a \not\downarrow_E^K B$ for $E = \text{acl}(E) \subseteq B$ there is a stable formula $\varphi(x, y, e) \in \mathcal{L}_e$ (i.e. the formula with fixed $e \in E$ is stable) and $b \in B$ such that $a \models \varphi(x, b, e)$ and $\varphi(x, b, e)$ Kim-forks over E .

Remark 5.16. By Fact 5.4 we know that if $\varphi(x, y)$ is a stable formula and $E = \text{acl}^{eq}(E) \subseteq B$ then any generalized φ -type $p_{\varphi^*} \in S_{\varphi^*}(E)$ has a unique extension $q \in S_{\varphi^*}(B)$ which is also definable over E . It is easy to check that this extension does not Kim-fork over E :

Write p as $p(x, B)$. If $(B_i)_{i < \omega}$ is an E -Morley sequence then by indiscernibility the unique extension $q \in S_{\varphi^*}((B_i)_{i < \omega})$ which is also definable over E implies $\cup_{i < \omega} p(x, B_i)$, so this type is consistent.

Remark 5.17. [KP01, Remark 1.3] If $\varphi(x, a) \equiv \psi(x, b)$ and $\psi(x, z)$ is stable, then for some $\mu(y) \in \text{tp}(a)$, $\varphi(x, y) \wedge \mu(y)$ is stable. In particular if $\psi(x, b)$ is a generalized φ -formula for some stable formula $\varphi(x, y)$, $\psi(x, y)$ might not be stable, but adding a fragment of $\text{tp}(b)$ will give a stable formula.

The proof of the following result works similarly as the one of Lemma 5.12. Here we need p to be a Kim-amalgamation base in order to apply the independence theorem, and Remark 5.17 in order to have stable formulas.

Lemma 5.18. *Assume that T is NSOP₁. If $p = \text{tp}(a/E)$ is a Kim-amalgamation base, $E \subseteq B$, and $\varphi(x, y, e)$ is a stable formula for some fixed $e \in E$, then there is a unique complete generalized φ -type p_{φ^*} extending p that does not Kim-fork over E and it is definable over E . In particular, for every generalized φ -formula $\psi(x, b)$ with $b \in B$, exactly one of $p \cup \{\psi(x, b)\}$ and $p \cup \{\neg\psi(x, b)\}$ Kim-forks over E .*

Proof. Assume that there are $a', a'' \models p$ such that $a' \not\downarrow_E^K B$, $a'' \not\downarrow_E^K B$ and that for some generalized φ -formula $\psi(x, b)$ with $b \in B$, $\models \psi(a', b)$ and $\models \neg\psi(a'', b)$. By Remark 5.17 there is a formula $\mu(y) \in \text{tp}(b/E)$ such that $\psi(x, y) \wedge \mu(y)$ is stable. Consider an E -Morley sequence $(b_i)_{i < \omega}$ such that $b = b_0$.

Let a'_i, a''_i be such that $a'_i b_i \equiv_E a' b_0$ and $a''_i b_i \equiv_E a'' b_0$. Then $\models \psi(a'_i, b_i) \wedge \mu(b_i)$ and $\models \neg\psi(a''_i, b_i) \wedge \mu(b_i)$ for every $i < \omega$. By applying the amalgamation theorem for Kim-forking over E along the sequence b_i we can construct a sequence $(a_i)_{i < \omega}$ such that $\models \psi(a_i, b_j) \wedge \mu(b_j)$ iff $i < j$, contradicting the stability of $\psi(x, y) \wedge \mu(y)$.

Consider some generalized φ -formula $\psi(x, b)$ with $b \in B$. $\psi(x, b) \in p_{\varphi^*}$ if and only if $p \cup \{\psi(x, b)\}$ does not Kim-fork over E , and by completeness $\psi(x, b) \notin p_{\varphi^*}$ if and only if $p \cup \{\neg\psi(x, b)\}$ does not Kim-fork over E . Both of these sets are E -type definable, so they are definable over E . \square

Using both results of uniqueness we get that Lemma 5.18 holds more generally for generalized φ -types over algebraically closed sets in T^{eq} .

Corollary 5.19. *Assume that T is NSOP₁. If $p = \text{tp}_{\varphi^*}(a/E)$ with $E = \text{acl}^{eq}(E)$, $E \subseteq B$, and $\varphi(x, y, e)$ is a stable formula for some fixed $e \in E$, then there is a unique complete generalized φ -type $p_{\varphi^*} \in S_{\varphi^*}(B)$ such that $p \cup p_{\varphi^*}$ does not Kim-fork over E , and such a type does not Kim-fork over E iff it is definable over E .*

Proof. Fix some completion $p' \in S(\text{bdd}(E))$, by Lemma 5.18 there is a unique $p'_{\varphi^*} \in S_{\varphi^*}(B)$ such that $p'_{\varphi^*} \cup p'$ does not Kim-fork over E (we recall that a type Kim-forks over E if and only if it Kim-forks over $\text{bdd}(E)$). So it coincides with the unique type p_{φ^*} over B that is definable over E from Remark 5.16. \square

Remark 5.20. Corollary 5.19 implies that in any NSOP₁ theory Kim-forking is transitive for complete generalized φ -types over algebraically closed sets in the imaginaries: Let $E = \text{acl}^{eq}(E) \subseteq$

$B = \text{acl}^{\text{eq}}(B) \subseteq D$, $p = \text{tp}_{\varphi^*}(a/E)$, $p' = \text{tp}_{\varphi^*}(a/B)$ and $p'' = \text{tp}_{\varphi^*}(a/D)$. If p'' does not Kim-fork over B and p' does not Kim-fork over E then p'' does not Kim-fork over E .

Definition 5.21. If $p = \text{tp}_{\varphi^*}(a/E)$ for $E = \text{acl}^{\text{eq}}(E)$, $E \subseteq B$, and $\varphi(x, y, e)$ is a stable formula for some fixed $e \in E$, then the unique global $\varphi(x, y, e)$ -type q_φ extending $p_\varphi := \text{tp}_{\varphi_e}(a/E)$ without forking over E is definable over E , we write $d_p^K x\varphi(x, y, e)$ its definition and $\text{Cb}(d_p^K x\varphi(x, y, e))$ the canonical parameter of this definition.

Remark 5.22. By uniqueness we get that when both the Shelah definition $d_p x\varphi(x, y, e)$ and the previous one exist they coincide. By considering a non-Kim-forking extension of $\text{tp}(a/E)$ to a model M we get that the definition $d_p^K x\psi(x, y, e)$, which is a formula with parameters in $\text{acl}^{\text{eq}}(E)$, is equivalent to a positive boolean combination of formulas of the form $\psi(a', y)$ with $a' \in M$.

Corollary 5.23. *Assume that T has weak stable Kim-forking. If $p = \text{tp}(a/B)$ with $B = \text{acl}^{\text{eq}}(B)$ and $E = \text{acl}(E) \subseteq B$, then $a \downarrow_E^K B$ if and only if $\text{Cb}(d_p x\varphi(x, y, e)) \in \text{acl}^{\text{eq}}(E)$ for every stable formula $\varphi(x, y, e)$ with $e \in E$.*

Proof. [\Leftarrow] Follows from a similar proof as the one of Lemma 5.9.

[\Rightarrow] Let $p' := \text{tp}(a/\text{acl}^{\text{eq}}(E))$. Assume that $a \downarrow_E^K B$ and consider some stable formula $\varphi(x, y, e)$ with $e \in E$. By Corollary 5.19 we know that there is a unique complete $\varphi(x, y, e)$ -type q_φ over B such that $p' \cup q_\varphi$ does not Kim-fork over E , so this type coincides with the $\varphi(x, y, e)$ -type of a over B since $a \downarrow_E^K B$, so they have the same definition and the same canonical parameter, which lies in $\text{acl}^{\text{eq}}(E)$. \square

Remark 5.24. [CP18, Remark 1.2] Let $\varphi(x, y, e)$ be a stable formula (with some parameters e). If $\theta(v, x, e) \vdash \exists^{=n} x\theta(v, x, e)$ then $\psi(v, y, e) = \exists x(\theta(v, x, e) \wedge \varphi(x, y, e))$ is also stable.

We will use this fact to get a stable formula from another one by replacing the variables by some ‘larger variables’ in the sense of the algebraic closure.

Remark 5.25. Corollary 5.23 implies that if T is a theory with existence and weak stable Kim-forking, given some sets $E = \text{acl}^{\text{eq}}(E) \subseteq B = \text{acl}^{\text{eq}}(B)$ and a tuple a there is a smallest subset $E \subseteq B_0 = \text{acl}^{\text{eq}}(B_0) \subseteq B$ such that $a \downarrow_{B_0}^K B$, in the sense that any $E \subseteq B' = \text{acl}^{\text{eq}}(B') \subseteq B$ such that $a \downarrow_{B'}^K B$ must contain B_0 . So T^{eq} has weak canonical bases in the sense of [Kim21, Definition 4.1].

Lemma 5.26. *Assume that T is NSOP₁ and has existence. Let $E \subseteq B \subseteq D$ be sets of parameters, a be a tuple such that $a \downarrow_B^K D$, $\varphi(x, y, e)$ be a stable formula with $e \in E$ and $d \in D$ be such that $\varphi(x, d, e)$ Kim-forks over E and $\models \varphi(a, d, e)$. Then there is a stable formula $\psi(x, y, e)$ and some $b \in B$ such that $\psi(x, b, e)$ Kim-forks over E and $\models \psi(a, b, e)$.*

Proof. Let $p = \text{tp}_{\varphi^*}(a/\text{acl}^{\text{eq}}(E))$, $p' := \text{tp}_{\varphi^*}(a/\text{acl}^{\text{eq}}(B))$ and $p'' = \text{tp}_{\varphi^*}(a/D)$.

Then $\text{Cb}(d_p^K x\psi(x, y, e)) \in \text{acl}^{\text{eq}}(B) \setminus \text{acl}^{\text{eq}}(E)$. In fact since $\varphi(x, d, e)$ Kim-forks over E the definition of a $\varphi(x, y, e)$ -type containing $\varphi(x, d, e)$ cannot lie in $\text{acl}^{\text{eq}}(E)$. On the other hand $\text{tp}_{\varphi^*}(a/D)$ does not Kim-fork over B , so by uniqueness and Definition 5.21 we get that the definition of $\text{tp}_{\varphi^*}(a/B)$ and the one of $\text{tp}_{\varphi^*}(a/D)$ coincide, so it lies in $\text{acl}^{\text{eq}}(B)$.

p' Kim-forks over E by Remark 5.20, so there is a generalized φ -formula $\psi(x, b')$ with $b' \in \text{acl}^{\text{eq}}(B)$ such that $\models \psi(a, b')$ and $\psi(a, b')$ Kim-forks over E . By Remark 5.17 we can assume that $\psi(x, y)$ is stable. Take $b \in B$ such that b' is algebraic over b , $\theta(z, y, e)$ a formula witnessing this. By considering a strengthening of θ we can assume that whenever $\models \theta(\beta', b)$ then $\varphi(x, \beta')$ Kim-divides over E . By Remark 5.24 the formula $\exists y(\theta(z, y, e) \wedge \psi(x, y, e))$ is stable, is satisfied by a, b and $\exists y(\theta(b, y, e) \wedge \psi(x, y, e))$ Kim-forks over E . \square

It is enough to show weak stable Kim forking for types over models:

Lemma 5.27. *Let T be a theory with existence such that whenever $a \not\downarrow_E^K N$ for N a model and $E = \text{acl}(E) \subseteq N$ there is $e \in E$, $n \in N$, $\varphi(x, y, e)$ a stable formula such that $\models \varphi(a, n, e)$ and $\varphi(x, n, e)$ Kim-forks over E . Then T has weak stable Kim-forking.*

Proof. Let $a, E \subseteq B$ such that $a \not\downarrow_E^K B$. Let N be a model containing B such that $a \not\downarrow_B^f N$. So $a \not\downarrow_E^K N$ and by assumption there is a stable formula $\psi(x, y, e)$ such that $\psi(x, n, e)$ Kim-forks over E for some $n \in N$. By Lemma 5.26 there is a stable formula $\varphi(x, y, e)$ and some $b \in B$ such that $\varphi(x, b, e)$ Kim-forks over E and $\models \varphi(x, y, e)$. The fact that T is NSOP₁, which we need to use Lemma 5.26, is a consequence of the fact that the hypothesis of this Lemma implies weak stable Kim-forking over models, which implies NSOP₁ by Proposition 5.10. \square

Examples of theories with weak stable Kim-forking:

1 - The first example we give here is the theory T_{feq}^* of parameterized equivalence relations. This theory has two sorts O for objects and P for parameters. The language is $\mathcal{L} = \{\equiv\}$, with $\equiv \subseteq P \times O^2$. We write $x \equiv_p y$ to mean that $(p, x, y) \in \equiv$. T_{feq}^* expresses that for any parameter $p \in P$ the relation \equiv_p is an equivalence relation on O with an infinity of classes all of which are infinite, and that for different parameters these relations interact randomly (see [CR16, Section 6.3] and [Bos23] for more details on this theory).

T_{feq}^* has existence, and for arbitrary sets $E \subseteq A, B$:

1. $A \not\downarrow_E^K B$ iff $A \cap B = E$ and $O(A) \not\downarrow_{O(E)}^p O(B)$ for every $p \in P(E)$ where $\not\downarrow^p$ is forking independence in the theory T_∞ of an equivalence relation with an infinite number of classes all infinite.
2. $A \not\downarrow_E^f B$ iff $A \cap B = E$ and $O(A) \not\downarrow_{O(E)}^p O(B)$ for every $p \in P(B)$.

The first condition is witnessed by stable formulas (namely equalities). For the second one, the difference between Kim-forking and forking here is that in the first case the parameter is inside of the basis.

For a given parameter p the formula $\varphi(x, y) = (x \equiv_p y)$ is stable since it defines an equivalence relation, so Kim-forking is always witnessed by stable formulas in this theory. However if we do not fix p , the formula $\psi(x, yz) := (x \equiv_z y)$ is unstable, so forking is not always witnessed by stable formulas in this theory (which we already knew from Proposition 5.8).

2 - The theory of vector spaces of infinite dimension over a field with weak stable Kim-forking (so in particular over a stable field) with a generic bilinear form sT_∞^K (which are studied in the Chapter 3) is an other example. A difference with the previous example is that in this case there is a non-trivial algebraic closure, and that Kim-independence is characterized only for algebraically closed structures: If $E = \text{acl}(E) \subseteq A = \text{acl}(A), B = \text{acl}(B)$ then $A \not\downarrow_E^K B$ if and only if $\langle A \rangle \cap \langle B \rangle = \langle E \rangle$ and $K(A) \not\downarrow_{K(E)}^K K(B)$.

The failure of one of these two conditions is witnessed by some stable formula, but for sT_∞^K to have weak stable Kim-forking we need this witness to exist even when A and B are not algebraically closed. This is what we prove now.

For this we use Remark 5.24 and the fact that in an NSOP₁ theory with existence $A \not\downarrow_E^K B$ if and only if $\text{acl}(EA) \not\downarrow_{\text{acl}(E)}^K \text{acl}(EB)$ for any sets E, A, B .

Assume that $a \not\downarrow_{\text{acl}(E)}^K b$, then $\text{acl}(Ea) \not\downarrow_{\text{acl}(E)}^K \text{acl}(Eb)$, so there is a stable formula $\varphi(x, y, e)$ with $e \in \text{acl}(E)$ and $\bar{a} \in \text{acl}(Ea), \bar{b} \in \text{acl}(Eb)$ such that $\models \varphi(\bar{a}, \bar{b}, e)$ and $\varphi(x, \bar{b}, e)$ Kim-divides over E . So we can assume that $\bar{a} \in \text{acl}(ea)$ and $\bar{b} \in \text{acl}(eb)$.

Let $\theta_a(\bar{x}, a, e)$ be a formula witnessing the fact that $\bar{a} \in \text{acl}(Ea)$. Let $\theta_b(\bar{y}, b, e)$ be a formula witnessing the fact that $\bar{b} \in \text{acl}(Eb)$, by strengthening θ_b we can assume that $\varphi(x, \bar{b}', e)$ Kim-divides over E whenever $\models \theta_b(\bar{b}', b, e)$. For this we eventually have to enlarge the tuple e .

By Remark 5.24 the formula $\psi(x, y, e) := \theta_a(\bar{x}, x, e) \wedge \theta_b(\bar{y}, y, e) \wedge \varphi(\bar{x}, \bar{y}, e)$ is stable, by construction $\models \psi(a, b, e)$ and the formula $\psi(x, b, e)$ Kim-forks over E .

3 - With a similar proof we can show that if an NSOP₁ theory satisfies that $\downarrow^K = \downarrow^a$ over arbitrary sets then it has weakly stable Kim-forking. In fact in that case $a \not\downarrow_E^K b$ is witnessed by $\exists x \exists y (\varphi(x, a, e) \wedge \psi(y, b, e) \wedge x = y)$ with $\varphi(x, a, e)$ and $\psi(y, b, e)$ algebraic formulas, and the stable formula we apply Remark 5.24 to is simply $(x = y)$. Examples of such a theories are given in the Chapter 2 of this manuscript.

5.3 The question of imaginaries

It is shown in section 2 of [CP18] that if a theory T has stable forking so does T^{eq} . We can ask the same question about weak stable Kim-forking. Here we adapt [CP18, Proposition 2.2] and then discuss the reasons why we can not adapt the proof of Casanovas in its entirety. In this section we work with a theory T with existence and weak stable Kim-forking.

Proposition 5.28. *If T has weak stable Kim-forking and existence, then T^{eq} has weak stable Kim-forking over real parameters: Whenever $a_F \not\downarrow_E^K B$ for E an algebraically closed set of real elements, $B \supseteq E$ possibly containing imaginaries and $a_F \in \mathbb{M}^{eq}$ there is a stable formula $\varphi(x, y, e)$ with $e \in E$ and $b \in B$ such that $a_F \models \varphi(x, b, e)$ and $\varphi(x, b, e)$ Kim-forks over E .*

Proof. Let a_F, E, B be as in the statement. We choose a saturated model $M \supseteq B$ such that $a_F \in \text{dcl}^{eq}(M)$ and a representative a of a_F such that $a \downarrow_{Ea_F}^f M$ (using existence). Then $a \not\downarrow_E^K M$. By assumption there is a stable formula $\varphi(x, y, e)$ for $e \in E$, and a tuple $m \in M$ such that $\models \varphi(a, m, e)$ and $\varphi(x, m, e)$ Kim-forks over E .

Let $p = \text{tp}_{\varphi^*}(a/M)$. By Lemma 5.26 there is a stable formula $\psi(x, y, e)$ and $b \in B$ such that $\models \psi(a, b, e)$ and $\psi(x, b, e)$ Kim-forks over E . Since $a \downarrow_{Ea_F}^K B$ the $\psi(x, y, e)$ -type of a over B is definable over Ea_F . Let $d \in \text{acl}^{eq}(Ea_F)$ be its canonical base and $\chi(d, y)$ be its definition, by Remark 5.17 we can assume that $\chi(w, y)$ is a stable formula.

Claim: If $q(w) := \text{tp}(d/E)$ then $q(w) \cup \{\chi(w, b)\}$ Kim-forks over E .

Proof: Consider an E -Morley sequence $(b_i)_{i < \omega}$ with $b_0 = b$. If $d' \models q(w) \cup \{\chi(w, b_i) : i < \omega\}$ by saturation there is some $(b'_i)_{i < \omega} \in M$ such that $d'(b_i)_{i < \omega} \equiv_E d(b'_i)_{i < \omega}$, so $\models \chi(d, b'_i)$ for all $i < \omega$, so $\psi(x, b'_i, e) \in \text{tp}_{\psi}(a/M)$, so $\{\psi(x, b'_i, e) : i < \omega\}$ is consistent, which contradicts the fact that $\psi(x, b, e)$ Kim-divides over E . \square

There is some $\mu(w, e) \in q(w, e)$ such that $\chi'(w, b, e) := \chi(w, b) \wedge \mu(w, e)$ Kim-forks over E , also $\chi'(w, y, e)$ is stable. We know that $d \in \text{acl}^{eq}(Ea_F)$, we can assume that $d \in \text{acl}^{eq}(ea_F)$, take $\theta(w, a_F, e)$ some formula witnessing this, i.e. $\models \theta(d, a_F, e)$ and $\models \exists^{-n} w \theta(w, x, z)$.

Let $\alpha(v, y, e) := \exists w (\theta(v, w, e) \wedge \chi(w, y, e))$. By Remark 5.24 $\alpha(v, y, e)$ is stable, $\models \alpha(a_F, b, e)$ and $\alpha(v, b, e)$ Kim-forks over E . For the last point if there were $a'_F \models \alpha(v, b, e)$ such that $a' \downarrow_E^K b$, then there is $d' \in \text{acl}(ea')$ such that $\models \chi'(d', b, e)$, which contradicts $d' \downarrow_E^K b$. \square

We would like to show that T^{eq} satisfies weak stable forking. Consider e_F a single imaginary, a tuple a and a model M containing e_F such that $a \not\downarrow_e^K M$. We want to find some stable formula $\varphi(x, y, e_F)$ and $m \in M$ such that $\models \varphi(a, m, e_F)$ and $\varphi(x, m, e_F)$ Kim-forks over e_F .

There is some representative e of e_F such that $a \downarrow_{e_F}^f Me$, so $a \not\downarrow_e M$. By assumption we can find a stable formula $\varphi(x, y, e)$ and $m \in M$ such that $\models \varphi(a, m, e)$ and $\varphi(x, m, e)$ Kim-forks over e . The problem now is that there is no obvious way of preserving these conditions when reducing e to e_F . The same difficulty appears if we want to deduce weak stable Kim-forking from weak stable Kim-forking over models.

5.4 Some questions

We conclude this chapter with the following questions:

1. Does existence and weak stable Kim-forking over models imply weak stable Kim-forking over arbitrary sets?
2. If T^{eq} has existence and weak stable Kim-forking, does T^{eq} have weak stable Kim-forking?
3. More specifically, can we prove that if a simple theory T has weak stable forking then T^{eq} also have weak stable forking?
4. Is there a theory that does not have weak stable Kim-forking? Either over models or algebraically closed sets if it has existence.

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Study of groups and fields in certain moderate theories

Abstract: This thesis is dedicated to the study of algebraic structures within a certain class of the model-theoretic classification of structures, namely that of NSOP₁ theories.

We begin by presenting how the classification developed in general and the study of NSOP₁ in particular, as well as the various model-theoretic tools used in this context, namely independence relations, forking-independence, and especially Kim-independence.

We then address the construction of NSOP₁ theories. In Chapter 2, we consider generic extensions of certain theories. We show that under certain assumptions, these generic extensions are NSOP₁, and in that case we characterize Kim-independence and forking-independence in them. This provides us, among other things, with an example of a definable group in an NSOP₁ theory.

Other examples of NSOP₁ theories are studied in Chapter 3. These concern the theory of a generic symmetric bilinear form over a field K . We show that if the field K is NSOP₁, then this theory is as well, and we characterize Kim-independence over models in it. We also demonstrate more precise results about this theory by adding conditions on the theory of the field K . We provide a counterexample to a conjecture concerning a link between Kim-independence and forking-independence in the case where K is an algebraically closed field.

In Chapter 4, we focus on model-theoretic tools for studying definable groups, and we aim to generalize them to the NSOP₁ context. We define the Kim-stabilizer and show that in several examples of NSOP₁ theories, we obtain a type-definable subgroup. We also define certain assumptions on a field K under which, using the Kim-stabilizer, we can establish a connection between a definable group in K and the K -rational points of an algebraic group over K .

In the appendix, we focus on a generalization of the stable deviation conjecture to the setting of NSOP₁ theories and Kim-deviation.

Keywords: Model theory; Classification; NSOP₁.

Étude des groupes et corps dans certaines théories modérées

Résumé: Cette thèse est dédiée à l'étude des structures algébriques dans une certaine classe de la classification modèle-théorique des structures, celle des théories NSOP_1 .

On commence par présenter comment s'est développée la classification en général et l'étude de NSOP_1 en particulier, ainsi que les différents outils modèle-théoriques que l'on utilise dans ce contexte, c'est-à-dire les relations d'indépendance, la Shelah-indépendance et surtout la Kim-indépendance.

On aborde ensuite la construction de théories NSOP_1 . Dans le Chapitre 2 on considère des extensions générique de certaines théories. On démontre que sous certaines hypothèses ces extensions génériques sont NSOP_1 et on caractérise la Kim-indépendance et la Shelah-indépendance dans celles-ci. Ceci nous fournit entre autres un exemple de groupe définissable dans une théorie NSOP_1 .

D'autres exemples de théories NSOP_1 sont étudiés dans le Chapitre 3. Il s'agit de la théorie d'une forme bilinéaire symétrique générique sur un espace vectoriel sur un corps K . On démontre que si le corps K est NSOP_1 alors cette théorie l'est aussi et on caractérise la Kim-indépendance sur des modèles dans celle-ci. On démontre aussi des résultats plus précis sur cette théorie en ajoutant des conditions sur la théorie du corps K . On fournit un contre-exemple à une conjecture portant sur un lien entre Kim-indépendance et Shelah-indépendance dans le cas où K est un corps algébriquement clos.

Dans le Chapitre 4 on s'intéresse aux outils modèle-théoriques pour étudier les groupes définissables, et on cherche à les généraliser au contexte de NSOP_1 . On définit le Kim-stabilisateur et on démontre que dans plusieurs exemples de théories NSOP_1 on obtient un sous-groupe type-définissable. On définit aussi certaines hypothèses sur un corps K sous lesquelles en utilisant le Kim-stabilisateur on peut établir un lien entre un groupe définissable dans K et les points K -rationnels d'un groupe algébrique sur K .

Dans l'appendice on s'intéresse à une généralisation de la conjecture de la déviation stable au cadre des théories NSOP_1 et de la Kim-déviabilité.

Mots clés: Théorie des modèles; Classification; NSOP_1 .

Image en couverture : Le pont de Narni. Crédit image : Jean-Baptiste Camille Corot

