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Thèse de doctorat



Perturbative and non-perturbative analysis of the renormalization group equations

Thèse de doctorat de l'Institut Polytechnique de Paris
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赠送给我的母亲，我的父亲和我的姐姐.

A ma mère, mon père et ma sœur.

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Résumé

La renormalisation en théorie quantique des champs est un des concepts importants en physique théorique. Par exemple, elle permet en théorie perturbative des champs de donner un sens à une théorie en ajoutant des contre-termes dans son lagrangien nu, de telle sorte que les quantités physiques de cette dernière soient libres des divergences ultraviolettes. Par exemple, en quatre dimensions, la théorie scalaire massive φ^4 comporte un terme d'interaction de la forme $\lambda \frac{\varphi^4}{4!}$ où $\lambda > 0$ est la constante de couplage. Les contre-termes requis pour renormaliser la théorie sont ceux pour la masse, le couplage et pour la fonction d'onde. La renormalisation est un procédé qui n'est pas limité à une seule méthode. En effet, des preuves de renormalisabilité de théories des champs incluent des analyses des intégrales apparaissant dans les diagrammes de Feynman tandis que d'autres preuves utilisent les équations de flot du groupe de renormalisation. Dans cette thèse, nous utilisons la méthode par les équations de flot. Au lieu d'analyser les diagrammes de Feynman et leur combinatoire, les fonctions de corrélation à n points $\mathcal{L}_n^{\alpha_0, \alpha}$ sont étudiées comme des fonctions d'un paramètre dit de flot $\alpha \in [\alpha_0, +\infty)$ où α_0 correspond à une coupure ultraviolette régularisant la théorie. La limite $\alpha \rightarrow +\infty$ correspond à l'échelle physique. Développées en série perturbative, elles sont les uniques solutions des équations de flot pour des conditions de bord imposées. Les équations de flot donnent un schéma inductif permettant de trouver des bornes sur les fonctions de corrélations $\mathcal{L}_n^{\alpha_0, \alpha}$. La théorie est dite renormalisable si les bornes obtenues permettent de montrer que les limites $\lim_{\alpha_0 \rightarrow 0} \lim_{\alpha \rightarrow +\infty} \mathcal{L}_n^{\alpha_0, \alpha}$ soient finies.

Une des questions majeures sur la théorie φ^4 concerne la trivialité en quatre dimensions. Elle correspond au fait qu'en partant d'une théorie regularisée, la limite que l'on trouve en enlevant la régularisation est la théorie libre. Des arguments heuristiques en faveur de la trivialité incluent la croissance du couplage effectif et la présence du pôle de Landau pour la théorie φ^4 en quatre dimensions. Des travaux dus à Aizenman, Fröhlich et Duminil-Copin [1, 2, 3] figurent parmi les résultats rigoureux les plus connus. Ils prouvent que la limite du continu de la théorie φ^4 en $d \geq 4$ dimensions à une ou deux composantes régularisée sur réseau est triviale. Elles reposent entre autres sur des inégalités de corrélation, bien connues pour des champs à une ou deux composantes. La preuve pour $d = 4$ est considérablement plus difficile que pour $d > 4$ car ces inégalités ne sont pas concluantes pour $d = 4$. Des inégalités plus fines et en utilisant une analyse multi-échelle ont néanmoins permis de prouver le cas $d = 4$ [3] mais les preuves restent limitées aux champs à une composante (modèle Ising). Plus récemment, Kopper a développé une approche avec l'aide des équations de flot mais en approximation de champ moyen [4]. Il a d'abord construit de manière générale les solutions des équations de flot non-perturbatives en champ moyen, en particulier des solutions asymptotiquement libres à partir d'un lagrangien nu non polynômial. Il a également analysé le problème de la trivialité sous l'hypothèse de faibles couplages nus. Les conditions de bord en champ moyen sont imposées à la coupure ultraviolette α_0 . Il a analysé un ansatz pour la fonction à deux points connectée et amputée. Des bornes inductives sur les coefficients de l'ansatz ont été obtenues avec les équations de

flot en champ moyen qui montrent que l'ansatz est bien défini. Les autres fonctions de corrélations à n points se déduisent des équations de flot et il suit que la théorie construite en champ moyen est triviale.

Cette thèse est dédiée à l'étude non-perturbative et perturbative des équations du groupe de renormalisation pour la théorie φ^4 en quatre dimensions, aussi notée théorie φ_4^4 . Dans la première partie de cette thèse, nous avons analysé à nouveau le problème de la trivialité. Nous avons étendu le résultat de Kopper à des couplages nus arbitraires. Nous avons obtenu des bornes plus fines sur les coefficients de l'ansatz qui généralisent celles établies par Kopper. Ensuite, nous nous sommes intéressés aux théories scalaires avec un nombre de composantes $N \geq 1$, qu'on appelle modèles $O(N)$. Une difficulté a été de dériver les équations de flot sous un schéma inductif permettant de répéter notre méthode. Nous avons utilisé les propriétés de symétrie du modèle $O(N)$ pour déterminer la structure tensorielle des fonctions de corrélation afin de réduire les équations de flot en champ moyen sous une forme généralisant celle trouvée par Kopper pour le champ à une composante. Nous avons à nouveau montré la trivialité du modèle $O(N)$ en suivant notre méthode. Nous avons également confronté nos résultats à ceux obtenus par Amit, Moshe et Zinn-Justin dans la limite $N \rightarrow +\infty$. L'unicité de la solution construite a aussi été étudiée. En effet, les arguments usuels d'analyticité ne fonctionnent pas puisqu'il n'a pas été possible de montrer que nos solutions obtenues soient analytiques en α_0 . En utilisant une formulation intégrale des équations de flot en champ moyen, d'abord pour le champ à une composante puis pour le modèle $O(N)$, nous prouvons l'unicité.

La deuxième partie de cette thèse est dédiée à l'étude du lien entre nos solutions triviales et la théorie des perturbations. Nous avons en particulier analysé le problème de la sommabilité au sens de Borel de la théorie des perturbations régularisée et renormalisée en champ moyen. Les développements perturbatifs en théorie des champs sont divergents. Les sources de cette divergence sont liées aux minima non triviaux de l'action et à la renormalisation, elles incluent donc les instantons et les renormalons. Cette dernière provient de la nécessité de la renormalisation et apparaît pour la théorie φ^4 en quatre dimensions. La sommabilité au sens de Borel est un concept permettant à nouveau de définir rigoureusement les fonctions de corrélation à partir de leur série perturbative. Des résultats sur la convergence locale de la transformée de Borel de la série perturbative et sur la sommabilité au sens de Borel ont été obtenus pour diverses théories des champs. Keller et Kopper ont en particulier établi l'existence locale de la transformée de Borel de la théorie φ^4 perturbative avec les équations de flot perturbatives [5, 6]. Nous avons analysé de manière générale les équations de flot perturbatives en champ moyen avec les conditions de renormalisation BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann). Des bornes pour les solutions perturbatives en champ moyen ont été obtenues. Ensuite, pour établir un lien entre nos solutions triviales et la théorie des perturbations, les solutions triviales ont été écrites sous forme d'une série perturbative tronquée en un couplage renormalisé convenablement choisi, ainsi qu'un reste. De plus, nous avons montré, à l'aide d'une méthode de point fixe, que des conditions de renormalisation spécifiques, incluant les conditions de renormalisation BPHZ, pouvaient être imposées pour nos solutions triviales. Ensuite, nous avons montré que nos solutions triviales ont un développement perturbatif en un couplage renormalisé convenablement choisi. En particulier, la fonction à deux points connectée amputée en champ moyen est analytique en le couplage renormalisé. Puis, nous avons obtenu des estimations sur les restes qui impliquent l'existence locale de la transformée de Borel de la théorie régularisée et renormalisée en champ moyen. Enfin, pour traiter le problème de la sommabilité au sens de Borel, nous avons utilisé un théorème dû à Nevanlinna et réactualisé par Sokal. Notre analyse sur le cas d'un couplage réel s'étend en fait dans le cas d'un couplage complexe. Toutes les hypothèses du

théorème de Nevanlinna-Sokal étant vérifiées, nous avons obtenu la sommabilité au sens de Borel.

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Introduction

Quantum field theory is a fundamental framework of theoretical physics that unifies both quantum mechanics and special relativity. It is a powerful tool to study systems with a large or an infinite number of degrees of freedom. For instance, Euclidean field theory is used in statistical mechanics to analyze the critical behavior of systems (critical exponents), while the degrees of freedom in high-energy physics are described using relativistic (Minkowski) field theory. In quantum field theory, one of the main obstacles for a rigorous formulation is the definition of the path integral. In most systems, the path integral is generally ill-defined. However, Euclidean field theory can in some cases be defined rigorously from a mathematical point of view. If the so-called Osterwalder-Schrader axioms [7, 8] can be verified, the Euclidean field theory can be analytically continued to Minkowski space. Also, the correlation functions of relativistic field theory are usually obtained via an analytic continuation of the Euclidean ones. Euclidean free field theories are well described using gaussian measures, while interacting field theories require a regularization to start on safe grounds. A motivation for regularization in quantum field theory arises from perturbative quantum field theory. The connected amputated correlation functions that appear in transition amplitudes are expanded in a power series w.r.t. a coupling constant λ . The coupling constant λ represents the strength of the interaction. Feynman diagrams constitute a useful diagrammatic representation of these perturbative contributions. The number of vertices V of these diagrams corresponds to the order λ^V of these contributions. Unfortunately, some contributions are apparently ill-defined. An example of theory where these divergences appear is the scalar φ^4 -theory in four dimensions. The interaction term is of the form $\frac{\lambda}{4!}\varphi^4$, where λ is the dimensionless coupling constant. In the Standard model, the Higgs field after spontaneous symmetry breaking typically behaves as a φ^4 -theory although the Higgs field is coupled to the fermionic degrees of freedom and the gauge bosons. For the φ^4 -theory in four dimensions, the first order correction to the two-point function is quadratically UV-divergent. A regularization of the theory by momentum cutoff renders all contributions finite. After regularizing the theory, renormalization is performed. Renormalization is a set of techniques to ensure that the physical quantities in the regularized theory are finite upon removing the regularization. Typically in φ^4 -theory in four dimensions, the parameters of the theory (mass, coupling constant) and the normalization of the propagator are adjusted by adding a finite number of counterterms to the lagrangian to cure the UV-divergences; φ^4 -theory in four dimensions is perturbatively renormalizable.

Perturbative renormalizability was first analyzed in Dyson's work [9, 10] and in the late sixties, the BPHZ-version (Bogoliubov-Parasyuk-Hepp-Zimmerman) [11, 12] was implemented with momentum space subtraction to heal the UV-divergences. Different regularization schemes were invented: Pauli-Villars regularization [13], analytical regularization [14], dimensional regularization [15, 16, 17] which is often used in gauge theory calculations. These different regularization schemes can be viewed to be equivalent, see e.g. [18] and they may impact on the symmetries of

the unregularized theories. For instance, the symmetries of the lattice regularized theory are different from those of the continuum limit, e.g. the lattice-regularized Euclidean field theories are not invariant under rotations. Nevertheless, the original symmetries should be restored upon removing the regularization. Most of the aforementioned methods include an exhaustive analysis of the integrals entering in Feynman diagrams. We can mention for instance Zimmermann's forest formula [19] which allows one to treat the overlapping divergences by ordering the subintegrations in a single Feynman diagram. First developed for massive theories, Lowenstein extended the BPHZ-renormalization for massless particles [20]. While these methods constitute powerful tools, the rigorous renormalizability proofs of quantum field theories based on Feynman diagrams are delicate because of their analytic and combinatorial complexity. In particular, at high order loop l , counterterms of loop order $l' < l$ actually enter into the diagrams of loop order l , meaning that the counterterms of order loop l depend on the counterterms of loop order $i \leq l - 1$.

A much simpler proof of renormalizability is based on the differential flow equations of the Wilson renormalization group [21, 22]. They were introduced by Wegner and Houghton [23]. In the Wilsonian renormalization group, the local effective action undergoes a continuous evolution. This non-perturbative point of view of renormalization is used in condensed matter physics to study critical points. In rigorous mathematical physics analysis, the renormalizable UV-asymptotically free Gross-Neveu model in two dimensions has been constructed [24, 25] by splitting the momentum range in the functional integral into a union of discrete and disjoint sets and successive integration of the quantum fluctuations; this idea is similar to Wilson's idea of block-spins partition in lattices in statistical physics. A new construction of the massive Euclidean Gross-Neveu model in infinite volume has been performed by Duch [26] by solving the flow equation without a discretization of phase-space, a tree expansion or a cluster expansion. Rigorous non-perturbative analysis of the renormalization flow has been treated in lecture notes, see e.g. [27] and in the monograph [28] where the question of formally summing perturbation series is dealt with. In perturbative renormalization, Gallavoti and Nicolò [29, 30] followed Wilson's ideas by decomposing the propagator of a free scalar field into disjoint momentum slices, the field is decomposed into a sum of independent random variables. Then the so-called tree expansion method has been developed by Gallavoti, Nicolò and Benfatto. This method has been adapted to prove renormalizability of quantum electrodynamics and has been used to analyze fermionic systems; see e.g. [31].

Polchinski in his seminal paper [32] proved the perturbative renormalizability of the scalar φ^4 -theory in four dimensions. The perturbative Schwinger functions are regularized by an UV-cutoff Λ_0 and an infrared cutoff Λ , also called flow parameter. This flow parameter interpolates the theory between the theory at the UV-cutoff (called bare theory) and the effective physical theory at low energy scale. The flow equations are differential equations whose solutions are the regularized connected amputated Schwinger functions. They describe how they evolve w.r.t. flow parameter Λ . The solutions of these equations are bounded by using an inductive scheme. These bounds are sufficiently strong to prove renormalizability [33]. The proof of perturbative renormalizability based on this method has been extended to the massless φ^4 -theory in four dimensions [34], to the non-linear σ -model [35], to the spontaneously broken $SU(2)$ Yang-Mills theory [36] and the theory without the Higgs mechanism [37], φ^4 -theory in four dimensions in Minkowski space [38]. Other results in mathematical physics established using the flow equations are the convergence of the operator product expansion in perturbation theory [39] and the local Borel summability of perturbative Euclidean massive φ^4 -theory in four dimensions [5, 6].

An important question about φ^4 -theory in d dimensions, also noted φ_d^4 -theory, is the question

of triviality. Triviality means that when one starts from a regularized interacting field theory, the only limiting theory upon removing the regularization is the free theory. Arguments in favor of triviality include the positivity of the beta function of φ^4 -theory and the existence of the so-called Landau pole, although these arguments were obtained using perturbative methods which become unreliable when the coupling constant becomes large. Numerical results seem to confirm triviality of φ^4 -theory; see e.g. [40, 41, 42]. Rigorous results on triviality of φ_d^4 -theory have been established; the most famous of these results are those established by Aizenman, Fröhlich and Duminil-Copin in [2, 43, 44]. They proved the triviality of the continuum limit of lattice regularized Euclidean one-component φ_d^4 -theory for $d > 4$ and for $d = 4$ with an infinite wavefunction renormalization, in the sense that the truncated four-point function of the theory, which is related to the renormalized coupling, vanishes in this limit. These proofs were only established for one or two component scalar fields because they relied on correlation inequalities, e.g. Lebowitz inequality [45]. See [46] for correlation inequalities for φ_d^4 -models. These correlation inequalities and their applicability have yet to be established for higher spin models. Aizenman and Duminil-Copin extended the proof to $d = 4$ in [3] using multi-scale analysis to sharpen the bounds obtained in [1] on the truncated four-point function for the one-component theory. The question whether the Standard model is trivial or not remains open because the aforementioned proofs are restricted to scalar fields. No rigorous framework has yet been completely developed to tackle this problem. However, we can mention that Lüscher analyzed lattice quantum electrodynamics (QED) in d dimensions in [47], and he obtained bounds on the renormalized charge. He concluded that for $d > 4$, the renormalized charge goes to zero in the continuum limit, meaning that the theory is trivial while the question in four dimensions remains open.

The first part of this thesis deals with the triviality problem of φ_4^4 . We tackle this problem using the non-perturbative renormalization group equations in the mean-field approximation. Kopper developed this approach in [6] to construct uniformly bounded asymptotically free scalar fields from a non-polynomial bounded bare effective action. He also analyzed the triviality problem. From a fixed bare interaction lagrangian for the φ_4^4 -theory, he constructed inductively mean-field trivial scalar fields using the mean-field flow equations from an ansatz for the two-point function. Nevertheless, his analysis is restricted to a one-component theory with small bare couplings. Furthermore, a technical simplification of the infrared cutoff was performed at the level of the regularized propagator in momentum space to simplify the calculations. Then, natural questions arise: Can we extend the triviality statement to arbitrarily large couplings? to the $O(N)$ -model, $N \geq 1$? to the theory without the technical infrared assumption? How can we adapt the mean-field flow equations to these cases? How can we adjust the ansatz for the two-point function?

We also tackled the analyticity of the mean-field flow equations. The trivial solutions constructed cannot be shown to be analytic with respect to the flow parameter at the bare scale. Therefore the uniqueness of the solutions cannot be proven using usual analyticity arguments. Nevertheless it is known that if the bare action is bounded from below, then the flow equations have an integral formulation. This integral formulation implies the uniqueness of the Schwinger functions as solutions of the flow equations for fixed boundary conditions.

Another aspect of this thesis is the perturbative analysis of the mean-field flow equations. Physicists often ask themselves whether triviality means that no perturbative theory should exist. Wilson pointed out that triviality and perturbation theory are not incompatible [48] by citing the example of the Lee model [49], in which the theory is trivial but the renormalized perturbation theory exists. In physics experiments, the maximum energy scale is related to an UV-cutoff, meaning that exper-

imental results are obtained for an effective theory while triviality is a limiting property. Before the discovery of the Higgs boson, consequences of triviality on the mass of the Higgs boson were discussed; see e.g. [50] for a discussion on the range of allowed values of the mass of the Higgs boson. In the second part of this thesis, we wanted to know the relation between our mean-field trivial solutions and perturbation theory. Can we obtain a renormalized perturbation theory from the mean-field trivial solution? Does this theory make sense?

The last question is of great importance. Indeed in quantum field theory, perturbative expansions are divergent. The sources of this divergence can be classified in two types:

- Instanton singularities when one analyzes non-trivial minima of the classical action in the complex coupling [51]. These singularities are related to the fact that carrying out an expansion in terms of Feynman diagrams leads to a number of diagram increasing very quickly at high orders in perturbation theory.
- In four dimensions, renormalon singularities appear. They have been introduced by 't Hooft [52]. They manifest through Feynman graphs with a number of renormalization subtractions proportional to the order of perturbation theory. For the φ_4^4 -theory, graphs with N insertions of bubble graphs contributing to the six-point function typically behave as $N!$, making the perturbative expansion apparently divergent.

In spite of these two sources of divergence, φ^4 Schwinger functions can in some cases be recovered from the perturbative expansion through Borel summation. In φ_2^4 models [53], the n -point Schwinger functions $S_n(g)$ have an asymptotic series in g

$$S_n(g) \sim \sum_{m \geq 0} a_m g^m \quad (0.0.1)$$

which is divergent, i.e. $|a_m| > \mathcal{O}(m!)$ [54]. The Borel transform of this series is defined by

$$h(t) := \sum_{n \geq 0} \frac{a_n}{n!} t^n . \quad (0.0.2)$$

The Borel transform (0.0.2) has a finite radius of convergence around $t = 0$ and it can be continued analytically for $t > 0$. Then, the Schwinger functions are recovered via the Laplace transform of the Borel transform

$$S_n(g) = \int_0^{+\infty} e^{-t} h(gt) dt . \quad (0.0.3)$$

Other results regarding the local Borel summability of perturbation theory include the seminal work of de Calan and Rivasseau [55]. They proved that even in presence of the two mentioned sources of divergence in φ_4^4 -theory, the Borel transform of the perturbative expansion has a finite radius of convergence, i.e. the perturbative amplitudes at order K do not grow more rapidly than $C^K K!$ where C is a constant. A main result is a bound on the number of graphs requiring $k \leq K$ renormalization subtractions. This number is bounded by

$$C^K \frac{K!}{k!} , \quad (0.0.4)$$

so that the bound on the amplitudes is of the form

$$C'^K K! , \tag{0.0.5}$$

where C' is another constant, thus implying the existence of the local Borel transform of perturbation theory. These bounds have been improved and generalized in [56]. Other results include the local existence of the Borel transform for QED [57] and construction and Borel summability of planar Euclidean φ_4^4 theory [58]. Keller [5] used for the first time the Wegner-Wilson-Polchinski flow equations to prove the local Borel summability. Then Kopper [6] analyzed the existence of the local Borel transform of perturbation theory with the flow equations at large order in perturbation theory and obtained bounds on the whole set of Schwinger functions and their behavior at large momenta. Most of the results obtained dealt with the local existence of the Borel transform (0.0.2). The stronger result of the reconstruction of the Schwinger function with the Borel sum (8.1.5) is more difficult to prove, since it requires explicit expressions of the correlation functions. Unfortunately, obtaining such expressions is often beyond reach. In our construction we have an explicit expression of the two-point function from which all the other mean-field n -point functions, $n \geq 4$, can be generated inductively using the mean-field flow equations. We can reformulate our initial questions on triviality and perturbation theory. How can we expand our mean-field trivial solutions in a perturbative series? What would be the renormalized coupling to perform such an expansion? Can we prove the local Borel summability of the mean-field regularized renormalized perturbation theory?

In Ch.1, we recall the perturbative renormalizability proof of φ_4^4 -theory by the flow equations in [59] as an introduction to the flow equations. In Sect.1.1, we introduce the notion of regularized propagator in momentum space, the first step towards the derivation of the flow equations. In Sect.1.2, we introduce the bare interaction lagrangian and we derive the flow equations, the main mathematical tool in this thesis. In Sect.1.3, we review the perturbative proof of renormalizability of φ_4^4 -theory by the flow equations to illustrate its main application. In Ch.2, we review the most prominent results on the triviality of φ_d^4 -theory. In Sect.2.1, we review Aizenman's proof for $d > 4$. He started by analyzing the Ising model. He used the random current representation to derive the tree-diagram bound on the truncated four-point function. This bound implies that the (dimensionless) renormalized coupling vanishes in the continuum limit for $d > 4$. The tree-diagram bound in [1] is inconclusive for $d = 4$. Aizenman's proof also applies to φ^4 -theory. In Sect.2.2, we review the results for $d = 4$. In Sect.2.2.1, we review briefly Fröhlich's proof. Starting from a φ^4 -theory on a lattice, he derived bounds on the renormalized coupling using random walk representation assuming an infinite wavefunction renormalization. These bounds imply triviality of the continuum limit of the theory. In Sect.2.2.2, we shortly review Aizenman and Duminil-Copin's result for $d = 4$ without the infinite wavefunction renormalization assumption. Through a multi-scale analysis, they obtained a sharper correction to the tree-diagram bound, implying triviality. In Ch.3, we review the construction of mean-field scalar fields using the mean-field flow equations. This work was done by Kopper in [4] and served as the starting point of this thesis. The mean-field flow equations are presented in Sect.3.1. In Sect.3.2.4, mean-field trivial solutions under the assumption of small bare couplings were constructed. In Ch.4, we analyze the $O(N)$ -model and we adapt the mean-field flow equations method to it, we derive them in Sect.4.2. In Ch.5, we prove the triviality statement. In Sect.5.1.1, we construct the mean-field trivial solution as in [4] but with arbitrarily large bare couplings. In Sect.5.1.2, we prove the uniqueness of the mean-field trivial solution for fixed mean-field bare boundary conditions. In Ch.6, we analyze the (one-component) theory with the natural IR cut-off; i.e. with a mass. Again we derive the mean-field flow equations in the new setting in Sect.6.1

and prove the triviality in Sect.6.2. In Ch.7, we establish the relation between the mean-field trivial solution and perturbation theory. We analyze the perturbative mean-field flow equations in Sect.7.1, and we obtain bounds on the mean-field perturbative Schwinger functions for fixed mixed mean-field boundary conditions in Sect.7.1.1. Mean-field flow equations for the remainders are derived in Sect.7.1.2. In Sect.7.2.1, we show that we can prescribe specific renormalization conditions, including the BPHZ renormalization conditions. This allows us to choose a renormalized coupling. In Sect.7.2.2, we expand our trivial solution in a perturbative series in a conveniently chosen renormalized coupling. In Ch.8, we prove the local Borel summability of the regularized renormalized perturbation theory. In Sect.8.2, we prove the local existence of its Borel transform for a positive renormalized coupling and show that it is asymptotic to the non-perturbative theory by deriving explicit estimates on the remainders. In Sect.8.3, we finally prove the local Borel summability of the regularized renormalized perturbation theory by extending our analysis to complex couplings with positive real parts. We end with a few conclusions and some future perspectives.

Chapter 1

Perturbative renormalizability of φ_4^4 -theory by the flow equations

Here we briefly review the perturbative proof of renormalizability of φ_4^4 -theory by the flow equations [59]. The purpose of this chapter is to introduce the flow equations, the central mathematical tool of this thesis.

We will expose the perturbative proof of renormalizability of non-symmetric φ^4 -theory in the Euclidean space \mathbb{R}^4 to illustrate how convenient the renormalization group equations are as a mathematical tool. We adopt the following convention and shorthand notation for the Fourier transform

$$f(x) = \int_p e^{ipx} \hat{f}(p), \quad \int_p := \int \frac{d^4p}{(2\pi)^4}. \quad (1.0.1)$$

This implies for the functional derivatives $\frac{\delta}{\delta\varphi(x)}$

$$\frac{\delta}{\delta\varphi(x)} = (2\pi)^4 \int_p e^{-ipx} \frac{\delta}{\delta\hat{\varphi}(p)}. \quad (1.0.2)$$

1.1 Regularized propagator in momentum space

To start on safe grounds, we define the theory on a finite volume in \mathbb{R}^4 . Let us consider a four-dimensional torus Ω of volume $|\Omega| = \prod_{i=1}^4 l_i$. A point $x \in \Omega$ has coordinates $x^i \in [-\frac{l_i}{2}, \frac{l_i}{2})$. The fields φ are assumed to be periodic $\varphi(x + n_l) = \varphi(x)$, where $x \in \Omega$ and

$$n_l = (n_1 l_1, \dots, n_4 l_4), \quad n_i \in \mathbb{Z}. \quad (1.1.1)$$

The momenta form a discrete set indexed by integers n_i

$$k := k(n) = \left(\frac{2\pi n_1}{l_1}, \dots, \frac{2\pi n_4}{l_4} \right), \quad n_i \in \mathbb{Z}. \quad (1.1.2)$$

Let m, Λ_0 be positive constants, with $0 < m \ll \Lambda_0$ and $0 \leq \Lambda \leq \Lambda_0$. First we define the covariance in finite volume by

$$C^{\Lambda, \Lambda_0}(x - y, m) = \frac{1}{|\Omega|} \sum_{n \in \mathbb{Z}^4} \frac{e^{ik(x-y)}}{k^2 + m^2} \left(e^{-\frac{k^2 + m^2}{\Lambda_0^2}} - e^{-\frac{k^2 + m^2}{\Lambda^2}} \right). \quad (1.1.3)$$

This covariance (1.1.3) has a well defined limit in the infinite volume limit whose expression is given by

$$C^{\Lambda, \Lambda_0}(x - y, m) = \int_k e^{ik(x-y)} C^{\Lambda, \Lambda_0}(k, m) \quad (1.1.4)$$

where we define the regularized propagator in momentum space

$$C^{\Lambda, \Lambda_0}(k, m) := \frac{1}{k^2 + m^2} \left(e^{-\frac{k^2 + m^2}{\Lambda_0^2}} - e^{-\frac{k^2 + m^2}{\Lambda^2}} \right). \quad (1.1.5)$$

The regularized propagator in momentum space (1.1.5) satisfies the following properties:

- $C^{\Lambda, \Lambda_0}(k, m)$ is analytic w.r.t. $\Lambda > 0$.
- $C^{\Lambda_0, \Lambda_0}(k, m) = 0$ ¹.
- At Λ fixed, $C^{\Lambda, \Lambda_0}(k, m)$ falls off more rapidly than any power of $|k|$.

Taking the limits $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow +\infty$, we obtain the Euclidean free scalar propagator

$$\lim_{\Lambda \rightarrow 0} \lim_{\Lambda_0 \rightarrow +\infty} C^{\Lambda, \Lambda_0}(k, m) = \frac{1}{k^2 + m^2}. \quad (1.1.6)$$

The role of the parameter flow Λ is to interpolate differentiably between the vanishing propagator at $\Lambda = \Lambda_0$ and the regularized propagator at $\Lambda = 0$. The Gaussian measure associated with the regularized propagator (1.1.3) is supported with probability one on $\mathcal{C}^\infty(\Omega)$, see e.g. [60]. Some properties of Gaussian measures tailored for our purposes, can be found in [59, 61] and in Appendix A.1. More generally a regularized propagator in momentum space as in (1.1.5) can be chosen by introducing a function $R^{\Lambda, \Lambda_0}(k, m)$ defined by

$$R^{\Lambda, \Lambda_0}(k, m) = \sigma_{\Lambda_0}(k^2) - \sigma_\Lambda(k^2) \quad (1.1.7)$$

and writing the regularized propagator in the more general form

$$C_{mod}^{\Lambda, \Lambda_0}(k, m) = \frac{R^{\Lambda, \Lambda_0}(k, m)}{k^2 + m^2}. \quad (1.1.8)$$

The modified propagator (1.1.8) is admissible as long as the function $\sigma_{\Lambda_0}(k^2)$ verifies the following properties

- **(A1)**: For fixed Λ , $\sigma_\Lambda(k^2)$ falls off more rapidly than any power of $|k|$.

¹The corresponding Gaussian measure corresponds in this case to a δ -type measure on function space.

- **(A2)**: At fixed k^2 , $\sigma_\Lambda(k^2)$ increases with Λ .

The regularized propagator (1.1.5) corresponds to

$$\sigma_\Lambda(k^2) = e^{-\frac{k^2+m^2}{\Lambda^2}}. \quad (1.1.9)$$

The function $\sigma_\Lambda(k^2)$ in (1.1.9) clearly satisfies properties **(A1)**-**(A2)**.

1.2 The flow equations in momentum space

We now derive the flow equations, which are the central tool in this thesis. First, we start from a local action in four-dimensional spacetime. The terms entering in the action usually take the form of a polynomial in the fields and their derivatives. The free part of the theory determines the propagator, which is bilinear in the fields. If we define the mass dimension by 1 for each derivative, then the scalar field φ has the mass dimension $\frac{d-2}{2}$ if d denotes the spacetime dimension. Since the action is dimensionless, the mass dimensions of the interaction couplings entering the interaction lagrangian are determined from the mass dimension of the corresponding interaction term. A local term in the action will be called *irrelevant* [21, 22, 48] if its mass dimension is greater than d and *relevant* if its mass dimension is lower than d . Terms with mass dimension d are called *marginal*. For instance in the φ_4^4 -theory with a quartic interaction term $g\varphi^4$, the coupling g is dimensionless. In perturbation theory, it turns out that in four dimensions, the UV-finite Schwinger functions are obtained by only considering relevant and marginal interaction terms and by prescribing a finite number of renormalization conditions. This number is equal to the number of relevant operators.

We consider the quantum field theory of a real scalar field φ with mass m on four-dimensional Euclidean spacetime. First we consider a four-dimensional torus Ω , i.e. a finite volume, before passing to the infinite-volume limit. We study bare interaction lagrangians of the form

$$\begin{aligned} L^{\Lambda_0, \Lambda_0}(\varphi) = & \int_{\Omega} d^4x \left(\frac{f}{3!} \varphi^3(x) + \frac{g}{4!} \varphi^4(x) \right) \\ & + \int_{\Omega} d^4x \left(v(\Lambda_0) \varphi(x) + \frac{a(\Lambda_0)}{2} \varphi(x)^2 + \frac{z(\Lambda_0)}{2} (\partial_\mu \varphi(x))^2 \right. \\ & \left. + \frac{b(\Lambda_0)}{3!} \varphi^3(x) + \frac{c(\Lambda_0)}{4!} \varphi^4(x) \right). \end{aligned} \quad (1.2.1)$$

Since the regularized Gaussian measure is supported on $C^\infty(\Omega)$, the quantities in L^{Λ_0, Λ_0} , i.e. $\varphi^2(x)$, $\varphi^4(x), \dots$ are well-defined. The first integral in (1.2.1) contains the field's self-interaction with real finite coupling constants f and g of respective mass dimensions 1 and 0. The remaining integral includes the counterterms. They depend on Λ_0 , and diverge when $\Lambda_0 \rightarrow +\infty$. They are required so that the physical quantities are finite when the UV-cutoff is removed, i.e. $\Lambda_0 \rightarrow +\infty$. The quantity $a(\Lambda_0)$ corresponds to the mass counterterm, $c(\Lambda_0)$ to the coupling counterterm, and $z(\Lambda_0)$ to the wavefunction counterterm. The bare interaction lagrangian has $O(4)$ -symmetry. If we set $f = 0$, the counterterms $v(\Lambda_0)$ and $b(\Lambda_0)$ are not required, and the theory is symmetric under $\varphi \rightarrow -\varphi$, i.e. the theory has a \mathbb{Z}_2 -symmetry.

We define the generating functional of the Schwinger functions of the regularized theory on a finite volume by

$$Z^{\Lambda, \Lambda_0}(J) := \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar} L^{\Lambda, \Lambda_0}(\phi)} e^{\frac{1}{\hbar} \langle \phi, J \rangle} . \quad (1.2.2)$$

where we set $\langle f, g \rangle := \int_{\Omega} d^4x f(x)g(x)$ and $J \in C^\infty(\Omega)$ is called the source. The parameter \hbar has been introduced to perform a loop expansion to be considered later. The measure μ_{Λ, Λ_0} in (1.2.2) is the Gaussian measure associated to $\hbar C^{\Lambda, \Lambda_0}$. In order for the functional integral (1.2.2) to be well defined, we assume that $g + c(\Lambda_0) > 0$ and $z(\Lambda_0) \geq 0$. Then, the bare interaction is continuous in any Sobolev norm of order $M \geq 1$ and is bounded from below, i.e. for some finite constant κ

$$-\infty < \kappa < L^{\Lambda_0, \Lambda_0}(\varphi) , \quad \varphi \in \text{supp}(\mu_{\Lambda, \Lambda_0}) . \quad (1.2.3)$$

Then, we obtain the uniform bound in Λ at fixed Λ_0

$$|Z^{\Lambda, \Lambda_0}(J)| \leq e^{-\kappa} \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{\frac{1}{\hbar} \langle \phi, J \rangle} = e^{-\kappa + \frac{1}{2\hbar} \langle J, C^{0, \Lambda_0} J \rangle} , \quad (1.2.4)$$

where we used (A.1.3).

We define the generating functional $W^{\Lambda, \Lambda_0}(J)$ of the truncated regularized Schwinger functions²

$$\frac{1}{\hbar} W^{\Lambda, \Lambda_0} = \ln(Z^{\Lambda, \Lambda_0}(J)) - \ln(Z^{\Lambda, \Lambda_0}(0)) . \quad (1.2.5)$$

The truncated Schwinger functions in position space $W_n^{\Lambda, \Lambda_0}(x_1, \dots, x_n)$ are obtained by successive functional derivatives

$$W_n^{\Lambda, \Lambda_0}(x_1, \dots, x_n) := \frac{\delta^n}{\delta J(x_1) \dots \delta J(x_n)} W^{\Lambda, \Lambda_0}(J) \Big|_{J=0} . \quad (1.2.6)$$

We perform a change of variable $\phi = \tilde{\varphi} + C^{\Lambda, \Lambda_0} * J$, where $C^{\Lambda, \Lambda_0} * J(x) = \int_{\Omega} d^4y C^{\Lambda, \Lambda_0}(x-y)J(y)$, and $J \in C^\infty(\Omega)$ is supposed to be compactly supported in momentum space. Using (A.1.5), the generating functional of the Schwinger functions $Z^{\Lambda, \Lambda_0}(J)$ reads

$$Z^{\Lambda, \Lambda_0}(J) = e^{\frac{1}{2\hbar} \langle J, C^{\Lambda, \Lambda_0} * J \rangle} \int d\mu_{\Lambda, \Lambda_0}(\tilde{\varphi}) e^{-\frac{1}{\hbar} L^{\Lambda_0, \Lambda_0}(\tilde{\varphi} + C^{\Lambda, \Lambda_0} * J)} . \quad (1.2.7)$$

For $\varphi = C^{\Lambda, \Lambda_0} * J$, we define

$$e^{-\frac{1}{\hbar} L^{\Lambda, \Lambda_0}(\varphi)} := \frac{1}{Z^{\Lambda, \Lambda_0}(0)} e^{-\frac{1}{2} \langle \varphi, (\hbar C^{\Lambda, \Lambda_0})^{-1} \varphi \rangle} Z^{\Lambda, \Lambda_0} \left((C^{\Lambda, \Lambda_0})^{-1} \varphi \right) . \quad (1.2.8)$$

From (1.2.7) we find

$$L^{\Lambda, \Lambda_0}(\varphi) = -\hbar \ln \left(\int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar} L^{\Lambda_0, \Lambda_0}(\phi + \varphi)} \right) + \hbar \ln(Z^{\Lambda, \Lambda_0}(0)) , \quad (1.2.9)$$

or equivalently

²In terms of a Feynman diagram expansion, only the connected diagrams contribute.

$$L^{\Lambda, \Lambda_0}(\varphi) = \frac{1}{2} \langle \varphi, (C^{\Lambda, \Lambda_0})^{-1} \varphi \rangle - W^{\Lambda, \Lambda_0} \left((C^{\Lambda, \Lambda_0})^{-1} \varphi \right). \quad (1.2.10)$$

L^{Λ, Λ_0} is called the Wilson effective action at scale Λ . We note from (1.2.8) that $L^{\Lambda, \Lambda_0}(0) = 0$. The moments of the Wilson effective action are the connected (free propagator) amputated Schwinger (CAS) functions³ which give access to the physical contents of the theory since in expressions of scattering amplitudes in quantum field theory, correlation functions factorize into CAS functions. The expression (1.2.9) can be extended to $\varphi \in \text{supp}(\mu_{\Lambda, \Lambda_0})$ so that the Wilson effective action is defined for arbitrary field φ in the support of the Gaussian measure μ_{Λ, Λ_0} . The term $Z^{\Lambda, \Lambda_0}(0)$ depends on the volume $|\Omega|$ and diverges when the volume exhausts \mathbb{R}^4 . We can define the vacuum part of the theory

$$I^{\Lambda, \Lambda_0} = -\hbar \ln(Z^{\Lambda, \Lambda_0}(0)) \quad (1.2.11)$$

so that the Wilson effective action can be rewritten as

$$e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\varphi) + I^{\Lambda, \Lambda_0})} = \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\phi + \varphi)}, \quad (1.2.12)$$

The flow equations are obtained by taking the Λ -derivative of the generating functional of the CAS functions. We denote $\dot{C}^\Lambda := \partial_\Lambda C^{\Lambda, \Lambda_0}$. We obtain using the change of covariance formula (A.1.7)

$$\begin{aligned} \frac{d}{d\Lambda} \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\phi + \varphi)} &= \frac{\hbar}{2} \int d\mu_{\Lambda, \Lambda_0}(\phi) \left\langle \frac{\delta}{\delta\phi}, \dot{C}^\Lambda \frac{\delta}{\delta\phi} \right\rangle e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\phi + \varphi)} \\ &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \frac{\delta}{\delta\varphi} \right\rangle \int d\mu_{\Lambda, \Lambda_0}(\phi) e^{-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\phi + \varphi)}, \end{aligned} \quad (1.2.13)$$

where in the second line, we used the dependence of the integrand on the field ϕ . The bare interaction lagrangian $L^{\Lambda_0, \Lambda_0}(\varphi)$ in (1.2.1) is twice differentiable w.r.t. φ . From (1.2.12) and (1.2.13) we obtain the functional differential equation

$$\frac{d}{d\Lambda} e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\varphi) + I^{\Lambda, \Lambda_0})} = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \frac{\delta}{\delta\varphi} \right\rangle e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\varphi) + I^{\Lambda, \Lambda_0})}. \quad (1.2.14)$$

Explicit differentiation w.r.t. Λ in (1.2.14) yields

$$\begin{aligned} \frac{d}{d\Lambda} (L^{\Lambda, \Lambda_0}(\varphi) + I^{\Lambda, \Lambda_0}) &= \frac{\hbar}{2} \left\langle \frac{\delta}{\delta\varphi}, \dot{C}^\Lambda \frac{\delta}{\delta\varphi} \right\rangle L^{\Lambda, \Lambda_0}(\varphi) \\ &\quad - \frac{1}{2} \left\langle \frac{\delta}{\delta\varphi} L^{\Lambda, \Lambda_0}(\varphi), \dot{C}^\Lambda \frac{\delta}{\delta\varphi} L^{\Lambda, \Lambda_0}(\varphi) \right\rangle. \end{aligned} \quad (1.2.15)$$

The functional differential equation (1.2.15) is called the Wilson flow equation. The functional differential equation (1.2.14) looks like a heat equation. The Wilson effective action can be formally recast as follows: we define a functional Laplacian operator $\Delta_{\Lambda, \Lambda_0}$ by

³The CAS functions are actually distributions.

$$\Delta_{\Lambda, \Lambda_0} := \frac{1}{2} \left\langle \frac{\delta}{\delta \varphi}, C^{\Lambda, \Lambda_0} \frac{\delta}{\delta \varphi} \right\rangle. \quad (1.2.16)$$

Then, we have

$$e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\varphi) + I^{\Lambda, \Lambda_0})} = e^{\hbar \Delta_{\Lambda, \Lambda_0}} e^{-\frac{1}{\hbar} L^{\Lambda_0, \Lambda_0}(\varphi)}. \quad (1.2.17)$$

By noting $\dot{\Delta}_{\Lambda, \Lambda_0} := \partial_{\Lambda} \Delta_{\Lambda, \Lambda_0}$ and the fact that $\Delta_{\Lambda, \Lambda_0}$ commutes with $\dot{\Delta}_{\Lambda, \Lambda_0}$, the r.h.s. of (1.2.17) also satisfies the Wilson flow equation and the initial condition at Λ_0 since $\Delta_{\Lambda_0, \Lambda_0} = 0$ and $I^{\Lambda_0, \Lambda_0} = 0$.

In [62], the generating functional is first defined on a finite spacetime lattice to begin on safe ground. Then, a perturbative flow equation is derived with finite-dimensional Gaussian integrals. Then the limit to continuous infinite spacetime is performed. The Wilson flow equation (1.2.15) has been analyzed beyond perturbation theory with convergent expansion techniques in [27]. See also [28] for a review of these techniques.

From the flow equation (1.2.12) for the generating functional $L^{\Lambda, \Lambda_0}(\varphi)$, a system of flow equations for its moments, i.e. the CAS functions can be obtained. To derive the flow equations, we have defined the theory on a finite volume. It turns out that when passing to the CAS functions, the infinite volume limit exists; see [33, 59] for more details. We will now treat the flow equations for the CAS functions in this limit. The generating functional is expanded in a formal power series in the Fourier transform of the field $\hat{\varphi}(p)$ (we suppose that $\varphi \in \mathcal{S}(\mathbb{R}^4)$). Due to translation invariance in position space, the total momentum is conserved; we can factorize the CAS functions, which are symmetric under any permutation of the set of the external momenta, in the infinite volume limit as

$$\bar{\mathcal{L}}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n) = \delta^4 \left(\sum_{i=1}^n p_i \right) \mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n), \quad p_n = -p_1 - \dots - p_{n-1}. \quad (1.2.18)$$

The term $\delta^4 \left(\sum_{i=1}^n p_i \right)$ also implies that we performed the infinite volume limit since it corresponds to the infinite volume limit of the factorized volume. The CAS functions $\mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n)$ are obtained by successive functional derivatives

$$(2\pi)^{4(n-1)} \frac{\delta}{\delta \hat{\varphi}(p_1)} \dots \frac{\delta}{\delta \hat{\varphi}(p_n)} L^{\Lambda, \Lambda_0}(\varphi)|_{\varphi=0} = \delta^4 \left(\sum_{i=1}^n p_i \right) \mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n). \quad (1.2.19)$$

They are completely symmetric under permutations of p_1, \dots, p_n . From the flow equations (1.2.15), we obtain the flow equations for the smooth n -point functions $\mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n)$

$$\begin{aligned} \partial_{\Lambda} \mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n) &= \frac{\hbar}{2} \int_k \partial_{\Lambda} C^{\Lambda, \Lambda_0}(k, m) \mathcal{L}_{n+2}^{\Lambda, \Lambda_0}(k, -k, p_1, \dots, p_n) \\ &\quad - \frac{1}{2} \sum_{r=0}^n \sum_{i_1 < \dots < i_r} \mathcal{L}_{r+1}^{\Lambda, \Lambda_0}(p_{i_1}, \dots, p_{i_r}, q) \partial_{\Lambda} C^{\Lambda, \Lambda_0}(q, m) \\ &\quad \cdot \mathcal{L}_{n-r+1}^{\Lambda, \Lambda_0}(-q, p_{j_1}, \dots, p_{j_{n-r}}), \end{aligned} \quad (1.2.20)$$

with $q = -p_{i_1} - \dots - p_{i_r}$ and $p_1 + \dots + p_n = 0$. In the quadratic term, for the given set of momenta $\{p_{i_1}, \dots, p_{i_r}\}$, $i_1 < \dots < i_r$, corresponds a unique set of momenta $\{p_{j_1}, \dots, p_{j_{n-r}}\}$, $j_1 < \dots < j_{n-r}$ such that

$$\{p_{i_1}, \dots, p_{i_r}\} \cup \{p_{j_1}, \dots, p_{j_{n-r}}\} = \{p_1, \dots, p_n\}. \quad (1.2.21)$$

The quadratic term can be rewritten in the more compact following form

$$\begin{aligned} & -\frac{1}{2} \sum_{r=0}^n \sum_{i_1 < \dots < i_r} \mathcal{L}_{r+1}^{\Lambda, \Lambda_0}(p_{i_1}, \dots, p_{i_r}, q) \partial_\Lambda C^{\Lambda, \Lambda_0}(q, m) \cdot \mathcal{L}_{n-r+1}^{\Lambda, \Lambda_0}(-q, p_{j_1}, \dots, p_{j_{n-r}}) \\ & = -\frac{1}{2} \sum_{\substack{n_1+n_2=n \\ n_i \geq 0}} \mathbb{S} \left(\mathcal{L}_{n_1+1}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n_1}, q) \partial_\Lambda C^{\Lambda, \Lambda_0}(q, m) \mathcal{L}_{n_2+1}^{\Lambda_0, \Lambda}(-q, p_{n_1+1}, \dots, p_n) \right), \end{aligned} \quad (1.2.22)$$

with $q = p_{n_1+1} + \dots + p_n = -p_1 - \dots - p_{n_1}$ and \mathbb{S} the symmetrisation operator summing over the permutations which do not leave invariant the two unordered sets of momenta $\{p_1, \dots, p_{n_1}\}$ and $\{p_{n_1+1}, \dots, p_n\}$.

We will now perform a perturbative analysis of the flow equations (1.2.20) through a loop expansion of the CAS functions as formal power series in \hbar

$$\mathcal{L}_n^{\Lambda, \Lambda_0}(p_1, \dots, p_n) = \sum_{l=0}^{+\infty} \hbar^l \mathcal{L}_{n,l}^{\Lambda, \Lambda_0}(p_1, \dots, p_n). \quad (1.2.23)$$

We introduce the following shorthand notation for the momentum derivatives

$$\begin{aligned} w &= (w_{1,1}, \dots, w_{n-1,4}), \quad w_{i,\mu} \in \mathbb{N}_0, \quad |w| := \sum_{i,\mu} w_{i,\mu} \\ \partial^w &:= \prod_{i=1}^{n-1} \prod_{\mu=1}^4 \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}}, \quad w! := \prod_{i=1}^{n-1} \prod_{\mu=1}^4 w_{i,\mu}!. \end{aligned} \quad (1.2.24)$$

From the flow equations (1.2.20) and the loop expansion (1.2.23), we obtain the following system of flow equations for the $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ and their momentum derivatives

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) &= \frac{1}{2} \int_k \partial_\Lambda C^{\Lambda, \Lambda_0}(k, m) \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}(k, -k, p_1, \dots, p_n) \\ & - \frac{1}{2} \sum_{\substack{n_1+n_2=n \\ n_i \geq 0}} \sum_{\substack{l_1+l_2=l \\ l_i \geq 0}} \sum_{\substack{w_1+w_2+w_3=w \\ w_i \geq 0}} c_{\{w_i\}} \mathbb{S} \left(\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n_1}, q) \right. \\ & \left. \cdot \partial^{w_3} \partial_\Lambda C^{\Lambda, \Lambda_0}(q, m) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda_0, \Lambda}(-q, p_{n_1+1}, \dots, p_n) \right), \end{aligned} \quad (1.2.25)$$

where we have introduced the combinatorial factor

$$c_{\{w_i\}} := \frac{w!}{w_1! w_2! w_3!}, \quad (1.2.26)$$

which originates from the Leibniz's rule. Note that for $l = 0$, the linear term in the r.h.s. of (1.2.25) is absent.

It is important to remark that thanks to translation invariance in position space, the CAS distributions can be factorized as in (1.2.18). Therefore the flow equations in momentum space allow us to directly analyze the smooth part of the CAS distributions, also called the CAS functions. The flow equations in position space have also been used to prove renormalizability of quantum field theories in spaces where translation invariance is broken: renormalizability of massive φ_4^4 -theory on Riemannian manifolds [63], renormalizability of φ^4 -theory on the half space $\mathbb{R}^+ \times \mathbb{R}^3$ [64, 65]. In these cases, bounds are obtained by smearing the CAS distributions with suitably chosen test functions.

1.3 Proof of perturbative renormalizability

Now we pass to the perturbative renormalizability proof of the regularized theory (1.2.1). We fix coupling constants f, g and perform a loop expansion of the coefficients of the counterterms

$v(\Lambda_0), a(\Lambda_0), z(\Lambda_0), c(\Lambda_0)$, i.e. for $v(\Lambda_0)$

$$v(\Lambda_0) = \sum_{l=1}^{+\infty} \hbar^l v_l(\Lambda_0). \quad (1.3.1)$$

These coefficients are chosen so that the following limits of the CAS functions at every loop order

$$\lim_{\Lambda_0 \rightarrow +\infty} \lim_{\Lambda \rightarrow 0} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) \quad (1.3.2)$$

are finite. To adjust the counterterms, we impose that the CAS functions at the physical scale $\Lambda = 0$ have specific values for a chosen set of external momenta. In our analysis, we will consider a massive theory $m > 0$ so that the renormalization conditions can be prescribed at zero external momenta. To take into account the Euclidean symmetry, we have for $l \geq 1$

$$\begin{cases} \mathcal{L}_{l,1}^{0, \Lambda_0}(0) = v_l^R, \\ \mathcal{L}_{l,2}^{0, \Lambda_0}(p, -p) = a_l^R + z_l^R p^2 + \mathcal{O}((p^2)^2), \\ \mathcal{L}_{l,3}^{0, \Lambda_0}(p_1, p_2, -p_1 - p_2) = b_l^R + \mathcal{O}(p_1^2, p_2^2), \\ \mathcal{L}_{l,4}^{0, \Lambda_0}(0, 0, 0, 0) = c_l^R. \end{cases} \quad (1.3.3)$$

In each loop order $l \geq 1$, the renormalization constants $v_l^R, a_l^R, z_l^R, b_l^R$ and c_l^R can be chosen freely. We can choose them to be independent of or to depend weakly on Λ_0 , i.e. the five constants have finite limits when $\Lambda_0 \rightarrow +\infty$. Here we choose the renormalization constants to be independent of Λ_0 . The relevant part of the theory is completely fixed by these constants along with the other constants at tree order $l = 0$. A simple choice is to set the five renormalization constants to zero.

First, we treat the tree order. The latter is determined by the classical part of the bare interaction lagrangian (1.2.1). This bare interaction lagrangian can be viewed as an initial condition at $\Lambda = \Lambda_0$

upon integrating the flow equations downwards from Λ_0 to $0 \leq \Lambda \leq \Lambda_0$, while going up in n . To start the proof of renormalizability, we also require

$$\mathcal{L}_{0,1}^{\Lambda,\Lambda_0} = 0, \quad \mathcal{L}_{0,2}^{\Lambda,\Lambda_0}(p, -p) = 0, \quad (1.3.4)$$

which can be deduced from (1.2.9)-(1.2.10). From (1.3.4) and (1.2.25) we find with the initial condition in the bare lagrangian (1.2.1) that

$$\mathcal{L}_{0,3}^{\Lambda,\Lambda_0}(p_1, p_2, -p_1 - p_2) = f, \quad (1.3.5)$$

and for $n = 4$ we have

$$\mathcal{L}_{0,4}^{\Lambda,\Lambda_0}(p_1, p_2, p_3, p_4) = g - f^2 \left(C^{\Lambda,\Lambda_0}(p_1 + p_2, m) + C^{\Lambda,\Lambda_0}(p_1 + p_3, m) + C^{\Lambda,\Lambda_0}(p_1 + p_4, m) \right). \quad (1.3.6)$$

We can then keep ascending in n to obtain the whole tree order. For $n > 4$, the initial conditions at Λ_0 are zero from the bare interaction lagrangian (1.2.1). First we prove

Proposition 1.1 (Boundedness). *For $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, w from (1.2.24) and for $0 \leq \Lambda \leq \Lambda_0$, the following bounds hold*

$$|\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n)| \leq (\Lambda + m)^{4-n-|w|} \mathcal{P}_1 \left(\ln \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\left\{ \frac{|p_i|}{\Lambda + m} \right\} \right), \quad (1.3.7)$$

where \mathcal{P}_k , $k = 1, 2$ are polynomials with nonnegative coefficients. The degree of these two polynomials and their coefficients depend on l, n, w but not on $\{p_i\}$, Λ, Λ_0 . For $l = 0$ the polynomials \mathcal{P}_1 are constants.

Proof. Using the shorthand notation for the momentum derivatives (1.2.24), the derivatives of the propagator (1.1.5) satisfy the bounds

$$|\partial^w \partial_\Lambda C^{\Lambda,\Lambda_0}(k, m)| \leq \Lambda^{-3-|w|} \mathcal{P} \left(\frac{|k|}{\Lambda} \right) e^{-\frac{k^2 + m^2}{\Lambda^2}}, \quad (1.3.8)$$

where \mathcal{P} denotes a polynomial of degree $|w|$ of the same type of \mathcal{P}_i . Weaker bounds are also used, namely

$$|\partial^w \partial_\Lambda C^{\Lambda,\Lambda_0}(k, m)| \leq (\Lambda + m)^{-3-|w|} \mathcal{P} \left(\frac{|k|}{\Lambda + m} \right) \quad (1.3.9)$$

and the crude bound

$$|C^{\Lambda,\Lambda_0}(k, m)| \leq \frac{2}{m^2}. \quad (1.3.10)$$

- We first treat the tree level, i.e. $l = 0$.

- $n + |w| \leq 4$: for $n \leq 3$, the bounds (1.3.7) obviously follow from (1.3.4) and (1.3.5). For $n = 4$ and $|w| = 0$, we use together (1.3.6) and (1.3.10) to obtain (1.3.7).

- $n + |w| > 4$: we go up in n and at n fixed and arbitrary w , we integrate the flow equations (1.2.25) downwards from Λ_0 with the initial conditions

$$\partial^w \mathcal{L}_{0,n}^{\Lambda_0, \Lambda_0}(p_1, \dots, p_n) = 0, \quad (1.3.11)$$

which can be inferred from (1.2.1). For $l = 0$, we remind that the r.h.s. of (1.2.25) is reduced to the quadratic term. Using the weaker bounds (1.3.9) and inserting the induction hypothesis, we have

$$\begin{aligned} |\partial^w \mathcal{L}_{0,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)| &\leq \int_{\Lambda}^{\Lambda_0} d\lambda |\partial^w \partial_{\lambda} \mathcal{L}_{0,n}^{\lambda, \Lambda_0}(p_1, \dots, p_n)| \\ &\leq \mathcal{P}'\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right) \int_{\Lambda}^{\Lambda_0} d\lambda (\lambda + m)^{4-n-|w|-1} \\ &\leq \mathcal{P}'\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right) \frac{1}{n + |w| - 4} (\Lambda + m)^{4-n-|w|}, \end{aligned} \quad (1.3.12)$$

where \mathcal{P}' is a polynomial with the required properties.

- Now we treat the higher loop orders, i.e. $l \geq 1$. We proceed as follows

- We go up in l .
- For fixed l , we go up in n .
- For fixed l, n , we descend with w to $w = 0$.

Through this scheme, the bounds (1.3.7) are produced inductively by successive integration of the flow equations (1.2.25), all the terms on the r.h.s. of (1.2.25) are prior to that on the l.h.s.:

- The linear term contains a term of loop order $l - 1$.
- In the quadratic term, contributing terms of loop order l have a smaller value n due to (1.3.4), the remaining terms have loop orders $l' < l$.

We integrate the flow equations as follows:

- For $n + |w| > 4$, since the bound (1.3.7) decreases as Λ increases, we integrate the flow equations (1.2.25) from Λ_0 downwards to $0 \leq \Lambda \leq \Lambda_0$ with the initial conditions

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}(p_1, \dots, p_n) = 0, \quad n + |w| > 4, \quad (1.3.13)$$

which can be inferred again from (1.2.1).

- For $n + |w| \leq 4$, the bound increases with Λ . Therefore, the flow equations (1.2.25) are integrated for a specific set of momenta (renormalization point) from the physical scale $\Lambda = 0$ upwards to Λ . Since we considered a massive theory, we have chosen the renormalization conditions at zero external momenta. Using the renormalization conditions (1.3.3), we get

$$\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0, \dots, 0) = \partial^w \mathcal{L}_{l,n}^{0, \Lambda_0}(0, \dots, 0) + \int_0^{\Lambda} d\lambda \partial^w \partial_{\lambda} \mathcal{L}_{l,n}^{\lambda, \Lambda_0}(0, \dots, 0). \quad (1.3.14)$$

Note that for $n = 1$, there is no momentum dependence. If a bound has been obtained using (1.3.14), then we can extend it to nonzero momenta using the Taylor formula with integral remainder

$$f(p) = f(0) + \sum_{i=1}^n p_i \int_0^1 dt (\partial_i f)(tp) \quad (1.3.15)$$

for a differentiable function f on \mathbb{R}^n . In the r.h.s. of (1.3.15), an additional term $(\Lambda + m)^{-1}$ arises from the induction hypothesis due to the momenta derivatives. This factor is combined with the momenta p_i to obtain a new polynomial \mathcal{P}_2 .

We can now proceed to bound successively the linear and the quadratic terms by distinguishing the different cases and following the strategies mentioned above. In the rest of the proof, we will denote by \mathcal{P}_i , $1 \leq i \leq 6$, polynomials of the same type as \mathcal{P}_1 or \mathcal{P}_2 according to the parity of i . First using the bounds (1.3.8), (1.3.9) and the induction hypothesis, we bound the r.h.s. of the flow equations (1.2.25)

$$\begin{aligned} & |\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)| \\ & \leq \int_k \frac{e^{-\frac{k^2+m^2}{\Lambda^2}}}{\Lambda^3} (\Lambda + m)^{4-n-2-|w|} \mathcal{P}_1\left(\ln \frac{\Lambda + m}{m}\right) \mathcal{P}_2\left(\frac{|k|}{\Lambda + m}, \left\{\frac{|p_i|}{\Lambda + m}\right\}\right) \\ & + (\Lambda + m)^{4-n-|w|-1} \mathcal{P}_3\left(\ln \frac{\Lambda + m}{m}\right) \mathcal{P}_4\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right). \end{aligned} \quad (1.3.16)$$

In the second term of the r.h.s. of (1.3.16), the sum of the polynomials have been combined into the product of two polynomials \mathcal{P}_3 and \mathcal{P}_4 . In the integral, we perform the integration by substituting $k \rightarrow \Lambda k$. Combining the result with the second term, we finally get

$$|\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)| \leq (\Lambda + m)^{4-n-|w|-1} \mathcal{P}_5\left(\ln \frac{\Lambda + m}{m}\right) \mathcal{P}_6\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right). \quad (1.3.17)$$

- First we treat the irrelevant terms, i.e. $n + |w| > 4$. We integrate from Λ_0 to Λ using the initial conditions (1.3.13) and the bound (1.3.17) to obtain

$$\int_\Lambda^{\Lambda_0} d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P}_5\left(\ln \frac{\lambda + m}{m}\right) < (\Lambda + m)^{4-n-|w|} \mathcal{P}_1\left(\ln \frac{\Lambda + m}{m}\right) \quad (1.3.18)$$

with a new polynomial \mathcal{P}_1 . Then to obtain (1.3.7), we use the fact that

$$\mathcal{P}_6\left(\left\{\frac{|p_i|}{\lambda + m}\right\}\right) \leq \mathcal{P}_6\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right), \quad \Lambda \leq \lambda \leq \Lambda_0. \quad (1.3.19)$$

- Now we treat the remaining cases. We first start with the special case $n = 1$ (momentum independent), then with $n = 2, |w| \leq 2$ and so on. We integrate the flow equations at the renormalization point (zero external momenta) from $\Lambda = 0$ to Λ . From (1.3.14) and the renormalization conditions (1.3.3) we have

$$\begin{aligned} |\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0, \dots, 0)| & \leq \int_0^\Lambda d\lambda (\lambda + m)^{4-n-|w|-1} \mathcal{P}_5\left(\ln \frac{\lambda + m}{m}\right) \\ & \leq (\Lambda + m)^{4-n-|w|} \mathcal{P}_1\left(\ln \frac{\Lambda + m}{m}\right), \end{aligned} \quad (1.3.20)$$

where \mathcal{P}_1 is again a polynomial with the required properties. Then, to obtain the bound (1.3.7) for general momenta, we use together the bound (1.3.20) and (1.3.15)

$$\begin{aligned}
|\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n)| &\leq |\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(0, \dots, 0)| \\
&+ \sum_{i=1}^n |p_i| \int_0^1 dt (\Lambda + m)^{4-n-|w|-1} \mathcal{P}_1\left(\ln \frac{\Lambda + m}{m}\right) \mathcal{P}_2\left(\left\{\frac{t|p_i|}{\Lambda + m}\right\}\right) \\
&\leq (\Lambda + m)^{4-n-|w|} \mathcal{P}'_1\left(\ln \frac{\Lambda + m}{m}\right) \mathcal{P}'_2\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right),
\end{aligned} \tag{1.3.21}$$

where we have new polynomials \mathcal{P}'_1 and \mathcal{P}'_2 which are obtained by combining the polynomials $\mathcal{P}_1, \mathcal{P}_2$ and the momenta p^i .

□

In Proposition 1.1, the bounds (1.3.7) do not rule out an oscillatory dependence in Λ_0 . To prove convergence, we have

Proposition 1.2 (Convergence). *For $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, w from (1.2.24) and for $0 \leq \Lambda \leq \Lambda_0$, the following bounds hold*

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n)| \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_3\left(\ln \frac{\Lambda_0 + m}{m}\right) \mathcal{P}_4\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right). \tag{1.3.22}$$

Proof. The proof is done by following the induction scheme stated in the proof of Proposition 1.1. We will integrate again the flow equations (1.2.20) and take the Λ_0 -derivative of the perturbative CAS functions afterwards. As a shorthand notation, we will note by $\partial^w \mathcal{R}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n)$ the r.h.s. of (1.2.25), so that

$$\partial_{\Lambda} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n) := \partial^w \mathcal{R}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n). \tag{1.3.23}$$

From (1.3.4), (1.3.5) and (1.3.6), the claim clearly holds for $l = 0$ and $n + |w| \leq 4$.

- For $n + |w| > 4$, integrating again the flow equations (1.2.25) from Λ_0 to Λ with the initial conditions (1.3.13), we get

$$\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n) = - \int_{\Lambda}^{\Lambda_0} d\lambda \partial^w \mathcal{R}_{l,n}^{\lambda,\Lambda_0}(p_1, \dots, p_n), \tag{1.3.24}$$

so that we have

$$\begin{aligned}
\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(p_1, \dots, p_n) &= -\partial^w \mathcal{R}_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \dots, p_n) \\
&- \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\lambda,\Lambda_0}(p_1, \dots, p_n).
\end{aligned} \tag{1.3.25}$$

Due to (1.3.13), only the quadratic part in the flow equations (1.2.25) contributes to first term in the r.h.s. of (1.3.25). Using Proposition 1.1 and (1.3.8) we get

$$\begin{aligned}
|\partial^w \mathcal{R}_{l,n}^{\Lambda_0,\Lambda_0}(p_1, \dots, p_n)| &\leq (\Lambda_0 + m)^{3-n-|w|} \mathcal{P}_1\left(\ln \frac{\Lambda_0 + m}{m}\right) \mathcal{P}_2\left(\left\{\frac{|p_i|}{\Lambda_0 + m}\right\}\right) \\
&\leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_1\left(\ln \frac{\Lambda_0 + m}{m}\right) \mathcal{P}_2\left(\left\{\frac{|p_i|}{\Lambda + m}\right\}\right),
\end{aligned} \tag{1.3.26}$$

since $0 \leq \Lambda \leq \Lambda_0$ and $n + |w| > 4$. To bound the second term in the r.h.s. of (1.3.25), we observe first that

$$\partial_{\Lambda_0} \partial_{\Lambda} C^{\Lambda, \Lambda_0}(k, m) = 0. \quad (1.3.27)$$

We bound the integrand by following the steps to derive (1.3.17), using together the induction hypothesis, Proposition 1.1 and the bounds (1.3.8) and (1.3.9), used in the linear term and in the quadratic term respectively to obtain

$$|\partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)| \leq \frac{(\Lambda + m)^{5-n-|w|-1}}{(\Lambda_0 + m)^2} \mathcal{P}_5 \left(\ln \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_6 \left(\left\{ \frac{|p_i|}{\Lambda + m} \right\} \right). \quad (1.3.28)$$

By integrating (1.3.28), we get

$$\begin{aligned} & \left| \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\lambda, \Lambda_0}(p_1, \dots, p_n) \right| \\ & \leq \int_{\Lambda}^{\Lambda_0} d\lambda \frac{(\lambda + m)^{5-n-|w|-1}}{(\Lambda_0 + m)^2} \mathcal{P}_5 \left(\ln \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_6 \left(\left\{ \frac{|p_i|}{\lambda + m} \right\} \right) \\ & \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_7 \left(\ln \frac{\Lambda_0 + m}{m} \right) \mathcal{P}_8 \left(\left\{ \frac{|p_i|}{\Lambda + m} \right\} \right). \end{aligned} \quad (1.3.29)$$

Combining the result (1.3.29) with (1.3.26), we obtain (1.3.22) for the irrelevant cases.

- Now we turn to the remaining cases. We proceed as in Proposition 1.1. We integrate again the flow equations (1.2.25) at the renormalization point from $\Lambda = 0$ to Λ and we take the Λ_0 -derivative. Since the renormalization conditions (1.3.3) do not depend on Λ_0 , we have

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0, \dots, 0)| \leq \int_0^{\Lambda} d\lambda |\partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\lambda, \Lambda_0}(0, \dots, 0)|. \quad (1.3.30)$$

We remark that (1.3.28) has been obtained without restriction on n, l, w . Therefore, using the bound (1.3.28) in the r.h.s. of (1.3.30), then we get after integration

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0, \dots, 0)| \leq \frac{(\Lambda + m)^{5-n-|w|}}{(\Lambda_0 + m)^2} \mathcal{P}_3 \left(\ln \frac{\Lambda_0 + m}{m} \right). \quad (1.3.31)$$

Then, we extend the bounds (1.3.22) obtained at zero momenta to general nonzero momenta using again (1.3.15) as in the proof of Proposition 1.1.

□

Then we can state

Theorem 1.1 (Perturbative renormalizability of φ_4^4 -theory). *Consider the massive φ_4^4 -theory in four dimensions of bare interaction lagrangian (1.2.1). Then, the perturbative CAS functions $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ which are solutions of the flow equations (1.2.25) with the renormalization conditions (1.3.3) and the initial conditions (1.3.11)-(1.3.13) are such that the limits*

$$\lim_{\Lambda_0 \rightarrow +\infty} \lim_{\Lambda \rightarrow 0} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n) \quad (1.3.32)$$

are finite.

Proof. For large values of Λ_0 , the bounds (1.3.22) can be rewritten as

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_n)| \leq \frac{(\Lambda + m)^{5-n-|w|}}{\Lambda_0^2} \left(\ln \frac{\Lambda_0}{m} \right)^{\nu(l,n,w)} \mathcal{P}_4 \left(\left\{ \frac{|p_i|}{\Lambda + m} \right\} \right), \quad (1.3.33)$$

for an integer $\nu(l, n, w)$. By integrating these bounds w.r.t. Λ_0 , for $\Lambda'_0 > \Lambda_0$

$$|\mathcal{L}_{l,n}^{0, \Lambda_0}(p_1, \dots, p_n) - \mathcal{L}_{l,n}^{0, \Lambda'_0}(p_1, \dots, p_n)| \leq \frac{m^{5-n}}{\Lambda_0} \left(\ln \frac{\Lambda_0}{m} \right)^{\nu(l,n,0)} \mathcal{P}_5 \left(\left\{ \frac{|p_i|}{m} \right\} \right). \quad (1.3.34)$$

Using the Cauchy criterion, the limits (1.3.2) exist and are finite. \square

Perturbative renormalizability of massive φ_4^4 -theory is proven with the aid of Proposition 1.2. Proposition 1.1 is purely technical but is required to prove Proposition 1.2. The bounds (1.3.7)-(1.3.22) are not sharp but sufficient to prove renormalizability. Optimal bounds w.r.t. the momentum behavior have been obtained by Kopper and Meunier [66]. Kopper [6] also analyzed the existence of the local Borel transform of perturbation theory with the flow equations at large order in perturbation theory and obtained sharper bounds on the whole set of Schwinger functions and their behavior at large momenta.

Chapter 2

Triviality results for the φ_4^4 -theory

In this thesis, we deal with the triviality of φ_4^4 -theory. As a result, it is useful to recall some rigorous triviality results on φ_d^4 -theory. We review in particular the triviality results established by Aizenman [1, 44], Fröhlich [2] and Duminil-Copin [3]. Other triviality results on φ_d^4 -theory have also been established, but they are still restricted to scalar fields with one or two components. For an exhaustive summary of these results, see e.g. [67] and the references therein. They proved that the lattice regularized truncated four-point function vanishes in the continuum limit. It is important to note that the triviality proofs for scalar φ_4^4 -theory are restricted to a single-component field or a two-component real field [2] and in $d \geq 4$ dimensions.

2.1 Triviality of the continuum limit of lattice φ_d^4 -theory in $d > 4$ dimensions

Here we will review Aizenman's result on triviality of the continuum limit of the lattice φ_d^4 -theory, $d > 4$. The proof is established for the Ising model in dimension $d > 4$, since the φ^4 -theory and the Ising model are closely related, see e.g. [68, 69, 3]. A more explicit proof for φ^4 -theory is also done in this paper, it follows similar steps albeit being more complicated. We fix $d > 4$. The starting point is a finite lattice $\hat{\Lambda} \subset \mathbb{Z}^d$. For the Ising model, at each site of the lattice, we consider a spin $\sigma_x = \pm 1$, $x \in \hat{\Lambda}$.

The hamiltonians corresponding to ferromagnetic pair interactions at zero magnetic field are of the form

$$H(\sigma) = -\frac{1}{2} \sum_{x,y \in \hat{\Lambda}} J_{x,y} \sigma_x \sigma_y, \quad J_{x,y} \geq 0. \quad (2.1.1)$$

In the case of the Ising model with nearest-neighbor interaction on \mathbb{Z}^d , we have $J_{x,y} = \delta_{|x-y|,1}$ where we have defined

$$|x| = \sum_{i=1}^d |x_i|. \quad (2.1.2)$$

The equilibrium states of the system are described by probability measures on the lattice. In particular, Gibbs states describe the thermodynamic equilibrium at inverse temperature $\beta = \frac{1}{T}$ (not

to be confused with the β function introduced in Sect.3.2). For a finite system, the expectation value of a quantity f is defined by

$$\langle f \rangle_{\hat{\Lambda}} := \frac{\text{Tr}_{\hat{\Lambda}}[f(\sigma)e^{-\beta H(\sigma)}]}{\text{Tr}_{\hat{\Lambda}}[e^{-\beta H(\sigma)}]}, \quad (2.1.3)$$

where the trace over $\hat{\Lambda}$ is defined by

$$\text{Tr}_{\hat{\Lambda}} := \prod_{x \in \hat{\Lambda}} \left(\frac{1}{2} \sum_{\sigma_x = \pm 1} \right). \quad (2.1.4)$$

For infinite systems, i.e. $\hat{\Lambda} = \mathbb{Z}^d$, the expressions are defined by suitable limits.

It is instructive to find an expression of expectation values $\langle f \rangle_{\hat{\Lambda}}$. First such a representation will be derived for finite systems. For infinite systems, relations involving spin correlations, namely $f = \prod_{x \in A} \sigma_x$, for a set $A \subset \hat{\Lambda}$, obtained on a finite lattice are obtained by the monotone convergence theorem.

First, we define the partition function

$$Z_{\hat{\Lambda}} := \text{Tr}_{\hat{\Lambda}}[e^{-\beta H(\sigma)}] = \text{Tr}_{\hat{\Lambda}} \left[\prod_{b=\{x,y\}} e^{\beta J_b \sigma_x \sigma_y} \right], \quad (2.1.5)$$

with $J_b := J_{x,y}$. Then, we introduce the weight

$$w(\mathbf{n}) := \prod_b \frac{(\beta J_b)^{n_b}}{n_b!}, \quad (2.1.6)$$

where a current \mathbf{n} assigns an integer n_b to the bond b of the lattice. We define the sum of two currents \mathbf{n}_1 and \mathbf{n}_2 by $\mathbf{m} := \mathbf{n}_1 + \mathbf{n}_2$ with $m_b = n_{1,b} + n_{2,b}$. By expanding the exponentials in the partition function (2.1.5)

$$e^{\beta J_b \sigma_x \sigma_y} = \sum_{n_b=0}^{+\infty} \frac{(\beta J_b)^{n_b} (\sigma_x \sigma_y)^{n_b}}{(n_b)!}, \quad (2.1.7)$$

we can rewrite the partition function

$$Z_{\hat{\Lambda}} = \sum_{\mathbf{n}} w(\mathbf{n}) \text{Tr}_{\hat{\Lambda}} \left[\prod_{x \in \hat{\Lambda}} \sigma_x^{\sum_{x \in b} n_b} \right]. \quad (2.1.8)$$

We define the set of sources $\partial \mathbf{n}$

$$\partial \mathbf{n} := \left\{ x \in \Lambda, \quad \sum_{x \in b} n_b \equiv 1 \pmod{2} \right\}. \quad (2.1.9)$$

In the r.h.s. of (2.1.8), the trace is either zero or one for a fixed current \mathbf{n} ; it is nonzero only if $\partial \mathbf{n} = \emptyset$. Therefore we have

$$Z_{\hat{\Lambda}} = \sum_{\partial \mathbf{n} = \emptyset} w(\mathbf{n}) . \quad (2.1.10)$$

Similarly, for correlation functions, if $\sigma_A = \prod_{x \in A} \sigma_x$, $A \subset \hat{\Lambda}$, we have, following the derivation of (2.1.10)

$$\langle \sigma_A \rangle = \frac{\sum_{\partial \mathbf{n} = A} w(\mathbf{n})}{\sum_{\partial \mathbf{n} = \emptyset} w(\mathbf{n})} . \quad (2.1.11)$$

It is possible to give a probabilistic interpretation of the spin correlations based on this representation. The probabilistic interpretation is called the random current representation. For a fixed current \mathbf{m} , we decompose $\hat{\Lambda}$ in clusters of connected bonds i.e. non-vanishing values of m_b . For $x \in \hat{\Lambda}$, we define $\mathcal{C}_{\mathbf{m}}(x)$ to be the connected cluster containing x . If y and x are connected then, $\mathcal{C}_{\mathbf{m}}(x) = \mathcal{C}_{\mathbf{m}}(y)$ or $y \in \mathcal{C}_{\mathbf{m}}(x)$.

Now, consider duplicated independent currents $\mathbf{n}_1, \mathbf{n}_2$ of sources A_1 and A_2 and a probability distribution proportional to $w(\mathbf{n})$. For F an event, the probability of F with sources A_1 and A_2 is denoted $\mathbb{P}[F \mid A_1, A_2] = \mathbb{P}[F \mid \partial \mathbf{n}_1 = A_1, \partial \mathbf{n}_2 = A_2]$ and we note

$$\mathcal{N}_2 = \sum_{\substack{\partial \mathbf{n}'_1 = A_1 \\ \partial \mathbf{n}'_2 = A_2}} w(\mathbf{n}'_1) w(\mathbf{n}'_2) . \quad (2.1.12)$$

We have

$$\mathbb{P}[\{\mathbf{n}_1, \mathbf{n}_2\} \mid A_1, A_2] = \frac{w(\mathbf{n}_1) w(\mathbf{n}_2)}{\mathcal{N}_2} = \frac{w(\mathbf{n}_1) w(\mathbf{n}_2)}{Z_{\hat{\Lambda}}^2 \langle \sigma_{A_1} \rangle \langle \sigma_{A_2} \rangle} . \quad (2.1.13)$$

The probability (2.1.13) can also be extended to a set of $k \geq 3$ independent currents

$$\mathbb{P}[\{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_k\} \mid A_1, A_2, \dots, A_k] = \frac{\prod_{j=1}^k w(\mathbf{n}_j)}{\mathcal{N}_k} , \quad (2.1.14)$$

where we defined

$$\mathcal{N}_k := \sum_{\substack{\partial \mathbf{n}'_j = A_j \\ 1 \leq j \leq k}} \prod_{a=1}^k w(\mathbf{n}'_a) = Z_{\hat{\Lambda}}^k \prod_{i=1}^k \langle \sigma_{A_i} \rangle . \quad (2.1.15)$$

In the rest of this section, $\mathbb{P}[K]$ will denote the measure of the currents \mathbf{n} such that the event K is true.

The connected four-point function $U_4(x_1, x_2, x_3, x_4)$, also called Ursell function is defined by

$$U_4(x_1, x_2, x_3, x_4) := \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle - \left[\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \text{permutations} \right] . \quad (2.1.16)$$

In [43], the idea is to bound U_4 so that it vanishes in the continuum limit. We will review here the important results that lead to the Gaussianity of the continuum limit. A preparatory and technical lemma is proven before finding bounds on U_4

Lemma 2.1 (Switching lemma). *For a finite $\hat{\Lambda}$, a set $A \subset \hat{\Lambda}$, $\{x, y\}$, $x, y \in \hat{\Lambda}$ and a set \mathcal{F} of flux configurations over bonds of $\hat{\Lambda}$*

$$\sum_{\substack{\partial \mathbf{n}_1 = A \\ \partial \mathbf{n}_2 = \{x, y\} \\ \mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}}} w(\mathbf{n}_1)w(\mathbf{n}_2) = \sum_{\substack{\partial \mathbf{n}_1 = A \Delta \{x, y\} \\ \partial \mathbf{n}_2 = \emptyset \\ \mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}}} w(\mathbf{n}_1)w(\mathbf{n}_2)X(\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y)), \quad (2.1.17)$$

where $A \Delta B$ designates the symmetric difference of the sets A and B . X designates the characteristic function, its value is one if its argument is true, zero otherwise.

Proof. See [1]. □

An immediate consequence is

Proposition 2.1. *For $x, y, z \in \hat{\Lambda}$,*

$$\langle \sigma_x \sigma_y \rangle^2 = \mathbb{P}[\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y) \mid \emptyset, \emptyset]. \quad (2.1.18)$$

and

$$\langle \sigma_x \sigma_y \rangle \langle \sigma_y \sigma_z \rangle = \langle \sigma_x \sigma_z \rangle \mathbb{P}[y \in \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) \mid \{x, z\}, \emptyset], \quad (2.1.19)$$

Proof. First we square (2.1.11), then we use Lemma 2.1 with $A = \{x, y\}$ so that $A \Delta \{x, y\} = \emptyset$

$$\begin{aligned} \langle \sigma_x \sigma_y \rangle^2 &= \frac{\sum_{\substack{\partial \mathbf{n}_1 = \{x, y\} \\ \partial \mathbf{n}_2 = \{x, y\}}} w(\mathbf{n}_1)w(\mathbf{n}_2)}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2)} \\ &= \frac{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2)X(\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y))}{\sum_{\substack{\partial \mathbf{n}_1 = \emptyset \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1)w(\mathbf{n}_2)}. \end{aligned} \quad (2.1.20)$$

The ratio corresponds to the probability that the event $\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y)$ is satisfied in the duplicate ensemble of the two currents, this yields (2.1.18). For (2.1.19), the same argument is repeated, the slight change is that the current \mathbf{n}_1 is constrained by $\partial \mathbf{n}_1 = \{x, z\}$. □

The size of the connected cluster containing x for a current \mathbf{m} is defined by

$$|\mathcal{C}_{\mathbf{m}}(x)| := \sum_y X_{x,y}(\mathbf{m}), \quad (2.1.21)$$

where $X_{x,y}(\mathbf{m})$ is defined by

$$X_{x,y}(\mathbf{m}) := X(\mathcal{C}_{\mathbf{m}}(x) = \mathcal{C}_{\mathbf{m}}(y)). \quad (2.1.22)$$

A useful consequence of Proposition 2.1 is an expression for the expectation of the size of the connected cluster

$$\mathbb{E}[|\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x)| \mid \emptyset, \emptyset] = \sum_y \mathbb{P}[\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(y) \mid \emptyset, \emptyset] = \sum_y \langle \sigma_x \sigma_y \rangle^2. \quad (2.1.23)$$

From Lemma 2.1, an expression of U_4 can be derived

Proposition 2.2. *For finite ferromagnetic systems,*

$$\begin{aligned} U_4(x_1, \dots, x_4) &= -2\langle \sigma_{x_1} \cdots \sigma_{x_4} \rangle \mathbb{P}[x_2, x_3, x_4 \in \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1) \mid \{x_1, \dots, x_4\}, \emptyset] \\ &= -2\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle \mathbb{P}[\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_3) \mid \{x_1, x_2\}, \{x_3, x_4\}]. \end{aligned} \quad (2.1.24)$$

If $x_1 = x_2$, then $\partial \mathbf{n} = \{x_1, x_2\}$ or $\{x_1, x_2, x_3, x_4\}$ becomes respectively $\partial \mathbf{n} = \emptyset$ and $\{x_3, x_4\}$.

Proof. The result is true if $x_1 = x_2$ or $x_1 = x_3$ so we can suppose that all points are distinct. From (2.1.11) and Lemma 2.1, the Ursell function can be rewritten as

$$\begin{aligned} Z_{\Lambda}^2 U_4(x_1, \dots, x_4) &= \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, \dots, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) \\ &\quad - \left[\sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2\} \\ \partial \mathbf{n}_2 = \{x_3, x_4\}}} + \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_3\} \\ \partial \mathbf{n}_2 = \{x_2, x_4\}}} + \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_4\} \\ \partial \mathbf{n}_2 = \{x_2, x_3\}}} \right] w(\mathbf{n}_1) w(\mathbf{n}_2) \\ &= \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, \dots, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) [1 - X_{x_1, x_2}(\mathbf{n}_1 + \mathbf{n}_2) \\ &\quad - X_{x_1, x_3}(\mathbf{n}_1 + \mathbf{n}_2) - X_{x_1, x_4}(\mathbf{n}_1 + \mathbf{n}_2)]. \end{aligned} \quad (2.1.25)$$

- For flux configurations $\mathbf{n}_1 + \mathbf{n}_2$ such that the source x_1 is connected to exactly one other source x_i , $i \neq 1$, the corresponding term in the r.h.s. of (2.1.25) is zero.
- The only remaining possibility is that the source x_1 is connected to all the other sources for flux configurations $\mathbf{n}_1 + \mathbf{n}_2$.

Therefore, we obtain

$$\begin{aligned} Z_{\Lambda}^2 U_4(x_1, \dots, x_4) &= -2 \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, \dots, x_4\} \\ \partial \mathbf{n}_2 = \emptyset}} w(\mathbf{n}_1) w(\mathbf{n}_2) X(x_2, x_3, x_4 \in \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)) \\ &= -2 \sum_{\substack{\partial \mathbf{n}_1 = \{x_1, x_2\} \\ \partial \mathbf{n}_2 = \{x_3, x_4\}}} w(\mathbf{n}_1) w(\mathbf{n}_2) X(\mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1) = \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_3)), \end{aligned} \quad (2.1.26)$$

where we used Lemma 2.1 with

$$\mathcal{F} = \{\mathbf{m} \mid x_2, x_3, x_4 \in \mathcal{C}_{\mathbf{n}_1 + \mathbf{n}_2}(x_1)\}. \quad (2.1.27)$$

By dividing each side by Z_{Λ}^2 and using (2.1.10), we obtain (2.1.24). \square

Proposition 2.1.24 implies the Lebowitz inequality [45], namely

$$U_4 \leq 0 \quad (2.1.28)$$

and the lower bound

$$-2\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle \leq U_4. \quad (2.1.29)$$

Correlation inequalities like the Lebowitz inequality are well-established for one-component or two-component fields. These inequalities are the main ingredient to prove triviality, therefore restricting the rigorous triviality proofs to one or two-component fields. Glimm and Jaffe derived a weaker upper bound for $|U_4|$ in [70] to bound the renormalized coupling constants. The following propositions provide upper and lower bounds on $|U_4|$

Proposition 2.3. *For a finite ferromagnetic system,*

$$2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle\mathbb{P}[\mathcal{C}_{\mathbf{n}_1}(x_1)\cap\mathcal{C}_{\mathbf{n}_2}(x_3)\neq\emptyset\mid\{x_1,x_2\},\{x_3,x_4\}]\leq|U_4(x_1,\dots,x_4)|\quad(2.1.30)$$

and

$$|U_4(x_1,\dots,x_4)|\leq 2\langle\sigma_{x_1}\sigma_{x_2}\rangle\langle\sigma_{x_3}\sigma_{x_4}\rangle\mathbb{P}[\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathcal{C}_{\mathbf{n}_2}(x_3)\neq\emptyset\mid\{x_1,x_2\},\{x_3,x_4\},\emptyset].\quad(2.1.31)$$

Proof. See [1]. □

Proposition 2.4 (Tree-diagram bound). *For a ferromagnetic system, we have*

$$|U_4(x_1,x_2,x_3,x_4)|\leq 2\sum_y\langle\sigma_{x_1}\sigma_y\rangle\langle\sigma_{x_2}\sigma_y\rangle\langle\sigma_{x_3}\sigma_y\rangle\langle\sigma_{x_4}\sigma_y\rangle.\quad(2.1.32)$$

Proof. First the tree-diagram bound (2.1.32) is proven for finite systems. For currents $(\mathbf{n}_1, \mathbf{n}_1)$ such that $\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathcal{C}_{\mathbf{n}_2}(x_2)\neq\emptyset$, the size of the intersection is surely larger than one. Then, using (2.1.19), (2.1.21), and the fact that $\mathcal{C}_{\mathbf{n}_2}(x_2)\subset\mathcal{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_2)$ we have

$$\begin{aligned} & \mathbb{P}[\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathcal{C}_{\mathbf{n}_2}(x_2)\neq\emptyset\mid\{x_1,x_2\},\{x_3,x_4\},\emptyset] \\ & \leq \mathbb{E}[|\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathcal{C}_{\mathbf{n}_2}(x_2)|\mid\{x_1,x_2\},\{x_3,x_4\},\emptyset] \\ & \leq \mathbb{E}[|\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1)\cap\mathcal{C}_{\mathbf{n}_2+\mathbf{n}_4}(x_2)|\mid\{x_1,x_2\},\{x_3,x_4\},\emptyset,\emptyset] \\ & = \sum_y\frac{\langle\sigma_{x_1}\sigma_y\rangle\langle\sigma_{x_2}\sigma_y\rangle}{\langle\sigma_{x_1}\sigma_{x_2}\rangle}\frac{\langle\sigma_{x_3}\sigma_y\rangle\langle\sigma_{x_4}\sigma_y\rangle}{\langle\sigma_{x_3}\sigma_{x_4}\rangle}. \end{aligned}\quad(2.1.33)$$

It was assumed that both $\langle\sigma_{x_1}\sigma_{x_2}\rangle$ and $\langle\sigma_{x_3}\sigma_{x_4}\rangle$ are nonzero. Inserting this result in the r.h.s. of (2.1.31), we obtain (2.1.32).

For infinite systems, the spin correlations $\langle\ \ \rangle_{\hat{\Lambda}}$ are defined by restriction of J to finite regions. Then, since the sequence $\langle\sigma_A\rangle_{\hat{\Lambda}}$ are increasing, the monotone convergence theorem is applied to obtain the limiting statement for infinite systems. □

The renormalized coupling g is defined by

$$g:=\frac{|\bar{U}_4|}{\chi^2\xi^d},\quad(2.1.34)$$

where $\xi(=m^{-1})$ is the correlation length and χ is the magnetic susceptibility

$$\chi=\sum_x\langle\sigma_O\sigma_x\rangle,\quad(2.1.35)$$

and $|\bar{U}_4|$ is defined by

$$|\bar{U}_4| = \sum_{x_2, x_3, x_4} |U_4(O, x_2, x_3, x_4)|. \quad (2.1.36)$$

Proposition 2.4 implies

$$|\bar{U}_4| \leq \chi^4. \quad (2.1.37)$$

For nearest-neighbor models, due to reflection positivity it has been proven that (see e.g. [71])

$$\chi \leq \frac{1}{\beta} \xi^2 \left(1 + \mathcal{O}\left(\frac{2}{\xi}\right)\right), \quad (2.1.38)$$

for $\beta < \beta_c$ where β_c is the inverse critical temperature. For $\beta < \beta_c$, the susceptibility behaves as

$$\chi \sim \left(\frac{\beta_c - \beta}{\beta_c}\right)^{-\gamma}, \quad \gamma > 0. \quad (2.1.39)$$

Aizenman proved in [1] that $\gamma = 1$ for translation-invariant models on \mathbb{Z}^d with $|J| := \sum_x J_{0,x} < +\infty$ for which

$$\sum_x \langle \sigma_O \sigma_x \rangle^2 < +\infty \quad (2.1.40)$$

at $\beta = \beta_c - \varepsilon$, $\varepsilon > 0$. This class of model includes the nearest-neighbor interaction Ising model in $d > 4$ dimensions.

Regarding the continuum limit, two points of view exist

- The spacing between two points goes to zero.
- The distance between two points is fixed and look at correlations at very large distances

$$S_{2n}^{cont}(x_1, \dots, x_{2n}) = \lim_{\substack{\beta \rightarrow \beta_c \\ \alpha, \eta \rightarrow +\infty}} \alpha^{2n} \langle \sigma_{[x_1 \eta]} \cdots \sigma_{[x_{2n} \eta]} \rangle, \quad (2.1.41)$$

where $[x_i \eta] \in \mathbb{Z}^d$ is a vector whose components are the integer part of those of $x_i \eta$. The parameters α, η are chosen so that the limits exist (at least in the weak sense). This point of view is closer to the statistical physics point of view since the lattice spacing a , which can be viewed as a UV-cutoff is fixed. For more details on the continuum limit, see [72].

In [1, 3], the second point of view is chosen while in [2], the first point of view is favored.

If we go back to the renormalized coupling, using (2.1.38), it is bounded by

$$g \leq \frac{\mathcal{O}(1 + \frac{1}{\xi})}{\beta^2 \xi^{d-4}}. \quad (2.1.42)$$

Since $\xi \rightarrow +\infty$ when $\beta \rightarrow \beta_c$ [73, 74], then $g \rightarrow 0$ when $\beta \rightarrow \beta_c$. We see that for $d = 4$, the bounds are inconclusive.

2.2 Triviality of the continuum limit of lattice φ_4^4 -theory

2.2.1 Under the assumption of infinite wavefunction renormalization

Here we review the results established by Fröhlich [2]. He proved triviality of the continuum limit of lattice regularized φ_d^4 -theory, $d \geq 4$. The proof is based on correlation inequalities. For $d = 4$, under the additional assumption of an infinite wave-function renormalization, triviality is also proven. The proof is valid for a one or two-component scalar field.

We denote \mathbb{Z}_a^d the d -dimensional simple hypercubic lattice of lattice spacing $a \in (0, 1]$. A real-valued lattice field φ_j is attached at each site $j \in \mathbb{Z}_a^d$ and we consider the corresponding distribution

$$d\lambda(\varphi_j) := e^{-\frac{1}{4}\lambda\varphi_j^4 + \frac{1}{2}\mu\varphi_j^2 - \varepsilon} d\varphi_j, \quad (2.2.1)$$

where $\lambda = \lambda(a)$, $\mu = \mu(a)$ and $\varepsilon = \varepsilon(a)$ are arbitrary functions of a with $\lambda(a) > 0$ and $d\varphi_j$ is the Lebesgue measure.

We consider interaction hamiltonians of the form

$$H(\varphi) = - \sum_{\langle j, j' \rangle} \varphi_j \varphi_{j'}, \quad (2.2.2)$$

where $\langle j, j' \rangle$ means that the sum ranges over the pairs of nearest neighbors in \mathbb{Z}_a^d . To start on safe ground, the theory is first defined on a finite lattice $\hat{\Lambda}$ of \mathbb{Z}_a^d . The infinite volume limit is performed afterwards when $\hat{\Lambda}$ exhausts \mathbb{Z}_a^d . Correlation inequalities [75, 76] imply that the infinite volume limit exists. The equilibrium state at inverse temperature β is described by the measure (in the infinite volume limit)

$$d\mu_{\beta,a}(\varphi) := Z^{-1} e^{-\beta(a)H(\varphi)} \prod_j d\lambda(\varphi_j), \quad (2.2.3)$$

where Z is the partition function and $\beta(a)$ is a positive function of a which will be specified later. The correlation functions are defined as the moments of the measure (2.2.3), i.e.

$$\langle \varphi_{x_1} \dots \varphi_{x_n} \rangle_{\beta,a} := \int \varphi_{x_1} \dots \varphi_{x_n} d\mu_{\beta,a}(\varphi). \quad (2.2.4)$$

It is convenient to work in momentum space. The two-point function in momentum space is

$$\langle |\hat{\varphi}(k)|^2 \rangle_{\beta,a} := \sum_j a^d e^{ik \cdot j} \langle \varphi_0 \varphi_j \rangle_{\beta,a}, \quad (2.2.5)$$

where $k \in \mathbb{R}^d$ and $k_\alpha \in [-\frac{\pi}{a}, \frac{\pi}{a}]$. It was shown in [77] that the two-point function in momentum space satisfies the bound

$$0 < \langle |\hat{\varphi}(k)|^2 \rangle_{\beta,a} \leq M_{\beta,a}^2 \delta(k) + \frac{1}{4} \pi^2 \frac{1}{\zeta(a) k^2}, \quad (2.2.6)$$

where $\zeta(a) := \beta(a) a^{2-d}$ and $M_{\beta,a}$ is the spontaneous magnetization. In particular from (2.2.6) and correlation inequalities

$$0 < \langle \varphi_0 \varphi_x \rangle_{\beta,a} \leq M_{\beta,a}^2 + \frac{c_\delta}{\zeta(a) |x|^{d-2}}, \quad (2.2.7)$$

where c_δ is a constant independent of β and a , see e.g. [67]. Under the assumption of an infinite wave-function renormalization, we have $\zeta(a) \rightarrow 0$ when $a \rightarrow 0$. The limit $a \rightarrow 0$ corresponds to removing an UV-cutoff in momentum space Λ_0 , i.e. $\Lambda_0 \rightarrow +\infty$.

The connected four-point function, called Ursell function is again given by

$$u_{\beta,a}^{(4)}(x_1, \dots, x_4) = \langle \varphi_{x_1} \dots \varphi_{x_4} \rangle_{\beta,a} - \left[\langle \varphi_{x_1} \varphi_{x_2} \rangle_{\beta,a} \langle \varphi_{x_3} \varphi_{x_4} \rangle_{\beta,a} + \text{permutations} \right]. \quad (2.2.8)$$

To prove bounds on the Ursell function, the following assumptions are chosen

$$\begin{cases} 0 < \zeta(a) \leq 1, \\ M_{\beta,a} = 0, \\ 0 < \overline{\lim}_{a \rightarrow 0} \langle \varphi_0 \varphi_x \rangle_{\beta,a} \leq \sup_a \langle \varphi_0 \varphi_x \rangle_{\beta,a} < +\infty, \end{cases} \quad (2.2.9)$$

where $x = ze_1$ with $0 < z < 1$, e_1 being the unit vector in the 1-direction of \mathbb{R}^d . The first two assumptions in (2.2.9) can be seen as renormalization conditions since they prescribe the values of $\zeta(a)$, and of $M_{\beta,a}$ since β is fixed. The last assumption in (2.2.9) ensures that the two-point function does not blow up in the continuum limit at non-coinciding points. In [2], Fröhlich comments on the compatibility of the renormalization conditions (2.2.9) for a theory in d dimensions.

Proposition 2.5. *Assume that $|x_i - x_j| \geq \delta$, for $\delta > 0$. Then for $d \geq 5$ we have*

$$-a^{d-4}(C_\delta \zeta(a) + C') \leq u_{\beta,a}^{(4)}(x_1, \dots, x_4) \leq 0, \quad (2.2.10)$$

for some positive constants C_δ and C' . Moreover for $d = 4$,

$$-(C_\delta \zeta(a) + C'' \zeta(a)^{\frac{\varepsilon}{2-\varepsilon}}) \leq u_{\beta,a}^{(4)}(x_1, \dots, x_4) \leq 0, \quad (2.2.11)$$

for another constant $C'' > 0$ and $0 < \varepsilon \leq 1$.

Proof. See [2]. □

The upper bound in (2.2.10) and (2.2.11) is the Lebowitz inequality. The lower bound is obtained using the random walk representation. The bounds (2.2.10) imply triviality for $d \geq 5$. For $d = 4$, the lower bound in (2.2.11) is sharper than the one in (2.2.10). It has been obtained using the bound

$$\langle \varphi_x \varphi_y \rangle_{\beta,a} \leq \frac{K}{|x - y|^\varepsilon}, \quad |x - y| > 1, \quad (2.2.12)$$

for $K > 0$ independent on a and $0 < \varepsilon < 1$. If $\lim_{a \rightarrow 0} \langle \varphi_x \varphi_y \rangle_{\beta,a}$ is euclidean invariant, then from the Källen-Lehman representation of the two-point function, we have

$$\langle \varphi_x \varphi_y \rangle_{\beta,a} \leq \frac{K}{|x - y|^\alpha}, \quad |x - y| > 1, \quad (2.2.13)$$

for some $\alpha \geq d - 2$, and for a constant K that does not depend on a . Therefore (2.2.12) in the limit $a = 0$ is satisfied with $0 < \varepsilon < 1$ since $|x - y| > 1$. From Proposition 2.5, at non-coinciding points,

$$\lim_{a \rightarrow 0} u_{\beta,a}^{(4)}(x_1, \dots, x_4) = 0, \quad (2.2.14)$$

assuming that

$$\lim_{a \rightarrow 0} \zeta(a) = 0, \quad (2.2.15)$$

implying triviality for $d = 4$.

2.2.2 Sharpened tree-diagram bound

Here we will briefly review the improvement of the tree-diagram bound (2.1.32) in [3] using multi-scale analysis. The proofs are quite long and we will not review them. The formalism is the same as the one introduced in [1, 43]. U_4^β denotes the truncated four-point function defined in (2.1.16) at inverse temperature β . The correlation length at inverse temperature is denoted $\xi(\beta)$ and is defined in [3] by

$$\xi(\beta) := \lim_{n \rightarrow +\infty} -\frac{n}{\ln(\langle \sigma_0 \sigma_{ne_1} \rangle_\beta)}, \quad (2.2.16)$$

where e_1 is the unit-vector in the 1-direction of \mathbb{Z}^d . $\langle \sigma_x \sigma_y \rangle_\beta$ denotes the spin-spin correlation introduced in Sect.2.1 at inverse temperature β . Then the following random variables are defined

$$T_{f,L}(\sigma) := \frac{1}{\sqrt{\Sigma_L}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \sigma_x, \quad (2.2.17)$$

where $f \in C_0(\mathbb{R}^d)$ is the set of compactly supported continuous functions, and Σ_L is the variance of the sum of spins over the box of size L , i.e.

$$\Sigma_L := \left\langle \sum_{x,y \in [-L,L]^d} \sigma_x \sigma_y \right\rangle_\beta \quad (2.2.18)$$

Aizenman and Duminil-Copin improved the tree-diagram bound (2.1.32)

Proposition 2.6. *For the nearest-neighbor ferromagnetic Ising model in $d = 4$, for $\beta \leq \beta_c$, $L \leq \xi(\beta)$ and $|x_i - x_j| \geq L$ for any $i \neq j$*

$$|U_4^\beta(x_1, x_2, x_3, x_4)| \leq \frac{C}{B_L(\beta)^c} \sum_y \langle \sigma_{x_1} \sigma_y \rangle_\beta \langle \sigma_{x_2} \sigma_y \rangle_\beta \langle \sigma_{x_3} \sigma_y \rangle_\beta \langle \sigma_{x_4} \sigma_y \rangle_\beta, \quad (2.2.19)$$

for constants $c, C > 0$ and $B_L(\beta)$ is the bubble diagram truncated at distance $L > 0$ defined by

$$B_L(\beta) := \sum_{x \in [-L,L]^4} \langle \sigma_O \sigma_x \rangle_\beta^2. \quad (2.2.20)$$

Proof. See [3]. □

The proof of (2.2.19) relies on the random current representation and a sharper analysis of the event $\mathcal{C}_{\mathbf{n}_1+\mathbf{n}_3}(x_1) \cap \mathcal{C}_{\mathbf{n}_2}(x_3) \neq \emptyset$ appearing in the r.h.s. of (2.1.31). The authors proceeded by partitioning the lattice in annuli $\text{Ann}(l_{k+1}, l_k) := \Lambda_{l_{k+1}} \setminus \Lambda_{l_k}$, with $\Lambda_n := [-n, n]^d$ and $(l_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence, with $l_1 > 1$ and $l_{k+1} = l_k^m$ for $m > 1$. The bound (2.2.19) is sharper than the tree-diagram bound (2.1.32) with the presence of the truncated bubble diagram $B_L(\beta)$. This correction is crucial in the triviality proof. From Proposition 2.6,

Proposition 2.7. *There exist positive constants c, C such that for the nearest neighbor ferromagnetic Ising model on \mathbb{Z}^4 , $\beta \leq \beta_c$, $L \leq \xi(\beta)$, $f \in C_0(\mathbb{R}^4)$ and $z \in \mathbb{R}$*

$$\left| \left\langle \exp[z T_{f,L}(\sigma) - \frac{z^2}{2} \langle (T_{f,L}(\sigma))^2 \rangle_\beta] \right\rangle_\beta - 1 \right| \leq \frac{C \|f\|_\infty^4 r_f^{12}}{(\ln L)^c} z^4, \quad (2.2.21)$$

where $\|f\|_\infty := \sup\{|f(x)|, x \in \mathbb{R}\}$ for any bounded function f and r_f is the diameter of the support of f .

Proof. See [3]. □

The factor $\ln L$ in the r.h.s. of (2.2.21) comes from the correction $B_L(\beta)$. In the limit $1 \ll L \leq \xi(\beta)$, the random variable $T_{f,L}(\sigma)$ is close to a Gaussian variable of variance $\langle (T_{f,L}(\sigma))^2 \rangle_\beta$. We recall here that in order to pass to the continuum limit, the point of view in [1] is adopted: the spacing of the lattice is fixed and correlations between spins at large distances are studied. The triviality result has been established

- At criticality $\beta = \beta_c$ under the assumption of a sufficiently regular two-point function.
- For $\beta < \beta_c$ without any further assumption. It turns out that they proved that the two-point function is regular at all scales up to the correlation length $\xi(\beta)$.

Chapter 3

Construction of asymptotically free/trivial scalar fields with the mean-field flow equations

In this chapter, we review the results from the paper [4]. The flow equations presented in Sect.1.2 allow to construct asymptotically free/trivial scalar fields in the mean-field approximation. We will again study the φ_4^4 -theory and employ the flow equations, but this time in the mean-field approximation. Not only can the flow equations be used to prove perturbative renormalizability of quantum field theories but they can also be used nonperturbatively to construct field theories with required properties (asymptotically free, trivial) at the cost of the mean-field approximation. We will especially highlight how they can be used to prove triviality of mean-field φ_4^4 -theory under the assumption of weak bare couplings.

3.1 Flow equations in the mean-field approximation

We will follow here the notations from [4] which differ slightly from [59]. We will follow the choice of the regularized propagator as in [4]

$$C^{\alpha_0, \alpha}(p, m) := \frac{1}{p^2 + m^2} \left(\exp(-\alpha_0(p^2 + m^2)) - \exp(-\alpha(p^2 + m^2)) \right), \quad (3.1.1)$$

where m is the mass of the field and $\alpha_0 > 0$ acts as an ultraviolet cutoff, and $\alpha \in [\alpha_0, +\infty)$ is the flow parameter. The regularized propagator (3.1.1) is positive, and locally analytic w.r.t. α . By taking the limits $\alpha_0 \rightarrow 0$ and $\alpha \rightarrow +\infty$ we recover the usual Euclidean propagator in momentum space, namely

$$\lim_{\alpha \rightarrow +\infty} \lim_{\alpha_0 \rightarrow 0} C^{\alpha_0, \alpha}(p, m) = \frac{1}{p^2 + m^2}. \quad (3.1.2)$$

Then we consider bare interaction lagrangians of the form

$$L_0^{\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \left(b_0(\alpha_0) (\partial\varphi(x))^2 + \sum_{n \in 2\mathbb{N}} c_{0,n}(\alpha_0) \varphi^n(x) \right), \quad c_{0,4}(\alpha_0) = g + \tilde{c}_{0,4}(\alpha_0), \quad (3.1.3)$$

where $(\partial\varphi(x))^2 = \sum_{\mu=0}^3 (\partial_\mu\varphi(x))^2$ and \mathcal{V} is a finite volume of \mathbb{R}^4 . The constants $b_0(\alpha_0)$, $c_{0,n}(\alpha_0)$ are called the bare couplings, g is the bare interaction coupling in the φ^4 interaction term and $b_0(\alpha_0)$, $\tilde{c}_{0,4}(\alpha_0)$ are counterterms. The quantities in the sum for $n \geq 6$ are the irrelevant terms while $b_0(\alpha_0)$, $c_{0,2}(\alpha_0)$ and $c_{0,4}(\alpha_0)$ are relevant and marginal terms. Generally they are required in order that the renormalized physical quantities such as the renormalized mass or the renormalized coupling constant are finite upon removing the UV cutoff. They should be such that for some constant $C^\mathcal{V} \in \mathbb{R}$, depending on \mathcal{V}

$$-\infty < C^\mathcal{V} < L_0^\mathcal{V}(\varphi) < +\infty, \quad \varphi \in \text{supp}(\mu^{\alpha_0, \alpha}), \quad (3.1.4)$$

where $\mu^{\alpha_0, \alpha}$ is the normalized Gaussian measure associated to the propagator $C^{\alpha_0, \alpha}$. The field φ is supposed to belong to the support of the Gaussian measure $\mu^{\alpha_0, \alpha}$. Since the regularized propagator $C^{\alpha_0, \alpha}(p, m)$ falls off more rapidly than any power of $|p|$ in momentum space, its support is contained in functions smooth in position space, see e.g. [78], so that the quantities in $L_0^\mathcal{V}$ i.e. $\varphi^2(x)$, $\varphi^4(x)$, \dots are well-defined.

If we assume that the bare interaction lagrangian (3.1.3) is twice differentiable w.r.t. φ , the flow equations for $\hat{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) := (n!)^{-1} \mathcal{L}_n^{\alpha_0, \alpha}(p_1, \dots, p_n)$ can be derived as before. They read

$$\begin{aligned} & \partial_\alpha \hat{\mathcal{L}}_n^{\alpha_0, \alpha}(p_1, \dots, p_n) \\ &= \binom{n+2}{2} \int_k \dot{C}^\alpha(k, m) \hat{\mathcal{L}}_{n+2}^{\alpha_0, \alpha}(k, -k, p_1, \dots, p_n) \\ & - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 \mathbb{S} \left(\hat{\mathcal{L}}_{n_1}^{\alpha_0, \alpha}(p_1, \dots, p_{n_1-1}, q) \dot{C}^\alpha(q, m) \hat{\mathcal{L}}_{n_2}^{\alpha_0, \alpha}(-q, p_{n_1}, \dots, p_n) \right), \end{aligned} \quad (3.1.5)$$

with $q = p_{n_1} + \dots + p_n = -p_1 - \dots - p_{n_1-1}$. \mathbb{S} is the symmetrisation operator averaging over the permutations σ such that $\sigma(1) < \sigma(2) < \dots < \sigma(n_1 - 1)$ and $\sigma(n_1) < \sigma(n_1 + 1) < \dots < \sigma(n)$. Since we considered a theory with a \mathbb{Z}_2 -symmetry, only even moments (n, n_1 and $n_2 \in 2\mathbb{N}$) are nonvanishing as the flow equations do not break this symmetry. They are an infinite system of non-linear differential equations, the solutions of which are the CAS functions. Here, we will follow a different approach. After performing the mean-field approximation we will fix the boundary conditions for those parameters at the bare scale α_0 instead of the physical scale $\alpha \rightarrow +\infty$.

The mean-field approximation is a tool to simplify the system (3.1.5). It is worth to note that in statistical physics the critical behavior is exactly obtained in $d > 4$ dimensions (Ginzburg criterion) [2, 1, 43] using the mean-field approximation. We hope that important aspects of the theory are preserved in this approximation. There the n -point functions are momentum independent and the mean-field flow equations are obtained by setting all momenta to zero [4]. We write

$$A_n^{\alpha_0, \alpha} = \hat{\mathcal{L}}_n^{\alpha_0, \alpha}(0, \dots, 0). \quad (3.1.6)$$

The mean field effective action $L_{mf}^{\alpha_0, \alpha}(\phi)$ takes the form of a formal power series (no assumption at this stage about the convergence of the series) in the mean-field variable $\phi \in \mathbb{R}$

$$L_{mf}^{\alpha_0, \alpha}(\phi) = \sum_{n \in 2\mathbb{N}} A_n^{\alpha_0, \alpha} \phi^n. \quad (3.1.7)$$

In [4] an additional technical simplification is to set $m = 0$ in $C^{\alpha_0, \alpha}(k, m)$, and then to analyze the theory in the interval $\alpha \in [\alpha_0, \alpha_{\max}]$, $\alpha_{\max} := \frac{1}{m^2}$ to avoid infrared problems. This technical simplification does not change the conclusions on triviality, see [79]. The regularized propagator then reads

$$\frac{e^{-\alpha_0 p^2} - e^{-\alpha p^2}}{p^2} \underset{p^2 \ll 1}{=} \alpha - \alpha_0 + O(p^2), \quad (3.1.8)$$

so that the infrared cutoff α_{\max} plays here the same role as $\frac{1}{m^2}$ in the original theory. We will choose units such that $m^2 = 1$ which corresponds to assuming $\alpha \in [\alpha_0, 1]$.

In the mean-field limit the flow equations (3.1.5) then become

$$\partial_\alpha A_n^{\alpha_0, \alpha} = \binom{n+2}{2} c_\alpha A_{n+2}^{\alpha_0, \alpha} - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha}, \quad \alpha \in [\alpha_0, 1], \quad (3.1.9)$$

where $c_\alpha := \frac{c}{\alpha^2}$ with $c := \frac{1}{16\pi^2}$. As in [4], we perform a change of function and variable to factor out a combinatorial factor and the power counting in α . We define

$$f_n(\mu) := \alpha^{2-\frac{n}{2}} c^{\frac{n}{2}-1} n A_n^{\alpha_0, \alpha}, \quad (3.1.10)$$

where $\mu := \ln\left(\frac{\alpha}{\alpha_0}\right)$. The mean-field flow equations can then be written in terms of the $f_n(\mu)$ in the following reduced form

$$f_{n+2}(\mu) = \frac{1}{n+1} \sum_{n_1+n_2=n+2} f_{n_1}(\mu) f_{n_2}(\mu) + \frac{n-4}{n(n+1)} f_n(\mu) + \frac{2}{n(n+1)} \partial_\mu f_n(\mu), \quad \mu \in [0, \mu_{\max}], \quad (3.1.11)$$

where $\mu_{\max} := \ln\left(\frac{1}{\alpha_0}\right)$. The mean-field flow equations (3.1.11) have been analyzed in [4, 79] as follows:

- Fix a bare interaction lagrangian (3.1.3).
- Impose the mean-field boundary conditions corresponding to (3.1.3)
- Define an ansatz for the two-point function $f_2(\mu)$ and use the mean-field flow equations (3.1.11) to construct inductively smooth solutions $f_n(\mu)$, $n \geq 4$.

3.2 Mean-field solutions of the mean-field flow equations

We will review here smooth mean-field solutions of the flow equations (3.1.11) analyzed by Kopper [4]. We are also concerned with the existence of the UV-limit of the mean-field solutions of the flow equations (3.1.11). Since taking the limit $\alpha_0 \rightarrow 0$ corresponds to taking the limit $\mu_{\max} \rightarrow +\infty$, the existence of the mean-field solutions of the flow equations (3.1.11) in this limit is equivalent to show that the limits

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}), \quad n \geq 2 \quad (3.2.1)$$

exist and are finite.

3.2.1 Bounded smooth mean-field solutions

The simplest solutions of (3.1.11) to construct are scale-invariant, i.e. $\partial_\mu f_n \equiv 0$. Then, the dynamical system (3.1.11) reads

$$\begin{aligned} f_4 &= \frac{1}{3}f_2(f_2 - 1) , \\ f_{n+2} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} f_{n_1}f_{n_2} + \frac{1}{n+1}f_n \left[2f_2 + 1 - \frac{4}{n} \right] . \end{aligned} \quad (3.2.2)$$

Upon fixing the value of f_2 , then one can analyse the behavior of f_n , $n \geq 4$. Felder [80] analyzed in great generality the global solutions of the partial differential equation

$$-\lambda \partial_\lambda u = \frac{1}{2} \partial_{xx} u - \frac{1}{2} (\partial_x u)^2 + du - \frac{d-2}{2} x \partial_x u \quad (3.2.3)$$

in $2 < d < 4$ dimensions. He studied in particular the fixed points that are global solutions of the ordinary differential equation

$$\frac{1}{2} \partial_{xx} u - \frac{1}{2} (\partial_x u)^2 + du - \frac{d-2}{2} x \partial_x u = 0 . \quad (3.2.4)$$

He concluded that in $d = 4 - \varepsilon$, the non-trivial fixed point solution u_4 in $3 < d < 4$ dimensions tends to zero. Nevertheless his analysis does not exclude the existence of fixed points other than those he found. The flow equations (3.1.11) can be formally obtained from (3.2.3) when $d \rightarrow 4$ by expanding a solution $u(x)$ in a power series in x

$$u(x) = \sum_{n \geq 0} \frac{f_n}{n} x^n . \quad (3.2.5)$$

We will go back to that remark later on. If the other fixed points are excluded, then we can conclude from [80, 4] that the moment expansion (3.2.5) is not valid for arbitrary $x \in \mathbb{R}$ or it does not ensure the regularity of the solution $u(x)$ that would be obtained from such an expansion.

We can analyze the fixed-point solutions of (3.2.2), by distinguishing different regimes according to the value of f_2 .

- $0 < |f_2| \leq \varepsilon \ll 1$: we see that $f_4 = \mathcal{O}(\varepsilon)$ with the opposite sign to that of f_2 , $f_6 = \mathcal{O}(\varepsilon^2)$ and $0 < f_n = \mathcal{O}(\varepsilon)$, $n \geq 8$. Therefore this regime is non-perturbative in the sense that f_n is not of increasing order in ε when n increases. Moreover, the action is bounded from below.
- $0 < f_2 < 1$: we have $f_4 < 0$ but this time we do not control the sign of f_n in general.
- $f_2 = 1$: we have $f_n = 0$, $n \geq 4$, we have the so-called free theory, also called high-temperature fixed point.
- $f_2 > 1$: by induction on n , we see that $f_n > 0$ and they are increasing as functions of f_2 and are bounded by a constant to the power of n . For instance, if one chooses $f_2 = 1 + \varepsilon$, with $0 < \varepsilon \ll 1$, we obtain $f_4 = \frac{1}{3}(1 + \varepsilon)\varepsilon$, $f_6 = \frac{1}{15}(1 + \varepsilon)^2\varepsilon$ and $f_n = \mathcal{O}(\varepsilon)$, $n \geq 4$.

- $f_2 < 0$: the signs of f_n are beyond control and their modulus $|f_n|$ can be arbitrarily large.

Now we suppose that $f_2(\mu)$ and its derivatives are uniformly bounded on $[0, \mu_{\max}]$

$$\begin{aligned} -K_1\delta &\leq f_2(0) \leq -\delta < 0, \\ |\partial_\mu^l f_2(\mu)| &\leq \frac{K_1^l \delta^{l+1}}{(l+1)^2} l!, \quad l \geq 0, \end{aligned} \quad (3.2.6)$$

with $0 < \delta \leq 1$ and $K_1 > 1$. The sign of $f_2(0)$ is in agreement with the sign of the mass counterterm in perturbation theory for φ_4^4 -theory at lowest order.

Proposition 3.1. *For $K \geq \sup(K_1, 4)$ and if $f_2(\mu)$ satisfies the bounds (3.2.6), then for $n \in 2\mathbb{N} + 2$, $l \geq 0$ and $\mu \in [0, \mu_{\max}]$*

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K^{n+l-2} \delta^{l+1}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!}. \quad (3.2.7)$$

Proof. See [4] or Appendix E.1. □

If we go back to the dynamical system (3.1.9), we can recover the mean-field solutions $A_n^{\alpha_0, \alpha}$ from the mean-field solutions $f_n(\mu)$. Especially if the mean-field solutions $f_n(\mu)$ satisfy the bounds (3.2.7) in Proposition 3.1, we obtain the estimate for the mean-field solutions $A_n^{\alpha_0, \alpha}$

$$|A_n^{\alpha_0, \alpha}| \leq \delta \left(\frac{\alpha K^2}{c} \right)^{\frac{n-2}{2}} \frac{1}{\alpha n}. \quad (3.2.8)$$

Therefore the mean-field effective action (4.2.2) is locally analytic w.r.t. $\phi \in \mathbb{R}$ and the radius of convergence of the series is greater than $\sqrt{\frac{c}{\alpha K^2}}$ for $0 < \alpha_0 \leq \alpha \leq \alpha_{\max}$ fixed.

3.2.2 Asymptotically free mean-field solutions

We recall that in perturbative quantum field theory, the $\beta(g)$ function is defined as the derivative of the effective coupling constant $g(\lambda)$ w.r.t. the logarithmic scale λ

$$\beta(g(\lambda)) := \lambda \frac{dg}{d\lambda}(\lambda). \quad (3.2.9)$$

Therefore, the β function controls the flow of the effective coupling $g(\lambda)$ with respect to the logarithmic energy scale. In practice $\beta(g(\lambda))$ can only be calculated to a finite order in the perturbative expansion. In asymptotically free theories, the leading term of the expansion of $\beta(g)$ in a formal power series in the renormalized coupling is negative so that the coupling constant vanishes at high energies, $g(\lambda) \rightarrow 0$ for $\lambda \rightarrow +\infty$. Since $g(0)$ is the renormalized coupling, in renormalized perturbation theory the Schwinger functions are expanded in a formal power series in $g(0)$ and the vanishing of the effective coupling (asymptotic freedom) implies that they vanish asymptotically at high energies.

In our setting we call the logarithmic energy scale μ , the UV-limit corresponds to $\mu = 0$. λ, μ are related by

$$\lambda = \mu_{\max} - \mu. \quad (3.2.10)$$

To remain consistent with the above discussion, we set

$$\tilde{f}_n(\lambda) := f_n(\mu)|_{\mu=\mu_{\max}-\lambda} . \quad (3.2.11)$$

It is important to note that $f_n(\lambda)$ depends on μ_{\max} which is the UV-cutoff. Taking the UV-limit amounts to take the limit $\mu_{\max} \rightarrow +\infty$ and keeping the physical couplings at $\lambda = 0$ fixed. Asymptotically free solutions satisfy

$$\lim_{\lambda \rightarrow +\infty} \tilde{f}_n(\lambda) = 0 , \quad (3.2.12)$$

keeping the physical couplings $\tilde{f}_n(0)$ fixed. Mean-field solutions satisfying

$$\lim_{\lambda \rightarrow +\infty} \tilde{f}_n(0) = 0 \quad (3.2.13)$$

while keeping the bare couplings $\tilde{f}_n(\mu_{\max})$ fixed are called trivial. We will come to the trivial solution later.

Among the class of the bounded smooth mean-field solutions of the flow equations (3.1.11) satisfying Proposition 3.1, a particular subclass of solutions which also define a mean-field theory in the UV-limit, i.e. the limits (3.2.1) are finite, are the asymptotically free scalar field theories. Smooth mean-field solutions $f_n(\mu)$ can be constructed by the following choice of $f_2(\mu)$

$$\begin{aligned} f_2(\mu) &= -\delta(\mu), \quad \delta(\mu_{\max}) = \delta > 0 , \\ \partial_\mu f_2(\mu) &= -\beta\delta^2(\mu), \quad 0 < \delta, \beta < \frac{1}{2} , \end{aligned} \quad (3.2.14)$$

where

$$\delta(\mu) = \frac{\delta}{1 + (\mu_{\max} - \mu)\beta\delta} . \quad (3.2.15)$$

We note that $\beta\delta^2$ plays the role of the β function in perturbation theory. We have

$$\partial_\mu^l f_2(\mu) = (-1)^l \beta^l l! (f_2(\mu))^{l+1} \quad (3.2.16)$$

so that $f_2(\mu)$ satisfies (3.2.6). In particular we have

$$\lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_2(\mu_{\max}) = -\beta^l l! \delta^{l+1} , \quad l \in \mathbb{N}_0 . \quad (3.2.17)$$

Proposition 3.2. *Among the solutions from Proposition 3.1, there exist nontrivial asymptotically free mean-field smooth solutions which verify*

$$\begin{aligned} f_2(\mu_{\max}) &< 0, \quad f_4(\mu_{\max}) > 0 , \\ \lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) &\text{ exist, } \quad n \in 2\mathbb{N}, \\ \lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_n(0) &= 0, \quad l \in \mathbb{N}_0, \quad n \in 2\mathbb{N} . \end{aligned} \quad (3.2.18)$$

Proof. See [4]. The proof proceeds similarly as the one of Proposition 3.1. We remark in particular that $f_4(\mu_{\max}) > 0 \Leftrightarrow \delta(\delta+1) - \beta\delta^2 > 0$. Choosing β, δ as in (3.2.14), this condition holds. Proceeding inductively in $n + l$, going up in n , all the $f_n(\mu_{\max})$ have limits upon removing the UV-cutoff, i.e. $\mu_{\max} \rightarrow +\infty$. It follows from (3.2.15) that $f_2(\mu)$ satisfies the last statement. \square

We remark that by taking $\beta < 0$ and fixing $\delta(0) > 0$ not too large, we obtain

$$\delta(\mu) = \frac{\delta(0)}{1 - \mu\beta\delta(0)}, \quad (3.2.19)$$

valid as long as $1 - \mu\beta\delta(0) \neq 0$ for $\mu \in [0, \mu_{\max}]$. This leads to trivial solutions with nonpolynomial bare action, in the sense that

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0, \quad (3.2.20)$$

by proceeding as in the proofs of Propositions 3.1-3.2. We will come back to the trivial solution.

An important remark is that the bare actions constructed from $f_n(0)$ in Proposition 3.2 are not necessarily bounded from below. Here we look for nonnegative and monotonic mean-field solutions $f_n(\mu)$, i.e.

$$f_n(\mu) \geq 0, \quad \partial_\mu f_n(\mu) \geq 0. \quad (3.2.21)$$

This ensures that the bare action is bounded from below so that the functional integral remains well-defined. We consider two-point functions $f_2(\mu)$ of the form

$$f_2(\mu) = 1 + \delta(\mu), \quad 0 < \delta(\mu) < 1, \quad \partial_\mu \delta(\mu) = \beta(\delta(\mu))^2, \quad 0 < \beta < 1. \quad (3.2.22)$$

We see that $f_2(\mu)$ is no longer of order δ if we compare it to (3.2.14). Writing $f_4(\mu)$ as

$$3f_4(\mu) = (1 + \delta(\mu))\delta(\mu) + \beta(\delta(\mu))^2 \quad (3.2.23)$$

we have from (3.2.22),

$$\partial_\mu^l f_4(\mu) \geq 0. \quad (3.2.24)$$

Proposition 3.3. *For a suitable $K > 1$ and $f_2(\mu)$ of the form (3.2.22), the mean-field solutions $f_n(\mu)$ are smooth and for $n \in 2\mathbb{N} + 2$, $l \in \mathbb{N}_0$*

$$0 < \partial_\mu^l f_n(\mu) \leq \delta(\mu) \frac{K^{n+l-2} (n+l-2)!}{(l+1)^2 (n-2)!}. \quad (3.2.25)$$

Proof. The proof is done again by induction in $n+l$, going up in n . The positivity follows from (3.2.22)-(3.2.24) and the mean-field flow equations (3.1.11). For the upper bounds, one proceeds as in the proof of Proposition 3.1. The difference is that the derivatives of $f_n(\mu)$ are no longer bounded by a power of $\delta(\mu)$ since $f_2(\mu)$ is not of order $\delta(\mu)$. Again, $K = 4$ is admissible. \square

Going back to the dynamical system (3.1.9), for the mean-field solutions $f_n(\mu)$ satisfying Proposition 3.3, the mean-field solutions $A_n^{\alpha_0, \alpha}$ verify

$$0 < A_n^{\alpha_0, \alpha} < (\delta_{n,2} + \delta(\mu)) \left(\frac{\alpha K^2}{c} \right)^{\frac{n-2}{2}} \frac{1}{\alpha n}, \quad 0 < \alpha_0 \leq \alpha \leq \alpha_{\max}. \quad (3.2.26)$$

Since we have $f_n(0) \geq 0$, the mean-field bare action has positive moments. From (3.2.26), the mean-field effective action (4.2.2) is analytic w.r.t. ϕ in a disc of radius $\geq \sqrt{\frac{c}{\alpha K^2}}$. These solutions turn out

to be asymptotically free as well. In fact by fixing $\delta = \delta(\mu_{\max}) > 0$ we have again (3.2.15) and we have statements similar to (3.2.17)-(3.2.18), namely

$$\begin{aligned} \lim_{\mu_{\max} \rightarrow +\infty} \partial_{\mu}^l f_2(\mu_{\max}) &= \delta_{l,0} + \beta^l l! \delta^{l+1}, & \lim_{\mu_{\max} \rightarrow +\infty} f_4(\mu_{\max}) &= \frac{1}{3}(1 + \delta)\delta + \beta\delta^2, \\ \lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) &\in \mathbb{R}_+, \\ \lim_{\mu_{\max} \rightarrow +\infty} \partial_{\mu}^l f_2(0) &= \delta_{l,0}, & \lim_{\mu_{\max} \rightarrow +\infty} \partial_{\mu}^l f_n(0) &= 0, \quad n \in 2\mathbb{N} + 2. \end{aligned} \quad (3.2.27)$$

It would be interesting to analyze these solutions more systematically.

3.2.3 Mean-field smooth solutions of bounded bare action

The mean-field effective actions $L_{mf}^{\alpha_0, \alpha}(\phi)$ corresponding to the solutions constructed in Sect.3.2.1, 3.2.2 are analytic w.r.t. the constant variable ϕ in a disc of radius $\geq \sqrt{\frac{c}{\alpha K^2}}$. In particular, at the end of Sect.3.2.2, nonnegative asymptotically free mean-field solutions have been constructed. But the mean-field effective bare action $L_0(\phi) = L_{mf}^{\alpha_0, \alpha_0}(\phi)$ is not well-defined for arbitrary $\phi \in \mathbb{R}$. If the mean-field effective bare action $L_{mf}^{\alpha_0, \alpha_0}(\phi)$ is obtained as the mean-field limit of a interaction functional $L_0^{\mathcal{Y}}(\varphi(x))$, it is in general not well-defined on the support of the Gaussian measure, despite the presence of the regularization. Now we present precise bounds for the mean-field solutions which ensure that the bare mean-field action functionals are well-defined. In particular, it has been shown that the mean-field solutions from this section are actually subclasses of those considered in Sect.3.2.1, 3.2.2.

To assure boundedness, we consider non-polynomial local bare action functionals of the form

$$L_0^{\mathcal{Y}}(\varphi(x)) = \sum_{n \in 2\mathbb{N}} \int_{\mathcal{V}} d^4x \tilde{c}_{0,n}(\alpha_0) \sin(\alpha_0^{\frac{n}{2}} \varphi^n(x)) \alpha_0^{-\frac{n}{2}}, \quad (3.2.28)$$

where $\tilde{c}_{0,n}(\alpha_0)$ denote the bare couplings of the theory. In the mean-field limit and in the infinite-volume limit, we have the mean-field bare action

$$L_{mf}^{\alpha_0, \alpha_0}(\phi) = \sum_{n \in 2\mathbb{N}} \tilde{A}_n^{\alpha_0, \alpha_0} \sin(\alpha_0^{\frac{n}{2}} \phi^n) \alpha_0^{-\frac{n}{2}}. \quad (3.2.29)$$

We define the dimensionless Schwinger functions $\tilde{a}(n)$ which are related to $\tilde{A}_n^{\alpha_0, \alpha_0}$ by

$$\tilde{A}_n^{\alpha_0, \alpha_0} := \frac{1}{n} \alpha_0^{\frac{n}{2}-2} \tilde{a}(n). \quad (3.2.30)$$

We also define the μ -derivatives of $\tilde{a}(n)$ at $\mu = 0$

$$\tilde{a}(n, l) := \partial_{\mu}^l \tilde{a}(n). \quad (3.2.31)$$

If (3.2.28) is expanded in moments of ϕ and mean-field limit is performed, then the mean-field flow equations for $\tilde{a}(n)$ are obtained by taking the value at $\alpha = \alpha_0$

$$\begin{aligned}
\tilde{a}(n+2) &= \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} (n'-1) \frac{(-1)^{\nu-1}}{(2\nu)!} \tilde{a}(n') \\
&+ \frac{1}{n+1} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n+2}} n' \frac{(-1)^{\nu-1}}{(2\nu-1)!} \tilde{a}(n') \\
&+ \frac{1}{(n+1)c} \sum_{\substack{n_1+n_2=n+2, \nu_1, \nu_2 \geq 0 \\ (1+2\nu_1)n' = n_1 \\ (1+2\nu_2)n'' = n_2}} \frac{(-1)^{\nu_1+\nu_2}}{(2\nu_1)! (2\nu_2)!} \tilde{a}(n') \tilde{a}(n'') \\
&+ \frac{2}{(n+1)c} \frac{n-4}{2n} \tilde{a}(n) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{n'-4}{2n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n') \\
&+ \frac{2}{(n+1)c} \frac{1}{n} \tilde{a}(n, 1) + \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} \frac{1}{n'} \frac{(-1)^\nu}{(2\nu+1)!} \tilde{a}(n', 1) \\
&+ \frac{2}{(n+1)c} \sum_{\substack{\nu \geq 1 \\ (1+2\nu)n' = n}} (-1)^\nu \frac{\nu}{(2\nu+1)!} \tilde{a}(n') .
\end{aligned} \tag{3.2.32}$$

In particular, we have

$$\tilde{a}(4) = \frac{1}{3c} (\tilde{a}(2)(\tilde{a}(2) - 1) + \tilde{a}(2, 1)) . \tag{3.2.33}$$

It is useful to introduce the following notation. For any positive integer n , its prime number decomposition is given by

$$n = \prod_{j \in \mathbb{P}} j^{p_j(n)} , \tag{3.2.34}$$

where \mathbb{P} denotes the set of prime numbers, $p_j(n)$ is the j -adic valuation of n , i.e. the multiplicity of the prime number j in the prime number decomposition of n . We define

$$\begin{aligned}
B(n) &:= \left[2^{\frac{p_2(n)}{4}} \cdot 3^{\frac{p_3(n)}{4}} \cdot 5^{\frac{p_5(n)}{2}} \prod_{j \in \mathbb{P}, j \geq 7} j^{\frac{9p_j(n)}{8}} \right]^{-1} , \\
B_\varepsilon(n, l) &:= B(n) \frac{(n+l)!}{n!} \varepsilon^{l+1} , \quad \varepsilon > 0 , l \geq 0 .
\end{aligned} \tag{3.2.35}$$

Lemma 3.1. *For $\varepsilon > 0$ sufficiently small, there exists $0 < \varepsilon' \leq \varepsilon$ such that if*

$$|\tilde{a}(2, l)| \leq B_{\varepsilon'}(2, l) , \quad l \geq 0 , \tag{3.2.36}$$

then for $n \geq 2$

$$|\tilde{a}(n, l)| \leq B_{\varepsilon'}(n, l) , \quad l \geq 0 . \tag{3.2.37}$$

Proof. See [4]. □

For ε' sufficiently small, one sees that the assumptions (3.2.36) imply those (3.2.6) in Proposition 3.1 for $\mu = 0$. If $f_2(\mu)$ is chosen as in (3.2.14) with β, δ close to ε then $f_2(0)$ satisfies (3.2.36). We can choose β, δ sufficiently small so that the assumptions (3.2.6) are satisfied by $f_2(\mu)$.

Then from consequence of Propostions 3.1, 3.2 and of Lemma 3.1

Theorem 3.1. *We consider the mean-field solutions of the mean-field flow equations for the bounded mean-field bare action (3.2.29). There exist nontrivial asymptotically free mean-field solutions of bounded mean field action of the mean-field flow equations. Moreover for $\varepsilon > 0$ sufficiently small, $0 < \varepsilon' \leq \varepsilon$, mean-field solutions which obey (3.2.36) have a bare mean field action $L_0(\phi)$ which is bounded by*

$$\varepsilon \sum_{n \in 2\mathbb{N}} \frac{1}{n^{\frac{5}{4}}} . \quad (3.2.38)$$

3.2.4 Mean-field trivial solution for weak bare couplings

Now we turn to the trivial solution constructed by Kopper [4] under the assumption that the bare couplings are small. The result in this section is weaker than the triviality results [43, 1, 2, 3] reviewed in Sect.2. As discussed before, a solution is called trivial if the couplings at the physical scale vanish for fixed bare couplings if the UV-cutoff is sent to infinity. In our setting, the logarithmic energy scale μ is used and we say that the φ^4 -theory is trivial if

$$\lim_{\mu_{\max} \rightarrow +\infty} f_{2n}(\mu_{\max}) = 0 , \quad 2n \geq 4 , \quad (3.2.39)$$

while keeping the bare value $f_4(0)$ fixed.

We study the following bare lagrangian without irrelevant terms i.e. $c_{0,n} = 0, n \geq 6$

$$L_0^{\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \left(c_{0,2}\varphi^2(x) + c_{0,4}\varphi^4(x) \right) \quad (3.2.40)$$

and the following (fixed) mean-field boundary conditions which can be read off from (3.2.40):

$$f_2(0) = 2(2\pi)^4 \alpha_0 c_{0,2}, \quad f_4(0) = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n \geq 6 . \quad (3.2.41)$$

A consequence of (3.2.41) is

Lemma 3.2. *For smooth solutions $f_n(\mu)$ of (3.1.11) with boundary conditions (3.2.41), we have*

$$\partial_{\mu}^l f_n(0) = 0 , \quad n \geq 6, \quad 0 \leq l \leq \frac{n}{2} - 3 . \quad (3.2.42)$$

Proof. The proof is done by induction in $n + 2l$, going up in l at a fixed value of $n + 2l$. We start at $n = 6$ and $l = 0$. For $n \geq 6$ and $l = 0$, the claim follows from the boundary conditions (3.2.41). For $0 \leq l \leq \frac{n}{2} - 4$, we derive (3.1.11) l times and we evaluate at $\mu = 0$. Then, solving $\partial_{\mu}^{l+1} f_n(0)$, we see that

$$\partial_{\mu}^{l+1} f_n(0) = 0 \quad (3.2.43)$$

because we used the induction hypothesis and the fact in the quadratic term, all the products

$$\partial_{\mu}^{l_1} f_{n_1}(0) \partial_{\mu}^{l_2} f_{n_2}(0) , \quad n_1 + n_2 = n + 2 , \quad l_1 + l_2 = l \quad (3.2.44)$$

are zero since we have either $l_1 \leq \frac{n_1}{2} - 3$ or $l_2 \leq \frac{n_2}{2} - 3$ due to the constraint $0 \leq l \leq \frac{n}{2} - 4$. \square

Then from Lemma 3.2, we can factorize $f_n(\mu)$ as follows

$$f_n(\mu) = \mu^{\frac{n}{2}-2} g_n(\mu), \quad n \geq 4, \quad (3.2.45)$$

where $g_n(\mu)$ are smooth. Therefore the flow equations for $g_n(\mu)$ are determined from (3.1.11)

$$\begin{aligned} \mu^2 g_{n+2} &= \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1} g_{n_2} + \mu \frac{1}{n+1} g_n \left(2f_2 + 1 - \frac{4}{n} \right) \\ &+ \frac{n-4}{n(n+1)} g_n + \frac{2}{n(n+1)} \mu \partial_\mu g_n. \end{aligned} \quad (3.2.46)$$

If we expand f_2 and g_n as a formal Taylor series around $\mu = 0$

$$f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k, \quad g_n(\mu) = \sum_{k \geq 0} g_{n,k} \mu^k, \quad (3.2.47)$$

we eventually have

$$f_{2,k+1} = \frac{1}{k+1} \left(3g_{4,k} + f_{2,k} - \sum_{\nu=0}^k f_{2,\nu} f_{2,k-\nu} \right), \quad (3.2.48)$$

$$\begin{aligned} g_{n,k+2} &= -\frac{n-4}{n+2k} g_{n,k+1} - \frac{2n}{n+2k} \sum_{\nu=0}^{k+1} g_{n,\nu} f_{2,k+1-\nu} - \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} g_{n_1,\nu} g_{n_2,k+2-\nu} \\ &+ \frac{n(n+1)}{n+2k} g_{n+2,k}. \end{aligned} \quad (3.2.49)$$

(3.2.48) corresponds to (3.2.46) at $n = 2$ while (3.2.49) corresponds to (3.2.46) for $n \geq 4$. Regularity at $\mu = 0$ implies for $n \geq 4$ the following inductive system

$$\frac{n-4}{n} g_{n,0} + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,0} = 0, \quad (3.2.50)$$

$$\frac{n-2}{n} g_{n,1} + 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,1} + g_{n,0} \left(2f_{2,0} + 1 - \frac{4}{n} \right) = 0. \quad (3.2.51)$$

If we start from arbitrary $f_{2,0}, g_{4,0}$ then from (3.2.50)-(3.2.51), all the $g_{n,0}$ and $g_{n,1}$ will be determined. The coefficients of the asymptotic series $f_{2,k'}$ and $g_{n,k'}$, $k' \geq 2$ are computed from (3.2.48)-(3.2.49).

Lemma 3.3. *We consider smooth mean-field solutions $f_n(\mu)$ of (3.2.48)-(3.2.51) which satisfy the mean-field boundary conditions (3.2.41). If for $0 < \varepsilon \leq 10^{-2}$*

$$|f_{2,0}| \leq \frac{\varepsilon}{4}, \quad 0 \leq f_{4,0} = g_{4,0} \leq \frac{\varepsilon}{32}, \quad (3.2.52)$$

then

$$|f_{2,1}| \leq \frac{\varepsilon}{2}, \quad |f_{4,1}| \leq \frac{\varepsilon^2}{32} \quad (3.2.53)$$

and for $n \geq 6$

$$|g_{n,0}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2}, \quad |g_{n,1}| \leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2} \quad (3.2.54)$$

with $g_{n,0} = (-1)^{\frac{n}{2}} |g_{n,0}|$.

Proof. From (3.2.48), we have

$$|f_{2,1}| \leq 3f_{4,0} + |f_{2,0}|(|f_{2,0}| + 1) \leq \frac{\varepsilon}{2}. \quad (3.2.55)$$

From (3.2.51) we have

$$|g_{4,1}| \leq 4g_{4,0}|f_{2,0}| \leq \frac{\varepsilon^2}{32}. \quad (3.2.56)$$

We proceed by induction in n . First we bound $g_{n,0}$ for $n \geq 6$. Using (3.2.50), we find

$$|g_{n,0}| \leq \frac{n}{n-4} \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 n_2^2} \leq \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2}. \quad (3.2.57)$$

For $g_{n,1}$ we use (3.2.51) to find

$$\begin{aligned} |g_{n,1}| &\leq \frac{2n}{n-2} \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{\varepsilon^{\frac{n}{2}-1}}{n_1^2 n_2^2} + \frac{n}{n-2} \frac{\varepsilon^{\frac{n}{2}-1}}{2n^2} \left(\frac{\varepsilon^2}{2} + 1 - \frac{4}{n} \right) \\ &\leq \frac{\varepsilon^{\frac{n}{2}-1}}{n^2}. \end{aligned} \quad (3.2.58)$$

The bounds on the sums over n_1, n_2 can be obtained by hand for $n \leq 10$ while for $n \geq 12$ we have

$$\begin{aligned} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} \frac{1}{n_1^2 n_2^2} &\leq \frac{1}{16} \sum_{\substack{n_1+n_2=\frac{n}{2}+1 \\ n_i \geq 2 \\ n_i \in \mathbb{N}}} \frac{1}{n_1^2 (\frac{n}{2} + 1 - n_1)^2} \\ &\leq \frac{1}{8} \sum_{3 \leq n_1 \leq \frac{n}{2}+1} \frac{4}{n_1^2 (n+2)^2} + \frac{1}{8} \frac{1}{(n-2)^2} \\ &\leq \frac{1}{2(n+2)^2} \left(\zeta(2) - \frac{5}{4} \right) + \frac{1}{8} \frac{1}{(n-2)^2}, \end{aligned} \quad (3.2.59)$$

where the ζ function is defined for $n \in \mathbb{N} \setminus \{1\}$ by

$$\zeta(n) = \sum_{k=1}^{+\infty} \frac{1}{k^n}. \quad (3.2.60)$$

In particular $\zeta(2) = \frac{\pi^2}{6}$. Since

$$\frac{n}{n-2} \left(\frac{1}{5} \frac{1}{(n+2)^2} + \frac{1}{8} \frac{1}{(n+2)^2} \right) \leq \frac{1}{2n^2} (1 - \varepsilon^2) \quad (3.2.61)$$

we obtain the bounds as claimed. The statement of the sign of $g_{n,0}$ is proven by induction in n , using (3.2.50). \square

Lemma 3.4. *Under the assumptions of Lemma 3.3*

$$|g_{n,k}| \leq 2^{k-2} \varepsilon^{\frac{n}{2}-1} \left(k + \frac{n-4}{2} \right)!, \quad |f_{2,k}| \leq 2^k \varepsilon |k-1|. \quad (3.2.62)$$

Proof. The proof is done by induction in $N = n + k$, going up in N . At a fixed value of N , we start at $n = 0$ or $n = 1$ and we go up in k . For $g_{n,0}$, $g_{n,1}$, $f_{2,0}$ and $f_{2,1}$ we use the bounds in Lemma 3.3. Using (3.2.49) and inserting the induction hypothesis, we obtain

$$\begin{aligned} |g_{n,k+2}| &\leq \frac{n-4}{n+2k} |g_{n,k+1}| + \frac{2n}{n+2k} \sum_{\nu=0}^{k+1} |g_{n,\nu} f_{2,k+1-\nu}| + \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} |g_{n_1,\nu} g_{n_2,k+2-\nu}| \\ &\quad + \frac{n(n+1)}{n+2k} |g_{n+2,k}| \\ &\leq 2^k \varepsilon^{\frac{n}{2}-1} \left[\frac{n-4}{2(n+2k)} \left(k+1 + \frac{n-4}{2} \right)! + \frac{n\varepsilon}{n+2k} \sum_{\nu=0}^{k+1} \left(\nu + \frac{n-4}{2} \right)! |k-\nu| \right. \\ &\quad + \frac{n}{4(n+2k)} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} \left(\nu + \frac{n_1-4}{2} \right)! \left(k+2-\nu + \frac{n_2-4}{2} \right)! \\ &\quad \left. + \frac{n(n+1)\varepsilon}{4(n+2k)} \left(k + \frac{n-2}{2} \right)! \right] \\ &\leq 2^k \varepsilon^{\frac{n}{2}-1} \left(k+2 + \frac{n-4}{2} \right)! \left[\frac{n-4}{n^2} + \frac{4\varepsilon}{n} + \frac{1}{4} + \frac{n+1}{2n} \varepsilon \right] \\ &\leq 2^k \varepsilon^{\frac{n}{2}-1} \left(k+2 + \frac{n-4}{2} \right)!. \end{aligned} \quad (3.2.63)$$

We used the following inequalities

$$\sum_{\nu=0}^{n-a} (n-\nu)! \nu! \leq 2 n!, \quad a \in \mathbb{N}, \quad a \leq n. \quad (3.2.64)$$

and

$$\sum_{\nu=0}^k (a+\nu)! (b+k-\nu)! \leq \sum_{\nu=0}^k (A+\nu)! (A+k-\nu)!, \quad A := \max(a, b), \quad a, b \in \mathbb{N}. \quad (3.2.65)$$

For $f_{2,k}$, we use (3.2.48) to obtain for $k \geq 1$

$$\begin{aligned}
|f_{2,k+1}| &\leq \frac{1}{k+1} \left(3|g_{4,k}| + 2^k \varepsilon (k-1)! + \sum_{\nu=0}^k 2^k \varepsilon^2 |\nu-1|! |k-\nu-1|! \right) \\
&\leq \frac{2^{k+1} \varepsilon k!}{k+1} \left(\frac{3}{8} + \frac{1}{2k} + 2\varepsilon \right) \\
&\leq 2^{k+1} \varepsilon k! .
\end{aligned} \tag{3.2.66}$$

The sum in (3.2.66) is bounded as follows

$$\begin{aligned}
\sum_{\nu=0}^k |\nu-1|! |k-\nu-1|! &= 2(k-1)! + \sum_{\nu=0}^{k-2} \nu! (k-2-\nu)! \\
&\leq 2(k-1)! + 2(k-2)! \leq 2k! .
\end{aligned} \tag{3.2.67}$$

□

Unlike the bounds derived in Proposition 3.1, the bounds in Lemma 3.4 are not sufficient to prove that the Taylor series around $\mu = 0$ converges. At this stage, the expansion (3.2.47) stands as a power series. The factorials in the bounds (3.2.62) are not optimal but seem close to optimality. We will come back to this remark. Nevertheless these bounds are sufficient to prove the existence of trivial solutions. In fact as shown in the following

Proposition 3.4. *There exist smooth mean-field solutions $f_n(\mu)$ of the mean-field flow equations (3.1.11) satisfying the mean-field boundary conditions (3.2.41) such that they vanish in the UV-limit, i.e. $\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0$.*

Proof. The key point of the proof is the construction of a two-point function $f_2(\mu)$ such that the mean-field smooth solutions $f_n(\mu)$ turn out to be trivial. We study an ansatz for the two-point function $f_2(\mu)$ of the form

$$f_2(\mu) = \sum_{q \geq 1} b_q \frac{x_q^{q-1}}{1+x_q^q}, \quad q \geq 1, \quad x_q := q\mu. \tag{3.2.68}$$

The ansatz (3.2.68) for the two-point function satisfies $f_2(0) = b_1$. If the series $\sum_{n \geq 1} b_n$ converges, then the ansatz (3.2.68) is well-defined. In that case, it is easy to see that

$$f_2(\mu_{\max}) = \mathcal{O}\left(\frac{1}{\mu_{\max}}\right), \tag{3.2.69}$$

meaning that $f_2(\mu_{\max}) \rightarrow 0$ when $\mu_{\max} \rightarrow +\infty$. If the derivatives of the ansatz (3.2.68) w.r.t. μ are well-defined, then we also have

$$\lim_{\mu_{\max} \rightarrow +\infty} \partial_{\mu}^l f_2(\mu_{\max}) = 0, \quad l \geq 0. \tag{3.2.70}$$

In that case, it follows from the mean-field flow equations (3.1.11) and (3.2.70) that $f_n(\mu_{\max}) \rightarrow 0$ for $n \geq 2$ when $\mu_{\max} \rightarrow +\infty$, i.e. the mean-field smooth solutions $f_n(\mu)$ constructed from the ansatz (3.2.68) are trivial.

By expanding $f_2(\mu)$ as in (3.2.47), its Taylor coefficients are

$$f_{2,k} = (k+1)^k \sum_{\rho=1}^{k+1} b_{\{\frac{k+1}{\rho}\}} (-1)^{\rho-1} \frac{1}{\rho^k}, \quad (3.2.71)$$

where by convention $b_0 = 0$ and

$$\left\{ \frac{m}{n} \right\} := \begin{cases} \frac{m}{n} & \text{if } \frac{m}{n} \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.72)$$

From (3.2.68)-(3.2.71), $f_{2,0} = b_1$ and $f_{2,1} = 2b_2 - b_1$ where $f_{2,1} = 3f_{4,0} - f_{2,0}(f_{2,0} - 1)$. Therefore the values of b_1 and b_2 are fixed by the free choice of $f_{2,0}$ and $f_{4,0}$. The b_n 's, $n \geq 3$ are then uniquely determined by (3.2.48)-(3.2.50) because of the boundary conditions (3.2.41) and the smoothness condition. From (3.2.71) we also have for $q \geq 1$

$$b_{q+1} = \frac{f_{2,q}}{(q+1)^q} - \sum_{\rho=2}^{q+1} b_{\{\frac{q+1}{\rho}\}} (-1)^{\rho-1} \frac{1}{\rho^q}. \quad (3.2.73)$$

We choose $f_{2,0}, f_{4,0}$ so that the assumptions of Lemma 3.3 are satisfied, meaning that the bounds on $f_{2,k}$ in Lemma 3.4 hold, namely

$$|f_{2,k}| \leq 2^k \varepsilon |k-1|!. \quad (3.2.74)$$

Now we prove by induction in $q \in \mathbb{N}$ that

$$|b_q| \leq 4 \left(\frac{3}{4} \right)^q \varepsilon. \quad (3.2.75)$$

The claim is easily verified for $q \leq 3$ using Lemmata 3.3, 3.4. For $q \geq 3$ we have

$$\begin{aligned} |b_{q+1}| &\leq 2^q \frac{(q-1)!}{(q+1)^q} \varepsilon + \sum_{\rho=2}^{q+1} |b_{\{\frac{q+1}{\rho}\}}| \frac{1}{\rho^q} \\ &\leq \left(\inf \left(\frac{1}{4}, \left(\frac{3}{4} \right)^q \right) + 4 \sum_{\rho=2}^{q+1} \left(\frac{3}{4} \right)^{\frac{q+1}{\rho}} \frac{1}{\rho^q} \right) \varepsilon \\ &\leq 4 \left(\frac{3}{4} \right)^q \varepsilon, \end{aligned} \quad (3.2.76)$$

where we used

$$\sum_{\rho=2}^{q+1} \left(\frac{3}{4} \right)^{\frac{q+1}{\rho}} \frac{1}{\rho^q} \leq \sum_{\rho=2}^{q+1} \frac{1}{\rho^q} \leq \zeta(q) - 1 \leq 2^{1-q}. \quad (3.2.77)$$

The bounds (3.2.75) show that the ansatz is well-defined, since the series (3.2.68) converges absolutely and uniformly in $\mu \in [0, \mu_{\max}]$. By uniform convergence of the series (3.2.68) on $[0, +\infty)$ and its derivatives

$$\sum_{q \geq 1} b_q \partial_\mu^l \frac{x_q^{q-1}}{1+x_q^q} \quad (3.2.78)$$

the mean-field two-point functions $f_2(\mu)$ and its derivatives vanish in the UV-limit

$$\lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_2(\mu_{\max}) = 0. \quad (3.2.79)$$

By induction in $n \geq 2$, we prove that $\partial_\mu^l f_n(\mu)$, $l \geq 0$ vanish in the UV-limit as well.

- The claim is verified for $n = 2$ and $l \geq 0$.
- Using the mean-flow equations (3.1.11), the claim follows for $n \geq 4$.

□

Collecting these findings, we get

Theorem 3.2 (Triviality in pure weakly-coupled mean-field φ^4 -theory). *We consider the mean-field smooth solutions of the flow equations (3.1.11). For boundary conditions (3.2.41) such that*

$$0 \leq c_{0,4} \leq \frac{\varepsilon}{2^7 \pi^2}, \quad |c_{0,2}| \leq \Lambda_0^2 \frac{\varepsilon}{2^7 \pi^4}, \quad 0 < \varepsilon \leq 10^{-2}, \quad \Lambda_0^{-2} = \alpha_0. \quad (3.2.80)$$

there exist smooth solutions of (3.1.11) $f_n \in C^\infty([0, \mu_{\max}])$ which vanish in the UV-limit, i.e., in the limit $\mu_{\max} \rightarrow +\infty$.

Proof. We have seen in Proposition 3.4 that such solutions exist. □

The main difference between the trivial solution constructed here and the solutions constructed in Sect.3.2.1, 3.2.2, 3.2.3 is that it cannot be shown that the trivial solution is analytic w.r.t. $\mu \in [0, \mu_{\max}]$. The point of non-analyticity is at $\mu = 0$. This is related to the fact that the Schwinger functions f_n , $n \geq 6$ have to be zero at $\mu = 0$, this infinite set of conditions is read off from the polynomial bare lagrangian (3.2.40). This infinite set of constraints does not appear in Sect.3.2.3 since Kopper considered a non-polynomial bare lagrangian. This constitutes a major difference w.r.t. the solutions constructed previously. Several remarks can be made about the constructed trivial solution. They served as motivation for this thesis.

- The Landau pole: if we fix the physical coupling

$$\tilde{g} = \tilde{g}(0) = f_2(\mu_{\max}), \quad (3.2.81)$$

then, using the notation introduced in Sect.3.2.2,

$$\tilde{g}(\lambda) := f_2(\mu_{\max} - \lambda), \quad \lambda = \mu_{\max} - \mu \quad (3.2.82)$$

diverges at a finite value of μ unless $\tilde{g}(0) \rightarrow 0$. In our case, if we truncate the ansatz to the lowest order

$$f_2(\mu) = a_1 \frac{1}{1 + \mu} \quad (3.2.83)$$

corresponding to the first term in (3.2.68), we obtain

$$\tilde{g}(\lambda) = \frac{\tilde{g}(0)}{1 - \beta\tilde{g}(0)\lambda}, \quad \beta = \frac{1}{a_1}. \quad (3.2.84)$$

Therefore the Landau pole is $\lambda_L = \frac{1}{\beta\tilde{g}(0)}$. Usually in perturbative φ_4^4 -theory, the physical coupling is defined by $\tilde{g} = f_4(\mu_{\max})$. But since $f_4(\mu)$ and $f_2(\mu)$ are related by (3.1.11) and $f_2(\mu)$ and $f_4(\mu)$ are of the same order when μ is large, we can re-express $f_2(\mu)$ in terms of $f_4(\mu)$ and vice versa so that the conclusion remains the same. If we take the full ansatz (3.2.68), we arrive at the same conclusion since we have an absolutely convergent series and all entries in the series behave similarly when $\mu \rightarrow +\infty$. It is important to note that perturbation truncations get out of control before reaching the Landau pole; perturbation theory does not allow one to make rigorous statements on triviality.

- It was also proposed to analyze the trivial solution and establish a comparison between them and the mean-field solutions of the Ising model and the correlation functions considered in [1, 44, 3, 2] in the mean-field limit. This was not done in this thesis.
- The triviality statement for the mean-field theory is restricted to the one-component and weakly coupled theory. In the triviality proofs [43, 2, 3], no assumption on the size of the bare coupling is made. Nevertheless the proofs are still restricted to one or two component scalar fields since no sufficiently strong correlation inequalities have been established for N -component scalar fields, $N > 2$. In this thesis, we extended the triviality proof to theories with arbitrarily large bare couplings and theories with N real scalar components ($N \geq 1$) but still in the mean-field approximation.
- The technical simplification $m = 0$ in the propagator and the introduction of an artificial IR cutoff could be dropped. In this thesis, we proved triviality of the mean-field theory in this new more physical setting.
- The approach is non-perturbative; the solutions constructed in Sect.3.2.1-3.2.3 are not bounded by a power proportional to n of δ or ε . For the trivial solution, the bounds (3.2.62) reveal a factor $\varepsilon^{\frac{n}{2}-1}$. It has been suggested that it would be interesting to recast the power series in ε as a power series in g , g being the renormalized coupling. Consequently, we analyzed in this thesis the link between the trivial solution and perturbation theory.

Chapter 4

The mean-field flow equations for the $O(N)$ -model

Now we turn to the main part in this thesis. In the next two chapters we will tackle two points mentioned in the previous chapter: extension to multi-component real scalar field theory, extension of the triviality result to arbitrarily large bare couplings. We analyze the self-interacting N -component vector scalar field theories on four-dimensional Euclidean space. The results we will review here are found in our paper [79]. The N -component vector model with $O(N)$ symmetry, called the $O(N)$ -model was first introduced by Stanley [81] to generalize the Ising model ($N = 1$), the XY model ($N = 2$) and the Heisenberg model ($N = 3$). In the continuum description the scalar field φ has now N real components $\varphi(x) = [\varphi_1(x), \dots, \varphi_N(x)]^T$, and the lagrangian has a global $O(N)$ symmetry. Since $\varphi \mapsto -\varphi$ is a $O(N)$ transformation, the theory has \mathbb{Z}_2 symmetry. Stanley [82] thought that the large N -limit of this model is linked to the spherical model introduced by Berlin and Kac[83]. This relation between the two models has been proved rigourously for fixed non-critical temperature [84] and for critical temperature [85, 86]. Through an expansion in powers of $\frac{1}{N}$, the behavior of the solutions of the $O(N)$ model in the large N limit has been studied in detail in [87] and by Moshe and Zinn-Justin [88]. We will go back to this remark in Sect.5.2.

To derive the mean-field flow equations for the $O(N)$ -model, we will

- follow the steps in [4] to derive the flow equations
- perform the mean-field approximation afterwards.

Using $O(N)$ symmetry, the form of the mean-field flow equations actually generalizes those of the single-component theory. Again, we will adopt the convention of the Fourier transform and its shorthand notation (1.0.1).

4.1 The flow equations for the $O(N)$ -model

To derive the flow equations for the $O(N)$ -model, we choose a regularized propagator of the N -component massive vector scalar field theory in momentum space which is given by a diagonal

$N \times N$ matrix to be called $C_N^{\alpha, \alpha_0}(x - y, m)$. The matrix elements of the regularized propagator in momentum space read

$$C_{N; ij}^{\alpha_0, \alpha}(p, m) = C^{\alpha_0, \alpha}(p, m) \delta_{ij} . \quad (4.1.1)$$

The quantity $C^{\alpha_0, \alpha}(p, m)$ is the propagator introduced in (3.1.1). This regularized propagator is positive and satisfies

- $C_{N; ij}^{\alpha_0, \alpha}(p, m)$ is analytic w.r.t. α .
- $C_{N; ij}^{\alpha_0, \alpha_0}(p, m) = 0$.
- At α and i fixed, $C_{N; ii}^{\alpha_0, \alpha}(p, m)$ falls off more rapidly than any power of $|p|$.

As in the one-component case, we study the following bare interactions lagrangians

$$L_{0, N}^{\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \left[\sum_{n \in \mathbb{N}} c_{0, n}(\alpha_0) (\varphi^2(x))^n \right] , \quad (4.1.2)$$

where $\varphi^2(x) := \sum_{i=1}^N \varphi_i^2(x)$ and \mathcal{V} is a finite volume in \mathbb{R}^4 . It is clear that this bare interaction lagrangian has $O(N)$ -invariance. The quantities $c_{0, n}(\alpha_0)$ are again the bare couplings. In the bare interaction lagrangian (4.1.2), we did not include this time the wavefunction renormalization counterterm because the field φ is constant in the mean-field limit. In four dimensions, the terms in the sum for $n \geq 6$ are irrelevant, while for $n = 2$ and $n = 4$ they are respectively relevant and marginal.

The functional integral with the bare lagrangian $L_{0, N}^{\mathcal{V}}(\varphi)$ is well-defined if for some constant $C_N^{\mathcal{V}} \in \mathbb{R}$, depending on \mathcal{V} and N

$$-\infty < C_N^{\mathcal{V}} < L_{0, N}^{\mathcal{V}}(\varphi) , \quad \varphi \in \text{supp}(\mu_N^{\alpha_0, \alpha}) , \quad (4.1.3)$$

where $\mu_N^{\alpha_0, \alpha}$ is the normalized Gaussian measure associated to the propagator $C_N^{\alpha_0, \alpha}$. We suppose that the field φ belongs to the support of the Gaussian measure $\mu_N^{\alpha_0, \alpha}$ associated to the propagator (4.1.1) so that it is smooth as in Sect.3.1. Again the infinite-volume limit of the connected Schwinger functions exist, we will drop the subscript \mathcal{V} .

We define the correlation or Schwinger functions for the $O(N)$ -model by

$$\langle \varphi_{i_1}(x_1) \varphi_{i_2}(x_2) \cdots \varphi_{i_n}(x_n) \rangle^{\alpha_0, \alpha} := \frac{1}{Z_N^{\alpha_0, \alpha}} \int d\mu_N^{\alpha_0, \alpha}(\varphi) e^{-L_{0, N}(\varphi)} \varphi_{i_1}(x_1) \varphi_{i_2}(x_2) \cdots \varphi_{i_n}(x_n) , \quad (4.1.4)$$

where $Z_N^{\alpha_0, \alpha}$ is a normalization factor.

The generating functional of the regularized connected amputated Schwinger functions (CAS) $e^{-L_N^{\alpha_0, \alpha}(\varphi)}$ at scale α satisfies

$$e^{-L_N^{\alpha_0, \alpha}(\varphi)} = \frac{1}{Z_N^{\alpha_0, \alpha}} \int d\mu_N^{\alpha_0, \alpha}(\phi) e^{-L_{0, N}(\phi + \varphi)} . \quad (4.1.5)$$

We expand $L_N^{\alpha_0, \alpha}(\varphi)$ in a formal power series in the components $\hat{\varphi}_i$

$$L_N^{\alpha_0, \alpha}(\varphi) = \sum_{n \in 2\mathbb{N}} \sum_{1 \leq i_1, \dots, i_n \leq N} \int_{p_1, p_2, \dots, p_n} \bar{\mathcal{L}}_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n) \hat{\varphi}_{i_1}(p_1) \cdots \hat{\varphi}_{i_n}(p_n) . \quad (4.1.6)$$

Again, the CAS distributions $\bar{\mathcal{L}}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n)$ or moments of $L_N^{\alpha_0, \alpha}$ can be factorized due to translation invariance in position space as

$$\bar{\mathcal{L}}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n) = \delta^4 \left(\sum_{i=1}^n p_i \right) \mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n), \quad p_n = -p_1 - \dots - p_{n-1}. \quad (4.1.7)$$

Through functional derivation, we obtain the CAS functions $\mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n)$

$$\frac{(2\pi)^{4n}}{n!} \frac{\delta}{\delta \hat{\varphi}_{i_1}(p_1)} \dots \frac{\delta}{\delta \hat{\varphi}_{i_n}(p_n)} L_N^{\alpha_0, \alpha}(\varphi)|_{\varphi=0} = \delta^4 \left(\sum_{i=1}^n p_i \right) \mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n). \quad (4.1.8)$$

The flow equations are obtained upon differentiation of (4.1.5) w.r.t. α using the change of covariance formula (A.1.7)

$$\partial_\alpha L_N^{\alpha_0, \alpha} = \frac{1}{2} \sum_{i=1}^N \left\langle \frac{\delta}{\delta \varphi_i}, \dot{C}^\alpha \frac{\delta}{\delta \varphi_i} \right\rangle L_N^{\alpha_0, \alpha} - \frac{1}{2} \sum_{i=1}^N \left\langle \frac{\delta}{\delta \varphi_i} L_N^{\alpha_0, \alpha}, \dot{C}^\alpha \frac{\delta}{\delta \varphi_i} L_N^{\alpha_0, \alpha} \right\rangle, \quad (4.1.9)$$

where $\dot{C}^\alpha = \partial_\alpha C^{\alpha_0, \alpha}$. Using (4.1.6), (4.1.7) in (4.1.9), the flow equations for the moments $\mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}$ read

$$\begin{aligned} \partial_\alpha \mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(p_1, \dots, p_n) &= \binom{n+2}{2} \sum_{j=1}^N \int_k \dot{C}^\alpha(k, m) \mathcal{L}_{n+2; i_1 i_2 \dots i_n j j}^{\alpha_0, \alpha}(p_1, \dots, p_n, k, -k) \\ &\quad - \frac{1}{2} \sum_{n_1+n_2=n+2} \sum_{j=1}^N n_1 n_2 \mathbb{S} \left[\mathcal{L}_{n_1; i_1 i_2 \dots i_{n_1-1} j}^{\alpha_0, \alpha}(p_1, \dots, p_{n_1-1}, q) \dot{C}^\alpha(q, m) \right. \\ &\quad \left. \mathcal{L}_{n_2; j i_{n_1+1} \dots i_n}^{\alpha_0, \alpha}(-q, p_{n_1}, \dots, p_n) \right], \end{aligned} \quad (4.1.10)$$

with $q := -p_1 - p_2 - \dots - p_{n_1-1}$. Here \mathbb{S} is a symmetrisation operator which permutes the pairs (i_j, p_j) . It averages over the permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(n_1 - 1)$ and $\sigma(n_1) < \sigma(n_1 + 1) < \dots < \sigma(n)$. Due to \mathbb{Z}_2 -symmetry, only even moments (in n, n_1 and n_2) are nonvanishing. One sees that the dynamical system (4.1.10) generalizes the one-component flow equations (3.1.5).

4.2 The mean-field approximation

In the mean field approximation, the n -point functions are assumed to be momentum independent. As in the single component-case, we consider the correlation functions per unit of volume in the infinite volume limit without fluctuations

$$A_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha} := \mathcal{L}_{n;i_1 i_2 \dots i_n}^{\alpha_0, \alpha}(0, \dots, 0). \quad (4.2.1)$$

We recall that the critical behavior of Ising type systems in $d > 4$ dimensions is exactly obtained [44], [2] in the mean-field approximation. Since in higher dimensions, the theory is non-renormalizable (this means that the theory requires an infinite number of counterterms when going

higher in perturbation theory), we conjecture that the only way to obtain finite physical quantities in the UV-limit would be to set to zero all the renormalized quantities, leading to a trivial theory. This has not been discussed and studied in this thesis. For $d < 4$, construction of nontrivial φ_d^4 -theory has been performed:

- $d = 3$: e.g. see [89, 90, 91]
- $d = 2$: see [92] for a semibounded polynomial P which includes the φ^4 model.

To obtain the mean-field flow equations for the $O(N)$ -model following [4], we expand the mean field effective action $L_{mf}^{\alpha_0, \alpha}(\phi)$ as a formal power series in the (constant) field $\phi \in \mathbb{R}$

$$L_{mf}^{\alpha_0, \alpha}(\phi) = \sum_{n \in 2\mathbb{N}} A_n^{\alpha_0, \alpha} \phi^n. \quad (4.2.2)$$

As in [4], we make the technical simplification from [4] and set $m = 0$ in $C^{\alpha_0, \alpha}(p, m)$, we then restrict the analysis of the theory to the interval $\alpha \in [\alpha_0, \frac{1}{m^2}]$ so that the upper limit on α takes the role of the infrared cutoff, thus replacing the mass of the original theory. In Sect.6 we will drop this simplification.

In the mean-field limit, the regularized propagator becomes

$$\frac{e^{-\alpha_0 p^2} - e^{-\frac{1}{m^2} p^2}}{p^2} \underset{p^2 \ll 1}{=} \frac{1}{m^2} - \alpha_0 + O(p^2). \quad (4.2.3)$$

Then the mean-field flow equations can be rewritten as

$$\begin{aligned} \partial_\alpha A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha} &= \binom{n+2}{2} \frac{c}{\alpha^2} \sum_{j=1}^N A_{n+2; i_1 i_2 \dots i_n j j}^{\alpha_0, \alpha} \\ &\quad - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 \sum_{j=1}^N \mathfrak{S} \left[A_{n_1; i_1 i_2 \dots i_{n_1-1} j}^{\alpha_0, \alpha} A_{n_2; j i_{n_1} i_{n_1+1} \dots i_n}^{\alpha_0, \alpha} \right], \quad c := \frac{1}{16\pi^2}. \end{aligned} \quad (4.2.4)$$

Without any further input, the flow equations (4.2.4) do not let us construct inductively (in n) the CAS functions since only the contraction of $A_{n+2; i_1 i_2 \dots i_{n+2}}^{\alpha_0, \alpha}$ w.r.t. its last two indices can be related to the CAS functions $A_{n'; i_1 i_2 \dots i_{n'}}^{\alpha_0, \alpha}$, $n' \leq n$. To overcome this difficulty, key properties must be satisfied by the mean-field solutions for the $O(N)$ model of the flow equations. These properties are

- **(P1)**: $A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha} = 0$ if n is odd (\mathbb{Z}_2 -symmetry).
- **(P2)**: $A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha}$ is symmetric under any permutation of its indices i_1, i_2, \dots, i_n (Bose symmetry).
- **(P3)**: $A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha}$ is $O(N)$ -invariant in the following sense: let O be an orthogonal matrix i.e. $O^T O = O^T O = I$, then

$$O_{i_1 j_1} O_{i_2 j_2} \dots O_{i_n j_n} A_{n; j_1 j_2 \dots j_n}^{\alpha_0, \alpha} = A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha}. \quad (4.2.5)$$

Properties **(P1)**, **(P2)** and **(P3)** require knowledge on the $O(N)$ -invariant symmetric tensors.

4.2.1 Isotropic Cartesian tensors

We define Isotropic Cartesian tensors and useful contraction properties which will be used to recast the dynamical system (4.2.4) into an inductive scheme.

Definition 4.1 (Cartesian tensors). *Let X be an Euclidean space of finite dimension $N \geq 1$. We identify X with its dual space X^* . A rank n -tensor \mathbf{T} is an element of $\bigotimes_{i=1}^n X$. Assuming that we work with an orthonormal basis, we do not need to distinguish the contravariant and the covariant components of a tensor. Then, $\mathbf{T} \in \bigotimes_{i=1}^n X$ is called a Cartesian rank n tensor. Its components are denoted by $T_{i_1 i_2 \dots i_n}$.*

Definition 4.2 (Isotropic Cartesian tensors). *A Cartesian rank n tensor \mathbf{T} is said to be isotropic if for any matrix $M \in SO(N)$*

$$M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_n j_n} T_{j_1 j_2 \dots j_n} = T_{i_1 i_2 \dots i_n} . \quad (4.2.6)$$

Proposition 4.1. *Let \mathbf{T} be a real rank n -tensor, $n \in 2\mathbb{N}$. If \mathbf{T} is symmetric and $O(N)$ -invariant, then it is of the form*

$$T_{i_1 i_2 \dots i_n} = A \sum_{\sigma \in S_n} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}} , \quad A \in \mathbb{R} , \quad (4.2.7)$$

where S_n denotes the set of permutations in $\{1, \dots, n\}$.

Proof. The most general forms of real isotropic Cartesian rank n tensors are

- $n < N$:

$$T_{i_1 i_2 \dots i_n} = \sum_{\sigma \in S_n} \lambda_{\sigma} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}} , \quad \lambda_{\sigma} \in \mathbb{R} . \quad (4.2.8)$$

- $n = N$:

$$T_{i_1 i_2 \dots i_n} = \sum_{\sigma \in S_n} \lambda_{\sigma} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}} + \mu \varepsilon_{i_1 i_2 \dots i_n} , \quad \lambda_{\sigma}, \mu \in \mathbb{R} . \quad (4.2.9)$$

- $n > N$ and N even:

$$T_{i_1 i_2 \dots i_n} = \sum_{\sigma \in S_n} \lambda_{\sigma} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}} + \sum_{\sigma \in S_n} \mu_{\sigma} \varepsilon_{i_{\sigma(1)} \dots i_{\sigma(N)}} \delta_{i_{\sigma(N+1)} i_{\sigma(N+2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}} , \quad (4.2.10)$$

where $\lambda_{\sigma}, \mu_{\sigma} \in \mathbb{R}$.

Here $\varepsilon_{i_1 i_2 \dots i_n}$ is the Levi-Civita tensor defined by

$$\varepsilon_{i_1 i_2 \dots i_n} = \begin{cases} 1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an even permutation of } (1, 2, \dots, n) \\ -1 & \text{if } (i_1 i_2 \dots i_n) \text{ is an odd permutation of } (1, 2, \dots, n) \\ 0 & \text{otherwise.} \end{cases} \quad (4.2.11)$$

For a proof see [93].

If \mathbf{T} is $O(N)$ -invariant, it is an isotropic Cartesian tensor. It then takes the form (4.2.8), (4.2.9), (4.2.10) depending on n . We consider the reflection R in the hyperplane through the origin, orthogonal to

\mathbf{e}_k , $1 \leq k \leq N$, where \mathbf{e}_k denotes a canonical basis vector of \mathbb{R}^N . The matrix expression of R is given by

$$R_{ij} = \delta_{ij} - 2\delta_{ik}\delta_{jk}. \quad (4.2.12)$$

Then we have

$$R_{i_1 j_1} \cdots R_{i_N j_N} \varepsilon_{j_1 \cdots j_N} = \det(R) \varepsilon_{i_1 \cdots i_N} = -\varepsilon_{i_1 \cdots i_N}. \quad (4.2.13)$$

Then from (4.2.13) and (4.2.8), (4.2.9), (4.2.10), symmetric and $O(N)$ -invariant tensors take the form (4.2.7). \square

Now we turn to contractions of $O(N)$ -invariant tensors. We recall the symmetric part of a rank n -tensor \mathbf{T} by

$$T_{(i_1 i_2 \cdots i_n)} := \frac{1}{n!} \sum_{\sigma \in S_n} T_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(n)}}. \quad (4.2.14)$$

and we define

$$F_{i_1 i_2 \cdots i_n} := \delta_{(i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{n-1} i_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(n-1)} i_{\sigma(n)}}. \quad (4.2.15)$$

Proposition 4.2. *We have the following identities:*

$$\sum_{j=1}^N F_{i_1 i_2 \cdots i_n j j} = \frac{N+n}{n+1} F_{i_1 i_2 \cdots i_n}, \quad \sum_{j=1}^N \mathfrak{S} \left[F_{i_1 i_2 \cdots i_{n-1} j} F_{i_{n-1} i_{n-1} \cdots i_n j} \right] = F_{i_1 i_2 \cdots i_n}. \quad (4.2.16)$$

Proof. Let $F(\mathbf{x})$ be the generating series of $F_{i_1 i_2 \cdots i_n}$ defined by

$$F(\mathbf{x}) := \sum_{n \in 2\mathbb{N}} \sum_{i_1, i_2, \dots, i_n} x_{i_1} \cdots x_{i_n} F_{i_1 i_2 \cdots i_n} = \sum_{n=1}^{+\infty} |\mathbf{x}|^{2n}, \quad |\mathbf{x}|^2 := \sum_{i=1}^N x_i^2, \quad \mathbf{x} \in \mathbb{R}^N.$$

The result of the action of the Laplacian on $F(\mathbf{x})$ is

$$\Delta F(\mathbf{x}) = \sum_{j=1}^N \partial_j^2 F(\mathbf{x}) = 2N + \sum_{n \in 2\mathbb{N}} \sum_{i_1, i_2, \dots, i_n} x_{i_1} \cdots x_{i_n} (n+2)(n+1) \sum_{j=1}^N F_{i_1 i_2 \cdots i_n j j}. \quad (4.2.17)$$

On the other hand we have for $n \in \mathbb{N}$

$$\sum_{j=1}^N \partial_j^2 |\mathbf{x}|^{2n} = 2n(N+2n-2) |\mathbf{x}|^{2n-2} \quad (4.2.18)$$

and therefore

$$\begin{aligned} \Delta F(\mathbf{x}) &= 2N + \sum_{n \in 2\mathbb{N}, n > 2} n(N+n-2) |\mathbf{x}|^{n-2} \\ &= 2N + \sum_{n \in 2\mathbb{N}} (n+2)(N+n) |\mathbf{x}|^n \\ &= 2N + \sum_{n \in 2\mathbb{N}} \sum_{i_1, \dots, i_n} x_{i_1} \cdots x_{i_n} (n+2)(N+n) F_{i_1 \cdots i_n}. \end{aligned} \quad (4.2.19)$$

Now (4.2.17) and (4.2.19) imply

$$\sum_{j=1}^N F_{i_1 i_2 \dots i_n j j} = \frac{N+n}{n+1} F_{i_1 i_2 \dots i_n} . \quad (4.2.20)$$

For the second identity in (4.2.16), we compute

$$\begin{aligned} \|\nabla F(x)\|^2 &:= \sum_{j=1}^N (\partial_j F(x))^2 = 4|x|^2 \sum_{n_1, n_2 \geq 1} n_1 n_2 |x|^{2(n_1+n_2-2)} = \sum_{n_1 \in 2\mathbb{N}, n_2 \in 2\mathbb{N}} n_1 n_2 |x|^{n_1+n_2-2} \\ &= 4|x|^2 + \sum_{n \in 2\mathbb{N}, n > 2} \sum_{n_1+n_2=n+2} n_1 n_2 \sum_{i_1, \dots, i_n} x_{i_1} \dots x_{i_n} F_{i_1 \dots i_n} . \end{aligned} \quad (4.2.21)$$

And on the other hand, we have

$$\|\nabla F(x)\|^2 = 4|x|^2 + \sum_{n \in 2\mathbb{N}, n > 2} \sum_{n_1+n_2=n+2} n_1 n_2 \sum_{i_1, \dots, i_n} x_{i_1} \dots x_{i_n} \sum_{j=1}^N F_{(i_1 \dots i_{n_1-1} j) j i_{n_1} \dots i_n} , \quad (4.2.22)$$

leading to

$$\sum_{j=1}^N F_{(i_1 \dots i_{n_1-1} j) j i_{n_1} \dots i_n} = F_{i_1 \dots i_n} . \quad (4.2.23)$$

From the definition of a symmetric part of a tensor \mathbf{T} in (4.2.14) and the fact that \mathbb{S} is an average operator, we obtain

$$\sum_{j=1}^N \mathbb{S} \left[F_{i_1 i_2 \dots i_{n_1-1} j} F_{j i_{n_1} i_{n_1+1} \dots i_n} \right] = F_{i_1 \dots i_n} . \quad (4.2.24)$$

□

4.2.2 The reduced mean-field flow equations for the $O(N)$ -model

With knowledge on $O(N)$ -invariant tensors, we can recast the flow equations (4.2.4) in a reduced form. From Proposition 4.1, we set

$$A_{n; i_1 i_2 \dots i_n}^{\alpha_0, \alpha} = A_n^{\alpha_0, \alpha} F_{i_1 i_2 \dots i_n} \quad (4.2.25)$$

with $A_n^{\alpha_0, \alpha}$ smooth.

The mean field flow equations (4.2.4) read

$$\begin{aligned} \partial_\alpha A_n^{\alpha_0, \alpha} F_{i_1 i_2 \dots i_n} &= \binom{n+2}{2} \frac{c}{\alpha^2} A_{n+2}^{\alpha_0, \alpha} \sum_{j=1}^N F_{i_1 i_2 \dots i_n j j} \\ &\quad - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha} \sum_{j=1}^N \mathbb{S} \left[F_{i_1 i_2 \dots i_{n_1-1} j} F_{j i_{n_1} i_{n_1+1} \dots i_n} \right] . \end{aligned} \quad (4.2.26)$$

Using Proposition 4.2, the flow equations for the coefficients $A_n^{\alpha_0, \alpha}$ are

$$\partial_\alpha A_n^{\alpha_0, \alpha} = \binom{n+2}{2} \frac{N+n}{n+1} \frac{c}{\alpha^2} A_{n+2}^{\alpha_0, \alpha} - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha}. \quad (4.2.27)$$

The factor $\binom{n+2}{2}$ leads to a combinatorial instability: iterated integration w.r.t. α of the mean-field solutions of the infinite dynamical system for the functions $A_n^{\alpha_0, \alpha}$ would produce at each step factors of order n^2 , meaning that the solutions blow up when $n \gg 1$. Therefore, we have to follow the strategy from [4, 79]:

- Start from a smooth two-point function $A_2^{\alpha_0, \alpha}$ ¹ and fix mean-field boundary conditions at the bare scale, i.e. $\alpha = \alpha_0$.
- Smooth solutions $A_n^{\alpha_0, \alpha}$ are constructed inductively (in n) using (4.2.27) whose properties depend on $A_2^{\alpha_0, \alpha}$.

Adopting the change of function and variable in (3.1.10)

$$f_n(\mu) := \alpha^{2-\frac{n}{2}} c^{\frac{n}{2}-1} n A_n^{\alpha_0, \alpha}, \quad \mu := \ln \left(\frac{\alpha}{\alpha_0} \right), \quad (4.2.28)$$

the mean-field flow equations now read

$$f_{n+2}(\mu) = \frac{1}{n+N} \sum_{n_1+n_2=n+2} f_{n_1}(\mu) f_{n_2}(\mu) + \frac{n-4}{n(n+N)} f_n(\mu) + \frac{2}{n(n+N)} \partial_\mu f_n(\mu), \quad n \geq 2, \quad (4.2.29)$$

for $\mu \in [0, \mu_{\max}]$ where $\mu_{\max} := \ln \left(\frac{1}{\alpha_0} \right)$. One sees that the reduced mean-field flow equations (4.2.29) generalize the dynamical system (3.1.11) ($N = 1$) to the $O(N)$ -model. From the flow equations (4.2.29), $f_{n+2}(\mu)$ is expressed in terms of a linear term in $f_n(\mu)$ and its derivative and a quadratic term in $f_{n'}(\mu)$, $n' \leq n$ as in Ch.3.

Remark. • The statements in Ch.3 on the local analyticity w.r.t. μ of uniformly bounded smooth solutions $f_n(\mu)$ and the analyticity of the bare-mean field action w.r.t. φ remain valid for $N > 1$.

4.3 Local analyticity w.r.t. α of the mean-field CAS functions

During the thesis, we came upon a non-trivial point regarding the analyticity of the CAS functions. Therefore it is useful to comment on the analyticity of the mean-field CAS functions w.r.t. α . The regularized propagator in momentum space $C^{\alpha_0, \alpha}(p, m)$ introduced in (3.1.1) is analytic w.r.t. α . If we construct the solutions $A_{n+2}^{\alpha_0, \alpha}$ of the flow equations (4.2.4) as indicated, we have the following local analyticity and uniqueness statements

Proposition 4.3. • Let $A_n^{\alpha_0, \alpha}$ be mean-field smooth solutions of (4.2.4). The boundary conditions are assumed to be induced by a two-point function $A_2^{\alpha_0, \alpha}$ and its derivatives at $\alpha = \alpha_0$ which is locally analytic w.r.t. α . Then $A_n^{\alpha_0, \alpha}$ is locally analytic w.r.t. α .

¹The properties of two-point function $A_2^{\alpha_0, \alpha}$ will be specified later.

- Let $A_2^{\alpha_0, \alpha}$ and $\tilde{A}_2^{\alpha_0, \alpha}$ be locally analytic w.r.t. α . If $A_n^{\alpha_0, \alpha}$ and $\tilde{A}_n^{\alpha_0, \alpha}$ are constructed from $A_2^{\alpha_0, \alpha}$ and $\tilde{A}_2^{\alpha_0, \alpha}$ respectively, using the flow equations (4.2.4) and if

$$\partial_\alpha^k \tilde{A}_2^{\alpha_0, \alpha} |_{\alpha=\alpha_0} = \partial_\alpha^k A_2^{\alpha_0, \alpha} |_{\alpha=\alpha_0}, \quad k \geq 0, \quad (4.3.1)$$

then we have for arbitrary α

$$\partial_\alpha^k \tilde{A}_n^{\alpha_0, \alpha} = \partial_\alpha^k A_n^{\alpha_0, \alpha}, \quad k \geq 0, \quad n \geq 2. \quad (4.3.2)$$

Proof. The proof of the first statement proceeds by induction in n . It obviously holds for $n = 2$. From (4.2.27), we have

$$A_{n+2}^{\alpha_0, \alpha} = \frac{2}{c(n+N)(n+2)} \alpha^2 \partial_\alpha A_n^{\alpha_0, \alpha} + \frac{\alpha^2}{c(n+N)(n+2)} \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\alpha_0, \alpha} A_{n_2}^{\alpha_0, \alpha}. \quad (4.3.3)$$

which implies the statement using the induction hypothesis.

The second statement is proven by induction in $N = n + 2k$, going up in n for fixed N . It then follows directly from the fact that locally analytic functions are uniquely defined by their Taylor expansions within their radius of convergence, and from the fact that sums of products of locally analytic functions are again locally analytic. \square

Proposition 4.3 has the following consequence:

- We start from a locally analytic two-point function $A_2^{\alpha_0, \alpha}$.
- We compute the successive CAS functions $A_n^{\alpha_0, \alpha}$ using (4.2.27).
- The locally analytic mean-field solutions $A_n^{\alpha_0, \alpha}$ are unique for fixed boundary conditions at the bare scale.

This result is non-trivial because in the next section, we will see that this uniqueness result cannot be used since the trivial solution we will construct is not analytic at $\mu = 0$. Unlike the trivial solution, the asymptotically free scalar field theories constructed in Sect.3.2.2 are not plagued by this obstruction to analyticity. The uniqueness of the trivial solution will be reviewed in Sect.5.1.2.

Chapter 5

Triviality of mean-field φ_4^4 -theories

5.1 Construction of the mean-field trivial solution

Now we turn to the triviality of the pure mean-field $O(N)$ φ_4^4 -theory. First we prove the existence of solutions of (4.2.29) which vanish in the UV-limit for fixed mean-field boundary conditions. Then we will prove the uniqueness of the mean-field trivial solution. We will again follow the method in Ch.3 and [4]. Our result is published in [79].

5.1.1 Existence of a mean-field trivial solution

We consider the following bare lagrangian without irrelevant terms i.e. $c_{0,n} = 0, n \geq 6$

$$L_{0,N}^{\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \left(c_{0,2}\varphi^2(x) + c_{0,4}\varphi^4(x) \right) \quad (5.1.1)$$

and the following (fixed) mean-field boundary conditions following from (4.2.1), (4.2.2), (3.1.10) and (5.1.1):

$$f_2(0) = 2(2\pi)^4 \alpha_0 c_{0,2}, \quad f_4(0) = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n \geq 6. \quad (5.1.2)$$

Lemma 3.2 remains valid for the mean-field smooth solutions $f_n(\mu)$ of the flow equations (4.2.29), we can set

$$f_n(\mu) = \mu^{\frac{n}{2}-2} g_n(\mu), \quad n \geq 4, \quad (5.1.3)$$

where $g_n(\mu)$ are smooth. Then, we follow the method in Sect.3.2.4: we rewrite (4.2.29) as

$$\begin{aligned} \mu^2 g_{n+2} &= \frac{1}{n+N} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1} g_{n_2} + \mu \frac{1}{n+N} g_n \left(2f_2 + 1 - \frac{4}{n} \right) \\ &+ \frac{n-4}{n(n+N)} g_n + \frac{2}{n(n+N)} \mu \partial_\mu g_n, \quad n \geq 4. \end{aligned} \quad (5.1.4)$$

As before, we expand f_2 and g_n as formal Taylor series around $\mu = 0$

$$f_2(\mu) = \sum_{k \geq 0} f_{2,k} \mu^k, \quad g_n(\mu) = \sum_{k \geq 0} g_{n,k} \mu^k, \quad (5.1.5)$$

so that we obtain the relations between the coefficients of the (Taylor) power series

$$f_{2,k+1} = \frac{1}{k+1} \left((N+2)g_{4,k} + f_{2,k} - \sum_{\nu=0}^k f_{2,\nu} f_{2,k-\nu} \right), \quad (5.1.6)$$

$$g_{n,k+2} = -\frac{n-4}{n+2k} g_{n,k+1} - \frac{2n}{n+2k} \sum_{\nu=0}^{k+1} g_{n,\nu} f_{2,k+1-\nu} - \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} g_{n_1,\nu} g_{n_2,k+2-\nu} \\ + \frac{n(n+N)}{n+2k} g_{n+2,k}. \quad (5.1.7)$$

(5.1.6) corresponds again to (5.1.4) at $n = 2$ while (5.1.7) corresponds to (5.1.4) for $n \geq 4$. Again, regularity at $\mu = 0$ implies for $n \geq 4$

$$\frac{n-4}{n} g_{n,0} + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,0} = 0, \quad (5.1.8)$$

$$\frac{n-2}{n} g_{n,1} + 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,1} + g_{n,0} \left(2f_{2,0} + 1 - \frac{4}{n} \right) = 0. \quad (5.1.9)$$

We will follow the steps in Ch.3

- Choose the two-point function as in (3.2.68).
- Derive bounds on $g_{n,k}$ and $f_{2,k}$ for given on $g_{4,0}$ and $f_{2,0}$.
- Derive bounds on b_n implying that $f_2(\mu)$ is well-defined on $[0, \mu_{\max}]$.
- Show that all the $f_n(\mu)$ are well-defined on $[0, \mu_{\max}]$.

5.1.1.1 Bounds on the coefficients $g_{n,k}, f_{2,k}$

We bound the coefficients $g_{n,k}, f_{2,k}$. Since we considered a theory with arbitrarily large bare couplings, the required bounds have to be sharper than the ones derived in Lemmata 3.3-3.4. We start proving

Lemma 5.1. *Let f_n be solutions of (4.2.29) which respect the boundary conditions (5.1.2). For given $f_{2,0}, f_{4,0}$ we choose $K > 1$ sufficiently large such that*

$$|f_{2,0}| \leq \frac{\sqrt{K}}{4}, \quad |f_{4,0}| = |g_{4,0}| \leq \frac{\sqrt{K}}{32}. \quad (5.1.10)$$

Then

$$|f_{2,1}| \leq \frac{KN}{2}, \quad |g_{4,1}| \leq \frac{K}{32}, \quad (5.1.11)$$

and for $n \geq 6$

$$|g_{n,0}| \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2}, \quad |g_{n,1}| \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{n^2} \left(1 + \frac{nK}{2}\right). \quad (5.1.12)$$

Proof. From (5.1.6) we have

$$|f_{2,1}| = |(N+2)g_{4,0} + f_{2,0} - f_{2,0}^2| \leq \frac{KN}{2}. \quad (5.1.13)$$

and

$$|g_{4,1}| = 4|g_{4,0}f_{2,0}| \leq \frac{K}{32}.$$

We proceed by induction in n . For $n \geq 6$, we find from (5.1.8) and for K large enough

$$|g_{n,0}| \leq \frac{n}{n-4} \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+1-\frac{3}{2}}}{n_1^2(n+2-n_1)^2} \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2}.$$

From (5.1.9), for $n \geq 6$ and choosing $K > 4$

$$\begin{aligned} |g_{n,1}| &\leq \frac{2n}{n-2} \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+1-\frac{3}{2}}}{n_1^2(n+2-n_1)^2} \left(1 + \frac{n_2K}{2}\right) + \frac{n}{n-2} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2} \left(\frac{\sqrt{K}}{2} + 1 - \frac{4}{n}\right) \\ &\leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{n^2} \left(1 + \frac{nK}{2}\right). \end{aligned}$$

The previous bounds for the sums over n_1 can be checked for $n \leq 10$. For $n \geq 12$, we use Lemma C.1 in Appendix C.1. \square

Now we state our main result on bounds on $g_{n,k}$, $f_{2,k}$.

Proposition 5.1. *Under the same assumptions as in Lemma 5.1, we have for $n \geq 4$, $k \geq 2$, $N \geq 1$, for K sufficiently large*

$$|g_{n,k}| \leq N^{\frac{n}{2}+k-2} K^{\frac{n}{2}+k-\frac{3}{2}} \left|\frac{n}{4} + k - 3\right| \frac{1}{(k!)^{\frac{1}{4}}}, \quad |f_{2,k}| \leq N^{k+1} K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}}}. \quad (5.1.14)$$

Proof. See Appendix F.2. \square

In Sect.3.2.4, the bounds (3.2.62) in Lemma 3.4 imply that the coefficients $g_{n,k}$ and $f_{2,k}$ grow at most as $2^k k!$ for a fixed value of n . In (3.2.73) and from Lemma 3.4, the first term in the r.h.s. of (3.2.73) is bounded as follows

$$\frac{|f_{2,k}|}{(k+1)^k} \leq 2^k \frac{(k-1)!}{(k+1)^k} \varepsilon \leq \left(\inf\left(\frac{1}{4}, \left(\frac{3}{4}\right)^k\right)\right) \varepsilon, \quad k \geq 3. \quad (5.1.15)$$

Then the series (3.2.68) can only be shown to be convergent for small bare couplings.

Here we have shown that the coefficients $g_{n,k}$ and $f_{2,k}$ grow at most as $N^k K^k [k!]^{\frac{3}{4}}$, sharpening the asymptotic behavior in k . This improved behavior turns out to be crucial to show that the function $f_2(\mu)$ and then also the $f_n(\mu)$ are well-defined for $\mu \in [0, \mu_{\max}]$, see (5.1.19) and Propositions 5.2-5.3. Indeed the first term in the r.h.s. of (3.2.73) is now bounded by

$$N^{k+1} K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}} (k+1)^k}. \quad (5.1.16)$$

Because K can be chosen to be arbitrarily large in the assumptions of Lemma 5.1, the factor $\frac{1}{(|k-1|!)^{\frac{1}{4}}}$ is necessary to ensure that this term is bounded by a term of a convergent series, regardless of the value of K . This behavior has been conjectured due to the presence of the denominators $\frac{1}{n+2k}$ and $\frac{1}{k+1}$ in the r.h.s. of (5.1.6)-(5.1.7) which fall off with k . They are in fact the most important terms in the proof of Proposition 5.1. Upon examining the proof of Proposition 5.1, any denominator of the form $(|k-1|!)^s$ with $s \leq \frac{1}{4}$ would fit because this would not completely exhaust the denominators $\frac{1}{n+2k}$. The limiting case $s = \frac{1}{4}$ exhausted them in the proof of Proposition 5.1.

The inductive bound in Proposition 5.1 also depends geometrically on K . We had to choose K sufficiently large so that the inductive proof works, and this does not pose a problem from the point of view of the convergence of the series $\sum_{q \geq 1} b_q$ for the reasons mentioned above. The technical part of the proof of Proposition 5.1 was to bound the quadratic terms, namely the second and third terms in the r.h.s. of (5.1.7) and the last term in the r.h.s. of (5.1.6). To bound them, we established inequalities on the sums of the form

$$\sum_{\nu=a}^{k+2-b} \frac{|\frac{n_1}{4} + \nu - 3|! |\frac{n_2}{4} + k - \nu - 1|!}{[(k+2-\nu)! \nu!]^l}, \quad n_1, n_2 \in 2\mathbb{N}, a, b, k \in \mathbb{N}_0, l \in (0, 1) \quad (5.1.17)$$

which include the aforementioned quadratic terms. These inequalities are collected in Appendix F.1. Moreover, we had to modify the arguments of the factorials in the numerator of the bounds (5.1.14). Indeed in Sect.3.2.4, the factorial $(\frac{n}{2} + k - 2)!$ is chosen so that the last term in the r.h.s. of (3.2.49) is combined with the combinatorial factors and the induction hypothesis to get

$$\frac{n(n+1)(\frac{n}{2} + k - 1)!}{n+2k} \leq 2 \left(\frac{n}{2} + k\right)!. \quad (5.1.18)$$

But doing so does not take into account the k -dependence of the denominator. Therefore, we substitute the factor $\frac{n}{2}$ by $\frac{n}{4}$ so that in the last term in the r.h.s. of (3.2.49), we can use the k -dependence of the denominators $\frac{1}{n+2k}$. Another point is that the factor -2 inside the arguments of the factorials in the bounds (3.2.62) has been chosen so that in the proof of Lemma 3.4, the analysis of the quadratic term yields the correct bound. Upon establishing our bounds (5.1.14), this factor -2 had to be replaced by a factor -3 to repeat this step. Despite these improvements, arguments in the modulus of the bounds (5.1.14) become rational numbers and can be negative. Consequently, we had to pay attention to the signs of the arguments of the moduli in the proof of Proposition 5.1.

5.1.1.2 Bounds on the coefficients b_n

If we use the bound (5.1.14) in (3.2.73) we have for $n \geq 1$

$$|b_{n+1}| \leq c_{n,N} + \sum_{\rho=2}^{n+1} |b_{\{\frac{n+1}{\rho}\}}| \frac{1}{\rho^n}, \quad c_{n,N} := N^{n+1} K^{n+\frac{1}{2}} \frac{|n-3|!}{(|n-1|!)^{\frac{1}{4}} (n+1)^n}. \quad (5.1.19)$$

We set $C_N := \sum_{n \geq 0} c_{n,N} < +\infty$. Now we establish bounds on the b_n 's. We prove the following

Proposition 5.2. *There exists $C(N, K) > 0$ such that*

$$|b_n| \leq C(N, K) \frac{n^2}{2^n}, \quad n \geq 1. \quad (5.1.20)$$

Proof. The proof is done by induction in $n \in \mathbb{N}$. For $n = 1$, $b_1 = f_{2,0}$ and the bound is obtained choosing any constant $C(N, K) \geq \frac{N\sqrt{K}}{2}$. For $n \in \mathbb{N}$, we use (5.1.19) to obtain

$$\begin{aligned} |b_{n+1}| &\leq c_{n,N} + \sum_{\rho=2}^{n+1} C(N, K) \left(\frac{n+1}{\rho} \right)^2 \frac{1}{2^{\frac{n+1}{\rho}} \rho^n} \\ &\leq c_{n,N} + C(N, K) \frac{(n+1)^2}{2^{n+3}} + (n+1)^2 C(N, K) \sum_{\rho=3}^{n+1} \frac{1}{2^{\frac{n+1}{\rho}} \rho^{n+2}}. \end{aligned} \quad (5.1.21)$$

There exists a constant $\tilde{C}(N, K) > 0$ such that

$$c_{n,N} \leq \tilde{C}(N, K) \frac{(n+1)^2}{2^{n+3}}, \quad n \geq 1. \quad (5.1.22)$$

Moreover we have

$$\sum_{\rho=3}^{n+1} \frac{1}{2^{\frac{n+1}{\rho}} \rho^{n+2}} \leq \sum_{\rho=3}^{n+1} \frac{1}{\rho^{n+2}} \leq \int_2^{n+1} \frac{dx}{x^{n+2}} = \frac{1}{2^{n+1}(n+1)} - \frac{1}{(n+1)^{n+2}} \leq \frac{1}{2^{n+2}}. \quad (5.1.23)$$

From (5.1.22) and (5.1.23), we get

$$|b_{n+1}| \leq \left[\frac{\tilde{C}(N, K)}{4} + \frac{C(N, K)}{4} + \frac{C(N, K)}{2} \right] \frac{(n+1)^2}{2^{n+1}} \leq C(N, K) \frac{(n+1)^2}{2^{n+1}}, \quad (5.1.24)$$

if we choose $C(N, K) \geq \tilde{C}(N, K)$. □

In Sect.3, Kopper obtained the geometric bounds (3.2.75). Since ε plays a marginal role in (3.2.75), we thought that the bounds (3.2.75) could be extended to large bare couplings. Here we found bounds of the form $C' n^2 r^n$, $C' > 1$ and $r < 1$, and K is recast into an unspecified constant $C(N, K)$ which is sufficiently large. This is only possible due to the sharp bounds (5.1.14). Again, the constant K plays a marginal role in the bounds (5.1.20) which will imply triviality.

We can comment the second term in the r.h.s. of (3.2.73). This term mixes the preceding b_q for $q < n + 1$ and inverse powers of q . Therefore, we are restricted to geometric bounds. Upon examining the proof of Proposition 5.2, the geometric bounds $\frac{1}{2^n}$ is not the only possible choice but it is sufficient here to prove that the ansatz (3.2.68) is well-defined.

5.1.1.3 On the mean-field solutions constructed from the ansatz (3.2.68)

In order to show that the mean-field theory is well-defined, we need to prove first that actually all the derivatives of $f_2(\mu)$ from (3.2.68) are well-defined. We proved the following

Lemma 5.2. *For $l \geq 0$, $n > l + 1$, we have*

$$\left| \partial_\mu^l \frac{x_n^{n-1}}{1 + x_n^n} \right| \leq \frac{n^{n+l-1} \mu^{n-l-1}}{1 + x_n^n} C_l, \quad \mu \in [0, \mu_{\max}], \quad (5.1.25)$$

where the integers C_l are defined by $C_0 = 1$ and

$$C_{l+1} = 1 + \sum_{j=0}^l \binom{l+1}{j} C_j \leq 4^{l+1} (l+1)!. \quad (5.1.26)$$

Proof. We prove the Lemma by induction in $l \geq 0$.

- The case $l = 0$ is obvious.
- We use the following formula

$$\left(\frac{f}{g} \right)^{(l)} = \frac{1}{g} \left[f^{(l)} - l! \sum_{j=1}^l \frac{g^{(l+1-j)}}{(l+1-j)!} \frac{1}{(j-1)!} \left(\frac{f}{g} \right)^{(j-1)} \right], \quad (5.1.27)$$

for f, g smooth functions with $g > 0$. See Appendix C.2 for a proof. We have for $n > l + 2$ and $\mu \geq 0$

$$\begin{aligned} \left| \partial_\mu^{l+1} \frac{x_n^{n-1}}{1 + x_n^n} \right| &\leq \frac{1}{1 + x_n^n} \left[n^{n-1} \prod_{m=0}^l (n-1-m) \mu^{n-2-l} \right. \\ &\quad \left. + (l+1)! \sum_{j=1}^{l+1} \frac{1}{(l+2-j)! (j-1)!} n^n \prod_{m=0}^{l+1-j} (n-m) \mu^{n-l-2+j} \frac{n^{n+j-2} \mu^{n-j}}{1 + x_n^n} C_{j-1} \right] \\ &\leq \frac{1}{1 + x_n^n} \left[n^{n-1} n^{l+1} \mu^{n-l-2} + \mu^{n-l-2} \sum_{j=1}^{l+1} \binom{l+1}{j-1} n^{l+2-j} n^{n+j-2} C_{j-1} \right] \\ &\leq \frac{n^{n+l} \mu^{n-l-2}}{1 + x_n^n} \left[1 + \sum_{j=0}^l \binom{l+1}{j} C_j \right] \leq \frac{n^{n+l} \mu^{n-l-2}}{1 + x_n^n} C_{l+1}. \end{aligned}$$

The bound on C_l can be straightforwardly proven by induction in l . □

Now we can prove the main feature of the constructed mean-field theory, namely the mean-field smooth solutions $f_n(\mu)$ vanish in the UV-limit.

Proposition 5.3. • $f_2(\mu)$ is well defined on $[0, \mu_{\max}]$ and

$$\lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_2(\mu_{\max}) = 0, \quad l \geq 0. \quad (5.1.28)$$

• The functions $\partial_\mu^l f_n(\mu)$, $l \geq 0$, $n \geq 4$, are well defined on $[0, \mu_{\max}]$ and

$$\lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_n(\mu_{\max}) = 0, \quad n \geq 4, \quad l \geq 0. \quad (5.1.29)$$

Proof. As a consequence of Lemma 5.2, for $l \geq 0$, $n > l + 1$ and $\mu \geq 0$

$$\left| \partial_\mu^l \frac{x_n^{n-1}}{1+x_n^n} \right| \leq n^{2l} \frac{x_n^{n-l-1}}{1+x_n^n} C_l. \quad (5.1.30)$$

For $0 \leq l < n - 1$, the function $g(\mu) = \frac{(n\mu)^{n-l-1}}{1+(n\mu)^n}$ defined on \mathbb{R}_+ reaches its maximum at $\tilde{\mu} = \frac{1}{n} \left(\frac{n}{l+1} - 1 \right)^{\frac{1}{n}}$ and

$$\|g\|_\infty = g(\tilde{\mu}) = \frac{l+1}{n} \left(\frac{n}{l+1} - 1 \right)^{1-\frac{l+1}{n}} \leq 1. \quad (5.1.31)$$

Proposition 5.2 and the bounds (5.1.20) imply the uniform convergence of f_2 and its derivatives on $[0, \mu_{\max}]$. As a result we have

$$\forall l \geq 0, \forall m \geq 1, \quad \lim_{\mu \rightarrow +\infty} \partial_\mu^l \frac{x_m^{m-1}}{1+x_m^m} = 0 \quad \implies \quad \forall l \geq 0, \quad \lim_{\mu_{\max} \rightarrow +\infty} \partial_\mu^l f_2(\mu_{\max}) = 0. \quad (5.1.32)$$

The proof of the second statement is done by induction in $n+l$, going up in n . We can then easily check the case $n = 4$ using (4.2.29) and (5.1.28). For $n \geq 4$, we obtain by differentiating (4.2.29) l times w.r.t. μ

$$\partial_\mu^l f_{n+2} = \frac{2}{n(n+N)} \partial_\mu^{l+1} f_n + \frac{n-4}{n(n+N)} \partial_\mu^l f_n + \frac{1}{n+N} \sum_{n_1+n_2=n+2} \sum_{l_1+l_2=l} \binom{l}{l_1} \partial_\mu^{l_1} f_{n_1} \partial_\mu^{l_2} f_{n_2}. \quad (5.1.33)$$

Using the induction hypothesis, $\partial_\mu^l f_{n+2}(\mu)$ are well defined on $[0, \mu_{\max}]$ and they vanish when $\mu_{\max} \rightarrow +\infty$. \square

Collecting our findings, we can now state our existence result.

Theorem 5.1. Consider an $O(N)$ vector model φ_4^4 -theory of bare interaction lagrangian (5.1.1). Let $f_n(\mu)$ be smooth solutions of the mean-field flow equations (4.2.29). We consider the mean-field boundary conditions (5.1.2) with

$$0 < c_{0,4} < +\infty, \quad |c_{0,2}| < +\infty. \quad (5.1.34)$$

There exist smooth solutions of (4.2.29) $f_n(\mu) \in C^\infty([0, \mu_{\max}])$ such that they vanish in the UV-limit, i.e.

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0, \quad n \geq 2. \quad (5.1.35)$$

Proof. For $0 < c_{0,4} < +\infty$ and $|c_{0,2}| < +\infty$ fixed, there exists K such that

$$0 \leq c_{0,4} \leq \frac{\sqrt{K}}{2^7 \pi^2}, \quad |c_{0,2}| \leq \Lambda_0^2 \frac{\sqrt{K}}{2^7 \pi^4}, \quad \Lambda_0^{-2} = \alpha_0. \quad (5.1.36)$$

Using Lemma 5.1 and Proposition 5.2, we choose K such that the bounds (5.1.14) hold. Then we choose a smooth two-point function $f_2(\mu)$ as in (3.2.68). From Proposition 5.2, Lemma 5.2 and Proposition 5.3, the obtained smooth solutions $f_n(\mu)$ vanish in the UV-limit. \square

We conclude with a few remarks.

- The limit $\mu_{\max} \rightarrow +\infty$ or $\alpha_0 \rightarrow 0$ is equivalent to removing the UV-cutoff.
- From the point of view of statistical mechanics, a lattice with a spacing h is fixed. This spacing h corresponds to a fixed UV-cutoff. We can then interpret $\mu_{\max} \rightarrow +\infty$ as the limit $\alpha \rightarrow +\infty$ at α_0 fixed.
- The bounds we have derived in Proposition 5.1 could be sharpened, but they are sufficient to prove triviality. As mentioned earlier, the choice of the bounds (5.1.14) and (5.1.20) is not the only possible one. We think that the denominator $|k-1|!^{\frac{1}{4}}$ is close to optimality upon examining the proof of Proposition 5.1. Moreover, the bounds (5.1.20) could be sharpened by improving the common ratio of the geometric bound.
- So far we have only established a theorem stating the existence of trivial solutions. The uniqueness of the mean-field solutions for fixed mean-field boundary conditions is proven in the next section.

5.1.2 Uniqueness of the mean-field trivial solution

It is important to note that the bounds derived in Propositions 3.4-5.1 do not show that the ansatz (3.2.68) is analytic at $\mu = 0$. They only imply that the ansatz (3.2.68) belongs to the Gevrey-class, i.e. there exists $M_1 > 0, \sigma > 0$ such that

$$|\partial_\mu^l f_2(\mu)| \leq M_1^l [l!]^{1+\sigma} . \quad (5.1.37)$$

Here we would have $\sigma = \frac{1}{4}$. We think that the infinite set of constraints $f_n(0) = 0, n \geq 6$ implies that the solutions are non-analytic at $\mu = 0$ but this has not been analyzed in depth. Therefore, we cannot use our statements from Sect.4.3 to prove the uniqueness of the smooth mean-field solutions. To prove uniqueness, note that the flow equations have an integral formulation through the definition of the Wilson effective action in (1.2.8). This integral formulation implies the uniqueness of the solutions of the flow equations for fixed boundary conditions. In our setting, we applied the mean-field approximation after deriving the flow equations. It is not obvious that such an approximation breaks or preserves the initial integral formulation. This will be proven in the following.

5.1.2.1 The case $N = 1$

For simplicity, we restrict to the one-component case, $N = 1$. The more general case $N > 1$ can be treated analogously. The mean-field flow equations (4.2.29) can be derived by following Felder's steps [80] for $d = 4$. He first considered the continuum limit of the hierarchical model, introduced by Dyson [94]. In this model, the effective action at the scale $L^{-1}\lambda$, where $L > 1$ is a number, is related to the effective action at the scale λ by

$$e^{-u(L^{-1}\lambda, x)} = \int d\mu_L(y) e^{-L^4 u(\lambda, L^{-1}x+y)} , \quad \lambda \in (0, \Lambda_0] , \quad (5.1.38)$$

where μ_L is the one-dimensional Gaussian measure defined by

$$d\mu_L(y) := \frac{1}{\sqrt{2\pi(L-1)}} e^{-\frac{y^2}{2(L-1)}} dy. \quad (5.1.39)$$

Both the r.h.s. and the l.h.s. of (5.1.38) have a well defined limit when $L \rightarrow 1$, since the Gaussian measure μ_L becomes a Dirac measure in the limit $L \rightarrow 1$. Differentiating the l.h.s. of (5.1.38) and evaluating at $L = 1$ yields

$$\partial_L e^{-u(L^{-1}\lambda, x)} \Big|_{L=1} = \partial_\lambda u(\lambda, x) \quad (5.1.40)$$

while the r.h.s. of (5.1.38) becomes

$$\begin{aligned} \partial_L \left[\int d\mu_L(y) e^{-L^4 u(\lambda, L^{-1}x+y)} \right] \Big|_{L=1} &= -\frac{1}{2} \partial_{xx} u(\lambda, x) + \frac{1}{2} (\partial_x u(\lambda, x))^2 - 4u(\lambda, x) \\ &\quad + x \partial_x u(\lambda, x). \end{aligned} \quad (5.1.41)$$

Felder obtained the partial differential equation (3.2.3) in d dimensions using this method. Here we have $d = 4$. We assume that the effective action $u(\lambda, x)$ is twice differentiable w.r.t. x . The next step is to expand $u(\lambda, x)$ as a power series in x

$$u(\lambda, x) = \sum_{n \in 2\mathbb{N}} \frac{(2)^{\frac{n}{2}} f_n(\lambda)}{n} x^n. \quad (5.1.42)$$

The moments or the coefficients of the power series $f_n(\lambda)$ are solutions of the dynamical system

$$-\frac{1}{n(n+1)} \lambda \partial_\lambda f_n = f_{n+2} - \frac{1}{n+1} \sum_{n_1+n_2=n+2} f_{n_1} f_{n_2} + \frac{4}{n(n+1)} f_n - \frac{1}{n+1} f_n. \quad (5.1.43)$$

If we set $\lambda = \Lambda_0 e^{-\frac{\mu}{2}}$, we obtain the flow equations (4.2.29). In [80], Felder derived (3.2.3) in two ways:

- Simplifying Wilson's renormalization group equations [21, 22] using the local potential approximation [95].
- Considering the continuum limit of the recursion relation of the hierarchical model introduced by Gallavotti [96]. Above we proceeded in this way.

It is important to note that the momentum expansion (5.1.42) may not be valid for arbitrarily large $x \in \mathbb{R}$. We prove that within the momentum expansion, the mean-field moments $f_n(\mu)$ constructed in Sect. 5.1.1 permit us to recover the mean-field effective action $u(\lambda, x)$ by this expansion, i.e. $u(\lambda, x)$ is locally analytic w.r.t. x . We proceed as follows:

- We bound the derivatives $\partial_\mu^l f_2(\mu)$.
- We bound inductively $\partial_\mu^l f_n(\mu)$ using the mean-field flow equations (4.2.29).

We introduce the new variable $X := n\mu$ and we define

$$p_n(X) = \frac{X^{n-1}}{1 + X^n}. \quad (5.1.44)$$

Lemma 5.3. We have for $l \in \mathbb{N}_0$ and $n \in \mathbb{N}$

$$|\partial_X^l p_n(X)| \leq \begin{cases} \frac{l! 3^{l+1} e^{3l}}{n^{l+1} \mu^{2l+1}} & X \in (0, 3) \\ \frac{3^{l+1} l!}{\mu^{l+1} n^{l+1}} & X \geq 3. \end{cases} \quad (5.1.45)$$

Proof. For $X \in (0, 3)$, the proof is done by induction in $l \in \mathbb{N}_0$. The case $l = 0$ is obvious. For $l > 0$, we use (5.1.27). Inserting the induction hypothesis in the r.h.s. of (5.1.27) gives

$$\begin{aligned} |\partial_X^l p_n(\mu)| &\leq \frac{1}{1 + X^n} \left[\prod_{i=0}^{l-1} (n-1-i) X^{n-1-l} + l! \sum_{j=1}^l \frac{\prod_{i=0}^{l-j} (n-i) X^{n-l-1+j} (j-1)! 3^j e^{3(j-1)}}{(l+1-j)! (j-1)! n^j \mu^{2j-1}} \right] \\ &\leq \frac{l!}{n^{l+1}} \left[\frac{3^l}{\mu^{2l+1} l!} + 3^{l+1} \sum_{j=1}^l \frac{e^{3(j-1)}}{(l+1-j)! \mu^{2l+1}} \right] \\ &\leq \frac{l! 3^{l+1} e^{3l}}{n^{l+1} \mu^{2l+1}}. \end{aligned} \quad (5.1.46)$$

For $X \geq 3$, we expand $p_n(X)$ as a power series in $\frac{1}{X}$, then we have

$$\begin{aligned} |\partial_X^l p_n(X)| &\leq \sum_{k=0}^{\infty} \frac{(nk+l)!}{(nk)!} \frac{1}{X^{nk+l+1}} \\ &\leq \frac{2^l l!}{X^{l+1}} \sum_{k=0}^{\infty} \frac{2^{nk}}{X^{nk}} \leq \frac{3^{l+1} l!}{\mu^{l+1} n^{l+1}}. \end{aligned} \quad (5.1.47)$$

□

Now we can derive bounds for the derivatives of $f_2(\mu)$

Proposition 5.4. We have for a constant $K_1(1, K)$

$$|\partial_\mu^l f_2(\mu)| \leq \frac{[K_1(1, K)]^{l+1} l!}{M_l(\mu)}, \quad l \geq 0, \quad \mu \in (0, \mu_{\max}], \quad (5.1.48)$$

where we defined

$$M_l(\mu) := \min\{\mu^{2l+1}, \mu^l\}. \quad (5.1.49)$$

Proof. From Proposition 5.3 and Proposition 5.2

$$\begin{aligned} |\partial_\mu^l f_2(\mu)| &\leq C(1, K) \left[\sum_{n < \frac{3}{\mu}} \frac{n! 3^{l+1} e^{3l}}{2^n \mu^{2l+1}} + \sum_{n \geq \frac{3}{\mu}} \frac{n^2 l! 3^{l+1}}{2^n \mu^l} \right] \\ &\leq \frac{K_1(1, K)^{l+1} l!}{M_l(\mu)}, \end{aligned} \quad (5.1.50)$$

if we choose $K_1(1, K) > 24 C(1, K)$.

□

Now we prove bounds for the derivatives of $f_n(\mu)$. It is convenient to distinguish the two regimes $\mu < 1$ and $\mu \geq 1$. We have

Proposition 5.5. *Let $f_n(\mu)$ be smooth mean-field solutions of the flow equations (4.2.29) and we assume that the derivatives of the two-point function $\partial_\mu^l f_2(\mu)$ satisfy the bounds (5.1.48). Then we have for a constant $K_2(1, K) > K_1(1, K)$*

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K_2(1, K)^{n+l-1} (n+l)!}{(l+1)^2} \frac{1}{n!} \frac{1}{\mu^{2l+n-1}}, \quad n \geq 2, \quad l \geq 0, \quad \mu < 1, \quad (5.1.51)$$

and

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K_2(1, K)^{n+l-1} (n+l)!}{(l+1)^2} \frac{1}{n!}, \quad n \geq 2, \quad l \geq 0, \quad \mu \geq 1. \quad (5.1.52)$$

Proof. The proof is done by induction in $n+l$, going up in n . We will prove our statement for $0 < \mu < 1$. For $n=2$, bounds follow from (5.1.48) since $M_l(\mu) = \mu^{2l+1}$. For $n > 2$ we differentiate (4.2.29) l times w.r.t. μ and we insert the induction hypothesis. We get

$$\begin{aligned} |\partial_\mu^l f_{n+2}(\mu)| &\leq \frac{2}{n(n+1)} \frac{K_2(1, K)^{n+l} (n+l+1)!}{(l+2)^2} \frac{1}{n!} \frac{1}{\mu^{2l+2+n-1}} \\ &+ \frac{1}{n+1} \frac{K_2(1, K)^{n+l-1} (n+l)!}{(l+1)^2} \frac{1}{n!} \frac{1}{\mu^{2l+n-1}} \\ &+ \frac{K_2(1, K)^{n+l}}{n+1} \sum_{n_1+n_2=n+2} \sum_{l_1+l_2=l} \binom{l}{l_1} \frac{(n_1+l_1)! (n_2+l_2)!}{(l_1+1)^2 (l_2+1)^2} \frac{1}{n_1!} \frac{1}{n_2!} \mu^{2l+n}. \end{aligned} \quad (5.1.53)$$

We will proceed term by term.

- First term: it is bounded by

$$\frac{K_2(1, K)^{n+l+1} (n+l+2)!}{(l+1)^2} \frac{1}{(n+2)!} \frac{2}{\mu^{2l+n+1} K_2(1, K)}. \quad (5.1.54)$$

- Second term: since $\mu < 1$, it is bounded by

$$\frac{K_2(1, K)^{n+l+1} (n+l+2)!}{(l+1)^2} \frac{1}{(n+2)!} \frac{1}{\mu^{2l+n+1} 4K_2(1, K)^2}. \quad (5.1.55)$$

- Third term: We use the following inequality, which follows from the Vandermonde identity

$$\binom{l}{l_1} \binom{n+2}{n_1} \leq \binom{n+l+2}{n_1+l_1}. \quad (5.1.56)$$

Then we obtain

$$\binom{l}{l_1} \frac{(n_1+l_1)! (n_2+l_2)!}{n_1! n_2!} = \binom{l}{l_1} \binom{n+2}{n_1} \frac{(n_1+l_1)! (n_2+l_2)!}{(n+2)!} \leq \frac{(n+l+2)!}{(n+2)!}. \quad (5.1.57)$$

From Lemma C.4 we obtain the following bound

$$\frac{K_2(1, K)^{n+l+1} (n+l+2)!}{(l+1)^2} \frac{1}{(n+2)!} \frac{6}{\mu^{2l+n+1} K_2(1, K)}, \quad (5.1.58)$$

using again the fact that $0 < \mu < 1$.

Summing the three bounds (5.1.54), (5.1.55) and (5.1.58) yields

$$\begin{aligned} |\partial_\mu^l f_n(\mu)| &\leq \left[\frac{8}{K_2(1, K)} + \frac{1}{4K_2(1, K)^2} \right] \frac{K_2(1, K)^{n+l+1} (n+l+2)!}{(l+1)^2 (n+2)!} \frac{1}{\mu^{2l+n+1}} \\ &\leq \frac{K_2(1, K)^{n+l+1} (n+l+2)!}{(l+1)^2 (n+2)!} \frac{1}{\mu^{2l+n+1}}. \end{aligned} \quad (5.1.59)$$

For $\mu \geq 1$, we can bound the negative power of μ in (5.1.48) by 1 and then we proceed as above. \square

From Proposition 5.5, the mean-field effective action $u(\mu, x)$ is locally analytic w.r.t. x for $\mu > 0$. Its radius of convergence $R(\mu)$ is such that

$$R(\mu) \geq \frac{\min\{\mu, 1\}}{\sqrt{2}K_2(1, K)}, \quad \mu \in (0, \mu_{\max}]. \quad (5.1.60)$$

For $\mu = 0$, the boundary conditions (5.1.2) imply that $u(0, x)$ is polynomial in x , the solution $u(\mu, x)$ is well-defined for $\mu \in [0, \mu_{\max}]$. We claim that for fixed mean-field boundary conditions, the solution $u(\mu, x)$ is unique. Note that the r.h.s. of (5.1.60) goes to zero in the limit $\mu \rightarrow 0$ while it is constant when $\mu \geq 1$. This is in agreement with our expectation although it does not prove the non-analyticity of the mean-field solutions at $\mu = 0$.

Collecting our finding we can state the uniqueness theorem

Theorem 5.2 (Uniqueness of the mean-field trivial solution for the pure φ_4^4 -theory). *We consider the bare interaction lagrangian of a pure φ_4^4 -theory (5.1.1). For fixed mean-field boundary conditions (5.1.2), consider smooth mean-field solutions $f_n(\mu)$ of the mean-field flow equations (4.2.29) which satisfy (5.1.2) such that the corresponding mean-field effective action $u(\lambda, x)$ is locally analytic w.r.t. $x \in \mathbb{R}$. Then they are unique.*

Proof. For fixed mean-field boundary conditions (5.1.2), let $f_n(\mu)$ be the mean-field solutions of (4.2.29) constructed in Sect.5.1.1 which satisfy (5.1.2). Let $\tilde{f}_n(\mu)$ be solutions of the mean-field flow equations (4.2.29) which satisfy $f_n(0) = \tilde{f}_n(0)$. We assume that the corresponding mean-field effective action $\tilde{u}(\lambda, x)$ is locally analytic w.r.t. x for $\lambda < \Lambda_0$. Then $u(\Lambda_0, z) = \tilde{u}(\Lambda_0, z)$ for any arbitrary $z \in \mathbb{R}$ since they are polynomial in z . It follows from (5.1.38) that for a fixed $\lambda < \Lambda_0$, $u(\lambda, x) = \tilde{u}(\lambda, x)$ for $0 < |x| \leq \varepsilon_\lambda$, $\varepsilon_\lambda > 0$. This implies that $f_n(\mu) = \tilde{f}_n(\mu)$. \square

5.1.2.2 The case $N > 1$

The extension to the more general case $N > 1$ is done by considering the mean-field effective action $u(\lambda, \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^N$. We recall the Euclidean scalar product in \mathbb{R}^N

$$(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^N x_i y_i, \quad |\mathbf{y}|^2 := (\mathbf{y}, \mathbf{y}) = \sum_{i=1}^N y_i^2. \quad (5.1.61)$$

The relation (5.1.38) is generalized as follows

$$e^{-u(L^{-1}\lambda, \mathbf{x})} = \int d\mu_{N,L}(\mathbf{y}) e^{-L^4 u(\lambda, L^{-1}\mathbf{x} + \mathbf{y})}, \quad \lambda \in (0, \Lambda_0], \quad (5.1.62)$$

where $\mu_{N,L}$ is the N -dimensional Gaussian measure defined by

$$d\mu_{N,L}(\mathbf{y}) := \frac{1}{\sqrt{2\pi(L-1)}} e^{-\frac{|\mathbf{y}|^2}{2(L-1)}} \prod_{i=1}^N dy_i . \quad (5.1.63)$$

The generalization of the partial differential equation (3.2.3) is

$$-\lambda \partial_\lambda u = \frac{1}{2} \Delta u - \frac{1}{2} |\nabla u|^2 + 4u - (\mathbf{x}, \nabla u) . \quad (5.1.64)$$

Now we expand $u(\lambda, \mathbf{x})$ as a power series in $|\mathbf{x}|$ to get

$$u(\lambda, \mathbf{x}) = \sum_{n \in 2\mathbb{N}} \frac{(2)^{\frac{n}{2}} f_n(\lambda)}{n} |\mathbf{x}|^n . \quad (5.1.65)$$

If we set $\lambda = \Lambda_0 e^{-\frac{\mu}{2}}$, then we find again (4.2.29). Propositions 5.3-5.5 and Theorem 5.2 can then be extended to $N > 1$ since in the proofs of Propositions 5.4-5.5, only the constants $K_1(1, K)$, $K_2(1, K)$ have to be modified and the factor $\frac{1}{n+N}$ in the mean-field flow equations (4.2.29) is bounded by $\frac{1}{n+1}$. It may be possible to include a N -dependent term in the bounds but the bounds we have derived in Propositions 5.4-5.5 are sufficient to prove local analyticity.

5.2 The large N -limit

In this short section, we comment on the behavior of the mean-field solutions in the large N -limit and as a crosscheck, we especially wanted to know whether we could recover the behavior in N established in [87, 88].

In situations where $N \gg 1$, the large N -expansion in powers of $\frac{1}{N}$ is complementary to the perturbative expansion in the coupling constant. This expansion is carried out first by rescaling the coupling constant as $g \rightarrow g/N$. Using this new expansion, the universal properties of critical systems obtained in an expansion in $\varepsilon = 4 - d$ can be obtained at fixed dimension but in this expansion w.r.t. $\frac{1}{N}$ [88].

We study the following bare interaction lagrangian with the aforementioned rescaling

$$L_0(\varphi) = \int d^4x \left(c_{0,2} \varphi^2(x) + \frac{c_{0,4}}{N} \varphi^4(x) \right) . \quad (5.2.1)$$

Lemma 5.4. *Let f_n be solutions of (4.2.29) which respect the boundary conditions (5.1.2). We assume that for some K sufficiently large:*

$$|f_{2,0}| \leq \frac{\sqrt{K}}{4}, \quad |f_{4,0}| = |g_{4,0}| \leq \frac{\sqrt{K}}{32N} . \quad (5.2.2)$$

Then

$$|f_{2,1}| \leq \frac{K}{2}, \quad |g_{4,1}| \leq \frac{K}{32N} , \quad (5.2.3)$$

$$|g_{n,0}| \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2N^{\frac{n}{2}-1}n^2}, \quad |g_{n,1}| \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{N^{\frac{n}{2}-1}n^2} \left(1 + \frac{nK}{2} \right), \quad n \geq 6, \quad (5.2.4)$$

Furthermore

$$|g_{n,k}| \leq \frac{1}{N^{\frac{n}{2}-1}} K^{\frac{n}{2}+k-\frac{3}{2}} \left| \frac{n}{4} + k - 3 \right|! \frac{1}{(k!)^{\frac{1}{4}}}, \quad |f_{2,k}| \leq K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}}}, \quad n \geq 4, k \geq 0. \quad (5.2.5)$$

Proof. We have

$$|f_{2,1}| \leq \frac{N+2}{N} \frac{\sqrt{K}}{32} + \frac{\sqrt{K}}{4} + \frac{K}{16} \leq \frac{K}{2}, \quad |g_{4,1}| \leq 4 \frac{\sqrt{K}}{4} \frac{\sqrt{K}}{32N} = \frac{K}{32N}, \quad (5.2.6)$$

since $N \geq 1$. We can then proceed by induction in n , the r.h.s. of (5.1.8)-(5.1.9) give the correct bound w.r.t. N on $g_{n,0}$, $g_{n,1}$ by following the power counting of $\frac{1}{N}$. The proof of (5.2.5) is identical to the proof of Proposition 5.1. To check the claimed behavior w.r.t. N , we plug the inverse powers of N in the r.h.s. of (5.1.6)-(5.1.7) and we analyze power counting of $\frac{1}{N}$ appearing in the different terms. Upon performing this analysis, the non-trivial terms are the following

- In the r.h.s. of (5.1.6), the factor $(n+N)g_{n+2,k}$ produces a term $\frac{n+N}{N}$ which is obviously bounded by $n+1$.
- For the bound on $f_{2,k+1}$, the term $(N+2)g_{4,k}$ is bounded by $\frac{N+2}{N} \leq 3$.

□

Due to these bounds, the bounds (5.1.20) still hold and using the results in Sect.5.1.1 allow us to construct the trivial solution as before. The behavior in N of the bounds derived in Lemma 5.4 is actually sharp, and we see that the two-point function f_2 does not blow up w.r.t. N while the four-point function behaves as $\frac{1}{N}$ in the large N limit. This result is in agreement with the results obtained from partial, resummed perturbation theory [87, 88].

Chapter 6

The case of a theory without the technical IR-cutoff

In Sect.5.1.1, the trivial solution has been constructed using a technical IR cutoff supposed to take the role of the mass and set the mass equal to zero in the propagator and its derivative w.r.t. α . Here we will work with the true propagator of the massive theory. We choose a regularized flowing propagator which also preserves the analyticity properties w.r.t. α . We will again prove triviality of the mean-field massive φ_4^4 -theory but without the artificial IR cutoff we have introduced in Sect.5.1.1. We restrict to the case $N = 1$ to avoid overloading the proof with technicalities but it also goes through for $N > 1$. Most of the content of this chapter is in our paper [79].

We adapt the flow equations to the new scheme and we will construct the trivial solution from the same ansatz (3.2.68). We follow the steps as from Sect.5.1.1 to prove triviality. We also add a result we did not publish, namely that we can obtain solutions such that the mean-field effective action is locally analytic w.r.t. the field variable ϕ .

6.1 The mean-field flow equations

6.1.1 The flow equations

We fix $\alpha_0 < \frac{1}{2m^2}$. We choose the following regularized propagator in momentum space

$$\tilde{C}^{\alpha_0, \alpha}(p, m) = \frac{e^{-\alpha_0(p^2+m^2)} - e^{-\alpha(p^2+m^2)}(\frac{1}{m^2} + \alpha_0 - \alpha)m^2}{p^2 + m^2}, \quad \alpha \in \left[\alpha_0, \frac{1}{m^2} + \alpha_0 \right]. \quad (6.1.1)$$

It satisfies the following properties

$$\tilde{C}^{\alpha_0, \alpha_0}(p, m) = 0, \quad \lim_{\alpha_0 \rightarrow 0} \lim_{\alpha \rightarrow \frac{1}{m^2} + \alpha_0} \tilde{C}^{\alpha_0, \alpha}(p, m) = \frac{1}{p^2 + m^2}, \quad \tilde{C}^{\alpha_0, \alpha}(p, m) \geq 0. \quad (6.1.2)$$

At fixed α , $\tilde{C}^{\alpha_0, \alpha}(p, m)$ falls off exponentially. Moreover it is smooth and analytic w.r.t. α . For convenience, we set $\tilde{\beta}_0 := \alpha_0 m^2$ and $\tilde{\beta} := \alpha m^2$ so that $\tilde{\beta}$ is dimensionless. Then we can rewrite the

regularized propagator as

$$\tilde{C}^{\tilde{\beta}_0, \tilde{\beta}}(p, m) = \frac{e^{-\frac{\tilde{\beta}_0}{m^2}(p^2+m^2)} - e^{-\frac{\tilde{\beta}}{m^2}(p^2+m^2)}(1 + \tilde{\beta}_0 - \tilde{\beta})}{p^2 + m^2}, \quad \tilde{\beta} \in [\tilde{\beta}_0, 1 + \tilde{\beta}_0]. \quad (6.1.3)$$

We can proceed as in Sect 3.1 to obtain the flow equations. The procedure remains identical we will not detail it again. For the CAS functions in momentum space, we can write the corresponding flow equations as

$$\begin{aligned} \partial_{\tilde{\beta}} \mathcal{L}_n^{\tilde{\beta}_0, \tilde{\beta}}(p_1, \dots, p_n) &= \binom{n+2}{2} \int_k \dot{\tilde{C}}^{\tilde{\beta}}(k, m) \mathcal{L}_{n+2}^{\tilde{\beta}_0, \tilde{\beta}}(k, -k, p_1, \dots, p_n) \\ &\quad - \frac{1}{2} \sum_{n_1+n_2=n+2} n_1 n_2 \mathfrak{S} \left(\mathcal{L}_{n_1}^{\tilde{\beta}_0, \tilde{\beta}}(p_1, \dots, p_{n_1-1}, q) \dot{\tilde{C}}^{\tilde{\beta}}(q, m) \mathcal{L}_{n_2}^{\tilde{\beta}_0, \tilde{\beta}}(-q, p_{n_1}, \dots, p_n) \right), \end{aligned} \quad (6.1.4)$$

where $\dot{\tilde{C}}^{\tilde{\beta}}(k, m) = \partial_{\tilde{\beta}} \tilde{C}^{\tilde{\beta}_0, \tilde{\beta}}(k, m)$. If we compare the flow equations (3.1.5) to (6.1.4), only the derivative of the propagator has to be modified.

6.1.2 The mean-field limit

In the mean field approximation, we replace $\mathcal{L}_n^{\tilde{\beta}_0, \tilde{\beta}}(p_1, \dots, p_n)$ with its value at zero momenta $A_n^{\tilde{\beta}_0, \tilde{\beta}} := \mathcal{L}_n^{\tilde{\beta}_0, \tilde{\beta}}(0, \dots, 0)$ as in Sect.3.1. From (6.1.4), we write down the mean-field flow equations

$$\begin{aligned} \partial_{\tilde{\beta}} A_n^{\tilde{\beta}_0, \tilde{\beta}} &= \binom{n+2}{2} I(\tilde{\beta}) A_{n+2}^{\tilde{\beta}_0, \tilde{\beta}} \\ &\quad - \frac{1}{2m^2} e^{-\tilde{\beta}} (2 + \tilde{\beta}_0 - \tilde{\beta}) \sum_{n_1+n_2=n+2} n_1 n_2 A_{n_1}^{\tilde{\beta}_0, \tilde{\beta}} A_{n_2}^{\tilde{\beta}_0, \tilde{\beta}}, \quad \tilde{\beta} \in [\tilde{\beta}_0, 1 + \tilde{\beta}_0], \end{aligned} \quad (6.1.5)$$

where $I(\tilde{\beta}) := m^2 c (1 + \tilde{\beta}_0 - \tilde{\beta}) \frac{e^{-\tilde{\beta}}}{\tilde{\beta}^2} + m^2 \int_k \frac{e^{-\tilde{\beta}(k^2+1)}}{k^2+1}$. We recall that $c = \frac{1}{16\pi^2}$.

If we chose the regularized propagator (3.1.1), then we would have to substitute $I(\tilde{\beta})$ with $c \frac{e^{-\tilde{\beta}}}{\tilde{\beta}^2}$. Moreover if we tried bounding $A_{n+2}^{\tilde{\beta}_0, \tilde{\beta}}$ inductively from $A_{n'}^{\tilde{\beta}_0, \tilde{\beta}}$, $n' \leq n$ and from $\partial_{\tilde{\beta}} A_n^{\tilde{\beta}_0, \tilde{\beta}}$, we would obtain bounds, by dividing by $I(\tilde{\beta})$, which blow up for $\tilde{\beta} \rightarrow +\infty$ because of the presence of the factor $e^{-\tilde{\beta}}$ in $I(\tilde{\beta})$. Here, since we have chosen the regularized propagator (6.1.1), $\tilde{\beta}$ is limited by $1 + \tilde{\beta}_0$ instead of $\tilde{\beta} \rightarrow +\infty$.

Again we factor out the scaling factor $\tilde{\beta}^{\frac{n}{2}-2}$ and the combinatorial factor setting

$$A_n^{\tilde{\beta}_0, \tilde{\beta}} := \tilde{\beta}^{\frac{n}{2}-2} e^{\frac{\tilde{\beta}n}{2}} \frac{1}{m^{n-4}} \frac{1}{n} a_n(\tilde{\beta}), \quad \tilde{\beta} \in [\tilde{\beta}_0, 1 + \tilde{\beta}_0]. \quad (6.1.6)$$

For shortness, we omitted the upper index $\tilde{\beta}_0$ on the r.h.s. of (6.1.6). Then the mean-field flow equations (6.1.5) become

$$\begin{aligned} G(\tilde{\beta}) a_{n+2}(\tilde{\beta}) &= \frac{2}{n(n+1)} \tilde{\beta} \partial_{\tilde{\beta}} a_n(\tilde{\beta}) + \frac{n-4}{n(n+1)} a_n(\tilde{\beta}) + \frac{1}{n+1} \tilde{\beta} a_n(\tilde{\beta}) \\ &\quad + \frac{1}{n+1} (2 + \tilde{\beta}_0 - \tilde{\beta}) \sum_{n_1+n_2=n+2} a_{n_1}(\tilde{\beta}) a_{n_2}(\tilde{\beta}), \end{aligned} \quad (6.1.7)$$

where $G(\tilde{\beta}) := c(1 + \tilde{\beta}_0 - \tilde{\beta}) + \tilde{\beta}^2 e^{\tilde{\beta}} \int_k \frac{e^{-\tilde{\beta}(k^2+1)}}{k^2+1}$. We can express the integral appearing in the expression of $G(\tilde{\beta})$ as

$$\tilde{\beta}^2 e^{\tilde{\beta}} \int_k \frac{e^{-\tilde{\beta}(k^2+1)}}{k^2+1} = 2c\tilde{\beta} \int_0^{+\infty} \frac{u^3 e^{-u^2}}{u^2 + \tilde{\beta}} du ,$$

meaning that the limit $\tilde{\beta} \rightarrow 0$ is finite. It is straightforward that $G(\tilde{\beta}_0) = c + \mathcal{O}(\tilde{\beta}_0)$ when $\tilde{\beta}_0 \rightarrow 0$; $G(\tilde{\beta})$ takes a role analogous to that of c in the theory for $m = 0$, as can be seen comparing (6.1.7) to (3.1.9).

To follow the method in from Ch.3, we perform the following change of variable

$$\tilde{\mu} := \ln \left(\frac{\tilde{\beta}}{\tilde{\beta}_0} \right) \quad (6.1.8)$$

so that $\tilde{\beta} \partial_{\tilde{\beta}} = \partial_{\tilde{\mu}}$. Setting $f_n(\tilde{\mu}) = a_n(\tilde{\beta})$ we obtain correspondingly

$$\begin{aligned} H(\tilde{\mu}) f_{n+2}(\tilde{\mu}) &= \frac{2}{n(n+1)} \partial_{\tilde{\mu}} f_n(\tilde{\mu}) + \frac{n-4}{n(n+1)} f_n(\tilde{\mu}) + \frac{1}{n+1} \tilde{\beta}_0 e^{\tilde{\mu}} f_n(\tilde{\mu}) \\ &+ \frac{1}{n+1} (2 + \tilde{\beta}_0 (1 - e^{\tilde{\mu}})) \sum_{n_1+n_2=n+2} f_{n_1}(\tilde{\mu}) f_{n_2}(\tilde{\mu}) , \quad \tilde{\mu} \in [0, \tilde{\mu}_{\max}] , \end{aligned} \quad (6.1.9)$$

with $\tilde{\mu}_{\max} := \ln \left(1 + \frac{1}{\tilde{\beta}_0} \right)$ and

$$H(\tilde{\mu}) := G(\tilde{\beta}_0 e^{\tilde{\mu}}) = c(1 + \tilde{\beta}_0) - c\tilde{\beta}_0 e^{\tilde{\mu}} + h(\tilde{\mu}) , \quad (6.1.10)$$

where we set

$$h(\tilde{\mu}) = \tilde{\beta}_0^2 e^{2\tilde{\mu}} \int_k \frac{e^{-\tilde{\beta}_0 e^{\tilde{\mu}} k^2}}{k^2+1} . \quad (6.1.11)$$

At this stage, we were wondering whether we could absorb the smooth factor $H(\tilde{\mu})$ in a new redefinition of the CAS functions. Due to Lemma D.1, the factor $H(\tilde{\mu})$ is not singular and can be absorbed through a new change of function

$$\tilde{f}_n(\tilde{\mu}) = (H(\tilde{\mu}))^{\frac{n}{2}-1} f_n(\tilde{\mu}) . \quad (6.1.12)$$

Then, the dynamical system (6.1.9) can be rewritten in terms of $\tilde{f}_n(\tilde{\mu})$

$$\begin{aligned} \tilde{f}_{n+2}(\tilde{\mu}) &= \frac{2}{n(n+1)} \partial_{\tilde{\mu}} \tilde{f}_n(\tilde{\mu}) + \frac{n-4}{n(n+1)} \tilde{f}_n(\tilde{\mu}) - \frac{n-2}{n(n+1)} \left[\partial_{\tilde{\mu}} \ln(H(\tilde{\mu})) \right] \tilde{f}_n(\tilde{\mu}) \\ &+ \frac{1}{n+1} \tilde{\beta}_0 e^{\tilde{\mu}} \tilde{f}_n(\tilde{\mu}) + \frac{1}{n+1} (2 + \tilde{\beta}_0 (1 - e^{\tilde{\mu}})) \sum_{n_1+n_2=n+2} \tilde{f}_{n_1}(\tilde{\mu}) \tilde{f}_{n_2}(\tilde{\mu}) . \end{aligned} \quad (6.1.13)$$

It is convenient to explicitly write the flow equations for $n = 2$ and for $n \geq 4$

$$\tilde{f}_4(\tilde{\mu}) = \frac{1}{3} \left(\partial_{\tilde{\mu}} \tilde{f}_2(\tilde{\mu}) - \tilde{f}_2(\tilde{\mu}) + \tilde{\beta}_0 e^{\tilde{\mu}} \tilde{f}_2(\tilde{\mu}) + (2 + \tilde{\beta}_0 (1 - e^{\tilde{\mu}})) \tilde{f}_2^2(\tilde{\mu}) \right) , \quad (6.1.14)$$

$$\begin{aligned}
\tilde{f}_{n+2}(\tilde{\mu}) &= \frac{2}{n(n+1)} \partial_{\tilde{\mu}} \tilde{f}_n(\tilde{\mu}) + \frac{n-4}{n(n+1)} \tilde{f}_n(\tilde{\mu}) - \frac{n-2}{n(n+1)} \left[\partial_{\tilde{\mu}} \ln(H(\tilde{\mu})) \right] \tilde{f}_n(\tilde{\mu}) \\
&+ \frac{1}{n+1} \tilde{\beta}_0 e^{\tilde{\mu}} \tilde{f}_n(\tilde{\mu}) + \frac{2}{n+1} (2 + \tilde{\beta}_0 (1 - e^{\tilde{\mu}})) \tilde{f}_2(\tilde{\mu}) \tilde{f}_n(\tilde{\mu}) \\
&+ \frac{1}{n+1} (2 + \tilde{\beta}_0 (1 - e^{\tilde{\mu}})) \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{f}_{n_1}(\tilde{\mu}) \tilde{f}_{n_2}(\tilde{\mu}), \quad n \geq 4.
\end{aligned} \tag{6.1.15}$$

If we compare the flow equations (6.1.14)-(6.1.15) to the flow equations (3.1.11), the former include additional $\tilde{\mu}$ -dependent terms. Nevertheless they retain a similar form, so we expect the triviality statement to remain valid as long as the $\tilde{\mu}$ -dependent terms are analytic w.r.t. $\tilde{\mu}$. We collect a few lemmas on the derivatives of these terms in Appendix D. They show that the functions $h(\tilde{\mu})$, $H(\tilde{\mu})$ and $\ln(H(\tilde{\mu}))$ are well-defined on $[0, \tilde{\mu}_{\max}]$ and locally analytic w.r.t. $\tilde{\mu}$.

6.1.3 The analyticity of the mean-field effective action in the new setting

Before turning to the triviality statement, we wanted first as a crosscheck, to verify that an analogous proposition to Proposition 3.1 can be stated in our setting. This would comfort our idea that the approximation $m = 0$ in the propagator is purely technical and should not affect the key properties of the theory. This result has not been published in our paper [79]. We will here study bounded mean field solutions of (6.1.9).

We have proved a first (weaker) version of Lemma D.2

Lemma 6.1. For $l \geq 0$ and $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$

$$|\partial_{\tilde{\mu}}^l h(\tilde{\mu})| \leq 5^l (l+1)^{l-1} c. \tag{6.1.16}$$

Proof. See Appendix D. □

Now we consider a smooth two-point function $f_2(\tilde{\mu})$ satisfying

$$\begin{aligned}
-K_1 \delta \leq f_2(0) \leq -\delta < 0, \\
|\partial_{\tilde{\mu}}^l f_2(\tilde{\mu})| \leq K_1^l \delta^{l+1} |l-2|^{l-2}, \quad l \geq 0.
\end{aligned} \tag{6.1.17}$$

for $0 < \delta \leq 1$ and $K_1 > 1$ some fixed constants.

Proposition 6.1. For a suitable $K > \sup(K_1, \frac{1}{\delta}, C)$ with C the constant in Lemma D.1 and $f_2(\tilde{\mu})$ satisfying (6.1.17), the mean field solutions $f_n(\tilde{\mu})$ of (6.1.9) are smooth and for $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$, $n \geq 2$ and $l \geq 0$,

$$|\partial_{\tilde{\mu}}^l f_n(\tilde{\mu})| \leq K^{n+l-2} 5^l \delta^{l+1} \frac{|n+l-4|^{n+l-4}}{(n-2)!}. \tag{6.1.18}$$

Remark. • We actually have

$$|\partial_{\tilde{\mu}}^l f_n(\tilde{\mu})| \leq \tilde{K}^{n+l} \frac{(n+l)!}{n!}, \tag{6.1.19}$$

for some constant \tilde{K} . They follow from the bounds (6.1.18) and the following bounds on $n!$ established by Robbins [97]

$$\sqrt{2\pi n} n^n e^{-n} e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n} n^n e^{-n} e^{\frac{1}{12n}}. \tag{6.1.20}$$

The bounds (6.1.19) are in close analogy to the bounds (3.2.7) which were established for $m = 0$.

- Assume that $f_2(\tilde{\mu})$ can be analytically continued to complex values of $\tilde{\mu}$ in a strip of width $\frac{2}{5K\delta e}$ around the real $\tilde{\mu}$ -axis $[0, \tilde{\mu}_{\max}]$. Then the same holds for $f_n(\tilde{\mu})$.

Proof. See Appendix E.2. □

There is an important consequence of Proposition 6.1. If we go back to the dynamical system (6.1.5) whose solutions are the $A_n^{\tilde{\beta}_0, \tilde{\beta}}$ and we assume that all the $f_n(\tilde{\mu})$ satisfy the bounds (6.1.18), then we obtain a uniform bound in $\tilde{\beta}_0$ for $A_n^{\tilde{\beta}_0, \tilde{\beta}}$

$$|A_n^{\tilde{\beta}_0, \tilde{\beta}}| \leq \frac{\delta e^\alpha}{n\alpha} (\alpha K^2 e^\alpha)^{\frac{n-2}{2}}, \quad \alpha \in [\tilde{\beta}_0, 1 + \tilde{\beta}_0], \quad n \geq 4. \quad (6.1.21)$$

Then the mean field action

$$L_{mf}^{\tilde{\beta}_0, \alpha}(\phi) = \sum_{n \in 2\mathbb{N}} A_n^{\tilde{\beta}_0, \alpha} \phi^n \quad (6.1.22)$$

is analytic w.r.t. ϕ in a disk of radius $\left[\frac{e^{-\alpha}}{\alpha K^2}\right]^{\frac{1}{2}}$.

6.2 The triviality of the mean-field φ_4^4 -theory

For the triviality proof we actually follow the method in Sect.5.1.1. We expect the triviality statement to be proven similarly despite the change of propagator and the modified flow equations. Due to Lemma D.4, $\partial_\mu \ln(H(\mu))$ is locally analytic around $\mu = 0$. For $|\mu|$ sufficiently small

$$\partial_{\tilde{\mu}} \ln(H(\tilde{\mu})) = \sum_{k \geq 0} h_k \tilde{\mu}^k, \quad h_k = \frac{\partial_{\tilde{\mu}}^{k+1} \ln(H(\tilde{\mu}))|_{\tilde{\mu}=0}}{k!}, \quad |h_k| \leq c C^{k+1} (5e)^{k+2} 2^{k+1}. \quad (6.2.1)$$

The bare mean-field boundary conditions for $\tilde{f}_n(\mu)$ become

$$\tilde{f}_2(0) = 2(2\pi)^4 \alpha_0 e^{-\beta_0} c_{0,2}, \quad \tilde{f}_4(0) = (2\pi)^4 e^{-2\beta_0} H(0) c_{0,4}, \quad \tilde{f}_n(0) = 0, \quad n \geq 6. \quad (6.2.2)$$

Again we will factor out a power of $\tilde{\mu}$ in $\tilde{f}_n(\tilde{\mu})$ for $n \geq 4$.

Lemma 6.2. For smooth solutions $\tilde{f}_n(\tilde{\mu})$ of (6.1.15) with boundary conditions (5.1.2), we have

$$\partial_{\tilde{\mu}}^l \tilde{f}_n(0) = 0, \quad n \geq 6, \quad 0 \leq l \leq \frac{n}{2} - 3. \quad (6.2.3)$$

Proof. The proof is done by induction in $N = n + 2l$, going up in l . We start at $N = 6$ and we have from the boundary conditions (6.2.2)

$$\tilde{f}_6(0) = 0.$$

For $0 \leq l < \frac{n}{2} - 3$, we use (6.1.13) and we solve it for $\partial_{\tilde{\mu}}^{l+1} \tilde{f}_n(0)$. Using the induction hypothesis, we obtain $\partial_{\tilde{\mu}}^{l+1} \tilde{f}_n(0) = 0$ since in the products

$$\partial_{\tilde{\mu}}^{l_1} \tilde{f}_{n_1}(0) \partial_{\tilde{\mu}}^{l_2} \tilde{f}_{n_2}(0),$$

the constraints $n_1 + n_2 = n + 2$ and $l_1 + l_2 \leq l$ and $l < \frac{n}{2} - 3$ imply that either $l_1 \leq \frac{n_1}{2} - 3$ or $l_2 \leq \frac{n_2}{2} - 3$. □

From Lemma 6.2, we can write

$$\tilde{g}_n(\tilde{\mu}) = \tilde{\mu}^{2-\frac{n}{2}} \tilde{f}_n(\tilde{\mu}), \quad n \geq 4, \quad (6.2.4)$$

where $\tilde{g}_n(\tilde{\mu})$ is smooth w.r.t. $\tilde{\mu}$. The flow equations (6.1.15) in terms of $\tilde{g}_n(\tilde{\mu})$ read

$$\begin{aligned} \tilde{\mu}^2 \tilde{g}_{n+2} &= \frac{n-4}{n(n+1)} \tilde{g}_n + \frac{2}{n(n+1)} \tilde{\mu} \partial_{\tilde{\mu}} \tilde{g}_n + \frac{1}{n+1} (2 + \beta_0 - \beta_0 e^{\tilde{\mu}}) \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{g}_{n_1} \tilde{g}_{n_2} \\ &+ \tilde{\mu} \frac{1}{n+1} \tilde{g}_n \left(2(2 + \beta_0 - \beta_0 e^{\tilde{\mu}}) \tilde{f}_2 + \beta_0 e^{\tilde{\mu}} + 1 - \frac{4}{n} - \left(1 - \frac{2}{n}\right) \partial_{\tilde{\mu}} \ln(H(\tilde{\mu})) \right). \end{aligned} \quad (6.2.5)$$

Again, we perform a formal Taylor expansion of \tilde{f}_2 and \tilde{g}_n around $\tilde{\mu} = 0$, namely

$$\tilde{f}_2(\tilde{\mu}) = \sum_{k \geq 0} \tilde{f}_{2,k} \tilde{\mu}^k, \quad \tilde{g}_n(\tilde{\mu}) = \sum_{k \geq 0} \tilde{g}_{n,k} \tilde{\mu}^k. \quad (6.2.6)$$

From (6.2.6), (6.2.1) and the flow equations (6.1.14), (6.1.15), the coefficients of the formal Taylor expansion of \tilde{f}_2 and \tilde{g}_n are related through the following inductive system

$$\begin{aligned} \tilde{f}_{2,k+1} &= \frac{1}{k+1} \left(3\tilde{g}_{4,k} + \tilde{f}_{2,k} - (2 + \beta_0) \sum_{\nu=0}^k \tilde{f}_{2,\nu} \tilde{f}_{2,k-\nu} - \beta_0 \sum_{\nu=0}^k \frac{1}{(k-\nu)!} \tilde{f}_{2,\nu} \right. \\ &\left. + \beta_0 \sum_{\nu=0}^k \frac{1}{\nu!} \sum_{\nu'=0}^{k-\nu} \tilde{f}_{2,\nu'} \tilde{f}_{2,k-\nu-\nu'} \right). \end{aligned} \quad (6.2.7)$$

$$\begin{aligned} \tilde{g}_{n,k+2} &= \frac{n(n+1)}{n+2k} \tilde{g}_{n+2,k} - \frac{n-4}{n+2k} \tilde{g}_{n,k+1} + \frac{n-2}{n+2k} \sum_{\nu=0}^{k+1} \tilde{g}_{n,\nu} h_{k+1-\nu} \\ &- \frac{\beta_0 n}{n+2k} \sum_{\nu=0}^{k+1} \frac{1}{(k+1-\nu)!} \tilde{g}_{n,\nu} - \frac{(2 + \beta_0)n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} \tilde{g}_{n_1,\nu} \tilde{g}_{n_2,k+2-\nu} \\ &+ \frac{\beta_0 n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} \frac{1}{\nu!} \sum_{\nu'=0}^{k+2-\nu-\nu'} \tilde{g}_{n_1,\nu'} \tilde{g}_{n_2,k+2-\nu-\nu'} \\ &- 2 \frac{(2 + \beta_0)n}{n+2k} \sum_{\nu=0}^{k+1} \tilde{g}_{n,\nu} \tilde{f}_{2,k+1-\nu} + \frac{2\beta_0 n}{n+2k} \sum_{\nu=0}^{k+1} \frac{1}{\nu!} \sum_{\nu'=0}^{k+1-\nu} \tilde{g}_{n,\nu'} \tilde{f}_{2,k+1-\nu-\nu'}. \end{aligned} \quad (6.2.8)$$

Regularity at $\tilde{\mu} = 0$ implies that

$$0 = \frac{n-4}{n} \tilde{g}_{n,0} + 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{g}_{n_1,0} \tilde{g}_{n_2,0}. \quad (6.2.9)$$

and

$$\begin{aligned}
0 = & \frac{n-2}{n} \tilde{g}_{n,1} + \frac{n-4}{n} \tilde{g}_{n,0} + \beta_0 \tilde{g}_{n,0} - \frac{n-2}{n} h_0 \tilde{g}_{n,0} + 4 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{g}_{n_1,0} \tilde{g}_{n_2,1} \\
& - \beta_0 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{g}_{n_1,0} \tilde{g}_{n_2,0} + 4 \tilde{g}_{n,0} \tilde{f}_{2,0}.
\end{aligned} \tag{6.2.10}$$

Using (6.2.9), (6.2.10) can be recast as

$$\frac{n-2}{n} \tilde{g}_{n,1} + 4 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \tilde{g}_{n_1,0} \tilde{g}_{n_2,1} + \tilde{g}_{n,0} \left[4 \tilde{f}_{2,0} + \left(1 - \frac{4}{n}\right) \left(1 + \frac{\beta_0}{2}\right) + \beta_0 - \left(1 - \frac{2}{n}\right) h_0 \right] = 0. \tag{6.2.11}$$

As in Sect.5.1.1.1, bounds on $\tilde{g}_{n,k}$ and $\tilde{f}_{2,k}$ will be derived.

6.2.0.1 Bounds on $\tilde{g}_{n,k}$ and $\tilde{f}_{2,k}$

Lemma 6.3. *Let \tilde{f}_n be smooth solutions of (6.1.14), (6.1.15). For given $\tilde{f}_{2,0}, \tilde{f}_{4,0}$ we choose K sufficiently large such that*

$$|\tilde{f}_{2,0}| \leq \frac{\sqrt{K}}{16}, \quad |\tilde{f}_{4,0}| = |\tilde{g}_{4,0}| \leq \frac{\sqrt{K}}{32}. \tag{6.2.12}$$

Then

$$|\tilde{f}_{2,1}| \leq \frac{K}{2}, \quad |\tilde{g}_{4,1}| \leq \frac{K}{32}, \tag{6.2.13}$$

and for $n \geq 4$

$$|\tilde{g}_{n,0}| \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2}, \quad |\tilde{g}_{n,1}| \leq \frac{K^{\frac{n}{2}-\frac{1}{2}}}{n}. \tag{6.2.14}$$

Proof. From (6.2.7) we have

$$|\tilde{f}_{2,1}| = |3\tilde{g}_{4,0} + (1 - \beta_0)\tilde{f}_{2,0} - 2\tilde{f}_{2,0}^2| \leq \frac{K}{2},$$

and from (6.2.11)

$$|\tilde{g}_{4,1}| = 2|\tilde{g}_{4,0}| \left| 4\tilde{f}_{2,0} + \beta_0 - \frac{1}{2}h_0 \right| \leq \frac{K}{32}$$

choosing $\sqrt{K} > 7cC > 4$ since we have the sharper bound

$$h_0 \leq 7cC,$$

which can be easily obtained from the explicit expression of H .

We proceed by induction in n . For $n \geq 6$, we find from (6.2.9) and from (6.2.11) for K large enough

$$|\tilde{g}_{n,0}| \leq \frac{n-1}{n-4} \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+1-\frac{3}{2}}}{n_1^2(n+2-n_1)^2} \leq \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2},$$

$$|\tilde{g}_{n,1}| \leq \frac{2n}{n-2}(n-2) \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+2-\frac{3}{2}}}{n_1^2(n+2-n_1)^2} + \frac{n}{n-2} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2} \left(\frac{\sqrt{K}}{4} + 2 + \frac{1}{2}7cC \right) \leq \frac{K^{\frac{n}{2}-\frac{1}{2}}}{n}.$$

For $n \leq 10$ the previous bounds for the sum over n_1 can be checked explicitly, for $n \geq 12$ we use Lemma C.1 in Appendix C.1. \square

Proposition 6.2. *Under the same assumptions as in Lemma 6.3, choosing K large enough we have*

$$|\tilde{g}_{n,k}| \leq K^{\frac{n}{2}+k-\frac{3}{2}} \left| \frac{n}{4} + k - 3 \right|! \frac{1}{(k!)^{\frac{1}{8}}}, \quad |\tilde{f}_{2,k}| \leq K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{8}}}, \quad n \geq 4, k \geq 0. \quad (6.2.15)$$

Proof. See Appendix G. \square

We see that the bounds (6.2.15) are similar to our bounds (5.1.14). The slight difference lies in the denominator, we have $(k!)^{\frac{1}{8}}$ instead of $(k!)^{\frac{1}{4}}$. When comparing the mean-field flow equations for the massive theory (6.2.8) and the mean-field flow equations (3.2.49) for $m = 0$:

- The mean-field dynamical system (6.2.8) include additional terms.
- Unlike the mean-field flow equations (3.2.49), the coefficients in (6.2.8) are $\tilde{\mu}$ -dependent.

Therefore, it is expected that the upper bounds on $\tilde{g}_{n,k}$ and $\tilde{f}_{2,k}$ to be greater than those of Proposition 5.1. Moreover, from (6.2.15), the Taylor coefficients $\tilde{g}_{n,k}$ and $\tilde{f}_{2,k}$ obey the simpler bounds

$$|\tilde{g}_{n,k}| \leq K^{\frac{n}{2}+k} \left(\frac{n}{4} + k \right)! \frac{1}{(k!)^{\frac{1}{8}}}, \quad |\tilde{f}_{2,k}| \leq K^k (k!)^{\frac{7}{8}}, \quad (6.2.16)$$

for a sufficiently large constant K . The bounds (6.2.16) are again sharper than the bounds (5.1.14) for the same reasons exposed in Sect.5.1.1.1. They allow us to prove that the ansatz (3.2.68) is well-defined for arbitrarily large bare couplings. However, as in Sect.5.1.1, the bounds (6.2.16) only imply that the ansatz (3.2.68) belongs to the Gevrey-class (5.1.37). Here we would have $\sigma = \frac{1}{8}$.

6.2.0.2 Existence of a mean-field trivial solution for the massive theory

We choose a smooth two-point function of the form (3.2.68). It turns out that

- The sequence $(b_n)_{n \geq 1}$ satisfies bounds of the same type as (5.1.20) ($N = 1$ in our setting).
- Both Lemma 5.2 and Proposition 5.3 hold in our setting.
- The solutions $\tilde{f}(\tilde{\mu})$ are all well-defined on $[0, \tilde{\mu}^{\max}]$ and they vanish again in the UV-limit.

Therefore the extension of Theorem 5.1 to the massive theory is straightforward. Collecting our findings we have

Theorem 6.1 (Triviality of pure mean-field φ^4 -theory for the theory with a physical IR cutoff). Consider the φ^4 -theory of bare interaction lagrangian (5.1.1) for $N = 1$. Let $\tilde{f}_n(\mu)$ be smooth solutions of the mean-field flow equations (6.1.13) and the corresponding mean-field boundary conditions (6.2.2) with

$$0 < c_{0,4} < +\infty, \quad |c_{0,2}| < +\infty . \quad (6.2.17)$$

There exist smooth solutions of (6.1.13) $\tilde{f}_n(\tilde{\mu}) \in C^\infty([0, \tilde{\mu}_{\max}])$ such that they vanish in the UV-limit, i.e.

$$\lim_{\tilde{\mu}_{\max} \rightarrow +\infty} \tilde{f}_n(\tilde{\mu}_{\max}) = 0, \quad n \geq 2 . \quad (6.2.18)$$

Proof. The proof is the same as for Theorem 5.1. □

The uniqueness of the solutions can be proven following the reasoning in Sect.5.1.2. The differences remain purely technical as the coefficients in the r.h.s. of (6.1.13) are analytic functions in the variable $\tilde{\mu}$.

Chapter 7

Triviality versus perturbation theory

Now we will focus on the second part of this thesis. So far, our approach was non-perturbative, i.e. the solutions $f_n(\mu)$ were not expanded as a formal power series in a renormalized coupling g . The purpose of this chapter is to relate the trivial solution we have constructed in Sect.5.1.1 to perturbation theory. First, we present the perturbative mean-field flow equations and find bounds on the perturbative mean-field CAS functions. In Sect.8.2, we use them to prove the local existence of the Borel transform of the regularized renormalized mean-field perturbative φ_4^4 -theory. We also show that specific renormalization conditions which include the BPHZ renormalization conditions can be imposed for our mean-field trivial solutions. Then, we prove that at a fixed UV-cutoff α_0 , the mean-field trivial smooth solutions we have constructed can be expanded in a power series in a renormalized coupling g . A preprint contains the results of the next two chapters [98].

7.1 The perturbative mean-field flow equations

We derive the perturbative mean-field flow equations. We keep the notations from Sect.3.1. Here we will restrict to the case of a one-component real scalar field.

7.1.1 The mean-field flow equations for the perturbative solutions

We recall that the mean-field flow equations (4.2.29) are non-perturbative, i.e. the Schwinger functions are not expanded in a formal power series in a coupling g . Before carrying out a formal power series expansion, it is instructive to first define the renormalized coupling g .

Going back to the ansatz (3.2.68) and the mean-field flow equations (3.1.11), we find generically in agreement with Proposition 5.3 for the asymptotic behavior of $f_2(\mu)$ and $f_4(\mu)$

$$f_2(\mu_{\max}) = \mathcal{O}\left(\frac{1}{\mu_{\max}}\right), \quad f_4(\mu_{\max}) = \mathcal{O}\left(\frac{1}{\mu_{\max}}\right), \quad f_{2n}(\mu_{\max}) = \mathcal{O}\left(\frac{1}{\mu_{\max}^2}\right), \quad n \geq 3. \quad (7.1.1)$$

Now we define

$$g := f_4(\mu_{\max}). \quad (7.1.2)$$

This corresponds to the standard definition of the renormalized coupling g in terms of the truncated four-point function. From the mean-field flow equations (3.1.9)-(3.1.11) and the ansatz (3.2.68), the renormalized CAS functions $f_n(\mu_{\max})$ can be expanded in powers of $\frac{1}{\mu_{\max}}$ or g . For $A_n^{\alpha_0, \alpha}$, the standard expansion in a power series w.r.t. g is

$$A_n^{\alpha_0, \alpha} = \sum_{j=1}^K g^j A_{n,j}^{\alpha_0, \alpha} + g^{K+1} \Delta A_{n, K+1}^{\alpha_0, \alpha}, \quad (7.1.3)$$

where $A_{n,j}^{\alpha_0, \alpha}$ are smooth w.r.t. α , and $g^{K+1} \Delta A_{n, K+1}^{\alpha_0, \alpha}$ is the remainder of the finite perturbative expansion of $A_n^{\alpha_0, \alpha}$. The mean-field CAS functions $A_{n,j}^{\alpha_0, \alpha}$ satisfy the following properties:

- $A_{n,j}^{\alpha_0, \alpha} \equiv 0$ if n is odd (\mathbb{Z}_2 -symmetry).
- $A_{n,j}^{\alpha_0, \alpha} \equiv 0$ if $n > 2j + 2$ since only connected amplitudes contribute (connectedness).

Note that $\mathcal{A}_{2,0}^{\alpha_0, \alpha} \equiv 0$ so that we can start an inductive scheme to derive bounds on $\partial_\alpha^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}$ for $m \geq 0$, $j \geq 1$, $n \geq 2$.

From the perturbative expansion (7.1.3), the mean-field flow equations (3.1.9) give a system of mean-field flow equations for the coefficients of the (formal) perturbative expansion $A_{n,j}^{\alpha_0, \alpha}$. They read

$$\partial_\alpha A_{n,j}^{\alpha_0, \alpha} = \binom{n+2}{2} c_\alpha A_{n+2,j}^{\alpha_0, \alpha} - \frac{1}{2} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i}} n_1 n_2 A_{n_1, j_1}^{\alpha_0, \alpha} A_{n_2, j_2}^{\alpha_0, \alpha}. \quad (7.1.4)$$

We can factor out combinatorial factors as follows

$$\mathcal{A}_{n,j}^{\alpha_0, \alpha} := c^{\frac{n}{2}-1} n A_{n,j}^{\alpha_0, \alpha}, \quad (7.1.5)$$

so that we obtain

$$\partial_\alpha \mathcal{A}_{n,j}^{\alpha_0, \alpha} = \frac{n(n+1)}{2\alpha^2} \mathcal{A}_{n+2,j}^{\alpha_0, \alpha} - \frac{n}{2} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i}} \mathcal{A}_{n_1, j_1}^{\alpha_0, \alpha} \mathcal{A}_{n_2, j_2}^{\alpha_0, \alpha}. \quad (7.1.6)$$

Now we proceed similarly as in [6]; we derive bounds on the mean-field perturbative CAS function $\mathcal{A}_{n,j}^{\alpha_0, \alpha}$ with the aid of the mean-field perturbative flow equations (7.1.4). In contrast to our previous non-perturbative analysis of the mean-field flow equations in [4, 79], we will impose boundary conditions at the bare scale ($\alpha = \alpha_0$) and ($\alpha = \alpha_{\max} = \frac{1}{m^2}$). This is the standard prescription of boundary conditions in perturbative analysis of the flow equations, see e.g. Sect.1.3. Here they are imposed in the mean-field approximation.

For $n \geq 6$, $j \geq 1$, we integrate the flow equations upwards from α_0 to α , the boundary conditions for the irrelevant part at the bare scale α_0 can be read from the bare interaction lagrangian 5.1.1

$$\mathcal{A}_{n,j}^{\alpha_0, \alpha} = 0, \quad n \geq 6, \quad j \geq 1. \quad (7.1.7)$$

For the relevant part, we will integrate the flow equations downwards from $\alpha = \frac{1}{m^2}$ to α , with renormalization conditions imposed at $\alpha = \alpha_{\max}$. We recall that they define the relevant parameters of the theory. We will choose the BPHZ renormalization conditions

$$\mathcal{A}_{2,j}^{\alpha_0, \alpha_{\max}} = 0, \quad \mathcal{A}_{4,j}^{\alpha_0, \alpha_{\max}} = \delta_{j,1}, \quad j \geq 1. \quad (7.1.8)$$

At this stage, we analyze the mean-field perturbative flow equations without relating them to the ansatz (3.2.68). The relation between the ansatz and perturbation theory will be established in Sect.7.2. We now derive bounds for the regularized renormalized mean-field amplitudes.

Proposition 7.1. *Let $\mathcal{A}_{n,j}^{\alpha_0, \alpha}$ be smooth solutions of the mean-field flow equations (7.1.6) for the boundary conditions (7.1.7) and the BPHZ renormalization conditions (7.1.8). For $\alpha \in [m^{-2}e^{-\frac{1}{2}}, \alpha_{\max}]$, $\alpha_{\max} = \frac{1}{m^2}$, they satisfy the bounds*

$$|\mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq \alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(\frac{n}{2})^2 (\frac{n}{2})!}, \quad |\partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq \alpha^{\frac{n}{2}-2-k} C^{j-\frac{n}{4}+k} \frac{(j+k+1)!}{(k+1)^2 (\frac{n}{2})^2 (\frac{n}{2})!}, \quad k \geq 1, \quad (7.1.9)$$

for a suitable constant $C > 1$.

Proposition 7.1 is sufficient to establish the local existence of the Borel transform. The subsequent bounds are stronger and imply those from Proposition 7.1

Proposition 7.2. *Let $\mathcal{A}_{n,j}^{\alpha_0, \alpha}$ be smooth solutions of the mean-field flow equations (7.1.6) for the boundary conditions (7.1.7) and the BPHZ renormalization conditions (7.1.8). For $\alpha \in [\alpha_0, \alpha_{\max}]$, $\alpha_{\max} = \frac{1}{m^2}$, they satisfy the bounds*

$$|\mathcal{A}_{2,j}^{\alpha_0, \alpha}| \leq \frac{C^{j-\frac{1}{2}}}{\alpha} \frac{j!}{(j+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \quad (7.1.10)$$

$$|\partial_\alpha^k \mathcal{A}_{2,j}^{\alpha_0, \alpha}| \leq \frac{C^{j-\frac{1}{2}+k}}{\alpha^{k+1}} \frac{(j+k+1)!}{(j+1)^2 (k+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \quad k \geq 1,$$

and for $n \geq 4$

$$|\mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq \alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda,$$

$$|\partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq \alpha^{\frac{n}{2}-2-k} C^{j-\frac{n}{4}+k} \frac{(j+k+1)!}{(j-\frac{n}{2}+2)^2 (k+1)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \quad k \geq 1, \quad (7.1.11)$$

for a suitable constant $C > 1$.

Proof. See Appendix H. □

The bounds (7.1.10)-(7.1.11) are similar to the bounds derived in [6] although we restrict ourselves to the mean-field approximation (zero momenta). However, they include derivatives w.r.t. α . The bounds (7.1.9) will be used in Sect.8. The proof follows the steps in [6]. Had we replaced the BPHZ renormalization conditions (7.1.8) by the more general renormalization conditions

$$\mathcal{A}_{2,j}^{\alpha_0, \alpha_{\max}} = \mathcal{A}_j, \quad \mathcal{A}_{4,j}^{\alpha_0, \alpha_{\max}} = B_j, \quad j \geq 1, \quad (7.1.12)$$

then the bounds (7.1.10),(7.1.11) and (7.1.9) still hold but the constant C in the bounds will depend on \mathcal{A}_j, B_j . We will return to this remark later. From the mean-field flow equations (3.1.9) and the expansion (7.1.3), we can identify the mean-field perturbative CAS functions $f_{n,j}(\mu)$ using the definition (3.1.10)

$$f_{n,j}(\mu) = \alpha^{2-\frac{n}{2}} c^{\frac{n}{2}-1} n A_{n,j}^{\alpha_0, \alpha} = \alpha^{2-\frac{n}{2}} \mathcal{A}_{n,j}^{\alpha_0, \alpha}, \quad \Delta f_n^{K+1}(\mu) = \alpha^{2-\frac{n}{2}} c^{\frac{n}{2}-1} n \Delta A_{n,K+1}^{\alpha_0, \alpha}. \quad (7.1.13)$$

The perturbative expansion of $f_n(\mu)$ corresponding to (7.1.3) is

$$f_n(\mu) = \sum_{j=1}^K g^j f_{n,j}(\mu) + g^{K+1} \Delta f_n^{K+1}(\mu). \quad (7.1.14)$$

From Proposition 7.2, bounds for $\partial_\mu^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}$ can be derived. They will be used in the proof of the local existence of the Borel transform of the regularized renormalized mean-field perturbation theory and in the proof of the local Borel summability of perturbation theory.

Proposition 7.3. *There exists a constant $C' > 1$ such that the smooth perturbative solutions $\mathcal{A}_{n,j}^{\alpha_0, \alpha}$ satisfy the bounds*

$$|\partial_\mu^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{(j+m+1)! C'^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu), \quad m \geq 1, \quad (7.1.15)$$

where we defined

$$\mathcal{F}(j, n, \mu) := \sum_{\lambda=0}^{j-\frac{n}{2}+\hat{\theta}(n)} \frac{1}{2^\lambda \lambda!} (1 + \mu_{\max} - \mu)^\lambda, \quad (7.1.16)$$

and

$$\hat{\theta}(n) := \begin{cases} 1 & \text{if } n \geq 4 \\ 0 & \text{if } n = 2. \end{cases} \quad (7.1.17)$$

Proof. From the Faà di Bruno's formula (see Appendix B.1) and Proposition 7.2 we get

$$\begin{aligned} |\partial_\mu^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}| &\leq \sum_{k=1}^m |\partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0, \alpha}| \sum_{p(m,k)} m! \prod_{j=1}^m \frac{(\alpha_0 e^\mu)^{\lambda_j}}{\lambda_j! (j!)^{\lambda_j}} \\ &\leq \sum_{k=1}^m (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{(j+k+1)! C^{j-\frac{n}{4}+k}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) S_m^k, \end{aligned} \quad (7.1.18)$$

where S_m^k is the Stirling number of the second kind defined by (see e.g. [99])

$$S_m^k := \sum_{p(m,k)} m! \prod_{j=1}^m \frac{1}{\lambda_j! (j!)^{\lambda_j}} . \quad (7.1.19)$$

Then we obtain

$$\begin{aligned} |\partial_\mu^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}| &\leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{C^{j-\frac{n}{4}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) \sum_{k=1}^m (j+k+1)! S_m^k \\ &\leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{C^{j-\frac{n}{4}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) \sum_{k=1}^m (j+1)! k! \binom{j+k+1}{k} S_m^k \\ &\leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{C^{j-\frac{n}{4}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) (j+1)! 2^{j+m+1} a(m) , \end{aligned} \quad (7.1.20)$$

where we introduced the ordered Bell number $a(n)$ (see e.g. [100]-[101])

$$a(n) := \sum_{k=0}^n k! S_n^k . \quad (7.1.21)$$

The ordered Bell numbers $a(n)$ satisfy the recursive formula [102]-[103]

$$a(n) = \sum_{i=1}^n \binom{n}{i} a(n-i) . \quad (7.1.22)$$

From (7.1.22), we have $|a(n)| \leq e^n n!$. Then

$$\begin{aligned} |\partial_\mu^m \mathcal{A}_{n,j}^{\alpha_0, \alpha}| &\leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{(j+1)! m! C'^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) \\ &\leq (\alpha_0 e^\mu)^{\frac{n}{2}-2} \frac{(j+m+1)! C'^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) , \end{aligned} \quad (7.1.23)$$

if we choose for example $C' = 2eC > C > 1$. □

Since we analyze the regularized renormalized mean-field perturbation theory, the bounds established for μ close to μ_{\max} will be sufficient. From Proposition 7.3 we get

Proposition 7.4. *For $\mu \in [0, \mu_{\max}]$, the smooth solutions $f_{n,j}(\mu)$ satisfy the bounds*

$$|\partial_\mu^m f_{n,j}(\mu)| \leq \frac{(j+m+1)! C''^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) , \quad (7.1.24)$$

for a constant $C'' > 1$.

Proof. From Leibniz' rule and Proposition 7.3, we have

$$\begin{aligned}
|\partial_\mu^m f_{n,j}(\mu)| &\leq \sum_{k=0}^m \binom{m}{k} (\alpha_0 e^\mu)^{2-\frac{n}{2}} \left| \frac{n}{2} - 2 \right|^k |\partial_\mu^{m-k} \mathcal{A}_{n,j}^{\alpha_0, \alpha}| \\
&\leq \frac{C'^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) \sum_{k=0}^m \binom{m}{k} \left| \frac{n}{2} - 2 \right|^k (j+m-k+1)! \\
&\leq \frac{(j+m+1)! 2^m C'^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu) \\
&\leq \frac{(j+m+1)! C''^{j+\frac{n}{2}+m}}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \mathcal{F}(j, n, \mu),
\end{aligned} \tag{7.1.25}$$

choosing for instance $C'' = 2C' > 1$. □

Proposition 7.4 implies

Proposition 7.5. For $\mu \in [\mu_{\max} - \frac{1}{2}, \mu_{\max}]$, the smooth solutions $f_{n,j}(\mu)$ satisfy the bounds

$$|\partial_\mu^m f_{n,j}(\mu)| \leq \frac{(j+m+1)! C'^{j+\frac{n}{2}+m}}{(\frac{n}{2})^2 (\frac{n}{2})!}, \tag{7.1.26}$$

for a constant $C' > 1$.

7.1.2 The mean-field flow equations for the remainders

At this stage, we have obtained bounds on the perturbative mean-field amplitudes $f_{n,j}(\mu)$ and their derivatives w.r.t. μ but we did not analyze the remainders yet. Using the perturbative expansion (7.1.3) in the flow equations (3.1.9), the mean-field remainders Δf_n^{K+1} are solutions of the flow equations

$$\begin{aligned}
\Delta f_{n+2}^{K+1}(\mu) &= \frac{2}{n(n+1)} \partial_\mu \Delta f_n^{K+1}(\mu) + \frac{n-4}{n(n+1)} \Delta f_n^{K+1}(\mu) \\
&+ \frac{1}{n+1} \sum_{n_1+n_2=n+2} \left[g^{K+1} \Delta f_{n_1}^{K+1}(\mu) \Delta f_{n_2}^{K+1}(\mu) + \Delta f_{n_1}^{K+1}(\mu) \sum_{j=1}^K g^j f_{n_2,j}(\mu) \right. \\
&+ \left. \Delta f_{n_2}^{K+1}(\mu) \sum_{j=1}^K g^j f_{n_1,j}(\mu) + \sum_{\substack{K < j_1+j_2 \leq 2K \\ 1 \leq j_i \leq K}} g^{j_1+j_2-(K+1)} f_{n_1,j_1}(\mu) f_{n_2,j_2}(\mu) \right].
\end{aligned} \tag{7.1.27}$$

The flow equations (7.1.27) are inconvenient for our analysis due to the last term which renders the dynamical system (7.1.27) inhomogeneous. We recast the flow equations (7.1.27) in a homogeneous form.

From the relation

$$g^{K+1} \Delta f_n^{K+1}(\mu) = \sum_{i=K+1}^{K'} g^i f_{n,i}(\mu) + g^{K'+1} \Delta f_n^{K'+1}(\mu), \quad K' > K \geq 0, \tag{7.1.28}$$

the quadratic term can be rewritten as

$$\begin{aligned}
\sum_{j=1}^K g^{K+1} f_{n_2, K+1-j}(\mu) \Delta f_{n_1}^j(\mu) &= \sum_{j=1}^K \left(\sum_{a=j}^K g^a f_{n_1, a}(\mu) + g^{K+1} \Delta f_{n_1}^{K+1}(\mu) \right) g^{K+1-j} f_{n_2, K+1-j}(\mu) \\
&= \sum_{j=1}^K \sum_{a=j}^K g^{K+1+a-j} f_{n_1, a}(\mu) f_{n_2, K+1-j}(\mu) \\
&\quad + g^{K+1} \Delta f_{n_1}^{K+1}(\mu) \sum_{j=1}^K g^{K+1-j} f_{n_2, K+1-j}(\mu).
\end{aligned} \tag{7.1.29}$$

Then, we sum over n_1, n_2 . Recalling the expansion (7.1.14) we get the homogeneous mean-field flow equations for the remainders

$$\begin{aligned}
\Delta f_{n+2}^{K+1}(\mu) &= \frac{2}{n(n+1)} \partial_\mu \Delta f_n^{K+1}(\mu) + \frac{n-4}{n(n+1)} \Delta f_n^{K+1}(\mu) \\
&\quad + \frac{1}{n+1} \sum_{n_1+n_2=n+2} \left[\sum_{j=1}^K f_{n_2, K+1-j}(\mu) \Delta f_{n_1}^j(\mu) + f_{n_1}(\mu) \Delta f_{n_2}^{K+1}(\mu) \right].
\end{aligned} \tag{7.1.30}$$

In the r.h.s. of (7.1.30), the quadratic term is split into two terms:

- The first term is built from the perturbative mean-field amplitudes and mean-field remainders of $n' \leq n$ -point functions of order $j \leq K$.
- The second term is built from the global mean-field solutions $f_{n'}(\mu)$, $n' \leq n$ and the mean-field remainders of $n' \leq n$ -point functions of order $K+1$.

The mean flow equations (7.1.30) will be used in Ch.8. The corresponding mean-field boundary conditions for the remainders are determined by the mean-field boundary conditions for $f_{n,j}(\mu)$ and for $f_n(\mu)$. To study the remainder of the mean-field CAS functions, we follow the inductive scheme:

- We start from the remainders $\Delta f_2^{K+1}(\mu)$, $K \in \mathbb{N}$.
- From (7.1.30), $\Delta f_{n+2}^{K+1}(\mu)$ is calculated from the remainders $\Delta f_{n'}^{K'}(\mu)$ for $n' \leq n$ and $K' \leq K+1$, from the perturbative solutions $f_{m,j}(\mu)$ for $m \leq n$ and $j \leq K+1$ and the global solutions $f_{n''}(\mu)$ for $n'' \leq n$.

The flow equations (7.1.30) allow us to derive inductive bounds on the remainders $\Delta f_n^{K+1}(\mu)$ and their derivatives w.r.t. μ from which we can prove the local Borel summability of the regularized renormalized mean-field perturbation theory.

7.2 The perturbative expansion of the ansatz

In this section, we establish the connection between the trivial solution constructed in [4, 79] to perturbation theory. Our main result is the following: For a fixed UV-cutoff α_0 , the mean-field smooth solutions of the flow equations (3.1.11) $f_n(\mu)$ have a perturbative expansion in powers of a renormalized coupling up to any order K in perturbation theory. We assume $\mu_{\max} > 6$.

7.2.1 Fixing specific renormalization conditions

We start by showing that we can prescribe specific renormalization conditions in our setting using a fixed-point method. For $b_1 \in \mathbb{R}$, we define $\mathcal{F}_{\mu_{\max}}(b_1)$ as the value of $f_2(\mu_{\max})$ starting from the initial condition $f_2(0) = b_1$ and the ansatz (3.2.68). We fix $|c| \leq \frac{1}{3}$. We show that we can choose, for $c_{0,4}$ fixed, b_1 such that

$$\mathcal{F}_{\mu_{\max}}(b_1) = \frac{c}{\mu_{\max}} =: g_c . \quad (7.2.1)$$

We choose the renormalized coupling $g = f_4(\mu_{\max})$. Then the BPHZ renormalization conditions compatibility amounts to show that there exists $b_1 \in \mathbb{R}$ such that

$$\mathcal{F}_{\mu_{\max}}(b_1) = 0 . \quad (7.2.2)$$

This corresponds to $c = 0$. We recall that

$$f_{2,0} = b_1 , \quad b_2 = \frac{f_{2,1} + b_1}{2} = \frac{3g_{4,0}}{2} + b_1 - \frac{b_1^2}{2} . \quad (7.2.3)$$

Therefore, we have

$$\mathcal{F}_{\mu_{\max}}(b_1) = \frac{b_1}{1 + \mu_{\max}} + b_1 \frac{2\mu_{\max}}{1 + 4\mu_{\max}^2} + (3g_{4,0} - b_1^2) \frac{\mu_{\max}}{1 + 4\mu_{\max}^2} + \sum_{q \geq 3} b_q \frac{x_{\max,q}^{q-1}}{1 + x_{\max,q}^q} , \quad (7.2.4)$$

where $x_{\max,q} := q\mu_{\max}$. Then, we define

$$\mathcal{G}_{\mu_{\max}}(b_1) := F(\mu_{\max}) \left[\frac{c}{\mu_{\max}} - (3g_{4,0} - b_1^2) \frac{\mu_{\max}}{1 + 4\mu_{\max}^2} - \sum_{q \geq 3} b_q \frac{x_{\max,q}^{q-1}}{1 + x_{\max,q}^q} \right] , \quad (7.2.5)$$

where

$$F(x) := \frac{(1+x)(1+4x^2)}{1+2x+6x^2} , \quad x \geq 0 . \quad (7.2.6)$$

The renormalization condition (7.2.1) is satisfied only if

$$\mathcal{G}_{\mu_{\max}}(b_1) = b_1 , \quad (7.2.7)$$

for some $b_1 \in \mathbb{R}$, meaning that $\mathcal{G}_{\mu_{\max}}(b_1)$ has a fixed point in \mathbb{R} .

The purpose of this subsection is dedicated to the proof of existence (and uniqueness) of a fixed point for $\mathcal{G}_{\mu_{\max}}(b_1)$ in \mathbb{R} under the assumption of small bare couplings. The compatibility of (7.2.1) boils down to the analysis of the dependence of the solutions $f_n(\mu)$ in terms of b_1 while $c_{0,4}$ remains free. In Chapters 3,5 bounds on the coefficients $g_{n,k}, f_{2,k}$ were derived, now we analyze their dependence on b_1 . We will use the Fuss-Catalan numbers of parameter $s > 0$

$$C_s(n) := \frac{1}{sn+1} \binom{(s+1)n}{n} . \quad (7.2.8)$$

First, we obtain closed expressions of $g_{n,0}$ and $g_{n,1}$.

Lemma 7.1. *We have for $n \geq 4$*

$$g_{n,0} = (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} C_2 \left(\frac{n}{2} - 1 \right), \quad (7.2.9)$$

$$g_{n,1} = (-1)^{\frac{n}{2}-1} g_{4,0}^{\frac{n}{2}-1} C_2 \left(\frac{n}{2} - 1 \right) \left(\frac{3n-4}{2} b_1 + \frac{n-4}{4} \right). \quad (7.2.10)$$

Proof. See Appendix I.1. □

The bounds on $g_{n,0}$ and $g_{n,1}$ previously derived in Chapters 3, 5 are satisfied as expected. Moreover, $g_{n,0}$ and $g_{n,1}$ are both polynomials in b_1 of respective degrees 0 and 1.

Lemma 7.2. *Let $f_n(\mu)$ be the smooth solutions of the flow equations (3.1.11) with the mean-field boundary conditions (5.1.2). If*

$$|f_{2,0}| \leq K, \quad 0 \leq g_{4,0} \leq \frac{K}{10}, \quad K \leq \frac{1}{30}, \quad (7.2.11)$$

we have

$$|g_{n,k}| \leq \left(\frac{3}{2} \right)^{k-2} K^{\frac{n}{2}-1} \left(\frac{n-4}{2} + k \right)!, \quad |f_{2,k}| \leq \left(\frac{3}{2} \right)^k K |k-1|. \quad (7.2.12)$$

Proof. The proof is done by induction in $N = n + 2k$. We follow the inductive scheme in the proof of Lemma 3.4. For $k \leq 1$, we use the bounds in Lemma 7.1 to successively get

$$|g_{n,0}| \leq \frac{K^{\frac{n}{2}-1}}{30^{\frac{n}{2}-1}} \frac{1}{n-1} \left(3 \left(\frac{n}{2} - 1 \right) \right) \leq \left(\frac{4K}{15} \right)^{\frac{n}{2}-1} \frac{1}{n-1} \leq \frac{4}{9} K^{\frac{n}{2}-1} \left(\frac{n-4}{2} \right)!. \quad (7.2.13)$$

$$\begin{aligned} |g_{n,1}| &\leq \frac{K^{\frac{n}{2}-1}}{30^{\frac{n}{2}-1}} \frac{1}{n-1} \left(3 \left(\frac{n}{2} - 1 \right) \right) \left(\frac{n-4}{4} + \frac{3n-4}{2} |b_1| \right) \\ &\leq \left(\frac{4K}{15} \right)^{\frac{n}{2}-1} K^{\frac{n}{2}-1} \left(\frac{1}{4} + \frac{3K}{2} \right) \leq \frac{2}{3} K^{\frac{n}{2}-1} \left(\frac{n-2}{2} \right)!. \end{aligned} \quad (7.2.14)$$

For $k \geq 0$, the induction hypothesis is inserted in the r.h.s. of (3.2.48) to get

$$\begin{aligned}
|g_{n,k+2}| &\leq \frac{n-4}{n+2k} |g_{n,k+1}| + \frac{2n}{n+2k} \sum_{\nu=0}^{k+1} |g_{n,\nu} f_{2,k+1-\nu}| + \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} |g_{n_1,\nu} g_{n_2,k+2-\nu}| \\
&\quad + \frac{n(n+1)}{n+2k} |g_{n+2,k}| \\
&\leq \left(\frac{3}{2}\right)^k K^{\frac{n}{2}-1} \left[\frac{2(n-4)}{3(n+2k)} \left(k+1+\frac{n-4}{2}\right)! + \frac{4nK}{3(n+2k)} \sum_{\nu=0}^{k+1} \left(\nu+\frac{n-4}{2}\right)! |k-\nu| \right] \\
&\quad + \frac{4n}{9(n+2k)} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} \left(\nu+\frac{n_1-4}{2}\right)! \left(k+2-\nu+\frac{n_2-4}{2}\right)! \\
&\quad + \frac{4n(n+1)K}{9(n+2k)} \left(k+\frac{n-2}{2}\right)! \Big] \\
&\leq \left(\frac{3}{2}\right)^k K^{\frac{n}{2}-1} \left(k+2+\frac{n-4}{2}\right)! \left[\frac{4(n-4)}{3n^2} + \frac{16K}{3n} + \frac{4}{9} + \frac{8(n+1)}{9n^2} K \right] \\
&\leq \left(\frac{3}{2}\right)^k K^{\frac{n}{2}-1} \left(k+2+\frac{n-4}{2}\right)! ,
\end{aligned} \tag{7.2.15}$$

where we used (3.2.64).

Now we bound $f_{2,k}$. The bound holds for $k = 0$. Moreover, the claim can be verified for $k = 1$

$$|f_{2,1}| \leq 3g_{4,0} + |f_{2,0}|(1 + |f_{2,0}|) \leq \frac{17}{15}K \leq \frac{3}{2}K . \tag{7.2.16}$$

For $k \geq 1$, we insert the induction hypothesis in the r.h.s. of (3.2.48) to get

$$\begin{aligned}
|f_{2,k+1}| &\leq \frac{1}{k+1} \left(3 \left(\frac{3}{2}\right)^{k-2} K k! + \left(\frac{3}{2}\right)^k K (k-1)! + \left(\frac{3}{2}\right)^k K^2 \sum_{\nu=0}^k |\nu-1|! |k-\nu-1|! \right) \\
&\leq \left(\frac{3}{2}\right)^{k+1} \frac{1}{2} \left(\frac{8}{9} K k! + \frac{2}{3} K k! + 4K^2 \frac{2}{3} k! \right) \\
&\leq \left(\frac{3}{2}\right)^{k+1} K k! .
\end{aligned} \tag{7.2.17}$$

□

The bounds in Lemma 7.2 look like the bounds (5.1.14) in Proposition 5.1. While the bounds (7.2.12) were derived under the assumptions of small bare couplings, and for $N = 1$, in Proposition 5.1, K had to be chosen sufficiently large to prove (5.1.14). Moreover Lemma 7.2 only provides bounds on $g_{n,k}$ and $f_{2,k}$ while Lemma 7.1 gives a closed expression of $g_{n,0}$ and $g_{n,1}$. We think it is not easy to find a closed expression of $g_{n,k}$, $f_{2,k}$ for $k \geq 2$ since they are coupled together in (3.2.48)-(3.2.49). Now we give bounds for the coefficients b_n .

Lemma 7.3. *Under the assumptions of Lemma 7.2, we have*

$$|b_n| \leq \frac{5}{2} \left(\frac{7}{10}\right)^{n-1} K , \quad n \geq 1 . \tag{7.2.18}$$

Proof. The claim holds for $n = 1$ obviously. We successively have

$$\begin{cases} |b_2| \leq \frac{3}{2}g_{4,0} + |b_1| + \frac{1}{2}|b_1|^2 \leq K \left(\frac{1}{20} + 1 + \frac{K}{2} \right) \leq \frac{7}{4}K, \\ |b_3| \leq \frac{|f_{2,2}| + |b_1|}{9} \leq K \left(\frac{1}{4} + \frac{1}{9} \right) \leq \frac{5}{2} \left(\frac{7}{10} \right)^2 K, \\ |b_4| \leq \frac{|f_{2,3}|}{64} + \frac{|b_2|}{8} \leq K \left(\frac{27}{256} + \frac{3}{16} \right) \leq \frac{5}{2} \left(\frac{7}{10} \right)^3 K. \end{cases} \quad (7.2.19)$$

For $n \geq 4$ we get by inserting the induction hypothesis in the r.h.s. of (3.2.73)

$$\begin{aligned} |b_{n+1}| &\leq \left(\frac{3}{2} \right)^n \frac{(n-1)!}{(n+1)^n} K + \frac{5}{2} K \sum_{\rho=2}^n \left(\frac{7}{10} \right)^{\frac{n+1}{\rho}-1} \frac{1}{\rho^n} + \frac{K}{(n+1)^n} \\ &\leq \left(\frac{7}{10} \right)^n \frac{3K}{10} + \frac{5}{2} \frac{2}{2^n} K + \frac{K}{(n+1)^n} \leq \frac{5}{2} \left(\frac{7}{10} \right)^n K, \end{aligned} \quad (7.2.20)$$

where we used the subsequent two inequalities whose proofs can be found in Appendix C.5.

$$\left(\frac{3}{2} \right)^n \frac{(n-1)!}{(n+1)^n} \leq \left(\frac{7}{10} \right)^n \frac{3}{10}, \quad n \geq 4 \quad (7.2.21)$$

□

and

$$\frac{3}{10} + 2x \left(\frac{5}{7} \right)^n + \left(\frac{10}{7(n+1)} \right)^n \leq x, \quad x \geq 1, \quad n \geq 4. \quad (7.2.22)$$

The bounds in Lemmata 7.2-7.3 are uniform in b_1 .

From Lemma 7.1, $g_{n,0}$ and $g_{n,1}$ are both polynomials in b_1 of degrees 0 and 1, respectively. We generalize this observation to the coefficients $g_{n,k}$, $k \geq 0$ with the following lemma

Lemma 7.4. *We have*

$$g_{n,k} = \mathcal{P}_{n,k}(b_1), \quad f_{2,k} = \mathcal{P}_k(b_1), \quad (7.2.23)$$

where $\mathcal{P}_{n,k}$ and \mathcal{P}_k are polynomials whose coefficients are real and depend, respectively, on $n, k, g_{4,0}$ and on $k, g_{4,0}$. We have also $\deg(\mathcal{P}_{n,k}) \leq k$ and $\deg(\mathcal{P}_k) = k + 1$.

Proof. Again, the induction scheme is the same as the one in the proof of Lemma 3.4. From Lemma 7.1, the claim holds obviously for $n \leq 1$. For $k \geq 0$, we insert the induction hypothesis in the r.h.s. of (3.2.48) and the claim follows.

For $f_{2,k}$, the statement holds for $k = 0$. For $k \geq 0$, the claim follows by inserting the induction hypothesis in the r.h.s. of (3.2.48). In particular, it is straightforward that the coefficient of the leading term of $f_{2,k}$ as a polynomial in b_1 equals $(-1)^k$. □

From Lemma 7.4, we write down the polynomial expansion of $g_{n,k}$ and $f_{2,k}$

$$g_{n,k} = \sum_{\nu=0}^k g_{n,k,\nu} b_1^\nu, \quad f_{2,k} = \sum_{\nu=0}^{k+1} f_{2,k,\nu} b_1^\nu. \quad (7.2.24)$$

From Lemma 7.1, we have in particular

$$g_{n,0,0} = g_{n,0}, \quad g_{n,1,0} = -g_{n,0} \frac{n-4}{4}, \quad g_{n,1,1} = -g_{n,0} \frac{3n-4}{2}. \quad (7.2.25)$$

From (3.2.48), we also get

$$f_{2,0,\nu} = \delta_{1,\nu}, \quad f_{2,1,0} = 3g_{4,0}, \quad f_{2,1,1} = -f_{2,1,2} = 1. \quad (7.2.26)$$

From (3.2.48) and (7.2.24), we have

$$f_{2,2,0} = \frac{3}{2}g_{4,0}, \quad f_{2,2,1} = -9g_{4,0} + \frac{1}{2}, \quad f_{2,2,2} = -\frac{3}{2}, \quad f_{2,2,3} = 1. \quad (7.2.27)$$

Inserting the polynomial expansion of $g_{n,k}, f_{2,k}$ (7.2.24) in (3.2.48), we now obtain two inductive systems for the coefficients $g_{n,k,\nu}$ and $f_{2,k,\nu}$

$$\begin{aligned} g_{n,k+2,\nu} = & -\frac{n-4}{n+2k}g_{n,k+1,\nu} - \frac{2n}{n+2k} \sum_{\rho=0}^{k+1} \sum_{\nu'=\max\{\nu-(k+2-\rho),0\}}^{\min\{\rho,\nu\}} g_{n,\rho,\nu'} f_{2,k+1-\rho,\nu-\nu'} \\ & - \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^{k+2} \sum_{\nu'=\max\{\nu-(k+2-\rho),0\}}^{\min\{\rho,\nu\}} g_{n_1,\rho,\nu'} g_{n_2,k+2-\rho,\nu-\nu'} \\ & + \frac{n(n+1)}{n+2k} g_{n+2,k,\nu} \end{aligned} \quad (7.2.28)$$

and

$$f_{2,k+1,\nu} = \frac{1}{k+1} \left(3g_{4,k,\nu} + f_{2,k,\nu} - \sum_{\rho=0}^k \sum_{\nu'=\max\{\nu-(k+1-\rho),0\}}^{\min\{\rho+1,\nu\}} f_{2,\rho,\nu'} f_{2,k-\rho,\nu-\nu'} \right), \quad (7.2.29)$$

where for convenience, we set $g_{n,k,\nu} = 0$ for $\nu > k$ and $f_{2,k,\nu} = 0$ for $\nu > k+1$. The proofs of the subsequent Lemmata 7.5 and 7.7 are similar to the proofs of Lemmata 7.2-7.3, therefore they are deferred to Appendix I.2.

Lemma 7.5. *Under the assumptions of Lemma (7.2)*

$$|g_{n,k,\nu}| \leq \frac{1}{4} K^{\frac{n}{2}-1} \left(\frac{n-4}{2} + k \right)! \binom{k}{\nu}, \quad |f_{2,k,\nu}| \leq |k-1|! \binom{k+1}{\nu}. \quad (7.2.30)$$

Proof. See Appendix I.2. □

Now we analyze the dependence of the coefficients b_q in terms of b_1 .

Lemma 7.6. *We have*

$$b_q = \mathcal{B}_q(b_1), \quad (7.2.31)$$

where \mathcal{B}_q is a polynomial of degree q whose coefficients are real and depend on $q, g_{4,0}$. In particular, the leading coefficient of \mathcal{B}_q is $\frac{(-1)^{q-1}}{q^{q-1}}$.

Proof. The proof is done by induction in q . The claim holds for $q = 1$, and it follows for $q \geq 1$ by inserting the induction hypothesis in the r.h.s. of (3.2.73). \square

From Lemma 7.6, we now write the polynomial expansion of b_q

$$b_q = \sum_{\nu=0}^q a_{q,\nu} b_1^\nu. \quad (7.2.32)$$

Then from (3.2.73) and (7.2.32) we get

$$a_{q+1,\nu} = \frac{f_{2,q,\nu}}{(q+1)^q} - \sum_{\rho=2}^{q+1} a_{\{\frac{q+1}{\rho}\},\nu} (-1)^{\rho-1} \frac{1}{\rho^q}. \quad (7.2.33)$$

Now we bound the coefficients of the polynomials \mathcal{B}_q

Lemma 7.7. *We have the following estimates*

$$|a_{q,\nu}| \leq \frac{1}{q} \left(\frac{3}{4}\right)^{q-2} \binom{q}{\nu}, \quad q \geq 1, \quad 0 \leq \nu \leq q. \quad (7.2.34)$$

Proof. See Appendix I.2. \square

Now we analyze $\mathcal{G}_{\mu_{\max}}(b_1)$ from (7.2.5). First we establish that $\mathcal{G}_{\mu_{\max}}(b_1)$ is differentiable on $[-K, K]$.

Proposition 7.6. *The function $\mathcal{G}_{\mu_{\max}}(b_1)$ is differentiable on $[-\kappa, \kappa]$, for a positive constant $\kappa \leq \frac{1}{30}$. Moreover*

$$|\mathcal{G}_{\mu_{\max}}(b_1)| < \kappa, \quad \left| \frac{\partial \mathcal{G}_{\mu_{\max}}}{\partial b_1}(b_1) \right| < 1, \quad b_1 \in [-\kappa, \kappa], \quad (7.2.35)$$

where we choose $\mu_{\max} > 6$.

Proof. From Lemma 7.6, the coefficients b_q are smooth in b_1 . From the bounds in Lemma 7.3 we have

$$\begin{aligned} |\mathcal{G}_{\mu_{\max}}(b_1)| &\leq |F(\mu_{\max})| \kappa \left[\frac{c}{\mu_{\max}} + \left(\frac{1}{10} + \kappa\right) \frac{1}{2\mu_{\max}} + \frac{1}{\mu_{\max}} \sum_{q \geq 3} \frac{|b_q|}{q} \right] \\ &\leq \frac{3\kappa}{4} \left[\frac{1}{3} + \frac{1}{15} + \frac{5}{2} \sum_{q \geq 3} \frac{1}{q} \left(\frac{7}{10}\right)^{q-1} \right] \\ &\leq \frac{3\kappa}{4} \left[\frac{1}{3} + \frac{1}{15} + \frac{5}{2} \frac{10}{7} \left(\ln \left(\frac{10}{3}\right) - \frac{189}{200} \right) \right] < \kappa. \end{aligned} \quad (7.2.36)$$

On the other hand, we have from Lemma 7.7

$$\left| \frac{\partial b_q}{\partial b_1} \right| \leq \left(\frac{3}{4}\right)^{q-2} \sum_{\nu=1}^q \frac{\nu}{q} \binom{q}{\nu} |b_1|^{\nu-1} = \left(\frac{3}{4}\right)^{q-2} \sum_{\nu=0}^{q-1} \binom{q}{\nu} |b_1|^\nu = \frac{4}{3} \left(\frac{3(1+|b_1|)}{4} \right)^{q-1}. \quad (7.2.37)$$

The sequence of functions

$$\delta_q(b_1) := \left(b_q \frac{x_{\max,q}^{q-1}}{1 + x_{\max,q}^q} \right)_{q \in \mathbb{N}} \quad (7.2.38)$$

depends on b_1 only through b_q . For $|b_1| \leq \frac{1}{30}$, the bounds (7.2.37) imply the uniform convergence on $[-\kappa, \kappa]$ of the series of functions

$$\left(\sum_{q=1}^N \frac{\partial b_q}{\partial b_1} \frac{x_{\max,q}^{q-1}}{1 + x_{\max,q}^q} \right)_{N \in \mathbb{N}}. \quad (7.2.39)$$

Therefore, $\mathcal{G}_{\mu_{\max}}(b_1)$ is differentiable w.r.t. $b_1 \in [-\kappa, \kappa]$. Then we can bound the derivative of $\mathcal{G}_{\mu_{\max}}(b_1)$

$$\left| \frac{\partial \mathcal{G}_{\mu_{\max}}}{\partial b_1}(b_1) \right| \leq |F(\mu_{\max})| \left[\frac{\kappa}{2\mu_{\max}} + \frac{4}{3\mu_{\max}} \sum_{q \geq 3} \frac{1}{q} \left(\frac{3(1+\kappa)}{4} \right)^{q-1} \right] < 1. \quad (7.2.40)$$

□

Now we collect our findings

Proposition 7.7. For $|c| \leq \frac{1}{3}$, for $g_{4,0} \leq \frac{\kappa}{30}$, $0 < \kappa \leq \frac{1}{30}$, there exists a unique $b_1 \in [-\kappa, \kappa]$ such that

$$\mathcal{F}_{\mu_{\max}}(b_1) = g_c = \frac{c}{\mu_{\max}}. \quad (7.2.41)$$

Proof. From Proposition 7.6, the function $\mathcal{G}_{\mu_{\max}}(b_1)$ satisfies the assumptions of Banach-Picard fixed point theorem [104, 105]. The unique fixed point of $\mathcal{G}_{\mu_{\max}}(b_1)$ is found through iterative procedure: choose an arbitrary $b \in [-\kappa, \kappa]$ and set $u_0 := b$. For $n \in \mathbb{N}_0$, $u_{n+1} = \mathcal{G}_{\mu_{\max}}(u_n)$. Then the sequence $(u_n)_{n \in \mathbb{N}_0}$ converges to the unique fixed point of $\mathcal{G}_{\mu_{\max}}(b_1)$ in $[-\kappa, \kappa]$. □

For $c \neq 0$, one sees using the mean-field flow equations (3.1.11), that $f_2(\mu_{\max}) = g_c$ and $f_4(\mu_{\max}) = -\frac{1}{3}g_c + \mathcal{O}\left(\frac{1}{\mu_{\max}^2}\right)$. This behavior is in agreement with (7.1.1). We can, for instance, expand the two-point function $f_2(\mu)$ in a power series w.r.t. $\frac{1}{\mu_{\max}}$. If such an expansion is possible, the mean-field flow equations (3.1.11) imply that all the $f_n(\mu)$ have such an expansion.

Proposition 7.7 implies that for $|c| \leq \frac{1}{3}$, $c_{0,4}$ and $c_{0,2}$ are related; i.e. $c_{0,2} = \phi_{c, \mu_{\max}}(c_{0,4})$ with $\phi_{c, \mu_{\max}}$ a bijection. Fixing the renormalization conditions in our construction corresponds to fix $c_{0,2}$ and $c_{0,4}$ with a non-trivial relation between them. We could ask ourselves how they evolve with different values of c or μ_{\max} .

The case $c = 0$ is special.

- We have

$$f_4(\mu_{\max}) = \mathcal{O}\left(\frac{1}{\mu_{\max}^2}\right).$$

We think this behavior is not generic for $|c| \leq \frac{1}{3}$. But we did not analyze it in detail.

- Proposition 7.7 not only proves that the BPHZ renormalization condition for the mass is compatible with our procedure. It also shows the uniqueness of the bare coupling associated with this condition.

- We do not know the sign of $g = f_4(\mu_{\max})$. We were unable to deduce its sign from the ansatz. Nevertheless, we think that $g > 0$ for $c_{0,4} > 0$. An argument in favor of positivity comes from the perturbative expansion (7.1.3), the definition (7.1.13) and the BPHZ renormalization conditions (7.1.8). Indeed, at first order in perturbation theory,

$$0 < 4\pi^2 c_{0,4} = g + \mathcal{O}(g^2) . \quad (7.2.42)$$

However, the l.h.s. of (7.2.42) does not depend on α_0 while the r.h.s. of (7.2.42) vanishes when $\mu_{\max} \rightarrow +\infty$, i.e. $\alpha_0 \rightarrow 0$. Another argument in favor of positivity comes from the analysis of the functional integral through discrete renormalization steps [106] between α_0 and α_{\max} . Through this analysis, the variation of the f_n at each step is controlled and one finds that the renormalized coupling is positive.

7.2.2 The perturbative expansion

Now we show that the ansatz for the mean-field two-point function has a perturbative expansion in a renormalized coupling. We note first that the ansatz (3.2.68) can be rewritten as follows for $\mu > 1$

$$f_2(\mu) = \frac{1}{\mu} \sum_{q \geq 1} \frac{b_q}{q} \frac{1}{1 + \frac{1}{\mu^q q^q}} . \quad (7.2.43)$$

We define the function

$$\tilde{f}_2(z) := z \sum_{q \geq 1} \frac{b_q}{q} \frac{1}{1 + \frac{z^q}{q^q}}, \quad z \in (-1, 1] , \quad (7.2.44)$$

From Proposition 5.2 $\tilde{f}_2(z)$ is well-defined for $z \in (0, 1]$ and $f_2(\mu) = \tilde{f}_2(\frac{1}{\mu})$.

Moreover, in Ch.5 we proved that $f_2(\mu)$ is locally analytic w.r.t. μ for $1 < \mu \leq \mu_{\max}$. It turns out that we can analytically continue $\tilde{f}_2(z)$ to the complex plane as follows

Proposition 7.8. \tilde{f}_2 is locally analytic w.r.t. z on the disk $\mathcal{D}(0, \frac{1}{2}) := \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$.

Proof. First we define

$$\tilde{f}_q(z) := \frac{b_q}{q} \frac{1}{1 + \frac{z^q}{q^q}} . \quad (7.2.45)$$

Clearly, $\tilde{f}_q(z)$ is locally analytic w.r.t. $z \in \mathcal{D}(0, \frac{1}{2})$. We can bound the partial sums

$$\left| \sum_{q \geq m+1} \frac{b_q}{q} \frac{1}{1 + \frac{z^q}{q^q}} \right| \leq \sum_{q \geq m+1} \frac{|b_q|}{q} \frac{1}{|1 + \frac{z^q}{q^q}|} \leq \sum_{q \geq m+1} \frac{|b_q|}{q} \frac{1}{1 - \frac{1}{2^q q^q}} \leq 2 \sum_{q \geq m+1} \frac{|b_q|}{q} . \quad (7.2.46)$$

Since the series $\sum_{q \geq 1} \frac{b_q}{q}$ is absolutely convergent, the series of functions $\sum_{1 \leq q \leq N} z \tilde{f}_q(z)$ uniformly converges to $\tilde{f}_2(z)$ on $\mathcal{D}(0, \frac{1}{2})$. Therefore $\tilde{f}_2(z)$ is locally analytic on the disk $\mathcal{D}(0, \frac{1}{2})$. \square

For $|z| < \frac{1}{2}$, we can expand $\tilde{f}_2(z)$ as a power series in z

$$\tilde{f}_2(z) = \sum_{m \geq 1} c_m z^m , \quad (7.2.47)$$

where we define

$$c_m := \sum_{\substack{k \geq 0, q \geq 1 \\ q^{k+1} = m}} \frac{(-1)^k b_q}{q^m}. \quad (7.2.48)$$

In particular, we have

$$c_1 = \sum_{q \geq 1} \frac{b_q}{q}, \quad (7.2.49)$$

while for c_m , $m \geq 2$, the sums in (7.2.48) are finite. From Proposition 5.2, it follows that for some constant $C > 0$ independent of m

$$|c_m| \leq C, \quad m \geq 1. \quad (7.2.50)$$

We fix μ so that $\varepsilon(\mu) := \mu_{\max} - \mu < 1$. We choose $c \neq 0$, for instance $c = \frac{1}{4}$, and the unique b_1 such that $f_2(\mu_{\max}) = g_c$. We will perform a perturbative expansion of $f_2(\mu)$ w.r.t.

$$\tilde{g} := \frac{1}{\mu_{\max}}. \quad (7.2.51)$$

Since

$$\frac{1}{\mu} = \frac{1}{\mu_{\max} - \varepsilon(\mu)} = \sum_{k=1}^{+\infty} \frac{\varepsilon(\mu)^{k-1}}{\mu_{\max}^k}, \quad (7.2.52)$$

we have (formally)

$$\begin{aligned} f_2(\mu) &= \sum_{m=1}^{+\infty} c_m \left(\sum_{k=1}^{+\infty} \frac{\varepsilon(\mu)^{k-1}}{\mu_{\max}^k} \right)^m \\ &= \sum_{m=1}^{+\infty} \sum_{\alpha=1}^m \sum_{\substack{k_1 + \dots + k_\alpha = m \\ k_i \geq 1}} c_\alpha \frac{\varepsilon(\mu)^{m-\alpha}}{\mu_{\max}^m} = F_2(\mu, \tilde{g}), \end{aligned} \quad (7.2.53)$$

where we defined for $y < \frac{1}{6}$

$$F_2(\mu, y) := \sum_{m=1}^{+\infty} a_m(\mu) y^m, \quad (7.2.54)$$

with

$$a_m(\mu) := \sum_{\alpha=1}^m c_\alpha \varepsilon(\mu)^{m-\alpha} \binom{m-1}{\alpha-1}, \quad m \geq 1. \quad (7.2.55)$$

From the perturbative expansion (7.2.53) and the mean-field flow equations (3.1.11), one sees that all the $f_n(\mu)$ have a (formal) perturbative expansion w.r.t. \tilde{g} . Had we performed the expansion (7.1.14) w.r.t. \tilde{g} , it would follow from the expansion (7.2.53) that the coefficients $a_j(\mu)$ correspond to the mean-field perturbative amplitudes.

Lemma 7.8. *The functions $a_m(\mu)$, $m \geq 1$, are analytic on $\Omega_1 := \{\mu > 6 \mid \varepsilon(\mu) < 1\}$ and*

$$|a_m(\mu)| \leq C(1 + \varepsilon(\mu))^{m-1}, \quad |\partial_\mu a_m(\mu)| \leq C(m-1)(1 + \varepsilon(\mu))^{m-2}, \quad (7.2.56)$$

where the constant C is the one introduced in (7.2.50).

Proof. First, the functions $a_m(\mu)$ are obviously analytic w.r.t $\mu \in \Omega_1$. Then, using (7.250), we have

$$|a_m(\mu)| \leq C \sum_{m=1}^{\alpha} \binom{m-1}{\alpha-1} \varepsilon(\mu)^{m-\alpha} = C(1 + \varepsilon(\mu))^{m-1}, \quad (7.257)$$

and

$$\begin{aligned} |\partial_{\mu} a_m(\mu)| &\leq \sum_{\alpha=1}^{m-1} \binom{m-1}{\alpha-1} |c_{\alpha}| (m-\alpha) (\varepsilon(\mu))^{m-\alpha-1} \leq C \sum_{\alpha=1}^{m-1} \binom{m-1}{\alpha} \alpha (\varepsilon(\mu))^{\alpha-1} \\ &= C(m-1)(1 + \varepsilon(\mu))^{m-2}. \end{aligned} \quad (7.258)$$

□

Since $\varepsilon(\mu) < 1$, we also obtain uniform bounds in μ

$$|a_m(\mu)| \leq C 2^{m-1}, \quad |\partial_{\mu} a_m(\mu)| \leq C(m-1)2^{m-2}. \quad (7.259)$$

From Lemma 7.8, the series (7.253) is convergent for $\tilde{g} < \frac{1}{6}$. Moreover, from Lemma 7.8, $F_2(\mu, y)$ is analytic w.r.t. $(\mu, y) \in \Omega_2 := \Omega_1 \times [0, \frac{1}{6})$. Note that the perturbative expansion (7.253) starts at $m = 1$ so that the inductive scheme in Sect.7.1.1 works. From (7.253), the renormalization conditions for the mean-field (connected) two-point function are

$$f_{2,j}(\mu_{\max}) = a_j(\mu_{\max}) = c_j = c \delta_{j,1}. \quad (7.260)$$

From the mean-field flow equations (3.1.11) and the perturbative expansion (7.253), we get the renormalization conditions for the mean-field (connected) four point function

$$f_{4,1}(\mu_{\max}) = -\frac{c}{3}, \quad f_{4,j}(\mu_{\max}) = \frac{1}{3} \left(\partial_{\mu} a_j(\mu_{\max}) + c^2 \delta_{m-1,1} \right), \quad j \geq 2. \quad (7.261)$$

From Lemma 7.8, we have for $j \geq 2$,

$$|f_{4,j}(\mu_{\max})| \leq \mathcal{B} j 2^j, \quad (7.262)$$

for a constant \mathcal{B} that does not depend on j . Since $F_2(\mu, y)$ is smooth in y , we obtain using the Taylor formula

$$f_2(\mu) = F_2(\mu, \tilde{g}) = \sum_{j=1}^K \tilde{g}^j a_j(\mu) + \tilde{g}^{K+1} \Delta f_2^{K+1}(\mu, \tilde{g}), \quad (7.263)$$

where

$$\Delta f_2^{K+1}(\mu, \tilde{g}) = \frac{1}{K!} \int_0^1 dt (1-t)^K \partial_y^{K+1} F_2(\mu, t\tilde{g}). \quad (7.264)$$

Proposition 7.9. *We have for $l \geq 0$*

$$|\partial_{\mu}^l \Delta f_2^{K+1}(\mu, \tilde{g})| \leq \frac{C_1^{K+1+l} (K+1+l)!}{(K+1)!}, \quad (7.265)$$

for a constant C_1 .

Proof. Since $F_2(\mu, y)$ is analytic w.r.t. $(\mu, y) \in \Omega_2$, and for $t \in [0, 1]$, $(\mu, t\tilde{g}) \in \Omega_2$, we have

$$|\partial_\mu^l \partial_{\tilde{g}}^{K+1} F_2(\mu, t\tilde{g})| \leq C_1^{K+1+l} (K+1+l)! \quad (7.2.66)$$

for a constant C_1 . From the uniform bounds (7.2.66), we have

$$|\partial_\mu^l \Delta f_2^{K+1}(\mu, \tilde{g})| \leq \frac{C_1^{K+1+l} (K+1+l)!}{K!} \int_0^1 dt (1-t)^K = \frac{C_1^{K+1+l} (K+1+l)!}{(K+1)!}. \quad (7.2.67)$$

□

From Proposition 7.9, the Borel transform of the perturbative series (7.2.53) w.r.t. \tilde{g} exists everywhere. However, we also want to analyze the remainders $\Delta f_n^{K+1}(\mu)$ for $n \geq 4$. The latter are constructed from the remainder $\Delta f_2^{K+1}(\mu, \tilde{g})$ using the mean-field flow equations for the remainders (7.1.30). From Proposition 7.5, we expect the terms containing the perturbative amplitudes $f_{n,j}(\mu)$ in the r.h.s. of (7.1.30) to spoil the global existence of the Borel transform of the perturbative series (7.1.14). A caveat is that the bounds in Proposition 7.5 were established after imposing the BPHZ renormalization conditions. We will go back to that remark in the next chapter where we show that the Borel transform of the perturbative series (7.1.14) exists locally. To simplify the notation, we will omit \tilde{g} in the remainders $\Delta f_n^{K+1}(\mu)$, since μ_{\max} is fixed and the variable \tilde{g} does not appear in the mean-field flow equations for the remainders (7.1.30).

Chapter 8

The local Borel-summability of the regularized renormalized mean-field perturbation theory

Here we prove the local Borel-summability of the regularized renormalized mean-field perturbative φ_4^4 -theory constructed from our ansatz (3.2.68). Before passing to the main statement, we make a short reminder on local Borel-summability in Sect.8.1. Then we will consider in Sect.8.2 the simple case of a real renormalized coupling to obtain the local existence of the Borel transform of our theory. We also show that at fixed order K in perturbation theory, we obtain estimates on the difference between the global solution and its perturbative expansion up to order K . They imply that the perturbative theory we constructed is asymptotic to the non-perturbative theory. Finally, we prove the local Borel summability of our theory in Sect.8.3.

8.1 Reminder on local Borel-summability

We recall the definition of the local Borel summability. Let $F(t)$ be a formal powers series

$$F(t) := \sum_{n \geq 0} a_n t^n . \quad (8.1.1)$$

The formal power series $F(t)$ is said to be locally Borel-summable if the series

- $B(t) := \sum_{n \geq 0} \frac{a_n}{n!} t^n$ has a non-zero radius of convergence $r > 0$. This is the local existence of the Borel transform of $F(t)$.
- $B(t)$ can be analytically continued to a neighborhood of the positive real axis.
- The function

$$g(z) := \frac{1}{z} \int_0^{+\infty} dt e^{-\frac{t}{z}} B(t) \quad (8.1.2)$$

converges for some $z \neq 0$.

$B(t)$ is called the Borel transform of the power series $F(t)$ while $g(z)$ is its Borel sum. The Borel sum is a Laplace transform of the Borel transform of $F(t)$, Laplace transforms converge in right half-planes [107]. Theorems on local Borel summability usually rely on Watson's theorem [108] which gives a sufficient condition for local Borel summability. Sokal pointed out that Nevanlinna proved an improved version [109]. Here we will state the theorem proven by Nevanlinna and Sokal [110], giving a necessary and sufficient condition for Borel summability.

Nevanlinna-Sokal theorem. *Let f be analytic in the circle $C_R := \{z \in \mathbb{C}, \operatorname{Re}(z^{-1}) > R^{-1}\}$ such that*

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z), \quad |R_N(z)| \leq A\sigma^N N! |z|^N, \quad z \in C_R, \quad (8.1.3)$$

uniformly in N and for some constants A, σ . Then the Borel transform $B(t)$ converges for $|t| \leq \frac{1}{\sigma}$ and can be continued analytically to the striplike region $S_\sigma := \{t \in \mathbb{C} \mid d(t, \mathbb{R}_+) < \frac{1}{\sigma}\}$. It also satisfies

$$|B(t)| \leq K e^{\frac{|t|}{R}} \quad (8.1.4)$$

uniformly in every strip $S_{\sigma'}$ with $\sigma' > \sigma$. Moreover, $f(z)$ can be recovered through the integral representation

$$f(z) = \frac{1}{z} \int_0^{+\infty} dt e^{-\frac{t}{z}} B(t), \quad z \in C_R. \quad (8.1.5)$$

Conversely, if $B(t)$ is analytic in a strip $S_{\sigma''}$ for $\sigma'' < \sigma$ and satisfies the bound (8.1.4), then the function $f(z)$ in (8.1.5) is analytic in the circle C_R , and the estimates (8.1.3) hold with $a_n = \frac{d^n}{dt^n} B(t)|_{t=0}$ uniformly in the circles $C_{R'}$ with $R' < R$.

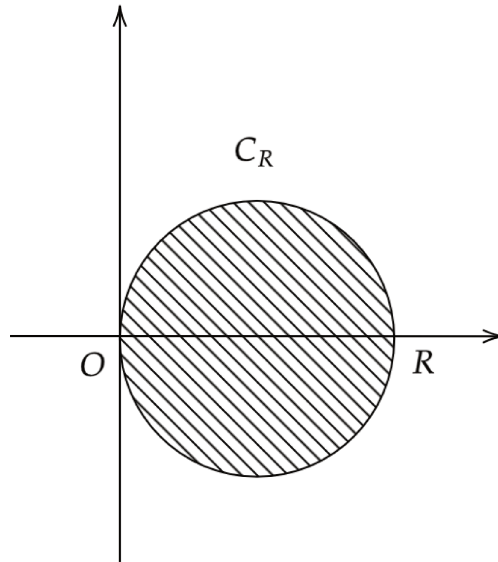


Figure 8.1: The region of analyticity of the Borel-summable function

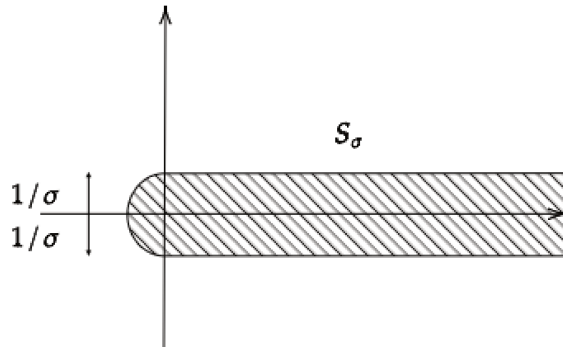


Figure 8.2: The region of analyticity of the Borel transform of a function satisfying the assumptions of the Nevanlinna-Sokal theorem.

8.2 Local existence of the Borel transform for a real coupling and estimates on $\Delta f_n^{K+1}(\mu)$

We prove that the Borel transform of the mean-field regularized renormalized perturbation theory has a non-zero radius of convergence w.r.t. the renormalized coupling \tilde{g} . We restrict ourselves to small bare couplings, i.e.

$$0 < c_{0,4} \leq \frac{K}{40\pi^2}, \quad |c_{0,2}| \leq \frac{K}{2^5\pi^4}\Lambda_0^2, \quad K \leq \frac{1}{30}. \quad (8.2.1)$$

Before passing to the bounds on the remainders, we remark that in our construction, we have the renormalization conditions (7.2.60)-(7.2.61). The corresponding renormalization constants \mathcal{A}_j in (7.1.12) are

$$\mathcal{A}_j = m^2 c \delta_{j,1}. \quad (8.2.2)$$

From (7.2.62), we have

$$|\mathcal{B}_j| \leq \mathcal{B} j 2^j. \quad (8.2.3)$$

The constants \mathcal{A}_j and \mathcal{B}_j are integration constants, meaning that in the proof of Proposition 7.2 one would obtain

$$\mathcal{A}_{2,j}^{\alpha_0,\alpha} = \mathcal{A}_j + \int_{\alpha}^{\alpha_{\max}} d\alpha' \partial_{\alpha} \mathcal{A}_{2,j}^{\alpha_0,\alpha'}, \quad \mathcal{A}_{4,j}^{\alpha_0,\alpha} = \mathcal{B}_j + \int_{\alpha}^{\alpha_{\max}} d\alpha' \partial_{\alpha} \mathcal{A}_{4,j}^{\alpha_0,\alpha'}. \quad (8.2.4)$$

From (8.2.2)-(8.2.3), the bounds (7.1.10)-(7.1.11) still hold by inspection.

Now we present the estimates on the remainders Δf_n^{K+1} . We recall that we assume $\mu_{\max} > 6$ and fix $\mu > \mu_{\max} - \frac{1}{2} := M_{\mu_{\max}}$.

Lemma 8.1. *The remainders $\Delta f_n^{K+1}(\mu)$ satisfy the following bounds*

$$|\partial_{\mu}^l \Delta f_n^{K+1}(\mu)| \leq C_2^{K+n+l-1} \frac{(n+K+l)!}{(n-1)!}, \quad n \geq 2, \quad l \geq 0, \quad K \geq 1, \quad (8.2.5)$$

for a constant C_2 .

Proof. The proof is done by induction in $n + K + l$, going up in n, K at a fixed value of $n + K + l$. For $n = 2$, the bounds follow from Proposition 7.9. The bounds (8.2.5) can be checked explicitly for $n = 4$. Differentiating (7.1.30) l times w.r.t. μ and solving $\partial_\mu^l \Delta f_{n+2}^{K+1}(\mu)$, we get

$$\begin{aligned} \partial_\mu^l \Delta f_{n+2}^{K+1}(\mu) &= \frac{2}{n(n+1)} \partial_\mu^{l+1} \Delta f_n^{K+1}(\mu) + \frac{n-4}{n(n+1)} \partial_\mu^l \Delta f_n^{K+1}(\mu) \\ &+ \frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ l_1+l_2=l}} \binom{l}{l_1} \left[\sum_{j=1}^K \partial_\mu^{l_2} f_{n_2, K+1-j}(\mu) \partial_\mu^{l_1} \Delta f_{n_1}^j(\mu) + \partial_\mu^{l_1} f_{n_1}(\mu) \partial_\mu^{l_2} \Delta f_{n_2}^{K+1}(\mu) \right]. \end{aligned} \quad (8.2.6)$$

We analyze each term in the r.h.s. of (8.2.6):

- The first term is bounded as follows

$$\frac{2}{n(n+1)} C_2^{K+n+l} \frac{(n+K+l+1)!}{(n-1)!} \leq C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!} \frac{2}{C_2(n+K+l+2)}. \quad (8.2.7)$$

- The second term is bounded by

$$\frac{n-4}{n(n+1)} C_2^{K+n+l} \frac{(n+K+l)!}{(n-1)!} \leq C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!} \frac{(n-4)}{C_2 n^2}. \quad (8.2.8)$$

- From Proposition 7.5 and the induction hypothesis, the third term is bounded by

$$\frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ l_1+l_2=l}} \binom{l}{l_1} \sum_{j=1}^K C_2^{j+n_1+l_1-1} C'^{K+1-j+\frac{n_2}{2}+l_2} \frac{(n_1+j+l_1-1)!}{(n_1-1)!} \frac{(K+1-j+l_2+1)!}{\left(\frac{n_2}{2}\right)^2 \left(\frac{n_2}{2}\right)!}, \quad (8.2.9)$$

where we used the (crude) inequality

$$\frac{1}{\left(\frac{n_2}{2}\right)!} \leq \frac{3}{2} \frac{(n_2-2)!}{(n_2-1)!}, \quad n_2 \in 2\mathbb{N}.$$

Using again the Vandermonde inequality (C.4.11) and $m! n! \leq (m+n)!$, we obtain

$$\frac{1}{n!} \binom{K}{j} \binom{l}{l_1} \binom{n}{n_1-1} (n_1+j+l_1-1)! (n_2+K-j+l_2)! \leq \frac{(n+K+l+1)!}{n!}. \quad (8.2.10)$$

Since

$$\sum_{j=1}^K \binom{K}{j}^{-1} \leq 6,$$

if we choose $C_2 > 2\pi C'$ the third term is bounded by

$$\frac{1}{4} C_2^{K+n+l+1} \frac{(n+K+l+2)! l}{(n+1)! (n+K+l+2)} \leq C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!} \frac{1}{4}. \quad (8.2.11)$$

- From Lemma 5.5 and the induction hypothesis, we obtain

$$\frac{1}{n+1} \sum_{\substack{n_1+n_2 \\ l_1+l_2=l}} \binom{l}{l_1} \frac{K_2^{n_1+l_1-1} (n_1+l_1)! C_2^{K+n_2+l_2-1} (n_2+K+l_2)!}{n_1! (l_1+1)^2 (n_2-1)!}. \quad (8.2.12)$$

We use again (C.4.11) to get

$$\begin{aligned} \binom{l}{l_1} \frac{(n_2+K+l_2)! (n_1+l_1)!}{n_1! (n_2-1)!} &= \binom{K}{0} \binom{l}{l_1} \binom{n+1}{n_1} \frac{1}{(n+1)!} (n_2+K+l_2)! (n_1+l_1)! \\ &\leq \frac{1}{(n+1)!} \binom{n+K+l+1}{n_1+l_1} (n_2+K+l_2)! (n_1+l_1)! \\ &\leq (n-2+K+l) \frac{(n+l+K+1)!}{(n+1)!} \leq \frac{(n+K+l+2)!}{(n+1)!}. \end{aligned} \quad (8.2.13)$$

Then, the fourth term is bounded by

$$C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!} \frac{\pi^2}{12C_2} \quad (8.2.14)$$

choosing $C_2 > K_2$.

Summing together (8.2.7), (8.2.8), (8.2.11) and (8.2.14) we get

$$\begin{aligned} |\partial_\mu^l \Delta f_{n+2}^{K+1}(\mu)| &\leq \left[\frac{1}{C_2} + \frac{1}{C_2} + \frac{1}{4} + \frac{\pi^2}{12C_2} \right] C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!} \\ &\leq C_2^{K+n+l+1} \frac{(n+K+l+2)!}{(n+1)!}, \end{aligned} \quad (8.2.15)$$

if we choose $C_2 > \max\{K_2, 4\}$. □

Lemma 8.1 not only implies the local Borel summability of the regularized renormalized mean-field perturbation theory but it also gives estimates on the remainders, i.e. the deviation between the global solution and its perturbative expansion up to order K .

- At a fixed order in perturbation theory, the estimates (8.2.5) imply that

$$f_n(\mu) \underset{\tilde{g} \rightarrow 0}{\sim} \sum_{j=1}^K \tilde{g}^j f_{n,j}(\mu), \quad K \geq 1. \quad (8.2.16)$$

This means that the regularized renormalized mean-field perturbation theory is asymptotic to our non-perturbative mean-field solutions.

- For fixed \tilde{g} , the remainders satisfy (8.1.3). In [6], Kopper established the local Borel summability of renormalized perturbation theory. His perturbative analysis was carried out for arbitrary external momenta by expanding the Schwinger functions as a power series in \hbar (loop expansion) instead of an expansion in powers of \tilde{g} . The two perturbative expansions are equivalent.

- Despite the fact that our mean-field smooth solutions $f_n(\mu)$ are trivial for $\mu_{\max} \rightarrow +\infty$, as long as the UV-cutoff is maintained, the $f_n(\mu_{\max})$ and the perturbative expansion are non-zero and the perturbative series is asymptotic to the full solution.

Finally, we collect our findings in

Theorem 8.1 (Local existence of the Borel transform of the regularized renormalized mean-field perturbative φ_4^4 -theory). *Consider the bare interaction lagrangian (5.1.1) and the smooth solutions $f_n(\mu)$ of the mean-field flow equations (3.1.11) for the mean-field boundary conditions (5.1.2). We assume that*

$$0 < c_{0,4} \leq \frac{K}{40\pi^2}, \quad |c_{0,2}| \leq \frac{K}{2^5\pi^4}\Lambda_0^2, \quad K \leq \frac{1}{30}. \quad (8.2.17)$$

These mean-field solutions $f_n(\mu)$ vanish in the UV-limit, i.e.

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0, \quad n \geq 2. \quad (8.2.18)$$

There exists a renormalized coupling satisfying $g(\alpha_0) \xrightarrow{\alpha_0 \rightarrow 0} 0$ such that for $\alpha_0 > 0$, the renormalized regularized mean-field CAS functions $f_n(\mu)$ have a perturbative expansion in powers of g

$$f_n(\mu) = \sum_{j=1}^K g^j f_{n,j}(\mu) + g^{K+1} \Delta f_n^{K+1}(\mu), \quad \mu \in (M_{\mu_{\max}}, \mu_{\max}]. \quad (8.2.19)$$

The Borel transform of the regularized renormalized mean-field perturbation exists locally. In particular we have the following estimates

$$\left| f_n(\mu) - \sum_{j=1}^K g^j f_{n,j}(\mu) \right| \leq g^{K+1} \tilde{C}^{K+n} \frac{(n+K)!}{(n-1)!}, \quad n \geq 2, \quad K \geq 0, \quad \mu \in (M_{\mu_{\max}}, \mu_{\max}], \quad (8.2.20)$$

for a constant $\tilde{C} > 0$.

Proof. We consider smooth solutions of the mean-field flow equations $f_n(\mu)$ constructed from the ansatz for the mean-field two point function (3.2.68). They are trivial. From Proposition 7.7 we choose the unique $c_{0,2}$ such that $f_2(\mu_{\max}) = \frac{c}{\mu_{\max}}$ for some $0 < c < \frac{1}{3}$. Then the mean-field smooth solutions have a perturbative expansion w.r.t. $\tilde{g} = \frac{1}{\mu_{\max}}$. The local existence of the Borel transform of the regularized renormalized mean-field perturbation theory and the estimates (8.2.20) follow from Lemma 8.1. \square

If we choose $c = 0$, namely the BPHZ renormalization conditions, the renormalized coupling is unexpectedly of order $\frac{1}{\mu_{\max}^2}$. But we think that this behavior is not generic. We conjecture that we can still perform a perturbative expansion of the trivial solutions as before and prove the local existence of the Borel transform of the regularized renormalized mean-field perturbation theory.

In the literature, proofs are usually limited to the local existence of the Borel transform of perturbation theory because explicit expressions of the Schwinger functions are not available. In contrast, we have an explicit expression of the mean-field two point function $f_2(\mu)$ and we derived some of its properties in Sect.7.2.2. In Sect.8.3, we will prove the local Borel summability of the regularized renormalized mean-field perturbation theory using the Nevanlinna-Sokal theorem.

8.3 Extension to a complex renormalized coupling and local Borel summability

Now we turn to the local Borel summability statement by extending our analysis to complex renormalized couplings. The variable μ remains real. From the perturbative expansion (7.2.53) and Lemma 7.8 in Sect.7.2.2, $F_2(\mu, y)$ has an analytic continuation to $\Omega_3 := \Omega_1 \times \mathcal{D}(0, \frac{1}{6})$. We fix $R > 0$ such that $\mathcal{C}_R \subset \mathcal{D}(0, \frac{1}{6})$ and $\tilde{g} \in \mathcal{C}_R$.

- The estimates on the mean-field CAS functions $f_n(\mu)$ established in Lemma 5.5 and the remainders in Lemma 8.1 still hold.
- The first part of the Taylor expansion in the r.h.s. of (7.2.63) is locally analytic w.r.t. \tilde{g} .
- To conclude, we need to verify that the remainder is analytic w.r.t. \tilde{g} . This is the purpose of the following lemma.

Lemma 8.2. *The remainder $\Delta f_2^{K+1}(\mu, \tilde{g})$ is analytic w.r.t. $\tilde{g} \in \mathcal{C}_R$.*

Proof. For $t \in [0, 1]$, the integrand in (7.2.64) is obviously analytic w.r.t. \tilde{g} . We fix a closed curve $\gamma \in \mathcal{C}_R$. From the uniform bounds (7.2.66) which remain valid in our setting, Fubini's theorem yields

$$\oint_{\gamma} d\tilde{g} \Delta f_2^{K+1}(\mu, \tilde{g}) = \frac{1}{K!} \int_0^1 dt (1-t)^K \oint_{\gamma} d\tilde{g} \partial_{\tilde{g}}^{K+1} F_2(\mu, t\tilde{g}) = 0. \quad (8.3.1)$$

We conclude with Morera's theorem. □

From the mean-field non-perturbative flow equations (3.1.11) and the mean-field flow equations for the remainders (7.1.30), the mean-field trivial solutions $f_n(\mu)$ satisfy the assumptions of the first statement of the Nevanlinna-Sokal theorem. We can now state

Theorem 8.2 (Local Borel summability of the regularized renormalized mean-field perturbative φ_4^4 -theory). *Consider the bare interaction lagrangian (5.1.1) and the smooth solutions $f_n(\mu)$ of the mean-field flow equations (3.1.11) for the mean-field boundary conditions (3.2.41). We assume that*

$$0 < c_{0,4} \leq \frac{K}{40\pi^2}, \quad |c_{0,2}| \leq \frac{K}{2^5\pi^4} \Lambda_0^2, \quad K \leq \frac{1}{30}. \quad (8.3.2)$$

Then these solutions of (3.1.11) $f_n(\mu)$ vanish in the UV-limit, i.e.

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0, \quad n \geq 2. \quad (8.3.3)$$

There exists a renormalized coupling satisfying $g(\alpha_0) \xrightarrow{\alpha_0 \rightarrow 0} 0$ such that for $\alpha_0 > 0$, the renormalized regularized mean-field Schwinger functions $f_n(\mu)$ have a perturbative expansion in powers of g . The renormalized regularized mean-field perturbative φ_4^4 -theory is locally Borel-summable.

Proof. We consider the smooth solutions of the mean-field flow equations $f_n(\mu)$ constructed from the ansatz for the mean-field two point function (3.2.68). They vanish in the UV-limit. We choose $g(\alpha_0) = \frac{1}{\mu_{\max}}$. The local Borel summability follows from Lemma 8.1, Lemma 8.2, and the Nevanlinna-Sokal theorem. □

Chapter 9

Conclusions

In this thesis, we analyzed non-perturbatively and perturbatively the φ_4^4 -theory with the renormalization group flow equations in the mean-field approximation. For the one-component theory, the bare interaction lagrangian is given by

$$L_0^{\mathcal{V}}(\varphi) = \int_{\mathcal{V}} d^4x \left(c_{0,2} \varphi^2(x) + c_{0,4} \varphi^4(x) \right).$$

We did not include the wavefunction renormalization counterterm since it vanishes in the mean-field approximation. The corresponding boundary conditions in the mean-field approximation for the rescaled mean-field Schwinger functions $f_n(\mu)$ are

$$f_2(0) = 2(2\pi)^4 \alpha_0 c_{0,2}, \quad f_4(0) = 4\pi^2 c_{0,4}, \quad f_n(0) = 0, \quad n \geq 6.$$

We proceeded as follows:

- The first step was to drop the weak bare couplings assumptions in [4]. We chose the more general assumptions on the bare couplings

$$0 \leq c_{0,4} \leq \frac{\sqrt{K}}{2^7 \pi^2}, \quad |c_{0,2}| \leq \Lambda_0^2 \frac{\sqrt{K}}{2^7 \pi^4}, \quad \Lambda_0^{-2} = \alpha_0,$$

for some finite constant K which is not bounded from above. We again analyzed an ansatz for the mean-field two-point function of the form

$$\sum_{q \geq 1} b_q \frac{x_q^{q-1}}{1 + x_q^q}, \quad x_q := q\mu.$$

With the mean-field flow equations, we proved that this ansatz is well defined on $[0, \mu_{\max}]$. for the mean-field boundary conditions we considered. The remaining mean-field solutions $f_n(\mu)$ of the mean-field equations satisfying the mean-field boundary conditions we imposed, are constructed inductively from the mean-field flow equations and the ansatz. They are all well-defined and vanish upon removing the UV-cutoff:

$$\lim_{\mu_{\max} \rightarrow +\infty} f_n(\mu_{\max}) = 0,$$

meaning that the mean-field theory we constructed is trivial. Then we also discovered that the uniqueness of the mean-field solutions $f_n(\mu)$ for fixed mean-field boundary conditions could not be solved using analyticity properties. This led to non-trivial aspects on analyticity of the CAS functions. The mean-field solutions $f_n(\mu)$ are thought not to be analytic at $\mu = 0$, this should be related to the infinite set of constraints $f_n(0) = 0$, $n \geq 6$, forcing all the trajectories of the Schwinger functions for $n \geq 6$ pass through zero at the bare scale.

- We managed to find a way to prove uniqueness, relying on an integral formulation of the mean-field flow equations. We were inspired by Felder's approach of deriving the following partial differential equation

$$-\lambda \partial_\lambda u = \frac{1}{2} \partial_{xx} u - \frac{1}{2} (\partial_x u)^2 + 4u - x \partial_x u ,$$

with an integral formulation. The integral formulation approach turns out to work if the bare interaction lagrangian is polynomial in the fields and bounded from below, and if the effective mean-field action is locally analytic w.r.t. the mean field variable.

- We extended our approach to the $O(N)$ -model, we considered the case of a N -component real scalar field $\varphi = [\varphi_1, \dots, \varphi_N]$. This case is not treated in the celebrated papers by Aizenman, Fröhlich and Duminil-Copin. We treated the model in close analogy to the one-component theory. We especially analyzed the tensor structure of the mean-field Schwinger functions to obtain simplified mean-field equations which allow us to apply our method from the one-component case. We again studied the same ansatz for the two-point function. We obtained bounds which generalizes those derived in the one-component case, again without any constraints on the size of the bare couplings. We draw the same conclusion as in the one-component case: the ansatz is well defined for $\mu \in [0, \mu_{\max}]$, leading again to triviality. The uniqueness proof can be straightforwardly extended to the $O(N)$ -model. As a crosscheck, we verified that our results in the large N limit results agree with the analysis carried out by Amit, Moshe and Zinn-Justin in which they resummed perturbation theory in powers of $\frac{1}{N}$.
- We also dropped the technical assumption that $m = 0$ in the propagator. We thought that this assumption should not affect the key properties of the theory. In order not to overload the proofs with technicalities, we analyzed the one-component theory. We chose another regularized propagator in momentum space

$$\tilde{C}^{\alpha_0, \alpha}(p, m) = \frac{e^{-\alpha_0(p^2+m^2)} - e^{-\alpha(p^2+m^2)} (\frac{1}{m^2} + \alpha_0 - \alpha) m^2}{p^2 + m^2} , \quad \alpha \in \left[\alpha_0, \frac{1}{m^2} + \alpha_0 \right] .$$

We derived the corresponding mean-field flow equations and studied again the ansatz (3.2.68). We found that we obtain triviality again. Furthermore, we verified that the mean-field effective action is locally analytic w.r.t. ϕ for asymptotically free mean-field solutions as in [4].

In the second part of the thesis, we analyzed the relation between triviality and perturbation theory. We analyzed the mean-field perturbative flow equations and obtained bounds on the mean-field perturbative CAS functions with BPHZ renormalization conditions.

- We checked that specific renormalization conditions, including the BPHZ renormalization conditions can be satisfied by our mean-field trivial solutions.

- We expanded the mean-field trivial solutions $f_n(\mu)$ perturbatively in a series in a conveniently chosen renormalized coupling $\tilde{g} > 0$, up to order K in perturbation theory

$$f_n(\mu) = \sum_{j=1}^K \tilde{g}^j f_{n,j}(\mu) + \tilde{g}^{K+1} \Delta f_n^{K+1}(\mu) .$$

- We used the bounds on the mean-field perturbative CAS functions to obtain estimates on the mean-field remainders $\Delta f_n^{K+1}(\mu)$ for a positive renormalized coupling \tilde{g} . They imply that for a fixed UV-cutoff, the perturbative expansion up to a fixed order in perturbation theory exists and is asymptotic to the global mean-field solutions in the limit $\tilde{g} \rightarrow 0$ and the estimates imply the local existence of the Borel transform of the regularized renormalized mean-field theory.
- Then, we finally proved the local Borel summability of the regularized renormalized mean-field theory. The estimates obtained in the real coupling case still hold, and it turns out that our analysis in the real case can be extended to complex couplings. The celebrated Nevanlinna-Sokal theorem implies the local Borel summability.

We end with remarks and possible perspectives:

- In this thesis, we considered $c_{0,4} > 0$ so that the bare lagrangian (3.2.40) is bounded from below. Nevertheless, the bounds we obtained in this thesis are valid for $c_{0,4} \leq 0$ or $c_{0,4} \in \mathbb{C}$. However, the uniqueness argument we came up with in Sect.5.1.2 does not work anymore. The case $c_{0,4} < 0$ is related to ultraviolet asymptotic freedom; see e.g. [111] and the literature. It could be interesting to investigate this case and compare the results with ours and those obtained in the literature.
- We could study the relation between $c_{0,2}$ and $c_{0,4}$ in our fixed-point method and analyze the specific case $c = 0$.
- So far we studied the mean-field $O(N)$ -model. The \mathbb{Z}_2 -symmetry is an underlying symmetry of the theory. It could be interesting to analyze theories breaking this symmetry; we expect the triviality result to hold. We could also analyze theories with a polynomial bare interaction lagrangian. This modifies the mean-field boundary conditions and aspects of our proof at a technical level, but we expect the triviality result to hold.
- So far, we studied real scalar field theories. We could also analyze theories with fermions or even gauge fields. Such theories have already been analyzed using the flow equations. In particular, quantum electrodynamics is thought to be trivial since its beta function is positive.
- Finally, it would be natural to drop the mean-field approximation. The construction of the CAS functions would be more complicated. Due to momentum conservation, the n -point functions actually depend on $n - 1$ external momenta while being symmetric under any permutation of the external momenta due to the Bose symmetry. Then there also arises the question whether the so-constructed CAS functions respect the axioms of a (Euclidean) quantum field theory.

Appendix A

Gaussian measures

A.1 Properties of Gaussian measures

We consider a Gaussian probability measure $d\mu$ on the space of continuous real-valued functions $C(\Omega)$, where Ω is a finite (simply connected compact) volume in \mathbb{R}^d , $d \geq 1$.

A.1.1 Covariance of a Gaussian measure

We recall here the definition of the covariance of a Gaussian measure, details can be found in [61]. A Gaussian measure of mean zero is uniquely characterized by its covariance

$$\int d\mu_C(\phi) \phi(x)\phi(y) = \tilde{C}(x, y) = \tilde{C}(y, x) . \quad (\text{A.1.1})$$

\tilde{C} is a positive non-degenerate bilinear form defined on $C^\infty(\Omega) \times C^\infty(\Omega)$. We assume that $\tilde{C}(x, y)$ is translation invariant, then $C(z) := \tilde{C}(x, y)$, $z = x - y$, is well defined. Using the notations

$$\langle \phi, J \rangle = \int_{\Omega} d^d x \phi(x) J(x) , \quad \langle J, C J \rangle = \int_{\Omega} d^d x d^d y J(x) C(x - y) J(y) \quad (\text{A.1.2})$$

with $J \in C^\infty(\Omega)$, the generating functional of the correlation functions is

$$\int d\mu_C(\phi) e^{\langle \phi, J \rangle} = e^{\frac{1}{2} \langle J, C J \rangle} . \quad (\text{A.1.3})$$

The generating functional is also called the characteristic functional of the Gaussian measure μ_C . For $C = (-\Delta + I)^{-1}$, where Δ denotes the Laplacian operator in \mathbb{R}^d , the corresponding Gaussian measure μ_C is supported on distributions with $1 - \frac{d}{2} - \varepsilon$ continuous derivatives, $\varepsilon > 0$. For a regularized propagator, the Fourier transform of which falls off rapidly in momentum space, the Gaussian measure is supported on smooth functions.

A.1.2 Properties of Gaussian measures

We list here some properties of Gaussian measures. Proofs can be found in [61].

- Integration by parts: Let $A(\phi)$ be a polynomial in $\phi(x)$ and its derivatives $\partial_\mu \phi(x)$.

$$\int d\mu_C(\phi) \phi(x) A(\phi) = \int d\mu_C(\phi) \int_\Omega d^d y C(x-y) \frac{\delta}{\delta \phi(y)} A(\phi) . \quad (\text{A.1.4})$$

- Translation of a Gaussian measure: Let C be a covariance. Under a change of variable $\phi = \varphi + \psi$ for $\varphi \in \text{supp}(\mu_C)$ and ψ such that its Fourier transform $\hat{\psi}(p)$ is compactly supported.

$$d\mu_C(\phi) = e^{-\frac{1}{2}\langle \psi, C^{-1} \psi \rangle} e^{-\langle C^{-1} \psi, \varphi \rangle} d\mu_C(\varphi) . \quad (\text{A.1.5})$$

- Decomposition of the covariance: Assume that

$$C = C_1 + C_2 , \quad C_i > 0 .$$

Then for $A(\phi)$ as in (A.1.4)

$$\int d\mu_C(\phi) A(\phi) = \int d\mu_{C_1}(\phi_1) \int d\mu_{C_2}(\phi_2) A(\phi_1 + \phi_2) . \quad (\text{A.1.6})$$

- Infinitesimal change of covariance: We assume the covariance depends on a parameter t , and is differentiable w.r.t. t

$$C(x-y) \equiv C_t(x-y) , \quad \dot{C}_t(x-y) := \frac{d}{dt} C_t(x-y) .$$

Let $F(\phi)$ be a smooth functional, integrable w.r.t. $\mu_{C_t} \forall t$. We have

$$\frac{d}{dt} \int d\mu_{C_t}(\phi) F(\phi) = \frac{1}{2} \int d\mu_{C_t}(\phi) \left\langle \frac{\delta}{\delta \phi}, \dot{C}_t \frac{\delta}{\delta \phi} \right\rangle F(\phi) . \quad (\text{A.1.7})$$

Appendix B

Faà di Bruno's formula

Here we recall the Faà di Bruno formula, discovered first by Faà di Bruno [112].

Proposition B.1. *Let I, J, K intervals in \mathbb{R} , $g : I \rightarrow J$ and $f : J \rightarrow K$ such that g has derivatives up to order n at $x \in I$, $y = g(x) \in J$ and f has derivatives up to order n at $y = g(x)$. Then $f \circ g$ has derivatives up to order n at x and*

$$\frac{d^n}{dx^n}(f \circ g)(x) = \sum_{k=1}^n \frac{d^k}{dy^k} f(y) \sum_{p(n,k)} n! \prod_{j=1}^{n-k+1} \frac{(g^{(j)}(x))^{\lambda_j}}{\lambda_j! (j!)^{\lambda_j}}, \quad (\text{B.0.1})$$

where $g^{(j)}(x)$ denotes $\frac{d^j}{dx^j} g(x)$ and the set $p(n, k)$ is defined as follows

$$p(n, k) := \left\{ (\lambda_1, \dots, \lambda_{n-k+1}) \in \mathbb{N}^{n-k+1}, \quad \sum_{j=1}^{n-k+1} \lambda_j = k, \quad \sum_{j=1}^{n-k+1} j \lambda_j = n \right\}. \quad (\text{B.0.2})$$

The formula (B.0.1) can be rewritten as

$$\frac{d^n}{dx^n}(f \circ g)(x) = \sum_{k=1}^n \frac{d^k}{dy^k} f(y) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)), \quad (\text{B.0.3})$$

where we introduced the Bell polynomials

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) := \sum_{p(n,k)} n! \prod_{j=1}^{n-k+1} \frac{x_j^{\lambda_j}}{\lambda_j! (j!)^{\lambda_j}}, \quad n \geq k. \quad (\text{B.0.4})$$

Appendix C

A collection of identities and inequalities

We collect here useful identities and inequalities.

C.1 Bound on a sum

Lemma C.1. For $n \geq 12$

$$\frac{n}{n-2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N}}} \frac{1}{n_1^2(n+2-n_1)^2} \leq \frac{1}{n^2}. \quad (\text{C.1.1})$$

Proof. First we have for $n \geq 12$

$$\sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N}}} \frac{1}{n_1^2(n+2-n_1)^2} \leq \frac{1}{16} \sum_{\substack{n_1+n_2=\frac{n}{2}+1 \\ n_i \geq 2, n_i \in \mathbb{N}}} \frac{1}{n_1^2(\frac{n}{2}+1-n_1)^2}.$$

We use the decomposition

$$\frac{1}{X^2(X-A)^2} = \frac{1}{A^2} \left(\frac{1}{X^2} + \frac{1}{(X-A)^2} + \frac{2}{AX} - \frac{2}{A(X-A)} \right), \quad A > 0.$$

We get

$$\begin{aligned} & \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N}}} \frac{1}{n_1^2(n+2-n_1)^2} \\ & \leq \frac{1}{4(n+2)^2} \sum_{2 \leq n_1 \leq \frac{n}{2}-1} \left(\frac{1}{n_1^2} + \frac{1}{(\frac{n}{2}+1-n_1)^2} + \frac{2}{(\frac{n}{2}+1)n_1} + \frac{2}{(\frac{n}{2}+1)(\frac{n}{2}+1-n_1)} \right) \\ & \leq \frac{1}{2(n+2)^2} \left(\zeta(2) - 1 + \frac{n-4}{n+2} \right) \leq \frac{5}{6(n+2)^2}, \end{aligned}$$

where we used the fact that $\sum_{2 \leq n_1 \leq \frac{n}{2}-1} \frac{1}{n_1} \leq \frac{n-4}{4}$. Therefore we have for $n \geq 12$

$$\frac{n}{n-2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{1}{n_1^2(n+2-n_1)^2} \leq \frac{5}{6(n+2)^2} \frac{n}{n-2} \leq \frac{5}{6n^2} \frac{n^2}{(n+2)^2} \frac{n}{n-2} \leq \frac{1}{n^2}.$$

□

C.2 Derivatives of $\frac{f}{g}$

We prove

Proposition C.1. *For f, g smooth with $g > 0$,*

$$\left(\frac{f}{g}\right)^{(l)} = \frac{1}{g} \left[f^{(l)} - l! \sum_{j=1}^l \frac{g^{(l+1-j)}}{(l+1-j)! (j-1)!} \left(\frac{f}{g}\right)^{(j-1)} \right]. \quad (\text{C.2.1})$$

Proof. The proof is done by induction in $l \in \mathbb{N}$. For $l = 1$, the statement is easily verified. Then differentiating (C.2.1) and using the induction hypothesis, we obtain

$$\begin{aligned} \left(\frac{f}{g}\right)^{(l+1)} &= \frac{f^{(l+1)}}{g} - \frac{g' f^{(l)}}{g^2} + \frac{g'}{g^2} \sum_{j=1}^l \binom{l}{j-1} g^{(l+1-j)} \left(\frac{f}{g}\right)^{(j-1)} \\ &\quad - \frac{1}{g} \sum_{j=1}^l \binom{l}{j-1} \left(g^{(l+2-j)} \left(\frac{f}{g}\right)^{(j-1)} + g^{(l+1-j)} \left(\frac{f}{g}\right)^{(j)} \right) \\ &= \frac{f^{(l+1)}}{g} - \frac{g'}{g} \left(\frac{f}{g}\right)^{(l)} - \frac{g^{(l+1)} f}{g^2} - l \frac{g'}{g} \left(\frac{f}{g}\right)^{(l)} \\ &\quad - \frac{1}{g} \sum_{j=2}^l \left[\binom{l}{j-1} + \binom{l}{j-2} \right] g^{(l+2-j)} \left(\frac{f}{g}\right)^{(j-1)} \\ &= \frac{1}{g} \left[f^{(l+1)} - (l+1)! \sum_{j=1}^{l+1} \frac{g^{(l+2-j)}}{(l+2-j)! (j-1)!} \left(\frac{f}{g}\right)^{(j-1)} \right], \end{aligned} \quad (\text{C.2.2})$$

where we used

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \quad n \in \mathbb{N}_0, k \in \mathbb{N}. \quad (\text{C.2.3})$$

□

C.3 Proofs of useful inequalities used in the proof of Proposition 5.1

Here we prove the inequalities stated without proof in the proof of Proposition 5.1.

C.3.1 Proof of (F.2.5)

We have

$$\frac{((k+1)(k+2))^{\frac{1}{4}}|k-2|!}{n^2} \leq k!, \quad n \geq 4, k \geq 0. \quad (\text{C.3.1})$$

Proof. • The bound is obvious for $0 \leq k \leq 1$.

• For $k \geq 2$, we have

$$\begin{aligned} \frac{((k+1)(k+2))^{\frac{1}{4}}(k-2)!}{n^2} &\leq \frac{\sqrt{k+2}(k-2)!}{n^2} \\ &\leq \frac{\sqrt{k+2}(k-1)!}{k-1} \frac{1}{n^2} \leq 2 \frac{(k-1)!}{n^2} \leq (k-1)!. \end{aligned} \quad (\text{C.3.2})$$

□

C.3.2 Proofs of (F.2.10),(F.2.12)

Lemma C.2. We have for $k \geq 1$ and $n \geq 4$

$$\left(k(k+1)(k+2)\right)^{\frac{1}{4}} \left|\frac{n}{4} + k - 4\right|! \leq C_a \left(\frac{n}{4} + k - 1\right)!, \quad (\text{C.3.3})$$

for a constant C_a which does not depend on n, k .

Proof. First we treat the cases such that $\frac{n}{4} + k < 4$. Since the number of these cases is finite, the inequality (C.3.3) stands for a constant $\mathcal{C} > 0$ which does not depend on \mathcal{C} . Then for the remaining cases we have

$$\begin{aligned} \left(k(k+1)(k+2)\right)^{\frac{1}{4}} \left(\frac{n}{4} + k - 4\right)! &\leq \frac{\left(k(k+1)(k+2)\right)^{\frac{1}{4}}}{\frac{n}{4} + k - 3} \left(\frac{n}{4} + k - 3\right)! \\ &\leq \mathcal{C}' \left(\frac{n}{4} + k - 1\right)!, \end{aligned} \quad (\text{C.3.4})$$

for a constant \mathcal{C}' . Then (C.3.3) follows upon choosing $C_a = \max(\mathcal{C}, \mathcal{C}')$. □

The inequality (F.2.12) is proven analogously.

C.3.3 Proof of (F.2.23)

We introduce the Beta function, also called Euler integral of the first kind, defined for complex numbers z_1, z_2 such that $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) > 0$ by

$$B(z_1, z_2) := \int_0^1 dt t^{z_1-1} (1-t)^{z_2-1}. \quad (\text{C.3.5})$$

The Beta function is related to the Gamma function through

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}. \quad (\text{C.3.6})$$

Then we have for $m \in \mathbb{N}_0$

$$\frac{m!}{\left(m + \frac{1}{2}\right)!} = \frac{\Gamma(m+1)}{\Gamma\left(m + \frac{3}{2}\right)} = \frac{1}{\sqrt{\pi}} B\left(m + \frac{1}{2}, \frac{1}{2}\right). \quad (\text{C.3.7})$$

Using the integral definition of the Beta function (C.3.5) we have

$$\begin{aligned} \frac{m!}{\left(m + \frac{1}{2}\right)!} &= \frac{1}{\sqrt{\pi}} \int_0^1 dt \frac{t^m}{\sqrt{1-t}} = \frac{2}{\sqrt{\pi}} \int_0^1 dt (1-t^2)^m \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} dt \sin^{2m+1}(t) = \frac{2}{\sqrt{\pi}} W_{2m+1}, \end{aligned} \quad (\text{C.3.8})$$

where $\{W_n\}_{n \in \mathbb{N}_0}$ designates the sequence of Wallis integrals. It is well known that

$$W_{2m+1} = \frac{1}{2m+1} \frac{4^m}{\binom{2m}{m}}. \quad (\text{C.3.9})$$

The inequality (F.2.23) is a consequence of

Lemma C.3.

$$\frac{\sqrt{\pi}}{2} \sqrt{m + \frac{1}{2}} \leq \frac{4^m}{\binom{2m}{m}}, \quad m \in \mathbb{N}_0. \quad (\text{C.3.10})$$

Proof. The proof is done by induction in $m \geq 0$. For $m = 0$, the claim is obvious. For $m \geq 1$, we have

$$\frac{\sqrt{\pi}}{2} \sqrt{m + \frac{1}{2}} = \frac{\sqrt{\pi}}{2} \sqrt{m - \frac{1}{2}} \sqrt{1 + \frac{1}{m - \frac{1}{2}}} \leq \sqrt{3} \frac{4^{m-1}}{\binom{2m-2}{m-1}} = \sqrt{3} \frac{4^{m-1}}{\binom{2m}{m}} \frac{m}{2(2m-1)} \leq \frac{4^m}{\binom{2m}{m}}.$$

□

C.4 Useful inequalities for the proof of Proposition 7.2

In order to derive bounds on the derivatives $\partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0, \alpha}$, we will first prove useful and elementary bounds which we will use in the proof of Proposition 7.2.

Lemma C.4. For $l \in \mathbb{N}_0$, $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{\substack{l_1+l_2=l \\ l_i \geq 0}} \frac{1}{(l_1+1)^2(l_2+1)^2} &\leq \frac{5}{(l+1)^2}, & \sum_{\substack{l_1+l_2=l \\ l_i \geq 1}} \frac{1}{(l_1+1)^2(l_2+1)^2} &\leq \frac{3}{(l+1)^2} \\ \sum_{\substack{n_1+n_2=n+1 \\ n_i \geq 1}} \frac{1}{n_1^3 n_2^3} &\leq \frac{4}{n^3}. \end{aligned} \quad (\text{C.4.1})$$

Proof. For $l \leq 5$, the inequality can be verified by hand. For $l > 5$, we have

$$\begin{aligned} \sum_{\substack{l_1+l_2=l \\ l_i \geq 0}} \frac{1}{(l_1+1)^2(l_2+1)^2} &= \frac{2}{(l+1)^2} + \sum_{k=1}^{l-1} \frac{1}{(k+1)^2(l-k+1)^2} \\ &\leq \frac{2}{(l+1)^2} + \int_0^l \frac{dx}{(x+1)^2(l-x+1)^2} = \frac{2}{(l+1)^2} \\ &\quad + \int_1^{l+1} dx \left(\frac{a+bx}{x^2} + \frac{c-bx}{(l+2-x)^2} \right), \end{aligned} \quad (\text{C.4.2})$$

where

$$a = \frac{1}{(l+2)^2}, \quad b = \frac{2}{(l+2)^3}, \quad c = \frac{3}{(l+2)^2}. \quad (\text{C.4.3})$$

Then the integral equals

$$\frac{1}{(l+2)^2} \left(2 \left[1 - \frac{1}{l+1} \right] + \frac{4}{l+2} \ln(l+1) \right) \leq \frac{3}{(l+1)^2}, \quad l > 5. \quad (\text{C.4.4})$$

The second statement in (C.4.1) follows directly from the first one, since a factor $\frac{2}{(l+1)^2}$ is subtracted on the l.h.s.

Again we can verify the inequality for $n \leq 5$. Assuming now that $n > 5$, we proceed as above and we obtain

$$\begin{aligned} \sum_{\substack{n_1+n_2=n+1 \\ n_i \geq 1}} \frac{1}{n_1^3 n_2^3} &= \sum_{\substack{n_i \geq 0 \\ n_1+n_2=n-1}} \frac{1}{(n_1+1)^3(n_2+1)^3} \\ &\leq \frac{2}{n^3} + \sup_{1 \leq n_1 \leq n-1} \frac{1}{(n_1+1)(n-n_1)} \sum_{\substack{1 \leq n_i \\ n_1+n_2=n-1}} \frac{1}{(n_1+1)^2(n_2+1)^2} \\ &\leq \frac{2}{n^3} + \frac{1}{2(n-1)} \sum_{1 \leq n_1 \leq n-2} \frac{1}{(n_1+1)^2(n-n_1)^2} \leq \frac{2}{n^3} + \frac{1}{2(n-1)} \frac{3}{n^2} \\ &\leq \frac{4}{n^3}, \end{aligned} \quad (\text{C.4.5})$$

where we used the second statement in (C.4.1). □

Lemma C.5. • For integers $n \geq 3, l \geq 0, \lambda \geq 0$

$$\sum_{\substack{n_1+n_2=n+1 \\ n_i \geq 1 \\ l_1+l_2=l \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2(l_2+1)^2 n_1^2 n_2^2} \frac{n!}{n_1! n_2!} \frac{\lambda!}{\lambda_1! \lambda_2!} \frac{(n_1+l_1-1)! (n_2+l_2-1)!}{(n+l-1)!} \leq K_0 \frac{1}{(l+1)^2} \frac{1}{n^2}, \quad (\text{C.4.6})$$

where we may choose $K_0 = 20$.

• For $n \geq 1, n_1 = 1, n_2 = n$

$$\sum_{\substack{l_1+l_2=l \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2(l_2+1)^2 n_1^2 n_2^2} \frac{n!}{n_1! n_2!} \frac{\lambda!}{\lambda_1! \lambda_2!} \frac{(n_1+l_1-1)! (n_2+l_2-1)!}{(n+l-1)!} \leq K'_0 \frac{1}{(l+1)^2} \frac{1}{n^2}, \quad (\text{C.4.7})$$

where we may choose $K'_0 = 5$.

- For integers $n \geq 3, l \geq 0, \lambda \geq 0, k \geq \alpha, \alpha \in \mathbb{N}_0$.

$$\begin{aligned} & \sum_{\substack{n_1+n_2=n+1 \\ n_i \geq 1 \\ l_1+l_2=l \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1+\lambda_2=\lambda \\ k_1+k_2=k-\alpha}} \frac{(k-\alpha)!}{k_1! k_2!} \frac{(n_1+l_1+k_1)! (n_2+l_2+k_2)!}{(l_1+1)^2 (l_2+1)^2 n_1^2 n_2^2} \frac{(n+1)!}{(n+l+k-\alpha+1)!} \frac{\lambda!}{n_1! n_2! \lambda_1! \lambda_2!} \frac{1}{(k_1+1)^2 (k_2+1)^2} \\ & \leq K''_0 \frac{1}{(l+1)^2} \frac{1}{n^2} \frac{1}{(k-\alpha+1)^2}, \end{aligned} \tag{C.4.8}$$

where we may choose $K''_0 = 75$.

- For integers $n \geq 1, k \geq 0, n_1 = 1, n_2 = n$

$$\begin{aligned} & \sum_{\substack{l_1+l_2=l \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1+\lambda_2=\lambda \\ k_1+k_2=k-\alpha}} \frac{(k-\alpha)!}{k_1! k_2!} \frac{(l_1+k_1+1)! (n+l_2+k_2)!}{(l_1+1)^2 (l_2+1)^2 n^2} \frac{(n+1)!}{(n+l+k-\alpha+1)!} \frac{\lambda!}{n! \lambda_1! \lambda_2!} \frac{1}{(k_1+1)^2 (k_2+1)^2} \\ & \leq K'''_0 \frac{1}{(l+1)^2} \frac{1}{n^2} \frac{1}{(k-\alpha+1)^2}, \end{aligned} \tag{C.4.9}$$

where we may choose $K'''_0 = 25$.

Proof. First for $n_1, n_2 \geq 1, l_1, l_2, \lambda_1, \lambda_2 \geq 0$

$$\begin{aligned} & \frac{n!}{n_1! n_2!} \frac{\lambda!}{\lambda_1! \lambda_2!} \frac{(n_1+l_1-1)! (n_2+l_2-1)!}{(n+l-1)!} \\ & = \frac{n}{n_1 n_2} \binom{n-1}{n_1-1} \binom{\lambda}{\lambda_1} \left[\binom{n+l-1}{n_1+l_1-1} \right]^{-1}. \end{aligned} \tag{C.4.10}$$

From the Vandermonde identity, we have the following inequality

$$\binom{a}{b} \binom{c}{d} \leq \binom{a+c}{b+d}, \quad a, b, c, d \in \mathbb{N}_0. \tag{C.4.11}$$

Then we show that for $l = l_1 + l_2$,

$$\sum_{\substack{\lambda_1 \leq l_1, \lambda_2 \leq l_2, \\ \lambda_1 + \lambda_2 = \lambda}} \frac{\lambda!}{\lambda_1! \lambda_2!} \leq \binom{l}{l_1}. \tag{C.4.12}$$

We proceed as follows: we assume that $l \geq 1$ and without loss $l_2 \leq l_1$. By induction on $0 \leq a \leq l_2$ we prove that

$$A_k := \left[\binom{l}{l_1} \right]^{-1} \sum_{\substack{\lambda_1 \leq l_1, \lambda_2 \leq l_2, \\ \lambda_1 + \lambda_2 = \lambda - a}} \frac{(l-a)!}{\lambda_1! \lambda_2!} \leq 1. \tag{C.4.13}$$

We start from $A_0 = 1$ since in the sum, only $\lambda_2 = l_2$ and $\lambda_1 = l_1$ are allowed when $a = 0$. Assuming that for $a \geq 1$, $A_{a-1} \leq 1$, we find

$$A_a = \frac{l_1 - (a-1)}{l - (a-1)} A_{a-1} + \left[\binom{l}{l_1} \right]^{-1} \binom{l-a}{l_1} \leq 1 - \frac{l_2}{l - (a-1)} + \frac{l_2 (l_2 - 1) \cdots (l_2 - (a-1))}{l (l-1) \cdots (l - (a-1))}. \quad (\text{C.4.14})$$

The latter expression equals 1 for $a = 1$. For $a > 1$, we can bound the upper bound in (C.4.14) by

$$1 - \frac{l_2}{l - (a-1)} \left(1 - \frac{(l_2 - 1)(l_2 - 2) \cdots (l_2 - (a-1))}{l(l-1) \cdots (l - (a-2))} \right) \leq 1. \quad (\text{C.4.15})$$

For $l_2 < a \leq l$, the sum in A_a does not contain more non-vanishing terms than the one in A_{a-1} and we can bound them as follows:

$$\frac{(l-a)!}{\lambda_1! \lambda_2!} \leq \frac{(l-(a-1))!}{(\lambda_1 + 1)! \lambda_2!}. \quad (\text{C.4.16})$$

Therefore, in that case, we have $A_a \leq A_{a-1}$. Now, from (C.4.11) and (C.4.12) we have

$$\sum_{\substack{\lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1 + \lambda_2 = \lambda}} \frac{n}{n_1 n_2} \frac{(n_1 + l_1 - 1)! (n_2 + l_2 - 1)! (n-1)! \lambda!}{(n_1 - 1)! \lambda_1! (n_2 - 1)! \lambda_2! (n + l - 1)!} \leq \frac{n}{n_1 n_2}. \quad (\text{C.4.17})$$

Using Lemma C.4 we obtain statement (C.4.6). Proof of statement (C.4.7) follows the proof of (C.4.6).

To prove statements (C.4.8)-(C.4.9), we note that for $n_1, n_2 \geq 1$, $k_1, k_2, l_1, l_2, \lambda_1, \lambda_2 \geq 0$ and $0 \leq \alpha \leq k$

$$\begin{aligned} & \frac{(k-\alpha)! (n+1)!}{k_1! k_2!} \frac{\lambda!}{n_1! n_2!} \frac{(n_1 + l_1 + k_1)! (n_2 + l_2 + k_2)!}{\lambda_1! \lambda_2!} \frac{1}{(n + l + k - \alpha + 1)!} \\ &= \binom{k-\alpha}{k_1} \binom{n+1}{n_1} \binom{\lambda}{\lambda_1} \left[\binom{n+l+k-\alpha+1}{n_1 + l_1 + k_1} \right]^{-1}. \end{aligned} \quad (\text{C.4.18})$$

Then from (C.4.11) we have

$$\binom{k-\alpha}{k_1} \binom{n+1}{n_1} \binom{l}{l_1} \leq \binom{n+l+k-\alpha+1}{n_1 + l_1 + k_1}. \quad (\text{C.4.19})$$

Then the rest of the proof is identical to the proof of (C.4.6). Proof of statement (C.4.9) follows from the proof of (C.4.8). \square

Lemma C.6. For $s \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $\alpha_0 \leq \alpha \leq \alpha_{\max} = \frac{1}{m^2}$,

$$\sum_{\lambda=0}^l \frac{1}{2^\lambda \lambda!} \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{s-1} (1 - \ln(m^2 \alpha'))^\lambda \leq \frac{2\alpha^s}{s} \sum_{\lambda=0}^l \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda. \quad (\text{C.4.20})$$

Proof. Through successive integration by parts, we obtain for $0 \leq \lambda \leq l$

$$\begin{aligned} \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{s-1} (1 - \ln(m^2 \alpha'))^\lambda &\leq \frac{\alpha^s}{s} (1 - \ln(m^2 \alpha))^\lambda + \frac{\lambda}{s} \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{s-1} (1 - \ln(m^2 \alpha'))^{\lambda-1} \\ &\leq \frac{\alpha^s}{s} \lambda! \sum_{\nu=0}^{\lambda} \frac{(1 - \ln(m^2 \alpha))^\nu}{\nu!} \frac{1}{s^{\lambda-\nu}}. \end{aligned} \quad (\text{C.4.21})$$

Summing over λ , we get

$$\begin{aligned}
\sum_{\lambda=0}^l \frac{1}{2^{\lambda} \lambda!} \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{s-1} (1 - \ln(m^2 \alpha'))^{\lambda} &\leq \frac{\alpha^s}{s} \sum_{\lambda=0}^l \frac{1}{2^{\lambda}} \sum_{\nu=0}^{\lambda} \frac{(1 - \ln(m^2 \alpha))^{\nu}}{\nu!} \frac{1}{s^{\lambda-\nu}} \\
&= \frac{\alpha^s}{s} \sum_{\nu=0}^l \frac{(1 - \ln(m^2 \alpha))^{\nu}}{2^{\nu} \nu!} \sum_{\lambda=0}^{l-\nu} \frac{1}{(2s)^{\lambda}} \\
&\leq \frac{2\alpha^s}{s} \sum_{\nu=0}^l \frac{(1 - \ln(m^2 \alpha))^{\nu}}{2^{\nu} \nu!}.
\end{aligned} \tag{C.4.22}$$

□

C.5 Proofs of (7.2.21) and (7.2.22)

C.5.1 Proof of (7.2.21)

We have

$$\left(\frac{3}{2}\right)^n \frac{(n-1)!}{(n+1)^n} \leq \left(\frac{7}{10}\right)^n \frac{3}{10}, \quad n \geq 4. \tag{C.5.1}$$

Proof. We set

$$a_n := \left(\frac{15}{7}\right)^n \frac{(n-1)!}{(n+1)^n}, \quad n \geq 1. \tag{C.5.2}$$

We have for $n \geq 1$

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{15}{7} \frac{n(n+1)^n}{(n+2)^{n+1}} \\
&\leq \frac{15}{7} \left(1 - \frac{1}{n+2}\right)^{n+1} \leq \frac{15}{7} e^{-\frac{5}{6}} \leq 1.
\end{aligned} \tag{C.5.3}$$

Therefore we have

$$a_n \leq a_4 \leq \frac{3}{10}. \tag{C.5.4}$$

□

C.5.2 Proof of (7.2.22)

We have

$$\frac{3}{10} + 2x \left(\frac{5}{7}\right)^n + \left(\frac{10}{7(n+1)}\right)^n \leq x, \quad x \geq 1, \quad n \geq 4. \tag{C.5.5}$$

Proof. It is sufficient to prove the claim for $n = 4$. For $n = 4$ we have

$$\frac{3}{10} + 2x \left(\frac{5}{7}\right)^4 + \left(\frac{2}{7}\right)^4 \leq x \Leftrightarrow x \geq \frac{7363}{11510}. \tag{C.5.6}$$

□

Appendix D

Bounds on the derivatives of functions

$H(\tilde{\mu}), h(\tilde{\mu})$

Lemma D.1.

$$0 < \frac{1}{H(\tilde{\mu})} \leq C, \quad \mu \in [0, \tilde{\mu}_{\max}]. \quad (\text{D.0.1})$$

Proof. We choose $\alpha_0 \leq \frac{1}{2m^2}$ so that $\tilde{\beta}_0 \leq \frac{1}{2}$. Obviously $H(\mu) > 0$ for $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$. We have successively

- For $\tilde{\mu} \in [0, -\ln(2\tilde{\beta}_0)]$, we have $H(\tilde{\mu}) \geq c(1 + \tilde{\beta}_0 - \frac{1}{2}) \geq \frac{c}{2} > 0$.
- For $\tilde{\mu} \in [-\ln(2\tilde{\beta}_0), \mu_{\max}]$ we have

$$H(\tilde{\mu}) \geq h(\tilde{\mu}) \geq \frac{1}{4} \int_k \frac{e^{-\frac{3k^2}{2}}}{k^2 + 1} \geq \frac{1}{4e^{\frac{3}{2}}} \int_{k, |k| \leq 1} \frac{1}{k^2 + 1} \geq \frac{1}{8e^{\frac{3}{2}}} \int_{k, |k| \leq 1} 1 = \frac{c}{16e^{\frac{3}{2}}}. \quad (\text{D.0.2})$$

Choosing $C := 256 e^{\frac{3}{2}} \pi^2$, the bound (D.0.1) is satisfied. \square

The next two Lemmata provide bounds on the derivatives of $h(\tilde{\mu})$ defined in (6.1.11).

Lemma 6.1. For $l \geq 0$ and $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$

$$|\partial_{\tilde{\mu}}^l h(\tilde{\mu})| \leq 5^l (l+1)^{l-1} c. \quad (\text{6.1.16})$$

Proof. The proof is done by induction in $l \geq 0$. First we note that $|h(\tilde{\mu})| \leq c$ for $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$. For $l = 1$ we have

$$|\partial_{\tilde{\mu}} h(\tilde{\mu})| = |(2 + \tilde{\beta}_0 e^{\tilde{\mu}})h(\tilde{\mu}) - c\tilde{\beta}_0 e^{\tilde{\mu}}| \leq (4 + 2\tilde{\beta}_0)c \leq 5c,$$

if $\tilde{\beta}_0 \leq \frac{1}{2}$. Using Leibniz's rule, we obtain for $l \geq 1$

$$\partial_{\tilde{\mu}}^l h(\tilde{\mu}) = \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} (2\delta_{l_1,0} + \tilde{\beta}_0 e^{\tilde{\mu}}) \partial_{\tilde{\mu}}^{l-1-l_1} h(\tilde{\mu}) - c\tilde{\beta}_0 e^{\tilde{\mu}}. \quad (\text{D.0.3})$$

Inserting the induction hypothesis in the r.h.s. of (D.0.6) we get

$$\begin{aligned}
|\partial_{\tilde{\mu}}^l h(\tilde{\mu})| &\leq \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} (3 + \tilde{\beta}_0) 5^{l-1-l_1} (l-l_1)^{l-2-l_1} c + c(1 + \tilde{\beta}_0) \\
&\leq (3 + \tilde{\beta}_0) 5^{l-1} \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} l^{l-1-l_1} c + c(1 + \tilde{\beta}_0) \\
&\leq (3 + \tilde{\beta}_0) 5^{l-1} (l+1)^{l-1} c + c(1 + \tilde{\beta}_0) \\
&\leq 5^l (l+1)^{l-1} c \left(\frac{7}{10} + \frac{3}{2 \cdot 5^l} \right) \leq 5^l (l+1)^{l-1} c.
\end{aligned} \tag{D.0.4}$$

□

Now we prove a stronger version of Lemma 6.1.

Lemma D.2. For $l \geq 0$ and $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$

$$|\partial_{\tilde{\mu}}^l h(\tilde{\mu})| \leq c(5e)^l |l-1|!. \tag{D.0.5}$$

Proof. The proof is done by induction in $l \geq 0$. To start the induction, we note $|h(\tilde{\mu})| \leq c$ for $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$. For $l = 1$ we have

$$|\partial_{\tilde{\mu}} h(\tilde{\mu})| = |(2 + \tilde{\beta}_0 e^{\tilde{\mu}})h(\tilde{\mu}) - c\tilde{\beta}_0 e^{\tilde{\mu}}| \leq (4 + 2\tilde{\beta}_0)c \leq c5e,$$

since $\tilde{\beta}_0 \leq \frac{1}{2}$. Using Leibniz' s rule, we obtain for $l \geq 1$

$$\partial_{\tilde{\mu}}^l h(\tilde{\mu}) = \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} (2\delta_{l_1,0} + \tilde{\beta}_0 e^{\tilde{\mu}}) \partial_{\tilde{\mu}}^{l-1-l_1} h(\tilde{\mu}) - c\tilde{\beta}_0 e^{\tilde{\mu}}. \tag{D.0.6}$$

Inserting the induction hypothesis in the r.h.s. of (D.0.6) we get

$$\begin{aligned}
|\partial_{\tilde{\mu}}^l h(\tilde{\mu})| &\leq \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} (3 + \tilde{\beta}_0) (5e)^{l-1-l_1} |l-2-l_1|! c + c(1 + \tilde{\beta}_0) \\
&\leq (3 + \tilde{\beta}_0) (5e)^{l-1} \sum_{l_1=0}^{l-1} \binom{l-1}{l_1} (l-1-l_1)! c + c(1 + \tilde{\beta}_0) \\
&\leq (3 + \tilde{\beta}_0) (5e)^{l-1} (l-1)! c + c(1 + \tilde{\beta}_0) \\
&\leq c(5e)^l (l-1)! \left(\frac{7}{10} + \frac{3}{2 \cdot 8^l} \right) \leq c(5e)^l (l-1)!.
\end{aligned} \tag{D.0.7}$$

□

Lemma D.3. For $l \geq 0$, $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$,

$$|\partial_{\tilde{\mu}}^l H(\tilde{\mu})| \leq 3c(5e)^l |l-1|!. \tag{D.0.8}$$

Proof. From Lemma D.2

$$|\partial_{\tilde{\mu}}^l H(\tilde{\mu})| \leq |c(1 + \tilde{\beta}_0) \delta_{l,0} - c \tilde{\beta}_0 e^{\tilde{\mu}}| + |\partial_{\tilde{\mu}}^l h(\tilde{\mu})| \leq 3c + c(5e)^l |l-1|! \leq 3c(5e)^l |l-1|!. \quad (\text{D.0.9})$$

□

Lemma D.4. For $l \geq 1$ and $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$,

$$|\partial_{\tilde{\mu}}^l \ln(H(\tilde{\mu}))| \leq c C^l (5e)^{l+1} 2^l (l-1)!. \quad (\text{D.0.10})$$

Proof. For $l = 1$ the bounds derived for $h, \partial_{\tilde{\mu}} h$ in Lemmas D.1 and D.3 yield :

$$|\partial_{\tilde{\mu}} \ln H(\tilde{\mu})| \leq c C 15 e \leq c C 50 e^2. \quad (\text{D.0.11})$$

For $l \geq 1$, using (5.1.27), we get

$$\partial_{\tilde{\mu}}^{l+1} \ln H(\tilde{\mu}) = \partial_{\tilde{\mu}}^l \left(\frac{\partial_{\tilde{\mu}} H(\tilde{\mu})}{H(\tilde{\mu})} \right) = \frac{1}{H(\tilde{\mu})} \left[\partial_{\tilde{\mu}}^{l+1} H(\tilde{\mu}) - l \sum_{j=1}^l \frac{\partial_{\tilde{\mu}}^{l+1-j} H(\tilde{\mu})}{(l+1-j)! (j-1)!} \partial_{\tilde{\mu}}^{j-1} \left(\frac{\partial_{\tilde{\mu}} H(\tilde{\mu})}{H(\tilde{\mu})} \right) \right]. \quad (\text{D.0.12})$$

Since $\partial_{\tilde{\mu}}^{j-1} \left(\frac{\partial_{\tilde{\mu}} H(\tilde{\mu})}{H(\tilde{\mu})} \right) = \partial_{\tilde{\mu}}^j \ln H(\tilde{\mu})$, we can then proceed by induction on the r.h.s. of (D.0.12).

From Lemma D.3, we obtain

$$\begin{aligned} |\partial_{\tilde{\mu}}^{l+1} \ln H(\tilde{\mu})| &\leq C \left[c(5e)^{l+1} l! + l! \sum_{j=1}^l \frac{4c(5e)^{l+1-j} (l-j)!}{(l+1-j)! (j-1)!} c C^j (5e)^{j+1} 2^j (j-1)! \right] \\ &\leq c C^{l+1} (5e)^{l+2} \left[\frac{l!}{5e C^l} + 4cl! \sum_{j=1}^l 2^j \right] \\ &\leq c C^{l+1} (5e)^{l+2} 2^{l+1} l! \left[\frac{1}{10eC} + \frac{1}{4\pi^2} \right] \leq c C^{l+1} (5e)^{l+2} 2^{l+1} l!. \end{aligned} \quad (\text{D.0.13})$$

□

From Lemmas D.2-D.4, one sees that $h(\tilde{\mu}), H(\tilde{\mu})$ and $\ln(H(\tilde{\mu}))$ are all analytic w.r.t. $\tilde{\mu}$.

Appendix E

Bounds on the mean-field solutions implying analyticity of the mean-field effective action w.r.t. the mean-field variable ϕ

E.1 Proof of Proposition 3.1

Proposition 3.1. *For $K \geq \sup(K_1, 4)$ and if $f_2(\mu)$ satisfies the bounds (3.2.6), then for $n \in 2\mathbb{N} + 2$, $l \geq 0$ and $\mu \in [0, \mu_{\max}]$*

$$|\partial_\mu^l f_n(\mu)| \leq \frac{K^{n+l-2} \delta^{l+1}}{(l+1)^2} \frac{(n+l-2)!}{(n-2)!}. \quad (3.2.7)$$

Proof. The proof is done by induction in $n+l \geq 2$, going up in n . Due to assumption (3.2.6), the bounds (3.2.7) are satisfied for $n=2$. For $n \geq 4$, we insert the induction hypothesis in (3.1.11) and it is straightforward to see that the bounds (3.2.7) hold. For $n \geq 4$, we differentiate (3.1.11) l times w.r.t. μ and we bound the r.h.s. with the induction hypothesis. We obtain

$$\begin{aligned} & |\partial_\mu^l f_{n+2}(\mu)| \\ & \leq \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{l_1+l_2=l} \binom{l}{l_1} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n_1+l_1-2)! (n_2+l_2-2)!}{(n_1-2)! (n_2-2)!} \\ & + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \binom{l}{l_1} \frac{K^{n+l_1-2}}{(l_1+1)^2} \frac{(n+l_1-2)!}{(n-2)!} \left[\delta \frac{K^{2+l_2-2}}{(l_2+1)^2} 2(2+l_2-2)! + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\ & + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}. \end{aligned} \quad (E.1.1)$$

We use (5.1.56) to obtain

$$\begin{aligned} \binom{l}{l_1} \frac{(n_1+l_1-2)! (n_2+l_2-2)!}{(n_1-2)! (n_2-2)!} & = \binom{l}{l_1} \binom{n-2}{n_1-2} \frac{(n_1+l_1-2)! (n_2+l_2-2)!}{(n-2)!} \\ & \leq \frac{(n+l-2)!}{(n-2)!} \end{aligned} \quad (E.1.2)$$

and

$$\binom{l}{l_1} \frac{(n+l_1-2)! (2+l_2-2)!}{(n-2)! (2-2)!} \leq \frac{(n+l-2)!}{(n-2)!}. \quad (\text{E.1.3})$$

Therefore we find that (E.1.1) is bounded by

$$\begin{aligned} & \frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{l_1+l_2=l} \frac{K^{n+l-2}}{(l_1+1)^2 (l_2+1)^2} \frac{(n+l-2)!}{(n-2)!} \\ & + \frac{\delta^{l+1}}{n+1} \sum_{l_1+l_2=l} \frac{K^{n+l-2}}{(l_1+1)^2} \frac{(n+l-2)!}{(n-2)!} \left[\frac{2\delta}{(l_2+1)^2} + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \\ & + \frac{2\delta^{l+2}}{n(n+1)} \frac{K^{n+l+1-2}}{(l+2)^2} \frac{(n+l+1-2)!}{(n-2)!}. \end{aligned} \quad (\text{E.1.4})$$

We have used the fact that in the second sum, the Kronecker symbol $\delta_{l_2,0}$ only selects the term corresponding to $l_1 = l$ so that we have the right combinatorial factor. We now choose K sufficiently large (we will detail at the end of the proof the possible values of K) such that for $n \geq 6$

$$\frac{\delta^{l+2}}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{l_1+l_2=l} \frac{1}{(l_1+1)^2 (l_2+1)^2} \leq \frac{1}{3} \frac{K^2}{(l+1)^2} \frac{(n+l)(n+l-1)}{n(n-1)} \quad (\text{E.1.5})$$

and

$$\frac{1}{n+1} \sum_{l_1+l_2=l} \frac{1}{(l_1+1)^2 (l_2+1)^2} \left[2\delta + \delta_{l_2,0} \left(1 - \frac{4}{n}\right) \right] \leq \frac{K^2}{2(l+1)^2} \frac{(n+l)(n+l-1)}{n(n-1)} \quad (\text{E.1.6})$$

and

$$\frac{2\delta}{n(n+1)} \frac{1}{(l+2)^2} \leq \frac{K}{6} \frac{1}{(l+1)^2} \frac{n+l}{n(n-1)}. \quad (\text{E.1.7})$$

Then (E.1.4) is finally bounded by

$$\left[\frac{1}{3} + \frac{1}{2} + \frac{1}{6} \right] \frac{\delta^{l+1} K^{n+l} (n+l)!}{(l+1)^2 n!}. \quad (\text{E.1.8})$$

If $\delta = 1$ and $K_1 \leq 4$, $K = 4$ is admissible and if $\delta < 1$ and $K_1 < 4$, we can choose smaller values of K . \square

E.2 Proof of Proposition 6.1

Proposition 6.1. *For a suitable $K > \sup(K_1, \frac{1}{\delta}, C)$ with C the constant in Lemma D.1 and $f_2(\tilde{\mu})$ satisfying (6.1.17), the mean field solutions $f_n(\tilde{\mu})$ of (6.1.9) are smooth and for $\tilde{\mu} \in [0, \tilde{\mu}_{\max}]$, $n \geq 2$ and $l \geq 0$,*

$$|\partial_{\tilde{\mu}}^l f_n(\tilde{\mu})| \leq K^{n+l-2} 5^l \delta^{l+1} \frac{|n+l-4|^{n+l-4}}{(n-2)!}. \quad (\text{6.1.18})$$

Proof. We prove the proposition by induction in $N = n + l \geq 2$. The bounds hold for the two-point function from the assumption (6.1.17). For $n \geq 2$ we derive l times w.r.t. $\tilde{\mu}$ (6.1.9) and we isolate $H(\tilde{\mu})\partial_{\tilde{\mu}}^l f_{n+2}(\tilde{\mu})$ in the l.h.s. of (6.1.9). We obtain

$$\begin{aligned}
H(\tilde{\mu})\partial_{\tilde{\mu}}^l f_{n+2}(\tilde{\mu}) &= \frac{2}{n(n+1)}\partial_{\tilde{\mu}}^{l+1} f_n(\tilde{\mu}) + \frac{n-4}{n(n+1)}\partial_{\tilde{\mu}}^l f_n(\tilde{\mu}) + \frac{1}{n+1}\tilde{\beta}_0\partial_{\tilde{\mu}}^l \left(e^{\tilde{\mu}} f_n(\tilde{\mu}) \right) \\
&\quad - \sum_{l_1=1}^l \binom{l}{l_1} \partial_{\tilde{\mu}}^{l_1} H(\tilde{\mu}) \partial_{\tilde{\mu}}^{l-l_1} f_{n+2}(\tilde{\mu}) \\
&\quad + \frac{1}{n+1} \sum_{l_1+l_2=l} \sum_{n_1+n_2=n+2} \binom{l}{l_1} \left((2 + \tilde{\beta}_0)\delta_{l_1,0} - \tilde{\beta}_0 e^{\tilde{\mu}} \right) \partial_{\tilde{\mu}}^{l_2} \left(f_{n_1}(\tilde{\mu}) f_{n_2}(\tilde{\mu}) \right).
\end{aligned} \tag{E.2.1}$$

Inserting the induction hypothesis in the r.h.s. of (E.2.1) and using Lemma 6.1 we get

$$\begin{aligned}
H(\tilde{\mu})|\partial_{\tilde{\mu}}^l f_{n+2}(\tilde{\mu})| &\leq \frac{2}{n(n+1)} K^{n+l-1} 5^{l+1} \delta^{l+2} \frac{|n+l-3|^{n+l-3}}{(n-1)!} \\
&\quad + \frac{n-4}{n(n+1)} K^{n+l-2} 5^l \delta^{l+1} \frac{|n+l-4|^{n+l-4}}{(n-2)!} \\
&\quad + \frac{1}{n+1} \tilde{\beta}_0 \sum_{l_1=0}^l \binom{l}{l_1} e^{\tilde{\mu}} K^{n+l-l_1-2} 5^{l-l_1} \delta^{l-l_1+1} \frac{|n+l-l_1-4|^{n+l-l_1-4}}{(n-2)!} \\
&\quad + \sum_{l_1=1}^l \binom{l}{l_1} c(\tilde{\beta}_0 e^{\tilde{\mu}} + 5^{l_1} (l_1+1)^{l_1-1}) K^{n+l-l_1} 5^{l-l_1} \delta^{l-l_1+1} \frac{(n+l-l_1-2)^{n+l-l_1-2}}{n!} \\
&\quad + \frac{1}{n+1} \sum_{n_1+n_2=n+2} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \binom{l-l_1}{l_2} \left((2 + \tilde{\beta}_0)\delta_{l_1,0} + \tilde{\beta}_0 e^{\tilde{\mu}} \right) \\
&\quad K^{n_1+l_2-2} 5^{l_2} \delta^{l_2+1} \frac{|n_1+l_2-4|^{n_1+l_2-4}}{(n_1-2)!} \\
&\quad \times K^{n_2+l-l_1-l_2-2} 5^{l-l_1-l_2} \delta^{l-l_1-l_2+1} \frac{|n_2+l-l_1-l_2-4|^{n_2+l-l_1-l_2-4}}{(n_2-2)!}.
\end{aligned} \tag{E.2.2}$$

We focus on the five terms in the r.h.s. of (E.2.2).

- For the first term, we get

$$\frac{2}{n(n+1)} K^{n+l-1} 5^{l+1} \delta^{l+2} \frac{|n+l-3|^{n+l-3}}{(n-1)!} \leq K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} \frac{5\delta}{K}, \tag{E.2.3}$$

where we used

$$|n+l-3|^{n+l-3} \leq (n+l-2)^{n+l-2}, \quad n \geq 2, l \geq 0. \tag{E.2.4}$$

- For the second term we obtain

$$\frac{n-4}{n(n+1)} K^{n+l-2} 5^l \delta^{l+1} \frac{|n+l-4|^{n+l-4}}{(n-2)!} \leq K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} \frac{2}{K^2}, \quad (\text{E.2.5})$$

where we used

$$n |n+l-4|^{n+l-4} \leq 2(n+l-2)^{n+l-2}, \quad n \geq 2, l \geq 0. \quad (\text{E.2.6})$$

- The third term is bounded by

$$\frac{1 + \tilde{\beta}_0}{n+1} K^{n+l-2} 5^l \delta^{l+1} \sum_{l_1=0}^l \binom{l}{l_1} \frac{|n+l-l_1-4|^{n+l-l_1-4}}{(n-2)!}, \quad (\text{E.2.7})$$

because $K\delta > 1$. For $n = 2$ we have

$$\sum_{l_1=0}^l \binom{l}{l_1} |l-l_1-2|^{l-l_1-2} \leq \sum_{l_1=0}^l \binom{l}{l_1} (l_1+1)^{l_1} \leq (l+2)^l \leq l^l e^2. \quad (\text{E.2.8})$$

For $n \geq 4$ we obtain

$$\sum_{l_1=0}^l \binom{l}{l_1} \frac{(n+l-l_1-4)^{n+l-l_1-4}}{(n-2)!} \leq \frac{(n+l-3)^{n+l-4}}{(n-2)!} \leq \frac{(n+l-2)^{n+l-2}}{(n-2)! (n+l-2)^2}. \quad (\text{E.2.9})$$

From (E.2.8) and (E.2.9), (E.2.7) is bounded by

$$K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} \frac{3e^2}{2K^2}. \quad (\text{E.2.10})$$

- The fourth term does not exist if $l = 0$. For $l \geq 1$ it is bounded by

$$cK^{n+l-1} 5^l \delta^{l-1+1} \left[2 \sum_{l_1=1}^l \binom{l}{l_1} \frac{(n+l-3)^{n+l-l_1-2}}{n!} + \sum_{l_1=1}^l \binom{l}{l_1} (l_1+1)^{l_1-1} \frac{(n+l-l_1-2)^{n+l-l_1-2}}{n!} \right], \quad (\text{E.2.11})$$

because $K\delta > 1$. The first sum in (E.2.11) is obviously bounded by $(n+l-2)^{n+l-2}$. For the other sum, we use the following formula from Sect. 3 in [113]

$$\sum_{r=0}^m \binom{m}{r} (r+1)^{r-1} (x-r)^{m-r} = (x+1)^m, \quad m \in \mathbb{N}, x \in \mathbb{N}_0. \quad (\text{E.2.12})$$

Using (E.2.12), it turns out that

$$\begin{aligned} & \sum_{l_1=1}^l \binom{l}{l_1} (l_1+1)^{l_1-1} (n+l-l_1-2)^{n+l-l_1-2} \\ & \leq \sum_{l_1=0}^{n+l-2} \binom{n+l-2}{l_1} (l_1+1)^{l_1-1} (n+l-l_1-2)^{n+l-l_1-2} \\ & \leq (n+l-1)^{n+l-2} \leq (n+l-2)^{n+l-2} e. \end{aligned}$$

Therefore, (E.2.11) is bounded by

$$K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2} (2+e)c}{n! K\delta}. \quad (\text{E.2.13})$$

- The fifth term is bounded by

$$\frac{3+2\tilde{\beta}_0}{n+1} K^{n+l-2} 5^l \delta^{l+2} \sum_{n_1+n_2=n+2} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, n_1, n_2).$$

where we introduce for simplicity and compactness

$$\mathcal{F}'(l, l_1, l_2, n_1, n_2) := \binom{l-l_1}{l_2} \frac{|n_1+l_2-4|^{n_1+l_2-4}}{(n_1-2)!} \frac{|n_2+l-l_1-l_2-4|^{n_2+l-l_1-l_2-4}}{(n_2-2)!}.$$

Let $(n_1, n_2) \in \{2, \dots, n\}^2$ such that $n_1 + n_2 = n + 2$ and $0 \leq l_1 \leq l$.

- First case $n_1 = 2, n_2 = n$: we distinguish two cases. If $n = 2$ we get, using (E.2.12)

$$\begin{aligned} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, 2, 2) &\leq \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \binom{l-l_1}{l_2} (l_2+1)^{l_2-1} (l-l_1-l_2)^{l-l_1-l_2} \\ &\leq \sum_{l_1=0}^l \binom{l}{l_1} (l+1-l_1)^{l-l_1} \leq l^2 e^2. \end{aligned}$$

For $n \geq 4$, using again (E.2.12), we obtain the following bound

$$\frac{1}{n+1} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, 2, n) \leq \frac{(n+l-2)^{n+l-2}}{n!} \frac{n}{(n+l-2)^2}. \quad (\text{E.2.14})$$

Overall we obtain the following bound for the double sums

$$\frac{1}{n+1} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, 2, n) \leq \frac{(n+l-2)^{n+l-2}}{n!} e^2. \quad (\text{E.2.15})$$

- The case $n_1 = n, n_2 = 2$ is symmetric with the previous case.
- The case $n_i \geq 4$ is relevant when $n \geq 6$. We distinguish the case where one of the n_i 's is equal to 4 and the other case $n_i > 4$ (this case is relevant for $n \geq 10$). The first case is treated the same way as in the previous cases. The computation gives the bound (we choose here $n_1 = 4$)

$$\frac{1}{n+1} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, 4, n-2) \leq \frac{(n+l-2)^{n+l-2}}{n!} 2.$$

For $n_i > 4$, we use the following inequalities

$$n^n e^{-n} \sqrt{n} \leq n! \leq e n^n e^{-n} \sqrt{n}, \quad n \geq 1. \quad (\text{E.2.16})$$

They obviously hold for $n = 1$ while for $n \geq 2$, they follow from (6.1.20).

Let $(n_1, n_2) \in \{6, n - 4\}^2$ such that $n_1 + n_2 = n + 2$ and $0 \leq l_1 \leq l$. Then for $0 \leq l_2 \leq l - l_1$ we use (E.2.16) and (5.1.56) to obtain

$$\begin{aligned} & \mathcal{F}'(l, l_1, l_2, n_1, n_2) \\ & \leq \binom{l-l_1}{l_2} \frac{(n_1+l_2-4)! (n_2+l-l_1-l_2-4)!}{(n_1-2)! (n_2-2)!} \frac{e^{n+l-l_1-6}}{\sqrt{(n_1+l_2-4)(n_2+l-l_1-l_2-4)}} \\ & \leq \frac{(n+l-l_1-2)!}{(n-2)!} \frac{e^{n+l-6}}{\left[(n_1+l_2-4)(n_2+l-l_1-l_2-4)\right]^{\frac{5}{2}}}. \end{aligned} \quad (\text{E.2.17})$$

Then, we sum over l_2 , and we use the Cauchy-Schwarz inequality and the following bounds (assume that $a > 0$ and $b > 0$) which are obtained through elementary partial fraction decomposition

$$\begin{aligned} \frac{1}{(a+l_2)^2(b-l_2)^2} & \leq \frac{1}{(a+b)^2} \left[\frac{1}{(a+l_2)^2} + \frac{1}{(b-l_2)^2} + \frac{2}{(a+b)(a+l_2)} \right], \quad 0 \leq l_2 < b, \\ \sum_{l_2 \geq 1} \frac{1}{l_2^2} & \leq 2, \quad \sum_{l_2 \geq 1} \frac{1}{l_2^3} \leq 2 \end{aligned} \quad (\text{E.2.18})$$

and

$$\begin{aligned} & \frac{1}{(a+l_2)^3(b-l_2)^3} \\ & \leq \frac{1}{(a+b)^3} \left[\frac{1}{(a+l_2)^3} + \frac{1}{(b-l_2)^3} + \frac{3}{(a+b)(a+l_2)^2} + \frac{12}{(a+b)^2(a+l_2)} \right. \\ & \quad \left. + \frac{12}{(a+b)^2(b-l_2)} \right], \quad 0 \leq l_2 < b \end{aligned} \quad (\text{E.2.19})$$

to get

$$\sum_{l_2=0}^{l-l_1} \mathcal{F}'(l, l_1, l_2, n_1, n_2) \leq \frac{(n+l-l_1-2)! e^{n+l-6}}{(n-2)!} \frac{\sqrt{60}}{(n+l-l_1-6)^{\frac{5}{2}}}. \quad (\text{E.2.20})$$

Then we have using (F.1.4)

$$\sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, n_1, n_2) \leq \frac{\sqrt{60}(n+l-1)! e^{n+l-6}}{(n-1)!} \sum_{l_1=0}^l \frac{1}{l_1! (n+l-l_1-6)^{\frac{5}{2}}}. \quad (\text{E.2.21})$$

We use the fact that

$$\sum_{l_1=0}^l \frac{1}{l_1! (n+l-l_1-6)^{\frac{5}{2}}} \leq \frac{3}{(n+l-9)^{\frac{5}{2}}} + \sum_{l_1=4}^l \frac{1}{(l_1-3)! \left[(l_1-2)(n+l-l_1-6)\right]^{\frac{5}{2}}}. \quad (\text{E.2.22})$$

For $l < 4$, we have using (E.2.16) and keeping only the first term in the r.h.s. of (E.2.22)

$$\frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i > 4}} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, n_1, n_2) \leq \frac{(n+l-2)^{n+l-2}}{n!} C_1, \quad (\text{E.2.23})$$

for a fixed constant $C_1 > 1$. For $l \geq 4$, (E.2.22) is bounded by

$$\frac{3}{(n+l-9)^{\frac{5}{2}}} + \frac{1}{4(n+l-10)^{\frac{5}{2}}} \sum_{l_1=4}^l \frac{1}{(l-3)!} \leq \frac{4}{(n+l-10)^{\frac{5}{2}}}.$$

As a result, we have

$$\frac{1}{n+1} \sum_{\substack{n_1+n_2=n+2 \\ n_i > 4}} \sum_{l_1=0}^l \sum_{l_2=0}^{l-l_1} \binom{l}{l_1} \mathcal{F}'(l, l_1, l_2, n_1, n_2) \leq \frac{(n+l-2)^{n+l-2}}{n!} C_2 \quad (\text{E.2.24})$$

for a fixed constant $C_2 > 1$.

Overall, the fifth term is bounded by

$$K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} \frac{C_3 \delta}{K^2}, \quad (\text{E.2.25})$$

for a fixed constant $C_3 > 1$.

Summing (E.2.3)-(E.2.5)-(E.2.10)-(E.2.13) and (E.2.25) yields

$$H(\tilde{\mu}) |\partial_{\tilde{\mu}}^l f_{n+2}(\tilde{\mu})| \leq K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} \left[\frac{5\delta}{K} + \frac{2}{K^2} + \frac{3e^2}{2K^2} + \frac{(2+e)c}{K\delta} + \frac{C_3\delta}{K^2} \right].$$

Using Lemma D.1 we get

$$\begin{aligned} |\partial_{\tilde{\mu}}^l f_{n+2}(\tilde{\mu})| &\leq K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!} C \left[\frac{5\delta}{K} + \frac{2}{K^2} + \frac{3e^2}{2K^2} + \frac{(2+e)c}{K\delta} + \frac{C_3\delta}{K^2} \right] \\ &\leq K^{n+l} 5^l \delta^{l+1} \frac{(n+l-2)^{n+l-2}}{n!}, \end{aligned} \quad (\text{E.2.26})$$

choosing $K > \sup(K_1, \frac{1}{\delta}, C)$ sufficiently large such that

$$K \geq C \left(5\delta + 2 + \frac{3e^2}{2} + \frac{(2+e)c}{\delta} + C_3\delta \right). \quad (\text{E.2.27})$$

□

Appendix F

Proof of Proposition 5.1

F.1 Preparatory Lemmata

It is useful to define for $n_1, n_2 \in 2\mathbb{N} \setminus \{2\}$ and $k, \nu \in \mathbb{N}_0$

$$g(n_1, n_2, k, \nu) := \frac{\left| \frac{n_1}{4} + \nu - 3 \right|! \left| \frac{n_2}{4} + k - \nu - 1 \right|!}{(k + 2 - \nu)! \nu!}, \quad (\text{F.1.1})$$

where for $n \in \mathbb{C} \setminus \mathbb{Z}_0^-$ we define $n! := \Gamma(n + 1)$ with Γ the Gamma function. From the definition of the factorial using the Gamma function, the definition of the binomial coefficient $\binom{n}{k}$ can be extended to $n, k \in \mathbb{C} \setminus \mathbb{Z}_0^-$ such that $n - k \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by

$$\binom{n}{k} = \frac{n!}{k! (n - k)!}.$$

We also define for $l \in (0, 1)$, $n_1, n_2 \in 2\mathbb{N} \setminus \{2\}$ and $k, a, b \in \mathbb{N}_0$

$$F(n_1, n_2, k, a, b, l) := \sum_{\nu=a}^{k+2-b} \frac{\left| \frac{n_1}{4} + \nu - 3 \right|! \left| \frac{n_2}{4} + k - \nu - 1 \right|!}{[(k + 2 - \nu)! \nu!]^l}. \quad (\text{F.1.2})$$

The quantities $g(n_1, n_2, k, \nu)$ and $F(n_1, n_2, k, a, b, l)$ both appear in the proofs of the bounds on $g_{n,k}$. Therefore it is useful to establish useful bounds on them which will be used in the inductive proofs of bounds on $g_{n,k}$.

Lemma F.1. *For $n_1, n_2, k, a, b \in \mathbb{N}_0$ such that*

- $a + b \leq k + 2$
- $3 - \frac{n_1}{4} \leq a$ and $3 - \frac{n_2}{4} \leq b$

we have

$$\mathcal{S}(n_1, n_2, k, a, b) := \sum_{\nu=a}^{k+2-b} g(n_1, n_2, k, \nu) \leq \frac{1}{\frac{n_1+n_2}{4} + a + b - 5} \frac{\left(\frac{n_1+n_2}{4} + k - 3 \right)!}{(k + 2 - (a + b))!}. \quad (\text{F.1.3})$$

Proof. We will use the following equality, found in Sect.1.10 of [113]

$$\sum_{\nu=0}^m \binom{a+\nu}{\nu} \binom{r+m-\nu}{m-\nu} = \binom{a+r+m+1}{m}, \quad a, r \geq 0, m \in \mathbb{N}. \quad (\text{F.1.4})$$

Using (F.1.4) we get

$$\begin{aligned} \mathcal{S}(n_1, n_2, k, a, b) &= \sum_{\nu=0}^{k+2-(a+b)} \frac{\left(\frac{n_1}{4} - 3 + a + \nu\right)! \left(\frac{n_2}{4} - 3 + b + k + 2 - (a+b) - \nu\right)!}{(\nu+a)! (k+2-a-\nu)!} \\ &\leq \sum_{\nu=0}^{k+2-(a+b)} \frac{\left(\frac{n_1}{4} - 3 + a + \nu\right)! \left(\frac{n_2}{4} - 3 + b + k + 2 - (a+b) - \nu\right)!}{\nu! (k+2-(a+b)-\nu)!} \\ &\leq \sum_{\nu=0}^{k+2-(a+b)} \left(\frac{n_1}{4} - 3 + a\right)! \left(\frac{n_2}{4} - 3 + b\right)! \binom{\frac{n_1}{4} + a - 3 + \nu}{\nu} \\ &\quad \binom{\frac{n_2}{4} + b - 3 + k + 2 - (a+b) - \nu}{k+2-(a+b)-\nu} \\ &\leq \left(\frac{n_1}{4} - 3 + a\right)! \left(\frac{n_2}{4} - 3 + b\right)! \binom{\frac{n_1+n_2}{4} + k - 3}{k+2-(a+b)} \\ &\leq \frac{1}{\frac{n_1+n_2}{4} + a + b - 5} \frac{\left(\frac{n_1+n_2}{4} + k - 3\right)!}{(k+2-(a+b))!}, \end{aligned} \quad (\text{F.1.5})$$

where we used

$$a! b! \leq (a+b)!, \quad a, b \geq 0.$$

□

A consequence of Lemma F.1 is

Lemma F.2. Let n_1, n_2, k, a and $l \in (0, 1)$ such that

- $2a \leq k + 2$
- $3 - \frac{n_1}{4} \leq a$ and $3 - \frac{n_2}{4} \leq a$

then:

$$F(n_1, n_2, k, a, a, l) \leq [a!(k+2-a)]^{1-l} \frac{1}{\frac{n_1+n_2}{4} + 2a - 5} \frac{\left(\frac{n_1+n_2}{4} + k - 3\right)!}{(k+2-2a)!}. \quad (\text{F.1.6})$$

Proof. We use the following identity

$$\nu! (k+2-\nu)! \leq a! (k+2-a)!, \quad a \leq \nu \leq k+2-a. \quad (\text{F.1.7})$$

Then from Lemma F.1, it follows that

$$\begin{aligned} F(n_1, n_2, k, a, a, l) &\leq [a! (k+2-a)]^{1-l} \mathcal{S}(n_1, n_2, k, a, a) \\ &\leq [a! (k+2-a)]^{1-l} \frac{1}{\frac{n_1+n_2}{4} + 2a - 5} \frac{\left(\frac{n_1+n_2}{4} + k - 3\right)!}{(k+2-2a)!}. \end{aligned} \quad (\text{F.1.8})$$

□

F.2 Proof of Proposition 5.1

Proposition 5.1. *Under the same assumptions as in Lemma 5.1, we have for $n \geq 4$, $k \geq 2$, $N \geq 1$, for K sufficiently large*

$$|g_{n,k}| \leq N^{\frac{n}{2}+k-2} K^{\frac{n}{2}+k-\frac{3}{2}} \left| \frac{n}{4} + k - 3 \right|! \frac{1}{(k!)^{\frac{1}{4}}}, \quad |f_{2,k}| \leq N^{k+1} K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}}}. \quad (5.1.14)$$

Proof. We proceed by induction going up in $M = n + k$. For a fixed value of $M = n + k$ we go up in k . To initialize the induction, the bounds for $k \leq 1$ for $g_{n,k}$ and $f_{2,k}$ follow from Lemma 5.1.

We will first bound $g_{n,k+2}$. Using (5.1.7) we can then bound $g_{n,k+2}$, knowing the bounds on the terms appearing on the r.h.s. We proceed term by term.

1. We see that for the cases where $\frac{n}{4} + k - 3 < 0$, namely $(n = 4, k = 0, 1)$, $(n = 6, k = 0, 1)$, $(n = 8, k = 0)$ and $(n = 10, k = 0)$, the bounds follow from Lemma 5.1.
2. We look at the different terms on the r.h.s. of (5.1.7).
 - First term: this term vanishes for $n = 4$. For $n \geq 6$ and $k = 0$ we use the bounds in Lemma 5.1 to obtain

$$\begin{aligned} \frac{n-4}{n+2} |g_{n,1}| &\leq N^{\frac{n}{2}-1} \frac{n-4}{n+2} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{n^2} \left(1 + \frac{nK}{2} \right) \\ &\leq N^{\frac{n}{2}-1} \frac{K^{\frac{n}{2}-\frac{3}{2}+1}}{n} \\ &\leq N^{\frac{n}{2}} K^{\frac{n}{2}+2-\frac{3}{2}} \left(\frac{n}{4} - 1 \right)! \frac{1}{K}. \end{aligned} \quad (F.2.1)$$

Then for $n \geq 6$ and $k \geq 1$, using the induction hypothesis we obtain

$$\begin{aligned} \frac{n-4}{n+2k} |g_{n,k+1}| &\leq N^{\frac{n}{2}+k-1} \frac{n-4}{n+2k} K^{\frac{n}{2}+k-\frac{1}{2}} \left| \frac{n}{4} + k - 2 \right|! \frac{1}{[(k+1)!]^{\frac{1}{4}}} \\ &\leq N^{\frac{n}{2}+k-1} K^{\frac{n}{2}+k-\frac{1}{2}} \left(\frac{n}{4} + k - 1 \right)! \frac{(k+2)^{\frac{1}{4}}}{[(k+2)!]^{\frac{1}{4}}} \frac{n-4}{n+2k} \frac{4}{(n+4k-4)} \\ &\leq N^{\frac{n}{2}+k-1} K^{\frac{n}{2}+k+2-\frac{3}{2}} \left(\frac{n}{4} + k - 1 \right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{1}{K}. \end{aligned} \quad (F.2.2)$$

- Second term: We can rewrite the second term as follows

$$\begin{aligned} \frac{2n}{n+2k} \sum_{\nu=0}^{k+1} |g_{n,\nu} f_{2,k+1-\nu}| &= \frac{2n}{n+2k} \left(|g_{n,0} f_{2,k+1}| + |g_{n,1} f_{2,k}| + |g_{n,k+1} f_{2,0}| \right. \\ &\quad \left. + |g_{n,k} f_{2,1}| + |g_{n,k-1} f_{2,2}| + \sum_{\nu=2}^{k-2} |g_{n,\nu} f_{2,k+1-\nu}| \right). \end{aligned} \quad (F.2.3)$$

For the terms with $\nu \leq 1$, we use the bounds in Lemma 5.1 to get:

- $\nu = 0$:

$$\begin{aligned}
\frac{2n}{n+2k} |g_{n,0} f_{2,k+1}| &\leq \frac{2n}{n+2k} N^{k+2} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2} K^{k+\frac{3}{2}} \frac{|k-2|!}{(k!)^{\frac{1}{4}}} \\
&\leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \frac{((k+1)(k+2))^{\frac{1}{4}} |k-2|!}{n^2 [(k+2)!]^{\frac{1}{4}}} \frac{1}{\sqrt{K}} \\
&\leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{1}{\sqrt{K}},
\end{aligned} \tag{F.2.4}$$

where we used (see Appendix C.3.1)

$$\frac{((k+1)(k+2))^{\frac{1}{4}} |k-2|!}{n^2} \leq \left(\frac{n}{4} + k - 1\right)!, \quad n \geq 4, k \geq 0. \tag{F.2.5}$$

- $\nu = 1$:

$$\begin{aligned}
\frac{2n}{n+2k} |g_{n,1} f_{2,k}| &\leq \frac{2n}{n+2k} N^{k+1} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{n^2} \left(1 + \frac{nK}{2}\right) K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}}} \\
&\leq \frac{2n}{n+2k} N^{k+1} \frac{3K^{\frac{n}{2}-\frac{1}{2}}}{4n} K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{4}}} \\
&\leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{6}{\sqrt{K}},
\end{aligned} \tag{F.2.6}$$

where we used the inequalities, valid for $n \geq 4, k \geq 0$,

$$\frac{K^{\frac{n}{2}-\frac{3}{2}}}{n^2} \left(1 + \frac{nK}{2}\right) \leq \frac{3K^{\frac{n}{2}-\frac{1}{2}}}{4n}, \quad \frac{|k-3|!}{4n(|k-1|!)^{\frac{1}{4}}} \leq \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}}, \tag{F.2.7}$$

which follow directly by inspection. From Lemma 5.1 and (5.1.6), we have

$$|f_{2,2}| \leq \frac{1}{2} ((N+2)|g_{4,1}| + |f_{2,1}| + 2|f_{2,0} f_{2,1}|) \leq NK\sqrt{K}. \tag{F.2.8}$$

- $k-1 \leq \nu \leq k+1$ implies that $k \geq 1$. We successively have

* $\nu = k-1$

$$\begin{aligned}
|g_{n,k-1} f_{2,2}| &\leq N^{\frac{n}{2}+k-3} K^{\frac{n}{2}+k-\frac{5}{2}} \left|\frac{n}{4} + k - 4\right|! \frac{1}{(k-1)!^{\frac{1}{4}}} NK\sqrt{K} \\
&\leq N^{\frac{n}{2}+k-2} K^{\frac{n}{2}+k-1} \frac{(k(k+1)(k+2))^{\frac{1}{4}}}{[(k+2)!]^{\frac{1}{4}}} \left|\frac{n}{4} + k - 4\right|! \\
&\leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{C_a}{\sqrt{K}},
\end{aligned} \tag{F.2.9}$$

where we have used the following inequality (see Appendix C.3.2)

$$(k(k+1)(k+2))^{\frac{1}{4}} \left|\frac{n}{4} + k - 4\right|! \leq C_a \left(\frac{n}{4} + k - 1\right)!, \tag{F.2.10}$$

for a constant C_a which does not depend on n, k .

* $\nu = k$: From Lemma 5.1, we have

$$\begin{aligned}
|g_{n,k}f_{2,1}| &\leq N^{\frac{n}{2}+k-2}K^{\frac{n}{2}+k-\frac{3}{2}}\left|\frac{n}{4}+k-3\right|!\frac{1}{(k!)^{\frac{1}{4}}}\frac{KN}{2} \\
&\leq N^{\frac{n}{2}+k-1}K^{\frac{n}{2}+k-\frac{1}{2}}\frac{\left((k+1)(k+2)\right)^{\frac{1}{4}}}{[(k+2)!]^{\frac{1}{4}}}\left|\frac{n}{4}+k-3\right|! \quad (\text{F.2.11}) \\
&\leq N^{\frac{n}{2}+k}K^{\frac{n}{2}+k+\frac{1}{2}}\left(\frac{n}{4}+k-1\right)!\frac{1}{[(k+2)!]^{\frac{1}{4}}}\frac{C_b}{\sqrt{K}},
\end{aligned}$$

where we have used (see Appendix C.3.2)

$$\left((k+1)(k+2)\right)^{\frac{1}{4}}\left|\frac{n}{4}+k-3\right|!\leq C_b\left(\frac{n}{4}+k-1\right)!, \quad (\text{F.2.12})$$

for a constant $C_b > 0$ which does not depend on n, k .

* $\nu = k + 1$: we have to bound $|g_{n,k+1}f_{2,0}|$, we go back to the first term to treat it. Therefore from (F.2.9),(F.2.11) and the bound on the first term, $|g_{n,\nu}f_{2,k+1-\nu}|$ are bounded by

$$N^{\frac{n}{2}+k}K^{\frac{n}{2}+k+\frac{1}{2}}\left(\frac{n}{4}+k-1\right)!\frac{1}{[(k+2)!]^{\frac{1}{4}}}\frac{C}{\sqrt{K}}, \quad (\text{F.2.13})$$

where C is a constant which does not depend on n, k .

- $2 \leq \nu \leq k - 2$ We now bound the remaining sum in (F.2.3). Note that this sum is non-zero only if $k \geq 4$, so we assume $k \geq 4$ from now on. The remaining sum is bounded by

$$N^{\frac{n}{2}+k}K^{\frac{n}{2}+k+\frac{1}{2}}\frac{n}{2\sqrt{K}(n+2k)}F\left(n, 4, k-2, 2, 2, \frac{1}{4}\right). \quad (\text{F.2.14})$$

Then from Lemma F.2 we have

$$\begin{aligned}
\frac{n}{n+2k}F\left(n, 4, k-2, 2, 2, \frac{1}{4}\right) &\leq \frac{n}{n+2k}[(k-2)!]^{\frac{3}{4}}\frac{4}{n}\frac{\left(\frac{n}{4}+k-4\right)!}{(k-4)!} \\
&\leq 4\frac{\left(\frac{n}{4}+k-1\right)!}{[(k+2)!]^{\frac{1}{4}}}\frac{\left(k(k+1)(k+2)\right)^{\frac{1}{4}}}{n+2k} \quad (\text{F.2.15}) \\
&\leq 4\frac{\left(\frac{n}{4}+k-1\right)!}{[(k+2)!]^{\frac{1}{4}}}.
\end{aligned}$$

Summing (F.2.4), (F.2.6), (F.2.13) and (F.2.15) we find

$$\frac{2n}{n+2k}\sum_{\nu=0}^{k+1}|g_{n,\nu}f_{2,k+1-\nu}| \leq N^{\frac{n}{2}+k}K^{\frac{n}{2}+k+\frac{1}{2}}\left(\frac{n}{4}+k-1\right)!\frac{1}{[(k+2)!]^{\frac{1}{4}}}\frac{C_2}{\sqrt{K}}, \quad (\text{F.2.16})$$

where $C_2 > 0$ is a suitable constant.

- Third term exists only if $n \geq 6$. It is bounded by

$$\begin{aligned} \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} |g_{n_1, \nu} g_{n_2, k+2-\nu}| &\leq \frac{2n}{n+2k} I + N^{\frac{n}{2}+k-1} \frac{2K^{\frac{n}{2}+k+\frac{1}{2}} n}{\sqrt{K}(n+2k)} \times \\ &\left[\sum_{\substack{4 \leq n_1 \leq 10 \\ n_1 \leq n_2 \\ n_1+n_2=n+2}} F\left(n_1, n_2, k, 2, 2, \frac{1}{4}\right) + \sum_{\substack{12 \leq n_1 \leq n_2 \\ n_1+n_2=n+2}} F\left(n_1, n_2, k, 0, 0, \frac{1}{4}\right) \right], \end{aligned} \quad (\text{F.2.17})$$

where we define

$$I := \sum_{\substack{4 \leq n_1 \leq 10 \\ n_1 \leq n_2 \\ n_1+n_2=n+2}} \left(|g_{n_1, 0} g_{n_2, k+2}| + |g_{n_1, 1} g_{n_2, k+1}| + |g_{n_1, k+2} g_{n_2, 0}| + |g_{n_1, k+1} g_{n_2, 1}| \right). \quad (\text{F.2.18})$$

The first and the second term in the r.h.s. of (F.2.17) contains a finite number of terms which does not depend on n . In I , we have for example

$$\begin{aligned} |g_{4,0} g_{n-2, k+2}| &\leq \frac{\sqrt{K}}{32} N^{\frac{n}{2}+k} K^{\frac{n}{2}+k-\frac{1}{2}} \left(\frac{n}{4} + k - \frac{3}{2}\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \\ &\leq \frac{A}{32} N^{\frac{n}{2}+k} K^{\frac{n}{2}+k} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}}, \end{aligned} \quad (\text{F.2.19})$$

for a constant $A > 0$ which does not depend on n, k . Since I contains a finite number of terms which can be bounded similarly, it follows from the bounds from Lemma 5.1 and the induction hypothesis, that we can bound I as follows

$$I \leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{C'}{\sqrt{K}}, \quad (\text{F.2.20})$$

where C' is a constant which does not depend on n, k .

Then we use Lemma F.2 to obtain respectively

$$\frac{n}{n+2k} \sum_{\substack{4 \leq n_1 \leq 10 \\ n_1 \leq n_2 \\ n_1+n_2=n+2}} F\left(n_1, n_2, k, 2, 2, \frac{1}{4}\right) \leq \frac{4n}{n+2k} \frac{[2k!]^{\frac{3}{4}} \left(\frac{n}{4} + k - \frac{5}{2}\right)!}{\frac{n}{4} - \frac{1}{2} (k-2)!} \quad (\text{F.2.21})$$

and¹

$$\frac{n}{n+2k} \sum_{\substack{12 \leq n_1 \leq n_2 \\ n_1+n_2=n+2}} F\left(n_1, n_2, k, 0, 0, \frac{1}{4}\right) \leq \frac{n}{2} \frac{2}{\frac{n}{4} - \frac{9}{2}} \frac{\left(\frac{n}{4} + k - \frac{5}{2}\right)!}{[(k+2)!]^{\frac{1}{4}}}. \quad (\text{F.2.22})$$

Using the inequality

$$\left(m + \frac{1}{2}\right)! \leq 2m! \sqrt{m + \frac{1}{2}}, \quad m \in \mathbb{N}_0, \quad (\text{F.2.23})$$

¹ $12 \leq n_1 \leq n_2$ implies $n \geq 22$, so that the denominators are non-zero.

we obtain finally

$$\begin{aligned} \frac{n}{n+2k} \sum_{\substack{4 \leq n_1 \leq 10 \\ n_1 \leq n_2 \\ n_1 + n_2 = n+2}} F\left(n_1, n_2, k, 2, 2, \frac{1}{4}\right) &\leq \frac{8n}{n+2k} \frac{2^{\frac{3}{4}}}{\frac{n}{4} - \frac{1}{2}} \frac{\left(\frac{n}{4} + k - 1\right)! \sqrt{\frac{n}{4} + k - \frac{5}{2}}}{[k!]^{\frac{1}{4}}} \\ &\leq C''' \frac{\left(\frac{n}{4} + k - 1\right)!}{[(k+2)!]^{\frac{1}{4}}}, \end{aligned} \quad (\text{F.2.24})$$

where C''' does not depend on n, k .

Summing over all contributions, we get the bound

$$\frac{n}{n+2k} \sum_{\substack{n_1 + n_2 = n+2 \\ n_i \geq 4}} \sum_{\nu=0}^{k+2} |g_{n_1, \nu} g_{n_2, k+2-\nu}| \leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{C_3}{\sqrt{K}}, \quad (\text{F.2.25})$$

for some finite positive constant C_3 .

- Fourth term: First, for $k \leq 1$, we use the bounds in Lemma 5.1 to obtain

$$- k = 0 \quad \frac{n(n+N)}{n} |g_{n+2,0}| \leq N^{\frac{n}{2}} K^{\frac{n}{2}+\frac{1}{2}} \left(\frac{n}{4} - 1\right)! \frac{1}{K}. \quad (\text{F.2.26})$$

$$- k = 1 \quad \frac{n(n+N)}{n+2} |g_{n+2,1}| \leq N^{\frac{n}{2}+1} K^{\frac{n}{2}+\frac{3}{2}} \left(\frac{n}{4}\right)! \frac{1}{K}. \quad (\text{F.2.27})$$

For $k \geq 2$ we get

$$\begin{aligned} \frac{n(n+N)}{n+2k} |g_{n+2,k}| &\leq \frac{n(n+N)}{n+2k} N^{\frac{n}{2}+k-1} K^{\frac{n}{2}+k-\frac{1}{2}} \left(\frac{n}{4} + k + \frac{1}{2} - 3\right)! \frac{1}{(k!)^{\frac{1}{4}}} \\ &\leq \frac{N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)!}{[(k+2)!]^{\frac{1}{4}}} \frac{n(1 + \frac{n}{N}) \sqrt{\frac{n}{4} + k - \frac{5}{2}} [(k+1)(k+2)]^{\frac{1}{4}}}{(n+2k) \left(\frac{n}{4} + k - 1\right) \left(\frac{n}{4} + k - 2\right) K} \\ &\leq N^{\frac{n}{2}+k} K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{4}}} \frac{4}{K}. \end{aligned} \quad (\text{F.2.28})$$

Summing (F.2.2), (F.2.16), (F.2.25) and (F.2.28), we obtain

$$\begin{aligned} |g_{n,k+2}| &\leq \left[\frac{1}{K} + \frac{C_2 + C_3}{\sqrt{K}} + \frac{4}{K} \right] N^{\frac{n}{2}+k} \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{[(k+2)!]^{\frac{1}{4}}} \left(\frac{n}{4} + k - 1\right)! \\ &\leq N^{\frac{n}{2}+k} \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{[(k+2)!]^{\frac{1}{4}}} \left(\frac{n}{4} + k - 1\right)!. \end{aligned} \quad (\text{F.2.29})$$

We will now bound the $f_{2,k}$. One can check term by term that the bound (5.1.14) is true for $k \leq 6$. For K large enough, and for $k > 6$ we obtain from (5.1.6) by induction

$$\begin{aligned}
|f_{2,k+1}| \leq & \frac{1}{k+1} \left((N+2)N^k \frac{K^{k+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}} + N^k K^{k+\frac{1}{2}} \frac{(k-3)!}{[(k-1)!]^{\frac{1}{4}}} \right. \\
& \left. + N^{k+1} K^{k+1} \sum_{\nu=0}^k \frac{|\nu-3|!|k-\nu-3|!}{[|\nu-1|!|k-\nu-1|!]^{\frac{1}{4}}} \right). \tag{F.2.30}
\end{aligned}$$

We will bound each term in the r.h.s. of (F.2.30). Again, we proceed term by term.

- First term: we trivially have

$$\frac{1}{(k+1)} \frac{K^{k+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}} \leq \frac{1}{K(k+1)} \frac{K^{k+1+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}}.$$

- Second term: obviously

$$\frac{K^{k+\frac{1}{2}}}{k+1} \frac{(k-3)!}{((k-1)!)^{\frac{1}{4}}} \leq \frac{k^{\frac{1}{4}}}{K(k+1)(k-2)} \frac{K^{k+1+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}}.$$

- Third term: we note that $[(\nu-1)!(k-\nu-1)!]^{\frac{3}{4}} \leq [(k-2)!]^{\frac{3}{4}} \leq \left(\frac{k!}{k(k-1)}\right)^{\frac{3}{4}}$ for $3 \leq \nu \leq k-3$. Then the terms summed over $3 \leq \nu \leq k-3$ are bounded by

$$\begin{aligned}
\frac{(k!)^{\frac{3}{4}}}{(k+1)(k(k-1))^{\frac{3}{4}}} \sum_{\nu=3}^{k-3} \frac{(\nu-3)!(k-\nu-3)!}{(\nu-1)!(k-\nu-1)!} & \leq \frac{(k!)^{\frac{3}{4}}}{(k-5)(k+1)(k(k-1))^{\frac{3}{4}}} \\
& \leq \frac{(k-2)!}{(k!)^{\frac{1}{4}}}. \tag{F.2.31}
\end{aligned}$$

The remaining terms are symmetric under $\nu' \mapsto k-\nu'$ so that we restrict ourselves to $0 \leq \nu \leq 2$. For $k > 6$, they are bounded by

- (a) $\nu = 0$:

$$\frac{6(k-3)!}{((k-1)!)^{\frac{1}{4}}} \leq 6 \frac{(k-2)!}{(k!)^{\frac{1}{4}}},$$

- (b) $\nu = 1$:

$$\frac{4(k-4)!}{((k-2)!)^{\frac{1}{4}}} \leq 4 \frac{(k-2)!}{(k!)^{\frac{1}{4}}},$$

- (c) $\nu = 2$:

$$\frac{(k-5)!}{((k-3)!)^{\frac{1}{4}}} \leq \frac{(k-2)!}{(k!)^{\frac{1}{4}}},$$

Then by summing we get

$$\frac{1}{k+1} \sum_{\nu=0}^k \frac{|\nu-3|!|k-\nu-3|!}{(|\nu-1|!|k-\nu-1|!)^{\frac{1}{4}}} \leq \left(1 + \frac{22}{7}\right) \frac{(k-2)!}{(k!)^{\frac{1}{4}}}.$$

Altogether we find

$$|f_{2,k+1}| \leq \left(\frac{1}{K} + \frac{1}{K} + \frac{5}{\sqrt{K}} \right) N^{k+2} \frac{K^{k+1+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}} \leq N^{k+2} \frac{K^{k+1+\frac{1}{2}}(k-2)!}{(k!)^{\frac{1}{4}}}. \quad (\text{F.2.32})$$

This ends the proof.

□

Appendix G

Proof of Proposition 6.2

Proposition 6.2. *Under the same assumptions as in Lemma 6.3, choosing K large enough we have*

$$|\tilde{g}_{n,k}| \leq K^{\frac{n}{2}+k-\frac{3}{2}} \left| \frac{n}{4} + k - 3 \right|! \frac{1}{(k!)^{\frac{1}{8}}}, \quad |\tilde{f}_{2,k}| \leq K^{k+\frac{1}{2}} \frac{|k-3|!}{(|k-1|!)^{\frac{1}{8}}}, \quad n \geq 4, k \geq 0. \quad (6.2.15)$$

Proof. The proof proceeds by induction in $N = n + k$ going up in k , as in the proof of Proposition 5.1. The bounds we obtain are expressed in terms of positive constants $C, C_i, i = 1, \dots, 11$ chosen sufficiently large and then for K sufficiently large, depending on these constants. For $k \leq 1$, the bounds follow from Lemma 6.3. In the r.h.s. of (6.2.8), the first, second, fifth and seventh terms can be treated as in the proof of Proposition 5.1. So we focus on the remaining terms.

- Third term: we separate the terms summed over $2 \leq \nu \leq k - 2$ and the remaining terms. Using Lemma 6.3 we have, choosing $K > 10 e c$

– $\nu = 0$:

$$\begin{aligned} \frac{n-2}{n+2k} |\tilde{g}_{n,0} h_{k+1}| &\leq \frac{n-2}{n+2k} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2} c C^{k+2} (5e)^{k+3} 2^{k+2} \\ &\leq K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1 \right)! \frac{1}{[(k+2)!]^{\frac{1}{8}}} 5 e c. \end{aligned} \quad (G.0.1)$$

– $\nu = 1$:

$$\begin{aligned} \frac{n-2}{n+2k} |\tilde{g}_{n,1} h_k| &\leq \frac{n-2}{n+2k} \frac{K^{\frac{n}{2}-\frac{1}{2}}}{n} c C^{k+1} (5e)^{k+2} 2^{k+1} \\ &\leq K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1 \right)! \frac{1}{[(k+2)!]^{\frac{1}{8}}} 5 e c. \end{aligned} \quad (G.0.2)$$

For $k-1 \leq \nu \leq k+1$, we have the following bounds

$$|\tilde{g}_{n,\nu}| \leq K^{\frac{n}{2}+\nu-\frac{3}{2}} \left(\frac{n}{4} + k - 1 \right)! \frac{1}{[(k+2)!]^{\frac{1}{8}}}. \quad (G.0.3)$$

Using (6.2.1) we obtain

$$\frac{n-2}{n+2k} \sum_{\nu=k-1}^{k+1} |\tilde{g}_{n,\nu} h_{k+1-\nu}| \leq 15ec K^{\frac{n}{2}+k+\frac{1}{2}} \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{8}}}.$$

The remaining sum can be bounded for $k \geq 4$ using Lemma F.2 and (6.2.1)

$$\begin{aligned} \frac{n-2}{n+2k} \sum_{\nu=2}^{k-2} |\tilde{g}_{n,\nu} h_{k+1-\nu}| &\leq \frac{n-2}{n+2k} K^{\frac{n}{4}+k+\frac{1}{2}} 5ec \sum_{\nu=2}^{k-2} \frac{(\frac{n}{4} + \nu - 3)!}{[\nu!]^{\frac{1}{8}}} \\ &\leq \frac{n-2}{n+2k} K^{\frac{n}{4}+k+\frac{1}{2}} 10ec F\left(n, 4, k-2, 2, 2, \frac{1}{8}\right) \\ &\leq \frac{n-2}{n+2k} K^{\frac{n}{4}+k+\frac{1}{2}} 10ec \frac{2^{\frac{7}{8}} [(k-2)!]^{\frac{7}{8}} (\frac{n}{4} + k - 3)!}{n (k-4)!} \\ &\leq K^{\frac{n}{4}+k+\frac{1}{2}} 20ec \left(\frac{n}{4} + k - 1\right)! \frac{1}{[(k+2)!]^{\frac{1}{8}}}. \end{aligned} \tag{G.0.4}$$

Then the third term is bounded by

$$K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4} + k - 1)!}{[(k+2)!]^{\frac{1}{8}}} \frac{2}{5}. \tag{G.0.5}$$

- Fourth term: we proceed similarly as for the third term, then the fourth term is bounded by

$$K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4} + k - 1)!}{[(k+2)!]^{\frac{1}{8}}} \frac{C_2}{\sqrt{K}}. \tag{G.0.6}$$

- Sixth term: Looking at the terms corresponding to $\nu \geq k+1$ and using the bounds from Lemma 6.3 we get

$$- \nu = k + 1$$

$$\begin{aligned} \frac{2}{(k+1)!} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} |\tilde{g}_{n_1,0} \tilde{g}_{n_2,1}| &\leq \frac{1}{(k+1)!} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+2-\frac{3}{2}}}{n_1^2 (n+2-n_1)} \\ &\leq \frac{1}{\sqrt{K}} \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{(k+1)!} \frac{1}{n} \leq \frac{1}{\sqrt{K}} K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4} + k - 1)!}{[(k+2)!]^{\frac{1}{8}}}. \end{aligned}$$

$$- \nu = k + 2$$

$$\begin{aligned} \frac{1}{(k+2)!} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} |\tilde{g}_{n_1,0} \tilde{g}_{n_2,0}| &\leq \frac{1}{4(k+2)!} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \frac{K^{\frac{n}{2}-\frac{3}{2}+1-\frac{3}{2}}}{n_1^2 (n+2-n_1)^2} \\ &\leq \frac{1}{4\sqrt{K}} \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{(k+2)!} \frac{1}{n^2} \leq \frac{1}{4\sqrt{K}} K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4} + k - 1)!}{[(k+2)!]^{\frac{1}{8}}}. \end{aligned}$$

For the remaining part of the sum, we substitute $F(n_1, n_2, k, c, c, \frac{1}{4})$, $c \in \{0, 2\}$ from the analysis of the third term in the proof of Proposition 5.1 by $F(n_1, n_2, k - \nu, c, c, \frac{1}{8})$, $c \in \{0, 2\}$. Then, the sixth term is bounded by

$$\frac{1}{2\sqrt{n+2k}} \frac{C_3}{\sqrt{K}} K^{\frac{n}{2}+k+\frac{1}{2}} \sum_{\nu=0}^k \frac{1}{\nu!} \frac{(\frac{n}{4}+k-1-\nu)!}{[(k+2-\nu)!]^{\frac{1}{8}}}. \quad (\text{G.0.7})$$

One can then extract the terms in the sum corresponding to $\nu \leq 1$ and bound them by $\frac{(\frac{n}{4}+k-1)!}{[(k+2)!]^{\frac{1}{8}}}$. The residual sum is non-zero for $k \geq 2$, it is bounded by

$$\sum_{\nu=2}^k \frac{1}{\nu!} \frac{(\frac{n}{4}+k-1-\nu)!}{[(k+2-\nu)!]^{\frac{1}{8}}} = \sum_{\nu=0}^{k-2} \frac{1}{(k-\nu)!} \frac{(\frac{n}{4}-1+\nu)!}{[(\nu+2)!]^{\frac{1}{8}}} \leq \frac{(\frac{n}{4}+k-3)!}{[(k-2)!]^{\frac{1}{8}}} e.$$

Then we obtain

$$\frac{1}{2\sqrt{n+2k}} \sum_{\nu=2}^k \frac{1}{\nu!} \frac{(\frac{n}{4}+k-1-\nu)!}{[(k+2-\nu)!]^{\frac{1}{8}}} \leq \frac{e}{2} \frac{1}{\sqrt{n+2k}} \frac{(\frac{n}{4}+k-3)!}{[(k-2)!]^{\frac{1}{8}}} \leq \frac{e}{2} \frac{(\frac{n}{4}+k-1)!}{[(k+2)!]^{\frac{1}{8}}}.$$

Finally the sixth term is bounded by

$$K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4}+k-1)!}{[(k+2)!]^{\frac{1}{8}}} \frac{C_4}{\sqrt{K}}. \quad (\text{G.0.8})$$

- Eighth term: first the term in the sum corresponding to $\nu = k + 1$ is

$$\frac{n}{n+2k} \frac{1}{(k+1)!} |\tilde{g}_{n,0} \tilde{f}_{2,0}| \leq \frac{\sqrt{K}}{16} \frac{K^{\frac{n}{2}-\frac{3}{2}}}{2n^2} \leq K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4}+k-1)!}{[(k+2)!]^{\frac{1}{8}}} \frac{1}{32nK}.$$

We follow the steps from the proof of Proposition 5.1 for the second term but substituting $F(n, 4, k-2, 2, 2, \frac{1}{4})$ by $F(n, 4, k-2-\nu, 2, 2, \frac{1}{8})$. Thus we have

$$\begin{aligned} \frac{n}{n+2k} F\left(n, 4, k-2-\nu, 2, 2, \frac{1}{8}\right) &\leq \frac{n}{n+2k} [(k-2)!]^{\frac{7}{8}} \frac{4}{n} \frac{(\frac{n}{4}+k-\nu-4)!}{(k-\nu-4)!} \\ &\leq 4 \frac{(\frac{n}{4}+k-\nu-1)!}{[(k+2-\nu)!]^{\frac{1}{8}}} \frac{(k(k+1)(k+2))^{\frac{1}{8}}}{n+2k} \leq \frac{4}{\sqrt{n+2k}} \frac{(\frac{n}{4}+k-1-\nu)!}{[(k+2-\nu)!]^{\frac{1}{8}}}. \end{aligned}$$

Therefore we can bound the eighth term by

$$\frac{K^{\frac{n}{2}+k+\frac{1}{2}} C_5}{\sqrt{K} \sqrt{n+2k}} \sum_{\nu=0}^k \frac{1}{\nu!} \frac{(\frac{n}{4}+k-1-\nu)!}{[(k+2-\nu)!]^{\frac{1}{8}}}. \quad (\text{G.0.9})$$

From the analysis of the sixth term, we deduce that the eighth term is finally bounded by

$$K^{\frac{n}{2}+k+\frac{1}{2}} \frac{(\frac{n}{4}+k-1)!}{[(k+2)!]^{\frac{1}{8}}} \frac{C_6}{\sqrt{K}}. \quad (\text{G.0.10})$$

Summing (G.0.5), (G.0.6), (G.0.8) and (G.0.10) with the already treated terms in the proof of Proposition 5.1, we obtain

$$|\tilde{g}_{n,k+2}| \leq \left[\frac{2}{5} + \frac{C_7}{K} + \frac{C_8}{\sqrt{K}} \right] \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{[(k+2)!]^{\frac{1}{8}}} \left(\frac{n}{4} + k - 1 \right)! \leq \frac{K^{\frac{n}{2}+k+\frac{1}{2}}}{[(k+2)!]^{\frac{1}{8}}} \left(\frac{n}{4} + k - 1 \right)!. \quad (\text{G.0.11})$$

The bounds for $\tilde{f}_{2,k}$, for $k \leq 1$, follow from Lemma 6.3. The bounds for $k \leq 6$ can be checked by hand noting that we can always factor out $\frac{1}{\sqrt{K}}$ in the r.h.s. of (6.2.7) using the induction hypothesis so that the bound holds choosing K sufficiently large. For $k > 6$ we will focus on the last two terms in the r.h.s. of (6.2.7) since the other terms are treated as in the proof of Proposition 5.1.

- Fourth term: it is bounded by

$$\frac{\beta_0}{(k+1)} \frac{K^{k+\frac{1}{2}+1}}{K} \sum_{\nu=0}^k \frac{|\nu-3|!}{(k-\nu)! [|\nu-1|!]^{\frac{1}{8}}}. \quad (\text{G.0.12})$$

The sum can be bounded as follows

$$\begin{aligned} \sum_{\nu=0}^k \frac{|\nu-3|!}{(k-\nu)! [|\nu-1|!]^{\frac{1}{8}}} &= \sum_{\nu=3}^k \frac{(\nu-3)!}{(k-\nu)! [(\nu-1)!]^{\frac{1}{8}}} + \frac{6}{k!} + \frac{2}{(k-1)!} + \frac{1}{(k-2)!} \\ &\leq [(k-1)!]^{\frac{7}{8}} \sum_{\nu=3}^k \frac{(\nu-3)!}{(k-\nu)! (\nu-1)!} \\ &\leq [(k-1)!]^{\frac{7}{8}} \frac{15}{k} \leq \frac{15(k-2)!}{[(k-1)!]^{\frac{1}{8}}}. \end{aligned} \quad (\text{G.0.13})$$

Taking into account the factor $\frac{1}{k+1}$, the fourth term is bounded by

$$K^{k+1+\frac{1}{2}} \frac{(k-2)! C_9}{[k!]^{\frac{1}{8}} K}. \quad (\text{G.0.14})$$

- Fifth term: we separate the sum as follows:

$$\frac{1}{2(k+1)} \sum_{\nu=0}^{k-2} \frac{1}{\nu!} \sum_{\nu'=0}^{k-\nu} |\tilde{f}_{2,\nu'} \tilde{f}_{2,k-\nu-\nu'}| + \frac{1}{(k+1)(k-1)!} |\tilde{f}_{2,0} \tilde{f}_{2,1}| + \frac{1}{2(k+1)!} \tilde{f}_{2,0}^2.$$

The bounds for the last two terms follow from Lemma 6.3. The sum is bounded as in the proof of Proposition 5.1 by a term proportional to

$$\frac{K^{k+1+\frac{1}{2}}}{\sqrt{K}} \sum_{\nu=0}^{k-2} \frac{1}{\nu!} \frac{(k-2-\nu)!}{[(k-\nu)!]^{\frac{1}{8}}}. \quad (\text{G.0.15})$$

This sum is then treated as in the fourth term. Therefore we have the final bound

$$K^{k+1+\frac{1}{2}} \frac{(k-2)! C_{10}}{[k!]^{\frac{1}{8}} \sqrt{K}}. \quad (\text{G.0.16})$$

Then, we obtain

$$|\tilde{f}_{2,k+1}| \leq K^{k+1+\frac{1}{2}} \frac{(k-2)! C_{11}}{[k!]^{\frac{1}{8}} \sqrt{K}} \leq K^{k+1+\frac{1}{2}} \frac{(k-2)!}{[k!]^{\frac{1}{8}}}. \quad (\text{G.0.17})$$

□

Appendix H

Proof of the bounds of the mean-field perturbative CAS-functions

Proposition 7.2. Let $\mathcal{A}_{n,j}^{\alpha_0,\alpha}$ be smooth solutions of the mean-field flow equations (7.1.6) for the boundary conditions (7.1.7) and the BPHZ renormalization conditions (7.1.8). For $\alpha \in [\alpha_0, \alpha_{\max}]$, $\alpha_{\max} = \frac{1}{m^2}$, they satisfy the bounds

$$\begin{aligned} |\mathcal{A}_{2,j}^{\alpha_0,\alpha}| &\leq \frac{C^{j-\frac{1}{2}}}{\alpha} \frac{j!}{(j+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \\ |\partial_\alpha^k \mathcal{A}_{2,j}^{\alpha_0,\alpha}| &\leq \frac{C^{j-\frac{1}{2}+k}}{\alpha^{k+1}} \frac{(j+k+1)!}{(j+1)^2 (k+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \quad k \geq 1, \end{aligned} \tag{7.1.10}$$

and for $n \geq 4$

$$\begin{aligned} |\mathcal{A}_{n,j}^{\alpha_0,\alpha}| &\leq \alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \\ |\partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0,\alpha}| &\leq \alpha^{\frac{n}{2}-2-k} C^{j-\frac{n}{4}+k} \frac{(j+k+1)!}{(j-\frac{n}{2}+2)^2 (k+1)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \quad k \geq 1, \end{aligned} \tag{7.1.11}$$

for a suitable constant $C > 1$.

Proof. We proceed by induction as follows:

- we go up in $j \in \mathbb{N}$.
- at a fixed value of j , we go downwards from $n = 2j + 2$ to $n = 2$.
- at a fixed value of j, n we go up in m .

We start the induction at $j = 1$. The non-linear term in the r.h.s. of (7.1.6) vanishes. Direct computation shows that

$$\mathcal{A}_{4,1}^{\alpha_0, \alpha} = 1, \quad \mathcal{A}_{2,1}^{\alpha_0, \alpha} = 3\left(m^2 - \frac{1}{\alpha}\right), \quad (\text{H.0.1})$$

therefore the bounds (7.1.10)-(7.1.11) are satisfied. For a fixed $j > 1$, we start at $n = 2j + 2$ and we go downwards to $n = 2$. The induction hypothesis holds for the set

$$\left\{ (j', n', k') \in \mathbb{N} \times (2\mathbb{N} \cap [1, 2j + 2]) \times \mathbb{N}_0, \right. \\ \left. \left(\left(\{j' = j\} \cap \{n' > n\} \right) \cup \left(\{j' < j\} \cap \{n' \in 2\mathbb{N} \cap [1, 2j' + 2]\} \right) \right) \cap \{k' \leq k\} \right\}. \quad (\text{H.0.2})$$

For $n > 2$, we proceed as follows

- $k = 0$: We integrate the l.h.s. of (7.1.6) upwards from α_0 to α for $n > 4$ and downwards from $\alpha = \frac{1}{m^2}$ to α for $n = 4$. We bound the r.h.s. of (7.1.6) with the induction hypothesis. We first start with the linear term.

- $n > 4$: The linear term is non-zero as long as $n + 2 \leq 2j + 2$. We use Lemma C.6 to obtain

$$\begin{aligned} & \frac{n(n+1)}{2} \int_{\alpha_0}^{\alpha} d\alpha' \frac{|\mathcal{A}_{n+2,j}^{\alpha_0, \alpha'}|}{\alpha'^2} \\ & \leq \frac{n(n+1)}{2} \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{\frac{n}{2}-3} \frac{C^{j-\frac{n}{4}-\frac{1}{2}} j!}{(j-\frac{n}{2}+1)^2 (\frac{n}{2}+1)^2 (\frac{n}{2}+1)!} \sum_{\lambda=0}^{j-\frac{n}{2}} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha'))^{\lambda} \\ & \leq \alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j-\frac{n}{2}+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} \frac{4(n+1)}{\sqrt{C}(n-4)}. \end{aligned} \quad (\text{H.0.3})$$

The non-linear term is always non-zero, we bound it first by

$$\frac{n}{2} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i}} \int_{\alpha_0}^{\alpha} d\alpha' |\mathcal{A}_{n_1, j_1}^{\alpha_0, \alpha'} \mathcal{A}_{n_2, j_2}^{\alpha_0, \alpha'}|. \quad (\text{H.0.4})$$

We distinguish $n_1 = 2$ or $n_1 = n$ from $n_i \geq 4$. We find for $n_i \geq 4$, $j_i \geq \frac{n_i}{2} - 1$,

$$\begin{aligned} & \int_{\alpha_0}^{\alpha} d\alpha' |\mathcal{A}_{n_1, j_1}^{\alpha_0, \alpha'}| |\mathcal{A}_{n_2, j_2}^{\alpha_0, \alpha'}| \\ & \leq \frac{C^{j-\frac{n}{4}-\frac{1}{2}} j_1! j_2!}{(j_1 - \frac{n_1}{2} + 2)^2 (j_2 - \frac{n_2}{2} + 2)^2 (\frac{n_1}{2})^2 (\frac{n_2}{2})^2 (\frac{n_1}{2})! (\frac{n_2}{2})!} \times \\ & \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{\frac{n}{2}-3} \sum_{\lambda_1=0}^{j_1-\frac{n_1}{2}+1} \frac{1}{2^{\lambda_1} \lambda_1!} (1 - \ln(m^2 \alpha'))^{\lambda_1} \sum_{\lambda_2=0}^{j_2-\frac{n_2}{2}+1} \frac{1}{2^{\lambda_2} \lambda_2!} (1 - \ln(m^2 \alpha'))^{\lambda_2}. \end{aligned} \quad (\text{H.0.5})$$

Setting the loop numbers $l_k = j_k - \frac{n_k}{2} + 1$ for $k = 1, 2$ and $l = j - \frac{n}{2} + 1$, and summing over the even integers $n_i \geq 4$, we have

$$C^{j-\frac{n}{4}-\frac{1}{2}} \sum_{\lambda=0}^l \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4, n_i \in 2\mathbb{N} \\ l_1+l_2=l \\ \lambda_1 \leq l_1, \lambda_2 \leq l_2 \\ \lambda_1+\lambda_2=\lambda}} \frac{(\frac{n_1}{2} + l_1 - 1)! (\frac{n_2}{2} + l_2 - 1)!}{(\frac{n_1}{2})! (\frac{n_2}{2})!} \frac{1}{(l_1 + 1)^2 (l_2 + 1)^2 (\frac{n_1}{2})^2 (\frac{n_2}{2})^2} \frac{\lambda!}{\lambda_1! \lambda_2!} \times \\ \int_{\alpha_0}^{\alpha} d\alpha' \alpha'^{\frac{n}{2}-3} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha'))^{\lambda}. \quad (\text{H.0.6})$$

Using Lemma C.5 (C.4.6) and Lemma C.6, we bound (H.0.6) by

$$\alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j - \frac{n}{2} + 2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} \frac{4K_0}{\sqrt{C}(n-4)}. \quad (\text{H.0.7})$$

For $n_1 = 2$ or $n_2 = 2$, we use again Lemma C.5 (C.4.7) and Lemma C.6 to get the following bound

$$\alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j - \frac{n}{2} + 2)^2 (\frac{n}{2})^2 (\frac{n}{2} + 1)!} \sum_{\lambda=0}^{j-\frac{n}{2}} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} \frac{4K'_0}{\sqrt{C}(n-4)}. \quad (\text{H.0.8})$$

Since $\ln(m^2 \alpha) < 0$, the summand is positive, therefore (H.0.4) is bounded by

$$\alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j - \frac{n}{2} + 2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} 2(K_0 + 2K'_0) \frac{n}{\sqrt{C}(n-4)}. \quad (\text{H.0.9})$$

Summing together (H.0.3) and (H.0.9), we have

$$|\mathcal{A}_{n,j}^{\alpha_0, \alpha}| \leq \left[\frac{14}{\sqrt{C}} + \frac{6(K_0 + 2K'_0)}{\sqrt{C}} \right] \frac{\alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} j!}{(j - \frac{n}{2} + 2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} \\ \leq \alpha^{\frac{n}{2}-2} C^{j-\frac{n}{4}} \frac{j!}{(j - \frac{n}{2} + 2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda}, \quad (\text{H.0.10})$$

choosing C sufficiently large. For instance, we may choose $\sqrt{C} = 194$.

- $n \leq 4$: We integrate the flow equations downwards from α_{\max} to α . We start with $n = 4$. The linear term is non-zero if $j \geq 2$. Inserting the induction hypothesis, the linear term is bounded by

$$10 \int_{\alpha}^{\alpha_{\max}} d\alpha' \frac{1}{\alpha'} \frac{C^{j-\frac{3}{2}} j!}{(j-1)^2 3^2 3!} \sum_{\lambda=0}^{j-2} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha'))^{\lambda} \\ \leq C^{j-1} \frac{j!}{j^2 2^2 2!} \sum_{\lambda=0}^{j-1} \frac{1}{2^{\lambda} \lambda!} (1 - \ln(m^2 \alpha))^{\lambda} \frac{80}{3\sqrt{C}}, \quad (\text{H.0.11})$$

where we used

$$\begin{aligned} \sum_{\lambda=0}^{j-2} \frac{1}{2^\lambda \lambda!} \int_{\alpha}^{\alpha_{\max}} d\alpha' \frac{(1 - \ln(m^2 \alpha'))^\lambda}{\alpha'} &\leq \sum_{\lambda=0}^{j-2} \frac{1}{2^\lambda (\lambda+1)!} (1 - \ln(m^2 \alpha))^{\lambda+1} \\ &\leq 2 \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda. \end{aligned} \quad (\text{H.0.12})$$

In the non-linear term, we have $n_1 = 2$, $n_2 = 4$ or $n_1 = 4$, $n_2 = 2$. The non-linear term is non-zero if $j \geq 2$. Therefore we get

$$\begin{aligned} &4 \sum_{\substack{j_1+j_2=j \\ j_i \geq 1}} \int_{\alpha}^{\alpha_{\max}} d\alpha' |\mathcal{A}_{2,j_1}^{\alpha_0, \alpha'} \mathcal{A}_{4,j_2}^{\alpha_0, \alpha'}| \\ &\leq 4C^{j-\frac{3}{2}} \sum_{\lambda=0}^{j-2} \sum_{\substack{j_1+j_2=j \\ j_i \geq 1 \\ \lambda_1+\lambda_2=\lambda}} \frac{j_1! j_2!}{(j_1+1)^2 j_2^2} \frac{\lambda!}{2! 2^2} \frac{1}{\lambda_1! \lambda_2! 2^\lambda \lambda!} \int_{\alpha}^{\alpha_{\max}} d\alpha' (1 - \ln(m^2 \alpha'))^\lambda. \end{aligned} \quad (\text{H.0.13})$$

Using Lemma C.5 (C.4.7) and (H.0.12), we bound the non-linear terms by

$$C^{j-1} \frac{4K'_0 j!}{\sqrt{C} m^2 j^2 2^2 2!} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda. \quad (\text{H.0.14})$$

Choosing $\sqrt{C} \geq 194$ such that

$$\frac{80}{3\sqrt{C}} + \frac{4K'_0}{m^2 \sqrt{C}} < 1 \quad (\text{H.0.15})$$

yields the claim for $n = 4$.

For $n = 2$, we use the bounds established for $n = 4$. We bound the linear term by

$$3C^{j-1} \frac{j!}{j^2 2^2 2!} \int_{\alpha}^{\alpha_{\max}} \frac{d\alpha'}{\alpha'^2} \sum_{\lambda=0}^{j-1} (1 - \ln(m^2 \alpha'))^\lambda \leq \frac{3}{2\alpha} C^{j-1} \frac{j!}{(j+1)^2} \sum_{\lambda=0}^{j-1} (1 - \ln(m^2 \alpha))^\lambda. \quad (\text{H.0.16})$$

The only terms in the non-linear term in the r.h.s. of (7.1.6) correspond to $n_1 = n_2 = 2$. Since for $n_k = 2$, $l_k = j_k$, the quadratic term is bounded by

$$C^{j-1} \sum_{\lambda=0}^{j-2} \sum_{\substack{j_1+j_2=j \\ \lambda_1 \leq j_1, \lambda_2 \leq j_2 \\ \lambda_1+\lambda_2=\lambda}} \frac{j_1! j_2! \lambda!}{(j_1+1)^2 (j_2+1)^2 \lambda_1! \lambda_2!} \int_{\alpha}^{\alpha_{\max}} \frac{d\alpha'}{\alpha'^2} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha'))^\lambda. \quad (\text{H.0.17})$$

We use Lemma C.5 (C.4.7) to bound (H.0.17) by

$$\begin{aligned} &C^{j-1} K'_0 \frac{j!}{(j+1)^2} \int_{\alpha}^{\alpha_{\max}} \frac{d\alpha'}{\alpha'^2} \sum_{\lambda=0}^{j-2} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha'))^\lambda \\ &\leq \frac{C^{j-1} K'_0}{\alpha} \frac{j!}{(j+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda, \end{aligned} \quad (\text{H.0.18})$$

because $\ln(m^2\alpha) < 0$. Choosing $\sqrt{C} > 194$ such that

$$\frac{3}{2\sqrt{C}} + \frac{K'_0}{\sqrt{C}} \leq 1, \quad (\text{H.0.19})$$

we obtain the claim for $n = 2$.

- $k \geq 1$

We multiply (7.1.6) by α^2 and differentiate k times w.r.t. α and we solve $\partial_\alpha^{k+1} \mathcal{A}_{n,j}^{\alpha_0,\alpha}$. We obtain

$$\begin{aligned} \partial_\alpha^{k+1} \mathcal{A}_{n,j}^{\alpha_0,\alpha} &= -\frac{2k}{\alpha} \partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0,\alpha} - \frac{k(k+1)}{\alpha^2} \partial_\alpha^{k-1} \mathcal{A}_{n,j}^{\alpha_0,\alpha} + \frac{n(n+1)}{2\alpha^2} \partial_\alpha^k \mathcal{A}_{n+2,j}^{\alpha_0,\alpha} \\ &\quad - \frac{n}{2} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i \\ k_1+k_2=k}} \frac{k!}{k_1! k_2!} \partial_\alpha^{k_1} \mathcal{A}_{n_1,j_1}^{\alpha_0,\alpha} \partial_\alpha^{k_2} \mathcal{A}_{n_2,j_2}^{\alpha_0,\alpha} \\ &\quad - \frac{nk}{\alpha} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i \\ k_1+k_2=k-1}} \frac{(k-1)!}{k_1! k_2!} \partial_\alpha^{k_1} \mathcal{A}_{n_1,j_1}^{\alpha_0,\alpha} \partial_\alpha^{k_2} \mathcal{A}_{n_2,j_2}^{\alpha_0,\alpha} \\ &\quad - \frac{nk(k+1)}{2\alpha^2} \sum_{\substack{n_1+n_2=n+2 \\ j_1+j_2=j \\ 2j_i+2 \geq n_i \\ k_1+k_2=k-2}} \frac{(k-2)!}{k_1! k_2!} \partial_\alpha^{k_1} \mathcal{A}_{n_1,j_1}^{\alpha_0,\alpha} \partial_\alpha^{k_2} \mathcal{A}_{n_2,j_2}^{\alpha_0,\alpha}. \end{aligned} \quad (\text{H.0.20})$$

We follow the convention that an empty sum is zero. We successively bound the terms in the r.h.s. of (H.0.20). For $n > 2$, we successively obtain

- First term:

$$\begin{aligned} &\left| \frac{2k}{\alpha} \partial_\alpha^k \mathcal{A}_{n,j}^{\alpha_0,\alpha} \right| \\ &\leq \frac{2k}{\alpha} \alpha^{\frac{n}{2}-2-k} \frac{C^{j-\frac{n}{4}+k} (j+k+1)!}{(j-\frac{n}{2}+2)^2 (k+1)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2\alpha))^\lambda \\ &\leq \alpha^{\frac{n}{2}-2-k-1} \frac{C^{j-\frac{n}{4}+k+1} (j+k+2)!}{(j-\frac{n}{2}+2)^2 (k+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2\alpha))^\lambda \frac{8}{C}. \end{aligned} \quad (\text{H.0.21})$$

- Second term¹:

$$\begin{aligned} &\left| \frac{k(k+1)}{\alpha^2} \partial_\alpha^{k-1} \mathcal{A}_{n,j}^{\alpha_0,\alpha} \right| \\ &\leq \frac{k(k+1)}{\alpha^2} \alpha^{\frac{n}{2}-2-k+1} \frac{C^{j-\frac{n}{4}+k-1} (j+k)!}{(j-\frac{n}{2}+2)^2 k^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2\alpha))^\lambda \\ &\leq \alpha^{\frac{n}{2}-2-k-1} \frac{C^{j-\frac{n}{4}+k+1} (j+k+2)!}{(j-\frac{n}{2}+2)^2 (k+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2\alpha))^\lambda \frac{9}{C^2}. \end{aligned} \quad (\text{H.0.22})$$

¹This term is non-zero if $k \geq 1$.

– Third term:

$$\begin{aligned}
& \left| \frac{n(n+1)}{2\alpha^2} \partial_\alpha^k \mathcal{A}_{n+2,j}^{\alpha_0, \alpha} \right| \\
& \leq \frac{n(n+1)}{2\alpha^2} \alpha^{\frac{n}{2}-1-k} \frac{C^{j-\frac{n}{4}-\frac{1}{2}+k} (j+k+1)!}{(j-\frac{n}{2}+1)^2 (k+1)^2 (\frac{n}{2}+1)^2 (\frac{n}{2}+1)!} \sum_{\lambda=0}^{j-\frac{n}{2}-1+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \\
& \leq \alpha^{\frac{n}{2}-2-k-1} \frac{C^{j-\frac{n}{4}+k+1} (j+k+2)!}{(j-\frac{n}{2}+2)^2 (k+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{32}{C^{\frac{3}{2}}},
\end{aligned} \tag{H.0.23}$$

since we have $j \geq \frac{n}{2}$.

– Fourth term: We proceed as in the case $k = 0$. We use Lemma C.5, inequalities (C.4.8)-(C.4.9) to obtain the following bound

$$\begin{aligned}
& \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{n(j+k+2)!}{2(j-\frac{n}{2}+2)^2 (k+1)^2 (\frac{n}{2})^2 (\frac{n}{2}+1)!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{(K_0'' + 2K_0''')}{C^{\frac{3}{2}}} \\
& \leq \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{(j+k+2)!}{(j-\frac{n}{2}+2)^2 (k+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{4(K_0'' + 2K_0''')}{C^{\frac{3}{2}}}.
\end{aligned} \tag{H.0.24}$$

– Fifth term²: We proceed as above using again Lemma (C.5) inequalities (C.4.8)-(C.4.9). We obtain the following bound

$$\begin{aligned}
& \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{nk(j+k+1)!}{(j-\frac{n}{2}+2)^2 k^2 (\frac{n}{2})^2 (\frac{n}{2}+1)!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{K_0'' + 2K_0'''}{C^{\frac{5}{2}}} \\
& \leq \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{(j+k+2)!}{(j-\frac{n}{2}+2)^2 (k+2)^2 (\frac{n}{2})^2 (\frac{n}{2})!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{18(K_0'' + 2K_0''')}{C^{\frac{5}{2}}}.
\end{aligned} \tag{H.0.25}$$

– Sixth term³: We proceed as above using again Lemma (C.5) inequalities (C.4.8)-(C.4.9).

²This term is non-zero if $k \geq 1$.

³This term is non-zero if $k \geq 2$.

We obtain the following bound

$$\begin{aligned}
& \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{nk(k+1)}{2} \frac{(j+k)!}{(j-\frac{n}{2}+2)^2(k-1)^2(\frac{n}{2})^2(\frac{n}{2}+1)!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda} \frac{(K_0''+2K_0''')}{C^{\frac{7}{2}}} \\
& \leq \alpha^{\frac{n}{2}-3-k} C^{j-\frac{n}{4}+k+1} \frac{(j+k+2)!}{(j-\frac{n}{2}+2)^2(k-1)^2(\frac{n}{2})^2(\frac{n}{2})!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda} \frac{16(K_0''+2K_0''')}{C^{\frac{7}{2}}}.
\end{aligned} \tag{H.0.26}$$

Adding (H.0.21)-(H.0.26), we find

$$\begin{aligned}
& |\partial_{\alpha}^{k+1} \mathcal{A}_{n,j}^{\alpha_0,\alpha}| \\
& \leq \left[\frac{8}{C} + \frac{9}{C^2} + \frac{32}{C^{\frac{3}{2}}} + \frac{38}{C^{\frac{3}{2}}} (K_0''+2K_0''') \right] \alpha^{\frac{n}{2}-2-k} \frac{C^{j-\frac{n}{4}+k+1} (j+k+2)!}{(j-\frac{n}{2}+2)^2(k+2)^2(\frac{n}{2})^2(\frac{n}{2})!} \times \\
& \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda} \\
& \leq \alpha^{\frac{n}{2}-2-k} \frac{C^{j-\frac{n}{4}+k+1} (j+k+2)!}{(j-\frac{n}{2}+2)^2(\frac{n}{2})^2(\frac{n}{2})! (\frac{n}{2}-1)!} \sum_{\lambda=0}^{j-\frac{n}{2}+1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda},
\end{aligned} \tag{H.0.27}$$

choosing C such that

$$\frac{8}{C} + \frac{1}{C^2} \left(9 + 32 + 38(K_0''+2K_0''') \right) \leq 1. \tag{H.0.28}$$

For $n = 2$, we repeat the same steps above. The essential difference w.r.t. the case $n > 2$ is that in the r.h.s. of (7.1.10), the sum runs over $0 \leq \lambda \leq j-1$. Here we will only explicit the bounds for the non-trivial terms.

- The first and second term in the r.h.s. of (H.0.20) are treated as above so that they are bounded by terms similar to (H.0.21) and (H.0.22) with the aforementioned change.
- Third term: We insert the induction hypothesis to obtain

$$\begin{aligned}
& \left| \frac{3}{\alpha^2} \partial_{\alpha}^k \mathcal{A}_{4,j}^{\alpha_0,\alpha} \right| \\
& \leq \frac{3}{\alpha^{k+2}} \frac{C^{j-1+k} (j+k+1)!}{j^2 (k+1)^2 4 \times 2} \sum_{\lambda=0}^{j-1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda} \\
& \leq \frac{1}{\alpha^{k+2}} \frac{C^{j+k+\frac{1}{2}} (j+k+2)!}{(j+1)^2 (k+2)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^{\lambda}\lambda!} (1-\ln(m^2\alpha))^{\lambda} \frac{6}{C^{\frac{3}{2}}},
\end{aligned} \tag{H.0.29}$$

- Fourth term: The non-linear term contains only contributions of the form

$$\partial_\alpha^{k_1} \mathcal{A}_{2,j_1}^{\alpha_0,\alpha} \partial_\alpha^{k_2} \mathcal{A}_{2,j_2}^{\alpha_0,\alpha}, \quad k_1 + k_2 = k, \quad j_1 + j_2 = j. \quad (\text{H.0.30})$$

Therefore, we bound the fourth term by

$$\begin{aligned} & \frac{1}{\alpha^{k+1}} C^{j-1+k} \sum_{\lambda=0}^{j-2} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \times \\ & \sum_{\substack{j_1+j_2=j \\ \lambda_1 \leq j_1-1, \lambda_2 \leq j_2-1 \\ k_1+k_2=k}} \frac{k!}{k_1! k_2!} \frac{\lambda!}{\lambda_1! \lambda_2!} \frac{(j_1 + k_1 + 1)! (j_2 + k_2 + 1)!}{(j_1 + 1)^2 (j_2 + 1)^2 (k_1 + 1)^2 (k_2 + 1)^2}. \end{aligned} \quad (\text{H.0.31})$$

Using Lemma C.5 (C.4.9), (H.0.31) is bounded by

$$\frac{1}{\alpha^{k+1}} C^{j+k+\frac{1}{2}} \frac{(j+k+2)!}{(j+1)^2 (k+1)^2} \sum_{\lambda=0}^{j-1} \frac{1}{2^\lambda \lambda!} (1 - \ln(m^2 \alpha))^\lambda \frac{K_0'''}{2C^{\frac{3}{2}}}. \quad (\text{H.0.32})$$

- The remaining terms in the r.h.s. of (H.0.20) can be treated analogously. They are bounded by terms similar to (H.0.25)-(H.0.26) with the aforementioned changes.

Summing the different bounds, we obtain the claim for $n = 2$. □

Appendix I

Useful Lemmata used to prove the renormalization conditions compatibility

I.1 Exact expressions of $g_{n,0}$ and $g_{n,1}$

We recall that

$$C_2(n) := \frac{1}{2n+1} \binom{3n}{n}, \quad n \geq 0. \quad (\text{I.1.1})$$

Lemma I.1. *We have*

$$\sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} n_2 C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) = \frac{(n-4)(n+2)}{2n} C_2\left(\frac{n}{2} - 1\right). \quad (\text{I.1.2})$$

Proof. We have

$$\begin{aligned} 2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} n_2 C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) &= \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} (n_2 + n_1) C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) \\ &= \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} (n+2) C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right). \end{aligned} \quad (\text{I.1.3})$$

We use the convolution identity [114]

$$\sum_{\substack{i_1+i_2=m \\ i_j \geq 0}} C_s(i_1) C_s(i_2) = \frac{2}{(s+1)m+2} \binom{(s+1)m+2}{m}, \quad s \geq 1, m \geq 0, \quad (\text{I.1.4})$$

to get

$$\begin{aligned}
\sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} n_2 C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) &= \frac{n+2}{2} \sum_{\substack{n_1+n_2=\frac{n}{2}-1 \\ n_i \geq 1}} C_2(n_1) C_2(n_2) \\
&= \frac{n+2}{2} \left[-2C_2\left(\frac{n}{2} - 1\right) + \frac{2}{\frac{3n}{2}-1} \left(\frac{\frac{3n}{2}-1}{\frac{n}{2}-1}\right) \right] \\
&= \frac{(n-4)(n+2)}{2n} C_2\left(\frac{n}{2} - 1\right).
\end{aligned} \tag{I.1.5}$$

□

Lemma 7.1. *We have for $n \geq 4$*

$$g_{n,0} = (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} C_2\left(\frac{n}{2} - 1\right), \tag{7.2.9}$$

$$g_{n,1} = (-1)^{\frac{n}{2}-1} g_{4,0}^{\frac{n}{2}-1} C_2\left(\frac{n}{2} - 1\right) \left(\frac{3n-4}{2} b_1 + \frac{n-4}{4}\right). \tag{7.2.10}$$

Proof. First we prove (7.2.9) by induction in $n \geq 4$. For $n = 4$, the result is obvious. For $n \geq 6$ we use (3.2.50) to obtain

$$\begin{aligned}
g_{n,0} &= -\frac{n}{n-4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} (-1)^{\frac{n}{2}-1} g_{4,0}^{\frac{n}{2}-1} C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) \\
&= (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} \frac{n}{n-4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4 \\ n_i \in 2\mathbb{N}}} C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) \\
&= (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} \frac{n}{n-4} \sum_{\substack{n_1+n_2=\frac{n}{2}-1 \\ n_i \geq 1}} C_2(n_1) C_2(n_2).
\end{aligned} \tag{I.1.6}$$

Using the convolution identity (I.1.4), we have

$$\begin{aligned}
g_{n,0} &= (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} \frac{n}{n-4} \left[-2C_2\left(\frac{n}{2} - 1\right) + \frac{2}{\frac{3n}{2}-1} \left(\frac{\frac{3n}{2}-1}{\frac{n}{2}-1}\right) \right] \\
&= (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} C_2\left(\frac{n}{2} - 1\right).
\end{aligned} \tag{I.1.7}$$

To prove (7.2.10) we proceed by induction in n . The claim is true for $n = 4$. Then we have using Lemma I.1

$$\begin{aligned}
g_{n,1} &= -\frac{2n}{n-2} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} g_{n_1,0} g_{n_2,1} - \frac{n}{n-2} g_{n,0} \left(2f_{2,0} + 1 - \frac{4}{n}\right) \\
&= (-1)^{\frac{n}{2}} g_{4,0}^{\frac{n}{2}-1} \frac{n}{n-2} \left[2 \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} C_2\left(\frac{n_1}{2} - 1\right) C_2\left(\frac{n_2}{2} - 1\right) \left(\frac{3n_2-4}{2} b_1 + \frac{n_2-4}{4}\right) \right. \\
&\quad \left. + \left(2b_1 + 1 - \frac{4}{n}\right) C_2\left(\frac{n}{2} - 1\right) \right] \\
&= (-1)^{\frac{n}{2}-1} g_{4,0}^{\frac{n}{2}-1} C_2\left(\frac{n}{2} - 1\right) \left(\frac{3n-4}{2} b_1 + \frac{n-4}{4}\right).
\end{aligned} \tag{I.1.8}$$

□

I.2 Behavior of the coefficients $g_{n,k}, f_{2,k}, b_n$ in terms of b_1

Lemma 7.5. *Under the assumptions of Lemma (7.2)*

$$|g_{n,k,\nu}| \leq \frac{1}{4} K^{\frac{n}{2}-1} \left(\frac{n-4}{2} + k \right)! \binom{k}{\nu}, \quad |f_{2,k,\nu}| \leq |k-1|! \binom{k+1}{\nu}. \quad (7.2.30)$$

Proof. The proof is done by induction in $N = n + 2k$, going up in N and at a fixed value of N , we go up in k . For $k \leq 1$, we use (7.2.25) to get

$$|g_{n,0,0}| \leq \frac{K^{\frac{n}{2}-1}}{30^{\frac{n}{2}-1}} \frac{1}{n-1} \binom{3(\frac{n}{2}-1)}{\frac{n}{2}-1} \leq \left(\frac{4K}{15} \right)^{\frac{n}{2}-1} \frac{1}{n-1} \leq \frac{1}{4} K^{\frac{n}{2}-1} \left(\frac{n-4}{2} \right)!, \quad n \geq 4 \quad (I.2.1)$$

and

$$|g_{n,1,0}| \leq \frac{K^{\frac{n}{2}-1}}{30^{\frac{n}{2}-1}} \frac{n-4}{4(n-1)} \binom{3(\frac{n}{2}-1)}{\frac{n}{2}-1} \leq \left(\frac{4K}{15} \right)^{\frac{n}{2}-1} \frac{1}{4} \leq \frac{1}{4} K^{\frac{n}{2}-1} \left(\frac{n-2}{2} \right)!, \quad n \geq 4. \quad (I.2.2)$$

We have as well

$$|g_{4,1,1}| = 4g_{4,0} \leq \frac{2K}{15} \leq \frac{K}{4}, \quad (I.2.3)$$

$$|g_{n,1,1}| \leq \frac{K^{\frac{n}{2}-1}}{30^{\frac{n}{2}-1}} \frac{3n-4}{2(n-1)} \binom{3(\frac{n}{2}-1)}{\frac{n}{2}-1} \leq \left(\frac{4K}{15} \right)^{\frac{n}{2}-1} \frac{3}{2} \leq \frac{1}{4} K^{\frac{n}{2}-1} \left(\frac{n-2}{2} \right)!, \quad n \geq 6. \quad (I.2.4)$$

We insert the induction hypothesis in the r.h.s. of (5.1.7)

- We treat the cases $k = 2$ and $n \geq 4$. We have

$$\begin{aligned} |g_{n,2,\nu}| &\leq \frac{n-4}{n} |g_{n,1,\nu}| + 2 \sum_{\rho=0}^1 \sum_{\nu'=\max\{\nu-(2-\rho),0\}}^{\min\{\rho,\nu\}} |g_{n,\rho,\nu'} f_{2,1-\rho,\nu-\nu'}| \\ &\quad + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 \sum_{\nu'=\max\{\nu-(2-\rho),0\}}^{\min\{\rho,\nu\}} |g_{n_1,\rho,\nu'} g_{n_2,2-\rho,\nu-\nu'}| \\ &\quad + (n+1) |g_{n+2,0,\nu}|. \end{aligned} \quad (I.2.5)$$

We use (7.2.25), (7.2.26) and (7.2.27) to get

- $\nu = 0$:

$$\begin{aligned}
|g_{n,2,0}| &\leq \frac{n-4}{n}|g_{n,1,0}| + 2|g_{n,0,0}f_{2,1,0}| + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 |g_{n_1,\rho,0}g_{n_2,2-\rho,0}| \\
&\quad + (n+1)|g_{n+2,0,0}| \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \left[\left(\frac{4}{15}\right)^{\frac{n}{2}-1} + 8\left(\frac{4}{15}\right)^{\frac{n}{2}-1} \frac{3}{n-1}g_{4,0} \right. \\
&\quad \left. + \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 \left(\rho + \frac{n_1-4}{2}\right)! \left(2-\rho + \frac{n_2-4}{2}\right)! + 4\left(\frac{4}{15}\right)^{\frac{n}{2}}K \right] \quad (I.2.6) \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{0} \left[\frac{2}{15} + \frac{2}{15} \frac{2K}{15} + \frac{1}{4} + \left(\frac{4}{15}\right)^2 \frac{2}{15} \right] \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{0}.
\end{aligned}$$

- $\nu = 1$

$$\begin{aligned}
|g_{n,2,1}| &\leq \frac{n-4}{4}|g_{n,1,1}| + 2|g_{n,0,0}| + 2|g_{n,1,0}| + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 \sum_{\nu'=0}^1 |g_{n_1,\rho,\nu'}g_{n_2,2-\rho,1-\nu'}| \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \left[6\left(\frac{4}{15}\right)^{\frac{n}{2}-1} + 2\left(\frac{4}{15}\right)^{\frac{n}{2}-1} \frac{4}{n-1} + 2\left(\frac{4}{15}\right)^{\frac{n}{2}-1} \right. \\
&\quad \left. + \frac{1}{4} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 \sum_{\nu'=0}^1 \left(\rho + \frac{n_1-4}{2}\right)! \left(2-\rho + \frac{n_2-4}{2}\right)! \binom{\rho}{\nu'} \binom{2-\rho}{1-\nu'} \right] \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{1} \left[\frac{2}{5} + \frac{2}{3} \frac{4}{15} + \frac{1}{2} \frac{4}{15} + \frac{1}{4} \right] \leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{1}, \quad (I.2.7)
\end{aligned}$$

where we used the Vandermonde formula

$$\sum_{\nu'=0}^{\nu} \binom{a}{\nu'} \binom{b}{\nu-\nu'} = \binom{a+b}{\nu}, \quad \nu, a, b \in \mathbb{N}_0, \nu \leq a+b. \quad (I.2.8)$$

- $\nu = 2$: we have first

$$|g_{4,2,2}| \leq 2g_{4,0,0}|f_{2,1,2}| + 2|g_{4,1,1}||f_{2,0,1}| \leq \frac{1}{3}K \leq \frac{1}{4}K 2!. \quad (I.2.9)$$

Then for $n \geq 6$ we have

$$\begin{aligned}
|g_{n,2,2}| &\leq 2|g_{n,0,0}| + 2|g_{n,1,1}| + \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^2 \sum_{\nu'=0}^2 |g_{n_1,\rho,\nu'}g_{n_2,2-\rho,2-\nu'}| \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{2} \left[\frac{4}{3(n-1)} \left(\frac{4}{15}\right)^{\frac{n}{2}-1} + 2\left(\frac{4}{15}\right)^{\frac{n}{2}-1} + \frac{1}{4} \right] \quad (I.2.10) \\
&\leq \frac{1}{4}K^{\frac{n}{2}-1} \binom{n}{2}! \binom{2}{2}.
\end{aligned}$$

- $n = 4$ and $k \geq 1$: First we see that $g_{4,1,\nu}$ satisfy the bounds as claimed. The case $k = 2$ is already treated. For $k = 3$, we have using (7.2.25), (7.2.26) and (7.2.27)

$$|g_{4,3,0}| \leq \frac{4}{3} \left[|g_{4,0,0}| |f_{2,2,0}| + \frac{5}{2} |g_{6,1,0}| \right] \leq \frac{4}{3} \left[\frac{K}{30} \frac{K}{20} + \frac{5}{2} \frac{1}{2} K^2 \frac{1}{300} \right] \leq \frac{1}{4} K^3, \quad (\text{I.2.11})$$

$$|g_{4,3,1}| \leq \frac{4}{3} \left[|g_{4,1,1}| |f_{2,1,0}| + |g_{4,2,0}| |f_{2,0,1}| + |g_{4,0,0}| |f_{2,2,1}| \right] + \frac{10}{3} |g_{6,1,1}| \leq \frac{1}{4} K^3 \binom{3}{1}, \quad (\text{I.2.12})$$

$$|g_{4,3,2}| \leq \frac{4}{3} \left[|g_{4,0,0}| |f_{2,2,2}| + |g_{4,1,1}| |f_{2,1,1}| + |g_{4,2,1}| |f_{2,0,1}| \right] \leq \frac{1}{4} K^3 \binom{3}{2}, \quad (\text{I.2.13})$$

$$|g_{4,3,3}| \leq \frac{4}{3} \left[|g_{4,0,0}| |f_{2,2,3}| + |g_{4,1,1}| |f_{2,1,2}| + |g_{4,2,2}| |f_{2,0,1}| \right] \leq \frac{1}{4} K^3 \binom{3}{3}. \quad (\text{I.2.14})$$

Then, for $k \geq 2$ we have, following the proof of Lemma 7.2 and (I.2.8)

$$\begin{aligned} |g_{4,k+2,\nu}| &\leq \frac{4}{4+2k} \frac{1}{4} \left[3k! + (k+1)! \right] \binom{k+2}{\nu} + \frac{10}{k+2} \frac{K^2}{4} (k+1)! \binom{k}{\nu} \\ &\leq \frac{1}{4} K^3 (k+2)! \binom{k+2}{\nu} \left[\frac{1}{4} + \frac{1}{4} + \frac{5K}{8} \right] \leq \frac{1}{4} K^3 (k+2)! \binom{k+2}{\nu}. \end{aligned} \quad (\text{I.2.15})$$

• $n \geq 6$ and $k \geq 1$: We obtain

$$\begin{aligned}
|g_{n,k+2,\nu}| &\leq \frac{n-4}{n+2k} |g_{n,k+1,\nu}| + \frac{2n}{n+2k} \sum_{\rho=0}^{k+1} \sum_{\nu'=\max\{\nu-(k+2-\rho),0\}}^{\min\{\rho,\nu\}} |g_{n,\rho,\nu'} f_{2,k+1-\rho,\nu-\nu'}| \\
&\quad + \frac{n}{n+2k} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^{k+2} \sum_{\nu'=\max\{\nu-(k+2-\rho),0\}}^{\min\{\rho,\nu\}} |g_{n_1,\rho,\nu'} g_{n_2,k+2-\rho,\nu-\nu'}| \\
&\quad + \frac{n(n+1)}{n+2k} |g_{n+2,k,\nu}| \\
&\leq \frac{1}{4} K^{\frac{n}{2}-1} \left[\frac{(n-4)}{(n+2k)} \left(k+1 + \frac{n-4}{2} \right)! \binom{k+1}{\nu} \right. \\
&\quad + \frac{2n}{(n+2k)} \sum_{\rho=0}^{k+1} \left(\rho + \frac{n-4}{2} \right)! |k-\nu|! \binom{k+2}{\nu} \\
&\quad + \frac{n}{4(n+2k)} \sum_{\substack{n_1+n_2=n+2 \\ n_i \geq 4}} \sum_{\rho=0}^{k+2} \left(\rho + \frac{n_1-4}{2} \right)! \left(k+2-\rho + \frac{n_2-4}{2} \right)! \binom{k+2}{\nu} \\
&\quad \left. + \frac{n(n+1)K}{(n+2k)} \left(k + \frac{n-2}{2} \right)! \binom{k}{\nu} \right] \\
&\leq \frac{K^{\frac{n}{2}-1}}{4} \left(k+2 + \frac{n-4}{2} \right)! \binom{k+2}{\nu} \\
&\quad \times \left[\frac{2(n-4)}{(n+2k)^2} + \frac{8n}{(n+2k)^2} + \frac{n^2}{4(n+2k)^2} + \frac{2n(n+1)}{(n+2k)^2} K \right] \\
&\leq \frac{1}{4} K^{\frac{n}{2}-1} \left(k+2 + \frac{n-4}{2} \right)! \binom{k+2}{\nu}.
\end{aligned} \tag{I.2.16}$$

For $f_{2,k}$, we proceed by induction in k . The bounds are satisfied for $k \leq 2$. For $k \geq 2$ we have

$$\begin{aligned}
|f_{2,k+1,\nu}| &\leq \frac{1}{k+1} \left(\frac{3}{4} K k! \binom{k}{\nu} + (k-1)! \binom{k+1}{\nu} + \sum_{\rho=0}^k |\rho-1|! |k-\rho-1|! \binom{k+2}{\nu} \right) \\
&\leq k! \binom{k+2}{\nu} \frac{1}{k+1} \left(\frac{3}{4} K + \frac{1}{k} + \frac{4}{k} \right) \\
&\leq k! \binom{k+2}{\nu}.
\end{aligned} \tag{I.2.17}$$

□

Lemma 7.7. *We have the following estimates*

$$|a_{q,\nu}| \leq \frac{1}{q} \left(\frac{3}{4} \right)^{q-2} \binom{q}{\nu}, \quad q \geq 1, \quad 0 \leq \nu \leq q. \tag{7.2.34}$$

Proof. The proof is done by induction in $q \geq 1$. For $q \leq 4$, the bounds can be checked by hand. They obviously hold for $q \leq 2$. We have from (7.2.33) and Lemma 7.5

$$\left\{ \begin{array}{l} |a_{3,0}| = \frac{g_{4,0}}{6} \leq \frac{1}{3} \left(\frac{3}{4}\right)^1 \\ |a_{3,1}| \leq \frac{2}{3}g_{4,0} + \frac{1}{18} \leq \frac{1}{3} \left(\frac{3}{4}\right)^1 3 \\ |a_{3,2}| = \frac{1}{6} \leq \frac{1}{3} \left(\frac{3}{4}\right)^1 3 \\ |a_{3,3}| = \frac{1}{9} \leq \frac{1}{3} \left(\frac{3}{4}\right)^1 \end{array} \right. , \quad \left\{ \begin{array}{l} |a_{4,0}| \leq \frac{4}{64} + \frac{3g_{4,0}}{128} \leq \frac{1}{4} \left(\frac{3}{4}\right)^2 \\ |a_{4,1}| \leq \frac{16+1+8}{64} \leq \frac{1}{4} \left(\frac{3}{4}\right)^2 4 \\ |a_{4,2}| \leq \frac{24+4}{64} \leq \frac{1}{4} \left(\frac{3}{4}\right)^2 6 \\ |a_{4,3}| \leq \frac{16}{64} \leq \frac{1}{4} \left(\frac{3}{4}\right)^2 4 \\ |a_{4,4}| \leq \frac{4}{64} \leq \frac{1}{4} \left(\frac{3}{4}\right)^2 \end{array} \right. . \quad (\text{I.2.18})$$

We insert the induction hypothesis in the r.h.s. of (3.2.73). For $q \geq 4$ we use Lemma 7.5 to obtain

$$\frac{(q-1)!}{(q+1)^{q-1}} \leq \left(\frac{3}{4}\right)^q \frac{1}{5}, \quad q \geq 4. \quad (\text{I.2.19})$$

We also have

$$\begin{aligned} \sum_{\rho=2}^{q+1} |a_{\{\frac{q+1}{\rho}, \nu\}}| \frac{1}{\rho^q} &\leq \binom{q+1}{\nu} \frac{1}{q+1} \left[\sum_{\rho=2}^q \frac{1}{\rho^{q-1}} + \frac{1}{(q+1)^{q-1}} \right] \\ &\leq \binom{q+1}{\nu} \frac{1}{q+1} \left[\zeta(q-1) - 1 + \frac{1}{(q+1)^{q-1}} \right] \\ &\leq \binom{q+1}{\nu} \frac{1}{q+1} \left[\frac{4}{2^q} + \frac{1}{(q+1)^{q-1}} \right]. \end{aligned} \quad (\text{I.2.20})$$

Therefore from (I.2.19) and (I.2.20) we have

$$\begin{aligned} |a_{q+1, \nu}| &\leq \frac{1}{q+1} \left(\frac{3}{4}\right)^{q-1} \binom{q+1}{\nu} \frac{3}{20} + \frac{1}{q+1} \binom{q+1}{\nu} \left[\frac{4}{2^q} + \frac{1}{(q+1)^{q-1}} \right] \\ &\leq \frac{1}{q+1} \left(\frac{3}{4}\right)^{q-1} \binom{q+1}{\nu}, \end{aligned} \quad (\text{I.2.21})$$

where we used

$$\frac{3}{20} + 3 \left(\frac{2}{3}\right)^q + \left(\frac{4}{3(q+1)}\right)^{q-1} \leq 1, \quad q \geq 4. \quad (\text{I.2.22})$$

□

References

- [1] M. Aizenman. “Geometric Analysis of ϕ^4 Fields and Ising Models. Part I and II”. In: *Commun. Math. Phys.* 86 (1982), pp. 1–48.
- [2] J. Fröhlich. “On the triviality of $\lambda\phi_4^4$ theories and the approach to the critical point in $d \geq 4$ dimensions”. In: *Nucl. Phys. B* 200 (1982), pp. 281–296.
- [3] M. Aizenman and H. Duminil-Copin. “Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models”. In: *Annals of Mathematics* 194.1 (2021).
- [4] Ch. Kopper. “Asymptotically Free Solutions of the Scalar Mean Field Flow Equations”. In: *Ann. Henri Poincaré* 23 (2022), pp. 3453–3492.
- [5] G. Keller. “Local Borel summability of Euclidean φ^4 in four-dimensions: A Simple proof via differential flow equations”. In: *Commun. Math. Phys.* 161 (1994), pp. 311–322.
- [6] Ch. Kopper. “On the Local Borel Transform of Perturbation Theory”. In: *Commun. Math. Phys.* 295 (2010), pp. 669–699.
- [7] K. Osterwalder and R. Schrader. “Axioms for Euclidean Green’s Functions”. In: *Commun. Math. Phys.* 31 (1973), pp. 83–112.
- [8] K. Osterwalder and R. Schrader. “Axioms for Euclidean Green’s Functions II”. In: *Commun. Math. Phys.* 42 (1975), pp. 281–305.
- [9] F.J. Dyson. “The Radiation theories of Tomonaga, Schwinger, and Feynman”. In: *Phys. Rev.* 75 (1949), pp. 486–502.
- [10] F.J. Dyson. “The S matrix in quantum electrodynamics”. In: *Phys. Rev.* 75 (1949), pp. 1736–1755.
- [11] N.N. Bogoliubov and O.S. Parasiuk. “Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder”. In: *Acta Math.* 97 (1957), pp. 227–266.
- [12] W. Zimmermann. *Lectures on Elementary Particles and Quantum Field Theory*. Vol. 1. Cambridge, M.I.T Press, Brandeis University Summer Institute in Theoretical Physics, 1970.
- [13] W. Pauli and F. Villars. “On invariant regularization in relativistic quantum theory”. In: *Rev. Mod. Phys.* 21 (1949), pp. 434–444.
- [14] E. Speer. *Generalized Feynman Amplitudes*. Princeton University, 1969.
- [15] G. ’t Hooft and M. Veltman. “Regularization and renormalization of Gauge fields”. In: *Nucl. Phys. B* 44 (1972), pp. 189–213.
- [16] C.G. Bollini and J.J. Giambiagi. “Lowest order "divergent" Graphs in ν -dimensional space”. In: *Phys. Lett. B* 40 (1972), pp. 566–568.

- [17] P. Breitenlohner and D. Maison. “Dimensional Renormalization and the Action Principle”. In: *Commun. Math. Phys.* 52 (1977), pp. 11–38.
- [18] A.S. Wightman and G. Velo, eds. *Renormalization Theory*. Erice 1975. Reidel, 1976.
- [19] W. Zimmermann. “Convergence of Bogoliubov’s method of renormalization in momentum space”. In: *Commun. Math. Phys.* 15 (1969), pp. 208–234.
- [20] J.H. Lowenstein. “Convergence theorems for renormalized Feynman integrals with zero-mass propagators”. In: *Commun. Math. Phys.* 47 (1976), pp. 53–68.
- [21] K.G. Wilson. “Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture”. In: *Phys. Rev. B* 4 (1971), pp. 3174–3183.
- [22] K.G. Wilson. “Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior”. In: *Phys. Rev. B* 4 (1971), pp. 3184–3205.
- [23] F.J. Wegner and A. Houghton. “Renormalization Group Equation for Critical Phenomena”. In: *Phys. Rev. A* 8 (1973), pp. 401–412.
- [24] K. Gawedzki and Kupiainen A. “Gross-Neveu model through convergent perturbation expansions”. In: *Commun. Math. Phys.* 102 (1985), pp. 1–30.
- [25] J. Feldman et al. “A renormalizable field theory: the massive Gross-Neveu model in two dimensions”. In: *Commun. Math. Phys.* 103 (1986), pp. 67–103.
- [26] P. Duch. “Construction of Gross-Neveu model using Polchinski flow equation”. In: *arXiv preprint:2403.18562 [math-ph]* (2024).
- [27] D.C. Brydges. “Functional Integrals and their Applications”. In: *Lausanne lectures 1992*, mp-arc 93-24.
- [28] V. Rivasseau. *From Perturbative to Constructive Renormalization*. Princeton Series in Physics. Princeton University Press, 2014.
- [29] G. Gallavotti and F. Nicolò. “Renormalization theory in four-dimensional scalar fields I”. In: *Commun. Math. Phys.* 100 (1985), pp. 545–590.
- [30] G. Gallavotti and F. Nicolò. “Renormalization theory in four-dimensional scalar fields II”. In: *Commun. Math. Phys.* 101 (1986), pp. 247–282.
- [31] G. Benfatto, A. Giuliani, and V. Mastropietro. “Low Temperature Analysis of Two-Dimensional Fermi Systems with Symmetric Fermi Surface”. In: *Ann. Henri Poincaré* 4 (2003), pp. 137–193.
- [32] J. Polchinski. “Renormalization And Effective Lagrangians”. In: *Nucl. Phys. B* 231 (1984), pp. 269–295.
- [33] G. Keller, Ch. Kopper, and M. Salmhofer. “Perturbative Renormalization and Effective Lagrangians in φ_4^4 ”. In: *Helv. Phys. Acta* 65 (1992), pp. 32–52.
- [34] G. Keller and Ch. Kopper. “Perturbative renormalization of massless φ^4 in four-dimensions with flow equations”. In: *Commun. Math. Phys.* 161 (1994), pp. 515–532.
- [35] P.K. Mitter and T.R. Ramadas. “The Two-dimensional $O(N)$ Nonlinear σ -Model: Renormalisation and Effective Actions”. In: *Commun. Math. Phys.* 122 (1989), pp. 575–596.
- [36] Ch. Kopper and V.F. Müller. “Renormalization of spontaneously broken $SU(2)$ Yang–Mills theory with flow equations”. In: *Reviews in Mathematical Physics* 21 (2009), pp. 781–820.

- [37] A.N. Efremov, R. Guida, and Ch. Kopper. “Renormalization of SU(2) Yang–Mills theory with flow equations”. In: *J. Math. Phys.* 58 (2017).
- [38] G. Keller, Ch. Kopper, and C. Schophaus. “Perturbative Renormalization with Flow Equations in Minkowski Space”. In: *Helv. Phys. Acta* 70 (1996), pp. 247–274.
- [39] S. Hollands and Ch. Kopper. “The Operator Product Expansion Converges in Perturbative Field Theory”. In: *Commun. Math. Phys.* 313 (2012), pp. 257–290.
- [40] M. Lüscher and P. Weisz. “Scaling laws and triviality bounds in the lattice φ_4^4 . I.” In: *Nucl. Phys. B* 290 (1987), pp. 25–60.
- [41] M. Lüscher and P. Weisz. “Scaling laws and triviality bounds in the lattice φ_4^4 . II.” In: *Nucl. Phys. B* 295 (1988), pp. 65–92.
- [42] M. Lüscher and P. Weisz. “Scaling laws and triviality bounds in the lattice φ_4^4 . III.” In: *Nucl. Phys. B* 318 (1989), pp. 705–741.
- [43] M. Aizenman. “Proof of the Triviality of φ_d^4 Field Theory and Some Mean-Field Features of Ising models for $d > 4$ ”. In: *Phys. Rev. Letters* 47 (1981), pp. 1–4.
- [44] M. Aizenman. “A geometric perspective on the scaling limits of critical Ising and ϕ_d^4 models”. In: *Current Developments in Mathematics*. 2021.
- [45] J. Lebowitz. “GHS and other inequalities”. In: *Commun. Math. Phys.* 35 (1974), pp. 87–92.
- [46] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. 2nd ed. Springer New York, 1987. Chap. 21.
- [47] M. Lüscher. “Charge screening and an upper bound on the renormalized charge in lattice QED”. In: *Nucl. Phys. B* 341 (1990), pp. 341–357.
- [48] K.G. Wilson and J. Kogut. “The Renormalization Group and the ε Expansion”. In: *Phys. Rep.* 12 (1974), pp. 75–199.
- [49] T.D. Lee. “Some Special Examples in Renormalizable Field Theory”. In: *Phys. Rev.* 95 (1954), pp. 1329–1334.
- [50] M. Lindner. “Implications of Triviality for the Standard Model”. In: *Z. Phys. C* 31 (1986), pp. 295–300.
- [51] L.N. Lipatov. “Divergence of the perturbation theory series and quasi-classical theory”. In: *Sov. Phys. JETP* 45 (1977), pp. 216–223.
- [52] G. ’t Hooft. “Can We Make Sense Out of “Quantum Chromodynamics”?” In: *The Whys of Subnuclear Physics*. Ed. by A. Zichichi. Boston, MA: Springer US, 1979, pp. 943–982.
- [53] J.-P. Eckman, J. Magnen, and R. Sénéor. “Decay properties and Borel summability for the Schwinger functions in $P(\phi)_2$ theories”. In: *Commun. Math. Phys.* 39 (1975), pp. 251–271.
- [54] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. 2nd ed. Springer New York, 1987. Chap. 23.
- [55] C. de Calan and V. Rivasseau. “Local existence of the Borel transform in Euclidean φ_4^4 ”. In: *Commun. Math. Phys.* 82 (1981), pp. 69–100.
- [56] J. Feldman et al. “Bounds on renormalized Feynman graphs”. In: *Commun. Math. Phys.* 100 (1985), pp. 23–55.

- [57] J. Feldman et al. *QED: A proof of renormalizability, Lecture notes in Physics*. Vol. 312. Berlin-Heidelberg-New York: Springer Verlag, 1988.
- [58] V. Rivasseau. “Construction and Borel summability of planar 4-dimensional Euclidean field theory”. In: *Commun. Math. Phys.* 95 (1984), pp. 445–486.
- [59] V.F. Müller. “Perturbative Renormalization by Flow Equations”. In: *Reviews in Mathematical Physics* 15.05 (2003), pp. 491–558.
- [60] T. Hida. *Stationary Stochastic Processes*. Princeton University Press, 1970. Chap. 4.
- [61] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. 2nd ed. Springer New York, 1987. Chap. 9,1.
- [62] M. Salmhofer. *Renormalization: An Introduction*. Theoretical and Mathematical Physics. Springer Berlin Heidelberg, 2013.
- [63] Ch. Kopper and V. Müller. “Renormalization Proof for Massive φ^4 Theory on Riemannian Manifolds”. In: *Commun. Math. Phys.* 275 (2007), pp. 331–372.
- [64] M. Borji and Ch. Kopper. “Perturbative renormalization of φ^4 theory on the half space $\mathbb{R}^+ \times \mathbb{R}^3$ with flow equations”. In: *J. Math. Phys.* 63 (2022).
- [65] M. Borji and Ch. Kopper. “The surface counter-terms of the φ^4 theory on the half space $\mathbb{R}^+ \times \mathbb{R}^3$ ”. In: *J. Math. Phys.* 65 (2024).
- [66] Ch. Kopper and F. Meunier. “Large Momentum bounds from the Flow Equations”. In: *Ann. Henri Poincaré*. 3 (2002), pp. 435–449.
- [67] A.D. Sokal. “An alternate constructive approach to the ϕ^4_3 quantum field theory, and a possible destructive approach to ϕ^4_4 ”. In: *Ann. Henri Poincaré* 37 (1982), pp. 317–398.
- [68] R. Griffiths. “Rigorous Results for Ising Ferromagnets of Arbitrary Spin”. In: *J. Math. Phys.* 10 (1969).
- [69] R. Griffiths and B. Simon. “The $(\Phi_2)^4$ Field Theory as a Classical Ising Model”. In: *Commun. Math. Phys.* 33 (1973), pp. 145–164.
- [70] J. Glimm and A. Jaffe. “Absolute bounds on vertices and couplings”. In: *Ann. Henri Poincaré A* 22 (1975), pp. 97–107.
- [71] J. Glimm and A. Jaffe. “Particles and scaling for lattice fields and Ising models”. In: *Commun. Math. Phys.* 51 (1976), pp. 1–13.
- [72] R. Fernández, J. Fröhlich, and A.D. Sokal. *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*. Springer-Verlag Berlin Heidelberg New York, 1992.
- [73] O.A. McBryan. “Existence of the critical point in ϕ^4 -field theory”. In: *Commun. Math. Phys.* 51 (1976), pp. 97–105.
- [74] B. Simon. “Correlation inequalities and the decay of correlations in ferromagnets”. In: *Commun. Math. Phys.* 77 (1980), pp. 111–126.
- [75] J. Ginibre. “General formulation of Griffiths’ inequalities”. In: *Commun. Math. Phys.* 16 (1970), pp. 310–328.
- [76] B. Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton Series in Physics (Princeton University Press), 1974.

- [77] J. Fröhlich, B. Simon, and T. Spencer. “Infrared bounds, phase transitions and continuous symmetry breaking”. In: *Commun. Math. Phys.* 50 (1976), pp. 79–95.
- [78] M.C. Reed. “Functional analysis and probability theory”. In: *Constructive Quantum Field Theory*. Ed. by G. Velo and A. Wightman. Vol. 25. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 2–43.
- [79] Ch. Kopper and P. Wang. “Triviality proof for mean-field φ_4^4 -theories”. In: *J. Math. Phys.* 66 (2025).
- [80] G. Felder. “Renormalization group in the local potential approximation”. In: *Commun. Math. Phys.* 111 (1987), pp. 101–121.
- [81] H.E. Stanley. “Dependence of Critical Properties on Dimensionality of Spins”. In: *Phys. Rev. Lett.* 20 (1968), pp. 589–592.
- [82] H.E. Stanley. “Spherical Model as the Limit of Infinite Spin Dimensionality”. In: *Phys. Rev.* 176 (1968), pp. 718–722.
- [83] T.H. Berlin and M. Kac. “The Spherical Model of a Ferromagnet”. In: *Phys. Rev.* 86 (1952), pp. 821–835.
- [84] M. Kac and C.J. Thompson. “Spherical model and the infinite spin dimensionality limit”. In: *Phys. Norv.* 5 (1971), pp. 163–168.
- [85] M.V. Shcherbina. “Spherical limit of n-vector correlations”. In: *Theor. Math. Phys.* 77 (1988), pp. 1323–1331.
- [86] J. Gough and J.V. Pulé. “The spherical model and Bose-Einstein condensation”. In: *Helv. Phys. Acta* 66 (1993), pp. 17–52.
- [87] D.J. Amit. *Field Theory, The Renormalization Group and Critical Phenomena*. McGraw-Hill, 1978. Chap. 6, 8.
- [88] M. Moshe and J. Zinn-Justin. “Quantum field theory in the large N limit: a review”. In: *Physics Reports* 385 (2003), pp. 69–228.
- [89] J. Glimm. “Boson fields with the $:\phi^4:$ Interaction in three dimensions”. In: *Comm. Math. Phys.* 10 (1968), pp. 1–47.
- [90] J.S. Feldman and K. Osterwalder. “The Wightman Axioms and the Mass Gap for Weakly Coupled $(\varphi^4)_3$ Quantum Field Theories”. In: *Ann. Phys.* 97 (1975), pp. 80–135.
- [91] J. Magnen and R. Sénéor. “The infinite volume limit of φ_3^4 model”. In: *Annales de l’I. H. P.* 24 (1976), pp. 95–159.
- [92] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. 2nd ed. Springer New York, 1987. Chap. 8.
- [93] P.G. Appleby, B.R. Duffy, and R.W. Ogden. “On the classification of isotropic tensors”. In: *Glasgow Mathematical Journal* 29.2 (1987), pp. 185–196.
- [94] J.F. Dyson. “Existence of a Phase-Transition in a One-Dimensional Ising Ferromagnet”. In: *Commun. Math. Phys.* 12 (1969), pp. 91–107.
- [95] G. Zumbach. “The renormalization group in the local potential approximation and its applications to the $O(N)$ model”. In: *Nucl. Phys. B* 413 (1994), pp. 754–770.

- [96] G. Gallavotti. “Some aspects of the renormalization problems in statistical mechanics and field theory”. In: *Memorie dell’Accademia Nazionale dei Lincei* 15 (1978), pp. 23–59.
- [97] H. Robbins. “A Remark on Stirling’s Formula”. In: *Am. Math. Mon.* 62 (1955), pp. 26–29.
- [98] Ch. Kopper and P. Wang. “Triviality versus perturbation theory: an analysis for mean-field φ_4^4 -theories”. In: *arXiv preprint: 2511.04509 [math-ph]* (2025).
- [99] L. Comtet. *Advanced Combinatorics*. 1970. Chap. 3, pp. 134–135.
- [100] I.J. Good. “The number of orderings of n candidates when ties are permitted”. In: *Fibonacci Q.* 13 (1975), pp. 11–18.
- [101] R. Sprugnoli. “Riordan arrays and combinatorial sums”. In: *Discrete Math.* 132 (1994), pp. 267–290.
- [102] O.A. Gross. “Preferential Arrangements”. In: *Am. Math. Mon.* 69 (1962), pp. 4–8.
- [103] N. Pippenger. “The hypercube of resistors, asymptotic expansions, and preferential arrangements”. In: *Mathematics Magazine* 83 (2010), pp. 331–346.
- [104] E. Picard. “Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives”. In: *J. Math. Pures et Appl.* § (1890), pp. 145–210.
- [105] S. Banach. “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales”. In: *Fund. Math.* 3 (1922), pp. 133–181.
- [106] R. Bauerschmidt, D.C. Brydges, and G. Slade. *Introduction to a Renormalisation Group Method*. Lecture Notes in Mathematics. Springer Singapore, 2019.
- [107] D.V. Widder. *The Laplace Transform*. Princeton U.P., 1941. Chap. 2.
- [108] G.N. Watson. “A theory of asymptotic series”. In: *Philos. Trans. Soc. London, Ser A* 211 (1912), pp. 279–313.
- [109] F. Nevanlinna. In: *Ann. Acad. Sci. Fenn. Ser. A* 12 (1918-1919).
- [110] A.D. Sokal. “An improvement of Watson’s theorem on Borel summability”. In: *J. Math. Phys.* 21 (1980), pp. 261–263.
- [111] K. Gawedzki and A. Kupiainen. “Nontrivial Continuum Limit of a ϕ^4 in Four-dimensions Model With Negative Coupling Constant”. In: *Nucl. Phys. B* 257 (1985), pp. 474–504.
- [112] C.F. Faà di Bruno. “Note sur une nouvelle formule de calcul différentiel”. In: *Quart. J. Math.* 1 (1857), pp. 359–360.
- [113] H.W. Gould. *Combinatorial Identities: Table I: Intermediate Techniques for Summing Finite Series*. Ed. by J. Quaintance. Vol. 4. From seven unpublished manuscript notebooks (1945-1990). Compiled by Quaintance J. 2010.
- [114] D. Birmajer, J.B. Gil, and M. Weiner. “Convolutions of Tribonacci, Fuss-Catalan, and Motzkin sequences”. In: *Fibonacci Quart.* 52 (2014), pp. 54–60.

Titre : Analyse perturbative et non-perturbative des équations du groupe de renormalisation

Mots clés : Théorie quantique des champs, Trivialité quantique, Equations du groupe de renormalisation, Théorie des perturbations, Champ moyen

Résumé : L'objectif de cette thèse est d'analyser de manière non-perturbative et perturbative les équations de flot du groupe de renormalisation. On s'intéresse plus particulièrement à la théorie scalaire réelle φ^4 massive en dimensions quatre sous l'approximation de champ moyen. Dans un premier moment, nous nous sommes intéressés au problème de la trivialité de cette théorie en champ moyen. A l'aide des équations de flot non-perturbatives en champ moyen, nous avons construit des solutions triviales en champ moyen pour des couplages nus quelconques. Nous avons fait une hypothèse technique d'une masse nulle dans le propagateur pour simplifier l'étude. L'unicité de cette solution pour des conditions de bord en champ moyen fixées est démontrée grâce à une formulation intégrale des équations de flot en champ moyen. Ensuite nous avons étudié le modèle $O(N)$, dans lequel le champ φ possède $N \geq 1$ composantes scalaires. Nous avons à nouveau démontré

la trivialité de la théorie en suivant notre méthode basée sur les équations de flot en champ moyen. En outre, nous avons relaxé l'hypothèse technique de masse nulle dans le propagateur régularisé dans l'espace des impulsions, nous avons établi que le résultat de trivialité reste vrai. Dans un second moment, nous avons confronté nos solutions triviales avec la théorie des perturbations en champ moyen. On a en particulier établi que sous une régularisation ultraviolette, nos solutions construites précédemment peuvent s'écrire comme une série formelle perturbative en un couplage renormalisé convenablement choisi. Nous montrons d'abord que la transformée de Borel de cette série perturbative renormalisée en champ moyen existe localement et que cette série est asymptotique à la théorie non-perturbative en champ moyen. Nous en déduisons que cette théorie perturbative régularisée et renormalisée en champ moyen est sommable au sens de Borel.

Title : Perturbative and non-perturbative analysis of the renormalization group equations

Keywords : Quantum Field Theory, Quantum triviality, Renormalization group equations, Perturbation theory, Mean Field

Abstract : This thesis focuses on a non-perturbative and perturbative analysis of the renormalization group flow equations. We analyze the real scalar massive φ^4 -theory in four dimensions in the mean-field approximation. First we tackled the triviality problem of the mean-field theory. Using the mean-field non-perturbative flow equations, we constructed, for arbitrary bare couplings, mean-field trivial solutions. In order to simplify the study, we made the technical assumption of a zero mass propagator. To prove the uniqueness of the mean-solutions for fixed mean-field boundary conditions, we derived an integral formulation of the mean-field flow equations. Then, we used our method to analyze the $O(N)$ -model in which the field φ has $N \geq 1$ scalar components. Again we pro-

ved the triviality result. Then, we dropped our technical assumption on the mass of the regularized propagator in momentum space, it turns out that the triviality result remains valid. In a second part, we confronted our trivial solutions to perturbation theory. We proved that under an ultraviolet regularization, our mean-field solutions have a perturbative expansion in a suitably chosen renormalized coupling. We show that the local Borel transform of the corresponding regularized renormalized mean-field perturbation theory exists and is asymptotic to the non-perturbative mean-field theory. Finally we prove the local Borel summability of the regularized renormalized mean-field perturbation theory.